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# A PERSPECTIVE ON METASURFACES, CIRCUITS, HOLOGRAMS AND INVISIBILITY 

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#### Abstract

Metamaterials are artificial materials, made by microscopic unit cells and projected to exhibit specific macroscopic properties, e.g. they can be designed in order to show a negative refractive index or a superluminal wave propagation. During the last decade, the interest in electromagnetic metamaterials has been grown because of their possible applications, such as for antennas, transmission lines, lenses, cloaking devices etcetera. In this work we deal specially with metasurfaces, i.e. thin artificial screens or "2D metamaterials". In particular, we analyze how to express the Huygens' Principle and the Boundary Conditions using the ElectroMagnetic Potentials, also considering the relativistic case. Starting from the Boundary Conditions we derive a circuit model for the project of a screen whose permittivity $\varepsilon$ and permeability $\mu$ are assigned. In the last chapters we wonder about the possibility of using metasurfaces in order to realize a holographic television or a hypothetical invisibility cloak.


Keywords: metamaterial; metasurface; holograms; invisibility cloak; Electromagnetic; Boundary Conditions; potential; Huygens principle; screen; circuit

## Sommario

I metamateriali sono materiali artificiali, costituiti da più elementi unitari microscopici e progettati per esibire specifiche proprietà macroscopiche. Ad esempio, essi possono essere caratterizzati da un indice di rifrazione negativo o da una velocità di propagazione delle onde superluminale. Nell'ultimo decennio, i metamateriali elettromagnetici hanno riscosso un interesse crescente, dovuto alle loro applicazioni per la realizzazione di antenne, linee di trasmissione, lenti, dispositivi di occultamento etc. In questo lavoro si tratterà soprattutto di metasuperfici, cioè di schermi sottili artificiali o "metamateriali 2D". In particolare, si analizzerà come riformulare il principio di Huygens e le Condizioni al Contorno usando i potenziali elettromagnetici, trattando anche il caso relativistico. A partire dalle Condizioni al Contorno si deriva un modello circuitale per il progetto di uno schermo con permittività $\varepsilon$ e permeabilità $\mu$ assegnate. Negli ultimi capitoli si considera la possibilità di usare le metasuperfici per la realizzazione di televisori olografici e per un ipotetico mantello dell'invisibilità.

Parole chiave: metamateriali; metasuperfici; ologrammi; invisibilità; elettromagnetismo; Condizioni al Contorno; potenziale; potenziali elettromagnetici; principio di Huygens; schermo; circuiti

## Acknowledgements

No act of kindness, no matter how small, is ever wasted.

Aesop, VI century b.C.
Scientific research is often a personal activity, like the study. One more time, it is quite probable that just few people truly know the contents of this thesis, at least now. However, many people have helped me, in different ways, maybe without being aware of that, even just with a thought or a kind act. Hence I wish to thank my family and all my friends, for their support and love, especially during the last frantic months.

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Have a nice reading!
Carlo Andrea Gonano,
Milan, Italy, $22^{\text {nd }}$ december 2015

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## Introduction

In a University we are especially bound to recognise not only the unity of science itself, but the communion of the workers in science. [...] We cannot, therefore, do better than improve the shining hour in helping forward the cross-fertilization of the sciences.
J.C. Maxwell, "The Telephone", Nature, 1878

## Science is one, but multiform

I started to actively study Electromagnetics about four year ago, then during the PhD course I focused on metamaterials and their paradoxes. I was soon interested by this topic, both for the unusual phenomena and the fantastic applications, like superluminal propagation effects and invisibility.

Who is writing has attended to Aeronautical Engineering, and perhaps this shift to the Electrical one could appear strange, but do not be too surprised. Think to an aircraft, to a car or to a mobile phone: all these objects are the outcomes of the integration among different systems and skills, ranging from Fluid-dynamics to Mechanics, from Thermo-dynamics to Electronics and so on. In a certain way, modern Engineering itself is an example of the benefits deriving from the collaboration among many knowledge areas, but it is not the only one. Also Medicine is a good example of multidisciplinary framework, with all its complexity, extended ramification and applications.

In the years I have become aware of the need, as well as the beauty, of the cross-fertilization of sciences. As a matter of fact, a different perspective can be truly helpful in the solution of problems looking each other distant. For example, let us consider James Clerk Maxwell: during his life this Scottish scientist worked on a multitude of subjects, ranging for Saturn's rings to automatic Control Theory, from Kinetic Theory of gases to color perception, arriving to unify the laws of Optics and Electromagnetics.

In brief, sometimes it could be useful to let our thoughts free to fly high, embracing many topics with sight, discovering their interconnections and exploring new horizons. For that we need to be open to curiosity and wonder: imagination and passion are extremely important in order to create something beautiful, not only in a scientific context.

## We cannot know everything

In theory, for a sufficient understanding of the EM metamaterial topic one should be already familiar with a plurality of subjects, for example: Circuit theory, Antennas and wave propagation, Optics and constitutive relation for EM materials, Electronics, Numerical Electromagnetic simulators, Special Relativity Theory, Quantum Mechanics, Photonics, Plasmonics, etcetera. Unfortunately, today the metamaterial subject seems to be yet too specialized and it is quite difficult to find texts (book, articles etc.) clear and accessible even for a justgraduated engineer.

Anyway, we cannot know everything, and we cannot even read everything, since our time and energy are limited. As a matter of fact, the research work consists also in the choice of which papers should be considered and which instead have to be left out, though these last should be not disdained. It could be they concern topics different from ours, maybe they are too specific, or more simply we must honestly admit we are not able to fully understand them. However, often I have observed that it is quite easier and faster to solve problems by your own, using pen and paper and trying to give meaning to equations.

Since it is not possible to read all the existing works about a topic, I cannot exclude that some of the results derived by me are already known, maybe in another form, or that somebody has anticipated me. I do not claim any priority, since I cannot be sure about that, but I state the autonomy of my research. In any case, I have to acknowledge that many of my results are based on the fundamental works of other people, on elegant articles and interesting ideas, which have been enlightened by mean of citations.

## Reading suggestions

As already anticipated, in order to appreciate this work the reader should be already familiar with many different topics. Even if I tried to write as clear as possible, I am afraid this doctoral thesis is some way more difficult to be read with respect to my previous one, i.e. my master thesis on N-Dimensional cross product. That difficulty is related either to the subject complexity and to the fact that chapters are quite entangled together. Moreover, I am not a native English speaker, thus I beg your pardon for the unavoidable errors and awkwardness.

When I was writing, I frequently asked myself: how much should I go deep in details? Should I demonstrate everything, starting from every basic principle in Electromagnetics, risking to be boring, or otherwise I had to proceed straight on rapidly, lightning, supposing all is obvious? As long as possible, I tried to find an equilibrium between those two extremes, either because my time was limited and because many results would turn out incomprehensible without effective explanations. Some concepts and equations are particularly important, so they are repeated in different places within this thesis. That fact is not casual: I cannot keep in memory all the equations I write by my own, and I cannot demand the reader to do it, or require that he/she could jump back tenths of pages to check every mathematical detail. All the figures have been inserted to make more pleasant and fluent both the reading and the comprehension of text. If not indicated differently, all the images have been created by the author.

As far as I am concerned, dear reader, I am not intentioned to waste your
time, thus I would suggest you to read just the parts looking interesting for you. If a chapter is boring or it sounds too complicated or abstract, skip it without any regret. Anyway the calculi remain available to whom they may interest.

## Contents outline

This thesis was written in hurry, too in hurry, in about three months. Actually, some results are truly recent and they could be not included in this work. Besides I had preferred to add some other examples and figures in order to better clarify some ideas, but once more, time works against me. Let us see now which are the contents of this thesis.

In the first part I introduce the concepts of metamaterial and metasurface, then we pass to a general analysis of the Huygens' Principle, applied in various contexts.

In the second part I deal with the Boundary Conditions for Electromagnetic fields. I know this topic could appear trivial to somebody, but during my research I often got into contradictions and paradoxes originated by the classic BCs written in terms of electric E and magnetic H fields. Just to bring few examples, the discontinuity for the electric field could require the introduction of magnetic currents, sometimes the magnetic field can become infinite, simple phenomena like the Volta Effect can be hardly described and so on.

I had already faced the same problems in a more severe form, while I was calculating the reflection and refraction coefficients between two different media. Finally, I decided I had to see clearly on that question and I resolved to derive autonomously the Boundary Conditions, adopting the Electromagnetic potential $V$ and $\vec{A}$ and getting a solid base in order to build up further theory and considerations. A posteriori, that approach has revealed to be very effective, either for interpreting physical quantities and for deriving Boundary Conditions in Space-Time, topic to which a specific chapter has been dedicated.

In the third part, starting from the Boundary Conditions, I illustrate the design of a circuit screen, whose properties like scattering, permittivity, permeability etc. can be assigned by the user.

The last chapters can be regarded as an intellectual challenge or speculation, a sort of imagineering. In those chapters we examine the possible applications of active metasurfaces in order to create a holographic television or a hypothetical invisibility cloak. I am quite skeptical about the practical realization of such wonderful devices, but I think that possibility should be seriously considered, without preconceptions. A scientist should be very careful before stating that something is "impossible".

In the conclusions I summarize the main achieved results and bring some final suggestions.

## Ars longa, vita brevis

Maybe this thesis could appear too much long for somebody: in some way, that is right, since the main achievements could be summarized in far fewer pages, perhaps around 30 or 40 . However, those results would be not demonstrated and explained, thus they would be much more hard to be understood. If in many cases I got deep into calculus details, it was just for a principle of rigour and completeness. Moreover, this could be my last scientific publications, therefore

I endeavoured to include at my best all the most interesting results I have developed during this last year.

Somebody else could say that this thesis does not contain many topics or details which would be worthy to be treated in depth: in some way, that also is right. As already said, this work was written rapidly: I had to make a choice on which parts I had to cut, on which examples could be included, on which figures could be prepared and which others I had to exclude.

During the last year I have also developed a Matlab program to simulate wave propagation through slab of material with assigned properties, in order to model a metamaterial through a succession of screens filled with currents. Unfortunately, the inclusion of those numerical simulations, the examples and their discussion would have required too much time.

In future, I hope to publish an updated version of this doctoral thesis. In the meanwhile, suggestions and reporting of obscure points are welcome.

Writing every chapter I tried to express myself as clearly as I can, but I am aware that this work could turn out to be quite difficult for a student in Engineering, and I am a bit sorry for that. Anyway, I hope this thesis could be inspiring for somebody, and in particular that the last chapters could arouse curiosity and passion for science and imagination.

Have a nice reading!
Carlo Andrea Gonano,
Milan, Italy, 22 ${ }^{\text {nd }}$ December 2015.

## Chapter 1

## Metamaterials and metasurfaces

The only way of discovering the limits of the possible is to venture a little way past them into the impossible.
A.C. Clarke, in Profiles of the Future: An
Enquiry into the Limits of the Possible, 1962

In this chapter we are going to give a rapid glance on the field of metamaterials (MTMs), on their basic concepts and applications. I cannot pretend to be exhaustive, since that's not the main task of this work.

## Definition of metamaterials

First of all, what is a "metamaterial" ? A possible definition 1 is:
A metamaterial is an artificial material explicitly designed to exhibit some specific properties. At a macroscopic scale, a metamaterial can be homogeneous (bulk material), while at a microscopic scale it is made by some ordered unit cells with a precise geometry.

That definition is quite general and can be applied to almost every artificial bulk material. Usually, a unit cell is regarded as "microscopic" if it is characteristic length $\Delta x$ is much smaller than the reference wavelength $\lambda_{0}$, so that homogenized material properties can be invoked.

$$
\begin{equation*}
\Delta x \ll \frac{\lambda_{0}}{2 \pi} \quad \text { condition for "small" unit cell } \tag{1.1}
\end{equation*}
$$

Usually the special properties of a metamaterial are determined by its microscopic structure rather than by its chemical composition.

## Metamaterials for different sectors

There are many kinds of MTM, for example:

- Acoustic metamaterials


Figure 1.1: Example of a generic periodic metamaterial. The unit cells are small, subwavelength, so that at macroscopic level the bulk metamaterial can be regarded as a homogeneous medium.

- Elastic metamaterials
- Thermal metamaterials
- Electromagnetic metamaterials
- etcetera...

In this thesis we are going to focus the attention on the Electromagnetic (EM) metamaterials. However, it is important to be aware that some ideas, methods and applications are not restricted just to one sector and but are common to many contexts. For example, you can have acoustic, elastic, thermal or electromagnetic waves propagating in a material, and you can describe those phenomena using the common concepts of impedance $\eta$, velocity $c$, wavelength $\lambda$, scattering, transmission and so on.

## EM metamaterial standard classification

In Electrical Engineering the metamaterials are usually classified on the basis of the real components of their permittivity $\varepsilon$ and permeability $\mu$ at different frequencies. Common dielectrics, like glass or ceramics, exhibit both positive


Figure 1.2: Standard $\varepsilon-\mu$ classification for electromagnetic materials.
Figure concept by R.W. Ziolkowsky and N. Engheta[1]
permittivity and permeability, $(\varepsilon>0, \mu>0)$, thus they are called Double

PoSitive (DPS) materials. At low frequencies metals and plasmas can exhibit negative permittivity $(\varepsilon<0, \mu>0)$, so they are called Epsilon NeGative (ENG) materials. Conversely, some gyrotropic magnetic materials are characterized by positive permittivity and negative permeability $(\varepsilon>0, \mu<0)$, thus they are called Mu NeGative (MNG) materials. Currently Double NeGative (DNG) materials are not found in Nature, but artificial ones have been designed and built exhibiting both negative permittivity and permeability $(\varepsilon<0, \mu<0)$. Such a DNG material allows negative refraction and backward propagation, though those effects could be achieved just at specific frequencies. Finally, metamaterials with relative permittivity and permeability close to zero are called Epsilon Near Zero (ENG) and Mu Near Zero (MNZ) respectively. This last kind of MTM is particularly interesting since they are characterized by a superluminal phase velocity $v_{\varphi}$.

Here we must specify that the $\varepsilon-\mu$ MTM classification is not a universal one, but anyway it is simple and effective in the many practical cases.

### 1.1 Very Brief Hystorical notes

In a certain way, the first optical MTMs ever realized are the stained glass windows of the $13^{\text {th }}$ century cathedrals, like Notre-Dame and La Sainte-Chapel in Paris[2]. In fact, the beautiful colours of the glass were originated by various nanometric, sub-wavelength metallic inclusions. However, the concept of artificial electromagnetic materials dates back just to 1898, when Jagadis Chunder Bose constructed a chiral material, made by twisted elements, which allowed to change the polarization of an incident microwave. In 1914 Karl Ferdinand Lindman worked on another "artificial" chiral media, using small wire helices in a host medium; the orientation of those inclusions had to be random in order to guarantee the isotropy of the bulk material. In the following decades the idea of controlling the propagation of EM waves in the materials has been separately investigated by many research group with different purposes. For example, in the 1950's and 1960's, Kock realized some artificial dielectric microwave lenses for satellite applications, actually tailoring the permittivity of the material, while in the 1980's the research in artificial EM materials was driven by the interest for the realization of stealth airplanes.

In 1968 the Russian physicist Victor Veselago published a theoretical work 3] in which he considered the possibility of a substance with both negative permeability and permittivity ( $\mu<0, \varepsilon<0$ ). Veselago predicted that such a material would have had paradoxical properties, allowing negative refraction, reversed Doppler Effect and reversed Cherenkov radiation. The work of Veselago has been almost ignored for many years, also because no materials with both $\mu<0$ and $\varepsilon<0$ were known to exist in Nature. Moreover, some of the anti-intuitive properties of such a Double Negative (DNG) material appeared to be contradictory or impossible.

In 2000 the first DNG material was realized and tested by David R. Smith and the group of the University of San Diego, California (U.S.C.D.) [4, 5]. The experiment was carried out at Boeing Phantom Works in Seattle, illuminating a MTM prism made of thin wires and Split Ring Resonators (SRR). The results confirmed the phenomenon of negative refraction in the microwave band, proving that the bulk material exhibited both negative permittivity and permeability.

After that experiment, the interest in MTMs has suddenly flourished, since the realization of EM materials with exotic and startling properties appeared not so difficult or impossible as before. Today the meta-material field is an active research area, with over 1500 scientific publications in $2015^{1}$.

### 1.2 Applications for EM metamaterials

In the last decade many research groups have explored the application of the metamaterials for various EM problems. Here we report just some hints.

As highlighted by John B. Pendry et al. [6, 7, 5] an ideal DNG material could be used to build a perfect flat lens with negative index of refraction, allowing to overcome the diffraction limits of the usual positive-index media. Such a device can be helpful for biomedical purposes, like sub-wavelength imaging and bio-sensing. The concepts of negative refraction and back propagation have been applied also to the project of waveguides. As a matter of fact, a circuit meta-material can be exploited in order to achieve compensation for distortion and phase delay in a Transmission Line. That topic have been investigated by many teams, like the ones lead by C. Caloz and G. Eleftheriades, which have effectively worked on the circuit modelling and experimentation [8, (9) [10. Moreover, meta-materials are currently exploited for antennas applications. The Sievenpiper "mushroom" reflector can be actually regarded as a meta-surface [11, explicitly designed to behave as a frequency selective mirror with zero reflection phase. More generally, the MTM concept has been used for the project of various high-impedance surfaces, Artificial Magnetic Conductors (AMC) for flat low profile antennas etc [12, 13, 14]. In addition, ENZ materials can be used for the synchronization of phased arrays, allowing beam steering and leaky-wave radiation [15].

For the moment, we stop here. We must point out that citing all the groups, published works and applications is impossible, even if you limit the choice to the most serious and important ones. In fact, the fields of MTM is a relative recent research area and the horizon of possibilities has still to be explored.

In my personal opinion, I would say we cannot really predict all the future applications for metamaterials, simply because our imagination is limited. We should remember Hertz, who experimentally verified the existence of electromagnetic waves in 1887. When people asked him for which purpose they could be useful, he answered: "It's of no use whatsoever. This is just an experiment that proves Maestro Maxwell was right". Well, today we see that Hertz was quite modest!

As scientists, we should remain open-minded, taking into account many possibilities. I'm telling that here for a precise reason, since now we are going to deal with amazing, mind-blasting applications.

### 1.2.1 Paradoxical applications

Some kinds of metamaterial are characterized by unusual or paradoxical properties, or else they can give rise to counter-intuitive phenomena, not predicted by the standard optics or electromagnetic theory.

[^0]
## Negative refraction and back propagation

We have already mentioned the DNG material with both negative permittivity and permeability. An EM wave entering from vacuum in such a medium would experience negative refraction. Besides, inside the DNG material the light seems to propagate backward. Actually, those effects has been considered as contradictory until 2000, when the first DNG material was realized and experimented.

## Superluminal propagation

Since in a metamaterial you can tailor the relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$, you can also tailor the speed of light inside that medium. Therefore, a metamaterial endowed with $\varepsilon_{r}$ or $\mu_{r}$ close to zero, less than 1 , would be characterized by a superluminal phase velocity $v_{\varphi}=c_{0} / \sqrt{\mu_{r} \varepsilon_{r}}$. In other words, at certain frequencies the light appears to propagate in the medium faster then in vacuum: $v_{\varphi}>c_{0}$. Formally, a superluminal phase velocity does not imply a contradiction with Einstein's Special Relativity, because it is different from the signal velocity. More explicitly, the phase velocity is not a defined as a space travelled per unit of time, but rather as:

$$
\begin{equation*}
v_{\varphi}(\omega) \triangleq \frac{\omega}{k} \quad \text { phase velocity } \tag{1.2}
\end{equation*}
$$

where $\omega$ and $k$ are the pulsation and the wavenumber respectively.
Usually ENZ metamaterials $\left(0<\varepsilon_{r}<1\right)$ or MNZ ones $\left(0<\mu_{r}<1\right)$ allows superlumination propagation just at few specific frequencies, in a very narrow band. However, if $\varepsilon_{r}$ and $\mu_{r}$ are dispersionless on a wide band, i.e. they are constant with frequency, then also the group velocity $v_{g}$ turns out be superluminal. Again, we remind that the group velocity is not the signal one, it is not defined as a space travelled per unit of time, but as:

$$
\begin{equation*}
v_{g}(\omega) \triangleq \frac{\partial \omega}{\partial k} \quad \text { group velocity } \tag{1.3}
\end{equation*}
$$

It should be noticed that the group of S. Hrabar has realized a MTM-inspired transmission lines, achieving superluminal broadband wave propagation [16, [17, 18, [19, thus exhibiting both $v_{\varphi}>c_{0}$ and $v_{g}>c_{0}$. For that purpose the transmission line was shunted with negative capacitor, so active elements were used.

In my opinion, the questions related to superlumination propagation and to its consistency with Relativity are extremely interesting, though unfortunately here I have not enough. . . "space-time" to analyse them in detail. However, the superluminal issue will be partially recovered in the last chapters, where we shall explore some applications for ENZ and MNZ metamaterials.

## Cloaking devices

Probably one of the most amazing and intriguing applications for metamaterials consists in their use for the project of cloaking devices. In principle, by mean of specific MTM you can construct a shell or a "mantle" capable to effectively conceal an object, by significantly reducing its scattering. In other words, such a cloaking device could be used to make an object "invisible" on a certain frequency band (not necessarily in the visible-light spectrum). Depending on the design
approach, the "invisibility" coating can made by different metamaterials, for example DNG ones $\left(\varepsilon_{r}<0, \mu_{r}<0\right)$ or else "superluminal " ones $\left(0 \leq \varepsilon_{r} \mu_{r}<1\right)$. Though that fantastic application seems plausible just in a Science-Fiction context, the invisibility quest is performed by many research teams, like the ones lead by J.B. Pendry, G. Eleftheriades, N. Engheta, A. Alú and others (see also chapter 14). It should be noticed that cloaking prototypes have been actually built and tested, though currently they work well just at specific frequencies.

The invisibility cloak problem will be explored more in detail in chapter 14 , which is entirely dedicated to it.

### 1.2.2 Technical difficulties

One of the main problems related to the EM metamaterials is their actual construction. A first difficulty consists in the fact that each constitutive unit cell should be much smaller than the operating wavelength $\lambda_{0}$, and that could be a severe limit at high frequency. For example, in the microwave band the frequency $f$ is in the $0.3 \div 300 \mathrm{GHZ}$ range, so the wavelength $\lambda_{0}=c_{0} / f$ can vary between 1 millimeter and 1 meter. That implies that a MTM operating at 300 GHZ should be composed by unit cell much smaller than 1 mm , and that could be a first technical problem. Obviously, at higher frequencies, e.g. in the visible spectrum, the unit cell's size should further decrease.

Another problem is that passive MTM are usually lossy and dispersive. In other words, they absorb power because of ohmic losses and they exhibit their particular features just in a narrow frequency band. As matter of fact, the first metamaterials were realized by assembling thin metallic wires and rings and they worked in proximity of resonant conditions, thus experiencing high sensibility to the selected frequency. Currently the metamaterial engineering is focusing on the use of active elements, like power-feed circuit devices, in order to obtain high performances on a wide frequency band. However, also the realization of active elements is characterized by technical limits, related for example to the maximum operating frequency and to the minimum size. Moreover, usually active elements are more expensive than passive ones, thus in some cases their mass production could result cost-prohibitive.

## Modelling problems

Beyond those technological issues, in the project of metamaterials various theoretical and modelling problems can arise. If they are treated superficially, they could bring to contradictory results. Without entering into details, we can mention two of those problems.

- Circuits VS antennas: usually the MTM unit cells are modelled or designed by mean of circuit elements, like resistors, capacitors, inductors etc. Yet, the lumped-element circuit theory can be rigorously applied just under specific hypotheses. In particular, the electric field has to be approximated as conservative and radiation is usually neglected. Both those conditions are not satisfied if you are dealing with the propagation of an EM wave which interacts with the circuit network.
- Micro VS macro: as already said, the macroscopic properties of a MTM are due mainly to its microscopic (sub-wavelength) structure rather than
to its chemical composition. Depending on the scale, different models are exploited in order to describe the material. For example, at a microscopic scale the material looks composed by discrete elements and Maxwell's Equations in vacuum are used, while at macroscopic scale the material is regarded as a homogeneous medium and hence Maxwell's equations in matter are used. Reconciling those two approach is not always an easy task, since quantities like polarization $P$ and magnetization $M$ can take different values depending on the adopted model and definition 20.

Last but not least, the interpretations of some theoretical solutions and actual phenomena are currently debated, like causality for backward propagation, consistency of superluminal effects with Relativity etc.

### 1.3 Metasurfaces

In this framework we shall focus our attention on a particular class of metamaterials: the meta-surfaces. Formally, a metasurface can be defined as a 2D metamaterial, made by a single layer of unit cells. In practice, it can be regarded as an artificial thin screen, whose width $\Delta x$ is much smaller than the operating wavelength $\lambda_{0}$.

The main advantage of metasurfaces is that they are thinner (so less bulky) and easier to build with respect to 3D MTMs. For example, a metasurface antenna reflector can be printed directly on a substrate, while to build a 3D MTM, thus assembling all the inclusions in the assigned geometry, is usually a more complicated job. Moreover, metasurfaces are suitable for local wave control, like beam steering, selective reflection and transmission and other antenna applications.

In this work we start analysing the problem of how to describe consistently the Boundary Conditions for a generic metasurface. Then we will consider its desired behaviour, hence on its scattering properties, in order to project the appropriate unit circuit cell. Finally, we show examples and possible applications for metasurfaces, highlighting some amazing consequences and paradoxes.

## Chapter 2

## Huygens' principle: mapping a 3D field on a 2D surface

O God, I could be bounded in a nutshell and count myself a king of infinite space. .

$$
\text { W.Shakespeare, Hamlet, Act II, scene } 2
$$

In this chapter we are going to analyze the so called Huygens-Fresnel Principle, showing that in reality it can be extended to describe a large variety of phenomena.

### 2.1 The Huygens-Fresnel principle

In 1690 the Dutch scientist Christiaan Huygens published its Treatise on the light [21, [22] in which he advanced the hypothesis of the wave nature of light. In the same book he studied the interference of the propagating waves and formulated its famous principle, which can be rephrased in this way:

Each point on a primary wavefront can be considered to be a new source of a secondary spherical wave and a secondary wavefront can be constructed as the envelope of these secondary spherical waves

In 1816 Augustin J. Fresnel showed that the Huygens' Principle was coherent with diffraction effects, while the production of regressive waves was forbidden by the destructive interference [23, 24]. In 1882 Gustav Kirchhoff rephrased the principle in mathematical form, using it to describe scalar wave propagation [25, 26 ,

Tipically, the Huygens' Principle can used to calculated the new wave front after a reflection or refraction. For example, it can be used to describe the diffraction produced by an obstacle or by an aperture.

### 2.1.1 An equivalence principle

Now, the Huygens' Principle can be considered as an equivalence principle, since it states that two different phenomena are equivalent or, in other words, they cannot be distinguished.

Imagine, for example, that there is a point-source of spherical acoustic waves and that you are hearing the sound at some distance $r$.

According to the Huygens' Principle, since every point of the wave front can be considered as a source of spherical waves itself, the original point-source can be substituted by a spherical surface of distributed acoustic sources. Thus, at some distance $r>R$, the fields produced by the point-source and by the surface sources are exactly the same.

### 2.2 A general Equivalence Principle for the fields

Now we are going to see how to extend the Huygens' Principle, formulating a general Equivalence Principle which has a larger validity and that can be applied also to stationary fields. Before starting to use mathematics, let us try to clarify this point.

Usually, fields can extent or propagate in the space and they are produced by some sources. For example, the gravitational field is generated by the masses ${ }^{1}$ and it can spread also in empty space. The music of a violin, so an acoustic field, is originated by the vibrations of the strings. A temperature field in an object can be produced by a source of heat, and so on.

It should be noticed that distant sources cannot interact directly, but they usually need the field to transmit the information far away. You can see the light emitted by the Sun, though you are not on the Sun. The oceans feel the Moon's gravity, which gives rise to the tides, but the Moon is not in the oceans' water. In other words, we cannot sense a source at distance, but we can sense the field produced by it. The information has to travel across the space.

### 2.2.1 Relation among fields and sources

Let us now consider a domain $\Omega_{1}$, containing some sources $\mathbf{J}_{1}$ inside it. Now we want to map the sources $\mathbf{J}_{1}$ on the boundary $\partial \Omega$ of the domain itself, in such a way that the external field remains unchanged. In other words, we want to find an equivalent distribution of sources on the boundary.

The basic procedure is:

1. divide the whole space $\Omega_{\infty}$ in two different domain $\Omega_{1}$ and $\Omega_{2}$
2. split the whole field $\mathbf{f}$ in two fields $\mathbf{f}_{1}$ and $\mathbf{f}_{2}$ on those two domains.
3. identify the source distributions $\mathbf{J}_{1}$ and $\mathbf{J}_{2}$ on the domains.
4. map sources $\mathbf{J}_{1}$ and field $\mathbf{f}_{1}$ on the boundary $\partial \Omega$ separating the two domains

In general, if you know the field $\mathbf{f}$ on the whole space (or space-time), you can determine its source distribution applying some operators $S($.$) to the field itself.$

$$
\begin{equation*}
S(\mathbf{f})=\mathbf{J} \tag{2.1}
\end{equation*}
$$

The arrays $\mathbf{f}$ and $\mathbf{J}$ can contain one or more variables, no matter if they are scalar or vectorial.

[^1]
### 2.2.2 Properties of the operator $S$

The operator $S($.$) is a local operator. Given the field \mathbf{f}(\vec{x}, t)$ and its derivatives on a point (or event) ( $\vec{x}_{0}, t_{0}$ ), the operator $S($.$) tells us which is the local density$ $\mathbf{J}\left(\vec{x}_{0}, t_{0}\right)$ of the considered source.

Let us consider, for example, the gravitational field $\vec{g}$, which is generated by masses. If you know the field $\vec{g}(\vec{x}, t)$ you can determine the density of mass $\rho$ through the equation:

$$
\begin{equation*}
\vec{\nabla}^{T} \cdot \vec{g}=-4 \pi G \rho \tag{2.2}
\end{equation*}
$$

In this case, the field is $\mathbf{f}=\{\vec{g}\}$, the source is $\mathbf{J}=\{\rho\}$ while $S($.$) is proportional$ to the divergence $\vec{\nabla}^{T}($.$) .$

## Linearity and non-linearity

In general, the operator $S($.$) can be non-linear. However in many contexts the$ fields and the wave propagation can be described or effectively approximated in terms of linear equations, thus it's possible to apply the superposition principle. More explicitly, an operator $S($.$) is linear if it satisfy the conditions:$

$$
\begin{array}{lrl}
S\left(\mathbf{f}_{A}+\mathbf{f}_{B}\right)=S\left(\mathbf{f}_{A}\right)+S\left(\mathbf{f}_{B}\right) & \forall \mathbf{f}_{A}, \mathbf{f}_{B} & \text { additivity } \\
S(\alpha \mathbf{f})=\alpha S(\mathbf{f}) & \forall \mathbf{f}, \forall \alpha \in \mathcal{C} & \text { homogeneity } \tag{2.4}
\end{array}
$$

Thus, if $S($.$) is linear, then the global field \mathbf{f}$ produced by two or more different source distributions $\mathbf{J}_{A}$ and $\mathbf{J}_{B}$ is equal to the sum of their respective fields

$$
\begin{equation*}
S\left(\mathbf{f}_{A}+\mathbf{f}_{B}\right)=S\left(\mathbf{f}_{A}\right)+S\left(\mathbf{f}_{B}\right)=\mathbf{J}_{A}+\mathbf{J}_{B} \tag{2.5}
\end{equation*}
$$

More generally:

$$
\begin{equation*}
S\left(\sum_{i=1}^{N}\left(\alpha_{i} \mathbf{f}_{i}\right)\right)=\sum_{i=1}^{N}\left(\alpha_{i} S\left(\mathbf{f}_{i}\right)\right)=\sum_{i=1}^{N}\left(\alpha_{i} \mathbf{J}_{i}\right) \tag{2.6}
\end{equation*}
$$

Anyway we remark that $S($.$) can be also a non linear operator.$

## No fields, no sources

If there are no sources, i.e. $\mathbf{J}=0$, the equation 2.1 reduces to:

$$
\begin{equation*}
S(\mathbf{f})=0 \tag{2.7}
\end{equation*}
$$

That is a homogeneous equation, stating that the field is propagating or extending autonomously. In fact, even if there are no sources, the field can be different for zero.

For example, the homogeneous 1-D D'Alembert's equation looks:

$$
\begin{equation*}
\frac{1}{c^{2}} \frac{\partial^{2} \rho}{\partial t^{2}}-\frac{\partial^{2} \rho}{\partial x^{2}}=0 \tag{2.8}
\end{equation*}
$$

The general well-known solution is the sum of a progressive and a regressive wave:

$$
\begin{equation*}
\rho(x, t)=\rho_{+}(\Delta x-c \Delta t)+\rho_{-}(\Delta x+c \Delta t) \tag{2.9}
\end{equation*}
$$

In this case the field $\mathbf{f}$ is $\rho(x, t)$, while the operator is the D'Alembert's one:

$$
\begin{equation*}
S(.)=\frac{1}{c^{2}} \frac{\partial^{2}(.)}{\partial t^{2}}-\frac{\partial^{2}(.)}{\partial x^{2}}=\square(.) \tag{2.10}
\end{equation*}
$$

If the field $\mathbf{f}$ is zero, usually it is required that there are no source. This is an arbitrary hypothesis, but it is quite reasonable since a source which generates nothing makes little sense. So for the rest of this work we assume that:

$$
\begin{equation*}
S(0)=0 \quad \text { if the field is zero everywhere, then there are no sources } \tag{2.11}
\end{equation*}
$$

### 2.2.3 Dividing the domains: the belonging function

Suppose we know the field $\mathbf{f}$ on the domain $\Omega_{\infty}$, which is divided in two distinct subdomains $\Omega_{1}$ and $\Omega_{2}$ :

$$
\left\{\begin{array}{l}
\Omega_{\infty}=\Omega_{1} \cup \Omega_{2}  \tag{2.12}\\
\Omega_{1} \cap \Omega_{2}=\emptyset
\end{array} \quad\right. \text { division in two distinct parts }
$$

We want to express the global field $\mathbf{f}$ as the sum of two fields $\mathbf{f}_{1}$ and $\mathbf{f}_{2}$ which are null outside their respective domains. In order to do that, we can define a


Figure 2.1: Fields and sources on domains $\Omega_{1}$ and $\Omega_{2}$.
belonging function $\in_{\Omega}$ such that:

$$
\epsilon_{\Omega}(\vec{x})= \begin{cases}1 & \text { for } \vec{x} \in \Omega  \tag{2.13}\\ 0 & \text { for } \vec{x} \notin \Omega\end{cases}
$$

This function is equal to 1 inside a domain $\Omega$, while it is zero outside. On the boundary $\partial \Omega(t)$ the belonging function is discontinuous, but it could be placed equal to $1 / 2$ if desired. Since $\Omega_{1}$ and $\Omega_{2}$ are complementary domain, it holds $\epsilon_{\Omega_{1}}+\epsilon_{\Omega_{2}}=1$ and the field $\mathbf{f}$ can be expressed as:

$$
\begin{gather*}
\mathbf{f}=\epsilon_{\Omega_{1}} \mathbf{f}+\epsilon_{\Omega_{2}} \mathbf{f}  \tag{2.14}\\
\mathbf{f}=\mathbf{f}_{\mathbf{1}}+\mathbf{f}_{\mathbf{2}} \tag{2.15}
\end{gather*}
$$

where:

$$
\mathbf{f}_{\mathbf{1}}(\vec{x})=\epsilon_{\Omega_{1}} \mathbf{f}= \begin{cases}\mathbf{f}(\vec{x}) & \text { for } \vec{x} \in \Omega_{1}  \tag{2.16}\\ 0 & \text { for } \vec{x} \notin \Omega_{1}\end{cases}
$$

and analogous for $\mathbf{f}_{\mathbf{2}}$. So the function $\mathbf{f}_{\mathbf{1}}$ is zero outside $\Omega_{1}$, while it is equal to $\mathbf{f}$ inside.


Figure 2.2: Belonging function $\in_{\Omega}$ for a generic domain $\Omega$.

### 2.2.4 Mapping a 3D field on a 2D surface

Let us now suppose we are in the domain $\Omega_{2}$ and that we are measuring the field $\mathbf{f}$. Since we are outside the domain $\Omega_{1}$, we cannot know the field $\mathbf{f}_{1}$ and so we cannot distinguish $\mathbf{f}$ from $\mathbf{f}_{2}$. However, the sources $\mathbf{J}_{1}$ inside $\Omega_{1}$ can generate a field extending also in $\Omega_{2}$, and that can be detected.

The question is: can we univocally reconstruct the distribution of sources $\mathbf{J}_{1}$ in $\Omega_{1}$ even if we stand outside? As we are going to verify, the general answer is "no".

Supposing that there are no sources on the boundary $\partial \Omega$, we can write:

$$
\begin{cases}\epsilon_{\Omega_{1}} S(\mathbf{f})=\mathbf{J}_{1}(\vec{x}) & \text { source inside } \Omega_{1}  \tag{2.17}\\ \epsilon_{\Omega_{2}} S(\mathbf{f})=\mathbf{J}_{2}(\vec{x}) & \text { source inside } \Omega_{2}\end{cases}
$$

Summing the equations together, we obtain the global source distribution:

$$
\begin{equation*}
S(\mathbf{f})=\mathbf{J}_{1}(\vec{x})+\mathbf{J}_{2}(\vec{x})=\mathbf{J}(\vec{x}) \tag{2.18}
\end{equation*}
$$

Since $S($.$) is a local operator, if we apply it on \Omega_{2}$ it will not give us enough information about the source distribution $\mathbf{J}_{1}$ on $\Omega_{2}$, in fact:

$$
\begin{array}{r}
\mathbf{f}_{2}=\mathbf{f} \quad \forall \vec{x} \in \Omega_{2} \quad \Longrightarrow \\
S\left(\mathbf{f}_{2}\right)=S(\mathbf{f}) \quad \forall \vec{x} \in \Omega_{2} \tag{2.20}
\end{array}
$$

If we calculate the sources just on the basis of $\mathbf{f}_{2}$, we get:

$$
S\left(\mathbf{f}_{2}\right)=S\left(\epsilon_{\Omega_{2}} \mathbf{f}\right)=\left\{\begin{align*}
\in_{\Omega_{2}} S(\mathbf{f}) & =\mathbf{J}_{2}(\vec{x}) & & \text { for } \vec{x} \in \Omega_{2}  \tag{2.21}\\
\left.S\left(\in_{\Omega_{2}} \mathbf{f}\right)\right|_{\partial \Omega} & =\mathbf{J}_{1, \partial \Omega}(\vec{x}) & & \text { for } \vec{x} \in \partial \Omega \\
S(0) & =0 & & \text { for } \vec{x} \in \Omega_{1}
\end{align*}\right.
$$

We can notice that the sources $\mathbf{J}_{2}$ are unchanged, while the sources $\mathbf{J}_{1}$ have disappeared and they have been replaced by an equivalent distribution $\mathbf{J}_{1, \partial \Omega}$ on the surface $\partial \Omega$. So, outer from the domain $\Omega_{1}$, you cannot distinguished if the field is generated by the original, volumetric distribution of sources $\mathbf{J}_{1}$ or by some equivalent surface sources distribution $\mathbf{J}_{1, \partial \Omega}$.

In practice, we have just taken functions $\mathbf{J}_{1}$ and $\mathbf{f}_{1}$ defined on 3 D domain $\Omega_{1}$ and we have mapped them on a 2 D surface $\partial \Omega$.

In the following equations we summarize this result in mathematical form:

$$
\mathbf{f}(\vec{x})=\left\{\begin{array}{ll}
\mathbf{f}_{2}(\vec{x}) & \text { for } \vec{x} \in \Omega_{2}  \tag{2.22}\\
\mathbf{f}_{1}(\vec{x}) & \text { for } \vec{x} \in \Omega_{1}
\end{array} \quad \mathbf{f}_{\mathbf{2}}(\vec{x})= \begin{cases}\mathbf{f}_{2}(\vec{x}) & \text { for } \vec{x} \in \Omega_{2} \\
0 & \text { for } \vec{x} \in \Omega_{1}\end{cases}\right.
$$



Figure 2.3: Huygens' principle. The original configuration of sources $\mathbf{J}_{1}$ inside domain $\Omega_{1}$ can be replaced by an equivalent boundary distribution $\vec{J}_{1, \partial \Omega}$ such that field $\mathbf{f}$ is unchanged outside and null inside.

$$
S(\mathbf{f})=\left\{\begin{array}{ll}
\mathbf{J}_{2}(\vec{x}) & \text { for } \vec{x} \in \Omega_{2}  \tag{2.23}\\
\left.\frac{1}{2}\left(\mathbf{J}_{1}+\mathbf{J}_{2}\right)\right|_{\partial \Omega} & \text { for } \vec{x} \in \partial \Omega \\
\mathbf{J}_{1}(\vec{x}) & \text { for } \vec{x} \in \Omega_{1}
\end{array} \quad S\left(\mathbf{f}_{2}\right)= \begin{cases}\mathbf{J}_{2}(\vec{x}) & \text { for } \vec{x} \in \Omega_{2} \\
\mathbf{J}_{1, \partial \Omega} & \text { for } \vec{x} \in \partial \Omega \\
0 & \text { for } \vec{x} \in \Omega_{1}\end{cases}\right.
$$

Though this result is quite general, since it can applied to any kind of field, it has a conceptual importance rather than a numerical or computational one.

In fact the belonging functions $\epsilon_{\Omega_{i}}$ are discontinuous on the boundary $\partial \Omega$ and so they are formally not derivable. Since $S($.) can contain derivatives, the value of sources $J_{1, \partial \Omega}$ on the boundary can go to infinity.

In the following chapters we are going to see that's not a great problem, since it is sufficient to define a suitable surface density, avoiding infinities.

### 2.2.5 Fields Equivalence Principle

Now we can formulate the Equivalence Principle for the fields:
Let $\mathbf{f}$ be a generic field whose sources are $\mathbf{J}$.
Let $\mathbf{J}_{1}$ be the source distribution inside a domain $\Omega_{1}$.
The field produced by $\mathbf{J}_{1}$ outside the domain $\Omega_{1}$ can be replaced by an equivalent distribution of sources $\mathbf{J}_{1, \partial \Omega}$ on the boundary $\partial \Omega$, in such a way that the external field remains unchanged. More generally, the whole field $\mathbf{f}_{1}$ on $\Omega_{1}$ can be exactly mapped on the boundary $\partial \Omega$ through the sources $\mathbf{J}_{1, \partial \Omega}$, in such a way that the external field remains unchanged.

The Huygens' Principle can be considered as a particular case of this Equivalence Principle, since the former formally does not apply to stationary fields.

### 2.3 Planets, Holograms and Waves

As already stated, the Equivalence Principle can be interpreted as a way to map the information from a 3 D volume to a 2 D surface ${ }^{2}$. In other words, the

[^2]information travelling outward can be rephrased in terms of boundary conditions for the problem. In this section we are going to show some example of the Equivalence Principle.

### 2.3.1 Gravity - some equivalent spherical fields

Imagine there are three spherical planets, which have all the same mass $m$ and the same radius $R$, but have a different internal structure.

The $1^{\text {st }}$ planet has a uniform density, so it holds:

$$
\rho_{1}(r)= \begin{cases}\frac{3}{4 \pi} \frac{1}{R^{3}} m & \text { for } r<R  \tag{2.24}\\ 0 & \text { for } r>R\end{cases}
$$

where $r$ is the distance from the center of the planet.
The $2^{\text {nd }}$ planet has a virtual surface, since all its mass is lumped in the center, so the density is:

$$
\begin{equation*}
\rho_{2}(r)=m \delta_{N}(r) \tag{2.25}
\end{equation*}
$$

where $\delta_{N}(r)$ is the Dirac delta in N dimensions (here $N=3$ ).
Finally, the $3^{\text {rd }}$ planet is hollow and all its mass is uniformly distributed on the surface. The density is so zero everywhere except on the spherical shell (i.e. for $r=R$ ):

$$
\begin{equation*}
\rho_{3}(r)=\frac{1}{4 \pi R^{2}} m \delta_{1}(r-R) \tag{2.26}
\end{equation*}
$$

The gravitation field $\vec{g}$ produced by each planet must obey to Newton's Law:

$$
\begin{equation*}
\vec{\nabla}^{T} \cdot \vec{g}=-4 \pi G \rho \tag{2.27}
\end{equation*}
$$

Since we are dealing with systems characterized by a spherical symmetry, we can rewrite the same equation in scalar form, in function of the distance $r$ :

$$
\begin{align*}
& \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} g\right)=-4 \pi G \rho(r)  \tag{2.28}\\
& \frac{\partial}{\partial r}\left(r^{2} g\right)=-4 \pi G r^{2} \rho(r) \tag{2.29}
\end{align*}
$$

To this law, we add the condition that $g(r)$ goes to zero at infinite distance.

$$
\begin{equation*}
\lim _{r \rightarrow \infty}(g(r))=0 \tag{2.30}
\end{equation*}
$$

The fields $g(r)$ generated by the planets can be calculated integrating 2.29, so we get:

$$
\begin{align*}
& g_{1}(r)= \begin{cases}-\left(G m \frac{1}{R^{3}}\right) r & \text { for } r<R \\
-G m \frac{1}{r^{2}} & \text { for } r>R\end{cases}  \tag{2.31}\\
& g_{2}(r)=-G m \frac{1}{r^{2}} \quad \forall r  \tag{2.32}\\
& g_{3}(r)=\left\{\begin{array}{cc}
0 & \text { for } r<R \\
-G m \frac{1}{r^{2}} & \text { for } r>R
\end{array}\right. \tag{2.33}
\end{align*}
$$

## CHAPTER 2. HUYGENS' PRINCIPLE: MAPPING A 3D FIELD ON A 2D SURFACE15



Figure 2.4: Huygens' principle applied to gravity. (a) Homogeneous planet; (b) Lumped planet: its mass is concentrated in the core; (c) Hollow planet: its mass is distributed on the surface. The three planets have all the same mass and outside they generate the same gravitational field.


Figure 2.5: Gravitational fields $-g(r)$ for the three planets

We can notice that for $r>R$ the gravitation field $g(r)$ is identical for all the planets, notwithstanding their different internal structure. Moreover, as predicted by the extended Huygens' Principle, all the mass for the $1^{\text {st }}$ and $2^{\text {nd }}$ planets can be mapped on the surface, as actually happens for the hollow planet.

Obviously that just a thought experiment, since an hollow planet would be unstable.

### 2.3.2 Holography - a perspective on 3D images

If you look at a picture or at a standard television, you can see a 2-dimensional (2-D) image. That means you can see the framed objects and the environment just from a single point of view. In other words, you cannot see the object in its full 3-Dimensionality, even if you look at the picture from different angles. On the contrary, a holographic image is a 3-D image, which can display an object from different angles and viewpoints, as it was actually in front of the observer.

Formally, a hologram consists in the recording of a 3-D light field on a 2-D surface support, for example a photographic plate. The recording of a standard hologram requires the use of a laser beam in order to illuminate an object and to create an interference pattern on a photographic plate. Moreover, usually the term "hologram" refers to the record support rather than to the virtual image produced by it.

In this thesis I will use the words "hologram" and "holographic "in a broad sense, though a bit improper, in order to refer to 3D images mapped on a 2D
surface.
For a long time I have been fascinated by the holograms, though I could not fully understand how they worked. I asked myself how could it be possible to compress the information of a 3D space on a 2D. In the XIX ${ }^{\text {th }}$ century the mathematician Georg Cantor had already demonstrated that a segment, a square and a cube have all the same infinite number of points (!) 27, 28, but that phenomenon was still strange and interesting at my eyes.

Studying the Huygens' Principle and developing its extension, finally I understood that it was the same concept at the basis of the holography. In fact, in both the cases you are mapping a 3D field on a 2D surface, in such a way that the external field generated by the sources is the same. Let us try to clarify that idea.

If you can see an object, that happens because it emits or reflects lights and colors. Since the visible light is an ElectroMagnetic wave, the object can be regarded as a source of waves. Exploiting the Extend Huygens' Principle it is so possible to reproduce the 3D object's light-field by substituting it with some equivalent source distribution on a 2D surface. In a standard 2D television (TV), images are originated by an ensemble of pixels. Every pixel can be considered as a source of light. Its hue and intensity consists in a combination of three primary color, usually Red, Green or Blue (RGB code). Every pixel is so associated to a single, global color, which is independent from the point of view. For example, if you are looking to a yellow pixel, it will appear yellow also if you change your viewing angle. In fact, each pixel can vary the intensity and the hue of the emitted light, but it cannot change its direction.

In a hypothetical 3D TV or holographic television, instead, each "pixel " is associated to a different viewpoint (on the screen), so that its color can change depending on the observer's viewing angle. In other words, each pixel contains


Figure 2.6: Comparison between 2D and 3D TV. (a) In a standard 2D television each pixel is characterized by a colour which does not depend by the viewpoint. (b) In a 3D holographic television each pixel can exhibit different colours depending by the viewpoint.
a whole image of the environment as it was seen from the position of the pixel itself.

The idea of creating a holographic television is quite intriguing, even if its realization could reveal to be difficult or practically impossible. Here we have just given a glance on the world of holograms. In chapter 13 we are going to see
more in detail how to realize a meta-surface acting as a holographic television.

### 2.3.3 Electromagnetism - surface equivalence for E and H

In ElectroMagnetics the concept of "equivalence" is adopted in many contexts, thanks to its effectiveness and usefulness.

The Thevenin's Theorem itself can be regarded as an equivalence principle for circuits. In fact, it tells how to substitute a complex (linear) network with few elementary circuits, allowing to drastically simplify many problems. In this section we are going to summarize some Equivalence Theorems applied to EM fields and propagation. Let us notice that we shall not demonstrate them, since we are going to explore the whole problem more in depth in chapters 5,6 and 7 .

## Love's equivalence Principle

In 1901 A.E.H. Love published a first surface equivalence theorem for the EM propagation [29, 30], which currently is known as the "Love's equivalence principle" 31:

The fields outside an imaginary closed surface are obtained by placing, over the closed surface, suitable electric and magnetic current densities that satisfy the Boundary Conditions. The current densities are selected so that the fields inside the closed surface are zero and outside are equal to the radiation produced by the actual sources.

In his work, Love supposed the space was divided in two regions, $\Omega_{1}$ and $\Omega_{2}$ (our notation), and that all the EM sources were contained in $\Omega_{1}$. In Love's interpretation, the electric $\vec{E}$ and magnetic $\vec{H}$ fields are generated by electric $\vec{J}_{e}$ and "magnetic" $\vec{J}_{m}$ currents, which are so regarded as the EM sources. In

(a) Original source configuration

(b) Equivalent surface source

Figure 2.7: Love's equivalence principle. The original configuration of electric $\vec{J}_{e 1}$ and magnetic $\vec{J}_{m 1}$ currents inside domain $\Omega_{1}$ can be replaced by equivalent surface distributions $\vec{J}_{s, e 1}, \vec{J}_{s, m 1}$ such that inner fields $\vec{E}_{1}$ and $\vec{H}_{1}$ are null
order to calculate fields $\vec{E}_{2}$ and $\vec{H}_{2}$ radiated outside $\Omega_{1}$, Love demonstrated that it possible to find a distribution of surface currents $\vec{J}_{s, e 1}$ and $\vec{J}_{s, m 1}$ which is equivalent to the original one. In the new configuration fields $\vec{E}_{1}$ and $\vec{H}_{1}$ are null inside $\Omega_{1}$, while the surface current distributions can be calculated as:

$$
\begin{align*}
& \vec{J}_{s, e 1}=\vec{n}_{21} \times \vec{H}_{2}  \tag{2.34}\\
& \vec{J}_{s, m 1}=-\vec{n}_{21} \times \vec{E}_{2} \tag{2.35}
\end{align*}
$$

where $\vec{n}_{21}$ is the normal to the boundary, pointing out from region 1 to region 2 .
Usually the magnetic current is indicated with the symbol $\vec{M}$, but we decided to use another notation, since in ElectroMagnetics the $\vec{M}$ is already adopted to express the magnetization field, which is a different quantity.

Love discovered that it was possible also to formulate an internal equivalent problem, mapping the external fields $\vec{E}_{2}$ and $\vec{H}_{2}$ on the boundary and substituting them with some surface currents $\vec{J}_{s, e 2} \vec{J}_{s, m 2}$.

In the final configuration the external fields $\vec{E}_{2}$ and $\vec{H}_{2}$ are null, while the sources on the boundary are:

$$
\begin{align*}
\vec{J}_{s, e 2} & =-\vec{n}_{21} \times \vec{H}_{1}  \tag{2.36}\\
\vec{J}_{s, m 2} & =\vec{n}_{21} \times \vec{E}_{1} \tag{2.37}
\end{align*}
$$

The Love's Equivalence Principle is particularly helpful if you have to calculate the fields on a finite domain, since it "transforms" far fields in near ones and vice versa. For that reason, it is implicitly exploited in EM numerical simulators, mapping the fields impinging from the infinity on a finite surface.

## Schelkunoff's equivalence theorem

In 1911 H.M. Mac Donald refined the Love's theorem proving its validity for non-dissipative media 32, 33. In 1936 S.A. Schelkunoff published an elegant paper [33] in which he re-demonstrated the surface Equivalence Principle for the EM fields, showing its application to some antennas problems.

In his work Schelkunoff extended the Love's theorem, considering the case of dissipative media filling the two regions. He also explicitly derived the boundary conditions linking the fields on the two domains:

$$
\begin{align*}
\vec{J}_{s, e} & =\vec{n}_{21} \times\left(\vec{H}_{2}-\vec{H}_{1}\right)  \tag{2.38}\\
\vec{J}_{s, m} & =-\vec{n}_{21} \times\left(\vec{E}_{2}-\vec{E}_{1}\right) \tag{2.39}
\end{align*}
$$

Actually, the equations $\sqrt{2.38}, \sqrt{2.39}$ can be considered as a generalization of the Love's BCs 2.34 (2.36).

Let us notice that the currents $\vec{J}_{s, e}$ and $\vec{J}_{s, m}$ are different from zero just if fields $\vec{E}$ and $\vec{H}$ are discontinuous across the surface.

Schelkunoff observed that if the fields are required to be zero in a region (so $\vec{E}=0, \vec{H}=0$ ), then that region can be replaced by a Perfect Electric Conductor (PEC) or by a Perfect Magnetic Conductor (PMC). For a PEC the tangential component of the electric field $\vec{E}$ must vanish on the surface. Analogously, for a PMC the tangential component of the magnetic field $\vec{H}$ must zero on the surface. In order to enforce the discontinuities, a magnetic current distribution $\vec{J}_{s, m}$ can be placed on the PEC surface. Conversely, an electric current distribution $\vec{J}_{e, m}$ can be placed on the PMC surface.

In his paper Schelkunoff wrote Maxwell's Equations in differential form, making use of the "magnetic vector potential" $\vec{A}$ and the "auxiliary electric vector potential" $\vec{F}$. However, in the end he wrote the BCs just in terms of tangential electric and magnetic fields.

## Chapter 3

## Maxwell's Equations in free space

> This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself [...] is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field
> James Clerk Maxwell, A Dynamical Theory of the Electromagnetic Field, 1864

In this chapter we briefly report the Maxwell's Equations and show how to write them with the EM potentials. Obviously this is supposed to be just a short memorandum on the groundwork of Electromagnetism, since the reader is probably already acquainted with them.

For our purpose, we shall consider mainly the Maxwell's Eq.s in free-space, and not those in media. That is not a serious limit, since Maxwell's Eq.s must be valid at any scale, and the macroscopic properties of a material are owned to its microscopic structure 20.

### 3.1 Maxwell's Equations in free space

The celebrated Maxwell's Equations are the fundamental laws of the electromagnetic theory. In modern notation, the Maxwell's Eq.s in free-space can be written as:

$$
\begin{align*}
\vec{\nabla}^{T} \cdot \vec{E} & =\frac{\rho_{e}}{\varepsilon_{0}}  \tag{3.1}\\
\vec{\nabla} \times \vec{B} & =\mu_{0} \vec{J}_{e}+\mu_{0} \varepsilon_{0} \frac{\partial \vec{E}}{\partial t}  \tag{3.2}\\
\vec{\nabla}^{T} \cdot \vec{B} & =0  \tag{3.3}\\
\vec{\nabla} \times \vec{E} & =-\frac{\partial \vec{B}}{\partial t} \tag{3.4}
\end{align*}
$$

where:

- $\vec{E}$ is the electric field
- $\vec{B}$ is the magnetic (induction) field
- $\rho_{e}$ is the electric charge density
- $\vec{J}_{e}$ is the electric current per unit of surface
- $\varepsilon_{0}$ is the free space electric permittivity
- $\mu_{0}$ is the free space magnetic permeability

The same set can be rearranged in a synoptic way, dividing the equations which contains the sources $\rho_{e}$ and $\vec{J}_{e}$ from the others. For sake of simplicity, we call equations (3.1), 3.2 the Wave set or Source set, while equations (3.3), 3.4) constitute the Faraday set.

$$
\left\{\begin{array} { l } 
{ \vec { \nabla } ^ { T } \cdot \vec { E } = \frac { \rho _ { e } } { \varepsilon _ { 0 } } }  \tag{3.5}\\
{ \vec { \nabla } \times \vec { B } = \mu _ { 0 } \vec { J } _ { e } + \mu _ { 0 } \varepsilon _ { 0 } \frac { \partial \vec { E } } { \partial t } }
\end{array} \quad \left\{\begin{array}{l}
\vec{\nabla}^{T} \cdot \vec{B}=0 \\
\vec{\nabla} \times \vec{E}=-\frac{\partial \vec{B}}{\partial t}
\end{array}\right.\right.
$$

### 3.2 Maxwell's Equations with the EM potentials

Equations (3.3), (3.4), i.e. the Faraday set, are automatically satisfied if we introduce the EM potentials $V$ and $\vec{A}$, such that:

$$
\left\{\begin{array}{l}
\vec{B}=\vec{\nabla} \times \vec{A}  \tag{3.6}\\
\vec{E}=-\frac{\partial \vec{A}}{\partial t}-\vec{\nabla} V
\end{array}\right.
$$

In general, the potentials $V$ and $\vec{A}$ are not uniquely defined, since you can apply a Gauge transformations to them, leaving fields $\vec{E}$ and $\vec{B}$ unchanged.

$$
\left\{\begin{align*}
V^{\prime} & =V-\frac{\partial}{\partial t} \phi  \tag{3.7}\\
\vec{A}^{\prime} & =\vec{A}+\vec{\nabla} \phi
\end{align*} \quad\right. \text { Gauge transformation }
$$

In Quantum Mechanics and in Relativity the EM potentials are usually fixed by the introduction of the so-called Lorentz Gauge:

$$
\begin{equation*}
\frac{1}{c_{0}^{2}} \frac{\partial V}{\partial t}+\vec{\nabla}^{T} \cdot \vec{A}=0 \quad \text { Lorentz Gauge } \tag{3.8}
\end{equation*}
$$

We remind that the speed of light $c_{0}$ in vacuum is linked to the permittivity $\varepsilon_{0}$ and permeability $\mu_{0}$ through the relation:

$$
\begin{equation*}
c_{0}^{2}=\frac{1}{\mu_{0} \varepsilon_{0}} \tag{3.9}
\end{equation*}
$$

By a direct substitution, Maxwell's Eq.s can be finally rewritten in terms of $V$ and $\vec{A}$ :

$$
\left\{\begin{array} { l } 
{ \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial ^ { 2 } V } { \partial t ^ { 2 } } - \nabla ^ { 2 } V = \mu _ { 0 } c _ { 0 } ^ { 2 } \rho _ { e } }  \tag{3.10}\\
{ \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial ^ { 2 } \vec { A } } { \partial t ^ { 2 } } - \nabla ^ { 2 } \vec { A } = \mu _ { 0 } \vec { J } _ { e } }
\end{array} \quad \left\{\begin{array}{l}
\frac{1}{c_{0}^{2}} \frac{\partial V}{\partial t}+\vec{\nabla}^{T} \cdot \vec{A}=0 \\
\vec{E}=-\frac{\partial \vec{A}}{\partial t}-\vec{\nabla} V \\
\vec{B}=\vec{\nabla} \times \vec{A}
\end{array}\right.\right.
$$

### 3.2.1 Charge conservation

The conservation of the electric charge can be deduced from Maxwell's Eq.s, no matter if they are written in terms of $\vec{E}$ and $\vec{B}$ or with EM potentials.

In the first case, the Wave set (3.1), (3.2) is considered. We apply the time derivative to the divergence of $E$ and the divergence to the "curl" of $B$ :

$$
\begin{align*}
& \left\{\begin{array}{l}
\frac{\partial}{\partial t} \vec{\nabla}^{T} \cdot \vec{E}=\frac{1}{\varepsilon_{0}} \frac{\partial \rho_{e}}{\partial t} \\
\vec{\nabla}^{T} \cdot(\vec{\nabla} \times \vec{B})=\mu_{0} \vec{\nabla}^{T} \cdot \vec{J}_{e}+\mu_{0} \varepsilon_{0} \vec{\nabla}^{T} \cdot \frac{\partial \vec{E}}{\partial t} \quad \Longrightarrow \\
\frac{\partial \rho_{e}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{J}_{e}=0 \quad \text { charge cons. }
\end{array} . \quad \begin{array}{l}
\end{array}\right] \tag{3.11}
\end{align*}
$$

Using the EM potentials, for proving the charge conservation its sufficient to apply the continuity operator $\frac{\partial \cdot}{\partial t}+\vec{\nabla}^{T}$. to the wave equations:

$$
\left\{\begin{align*}
\frac{\partial}{\partial t}\left(\frac{1}{c_{0}^{2}} \frac{\partial^{2} V}{\partial t^{2}}-\nabla^{2} V\right) & =\mu_{0} c_{0}^{2} \frac{\partial}{\partial t} \rho_{e}  \tag{3.13}\\
\vec{\nabla}^{T} \cdot\left(\frac{1}{c_{0}^{2}} \frac{\partial^{2} \vec{A}}{\partial t^{2}}-\nabla^{2} \vec{A}\right) & =\mu_{0} \vec{\nabla}^{T} \cdot \vec{J}_{e}
\end{align*}\right.
$$

Summing the equations together, holding the Lorentz Gauge 3.8, finally we get:

$$
\begin{equation*}
\frac{\partial \rho_{e}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{J}_{e}=0 \tag{3.14}
\end{equation*}
$$

This continuity equations is quite important because it tell us that, for non stationary problem, we can treat the charge density $\rho_{e}$ as a dependent variable. In fact, once the current field $\vec{J}_{e}$ is known, we can calculate the charge, while the opposite could be impossible. In the Laplace's domain the charge density $\rho_{e}$ will be equal to:

$$
\begin{equation*}
\rho_{e}=-\frac{1}{s} \vec{\nabla}^{T} \cdot \vec{J}_{e} \tag{3.15}
\end{equation*}
$$

More generally, once the currents are known and the initial conditions are provided, then the charges can be completely determined.

### 3.3 N Dimensional notation

The magnetic field $\vec{B}$ is not a vector, strictly speaking, but a pseudovector. In fact, it does not respect reflection rules and it is associated to the concept of
rotation rather than to translation [34, 35, 36]. Actually, the magnetic field is better described by an anti-symmetric tensor $\overline{\bar{B}}$. In 3D holds:

$$
\overline{\bar{B}}=[B \times]=\left[\begin{array}{ccc}
0 & -B_{z} & B_{y}  \tag{3.16}\\
B_{z} & 0 & -B_{x} \\
-B_{y} & B_{x} & 0
\end{array}\right]
$$

In Quantum Mechanics and in Relativity the tensorial nature of the magnetic field $\overline{\bar{B}}$ is quite well known. In fact, it appears in the Electromagnetic tensor $F^{\mu \nu}$ together with the electric field.

$$
F^{\mu \nu}=\left[\begin{array}{cc}
0 & -\vec{E}^{T} / c_{0}  \tag{3.17}\\
\vec{E} / c_{0} & \overline{\bar{B}}
\end{array}\right]=\left[\begin{array}{cccc}
0 & -E_{1} / c_{0} & -E_{2} / c_{0} & -E_{3} / c_{0} \\
E_{1} / c_{0} & 0 & -B_{3} & B_{2} \\
E_{2} / c_{0} & B_{3} & 0 & -B_{1} \\
E_{3} / c_{0} & -B_{2} & B_{1} & 0
\end{array}\right]
$$

In this work preferably I will adopt a notation which is valid in any number of Dimensions (ND notation). For further details, I suggest to read the compendium [35] on the multidimensional cross product $\hat{\wedge}$.

Since $B$ is the curl of $\vec{A}$, in ND that relation will look:

$$
\begin{align*}
\overline{\bar{B}} & =\vec{\nabla} \hat{\wedge} \vec{A}=\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]-\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]^{T}  \tag{3.18}\\
B_{i j} & =A_{i / j}-A_{j / i}=\frac{\partial A_{i}}{\partial x_{j}}-\frac{\partial A_{j}}{\partial x_{i}} \tag{3.19}
\end{align*}
$$

It can be verified that the "curl" of $B$ can be rephrased as follows:

$$
\begin{gather*}
\vec{\nabla} \times \vec{B}=-\vec{B} \times \vec{\nabla}=-[B \times] \cdot \vec{\nabla}=-\overline{\bar{B}} \cdot \vec{\nabla}  \tag{3.20}\\
\vec{\nabla} \times \vec{B}=-\overline{\bar{B}} \cdot \vec{\nabla}  \tag{3.21}\\
(\vec{\nabla} \times \vec{B})_{i}=-\sum_{j=1}^{N} B_{i j / j} \tag{3.22}
\end{gather*}
$$

The complete expression of N-dimensional Maxwell's Equations written in terms of $\vec{E}$ and $\overline{\bar{B}}$ can be found in 36.

### 3.3.1 Notation for the scalar EM potential

The scalar EM potential $V$ is sometimes indicated with other symbols, for example with $\varphi$. In this work I will privilege the interpretation of $V$ as a kind of electromagnetic pressure. In addition, the Lorentz Gauge will be regarded as a continuity equation, stating the conservation of some physical variable. That interpretation is often adopted also in Quantum Mechanics 37, 38.

Now I introduce the density $\rho_{A}$ and the symbol $P_{A}$ instead of $V$ for the scalar potential. These quantities are related in this way:

$$
\begin{equation*}
V=P_{A}=\rho_{A} c_{0}^{2} \tag{3.23}
\end{equation*}
$$

Preferably, I will use the symbol $V$ in a circuit context. Let's notice that (3.23) can be considered as a constitutive relation linking a density $\rho_{A}$ to a "pressure" $P_{A}$.

### 3.3.2 Maxwell's Equations in N-Dimensions

Finally, we can write the complete set of Maxwell's equations with the EM potentials, in a form which is valid in any number of spatial dimensions:

$$
\left\{\begin{array} { l } 
{ \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial ^ { 2 } \rho _ { A } } { \partial t ^ { 2 } } - \nabla ^ { 2 } \rho _ { A } = \mu _ { 0 } \rho _ { e } }  \tag{3.24}\\
{ \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial ^ { 2 } \vec { A } } { \partial t ^ { 2 } } - \nabla ^ { 2 } \vec { A } = \mu _ { 0 } \vec { J } _ { e } }
\end{array} \quad \left\{\begin{array}{l}
\frac{\partial \rho_{A}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{A}=0 \\
\vec{E}=-\frac{\partial \vec{A}}{\partial t}-\vec{\nabla}\left(\rho_{A} c_{0}^{2}\right) \\
\overline{\bar{B}}=\vec{\nabla} \hat{\wedge} \vec{A}
\end{array}\right.\right.
$$

### 3.4 Essential Maxwell's Equations in Laplace's Domain

If a system is time-varying, i.e it is non-stationary, then the densities $\rho_{A}$ and $\rho_{e}$ are dependent variables, because they may be calculated from fields $\vec{A}$ and $\vec{J}_{e}$. In fact, thanks to the continuity equations, $\rho_{A}$ and $\rho_{e}$ can be express in the Laplace domain as:

$$
\left\{\begin{array} { l } 
{ \frac { \partial \rho _ { A } } { \partial t } + \vec { \nabla } ^ { T } \cdot \vec { A } = 0 }  \tag{3.25}\\
{ \frac { \partial \rho _ { e } } { \partial t } + \vec { \nabla } ^ { T } \cdot \vec { J } _ { e } = 0 }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\rho_{A}=-\frac{1}{s} \vec{\nabla}^{T} \cdot \vec{A} \\
\rho_{e}=-\frac{1}{s} \vec{\nabla}^{T} \cdot \vec{J}_{e}
\end{array}\right.\right.
$$

So, for non stationary problem, Maxwell's Eq.s can be written in the Laplace domain as:

$$
\left\{\begin{array} { l } 
{ \rho _ { e } = - \frac { 1 } { s } \vec { \nabla } ^ { T } \cdot \vec { J } _ { e } }  \tag{3.26}\\
{ \mu _ { 0 } \vec { J } _ { e } = \frac { s ^ { 2 } } { c _ { 0 } ^ { 2 } } \vec { A } - \nabla ^ { 2 } \vec { A } }
\end{array} \left\{\begin{array}{l}
\rho_{A}=-\frac{1}{s} \vec{\nabla}^{T} \cdot \vec{A} \\
s \vec{E}=-s^{2} \vec{A}+c_{0}^{2} \vec{\nabla}\left(\vec{\nabla}^{T} \cdot \vec{A}\right) \\
\overline{\bar{B}}=\vec{\nabla} \hat{\wedge} \vec{A}
\end{array}\right.\right.
$$

This form is particularly helpful when you are dealing with propagation problems. You can also notice that all the fields can be deduced just by the vector potential $\vec{A}$, which turns out to be the fundamental independent field.

## Chapter 4

## Surface Equivalence Theorem for EM Potentials

> | It is natural to regard it $[\vec{A}]$ as the velocity |
| :--- |
| of some real physical thing. Thus with the |
| new theory of electrodynamics we are |
| rather forced to have an aether |
| P.A.M. Dirac, "Is there an aether?", |
| Nature, 1951 |

In this chapter we are going to see how to apply the extended Huygens' Principle (see sec 2.2.5) to the electromagnetic fields. This time, however, we are going to use the EM scalar and vector potentials $V$ and $\vec{A}$ instead of the "classic" electric and magnetic fields $\vec{E}$ and $\vec{H}$.

In particular, in this thesis we demonstrate rigorously a Surface Equivalence Theorem for the EM potentials, explaining how to write the Boundary Conditions (BC) for an EM problem in terms of $V, \vec{A}$ and their gradients. Let's notice that, if not specified otherwise, the boundary $\partial \Omega$ is always supposed to be still, at rest, fixed in time. The Relativistic BCs will be treated in chapter 8 .

### 4.1 Why to use EM potentials?

Probably, the first question an electrical engineer could now ask is:
Why should we work with the EM potentials rather than with $\vec{E}$ and $\vec{H}$ ? Usually we use $\vec{E}$ and $\vec{H}$ to describe antennas propagation, radiation patterns, etc., and they work well.

Actually, most of time electrical engineers are inclined to think the EM field in terms of $\vec{E}$ and $\vec{H}$ and normally they are not used to work with potentials $V$ and $\vec{A}$. Here we report a short list of reasons of why the EM potentials can be of great utility, not only for electrical engineers.

1. In Quantum Mechanics the EM potentials are fundamental, since they allow to write the equations in terms of action. In fact physicist are usually more familiar with $V$ and $\vec{A}$ with the respect to electrical engineers.
2. In the Relativity Theory, $V$ and $\vec{A}$ are easier to be transformed from a reference frame to another, especially if you compare them to electric $\vec{E}$ and magnetic $\vec{B}$ fields. In fact, it is possible to define a 4 -vector $A^{\mu}=\left[V / c_{0} ; \vec{A}\right]$ which transforms through a simple Lorentz Boost:

$$
\begin{equation*}
A^{\mu^{\prime}}=\Lambda_{\nu}^{\mu} A^{\nu} \tag{4.1}
\end{equation*}
$$

where $\Lambda_{\nu}^{\mu}$ is the Lorentz boost tensor associated to a frame change. On the contrary, if you want to transform $\vec{E}$ and $\vec{B}$ to another reference frame you have to deal with a whole tensor $F^{\mu \nu}$ instead that with a 4 -vector $A^{\mu}$.

$$
\begin{equation*}
F^{\prime \alpha \beta}=\Lambda_{\mu}^{\alpha} F^{\mu \nu} \Lambda_{\nu}^{\beta} \tag{4.2}
\end{equation*}
$$

3. $V$ and $\vec{A}$ allow to remove the magnetic field $\vec{B}$, which is not a true vector but a pseudovector, thus it has to be described by a skew-symmetric tensor [34, 35, 36]

$$
\begin{equation*}
\overline{\bar{B}}=\vec{\nabla} \wedge \vec{A} \tag{4.3}
\end{equation*}
$$

4. The use of EM potential allows analogies with Mechanics and FluidDynamics, which are not so simple or evident adopting $\vec{E}$ and $\vec{B}$. For example, the Lorentz Gauge can be interpreted as continuity equation while $\vec{A}$ is often regarded as momentum per unit of charge.
5. They seems to have objective existence, as showed by the Aharonov-Bohm experiment [39] and by the Casimir effect. If that is true, then $V$ and $\vec{A}$ have a physical meaning and are not a mere, artificial mathematical construction [37, 38].
6. In my opinion the Maxwell's Equations have a more elegant aspect if they are written in terms of EM potential.

Beyond those "aesthetic" considerations, $V$ and $\vec{A}$ are helpful also for numerical and computational reasons.

For example, many numerical methods are based on the $\vec{E}-\vec{H}$ formulation, and they suffer for the so-called low-frequency breakdown or catastrophe 40. If the scale $\Delta x$ of the problem is much smaller than the operating wavelength $\lambda_{0}$, then the method becomes rapidly unstable. On the contrary, the solution in terms of EM potentials $V$ and $\vec{A}$ can be easily calculated also for static or low-frequency problem. Afterwards, fields $\vec{E}$ and $\vec{H}$ can be determined applying gradients and derivatives to $V$ and $\vec{A}$. In addition, as we are going to check, EM potentials are quite regular fields and we can write the Boundary Conditions for a problem demanding that $V$ and $\vec{A}$ have finite values:

$$
\begin{equation*}
|V|<+\infty \quad \quad\|\vec{A}\|<+\infty \tag{4.4}
\end{equation*}
$$

On the contrary, we cannot guarantee the same condition for fields $\vec{E}$ and $\vec{H}$, which can vary abruptly across the boundary and reach infinite values. Moreover, we are going to verify that the standard surface equivalence principle for $\vec{E}$ and $\vec{H}$ is not sufficient for all the possible electromagnetic problems. In particular, it will fail in the characterization of the Volta's effect, and in general in all the static EM configurations.

### 4.1.1 Problems with the classic formulation in E and H

In summary, the classic form of the Surface Equivalence Principle, involving $\vec{E}$ and $\vec{H}$, has some serious drawbacks:

- Numerical instability for static or low-frequency problems.
- Not suitable for Quantum Mechanics, and sometimes also for Relativity.
- Boundary Conditions could not be valid if $\vec{E}$ and $\vec{H}$ are not enough regular on the surface.
- Introduction of magnetic currents $\vec{J}_{m}$

The last point is worthy of some considerations. If you look at the Maxwell's Equations, you can notice that the only sources are the electric charge $\rho_{e}$ and current $\vec{J}_{e}$ densities, while there are neither magnetic charge $\rho_{m}$ nor magnetic current $\vec{J}_{m}$. In other words, there is no evidence of magnetic monopoles. The problem of the possible existence of isolated magnetic charges has been debated for a long time, and here it is not the right place to discuss about it. However, we have to mention that Dirac wrote an intriguing article [41] on that question, showing that the existence of magnetic charge would allow to symmetrize the Maxwell's Equations as follows:

$$
\left\{\begin{array} { l } 
{ \vec { \nabla } ^ { T } \cdot \vec { E } = \frac { \rho _ { e } } { \varepsilon _ { 0 } } }  \tag{4.5}\\
{ \vec { \nabla } ^ { T } \cdot \vec { B } = \rho _ { m } }
\end{array} \quad \left\{\begin{array}{l}
\vec{\nabla} \times \vec{E}=-\vec{J}_{m}-\frac{\partial \vec{B}}{\partial t} \\
\vec{\nabla} \times \vec{B}=\mu_{0} \vec{J}_{e}+\frac{1}{c_{0}^{2}} \frac{\partial \vec{E}}{\partial t}
\end{array}\right.\right.
$$

Although this formulation is doubtlessly elegant, it seems to work just in 3 Dimensions. Moreover, until today there is no evidence of magnetic charges, notwithstanding many experiments have been performed to detect them.

As we have already seen, in Schelkunoff's view the introduction of a surface magnetic current $\vec{J}_{s, m}$ is mandatory in order to enforce a discontinuity for the tangential electric field:

$$
\begin{equation*}
\vec{n}_{21} \times\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\vec{J}_{s, m} \tag{4.6}
\end{equation*}
$$

Schelkunoff himself wrote [33]:
The physical sources of electromagnetic fields are electric and magnetic charges in motion, that is electric and magnetic currents. The radio engineer has never been interested in shaking magnets for the purpose of radiating energy and has settled into a habit of ignoring magnetic currents altogether as if they were non-existent. [...] We shall find it convenient, at least for analytical purposes, to employ the concept of magnetic current on a par with the concept of electric current.

Even if we work well with magnetic currents, we are rather forced to wonder about their physical meaning, or if otherwise they are just fictitious mathematical quantities.

In the end I have to warn that, like $\vec{B}$, magnetic current also is a pseudovector quantity, and so it is described by an anti-symmetric tensor $\overline{\bar{J}}_{m}$. In N Dimensional notation, the eq. (4.6) will look:

$$
\begin{equation*}
\vec{n}_{21} \wedge\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\overline{\bar{J}}_{s, m} \tag{4.7}
\end{equation*}
$$

Shortly, we are going to write the Boundary Conditions for the EM field without invoking the existence of magnetic current.

### 4.2 Wave equations for EM potentials

The electromagnetic field is completely described by the EM potentials $\rho_{A}=V / c_{0}^{2}$ and $\vec{A}$. In order to extend the Huygens's Principle to those fields, we have to known how they are generated. The sources of the electromagnetic fields are the electric charge $\rho_{e}$ and current $\vec{J}_{e}$ densities. Those quantities are linked to the EM potential through two wave equations:

$$
\begin{align*}
& \frac{1}{c_{0}^{2}} \frac{\partial^{2} \rho_{A}}{\partial t^{2}}-\nabla^{2} \rho_{A}=\mu_{0} \rho_{e}  \tag{4.8}\\
& \frac{1}{c_{0}^{2}} \frac{\partial^{2} \vec{A}}{\partial t^{2}}-\nabla^{2} \vec{A}=\mu_{0} \vec{J}_{e} \tag{4.9}
\end{align*}
$$

In this form, the $S($.$) operator is the D'Alembert's one.$

### 4.2.1 Green's function for the wave equation

Since the $S\left(\right.$. ) operator is linear, it is possible to find a Green's function $\varphi_{G}$ such that:

$$
\begin{equation*}
\frac{1}{c_{0}^{2}} \frac{\partial^{2} \varphi_{G}}{\partial t^{2}}-\nabla^{2} \varphi_{G}=\delta_{N+1}\left(\vec{x}-\vec{x}_{0}, t-t_{0}\right) \tag{4.10}
\end{equation*}
$$

where $\left(\vec{x}_{0}, t_{0}\right)$ is the space-time position for the instant source. The wave equation is symmetric with respect to space and time reversal. Moreover, it has a spherical symmetry in space, since there is no privileged direction.

$$
\begin{gather*}
\left\{\begin{array} { l l } 
{ \Delta t ^ { \prime } = - \Delta t } & { \text { time reversal } } \\
{ \Delta \vec { x } ^ { \prime } = \overline { \overline { R } } \Delta \vec { x } } & { \text { arbitrary rotation } }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\frac{\partial^{2} .}{\partial t^{\prime 2}}=\frac{\partial^{2}}{\partial t^{2}} \\
\nabla^{\prime 2}(.)=\nabla^{2}(.)
\end{array}\right.\right.  \tag{4.11}\\
\frac{1}{c_{0}^{2}} \frac{\partial^{2} \varphi_{G}}{\partial t^{\prime 2}}-\nabla^{\prime 2} \varphi_{G}=\delta_{N+1}\left(\vec{x}^{\prime}-\vec{x}_{0}^{\prime}, t^{\prime}-t_{0}^{\prime}\right) \quad \text { equation unchanged } \tag{4.12}
\end{gather*}
$$

There are so two different spherical solutions, one associated to waves propagating away from the source, the other propagating backward (regressive wave). In 3-D the solutions are:

$$
\begin{array}{rlrl}
\varphi_{G+}(r, t) & =\frac{1}{4 \pi r} \delta_{1}\left(r-c_{0} \Delta t\right) & \text { progressive wave } \\
\varphi_{G-}(r, t) & =\frac{1}{4 \pi r} \delta_{1}\left(r+c_{0} \Delta t\right) & & \text { regressive wave } \tag{4.14}
\end{array}
$$

where $r=\left\|\vec{x}-\vec{x}_{0}\right\|$ and $\Delta t=t-t_{0}$. If we require the source to be causal, i.e. it cannot send waves in the past, then the regressive solution will be excluded and just the progressive one shall be considered:

$$
\begin{equation*}
\varphi_{G}(r, t)=\frac{1}{4 \pi r} \delta_{1}\left(r-c_{0} \Delta t\right) \quad \text { causal solution for } r>c_{0} \Delta t \tag{4.15}
\end{equation*}
$$

Working in the time domain could be difficult. In fact, if you want to know the field $\mathbf{f}(\vec{x}, t)$ at a certain time $t_{0}$, you have to know also the past configuration of sources $\mathbf{J}$ which generated it, since the information could travel at a finite speed.

For our purpose it is better to rephrase D'Alembert equations 4.8, 4.9 in the frequency domain, thus transforming them in Helmholtz equations.

### 4.2.2 Helmoltz equations

In order to deal with frequencies rather than with time, we perform the Laplace transformation on all the variables. For a function $f(t)$, its Laplace equivalent $f(s)$ is so defined:

$$
\begin{equation*}
f(s)=\int_{-\infty}^{+\infty}\left(f(t) e^{-s t}\right) d t \tag{4.16}
\end{equation*}
$$

where $s$ is the complex variable, defined as:

$$
\begin{equation*}
s=\sigma-i \omega \quad \text { with } \sigma, \omega \in \mathbb{R} \tag{4.17}
\end{equation*}
$$

Important note: in this thesis we adopt the phasor convention $e^{i(k x-\omega t)}$ for propagating field, though in Electrical Engineer the form $e^{i(\omega t-k x)}$ is more commonly adopted. We decided to use the former one because the spatial term $e^{i k x}$ will appear very frequently, instead of $e^{-i \omega t}$. In addition, that convention is largely used in Physics.

$$
\begin{equation*}
\psi(\vec{x}, t)=\psi_{0} e^{i\left(\vec{k}^{T} \cdot \vec{x}-\omega t\right)} \quad \text { propagating wavefunction } \tag{4.18}
\end{equation*}
$$

Applying the Laplace transformation to the wave equations 4.8, 4.9), it yields:

$$
\begin{align*}
& \frac{s^{2}}{c_{0}^{2}} \rho_{A}-\nabla^{2} \rho_{A}=\mu_{0} \rho_{e}  \tag{4.19}\\
& \frac{s^{2}}{c_{0}^{2}} \vec{A}-\nabla^{2} \vec{A}=\mu_{0} \vec{J}_{e} \tag{4.20}
\end{align*}
$$

We can now define a reference wavenumber $k_{0}$ as:

$$
\begin{equation*}
k_{0}=i \frac{s}{c_{0}} \quad \Longrightarrow \quad k_{0}=\frac{\omega}{c_{0}} \quad \text { if } \sigma=0 \tag{4.21}
\end{equation*}
$$

Finally we obtain the Helmholtz equations for the EM potentials:

$$
\begin{align*}
\nabla^{2} \rho_{A}+k_{0}^{2} \rho_{A} & =-\mu_{0} \rho_{e}  \tag{4.22}\\
\nabla^{2} \vec{A}+k_{0}^{2} \vec{A} & =-\mu_{0} \vec{J}_{e} \tag{4.23}
\end{align*}
$$

The advantage of this formulation resides in the fact that it does not contain time derivatives.

## Solutions for the Helmholtz equations

The Green's function $\varphi_{G}$ for the Helmholtz equation is such that:

$$
\begin{equation*}
\nabla^{2} \varphi_{G}+k_{0}^{2} \varphi_{G}=-\delta_{N}\left(\vec{x}-\vec{x}_{0}\right) \tag{4.24}
\end{equation*}
$$

Actually, it could be verified that this is the Laplace equivalent of eq. 4.10 for $t=t_{0}$. Thus, in 3-D its "causal" solution $\varphi_{G}$ is the well-known:

$$
\begin{equation*}
\varphi_{G}\left(\vec{x}-\vec{x}_{0}, k_{0}\right)=\frac{1}{4 \pi} \frac{1}{r} e^{i k_{0} r} \tag{4.25}
\end{equation*}
$$

Finally, the general solution for $\rho_{A}$ and $\vec{A}$ can be expressed as a convolution of the Green's function with the source terms:

$$
\begin{align*}
& \rho_{A}(\vec{x})=\mu_{0} \int_{\Omega_{\infty}} \varphi_{G}(\vec{x}-\vec{y}) \rho_{e}(\vec{y}) d \Omega_{y}+\rho_{A, h}(\vec{x})  \tag{4.26}\\
& \vec{A}(\vec{x})=\mu_{0} \int_{\Omega_{\infty}} \varphi_{G}(\vec{x}-\vec{y}) \vec{J}_{e}(\vec{y}) d \Omega_{y}+\vec{A}_{h}(\vec{x}) \tag{4.27}
\end{align*}
$$

where:

- $\vec{y}$ is the integration variable
- $\Omega_{\infty}$ is the whole infinite space domain
- $\rho_{A, h}$ and $\vec{A}_{h}$ are solutions for the homogeneous equations (no sources), and are fixed by the Boundary Conditions:

$$
\begin{array}{lll}
\nabla^{2} \rho_{A, h}+k_{0}^{2} \rho_{A, h} & =0 & + \text { B.C.s } \\
\nabla^{2} \vec{A}_{h}+k_{0}^{2} \vec{A}_{h} & =0 & + \text { B.C.s } \tag{4.29}
\end{array}
$$

Now we have the basic tools for finding the boundary sources of the EM field.

### 4.3 Surface Equivalence for Scalar Potential $\rho_{A}$

Let's suppose we know the scalar potential $\rho_{A}$ on a space region $\Omega_{x}$. The


Figure 4.1: Scalar potential $\rho_{A}$ on domain $\Omega_{x}$
belonging function $\epsilon_{\Omega}$ for $\Omega_{x}$ is so defined:

$$
\epsilon_{\Omega}(\vec{x})= \begin{cases}1 & \text { for } \vec{x} \in \Omega_{x}  \tag{4.30}\\ 0 & \text { for } \vec{x} \notin \Omega_{x}\end{cases}
$$

So we know just $\epsilon_{\Omega}(\vec{x}) \rho_{A}(\vec{x})$, that is a field which is null outside $\Omega_{x}$. We want to determine:

- the sources $\rho_{e}$ inside the domain $\Omega_{x}$.
- the eventual surface sources on the domain's boundary $\partial \Omega_{x}$.

For our task, we shall tackle the problem with an integral approach.

### 4.3.1 Surface Theorem for scalar fields

Given a generic scalar field $f$, we can write:

$$
\begin{equation*}
\epsilon_{\Omega}(\vec{x}) f(\vec{x})=\int_{\Omega_{y}} \delta_{N}(\vec{x}-\vec{y}) f(\vec{y}) d \Omega_{y} \tag{4.31}
\end{equation*}
$$

Let the field $f$ be generated by some source $J$ through a linear equation:

$$
\begin{equation*}
S(f)=J \tag{4.32}
\end{equation*}
$$

Since the $S($.$) operator is linear, it will by associated to a Green's function \varphi$ such that:

$$
\begin{equation*}
S(\varphi)=-\delta\left(\vec{x}-\vec{x}_{0}\right) \tag{4.33}
\end{equation*}
$$

Substituting in 4.31, we get:

$$
\begin{equation*}
\epsilon_{\Omega}(\vec{x}) f(\vec{x})=-\int_{\Omega_{y}} S(\varphi(\vec{x}-\vec{y})) f(\vec{y}) d \Omega_{y} \tag{4.34}
\end{equation*}
$$

More compactly:

$$
\begin{equation*}
\epsilon_{\Omega} f=-\int_{\Omega_{y}} S_{y}(\varphi) f_{y} d \Omega_{y} \tag{4.35}
\end{equation*}
$$

That's quite a general starting point. Now we are interested to obtain an expression of this kind:

$$
\begin{equation*}
\epsilon_{\Omega}(\vec{x}) f(\vec{x})=-\int_{\Omega_{y}} S(f(\vec{y})) \varphi(\vec{x}-\vec{y}) d \Omega_{y}+\text { int. on } \partial \Omega \tag{4.36}
\end{equation*}
$$

That is, we desire to apply the operator $S($.$) to the field \mathbf{f}$ and to sum the volume integral with a boundary one.

More compactly, we can also write:

$$
\begin{equation*}
\epsilon_{\Omega} f=-\int_{\Omega_{y}} S_{y}(f) \varphi d \Omega_{y}+\quad \text { int. on } \partial \Omega \tag{4.37}
\end{equation*}
$$

Supposing $S($.$) is the Helmholtz operator, we shall require that:$

$$
\begin{equation*}
\epsilon_{\Omega} f=-\int_{\Omega_{y}}\left(\nabla_{y}^{2} f_{y}+k_{0}^{2} f_{y}\right) \cdot \varphi d \Omega_{y}+\quad \text { int. on } \partial \Omega \tag{4.38}
\end{equation*}
$$

We have to pass from (4.31) to this last one. The first step consists in replacing the Dirac delta with the Green's function, so that:

$$
\begin{equation*}
\epsilon_{\Omega} f=-\int_{\Omega_{y}}\left(\nabla_{y}^{2} \varphi+k_{0}^{2} \varphi\right) \cdot f_{y} d \Omega_{y}+\quad \text { int. on } \partial \Omega \tag{4.39}
\end{equation*}
$$

## Green's lemma

In order to calculate the sources on $\Omega_{x}$, we can exploit the Green's lemma:

$$
\begin{equation*}
a \nabla^{2} b-b \nabla^{2} a=\vec{\nabla}^{T} \cdot(a \vec{\nabla} b-b \vec{\nabla} a) \tag{4.40}
\end{equation*}
$$

For the current case, it holds:

$$
\begin{equation*}
f_{y} \nabla_{y}^{2} \varphi-\varphi \nabla_{y}^{2} f_{y}=\vec{\nabla}_{y}^{T} \cdot\left(f_{y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{y}\right) \tag{4.41}
\end{equation*}
$$

So we obtain the desired relation:

$$
\begin{equation*}
f_{y}\left(\nabla_{y}^{2} \varphi\right)=\varphi\left(\nabla^{2} f_{y}\right)+\vec{\nabla}_{y}^{T} \cdot\left(f_{y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{y}\right) \tag{4.42}
\end{equation*}
$$

Now we can substitute this term in the integral 4.39), finding:

$$
\begin{align*}
\epsilon_{\Omega} f= & -\int_{\Omega_{y}}\left(\nabla_{y}^{2} f_{y}+k_{0}^{2} f_{y}\right) \cdot \varphi d \Omega_{y}+ \\
& -\int_{\Omega_{y}} \vec{\nabla}_{y}^{T} \cdot\left(f_{y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{y}\right) d \Omega_{y} \tag{4.43}
\end{align*}
$$

The last integral term can be transformed in a boundary integral thanks to the Gauss' Divergence law:

$$
\begin{equation*}
\int_{\Omega} \vec{\nabla}^{T} \cdot \vec{v} d \Omega=\oint_{\partial \Omega} \vec{n}^{T} \cdot \vec{v} d S \tag{4.44}
\end{equation*}
$$

where $\vec{n}$ is the normal pointing outward $\Omega$. So it follows:

$$
\begin{align*}
& \int_{\Omega_{y}} \vec{\nabla}_{y}^{T} \cdot\left(f_{y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{y}\right) d \Omega_{y}=  \tag{4.45}\\
& \oint_{\partial \Omega_{y}} \vec{n}_{y}^{T} \cdot\left(f_{y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{y}\right) d S_{y}
\end{align*}
$$

Substituting in 4.43, finally we obtain:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) f(\vec{x})= & -\int_{\Omega_{y}}\left(\nabla_{y}^{2} f_{y}+k_{0}^{2} f_{y}\right) \cdot \varphi d \Omega_{y}+ \\
& -\oint_{\partial \Omega_{y}} \vec{n}_{y}^{T} \cdot\left(f_{y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{y}\right) d S_{y} \tag{4.46}
\end{align*}
$$

This is quite an important mathematical result. Now we shall give a physical interpretation of it.

### 4.3.2 Sources for the scalar EM potential

The EM potential $\rho_{A}$ is a scalar field so, holding the result 4.46, it can be expressed as:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \rho_{A}(\vec{x}) & =-\int_{\Omega_{y}}\left(\nabla_{y}^{2} \rho_{A}(\vec{y})+k_{0}^{2} \rho_{A}(\vec{y})\right) \cdot \varphi d \Omega_{y}+ \\
& -\oint_{\partial \Omega_{y}} \vec{n}_{y}^{T} \cdot\left(\rho_{A}(\vec{y}) \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} \rho_{A}(\vec{y})\right) d S_{y} \tag{4.47}
\end{align*}
$$

The integral on $\Omega$ depends just on the distribution of charge $\rho_{e}$ inside it, while the surface integral is related to the boundary sources.

## Elementary sources

Let us analyze the single terms in detail and properly define the sources:

- $\rho_{e}$ is the density of electric charge on $\Omega$.

$$
\begin{equation*}
\rho_{e}=-\frac{1}{\mu_{0}}\left(\nabla^{2} \rho_{A}+k_{0}^{2} \rho_{A}\right) \tag{4.48}
\end{equation*}
$$

- $\sigma_{e}$ is the charge per unit of surface on $\partial \Omega$ :

$$
\begin{equation*}
\sigma_{e}=\frac{1}{\mu_{0}}\left(\vec{n}^{T} \cdot \vec{\nabla} \rho_{A}\right) \tag{4.49}
\end{equation*}
$$

- $\vec{d}_{e}$ are the dipoles or "doublets" per unit of surface on $\partial \Omega$

$$
\begin{equation*}
\vec{d}_{e}=-\frac{1}{\mu_{0}}\left(\rho_{A} \vec{n}\right) \tag{4.50}
\end{equation*}
$$



Figure 4.2: Elementary sources for the scalar EM potential (a) Electric charge density $\rho_{e}$; (b) Electric charge per surface unit $\sigma_{e}$; (c) Electric dipole per surface unit $\vec{d}_{e}$.

Let's notice that $\vec{d}_{e}$ is parallel to the local normal $\vec{n}$, which points outward $\Omega$. After a substitution of $\rho_{e}, \sigma_{e}$ and $\vec{d}_{e}$ in the integrals we find:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \frac{1}{\mu_{0}} \rho_{A}(\vec{x})= & \int_{\Omega_{y}} \rho_{e}(\vec{y}) \cdot \varphi_{G}(\vec{x}-\vec{y}) d \Omega_{y}+ \\
& \oint_{\partial \Omega_{y}} \sigma_{e}(\vec{y}) \cdot \varphi_{G}(\vec{x}-\vec{y}) d S_{y}+  \tag{4.51}\\
& \oint_{\partial \Omega_{y}} \overrightarrow{d_{e}^{T}}(\vec{y}) \cdot\left(\vec{\nabla}_{y} \varphi_{G}(\vec{x}-\vec{y})\right) d S_{y}
\end{align*}
$$

More compactly:

$$
\begin{align*}
\epsilon_{\Omega} \frac{1}{\mu_{0}} \rho_{A}= & \int_{\Omega_{y}} \rho_{e, y} \cdot \varphi d \Omega_{y}+ \\
& \oint_{\partial \Omega_{y}} \sigma_{e, y} \cdot \varphi d S_{y}+  \tag{4.52}\\
& \oint_{\partial \Omega_{y}} \vec{d}_{e, y}^{T} \cdot \vec{\nabla}_{y} \varphi d S_{y}
\end{align*}
$$

Actually, this equation states the Surface Equivalence Principle for the scalar potential $\rho_{A}$. In fact, all the sources external to $\Omega_{x}$ have been mapped on the boundary $\partial \Omega_{x}$, and we have demonstrated that they are of two types: charges $\sigma_{e}$ and dipoles $\vec{d}_{e}$ per unit of surface.

### 4.4 Elementary fields and sources

In this section we report the fields generated by the elementary sources of the scalar potential.

### 4.4.1 Field for a lumped charge

For a net electric charge $Q_{e}$ on a point $\vec{x}_{0}$ the density distribution $\rho_{e}$ is:

$$
\begin{equation*}
\rho_{e}(\vec{x})=Q_{e} \delta_{N}\left(\vec{x}-\vec{x}_{0}\right) \tag{4.53}
\end{equation*}
$$

Hence, in the Laplace's domain the field $\rho_{A}$ produced by $Q_{e}$ will be:

$$
\begin{equation*}
\rho_{A}(\vec{x})=\mu_{0} Q_{e} \varphi_{G}\left(\vec{x}-\vec{x}_{0}\right) \tag{4.54}
\end{equation*}
$$

In a 3D space:

$$
\begin{equation*}
\rho_{A}(\vec{x})=\mu_{0} \frac{1}{4 \pi r} e^{i k_{0} r} Q_{e} \tag{4.55}
\end{equation*}
$$

### 4.4.2 Field for a lumped dipole

Given two equal and opposite charges $Q_{e}$ and $-Q_{e}$, the dipole $\vec{p}_{e}$ associated to them is:

$$
\begin{equation*}
\vec{p}_{e}=Q_{e} \Delta \vec{x}_{21} \tag{4.56}
\end{equation*}
$$

where $\Delta \vec{x}_{21}=\vec{x}_{2}-\vec{x}_{1}$ is difference of positions between the charges. More generally, if the charges have different magnitudes, the dipole can be calculated as:

$$
\begin{equation*}
\vec{p}_{e}=\frac{1}{2}\left(Q_{e 2}-Q_{e 1}\right) \Delta \vec{x}_{21} \tag{4.57}
\end{equation*}
$$

Supposing the charges $Q_{e}$ and $-Q_{e}$ are placed on points $\vec{x}_{2}$ and $\vec{x}_{1}$ respectively, the scalar field produced by them will be:

$$
\begin{align*}
\frac{1}{\mu_{0}} \rho_{A}(\vec{x}) & =Q_{e} \varphi_{G}\left(\vec{x}-\vec{x}_{2}\right)-Q_{e} \varphi_{G}\left(\vec{x}-\vec{x}_{1}\right)  \tag{4.58}\\
\frac{1}{\mu_{0}} \rho_{A}(\vec{x}) & =Q_{e}\left(\varphi_{G}\left(\vec{x}-\vec{x}_{2}\right)-\varphi_{G}\left(\vec{x}-\vec{x}_{1}\right)\right) \tag{4.59}
\end{align*}
$$

We can express the center $\vec{x}_{0}$ of the dipole as:

$$
\begin{equation*}
\vec{x}_{0}=\frac{1}{2}\left(\vec{x}_{1}+\vec{x}_{2}\right) \tag{4.60}
\end{equation*}
$$

In the limit of a lumped dipole $\vec{p}_{e}$, the charges are infinitely close one to each other, while the product $Q_{e} \Delta \vec{x}_{21}$ has still a finite value. So it follows:

$$
\begin{gather*}
\lim _{\Delta x \rightarrow 0}\left(Q_{e} \Delta \vec{x}_{21}\right)=\vec{p}_{e} \quad \text { finite value }  \tag{4.61}\\
\left\{\begin{array}{c}
\lim _{\Delta x \rightarrow 0}\left(\varphi_{G}\left(\vec{x}-\vec{x}_{2}\right)\right)=\varphi_{G}\left(\vec{x}-\vec{x}_{0}\right)+\frac{\partial \varphi_{G}}{\partial \vec{x}} \cdot \frac{1}{2} \Delta \vec{x}_{21} \\
\lim _{\Delta x \rightarrow 0}\left(\varphi_{G}\left(\vec{x}-\vec{x}_{1}\right)\right)=\varphi_{G}\left(\vec{x}-\vec{x}_{0}\right)-\frac{\partial \varphi_{G}}{\partial \vec{x}} \cdot \frac{1}{2} \Delta \vec{x}_{21}
\end{array} \Longrightarrow\right.  \tag{4.62}\\
\lim _{\Delta x \rightarrow 0}\left(\varphi_{G}\left(\vec{x}-\vec{x}_{2}\right)-\varphi_{G}\left(\vec{x}-\vec{x}_{1}\right)\right)=\frac{\partial \varphi_{G}}{\partial \vec{x}} \cdot \Delta \vec{x}_{21}=\vec{\nabla} \varphi_{G} \cdot \Delta \vec{x}_{21} \tag{4.63}
\end{gather*}
$$

Finally the field $\rho_{A}$ produced by the lumped dipole $\vec{p}_{e}$ is:

$$
\begin{align*}
\frac{1}{\mu_{0}} \rho_{A}(\vec{x}) & =Q_{e} \vec{\nabla} \varphi_{G}^{T} \cdot \Delta \vec{x}_{21}  \tag{4.64}\\
\rho_{A}(\vec{x}) & =\mu_{0} \vec{p}_{e}^{T} \cdot \vec{\nabla} \varphi_{G} \tag{4.65}
\end{align*}
$$

This result is valid also in the frequency domain.

### 4.5 Surface Equivalence for Vector Potential A

Till now we have faced the problem of determining the sources for the scalar field $\rho_{A}$. In the same way, we want to find and map the sources for the vector potential $\vec{A}$. The procedure is quite similar to the one adopted for $\rho_{A}$. We have just to consider each $\mathrm{i}^{\text {th }}$ vector's component $A_{i}$ and $J_{e, i}$ as a scalar fields. For example, the Helmholtz equation for $\vec{A}$ can be written as:

$$
\begin{equation*}
\nabla^{2} A_{i}+k_{0}^{2} A_{i}=-\mu_{0} J_{e, i} \quad \forall i \in\{1 ; 2 ; \cdots ; N\} \tag{4.66}
\end{equation*}
$$

Let's now resume the Surface Equivalence Theorem for scalar fields, extending it to vectorial ones.

### 4.5.1 Surface Theorem for vector fields

Let $\mathbf{f}$ be a vector field, generated by some source $\mathbf{J}$ through a linear equation:

$$
\begin{equation*}
S(\mathbf{f})=\mathbf{J} \tag{4.67}
\end{equation*}
$$

Thanks to the linearity of $S($.$) , for each \mathrm{i}^{\text {th }}$ component of $\mathbf{f}$ and $\mathbf{J}$ it holds:

$$
\begin{equation*}
S\left(f_{i}\right)=J_{i} \quad \forall i \in\{1 ; 2 ; \cdots ; N\} \tag{4.68}
\end{equation*}
$$

If $S($.$) is the Helmholtz operator, we can exploit the Surface Theorem for Scalar$ Fields 4.31) in sec. 4.3.1, so it yields:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) f_{i}(\vec{x})= & -\int_{\Omega_{y}}\left(\nabla_{y}^{2} f_{i, y}+k_{0}^{2} f_{i, y}\right) \cdot \varphi d \Omega_{y}+  \tag{4.69}\\
& -\oint_{\partial \Omega_{y}} \vec{n}_{y}^{T} \cdot\left(f_{i, y} \vec{\nabla}_{y} \varphi-\varphi \vec{\nabla}_{y} f_{i, y}\right) d S_{y}
\end{align*}
$$

Since the equation holds for each $\mathrm{i}^{\text {th }}$ component, we can simply write:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \mathbf{f}(\vec{x})= & -\int_{\Omega_{y}}\left(\nabla_{y}^{2} \mathbf{f}_{y}+k_{0}^{2} \mathbf{f}_{y}\right) \cdot \varphi d \Omega_{y}+ \\
& -\oint_{\partial \Omega_{y}}\left(\mathbf{f}_{y} \frac{\partial \varphi}{\partial \vec{y}}-\varphi \frac{\partial \mathbf{f}_{y}}{\partial \vec{y}}\right) \cdot \vec{n}_{y} d S_{y} \tag{4.70}
\end{align*}
$$

This result is quite general and is valid for any kind of vector space. Now we shall give a physical interpretation of it, applying it to the vector potential.

### 4.5.2 Sources for the vector EM potential

The EM vector potential $\vec{A}$ is a vector field so, holding eq. 4.70, it can be expressed as:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \vec{A}(\vec{x}) & =-\int_{\Omega_{y}}\left(\nabla_{y}^{2} \vec{A}(\vec{y})+k_{0}^{2} \vec{A}(\vec{y})\right) \cdot \varphi d \Omega_{y}+ \\
& -\oint_{\partial \Omega_{y}}\left(\vec{A}(\vec{y}) \vec{\nabla}_{y} \varphi-\varphi \frac{\partial \vec{A}_{y}}{\partial \vec{y}}\right) \cdot \vec{n}_{y} d S_{y} \tag{4.71}
\end{align*}
$$

More compactly:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \vec{A}(\vec{x}) & =-\int_{\Omega_{y}}\left(\nabla_{y}^{2} \vec{A}_{y}+k_{0}^{2} \vec{A}_{y}\right) \cdot \varphi d \Omega_{y}+ \\
& -\oint_{\partial \Omega_{y}}\left(\overline{\overline{A_{y} \nabla_{y} \varphi}}-\varphi \frac{\partial \vec{A}_{y}}{\partial \vec{y}}\right) \cdot \vec{n}_{y} d S_{y} \tag{4.72}
\end{align*}
$$

The dyad involving $\vec{A}_{y}$ and $\vec{\nabla}_{y} \varphi$ can be rephrased as:

$$
\begin{equation*}
\overline{\overline{A_{y} \nabla_{y} \varphi}} \cdot \vec{n}_{y}=\vec{A}_{y}\left(\vec{\nabla}_{y} \varphi^{T} \cdot \vec{n}_{y}\right)=\overline{\overline{A_{y} n_{y}}} \cdot \vec{\nabla}_{y} \varphi \tag{4.73}
\end{equation*}
$$

So it follows:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \vec{A}(\vec{x}) & =-\int_{\Omega_{y}}\left(\nabla_{y}^{2} \vec{A}_{y}+k_{0}^{2} \vec{A}_{y}\right) \cdot \varphi d \Omega_{y}+ \\
& -\oint_{\partial \Omega_{y}}\left(\overline{\overline{A_{y} n_{y}}}\right) \cdot \vec{\nabla}_{y} \varphi-\varphi\left(\frac{\partial \vec{A}_{y}}{\partial \vec{y}} \cdot \vec{n}_{y}\right) d S_{y} \tag{4.74}
\end{align*}
$$

The integral on $\Omega$ depends just on the distribution of currents $\vec{J}_{e}$ inside it, while the surface integral is related to the boundary sources.

## Elementary sources

Let us analyze the single terms in detail and properly define the sources:

- $\vec{J}_{e}$ is the density of electric current on $\Omega$ :

$$
\begin{equation*}
\vec{J}_{e}=-\frac{1}{\mu_{0}}\left(\nabla^{2} \vec{A}+k_{0}^{2} \vec{A}\right) \tag{4.75}
\end{equation*}
$$

- $\vec{J}_{s, e}$ is the surface current on $\partial \Omega$ :

$$
\begin{equation*}
\vec{J}_{s, e}=\frac{1}{\mu_{0}}\left(\frac{\partial \vec{A}_{y}}{\partial \vec{y}} \cdot \vec{n}_{y}\right) \tag{4.76}
\end{equation*}
$$

- $\overline{\bar{D}}_{e}$ are the doublets of currents per unit of surface on $\partial \Omega$

$$
\begin{align*}
& \overline{\bar{D}}_{e}=-\frac{1}{\mu_{0}}[\overline{\overline{A n}}]  \tag{4.77}\\
& \overline{\bar{D}}_{e}=-\frac{1}{\mu_{0}} \vec{A} \otimes \vec{n}  \tag{4.78}\\
& D_{e, i j}=-\frac{1}{\mu_{0}} A_{i} n_{j} \tag{4.79}
\end{align*}
$$

After a substitution of $\vec{J}_{e}, \vec{J}_{s, e}$ and $\overline{\bar{D}}_{e}$ in the integrals, we find:

$$
\begin{align*}
\epsilon_{\Omega}(\vec{x}) \frac{1}{\mu_{0}} \vec{A}(\vec{x})= & \int_{\Omega_{y}} \vec{J}_{e}(\vec{y}) \cdot \varphi_{G}(\vec{x}-\vec{y}) d \Omega_{y}+ \\
& \oint_{\partial \Omega_{y}} \vec{J}_{s, e}(\vec{y}) \cdot \varphi_{G}(\vec{x}-\vec{y}) d S_{y}+  \tag{4.80}\\
& \oint_{\partial \Omega_{y}} \overline{\bar{D}}_{e}(\vec{y}) \cdot\left(\vec{\nabla}_{y} \varphi_{G}(\vec{x}-\vec{y})\right) d S_{y}
\end{align*}
$$

More compactly:

$$
\begin{align*}
\epsilon_{\Omega} \frac{1}{\mu_{0}} \vec{A}= & \int_{\Omega_{y}} \vec{J}_{e, y} \cdot \varphi d \Omega_{y}+ \\
& \oint_{\partial \Omega_{y}} \vec{J}_{s, y} \cdot \varphi d S_{y}+  \tag{4.81}\\
& \oint_{\partial \Omega_{y}} \overline{\bar{D}}_{e} \cdot \vec{\nabla}_{y} \varphi d S_{y}
\end{align*}
$$

Actually, this equation states the Surface Equivalence Principle for the vector potential $\vec{A}$. In fact, all the sources external to $\Omega_{x}$ have been mapped on the boundary $\partial \Omega_{x}$, and we have demonstrated that they are of two types: currents $\vec{J}_{s, e}$ and current doublets $\overline{\bar{D}}_{e}$ per unit of surface.

### 4.6 Elementary fields and sources

In this section we report the fields generated by the elementary sources of the vector potential.

### 4.6.1 Field for a lumped current

Let $\vec{I}_{e}$ be a lumped, directional electric current, defined across a length $\Delta x$ centred on a point $\vec{x}_{0}$. The density distribution $\vec{J}_{e}$ associated to that current is:

$$
\begin{equation*}
\vec{J}_{e}(\vec{x})=\left(\vec{I}_{e} \Delta x\right) \delta_{N}\left(\vec{x}-\vec{x}_{0}\right) \tag{4.82}
\end{equation*}
$$

Hence, in the Laplace's domain the field $\vec{A}$ produced by $\vec{J}_{e}$ will be:

$$
\begin{equation*}
\vec{A}(\vec{x})=\mu_{0}\left(\vec{I}_{e} \Delta x\right) \varphi_{G}\left(\vec{x}-\vec{x}_{0}\right) \tag{4.83}
\end{equation*}
$$

In a 3 D space:

$$
\begin{equation*}
\vec{A}(\vec{x})=\mu_{0} \frac{1}{4 \pi r} e^{i k_{0} r}\left(\vec{I}_{e} \Delta x\right) \tag{4.84}
\end{equation*}
$$

It is quite easy to verify that a lumped oscillating current $\vec{I}_{e}$ correspond to an oscillating dipole $\vec{p}_{e}$.

$$
\begin{gather*}
\frac{\partial \rho_{e}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{J}_{e}=0 \quad \Longrightarrow  \tag{4.85}\\
\frac{\partial \vec{p}_{e}}{\partial t}=\vec{I}_{e} \Delta x  \tag{4.86}\\
\frac{\partial Q_{e}}{\partial t}=I_{e} \tag{4.87}
\end{gather*}
$$

More precisely, the charge conservation can be deduced by the Lorentz Gauge (see also sec. 3.2.1).

### 4.6.2 Field for a lumped current doublet

Given two equal and opposite current $\vec{I}_{e}$ and $-\vec{I}_{e}$, the net doublet $\overline{\bar{D}}_{E}$ associated to them is:

$$
\begin{gather*}
\overline{\bar{D}}_{E}=\vec{I}_{e} \otimes \Delta \vec{x}_{21}  \tag{4.88}\\
D_{E, i j}=I_{e, i} \Delta x_{j} \tag{4.89}
\end{gather*}
$$

where $\Delta \vec{x}_{21}=\vec{x}_{2}-\vec{x}_{1}$ is difference of positions between the charges. More generally, if the currents have different magnitudes and directions, the doublet can be calculated as:

$$
\begin{equation*}
\overline{\bar{D}}_{E}=\frac{1}{2}\left(\vec{I}_{e, 2}-\vec{I}_{e, 1}\right) \otimes \Delta \vec{x}_{21} \tag{4.90}
\end{equation*}
$$

Supposing the currents $\vec{I}_{e}$ and $-\vec{I}_{e}$ are placed on points $\vec{x}_{2}$ and $\vec{x}_{1}$ respectively, the vector field produced by them will be:

$$
\begin{equation*}
\frac{1}{\mu_{0}} \vec{A}(\vec{x})=\left(\vec{I}_{e} \Delta x\right)\left(\varphi_{G}\left(\vec{x}-\vec{x}_{2}\right)-\varphi_{G}\left(\vec{x}-\vec{x}_{1}\right)\right) \tag{4.91}
\end{equation*}
$$

Following the same steps of section 4.4.2 we find the field $\vec{A}$ produced by a lumped current doublet $\vec{D}_{E}$ :

$$
\begin{gather*}
\frac{1}{\mu_{0}} \vec{A}(\vec{x})=\vec{I}_{e}\left(\Delta \vec{x}_{21}^{T} \cdot \vec{\nabla} \varphi_{G}\right)  \tag{4.92}\\
\frac{1}{\mu_{0}} \vec{A}(\vec{x})=\left[\vec{I}_{e} \otimes \Delta \vec{x}_{21}\right] \cdot \vec{\nabla} \varphi_{G}  \tag{4.93}\\
\vec{A}(\vec{x})=\mu_{0} \overline{\bar{D}}_{E} \cdot \vec{\nabla} \varphi_{G} \tag{4.94}
\end{gather*}
$$

This result is valid also in the frequency domain.

### 4.6.3 Field for a generic doublet ensemble

A generic matrix $\overline{\bar{G}}$ can be expressed as the sum of many dyads, in fact:

$$
\begin{equation*}
\overline{\bar{G}}=\sum_{i=1}^{N} \sum_{j=1}^{N}\left(G_{i j} \overline{\overline{e_{i} e_{j}}}\right)=\sum_{i=1}^{N} \sum_{j=1}^{N}\left(G_{i j}\left[\vec{e}_{i} \otimes \vec{e}_{j}\right]\right) \tag{4.95}
\end{equation*}
$$

where $\vec{e}_{i}$ is the unitary vector for the $i^{\text {th }}$ direction. Therefore, by summing together many different doublets it is possible to construct the desired matrix $\overline{\bar{G}}$. In this way, the vector field $\vec{A}$ generated by the doublet ensemble can be compactly expressed as:

$$
\begin{equation*}
\vec{A}(\vec{x})=\overline{\bar{G}} \cdot \vec{\nabla} \varphi_{G} \tag{4.96}
\end{equation*}
$$

This is a very general form for describing dipoles. Let us see how to interpret $\overline{\bar{G}}$. That matrix can be split in its symmetric and anti-symmetric components $\overline{\bar{G}}_{D}$ and $\overline{\bar{G}}_{R}$, in fact:

$$
\begin{array}{lr}
\overline{\bar{G}}_{D}=\frac{1}{2}\left(\overline{\bar{G}}+\overline{\bar{G}}^{T}\right) & \text { symmetric matrix } \\
\overline{\bar{G}}_{R}=\frac{1}{2}\left(\overline{\bar{G}}-\overline{\bar{G}}^{T}\right) & \text { anti-symmetric matrix } \tag{4.98}
\end{array}
$$

The symmetric matrix $\overline{\bar{G}}_{D}$ can be diagonalized through a rotation $\overline{\bar{R}}$, so that:

$$
\overline{\bar{R}} \overline{\bar{G}}_{D} \overline{\bar{R}}^{T}=\overline{\bar{\Lambda}}=\left[\begin{array}{cccc}
\lambda_{1} & 0 & \cdots & 0  \tag{4.99}\\
0 & \lambda_{2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \lambda_{N}
\end{array}\right]
$$

Once the eigenvalues $\lambda_{i}$ are known, it possible to calculate the associate doublet $\overline{\bar{D}}_{d, i}$ as:

$$
\begin{equation*}
\overline{\bar{D}}_{d, i}=\frac{1}{\mu_{0}} \frac{1}{2}\left(\Delta I_{d, i} \Delta x\right) \overline{\overline{e_{i} e_{i}}} \tag{4.100}
\end{equation*}
$$

where $\left(\Delta I_{d, i} \Delta x\right) / 2=\lambda_{i}$. That can be interpreted as a quadrupole oscillating in the $i^{\text {th }}$ direction. On the contrary, the anti-symmetric $\overline{\bar{G}}_{R}$ is associated to magnetic "dipoles" $\overline{\bar{m}}$ or lumped vortices:

$$
\begin{equation*}
\overline{\bar{m}}=\overline{\bar{G}}_{R} \tag{4.101}
\end{equation*}
$$

The rotational field $\vec{A}_{R}$ can be so expressed as:

$$
\begin{array}{lrl}
\vec{A}_{R}(\vec{x}) & =\overline{\bar{m}} \cdot \vec{\nabla} \varphi_{G} & \text { ND notation } \\
\vec{A}_{R}(\vec{x})=\vec{m} \times \vec{\nabla} \varphi_{G} & \text { 3D notation } \tag{4.103}
\end{array}
$$

We remind that in 3D notation anti-symmetric matrices like $\overline{\bar{m}}$ can be represented by a pseudo-vector, but that is just a particular case 34, 35.

## Chapter 5

## Boundary Conditions for EM Potentials

> Maybe the same observations, and the same ideas have come to mind to others before me, and I have been preceded also in them, since the phenomena, i.e. the achieved effects, have to be the same in essence.
A. Volta, letter to the Counselor M. Landriani, 1801

In this chapter the Boundary Conditions (BCs) for the EM potentials $\rho_{A}$ and $\vec{A}$ will be considered, starting from the vector potential $\vec{A}$, since the procedure for $\rho_{A}$ is analogous.

### 5.1 Boundary Conditions for vector potential A

Let the field $\vec{A}$ be known on two different regions $\Omega_{1}$ and $\Omega_{2}$ of the space. The union of the domains $\Omega_{1}$ and $\Omega_{2}$ is equal to the whole space $\Omega_{\infty}$, and they are separated by a boundary $\partial \Omega$. The field $\vec{A}$ can be discontinuous across the


Figure 5.1: Vector potentials $\vec{A}_{1}, \vec{A}_{2}$ on domains $\Omega_{1}, \Omega_{2}$. The field's discontinuities are related to the boundary sources $\vec{J}_{s}, \overline{\bar{D}}_{e}$
boundary $\partial \Omega$, so we define the fields $\vec{A}_{1}$ and $\vec{A}_{2}$ on their respective domains:

$$
\begin{equation*}
\overrightarrow{A_{1}}=\in_{\Omega_{1}} \vec{A} ; \quad \overrightarrow{A_{2}}=\in_{\Omega_{2}} \vec{A} \tag{5.1}
\end{equation*}
$$

Remind that $\vec{A}_{1}$ is equal to $\vec{A}$ inside $\Omega_{1}$, but it is null outside.


Figure 5.2: Subdivision of fields $\vec{A}_{1}, \vec{A}_{2}$ on domains $\Omega_{1}, \Omega_{2}$. (a) Field $\vec{A}_{1}=\in_{\Omega_{1}} \vec{A}$; (b) Field $\vec{A}_{2}=\in_{\Omega_{2}} \vec{A}$.

Now we want to determine the Boundary Conditions for $\vec{A}_{1}$, calculating the equivalent sources on $\partial \Omega$.

The normal $\vec{n}_{21}$ to the boundary points from $\Omega_{1}$ to $\Omega_{2}$, so that it is antisymmetric with respect to the exchange of the subscripts 1 and 2 :

$$
\begin{equation*}
\vec{n}_{21}=-\vec{n}_{21} \tag{5.2}
\end{equation*}
$$

Thanks to 4.76, 4.77), the Boundary Conditions for $\overrightarrow{A_{1}}$ are:

$$
\left\{\begin{array}{c}
\vec{J}_{s, 1}=\frac{1}{\mu_{0}} \frac{\partial \vec{A}_{1}}{\partial \vec{x}} \cdot \vec{n}_{21}  \tag{5.3}\\
\overline{\bar{D}}_{e, 1}=-\frac{1}{\mu_{0}}\left[\overline{\overline{A_{1} n_{21}}}\right]
\end{array}\right.
$$

Analogously, the BCs for field $\vec{A}_{2}$ on $\Omega_{2}$ will be:

$$
\left\{\begin{array}{c}
\vec{J}_{s, 2}=\frac{1}{\mu_{0}} \frac{\partial \vec{A}_{2}}{\partial \vec{x}} \cdot \vec{n}_{12}  \tag{5.4}\\
\overline{\bar{D}}_{e, 2}=-\frac{1}{\mu_{0}}\left[\overline{\overline{A_{2} n_{12}}}\right]
\end{array}\right.
$$

Summing together the fields $\vec{A}_{1}$ and $\vec{A}_{2}$ implies also the sum of the boundary sources, since the Helmholtz operator is linear.

$$
\begin{equation*}
\vec{A}=\vec{A}_{1}+\vec{A}_{2} \tag{5.5}
\end{equation*}
$$

So we can define the global sources on the boundary $\partial \Omega$ as:

$$
\begin{align*}
\vec{J}_{s} & =\vec{J}_{s, 1}+\vec{J}_{s, 2}  \tag{5.6}\\
\overline{\bar{D}}_{e} & =\overline{\bar{D}}_{e, 1}+\overline{\bar{D}}_{e, 2} \tag{5.7}
\end{align*}
$$

Summing systems (5.3) and (5.4), we finally obtain the complete set of Boundary Conditions for field $\vec{A}$ :

$$
\begin{align*}
\vec{J}_{s} & =-\frac{1}{\mu_{0}}\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}  \tag{5.8}\\
\overline{\bar{D}}_{e} & =\frac{1}{\mu_{0}}\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21} \tag{5.9}
\end{align*}
$$

### 5.2 Boundary Conditions for scalar potential $\rho_{A}$

In the case we consider the scalar field $\rho_{A}$ the procedure for deriving the BCs is analogous to the one for $\vec{A}$, but a little easier, Briefly:

1. Define the fields on the domains $\Omega_{1}$ and $\Omega_{2}$ :

$$
\begin{equation*}
\rho_{A 1}=\in_{\Omega_{1}} \rho_{A} ; \quad \quad \rho_{A 2}=\in_{\Omega_{2}} \rho_{A} \tag{5.10}
\end{equation*}
$$

2. Calculate the boundary sources for both the fields:

$$
\left\{\begin{array} { l } 
{ \sigma _ { e 1 } = \frac { 1 } { \mu _ { 0 } } \frac { \partial \rho _ { A 1 } } { \partial \vec { x } } \cdot \vec { n } _ { 2 1 } }  \tag{5.11}\\
{ \vec { d } _ { e 1 } = - \frac { 1 } { \mu _ { 0 } } \rho _ { A 1 } \vec { n } _ { 2 1 } }
\end{array} \left\{\begin{array}{l}
\sigma_{e 2}=\frac{1}{\mu_{0}} \frac{\partial \rho_{A 2}}{\partial \vec{x}} \cdot \vec{n}_{12} \\
\vec{d}_{e 2}=-\frac{1}{\mu_{0}} \rho_{A 2} \vec{n}_{12}
\end{array}\right.\right.
$$

3. Sum the fields and their sources, finding so the Boundary Conditions:

$$
\begin{gather*}
\rho_{A}=\rho_{A 1}+\rho_{A 2}  \tag{5.12}\\
\sigma_{e}=-\frac{1}{\mu_{0}}\left(\frac{\partial \rho_{A 2}}{\partial \vec{x}}-\frac{\partial \rho_{A 1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}  \tag{5.13}\\
\vec{d}_{e}=\frac{1}{\mu_{0}}\left(\rho_{A 2}-\rho_{A 1}\right) \vec{n}_{12} \tag{5.14}
\end{gather*}
$$

### 5.3 Summary for the wave BC

As we already pointed out, the charges $\sigma_{e}$ and the currents $\vec{J}_{s}$ generate discontinuities for the gradients of $\rho_{A}$ and $\vec{A}$ respectively. Conversely, dipoles $\overrightarrow{d_{e}}$ and current doublets $\overline{\bar{D}}_{e}$ generate the discontinuities for the fields themselves, without changing their gradient.

In table 5.1 we report the set of BC for the EM potentials:

## Table 5.1: Boundary Condition for EM potentials

|  | Gradient disc. | Field disc. |  |
| :--- | :---: | :---: | :---: |
| Charges | $\mu_{0} \sigma_{e}=-\left(\frac{\partial \rho_{A 2}}{\partial \vec{x}}-\frac{\partial \rho_{A 1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}$ | $\mu_{0} \vec{d}_{e}=$ | $\left(\rho_{A 2}-\rho_{A 1}\right) \vec{n}_{12}$ |
| Currents | $\mu_{0} \vec{J}_{s}=-\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}$ | $\mu_{0} \overline{\bar{D}}_{e}=$ | $\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21}$ |


(a) Discontinuity for the gradient of $\rho_{A}$. In the example the gradient is null on domain $\Omega_{1}$, while it is positive on domain $\Omega_{2}$. The surface charge $\sigma_{e}$ is negative.

(c) Discontinuity for the gradient of $\vec{A}_{t}$. In the example the gradient is null on domain $\Omega_{1}$, while it is positive on domain $\Omega_{2}$. The surface current $\vec{J}_{s, t}$ is tangential to the boundary.

(b) Discontinuity for scalar potential $\rho_{A}$. In the example the gradients are null on both domains $\Omega_{1}$ and $\Omega_{2}$. The electric surface dipoles $\vec{d}_{e}$ point from side 1 to 2 .

(d) Discontinuity for vector potential $\vec{A}_{t}$. In the example the gradients are null on both domains $\Omega_{1}$ and $\Omega_{2}$. The surface current doublets $\bar{D}_{e} \cdot \vec{n}_{21}$ are tangential to the boundary.

Figure 5.3: Boundary Conditions for the EM potentials and their gradients.

Some graphical examples (fig 5.3) could be useful to better understand the physical meaning of those variables. In figure 5.3d we have guessed that $\vec{A}$ is perpendicular to $\vec{n}$, but we cannot exclude that the vector potentials is endowed with a component $\vec{A}_{n} / / \vec{n}$. In order to enforce discontinuities for $\vec{A}_{n}$ and its gradient, currents orthogonal to the surface are needed.

(a) Discontinuity for the gradient of $\vec{A}_{n}$. In the example the gradient is null on domain $\Omega_{1}$, while it is positive on domain $\Omega_{2}$. The surface current $\vec{J}_{s, n}$ is perpendicular to the boundary.

(b) Discontinuity for vector potential $\vec{A}_{n}$. In the example the gradients are null on both domains $\Omega_{1}$ and $\Omega_{\underline{2}}$. The surface current doublets $\bar{D}_{e} \vec{n}_{21}$ are perpendicular to the boundary.

Figure 5.4: Boundary Conditions for the vector potential normal component $\vec{A}_{n}$ and its gradient.

### 5.3.1 A first circuit interpretation

In order to make the gradients $\frac{\partial \rho_{A}}{\partial \vec{x}}$ and $\frac{\partial \vec{A}}{\partial \vec{x}}$ discontinuous, we need just 1 layer of charges $\sigma_{e}$ and currents $\vec{J}_{s}$. Otherwise, if we are interested to have a discontinuity in the fields $\rho_{A}$ and $\vec{A}$ we need doublets of charges $\vec{d}_{e}$ and currents $\overline{\bar{D}}_{e}$, which are associated to 2 layers of charges and currents.

In other words, if we want to construct a metasurface or a screen, we should take into account that it will be made of at least 2 layers. That is particularly important also if we model the screen with circuits. I will come back later on that point: now I leave just of suggestion about it.

Since the screen is made of 2 layers, we can guess there are two distributions of charges $\sigma_{e, 1}, \sigma_{e, 2}$ and currents $\vec{J}_{s, 1}, \vec{J}_{s, 2}$ on them, such that:

$$
\left\{\begin{array} { l } 
{ \sigma _ { e } = \sigma _ { e 2 } + \sigma _ { e 1 } }  \tag{5.15}\\
{ \vec { d } _ { e } = \frac { 1 } { 2 } ( \sigma _ { e 2 } - \sigma _ { e 1 } ) \Delta x \vec { n } _ { 2 1 } }
\end{array} \quad \left\{\begin{array}{l}
\vec{J}_{s}=\vec{J}_{s 2}+\vec{J}_{s 1} \\
\overline{\bar{D}}_{e}=\frac{1}{2}\left(\vec{J}_{s 2}-\vec{J}_{s 1}\right) \otimes \vec{n}_{21} \Delta x
\end{array}\right.\right.
$$

where $\Delta x$ is the screen's width. If the screen is modeled in terms of circuit network, then charges $\sigma_{e 1}$ and $\sigma_{e 2}$ will be node variables, while currents $J_{s 1}$ and $J_{s 2}$ will flow through the edges. Once the Boundary Conditions are known,


Figure 5.5: First circuit model for 2-layer screen. Surface charges $\sigma_{e 1}$ and $\sigma_{e 2}$ are associated to nodes, while surface currents $\vec{J}_{s 1}, \vec{J}_{s 2}$ flow through the edges.
charges and currents on layers 1 and 2 can be calculated as:

$$
\left\{\begin{array} { l } 
{ \sigma _ { e 1 } = \frac { 1 } { 2 } \sigma _ { e } - \frac { 1 } { \Delta x } \vec { d } _ { e } ^ { T } \cdot \vec { n } _ { 2 1 } }  \tag{5.16}\\
{ \sigma _ { e 2 } = \frac { 1 } { 2 } \sigma _ { e } + \frac { 1 } { \Delta x } \vec { d } _ { e } ^ { T } \cdot \vec { n } _ { 2 1 } }
\end{array} \quad \left\{\begin{array}{l}
\vec{J}_{s 1}=\frac{1}{2} \vec{J}_{s}-\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \\
\vec{J}_{s 2}=\frac{1}{2} \vec{J}_{s}+\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right.\right.
$$

### 5.4 Conditions required by Lorentz and Faraday

If we look just at the wave equations (4.8 (4.9), the potential $\rho_{A}$ and $\vec{A}$ would appear to be two distinct, non-interacting fields. However, they are linked each other by the Lorentz Gauge and by the Faraday's Law:

$$
\begin{array}{ll}
\frac{\partial \rho_{A}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{A}=0 & \text { Lorentz Gauge } \\
\frac{\partial \vec{A}}{\partial t}=-\vec{\nabla}\left(\rho_{A} c_{0}^{2}\right)-\vec{E} & \text { Faraday's Law } \tag{5.18}
\end{array}
$$

Let's see which are the additional Boundary Conditions related to them.

### 5.4.1 BC for Lorentz Gauge

As previously said, the Lorentz Gauge can be interpreted as a continuity equation for the scalar potential $\rho_{A}$. In integral form, it will look:

$$
\begin{equation*}
\frac{\partial}{\partial t} \int_{\Omega} \rho_{A} d \Omega_{x}+\oint_{\partial \Omega} \vec{n}^{T} \cdot \vec{A} d S_{x}=0 \tag{5.19}
\end{equation*}
$$

We can consider a thin screen of width $\Delta x$ and an elementary surface $S$, thus the integral can be approximated as:

$$
\begin{align*}
& \frac{\partial}{\partial t}\left(\left\langle\rho_{A}\right\rangle S \Delta x\right)+S \cdot\left(\vec{A}_{2}-\vec{A}_{1}\right)^{T} \cdot \vec{n}_{21}=0  \tag{5.20}\\
& \frac{\partial}{\partial t}\left\langle\rho_{A}\right\rangle \Delta x+\left(\overrightarrow{A_{2}}-\vec{A}_{1}\right)^{T} \cdot \vec{n}_{21}=0 \tag{5.21}
\end{align*}
$$

where:

- $\Delta x \rightarrow 0^{+}$is the screen's width.
- $\left\langle\rho_{A}\right\rangle \approx \frac{1}{2}\left(\rho_{A 2}+\rho_{A 1}\right)$ is the average potential on the unit surface.
- $\vec{A}_{2}$ and $\vec{A}_{1}$ are vectors on the two sides of the surface

Since $\Delta x$ tends to zero, if we require that $\rho_{A}$ and its time derivative had a finite value, then it must hold:

$$
\begin{align*}
& \left|\frac{\partial}{\partial t}\left\langle\rho_{A}\right\rangle\right|<+\infty \quad \Longrightarrow \quad \lim _{\Delta x \rightarrow 0^{+}}\left(\frac{\partial}{\partial t}\left\langle\rho_{A}\right\rangle \Delta x\right)=0 \quad \Longrightarrow  \tag{5.22}\\
& \left(\vec{A}_{2}-\vec{A}_{1}\right)^{T} \cdot \vec{n}_{21}=0 \quad \text { BC from the Lorentz Gauge }  \tag{5.23}\\
& A_{n 2}=A_{n 1} \tag{5.24}
\end{align*}
$$

Thus the Lorentz Gauge requires the normal component of $\vec{A}$ to be continuous across the surface.

In other words, there are no "shock waves" or violent compression across the screen. Let's notice that the condition 5.23 could be derived also using the Coulomb's Gauge ( $\left.\vec{\nabla}^{T} \cdot \vec{A}=0\right)$.

That condition on the normal component of $\vec{A}$ joins to the one imposed by the wave equation, in fact:

$$
\left\{\begin{array}{l}
\left(\vec{A}_{2}-\vec{A}_{1}\right)^{T} \cdot \vec{n}_{21}=0  \tag{5.25}\\
\mu_{0} \overline{\bar{D}}_{e}=\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21}
\end{array}\right.
$$

## Current doublets and magnetic moments

If we know the discontinuity for $\vec{A}$, then we can easily construct the matrix $\overline{\bar{D}}_{e}$ as a dyad. On the contrary, if we want to determine $\vec{A}_{2}-\vec{A}_{1}$, we multiply $\overline{\bar{D}}_{e}$ for the normal $\vec{n}$.

$$
\begin{align*}
\vec{A}_{2}-\vec{A}_{1} & =\mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{5.26}\\
\Delta \vec{A} & =\mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n} \quad \text { compact form }  \tag{5.27}\\
\Delta A_{i} & =\mu_{0} \sum_{j=1}^{N} D_{e, i j} \cdot \vec{n}_{j} \tag{5.28}
\end{align*}
$$

Since the normal component of $\vec{A}$ must be continuous, in some cases the dyad $\overline{\bar{D}}_{e}$ has the same effect of an anti-symmetric matrix, in fact:

$$
\begin{align*}
& \Delta \vec{A}^{T} \cdot \vec{n}=0 \quad \Longrightarrow \quad \vec{n} \cdot\left(\Delta \vec{A}^{T} \cdot \vec{n}\right)=\overrightarrow{0} \quad \Longrightarrow  \tag{5.29}\\
& \overline{\overline{n \Delta A}} \cdot \vec{n}=\overrightarrow{0}  \tag{5.30}\\
& \mu_{0} \overline{\bar{D}}_{e}=\overline{\overline{\Delta A n}}  \tag{5.31}\\
& \mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n}=(\overline{\overline{\Delta A n}}-\overline{\overline{n \Delta A}}) \cdot \vec{n} \tag{5.32}
\end{align*}
$$

The difference of the two dyads is an anti-symmetric matrix and it is exactly the ND cross product between $\vec{n}$ and $\Delta \vec{A}$ :

$$
\begin{equation*}
\vec{n} \wedge \Delta \vec{A}=\overline{\overline{\Delta A n}}-\overline{\overline{n \Delta A}} \quad \text { ND cross product } \tag{5.33}
\end{equation*}
$$

So we can also write:

$$
\begin{array}{ll}
\mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n}=[\vec{n} \wedge \Delta \vec{A}] \cdot \vec{n} & \text { ND notation } \\
\mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n}=(\vec{n} \times \Delta \vec{A}) \times \vec{n} & \text { 3D notation } \tag{5.35}
\end{array}
$$

The anti-symmetric matrix can be interpreted as a magnetic moment per unit of surface $\overline{\bar{m}} / S$ such that:

$$
\begin{array}{rlr}
\frac{1}{S} \overline{\bar{m}} & =[\vec{n} \wedge \Delta \vec{A}] \\
\overline{\bar{m}} & =-\overline{\bar{m}}^{T} & \\
\Delta \vec{A} & =\frac{1}{S} \overline{\bar{m}} \cdot \vec{n} \quad & \text { ND notation } \\
\Delta \vec{A} & =\frac{1}{S} \vec{m} \times \vec{n} & \text { 3D notation } \tag{5.39}
\end{array}
$$

More precisely, the magnetic moment can be expressed as the difference of two currents doublets:

$$
\begin{equation*}
\overline{\bar{m}}=S \mu_{0}\left(\overline{\bar{D}}_{e}-\overline{\bar{D}}_{e}^{T}\right) \tag{5.40}
\end{equation*}
$$

By construction, $\vec{m}$ will automatically result tangent to the surface.
Shortly, in many contexts the effect of a current doublet $\overline{\bar{D}}_{e}$ is the same of a magnetic "dipole" $\vec{m}$ tangent on the surface. In both the cases a discontinuity is produced in the tangential component $\vec{A}_{t}$ of vector potential.

## Tangential component for A

The tangential component $\vec{A}_{t}$ of the vector potential can be expressed as:

$$
\begin{array}{rlr}
\overrightarrow{A_{t}} & =[\vec{n} \wedge \vec{A}] \cdot \vec{n} & \text { ND notation } \\
\overrightarrow{A_{t}} & =(\vec{n} \times \vec{A}) \times \vec{n} & \text { 3D notation } \tag{5.42}
\end{array}
$$

Since the normal component $\vec{A}_{n}$ is continuous across the surface, the discontinuity for $\vec{A}$ involves just the tangential component:

$$
\begin{equation*}
\Delta \vec{A}=\vec{A}_{2}-\vec{A}_{1}=\vec{A}_{2 t}-\vec{A}_{1 t} \tag{5.43}
\end{equation*}
$$

So we can express that difference in function of the current doublets:

$$
\begin{equation*}
\vec{A}_{2 t}-\vec{A}_{1 t}=\mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \tag{5.44}
\end{equation*}
$$

In other form:

$$
\begin{align*}
& \vec{A}_{2 t}-\vec{A}_{1 t}=\mu_{0}\left(\overline{\bar{D}}_{e}-\overline{\bar{D}}_{e}^{T}\right) \cdot \vec{n}_{21}  \tag{5.45}\\
& \vec{A}_{2 t}-\vec{A}_{1 t}=\frac{1}{S} \overline{\bar{m}} \cdot \vec{n}_{21} \tag{5.46}
\end{align*}
$$

Conversely, the doublet distribution and the magnetic moment can be calculated as:

$$
\begin{align*}
\mu_{0} \overline{\bar{D}}_{e} & =\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right) \otimes \vec{n}_{21}  \tag{5.47}\\
\frac{1}{S} \overline{\bar{m}} & =\vec{n}_{21} \hat{\wedge}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right) \tag{5.48}
\end{align*}
$$

### 5.4.2 BC for Faraday's Law

The Faraday's Law can be regarded as a balance equation for the electromagnetic momentum.

$$
\begin{align*}
& \frac{\partial \vec{A}}{\partial t}=-\vec{\nabla}\left(\rho_{A} c_{0}^{2}\right)-\vec{E}  \tag{5.49}\\
& \frac{\partial \vec{A}}{\partial t}=-\vec{\nabla} P_{A}-\vec{E} \tag{5.50}
\end{align*}
$$

In integral form, it will look:

$$
\begin{equation*}
\frac{\partial}{\partial t} \int_{\Omega} \vec{A} d \Omega_{x}+\int_{\Omega} \vec{E} d \Omega_{x}=-\oint_{\partial \Omega} P_{A} \vec{n} d S_{x} \tag{5.51}
\end{equation*}
$$

We can consider a thin screen of width $\Delta x$ and an elementary surface $S$, thus the integral can be approximated as:

$$
\begin{align*}
& \left(\frac{\partial}{\partial t}\langle\vec{A}\rangle+\langle\vec{E}\rangle\right) S \Delta x=-S\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21}  \tag{5.52}\\
& \left(\frac{\partial}{\partial t}\langle\vec{A}\rangle+\langle\vec{E}\rangle\right) \Delta x=-\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21} \tag{5.53}
\end{align*}
$$

where:

- $\Delta x \rightarrow 0^{+}$is the screen's width.
- $\langle\vec{A}\rangle \approx \frac{1}{2}\left(\vec{A}_{2}+\vec{A}_{1}\right)$ is the average vector potential on the unit surface.
- $\langle\vec{E}\rangle$ is the average electric field on the unit surface.
- $P_{A 2}$ and $P_{A 1}$ are the potentials on the two sides of the surface

Since $\Delta x$ tends to zero, if we require that $\vec{A}$ and its time derivative had a finite value, then it must hold:

$$
\begin{align*}
&\left\|\frac{\partial}{\partial t}\langle\vec{A}\rangle\right\|<+\infty \quad \Longrightarrow \quad \lim _{\Delta x \rightarrow 0^{+}}\left(\frac{\partial}{\partial t}\langle\vec{A}\rangle \Delta x\right)=0 \quad \Longrightarrow  \tag{5.54}\\
&\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21}=-\lim _{\Delta x \rightarrow 0^{+}}(\langle\vec{E}\rangle \Delta x) \quad \text { BC from the Faraday's Law } \tag{5.55}
\end{align*}
$$

## Case of finite electric field

If we also require that the electric field has a finite value on the screen, then we get:

$$
\begin{align*}
& \|\langle\vec{E}\rangle\|<+\infty \quad \Longrightarrow \quad \lim _{\Delta x \rightarrow 0^{+}}(\langle\vec{E}\rangle \Delta x)=0 \quad \Longrightarrow  \tag{5.56}\\
& \left(P_{A 2}-P_{A 1}\right) \vec{n}_{21}=\overrightarrow{0}  \tag{5.57}\\
& P_{A 2}=P_{A 1} \quad \text { continuity of the scalar potential } \tag{5.58}
\end{align*}
$$

So the Faraday's Law seems to state the continuity of the scalar potential $P_{A}=\rho_{A} c_{0}^{2}$ across the screen, except if $\vec{A}$ and $\vec{E}$ are not finite on the boundary.

Let's notice that if the potential $\rho_{A}$ is continuous, then there are no orthogonal electric dipoles on the surface, in fact:

$$
\begin{align*}
\rho_{A 2}=\rho_{A 1} & \text { and } \quad \mu_{0} \vec{d}_{e}=\left(\rho_{A 2}-\rho_{A 1}\right) \vec{n}_{21} \quad \Longrightarrow  \tag{5.59}\\
& \vec{d}_{e}=\overrightarrow{0} \quad \text { no electric dipoles } \tag{5.60}
\end{align*}
$$

### 5.4.3 A counterexample: the Volta Effect

Apparently, the requests that the potential $V=P_{A}$ is continuous and that the electric field $\vec{E}$ has a finite value seem quite reasonable. However, there is a counterexample to that situation, since actually in Nature sharp "jump" can occur. That counterexample is the Volta Effect, named after Alessandro Volta who discovered and exploited it to invent the first electric battery in 1801.

In a metal the electrons can move quite freely through the crystal lattice. Therefore, in order to extract an electron from the metal you should provide an extraction work equal to $(-e) \Delta V$, where $\Delta V$ is the extraction potential. Each metal or conductor is characterized by its own extraction potential $\Delta V_{i}$, which depends by the temperature, oxidation etc. For example, at $\mathrm{T}=20^{\circ} \mathrm{C}$ the potentials for the copper Cu and the zinc Zn are:

$$
\begin{equation*}
\Delta V_{C u}=4,4 V \quad \Delta V_{Z n}=3,4 V \tag{5.61}
\end{equation*}
$$

Volta discovered that, once two different metals are in contact, a difference of potential occurs between them. That voltage $\Delta V$ is equal to the difference of


Figure 5.6: Volta Effect. Between two different metals at contact a voltage $\Delta V$ arise, which is equal to the difference of extraction potentials for the single metals.
the extraction potentials for the two metals.

$$
\begin{equation*}
\Delta V_{Z n-C u}=\Delta V_{Z n}-\Delta V_{C u}=1,0 V \tag{5.62}
\end{equation*}
$$

The Volta Effect is due to the electrons which tend to migrate towards the metal with the higher extraction potential, till the equilibrium is attained.

In our example, the copper Cu will store a negative charge, while the zinc Zn will store a positive one. Let us now analyze the system and the Boundary Conditions:

- the metals are conductors and the system is static. Therefore inside any single metal the electric field is null:

$$
\begin{equation*}
\vec{E}=-\vec{\nabla} V=\overrightarrow{0} \quad \text { for } \vec{x} \in \Omega_{1} \text { or } \vec{x} \in \Omega_{2} \tag{5.63}
\end{equation*}
$$

- on the interface between the two metals there is a potential's jump, hence from the BCs we find a layer of dipoles $\vec{d}_{e}$ on the boundary.

$$
\begin{align*}
V_{2}-V_{1} & =c_{0}^{2}\left(\rho_{A 2}-\rho_{A 1}\right)=c_{0}^{2} \mu_{0} \vec{d}_{e}^{T} \cdot \vec{n}_{21} \quad \Longrightarrow  \tag{5.64}\\
\vec{d}_{e} & =\varepsilon_{0}\left(V_{2}-V_{1}\right) \vec{n}_{21} \tag{5.65}
\end{align*}
$$

In static conditions $\vec{E}=-\vec{\nabla} V$, so formally the electric field should be infinite on the boundary $\partial \Omega$. In reality that does not happen, but it is true that locally $\vec{E}$ can have very high values:

$$
\begin{equation*}
\|\langle\vec{E}\rangle\| \rightarrow+\infty \tag{5.66}
\end{equation*}
$$



Figure 5.7: Volta Effect. In static case, the scalar potentials $\rho_{A 1}$ and $\rho_{A 2}$ are constant inside each metal, thus the electric fields $\vec{E}_{1}$ and $\vec{E}_{2}$ are null. At the interface the average field $\langle\vec{E}\rangle$ tends to infinity because of the potential discontinuity $\Delta V_{21}$ generated by surface dipoles $\vec{d}_{e}$.

## Computational aspects

Modelling a potential discontinuity $V_{2}-V_{1}$ could be quite simple from a numerical point of view. For example, in a Finite Volume Method (FVM) it is possible to associate a certain potential $V$ to each cell of the mesh. Instead, modelling the Volta Effect using the electric field $\vec{E}$ could be computationally difficult, because on the cell's boundary $\vec{E}$ tends to infinity.

### 5.4.4 Normal and tangential components for Faraday

For further tasks it is useful to separate the normal and tangential components for the Faraday's Law. Taking eq. 5.53), it can be decomposed or projected as:

$$
\left\{\begin{array}{rlr}
\left(\frac{\partial}{\partial t}\left\langle\vec{A}_{n}\right\rangle+\left\langle\vec{E}_{n}\right\rangle\right) \Delta x=-\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21} & \text { normal component }  \tag{5.67}\\
\left(\frac{\partial}{\partial t}\left\langle\vec{A}_{t}\right\rangle+\left\langle\vec{E}_{t}\right\rangle\right) \Delta x=\overrightarrow{0} & \text { tangential component }
\end{array}\right.
$$

Requiring that $\vec{A}$ and its derivative have finite values, it yields:

$$
\left\{\begin{align*}
\lim _{\Delta x \rightarrow 0^{+}}\left(\left\langle\vec{E}_{n}\right\rangle \Delta x\right) & =-\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21}  \tag{5.68}\\
\lim _{\Delta x \rightarrow 0^{+}}\left(\left\langle\vec{E}_{t}\right\rangle \Delta x\right) & =\overrightarrow{0}
\end{align*}\right.
$$

From the last equation it follows that the tangential component of $\vec{E}$ has a finite value, so it can be set equal to the average between the fields on the two sides of the screen:

$$
\begin{equation*}
\left\|\left\langle\vec{E}_{t}\right\rangle\right\|<+\infty \quad \Longrightarrow \quad\left\langle\vec{E}_{t}\right\rangle=\frac{1}{2}\left(\vec{E}_{2 t}+\vec{E}_{1 t}\right) \tag{5.69}
\end{equation*}
$$

The normal component of $\vec{E}_{n}$ instead can have an infinite value and it is associate to the local surface dipole $\vec{d}_{e}$ :

$$
\begin{gather*}
\left\langle\vec{E}_{n}\right\rangle \Delta x=-\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21}=-\mu_{0} c_{0}^{2} \vec{d}_{e}  \tag{5.70}\\
\mu_{0} \vec{d}_{e}=-\frac{1}{c_{0}^{2}}\left\langle\vec{E}_{n}\right\rangle \Delta x \tag{5.71}
\end{gather*}
$$

In general, $\left\langle\vec{E}_{n}\right\rangle$ could be different from the average between the field on the two sides of the screen:

$$
\begin{equation*}
\left\langle\vec{E}_{n}\right\rangle \neq \frac{1}{2}\left(\vec{E}_{2 n}+\vec{E}_{1 n}\right) \tag{5.72}
\end{equation*}
$$

## Direct projection

The Faraday's Law provides another Boundary Condition for the electric field, which involves the gradient of the potential $P_{A}$. The electric field can be expressed as:

$$
\begin{equation*}
\vec{E}=-\left(\frac{\partial \vec{A}}{\partial t}+\vec{\nabla} P_{A}\right) \tag{5.73}
\end{equation*}
$$

This differential equation must be valid on both the boundary sides 1 and 2 , thus we can write:

$$
\begin{gather*}
\left\{\begin{aligned}
\vec{E}_{1} & =-\left(\frac{\partial \vec{A}_{1}}{\partial t}+\vec{\nabla} P_{A 1}\right) \\
\vec{E}_{2} & =-\left(\frac{\partial \vec{A}_{2}}{\partial t}+\vec{\nabla} P_{A 2}\right)
\end{aligned}\right.  \tag{5.74}\\
\vec{E}_{2}-\vec{E}_{1}=-\left(\frac{\partial}{\partial t}\left(\vec{A}_{2}-\overrightarrow{A_{1}}\right)+\left(\vec{\nabla} P_{A 2}-\vec{\nabla} P_{A 1}\right)\right) \tag{5.75}
\end{gather*}
$$

The eq. 5.75 can be used to calculate the discontinuity for $\vec{E}$ once the potentials and their derivatives are known. We can separate the normal and tangential component for the vectors in 5.75, obtaining so:

$$
\left\{\begin{align*}
\vec{E}_{2 n}-\vec{E}_{1 n} & =-\left(\frac{\partial}{\partial t}\left(\vec{A}_{2 n}-\vec{A}_{1 n}\right)+\left(\vec{\nabla}_{n} P_{A 2}-\vec{\nabla}_{n} P_{A 1}\right)\right)  \tag{5.76}\\
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\left(\frac{\partial}{\partial t}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right)+\left(\vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}\right)\right)
\end{align*}\right.
$$

From the BC for Lorentz Gauge 5.23 we know that the normal component $\vec{A}_{n}$ is continuous across the surface, so:

$$
\begin{align*}
& \vec{A}_{2 n}-\vec{A}_{1 n}=\overrightarrow{0} \quad \Longrightarrow  \tag{5.77}\\
& \vec{E}_{2 n}-\vec{E}_{1 n}=-\left(\vec{\nabla}_{n} P_{A 2}-\vec{\nabla}_{n} P_{A 1}\right)  \tag{5.78}\\
& E_{2 n}-E_{1 n}=-\left(\vec{\nabla}_{A 2}-\vec{\nabla} P_{A 1}\right)^{T} \cdot \vec{n}_{21} \quad \text { scalar form } \tag{5.79}
\end{align*}
$$

The last equation can be rephrased to link the normal electric field $E_{n}$ to the surface charge $\sigma_{e}$. In fact, holding the BC 5.13 we can write:

$$
\left\{\begin{array}{c}
E_{2 n}-E_{1 n}=-c_{0}^{2}\left(\vec{\nabla} \rho_{A 2}-\vec{\nabla} \rho_{A 1}\right)^{T} \cdot \vec{n}_{21} \\
\mu_{0} \sigma_{e}=-\left(\vec{\nabla} \rho_{A 2}-\vec{\nabla} \rho_{A 1}\right)^{T} \cdot \vec{n}_{21}  \tag{5.81}\\
E_{2 n}-E_{1 n}=\mu_{0} c_{0}^{2} \sigma_{e}
\end{array} \Longrightarrow\right.
$$

More explicitly:

$$
\begin{equation*}
\vec{n}_{21}^{T} \cdot\left(\vec{E}_{2}-\vec{E}_{1}\right)=\frac{1}{\varepsilon_{0}} \sigma_{e} \tag{5.82}
\end{equation*}
$$

### 5.5 Summary for the BC from Lorentz and Faraday

In this section we report the set of Boundary Conditions related to the Lorentz Gauge and to the Faraday's Law. In all the cases we require that the EM potentials $P_{A}=\rho_{A} c_{0}^{2}, \vec{A}$ and their derivatives have finite values.

## Normal component for A

The continuity for the normal component $\vec{A}_{n}$ of the vector potential follows from the Lorentz Gauge:

$$
\begin{equation*}
\vec{A}_{2 n}-\vec{A}_{1 n}=\overrightarrow{0} \tag{5.83}
\end{equation*}
$$

## Average electric field on the surface

For a screen of width $\Delta x \rightarrow 0^{+}$, the normal component $\vec{E}_{n}$ of the electric field can go to infinity. Instead, the tangential component $\vec{E}_{t}$ has a finite value. Those conditions come from the integral Faraday's Law (5.51).

$$
\left\{\begin{array} { r l } 
{ \operatorname { l i m } _ { \Delta x \rightarrow 0 ^ { + } } ( \langle \vec { E } _ { n } \rangle \Delta x ) } & { = - ( P _ { A 2 } - P _ { A 1 } ) \vec { n } _ { 2 1 } }  \tag{5.84}\\
{ \operatorname { l i m } _ { \Delta x \rightarrow 0 ^ { + } } ( \langle \vec { E } _ { t } \rangle \Delta x ) } & { = \vec { 0 } }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\left\langle\vec{E}_{n}\right\rangle \neq \frac{1}{2}\left(\vec{E}_{2 n}+\vec{E}_{1 n}\right) \\
\left\langle\vec{E}_{t}\right\rangle=\frac{1}{2}\left(\vec{E}_{2 t}+\vec{E}_{1 t}\right)
\end{array}\right.\right.
$$

## Discontinuity for the electric field

The discontinuities of the electric fields are related to the EM potentials through the Faraday's Law.

$$
\begin{cases}\vec{E}_{2 n}-\vec{E}_{1 n}= & -\left(\vec{\nabla}_{n} P_{A 2}-\vec{\nabla}_{n} P_{A 1}\right)  \tag{5.85}\\ \vec{E}_{2 t}-\vec{E}_{1 t}=-\frac{\partial}{\partial t}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right) & -\left(\vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}\right)\end{cases}
$$

Furthermore, the discontinuities are related to the distribution of charge and current doublets (see (5.44), (5.81)):

$$
\left\{\begin{array}{l}
\vec{E}_{2 n}-\vec{E}_{1 n}=\mu_{0} c_{0}^{2} \sigma_{e} \vec{n}_{21}  \tag{5.86}\\
\vec{E}_{2 t}-\vec{E}_{1 t}=-\frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}-\left(\vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}\right)
\end{array}\right.
$$

## Chapter 6

## Boundary Conditions for E and $B$ fields

- But, after all, what use is it?
- Why, sir, there is every probability that you will soon be able to tax it!
M. Faraday to W. Gladstone, asking of
practical value of electricity (as quoted
by W.H.Lecky, 1899)

Till now we have calculated the Boundary Conditions in terms of EM potentials $\rho_{A}$ and $\vec{A}$. In this section we derive the BCs for the electric $\vec{E}$ and magnetic $\overline{\bar{B}}$ field. Practically, once $\vec{E}, \overline{\bar{B}}$ are known we want to determine the distributions of sources $\left(\sigma_{e}, \vec{J}_{s}, \vec{d}_{e}, \overline{\bar{D}}_{e}\right)$ on the surface.

For clarity, we report here the set of Maxwell's Eq.s in empty space:

$$
\left\{\begin{array} { l } 
{ \vec { \nabla } ^ { T } \cdot \vec { E } = \mu _ { 0 } c _ { 0 } ^ { 2 } \rho _ { e } }  \tag{6.1}\\
{ - \overline { \overline { B } } \cdot \vec { \nabla } = \mu _ { 0 } \vec { J } _ { e } + \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial \vec { E } } { \partial t } }
\end{array} \quad \left\{\begin{array}{l}
\nabla_{i j k} \overline{\bar{B}}=0 \\
\vec{\nabla} \hat{\wedge} \vec{E}=-\frac{\partial \overline{\bar{B}}}{\partial t}
\end{array}\right.\right.
$$

The first pair of equations is the Wave set, while the second one is the Faraday set. In order to retrieve the Boundary Conditions for those equations we should rephrase them in an integral way. For completeness, we have to signal that it is possible to adopt also a differential approach. As a matter of fact, in a nice article 42 Lindell derived the Boundary Conditions and the Huygens' principle for an electromagnetic problem adopting a pure differential method. However, he had to deal with the derivation of discontinuous functions and he was almost forced to introduce magnetic charge and current in order to retrieve the BCs for $\vec{E}$ and $\vec{B}$. Here we prefer to exploit an integral approach, avoiding to derive discontinuous fields and to make any specific hypothesis on the magnetic monopoles existence.

### 6.1 BC: from Nabla to the normal vector

As you can see, the nabla operator $\vec{\nabla}()=.\frac{\partial(.)}{\partial \vec{x}}$ frequently occurs in Maxwell's Eq.s. It can act on a field as a divergence, a curl, a gradient, etc.

Let us suppose there are two generic fields $\mathbf{f}$ and $\mathbf{v}$, which are linked by a differential equation in the form:

$$
\begin{equation*}
\vec{\nabla} \star \mathbf{v}=\mathbf{f} \tag{6.2}
\end{equation*}
$$

where $\vec{\nabla} \star$ is a generic differential operator on space.
Now we want to write the same equation in integral form for a surface element $S$ of infinitesimal width $\Delta x$.

$$
\begin{equation*}
\operatorname{Vol}(\Omega)=S \Delta x \quad \text { volume for the elementary surface } \tag{6.3}
\end{equation*}
$$

We can integrate the equation on the volume's slab, so:

$$
\begin{equation*}
\int_{\Omega} \vec{\nabla} \star \mathbf{v} d \Omega_{x}=\int_{\Omega} \mathbf{f} d \Omega_{x} \tag{6.4}
\end{equation*}
$$

Now the task is to transform the integral on $\Omega$ in a surface integral.

### 6.1.1 Theorem for the Generalized Gradient

Thanks to the Theorem for the Generalized Gradient proved in [34] it is possible to rephrase the volume integral of a gradient into a surface integral, holding:

$$
\begin{equation*}
\int_{\Omega} \frac{\partial v_{i}}{\partial x_{j}} d \Omega_{x}=\oint_{\partial \Omega} v_{i} n_{j} d S_{x} \quad \forall i, j \tag{6.5}
\end{equation*}
$$

This theorem is comprehensive of many other ones, for example:

$$
\begin{align*}
\int_{\Omega} \vec{\nabla}^{T} \cdot \vec{v} d \Omega_{x} & =\oint_{\partial \Omega} \vec{n}^{T} \cdot \vec{v} d S_{x}  \tag{6.6}\\
\int_{\Omega} \vec{\nabla} \hat{\wedge} \vec{v} d \Omega_{x} & =\oint_{\partial \Omega} \vec{n} \wedge \vec{v} d S_{x}  \tag{6.7}\\
\int_{\Omega} \vec{\nabla} \phi d \Omega_{x} & =\oint_{\partial \Omega} \vec{n} \phi d S_{x}  \tag{6.8}\\
\int_{\Omega}\left[\frac{\partial \vec{v}}{\partial \vec{x}}\right] d \Omega_{x} & =\oint_{\partial \Omega} \overline{\overline{v n}} d S_{x} \tag{6.9}
\end{align*}
$$

For the complete theory, see [34]. Hence we can write:

$$
\begin{equation*}
\int_{\Omega} \vec{\nabla} \star \mathbf{v} d \Omega_{x}=\oint_{\partial \Omega} \vec{n} \star \mathbf{v} d S_{x} \tag{6.10}
\end{equation*}
$$

The basic concept is always the same: passing from a volume integral on $\Omega$ to an integral on the boundary $\partial \Omega$, we have to transform $\vec{\nabla}$ in the normal vector $\vec{n}$.


Figure 6.1: Elementary surface element. The discontinuity for field $\mathbf{v}$ is related to the average value $\langle f\rangle$.

### 6.1.2 $\quad \mathrm{BC}$ for the surface element

Finally, the relation between the fields $\mathbf{v}$ and $\mathbf{f}$ can be written in integral form as:

$$
\begin{equation*}
\oint_{\partial \Omega} \vec{n} \star \mathbf{v} d S_{x}=\int_{\Omega} \mathbf{f} d \Omega_{x} \tag{6.11}
\end{equation*}
$$

For an infinitesimal surface element we can define the average fields such that:

$$
\begin{align*}
\langle\mathbf{f}\rangle S \Delta x & =\int_{\Omega} \mathbf{f} d \Omega_{x}  \tag{6.12}\\
\mathbf{v}_{1} S & =\int_{\partial \Omega 1} \vec{n}_{12} \star \mathbf{v} d S_{x} \tag{6.13}
\end{align*} \quad \text { side } 1 .
$$

Since $\vec{n}_{21}=-\vec{n}_{12}$, the eq. 6.11 can be rewritten as:

$$
\begin{equation*}
\vec{n}_{21} \star\left(\mathbf{v}_{2}-\mathbf{v}_{1}\right)=\langle\mathbf{f}\rangle \Delta x \tag{6.16}
\end{equation*}
$$

That exactly the Boundary Condition for $\mathbf{v}$ and $\mathbf{f}$ we were looking for. Shortly:

$$
\begin{equation*}
\nabla \star \mathbf{v}=\mathbf{f} \quad \leftrightarrows \quad \vec{n}_{21} \star\left(\mathbf{v}_{2}-\mathbf{v}_{1}\right)=\langle\mathbf{f}\rangle \Delta x \tag{6.17}
\end{equation*}
$$

### 6.2 BC for Maxwell's equations in E and B

Thanks to the integral approach, now we can write the BCs for the Maxwell's Eq.s in terms of $\vec{E}$ and $\overline{\bar{B}}$ :

$$
\left\{\begin{array} { c } 
{ \vec { n } _ { 2 1 } ^ { T } \cdot ( \vec { E } _ { 2 } - \vec { E } _ { 1 } ) = \mu _ { 0 } c _ { 0 } ^ { 2 } \langle \rho _ { e } \rangle \Delta x } \\
{ - ( \overline { \overline { B } } _ { 2 } - \overline { \overline { B } } _ { 1 } ) \cdot \vec { n } _ { 2 1 } = \mu _ { 0 } \langle \vec { J } _ { e } \rangle \Delta x + \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial \langle \vec { E } \rangle } { \partial t } \Delta x }
\end{array} \left\{\begin{array}{l}
\vec{n}_{21} \dot{i j k}\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right)=0 \\
\vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \Delta x
\end{array}\right.\right.
$$

We immediately define the charge $\sigma_{e}$ and current $\vec{J}_{s}$ per unit of surface:

$$
\begin{equation*}
\sigma_{e}=\left\langle\rho_{e}\right\rangle \Delta x ; \quad \vec{J}_{s}=\left\langle\vec{J}_{e}\right\rangle \Delta x \tag{6.19}
\end{equation*}
$$

The equation for the "divergence" of $B$ is particularly uncomfortable, and that is because the magnetic field is a tensor. It is helpful to replace it with the curl of $\vec{A}$, in fact:

$$
\begin{align*}
& \nabla_{i j k} \overline{\bar{B}}=0 \longleftrightarrow \overline{\bar{B}}=\vec{\nabla} \wedge \vec{A}  \tag{6.20}\\
& \vec{n}_{21} \wedge\left(\vec{A}_{2}-\vec{A}_{1}\right)=\langle\overline{\bar{B}}\rangle \Delta x \tag{6.21}
\end{align*}
$$

Finally, the BCs for the Maxwell's Eq.s are:

$$
\left\{\begin{array} { c } 
{ \vec { n } _ { 2 1 } ^ { T } \cdot ( \vec { E } _ { 2 } - \vec { E } _ { 1 } ) = \mu _ { 0 } c _ { 0 } ^ { 2 } \sigma _ { e } }  \tag{6.22}\\
{ - ( \overline { \overline { B } } _ { 2 } - \overline { \overline { B } } _ { 1 } ) \cdot \vec { n } _ { 2 1 } = \mu _ { 0 } \vec { J } _ { s } + \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial \langle \vec { E } \rangle } { \partial t } \Delta x }
\end{array} \left\{\begin{array}{l}
\vec{n}_{21} \wedge\left(\vec{A}_{2}-\vec{A}_{1}\right)=\langle\overline{\bar{B}}\rangle \Delta x \\
\vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \Delta x
\end{array}\right.\right.
$$

Let's analyze them more in detail, in order to verify if they are coherent with the ones for the EM potentials.

### 6.2.1 E normal component

The normal component $\vec{E}_{n}$ of the electric field is involved in just one BC, that is:

$$
\begin{align*}
\vec{n}_{21}^{T} \cdot\left(\vec{E}_{2}-\vec{E}_{1}\right) & =\mu_{0} c_{0}^{2} \sigma_{e}  \tag{6.23}\\
\vec{E}_{2 n}-\vec{E}_{1 n} & =\mu_{0} c_{0}^{2} \sigma_{e} \vec{n}_{21} \tag{6.24}
\end{align*}
$$

So the discontinuity of $\vec{E}_{n}$ is associated to a layer of charge $\sigma_{e}$. That result is


Figure 6.2: The discontinuity for the normal electric field $\vec{E}_{n}$ is associated to the surface charge $\sigma_{e}$
fully coherent with the EM potential's BC, in fact it was already derived in sec. 5.4 .4 eq. 5.81.

### 6.2.2 B normal component

In 3D notation, the Gauss Law for the magnetic field states that $B$ is solenoidal. It follows that the related BC imposes the continuity of the normal component:

$$
\begin{align*}
& \vec{\nabla}^{T} \cdot \vec{B}=0 \quad \Longrightarrow \quad \vec{n}_{21} \cdot\left(\vec{B}_{2}-\vec{B}_{1}\right)=0  \tag{6.25}\\
B_{2 n}= & B_{1 n} \quad \text { continuity for normal component } \tag{6.26}
\end{align*}
$$

Anyway, since $\vec{B}$ is a pseudovector we must be very careful dealing with it.
In ND notation, the same BC can be rewritten in the form:

$$
\begin{array}{cll}
\nabla_{i j k} \overline{\bar{B}}=0 & \Longrightarrow & \vec{n}_{21}{ }_{i j k}\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right)=0 \\
n_{k} \Delta B_{i j}+n_{i} \Delta B_{j k}+n_{j} \Delta B_{k i}=0 & \forall i, j, k \in\{1 ; 2 ; \cdots ; N\} \tag{6.28}
\end{array}
$$

The continuity of the normal component $B_{n}$ descend by $\vec{B}=\vec{\nabla} \times \vec{A}$ and by the absence of magnetic monopoles. However, be aware that the magnetic field $B$ could be quite irregular on a screen. In the following chapter 7 we are going to see how to bypass that constraint on $B_{n}$.

### 6.2.3 E tangential component

The discontinuity for the tangential electric field $\vec{E}_{t}$ is associated to the Faraday's Law, written in terms of $\vec{E}$ and $\overline{\bar{B}}$. In fact, the Boundary Conditions for it is:

$$
\begin{equation*}
\vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \Delta x \tag{6.29}
\end{equation*}
$$

The cross product is sensible just to the perpendicular component of a vector on another one, in fact:

$$
\begin{align*}
& \vec{n} \wedge \vec{v}=\vec{n} \wedge \vec{v}_{t} \quad \Longrightarrow  \tag{6.30}\\
& \vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2}-\vec{E}_{1}\right)=\vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2 t}-\vec{E}_{1 t}\right) \tag{6.31}
\end{align*}
$$

So the associated BC becomes:

$$
\begin{equation*}
\vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2 t}-\vec{E}_{1 t}\right)=-\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \Delta x \tag{6.32}
\end{equation*}
$$

The tangential component of $\vec{E}_{t}$ can be calculated as:

$$
\begin{equation*}
\vec{E}_{t}=[\vec{n} \wedge \vec{E}] \cdot \vec{n} \tag{6.33}
\end{equation*}
$$

Thus, if we want to obtain $\vec{E}_{2 t}-\vec{E}_{1 t}$, it is sufficient multiply for $\vec{n}_{21}$ :

$$
\begin{equation*}
\vec{E}_{2 t}-\vec{E}_{1 t}=-\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \cdot \vec{n}_{21} \Delta x \quad \text { discontinuity for } \vec{E}_{t} \tag{6.34}
\end{equation*}
$$

This result is quite meaningful and is worthy of some consideration. First of all, we notice that, in the limit of a thin surface $\Delta x \rightarrow 0^{+}$, if the average magnetic field $\langle\overline{\bar{B}}\rangle$ has a finite value, then a discontinuity for $\vec{E}_{t}$ would be impossible. In fact:

$$
\begin{align*}
&\|\langle\overline{\bar{B}}\rangle\|,\left\|\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t}\right\|<+\infty \Longrightarrow \lim _{\Delta x \rightarrow 0^{+}}\left(\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \cdot \vec{n}_{21} \Delta x\right)=\overline{\overline{0}} \Longrightarrow  \tag{6.35}\\
& \vec{E}_{2 t}-\vec{E}_{1 t}=\overrightarrow{0} \tag{6.36}
\end{align*}
$$

Therefore, if you want to impose a discontinuity for $\vec{E}_{t}$, you must also accept that the magnetic field $\langle\overline{\bar{B}}\rangle$ could tend to the infinite on the surface.

(a) The discontinuity for the tangential electric field $\vec{E}_{t}$ is proportional to time-derivative of surface doublet $\overline{\bar{D}}_{e}$.

(b) The average magnetic field $\langle B\rangle_{\text {int }}$ on the interface can be extremely high, even if $B_{1}$ and $B_{2}$ on the two sides are small or null.

Figure 6.3: A discontinuity for the tangential electric field $\vec{E}_{t}$ implies that the time-derivative of magnetic field $\overline{\bar{B}}$ can reach infinite values on the interface. In general, the magnetic field could abruptly change across and inside a finite-thickness boundary.

In other words, the magnetic field $B$ could be not regular on the screen, even if the fields $B_{1}$ and $B_{2}$ on the two sides are continuous and derivable. In general, $\langle\overline{\bar{B}}\rangle$ could be different from the average between the field on the two sides of the screen:

$$
\begin{equation*}
\langle\overline{\bar{B}}\rangle \neq \frac{1}{2}\left(\overline{\bar{B}}_{2}+\overline{\bar{B}}_{1}\right) \tag{6.37}
\end{equation*}
$$

## Magnetic current

According to Schelkunoff[33] (see also sec. 2.3.3), the BC for the tangential electric field can be expressed in function of a hypothetical surface magnetic current $\vec{J}_{s, m}$ :

$$
\begin{equation*}
\vec{n}_{21} \times\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\vec{J}_{s, m} \tag{6.38}
\end{equation*}
$$

Rephrasing it with the ND notation, we get:

$$
\begin{equation*}
\vec{n}_{21} \wedge\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\overline{\bar{J}}_{s, m} \tag{6.39}
\end{equation*}
$$

Comparing 6.39 with 6.29, we can express the surface magnetic current as:

$$
\begin{array}{ll}
\overline{\bar{J}}_{s, m}=\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \Delta x & \text { ND notation } \\
\vec{J}_{s, m}=\frac{\partial\langle\vec{B}\rangle}{\partial t} \Delta x & \text { 3D notation } \tag{6.41}
\end{array}
$$

We observe that in this framework the "magnetic current" can be effectively regarded as a fictitious quantity. Moreover, taking the divergence for $\vec{J}_{s, m}$ we
find it is solenoidal, since $\vec{\nabla}^{T} \cdot \vec{B}=0$ :

$$
\begin{align*}
\vec{\nabla}_{i j k} \cdot \overline{\bar{J}}_{s, m}=0 & \text { ND notation }  \tag{6.42}\\
\vec{\nabla}^{T} \cdot \vec{J}_{s, m}=0 & \text { 3D notation } \tag{6.43}
\end{align*}
$$

In this context, that simply means there are no magnetic charges $\left(\rho_{m}=0\right)$.

## Sources for the discontinuity

Our task is to express the discontinuity of $\vec{E}_{t}$ in function of the electric charges and currents. Let's observe again the BC coming from the Faraday's set:

$$
\left\{\begin{array}{l}
\vec{n}_{21} \wedge\left(\vec{A}_{2}-\vec{A}_{1}\right)=\langle\overline{\bar{B}}\rangle \Delta x  \tag{6.44}\\
\vec{n}_{21} \wedge\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\frac{\partial\langle\overline{\bar{B}}\rangle}{\partial t} \Delta x
\end{array}\right.
$$

The discontinuity for $\vec{A}$ is related to the current doublets, in fact:

$$
\begin{align*}
& \left\{\begin{array}{l}
\mu_{0} \overline{\bar{D}}_{e}=\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21} \\
\mu_{0} \overline{\bar{D}}_{e}^{T}=\vec{n}_{21} \otimes\left(\vec{A}_{2}-\vec{A}_{1}\right)
\end{array} \Longrightarrow\right.  \tag{6.45}\\
& \vec{n}_{21} \wedge\left(\vec{A}_{2}-\vec{A}_{1}\right)=\mu_{0}\left(\overline{\bar{D}}_{e}-\overline{\bar{D}}_{e}^{T}\right)  \tag{6.46}\\
& \vec{A}_{2 t}-\vec{A}_{1 t}=\mu_{0}\left(\overline{\bar{D}}_{e}-\overline{\bar{D}}_{e}^{T}\right) \cdot \vec{n}_{21} \tag{6.47}
\end{align*}
$$

So the average magnetic field on the screen results to be proportional to the local current doublets:

$$
\begin{equation*}
\mu_{0}\left(\overline{\bar{D}}_{e}-\overline{\bar{D}}_{e}^{T}\right)=\langle\overline{\bar{B}}\rangle \Delta x \tag{6.48}
\end{equation*}
$$

With a last substitution, we finally can express $\vec{E}_{t}$ in function of $\vec{A}_{t}$ and $\overline{\bar{D}}_{e}$ :

$$
\begin{align*}
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\frac{\partial}{\partial t}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right)  \tag{6.49}\\
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\frac{\partial}{\partial t}\left(\overline{\bar{D}}_{e}-\overline{\bar{D}}_{e}^{T}\right) \cdot \vec{n}_{21}  \tag{6.50}\\
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \tag{6.51}
\end{align*}
$$

Conversely, we can calculate the current doublets $\overline{\bar{D}}_{e}$ as:

$$
\begin{equation*}
\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e}=-\left(\vec{E}_{2 t}-\vec{E}_{1 t}\right) \otimes \vec{n}_{21} \tag{6.52}
\end{equation*}
$$

Important note: even if we have derived them directly from the Maxwell's Eq.s, these last equations are not always valid (see also (5.76). That is due to the fact that the electric and magnetic fields are less regular than the EM potentials.

We are going to verify that eq.s (6.49 (6.51) come true just if the Condition for continuous scalar potential is imposed: $P_{A 1}=P_{A 2}$.

### 6.2.4 $\quad \mathrm{B}$ tangential component

The last Boundary Condition to be considered is the one for the Maxwell-Ampére equation:

$$
\begin{equation*}
-\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21}=\mu_{0} \vec{J}_{s}+\frac{1}{c_{0}^{2}} \frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x \tag{6.53}
\end{equation*}
$$

We observe that the discontinuity for $B$ is related just to the tangential component of $\vec{J}_{s}$ and $\vec{E}$. In fact, $\overline{\bar{B}}$ is an anti-symmetric tensor, so it holds:

$$
\begin{equation*}
\overline{\bar{B}}=-\overline{\bar{B}}^{T} \quad \Longrightarrow \quad \vec{n}^{T} \cdot \overline{\bar{B}} \cdot \vec{n}=0 \quad \Longrightarrow \quad \vec{n} \perp(\overline{\bar{B}} \cdot \vec{n}) \quad \forall \vec{n} \tag{6.54}
\end{equation*}
$$

In 3D notation:

$$
\begin{equation*}
\overline{\bar{B}} \cdot \vec{n}=\vec{B} \times \vec{n} \quad \Longrightarrow \quad \vec{n} \perp(\vec{n} \times \vec{B}) \quad \forall \vec{n} \tag{6.55}
\end{equation*}
$$

So we can subdivide the BC in its normal and tangential components:

$$
\left\{\begin{align*}
\overrightarrow{0} & =\mu_{0} \vec{J}_{s, n}+\frac{1}{c_{0}^{2}} \frac{\partial\left\langle\vec{E}_{n}\right\rangle}{\partial t} \Delta x  \tag{6.56}\\
-\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21} & =\mu_{0} \vec{J}_{s, t}+\frac{1}{c_{0}^{2}} \frac{\partial\left\langle\vec{E}_{t}\right\rangle}{\partial t} \Delta x
\end{align*}\right.
$$

In the limit of $\Delta x \rightarrow 0^{+}$, so for an infinitesimal thin screen, we have:

$$
\left\{\begin{array}{l}
\lim _{\Delta x \rightarrow 0^{+}}\left(\left\langle\vec{E}_{n}\right\rangle \Delta x\right)=-\left(P_{A 2}-P_{A 1}\right) \vec{n}_{21}=-\mu_{0} c_{0}^{2} \vec{d}_{e}  \tag{6.57}\\
\lim _{\Delta x \rightarrow 0^{+}}\left(\left\langle\vec{E}_{t}\right\rangle \Delta x\right)=\overrightarrow{0}
\end{array}\right.
$$

Removing the electric fields, finally we obtain:


Figure 6.4: The discontinuity for the tangential magnetic field $\overline{\bar{B}} \cdot \vec{n}_{21}$ is generated by the tangential surface current $\vec{J}_{s, t}$

$$
\left\{\begin{array}{l}
\frac{\partial}{\partial t} \vec{d}_{e}=\vec{J}_{s, n}  \tag{6.58}\\
\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21}=-\mu_{0} \vec{J}_{s, t}
\end{array}\right.
$$

The first equation involves the normal electric current $\vec{J}_{s, n}$ and dipole $\vec{d}_{e}$ and it can be interpreted as the conservation of charge for an isolated dipole. The second equation tell us that, in order to enforce a discontinuity for the magnetic field $B$, we need a current sheet $\vec{J}_{s, t}$ parallel to the surface.

### 6.3 BCs Summary for $E$ and $B$

In this section we report the set of Boundary Conditions related to the electric $\vec{E}$ and magnetic $\overline{\bar{B}}$ fields, as they can be inferred from the Maxwell's Eq.s. In all the cases the surface sources are expressed in function of the fields on the boundary.

$$
\begin{align*}
\sigma_{e} & =\varepsilon_{0} \vec{n}_{21}^{T} \cdot\left(\vec{E}_{2}-\vec{E}_{1}\right)  \tag{6.59}\\
\vec{d}_{e} & =-\varepsilon_{0}\left(\left\langle\vec{E}_{n}\right\rangle \Delta x\right)  \tag{6.60}\\
\vec{J}_{s, n} & =\frac{\partial}{\partial t} \vec{d}_{e}=-\varepsilon_{0} \frac{\partial}{\partial t}\left(\left\langle\vec{E}_{n}\right\rangle \Delta x\right)  \tag{6.61}\\
\vec{J}_{s, t} & =-\frac{1}{\mu_{0}}\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21}  \tag{6.62}\\
\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} & =-\left(\vec{E}_{2 t}-\vec{E}_{1 t}\right) \otimes \vec{n}_{21} \tag{6.63}
\end{align*}
$$

As it emerges from these conditions, the electric field can be discontinuous for both the normal $\vec{E}_{n}$ and the tangential $\vec{E}_{t}$ components. On the contrary, the magnetic field $\vec{B}$ seems to be discontinuous just in the "tangential" component, while the "normal" one appears continuous.

## Chapter 7

## Harmonizing the Boundary Conditions

- The harmony always conceals an illusion.
- As much as there is in your algebra.
- I would like the man was designed by Euclid.
- And I would prefer the one conceived by Homer.

Cimourdain and Gauvain,
in V. Hugo's Ninety-Three, 1874
In this chapter we check if the BCs found for the EM potentials $\rho_{A}$ and $\vec{A}$ and those for $E$ and $B$ are equivalent or not. As we are going to verify, that can come true just if we impose the additional condition for the continuity of scalar potential.

### 7.1 Incompatibility for the tangential E

The discontinuity for the tangential component $\vec{E}_{t}$ has been calculated in two ways, which bring to different results. Writing the Faraday's Law in terms of $E$ and $B$ fields, we found:

$$
\begin{align*}
& \vec{E}_{2 t}-\vec{E}_{1 t}=-\frac{\partial}{\partial t}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right)  \tag{7.1}\\
& \vec{E}_{2 t}-\vec{E}_{1 t}=-\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \tag{7.2}
\end{align*}
$$

The problem is that this conditions could be not always valid. In fact, if we write the Faraday's Law in terms of EM potentials, we find a different result:

$$
\begin{align*}
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\frac{\partial}{\partial t}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right)+\left(\vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}\right)  \tag{7.3}\\
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}-\left(\vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}\right) \tag{7.4}
\end{align*}
$$

where $\vec{\nabla}_{t}($.$) is the gradient's tangential component:$

$$
\begin{equation*}
\vec{\nabla}_{t} \phi=(\overline{\bar{I}}-\overline{\overline{n n}}) \cdot \vec{\nabla} \phi \tag{7.5}
\end{equation*}
$$

The difference between the two expressions $\sqrt{7.1}(7.3)$ is due to the discontinuity for the tangential component of the potential's gradient.

In order to harmonize eq.s 7.1 7.3, to we require that:

$$
\begin{equation*}
\vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}=\overrightarrow{0} \tag{7.6}
\end{equation*}
$$

This condition is equivalent to imposing the continuity of potential $P_{A}$ across the surface.

### 7.2 Continuity for the scalar potential

Let $f$ be a scalar function on a point $\vec{x}$ on the surface. The function on a point $\vec{x}+d \vec{x}_{t}$, close to the first one, can be calculated as:

$$
\begin{equation*}
f\left(\vec{x}+d \vec{x}_{t}\right)=f(\vec{x})+d \vec{x}_{t}^{T} \cdot \vec{\nabla} f \quad \text { with } \quad d \vec{x}_{t} \perp \vec{n} \tag{7.7}
\end{equation*}
$$

Since $d \vec{x}_{t}$ is tangent to the surface, the normal component of the gradient has not effect, so

$$
\begin{align*}
d \vec{x}_{t}^{T} \cdot \vec{\nabla} f & =d \vec{x}_{t}^{T} \cdot \vec{\nabla}_{t} f  \tag{7.8}\\
f\left(\vec{x}+d \vec{x}_{t}\right)-f(\vec{x}) & =\left(\vec{\nabla}_{t} f\right)^{T} \cdot d \vec{x}_{t} \tag{7.9}
\end{align*}
$$

Writing the same equation on the two sides of the surface, we find:

$$
\begin{equation*}
\left(\vec{\nabla}_{t} f_{2}-\vec{\nabla}_{t} f_{1}\right)^{T} \cdot d \vec{x}_{t}=\left(f_{2}\left(\vec{x}+d \vec{x}_{t}\right)-f_{1}\left(\vec{x}+d \vec{x}_{t}\right)\right)-\left(f_{2}(\vec{x})-f_{1}(\vec{x})\right) \tag{7.10}
\end{equation*}
$$

Requiring the continuity of the gradient $\vec{\nabla}_{t} f$, we get:

$$
\begin{align*}
& 0=\left(f_{2}\left(\vec{x}+d \vec{x}_{t}\right)-f_{1}\left(\vec{x}+d \vec{x}_{t}\right)\right)-\left(f_{2}(\vec{x})-f_{1}(\vec{x})\right)  \tag{7.11}\\
& \left.\left(f_{2}-f_{1}\right)\right|_{\vec{x}+d \vec{x}_{t}}=\left.\left(f_{2}-f_{1}\right)\right|_{\vec{x}} \tag{7.12}
\end{align*}
$$

In order to guarantee this condition is sufficient the function is continuous:

$$
\begin{align*}
f_{2} & =f_{1} \quad \forall \vec{x} \in \partial \Omega \quad \Longrightarrow  \tag{7.13}\\
\left.\left(f_{2}-f_{1}\right)\right|_{\vec{x}+d \vec{x}_{t}} & =\left.\left(f_{2}-f_{1}\right)\right|_{\vec{x}}=0 \quad \Longrightarrow  \tag{7.14}\\
\frac{1}{d x_{t}}\left(f_{2}\left(\vec{x}+d \vec{x}_{t}\right)-f_{2}(\vec{x})\right) & =\frac{1}{d x_{t}}\left(f_{1}\left(\vec{x}+d \vec{x}_{t}\right)-f_{1}(\vec{x})\right) \quad \Longrightarrow  \tag{7.15}\\
\vec{\nabla}_{t} f_{2} & =\vec{\nabla}_{t} f_{1} \tag{7.16}
\end{align*}
$$

If our function is the scalar potential $P_{A}$, then it holds:

$$
\begin{equation*}
P_{A 2}=P_{A 1} \quad \Longrightarrow \quad \vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}=\overrightarrow{0} \tag{7.17}
\end{equation*}
$$

More generally, if the difference $P_{A 2}-P_{A 1}$ is constant all over the surface, then the difference of the tangential gradients is zero and viceversa:

$$
\begin{equation*}
P_{A 2}-P_{A 1}=\text { const. } \forall \vec{x} \in \partial \Omega \quad \Longleftrightarrow \quad \vec{\nabla}_{t} P_{A 2}-\vec{\nabla}_{t} P_{A 1}=\overrightarrow{0} \quad \forall \vec{x} \in \partial \Omega \tag{7.18}
\end{equation*}
$$

So, in order to council eq.s (7.1) and (7.3), it's sufficient to impose the Condition for the Continuity of the Scalar Potential:

$$
\begin{align*}
P_{A 2} & =P_{A 1} \quad \Longrightarrow  \tag{7.19}\\
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\frac{\partial}{\partial t}\left(\vec{A}_{2 t}-\vec{A}_{1 t}\right)  \tag{7.20}\\
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \tag{7.21}
\end{align*}
$$

### 7.2.1 Consequences from the potential's continuity

The condition of a continuous scalar potential across a surface implies some consequences. The discontinuity of $P_{A}$ is related to the surface distribution of electric dipole $\vec{d}_{e}$, in fact:

$$
\begin{equation*}
\left(P_{A 2}-P_{A 1}\right) \cdot \vec{n}_{21}=\mu_{0} c_{0}^{2} \vec{d}_{e}=-\Delta x\left\langle\vec{E}_{n}\right\rangle \tag{7.22}
\end{equation*}
$$

If the scalar potential is continuous, then there are no orthogonal dipoles $\vec{d}_{e}$ and the normal electric field $\left\langle\vec{E}_{n}\right\rangle$ has a finite value:

$$
\begin{align*}
& P_{A 2}=P_{A 1} \Longrightarrow\left\{\begin{array}{l}
\vec{d}_{e}=\overrightarrow{0} \\
\lim _{\Delta x \longrightarrow 0^{+}}\left(\Delta x\left\langle\vec{E}_{n}\right\rangle\right)=\overrightarrow{0}
\end{array}\right.  \tag{7.23}\\
& \left\langle\vec{E}_{n}\right\rangle=\frac{1}{2}\left(\vec{E}_{n 2}+\vec{E}_{n 1}\right) \tag{7.24}
\end{align*}
$$

That is an important simplification, but we must be aware that it is not always allowed. For example, it cannot be adopted if you are dealing with the Volta Effect. Moreover, if you are projecting a thin screen or a meta-surface, you have to check if your surface can be orthogonally polarized. If that's true (so $\vec{d}_{e} \neq \overrightarrow{0}$ ), you should be aware that the usual BCs for the electric and magnetic field could not work properly. In fact, as already said, $\langle\vec{E}\rangle$ and $\langle\overline{\bar{B}}\rangle$ could tend to infinite values inside the screen itself. On the contrary, the EM potentials $P_{A}$ and $\vec{A}$ can be chosen to be finite in value.

## Tangential components for $\mathbf{E}$

If the potential $P_{A}$ is continuous across the surface, then it is possible to calculate the tangential component $\vec{E}_{1 t}, \vec{E}_{2 t}$ as:

$$
\begin{align*}
\vec{E}_{1 t} & =-\frac{\partial}{\partial t}\left(\vec{A}_{1 t}\right)-\left\langle\vec{\nabla}_{t} P_{A}\right\rangle  \tag{7.25}\\
\vec{E}_{2 t} & =-\frac{\partial}{\partial t}\left(\vec{A}_{2 t}\right)-\left\langle\vec{\nabla}_{t} P_{A}\right\rangle \tag{7.26}
\end{align*}
$$

### 7.3 Maxwell-Ampére BCs coherence

The Boundary Condition associated to the Maxwell-Ampére equation is:

$$
\begin{equation*}
-\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21}=\mu_{0} \vec{J}_{s}+\frac{1}{c_{0}^{2}} \frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x \tag{7.27}
\end{equation*}
$$

Splitting it in the normal and tangent components, two equations are obtained:

$$
\begin{align*}
\frac{1}{c_{0}^{2}} \frac{\partial\left\langle\vec{E}_{n}\right\rangle}{\partial t} \Delta x & =-\mu_{0} \vec{J}_{s, n}  \tag{7.28}\\
\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21} & =-\mu_{0} \vec{J}_{s, t} \tag{7.29}
\end{align*}
$$

We have to verify if they are coherent with the BC for the gradient of $\vec{A}$ :

$$
\begin{equation*}
\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}=-\mu_{0} \vec{J}_{s, e} \tag{7.30}
\end{equation*}
$$

In order to do that, we restart from the differential equation:

$$
\begin{equation*}
-\overline{\bar{B}} \cdot \vec{\nabla}=\mu_{0} \vec{J}_{e}+\frac{1}{c_{0}^{2}} \frac{\partial \vec{E}}{\partial t} \tag{7.31}
\end{equation*}
$$

We have to express the magnetic field $B$ in terms of $\vec{A}$ :

$$
\begin{align*}
& \overline{\bar{B}}=\vec{\nabla} \wedge \vec{A} \Longrightarrow  \tag{7.32}\\
& \overline{\bar{B}} \cdot \vec{\nabla}=[\vec{\nabla} \wedge \vec{A}] \cdot \vec{\nabla}=\left(\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]-\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]^{T}\right) \cdot \vec{\nabla}  \tag{7.33}\\
& \overline{\bar{B}} \cdot \vec{\nabla}=\nabla^{2} \vec{A}-\vec{\nabla}\left(\vec{\nabla}^{T} \cdot \vec{A}\right)=\left(\frac{\partial \vec{A}}{\partial \vec{x}}-\left(\vec{\nabla}^{T} \cdot \vec{A}\right) \overline{\bar{I}}\right) \cdot \vec{\nabla} \tag{7.34}
\end{align*}
$$

Thus:

$$
\begin{equation*}
-\left(\frac{\partial \vec{A}}{\partial \vec{x}}-\left(\vec{\nabla}^{T} \cdot \vec{A}\right) \overline{\bar{I}}\right) \cdot \vec{\nabla}=\mu_{0} \vec{J}_{e}+\frac{1}{c_{0}^{2}} \frac{\partial \vec{E}}{\partial t} \tag{7.35}
\end{equation*}
$$

The Boundary Condition associated to this equations will be so:

$$
\begin{gather*}
-\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}+\left(\vec{\nabla}^{T} \cdot \vec{A}_{2}-\vec{\nabla}^{T} \cdot \vec{A}_{1}\right) \cdot \vec{n}_{21}=  \tag{7.36}\\
\mu_{0} \vec{J}_{s}+\frac{1}{c_{0}^{2}} \frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x
\end{gather*}
$$

From the Lorentz Gauge we know that the divergence of $\vec{A}$ is related to the time derivative of $P_{A}$ :

$$
\begin{align*}
& \frac{1}{c_{0}^{2}} \frac{\partial P_{A}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{A}=0 \Longrightarrow  \tag{7.37}\\
& \frac{1}{c_{0}^{2}} \frac{\partial}{\partial t}\left(P_{A 2}-P_{A 1}\right)=-\left(\vec{\nabla}^{T} \cdot \vec{A}_{2}-\vec{\nabla}^{T} \cdot \overrightarrow{A_{1}}\right) \cdot \vec{n}_{21} \tag{7.38}
\end{align*}
$$

So by substitution we obtain:

$$
\begin{gather*}
-\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}-\frac{1}{c_{0}^{2}} \frac{\partial}{\partial t}\left(P_{A 2}-P_{A 1}\right) \cdot \vec{n}_{21}=  \tag{7.39}\\
\mu_{0} \vec{J}_{s}+\frac{1}{c_{0}^{2}} \frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x
\end{gather*}
$$

If we require the equivalence of the $\mathrm{BCs}(7.27$ and 7.30 , then it must hold:

$$
\begin{array}{rlr}
-\frac{1}{c_{0}^{2}} \frac{\partial}{\partial t}\left(P_{A 2}-P_{A 1}\right) \cdot \vec{n}_{21}=\frac{1}{c_{0}^{2}} \frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x & \text { for } \Delta x \rightarrow 0^{+} \\
-\frac{\partial}{\partial t}\left(P_{A 2}-P_{A 1}\right) \cdot \vec{n}_{21}=\frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x & \text { for } \Delta x \rightarrow 0^{+} \tag{7.41}
\end{array}
$$

In the hypothesis of a continuous scalar potential $P_{A 2}=P_{A 1}$ and a finite electric field $\langle\vec{E}\rangle$, this last condition is automatically satisfied. Otherwise, 7.27 could be not valid, since it was deduced in the hypothesis of a finite electric field.

### 7.4 Essential Boundary Conditions

If you are dealing with a stationary or static problem, Maxwell's Eq.s reduce to:

$$
\left\{\begin{array} { l } 
{ - \nabla ^ { 2 } \rho _ { A } = \mu _ { 0 } \rho _ { e } }  \tag{7.42}\\
{ - \nabla ^ { 2 } \vec { A } = \mu _ { 0 } \vec { J } _ { e } }
\end{array} \quad \left\{\begin{array}{l}
\vec{\nabla}^{T} \cdot \vec{A}=0 \\
\vec{E}=-\vec{\nabla}\left(\rho_{A} c_{0}^{2}\right) \\
\overline{\bar{B}}=\vec{\nabla} \hat{\rightharpoonup} \vec{A}
\end{array}\right.\right.
$$

In that case the potentials $\rho_{A}$ and $\vec{A}$ are reciprocally independent, and the same happens for the electric $\vec{E}$ and magnetic $\overline{\bar{B}}$ fields. In fact the equations are completely decoupled, and the electric charge $\rho_{e}$ and currents $\vec{J}_{e}$ are not explicitly linked each other.

On the contrary, if you are dealing with a non-stationary problem, then $\rho_{e}$ and $\vec{J}_{e}$ are explicitly related by the continuity equation:

$$
\begin{align*}
\frac{\partial \rho_{e}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{J}_{e}=0 & \text { time domain }  \tag{7.43}\\
s \rho_{e}+\vec{\nabla}^{T} \cdot \vec{J}_{e}=0 & \text { Laplace domain } \tag{7.44}
\end{align*}
$$

Therefore, if you know the current's distribution you can calculate the density charge, which reveals to be a dependent variable.

Extending that concept, it means that all the "charges" $\rho_{e}, \sigma_{e}, \vec{d}_{e}$ depends on the "currents" $\vec{J}_{e}, \vec{J}_{s}, \overline{\bar{D}}_{e}$, which are independent variables. For the same reason, we can regard the Boundary Condition involving the currents as the essential BCs , since the other ones will be automatically satisfied thanks to the fields' equations. Thus, for the EM potentials the essential BCs are:

$$
\begin{align*}
\vec{A}_{2 t}-\vec{A}_{1 t} & =\mu_{0} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{7.45}\\
\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21} & =-\mu_{0} \vec{J}_{s, e} \tag{7.46}
\end{align*}
$$

The same BCs can be rephrased for the electric and magnetic fields, providing the Condition of continuity for the scalar potential $\left(\rho_{A 2}=\rho_{A 1}\right)$ :

$$
\begin{align*}
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{7.47}\\
\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21} & =-\mu_{0} \vec{J}_{s, t} \tag{7.48}
\end{align*}
$$

where $\vec{J}_{s, t}$ is the tangential component of the surface current. Adopting the magnetic field $H=B / \mu_{0}$, the essential BCs will look:

$$
\begin{align*}
& \vec{E}_{2 t}-\vec{E}_{1 t}=-\overline{\bar{J}}_{s, m} \cdot \vec{n}_{21}=-\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{7.49}\\
& \left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21}=-\vec{J}_{s, t} \tag{7.50}
\end{align*}
$$

In 3D notation:

$$
\begin{align*}
\vec{n}_{21} \times\left(\vec{E}_{2 t}-\vec{E}_{1 t}\right) & =-\vec{J}_{s, m}  \tag{7.51}\\
\vec{n}_{21} \times\left(\vec{H}_{2}-\vec{H}_{1}\right) & =\vec{J}_{s, e} \tag{7.52}
\end{align*}
$$

These are exactly the BCs $2.38,(2.39$ found by Schelkunoff.

### 7.5 A surface of "magnetic monopoles"

Let's now see one example which well highlights the limits of classic BCs and the advantages of EM potentials. Suppose there is space region $\Omega_{2}$ characterized by a static magnetic field $\overline{\bar{B}}_{\text {inc }}$, constant in time and assigned. We also suppose that the electric field $\vec{E}$ is zero. Suppose we want to shield a certain region $\Omega_{1}$ in such a way that inside it the magnetic field is null $\left(\overline{\bar{B}}_{1}=\overline{\overline{0}}\right)$. Above all, we also require that the magnetic field outside remains unchanged. (This last condition distinguishes this hypothetical "magnetic cloaking" from the Meissner Effect and other shielding techniques.) Mathematically, the global fields will be so:

$$
\vec{E}(\vec{x})=\overrightarrow{0} ; \quad \overline{\bar{B}}(\vec{x})=\left\{\begin{array}{cl}
\overline{\overline{0}} & \text { for } \vec{x} \in \Omega_{1}  \tag{7.53}\\
\overline{\bar{B}}_{\text {inc }} & \text { for } \vec{x} \in \Omega_{2}
\end{array}\right.
$$



Figure 7.1: Ideal magnetic shielding - static case

### 7.5.1 Use of classic 3D BCs for electric and magnetic fields

Using the 3D Boundary Conditions proposed by Schelkunoff, we find the relations for the field discontinuities across the interface:

$$
\left\{\begin{array} { r l } 
{ \vec { n } _ { 2 1 } \times ( \vec { E } _ { 2 } - \vec { E } _ { 1 } ) = - \vec { J } _ { s , m } }  \tag{7.54}\\
{ \vec { n } _ { 2 1 } \times ( \vec { B } _ { 2 } - \vec { B } _ { 1 } ) = \mu _ { 0 } \vec { J } _ { s , e } }
\end{array} \quad \Longrightarrow \quad \left\{\begin{array}{r}
\overrightarrow{0}=-\vec{J}_{s, m} \\
\vec{n}_{21} \times \vec{B}_{i n c}=\mu_{0} \vec{J}_{s, e}
\end{array}\right.\right.
$$

Since the electric field is continuous, no magnetic currents $\vec{J}_{s, m}$ are expected, while the discontinuity for $\vec{B}$ is related to electric ones $\vec{J}_{s, e}$. However, if we diligently obey to the Boundary Conditions for $\vec{E}$ and $\vec{B}$, we should conclude that our request on shielding is impossible to satisfy. As a matter of fact, in 3D notation the Gauss Law for the magnetic field states that $\vec{B}$ is solenoidal, hence implying there are no magnetic monopoles (see also sec. 6.2.2). In 3D, that Law is associated to the continuity for normal component of $\vec{B}$ :

$$
\begin{array}{lc}
\vec{\nabla}^{T} \cdot \vec{B}=0 & \text { Gauss Law for magnetic field } \\
\vec{n}_{21}^{T} \cdot\left(\vec{B}_{2}-\vec{B}_{1}\right)=0 & \text { associated BC } \tag{7.56}
\end{array}
$$

Since we required that $\vec{B}_{1}=\overrightarrow{0}$ and $\vec{B}_{2}=\vec{B}_{\text {inc }}$, it could happen that somewhere


Figure 7.2: Apparent violation of Gauss Law for magnetism. The normal component of $\vec{B}$ is discontinuous across the boundary, i.e. $\vec{n}_{21} \cdot\left(\vec{B}_{2}-\vec{B}_{1}\right) \neq 0$, hence in some interpretations magnetic charges should be distributed on the surface.
the normal component for $\vec{B}$ is discontinuous, in fact:

$$
\begin{align*}
& \vec{n}_{21}^{T} \cdot\left(\vec{B}_{i n c}-\overrightarrow{0}\right) \neq 0  \tag{7.57}\\
& \vec{n}_{21}^{T} \cdot\left(\vec{B}_{2}-\vec{B}_{1}\right) \neq 0 \tag{7.58}
\end{align*}
$$

Somebody could say that, in order to overcome that constraint, we could place a distribution of magnetic monopoles on the surface. However, we are going to verify that is superfluous, since we can achieve the same result just by mean of EM potentials and electric sources.

### 7.5.2 Use of ND BCs for EM potentials

Let us remind that the magnetic field can reach infinite or very high values on an interface, as we have seen in sec. 6.2.3. That fact should make us suspicious, since it is a warning about the field irregularity. For that reason is convenient to rewrite the Gauss Law for magnetism in terms of vector potential $\vec{A}$, using the ND notation:

$$
\begin{array}{lr}
\vec{B}=\vec{\nabla} \times \vec{A} & \text { 3D notation } \\
\overline{\bar{B}}=\vec{\nabla} \hat{\wedge} \vec{A} & \text { ND notation } \tag{7.60}
\end{array}
$$

As we have already verified in $\sec 6.2$, the BC associated to this last equation is:

$$
\begin{equation*}
\vec{n}_{21} \wedge\left(\vec{A}_{2}-\vec{A}_{1}\right)=\langle\overline{\bar{B}}\rangle_{i n t} \Delta x \tag{7.61}
\end{equation*}
$$

where $\langle\overline{\bar{B}}\rangle_{\text {int }}$ is the average magnetic field inside the interface itself. If the discontinuity for $\vec{A}$ is assigned and the surface is very thin $\left(\Delta x \rightarrow 0^{+}\right)$, the magnetic field $\langle\overline{\bar{B}}\rangle_{i n t}$ can abruptly increase.

Now let us see how to calculate the distribution of (electric) sources $\vec{J}_{t}$ and $\overline{\bar{D}}_{e}$ necessary for the required "magnetic cloaking". For sake of clarity, here we shall consider a simple example. In particular, we make the hypotheses that the shielded region $\Omega_{1}$ is spherical, centred in $\vec{x}_{0}$, with radius $R$, and that magnetic field $\overline{\bar{B}}_{i n c}$ is constant on $\Omega_{2}$.

## Calculating the vector potential $\vec{A}$

Since we required a null magnetic field on $\Omega_{1}$, i.e. $\overline{\bar{B}}_{1}=\overline{\overline{0}}$, the field generated by the boundary sources must be null outside, on $\Omega_{2}$, and opposite to $\overline{\bar{B}}_{\text {inc }}$ inside, on $\Omega_{1}$.

In fact, the global field $\overline{\bar{B}}$ can be expressed as the sum of the assigned field $\overline{\bar{B}}_{i n c}$ and the one radiated $\overline{\bar{B}}_{i r r}$ by the boundary sources:

$$
\begin{equation*}
\overline{\bar{B}}=\overline{\bar{B}}_{i n c}+\overline{\bar{B}}_{i r r} \tag{7.62}
\end{equation*}
$$

Holding eq. 7.53 , the radiated field will be so:

$$
\overline{\bar{B}}_{i r r}(\vec{x})=\left\{\begin{array}{cl}
-\overline{\bar{B}}_{\text {inc }} & \text { for } \vec{x} \in \Omega_{1}  \tag{7.63}\\
\overline{\overline{0}} & \text { for } \vec{x} \in \Omega_{2}
\end{array}\right.
$$

If, as required, $\overline{\bar{B}}_{i n c}$ does not depend on $\vec{x}$, then the radiated vector potential $\vec{A}_{i r r}$ can be calculated as:

$$
\vec{A}_{i r r}(\vec{x})=\left\{\begin{array}{cc}
-\frac{1}{2} \overline{\bar{B}}_{i n c} \cdot\left(\vec{x}-\vec{x}_{0}\right) & \text { for } \vec{x} \in \Omega_{1}  \tag{7.64}\\
\vec{A}_{i r r, 0}(\vec{x}) & \text { for } \vec{x} \in \Omega_{2}
\end{array}\right.
$$

It's quite easy the verify that inside the sphere, i.e. for $\vec{x} \in \Omega_{1}$, it holds:

$$
\begin{equation*}
\vec{\nabla} \wedge \vec{A}_{i r r}=-\overline{\bar{B}}_{i n c} \quad \text { for }\left\|\vec{x}-\vec{x}_{0}\right\| \leq R \tag{7.65}
\end{equation*}
$$

We remember that $\overline{\bar{B}}$ is the curl of $\vec{A}$ :

$$
\begin{align*}
\overline{\bar{B}} & =\vec{\nabla} \hat{\wedge} \vec{A}=\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]-\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]^{T}  \tag{7.66}\\
B_{i j} & =A_{i / j}-A_{j / i}=\frac{\partial A_{i}}{\partial x_{j}}-\frac{\partial A_{j}}{\partial x_{i}} \tag{7.67}
\end{align*}
$$

In 3D notation, the equations for $\vec{A}_{i r r}$ would look:

$$
\begin{align*}
& \vec{A}_{i r r}(\vec{x})=\left\{\begin{array}{cc}
-\frac{1}{2} \vec{B}_{i n c} \times\left(\vec{x}-\vec{x}_{0}\right) & \text { for } \vec{x} \in \Omega_{1} \\
\vec{A}_{i r r, 0}(\vec{x}) & \text { for } \vec{x} \in \Omega_{2}
\end{array}\right.  \tag{7.68}\\
& \vec{\nabla} \times \vec{A}_{i r r}=-\vec{B}_{i n c} \quad \text { for }\left\|\vec{x}-\vec{x}_{0}\right\| \leq R \tag{7.69}
\end{align*}
$$

More explicitly, we have to generate a kind of vector potential "vortex", opposing to the external magnetic field.

Let us now consider the field $\vec{A}_{i r r, 0}(\vec{x})$ radiated outside. Since the radiated field $\overline{\bar{B}}_{i r r}(\vec{x})$ must be null outside the sphere, it must hold:

$$
\begin{align*}
\overline{\bar{B}}_{i r r}= & \vec{\nabla} \wedge \vec{A}_{i r r}=\overline{\overline{0}} \quad \text { for } \vec{x} \in \Omega_{2} \quad \Longrightarrow  \tag{7.70}\\
& \vec{\nabla} \wedge \vec{A}_{i r r, 0}=\overline{\overline{0}} \tag{7.71}
\end{align*}
$$

Here the simplest solution is to place $\vec{A}_{i r r, 0}=\overrightarrow{0}$, so that the global radiated field $\vec{A}_{i r r}$ turns out to be:

$$
\vec{A}_{i r r}(\vec{x})=\left\{\begin{array}{cl}
-\frac{1}{2} \overline{\bar{B}}_{i n c} \cdot\left(\vec{x}-\vec{x}_{0}\right) & \text { for } \vec{x} \in \Omega_{1}  \tag{7.72}\\
\overrightarrow{0} & \text { for } \vec{x} \in \Omega_{2}
\end{array}\right.
$$

The vector potential $\vec{A}_{\text {inc }}$ associated to the constant incident field $\vec{B}_{\text {inc }}$ is easier to calculate, in fact:

$$
\begin{align*}
& \vec{\nabla} \wedge \vec{A}_{e x t, 0}=\overline{\bar{B}}_{i n c} \Longrightarrow  \tag{7.73}\\
& \vec{A}_{i n c}=\frac{1}{2} \overline{\bar{B}}_{i n c} \cdot\left(\vec{x}-\vec{x}_{0}\right)+\vec{A}_{i n c, 0} \tag{7.74}
\end{align*}
$$

where $\vec{A}_{i n c, 0}$ is an irrotational term which can be set equal to zero.
Finally, the global vector potential $\vec{A}(\vec{x})$ can be calculated as:

$$
\begin{equation*}
\vec{A}=\vec{A}_{i n c}+\vec{A}_{i r r} \tag{7.75}
\end{equation*}
$$

More explicitly:

$$
\vec{A}(\vec{x})=\left\{\begin{array}{cl}
\overrightarrow{0} & \text { for } \vec{x} \in \Omega_{1}  \tag{7.76}\\
\frac{1}{2} \overline{\bar{B}}_{i n c} \cdot\left(\vec{x}-\vec{x}_{0}\right) & \text { for } \vec{x} \in \Omega_{2}
\end{array}\right.
$$

Once we know the field, we can calculate the distribution of source on the sphere's surface.

## Calculating currents and doublets

Now we want to determine which surface currents $\vec{J}_{s, e}$ and doublet $\overline{\bar{D}}_{e}$ are needed to achieve the desired shielding or "magnetic cloaking".

From chapter 5 we know that the Boundary Conditions can be written as:

$$
\begin{align*}
\mu_{0} \overline{\bar{D}}_{e} & =\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21}  \tag{7.77}\\
\mu_{0} \vec{J}_{s, e} & =-\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21} \tag{7.78}
\end{align*}
$$

For the examined case, we can calculate fields $\vec{A}_{2}$ and $\vec{A}_{1}$ and their gradient on the sphere's boundary, therefore, holding 7.76, we can write:

$$
\begin{gather*}
\vec{x}-\vec{x}_{0}=R \vec{n}_{21}  \tag{7.79}\\
\begin{cases}\vec{A}_{1}=\quad \overrightarrow{0} \\
\vec{A}_{2}=\frac{1}{2} R \overline{\bar{B}}_{i n c} \cdot \vec{n}_{21}\end{cases}  \tag{7.80}\\
\left\{\begin{array}{l}
\frac{\partial \vec{A}_{1}}{\partial \vec{x}}=\overline{\overline{0}} \\
\frac{\partial \vec{A}_{2}}{\partial \vec{x}}=\frac{1}{2} \overline{\bar{B}}_{i n c}
\end{array}\right.
\end{gather*}
$$

After few calculi, we finally obtain the distribution of surface doublets and currents:

$$
\begin{align*}
& \overline{\bar{D}}_{e}=\frac{1}{\mu_{0}} \frac{1}{2} R \overline{\bar{B}}_{i n c} \cdot \overline{\overline{n n}}  \tag{7.81}\\
& \vec{J}_{s, e}=-\frac{1}{\mu_{0}} \frac{1}{2} \overline{\bar{B}}_{i n c} \cdot \vec{n}_{21} \tag{7.82}
\end{align*}
$$

Let us observe that this result is quite different from the one obtained using the classic BCs by Schelkunoff. In fact, we have doublets $\overline{\bar{D}}_{e}$, constant in time, and the electric surface current $\vec{J}_{s, e}$ turns out to be half of the predicted value. We can also notice that $\vec{J}_{s, e}$ is tangent to the sphere and it does not depend on the radius $R$, while $\overline{\bar{D}}_{e}$ does. Those quantities can be also related as:

$$
\begin{equation*}
\overline{\bar{D}}_{e} \cdot \vec{n}_{21}=-R \vec{J}_{s, e} \tag{7.83}
\end{equation*}
$$

In 3D notation, the source distributions can be so expressed:

$$
\begin{align*}
\overline{\bar{D}}_{e} \cdot \vec{n}_{21} & =\frac{1}{\mu_{0}} \frac{1}{2} R \vec{B}_{i n c} \times \vec{n}_{21}  \tag{7.84}\\
\vec{J}_{s, e} & =-\frac{1}{\mu_{0}} \frac{1}{2} \vec{B}_{i n c} \times \vec{n}_{21} \tag{7.85}
\end{align*}
$$

Now let us see how to interpret the achieved results.

### 7.5.3 Concentric current shells

In the previous chapters we have often observed that doublets $\overline{\bar{D}}_{e}$ are associated to 2 current layers, and sometimes they can be regarded as current vortices. More precisely, given two surface current $\vec{J}_{1}$ and $\vec{J}_{2}$ at distance $\Delta x$, their doublet is such that:

$$
\begin{equation*}
\overline{\bar{D}}_{e} \cdot \vec{n}_{21}=\frac{1}{2} \Delta x\left(\vec{J}_{2}-\vec{J}_{1}\right) \tag{7.86}
\end{equation*}
$$

As a matter of fact, we could obtain the same shielding effect by mean of two different concentric currents shells, instead of a single one. We call $\vec{J}_{2}$ and $\vec{J}_{1}$ the surface current distributions for two spheres whose radii are $R_{2}$ and $R_{1}$ respectively. The distance between the two shells is so $\Delta x=R_{2}-R_{1}$.

It can be demonstrated (but here we shall not do it) that the inner magnetic field is null if it holds:

$$
\left\{\begin{align*}
R_{2} \vec{J}_{2}+R_{1} \vec{J}_{1} & =\overrightarrow{0}  \tag{7.87}\\
\vec{J}_{2}+\vec{J}_{1} & =\vec{J}_{s, e}
\end{align*}\right.
$$

where $\vec{J}_{s, e}$ is the total surface current calculated in 7.82 . Thus, the currents flowing in the two shells turns out to be:

$$
\left\{\begin{array}{l}
\vec{J}_{1}=\frac{R_{2}}{R_{2}-R_{1}} \vec{J}_{s, e}  \tag{7.88}\\
\vec{J}_{2}=-\frac{R_{1}}{R_{2}-R_{1}} \vec{J}_{s, e}
\end{array}\right.
$$

It's important to notice that for $R_{1} \rightarrow R_{2}$ both $\vec{J}_{2}$ and $\vec{J}_{1}$ can reach extremely high values if compared to $\vec{J}_{s, e}$. Moreover, they have different signs and the whole system can be interpreted as made by two concentric spherical solenoids.

Finally, the equivalent doublet $\overline{\bar{D}}_{e}$ is related to the global current $\vec{J}_{s, e}$ through the average radius:

$$
\begin{equation*}
\overline{\bar{D}}_{e} \cdot \vec{n}_{21}=-\frac{1}{2}\left(R_{2}+R_{1}\right) \vec{J}_{s, e} \tag{7.89}
\end{equation*}
$$

### 7.5.4 Concluding remark

As already stated, here we meant just to bring an example of the difficulties related to the classic Boundary Conditions written in terms of $\vec{E}$ and $\vec{B}$. On the contrary, using the EM potentials we solved the "magnetic cloaking" problem without invoking the existence of magnetic monopoles or currents, problem which initially looked as impossible. In my personal opinion, this example and the Volta Effect well illustrate the advantages of thinking to electromagnetic problems in terms of potentials $\rho_{A}$ and $\vec{A}$.

Last but not least, the examined case of "magnetic cloak" can be regarded as a static... appetizer before the general case of the invisibility cloak, which will be treated in chapter 14.

## Chapter 8

## Space-time Boundary Conditions for EM fields

They're cheering us both, you because nobody understands you, and me because everybody understands me

Charlie Chaplin to Albert Einstein, speaking about people, 1931

In this chapter the relativistic formulation for the Huygens' Principle will be presented, then it will be applied to the electromagnetic fields. This is a very recent result, which emerged almost spontaneously while I was writing the Boundary Conditions down on papers. This chapter could be interesting for physicists studying plasmas or relativistic Quantum Mechanics. Besides, this topic could be useful also for electrical and telecommunication engineers who work on scattering for moving bodies, echo-Doppler and so on. The common problem consists in the coherent description of the electromagnetic fields (and their BCs ) in different reference frames.

I was in doubt if I had (time) to write this chapter or not, but finally I decided to insert it since the results look quite plain and consistent. Here the standard relativistic notation will be adopted, though somewhere I will try to make it explicit. Moreover, here I report just a summary: the complete discussion of the details and consequences would require a whole book.

### 8.1 Relativistic domains

Till now we investigated how to apply the extended Huygens' Principle on space domains. However, nothing forbids us to apply it also to space-time domains. The basic idea is always the same: we have to map a field $\mathbf{f}$ from a domain $\Omega$ on its boundary $\partial \Omega$, determining the surface sources.

(a) Generic vector field $\vec{v}$ on space domain $\Omega_{x}$

(b) Generic ( $\mathbf{N}+\mathbf{1}$ )-vector field $v^{\mu}$ on space-time domain $\Omega$

Figure 8.1: The Huygens' principle can be applied also to space-time fields and domains.

In a N-Dimensional space we can exploit some theorems 34, like:

$$
\begin{align*}
\int_{\Omega} \vec{\nabla} \phi d \Omega_{x} & =\oint_{\partial \Omega} \vec{n} \phi d S_{x}  \tag{8.1}\\
\int_{\Omega} \frac{\partial v_{i}}{\partial x_{j}} d \Omega_{x} & =\oint_{\partial \Omega} v_{i} n_{j} d S_{x} \quad \forall i, j \tag{8.2}
\end{align*}
$$

In all those cases, the integral on a "volume" $\Omega_{x}$ is transformed in a "surface" $\partial \Omega_{x}$ where a normal $\vec{n}$ replaces the gradient.

### 8.1.1 Space-time normal ( $\mathrm{N}+1$ )-vector

Let us consider a domain $\Omega$ in a ( $\mathrm{N}+1$ )-Dimensional space-time. On its boundary $\partial \Omega$ we can define a $(N+1)$-normal $n_{\mu}$ such that it is perpendicular to the boundary itself, thus:

$$
\begin{equation*}
n_{\mu} d x^{\mu}=0 \quad \forall d x^{\mu} \text { tangent to } \partial \Omega \tag{8.3}
\end{equation*}
$$

where:

$$
d x^{\mu}=\left[\begin{array}{c}
d\left(c_{0} t\right)  \tag{8.4}\\
d \vec{x}
\end{array}\right]
$$

Let us notice that the product $n_{\mu} d x^{\mu}$ is invariant with the reference frame.


Figure 8.2: Space-time boundary element. The ( $\mathbf{N}+\mathbf{1}$ ) normals $n_{\mu}$ is perpendicular to the $(\mathbf{N}+1)$-vector $d x^{\mu}$ tangential to the boundary.

Since $n_{\mu}$ has to be "normal", we require that its Minkowski modulus is unitary:

$$
\begin{align*}
& n_{\mu} n^{\mu}= \pm 1  \tag{8.5}\\
& n^{\mu}=\eta^{\mu \nu} n_{\nu}  \tag{8.6}\\
& n_{\mu} \eta^{\mu \nu} n_{\nu}= \pm 1 \tag{8.7}
\end{align*}
$$

where:

$$
n_{\mu}=\left[\begin{array}{l}
n_{0}  \tag{8.8}\\
\vec{n}_{x}
\end{array}\right] ; \quad \eta^{\mu \nu}=\left[\begin{array}{cc}
1 & \overrightarrow{0}^{T} \\
\overrightarrow{0} & -\overline{\bar{I}}
\end{array}\right]
$$

Let us now consider a "rectangular" domain $\Omega$ in the space-time. It is limited by a spatial boundary and by two instants $t_{0}$ and $t_{f}$. The initial and final

(a) Domain "at rest" (not moving). The initial and final instants are well defined.

(b) Domain in uniform rectilinear motion. The initial and final instants are not explicitly defined.

Figure 8.3: Lorentz boost for a "parallelogram" space-time domain. The $(\mathbf{N}+1)$-normal $n_{\mu}$ are always perpendicular to the boundaries, which can be space-like or time-like ones.
conditions, are associated to time-like ( $\mathrm{N}+1$ )-normals:

$$
\begin{equation*}
n_{\mu} n^{\mu}=+1 \quad \text { time-like B.C. } \tag{8.9}
\end{equation*}
$$

For the considered case their form is:

$$
n_{\nu}=\left[\begin{array}{c}
-1  \tag{8.10}\\
\overrightarrow{0}
\end{array}\right] \quad \text { initial condition; } \quad n_{\nu}=\left[\begin{array}{l}
1 \\
\overrightarrow{0}
\end{array}\right] \quad \text { final condition; }
$$

On the contrary, the spatial boundary conditions are associated to space-like ( $\mathrm{N}+1$ )-normals:

$$
\begin{equation*}
n_{\mu} n^{\mu}=-1 \quad \text { space-like B.C. } \tag{8.11}
\end{equation*}
$$

For the considered "rectangular" domain their form is:

$$
n_{\nu}=\left[\begin{array}{c}
0  \tag{8.12}\\
\vec{n}
\end{array}\right] \quad \text { with } \vec{n}^{T} \cdot \vec{n}=1
$$

Equations 8.10, 8.12 are valid if you take into account a still, not moving domain, whose initial and final instants are fixed.

## Changing the reference frame

Now we desire to determine the ( $\mathrm{N}+1$ )-normal structure for a generic space-time boundary. Let us suppose we change the reference frame applying a Lorentz boost with velocity $\vec{v}$ :

$$
\Lambda_{\nu}^{\mu}=\overline{\bar{\Lambda}}(-\vec{v})=\left[\begin{array}{cc}
\gamma & \gamma \vec{\beta}^{T}  \tag{8.13}\\
\gamma \vec{\beta} & \overline{\bar{\gamma}}
\end{array}\right]
$$

where:

$$
\begin{equation*}
\vec{\beta}=\frac{\vec{v}}{c_{0}} ; \quad \gamma=\frac{1}{\sqrt{1-\beta^{2}}} ; \quad \overline{\bar{\gamma}}=\overline{\bar{I}}+(\gamma-1)\left[\overline{\overline{\beta \beta^{T}}}\right] / \beta^{2} \tag{8.14}
\end{equation*}
$$

The ( $\mathrm{N}+1$ )-normal $n^{\prime \mu}$ in the new frame can be so calculated as:

$$
\begin{equation*}
n^{\prime \mu}=\Lambda_{\nu}^{\mu} n^{\nu} \tag{8.15}
\end{equation*}
$$

If the $(\mathrm{N}+1)$-normal is a time-like one, then we find:

$$
\begin{align*}
& n^{\nu}= \pm\left[\begin{array}{l}
1 \\
\overrightarrow{0}
\end{array}\right] \Longrightarrow n^{\prime \mu}= \pm \gamma\left[\begin{array}{l}
1 \\
\vec{\beta}
\end{array}\right] \Longrightarrow  \tag{8.16}\\
& n_{\mu}^{\prime}= \pm \gamma\left[\begin{array}{c}
1 \\
-\vec{\beta}
\end{array}\right]= \pm \gamma\left[\begin{array}{c}
1 \\
-\frac{\vec{v}}{c_{0}}
\end{array}\right] \quad \text { time-like }(\mathrm{N}+1) \text {-normal } \tag{8.17}
\end{align*}
$$

So the general structure for time-like normals is:

$$
n_{\mu}=\gamma\left[\begin{array}{c}
-1  \tag{8.18}\\
\frac{\vec{v}}{c_{0}}
\end{array}\right] \quad \text { initial cond.; } \quad n_{\mu}=\gamma\left[\begin{array}{c}
1 \\
-\frac{\vec{v}}{c_{0}}
\end{array}\right] \quad \text { final cond. }
$$

Conversely, if the ( $\mathrm{N}+1$ )-normal is a space-like one, then we find:

$$
n^{\nu}=\left[\begin{array}{c}
0  \tag{8.19}\\
-\vec{n}
\end{array}\right] \quad \Longrightarrow \quad n^{\prime \mu}=\left[\begin{array}{c}
-\gamma \vec{\beta}^{T} \cdot \vec{n} \\
-\overline{\bar{\gamma}} \cdot \vec{n}
\end{array}\right]
$$

So the general structure for space-like normals is:

$$
n_{\mu}=\left[\begin{array}{c}
-\gamma \frac{\vec{v}}{c_{0}} \cdot \vec{n}  \tag{8.20}\\
\overline{\bar{\gamma}} \cdot \vec{n}
\end{array}\right] \quad \text { space-like }(\mathrm{N}+1) \text {-normal }
$$

Finally, if you know the boundary velocity $\vec{v}$ and the local normal $\vec{n}$ then you can directly construct $n^{\mu}$.

### 8.1.2 Theorems for the space-time gradient

The theorems 8.1, (8.2) can be extended to space-time in a quite immediate and intuitive way. Let us introduce the $(\mathrm{N}+1)$-gradient $\partial_{\nu}($.$) , such that:$

$$
\begin{gather*}
d \phi=\frac{\partial \phi}{\partial x^{\nu}} d x^{\nu}=\left(\partial_{\nu} \phi\right) d x^{\nu}  \tag{8.21}\\
\partial_{\nu}(.)=\frac{\partial .}{\partial x^{\nu}}=\left[\begin{array}{c}
\frac{1}{c_{0}} \frac{\partial .}{\partial t} \\
-\frac{\partial .}{\partial \vec{x}}
\end{array}\right] \quad \partial^{\nu}(.)=\frac{\partial .}{\partial x_{\nu}}=\left[\begin{array}{c}
\frac{1}{c_{0}} \frac{\partial .}{\partial t} \\
\frac{\partial .}{\partial \vec{x}}
\end{array}\right] \tag{8.22}
\end{gather*}
$$


(a) Reference frame at "rest". The generic ( $\mathbf{N}+1$ )-vector $v^{\mu}$ has zero space-components.

(b) Galileo transformation. The time-component for the ( $\mathrm{N}+1$ )vector $v^{\mu}$ is unchanged after the boost, since time intervals are invariant. The light-cone has been deformed, thus the light can propagate with a speed different from $c_{0}$.

(c) Lorentz transformation. The red hyperbole and the cyan light-cone are unchanged, so that the speed of light is always $c_{0}$ (i.e. it is invariant). Time intervals are not invariant, hence simultaneity among events is lost.

Figure 8.4: Visual representation for Galileo transformation and Lorentz one. In both the cases the reference frames are moving with constant velocity, thus the transformations are linear (straight lines remain straight ones).

For a scalar field, the space-time gradient theorem will look so:

$$
\begin{align*}
\int_{\Omega} \frac{\partial \phi}{\partial x^{\nu}} d \Omega & =\oint_{\partial \Omega} n_{\nu} \phi d S & \forall \nu \in\{0 ; 1 ; \cdots ; N\}  \tag{8.23}\\
\int_{\Omega} \partial_{\nu} \phi d \Omega & =\oint_{\partial \Omega} n_{\nu} \phi d S & \forall \nu \in\{0 ; 1 ; \cdots ; N\} \tag{8.24}
\end{align*}
$$

Thus, analogously to the space case, the $(\mathrm{N}+1)$-normal $n_{\nu}$ is associated to the $(\mathrm{N}+1)$-gradient $\partial_{\nu}$.

## Vector case

The same reasoning can be applied to ( $\mathrm{N}+1$ )-vectors in space-time, in fact:

$$
\begin{align*}
d v^{\mu} & =\frac{\partial v^{\mu}}{\partial x^{\nu}} d x^{\nu}=\left(\partial_{\nu} v^{\mu}\right) d x^{\nu}  \tag{8.25}\\
d v^{\mu} & =\frac{\partial v^{\mu}}{\partial x_{\nu}} d x_{\nu}=\left(\partial^{\nu} v^{\mu}\right) d x_{\nu} \tag{8.26}
\end{align*}
$$

Thus the generalized theorem for gradient will look:

$$
\begin{align*}
\int_{\Omega} \frac{\partial v^{\mu}}{\partial x^{\nu}} d \Omega & =\oint_{\partial \Omega} n_{\nu} v^{\mu} d S & \forall \mu, \nu \in\{0 ; 1 ; \cdots ; N\}  \tag{8.27}\\
\int_{\Omega} \partial_{\nu} v^{\mu} d \Omega & =\oint_{\partial \Omega} n_{\nu} v^{\mu} d S & \forall \mu, \nu \in\{0 ; 1 ; \cdots ; N\} \tag{8.28}
\end{align*}
$$

I like this theorem, either for its simplicity and generality. As we are going to see, it can be effectively used to extend the Huygens' Principle in space-time.

### 8.2 Relativistic surface equivalence theorem

The steps for the closed form derivation in the relativistic case are analogous to ones already explained for the spatial case (see sec 4.3.1 and 4.5.1), therefore we shall not comment every equation in detail.

### 8.2.1 Elementary solution for the wave equation

If we are interested to determine the source for wave equation in space-time, then we have to find the corresponding operator, which has to be invariant with the reference frame. It can be easily verified that the D'Alembert operator satisfy those requirements. As a matter of fact, it can be directly expressed in terms of ( $\mathrm{N}+1$ )-gradients:

$$
\begin{equation*}
\partial_{\nu} \partial^{\nu}(.)=\partial^{\nu} \partial_{\nu}(.)=\frac{1}{c_{0}^{2}} \frac{\partial^{2} .}{\partial t^{2}}-\nabla^{2}(.) \tag{8.29}
\end{equation*}
$$

The associated Green's function $\varphi_{G}$ is such that:

$$
\begin{equation*}
\frac{1}{c_{0}^{2}} \frac{\partial^{2} \varphi_{G}}{\partial t^{2}}-\nabla^{2} \varphi_{G}=\delta_{N}\left(\vec{x}-\vec{x}_{0}\right) \delta_{1}\left(t-t_{0}\right) \tag{8.30}
\end{equation*}
$$

Translating it in relativistic notation, we achieve a compact expression:

$$
\begin{equation*}
\partial_{\nu} \partial^{\nu} \varphi_{G}=\delta_{N+1}\left(x^{\mu}-x_{, 0}^{\mu}\right) \tag{8.31}
\end{equation*}
$$

### 8.2.2 Theorem demonstration

Let $\mathbf{f}\left(x^{\mu}\right)$ be a generic field defined on space-time. Let $\Omega$ be space-time domain its belonging function $\in_{\Omega}$ determines if an event $x^{\mu}=\left[c_{0} t ; \vec{x}\right]$ belongs or not to $\Omega$ :

$$
\epsilon_{\Omega}\left(x^{\mu}\right)= \begin{cases}1 & \text { for } x^{\mu} \in \Omega  \tag{8.32}\\ 0 & \text { for } x^{\mu} \notin \Omega\end{cases}
$$

Now we desire to determine the "sources" for field $\mathbf{f}$ inside the domain $\Omega$. In other words, we have to set to zero the field outside $\Omega$. Following an analogous procedure to the one described in 4.3.14.5.1, we find:

$$
\begin{align*}
& \epsilon_{\Omega}\left(x^{\mu}\right) \mathbf{f}\left(x^{\mu}\right)=\int_{\Omega} \mathbf{f}\left(y^{\mu}\right) \delta_{N+1}\left(x^{\mu}-y^{\mu}\right) d \Omega_{y}  \tag{8.33}\\
& \epsilon_{\Omega}\left(x^{\mu}\right) \mathbf{f}\left(x^{\mu}\right)=\int_{\Omega} \mathbf{f}\left(y^{\mu}\right) \partial_{\nu} \partial^{\nu} \varphi_{G}\left(x^{\mu}-y^{\mu}\right) d \Omega_{y} \tag{8.34}
\end{align*}
$$

where $y^{\mu}$ is the integration variable. More compactly, we can write:

$$
\begin{equation*}
\epsilon_{\Omega}\left(x^{\mu}\right) \mathbf{f}\left(x^{\mu}\right)=\int_{\Omega} \mathbf{f} \partial_{\nu} \partial^{\nu} \varphi d \Omega_{y} \tag{8.35}
\end{equation*}
$$

In sec. 4.3.1 we have seen that the Green's lemma in space is:

$$
\begin{equation*}
a \nabla^{2} b-b \nabla^{2} a=\vec{\nabla}^{T} \cdot(a \vec{\nabla} b-b \vec{\nabla} a) \tag{8.36}
\end{equation*}
$$

Its relativistic version is:

$$
\begin{equation*}
a \partial_{\nu} \partial^{\nu} b-b \partial_{\nu} \partial^{\nu} a=\partial_{\nu}\left(a \partial^{\nu} b-b \partial^{\nu} a\right) \tag{8.37}
\end{equation*}
$$

Therefore it holds:

$$
\begin{equation*}
\mathbf{f} \partial_{\nu} \partial^{\nu} \varphi=\varphi \partial_{\nu} \partial^{\nu} \mathbf{f}+\partial_{\nu}\left(\mathbf{f} \partial^{\nu} \varphi-\varphi \partial^{\nu} \mathbf{f}\right) \tag{8.38}
\end{equation*}
$$

Substituting this last expression in the integral in 8.35, we find:

$$
\begin{equation*}
\epsilon_{\Omega}\left(x^{\mu}\right) \mathbf{f}\left(x^{\mu}\right)=\int_{\Omega} \varphi \partial_{\nu} \partial^{\nu} \mathbf{f} d \Omega+\int_{\Omega} \partial_{\nu}\left(\mathbf{f} \partial^{\nu} \varphi-\varphi \partial^{\nu} \mathbf{f}\right) d \Omega \tag{8.39}
\end{equation*}
$$

Thanks to the generalized theorem for gradient 8.28), the last term can be transformed in a boundary integral, in fact:

$$
\begin{equation*}
\int_{\Omega} \partial_{\nu}\left(\mathbf{f} \partial^{\nu} \varphi-\varphi \partial^{\nu} \mathbf{f}\right) d \Omega=\oint_{\partial \Omega} n_{\nu}\left(\mathbf{f} \partial^{\nu} \varphi-\varphi \partial^{\nu} \mathbf{f}\right) d S_{y} \tag{8.40}
\end{equation*}
$$

Let us notice that $\mathbf{f}$ can be indifferently a scalar field or rather a vector one. Also, it is not required to be invariant.

## Relativistic surface equivalence theorem

Finally, the relativistic surface equivalence theorem results to be:

$$
\begin{equation*}
\epsilon_{\Omega}\left(x^{\mu}\right) \mathbf{f}\left(x^{\mu}\right)=\int_{\Omega} \varphi \partial_{\nu} \partial^{\nu} \mathbf{f} d \Omega_{y}+\oint_{\partial \Omega} n_{\nu}\left(\mathbf{f} \partial^{\nu} \varphi-\varphi \partial^{\nu} \mathbf{f}\right) d S_{y} \tag{8.41}
\end{equation*}
$$

where:

$$
\begin{align*}
& \varphi=\varphi_{G}\left(x^{\mu}-y^{\mu}\right)  \tag{8.42}\\
& \partial_{\nu} \partial^{\nu} \varphi_{G}=\delta_{N+1}\left(x^{\mu}-y^{\mu}\right)  \tag{8.43}\\
& \mathbf{f}=\mathbf{f}\left(y^{\mu}\right) \tag{8.44}
\end{align*}
$$

### 8.3 Relativistic notation for Maxwell's

In order to write Maxwell's equations in relativistic notation, we introduce the $(\mathrm{N}+1)$-vectors $A^{\mu}$ and $J^{\mu}$, which are respectively the $(N+1)$-potential and the ( $N+1$ )-current.

$$
\begin{array}{ll}
A^{\mu}=\left[\begin{array}{c}
\rho_{A} c_{0} \\
\vec{A}
\end{array}\right] & J^{\mu}=\left[\begin{array}{c}
\rho_{e} c_{0} \\
\vec{J}_{e}
\end{array}\right] \\
A_{\mu}=\left[\begin{array}{c}
\rho_{A} c_{0} \\
-\vec{A}
\end{array}\right] & J_{\mu}=\left[\begin{array}{c}
\rho_{e} c_{0} \\
-\vec{J}_{e}
\end{array}\right] \tag{8.46}
\end{array}
$$

In a non-relativistic notation, Maxwell's Eq.s can be expressed as:

$$
\left\{\begin{array} { l } 
{ \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial ^ { 2 } \rho _ { A } } { \partial t ^ { 2 } } - \nabla ^ { 2 } \rho _ { A } = \mu _ { 0 } \rho _ { e } }  \tag{8.47}\\
{ \frac { 1 } { c _ { 0 } ^ { 2 } } \frac { \partial ^ { 2 } \vec { A } } { \partial t ^ { 2 } } - \nabla ^ { 2 } \vec { A } = \mu _ { 0 } \vec { J } _ { e } }
\end{array} \quad \left\{\begin{array}{l}
\frac{\partial \rho_{A}}{\partial t}+\vec{\nabla}^{T} \cdot \vec{A}=0 \\
\vec{E}=-\frac{\partial \vec{A}}{\partial t}-\vec{\nabla}\left(\rho_{A} c_{0}^{2}\right) \\
\overline{\bar{B}}=\vec{\nabla} \hat{\wedge} \vec{A}
\end{array}\right.\right.
$$

The first set contains just wave equations (Wave set), while the second one includes the Lorentz Gauge and the expressions for the electric $\vec{E}$ and magnetic $\bar{B}$ fields. Here we will not demonstrate that, but it can be verified that those sets can be compacted in relativistic form, thus they look:

$$
\partial_{\nu} \partial^{\nu}\left(A^{\mu}\right)=\mu_{0} J^{\mu} \quad\left\{\begin{array}{l}
\partial_{\mu} A^{\mu}=0  \tag{8.48}\\
F^{\mu \nu}=\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}
\end{array}\right.
$$

For sake of clarity, here we explicitly calculate the EM tensor $F^{\mu \nu}$ :

$$
\begin{align*}
F^{\mu \nu} & =\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}  \tag{8.49}\\
F^{\mu \nu} & =\frac{\partial A^{\nu}}{\partial x_{\mu}}-\frac{\partial A^{\mu}}{\partial x_{\nu}} \tag{8.50}
\end{align*}
$$

where:

$$
\begin{array}{ll}
\frac{\partial A^{\nu}}{\partial x_{\mu}}=\left[\begin{array}{ll}
\frac{\partial\left(\rho_{A} c_{0}\right)}{\partial x_{\mu}}, & \frac{\partial \vec{A}^{T}}{\partial x_{\mu}}
\end{array}\right]= & {\left[\begin{array}{cc}
\frac{\partial\left(\rho_{A} c_{0}\right)}{\partial\left(c_{0} t\right)} & \frac{\partial \vec{A}^{T}}{\partial\left(c_{0} t\right)} \\
-\frac{\partial\left(\rho_{A} c_{0}\right)}{\partial \vec{x}} & -\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]^{T}
\end{array}\right]} \\
\frac{\partial A^{\mu}}{\partial x_{\nu}}=\left[\begin{array}{ll}
\frac{\partial A^{\mu}}{\partial\left(c_{0} t\right)}, & \left.-\frac{\partial A^{\mu}}{\partial \vec{x}}\right]=
\end{array}\right. & {\left[\begin{array}{cc}
\frac{\partial\left(\rho_{A} c_{0}\right)}{\partial\left(c_{0} t\right)} & -\frac{\partial\left(\rho_{A} c_{0}\right)}{\partial \vec{x}} \\
\frac{\partial \vec{A}}{\partial\left(c_{0} t\right)} & -\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]
\end{array}\right]} \tag{8.52}
\end{array}
$$

Thus the EM tensor turns out to be:

$$
\begin{align*}
& F^{\mu \nu}=\left[\begin{array}{cc}
0 & \left(\frac{\partial \vec{A}}{\partial\left(c_{0} t\right)}+\vec{\nabla}\left(\rho_{A} c_{0}\right)\right)^{T} \\
-\left(\frac{\partial \vec{A}}{\partial\left(c_{0} t\right)}+\vec{\nabla}\left(\rho_{A} c_{0}\right)\right) & {\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]-\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]^{T}}
\end{array}\right]  \tag{8.53}\\
& F^{\mu \nu}=\left[\begin{array}{cc}
0 & -\vec{E}^{T} / c_{0} \\
\vec{E} / c_{0} & \overline{\bar{B}}
\end{array}\right] \tag{8.54}
\end{align*}
$$

### 8.3.1 Charges and currents in space-time

Now let us see how we can describe charges and currents in a space-time context. For a net charge $Q_{e}$ it is possible to construct a net $(\mathrm{N}+1)$-current $I^{\mu}$, such that:

$$
I^{\mu}=Q_{e, 0} v^{\mu}=Q_{e, 0} \gamma\left[\begin{array}{c}
c_{0}  \tag{8.55}\\
\vec{v}
\end{array}\right]
$$

where $v^{\mu}$ is the $(\mathrm{N}+1)$-velocity, while $Q_{e, 0}$ can be regarded as the rest charge. In space-time the $(\mathrm{N}+1)$-current $I^{\mu}$ will be tangent the particle's world line. Let


Figure 8.5: Charges and currents in space-time.
us notice that the temporal component $I^{0}$ for the ( $\mathrm{N}+1$ )-current can be either positive or negative, depending on the charge sign. Thus it is possible to deal with $(\mathrm{N}+1)$-currents $J^{\mu}$ which seem to travel back in time.

### 8.3.2 Closed causal loops

Actually, some phenomena like the creation and the annihilation for particleantiparticle pairs are interpreted as "time travels". For example, let us consider
two gamma-ray photons. At instant $t_{1}$ they collide and disappear, giving raise to an electron-positron pair, thus to two particles with opposite charges. At instant $t_{2}$ the particles annihilate each other, re-emitting two gamma-ray photons. According to some interpretations, e.g. by R.P. Feynman [43] and R. Rucker[44],


Figure 8.6: Closed causal loop. In this Feynman diagram two gamma-ray photons collide at instant $t_{1}$, creating an electron-positron pair which annihilates at instant $t_{2}$, re-emitting the two photons. The ( $\mathbf{N}+1$ )-current $J^{\mu}$ looks to flow along a closed space-time path, hence in some interpretations the electron is regarded as a positron travelling backward in time.
the positron can be regarded as the electron which is travelling back in time, or vice-versa. As a matter of fact, observing the space-time diagram, we notice that:

- the electric charge is always conserved, since $\partial_{\mu} J^{\mu}=0$
- the ( $\mathrm{N}+1$ )-current $J^{\mu}$ looks to travel along a closed causal loop in spacetime.

We could object that, if a particle is really travelling back in time, then also its mass should become negative. More precisely, it should be demonstrated that the time-component $v^{0}$ for the ( $\mathrm{N}+1$ )-velocity is effectively negative. Anyway, this is not the right place for discussing that question. The aim of this last example was only to show that $(\mathrm{N}+1)$-vectors, like $J^{\mu}$ and $A^{\mu}$, can point "backward in time ".

### 8.3.3 Dipoles and doublets in space-time

An electric dipoles is composed by two charges, usually equal and opposite. In space-time the dipoles, but also the doublets, can be thus described by mean of a pair of $(\mathrm{N}+1)$-currents $I_{, 1}^{\mu}$ and $I_{, 2}^{\mu}$. For that purpose, we can define the $(\mathrm{N}+1)$-distance $\Delta x_{, 21}^{\mu}$ between two charges in space time, so that:

$$
\begin{equation*}
\Delta x_{, 21}^{\nu}=x_{, 2}^{\nu}-x_{, 1}^{\nu} \tag{8.56}
\end{equation*}
$$

Therefore, the net-relativistic doublet $D_{n e t}^{\mu \nu}$ associated to the two ( $\mathrm{N}+1$ )-currents $I_{, 1}^{\mu}$ and $I_{, 2}^{\mu}$ is equal to:

$$
\begin{equation*}
D_{n e t}^{\mu \nu}=\frac{1}{2}\left(I_{, 2}^{\mu}-I_{, 1}^{\mu}\right) \Delta x_{, 21}^{\nu} \quad \text { net relativistic doublet } \tag{8.57}
\end{equation*}
$$



Figure 8.7: Dipoles and doublets in space-time.

More compactly:

$$
\begin{equation*}
D_{n e t}^{\mu \nu}=\frac{1}{2} \Delta I^{\nu} \Delta x^{\nu} \tag{8.58}
\end{equation*}
$$

Let us notice that this definition is analogous to the one for the net space doublet $\overline{\bar{D}}_{E}$ reported in 4.90.

$$
\begin{equation*}
\overline{\bar{D}}_{E}=\frac{1}{2}\left(\vec{I}_{2}-\vec{I}_{1}\right) \otimes \Delta \vec{x}_{21} \tag{8.59}
\end{equation*}
$$



Figure 8.8: Net relativistic doublet $D_{n e t}^{\mu \nu}$. In this example the ( $\mathbf{N}+\mathbf{1}$ )currents $I_{, 1}^{\mu}$ and $I_{, 2}^{\mu}$ are equal and opposite. Moreover, the two charges are considered at the same instant, thus the ( $\mathbf{N}+\mathbf{1}$ )-distance vector $\Delta x^{\nu}{ }_{21}$ is purely space-like (i.e., its time-component is zero).

### 8.4 Space-time boundary sources for EM fields

Now we are going to apply the relativistic surface equivalence theorem 8.41 to the electromagnetic field. The general form of the theorem is:

$$
\begin{equation*}
\epsilon_{\Omega}\left(x^{\mu}\right) \mathbf{f}\left(x^{\mu}\right)=\int_{\Omega} \varphi \partial_{\nu} \partial^{\nu} \mathbf{f} d \Omega_{y}+\oint_{\partial \Omega} n_{\nu}\left(\mathbf{f} \partial^{\nu} \varphi-\varphi \partial^{\nu} \mathbf{f}\right) d S_{y} \tag{8.60}
\end{equation*}
$$

For an electromagnetic problem it is quite convenient to adopt the ( $\mathrm{N}+1$ )potential $A^{\mu}$ in the role of fundamental field $\mathbf{f}$. Therefore, after a simple substitution we can calculate $A^{\mu}$ inside the space-time domain $\Omega$ :

$$
\begin{equation*}
\in_{\Omega}\left(x^{\mu}\right) A^{\mu}\left(x^{\mu}\right)=\int_{\Omega} \varphi\left(\partial_{\nu} \partial^{\nu} A^{\mu}\right) d \Omega_{y}+\oint_{\partial \Omega} n_{\nu}\left(A^{\mu} \partial^{\nu} \varphi-\varphi \partial^{\nu} A^{\mu}\right) d S_{y} \tag{8.61}
\end{equation*}
$$

The EM sources inside the "volume" $\Omega$ are simply the ( $\mathrm{N}+1$ )-current densities $J^{\mu}$, in fact the wave equation holds:

$$
\begin{equation*}
\partial_{\nu} \partial^{\nu}\left(A^{\mu}\right)=\mu_{0} J^{\mu} \tag{8.62}
\end{equation*}
$$

On the boundary $\partial \Omega$ we can instead define the $(\mathrm{N}+1)$-surface current $J_{, s}^{\mu}$ and the doublet tensor $D^{\mu}{ }_{\nu}$ :

$$
\begin{align*}
n_{\nu}\left(\partial^{\nu} A^{\mu}\right) & =-\mu_{0} J_{, s}^{\mu}  \tag{8.63}\\
n_{\nu} A^{\mu} & =\mu_{0} D_{\nu}^{\mu} \tag{8.64}
\end{align*}
$$

The theorem 8.61 so becomes:

$$
\begin{equation*}
\frac{1}{\mu_{0}} \in_{\Omega}\left(x^{\mu}\right) A^{\mu}\left(x^{\mu}\right)=\int_{\Omega} J^{\mu} \varphi d \Omega_{y}+\oint_{\partial \Omega} J_{, s}^{\mu} \varphi d S_{y}+\oint_{\partial \Omega} D_{\nu}^{\mu} \partial^{\nu} \varphi d S_{y} \tag{8.65}
\end{equation*}
$$

If desired, the terms associated to the doublet can be rearranged in another way, in fact:

$$
\begin{align*}
D_{\nu}^{\mu} \partial^{\nu} \varphi & =D^{\mu \nu} \partial_{\nu} \varphi  \tag{8.66}\\
A^{\mu} n^{\nu} & =\mu_{0} D^{\mu \nu} \tag{8.67}
\end{align*}
$$

Shortly, we can write:

$$
\begin{align*}
\frac{1}{\mu_{0}} \in_{\Omega}\left(x^{\mu}\right) A^{\mu}\left(x^{\mu}\right)= & \int_{\Omega} J^{\mu} \varphi d \Omega_{y}+ \\
& \oint_{\partial \Omega} J_{, s}^{\mu} \varphi d S_{y}+  \tag{8.68}\\
& \oint_{\partial \Omega} D^{\mu \nu} \partial_{\nu} \varphi d S_{y}
\end{align*}
$$

where the source are so related to the Boundary Conditions for $A^{\mu}$ :

$$
\begin{align*}
\partial_{\nu} \partial^{\nu}\left(A^{\mu}\right) & =\mu_{0} J^{\mu}  \tag{8.69}\\
n_{\nu}\left(\partial^{\nu} A^{\mu}\right) & =-\mu_{0} J_{, s}^{\mu}  \tag{8.70}\\
A^{\mu} n^{\nu} & =\mu_{0} D^{\mu \nu} \tag{8.71}
\end{align*}
$$


(a) World line for a $(\mathrm{N}+1)$-current $J^{\mu}$. A light-cone associated to an event is also represented.

(b) Space-time streamlines for ( $\mathrm{N}+1$ )-vector potential $A^{\mu}$ on domain $\Omega$.

Figure 8.9: Examples of sources and fields word lines.

### 8.4.1 Space-time BC for wave equation

With a procedure analogous to the one adopted for the spatial case (see 5.1), we can derive the Boundary Conditions for an interface $\partial \Omega$ dividing two space-time domains $\Omega_{1}$ and $\Omega_{2}$ :

$$
\left\{\begin{array} { r l } 
{ n _ { \nu , 2 1 } ( \partial ^ { \nu } A _ { , 1 } ^ { \mu } ) } & { = - \mu _ { 0 } J _ { , s 1 } ^ { \mu } }  \tag{8.72}\\
{ A _ { , 1 } ^ { \mu } n _ { , 2 1 } ^ { \nu } } & { = \mu _ { 0 } D _ { , 1 } ^ { \mu \nu } }
\end{array} \quad \left\{\begin{array}{rl}
n_{\nu, 12}\left(\partial^{\nu} A_{, 2}^{\mu}\right) & =-\mu_{0} J_{,, 2}^{\mu} \\
A_{, 2}^{\mu} n_{, 12}^{\nu} & =\mu_{0} D_{, 2}^{\mu \nu}
\end{array}\right.\right.
$$

Since $n^{\nu}{ }_{, 12}=-n^{\nu}{ }_{, 21}$, summing together the two systems we find the complete set of BC associated to the relativistic wave equation 8.62):

$$
\left\{\begin{align*}
n_{\nu, 21}\left(\partial^{\nu} A_{, 2}^{\mu}-\partial^{\nu} A_{,, 1}^{\mu}\right) & =\mu_{0} J_{, s}^{\mu}  \tag{8.73}\\
\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right) n_{, 21}^{\nu} & =-\mu_{0} D^{\mu \nu}
\end{align*}\right.
$$

This result is analogous to the one already found for the spatial case, in fact:

$$
\left\{\begin{array}{r}
\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{1}}{\partial \vec{x}}\right) \cdot \vec{n}_{21}=-\mu_{0} \vec{J}_{s}  \tag{8.74}\\
\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21}=\mu_{0} \overline{\bar{D}}_{e}
\end{array}\right.
$$

Now let us analyse more in detail the structure for the BCs in 8.73.

## BC for surface ( $\mathrm{N}+1$ )-current

The Boundary Condition associated to the ( $\mathrm{N}+1$ )-gradient for $A^{\mu}$ is:

$$
\begin{equation*}
n_{\nu, 21}\left(\partial^{\nu} A_{, 2}^{\mu}-\partial^{\nu} A_{, 1}^{\mu}\right)=\mu_{0} J_{, s}^{\mu} \tag{8.75}
\end{equation*}
$$

where $n_{\nu, 21}$ is the ( $\mathrm{N}+1$ )-normal pointing from $\Omega_{1}$ to $\Omega_{2}$ and $J_{, s}^{\mu}$ is the surface $(\mathrm{N}+1)$-current. Making the single terms explicit, we find:

$$
\begin{align*}
& J_{, s}^{\mu}=\left[\begin{array}{c}
\sigma_{e} c_{0} \\
\vec{J}_{s, e}
\end{array}\right]  \tag{8.76}\\
& \left(\partial^{\nu} A^{\mu}\right) n_{\nu}=\left[\begin{array}{cc}
\frac{\partial \rho_{A}}{\partial t} & -\frac{\partial\left(\rho_{A} c_{0}\right)}{\partial \vec{x}} \\
\frac{1}{c_{0}} \frac{\partial \vec{A}}{\partial t} & -\left[\frac{\partial \vec{A}}{\partial \vec{x}}\right]
\end{array}\right] \cdot\left[\begin{array}{l}
n_{0} \\
\vec{n}_{x}
\end{array}\right] \tag{8.77}
\end{align*}
$$

Supposing that (N+1)-normal is a space-like one, after some calculi we obtain the relativistic BC for the surface charge $\sigma_{e}$ and current $\vec{J}_{s, e}$ :
where $\vec{v}$ is the boundary's velocity. For a slow-moving body, so for $v / c_{0} \ll 1$, we can approximate:

$$
\begin{array}{rlr}
\gamma=\frac{1}{\sqrt{1-\frac{v^{2}}{c_{0}^{2}}}} & \approx 1+\frac{1}{2} \frac{v^{2}}{c_{0}^{2}} & \text { for } \frac{v}{c_{0}} \ll 1 \\
\overline{\bar{\gamma}}=\overline{\bar{I}}+(\gamma-1)\left[\overline{\overline{v v^{T}}}\right] / v^{2} & \approx \overline{\bar{I}}+\frac{1}{2} \frac{1}{c_{0}^{2}}\left[\overline{\overline{v v^{T}}}\right] & \text { for } \frac{v}{c_{0}} \ll 1 \tag{8.80}
\end{array}
$$

Neglecting the $2^{\text {nd }}$ order terms, thus linearizing, the BCs 8.78 can be approximated as:

In the hypothesis of a standing boundary, so for $\vec{v}=\overrightarrow{0}$, we retrieve the BCs previously derived in chapter 5

$$
\left\{\begin{array}{l}
\left(\frac{\partial \rho_{A 2}}{\partial \vec{x}}-\frac{\partial \rho_{A 1}}{\partial \vec{x}}\right) \vec{n}_{21}=-\mu_{0} \sigma_{e}  \tag{8.82}\\
\left(\frac{\partial \vec{A}_{2}}{\partial \vec{x}}-\frac{\partial \vec{A}_{2}}{\partial \vec{x}}\right) \vec{n}_{21}=-\mu_{0} \vec{J}_{s, e}
\end{array}\right.
$$

## $B C$ for relativistic surface doublet

Let us analyze the Boundary Condition for the relativistic surface doublet $D^{\mu \nu}$, which is associated to the discontinuity of $A^{\mu}$ across the boundary:

$$
\begin{equation*}
\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right) n_{, 21}^{\nu}=-\mu_{0} D^{\mu \nu} \tag{8.83}
\end{equation*}
$$

Making the product $A^{\mu} n^{\nu}$ explicit, we find:

$$
A^{\mu} n^{\nu}=\left[\begin{array}{c}
\rho_{A} c_{0}  \tag{8.84}\\
\vec{A}
\end{array}\right] \otimes\left[\begin{array}{ll}
n_{0}, & -\vec{n}_{x}
\end{array}\right]=\left[\begin{array}{cc}
\left(\rho_{A} c_{0}\right) n_{0} & -\left(\rho_{A} c_{0}\right) \vec{n}_{x}^{T} \\
\vec{A} n_{0} & -\left[\overline{\overline{A n_{x}^{T}}}\right]
\end{array}\right]
$$

Supposing that ( $\mathrm{N}+1$ )-normal is a space-like one, after some calculi we obtain:

$$
\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right) n_{, 21}^{\nu}=-\left[\begin{array}{cc}
\gamma\left(\rho_{A 2}-\rho_{A 1}\right)\left(\vec{v}^{T} \cdot \vec{n}_{21}\right) & \left(\rho_{A 2}-\rho_{A 1}\right) c_{0}\left(\overline{\bar{\gamma}} \cdot \vec{n}_{21}\right)^{T}  \tag{8.85}\\
\gamma\left(\frac{\vec{v}}{}^{c_{0}} \cdot \vec{n}_{21}\right)\left(\vec{A}_{2}-\vec{A}_{1}\right) & {\left[\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21}\right] \overline{\bar{\gamma}}}
\end{array}\right]
$$

Now it is convenient to introduce the electric dipoles $\vec{d}_{e}$ and doublets $\overline{\bar{D}}_{e}$ :

$$
\left\{\begin{align*}
\mu_{0} \vec{d}_{e} & =\left(\rho_{A 2}-\rho_{A 1}\right) \cdot \vec{n}_{21}  \tag{8.86}\\
\mu_{0} \overline{\bar{D}}_{e} & =\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21}
\end{align*}\right.
$$

By substitution, we find:

$$
\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right) n_{, 21}^{\nu}=-\mu_{0}\left[\begin{array}{cc}
\gamma \vec{d}_{e}^{T} \cdot \vec{v} & c_{0} \vec{d}_{e}^{T} \cdot \overline{\bar{\gamma}}  \tag{8.87}\\
\gamma \bar{D}_{e} \cdot \frac{\vec{v}}{c_{0}} & \overline{\bar{D}}_{e} \cdot \overline{\bar{\gamma}}
\end{array}\right]
$$

Thus, we have just deduced the expression for the relativistic surface doublet $D^{\mu \nu}$ :

$$
D^{\mu \nu}=\left[\begin{array}{cc}
\gamma \vec{d}_{e}^{T} \cdot \vec{v} & c_{0} \vec{d}_{e}^{T} \cdot \overline{\bar{\gamma}}  \tag{8.88}\\
\gamma \overline{\bar{D}}_{e} \cdot \frac{\vec{v}}{c_{0}} & \overline{\bar{D}}_{e} \cdot \overline{\bar{\gamma}}
\end{array}\right] \quad \text { space-like relativistic doublet }
$$

For a slow-moving body, so for $v / c_{0} \ll 1$, the doublet $D^{\mu \nu}$ can be approximated as:

$$
D^{\mu \nu} \approx\left[\begin{array}{cc}
\vec{d}_{e}^{T} \cdot \vec{v} & c_{0} \vec{d}_{e}^{T}  \tag{8.89}\\
\overline{\bar{D}}_{e} \cdot \frac{\vec{v}}{c_{0}} & \overline{\bar{D}}_{e}
\end{array}\right]
$$

In the hypothesis of a standing boundary, so for $\vec{v}=\overrightarrow{0}$, we obtain:

$$
D^{\mu \nu}=\left[\begin{array}{cc}
0 & c_{0} \vec{d}_{e}^{T}  \tag{8.90}\\
\overrightarrow{0} & \overline{\bar{D}}_{e}
\end{array}\right]
$$

This result is consistent with the definition 5.15 of $\vec{d}_{e}$ and $\overline{\bar{D}}_{e}$ in function of charges and currents, in fact:

$$
\begin{gather*}
\left\{\begin{array}{c}
\vec{d}_{e}=\frac{1}{2}\left(\sigma_{e 2}-\sigma_{e 1}\right) \Delta \vec{x}_{21} \\
\overline{\bar{D}}_{e}=\frac{1}{2}\left(\vec{J}_{s, e 2}-\vec{J}_{s, e 1}\right) \otimes \Delta \vec{x}_{21}
\end{array} \quad \Longrightarrow\right.  \tag{8.91}\\
{\left[\begin{array}{cc}
0 & c_{0} \vec{d}_{e}^{T} \\
\overrightarrow{0} & \overline{\bar{D}}_{e}
\end{array}\right]=\frac{1}{2}\left[\begin{array}{c}
\left(\left(\sigma_{e 2}-\sigma_{e 1}\right) c_{0}\right. \\
\left(\vec{J}_{s, e 2}-\vec{J}_{s, e 1}\right)
\end{array}\right] \otimes\left[\begin{array}{ll}
0, & \left.\Delta \vec{x}_{21}\right]
\end{array}\right.} \tag{8.92}
\end{gather*}
$$

So the tensor $D^{\mu \nu}$ clearly appears to be a dyadic product between a surface $(\mathrm{N}+1)$-current $J_{, s}^{\mu}$ difference and a space-like ( $\mathrm{N}+1$ )-position $x^{\nu}$ difference. Let us notice that for the considered case the two events $x_{, 1}^{\nu}$ and $x_{, 2}^{\nu}$ are simultaneous, so the temporal component of $\Delta x^{\nu}{ }_{, 12}$ turns out to be null.

$$
J_{, s}^{\mu}=\left[\begin{array}{c}
\sigma_{e} c_{0}  \tag{8.93}\\
\vec{J}_{s, e}
\end{array}\right] ; \quad \Delta x_{, 12}^{\nu}=\left[\begin{array}{c}
0 \\
\Delta \vec{x}_{21}
\end{array}\right]
$$

Thus it follows:

$$
\begin{equation*}
D^{\mu \nu}=\frac{1}{2}\left(J_{, s 2}^{\mu}-J_{, s 1}^{\mu}\right) \Delta x_{, 21}^{\nu} \tag{8.94}
\end{equation*}
$$

We specify that this last result is general, not restricted to the particular case of boundary at rest.

### 8.4.2 Lorentz Gauge

As already stated in eq. 8.48), the Lorentz Gauge can be rephrased in relativistic notation as:

$$
\begin{align*}
& \frac{\partial\left(\rho_{A} c_{0}\right)}{\partial\left(c_{0} t\right)}+\vec{\nabla}^{T} \cdot \vec{A}=0  \tag{8.95}\\
& \partial_{\mu} A^{\mu}=0 \tag{8.96}
\end{align*}
$$

The Boundary Condition associated to it will be so:

$$
\begin{equation*}
n_{\mu, 12}\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right)=0 \tag{8.97}
\end{equation*}
$$

Supposing the (N+1)-normal $n_{, 12}^{\mu}$ is space-like, after some calculi we can rewrite the same BC in the form:

$$
\begin{equation*}
-\gamma\left(\rho_{A 2}-\rho_{A 1}\right) \vec{n}_{21}^{T} \cdot \vec{v}+\vec{n}_{21}^{T} \cdot \overline{\bar{\gamma}} \cdot\left(\vec{A}_{2}-\vec{A}_{1}\right)=0 \tag{8.98}
\end{equation*}
$$

For a slow-moving body, so for $v / c_{0} \ll 1$, we can approximate:

$$
\begin{align*}
& -\left(\rho_{A 2}-\rho_{A 1}\right) \vec{n}_{21}^{T} \cdot \vec{v}+\vec{n}_{21}^{T} \cdot\left(\vec{A}_{2}-\vec{A}_{1}\right) \approx 0  \tag{8.99}\\
& \vec{n}_{21}^{T} \cdot\left(-\left(\rho_{A 2}-\rho_{A 1}\right) \vec{v}+\left(\vec{A}_{2}-\vec{A}_{1}\right)\right) \approx 0 \tag{8.100}
\end{align*}
$$

In the hypothesis of a standing boundary, so for $\vec{v}=\overrightarrow{0}$, we obtain:

$$
\begin{equation*}
\vec{n}_{21}^{T} \cdot\left(\vec{A}_{2}-\vec{A}_{1}\right)=0 \tag{8.101}
\end{equation*}
$$

Again, we retrieve exactly the BC already drawn for the spatial case (see eq. (5.23).

## Condition on relativistic doublet

The Lorentz Gauge imposes a constraint also on the structure of the tensor $D^{\mu \nu}$, in fact:

$$
\left\{\begin{array}{l}
\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right) n_{\mu, 12}=0  \tag{8.102}\\
\left(A_{, 2}^{\mu}-A_{, 1}^{\mu}\right) n_{\nu, 12}=-\mu_{0} D_{\nu}^{\mu}
\end{array} \quad \Longrightarrow \quad D_{\mu}^{\mu}=0 \quad\right. \text { null trace }
$$

Alternatively, we can write:

$$
\begin{array}{ll}
n_{\mu} D^{\mu}{ }_{\nu}=0 & \forall \mu, \nu \in\{0 ; 1 ; \cdots ; N\} \\
n_{\mu} D^{\mu \nu}=0 & \forall \mu, \nu \in\{0 ; 1 ; \cdots ; N\} \tag{8.104}
\end{array}
$$

That form is analogous to the one derived for the spatial case (see also eq. 5.25):

$$
\begin{equation*}
\vec{n}^{T} \cdot \overline{\bar{D}}_{e}=\overrightarrow{0}^{T} \tag{8.105}
\end{equation*}
$$

### 8.4.3 Charge conservation

The conservation for the electric charge can be deduced from the union of the Lorentz Gauge 8.96) and the Wave equation 8.62, in fact:

$$
\begin{align*}
& \left\{\begin{array}{c}
\partial_{\mu} A^{\mu}=0 \\
\partial_{\nu} \partial^{\nu}\left(A^{\mu}\right)=\mu_{0} J^{\mu}
\end{array} \Longrightarrow\right.  \tag{8.106}\\
& \begin{cases}\partial_{\nu} \partial^{\nu}\left(\partial_{\mu} A^{\mu}\right)=0 \\
\partial_{\mu}\left(\partial_{\nu} \partial^{\nu} A^{\mu}\right)=\mu_{0} \partial_{\mu} J^{\mu}\end{cases}  \tag{8.107}\\
& \partial_{\mu} J^{\mu}=0 \quad \text { charge conservation } \tag{8.108}
\end{align*}
$$

Since that is a continuity equation, like the Lorentz Gauge, the Boundary Conditions associated to it will have an analogous form:

$$
\begin{gather*}
n_{\mu, 12}\left(J_{, 2}^{\mu}-J_{, 1}^{\mu}\right)=0  \tag{8.109}\\
-\gamma\left(\rho_{e 2}-\rho_{e 1}\right) \vec{n}_{21}^{T} \cdot \vec{v}+\vec{n}_{21}^{T} \cdot \overline{\bar{\gamma}} \cdot\left(\vec{J}_{e 2}-\vec{J}_{e 1}\right)=0 \tag{8.110}
\end{gather*}
$$

Obviously, the considerations are similar to the ones for Lorentz Gauge.

### 8.4.4 Space-time BC for EM tensor

Till now we have treated just the Boundary Conditions involving the field $A^{\mu}$ and $J^{\mu}$. Now we are going to consider the BCs for the electromagnetic tensor $F^{\mu \nu}$, which contains the electric $\vec{E}$ and magnetic $\overline{\bar{B}}$ fields.

$$
\begin{align*}
F^{\mu \nu} & =\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}  \tag{8.111}\\
F^{\mu \nu} & =\left[\begin{array}{cc}
0 & -\vec{E}^{T} / c_{0} \\
\vec{E} / c_{0} & \overline{\bar{B}}
\end{array}\right] \tag{8.112}
\end{align*}
$$

If we are interested just to the discontinuity of $F^{\mu \nu}$, without invoking neither $(\mathrm{N}+1)$-currents $J^{\mu}$ nor $(\mathrm{N}+1)$-normals $n_{\nu}$, then we can simply write:

$$
\begin{equation*}
F_{, 2}^{\mu \nu}-F_{, 1}^{\mu \nu}=\left(\partial^{\mu} A_{, 2}^{\nu}-\partial^{\nu} A_{, 2}^{\mu}\right)-\left(\partial^{\mu} A_{, 1}^{\nu}-\partial^{\nu} A_{, 1}^{\mu}\right) \tag{8.113}
\end{equation*}
$$

Otherwise, we have to calculate the ( $\mathrm{N}+1$ )-divergence for the tensor $F^{\mu \nu}$, in order to link it to the $(\mathrm{N}+1)$-currents $J^{\mu}$ :

$$
\begin{align*}
& \partial_{\nu} F^{\mu \nu}=\partial_{\nu}\left(\partial^{\mu} A^{\nu}-\partial^{\nu} A^{\mu}\right)  \tag{8.114}\\
& \partial_{\nu} F^{\mu \nu}=\partial_{\nu}\left(\partial^{\mu} A^{\nu}\right)-\partial_{\nu}\left(\partial^{\nu} A^{\mu}\right) \tag{8.115}
\end{align*}
$$

Thanks to the Lorentz Gauge, the first term on the right of the equal is null:

$$
\begin{gather*}
\partial_{\nu}\left(\partial^{\mu} A^{\nu}\right)=\partial_{\mu}\left(\partial^{\nu} A^{\nu}\right)=\partial_{\mu}(0)=0 \quad \Longrightarrow  \tag{8.116}\\
\partial_{\nu} F^{\mu \nu}=-\partial_{\nu} \partial^{\nu} A^{\mu} \tag{8.117}
\end{gather*}
$$

Thanks to the wave equation 8.62 , finally we find:

$$
\begin{equation*}
\partial_{\nu} F^{\mu \nu}=-\mu_{0} J^{\mu} \tag{8.118}
\end{equation*}
$$

Actually, this last equation can be interpreted as a gauge-invariant formulation of the laws relating the EM field to their sources. Exchanging the indices $\mu$ and $\nu$, it can be also rephrased as:

$$
\begin{align*}
\partial_{\mu} F^{\nu \mu} & =-\mu_{0} J^{\nu}  \tag{8.119}\\
\partial_{\mu} F^{\mu \nu} & =\mu_{0} J^{\nu} \tag{8.120}
\end{align*}
$$

We remind that $F^{\mu \nu}$ is a anti-symmetric tensor, thus:

$$
\begin{equation*}
F^{\mu \nu}=-F^{\nu \mu} \tag{8.121}
\end{equation*}
$$

Through the usual method, i.e. by substituting the gradient with the normal, we obtain the BC associated to the EM tensor:

$$
\begin{equation*}
n_{\nu, 21}\left(F_{, 2}^{\mu \nu}-F_{, 1}^{\mu \nu}\right)=-\mu_{0} J_{, s}^{\mu} \tag{8.122}
\end{equation*}
$$

More compactly:

$$
\begin{equation*}
\Delta F^{\mu \nu} n_{\nu}=-\mu_{0} J_{, s}^{\mu} \tag{8.123}
\end{equation*}
$$

Now let us explain this result in terms of fields $\vec{E}$ and $\overline{\bar{B}}$. Taking into account a space-like ( $\mathrm{n}+1$ ) normal, we get:

$$
\Delta F^{\mu \nu} n_{\nu}=\left[\begin{array}{cc}
0 & -\Delta \vec{E}^{T} / c_{0}  \tag{8.124}\\
\Delta \vec{E} / c_{0} & \bar{B}
\end{array}\right] \cdot\left[\begin{array}{c}
-\gamma \frac{\vec{v}^{T}}{c_{0}} \cdot \vec{n} \\
\overline{\bar{\gamma}} \cdot \vec{n}
\end{array}\right]=-\mu_{0}\left[\begin{array}{c}
\sigma_{e} c_{0} \\
\vec{J}_{s, e}
\end{array}\right]
$$

After some calculi, we find the Boundary Conditions:

$$
\left\{\begin{array}{l}
\vec{n}_{21}^{T} \cdot \overline{\bar{\gamma}} \cdot\left(\vec{E}_{2}-\vec{E}_{1}\right)=\mu_{0} c_{0}^{2} \sigma_{e}  \tag{8.125}\\
\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \overline{\bar{\gamma}} \cdot \vec{n}_{21}-\gamma\left({\frac{\vec{v}}{c_{0}}}^{T} \cdot \vec{n}_{21}\right) \frac{1}{c_{0}^{2}}\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\mu_{0} \vec{J}_{s, e}
\end{array}\right.
$$

For a slow-moving body, so for $v / c_{0} \ll 1$, we can approximate:

$$
\left\{\begin{array}{l}
\vec{n}_{21}^{T} \cdot\left(\vec{E}_{2}-\vec{E}_{1}\right)=\mu_{0} c_{0}^{2} \sigma_{e}  \tag{8.126}\\
\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21}-\left({\frac{\vec{v}}{c_{0}}}^{T} \cdot \vec{n}_{21}\right) \frac{1}{c_{0}^{2}}\left(\vec{E}_{2}-\vec{E}_{1}\right)=-\mu_{0} \vec{J}_{s, e}
\end{array}\right.
$$

In the hypothesis of a standing boundary, so for $\vec{v}=\overrightarrow{0}$, we retrieve the BCs previously derived in chapter 6 (see also eq.s 6.23), 6.58):

$$
\left\{\begin{array}{c}
\vec{n}_{21}^{T} \cdot\left(\vec{E}_{2}-\vec{E}_{1}\right)=\mu_{0} c_{0}^{2} \sigma_{e}  \tag{8.127}\\
\left(\overline{\bar{B}}_{2}-\overline{\bar{B}}_{1}\right) \cdot \vec{n}_{21}=-\mu_{0} \vec{J}_{s, e}
\end{array}\right.
$$

## Chapter 9

## Plane waves

To a man with a hammer, everything looks like a nail.

Modern proverb, usually attributed to Mark Twain.

In this chapter we show how to calculate the fields irradiated by an ideal, infinite, plane screen or metasurface. In all the cases, we shall consider plane waves at a single frequency, so preferably we will work in the Laplace's domain and use phasors in order to describe propagation.

### 9.1 Plane waves propagating in free space

In the empty space the vector potential $\vec{A}$ propagates obeying to the homogeneous Helmholtz equation:

$$
\begin{equation*}
\nabla^{2} \vec{A}+k_{0}^{2} \vec{A}=0 \tag{9.1}
\end{equation*}
$$

where $k_{0}$ is the reference wave number:

$$
\begin{equation*}
k_{0}=\frac{\omega}{c_{0}}=\frac{\lambda_{0}}{2 \pi} \tag{9.2}
\end{equation*}
$$

The general solution to the homogeneous equation has form:

$$
\begin{align*}
& \vec{A}(\vec{x})=\vec{A}_{0} e^{i \vec{k}^{T} \cdot \Delta \vec{x}} \quad \Longrightarrow  \tag{9.3}\\
& \vec{A}_{0} \cdot\left(-\vec{k}^{T} \cdot \vec{k}+k_{0}^{2}\right)=0 \quad \forall \vec{A}_{0} \quad \Longrightarrow  \tag{9.4}\\
& \vec{k}^{T} \cdot \vec{k}=k_{0}^{2} \tag{9.5}
\end{align*}
$$

So we find a condition on the module of wavevector $\vec{k}$. We separate the real and imaginary parts or $\vec{k}$ :

$$
\begin{align*}
& \vec{k}=\vec{k}^{\prime}+i \vec{k}^{\prime \prime} \quad \text { with } \vec{k}^{\prime}, \vec{k}^{\prime \prime} \in \mathbb{R}^{N}  \tag{9.6}\\
& \vec{k}^{T} \cdot \vec{k}=\left(k^{\prime 2}-k^{\prime \prime 2}\right)+i 2\left(\vec{k}^{T} \cdot \vec{k}^{\prime \prime}\right) \tag{9.7}
\end{align*}
$$

Since $k_{0}$ is a real scalar quantity, there are two equations to be satisfied:

$$
\left\{\begin{array} { c } 
{ k ^ { \prime 2 } - k ^ { \prime \prime 2 } = k _ { 0 } ^ { 2 } }  \tag{9.8}\\
{ \vec { k } ^ { \prime T } \cdot \vec { k } ^ { \prime \prime } = 0 }
\end{array} \Longrightarrow \left\{\begin{array}{c}
k^{\prime 2}=\cosh (\alpha) k_{0} \\
k^{\prime \prime 2}=\sinh (\alpha) k_{0} \\
\vec{k}^{\prime} \perp \vec{k}^{\prime \prime}
\end{array}\right.\right.
$$

So we find that the imaginary component $\vec{k}^{\prime \prime}$ must be orthogonal to the real one. Besides, the magnitudes of the two components are related by a hyperbolic dispersion law ( $\alpha$ is a free parameter). It should be noticed that the wavevector's imaginary component $\vec{k}^{\prime \prime}$ is associated to the transverse exponential decay occurring in leaky waves, e.g. evanescent or plasmonic ones.

### 9.1.1 Electric field

The electric field $\vec{E}$ for a plane wave can be determined thanks to the Faraday's Law in the Laplace's Domain (see 3.26):

$$
\begin{equation*}
s \vec{E}=-s^{2} \vec{A}+c_{0}^{2} \vec{\nabla}\left(\vec{\nabla}^{T} \cdot \vec{A}\right) \tag{9.9}
\end{equation*}
$$

where $s=-i \omega$. For a plane wave, the same equations can be rephrased in function of the wavevector $\vec{k}$ :

$$
\begin{equation*}
\vec{E}=-s\left(\overline{\bar{I}}-\frac{1}{k_{0}^{2}} \overline{\overline{k k}}\right) \cdot \vec{A} \tag{9.10}
\end{equation*}
$$

So the electric field $\vec{E}$ turns out to be perpendicular to $\vec{k}$ (transverse wave), in fact:

$$
\begin{equation*}
\vec{k}^{T} \cdot \vec{E}=-s \vec{k}^{T} \cdot\left(\overline{\bar{I}}-\frac{1}{k_{0}^{2}} \overline{\overline{k k}}\right) \cdot \vec{A}=0 \tag{9.11}
\end{equation*}
$$

This result is coherent with the fact the divergence of $\vec{E}$ is zero in absence of charge $\left(\rho_{e}=0\right)$ :

$$
\begin{equation*}
\vec{\nabla}^{T} \cdot \vec{E}=0 \quad \longleftrightarrow \quad i \vec{k}^{T} \cdot \vec{E}=0 \tag{9.12}
\end{equation*}
$$

Moreover, the longitudinal component $\vec{A}_{C}$ of the vector potential does not affect the electric field, in fact:

$$
\begin{equation*}
\vec{A}_{C} / / \vec{k} \quad \Longrightarrow \quad\left(\overline{\bar{I}}-\frac{1}{k_{0}^{2}} \overline{\overline{k k}}\right) \cdot \vec{A}_{C}=\overrightarrow{0} \tag{9.13}
\end{equation*}
$$

For that reason, $\vec{E}$ can be expressed also in another way, involving the cross product between $\vec{k}$ and $\vec{A}$, in fact:

$$
\begin{align*}
& {[\vec{k} \wedge \vec{A}] \cdot \vec{k}=(\overline{\overline{A k}}-\overline{\overline{k A}}) \cdot \vec{k}=\vec{A} k_{0}^{2}-\vec{k}\left(\vec{A}^{T} \cdot \vec{k}\right)}  \tag{9.14}\\
& {[\vec{k} \wedge \vec{A}] \cdot \vec{k}=\left(k_{0}^{2} \overline{\bar{I}}-\overline{\overline{k k}}\right) \cdot \vec{A}} \tag{9.15}
\end{align*}
$$

Hence:

$$
\begin{equation*}
\vec{E}=-s \frac{1}{k_{0}^{2}}[\vec{k} \wedge \vec{A}] \cdot \vec{k} \tag{9.16}
\end{equation*}
$$

### 9.1.2 Magnetic field

The magnetic field $\overline{\bar{H}}$ for a plane wave can be directly calculated as the curl of $\vec{A}$ :

$$
\begin{align*}
& \mu_{0} \overline{\bar{H}}=\vec{\nabla} \hat{\wedge} \vec{A}  \tag{9.17}\\
& \mu_{0} \overline{\bar{H}}=\vec{\nabla} \hat{\wedge}\left(\vec{A}_{0} e^{i \vec{k}^{T} \cdot \Delta \vec{x}}\right)=i\left(\overline{\overline{A_{0} k}}-\overline{\overline{k A_{0}}}\right) e^{i \vec{k}^{T} \cdot \Delta \vec{x}}  \tag{9.18}\\
& \mu_{0} \overline{\bar{H}}=i \vec{k} \wedge \vec{A} \quad \text { ND notation }  \tag{9.19}\\
& \mu_{0} \vec{H}=i \vec{k} \times \vec{A} \quad \text { 3D notation } \tag{9.20}
\end{align*}
$$

In 3D notation, the magnetic field $\vec{H}$ is perpendicular to the wavevector $\vec{k}$ and to the electric field $\vec{E}$.

### 9.1.3 Direct link for electric and magnetic fields

The electric field and the magnetic one can be calculated one in function of the other and viceversa, without explicitly using the vector potential.

Looking at 9.16 and 9.19 , we notice that in both the equations the cross product appears:

$$
\left\{\begin{array} { r l } 
{ \vec { E } } & { = - s \frac { 1 } { k _ { 0 } ^ { 2 } } [ \vec { k } \wedge \vec { A } ] \cdot \vec { k } }  \tag{9.21}\\
{ \mu _ { 0 } \overline { \overline { H } } } & { = i \vec { k } \wedge \vec { A } }
\end{array} \Longrightarrow \left\{\begin{array}{rl}
i \vec{k} \wedge \vec{E} & =-s \mu_{0} \overline{\bar{H}} \\
-\mu_{0} \overline{\bar{H}} \cdot(i \vec{k}) & =\frac{1}{c_{0}^{2}} s \vec{E}
\end{array}\right.\right.
$$

These last equations exactly correspond the Faraday's Law and to the MaxwellAmpére Law respectively. In fact, in absence of currents $\left(\vec{J}_{e}=\overrightarrow{0}\right)$, they look:

$$
\left\{\begin{align*}
\vec{\nabla} \hat{\wedge} \vec{E} & =-\mu_{0} \frac{\partial \overline{\bar{H}}}{\partial t}  \tag{9.22}\\
-\mu_{0} \overline{\bar{H}} \cdot \vec{\nabla} & =\frac{1}{c_{0}^{2}} \frac{\partial \vec{E}}{\partial t}
\end{align*}\right.
$$

In synthesis, there is a direct correspondence among the operators in space-time and those in the wavevector-frequency domain:

$$
\begin{align*}
& \frac{\partial \cdot}{\partial t} \longleftrightarrow \quad-i \omega=s  \tag{9.23}\\
& \vec{\nabla} \quad \longleftrightarrow \quad i \vec{k}
\end{align*}
$$

That concept is widely used in Quantum Mechanics.

### 9.2 Plane currents

Suppose we know the distribution of sources on an infinite plane screen, whose thickness is zero or anyway much smaller that the operating wavelength $\lambda_{0}$.

The plane's equations is:

$$
\begin{equation*}
\vec{n}_{21}^{T} \cdot\left(\vec{x}-\vec{x}_{0}\right)=\vec{n}_{21}^{T} \cdot \Delta \vec{x}=0 \tag{9.24}
\end{equation*}
$$

where $\vec{n}_{21}$ is the normal to the screen and $\vec{x}_{0}$ is a point on it.
We consider surface currents $\vec{J}_{s}$ and doublets $\overline{\bar{D}}_{e}$ for a given tangential wavevector $\vec{k}_{t} \perp \vec{n}_{21}$, so that:

$$
\left\{\begin{array}{c}
\vec{J}_{s}=\vec{J}_{s 0} e^{i \vec{k}_{t}^{T} \cdot \Delta \vec{x}}  \tag{9.25}\\
\overline{\bar{D}}_{e}=\overline{\bar{D}}_{e 0} e^{i \vec{k}_{t}^{T} \cdot \Delta \vec{x}}
\end{array} \quad \forall \vec{x} \in \partial \Omega\right.
$$

Let us notice that the current $\vec{J}_{s}$ could have a component normal to the screen itself. Effectively that could happen for an anti-symmetric surface plasmonic waves.

Once the currents $\vec{J}_{s}$ and $\overline{\bar{D}}_{e}$ on the screen are known, then it is possible to calculate the radiated fields $\vec{A}_{1}$ and $\vec{A}_{2}$.

(a) Normal surface current $\vec{J}_{s, n}$.

(b) Tangential surface current $\vec{J}_{s, t}$.

(c) Tangential surface doublet $\overline{\bar{D}}_{e, t}$.

Figure 9.1: Plane surface waves for currents and doublets. In this example a real tangential wavevector $\vec{k}_{t}$ is considered.

### 9.3 Boundary Conditions for radiated vector potential

Supposing the screen is radiating plane waves, the fields $\vec{A}_{1}$ and $\vec{A}_{2}$ on the two sides will have the form:

$$
\begin{array}{ll}
\vec{A}_{1}(\vec{x})=\vec{A}_{10} e^{i \vec{k}_{1}^{T} \cdot \Delta \vec{x}} & \text { for } \vec{x} \in \Omega_{1} \\
\vec{A}_{2}(\vec{x})=\vec{A}_{20} e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}} & \text { for } \vec{x} \in \Omega_{2} \tag{9.27}
\end{array}
$$

Calculating the gradients brings to:

$$
\begin{align*}
& \frac{\partial \vec{A}_{1}}{\partial \vec{x}}=\vec{A}_{1} \otimes\left(i \vec{k}_{1}\right)=i \overline{\overline{A_{1} k_{1}}}  \tag{9.28}\\
& \frac{\partial \vec{A}_{2}}{\partial \vec{x}}=\vec{A}_{2} \otimes\left(i \vec{k}_{2}\right)=i \overline{\overline{A_{2} k_{2}}} \tag{9.29}
\end{align*}
$$

### 9.3.1 $\quad \mathrm{BC}$ for tangentials wavevectors

In order to determine the amplitudes $\vec{A}_{10}, \vec{A}_{20}$ and the wavevectors $\vec{k}_{1}, \vec{k}_{2}$, the Boundary Conditions 7.46 have to be imposed:

$$
\begin{align*}
i\left(\overline{\overline{A_{2} k_{2}}}-\overline{\overline{A_{1} k_{1}}}\right) \cdot \vec{n}_{21} & =-\mu_{0} \vec{J}_{s}  \tag{9.30}\\
\left(\vec{A}_{2}-\vec{A}_{1}\right) \otimes \vec{n}_{21} & =\quad \mu_{0} \overline{\bar{D}}_{e} \tag{9.31}
\end{align*}
$$

Making the amplitudes explicit, we obtain:

$$
\begin{array}{r}
i\left(\overline{\overline{A_{20} k_{2}}} e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}}-\overline{\overline{A_{10} k_{1}}} e^{i \vec{k}_{1}^{T} \cdot \Delta \vec{x}}\right) \cdot \vec{n}_{21}=-\mu_{0} \vec{J}_{s 0} e^{i \vec{k}_{t}^{T} \cdot \Delta \vec{x}} \\
\left(\vec{A}_{20} e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}}-\vec{A}_{10} e^{i \vec{k}_{1}^{T} \cdot \Delta \vec{x}}\right) \otimes \vec{n}_{21}=\mu_{0} \overline{\bar{D}}_{e 0} e^{i \vec{k}_{t}^{T} \cdot \Delta \vec{x}} \tag{9.33}
\end{array}
$$

Those equations must be valid for any $\vec{x} \in \partial \Omega$. That is possible just if all the complex exponentials, and their phases, are equal:

$$
\begin{array}{rll}
e^{i \vec{k}_{1}^{T} \cdot \Delta \vec{x}}=e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}}=e^{i \vec{k}_{t}^{T} \cdot \Delta \vec{x}} & \forall \vec{x} \in \partial \Omega & \Longrightarrow \\
\vec{k}_{1}^{T} \cdot \Delta \vec{x}=\vec{k}_{2}^{T} \cdot \Delta \vec{x}=\vec{k}_{t}^{T} \cdot \Delta \vec{x} & \forall \vec{x} \in \partial \Omega & \tag{9.35}
\end{array}
$$

Since $\Delta \vec{x} \perp \vec{n}_{21}$ and $\vec{k}_{t} \perp \vec{n}_{21}$, the last equation involves just the tangential component of the wavevectors, in fact:

$$
\begin{equation*}
\vec{k}^{T} \cdot \Delta \vec{x}=\left(\vec{k}_{n}+\vec{k}_{t}\right)^{T} \cdot \Delta \vec{x}=\vec{k}_{t}^{T} \cdot \Delta \vec{x} \tag{9.36}
\end{equation*}
$$

Finally we obtain the phase condition for the tangential wavevectors:

$$
\begin{equation*}
\vec{k}_{1 t}=\vec{k}_{2 t}=\vec{k}_{t} \tag{9.37}
\end{equation*}
$$

Remind that all these wavevectors can be complex.


Figure 9.2: Components for $\vec{k}_{1}$ and $\vec{k}_{2}$. The tangential wavevector $\vec{k}_{t}$ is the same on the two sides, while the normal wavevectors $\vec{k}_{n 1}$ and $\vec{k}_{n 2}$ are one the opposite of the other.

### 9.3.2 BC for normal wavevectors

Considering that the propagators $e^{i k x}$ are the same, the Boundary Conditions for the amplitudes can be written as:

$$
\begin{align*}
i\left(\overline{\overline{A_{20} k_{2}}}-\overline{\overline{A_{10} k_{1}}}\right) \cdot \vec{n}_{21} & =-\mu_{0} \vec{J}_{s 0}  \tag{9.38}\\
\left(\vec{A}_{20}-\vec{A}_{10}\right) \otimes \vec{n}_{21} & =\quad \mu_{0} \overline{\bar{D}}_{e 0} \tag{9.39}
\end{align*}
$$

The wavevectors appear just in the first equation and they are scalarly multiplied for $\vec{n}_{21}$, but that is not a problem. Once $k_{t}$ is known, it is possible to determine the normal component $k_{n}$, in fact:

$$
\begin{align*}
\vec{k}^{T} \cdot \vec{k} & =k_{0}^{2}  \tag{9.40}\\
\vec{k}_{n}^{T} \cdot \vec{k}_{n}+\vec{k}_{t}^{T} \cdot \vec{k}_{t} & =k_{0}^{2}  \tag{9.41}\\
k_{n}^{2}+\vec{k}_{t}^{T} \cdot \vec{k}_{t} & =k_{0}^{2} \tag{9.42}
\end{align*}
$$

Afters some calculi, we obtain:

$$
\begin{equation*}
k_{1 n}^{2}=k_{2 n}^{2}=k_{0}^{2}-\vec{k}_{t}^{T} \cdot \vec{k}_{t} \tag{9.43}
\end{equation*}
$$

There are so two possible solutions:

1. If $\vec{k}_{1 n}=\vec{k}_{2 n}$, then the two wavevectors are identical: $\vec{k}_{1}=\vec{k}_{2}$. However, it could be verified that is possible just under the condition:

$$
\begin{equation*}
\vec{J}_{s}=-i\left(\vec{k}^{T} \cdot \vec{n}_{21}\right) \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \tag{9.44}
\end{equation*}
$$

That means $\vec{J}_{s}$ and $\overline{\bar{D}}_{e}$ cannot be chosen freely, because they depend from each other.
2. If $\vec{k}_{1 n}=-\vec{k}_{2 n}$, then the two wavevectors are one the specular version of the other.

$$
\left\{\begin{array}{l}
\vec{k}_{2 n}=-\vec{k}_{1 n}  \tag{9.45}\\
\vec{k}_{2 t}=\vec{k}_{1 t}
\end{array} \quad \Longrightarrow \quad \vec{k}_{2}=(\overline{\bar{I}}-2 \overline{\overline{n n}}) \vec{k}_{1} \quad\right. \text { and viceversa }
$$

For the rest of this work we assume $\vec{k}_{n 1}=-\vec{k}_{n 2}$
For the further calculi we define the scalar normal component $k_{n}$ as:

$$
\begin{equation*}
k_{n}=\vec{k}_{2}^{T} \cdot \vec{n}_{21}=\vec{k}_{1}^{T} \cdot \vec{n}_{12} \tag{9.46}
\end{equation*}
$$

Finally, once $\vec{k}_{t}$ is assigned, the wavevectors can be calculated as

$$
\left\{\begin{array}{l}
\vec{k}_{1}=\vec{k}_{t}-k_{n} \vec{n}_{21}=\vec{k}_{t} \mp\left(\sqrt{k_{0}^{2}-\vec{k}_{t}^{T} \cdot \vec{k}_{t}}\right) \vec{n}_{21}  \tag{9.47}\\
\vec{k}_{2}=\vec{k}_{t}+k_{n} \vec{n}_{21}=\vec{k}_{t} \pm\left(\sqrt{k_{0}^{2}-\vec{k}_{t}^{T} \cdot \vec{k}_{t}}\right) \vec{n}_{21}
\end{array}\right.
$$

The sign of $k_{n}$ is undefined, but it can be fixed if the screen is emitting or absorbing:

$$
\begin{array}{lll}
\operatorname{Re}\left(k_{n}\right)>0 & \Longrightarrow \quad \text { emitted wave } \\
\operatorname{Re}\left(k_{n}\right)<0 & \Longrightarrow \quad \text { absorbed wave } \tag{9.49}
\end{array}
$$

### 9.4 Radiated waves

Finally, the BCs for the amplitudes $\vec{A}_{10}, \vec{A}_{20}$ are reduced to:

$$
\begin{array}{lr}
\vec{A}_{20}+\vec{A}_{10}=i \frac{1}{k_{n}} \mu_{0} \vec{J}_{s 0} & \text { symmetric component } \\
\vec{A}_{20}-\vec{A}_{10}=\mu_{0} \overline{\bar{D}}_{e 0} \cdot \vec{n}_{21} & \text { anti-symmetric component } \tag{9.51}
\end{array}
$$

After few calculi, we obtain the expression for the whole radiated field:

$$
\begin{align*}
& \vec{A}_{1}=\frac{1}{2} \mu_{0}\left(i \frac{1}{k_{n}} \vec{J}_{s 0}-\overline{\bar{D}}_{e 0} \cdot \vec{n}_{21}\right) e^{i \vec{k}_{1}^{T} \cdot \Delta \vec{x}}  \tag{9.52}\\
& \vec{A}_{2}=\frac{1}{2} \mu_{0}\left(i \frac{1}{k_{n}} \vec{J}_{s 0}+\overline{\bar{D}}_{e 0} \cdot \vec{n}_{21}\right) e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}} \tag{9.53}
\end{align*}
$$

From these two equations we can determine the other fields.


Figure 9.3: Symmetric radiated plane waves. Here $k_{t}=0.60 k_{0}, \vec{J}_{s, n}=\overrightarrow{0}$, $\vec{J}_{s, t} / / \vec{k}_{t}, \overline{\bar{D}}_{e}=\overline{\overline{0}}$.


Figure 9.4: Anti-symmetric radiated plane waves. Here $k_{t}=0.60 k_{0}, \vec{J}_{s, n}=\overrightarrow{0}$, $\vec{J}_{s, t}=\overrightarrow{0}, \overline{\bar{D}}_{e} \cdot \vec{n}_{21} / / \vec{k}_{t}$.

### 9.4.1 Radiated scalar potential

Thanks to the Lorentz Gauge, the scalar potential $P_{A}$ can be calculated from the vector one:

$$
\begin{equation*}
\frac{1}{c_{0}^{2}} s P_{A}+\vec{\nabla}^{T} \cdot \vec{A}=0 \quad \Longrightarrow \quad P_{A}=-\frac{1}{s} c_{0}^{2} \vec{\nabla}^{T} \cdot \vec{A} \tag{9.54}
\end{equation*}
$$



Figure 9.5: Symmetric leaky wave. Here $k_{t}=1.02 k_{0}, \vec{J}_{s, n}=\overrightarrow{0}, \vec{J}_{s, t} / / \vec{k}_{t}$, $\overline{\bar{D}}_{e}=\overline{\overline{0}}$.


Figure 9.6: Anti-symmetric leaky wave. Here $k_{t}=1.02 k_{0}, \vec{J}_{s, n}=\overrightarrow{0}, \vec{J}_{s, t}=\overrightarrow{0}$, $\overline{\bar{D}}_{e} \cdot \vec{n}_{21} / / \vec{k}_{t}$.
where $s=-i \omega=-i k_{0} c_{0}$, so for a propagating wave holds:

$$
\begin{equation*}
P_{A}=c_{0} \frac{1}{k_{0}} \vec{k}^{T} \cdot \vec{A} \tag{9.55}
\end{equation*}
$$

We can scalarly multiply $\vec{A}$ for $\vec{k}$, getting:

$$
\begin{equation*}
\vec{k}^{T} \cdot \vec{A}=\frac{1}{2} \mu_{0}\left(i \frac{1}{k_{n}} \vec{k}^{T} \cdot \vec{J}_{s 0}+\vec{k}^{T} \cdot \overline{\bar{D}}_{e 0} \cdot \vec{n}\right) e^{i \vec{k}^{T} \cdot \Delta \vec{x}} \tag{9.56}
\end{equation*}
$$

We remind that from the continuity of $\vec{A}_{n}$ it follows:

$$
\begin{equation*}
\vec{n}^{T} \cdot \overline{\bar{D}}_{e} \cdot \vec{n}=0 \quad \Longrightarrow \quad \vec{k}^{T} \cdot \overline{\bar{D}}_{e} \cdot \vec{n}=\vec{k}_{t}^{T} \cdot \overline{\bar{D}}_{e} \cdot \vec{n} \tag{9.57}
\end{equation*}
$$

After some counts, we obtain:

$$
\begin{equation*}
\vec{k}^{T} \cdot \vec{A}=\frac{1}{2} \mu_{0}\left(i \vec{k}^{T} \cdot \vec{J}_{s 0}+\vec{k}_{t}^{T} \cdot \bar{D}_{e 0} \cdot \vec{n}+i \frac{1}{k_{n}} \vec{k}_{t}^{T} \cdot \vec{J}_{s, t}\right) e^{i \vec{k}^{T} \cdot \Delta \vec{x}} \tag{9.58}
\end{equation*}
$$

Hence on the two sides holds:

$$
\begin{align*}
& \vec{k}_{1}^{T} \cdot \vec{A}_{1}=\frac{1}{2} \mu_{0}\left(i \frac{1}{k_{n}}\left(\vec{k}_{t}^{T} \cdot \vec{J}_{s, t}\right)\right)-\frac{1}{2} \mu_{0}\left(i \vec{J}_{s, n}+\overline{\bar{D}}_{e}^{T} \cdot \vec{k}_{t}\right)^{T} \cdot \vec{n}_{21}  \tag{9.59}\\
& \vec{k}_{2}^{T} \cdot \vec{A}_{2}=\frac{1}{2} \mu_{0}\left(i \frac{1}{k_{n}}\left(\vec{k}_{t}^{T} \cdot \vec{J}_{s, t}\right)\right)+\frac{1}{2} \mu_{0}\left(i \vec{J}_{s, n}+\overline{\bar{D}}_{e}^{T} \cdot \vec{k}_{t}\right)^{T} \cdot \vec{n}_{21} \tag{9.60}
\end{align*}
$$

Finally, the radiated potentials $P_{A 1}$ and $P_{A 2}$ are:

$$
\left\{\begin{array} { l } 
{ P _ { A 1 } = c _ { 0 } \frac { 1 } { k _ { 0 } } \vec { k } _ { 1 } ^ { T } \cdot \vec { A } _ { 1 } }  \tag{9.61}\\
{ P _ { A 2 } = c _ { 0 } \frac { 1 } { k _ { 0 } } \vec { k } _ { 2 } ^ { T } \cdot \vec { A } _ { 2 } }
\end{array} \quad \left\{\begin{array}{l}
\rho_{A 1}=\frac{1}{\omega} \vec{k}_{1}^{T} \cdot \vec{A}_{1} \\
\rho_{A 2}=\frac{1}{\omega} \vec{k}_{2}^{T} \cdot \vec{A}_{2}
\end{array}\right.\right.
$$

## Condition of continuity

In order to use the standard Boundary Conditions for $\vec{E}$ and $\vec{H}$ the scalar potential must be continuous across the surface. The condition of continuity $P_{A 1}=P_{A 2}$ is satisfied if and just if:

$$
\begin{gather*}
\left(i \vec{J}_{s, n}+\overline{\bar{D}}_{e}^{T} \cdot \vec{k}_{t}\right)^{T} \cdot \vec{n}=0  \tag{9.62}\\
J_{s, n}=i \vec{k}_{t}^{T} \cdot \overline{\bar{D}}_{e} \cdot \vec{n} \tag{9.63}
\end{gather*}
$$

So the normal component $\vec{J}_{s, n}$ of the surface current is related to the doublet distribution. If that condition is satisfied, then there are no dipoles $\vec{d}_{e}$ orthogonal to the surface.

### 9.4.2 Radiated Electric and Magnetic fields

Once the radiated vector potentials $\vec{A}_{1}, \vec{A}_{2}$ are known, the electric $\vec{E}$ and magnetic $\overline{\bar{H}}$ fields on the two screen's side can be calculated. Unfortunately,
the complete expression of both $\vec{E}$ and $\overline{\bar{H}}$ in function of $\vec{J}_{s}$ and $\overline{\bar{D}}_{e}$ turns out to be very complex and long. Furthermore, the expansion of the single terms, highlighting tangent components, polarizations, average values, discontinuities etc. is quite boring and does not add any particular advantage from a numerical point of view. Therefore, we report just the basic equations which allow to directly calculate the radiated fields $\vec{E}$ and $\overline{\bar{H}}$ on sides 1 and 2 .

The electric fields $\vec{E}_{1}$ and $\vec{E}_{2}$ are:

$$
\left\{\begin{array}{l}
\vec{E}_{1}=-s\left(\overline{\bar{I}}-\frac{1}{k_{0}^{2}} \overline{\overline{k_{1} k_{1}}}\right) \cdot \vec{A}_{1}  \tag{9.64}\\
\vec{E}_{2}=-s\left(\overline{\bar{I}}-\frac{1}{k_{0}^{2}} \overline{\overline{k_{2} k_{2}}}\right) \cdot \vec{A}_{2}
\end{array}\right.
$$

Conversely, the magnetic fields $\overline{\bar{H}}_{1}$ and $\overline{\bar{H}}_{2}$ can be calculated as:

$$
\left\{\begin{array}{l}
\overline{\bar{H}}_{1}=\frac{1}{\mu_{0}} i \vec{k}_{1} \wedge \vec{A}_{1}  \tag{9.65}\\
\overline{\bar{H}}_{2}=\frac{1}{\mu_{0}} i \vec{k}_{2} \wedge \vec{A}_{2}
\end{array}\right.
$$

## Chapter 10

## Designing a 2-layer screen

Just a little theory and calculation would have saved him 90 per cent of the labor.

> N. Tesla about T. A. Edison's empiric method, The New York Times, 19 Oct. 1931

Till now we worked within the hypothesis that surfaces have zero thickness. However, if we want to project and construct a real screen or meta-surface, we must take into account that it has a finite thickness.

In this transition chapter we begin to investigate the properties of a two-layer screen, using circuits in order to model it.

### 10.1 Towards a circuit network

As we have anticipated in sec 5.3.1, the presence of dipoles $\vec{d}_{e}$ and doublets $\overline{\bar{D}}_{e}$ suggests that the screen has to be made by 2 layers, at least. Each $\mathrm{i}^{\text {th }}$ layer can be characterized by surface charge $\sigma_{e, i}$ and current $\vec{J}_{e, i}$.

So, if we desire to implement a circuit model for the screen, we would need at least 2 superimposed circuit nets or layers, linked each other node by node. After


Figure 10.1: Topological scheme for a 2-layer circuit network. The two grids are linked each other node by node.
all, building a screen of zero-thickness is quite difficult, thus we shall consider a finite width $\Delta x$ for it.

On each node of layer 1 we can have charge $\sigma_{e 1}$, while a tangent current $\vec{J}_{t 1}$ can flow in each edge. Analogously, the layer 2 will be characterized by charge


Figure 10.2: Elementary circuit. Surface charges $\sigma_{1}$ and $\sigma_{2}$ are associated to nodes, while tangential surface currents $J_{t 1}$ and $J_{t 2}$ flow in layer's edges. The normal surface current $J_{n 21}$ flows from one layer to the other one through the orthogonal link.
$\sigma_{e 2}$ and current $\vec{J}_{t 2}$. Finally, the normal current $\vec{J}_{n}$ flows in the edges connecting the two layers.

### 10.1.1 Circuit variables from Boundary Conditions

We have 5 variables, which are associated to the two circuit layers:

$$
\begin{equation*}
\sigma_{e 1}, \sigma_{e 2}, \vec{J}_{t 2}, \vec{J}_{t 1}, \vec{J}_{n} \tag{10.1}
\end{equation*}
$$

Now we want to link them to the BCs variables, which are:

$$
\begin{equation*}
\sigma_{e}, \vec{d}_{e}, \vec{J}_{t}, \vec{J}_{n}, \overline{\bar{D}}_{e} \tag{10.2}
\end{equation*}
$$

Both the total charge $\sigma_{e}$ and the total tangent current $\vec{J}_{t}$ are simply equal to the sum of those on the two layers, so:

$$
\begin{align*}
\sigma_{e} & =\sigma_{e 2}+\sigma_{e 1}  \tag{10.3}\\
\vec{J}_{t} & =\vec{J}_{t 2}+\vec{J}_{t 1} \tag{10.4}
\end{align*}
$$

The normal component of the current $\vec{J}_{n}$ remains unchanged.
The electric dipole can be calculated on the basis of its definition, in fact:

$$
\begin{equation*}
\vec{d}_{e}=\frac{1}{2}\left(\sigma_{e 2}-\sigma_{e 1}\right) \Delta x \vec{n}_{21} \tag{10.5}
\end{equation*}
$$

For what concern the current doublet $\overline{\bar{D}}_{e}$, we have seen it is associated to magnetic moment and to the concept of current "vortex". Shortly, it is related to the anti-symmetric component of the current. So it can be set equal to:

$$
\begin{equation*}
\overline{\bar{D}}_{e}=\frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \otimes \vec{n}_{21} \Delta x \tag{10.6}
\end{equation*}
$$

## Summary

In summary, the BCs variables can be calculated as:

$$
\left\{\begin{array} { l } 
{ \sigma _ { e } = \sigma _ { e 2 } + \sigma _ { e 1 } }  \tag{10.7}\\
{ \vec { d } _ { e } = \frac { 1 } { 2 } ( \sigma _ { e 2 } - \sigma _ { e 1 } ) \Delta x \vec { n } _ { 2 1 } }
\end{array} \quad \left\{\begin{array}{l}
\vec{J}_{t}=\vec{J}_{t 2}+\vec{J}_{t 1} \\
\overline{\bar{D}}_{e}=\frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \otimes \vec{n}_{21} \Delta x
\end{array}\right.\right.
$$

On the contrary, charges and currents on the layers 1 and 2 can be calculated as:

$$
\left\{\begin{array} { l } 
{ \sigma _ { e 1 } = \frac { 1 } { 2 } \sigma _ { e } - \frac { 1 } { \Delta x } \vec { d } _ { e } ^ { T } \cdot \vec { n } _ { 2 1 } }  \tag{10.8}\\
{ \sigma _ { e 2 } = \frac { 1 } { 2 } \sigma _ { e } + \frac { 1 } { \Delta x } \vec { d } _ { e } ^ { T } \cdot \vec { n } _ { 2 1 } }
\end{array} \quad \left\{\begin{array}{l}
\vec{J}_{t 1}=\frac{1}{2} \vec{J}_{t}-\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \\
\vec{J}_{t 2}=\frac{1}{2} \vec{J}_{t}+\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right.\right.
$$

### 10.1.2 Circuit network in the continuous potential hypothesis

In chapter 7 we saw that, in order to use the classic BCs for $\vec{E}$ and $\vec{H}$, we need to impose the condition of continuous potential across the surface: $P_{A 1}=P_{A 2}$. That hypothesis implies some consequences, in particular we refer to the absence of orthogonal electric dipoles $\vec{d}_{e}$. If there are no such dipoles, then the charges $\sigma_{e 1}, \sigma_{e 2}$ on the two layers must be equal.

$$
\begin{gather*}
\text { if } P_{A 1}=P_{A 2} \quad \Longrightarrow \vec{d}_{e}=\overrightarrow{0} \quad \Longrightarrow  \tag{10.9}\\
\sigma_{e 1}=\sigma_{e 2}=\frac{1}{2} \sigma_{e} \tag{10.10}
\end{gather*}
$$

That means that each pair of adjacent nodes on layers 1 and 2 is equipotential, so they are linked by a short-circuit. Therefore, it is possible to simplify the circuit


Figure 10.3: If the potential is continuous across the interface, then there are no dipoles and thus the charges on opposite nodes must be equal. Since $P_{A 1}=P_{A 2}$, each node on layer 1 is equipotential to its adjacent on layer 2, so they can be short-circuited and merged in a single node.
merging nodes 1 and 2 in a single one, shortening the normal link between them. The two layers are so strictly joined together and the screen looks made of rings connecting close nodes.

The charge can accumulate on each node, so the Extended Kirchoff Current Law (KCL) 45, 46 holds:

$$
\begin{equation*}
\frac{d Q_{e, k}}{d t}+\sum_{\text {out }} I=0 \quad \forall \mathrm{k}^{\mathrm{th}} \text { node } \tag{10.11}
\end{equation*}
$$

where $Q_{e, k}$ is the net charge on the $\mathrm{k}^{\text {th }}$ node and $I$ is the net currents flowing out.

If we consider an elementary surface $S$ and edges with length $\Delta z$, we can rewrite the conservation of charge 10.11 in terms of $\sigma_{e}$ and $J_{t}$ :

$$
\left\{\begin{array}{c}
\sigma_{e}=Q_{e} / S  \tag{10.12}\\
J_{t}=I / \Delta z
\end{array} \quad \Longrightarrow \quad \frac{d \sigma_{e, k}}{d t}+\frac{\Delta z}{S} \sum_{\text {out }} J_{t}=0 \quad \forall \mathrm{k}^{\text {th }}\right. \text { node }
$$

Be aware that till now we have developed just a topological circuit model, so a description of the structure. It suggests us how to connect the circuit elements, but it does not tell what actually we should assemble. We can mount different devices, like impedances, voltage generators, transistors etc., but that choice will depend on the properties required for the metasurface.

### 10.2 BCs for a thin 2-layer screen

In the previous chapters (e.g. see 7.4) we have derived the Boundary Conditions for fields $\vec{E}, \overline{\bar{H}}$ under the hypothesis of continuous potential and zero-thickness surface. Those BCs are:

$$
\begin{align*}
\vec{E}_{2 t}-\vec{E}_{1 t} & =-\mu_{0} \frac{\partial}{\partial t} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{10.13}\\
\left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21} & =-\vec{J}_{t} \tag{10.14}
\end{align*}
$$

Rigorously, these BCs are valid just for a surface with an infinitesimal width. Now we want to determine the BCs for a thin 2-layer screen, whose width is $\Delta x$.

A screen can be considered thin, hence a metasurface, if its thickness is much smaller than the operating wavelength $\lambda_{0}$. More precisely, the condition is:

$$
\begin{equation*}
k_{0} \Delta x \ll 1 \quad \Longrightarrow \quad \Delta x \ll \frac{\lambda_{0}}{2 \pi} \quad \text { thin screen } \tag{10.15}
\end{equation*}
$$

Suppose that a surface current $\vec{J}_{i, t}$ flows in each $i^{\text {th }}$ layer and that fields $\vec{E}_{t}$ and $\overline{\bar{H}}_{2}$ linearly vary between the two layers. The tangential current $\vec{J}_{t}$ can make the magnetic field $\overline{\bar{H}}$ discontinuous, but not the electric one $\vec{E}$. Thus the BCs for the two layer are:

$$
\begin{align*}
& \left(\overline{\bar{H}}_{1, i n t}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21}=-\vec{J}_{t 1}  \tag{10.16}\\
& \left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{2, i n t}\right) \cdot \vec{n}_{21}=-\vec{J}_{t 2} \tag{10.17}
\end{align*}
$$

where $H_{1, \text { int }}$ and $H_{2, \text { int }}$ are the internal magnetic fields, on the inner sides of the two layers. The electric field is instead continuous across each single layer.

(a) Electric field $\vec{E}_{t}$ across the screen. The field is continuous across any single layer.

(b) Magnetic field $H$ across the screen. The field can be discontinuous across both the layers.

Figure 10.4: Linear approx for the tangential electric $\vec{E}_{t}$ and magnetic $\overline{\bar{H}} \cdot \vec{n}_{21}$ fields. The surface currents $\vec{J}_{t 1}$ and $\vec{J}_{t 2}$ determine the discontinuities across the whole interface. This is just an example, since fields depend also on time-derivatives.

Inside the screen, between the layers, there are no currents, so it holds:

$$
\left\{\begin{align*}
\vec{\nabla} \hat{\wedge} \vec{E} & =-\mu_{0} \frac{\partial \overline{\bar{H}}}{\partial t}  \tag{10.18}\\
-\overline{\bar{H}} \cdot \vec{\nabla} & =\varepsilon_{0} \frac{\partial \vec{E}}{\partial t}
\end{align*}\right.
$$

Hence, the correspondent Boundary Conditions for the interior are:

$$
\left\{\begin{array}{cl}
\vec{n}_{21} \hat{\wedge}\left(\vec{E}_{2}-\vec{E}_{1}\right) & =-\mu_{0} \frac{\partial\langle\overline{\bar{H}}\rangle_{\text {int }}}{\partial t} \Delta x  \tag{10.19}\\
-\left(\overline{\bar{H}}_{2, i n t}-\overline{\bar{H}}_{1, i n t}\right) \cdot \vec{n}_{21} & =\varepsilon_{0} \frac{\partial\langle\vec{E}\rangle}{\partial t} \Delta x
\end{array}\right.
$$

Thanks to the hypothesis of thin screen ( $\operatorname{small} \Delta x$ ), we can linearly approximate the internal fields, so we find:

$$
\begin{align*}
\langle\vec{E}\rangle & =\frac{1}{2}\left(\vec{E}_{2}+\vec{E}_{1}\right)  \tag{10.20}\\
\langle\overline{\bar{H}}\rangle_{i n t} & =\frac{1}{2}\left(\overline{\bar{H}}_{2}+\overline{\bar{H}}_{1}\right)_{i n t} \tag{10.21}
\end{align*}
$$

In the Laplace's Domain, the complete set of BCs will look so:

$$
\begin{align*}
& \left\{\begin{array}{l}
\left(\overline{\bar{H}}_{1, \text { int }}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21}=-\vec{J}_{t 1} \\
\left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{2, i n t}\right) \cdot \vec{n}_{21}=-\vec{J}_{t 2}
\end{array}\right.  \tag{10.22}\\
& \left\{\begin{array}{c}
\vec{E}_{2 t}-\vec{E}_{1 t}=-s \mu_{0} \frac{1}{2}\left(\overline{\bar{H}}_{2}+\overline{\bar{H}}_{1}\right)_{\text {int }} \cdot \vec{n}_{21} \Delta x \\
\left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right)_{\text {int }} \cdot \vec{n}_{21}=-s \varepsilon_{0} \frac{1}{2}\left(\vec{E}_{2 t}+\vec{E}_{1 t}\right) \Delta x
\end{array}\right. \tag{10.23}
\end{align*}
$$

The internal fields can be eliminated by substitution, so we get:

$$
\left\{\begin{array}{cc}
\vec{E}_{2 t}-\vec{E}_{1 t}=-s \mu_{0} \frac{1}{2}\left(\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right)+\left(\overline{\bar{H}}_{2}+\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21}\right) \Delta x  \tag{10.24}\\
-\left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21} & = \\
\left(\vec{J}_{t 2}+\vec{J}_{t 1}\right)+s \varepsilon_{0} \frac{1}{2}\left(\vec{E}_{2 t}+\vec{E}_{1 t}\right) \Delta x
\end{array}\right.
$$

In a more compact, scalar form:

$$
\begin{align*}
& \Delta E_{21}=-s \mu_{0} \Delta x\left(\frac{1}{2}\left(J_{2}-J_{1}\right)+\langle H\rangle\right)  \tag{10.25}\\
& \Delta H_{21}=-\left(J_{2}+J_{1}\right)-s \varepsilon_{0} \Delta x\langle E\rangle \tag{10.26}
\end{align*}
$$

This is quite an important result.

### 10.2.1 Calculating currents and doublets

In the limit of $\Delta x \rightarrow 0^{+}$, the contribution of the "average" fields $\langle E\rangle$ and $\langle H\rangle$ must vanish, so it yields:

$$
\left\{\begin{array}{rlr}
\vec{E}_{2 t}-\vec{E}_{1 t} & =-s \mu_{0} \frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \Delta x  \tag{10.27}\\
-\left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21} & = & \left(\vec{J}_{t 2}+\vec{J}_{t 1}\right)
\end{array} \quad \text { for } \Delta x \rightarrow 0^{+}\right.
$$

Imposing that the classic BCs 10.13 10.14 are satisfied, we find the expression for the global current $\vec{J}_{t}$ and the doublet $\bar{D}_{e}$. We assign so:

$$
\begin{align*}
\vec{J}_{t} & =\vec{J}_{t 2}+\vec{J}_{t 1}  \tag{10.28}\\
\overline{\bar{D}}_{e} \cdot \vec{n}_{21} & =\frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \Delta x \tag{10.29}
\end{align*}
$$

Finally, the BCs for a 2-layer screen with finite width become:

$$
\begin{align*}
& \vec{E}_{2 t}-\vec{E}_{1 t}=-s \mu_{0}\left(\overline{\bar{D}}_{e}+\langle\overline{\bar{H}}\rangle \Delta x\right) \cdot \vec{n}_{21}  \tag{10.30}\\
& \left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21}=-\vec{J}_{t} \quad-s \varepsilon_{0} \Delta x\left\langle\vec{E}_{t}\right\rangle \tag{10.31}
\end{align*}
$$

Let us notice that all the variables have finite values.

### 10.2.2 Dimensional analysis

The permeability $\mu_{0}$ and the permittivity $\varepsilon_{0}$ appear in the systems 10.24 and in the following equations. In order to compare the "weight" of the single terms, it is convenient to multiply the quantities $J$ and $H$ for the vacuum impedance $\eta_{0}$. In this way, all the variable will have the same physical dimensions of the electric field (e.g. $[V / m]$ ).

Now we desire to express $\mu_{0}$ and $\varepsilon_{0}$ in function of the wavenumber $k_{0}$ and of the vacuum impedance $\eta_{0}$. We know that:

$$
\begin{align*}
s & =-i \omega ; & k_{0} & =\frac{\omega}{c_{0}}  \tag{10.32}\\
c_{0} & =\frac{1}{\sqrt{\mu_{0} \varepsilon_{0}}} ; & \eta_{0} & =\sqrt{\frac{\mu_{0}}{\varepsilon_{0}}}
\end{align*}
$$

So, after some calculi, we find:

$$
\begin{align*}
& s \mu_{0}=-i k_{0} \eta_{0}  \tag{10.34}\\
& s \varepsilon_{0}=-i k_{0} \frac{1}{\eta_{0}} \tag{10.35}
\end{align*}
$$

Multiplying for the screen's width $\Delta x$, we get:

$$
\begin{align*}
\theta_{0} & =k_{0} \Delta x \quad \text { adim. thickness }  \tag{10.36}\\
s \mu_{0} \Delta x & =-\left(i \theta_{0}\right) \eta_{0}  \tag{10.37}\\
s \varepsilon_{0} \Delta x & =-\left(i \theta_{0}\right) \frac{1}{\eta_{0}} \tag{10.38}
\end{align*}
$$

Substituting in 10.30 and 10.31, finally we obtain:

$$
\begin{align*}
\vec{E}_{2 t}-\vec{E}_{1 t} & =\quad i \eta_{0}\left(k_{0} \overline{\bar{D}}_{e}+\theta_{0}\langle\overline{\bar{H}}\rangle\right) \cdot \vec{n}_{21}  \tag{10.39}\\
\eta_{0}\left(\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}\right) \cdot \vec{n}_{21} & =-\eta_{0} \vec{J}_{t} \quad-i \theta_{0}\left\langle\vec{E}_{t}\right\rangle \tag{10.40}
\end{align*}
$$

For a thin screen $\theta_{0}=k_{0} \Delta x \ll 1$ the contribution of the average fields can be neglected or at least linearized.

### 10.2.3 Matrix equation for Boundary Conditions

In order to highlight the different terms it is better to rewrite the eq.s 10.39, (10.40) in matrix form. For sake of simplicity, we drop the subscript $t$ and the vector notation, but obviously the relation still remains valid:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{10.41}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]\left(-\eta_{0}\left\{\begin{array}{c}
J \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}+i \theta_{0}\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}\right)
$$

If we want further compact the notation, we can define a field array $\mathbf{f}_{h}$ and a source array $\mathbf{J}_{h}$, so that:

$$
\begin{align*}
& \mathbf{f}_{h}=\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} \quad \mathbf{J}_{h}=\left\{\begin{array}{c}
J \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}  \tag{10.42}\\
& \Delta \mathbf{f}_{h, 21}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left(-\eta_{0} \mathbf{J}_{h}+i \theta_{0}\left\langle\mathbf{f}_{h}\right\rangle\right) \tag{10.43}
\end{align*}
$$

### 10.2.4 Sources in function of EM fields

If we are interested to determine the sources $\mathbf{J}_{h}$ in function of the EM fields $\mathbf{f}_{h}$, we have to invert the relation (10.41). After few calculi, the sources $J$ and $D_{e}$ can be expressed as:

$$
\left\{\begin{array}{c}
J  \tag{10.44}\\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\Delta E_{21} \\
\eta_{0} \Delta H_{21}
\end{array}\right\}+i \theta_{0}\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}\right)
$$

Adopting the source $\mathbf{J}_{h}$ and field $\mathbf{f}_{h}$ arrays defined in eq. 10.42 , we can write in a compacter form:

$$
\mathbf{J}_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{10.45}\\
1 & 0
\end{array}\right] \Delta \mathbf{f}_{h, 21}+i \theta_{0}\left\langle\mathbf{f}_{h}\right\rangle\right)
$$

It should be noticed that these last relations are valid independently from the constitutive relation describing the screen's specific properties.

### 10.3 Propagating fields on the two screen's sides

In this section we analyse the propagation of field $E$ and $H$ across a thin screen, then we rephrase them in terms of electrical progressive $E_{+}$and regressive $E_{-}$waves. This is intended to be just an introduction to next chapter, which concerns the scattering and the constitutive relations.

For clarity and simplicity, we shall adopt a scalar notation, but all the results are valid also for vectors.

### 10.3.1 Free propagation

Let us consider a region of space without sources, thus:

$$
\begin{equation*}
\mathbf{J}_{h}=0 \quad \Longrightarrow \quad J=0 ; \quad D_{e}=0 \tag{10.46}
\end{equation*}
$$

That is the case of a perfectly transparent screen, or else a virtual screen. Thus, the relation between the fields' discontinuities and their average values is:

$$
\begin{align*}
\Delta \mathbf{f}_{h, 21} & =i \theta_{0}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\langle\mathbf{f}_{h}\right\rangle  \tag{10.47}\\
\left\{\begin{array}{c}
\Delta E_{21} \\
\eta_{0} \Delta H_{21}
\end{array}\right\} & =i \theta_{0}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \tag{10.48}
\end{align*}
$$

In the limit of $\theta_{0} \ll 1$, so in the thin screen hypothesis, we can approximate:

$$
\begin{align*}
& \Delta f \approx \frac{\partial f}{\partial \theta_{0}} \theta_{0} \quad \Longrightarrow  \tag{10.49}\\
& \frac{\partial}{\partial \theta_{0}}\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\}=i\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} \tag{10.50}
\end{align*}
$$

We obtain so a first order differential equation: now we desire to determine the fields $E$ and $H$ in function of the angle $\theta_{0}$. Even without an explicit diagonalization, we transform the variables defining two equivalent electric fields $E_{+}$and $E_{-}$:

$$
\left\{\begin{array} { r l } 
{ E _ { + } } & { = \frac { 1 } { 2 } ( E + \eta _ { 0 } H ) }  \tag{10.51}\\
{ E _ { - } } & { = \frac { 1 } { 2 } ( E - \eta _ { 0 } H ) }
\end{array} \quad \left\{\begin{array}{r}
E=E_{+}+E_{-} \\
\eta_{0} H
\end{array}=E_{-}-E_{-}\right.\right.
$$

After few calculi, we get:

$$
\begin{align*}
\left\{\begin{array}{r}
\frac{\partial}{\partial \theta_{0}}\left(E+\eta_{0} H\right)= \\
\frac{\partial}{\partial \theta_{0}}\left(E-\eta_{0} H\right)= \\
\\
-
\end{array}\left(E+\eta_{0} H\right)\right.
\end{aligned} \Longrightarrow\left\{\begin{array}{l}
\frac{\partial}{\partial \theta_{0}} E_{+}=+i E_{+}  \tag{10.52}\\
\frac{\partial}{\partial \theta_{0}} E_{-}=-i E_{-}
\end{array}\right] \begin{aligned}
& \frac{\partial}{\partial \theta_{0}}\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\}=i\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\} \tag{10.53}
\end{align*}
$$

So the electric waves propagating in free space will be:

$$
\left\{\begin{array}{l}
E_{+}=E_{+, 0} e^{i \theta_{0}}  \tag{10.54}\\
E_{-}=E_{-, 0} e^{-i \theta_{0}} \quad \text { equivalent electric fields }, \text { 竍 }
\end{array}\right.
$$

The global electric and magnetic fields result to be:

$$
\left\{\begin{align*}
E=E_{+}+E_{-} & =E_{+, 0} e^{i \theta_{0}}+E_{-, 0} e^{-i \theta_{0}}  \tag{10.55}\\
\eta_{0} H=E_{+}-E_{-} & =E_{+, 0} e^{i \theta_{0}}-E_{-, 0} e^{-i \theta_{0}}
\end{align*}\right.
$$

If we want to express the incident magnetic field $H$ fields with its own phasors, we get:

$$
\left\{\begin{array}{l}
\eta_{0} H_{+}=E_{+}  \tag{10.56}\\
\eta_{0} H_{-}=-E_{-}
\end{array}\right.
$$

### 10.3.2 Transformation in equivalent electric fields

In conclusion, once the global fields $E H$ impinging on the screen are known, then it is possible to define some equivalent incident electric amplitudes $E_{+}$and $E_{-}$. Here we resume those transformations in a compact matrix form:

$$
\begin{align*}
\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\} & =\frac{1}{2}\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\}  \tag{10.57}\\
\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} & =\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\} \tag{10.58}
\end{align*}
$$

It can be easily verified that those transformations still remain valid for vectors. More precisely, they are:

$$
\begin{align*}
\left\{\begin{array}{l}
\vec{E}_{t+} \\
\vec{E}_{t-}
\end{array}\right\} & =\frac{1}{2}\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \overline{\bar{H}} \cdot \vec{n}_{21}
\end{array}\right\}  \tag{10.59}\\
\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \overline{\bar{H}} \cdot \vec{n}_{21}
\end{array}\right\} & =\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{l}
\vec{E}_{t+} \\
\vec{E}_{t-}
\end{array}\right\} \tag{10.60}
\end{align*}
$$

In a very compact form:

$$
\begin{align*}
\mathbf{E} & =\overline{\bar{T}}_{E} \cdot \mathbf{f}_{h}  \tag{10.61}\\
\mathbf{f}_{h} & =\overline{\bar{T}}_{E}^{-1} \cdot \mathbf{E} \tag{10.62}
\end{align*}
$$

This transformation is important because it allows us to convert the magnetic field $H$ in some equivalent electric one. That is a fundamental step in order to built the circuit equivalent. In fact, the electric field $\vec{E}$ can be rephrased as voltage $\Delta V$, while the magnetic field $H$ cannot be directly interpreted as a circuit variable. Let us notice that here $H$ cannot be strictly regarded as current, since we already have the surface currents $J$.

### 10.3.3 Radiated electric field

Let us consider the fields irradiated by the screen. We call $E_{+}$the electric field propagating forward (1-to-2 direction) and $E_{-}$the field propagating backward (2-to-1 direction).

For each coordinate $\theta_{0}=k_{0}\left(x-x_{0}\right)$, we can express the electric field as:

$$
\begin{align*}
& E=E_{+, 0} e^{i \theta_{0}}+E_{-, 0} e^{-i \theta_{0}}  \tag{10.63}\\
& \mathbf{E}=\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\} \quad \text { field array } \tag{10.64}
\end{align*}
$$

Since the radiated waves travel away from the screen, it follows that:

$$
E_{i r r}\left(\theta_{0}\right)=\left\{\begin{array}{ll}
E_{1-} e^{-i \theta_{0}} & \text { on side 1 }  \tag{10.65}\\
E_{2+} e^{i \theta_{0}} & \text { on side 2 }
\end{array}\right. \text { radiated electric fields }
$$

So the electric field arrays $\mathbf{E}_{1, \text { irr }}$ and $\mathbf{E}_{2, \text { irr }}$ on the two sides can be expressed as:

$$
\mathbf{E}_{1, \text { irr }}=\left\{\begin{array}{c}
0  \tag{10.66}\\
E_{1-}
\end{array}\right\} \quad \mathbf{E}_{2, \text { irr }}=\left\{\begin{array}{c}
E_{2+} \\
0
\end{array}\right\}
$$



Figure 10.5: Electric waves radiated by the screen. In this example no incident fields are considered.

## Radiated magnetic fields

Thanks to the transformation 10.58, we can calculate the radiated electric and magnetic field on the two screen's sides:

$$
\begin{align*}
\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} & =\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\}  \tag{10.67}\\
\mathbf{f}_{h} & =\overline{\bar{T}}_{E}^{-1} \cdot \mathbf{E} \quad \text { compact form } \tag{10.68}
\end{align*}
$$

So we get:

$$
\left\{\begin{array}{l}
\eta_{0} H_{1, i r r}=-E_{1, i r r}=-E_{1-}  \tag{10.69}\\
\eta_{0} H_{2, i r r}=E_{2, i r r}=E_{2+}
\end{array}\right.
$$

## Average radiated fields

The average radiated fields are nicely related to their discontinuities, in fact:

$$
\begin{align*}
\langle E\rangle_{i r r} & =\frac{1}{2}\left(E_{2+}+E_{1-}\right)=\frac{1}{2} \eta_{0}\left(H_{2}-H_{1}\right)_{i r r}  \tag{10.70}\\
\eta_{0}\langle H\rangle_{i r r} & =\frac{1}{2} \eta_{0}\left(H_{2}+H_{1}\right)_{i r r}=\frac{1}{2}\left(E_{2}-E_{1}\right)_{i r r} \tag{10.71}
\end{align*}
$$

So we can compactly write:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{10.72}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}_{i r r}=2\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}_{i r r}
$$

Be aware that these relations are valid just for the radiated fields.

### 10.3.4 Radiation Law for thin screen

Now we desire to determine the fields radiated by the currents on a screen element. In other words, we are looking for a radiation law in the form:

$$
\begin{equation*}
\mathbf{f}_{i r r}=-Z_{i r r} \mathbf{J} \tag{10.73}
\end{equation*}
$$

where $\mathbf{f}_{i r r}$ is an array containing the radiated fields, while $\mathbf{J}$ contains the sources. Remembering eq.s 10.42 and 10.43 , we can write the BCs for the radiated fields:

$$
\begin{align*}
& \mathbf{f}_{h, i r r}=\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\}_{i r r} \quad \mathbf{J}_{h}=\left\{\begin{array}{c}
J \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}  \tag{10.74}\\
& \Delta \mathbf{f}_{h, 21, i r r}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left(-\eta_{0} \mathbf{J}_{h}+i \theta_{0}\left\langle\mathbf{f}_{h}\right\rangle_{i r r}\right) \tag{10.75}
\end{align*}
$$

We can express the discontinuities in function of the average radiated fields (see (10.72), so:

$$
\Delta \mathbf{f}_{h, 21, i r r}=2\left[\begin{array}{ll}
0 & 1  \tag{10.76}\\
1 & 0
\end{array}\right]\left\langle\mathbf{f}_{h}\right\rangle_{i r r}
$$

After some calculi, finally we draw the radiation law:

$$
\begin{equation*}
\left\langle\mathbf{f}_{h}\right\rangle_{i r r}=-\frac{1}{2} \eta_{0} \frac{1}{1-\frac{1}{2}\left(i \theta_{0}\right)} \mathbf{J}_{h} \tag{10.77}
\end{equation*}
$$

More explicitly:

$$
\left\{\begin{array}{c}
\langle E\rangle  \tag{10.78}\\
\eta_{0}\langle H\rangle
\end{array}\right\}_{i r r}=-\frac{1}{2} \eta_{0} \frac{1}{1-\frac{1}{2}\left(i \theta_{0}\right)}\left\{\begin{array}{c}
J \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}
$$

In the limit of a zero-thickness screen $\left(\theta_{0}=0\right)$ we simply obtain:

$$
\left\{\begin{array}{c}
\langle E\rangle  \tag{10.79}\\
\eta_{0}\langle H\rangle
\end{array}\right\}_{i r r}=-\frac{1}{2} \eta_{0}\left\{\begin{array}{c}
J \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}
$$

## Radiation for currents J1 and J2

The radiation law can be written also for other pairs of fields and sources. In order to better rephrase it for circuit variable, we choose:

$$
\mathbf{f}_{i r r}=\mathbf{E}_{i r r}=\left\{\begin{array}{l}
E_{1}  \tag{10.80}\\
E_{2}
\end{array}\right\}_{i r r} \quad \mathbf{J}=\left\{\begin{array}{l}
J_{1} \\
J_{2}
\end{array}\right\}
$$

where $E_{1, i r r}$ and $E_{2, i r r}$ are the radiated electric fields, while $J_{1}$ and $J_{2}$ are the surface currents on the two sides of the screen. We know how to express $\mathbf{E}_{\text {irr }}$
and $\mathbf{J}$ in function of $\mathbf{f}_{h, i r r}$ and $\mathbf{J}_{h}$ respectively, in fact:

$$
\begin{gather*}
\mathbf{E}_{i r r}=\left\{\begin{array}{l}
E_{1} \\
E_{2}
\end{array}\right\}_{i r r}=\left\{\begin{array}{l}
E_{1-} \\
E_{2+}
\end{array}\right\}=\left[\begin{array}{rr}
1 & -1 \\
1 & 1
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}_{i r r}  \tag{10.81}\\
\mathbf{J}=\left\{\begin{array}{l}
J_{1} \\
J_{2}
\end{array}\right\}=\frac{1}{2}\left[\begin{array}{rr}
1 & \frac{2}{i \theta_{0}} \\
1 & -\frac{2}{i \theta_{0}}
\end{array}\right]\left\{\begin{array}{c}
J \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\} \tag{10.82}
\end{gather*}
$$

After some calculi, finally we get the radiation law in the form:

$$
\left\{\begin{array}{l}
E_{1}  \tag{10.83}\\
E_{2}
\end{array}\right\}_{i r r}=-\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & \frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} \\
\frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} & 1
\end{array}\right]\left\{\begin{array}{l}
J_{1} \\
J_{2}
\end{array}\right\}
$$

More compactly:

$$
\begin{align*}
\mathbf{E}_{i r r} & =-Z_{i r r} \mathbf{J}  \tag{10.84}\\
Z_{i r r} & =\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & b\left(\theta_{0}\right) \\
b\left(\theta_{0}\right) & 1
\end{array}\right]  \tag{10.85}\\
b\left(\theta_{0}\right) & =\frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} \tag{10.86}
\end{align*}
$$

## Comparison with the exact solution

It can be demonstrated that the radiated electric fields for two parallel screen filled with currents are equal to:

$$
\left\{\begin{array}{l}
E_{1}  \tag{10.87}\\
E_{2}
\end{array}\right\}_{i r r}=-\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & e^{i \theta_{0}} \\
e^{i \theta_{0}} & 1
\end{array}\right]\left\{\begin{array}{l}
J_{1} \\
J_{2}
\end{array}\right\}
$$

This is the exact solution, and it is quite to similar to the one 10.83 we drew within the hypothesis of thin screen and linearized fields.

Actually, for $\theta_{0} \ll 1$, the extra-diagonal term $b(\theta)$ is a good approximation of the phasor $e^{i \theta_{0}}$ :

$$
\begin{equation*}
b(\theta)=\frac{1+\frac{1}{2}(i \theta)}{1-\frac{1}{2}(i \theta)} \approx e^{i \theta} \quad \text { for } \theta \ll 1 \tag{10.88}
\end{equation*}
$$

In fact:

$$
\begin{align*}
\frac{1+\frac{1}{2}(i \theta)}{1-\frac{1}{2}(i \theta)} & =\frac{\left(1+\frac{1}{2}(i \theta)\right)^{2}}{1-\frac{1}{4}(i \theta)^{2}}=\frac{1+i \theta-\frac{1}{4} \theta^{2}}{1+\frac{1}{4} \theta^{2}} \approx 1+i \theta+o\left(\theta^{2}\right)  \tag{10.89}\\
e^{i \theta} & \approx 1+i \theta+o\left(\theta^{2}\right) \quad \text { Taylor expansion } \tag{10.90}
\end{align*}
$$

That approximation is quite elegant, since it guarantees the unitary modulus for $b(\theta)$, like $e^{i \theta}$ :

$$
\begin{align*}
& |1+i \theta| \neq 1  \tag{10.91}\\
& \left|e^{i \theta}\right|=\left|\frac{1+\frac{1}{2}(i \theta)}{1-\frac{1}{2}(i \theta)}\right|=1 \quad \text { unitary modulus } \tag{10.92}
\end{align*}
$$

More over:

$$
\begin{equation*}
b(-\theta)=\frac{1}{b(\theta)} \quad \text { like } \quad e^{-i \theta}=\frac{1}{e^{i \theta}} \tag{10.93}
\end{equation*}
$$

### 10.4 Permittivity and permeability for the screen

Suppose we want to build a thin circuit screen, assigning both the relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$.

In order to do that, we are going to:

1. analyze the propagation of the electric $E$ and magnetic $H$ fields in a thick homogeneous slab with assigned $\varepsilon_{r}$ and $\mu_{r}$.
2. extract a constitutive relation for a thin slab, involving $E$ and $H$
3. convert the constitutive relation in circuit form, removing the magnetic field.
4. calculate the impedances to be effectively installed on the screen.

In this section we are going to concentrate on the first two points, while the following ones will be treated in chapter 12 .

### 10.4.1 Thick homogeneous slab

Consider an infinite, plane, homogeneous slab made of a material with known characteristic impedance $\eta$ and phase velocity $v_{\varphi}$. The thickness $\Delta x_{L}$ can be great or small at pleasure.

For the propagation of EM waves, $\eta$ and $v_{\varphi}$ are strictly related to the relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$ :

$$
\begin{align*}
\eta & =\sqrt{\frac{\mu}{\varepsilon}}=\sqrt{\frac{\mu_{r}}{\varepsilon_{r}}} \eta_{0}  \tag{10.94}\\
v_{\varphi} & =\frac{1}{\sqrt{\mu_{r} \varepsilon_{r}}} c_{0} \tag{10.95}
\end{align*}
$$

Be aware that, since we are working in the frequency domain, both $\mu_{r}$ and $\varepsilon_{r}$ can be complex quantities.

At a given frequency, the wavenumber $k$ inside the material can be calculated as:

$$
\begin{equation*}
k=\frac{\omega}{v_{\varphi}}=\frac{\omega}{c_{0}} \sqrt{\mu_{r} \varepsilon_{r}}=k_{0} \sqrt{\mu_{r} \varepsilon_{r}} \tag{10.96}
\end{equation*}
$$

In absence of free charge and current, the electric and magnetic fields propagates inside the slab obeying to the Helmholtz equations:

$$
\begin{align*}
& \nabla^{2} \vec{E}+k^{2} \vec{E}=\overrightarrow{0}  \tag{10.97}\\
& \nabla^{2} \overline{\bar{H}}+k^{2} \overline{\bar{H}}=\overline{\overline{0}} \tag{10.98}
\end{align*}
$$

So there general solution for the field propagating inside will be:

$$
\begin{equation*}
A=A_{+} e^{i \vec{k}^{T} \cdot \Delta \vec{x}}+A_{-} e^{-i \vec{k}^{T} \cdot \Delta \vec{x}} \tag{10.99}
\end{equation*}
$$

For sake of simplicity, we shall consider the case of waves propagating orthogonally to the slab's surface $(\vec{k} / / \vec{n})$ and will drop the vector notation.

So the fields inside the slab can be expressed as:

$$
\begin{align*}
& E=E_{+, 0} e^{i \theta}+E_{-, 0} e^{-i \theta}  \tag{10.100}\\
& H=H_{+, 0} e^{i \theta}+H_{-, 0} e^{-i \theta} \tag{10.101}
\end{align*}
$$

where $\theta$ is the phase angle with respect to the center $\vec{x}_{c}$ of the slab:

$$
\begin{align*}
& \theta=k \Delta x=k_{0} \Delta x \sqrt{\mu_{r} \varepsilon_{r}}  \tag{10.102}\\
& \theta=\theta_{0} \sqrt{\mu_{r} \varepsilon_{r}} \tag{10.103}
\end{align*}
$$

The electric and magnetic field can be related by the Faraday's Law in the form:

$$
\begin{equation*}
\vec{\nabla} \wedge \vec{E}=i \omega(\mu \overline{\bar{H}}) \tag{10.104}
\end{equation*}
$$

After some calculi, we can write the magnetic field $H$ in function of the electric amplitudes:

$$
\begin{align*}
E & =E_{+, 0} e^{i \theta}+E_{-, 0} e^{-i \theta}  \tag{10.105}\\
\eta H & =E_{+, 0} e^{i \theta}-E_{-, 0} e^{-i \theta} \tag{10.106}
\end{align*}
$$

### 10.4.2 Boundary conditions

Now we desire to calculate the fields on the two sides of the slab and draw a constitutive relation in the form:

$$
\begin{equation*}
\Delta \mathbf{f}_{21}=C\langle\mathbf{f}\rangle \tag{10.107}
\end{equation*}
$$

From the standard boundary conditions, the tangent component of $E$ and $H$ must be continuous on each side, because there are no free sources:

$$
\left\{\begin{array} { l } 
{ E _ { 1 } = E _ { \text { int } } ( x _ { 1 } ) }  \tag{10.108}\\
{ H _ { 1 } = H _ { \text { int } } ( x _ { 1 } ) }
\end{array} \quad \left\{\begin{array}{l}
E_{2}=E_{\text {int }}\left(x_{2}\right) \\
H_{2}=H_{\text {int }}\left(x_{2}\right)
\end{array} \quad\right.\right. \text { tangent fields continuity }
$$

where $x_{1}$ and $x_{2}$ are the coordinate of the slab's faces.
We define the discontinuities and the "average" fields for the slab as:

$$
\left\{\begin{array} { l } 
{ \Delta E _ { 2 1 } = E _ { 2 } - E _ { 1 } }  \tag{10.109}\\
{ \Delta H _ { 2 1 } = H _ { 2 } - H _ { 1 } }
\end{array} \quad \left\{\begin{array}{l}
\langle E\rangle=\frac{1}{2}\left(E_{1}+E_{2}\right) \\
\langle H\rangle=\frac{1}{2}\left(H_{1}+H_{2}\right)
\end{array}\right.\right.
$$

Substituting in 10.105, 10.106), after some counts we find:

$$
\left\{\begin{array}{c}
\langle E\rangle=\left(E_{+, 0}+E_{-, 0}\right) \cos \left(\frac{1}{2} \theta_{21}\right) \\
\eta \Delta H_{21}=\left(E_{+, 0}+E_{-, 0}\right) 2 i \sin \left(\frac{1}{2} \theta_{21}\right)  \tag{10.111}\\
\eta \frac{\partial \Delta H_{21}}{\partial \theta}=i\langle E\rangle
\end{array} \Longrightarrow\right.
$$

where $\theta_{21}=k\left(x_{2}-x_{1}\right)$. Similarly, we can link $\langle H\rangle$ to $\Delta E_{21}$ :

$$
\left\{\begin{array}{l}
\eta\langle H\rangle=\left(E_{+, 0}-E_{-, 0}\right) \cos \left(\frac{1}{2} \theta_{21}\right) \\
\Delta E_{21}=\left(E_{+, 0}-E_{-, 0}\right) 2 i \sin \left(\frac{1}{2} \theta_{21}\right)  \tag{10.113}\\
\frac{\partial \Delta E_{21}}{\partial \theta}=i \eta\langle H\rangle
\end{array} \Longrightarrow\right.
$$

Finally, we can summarize this two results in a single differential matrix equation:

$$
\frac{\partial}{\partial \theta}\left\{\begin{array}{c}
\Delta E_{21}  \tag{10.114}\\
\eta \Delta H_{21}
\end{array}\right\}=i\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta\langle H\rangle
\end{array}\right\}
$$

A similar expression could be directly obtained deriving 10.105, 10.106 with respect to $\theta$

$$
\frac{\partial}{\partial \theta}\left\{\begin{array}{c}
E  \tag{10.115}\\
\eta H
\end{array}\right\}=i\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
E \\
\eta H
\end{array}\right\}
$$

Now we desire to make explicit the role of permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$

### 10.4.3 Deriving the constitutive relation for E and H

Both the impedance $\eta$ and the phase $\theta$ depends on the relative permittivity and permeability, in fact:

$$
\begin{equation*}
\eta=\sqrt{\frac{\mu_{r}}{\varepsilon_{r}}} \eta_{0} ; \quad \theta=\sqrt{\mu_{r} \varepsilon_{r}} \theta_{0} \tag{10.116}
\end{equation*}
$$

Thanks to that, we can rewrite eq.s 10.114 10.115) in the form:

$$
\begin{array}{rlr}
\frac{\partial}{\partial \theta_{0}}\left\{\begin{array}{c}
\Delta E_{21} \\
\eta_{0} \Delta H_{21}
\end{array}\right\} & =i\left[\begin{array}{cc}
0 & \mu_{r} \\
\varepsilon_{r} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \\
\frac{\partial}{\partial \theta_{0}}\left\{\begin{array}{c}
E \\
\eta H
\end{array}\right\} & =i\left[\begin{array}{cc}
0 & \mu_{r} \\
\varepsilon_{r} & 0
\end{array}\right]\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} \tag{10.118}
\end{array}
$$

Those equations are valid for slabs of any thickness and that formulation is analogous to the one often adopted for the Transmission Lines, in fact:

$$
\frac{\partial}{\partial x}\left\{\begin{array}{l}
V  \tag{10.119}\\
I
\end{array}\right\}=\left[\begin{array}{cc}
0 & -z(s) \\
-y(s) & 0
\end{array}\right]\left\{\begin{array}{l}
V \\
I
\end{array}\right\}
$$

where $z$ and $y$ are the impedance and admittance per unit length respectively:

$$
\begin{align*}
& z=\frac{Z}{\Delta x}=s \mu(s)  \tag{10.120}\\
& y=\frac{Y}{\Delta x}=s \varepsilon(s) \tag{10.121}
\end{align*}
$$



Figure 10.6: Basic circuit Transmission Line. The series impedance p.u.l. $z$ is analogous to permeability $\mu$, while the shunt admittance p.u.l. $y$ is analogous to permittivity $\varepsilon$.

## Thin slab, thin screen

If we require that the slab has a small thickness if compared to the wavelength $\lambda_{0}$, then we can well approximate eq. 10.117) as:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{10.122}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=i \theta_{0}\left[\begin{array}{cc}
0 & \mu_{r} \\
\varepsilon_{r} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \quad \text { for } \theta_{0} \ll 1
$$

This last equations can be effectively regarded as the constitutive relation for a thin screen with assigned permittivity and permeability.

## Chapter 11

## Scattering and constitutive relations


#### Abstract

E come l'aere, quand'è ben pïorno, per l'altrui raggio che 'n sé si reflette, di diversi color diventa addorno; così l'aere vicin quivi si mette in quella forma ch'è in lui suggella virtüalmente l'alma che ristette


Till now we calculated the field's sources necessary to create a discontinuity in the field itself. However, we did not yet faced some practical aspects, like:

- Choosing the properties for our system or device.

Since we desire a precise behaviour for our system, we must define the relation among the input variables and the output ones. In other words, we should express the transmitted fields in function of the incident ones. That means to assign the desired properties for the systems.

- Generating the required sources (e.g., currents) in order to achieve the desired behaviour. For example, we could choose some generators or impedances to be mounted on a screen, giving rise to the needed currents.

In this chapter we are going to focus the attention on the desired behaviour of our systems, trying to use a very general approach to the local scattering theory. For the circuit case we found a nice compendium [47] by P.Young, though here we are going to consider mainly EM wave propagation in free space.

### 11.1 Choosing the appropriate fields

Given a generic system, it can take in input some variables $\mathbf{f}_{i n}$ and produce some output variables $\mathbf{f}_{t r}$.

In our case the system is a screen or metasurface, while the variables are the electromagnetic fields. The EM waves can be identified with different parameters,
depending on the user's needs. For example, we can describe a plane wave through the EM potentials:

$$
\mathbf{f}=\left\{\begin{array}{c}
\vec{A}  \tag{11.1}\\
\frac{1}{k_{0}} \frac{\partial \vec{A}}{\partial \vec{x}}
\end{array}\right\} \quad \text { or } \quad \mathbf{f}=\left\{\begin{array}{c}
\vec{A} \\
\vec{k}
\end{array}\right\}
$$

Another possibility consists in using the electric and magnetic fields as state variables:

$$
\mathbf{f}=\left\{\begin{array}{c}
\vec{E}  \tag{11.2}\\
\eta_{0} \overline{\bar{H}}
\end{array}\right\} \quad \text { or } \quad \mathbf{f}=\left\{\begin{array}{l}
\vec{E} \\
\vec{k}
\end{array}\right\}
$$

Therefore, we have succeeded in writing the Boundary Conditions in terms of tangent fields, so another possible choice is:

$$
\mathbf{f}=\left\{\begin{array}{c}
\vec{E}_{t}  \tag{11.3}\\
\eta_{0} \overline{\bar{H}} \cdot \vec{n}_{21}
\end{array}\right\}
$$

Otherwise, if we prefer to avoid the magnetic field $H$, we can define some equivalent electric amplitudes:

$$
\mathbf{f}=\left\{\begin{array}{l}
\vec{E}_{1}  \tag{11.4}\\
\vec{E}_{2}
\end{array}\right\}
$$

Shortly, the basic idea is to assemble an array $\mathbf{f}$ containing the needful data on the input or output signals. That choice is quite arbitrary and strongly depends on the problem you are handling.

### 11.2 Linear Scattering

We suppose that our system is linear, so we can express the scattering relation as:

$$
\begin{equation*}
\mathbf{f}_{t r}=S \mathbf{f}_{i n}+\mathbf{f}_{t r, 0} \quad \text { Scattering relation } \tag{11.5}
\end{equation*}
$$

where:

- $\mathbf{f}_{t r}$ is the array of the transmitted or output fields.
- $S$ is the so-called "Scattering matrix", linking the input fields to the output ones. In other words, $S$ is a transfer function.
- $\mathbf{f}_{i n}$ is the array of the incident or input fields.
- $\mathbf{f}_{t r, 0}$ is the array of the emitted fields in absence of input fields.


### 11.2.1 Two ports system

For a 2 -ports system, e.g. a screen dividing two regions $\Omega_{1}$ and $\Omega_{2}$, the scattering relation would look:

$$
\left\{\begin{array}{l}
\vec{f}_{1}  \tag{11.6}\\
\vec{f}_{2}
\end{array}\right\}_{t r}=\left[\begin{array}{ll}
\overline{\bar{S}}_{11} & \overline{\bar{S}}_{12} \\
\overline{\bar{S}}_{21} & \overline{\bar{S}}_{22}
\end{array}\right] \cdot\left\{\begin{array}{l}
\vec{f}_{1} \\
\vec{f}_{2}
\end{array}\right\}_{i n}+\left\{\begin{array}{l}
\vec{f}_{1,0} \\
\vec{f}_{2,0}
\end{array}\right\}_{t r}
$$

Normally the elements $S_{11}, S_{22}$ are the reflection coefficients, while $S_{21}, S_{12}$ are the transmission ones. Usually the scattering matrix $S$ should be assigned by the user, because it describes the properties of the systems to be realized.


Figure 11.1: Scattering for a generic linear 2-port system. The incident fields $f_{i n, 1}, f_{i n, 2}$ are the input and enter in the system, while the transmitted ones $f_{t r, 1}, f_{t r, 2}$ are the output and exit from the system.

### 11.2.2 Control Law

Even if you know exactly which sources $\mathbf{J}$ you need to generate a certain fields f, you have yet to find how to produce those sources. In many contexts, the sources are induced by the fields themselves. For example, an electric field can produce a current in material.

Now our task is to find a control law linking the sources $\mathbf{J}$ to the global fields f on the system.

$$
\begin{equation*}
\mathbf{J}=Y \mathbf{f}+\mathbf{J}_{\text {res }} \quad \text { control law } \tag{11.7}
\end{equation*}
$$

The matrix $Y$ is a "gain" matrix, but in an electromagnetic context it can be regarded as an admittance matrix. In general, it could be not square, depending on the number of sources.

The sources $\mathbf{J}$ can be interpreted as control variables, while the fields $\mathbf{f}$ as state variables. The term $\mathbf{J}_{\text {res }}$ is related to possible internal resonances for the system.

Also in this case, like for fields, the choice of the control variables $\mathbf{J}$ is quite arbitrary. Some options are:

$$
\begin{align*}
& \mathbf{J}=\left\{\begin{array}{c}
\vec{J}_{t} \\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\} \quad \text { currents and doublets }  \tag{11.8}\\
& \mathbf{J}=\left\{\begin{array}{l}
\vec{J}_{t 1} \\
\vec{J}_{t 2}
\end{array}\right\} \quad \text { currents on the screen's layers }  \tag{11.9}\\
& \mathbf{J}=\frac{1}{S}\left\{\begin{array}{c}
\vec{p}_{t} \\
\bar{m} \cdot \vec{n}_{21}
\end{array}\right\} \quad \text { electric and magnetic dipoles p.u. surface }  \tag{11.10}\\
& \text { etc. } \tag{11.11}
\end{align*}
$$

Preferably, we think to $\mathbf{J}$ as an array of currents flowing in the screen's layers.

### 11.2.3 Radiated fields

The transmitted fields $\mathbf{f}_{t r}$ are equal to the sum of the incident fields $\mathbf{f}_{i n}$ with those radiated or emitted by the system itself. More precisely, for a zero-thickness system holds:

$$
\begin{cases}\vec{f}_{t r, 1}=\vec{f}_{i n, 2}+\vec{f}_{i r r, 1} & \text { field exiting from port } 1  \tag{11.12}\\ \vec{f}_{t r, 2}=\vec{f}_{i n, 1}+\vec{f}_{i r r, 2} & \text { field exiting from port } 2\end{cases}
$$

In that case, if the system does not radiate $\left(\mathbf{f}_{i r r}=0\right)$, then the incident fields pass unaltered through it.

However, if the system has a certain extension, then the propagating fields can vary inside it. For example, the phase can change of a value $\theta$, so the transmitted fields would be:

$$
\begin{cases}\vec{f}_{t r, 1}=e^{i \theta} \vec{f}_{i n, 2}+\vec{f}_{i r r, 1} & \text { field exiting from port } 1  \tag{11.13}\\ \vec{f}_{t r, 2}=e^{i \theta} \vec{f}_{i n, 1}+\vec{f}_{i r r, 2} & \text { field exiting from port } 2\end{cases}
$$

In matrix form:

$$
\begin{align*}
& \mathbf{f}_{t r}=e^{i \theta}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \mathbf{f}_{i n}+\mathbf{f}_{i r r}  \tag{11.14}\\
& \mathbf{f}_{t r}=S_{0} \mathbf{f}_{i n}+\mathbf{f}_{i r r} \quad \text { general form } \tag{11.15}
\end{align*}
$$

Now we want to express the radiated fields $\mathbf{f}_{i r r}$ in function of the incident ones. By substituting in the scattering relation 11.5, we find:

$$
\begin{equation*}
\mathbf{f}_{i r r}=\left(S-S_{0}\right) \mathbf{f}_{i n}+\mathbf{f}_{t r, 0} \tag{11.16}
\end{equation*}
$$

For sake of simplicity, we place:

$$
\begin{equation*}
\mathbf{f}_{i r r, 0}=\mathbf{f}_{t r, 0} \quad \text { fixed radiation } \tag{11.17}
\end{equation*}
$$

So we can write:

$$
\begin{equation*}
\mathbf{f}_{i r r}=\left(S-S_{0}\right) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0} \tag{11.18}
\end{equation*}
$$

### 11.2.4 Radiation Law

The radiated fields $\mathbf{f}_{i r r}$ are exactly those generated by the sources J. In fact, lacking the sources, there are no radiated fields. We suppose that the radiation law is linear, so that:

$$
\begin{equation*}
\mathbf{f}_{i r r}=-Z_{i r r} \mathbf{J} \quad \text { radiation law } \tag{11.19}
\end{equation*}
$$

For example, for two parallel layers filled with currents $J_{1}$ and $J_{2}$ the radiated


Figure 11.2: Radiated fields for a generic linear 2-port system. $Z_{i r r}$ is the radiation matrix impedance, while $J$ is the source array.
electric waves are:

$$
\left\{\begin{array}{l}
E_{1}  \tag{11.20}\\
E_{2}
\end{array}\right\}_{i r r}=-\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & e^{i \theta_{0}} \\
e^{i \theta_{0}} & 1
\end{array}\right]\left\{\begin{array}{l}
J_{1} \\
J_{2}
\end{array}\right\}
$$

That kind of relation allows us to deduce the control law in the form $\mathbf{J}=Y \mathbf{f}$. Thus we have to express the global fields $\mathbf{f}$ in function of the incident ones $\mathbf{f}_{i n}$

### 11.2.5 Global fields

For a 2-port system, the global fields $\mathbf{f}$ on the two ports are equal to the sum of the incident and transmitted ones:

$$
\begin{cases}\vec{f}_{1}=\vec{f}_{i n, 1}+\vec{f}_{t r, 1} & \text { global field on port } 1  \tag{11.21}\\ \vec{f}_{2}=\vec{f}_{i n, 2}+\vec{f}_{t r, 2} & \text { global field on port } 2\end{cases}
$$

More generally, we can write:

$$
\begin{equation*}
\mathbf{f}=\mathbf{f}_{i n}+\mathbf{f}_{t r} \tag{11.22}
\end{equation*}
$$

Thanks to the scattering relation 11.5 , we can express the global fields in function of the incident ones:

$$
\begin{equation*}
\mathbf{f}=(I+S) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0} \tag{11.23}
\end{equation*}
$$

Now we have all the data we need to determine the control law.

### 11.3 Calculating the control law

Our task is to calculate the admittance matrix $Y$ in function of the scattering one $S$. Let us summarize the main useful equations:

$$
\begin{array}{rlr}
\mathbf{f}_{i r r} & =-Z_{i r r} \mathbf{J} & \text { radiation law } \\
\mathbf{f}_{i r r} & =\left(S-S_{0}\right) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0} & \text { radiated fields } \\
\mathbf{f} & =(I+S) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0} & \text { global fields } \\
\mathbf{J} & =Y \mathbf{f}+\mathbf{J}_{\text {res }} & \text { control law } \tag{11.27}
\end{array}
$$

We can start eliminating the global field by substitution:

$$
\begin{align*}
\mathbf{J} & =Y\left((I+S) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0}\right)+\mathbf{J}_{r e s}  \tag{11.28}\\
\mathbf{J} & =Y(I+S) \mathbf{f}_{i n}+\left(Y \mathbf{f}_{i r r, 0}+\mathbf{J}_{r e s}\right) \tag{11.29}
\end{align*}
$$

We define the matrix $Y_{\text {eff }}$ and the constant current $\mathbf{J}_{0}$ as:

$$
\begin{align*}
Y_{e f f} & =Y(I+S)  \tag{11.30}\\
\mathbf{J}_{0} & =Y \mathbf{f}_{i r r, 0}+\mathbf{J}_{r e s} \tag{11.31}
\end{align*}
$$

Thus it follows:

$$
\begin{equation*}
\mathbf{J}=Y_{e f f} \mathbf{f}_{i n}+\mathbf{J}_{0} \tag{11.32}
\end{equation*}
$$

The matrix $Y_{\text {eff }}$ can be interpreted as the screen's polarizability $\overline{\bar{\alpha}}$. For example, if we would choose as source variables the electric and magnetic dipoles $\vec{p}$ and $\overline{\bar{m}}$, we had:

$$
\begin{align*}
& \mathbf{J}=\left\{\begin{array}{c}
\vec{p}_{t} \\
\overline{\bar{m}} \cdot \vec{n}_{21}
\end{array}\right\} ; \quad \mathbf{f}=\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \overline{\bar{H}} \cdot \vec{n}_{21}
\end{array}\right\} \quad \Longrightarrow  \tag{11.33}\\
& \left\{\begin{array}{c}
\vec{p}_{t} \\
\vec{m}_{t}
\end{array}\right\}=\left[\begin{array}{ll}
\overline{\bar{\alpha}}_{E E} & \overline{\bar{\alpha}}_{E H} \\
\overline{\bar{\alpha}}_{H E} & \overline{\bar{\alpha}}_{H H}
\end{array}\right]\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}_{i n} \tag{11.34}
\end{align*}
$$

This approach was exploited by Ra'di et al. in a beautiful paper 48

### 11.3.1 Admittance matrix

The number of equations has been reduced to three:

$$
\left\{\begin{align*}
\mathbf{f}_{i r r} & =-Z_{i r r} \mathbf{J}  \tag{11.35}\\
\mathbf{f}_{i r r} & =\left(S-S_{0}\right) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0} \\
\mathbf{J} & =Y_{e f f} \mathbf{f}_{i n}+\mathbf{J}_{0}
\end{align*}\right.
$$

Now we can eliminate the sources $\mathbf{J}$ by substitution in the first eq., obtaining so:

$$
\left\{\begin{array}{l}
\mathbf{f}_{i r r}=-Z_{i r r}\left(Y_{e f f} \mathbf{f}_{i n}+\mathbf{J}_{0}\right)  \tag{11.36}\\
\mathbf{f}_{i r r}=\left(S-S_{0}\right) \mathbf{f}_{i n}+\mathbf{f}_{i r r, 0}
\end{array}\right.
$$

Both the equations must be valid for any incident and radiated field, so the final constraints to be satisfied are:

$$
\begin{align*}
& Z_{i r r} Y_{e f f}=-\left(S-S_{0}\right)  \tag{11.37}\\
& Z_{i r r} \mathbf{J}_{0}=-\mathbf{f}_{i r r, 0} \tag{11.38}
\end{align*}
$$

Since we are interested in finding $Y$, we explicitly write the effective admittance $Y_{e f f}$, getting so:

$$
\begin{align*}
& Z_{i r r} Y(I+S)=-\left(S-S_{0}\right)  \tag{11.39}\\
& Z_{i r r} Y=\left(S_{0}-S\right)(I+S)^{-1} \tag{11.40}
\end{align*}
$$

This last relation is exactly the one we was looking for. In fact it links the admittance matrix $Y$ to the scattering one. Since in general $\mathbf{f}$ and $\mathbf{J}$ can have a different number of elements, both $Z_{i r r}$ and $Y$ can be rectangular and so they could not be inverted.

On the contrary, if $Z_{i r r}$ and $Y$ are square and invertible, we can write:

$$
\begin{array}{ll}
Y_{i r r}=Z_{i r r}^{-1} ; \quad Z=Y^{-1} & \\
Y=Y_{i r r}\left(S_{0}-S\right)(I+S)^{-1} & \text { admittance matrix } \\
Z=\quad(I+S)\left(S_{0}-S\right)^{-1} Z_{i r r} & \text { impedance matrix } \tag{11.43}
\end{array}
$$

Therefore, the most general form still remains:

$$
\begin{equation*}
Z_{i r r} Y=\left(S_{0}-S\right)(I+S)^{-1} \tag{11.44}
\end{equation*}
$$

Let us notice that the product $Z_{i r r} Y$ is independent from the choice of the sources $\mathbf{J}$. That is particularly important if you want to change or transform your control variables.

### 11.3.2 Calculating the scattering matrix

In some cases the admittance matrix $Y$, or some other kind of constitutive relation, is assigned, while the scattering matrix $S$ is to be determined. For that purpose, just a single equation has to be handled:

$$
\begin{equation*}
Z_{i r r} Y(I+S)=-\left(S-S_{0}\right) \tag{11.45}
\end{equation*}
$$

After few calculi, we obtain:

$$
\begin{equation*}
S=\left(S_{0}-Z_{i r r} Y\right)\left(I+Z_{i r r} Y\right)^{-1} \tag{11.46}
\end{equation*}
$$

If you do not install any devices, i.e. the admittance matrix $Y$ is zero, then the scattering matrix is equal to the one for free propagation $S_{0}$ :

$$
\begin{equation*}
\text { if } Y=0 \quad \Longrightarrow \quad S=S_{0} \tag{11.47}
\end{equation*}
$$

### 11.4 Other constitutive relations

In some contexts neither the scattering matrix $S$ nor the admittance one $Y$ are assigned and the constitutive relations are known in some other form. Moreover, the variables involved in a same relation can be changed or transformed in other ones, without affecting the actual behaviour of the system.

Here we report just a pair of examples for different constitutive relations, showing the basic rules in order to translate one in another. This section is not mandatory to be read: it is simply a help for calculus and an aid against the confusion rising from different approaches and notations.

### 11.4.1 Trasmission matrix

For a 2-port system, the variables $\mathbf{f}_{1}$ on port 1 can be related to the ones $\mathbf{f}_{2}$ by mean of a transmission matrix $T_{21}$ such that:

$$
\begin{equation*}
\mathbf{f}_{2}=T_{21} \mathbf{f}_{1} \tag{11.48}
\end{equation*}
$$

For example, if we choose an array of electric amplitudes $\left\{E_{+} ; E_{-}\right\}$, then the relation for a transparent screen would be:

$$
\left\{\begin{array}{l}
E_{+}  \tag{11.49}\\
E_{-}
\end{array}\right\}_{2}=\left[\begin{array}{cc}
e^{i \theta_{0}} & 0 \\
0 & e^{-i \theta_{0}}
\end{array}\right]\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\}_{1}=
$$

where $\theta_{0}$ is the screen's adimensional thickness.


Figure 11.3: Convention for incident and transmitted electric wave amplitudes. The screen's scattering is related to the surface currents $\vec{J}_{t 1}, \vec{J}_{t 2}$.

It can be demonstrated that the scattering matrix $S$ can be calculated from the transmission one $T_{21}$. For example, the equations to be compared could be:

$$
\begin{align*}
& \left\{\begin{array}{l}
E_{1-} \\
E_{2+}
\end{array}\right\}=\left[\begin{array}{ll}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{array}\right] \cdot\left\{\begin{array}{l}
E_{1+} \\
E_{2-}
\end{array}\right\} \tag{11.50}
\end{align*} \text { scattering relation } \quad \text { (ransmission relation }
$$

Requiring that these two equations are valid for any incident and transmitted field, after some counts the scattering matrix results to be:

$$
\overline{\bar{S}}=\frac{1}{T_{22}}\left[\begin{array}{cc}
-T_{21} & 1  \tag{11.52}\\
\operatorname{det}(\overline{\bar{T}}) & T_{12}
\end{array}\right]
$$

On the contrary, if we know the scattering matrix, then the transmission one can be calculated as:

$$
\overline{\bar{T}}=\frac{1}{S_{12}}\left[\begin{array}{cc}
-\operatorname{det}(\overline{\bar{S}}) & S_{22}  \tag{11.53}\\
-S_{11} & 1
\end{array}\right]
$$

## Matrix case

If we consider field arrays containing vectors rather than scalar amplitudes, we can still calculate scattering and transmission matrices. However, this time the elements in $S$ and $T_{21}$ are no more scalar quantities but matrix ones:

$$
\begin{align*}
& \left\{\begin{array}{l}
\vec{E}_{1-} \\
\vec{E}_{2+}
\end{array}\right\}=\left[\begin{array}{ll}
\overline{\bar{S}}_{11} & \overline{\bar{S}}_{12} \\
\overline{\bar{S}}_{21} & \overline{\bar{S}}_{22}
\end{array}\right] \cdot\left\{\begin{array}{l}
\vec{E}_{1+} \\
\vec{E}_{2-}
\end{array}\right\}
\end{align*} \text { scattering relation } \quad \begin{array}{ll}
\left\{\begin{array}{ll}
\vec{E}_{2+} \\
\vec{E}_{2-}
\end{array}\right\}=\left[\begin{array}{ll}
\overline{\bar{T}}_{11} & \overline{\bar{T}}_{12} \\
\overline{\bar{T}}_{21} & \overline{\bar{T}}_{22}
\end{array}\right] \cdot\left\{\begin{array}{l}
\vec{E}_{1+} \\
\vec{E}_{1-}
\end{array}\right\} & \text { transmission relation } \tag{11.54}
\end{array}
$$

After some calculi, the scattering matrix turns out to be:

$$
\overline{\bar{S}}=\left[\begin{array}{cc}
-\overline{\bar{T}}_{22}^{-1} \overline{\bar{T}}_{21} & \overline{\bar{T}}_{22}^{-1}  \tag{11.56}\\
\overline{\bar{T}}_{11}-\overline{\bar{T}}_{12} \overline{\bar{T}}_{22}^{-1} \overline{\bar{T}}_{21} & \overline{\bar{T}}_{12} \overline{\bar{T}}_{22}^{-1}
\end{array}\right]
$$

Conversely, the transmission matrix can be expressed as:

$$
\overline{\bar{T}}=\left[\begin{array}{cc}
\overline{\bar{S}}_{21}-\overline{\bar{S}}_{12}^{-1} \overline{\bar{S}}_{22} \overline{\bar{S}}_{11} & \overline{\bar{S}}_{12}^{-1} \overline{\bar{S}}_{22}  \tag{11.57}\\
-\overline{\bar{S}}_{12}^{-1} \overline{\bar{S}}_{11} & \overline{\bar{S}}_{12}^{-1}
\end{array}\right]
$$

More compactly:

$$
\overline{\bar{T}}=\overline{\bar{S}}_{12}^{-1}\left[\begin{array}{cc}
\overline{\bar{S}}_{12} \overline{\bar{S}}_{21}-\overline{\bar{S}}_{22} \overline{\bar{S}}_{11} & \overline{\bar{S}}_{22}  \tag{11.58}\\
-\overline{\bar{S}}_{11} & \overline{\bar{I}}
\end{array}\right]
$$

Obviously, the "scalar" transformations (11.53), 11.52) are just particular cases for the more general matrix transformations, respectively (11.57), 11.58).

### 11.4.2 Discontinuity-average relation

As we have already seen in sec. 10.4.2, linking the field's discontinuities to their average values could be helpful, because of the BC's structure. Hence, in some cases the constitutive relation can be expressed as:

$$
\begin{equation*}
\Delta \mathbf{f}_{21}=C_{21}\langle\mathbf{f}\rangle \tag{11.59}
\end{equation*}
$$

where $\langle\mathbf{f}\rangle$ is the array containing the "average" fields on the system, while $\Delta \mathbf{f}_{21}$ contains the field differences between the two sides.

Let us consider, for example, a thin screen with relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$. It can be verified (see 10.4 ) that the constitutive relation can be written as:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{11.60}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=i \theta_{0}\left[\begin{array}{cc}
0 & \mu_{r} \\
\varepsilon_{r} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \quad \text { for } \theta_{0} \ll 1
$$

However, sometimes that form could be inconvenient, and so it is necessary to translate it in another way. For example, suppose we desire to determine the transmission matrix for that system. Making explicit the terms in eq. 11.59, we get:

$$
\begin{align*}
\Delta \mathbf{f}_{21} & =C_{21}\langle\mathbf{f}\rangle  \tag{11.61}\\
\mathbf{f}_{2}-\mathbf{f}_{1} & =C_{21} \frac{1}{2}\left(\mathbf{f}_{2}+\mathbf{f}_{1}\right)  \tag{11.62}\\
\left(I-\frac{1}{2} C_{21}\right) \mathbf{f}_{2} & =\left(I+\frac{1}{2} C_{21}\right) \mathbf{f}_{1}  \tag{11.63}\\
\mathbf{f}_{2} & =\left(I-\frac{1}{2} C_{21}\right)^{-1} \cdot\left(I+\frac{1}{2} C_{21}\right) \mathbf{f}_{1} \tag{11.64}
\end{align*}
$$

So we have linked the fields on side 1 to fields on side 2. Hence the transmission matrix $T_{21}$ turns out to be:

$$
\begin{equation*}
T_{21}=\left(I-\frac{1}{2} C_{21}\right)^{-1} \cdot\left(I+\frac{1}{2} C_{21}\right) \tag{11.65}
\end{equation*}
$$

Vice-versa, if we know $T_{21}$, the matrix $C_{21}$ can be calculated as:

$$
\begin{align*}
\left(I-\frac{1}{2} C_{21}\right) T_{21} & =\left(I+\frac{1}{2} C_{21}\right)  \tag{11.66}\\
T_{21}-I & =\frac{1}{2} C_{21} \cdot T_{21}+\frac{1}{2} C_{21}  \tag{11.67}\\
\frac{1}{2} C_{21} & =\left(T_{21}-I\right)\left(T_{21}+I\right)^{-1} \tag{11.68}
\end{align*}
$$

Finally:

$$
\begin{equation*}
C_{21}=2\left(T_{21}-I\right)\left(T_{21}+I\right)^{-1} \tag{11.69}
\end{equation*}
$$

Therefore, we can pass from an approach to the other.

### 11.4.3 Sources and fields relation

In sections 10.2 .310 .2 .4 we have extracted the BCs linking fields $E$ and $H$ to the sources $J$ and $D_{e}$. In particular, the sources can be calculated as:

$$
\left\{\begin{array}{c}
\vec{J}_{t}  \tag{11.70}\\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\Delta \vec{E}_{t, 21} \\
\eta_{0} \Delta \vec{H}_{t, 21}
\end{array}\right\}+i \theta_{0}\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\left\langle\vec{H}_{t}\right\rangle
\end{array}\right\}\right)
$$

More compactly:

$$
\mathbf{J}_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{11.71}\\
1 & 0
\end{array}\right] \Delta \mathbf{f}_{h, 21}+i \theta_{0}\left\langle\mathbf{f}_{h}\right\rangle\right)
$$

Those basic Boundary Conditions tell us which currents should be present on the screen, but they do not tell us anything about how to produce those sources. Let us try to express the field sources in function of the field average values $\left\langle\vec{E}_{t}\right\rangle$ and $\left\langle\vec{H}_{t}\right\rangle$ on the surface. Suppose we already know the discontinuity-average relation in the form:

$$
\begin{equation*}
\Delta \mathbf{f}_{h, 21}=C_{h}\left\langle\mathbf{f}_{h}\right\rangle \tag{11.72}
\end{equation*}
$$

More explicitly:

$$
\left\{\begin{array}{c}
\Delta \vec{E}_{t, 21}  \tag{11.73}\\
\eta_{0} \Delta \vec{H}_{t, 21}
\end{array}\right\}=\left[\begin{array}{cc}
\overline{\bar{C}}_{E E} & \overline{\bar{C}}_{E H} \\
\overline{\bar{C}}_{H E} & \overline{\bar{C}}_{H H}
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\left\langle\vec{H}_{t}\right\rangle
\end{array}\right\}
$$

By substituting eq. 11.72 in 11.71 , we can calculate the currents in function of the average fields:

$$
\mathbf{J}_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{11.74}\\
1 & 0
\end{array}\right] C_{h}+i \theta_{0} I\right)\left\langle\mathbf{f}_{h}\right\rangle
$$

For a short-hand notation, we can introduce an admittance matrix $Y_{h}$ such that:

$$
Y_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{11.75}\\
1 & 0
\end{array}\right] C_{h}+i \theta_{0} I\right)
$$

So we get:

$$
\begin{equation*}
\mathbf{J}_{h}=Y_{h}\left\langle\mathbf{f}_{h}\right\rangle \tag{11.76}
\end{equation*}
$$

This last equation can be interpreted either as a control law or as a constitutive relation for the surface material.

### 11.5 Standard Procedure

Here we show a possible algorithm for calculating the admittance matrix $Y$ starting from a constitutive relation. This is just an example for a procedure to be implemented in a computer program. We report it just for clarity.

Suppose the local constitutive relation for a screen is given in the form:

$$
\begin{equation*}
\Delta \mathbf{f}_{h, 21}=C_{h}\left\langle\mathbf{f}_{h}\right\rangle \tag{11.77}
\end{equation*}
$$

where:

$$
\mathbf{f}_{h}=\left\{\begin{array}{c}
\vec{E}_{t}  \tag{11.78}\\
\eta_{0} \vec{H}_{t}
\end{array}\right\}
$$

We desire to determine the admittances $Y$ to be installed to the screen, such that:

$$
\begin{equation*}
\mathbf{J}=Y \mathbf{E} \tag{11.79}
\end{equation*}
$$

More explicitly:

$$
\left\{\begin{array}{l}
\vec{J}_{1 t}  \tag{11.80}\\
\vec{J}_{2 t}
\end{array}\right\}=\left[\begin{array}{ll}
\overline{\bar{Y}}_{11} & \overline{\bar{Y}}_{12} \\
\overline{\bar{Y}}_{21} & \overline{\bar{Y}}_{22}
\end{array}\right]\left\{\begin{array}{l}
\vec{E}_{1 t} \\
\vec{E}_{2 t}
\end{array}\right\}
$$

### 11.5.1 Basic procedure

1. Transform the variables $\mathbf{f}_{h}$ in their equivalent $\mathbf{f}=\mathbf{E}$ :

$$
\begin{align*}
& \mathbf{E}=T_{E} \mathbf{f}_{h}  \tag{11.81}\\
& T_{E}=\frac{1}{2}\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right] \tag{11.82}
\end{align*}
$$

So the eq. 11.77 can be rephrased as:

$$
\begin{align*}
& \Delta \mathbf{E}_{21}=T_{E} \cdot C_{h} \cdot T_{E}^{-1}\langle\mathbf{E}\rangle  \tag{11.83}\\
& \Delta \mathbf{E}_{21}=C_{E}\langle\mathbf{E}\rangle \tag{11.84}
\end{align*}
$$

Shortly, the matrix $C_{E}$ is calculated as:

$$
\begin{equation*}
C_{E}=T_{E} \cdot C_{h} \cdot T_{E}^{-1} \tag{11.85}
\end{equation*}
$$

2. Calculate the transmission matrix $\overline{\bar{T}}$

$$
\begin{equation*}
\overline{\bar{T}}=\left(I-\frac{1}{2} C_{E}\right)^{-1} \cdot\left(I+\frac{1}{2} C_{E}\right) \tag{11.86}
\end{equation*}
$$

3. Calculate the scattering matrix $\overline{\bar{S}}$.

$$
\begin{align*}
& \overline{\bar{S}}=\frac{1}{T_{22}}\left[\begin{array}{cc}
-T_{21} & 1 \\
\operatorname{det}(\overline{\bar{T}}) & T_{12}
\end{array}\right]  \tag{11.87}\\
& \text { (scalar case) }  \tag{11.88}\\
& \overline{\bar{S}}=\left[\begin{array}{cc}
-\overline{\bar{T}}_{22}^{-1} \overline{\bar{T}}_{21} & \overline{\bar{T}}_{22}^{-1} \\
\overline{\bar{T}}_{11}-\overline{\bar{T}}_{12} \overline{\bar{T}}_{22}^{-1} \overline{\bar{T}}_{21} & \overline{\bar{T}}_{12} \overline{\bar{T}}_{22}^{-1}
\end{array}\right]
\end{align*}
$$

4. Calculate the admittance matrix $Y$ such that:

$$
\begin{equation*}
Y=-Z_{i r r}^{-1}\left(S-S_{0}\right)(I+S)^{-1} \tag{11.89}
\end{equation*}
$$

5. Calculate the impedance matrix $Z$, if needed:

$$
\begin{equation*}
Z=Y^{-1} \tag{11.90}
\end{equation*}
$$

This algorithm can be reversed: in fact, if you know the impedance matrix you can go back to the constitutive relation. Obviously that is allowed only if matrices are not singular.

### 11.5.2 Scattering for transparent screen

Now let us see an example of how to apply the calculus procedure just described.
Let us consider a perfectly transparent screen, which does not alter the propagating fields. From section 10.3.1 we know that the "constitutive relation" for free propagation can be written as:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{11.91}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=i \theta_{0}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \quad \text { for } \theta_{0} \ll 1
$$

Now we desire to determine the scattering matrix $S_{0}$ and the impedance matrix $Z$ within the hypothesis of thin screen $\left(\theta_{0} \ll 1\right)$. So we apply the algorithm:

1. Transform the constitutive relation for electric amplitudes $\mathbf{E}$ :

$$
\begin{align*}
& C_{E}=T_{E} \cdot C_{h} \cdot T_{E}^{-1}  \tag{11.92}\\
& C_{E}=\frac{1}{2}\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right] \cdot i \theta_{0}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \cdot\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]  \tag{11.93}\\
& C_{E}=i \theta_{0}\left[\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right] \tag{11.94}
\end{align*}
$$

2. Calculate the transmission matrix $\overline{\bar{T}}_{0}$ :

$$
\begin{align*}
& \overline{\bar{T}}_{0}=\left(I-\frac{1}{2} C_{E}\right)^{-1} \cdot\left(I+\frac{1}{2} C_{E}\right)  \tag{11.95}\\
& \overline{\bar{T}}_{0}=\left[\begin{array}{cc}
\frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} & 0 \\
0 & \frac{1-\frac{1}{2}\left(i \theta_{0}\right)}{1+\frac{1}{2}\left(i \theta_{0}\right)}
\end{array}\right]  \tag{11.96}\\
& \overline{\bar{T}}_{0}=\left[\begin{array}{cc}
b\left(\theta_{0}\right) & 0 \\
0 & b\left(-\theta_{0}\right)
\end{array}\right] \quad \text { with } \quad b\left(\theta_{0}\right)=\frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} \approx e^{i \theta_{0}} \tag{11.97}
\end{align*}
$$

3. Calculate the scattering matrix $\overline{\bar{S}}_{0}$ :

$$
\begin{align*}
& \overline{\bar{S}}_{0}=\frac{1}{T_{22}}\left[\begin{array}{cc}
-T_{21} & 1 \\
\operatorname{det}(\overline{\bar{T}}) & T_{12}
\end{array}\right]  \tag{11.98}\\
& \overline{\bar{S}}_{0}=b\left(\theta_{0}\right)\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \approx e^{i \theta_{0}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \tag{11.99}
\end{align*}
$$

As expected, the reflection coefficients $S_{11}$ and $S_{22}$ result to be null, while the transmission ones $S_{12}$ and $S_{21}$ are equal and take into account the phase shift across the system.
4. Calculate the admittance matrix $Y$ such that:

$$
\begin{align*}
& Y=-Z_{i r r}^{-1}\left(S-S_{0}\right)(I+S)^{-1}  \tag{11.100}\\
& Y=-Z_{i r r}^{-1}\left(S_{0}-S_{0}\right)\left(I+S_{0}\right)^{-1}  \tag{11.101}\\
& Y=0 \tag{11.102}
\end{align*}
$$

The admittance matrix $Y$ turns out to be zero, therefore no devices have to be installed on the screen, which can be effectively regarded as empty space.

### 11.6 Summary of basic matrices for thin screen

Here we summarize the fundamental matrices necessary to characterize a thin screen. In all the cases we consider the field array made by electric amplitudes. Moreover, since $\theta_{0} \ll 1$, we approximate the propagating phasor $e^{i \theta_{0}}$ as:

$$
\begin{equation*}
b\left(\theta_{0}\right)=\frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} \approx e^{i \theta_{0}} \tag{11.103}
\end{equation*}
$$

If you substitute $b\left(\theta_{0}\right)$ with $e^{i \theta_{0}}$ you can retrieve the exact solution. It should be noticed that the approximation $b\left(\theta_{0}\right)$ is particularly useful if you want to determine the impedance to be installed on the screen. On the contrary, using the exact solution with $e^{i \theta_{0}}$ could bring to "retarded" impedances or to complicated phase shifters.

That concept will appear clearer in chapter 12 where we explicitly calculate the impedances to be mounted on the screen.

### 11.6.1 Radiation impedance matrix

The radiation law for a 2-layer screen can be written as:

$$
\begin{align*}
\mathbf{E}_{i r r} & =-Z_{i r r} \mathbf{J}  \tag{11.104}\\
\left\{\begin{array}{l}
E_{1-} \\
E_{2+}
\end{array}\right\}_{i r r} & =-\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & \frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} \\
\frac{1+\frac{1}{2}\left(i \theta_{0}\right)}{1-\frac{1}{2}\left(i \theta_{0}\right)} & 1
\end{array}\right]\left\{\begin{array}{l}
J_{1} \\
J_{2}
\end{array}\right\} \tag{11.105}
\end{align*}
$$

Thus the radiation impedance matrix $Z_{i r r}$ is:

$$
Z_{i r r}=\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & b\left(\theta_{0}\right)  \tag{11.106}\\
b\left(\theta_{0}\right) & 1
\end{array}\right]=\frac{1}{2} \eta_{0}\left(I+S_{0}\right)
$$

See also section 10.3 .4 for numerical details.

### 11.6.2 Scattering matrix for empty space

The scattering matrix $S_{0}$ associated to an empty system can be expressed as:

$$
\overline{\bar{S}}_{0}=b\left(\theta_{0}\right)\left[\begin{array}{ll}
0 & 1  \tag{11.107}\\
1 & 0
\end{array}\right]
$$

### 11.6.3 Transmission matrix for empty space

The transmission matrix $T_{0}$ associated to an empty system can be expressed as:

$$
\overline{\bar{T}}_{0}=\left[\begin{array}{cc}
b\left(\theta_{0}\right) & 0  \tag{11.108}\\
0 & b\left(-\theta_{0}\right)
\end{array}\right]
$$

### 11.6.4 Discontinuity-average relations for empty space

The constitutive relation for an empty, thin screen can be written as:

$$
\begin{equation*}
\Delta \mathbf{E}_{21}=C_{E}\langle\mathbf{E}\rangle \tag{11.109}
\end{equation*}
$$

More explicitly:

$$
\left\{\begin{array}{l}
\Delta E_{+}  \tag{11.110}\\
\Delta E_{-}
\end{array}\right\}_{21}=i \theta_{0}\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left\{\begin{array}{l}
\langle E\rangle_{+} \\
\langle E\rangle_{-}
\end{array}\right\}
$$

If we choose a different field array, for example $\mathbf{f}_{h}=\left\{E ; \eta_{0} H\right\}$, then the same relation can be rephrased as:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{11.111}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=i \theta_{0}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}
$$

This last two relations can be regarded as a discretization for the differential equations:

$$
\begin{align*}
\frac{\partial}{\partial \theta_{0}}\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\} & =i\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left\{\begin{array}{l}
E_{+} \\
E_{-}
\end{array}\right\}  \tag{11.112}\\
\frac{\partial}{\partial \theta_{0}}\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} & =i\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
E \\
\eta_{0} H
\end{array}\right\} \tag{11.113}
\end{align*}
$$

For numerical details, see also section 10.3 .

### 11.7 Linear Bianisotropic material

In the previous pages we have seen that the properties for a system are determined once the scattering matrix $S$ is assigned, or otherwise the admittance matrix $Y$ is known. In particular, since we are dealing with electromagnetic waves, it could be convenient to describe the system's properties in terms of permittivity $\varepsilon$ and permeability $\mu$.

Without loss of generality, let us suppose a meta-surface can be modelled as made of a linear bianisotropic material. Adopting the standard 3D notation, the constitutive relation for such a material will look so:

$$
\left\{\begin{array}{l}
\vec{D}=\overline{\bar{\varepsilon}} \vec{E}+\overline{\bar{\xi}} \vec{H}  \tag{11.114}\\
\vec{B}=\overline{\bar{\zeta}} \vec{E}+\overline{\bar{\mu}} \vec{H}
\end{array}\right.
$$

where $\overline{\bar{\varepsilon}}$ and $\overline{\bar{\mu}}$ are the permittivity and permeability tensors respectively, while $\overline{\bar{\xi}}$ and $\overline{\bar{\zeta}}$ are the coupling terms. Till now we have described the field's sources using currents $\vec{J}_{t}$ and doublets $\overline{\bar{D}}_{e}$, but now it is more convenient to express the constitutive relations in terms of polarization $\vec{P}$ and magnetization $\vec{M}$. From the definition of fields $\vec{D}$ and $\vec{H}$, we know that:

$$
\left\{\begin{array}{l}
\vec{D}=\varepsilon_{0} \vec{E}+\vec{P}  \tag{11.115}\\
\vec{B}=\mu_{0}(\vec{H}+\vec{M})
\end{array}\right.
$$

In order to deal with homogeneous physical dimensions, we convert all the fields so that they can be summed with an electric one:

$$
\left\{\begin{align*}
\frac{1}{\varepsilon_{0}} \vec{D} & =\vec{E}+\frac{1}{\varepsilon_{0}} \vec{P}  \tag{11.116}\\
c_{0} \vec{B} & =\eta_{0}(\vec{H}+\vec{M})
\end{align*}\right.
$$

Besides we have seen that the fundamental Boundary Conditions involve just the tangent components for $\vec{E}$ and $\vec{H}$, so we shall limit our analysis to them. Shortly, we can write:

$$
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{D}_{t}  \tag{11.117}\\
c_{0} \vec{B}_{t}
\end{array}\right\}=\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}+\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{P}_{t} \\
\eta_{0} \vec{M}_{t}
\end{array}\right\}
$$

As usual, the tangent components for the magnetic fields are calculated as:

$$
\left\{\begin{array}{l}
\vec{B}_{t}=\overline{\bar{B}} \cdot \vec{n}_{21}=\vec{B} \times \vec{n}_{21}=-\vec{n}_{21} \times \vec{B}  \tag{11.118}\\
\vec{H}_{t}=\overline{\bar{H}} \cdot \vec{n}_{21}=\vec{H} \times \vec{n}_{21}=-\vec{n}_{21} \times \vec{H} \\
\vec{M}_{t}=\overline{\bar{M}} \cdot \vec{n}_{21}=\vec{M} \times \vec{n}_{21}=-\vec{n}_{21} \times \vec{M}
\end{array}\right.
$$

Now we can introduce the adimensional susceptibility matrix $\chi$, relating the electric $E$ and magnetic $H$ fields to $P$ and $M$ :

$$
\begin{align*}
& \left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{P}_{t} \\
\eta_{0} \vec{M}_{t}
\end{array}\right\}=\left[\begin{array}{ll}
\overline{\bar{\chi}}_{E E} & \overline{\bar{\chi}}_{E H} \\
\overline{\bar{\chi}}_{H E} & \overline{\bar{\chi}}_{H H}
\end{array}\right] \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\} \Longrightarrow  \tag{11.119}\\
& \left\{\begin{array}{c}
\frac{1}{\varepsilon_{0} \vec{D}_{t}} \\
c_{0} \vec{B}_{t}
\end{array}\right\}=\left(\left[\begin{array}{cc}
\overline{\bar{I}} & \overline{\overline{0}} \\
\overline{\overline{0}} & \overline{\bar{I}}
\end{array}\right]+\left[\begin{array}{ll}
\overline{\bar{\chi}}_{E E} & \overline{\bar{\chi}}_{E H} \\
\overline{\bar{\chi}}_{H E} & \overline{\bar{\chi}}_{H H}
\end{array}\right]\right) \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}  \tag{11.120}\\
& \left\{\begin{array}{l}
\frac{1}{\varepsilon_{0}} \vec{D}_{t} \\
c_{0} \vec{B}_{t}
\end{array}\right\}=[I+\chi] \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\} \tag{11.121}
\end{align*}
$$

The permittivity, permeability etc. can be expressed in function of the various susceptibilities $\chi$ :

$$
\begin{gather*}
\left\{\begin{array}{l}
\vec{D}_{t} \\
\vec{B}_{t}
\end{array}\right\}=\left[\begin{array}{ll}
\overline{\bar{\varepsilon}}_{t} & \overline{\bar{\xi}}_{t} \\
\overline{\bar{\zeta}}_{t} & \overline{\bar{\mu}}_{t}
\end{array}\right] \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\} \Longrightarrow  \tag{11.122}\\
\left\{\begin{array} { l } 
{ \overline { \overline { \varepsilon } } _ { t } = \varepsilon _ { 0 } ( \overline { \overline { I } } + \overline { \overline { \chi } } _ { E E } ) } \\
{ \overline { \overline { \zeta } } _ { t } = \frac { 1 } { c _ { 0 } } \overline { \overline { \chi } } _ { H E } }
\end{array} \left\{\begin{array}{l}
\overline{\bar{\xi}}_{t}=\frac{1}{c_{0}} \overline{\bar{\chi}}_{E H} \\
\overline{\bar{\mu}}_{t}=\mu_{0}\left(\overline{\bar{I}}+\overline{\bar{\chi}}_{H H}\right)
\end{array}\right.\right. \tag{11.123}
\end{gather*}
$$

Now we desire to determine the susceptibility matrix $\chi$ for our screen. Suppose we have already calculated the constitutive relation linking the source $\mathbf{J}_{h}$ to the average fields $\mathbf{f}_{h}$ through an admittance matrix $Y_{h}$ (for details, see also sec. 11.4.3):

$$
\begin{equation*}
\mathbf{J}_{h}=Y_{h}\left\langle\mathbf{f}_{h}\right\rangle \tag{11.124}
\end{equation*}
$$

We remind that the arrays $\mathbf{J}_{h}$ and $\mathbf{f}_{h}$ are:

$$
\mathbf{J}_{h}=\left\{\begin{array}{c}
\vec{J}_{t}  \tag{11.125}\\
\frac{s}{c_{0}} \\
\bar{D}_{e} \cdot \vec{n}_{21}
\end{array}\right\} \quad \mathbf{f}_{h}=\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}
$$

Now, in order to express the susceptibilities $\chi$ in function of the admittances $Y_{h}$ we have to calculate $\vec{P}_{t}$ and $\vec{M}_{t}$ in function of sources $\vec{J}_{t}, \overline{\bar{D}}_{e}$.

For this time, we shall calculate the polarization $P$ and magnetization $M$ using their widely-adopted classic definition, though in some cases it can bring to results not fully consistent with Maxwell's equations [20].

### 11.7.1 Polarization per unit of surface

Given a surface element whose area is $S$ and width is $\Delta x$, the average polarization $\vec{P}$ can be calculated as:

$$
\begin{equation*}
\vec{P}=\frac{1}{S \Delta x} \vec{p} \quad \text { classic definition } \tag{11.126}
\end{equation*}
$$

where $S=\Delta y \Delta z$ and $\vec{p}=Q_{e} \Delta \vec{y}$ is the electric dipole.
Naming $I_{y}$ the current flowing in the $y$ direction, tangent to the surface, because of charge conservation we get:

$$
\begin{align*}
\frac{d Q_{e}}{d t} & =I_{y}=\left(J_{y} \Delta z\right) \quad \Longrightarrow  \tag{11.127}\\
\frac{d \vec{p}_{t}}{d t} & =\Delta y\left(\vec{J}_{t} \Delta z\right)=S \vec{J}_{t} \tag{11.128}
\end{align*}
$$

In the Laplace's domain, this last relation will look:

$$
\begin{equation*}
s \vec{p}_{t}=S \vec{J}_{t} \tag{11.129}
\end{equation*}
$$

Now we can express the polarization $P$ in function of the surface current $J_{t}$ :

$$
\begin{equation*}
\vec{P}_{t}=\frac{1}{S \Delta x} \vec{p}_{t}=\frac{1}{S \Delta x}\left(\frac{1}{s} S \vec{J}_{t}\right)=\frac{1}{s \Delta x} \vec{J}_{t} \tag{11.130}
\end{equation*}
$$

Finally:

$$
\begin{equation*}
\frac{1}{\varepsilon_{0}} \vec{P}_{t}=\frac{1}{s \varepsilon_{0} \Delta x} \vec{J}_{t} \tag{11.131}
\end{equation*}
$$

Be aware that this relation is valid just if the classic definition for $P$ is adopted.

(c) Calculating polarization for a surface element.
(d) Calculating magnetization for a surface element.

Figure 11.4: Scheme for calculating the polarization $P$ and magnetization $M$ within their classic definition as "dipoles per unit of volume"

### 11.7.2 Magnetization per unit of surface

Still referring to the surface element whose sizes are $\Delta x, \Delta y, \Delta z$, we can calculate the average magnetization $\overline{\bar{M}}$ as:

$$
\begin{equation*}
\overline{\bar{M}}=\frac{1}{S \Delta x} \overline{\bar{m}} \quad \text { classic definition } \tag{11.132}
\end{equation*}
$$

where $\overline{\bar{m}}$ is the net magnetic moment, which can be expressed in function of the anti-symmetric current flowing inside the surface element:

$$
\begin{align*}
\overline{\bar{m}} & =\overline{\bar{S}}_{x y} I_{s y m m}^{a n t i}  \tag{11.133}\\
\overline{\bar{m}} \cdot \vec{n}_{21} & =S_{x y}\left(\Delta z \frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right)\right)  \tag{11.134}\\
\overline{\bar{m}} \cdot \vec{n}_{21} & =(\Delta x \Delta y \Delta z) \frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \tag{11.135}
\end{align*}
$$

Hence the tangent component $\vec{M}_{t}$ can be calculated as:

$$
\begin{align*}
\vec{M}_{t} & =\frac{1}{S \Delta x} \overline{\bar{m}} \cdot \vec{n}_{21}  \tag{11.136}\\
\vec{M}_{t} & =\frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \tag{11.137}
\end{align*}
$$

We know that the doublet $\overline{\bar{D}}_{e}$ is related to the anti-symmetric current component, in fact:

$$
\begin{equation*}
\overline{\bar{D}}_{e} \cdot \vec{n}_{21}=\frac{1}{2}\left(\vec{J}_{t 2}-\vec{J}_{t 1}\right) \Delta x \tag{11.138}
\end{equation*}
$$

So we find:

$$
\begin{gather*}
\overline{\bar{m}} \cdot \vec{n}_{21}=S \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{11.139}\\
\vec{M}_{t}=\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21} \tag{11.140}
\end{gather*}
$$

Finally, after few steps we get:

$$
\begin{equation*}
\eta_{0} \vec{M}_{t}=\frac{1}{s \varepsilon_{0} \Delta x}\left(\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}\right) \tag{11.141}
\end{equation*}
$$

Be aware that this relation is valid just if the classic definition for $M$ is adopted.

### 11.7.3 Calculating the susceptibility

Holding eq.s 11.131, 11.141, the polarization $\vec{P}$ and magnetization $\vec{M}$ can be compactly expressed as:

$$
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{P}_{t}  \tag{11.142}\\
\eta_{0} \vec{M}_{t}
\end{array}\right\}=\frac{1}{s \varepsilon_{0} \Delta x}\left\{\begin{array}{c}
\vec{J}_{t} \\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\} \quad \text { from classic definition }
$$

We remind that:

$$
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{P}_{t}  \tag{11.143}\\
\eta_{0} \vec{M}_{t}
\end{array}\right\}=\chi\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\} ; \quad\left\{\begin{array}{c}
\vec{J}_{t} \\
\frac{s}{c_{0}} \\
\bar{D}_{e} \cdot \vec{n}_{21}
\end{array}\right\}=Y_{h}\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}
$$

Thus we can calculate the susceptibility matrix $\chi$ on the basis of the admittance one $Y_{h}$ :

$$
\begin{equation*}
\chi=\frac{1}{s \varepsilon_{0} \Delta x} Y_{h} \tag{11.144}
\end{equation*}
$$

The same relation can be rephrased adopting the adimensional thickness $\theta_{0}$, in fact:

$$
\begin{equation*}
s \varepsilon_{0} \Delta x=-i\left(k_{0} \Delta x\right) \frac{1}{\eta_{0}}=-\frac{i \theta_{0}}{\eta_{0}} \tag{11.145}
\end{equation*}
$$

So it follows:

$$
\begin{equation*}
\chi=\frac{1}{-i \theta_{0}} \eta_{0} Y_{h} \tag{11.146}
\end{equation*}
$$

Therefore, within the thin screen hypothesis $\left(\theta_{0} \ll 1\right)$, the required susceptibility $\chi$ could turn out to be numerically very high if compared to $\eta_{0} Y_{h}$. Finally, the permittivity, permeability etc. can be expressed in function of the admittance matrix $Y_{h}$ :

$$
Y_{h}=\left[\begin{array}{cc}
\overline{\bar{Y}}_{E E} & \overline{\bar{Y}}_{E H}  \tag{11.147}\\
\overline{\bar{Y}}_{H E} & \overline{\bar{Y}}_{H H}
\end{array}\right]
$$

$$
\left\{\begin{array} { l } 
{ \overline { \overline { \varepsilon } } _ { t } = \varepsilon _ { 0 } ( \overline { \overline { I } } + \frac { 1 } { s \varepsilon _ { 0 } \Delta x } \overline { \overline { Y } } _ { E E } ) }  \tag{11.148}\\
{ \overline { \overline { \zeta } } _ { t } = \frac { 1 } { c _ { 0 } } \frac { 1 } { s \varepsilon _ { 0 } \Delta x } \overline { \overline { Y } } _ { H E } }
\end{array} \quad \left\{\begin{array}{l}
\overline{\bar{\xi}}_{t}=\frac{1}{c_{0}} \overline{\bar{Y}}_{E H} \\
\overline{\bar{\mu}}_{t}=\mu_{0}\left(\overline{\bar{I}}+\frac{1}{s \varepsilon_{0} \Delta x} \overline{\bar{Y}}_{H H}\right)
\end{array}\right.\right.
$$

## Alternative calculation

If desired, the susceptibility $\chi$ can be calculated in function of the matrix $C_{h}$, which relates the average fields to their discontinuities. From section 11.4.3 we know that:

$$
Y_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{11.149}\\
1 & 0
\end{array}\right] C_{h}+i \theta_{0} I\right)
$$

Hence, holding 11.146, after few calculi the susceptibility matrix $\chi$ results:

$$
\chi=-I+\frac{1}{i \theta_{0}}\left[\begin{array}{ll}
0 & 1  \tag{11.150}\\
1 & 0
\end{array}\right] C_{h}
$$

Therefore, with a last substitution in 11.121 we obtain:

$$
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{D}_{t}  \tag{11.151}\\
c_{0} \vec{B}_{t}
\end{array}\right\}=\frac{1}{i \theta_{0}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] C_{h} \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}
$$

More explicitly, the permittivity, permeability, the coupling terms etc. can be calculated as:

$$
\left\{\begin{array} { l } 
{ \overline { \overline { \varepsilon } } _ { t } = \frac { 1 } { i \theta _ { 0 } } \varepsilon _ { 0 } \overline { \overline { C } } _ { H E } }  \tag{11.152}\\
{ \overline { \overline { \zeta } } _ { t } = \frac { 1 } { i \theta _ { 0 } } \frac { 1 } { c _ { 0 } } \overline { \overline { C } } _ { H H } }
\end{array} \quad \left\{\begin{array}{l}
\overline{\bar{\xi}}_{t}=\frac{1}{i \theta_{0}} \frac{1}{c_{0}} \overline{\bar{C}}_{E E} \\
\overline{\bar{\mu}}_{t}=\frac{1}{i \theta_{0}} \mu_{0} \overline{\bar{C}}_{E H}
\end{array}\right.\right.
$$

Where:

$$
C_{h}=\left[\begin{array}{ll}
\overline{\bar{C}}_{E E} & \overline{\bar{C}}_{E H}  \tag{11.153}\\
\overline{\bar{C}}_{H E} & \overline{\bar{C}}_{H H}
\end{array}\right]
$$

This last formulation is particularly useful if the constitutive relation is given in the discontinuity-average form.

### 11.7.4 Linear isotropic material

Here we report a simple example in order to check the consistency of our method.
Suppose we desire to describe a linear and isotropic material with assigned relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$. From section 10.4.3. we know the discontinuity-average relation for the electric and magnetic fields, that is:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{11.154}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=i \theta_{0}\left[\begin{array}{cc}
0 & \mu_{r} \\
\varepsilon_{r} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \quad \text { for } \theta_{0} \ll 1
$$

So the matrix $C_{h}$ is simply:

$$
C_{h}=i \theta_{0}\left[\begin{array}{cc}
0 & \mu_{r}  \tag{11.155}\\
\varepsilon_{r} & 0
\end{array}\right]
$$

Thanks to eq. 11.75, we can directly calculate the admittance matrix:

$$
\begin{align*}
Y_{h} & =\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] C_{h}+i \theta_{0} I\right) \quad \Longrightarrow  \tag{11.156}\\
Y_{h} & =-\frac{1}{\eta_{0}}\left(i \theta_{0}\right)\left[\begin{array}{cc}
\varepsilon_{r}-1 & 0 \\
0 & \mu_{r}-1
\end{array}\right] \tag{11.157}
\end{align*}
$$

More explicitly, the relation between the sources $\mathbf{J}$ and the fields $\mathbf{f}_{h}$ is:

$$
\left\{\begin{array}{c}
\vec{J}_{t}  \tag{11.158}\\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\}=-\frac{1}{\eta_{0}} i \theta_{0}\left[\begin{array}{cc}
\varepsilon_{r}-1 & 0 \\
0 & \mu_{r}-1
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{21}
\end{array}\right\}
$$

Thanks to eq. 11.146, the susceptibility matrix $\chi$ turns out to be:

$$
\begin{align*}
\chi & =\frac{1}{-i \theta_{0}} \eta_{0} Y_{h}  \tag{11.159}\\
\chi & =\left[\begin{array}{cc}
\varepsilon_{r}-1 & 0 \\
0 & \mu_{r}-1
\end{array}\right] \tag{11.160}
\end{align*}
$$

Holding eq. 11.121), finally we obtain the usual constitutive relation expected for a linear isotropic material:

$$
\begin{align*}
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{D}_{t} \\
c_{0} \vec{B}_{t}
\end{array}\right\} & =\left[\begin{array}{cc}
\varepsilon_{r} & 0 \\
0 & \mu_{r}
\end{array}\right] \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}  \tag{11.161}\\
\left\{\begin{array}{l}
\vec{D}_{t} \\
\vec{B}_{t}
\end{array}\right\} & =\left[\begin{array}{cc}
\varepsilon & 0 \\
0 & \mu
\end{array}\right] \cdot\left\{\begin{array}{l}
\vec{E}_{t} \\
\vec{H}_{t}
\end{array}\right\} \tag{11.162}
\end{align*}
$$

## Chapter 12

## Circuit screen

There's a powerful, obedient, swift, and effortless force that can be bent to any use and which reigns supreme aboard my vessel. It does everything. It lights me, it warms me, it's the soul of my mechanical equipment. This force is electricity.

Captain Nemo, in J.Verne's Twenty
thousand leagues under the sea, 1870

This chapter is dedicated to the development of a circuit model for a thin screen or meta-surface, and it is firmly founded on the Boundary Conditions previously derived for the EM fields. That is a fundamental point, since many Metamaterials are designed or modeled by mean of circuits. The problem is that in Circuit Theory formally the radiation is not allowed. In fact, the electric field $\vec{E}$ is usually required to be conservative in order to define a potential $V$, but that condition is not satisfied for a Transverse EM wave 49, 46].

So, if we want to continue to project MTM through circuits, we need to verify the consistency of our results with Maxwell's Equations. In other words, we must be careful to avoid contradictions between Circuit and Antenna's theories.

### 12.1 Basic circuits and constitutive relations

Usually the constitutive relation for circuits consists in a link among voltages $V$ and currents $I$. If the circuit is linear, the most general relation among $V$ and $I$ can be written as:

$$
\begin{equation*}
\overline{\bar{A}}_{V}\{V\}+\overline{\bar{A}}_{I}\{I\}+\{b\}_{0}=0 \tag{12.1}
\end{equation*}
$$

where $\overline{\bar{A}}_{V}, \overline{\bar{A}}_{I}$ and $\{b\}_{0}$ are generic coefficients. That relation can be declined in may ways, depending on the need.

Hereafter we report just a summary of some well-known, basic circuits, together with their mathematical constitutive relation. In all the cases, we shall consider a 2-port linear system, thus two voltages $V_{1}, V_{2}$ and two currents $I_{1}, I_{2}$ which have to be linked together. Anyway, the reader is supposed to be already familiar with those concepts.

### 12.1.1 Thevenin equivalent

In the Thevenin approach, voltages $V_{1}, V_{2}$ depend on the currents $I_{1}, I_{2}$ flowing inward the circuit. The most general expression is:

$$
\begin{equation*}
\{V\}=\overline{\bar{Z}}\{I\}+\left\{V_{0}\right\} \tag{12.2}
\end{equation*}
$$

For a 2-port system:

$$
\left\{\begin{array}{l}
V_{1}  \tag{12.3}\\
V_{2}
\end{array}\right\}=\left[\begin{array}{ll}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{array}\right]\left\{\begin{array}{c}
I_{1} \\
I_{2}
\end{array}\right\}+\left\{\begin{array}{l}
V_{1,0} \\
V_{2,0}
\end{array}\right\}
$$

The equivalent circuit is depicted in fig 12.1. The impedances $Z_{1}, Z_{2}$ are often


Figure 12.1: General Thevenin equivalent for a 2-port circuit system
called "series" impedances. $Z_{S}$ is the shunted one, while $Z_{G}$ is the gyration impedance, associated to the dependent voltage generators. It can be demonstrated that it holds:

$$
\left\{\begin{array}{l}
V_{1}  \tag{12.4}\\
V_{2}
\end{array}\right\}=\left[\begin{array}{rr}
Z_{1}+Z_{S} & -Z_{G}+Z_{S} \\
Z_{G}+Z_{S} & Z_{2}+Z_{S}
\end{array}\right]\left\{\begin{array}{l}
I_{1} \\
I_{2}
\end{array}\right\}+\left\{\begin{array}{l}
V_{1,0} \\
V_{2,0}
\end{array}\right\}
$$

More explicitly, once the matrix $\overline{\bar{Z}}$ is known the impedances $Z_{1}, Z_{2}, Z_{S}, Z_{G}$ can be calculated as:

$$
\left\{\begin{array} { l } 
{ Z _ { S } = \frac { 1 } { 2 } ( Z _ { 2 1 } + Z _ { 1 2 } ) }  \tag{12.5}\\
{ Z _ { G } = \frac { 1 } { 2 } ( Z _ { 2 1 } - Z _ { 1 2 } ) }
\end{array} \quad \left\{\begin{array}{l}
Z_{1}=Z_{11}-\frac{1}{2}\left(Z_{21}+Z_{12}\right)=Z_{11}-Z_{S} \\
Z_{2}=Z_{22}-\frac{1}{2}\left(Z_{21}+Z_{12}\right)=Z_{22}-Z_{S}
\end{array}\right.\right.
$$

### 12.1.2 Norton equivalent

In the Norton approach, currents $I_{1}, I_{2}$ depend on voltages $V_{1}, V_{2}$. The most general expression is:

$$
\begin{equation*}
\{I\}=\overline{\bar{Y}}\{V\}+\left\{I_{0}\right\} \tag{12.6}
\end{equation*}
$$

For a 2-port system:

$$
\left\{\begin{array}{l}
I_{1}  \tag{12.7}\\
I_{2}
\end{array}\right\}=\left[\begin{array}{ll}
Y_{11} & Y_{12} \\
Y_{21} & Y_{22}
\end{array}\right]\left\{\begin{array}{l}
V_{1} \\
V_{2}
\end{array}\right\}+\left\{\begin{array}{l}
I_{1,0} \\
I_{2,0}
\end{array}\right\}
$$

The equivalent circuit is depicted in fig 12.2 . The admittances $Y_{1}, Y_{2}$ are shunted, $Y_{S}$ is in series while $Y_{G}$ is associated to the dependent current generators. It can be demonstrated that it holds:

$$
\left\{\begin{array}{c}
I_{1}  \tag{12.8}\\
I_{2}
\end{array}\right\}=\left[\begin{array}{cc}
Y_{1}+Y_{S} & -\left(Y_{S}-Y_{G}\right) \\
-\left(Y_{S}+Y_{G}\right) & Y_{2}+Y_{S}
\end{array}\right]\left\{\begin{array}{c}
V_{1} \\
V_{2}
\end{array}\right\}+\left\{\begin{array}{c}
I_{1,0} \\
I_{2,0}
\end{array}\right\}
$$



Figure 12.2: General Norton equivalent for a 2-port circuit system

More explicitly, once the matrix $\overline{\bar{Y}}$ is known the admittances $Y_{1}, Y_{2}, Y_{S}, Y_{G}$ can be calculated as:

$$
\left\{\begin{array} { l } 
{ Y _ { S } = - \frac { 1 } { 2 } ( Y _ { 1 2 } + Y _ { 2 1 } ) }  \tag{12.9}\\
{ Y _ { G } = - \frac { 1 } { 2 } ( Y _ { 1 2 } - Y _ { 2 1 } ) }
\end{array} \quad \left\{\begin{array}{l}
Y_{1}=Y_{11}+\frac{1}{2}\left(Y_{12}+Y_{21}\right)=Y_{11}-Y_{S} \\
Y_{2}=Y_{22}+\frac{1}{2}\left(Y_{12}+Y_{21}\right)=Y_{22}-Y_{S}
\end{array}\right.\right.
$$

Let us note that the admittance matrix $\overline{\bar{Y}}$ for the Norton equivalent is the inverse of the impedance one $\overline{\bar{Z}}$ adopted in the Thevenin approach.

$$
\begin{equation*}
\overline{\bar{Z}} \overline{\bar{Y}}=\overline{\bar{I}} \tag{12.10}
\end{equation*}
$$

Therefore, if they are not singular then it is possible to calculate the former in function of the latter one and vice-versa.

### 12.1.3 Transmission matrix $T$

In some cases the quantities on a port are known and those on the other port have to be calculated. Thus it is possible to express $V_{2}$ and $I_{2}$ in function of $V_{1}$ and $I_{1}$ by mean of a transmission matrix $\overline{\bar{T}}_{21}$ :

$$
\left\{\begin{array}{c}
V_{2}  \tag{12.11}\\
-I_{2}
\end{array}\right\}=\overline{\bar{T}}_{21} \cdot\left\{\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right\}+\left\{\begin{array}{c}
V_{2,0} \\
-I_{2,0}
\end{array}\right\}
$$

The sign minus before $I_{2}$ is needed in order to ensure cascade multiplication for a series of transmission matrices. For a 2-port circuit network the transmission


Figure 12.3: The transmission matrix $\overline{\bar{T}}_{21}$ links the circuit variables $V_{1}, I_{1}$ on port 1 to variables $V_{2},-I_{2}$ on port 2 .
matrix $\overline{\bar{T}}_{21}$ can be calculated on the basis of the impedance one $\overline{\bar{Z}}$ :

$$
\overline{\bar{T}}=\left[\begin{array}{ll}
T_{11} & T_{12}  \tag{12.12}\\
T_{21} & T_{22}
\end{array}\right]=\frac{1}{Z_{12}}\left[\begin{array}{cc}
Z_{22} & -\left(Z_{22} Z_{11}-Z_{21} Z_{12}\right) \\
1 & -Z_{11}
\end{array}\right]
$$

Analogously, the impedance matrix $\overline{\bar{Z}}$ can be calculated as:

$$
\overline{\bar{Z}}=\left[\begin{array}{ll}
Z_{11} & Z_{12}  \tag{12.13}\\
Z_{21} & Z_{22}
\end{array}\right]=\frac{1}{T_{21}}\left[\begin{array}{cc}
-T_{22} & -1 \\
T_{22} T_{11}-T_{21} T_{12} & T_{11}
\end{array}\right]
$$

### 12.1.4 Negative Impedance Converter

As we are going to verify in sec 12.5 .3 , in order to realize specific meta-materials or meta-surfaces the implementation of negative impedances could be required. More generally, in order to achieve some desired features, active circuits are needed. Here we report the basic scheme for an ideal Negative Impedance Converter (NIC), modeled through an ideal Linear Operational Amplifier. The equivalent network is depicted in fig 12.4. It can be easily demonstrated that


Figure 12.4: Scheme for a Negative Impedance Converter mounting a Linear Operational Amplifier.
the quantities on the two ports are so related:

$$
\begin{gather*}
\left\{\begin{array}{c}
I_{1} Z_{1}=I_{2} Z_{2} \\
V_{1}=V_{2}
\end{array} \Longrightarrow\right.  \tag{12.14}\\
\left\{\begin{array}{c}
V_{2} \\
-I_{2}
\end{array}\right\}=\left[\begin{array}{cc}
1 & 0 \\
0 & -\frac{Z_{1}}{Z_{2}}
\end{array}\right]\left\{\begin{array}{c}
V_{1} \\
I_{1}
\end{array}\right\} \tag{12.15}
\end{gather*}
$$

Therefore, if a generic impedance $Z$ is connected to port 2 , it will follow:

$$
\begin{align*}
& V_{2}=Z\left(-I_{2}\right) \quad \Longrightarrow  \tag{12.16}\\
& V_{1}=-\left(\frac{Z_{1}}{Z_{2}}\right) Z I_{1} \tag{12.17}
\end{align*}
$$

Thus, if $Z_{1}=Z_{2}$ the impedance $Z$ seems to have its own sign reversed, becoming negative. Be aware that a NIC is an active device, since it needs a power supply to work.

### 12.1.5 Gyrator

A gyrator is a non-reciprocal 2-port device which changes the input voltage and current without absorbing nor generating power. More precisely, its constitutive law is:

$$
\left\{\begin{array}{c}
V_{1}=-Z_{G} I_{2}  \tag{12.18}\\
V_{2}=Z_{G} I_{1}
\end{array}\right.
$$



Figure 12.5: A NIC can be used to change the sign of a generic impedance $Z$, thus making it to appear "negative" on the other port.
where $Z_{G}$ is the gyration impedance. The gyrator can be interpreted as made by two voltage generators controlled by the currents flowing in the opposite ports. A gyrator is a perfectly anti-reciprocal device, and in fact it is characterized by


Figure 12.6: Gyrator
anti-symmetric impedance $\overline{\bar{Z}}$ and admittance $\overline{\bar{Y}}$ matrices:

$$
\overline{\bar{Z}}=\left[\begin{array}{cc}
0 & -Z_{G}  \tag{12.19}\\
Z_{G} & 0
\end{array}\right] ; \quad \overline{\bar{Y}}=\left[\begin{array}{cc}
0 & Y_{G} \\
-Y_{G} & 0
\end{array}\right] ; \quad Y_{G}=\frac{1}{Z_{G}}
$$

Hence, the transmission matrix $\overline{\bar{T}}_{21}$ turns out to be:

$$
\overline{\bar{T}}_{21}=\left[\begin{array}{cc}
0 & -Z_{G}  \tag{12.20}\\
\frac{1}{Z_{G}} & 0
\end{array}\right]
$$

## Power inside the gyrator

Even if a real gyrator is made by active elements and so it must be power-feed, formally it is lossless and the power inside that is identically zero. In other words, globally the gyrator does not absorb nor generate power, in fact:

$$
\begin{equation*}
\text { Pow }_{I N}=V_{2} I_{2}+V_{1} I_{1}=\left(Z_{G} I_{1}\right) I_{2}+\left(-Z_{G} I_{2}\right) I_{1}=0 \tag{12.21}
\end{equation*}
$$

In that sense, the gyrator is analogous to a mechanical gyroscope.

## Impedance inverter

An interesting property of the gyrator is that it can be used to invert an impedance. Actually, if a generic impedance $Z$ is connected to port 2 , then it
holds:

$$
\begin{align*}
& V_{2}=Z\left(-I_{2}\right) \quad \Longrightarrow  \tag{12.22}\\
& V_{1}=-Z_{G} I_{2}=Z_{G} \frac{V_{2}}{Z}=Z_{G} \frac{\left(Z_{G} I_{1}\right)}{Z}  \tag{12.23}\\
& V_{1}=\frac{Z_{G}^{2}}{Z} I_{1} \tag{12.24}
\end{align*}
$$

For example, if $Z_{G}$ is a resistor $R$, then a capacitor $C$ connected to port 2 will appear as an inductor $L$ on port 1 , and vice-versa:

$$
\begin{equation*}
s L=s\left(R^{2} C\right) \tag{12.25}
\end{equation*}
$$

As a matter of fact, in many integrated circuits the inductor is simulated by connecting a real capacitor to a gyrator.

## Antoniou Gyrator

A gyrator can be built using Operational Amplifiers and impedances, assembled in many ways. As a matter of example, here we report the scheme of a gyrator based on the Antoniou configuration[50]: As long as I know, currently all the


Figure 12.7: Gyrator in the Antoniou configuration
real gyrator have limited bandwidth and are made by active elements, like the Operational Amplifier.

### 12.2 Circuit variables

If we desire to realize a circuit model for a screen, we have to translate the field variables in circuit ones.

A circuit network is made of nodes, connected by edges, thus we have to pass from a continuous space to a discrete one. Moreover, we have to transform distributed variables (fields) in lumped ones. Usually scalar fields are associated to nodes, while vector fields are associated to edges. On the contrary, pseudovector fields are associated to loops [35, 46].

### 12.2.1 Discrete circuit surface

As we already anticipated (see sec. 10.1), a screen can be well modeled as made of two superimposed circuit grids. However, that implies some kind of surface discretization. We can approximate a surface element with circuits if its sizes are much smaller than the operating wavelength $\lambda_{0}$. In this way, the incident waves cannot "distinguish" if the surface if continuous or discrete.

For the rest of this work we shall consider an elementary surface element of sizes $\Delta x, \Delta y, \Delta z$, where $\Delta x$ is the screen's thickness:

$$
\begin{equation*}
\Delta x \ll \frac{\lambda_{0}}{2 \pi} \quad \text { condition for thin screen } \tag{12.26}
\end{equation*}
$$

Since the screen has two layers, there are two surface currents, $\vec{J}_{t 1}$ and $\vec{J}_{t 2}$, one for each side.

### 12.2.2 Currents

The global current flowing in a surface element can be expressed as:

$$
I=\Delta z J_{t} \quad \Longrightarrow \quad\left\{\begin{array}{l}
I_{1}=\Delta z J_{t 1}  \tag{12.27}\\
I_{2}=\Delta z J_{t 2}
\end{array}\right.
$$

Now we desire to rephrase the currents $I_{1}$ and $I_{2}$ in function of the distributed variables $J_{t}$ and $D_{e}$. We know that the surface currents $\vec{J}_{t 1}$ and $\vec{J}_{t 2}$ can be calculated as:

$$
\left\{\begin{array}{l}
\vec{J}_{t 1}=\frac{1}{2} \vec{J}_{t}-\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{12.28}\\
\vec{J}_{t 2}=\frac{1}{2} \vec{J}_{t}+\frac{1}{\Delta x} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right.
$$

So, in scalar form:

$$
\left\{\begin{array}{l}
J_{t 1}=\frac{1}{2} J_{t}-\frac{1}{\Delta x} D_{e}  \tag{12.29}\\
J_{t 2}=\frac{1}{2} J_{t}+\frac{1}{\Delta x} D_{e}
\end{array}\right.
$$

The terms related to the doublet can be rearranged in function of the adimensional thickness $\theta_{0}$ :

$$
\begin{equation*}
\frac{1}{\Delta x} D_{e}=\frac{c_{0}}{s \Delta x}\left(\frac{s}{c_{0}} D_{e}\right)=-\frac{1}{i \theta_{0}}\left(\frac{s}{c_{0}} D_{e}\right) \tag{12.30}
\end{equation*}
$$

After some substitutions, we finally find the currents $I_{1}$ and $I_{2}$

$$
\left\{\begin{array}{l}
I_{1}  \tag{12.31}\\
I_{2}
\end{array}\right\}=\Delta z \frac{1}{\left(i \theta_{0}\right)}\left[\begin{array}{rr}
\frac{1}{2}\left(i \theta_{0}\right) & 1 \\
\frac{1}{2}\left(i \theta_{0}\right) & -1
\end{array}\right]\left\{\begin{array}{c}
J_{t} \\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}
$$

So, once the current $\vec{J}_{t}$ and the doublet $\overline{\bar{D}}_{e}$ are known, we can calculate the currents $I_{1}$ and $I_{2}$ for each direction. Inverting the previous relation, we can determine $J_{t}$ and $D_{e}$ :

$$
\left\{\begin{array}{rlr}
J_{t} & =\frac{1}{\Delta z}\left(I_{2}+I_{1}\right)  \tag{12.32}\\
\frac{s}{c_{0}} D_{e} & =-\frac{1}{2}\left(i \theta_{0}\right) \frac{1}{\Delta z}\left(I_{2}-I_{1}\right)
\end{array}\right.
$$

In matrix form:

$$
\left\{\begin{array}{c}
J_{t}  \tag{12.33}\\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}=\frac{1}{\Delta z}\left[\begin{array}{cc}
1 & 1 \\
\frac{1}{2}\left(i \theta_{0}\right) & -\frac{1}{2}\left(i \theta_{0}\right)
\end{array}\right]\left\{\begin{array}{l}
I_{1} \\
I_{2}
\end{array}\right\}
$$

### 12.2.3 Voltages

Most of times the currents are moved by some kind of generator, which gives rise to an electromotive force (e.m.f.).

If we know the tangent electric fields $\vec{E}_{t 1}$ and $\vec{E}_{t 2}$ on the two sides of the screen, we can define two voltages $V_{1}$ and $V_{2}$ :

$$
\begin{align*}
& V_{1}=\Delta y E_{1}  \tag{12.34}\\
& V_{2}=\Delta y E_{2} \tag{12.35}
\end{align*}
$$

That can be done for every tangent direction, so that we can consider each vector component of $\vec{E}_{t}$ as a scalar quantity. We observe that:

- $V_{1}$ and $V_{2}$ can be interpreted as voltage generators placed on sides 1 and 2 respectively.
- $V_{1}$ and $V_{2}$ comprehend either the effects of the incident waves and the radiated ones:

$$
\begin{align*}
& V_{1}=\Delta y\left(E_{1, i n}+E_{1, i r r}\right)=V_{1, i n}+V_{1, i r r}  \tag{12.36}\\
& V_{2}=\Delta y\left(E_{2, i n}+E_{2, i r r}\right)=V_{2, i n}+V_{2, i r r} \tag{12.37}
\end{align*}
$$

- actually $V_{1}$ and $V_{2}$ are not potentials' differences, but voltages: they are defined on edges rather then on couple of nodes. In a heavier but more precise notation I would have to indicate them with $\Delta V_{E, 10}$ and $\Delta V_{E, 20}$. Since the electric field can be not conservative, formally the Kirchhoff Voltage Law (KVL) does not apply to "voltages".


### 12.3 Non-conservative fields

In Circuit Theory the potentials are defined on nodes and the electric field is approximated as conservative:

$$
\begin{equation*}
\vec{E}=-\vec{\nabla} P_{A} \quad \Longrightarrow \quad \vec{\nabla} \times \vec{E}=\overrightarrow{0} \tag{12.38}
\end{equation*}
$$

Actually, that hypothesis is associated to the Kirchhoff Voltage Law, which applies just to the differences of potentials along a closed loop:

$$
\begin{equation*}
\sum_{\text {loop }}\left(\Delta P_{A, i j}\right)=0 \tag{12.39}
\end{equation*}
$$

However, if we are dealing with transverse EM waves, or anyway with a nonstationary problem, then the electric field is not conservative and it cannot be regarded as a "difference of potential".

$$
\begin{equation*}
\vec{E}=-\frac{\partial \vec{A}}{\partial t}-\vec{\nabla} P_{A} \quad \Longrightarrow \quad \vec{\nabla} \times \vec{E} \neq \overrightarrow{0} \tag{12.40}
\end{equation*}
$$

### 12.3.1 Ground reference

As we have seen in chapter 10, it is possible to define a potential $P_{A}$ on every node in our circuit surface. That is valid also for non-conservative fields: in fact we have derived the Boundary Conditions for the very general non-stationary case. However, we have not defined any ground reference for our circuit network, but that is not a problem. In fact, thanks to the Gauge Invariance, we can fix an arbitrary reference potential $P_{A \infty}$.

Anyway, as we are going to check, the concept of "ground" is not so mandatory.

### 12.3.2 What really moves the currents?

Now we have to answer to the question:

> What does really moves the currents in a circuit?
> It is a potential difference? The electric field?
> Or maybe also the magnetic one?

We need to ascertain the right answer in order to model the e.m.f. in the circuit net. We can start considering the actual EM force exerted on a current, that is the so-called Lorentz force:

$$
\begin{array}{lr}
\vec{f}=\rho_{e} \vec{E}-\overline{\bar{B}} \vec{J}_{e} & \text { ND notation } \\
\vec{f}=\rho_{e} \vec{E}-\vec{B} \times \vec{J}_{e} & \text { 3D notation } \tag{12.42}
\end{array}
$$

If the current is flowing in a wire, it will be sensible just to the tangent force and so to $\vec{E}_{t}$. In fact, since the magnetic force $\vec{J}_{e} \times \vec{B}$ is always perpendicular to the current, it cannot move or accelerate it in the wire direction. Obviously, the wire can be stretched on a side by the magnetic field, but that effect does not imply the induction of a current.

For the same reason, the electric field $\vec{E}_{n}$ normal to the wire cannot move the charges in the wire direction. So we can exclude the magnetic field as the actual electro motive force.

## Conductive ring

Now let us consider a simple example in order to distinguish the difference of potentials from the voltages.

Imagine you have a circular conducting ring and a common permanent magnet. If you accost the magnet perpendicularly to the ring, it shall induce a current. In fact, you are varying the magnetic flux through the ring, giving raise to a rotational electric field. That is implicitly stated by the Faraday's Law:

$$
\begin{equation*}
\vec{\nabla} \wedge \vec{E}=-\frac{\partial \overline{\bar{B}}}{\partial t} \tag{12.43}
\end{equation*}
$$

We can discretize the ring as made of two or more circuit edges. Each edge links two adjacent nodes and it can be characterized by a certain resistance $R$ and inductance $L$. Since the problem has a circular symmetry, all the nodes are characterized by the same potential. Even so, a current $I$ flows in the ring, so we have to model the tangent electric field $E_{t}$ by mean of a voltage generator in series with the impedances $R$ and $L$.


Figure 12.8: Non-conservative electric field in circuits. (a) The moving magnet induces a rotational electric field $\vec{E}$ in the conductive ring, as stated by the Faraday's Law. (b) The ring can be discretized with identical circuit elements. The effects of non-conservative electric field $\vec{E}$ are reproduced by mean of voltage generators. Thanks to the rotational symmetry, all the nodes are equipotential.

Finally, we have checked that circuit currents are moved neither by the magnetic field nor by potentials' differences, but on the contrary they are induced by the tangent electric field $E_{t}$. I have to signal that this last example is not completely mine: I borrowed the idea by Walter Lewin, who also realized an experiment to explain it.

### 12.4 Circuit Boundary Conditions and constitutive relations

Let us resume the Boundary Conditions for $\vec{E}$ and $\overline{\bar{H}}$ in scalar form (see also $\sec 10.2 .1$ ):

$$
\left\{\begin{array}{l}
E_{2}-E_{1}=-s \mu_{0} D_{e}-s \mu_{0} \frac{1}{2}\left(H_{1}+H_{2}\right) \Delta x  \tag{12.44}\\
H_{2}-H_{1}=-J_{t}-s \varepsilon_{0} \frac{1}{2}\left(E_{1}+E_{2}\right) \Delta x
\end{array}\right.
$$

Remember that we are working in the hypothesis of thin screen $\left(k_{0} \Delta x \ll 1\right)$. Now we have to re-write the BCs in circuit form, so we place:

$$
\left\{\begin{array} { l } 
{ J _ { t } = \frac { 1 } { \Delta z } ( I _ { 2 } + I _ { 1 } ) }  \tag{12.45}\\
{ D _ { e } = \frac { 1 } { 2 } \frac { 1 } { \Delta z } ( I _ { 2 } - I _ { 1 } ) \Delta x }
\end{array} \quad \left\{\begin{array}{l}
V_{1}=\Delta y E_{1} \\
V_{2}=\Delta y E_{2}
\end{array}\right.\right.
$$

After some substitutions, we find:

$$
\left\{\begin{align*}
V_{2}-V_{1} & =-s \mu_{0}\left(\frac{\Delta x \Delta y}{\Delta z}\right)\left(\frac{1}{2}\left(I_{2}-I_{1}\right)+\frac{1}{2}\left(H_{1}+H_{2}\right) \Delta z\right)  \tag{12.46}\\
\Delta z\left(H_{2}-H_{1}\right) & =-\left(I_{2}+I_{1}\right) \quad-s \varepsilon_{0}\left(\frac{\Delta x \Delta z}{\Delta y}\right) \frac{1}{2}\left(V_{1}+V_{2}\right)
\end{align*}\right.
$$

In order to lighten the notation and to enforce the circuit interpretation, we define some reference impedances:

$$
\begin{array}{ll}
L_{0}=\mu_{0} \frac{\Delta x \Delta y}{\Delta z}=\mu_{0} \frac{S_{x y}}{\Delta z} & \text { reference inductance } \\
C_{0}=\varepsilon_{0} \frac{\Delta x \Delta z}{\Delta y}=\varepsilon_{0} \frac{S_{x z}}{\Delta y} & \text { reference capacitance } \tag{12.48}
\end{array}
$$

Now we can write compactly:

$$
\left\{\begin{align*}
\Delta V_{21} & =-s L_{0}\left(\frac{1}{2}\left(I_{2}-I_{1}\right)+\langle H\rangle \Delta z\right)  \tag{12.49}\\
\Delta H_{21} \Delta z & =-\left(I_{2}+I_{1}\right)-s C_{0}\langle V\rangle
\end{align*}\right.
$$

These are the circuit Boundary Conditions we were looking for. The problem is that the magnetic field $H$ is still present and formally we cannot interpret it neither as a voltage nor as a current.

### 12.4.1 Constitutive relations for the screen

The Boundary Conditions drawn from Maxwell's Equations are not sufficient to describe the behaviour of the screen. In fact, we need also some kind of constitutive relation in order to define the properties of our system.

With the words "constitutive relation" we are speaking in a general way, since you can describe your system with different approaches. For example, you can assign the scattering matrix $S$, or otherwise the admittance matrix $Y$. In some cases it is useful to adopt a transmission matrix $T_{21}$, linking the quantities on port 1 to the ones on port 2 . As we are going to see, in many contexts the constitutive relation is assigned in the form:

$$
\begin{equation*}
\Delta \mathbf{f}_{21}=C\langle\mathbf{f}\rangle \tag{12.50}
\end{equation*}
$$

where $\langle\mathbf{f}\rangle$ is the array containing the "average" fields on the system, while $\Delta \mathbf{f}_{21}$ contains the field differences between the two sides.

For example, if we choose the electric and magnetic fields, the constitutive relation could look:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{12.51}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=\overline{\bar{C}}_{h}\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}
$$

I don't like that relation, but it reveals to be quite helpful for the description of materials with assigned permittivity $\varepsilon$ and permeability $\mu$.

### 12.5 Equivalent symmetric circuit

Suppose we want to build a thin circuit screen, assigning both the relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$. In the previous chapters (see sec 10.4) we have derived the constitutive relation in $E$ and $H$, but now we have to search the equivalent circuit ensuring the desired $\varepsilon_{r}$ and $\mu_{r}$, hence now we are going to see which impedances have to be assembled on each screen surface element. This time I decided to start from the "final" result, since I believe that way of explanation is more intuitive instead of the direct calculation, which is unfortunately a bit heavy.

For this part I must award my inspiration to a beautiful paper by Selvanayagam and Eleftheriades [51], which deals exactly with the circuit modeling of Huygens Surface. For its clearness and elegance I suggest you to read it aside. Anyway, as you can easily verify, my approach to the problem is quite different, since I do not interpret the magnetic field as a current and take into account the finite thickness of the screen. Moreover, I am going to express the impedances in function of $\varepsilon_{r}$ and $\mu_{r}$ and vice-versa.

### 12.5.1 Constitutive relation for symmetric circuit

Suppose we link voltages $V$ and currents $I$ by mean of just two impedances $Z_{e}$ and $Z_{m}$. We guess that the symmetric components for the voltages and the currents are proportional, and the same happens for the anti-symmetric components:

$$
\left\{\begin{array}{rlr}
\frac{1}{2}\left(V_{2}+V_{1}\right) & =Z_{e} \quad\left(I_{2}+I_{1}\right) & \text { symmetric component }  \tag{12.52}\\
\left(V_{2}-V_{1}\right) & =Z_{m} \frac{1}{2}\left(I_{2}-I_{1}\right) & \text { anti-symmetric component }
\end{array}\right.
$$

In practice, we are supposing to build a circuit made by a straight wire and a ring, whose impedances are respectively $Z_{e}$ and $Z_{m}$. The straight wire will be


(b)

Figure 12.9: Elementary circuits. (a) Straight wire $Z_{e}$ : related to permittivity and to the symmetric components of voltages and currents. (b) Ring $Z_{m}$ : related to permeability and to the anti-symmetric components of voltages and currents.
associated to the net electric dipole and so to the average electric field. The ring will be related to the magnetic dipole and so to the average magnetic field.

If we wish to draw the Thevenin equivalent for that system, we would get:

$$
\left\{\begin{array}{l}
V_{1}  \tag{12.53}\\
V_{2}
\end{array}\right\}=\left[\begin{array}{ll}
Z_{e}+\frac{1}{4} Z_{m} & Z_{e}-\frac{1}{4} Z_{m} \\
Z_{e}-\frac{1}{4} Z_{m} & Z_{e}+\frac{1}{4} Z_{m}
\end{array}\right]\left\{\begin{array}{c}
I_{1} \\
I_{2}
\end{array}\right\}
$$

Once the relations $\sqrt{12.52}$ among voltages and currents are known, we can combine them with the circuit Boundary Conditions (see eq. 12.49 ).


Figure 12.10: Unit circuit cell for a reciprocal screen with assigned permittivity and permeability. Let us notice that the voltage generators reproduce the effects of both the incident and the radiated electric field and that just the impedances $Z_{e}$ and $Z_{m}$ have to be actually mounted.

The two sets of equations can be rephrased compactly as:

$$
\begin{align*}
& \left\{\begin{array}{l}
\langle V\rangle=2 Z_{e}\langle I\rangle \\
\Delta V_{21}=\frac{1}{2} Z_{m} \Delta I_{21}
\end{array}\right.  \tag{12.54}\\
& \left\{\begin{array}{c}
\Delta V_{21}=-s L_{0}\left(\frac{1}{2} \Delta I_{21}+\langle H\rangle \Delta z\right) \\
\Delta H_{21} \Delta z=-2\langle I\rangle \quad-s C_{0}\langle V\rangle
\end{array}\right. \tag{12.55}
\end{align*}
$$

We desire to re-construct the constitutive relation in terms of $E$ and $H$ field, so we substitute the currents $\langle I\rangle$ and $\Delta I_{21}$ in the equations. After some calculi, we obtain so:

$$
\left\{\begin{align*}
\Delta V_{21} & =-\left(Z_{m} / / s L_{0}\right)\langle H\rangle \Delta z  \tag{12.56}\\
\Delta H_{21} \Delta z & =-\left(\frac{1}{Z_{e}}+s C_{0}\right)\langle V\rangle
\end{align*}\right.
$$

In matrix form

$$
\left\{\begin{array}{c}
\Delta V_{21}  \tag{12.57}\\
\Delta H_{21} \Delta z
\end{array}\right\}=-\left[\begin{array}{cc}
0 & \left(Z_{m} / / s L_{0}\right) \\
\frac{1}{Z_{e}}+s C_{0} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle V\rangle \\
\langle H\rangle \Delta z
\end{array}\right\}
$$

This constitutive relation has a structure quite similar to the one we found for $E$ and $H$ (see eq. 10.122 ) . We remind that:

$$
\begin{gather*}
V=\Delta y E  \tag{12.58}\\
\left\{\begin{array} { l } 
{ ( i \theta _ { 0 } ) \eta _ { 0 } = - s \mu _ { 0 } \Delta x } \\
{ ( i \theta _ { 0 } ) \frac { 1 } { \eta _ { 0 } } = - s \varepsilon _ { 0 } \Delta x }
\end{array} \left\{\begin{array}{l}
\mu_{0}=\frac{\Delta z}{\Delta x \Delta y} L_{0} \\
\varepsilon_{0}=\frac{\Delta y}{\Delta x \Delta z} C_{0}
\end{array}\right.\right. \tag{12.59}
\end{gather*}
$$

Hence:

$$
\left\{\begin{align*}
\left(i \theta_{0}\right) \eta_{0} & =-\frac{\Delta z}{\Delta y} s L_{0}  \tag{12.60}\\
\left(i \theta_{0}\right) \frac{1}{\eta_{0}} & =-\frac{\Delta y}{\Delta z} s C_{0}
\end{align*}\right.
$$

After some substitution, the equation 10.122 can be so rewritten as:

$$
\left\{\begin{array}{c}
\Delta V_{21}  \tag{12.61}\\
\Delta H_{21} \Delta z
\end{array}\right\}=-s\left[\begin{array}{cc}
0 & \mu_{r} L_{0} \\
\varepsilon_{r} C_{0} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle V\rangle \\
\langle H\rangle \Delta z
\end{array}\right\}
$$

Matching this equation with 12.57, we find so the conditions:

$$
\begin{align*}
& \mu_{r} s L_{0}=\left(Z_{m} / / s L_{0}\right)  \tag{12.62}\\
& \varepsilon_{r} s C_{0}=\frac{1}{Z_{e}}+s C_{0} \tag{12.63}
\end{align*}
$$

Now we are able to express the impedances $Z_{e}$ and $Z_{m}$ in function of $\varepsilon_{r}$ and $\mu_{r}$ respectively, and viceversa.

### 12.5.2 Impedances, permittivity and permeability

If we know the impedances $Z_{e}$ and $Z_{m}$ mounted on the screen, them the relative permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$ can be calculated as:

$$
\begin{align*}
& \varepsilon_{r}=1+\frac{1}{s C_{0} Z_{e}}  \tag{12.64}\\
& \mu_{r}=\frac{1}{1+\frac{s L_{0}}{Z_{m}}} \tag{12.65}
\end{align*}
$$

where:

$$
\begin{array}{ll}
L_{0}=\mu_{0} \frac{\Delta x \Delta y}{\Delta z}=\mu_{0} \frac{S_{x y}}{\Delta z} & \text { reference inductance } \\
C_{0}=\varepsilon_{0} \frac{\Delta x \Delta z}{\Delta y}=\varepsilon_{0} \frac{S_{x z}}{\Delta y} & \text { reference capacitance } \tag{12.67}
\end{array}
$$

On the contrary, if $\varepsilon_{r}$ and $\mu_{r}$ are assigned, then the impedances $Z_{e}$ and $Z_{m}$ to be installed are:

$$
\begin{align*}
Z_{e} & =\frac{1}{\varepsilon_{r}-1} \frac{1}{s C_{0}}  \tag{12.68}\\
Z_{m} & =\frac{1}{\frac{1}{\mu_{r}}-1} s L_{0} \quad=\quad-\frac{\mu_{r}}{\mu_{r}-1} s L_{0} \tag{12.69}
\end{align*}
$$

We remember that the equivalent circuit is made of a straight wire and a ring, the former endowed with impedance $Z_{e}$, the latter with $Z_{m}$.

## Non-dispersive screen

In general, both permittivity $\varepsilon_{r}(s)$ and $\mu_{r}(s)$ can vary with the frequency, as happens for a dispersive material. If we require that $\varepsilon_{r}$ and $\mu_{r}$ are real (so, lossless) and constant with frequency, or in other words that the material is dispersionless, then:

- $Z_{e}$ has to be a capacitor $C=\left(\mu_{r}-1\right) C_{0}$.
- $Z_{m}$ has to be an inductor $L=\frac{\mu_{r}}{1-\mu_{r}} L_{0}$.


Figure 12.11: Capacitive wire and inductive ring

Accordingly, the screen can be made by a capacitive grid and an ordered array of inductive rings. Let us notice that if $\varepsilon_{r}$ and $\mu_{r}$ are equal to one, then the impedances tends to infinite values, so they consist of open circuits. That exactly correspond to the case in which you do not install any device, or else there is no screen and thus waves propagate freely.

$$
\text { if }\left\{\begin{array} { l } 
{ \varepsilon _ { r } \rightarrow 1 }  \tag{12.70}\\
{ \mu _ { r } \rightarrow 1 }
\end{array} \Longrightarrow \left\{\begin{array}{c}
Z_{e} \rightarrow \infty \\
Z_{m} \rightarrow \infty
\end{array}\right.\right. \text { open circuits }
$$

### 12.5.3 Negative inductors and capacitors

Depending on the values assumed by $\varepsilon_{r}$ and $\mu_{r}$, the capacity or the inductance associated to $Z_{e}$ and $Z_{m}$ can result to be negative.

In theory, realizing a negative impedance is not impossible, at least at a macroscopic scale and below a certain frequency. Effectively, using a Negative Impedance Converter (NIC) it is possible to change the sign of an impedance within a broad-band, without violating causality [16, 19]. From a theoretical point of view the concepts of negative resistance, capacitance or inductance seem to not imply any contradiction. However, we must be aware that building an ideal Negative Impedance Converter could be quite difficult, in fact:

- Usually a NIC is made by an ensemble of integrated, electronic devices, like transistors, Linear Operational Amplifier, diodes and so on. All those electronic elements have a finite size and work at a finite frequency.
For example, a transistor in a mobile phone can be 1000 nm long, with a switching frequency up to 500 GHz . That performances could be insufficient if you need smaller devices (few nanometers) working at higher frequencies (visible spectrum, Tera Hertz regime).
Shortly, the NIC's size and bandwidth are limited.
- The NIC needs a power supply to work, since it contains active elements.
- For many configurations negative resistors, capacitors or inductors are unstable, since they pump rather than absorb power in the network. For that reason, those devices can be approximated as linear impedances just in limited operating range.
More generally, the stability of any active non-Foster element [52, 18, 19] must be ensured by mean of some control method, for example connecting it to passive elements.

Let us now analyze in which cases a Negative Impedance Converter is needed. If $Z_{e}$ is a negative capacitor, with capacity $C_{e}<0$, then holds:

$$
\begin{align*}
& Z_{e}=-\frac{1}{s\left|C_{e}\right|}=\frac{1}{\varepsilon_{r}-1} \frac{1}{s C_{0}} \quad \Longrightarrow \quad \frac{1}{\varepsilon_{r}-1}<0 \quad \Longrightarrow  \tag{12.71}\\
& \varepsilon_{r}<1 \quad \text { condition for negative capacitor } \tag{12.72}
\end{align*}
$$

If $Z_{m}$ is a negative inductor, with inductance $L_{m}<0$, then holds:

$$
\begin{align*}
& Z_{m}=-s\left|L_{m}\right|=\frac{1}{\frac{1}{\mu_{r}}-1} s L_{0} \quad \Longrightarrow  \tag{12.73}\\
& \frac{1}{\frac{1}{\mu_{r}}-1}<0 \quad \Longrightarrow \quad \frac{1}{\mu_{r}}<1 \quad \Longrightarrow  \tag{12.74}\\
& \mu_{r}<0 \text { or } \mu_{r}>1 \quad \text { condition for negative inductor } \tag{12.75}
\end{align*}
$$

In summary:

$$
\begin{array}{rccc}
Z_{e}=-\frac{1}{s\left|C_{e}\right|} & \Longleftrightarrow & \varepsilon_{r}<1 & \text { negative capacitor } \\
Z_{m}=-s\left|L_{m}\right| & \Longleftrightarrow & \mu_{r}<0 \text { or } \mu_{r}>1 & \text { negative inductor } \tag{12.77}
\end{array}
$$

### 12.5.4 Narrow band screen

In many contexts installing a negative capacitor or inductor could be very difficult. For example, if you want to construct an optical meta-surface, so in the visible spectrum, the operating wavelength $\lambda_{0}$ is in the $380 \div 750 \mathrm{~nm}$ range. Since the unit cell must be sub-wavelength $\left(\Delta x \ll \lambda_{0}\right)$, you have to use integrated nano-circuits. Currently, active nano-circuits are very difficult to be constructed and anyway their cost is still quite high.

For those reasons, often the engineers renounce to the requirement of a broadband, non-dispersive metasurface. They accept that the screen is characterized by the assigned $\varepsilon_{r}$ and $\mu_{r}$ just at a single "frequency" $\omega_{0}$. Within that hypothesis, the behaviour of a negative capacitor can be reproduced by a standard positive inductor. Vice-versa, a negative inductor can be imitated by a common positive capacitor. The only condition is that, for the selected frequency, the alternative element has the same reactance $X\left(\omega_{0}\right)$ of the replaced one.

Now we desire to calculate the value of the equivalent inductors and capacitors. For sake of clarity, this time I will adopt the circuit convention $s=+i \omega$ instead of $s=-i \omega$, but that choice does not affect the final result.

For a negative capacity $C_{e}<0$ the equivalent inductance $L_{e}$ at $\omega_{0}$ can be calculated as:

$$
\begin{align*}
& Z_{e}\left(\omega_{0}\right)=-\frac{1}{\left(i \omega_{0}\right)\left|C_{e}\right|}=\left(i \omega_{0}\right) L_{e} \quad \Longrightarrow  \tag{12.78}\\
& L_{e}=\frac{1}{\omega_{0}^{2}} \frac{1}{\left|C_{e}\right|} \tag{12.79}
\end{align*}
$$

Similarly, for a negative inductance $L_{m}<0$, the equivalent capacity $C_{m}$ at $\omega_{0}$
can be calculated as:

$$
\begin{align*}
& Z_{m}\left(\omega_{0}\right)=-\left(i \omega_{0}\right) L_{m}=\frac{1}{\left(i \omega_{0}\right)\left|C_{e}\right|} \Longrightarrow  \tag{12.80}\\
& C_{m}=\frac{1}{\omega_{0}^{2}} \frac{1}{\left|L_{m}\right|} \tag{12.81}
\end{align*}
$$

Remind that, after those substitutions, the screen will exhibit frequency-dependent permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$. That trick works just at a single pulsation $\omega_{0}$

### 12.6 3D Bulk Material

Till now we have treated the screen as a thin quasi 1-D system: in fact we have always considered just the tangent field. Moreover, in order to derive $\varepsilon_{r}$ and $\mu_{r}$, we worked within the hypothesis of waves propagating orthogonally to the screen itself.

Furthermore, our thin screen can be regarded as the basic element for more complex structures. Joining together many thin screens we can build up a 3D bulk material with some desired properties. For example, you can stack up a succession of $n$ identical screens obtaining so a thick homogeneous - though non isotropic - slab. In other words, you can construct an artificial bulk material


Figure 12.12: A bulk slab with assigned properties can be modelled by superimposing many thin screens.
with some assigned features, e.g. permittivity $\varepsilon_{r}$ and permeability $\mu_{r}$.

### 12.6.1 3D homogeneous isotropic Metamaterial

In order to realize a bulk metamaterial with circuits we have to connect together many impedance grid in the three spatial dimensions. If we require the material to be macroscopically homogeneous and isotropic, then the grids have to be made by periodic unit cells. The structure of each cell must be the same for every selected direction.

Ensuring a perfect, spherical isotropy could be quite difficult, but that feature is often well reproduced by some crystal lattice, for example with cubic symmetry.

Usually, in electromagnetics a material is considered isotropic if the permittivity and permeability tensors are proportional to the identity matrix:

$$
\begin{equation*}
\overline{\bar{\varepsilon}}=\varepsilon \overline{\bar{I}} ; \quad \overline{\bar{\mu}}=\mu \overline{\bar{I}} \tag{12.82}
\end{equation*}
$$

In other words, $\varepsilon$ and $\mu$ are the same for all the directions. That condition can be reproduced at micro-scale, assembling impedances $Z_{e}$ and $Z_{m}$ in two separated orthogonal lattices. The "electric" impedances $Z_{e}$ can be linked in

(a) Isotropic star of wire impedances $Z_{e}$.

(b) Isotropic octahedron of ring impedances $Z_{m}$.

Figure 12.13: Circuit cells for a 3D orthogonal lattice with assigned permittivity and permeability. For sake of simplicity just three impedances $Z_{m}$ are shown on the octahedron rings. These latter can be independent (not linked).
a star configuration. On the contrary, the "magnetic" impedances $Z_{m}$ should be installed on structure made by perpendicular loops. Probably the simplest configurations are the octahedron and the cube of rings.

I would say it is better to use square rings rather than circular ones, because the formers allows a more effective filling of the space. I cannot exclude the


Figure 12.14: Example of a 3D orthogonal circuit lattice (impedances $Z_{e}$ and $Z_{m}$ are not shown).
possibility of building a tetrahedral lattice or the use of hexagonal rings, though probably such a network would be more difficult to be modelled.

The bulk material is homogeneous and isotropic on macro-scale ( $\lambda_{0} \gg$ $2 \pi \Delta x)$, therefore the "electric" grid and the array of magnetic rings can be suitably shifted, allowing a better filling of the space.

### 12.6.2 3D non-homogeneous anisotropic Metamaterial

If the bulk metamaterial is required to be non-homogeneous, then it can be made by different unit cells, though always sub-wavelength. If the metamaterial is required to be locally non-isotropic, too, then each unit cell can be made by many different impedances associated to every specific direction.

For example, you could require a permittivity $\varepsilon_{i i}$ and a permeability $\mu_{i i}$ for the $i^{\text {th }}$ direction, so the tensors could look:

$$
\overline{\bar{\varepsilon}}=\left[\begin{array}{ccc}
\varepsilon_{11} & 0 & 0  \tag{12.83}\\
0 & \varepsilon_{22} & 0 \\
0 & 0 & \varepsilon_{33}
\end{array}\right] \quad \overline{\bar{\mu}}=\left[\begin{array}{ccc}
\mu_{11} & 0 & 0 \\
0 & \mu_{22} & 0 \\
0 & 0 & \mu_{33}
\end{array}\right]
$$

For each $i^{\text {th }}$ direction, the impedance $Z_{e, i}$ to be installed on the "electric" star can be calculated as:

$$
\begin{equation*}
Z_{e, i}=\frac{1}{\varepsilon_{r, i i}-1} \frac{1}{s C_{0}} \tag{12.84}
\end{equation*}
$$

In a similar way, the impedance $Z_{m, i}$ to be installed on each magnetic ring perpendicular to the $i^{\text {th }}$ direction is:

$$
\begin{equation*}
Z_{m, i}=-\frac{\mu_{r, i i}}{\mu_{r, i i}-1} s L_{0} \tag{12.85}
\end{equation*}
$$


(a) Anisotropic star of wire impedances $Z_{e, i}$.

(b) Anisotropic octahedron of ring impedances $Z_{m, i}$.

Figure 12.15: Circuit cells for a 3D orthogonal lattice with assigned anisotropic permittivity and permeability. For sake of simplicity just three impedances $Z_{m, i}$ are shown on the octahedron rings. These latter can be independent (not linked).

### 12.7 Examples of taylored screens

In this section we show how to determine the circuit structure for a metasurface with some desired properties. In all the following examples we are going to consider a unit circuit cell for a surface with assigned permittivity and permeability. As you can verify, all the circuit will reveal to be symmetric or reciprocal. In other words, the two surface's sides can be exchanged without that operation can be noticed.

### 12.7.1 Reflectionless surface

For many applications a metasurface is required to be reflectionless. In that case the surface intensive impedance $\eta$ must be equal to the vacuum's one $\eta_{0}$.

$$
\begin{equation*}
\eta=\eta_{0} \tag{12.86}
\end{equation*}
$$

In principle, that of condition of perfect matching should be valid for any incidence angle. In other words, the surface's impedance should be isotropic and equal to $\eta_{0}$. Here we consider just the case of an orthogonal impinging wave, though the idea can be generalized, making the surface locally isotropic by mean of orthogonal unit cells (see sec. 12.6.1). In order to remove the reflection, the relative permittivity $\mu_{r}$ and $\varepsilon_{r}$ must be equal, in fact:

$$
\begin{array}{ll}
\eta=\eta_{0} ; & \eta=\sqrt{\frac{\mu_{r}}{\varepsilon_{r}}} \eta_{0} \quad \Longrightarrow \quad \sqrt{\frac{\mu_{r}}{\varepsilon_{r}}}=1 \quad \Longrightarrow \\
\mu_{r}=\varepsilon_{r} \quad & \text { no-reflection condition } \tag{12.88}
\end{array}
$$

### 12.7.2 Electric mirror

An ideal mirror can perfectly reflect any incident EM wave. More precisely, an electric mirror reflects in such a way that the tangent electric field $\vec{E}_{t}$ is null on the surface. In order to find the circuit equivalent for a symmetric electric mirror, we have so to impose the condition that $\vec{E}_{t}=\overrightarrow{0}$ on both the sides of the screen. That must hold for any incident wave and for any magnetic field.

$$
\left\{\begin{array} { l } 
{ \vec { E } _ { 1 t } = \vec { 0 } }  \tag{12.89}\\
{ \vec { E } _ { 2 t } = \vec { 0 } }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\langle E\rangle=0 \\
\Delta E_{21}=0
\end{array}\right.\right.
$$

Inserting $\langle E\rangle$ and $\Delta E_{21}$ in the constitutive relation 10.122 involving $\mu_{r}$ and $\varepsilon_{r}$, we find the limit condition:

$$
\left\{\begin{array}{rl}
0 & =i \theta_{0} \mu_{r} \eta_{0}\langle H\rangle  \tag{12.90}\\
\eta_{0} \Delta H_{21} & =i \theta_{0} \varepsilon_{r} \cdot 0
\end{array} \quad \forall\langle H\rangle, \Delta H_{21}\right.
$$

Since the magnetic fields should be not constrained, we impose that this last condition is valid in the limit of zero permeability and infinite permittivity:

$$
\begin{equation*}
\mu_{r} \rightarrow 0 ; \quad\left|\varepsilon_{r}\right| \rightarrow+\infty \tag{12.91}
\end{equation*}
$$

The impedances $Z_{e}$ and $Z_{m}$ will be so:

$$
\begin{gather*}
Z_{e}=\lim _{\varepsilon_{r} \rightarrow \infty}\left(\frac{1}{\varepsilon_{r}-1} \frac{1}{s C_{0}}\right)=0  \tag{12.92}\\
Z_{m}=\lim _{\mu_{r} \rightarrow 0}\left(-\frac{\mu_{r}}{\mu_{r}-1} s L_{0}\right)=0 \tag{12.93}
\end{gather*}
$$

So both $Z_{e}$ and $Z_{m}$ are zero, meaning that just short circuits have to be installed. However, in this case the circuit configuration can be simplified, removing the ring which is not strictly mandatory. In fact, the electric mirror can be modelled with two parallel short circuit, or even more simply with a single short circuit layer.

Shortly, the electric mirror can be modelled by mean of short circuits, which are equivalent to a Perfect Electric Conductor (PEC). As a matter of fact, common mirrors are made by a silver layer which well approximates a PEC at visible frequencies.

### 12.7.3 Magnetic mirror

An ideal magnetic mirror perfectly reflects the incident EM waves. Differently from an electric one, for a magnetic mirror the tangent magnetic field $\vec{H}_{t}$ is required to be null on the surface. In order to find the circuit equivalent for a symmetric magnetic mirror, we have so to impose the condition that $\vec{H}_{t}=\overrightarrow{0}$ on both the sides of the screen. That must hold for any incident wave and for any electric field.

$$
\left\{\begin{array} { l } 
{ \vec { H } _ { 1 t } = \vec { 0 } }  \tag{12.94}\\
{ \vec { H } _ { 2 t } = \vec { 0 } }
\end{array} \Longrightarrow \left\{\begin{array}{l}
\langle H\rangle=0 \\
\Delta H_{21}=0
\end{array}\right.\right.
$$

Inserting $\langle H\rangle$ and $\Delta H_{21}$ in the constitutive relation 10.122 involving $\mu_{r}$ and $\varepsilon_{r}$, we find the limit condition:

$$
\left\{\begin{array}{rl}
\Delta E_{21} & =i \theta_{0} \mu_{r} \cdot 0  \tag{12.95}\\
0 & =i \theta_{0} \varepsilon_{r}\langle E\rangle
\end{array} \quad \forall\langle E\rangle, \Delta E_{21}\right.
$$

Since the electric fields should be not constrained, we impose that this last condition is valid in the limit of infinite permeability and zero permittivity:

$$
\begin{equation*}
\left|\mu_{r}\right| \rightarrow+\infty ; \quad \varepsilon_{r} \rightarrow 0 \tag{12.96}
\end{equation*}
$$

The impedances $Z_{e}$ and $Z_{m}$ will be so:

$$
\begin{align*}
Z_{e} & =\lim _{\varepsilon_{r} \rightarrow 0}\left(\frac{1}{\varepsilon_{r}-1} \frac{1}{s C_{0}}\right)=-\frac{1}{s C_{0}}  \tag{12.97}\\
Z_{m} & =\lim _{\mu_{r} \rightarrow \infty}\left(-\frac{\mu_{r}}{\mu_{r}-1} s L_{0}\right)=-s L_{0} \tag{12.98}
\end{align*}
$$

Finally, the magnetic mirror appears to be made of negative capacitive wires and negative inductance rings:

$$
\begin{align*}
Z_{e} & =-\frac{1}{s C_{0}}  \tag{12.99}\\
Z_{m} & =-s L_{0} \tag{12.100}
\end{align*}
$$

That structure is the circuit equivalent for a Perfect Magnetic Conduction (PMC), which currently is difficult to be found in Nature.

## Narrow band magnetic mirror

Differently from the electric mirror, the magnetic one seems to require the use of active elements, like negative capacitors and inductors.

However, if you are working at a single "frequency" $\omega_{0}$, you can substitute the negative capacitor with a positive inductor, and viceversa the negative
inductive ring can be replaced by a positive capacitive ring. The effective positive inductance $L_{e}$ and capacitance $C_{m}$ can be calculated as:

$$
\begin{align*}
L_{e} & =\frac{1}{\omega_{0}^{2} C_{0}}  \tag{12.101}\\
C_{m} & =\frac{1}{\omega_{0}^{2} L_{0}} \tag{12.102}
\end{align*}
$$

For details, see section 12.5 .4
Currently an application for the magnetic mirrors is their use as ground plane or back-reflector for flat antenna. As a matter of fact, the mushroom high-impedance surface by Sievenpiper 11 can be modelled as a non-symmetric magnetic mirror operating in a narrow band.

### 12.7.4 Superluminal screen

In principle, if the relative permittivity and permeability are small, then the light propagates in the medium faster than in empty space. More precisely, if $\left|\mu_{r} \varepsilon_{r}\right|<1$ then the phase velocity $v_{\varphi}$ is superluminal:

$$
\begin{align*}
& v_{\varphi}=\frac{1}{\sqrt{\mu_{r} \varepsilon_{r}}} c_{0}  \tag{12.103}\\
& v_{\varphi}>c_{0} \quad \Longleftrightarrow \quad\left|\mu_{r} \varepsilon_{r}\right|<1 \tag{12.104}
\end{align*}
$$

Formally a superluminal phase velocity is not in contradiction with Einstein's Relativity Theory, since it does not transfer any information. More explicitly, the phase velocity is different from the speed of propagation of an impulse or anyway of a discontinuity. The interpretation of superluminal effects is very complex and subtle and that issue is currently at debate[17, 19, [53, 54, 55]. It cannot to be treated here, though it is one of my favourite topics.

Now let suppose we wish an infinite phase velocity for a reflectionless screen. In order to obtain those features, we impose the conditions:

$$
\begin{align*}
& v_{\varphi} \rightarrow \infty \quad \Longrightarrow \quad\left|\mu_{r} \varepsilon_{r}\right| \rightarrow 0^{+}  \tag{12.105}\\
& \mu_{r}=\varepsilon_{r} \tag{12.106}
\end{align*}
$$

So it follows that both permittivity and permeability should be equal and near to zero:

$$
\begin{equation*}
\mu_{r}=\varepsilon_{r}=0 \tag{12.107}
\end{equation*}
$$

Thus the corresponding impedance $Z_{e}$ and $Z_{m}$ are:

$$
\begin{align*}
Z_{e} & =\lim _{\varepsilon_{r} \rightarrow 0}\left(\frac{1}{\varepsilon_{r}-1} \frac{1}{s C_{0}}\right)=-\frac{1}{s C_{0}}  \tag{12.108}\\
Z_{m} & =\lim _{\mu_{r} \rightarrow 0}\left(-\frac{\mu_{r}}{\mu_{r}-1} s L_{0}\right)=0 \tag{12.109}
\end{align*}
$$

That means the screen has to be made by negative capacitive wires and by short circuit rings. If we had asked a finite, superluminal phase velocity, for example 10 times greater than $c_{0}$, and a unit permittivity $\varepsilon_{r}$, we had found:

$$
\begin{align*}
& \left\{\begin{array} { l } 
{ v _ { \varphi } = 1 0 c _ { 0 } } \\
{ \varepsilon _ { r } = 1 }
\end{array} \Longrightarrow \left\{\begin{array}{c}
\mu_{r} \varepsilon_{r}=10^{-2} \\
\varepsilon_{r}=1
\end{array} \Longrightarrow\right.\right.  \tag{12.110}\\
& \mu_{r}=10^{-2} ; \quad \varepsilon_{r}=1 \tag{12.111}
\end{align*}
$$

So the impedances $Z_{e}$ and $Z_{m}$ are:

$$
\begin{align*}
Z_{e} & =\lim _{\varepsilon_{r} \rightarrow 1}\left(\frac{1}{\varepsilon_{r}-1} \frac{1}{s C_{0}}\right)=\infty  \tag{12.112}\\
Z_{m} & =\frac{10^{-2}}{1-10^{-2}} s L_{0}=\frac{1}{99} s L_{0} \tag{12.113}
\end{align*}
$$

In other words, there is no need to install any straight wire, since it would be an open circuit. Furthermore, a slightly inductive ring is needed.

Differently from the previous case, this time the screen exhibit a strong reflection (in fact $\eta \gg \eta_{0}$ ) but it does not mount any active element.

### 12.7.5 Negative refraction - DNG screen

Inside a material with negative refractive index the waves seem to propagate backward and exhibit a negative angle of refraction. In 1968 V. Veselago 3 verified that phenomena can happen for a material with both negative $\varepsilon_{r}$ and $\mu_{r}$ :

$$
\begin{equation*}
\varepsilon_{r}<0 \quad \mu_{r}<0 \tag{12.114}
\end{equation*}
$$

The concept of Double Negative (DNG) metamaterial was later investigated by J.B. Pendry and D.R. Smith [6, 7, 5].

Suppose we desire to construct a reflectionless screen allowing backward propagation. For sake of simplicity we require that:

$$
\begin{equation*}
\varepsilon_{r}=\mu_{r}=-1 \tag{12.115}
\end{equation*}
$$

More rigorously, we should also consider the eventual losses within our DNG material, so also the imaginary part of $\varepsilon_{r}$ and $\mu_{r}$.

Calculating the impedances $Z_{e}$ and $Z_{m}$ to be installed, we get:

$$
\begin{align*}
Z_{e} & =-\frac{1}{2} \frac{1}{s C_{0}}  \tag{12.116}\\
Z_{m} & =-\frac{1}{2} s L_{0} \tag{12.117}
\end{align*}
$$

Let us notice that in order to achieve a broadband back-propagation both negative inductors and capacitors are needed. In particular, the negative capacitor is mounted on the straight wire, while the negative inductor is on the ring.

## Narrow band DNG screen

If you accept the screen to exhibit negative refraction at a single frequency, then it is possible to replace the previous circuits with a straight inductive wire and a capacitive ring. The effective positive inductance $L_{e}$ and capacitance $C_{m}$ can be calculated as:

$$
\begin{align*}
L_{e} & =\frac{1}{2} \frac{1}{\omega_{0}^{2} C_{0}}  \tag{12.119}\\
C_{m} & =2 \frac{1}{\omega_{0}^{2} L_{0}} \tag{12.120}
\end{align*}
$$

As a matter of fact, the first DNG material realized by David R. Smith et al. (4) was made by thin inductive wires and by capacitive split rings.

## Chapter 13

## Holographic Television


#### Abstract

...but there are times when a critic truly risks something, and that is in the discovery and defense of the new.


Anton Ego's intellectual honesty, Ratatouille, 2007

In this chapter we examine how to exploit the Huygens' Principle in order to realize a 3D TV or "holographic" television. I cannot guarantee such a device can be effectively built, because there many technical difficulties related to the nano-scale elements' size. Of course, this is just a first-concept proposal for an application, a nice theoretic speculation, or, better, a challenge. We start from a "fantastic" phenomenon and we try to come backward to its origin in order to reproduce it.

### 13.1 Holographic television concept

As already anticipated in sec 2.3.2, in a common 2D television each pixel is characterized by a color and a light intensity. If you look at the screen from different viewpoints, you will see always the same flat image. On the contrary, a holographic television or 3D TV allows to observe the framed objects from different perspective, in their full 3-Dimensionality. Therefore, in a 3D TV each "pixel" is associated not only to the color and intensity, but also to the direction of light.

### 13.1.1 3D videocamera

Thanks to computer graphics, today it is possible to create 3D virtual reality. In fact, many movies and videogames are developed in a fictitious 3D space, in which the objects are described through their coordinates and colors.

Shortly, in such a virtual reality the environments and the objects are known in their full 3-Dimensionality, so you can create a 3D virtual image or movie using just the computer and then you can display it through a holographic screen.

However, if you desire to frame a real object, then some kind of 3D videocamera is needed. That holo-camera should measure, point by point, the light field emitted or reflected by the shot object. In other words, it must record the incident EM fields. Differently from a common 2D videocamera, a 3D one had to map many viewpoints, so probably it would not need any objective or photographic lens. Rather, a 3D videocamera would exploit an en-plein air sensor-retina. More simply, every point or pixel on the 3D videocamera should be exposed to the whole light-field. On the contrary, in a 2D videocamera each sensor can "see" just a single light-ray passing through the objective hole.

### 13.1.2 Holographic pixel

Every pixel for the holographic videocamera or holo-camera must be composed by some sensors able to detect the incident, tangential electric and magnetic fields. Those sensors could be made by thin wires and rings, sensible to $\vec{E}_{t}$ and $\vec{H}_{t}$ respectively, but other solutions could be available. The incident $\vec{E}_{t}$ and $\vec{H}_{t}$ would induce currents $\vec{J}_{t}$ and "vortices" $\overline{\bar{D}}_{e}$ in each pixel. Those currents can be recorded and the information can be transmitted elsewhere. Alternatively, the values of fields $\vec{E}_{t}$ and $\vec{H}_{t}$ could be memorized.

### 13.1.3 Byte per pixel

Both $\vec{J}_{t}$ and $\overline{\bar{D}}_{e} \cdot \vec{n}_{21}$ are vectors tangent to the pixel's surface. Supposing to use 1 Byte to describe each vector component, globally every pixel will require 4 Byte per each color. As a matter of fact, each color is associated to a frequency, so if we want to reproduce 3 primary colors (Red, Green, Blue), we need 3 currents $\vec{J}_{t}$ and 3 "vortices" $\overline{\bar{D}}_{e} \cdot \vec{n}_{21}$. Finally, 12 Bytes have to be assigned to each pixel.

$$
\begin{equation*}
3 \times 4 \text { Byte }=12 \text { Byte } \quad \text { per each pixel } \tag{13.1}
\end{equation*}
$$

Usually, in a standard 2D TV a pixel accounts for just 3 Bytes, one per each color.

### 13.1.4 Holographic screen

The screen of a holographic television must reproduce the light field generated by the framed objects. In order to make the image 3-Dimensional, each emitting pixel should be crossed by currents $\vec{J}_{t}$ and vortices $\overline{\bar{D}}_{e}$ so that:

$$
\begin{align*}
\vec{E}_{t, i r r} & =-\left(\frac{s}{c_{0}}\right) \overline{\bar{D}}_{e} \cdot \vec{n}_{21}  \tag{13.2}\\
\eta_{0} \vec{H}_{t, i r r} & =-\eta_{0} \vec{J}_{t} \tag{13.3}
\end{align*}
$$

where $\vec{E}_{t}$ and $\vec{H}_{t}$ are the tangential fields produced by the framed objects. Those fields have thus to be re-irradiated by the screen. Let us notice that we are guessing that the screen is projecting just on a side, towards the observer, while on the other side the screen is opaque $\left(E_{t}=0, H_{t}=0\right)$. In synthesis, an active circuit must be present on each pixel, allowing to generate the desired currents and vortices. Thanks to the Boundary Conditions previously derived, it is possible to compute these EM sources.


Figure 13.1: 3D holographic TV. Each pixel is characterized by currents $\vec{J}_{t}, \overline{\bar{D}}_{e}$ and can exhibit a different color depending on the viewpoint.

### 13.1.5 Curved holographic screen

In order to ensure an optimal 3D vision of a scene set, a curved holographic screen can be suggested instead of a flat one. For example, it could be half-cylindrical or spherical. Let us notice that till now we have not required the holo-camera or the television to be flat, and anyway it is not mandatory that they have the same geometry or size. Actually, it is sufficient that the screen re-emits the fields produced by the objects as they were directly behind the screen itself.

Generally speaking, it is convenient the holo-camera to be convex or at least flat, in such a way that each pixel/sensor can see just the surrounding environment and not the other pixels. On the contrary, for the projecting screen many geometric configurations are possible, depending on the demanded type of vision. For example, a convex, cylindrical screen could fit well in a museum or if the spectator walks around the virtual image. A concave cylindrical screen is instead more indicated if the observer is supposed to feel inside an environment.

### 13.2 Resolution for a 3D TV

In this section we face the question of how much information is stored in a holographic image and if it can be compressed or digitally elaborated.

As we have seen, each holographic pixel can be regarded as a source of the EM fields, thus we can invoke the Huygens' Principle in order to map a 3D light field on a 2D surface. However, a pixel has a finite size and if we require that the radiated fields are almost uniform on it, then the pixel must be much smaller than the operating wavelength:

$$
\begin{equation*}
\Delta y_{P I X} \ll \frac{\lambda_{0}}{2 \pi} \tag{13.4}
\end{equation*}
$$

where $\Delta y_{P I X}$ is the edge length for a single pixel.
A human being is able to see colours in the visible spectrum, so with a wavelength $\lambda_{0}$ in the range between 380 nanometers (blue-violet) and 750 nanometers (red). The maximum pixel's size could be so of about 40 nanometers:

$$
\begin{equation*}
\Delta y_{P I X}=40 \mathrm{~nm}=4 \cdot 10^{-8} \mathrm{~m} \tag{13.5}
\end{equation*}
$$

The number of pixels on a $30 \mathrm{~cm} \times 40 \mathrm{~cm}$ screen would be so:

$$
\begin{align*}
& \text { \#pixels }=\frac{30 \cdot 10^{-2} \mathrm{~m}}{4 \cdot 10^{-8} \mathrm{~m}} \cdot \frac{40 \cdot 10^{-2} \mathrm{~m}}{4 \cdot 10^{-8} \mathrm{~m}}=7,5 \cdot 10^{13}  \tag{13.6}\\
& \text { \#pixels }=7,5 \cdot 10^{13} \tag{13.7}
\end{align*}
$$

So the number of pixels would be 75 thousands of billions! Moreover, considering that 12 Bytes are need per each pixel, we can guess the information stored in a single holographic image would be enormous, equal to 900 Tera Bytes!

$$
\begin{equation*}
12 \text { Bytes/pixel } \times 7,5 \cdot 10^{13} \text { pixels }=9 \cdot 10^{14} \text { Bytes } \tag{13.8}
\end{equation*}
$$

Frankly I think that currently it is not possible to construct a digital holographic television with that resolution. An analogical prototype would be probably much easier to be realized. As a matter of fact, in the $\mathrm{XX}^{\text {th }}$ century the picture resolution for the analogical photography was much higher if compared to the
digital one. Besides, the classic holograms generated by laser beam interference are impressed on analogical photographic plates, so in practice a high-density information recording is not so impossible.

### 13.2.1 Compressing the information

Even if the pixel size cannot be greater than $\lambda_{0}=380 \mathrm{~nm}$, though I guess it is possible to effectively compress the information for 3D image without a significant quality loss.

Actually, mankind can manage a limited amount of information. For example, a human being can hardly distinguish details below 0.1 millimeters $\left(10^{-4} m\right)$ and usually 3 Bytes - which correspond to $2^{24}=16,777,216$ combinations - are plenty sufficient to describe all visible colours (and other more!). Shortly, the quality for a 2D image is related to its spatial and cromatic resolutions. Human sensibility however is limited and it cannot appreciate High-Definition details over a certain level. Therefore, we can conceive a holographic screen made by macro-pixels with size $\Delta y \approx 10^{-4} \mathrm{~m}$. Every macro-pixel would be composed by an array of elementary nano-pixels with size $\Delta y_{P I X} \approx 4 \cdot 10^{-8} \mathrm{~m}$.

Every macro-pixel can exhibit a different color or intensity depending from the viewpoint, so we have to take in account the angular resolution demanded for our screen. Mankind have a minimal angle of resolution of about half minute


Figure 13.2: Angular resolution.
degree ( $0.5^{\prime}$ ). If the holographic screen can be observed within a visual range of $180^{\circ}$, then the total number of view angles is:

$$
\begin{equation*}
180^{\circ} /\left(0.5^{\circ} / 60\right)=21,600 \approx 2 \cdot 10^{4} \quad \text { view angles } \tag{13.9}
\end{equation*}
$$

Honestly I think that angular resolution is exaggerated: probably just $2 \cdot 10^{2}$ view angles could be sufficient for a realistic 3D effect, specially if the observer is quite distant from the screen.

Anyway, the basic idea is that for a holographic television three resolution types have to be guaranteed:

- Spatial resolution.

Example: $\Delta y \approx 10^{-4} \mathrm{~m}$

- Chromatic resolution.

Example: 12 Bytes for RGB colors and polarization.

- Angular resolution.

Example: $1^{\circ}$.
Let us analyze more in detail how to compress the information.

### 13.2.2 Macro-pixel: phased array

Suppose a macro-pixel can be seen just from one single direction. In that case, it is emitting a light with a wavevector $\vec{k}$ We call $\vec{k}_{t}$ the tangential component


Figure 13.3: Each macropixel works as a linear phased array, emitting many wavefronts in different directions.


Figure 13.4: Each macropixel is made by many phased nanopixel. For each color and tangent wavevector $\vec{k}_{t}$ is possible to define some surface source distribution $\vec{J}(x, y), \overline{\bar{D}}_{e}(x, y)$ on the whole macropixel.
of $\vec{k}$, so the current distribution on the macro-pixel is:

$$
\begin{equation*}
\vec{J}(x, y)=\vec{J}_{0} e^{i\left(k_{t x} x+k_{t y} y\right)}=\vec{J}_{0} e^{i \vec{k}_{t} \cdot \Delta \vec{x}} \tag{13.10}
\end{equation*}
$$

and analogous for the "vortices" $\overline{\bar{D}} \cdot{ }_{e} \cdot \vec{n}_{21}$.
Even if the macro-pixel is much larger than the single nano-pixels, though the source distribution is known on each of them and it is completely described by few parameters, which are:

$$
\begin{equation*}
\vec{J}_{0} ; \quad \overline{\bar{D}}_{e, 0} \cdot \vec{n}_{21} ; \quad \vec{k}_{t} \tag{13.11}
\end{equation*}
$$

Those three variables should be defined for each primary color.

Shortly, each tangent wavevector $\vec{k}_{t}$ is associated to a certain viewpoint.

$$
\begin{align*}
\vec{k}_{t} & =k_{t x} \vec{e}_{x}+k_{t y} \vec{e}_{y}  \tag{13.12}\\
k_{t} & =\sin (\theta) k_{0} \tag{13.13}
\end{align*}
$$

where $\theta$ is the angle of irradiation. Since $k_{t}$ is usually in the range $\left[-k_{0} ; k_{0}\right]$, we


Figure 13.5: Chebyshev's spacing for $k_{t}$ : the interval [ $-k_{0} ; k_{0}$ ] is divided in equals parts, so that the tangent wavenumber $k_{t}$ is linearly spaced. Let us notice that angles are not equally spaced, since $k_{t}=\sin (\theta) k_{0}$.
can discretize that interval in $N_{\theta}$ equal parts.

$$
\begin{align*}
& \vec{k}_{t}\left(n_{x}, n_{y}\right)=n_{x}\left(\frac{2 k_{0}}{N_{\theta}-1}\right) \vec{e}_{x}+n_{y}\left(\frac{2 k_{0}}{N_{\theta}-1}\right) \vec{e}_{y}  \tag{13.14}\\
& \text { with } n_{x}, n_{y} \in \mathbb{Z} \quad \text { and } \quad-\frac{1}{2}\left(N_{\theta}-1\right) \leq n_{x}, n_{y} \leq \frac{1}{2}\left(N_{\theta}-1\right)
\end{align*}
$$

That is just an option. Be aware that in this way the view angles will be no more equally spaced: in fact, the resolution will appear finer for an observer looking perpendicularly to the screen.

Finally, if we require that the macro-pixel is visible from many directions, then the local source distribution can be approximated by mean of Fourier expansion, e.g.:

$$
\begin{equation*}
\vec{J}(x, y)=\sum_{n_{x}} \sum_{n_{y}} \vec{J}_{0}\left(n_{x}, n_{y}\right) e^{i \Delta \vec{x}^{T} \cdot \vec{k}_{t}\left(n_{x}, n_{y}\right)} \tag{13.15}
\end{equation*}
$$

and analogous for the "vortices" $\overline{\bar{D}}_{e} \cdot \vec{n}_{21}$. In summary:

1. For each macro-pixel, each direction $\vec{k}_{t}$ and each primary color, the amplitudes $\vec{J}_{0}$ and $\overline{\bar{D}}_{e, 0} \cdot \vec{n}_{21}$ must be assigned. If every (complex) component of $\vec{J}_{0}$ is described with 1 Byte, and there are 3 primary colors (RGB), then for each direction $\vec{k}_{t}$ on each macro-pixel 12 Bytes are needed.
2. Each direction $\vec{k}_{t}$ can be identified by two integer number $n_{x}$ and $n_{y}$. If every number can take $N_{\theta}$ different values, then the number of possible directions is $N_{\theta}^{2}$.
3. Hence, each macro-pixel is associated to $12 \cdot N_{\theta}^{2}$ Bytes. If we use 1 Byte to identify $n_{x}$, that means $N_{\theta}=2^{8}=256$, so each macro-pixel would require:

$$
\begin{align*}
& 12 \cdot 2^{16} \text { Bytes } \approx 12 \cdot 6.6 \cdot 10^{4} \text { Bytes } \approx 7.9 \cdot 10^{5} \text { Bytes }  \tag{13.16}\\
& 12 \cdot 2^{16} \text { Bytes } \approx 790 k B \approx 1 \mathrm{MB} \tag{13.17}
\end{align*}
$$

Thus, each macro-pixel on a holographic TV accounts for about 1 MegaByte, instead of the 3 Bytes associated to a 2D television pixel.
Let us assume the macro-pixel size is around $1 \mathrm{~mm}=10^{-3} \mathrm{~m}$. In that case, the total number of macro-pixels for a $30 \mathrm{~cm} \times 40 \mathrm{~cm}$ screen is:

$$
\begin{align*}
& \text { \#macropixels }=\frac{30 \cdot 10^{-2} \mathrm{~m}}{10^{-3} \mathrm{~m}} \cdot \frac{40 \cdot 10^{-2} \mathrm{~m}}{10^{-3} \mathrm{~m}}=1.2 \cdot 10^{5}  \tag{13.18}\\
& \# \text { macropixels }=1.2 \cdot 10^{5} \tag{13.19}
\end{align*}
$$

Since each macropixel accounts for $10^{6}$ Bytes, the information associated to a single holographic image would be:

$$
\begin{equation*}
1.2 \cdot 10^{11} \text { Bytes }=120 \mathrm{~GB} \tag{13.20}
\end{equation*}
$$

120 GigaBytes per image are still too many, but comparing this result the previous 900 Tera Bytes ( $\approx 10^{15}$ Bytes), you can notice that we have compressed the image of a factor close to $10^{4}$. In other words, we have reduced the essential information 10 thousand times.

### 13.2.3 Other compression techniques

Currently, managing 120 GB per image is still quite a hard task, specially if you are interested in a television with a high temporal resolution (e.g. 60 Hz ). Anyway, the procedure just shown is only an example of how to compress the information. Actually there are many other methods which are already used in the Digital Image Processing and usually they are based on the Fourier analysis. For example, videos on Youtube can be characterized by low spatial and chromatic resolutions, but that fact allows a rapid transmission of the movie through the internet.

### 13.2.4 Comparison between 2D TV and 3D holographic TV

Let us now consider two televisions, with the same pixel number, and the same spatial and chromatic resolutions. Even if all those parameters are identical, anyway a 3D holographic TV will require much more information with respect to a classic 2D TV. In fact, for a 2D TV the total information associated to an image is:

$$
\begin{equation*}
3 \text { Bytes • \# pixels } \tag{13.21}
\end{equation*}
$$

That amount of data is associated to just 1 point of view. A holographic TV, instead, must transmit the information related to $N_{\theta}^{2}$ viewpoints, namely it is characterized also by an angular resolution. For each viewpoint 12 Bytes could be necessary in order to describe the local EM sources (like currents).

$$
\begin{array}{ll}
\text { 2D TV : } & 3 \text { Bytes } \cdot 1 \cdot \# \text { pixels } \\
3 \mathrm{D} \mathrm{TV} \mathrm{:} & 12 \text { Bytes } \cdot N_{\theta}^{2} \cdot \# \text { pixels } \tag{13.23}
\end{array}
$$

So, for the considered case, a 3D TV need an amount of data $4 N_{\theta}^{2}$ times greater than the one required by a 2 D TV.

### 13.3 Advantages and drawbacks

A holographic TV like the one just described is quite different from the classic concept of hologram. In fact, usually holographic pictures are obtained by illuminating a static subject with a laser beam, making it to interfere with a reference one. Hence the interference pattern is recorded on a photographic plate. In that sense, the production of the hologram is quite "analogical" and static. Thus, the main advantage of a 3D TV over a classic laser-produced hologram would be the possibility to change dynamically. In fact, a 3D TV could display different images or movies, since its pixels are reconfigurable. On the contrary, in a standard hologram pictures are statically recorded on the support. Moreover, usually laser-produced holograms are mono-chromatic, while the pixels in 3D TV can be controlled to emit various colors.

Unfortunately, today the production of a holographic TV seems to be very hard because of technical difficulties. As already said, pixels have to be constructed at nanoscale and specially they have to be controlled and that would require a great computational power. Currently those two conditions seems very far to realized, but I cannot exclude that some nano-tech company is already working on that topic.

In my opinion, it is quite probable that holographic TV will be anticipated by some other "augmented reality" device - like 3D glasses - which are easier to make. However, 3D glasses are conceived to be used by a single person, while a 3D TV would be intended to be watched by many people at the same time. Another possible application, coming now in my mind, consists in holographic projections at cinemas, for very realistic 3D movies. Actually, that could be an alternative way to overcome some technological limits, but this is just a hypothesis and I am not sure about its effective realizability or convenience.

## Chapter 14

## Invisibility cloak

"And don't worry. I've got this." Emmerich reaches up to his shoulder and activates the optical camouflage. "It's the same stealth technology as the ninja."

Metal Gear Solid, Konami, 1998
The idea of achieving invisibility is quite widespread in popular culture: myths, literature, comics, TV series, movies and also videogames. The number of characters who become invisible by mean of some magic device or fantastic technology is very high. I can just mention the Greek hero Perseus, the princess Angelica in the Orlando Furioso, the knight Siegfried in the Nibelungenlied saga, Harry Potter in fantasy novels by J.K. Rowling and so on. Optical stealth technologies are often shown or described also in science fictions like The Invisible Man by H.G. Wells, the Star Trek series, videogames like Metal Gear Solid and Halo etc. Clearly, such a wonderful possibility of getting invisible must appear very fascinating to a lot of people.

In this chapter we are going to analyze the question of invisibility under a scientific point of view, trying to be realistic and as rigorous as we can, without concealing practical difficulties. Anyway, this can be regarded also as a challenge to imagination, an encouragement to explore the limits of engineering and to open the mind to new possibilities.

### 14.1 Cloaking techniques for invisibility

Let us now analyze which are the conditions required for an invisibility cloak or cloaking device. Generally speaking, in order to make an object invisible (without destroying it), you have to ensure that it cannot disturb the light field outside the volume occupied by the object itself.

In other words, an ideal cloaking device should be endowed with the following features:

- no reflection: waves impinging on the cloak must be no reflected.
- no shading: the system must not project any shadow and it must be absorption-less.
- no refraction: incident waves must pass unaltered beyond the cloaked system. More precisely, from the exterior you cannot detected any deviation or phase shift owned to refraction.
- no emission: The cloaked system should not emit any waves, i.e. it cannot radiate. However, that is almost impossible because of the black-body radiation.

In general, the "invisible" object should not produce any effect that could be detected outside the cloaking device.

Those principles have a quite large application, since they can be applied also to different kinds of wave, for example acoustic or mechanical ones. Here we shall focus the attention on the electromagnetic (EM) waves, and in particular on the visible light.

### 14.1.1 Partial invisibility

As matter of fact, in the every day life it is possible to observe some kind of partial invisibility, though not suitable for cloaking.

For example, the common glass is quite a transparent medium, and it can happen that someone clash against a polished glass because "he didn't see it". However, the common glass is not perfectly invisible, in fact:

- it partially reflects the incident light
- it is not transparent for the infra-red waves
- because of refraction, the glass deflects the light, and that effect is exploited for the designing of lenses. Thus, even if the glass was be perfectly transparent, it could be anyway detected thanks to refraction.

Probably the best example of an invisible medium is the air, whose relative permittivity and permeability are quite near to 1 . However, surrounding an object with air will not make it invisible. Actually, our problem is not to find a transparent medium, but to cloak an arbitrary object.

In a certain way, some stealth aircraft like the F-117 can be regarded as "invisible", since they cannot be detected by radar. However, their camouflage is based on the minimization of the echo (i.e. the reflection) in the radar direction. That effect can be obtained thanks to the particular aircraft shape and to the partial absorption of the impinging energy. So the echo can be very low at certain frequency, but anyway the aircraft will project a shadow behind its tail.

### 14.1.2 Basic methodology

In synthesis, as long as I known there are only two physical methods to render an object invisible:

1. making the object globally transparent, in such a way that light can pass through it without being disturbed.
2. bending the light around the object, in such a way that outside the light field is unperturbed.


Figure 14.1: The two main methods to make an object invisible. Combinations of them are also possible.

That is just a basic classification, since other methods are possible, like a combination of the two ones just mentioned. Now let us analyze them more in detail.

### 14.1.3 Transparency cloak

If we want to achieve invisibility by mean of transparency, then the incident light must pass unaltered through the object, as this latter was made of vacuum. In practice, we have to make the object optically transparent and non-refractive. That is possible just if:

- the object's impedance $\eta$ is equal to the one of the surrounding medium, i.e.:

$$
\begin{equation*}
\eta=\eta_{0} \tag{14.1}
\end{equation*}
$$

That ensures the absence of reflection.

- the object's refractive index $n$ is equal to the one of the surrounding medium, i.e.:

$$
\begin{equation*}
n=n_{0} \tag{14.2}
\end{equation*}
$$

Since the refractive index is relate to the wave speed, that condition can be replaced by an equivalent one:

$$
\begin{equation*}
c=c_{0} \quad \text { equal phase velocity } \tag{14.3}
\end{equation*}
$$

So the wave should propagate with the same velocity outside and inside the object. That ensures the absence of refraction.

Those two conditions can be extended also to other types of waves, e.g. acoustic or mechanical ones, once you have well defined the characteristic impedance $\eta$ and speed $c$.

It is quite important to notice that, in order to achieve perfect invisibility, both the conditions have to be satisfied. The condition $\eta=\eta_{0}$ removes the reflection, while $c=c_{0}$ ensures the absence of refraction.

The classic example for the transparency cloaking is the one of the pyrex bottle dipped in the glycerine. If you submerge a pirex glass in glycerine, the bottle seems to disappear. In fact, the two media have almost the same refractive
index $n \approx 1.47$, and since they are not magnetic ( $\mu \approx \mu_{0}$ ) they have also the same optical impedance $\eta$.

The idea of achieving "invisibility" through that method is not new, and it is well described in the novel The invisible man[56], by H.G. Wells. Griffin, the scientist who became transparent, tells Dr. Kemp:

You make the glass invisible by putting it into a liquid of nearly the same refractive index; a transparent thing becomes invisible if it is put in any medium of almost the same refractive index. And if you will consider only a second, you will see also that the powder of glass might be made to vanish in air, if its refractive index could be made the same as that of air; for then there would be no refraction or reflection as the light passed from glass to air.

Let us point out that in order to achieve a perfect invisibility the cloaked object should be optically homogeneous, i.e. $\eta$ and $c$ should be constant everywhere inside the object itself. In fact, even a small imperfection could make it detectable.

### 14.1.4 Scattering cancellation

Even if an object has not the same impedance and refractive index of the surrounding medium, it is still possible to make it globally invisible. The basic concept is to cancel the light scattered by the object itself. That can be done producing a destructive interference. For example, you can shield an object with a material shell cancelling both the reflection and the refraction. The light still pass through the object, but this latter cannot be detected because the shell generates a wave in phase opposition, which neutralizes the scattering. The method was deeply investigated by the groups of Nader Engheta and Andrea Alú $57,58,59,60$, which carefully analyzed the features required for the cloaking shell. The light scattered by the object is proportional to its oscillating electric $\vec{p}$ and magnetic $\vec{m}$ dipoles. Thus, in order to achieve a bulk invisibility, the global polarization and magnetization for the system must be zero. In other words,


Figure 14.2: Scattering cancellation. The external shell electrically oscillates with a phase opposite to that of the cloaked body, thus creating a destructive interference. As a result, the scattering is partially or totally suppressed.
the object and the shell should be characterized by equal and opposite dipole
moment $\vec{p}$ and $\vec{m}$, so that the global radiation is cancelled.

$$
\begin{align*}
\vec{p}_{1}+\vec{p}_{2} & =\overrightarrow{0}  \tag{14.4}\\
\vec{m}_{1}+\vec{m}_{2} & =\overrightarrow{0} \tag{14.5}
\end{align*}
$$

These last conditions are just a first-order approximation. In fact, if you are interested to eliminate the whole radiation, you should also consider the higherorder terms, e.g. quadrupole moments and so on.

Supposing that the object and the shell are made by homogeneous, isotropic, linear material, we can express their respective electric and magnetic dipole in function of polarization and magnetization:

$$
\begin{align*}
& \vec{p}=V\langle\vec{P}\rangle=V \chi_{E E}\langle\vec{E}\rangle=V\left(\varepsilon_{r}-1\right)\langle\vec{E}\rangle  \tag{14.6}\\
& \vec{m}=V\langle\vec{M}\rangle=V \chi_{H H}\langle\vec{B}\rangle=V\left(\mu_{r}-1\right)\langle\vec{B}\rangle \tag{14.7}
\end{align*}
$$

where $V$ is the volume, $\langle$.$\rangle is the average on space, and the \chi \mathrm{s}$ are the relative susceptibilities. The condition of dipole cancellation for a 2-bodies system can be so rewritten in the form:

$$
\begin{align*}
V_{1}\left(\varepsilon_{r 1}-1\right)\left\langle\vec{E}_{1}\right\rangle+V_{2}\left(\varepsilon_{r 2}-1\right)\left\langle\vec{E}_{2}\right\rangle & =\overrightarrow{0}  \tag{14.8}\\
V_{1}\left(\mu_{r 1}-1\right)\left\langle\vec{B}_{1}\right\rangle+V_{2}\left(\mu_{r 2}-1\right)\left\langle\vec{B}_{2}\right\rangle & =\overrightarrow{0} \tag{14.9}
\end{align*}
$$

Supposing that the average fields $\langle\vec{E}\rangle$ and $\langle\vec{B}\rangle$ are the same for the two bodies (but that could be not guaranteed), then it follows:

$$
\begin{gather*}
\left\{\begin{array}{l}
\left\langle\vec{E}_{1}\right\rangle=\left\langle\vec{E}_{2}\right\rangle \\
\left\langle\vec{B}_{1}\right\rangle=\left\langle\vec{B}_{2}\right\rangle
\end{array} \Longrightarrow\right.  \tag{14.10}\\
V_{1}\left(\varepsilon_{r 1}-1\right)+V_{2}\left(\varepsilon_{r 2}-1\right)=\overrightarrow{0}  \tag{14.11}\\
V_{1}\left(\mu_{r 1}-1\right)+V_{2}\left(\mu_{r 2}-1\right)=\overrightarrow{0} \tag{14.12}
\end{gather*}
$$

Therefore, once the object's permittivity $\varepsilon_{r 1}$ and permeability $\mu_{r 1}$ are given, the ones for the shell $\varepsilon_{r 2}$ and $\mu_{r 2}$ can be calculated as:

$$
\begin{align*}
& \varepsilon_{r 2}=1-\frac{V_{1}}{V_{2}}\left(\varepsilon_{r 1}-1\right)  \tag{14.13}\\
& \mu_{r 2}=1-\frac{V_{1}}{V_{2}}\left(\mu_{r 1}-1\right) \tag{14.14}
\end{align*}
$$

It should be noticed that if you want to cloak an object with positive $\varepsilon_{r 1}$ and/or $\mu_{r 1}$ - which is quite a common request - then a shell exhibiting negative $\varepsilon_{r 2}$ and/or $\mu_{r 2}$ could be mandatory in some cases. In fact, the cloaking shell could be made of a Double Negative (DNG) metamaterial or support plasmonic wave propagation, depending on the required properties. For further details I suggest to read some beautiful articles by A. Alù, F. Monticone et al. [57, 58, 60, 61], which deal exactly with the design of cloaking shell devices.

### 14.1.5 Limits for the "bulk" invisibility

Whether an object is made transparent or you cancel its scattering by covering it with a cloaking shell, in both the cases light propagates across the object itself. Hence the methods just described in the previous section have some important limits.

In the first case, the object should be made optically homogeneous and isotropic. That task could be extremely difficult if you are considering an object composed by different materials. The human body itself is made by different tissues, so it is quite hard to believe that Griffin, the Invisibile man by H.G. Wells, truly became transparent.

In the second case, i.e. if you intend to cancel the scattering, you should know in advance the object's scattering itself. In fact, different objects will reflect and refract light in different ways, so you have to known some of the object's properties, like the shape and the constitutive materials. In practice, you can project a shell which is able to cloak very well one specific object, but not another one. Actually, the shell is tailored to work just for a single object. For that reason, that coverage technique is mainly adopted to cloak objects with simple-geometry, like spheres or cylinders.

### 14.1.6 Camouflage Invisibility

Another technique for making an object invisible is to disguise it within the environment or to deflect the light around it. The former method consists in some kind of optical camouflage. Let us consider for example a system like the one in fig. 14.3. A man is standing between two screens, while an external


Figure 14.3: Invisibility by video retro-projection.
observer is looking at the first one. A videocamera is shooting the environment behind the system, and the image is projected on the front screen. In that way the observer can see the objects beyond the standing man, but not the man himself, who turns out to be invisible.

That retro-projective technique has been effectively implemented by Tachiet al. [62, 63] in 2003: the Japanese researchers have developed a rendering method such that the images behind a mantle are displayed on its front side.

A similar method is also shown in the 2011 movie Mission Impossible: Ghost Protocol, when the agent Ethan Hunt steals into the Kremlin making use of an active-rendering screen which mimics the corridor around.

That principle of imitating the surrounding environment for stealth purpose is quite widespread in Nature. For example, the chameleon can change its skin
color thanks to chromatophores. Moreover, standard uniforms dressed by modern soldiers are designed to reproduce the ambient colors. Anyway, that kind of camouflage is quite far to be perfect.

## Perspective limits

Notwithstanding the possible technological refining, the main drawback for the retro-projective or camouflage techniques is that they work well just for one or few viewpoints. For example, if you look to the 2D screen from a different position, you can easily notice that you are looking to a projected image. In fact, in Mission Impossible: Ghost Protocol the screen trick is discovered because two guards are looking at the projection from 2 different points of view.

That problem occurs also if you try to achieve invisibility through some hall of mirrors: you can deflect the light around the region to be cloaked, but


Figure 14.4: Example of "invisibility" achieved through a hall of mirrors. An object placed in the concealed region, between the observer and the apple, would be not visible.
usually there is at least one position from where the device can be noticed. Besides, bending the light implies it will travel for a longer path, therefore the environment behind the cloaked region could appear farther than in reality.

### 14.1.7 Invisibility by light bending

At a theoretical level, invisibility by deflection can be achieved if you perfectly curve the light around an object, so that an external observer cannot detect it.

That possibility of bending light guiding it through some metamaterial coating was seriously considered by J.B. Pendry et al., who explained it in a beautiful paper [64]. The proposed method is based on the Coordinate-transformation technique, which can be summarized in this way:

1. Consider a single case of EM wave propagation for which you know exactly the EM fields.
2. Distort your space domain applying a coordinate-transformation, in such a way that light will travel along the desired path.
3. Apply the same coordinate-transformation to Maxwell's Equations, so that you can retrieve the needed permittivity $\varepsilon$ and permeability $\mu$ in every point of the transformed space.

Let us clarify that concept with an example. Suppose you want to cloak an object contained in a spherical shell, which has to be designed to guide away the incident light. Therefore you have to determine which kind of (meta-)material your coating should be made of.


Figure 14.5: Transformation optics technique.
With the coordinate transformation technique, you can start considering a plane wave propagating in vacuum $\left(\varepsilon=\varepsilon_{0}, \mu=\mu_{0}\right)$. Then you curve the original space domain, creating a hole of radius $R_{1}$ where light cannot propagate. A simple, linear transformation proposed by Pendry et al. allows to mimic a spherical cloaking shell whose external radius is $R_{2}$ :

$$
\begin{align*}
r^{\prime} & = \begin{cases}R_{1}+\frac{R_{2}-R_{1}}{R_{2}} r & \text { for } 0 \leq r \leq R_{2} \\
r & \text { for } r>R_{2}\end{cases}  \tag{14.15}\\
\theta^{\prime} & =\theta  \tag{14.16}\\
\phi^{\prime} & =\phi \tag{14.17}
\end{align*}
$$

where $r^{\prime}, \theta^{\prime}$ and $\phi^{\prime}$ are the transformed spherical coordinates. Let us notice that for $r>R_{2}$, so outside the cloaking device, there is no domain's distortion.

After that transformation, the EM waves are bent around the hole, which did not exist in the original space. For the considered case, the relative permittivity and permeability within the coating $\left(R_{1}<r^{\prime}<R_{2}\right)$ turn out to be 64:

$$
\begin{align*}
& \varepsilon_{r^{\prime}}^{\prime}=\mu_{r^{\prime}}^{\prime}=\frac{R_{2}}{R_{2}-R_{1}} \frac{\left(r^{\prime}-R_{1}\right)^{2}}{r^{\prime 2}}  \tag{14.18}\\
& \varepsilon_{\theta^{\prime}}^{\prime}=\mu_{\theta^{\prime}}^{\prime}=\frac{R_{2}}{R_{2}-R_{1}}  \tag{14.19}\\
& \varepsilon_{\phi^{\prime}}^{\prime}=\mu_{\phi^{\prime}}^{\prime}=\frac{R_{2}}{R_{2}-R_{1}} \tag{14.20}
\end{align*}
$$

That is just a possible transformation, since other solutions are available. In practice, this cloaking technique is based on two main requirements:

- The cloaked object should be not hit by the waves impinging from outside, so that it is completely concealed.
- The incident waves enter into the coating without reflection, then they are guided through the coating itself, curving around the cavity, and finally emerge outside returning to their original trajectory.

In order to avoid reflection the external shell surface must be perfectly matched with the surrounding environment. In other words, the condition for matched impedance $\eta$ must be satisfied.

The light-rays which would struck against the object have to be strongly deflected: that effect can be obtained by diminishing the refractive index towards the inner layers. Since $n=c_{0} / c=c_{0} / v_{\varphi}$, if $n$ is less then 1 then a superluminal phase velocity is required. In fact, the internal waves have to travel across a longer path in the same time, so that they exit in phase with external waves. If a superluminal phase velocity is required, then permittivity and permeability smaller than 1 or near to zero could be mandatory:

$$
\begin{equation*}
v_{\varphi}>c_{0} \quad \Longrightarrow \quad\left|\mu_{r} \varepsilon_{r}\right|<1 \tag{14.21}
\end{equation*}
$$

For that reason, an Epsilon Near Zero (ENZ) or a Mu Near Zero (MNZ) metamaterial could be useful for the construction of such an invisibility mantle.

## Technical difficulties

From a theoretical point of view, the method proposed by Pendry et al. is truly close to be perfect, since it works for any arbitrary object. Unfortunately, the construction of a cloaking device in the visible spectrum appears still far to be a reality. Indeed, the production of the ideal meta-material (MTM) coating presents some very severe technological limits:

- the unit cell for a MTM must be sub-wavelength, and for the visible light the wavelength $\lambda_{0}$ is within the $380 \div 750 \mathrm{~nm}$ range. Therefore, for the realization of a bulk optical MTM the mass-production of nanoscopic devices would be mandatory. Currently that is an active research area.
- a cloaking device like to one conceived by Pendry et al. is inhomogeneous and anisotropic. Building a bulk material with those features is an additional technical challenge.
- usually passive MTM are dispersive $\left(\frac{\partial \varepsilon}{\partial \omega} \neq 0, \frac{\partial \mu}{\partial \omega} \neq 0\right)$ and lossy $(\operatorname{Im}(\varepsilon) \neq$ $0, \operatorname{Im}(\mu) \neq 0)$, so that they are narrow-band. In other words, they exhibit their amazing properties just at few frequencies [6].
- Attaining a multi-frequency invisibility would require the use of active element, e.g. powered circuits, transistors, operational amplifier etc. Those devices could be needed if you have to achieve a broad-band superluminal phase and group velocity: $v_{\varphi}>c_{0}, v_{g}>c_{0}$.
Today the manufacturing of passive nanoscale elements is already difficult and expensive, so the mass production of active (non-Foster) nanoscale devices looks as a still harder task.

However, those are just technological limits, not physical ones: we cannot exclude someday they will be overcome. Anyway, even if artificial invisibility would be never realized, in my personal opinion the method proposed by Pendry et al.
is conceptually very elegant, since it is mathematically plain and clear, though rigorous, and it could be extended also to other types of waves, like acoustic or mechanical ones.

### 14.2 Invisibility screen

In this section we are going to analyze which properties are required for a perfect invisibility screen or mantle. With that expression we mean a thin, artificial coating ensuring the optical cloaking for an arbitrary object. Obviously, we start speaking of an ideal screen, maybe not actually realizable or not endowed with all the wished properties. Here our task is to verify if it is possible, at least in principle, to project a meta-surface allowing that fantastic effect.

## Properties for an ideal invisibility screen:

1. Zero scattering: the coating does not alter or disturb the outer light field. Therefore, globally the cloaking device neither reflects, absorbs or emits.
2. Internal shielding: the coating avoids that the waves generated by the cloaked object can propagate outside.
3. Broad-band: the coating must ensure invisibility on a wide range of frequency, at least an octave $\left[f_{0} ; 2 f_{0}\right]$.
4. Arbitrary geometry: the coating shape should be quite arbitrary, not limited to a spherical or cylindrical shell configuration.
5. Small thickness: the coating should be thin, i.e. its thickness $\Delta x$ should be sub-wavelength:

$$
\begin{equation*}
\Delta x \ll \frac{\lambda_{0, M I N}}{2 \pi} \tag{14.22}
\end{equation*}
$$

Hence, the invisibility screen can be regarded and designed as a metasurface.

The last two requirements are dictated more by convenience rather than by a strict necessity. In fact, it is easier to build a 2D meta-surface rather than a 3D bulk meta-material, while the arbitrary geometry is a target to reach.

As already stated, we cannot guarantee in advance that all those properties can be effectively satisfied. What we are doing is to start from the desired result and, reasoning backward, try to determine how that can be achieved. In other words, we know the final result and try find the conditions for reproducing it. That approach can be regarded as a kind of top-down project strategy.

### 14.2.1 Boundary Conditions for invisibility

Let us start to see which are the Boundary Conditions associated to a hypothetical invisibility screen. Here we require that our metasurface consists in the interface dividing an inner domain $\Omega_{1}$ and the rest of the space $\Omega_{2}$. The object to be cloaked should be place inside $\Omega_{1}$. Suppose the object does not spontaneously emit light, and that inside $\Omega_{1}$ there is complete darkness, so that the object


Figure 14.6: Invisibility screen. Outside the EM fields $\vec{E}_{2}, \overline{\bar{B}}_{2}$ are unperturbed and are equal to the incident ones. Here the cloaked object (an apple) does not emit light, thus inside there is darkness and the EM fields $\vec{E}_{1}$ and $\overline{\bar{B}}_{1}$ are zero. From the Boundary Conditions it is possible to calculate the surface current distributions $\vec{J}_{t}, \overline{\bar{D}}_{e} \cdot \vec{n}_{21}$.
cannot interact with the EM fields. Mathematically, that is equivalent to ask that the electric $\vec{E}_{1}$ and magnetic $\overline{\bar{B}}_{1}$ fields are null inside:

$$
\begin{equation*}
\vec{E}_{1}=\overrightarrow{0} ; \quad \overline{\bar{B}}_{1}=\overline{\overline{0}} \quad \forall \vec{x} \in \Omega_{1} \tag{14.23}
\end{equation*}
$$

Outside, the screen can be hit from waves impinging on different points and from many directions. Since outside the EM fields must be not perturbed, they must be equal to the incident ones, hence:

$$
\begin{align*}
& \vec{E}_{2}=\epsilon_{\Omega_{2}} \vec{E}_{i n c}  \tag{14.24}\\
& \overline{\bar{B}}_{2}=\epsilon_{\Omega_{2}} \overline{\bar{B}}_{i n c} \tag{14.25}
\end{align*}
$$

where $\vec{E}_{i n c}$ and $\overline{\bar{B}}_{i n c}$ are the assigned incident fields.

## Surface currents and constitutive relation

The electric $\vec{E}$ and magnetic $\overline{\bar{B}}$ fields are already known on the domains $\Omega_{1}$ and $\Omega_{2}$. Thanks to the Boundary Conditions derived in sec.s 10.2 .4 and 11.4.3, we can exactly calculate the sources on the interface $\partial \Omega$. More explicitly, we can determine the surface currents $\vec{J}_{t}$ and "vortices" $\bar{D}_{e} \cdot \vec{n}_{21}$ required to achieve the desired effect. For sake of clarity, we re-write eq. 11.70 :

$$
\left\{\begin{array}{c}
J  \tag{14.26}\\
\frac{s}{c_{0}} D_{e}
\end{array}\right\}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\Delta E_{21} \\
\eta_{0} \Delta H_{21}
\end{array}\right\}+i \theta_{0}\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\}\right)
$$

For convenience, here we replaced the (induction) magnetic field with $H=B / \mu_{0}$. More compactly, we can write:

$$
\mathbf{J}_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{14.27}\\
1 & 0
\end{array}\right] \Delta \mathbf{f}_{h, 21}+i \theta_{0}\left\langle\mathbf{f}_{h}\right\rangle\right)
$$



Figure 14.7: Boundary Conditions for the invisibility screen. The exterior fields $\vec{E}_{2}, \overline{\bar{H}}_{2}$ are assigned and the interior ones $\vec{E}_{1}, \overline{\bar{H}}_{1}$ are zero. The surface currents $\vec{J}_{t}$ and "vortices" $\overline{\bar{D}}_{e} \cdot \vec{n}_{21}$ can be calculated from the field's discontinuities.

For details, see sec.s $10.2 .3,10.2 .4$ It should be noticed that these last relations are valid independently from the invisibility constraint. In other words, they are not yet a constitutive relation. The basic Boundary Conditions tell us which currents should be present on the interface, but they do not tell us anything about how to produce those sources.

Let us try to express the field sources in function of the field average values $\left\langle\vec{E}_{t}\right\rangle$ and $\left\langle\overline{\bar{H}}_{t}\right\rangle$ on the surface. For the current case, we can exploit the invisibility constraints 14.23, 14.24, so we find:

$$
\left\{\begin{array}{l}
\langle\vec{E}\rangle=\frac{1}{2}\left(\vec{E}_{2}+\vec{E}_{1}\right)=\frac{1}{2}\left(\vec{E}_{2}+\overrightarrow{0}\right)=\frac{1}{2} \vec{E}_{2}  \tag{14.28}\\
\langle\overline{\bar{H}}\rangle=\frac{1}{2}\left(\overline{\bar{H}}_{2}+\overline{\bar{H}}_{1}\right)=\frac{1}{2}\left(\overline{\bar{H}}_{2}+\overline{\overline{0}}\right)=\frac{1}{2} \overline{\bar{H}}_{2}
\end{array}\right.
$$

So we have expressed the average fields on the screen in function of the incident ones $\vec{E}_{2}, \overline{\bar{H}}_{2}$. With an analogous reasoning we can write the field discontinuities:

$$
\left\{\begin{array}{l}
\Delta \vec{E}_{21}=\vec{E}_{2}-\vec{E}_{1}=\vec{E}_{2}-\overrightarrow{0}=\vec{E}_{2}  \tag{14.29}\\
\Delta \overline{\bar{H}}_{21}=\overline{\bar{H}}_{2}-\overline{\bar{H}}_{1}=\overline{\bar{H}}_{2}-\overline{\overline{0}}=\overline{\bar{H}}_{2}
\end{array}\right.
$$

Combining together sets 14.28 and 14.29, we finally achieve a constitutive relation for the screen:

$$
\left\{\begin{array}{l}
\Delta \vec{E}_{21}=2 \vec{E}_{2}  \tag{14.30}\\
\Delta \overline{\bar{H}}_{21}=2 \overline{\bar{H}}_{2}
\end{array}\right.
$$

Since we are interested to the field tangent component, we can also write:

$$
\left\{\begin{array}{c}
\Delta \vec{E}_{t, 21}  \tag{14.31}\\
\eta_{0} \Delta \vec{H}_{t, 21}
\end{array}\right\}=2\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\left\langle\vec{H}_{t}\right\rangle
\end{array}\right\} \quad \forall \vec{x} \in \partial \Omega
$$

More compactly:

$$
\begin{align*}
& \Delta \mathbf{f}_{h, 21}=C_{h}\left\langle\mathbf{f}_{h}\right\rangle  \tag{14.32}\\
& C_{h}=2\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \tag{14.33}
\end{align*}
$$

The equation 14.31 can be actually regarded as constitutive relation which must be valid for each point $\vec{x}$ on the boundary $\partial \Omega$. By substituting (14.32) in (14.27), we can calculate the currents in function of the average fields:

$$
\mathbf{J}_{h}=\frac{1}{\eta_{0}}\left(-\left[\begin{array}{ll}
0 & 1  \tag{14.34}\\
1 & 0
\end{array}\right] C_{h}+i \theta_{0} I\right)\left\langle\mathbf{f}_{h}\right\rangle
$$

Since $C_{h}=2 I$, after few calculi we get:

$$
\left\{\begin{array}{c}
\vec{J}_{t}  \tag{14.35}\\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\}=-\frac{1}{\eta_{0}}\left[\begin{array}{cc}
-i \theta_{0} & 2 \\
2 & -i \theta_{0}
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{21}
\end{array}\right\}
$$

This last equation can be interpreted either as a control law or as a constitutive relation for the surface material. Since the impinging waves must propagate undisturbed above the surface, that conditions must be guaranteed on every screen's point. More compactly, it can be written as:

$$
\begin{align*}
\mathbf{J}_{h} & =Y_{h} \mathbf{f}_{h}  \tag{14.36}\\
Y_{h} & =\frac{1}{\eta_{0}}\left[\begin{array}{ll}
i \theta_{0} & -2 \\
-2 & i \theta_{0}
\end{array}\right] \tag{14.37}
\end{align*}
$$

where $Y_{h}$ is the equivalent admittance matrix.

### 14.2.2 An "odd" material for non-reciprocal screen

Now we ask: what is the material the screen is made of? We have determined a constitutive relation for that, so it should be simple to draw the permittivity $\varepsilon$ and the permeability $\mu$. However, that task is not so an easy one.

Let us consider a "common" material with assigned $\varepsilon_{r}$ and $\mu_{r}$. We have seen in sec. 10.4.3 that for such a material the constitutive relation can be written in the form:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{14.38}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=i \theta_{0}\left[\begin{array}{cc}
0 & \mu_{r} \\
\varepsilon_{r} & 0
\end{array}\right]\left\{\begin{array}{c}
\langle E\rangle \\
\eta_{0}\langle H\rangle
\end{array}\right\} \quad \text { for } \theta_{0} \ll 1
$$

So here the matrix $C_{h}$ is completely extra-diagonal, instead of the one for the invisibility screen which is diagonal. Using eq. 14.34, after few calculi we can express the relation among the sources and the fields as:

$$
\left\{\begin{array}{c}
\vec{J}_{t}  \tag{14.39}\\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\}=-\frac{1}{\eta_{0}} i \theta_{0}\left[\begin{array}{cc}
\varepsilon_{r}-1 & 0 \\
0 & \mu_{r}-1
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{21}
\end{array}\right\}
$$

So we have obtained another diagonal matrix: the electric field $\vec{E}$ is associated to a net surface current $\vec{J}_{t}$ while the magnetic field $\overline{\bar{H}}$ is associated to current doublets or vortices $\overline{\bar{D}}_{e}$. Moreover, the system is reciprocal: you can exchange
subscripts 1 and 2 and the equation will be still valid. In fact, the inner and the outer sides of such a screen behave in the same way: you cannot distinguish one from the other.

Let us now analyze the relation 14.35 for the invisibility screen: it is not diagonal, and in the thin-screen limit (so for $\theta \rightarrow 0^{+}$) it can be approximated as:

$$
\left\{\begin{array}{c}
\vec{J}_{t}  \tag{14.40}\\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{21}
\end{array}\right\}=-\frac{2}{\eta_{0}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{21}
\end{array}\right\} \quad \text { for } \theta \rightarrow 0^{+}
$$

This constitutive relation is quite "strange", in fact:

- the magnetic field $\overline{\bar{H}}$ generates a net current $\vec{J}_{t}$.
- the electric field $\vec{E}$ generates a current vortex $\overline{\bar{D}}_{e}$.

Usually in Nature the opposite happens! In addition, the system is non-reciprocal: if you exchange subscripts 1 and 2 the equation will change form. That reveals the inner and the outer sides of such a screen behaves differently: this time you can distinguish one from the other.

$$
\begin{gather*}
\vec{n}_{21}=-\vec{n}_{12} \quad \Longrightarrow  \tag{14.41}\\
\vec{H}_{t}=\overline{\bar{H}} \cdot \vec{n}_{21}=-\overline{\bar{H}} \cdot \vec{n}_{12}  \tag{14.42}\\
\overline{\bar{D}}_{e} \cdot \vec{n}_{21}=-\overline{\bar{D}}_{e} \cdot \vec{n}_{12}  \tag{14.43}\\
\left\{\begin{array}{c}
\vec{J}_{t} \\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{12}
\end{array}\right\}=-\frac{2}{\eta_{0}}\left[\begin{array}{cc}
0 & -1 \\
-1 & 0
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{12}
\end{array}\right\} \quad \text { for } \theta \rightarrow 0^{+} \tag{14.44}
\end{gather*}
$$

In the limit of $\theta \rightarrow 0^{+}$the screen turns out to be perfectly anti-symmetric or antireciprocal, as witnessed by the change of sign for the matrix. In a certain sense, the inner side and the outer side are one the negative of the other. However, if we consider a finite-thickness screen the constitutive relation will be not perfectly anti-reciprocal, but just non-reciprocal:

$$
\left\{\begin{array}{c}
\vec{J}_{t}  \tag{14.45}\\
\frac{s}{c_{0}} \overline{\bar{D}}_{e} \cdot \vec{n}_{12}
\end{array}\right\}=\frac{1}{\eta_{0}}\left[\begin{array}{cc}
i \theta_{0} & 2 \\
2 & i \theta_{0}
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{12}
\end{array}\right\}
$$

Be aware that the only signs to be reversed are those for the extra-diagonal terms.

## Tellegen material

Suppose now we desire to describe the surface material as a classic bianisotropic linear material. In other words, we want to link fields $E$ and $H$ to $D$ and $B$ through some coefficients:

$$
\left\{\begin{array}{l}
\vec{D}=\overline{\bar{\varepsilon}} \vec{E}+\overline{\bar{\xi}} \vec{H}  \tag{14.46}\\
\vec{B}=\overline{\bar{\zeta}} \vec{E}+\overline{\bar{\mu}} \vec{H}
\end{array}\right.
$$

where $\overline{\bar{\varepsilon}}$ and $\overline{\bar{\mu}}$ are the permittivity and permeability tensors respectively, while $\overline{\bar{\xi}}$ and $\overline{\bar{\zeta}}$ are the coupling terms. Be aware that here we are using the standard non-relativistic 3D notation.

From section 11.7.3 we know how to relate the tangential field components through the matrix $C_{h}$ :

$$
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{D}_{t}  \tag{14.47}\\
c_{0} \vec{B}_{t}
\end{array}\right\}=\frac{1}{i \theta_{0}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] C_{h} \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}
$$

Since for the invisibility cloak it must hold $C_{h}=2 I$, we find:

$$
\left\{\begin{array}{c}
\frac{1}{\varepsilon_{0}} \vec{D}_{t}  \tag{14.48}\\
c_{0} \vec{B}_{t}
\end{array}\right\}=\frac{2}{i \theta_{0}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \cdot\left\{\begin{array}{c}
\vec{E}_{t} \\
\eta_{0} \vec{H}_{t}
\end{array}\right\}
$$

where:

$$
\begin{equation*}
i \theta_{0}=i k_{0} \Delta x=\frac{s}{c_{0}} \Delta x \tag{14.49}
\end{equation*}
$$

More explicitly, the tangent component for permittivity, permeability etc. can be calculated as:

$$
\left\{\begin{array} { l } 
{ \overline { \overline { \varepsilon } } _ { t } = \overline { \overline { 0 } } }  \tag{14.50}\\
{ \overline { \overline { \zeta } } _ { t } = \frac { 1 } { c _ { 0 } } \frac { 2 } { i \theta _ { 0 } } \overline { \overline { I } } }
\end{array} \quad \left\{\begin{array}{l}
\overline{\bar{\xi}}_{t}=\frac{1}{c_{0}} \frac{2}{i \theta_{0}} \overline{\bar{I}} \\
\overline{\bar{\mu}}_{t}=\overline{\overline{0}}
\end{array}\right.\right.
$$

Let us notice that both the permittivity and the permeability tensors results to be null, while the coupling terms $\overline{\bar{\zeta}}_{t}$ and $\overline{\bar{\xi}}_{t}$ are equal. Therefore the tangent fields are so related:

$$
\left\{\begin{array}{rl}
\vec{D}_{t} & =\frac{2}{i \theta_{0}} \frac{1}{c_{0}} \overline{\bar{H}} \cdot \vec{n}_{21}  \tag{14.51}\\
=\frac{2}{s \Delta x} \overline{\bar{H}} \cdot \vec{n}_{21} \\
\overline{\bar{B}} \cdot \vec{n}_{21} & =\frac{2}{i \theta_{0}} \frac{1}{c_{0}} \vec{E}_{t}
\end{array}=\frac{2}{s \Delta x} \vec{E}_{t}\right.
$$

Here we find a very unusual result, since the displacement field $D$ is associated to the magnetic field $H$, while the magnetic (induction) field $B$ is linked to the electric field $E$. I discovered, almost casually, that materials with non-zero coupling terms $\overline{\bar{\zeta}}_{t}$ and $\overline{\bar{\xi}}_{t}$ are called Tellegen media [65, 66], but honestly I do not know anything more, since these outcomes are very recent.

Moreover, we can observe that the coupling terms $\overline{\bar{\zeta}}_{t}$ and $\overline{\bar{\xi}}_{t}$ in the constitutive relation depends on the surface thickness $\Delta x$, That fact could appear very strange, since the thickness is an extensive property, like mass or volume, not characteristic for a material but for a whole system. On the contrary, permittivity, permeability etc. are intensive properties, like pressure or temperature, which are not dependent on the system's dimensions. However, that paradox can be resolved if we take into account that an ideal meta-surface should be characterized by a null thickness, so the coupling terms would tend to infinity:

$$
\begin{equation*}
\lim _{\Delta x \rightarrow 0}\left(\zeta_{t}\right)=\lim _{\Delta x \rightarrow 0}\left(\xi_{t}\right)=\infty \tag{14.52}
\end{equation*}
$$

Besides, if we guess to build a bulk meta-material shell by superimposing many meta-surface layers, we obtain that the EM waves will propagate just on the external interface, since they cannot penetrate beyond the first layer.

Shortly, we can say that for an ideal, zero-thickness invisibility screen the coupling terms $\overline{\bar{\zeta}}_{t}$ and $\overline{\bar{\xi}}_{t}$ should be tending to infinity, but since real meta-surface have finite width, $\overline{\bar{\zeta}}_{t}$ and $\overline{\bar{\xi}}_{t}$ also can be finite.

## Normal component

We remind that we have calculated permittivity, permeability etc. just for the tangential fields components. From the hypothesis of continuous potential (see sec. 7.2.1, we know that in principle the meta-surface should be not normally polarized. As a consequence, the normal permittivity $\varepsilon_{n}$ will be equal to the vacuum's one:

$$
\begin{align*}
\vec{d}_{e} & =\overrightarrow{0} \text { no normal dipoles } \quad \Longrightarrow  \tag{14.53}\\
\vec{P}_{n} & =\overrightarrow{0} \text { null normal polarization } \quad \Longrightarrow  \tag{14.54}\\
\vec{D}_{n} & =\varepsilon_{0} \vec{E}_{n} \tag{14.55}
\end{align*}
$$

Therefore the complete expression for the displacement $\vec{D}$ could be:

$$
\begin{align*}
& \vec{D}=\varepsilon_{0} \vec{E}_{n}+\overline{\bar{\xi}}_{t, 21} \cdot \overline{\bar{H}} \cdot \vec{n}_{21}  \tag{14.56}\\
& \vec{D}=\varepsilon_{0} \vec{E}_{n}+\overline{\bar{\xi}}_{t, 21} \cdot\left(\vec{H} \times \vec{n}_{21}\right) \tag{14.57}
\end{align*}
$$

For what concerns the "normal" permeability $\mu_{n}$, the question looks some way more complex and tricky. Remember that the magnetic fields $\vec{B}$ and $\vec{H}$ are pseudo-vector quantities and that we have not derived a condition equivalent to the one for potential's continuity. Here I prefer not to advance any further constraint: hypotheses non fingo.

## Limit case for Pendry's invisibility cloak

I have not yet proved it by calculus, but I am quite confident that the invisibility meta-surface previously described can be considered as a limit case for the cloaking shell 14.1 .7 conceived by Pendry. In fact, for $R_{1} \rightarrow R_{2}$ the shell reduces to be an anisotropic, non-reciprocal surface like to one we have taken into account. Again, I am not completely sure about that, but perhaps it is possible to project a non-reciprocal invisibility screen by superimposing two different reciprocal meta-surfaces, each one characterized by appropriate permittivity and permeability. Actually, that could be regarded as a two-layer approximation of the Pendry's cloaking shell.

### 14.2.3 Shielding the inner side

Let us suppose we succeeded in finding or building that amazing non-reciprocal material ensuring that:

$$
\left\{\begin{array}{c}
\Delta \vec{E}_{t, 21}  \tag{14.58}\\
\eta_{0} \Delta \vec{H}_{t, 21}
\end{array}\right\}=2\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\left\langle\vec{H}_{t}\right\rangle
\end{array}\right\} \quad \forall \vec{x} \in \partial \Omega
$$

Now, that constitutive relation was derived supposing that inside the cloaking device there was darkness, i.e. that the electric and magnetic field were null:

$$
\begin{equation*}
\vec{E}_{1}=\overrightarrow{0} ; \quad \overline{\bar{H}}_{1}=\overline{\overline{0}} \quad \forall \vec{x} \in \Omega_{1} \tag{14.59}
\end{equation*}
$$

That hypothesis is quite reasonable if the object inside does not emit light or anyway if it does not radiate.

Otherwise, if you have to cloak a luminous or radiating object, then you have to add an internal screen, in order to avoid that waves emitted by the body could propagate outside.

The internal screen can be chosen quite freely: for example, it can be a mirror - not matter if an electric or a magnetic one - reflecting the emitted waves toward the interior. It could be also another invisibility screen, though oriented to face the inner side. Anyway, probably the simplest solution consists in using a common metallic mirror. Now, if we look at the whole cloaking system


Figure 14.8: If the cloaked object emits light, then an internal shield should be added to confine the EM field inside. The external screen can be a nonreciprocal invisibility meta-surface, thus the two screens are separated by a dark region $(\vec{E}=\overrightarrow{0}, \overline{\bar{H}}=\overline{\overline{0}})$ and do not interact.
we note that it includes 2 screens, separated by a dark region $(\vec{E}=\overrightarrow{0}, \overline{\bar{H}}=\overline{\overline{0}})$. Formally, the two screens are one independent from the other, since they cannot "communicate" exchanging any information.

Reminding that each screen is made of 2 layers, a couple of screen correspond to 4 layers and so to 4 currents. If we wish to build a symmetric or reciprocal system, we can assemble the outer and the inner screens so that they are very close each other, or rather at contact. However, it could be not possible to reduce the number of "layers" from 4 to 2 , because:

1. The incident fields are 4 ( 2 per each side) and they are assigned from the surrounding environment, or more precisely by the sources on $\Omega_{1}$ and $\Omega_{2}$.

$$
\mathbf{f}_{h, 1 i n c}=\left\{\begin{array}{c}
\vec{E}_{1 t}  \tag{14.60}\\
\eta_{0} \overline{\bar{H}}_{1} \cdot \vec{n}_{21}
\end{array}\right\}_{i n c} \quad \mathbf{f}_{h, 2 i n c}=\left\{\begin{array}{c}
\vec{E}_{2 t} \\
\eta_{0} \overline{\bar{H}}_{2} \cdot \vec{n}_{12}
\end{array}\right\}_{i n c}
$$

2. the fields transmitted or reflected are always 4 and they have to be controlled.
3. For each double-layer screen there are just 2 control variables, i.e. $\vec{J}_{t}$ and $\overline{\bar{D}}_{e}$, or equivalently $\vec{J}_{t 1}$ and $\vec{J}_{t 2}$.

Therefore, for a complete control of the transmitted/reflected fields using only 2 control variables could be not sufficient.

In addition, let us notice that the constitutive relation for each screen can be expressed in the form:

$$
\begin{align*}
\left\langle\mathbf{f}_{h}\right\rangle & =\overline{\bar{\eta}} \mathbf{J}_{h}  \tag{14.61}\\
\left\{\begin{array}{c}
\left\langle\vec{E}_{t}\right\rangle \\
\eta_{0}\langle\overline{\bar{H}}\rangle \cdot \vec{n}_{21}
\end{array}\right\} & =\overline{\bar{\eta}}\left\{\begin{array}{c}
\vec{J}_{t} \\
\frac{s}{c_{0}} \\
\bar{D}_{e} \cdot \vec{n}_{12}
\end{array}\right\} \tag{14.62}
\end{align*}
$$

where there are just 2 fields and 2 control variables.

### 14.2.4 Fast surface wave current

Before to deal with circuit model, it is well to highlight some other strange features of the invisibility screen. Let us consider a plane wave investing the cloaking shell. The equation for the incident electric field could be:

$$
\begin{equation*}
\vec{E}_{2 \text { inc }}=\vec{E}_{2,0} e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}} \tag{14.63}
\end{equation*}
$$

The wave travels beyond the meta-surface globally undisturbed, though anyway the wave locally interacts with the surface itself. For example, the charge distribution $\sigma_{e}$ on the boundary must neutralize the effects of the normal electric field $\vec{E}_{2 n}$. Since the inner field is zero $\left(\vec{E}_{1 n}=\overrightarrow{0}\right)$, the surface charge can be calculated as:

$$
\begin{equation*}
\frac{\sigma_{e}}{\varepsilon_{0}}=\vec{n}_{21}^{T} \cdot \vec{E}_{2 i n c}=\left(\vec{n}_{21}^{T} \cdot \vec{E}_{2,0}\right) e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}} \quad \forall \vec{x} \in \partial \Omega \tag{14.64}
\end{equation*}
$$

Hence on the boundary a charge wave should arise. The current distribution $\vec{J}_{t}$ can be calculated thanks to the Boundary Conditions:

$$
\begin{align*}
& \vec{J}_{t}=-\overline{\bar{H}}_{2 \text { inc }} \cdot \vec{n}_{21}  \tag{14.65}\\
& \vec{J}_{t}=\frac{1}{\eta_{0} k_{0}}\left[\vec{k}_{2} \wedge \vec{E}_{2,0}\right] \cdot \vec{n}_{21} e^{i \vec{k}_{2}^{T} \cdot \Delta \vec{x}} \quad \forall \vec{x} \in \partial \Omega \tag{14.66}
\end{align*}
$$

In practice, we are dealing with a surface wave current which slides on the screen. In a certain way, that phenomenon is similar to the propagation of evanescent or plasmonic waves, though there are some fundamental differences. In particular, we are going to verify that surface waves for the invisibility screen are very fast. More precisely, again they are characterized by a superluminal phase velocity: $v_{\varphi}>c_{0}$

## Comparison with evanescent and plasmonic waves

Here we make a rapid comparison among the propagations of "fast", evanescent and plasmonic waves.

The evanescent waves occur at the interface between two dielectrics, and usually they are originated by internal frustrated reflection. They are called "evanescent " because of the exponential decay in the direction perpendicular to the surface. More precisely, the normal component $k_{n}$ of the wavevector is purely imaginary.

For example, if we consider a plane evanescent wave propagating in the $x$ direction, its equation could be:

$$
\begin{equation*}
\vec{E}=\vec{E}_{0} e^{i k_{t} x} e^{-\left|k_{n}\right| z} \tag{14.67}
\end{equation*}
$$

where:

$$
\begin{cases}k_{t} \in \mathbb{C} & \text { complex tangent wavevector }  \tag{14.68}\\ k_{n} \in \mathbb{I} & \text { imaginary normal wavevector }\end{cases}
$$

Analogously, plasmonic waves too exhibit an exponential decay in the direction normal to the surface. However, they occur at the interface between a metal (or conductor) and a dielectric (e.g., air ). In practice they consist of collective oscillation of charge.

For both the evanescent and the plasmonic waves the normal component of the wavevector is purely imaginary, i.e.:

$$
\begin{equation*}
k_{n} \in \mathbb{I} \quad \Longrightarrow \quad k_{n}^{2}=-\left|k_{n}\right|^{2} \tag{14.69}
\end{equation*}
$$

Supposing that the wave is propagating in vacuum, the modulus of the tangent wavevector $\vec{k}_{t}$ will turn out to be greater than $k_{0}=\omega / c_{0}$ :

$$
\begin{align*}
& \vec{k}_{t}^{T} \cdot \vec{k}_{t}=k_{t}^{2}=k_{0}^{2}-k_{n}^{2}  \tag{14.70}\\
& k_{t}^{2}=k_{0}^{2}+\left|k_{n}\right|^{2} \quad \Longrightarrow \quad k_{t}^{2}>k_{0}^{2} \tag{14.71}
\end{align*}
$$

For details, see also sec. 9.1. Supposing that $\vec{k}_{t}$ is real $\left(\vec{k}_{t} \in \mathbb{R}^{N}\right)$, both the tangent wavelength $\lambda_{t}$ and the phase velocity $v_{\varphi t}$ will be smaller with respect to those in free-space:

$$
\begin{align*}
& k_{t}>k_{0}=\frac{\omega}{c_{0}}=\frac{2 \pi}{\lambda_{0}}  \tag{14.72}\\
& \lambda_{t}=\frac{2 \pi}{k_{t}} \quad \Longrightarrow \quad \lambda_{t}<\lambda_{0} \quad \text { superlocalization }  \tag{14.73}\\
& v_{\varphi t}=\frac{\omega}{k_{t}} \quad \Longrightarrow \quad v_{\varphi t}<c_{0} \quad \text { subluminal phase velocity } \tag{14.74}
\end{align*}
$$

Thus the tangent wavelength $\lambda_{t}$ associated to a leaky wave, no matter if evanescent or plasmonic one, will be smaller than the vacuum wavelength $\lambda_{0}$. That "superlocalization" effect is quite interesting because it can be exploited to improve the resolution for optical devices without increasing the frequency. This analysis also tells us that the surface current waves on the invisibility metasurface are neither evanescent nor plasmonic. Indeed the normal component of the wavevector $\vec{k}_{2}$ turns out to be real:

$$
\begin{align*}
& \vec{k}_{2} \in \mathbb{R}^{N}  \tag{14.75}\\
& k_{n}=\vec{k}_{2}^{T} \cdot \vec{n}_{21} \quad \Longrightarrow \quad k_{n} \in \mathbb{R} \tag{14.76}
\end{align*}
$$

As a consequence, $k_{t}$ will be smaller than $k_{0}$ :

$$
\begin{equation*}
k_{t}^{2}=k_{0}^{2}-k_{n}^{2}=k_{0}^{2}-\left|k_{n}\right|^{2} \quad \Longrightarrow k_{t}^{2}<k_{0}^{2} \quad 0 \leq k_{t}<k_{0} \tag{14.77}
\end{equation*}
$$

Hence, both the tangent wavelength $\lambda_{t}$ and the phase velocity $v_{\varphi t}$ on the surface will be greater than those in free-space:

$$
\begin{align*}
& \lambda_{t}>\lambda_{0} \quad \text { sublocalization }  \tag{14.78}\\
& v_{\varphi t}=\frac{\omega}{k_{t}} \quad \Longrightarrow \quad v_{\varphi t}>c_{0} \quad \text { superluminal phase velocity } \tag{14.79}
\end{align*}
$$

So we see that we are dealing with fast surface wave.
This comparison was performed just to avoid confusion between fast and leaky (evanescent or plasmonic) waves, since those phenomena share some similarities but they are not identical.

### 14.2.5 Perfect absorber, guide and emitter

Let us now consider the scattering for the material constituting the invisibility meta-surface: its behaviour will appear different depending on the considered screen's zone.

If in a certain region the EM wave is impinging on the screen, then the material should behave like a perfect absorber. In fact, it must not reflect the incident wave nor transmit it on the other side.

$$
\begin{equation*}
\text { if } \vec{k}_{2}^{T} \cdot \vec{n}_{21}=k_{n}<0 \quad \Longrightarrow \quad \text { perfect absorber required } \tag{14.80}
\end{equation*}
$$



Figure 14.9: Perfect absorbing screen.
If in another region the EM wave is propagating tangentially on the surface, then the material should support the propagation for the appropriate current oscillations. In other, it behaves like a waveguide

$$
\begin{equation*}
\text { if } \vec{k}_{2} \perp \vec{n}_{21} \quad \Longrightarrow \quad k_{n}=0 \quad \Longrightarrow \quad \text { wave guide required } \tag{14.81}
\end{equation*}
$$



Figure 14.10: Perfect wave-guiding screen.
Finally, if in a certain region the wave assigned from outside seems to emerge from the screen, then the material should behave like a perfect emitter. Apparently, it must spontaneously radiate outward the required wave, without any power supply.

$$
\begin{equation*}
\text { if } \vec{k}_{2}^{T} \cdot \vec{n}_{21}=k_{n}>0 \quad \Longrightarrow \quad \text { perfect emitter required } \tag{14.82}
\end{equation*}
$$



Figure 14.11: Perfect emitting screen.

On the contrary, we know that globally the screen does not absorb power nor produce it, but it just transmit it making the wave to "slip" on the boundary. That happens by mean of the surface current waves.

At a first sight, building a material which is a perfect absorber, a wave guide and a perfect emitter at the same time sounds like a close-to-absurd idea. However, also the bulk cloaking shell by Pendry et al. (see sec. 14.1.7) is endowed with those features. More precisely, the screen or shell must behave as a reflectionless perfect wave guide, capable of absorbing and re-emitting all the incident waves.

### 14.2.6 Circuit invisibility screen

Now let us see if it is possible to model with circuits the material for the invisibility screen. In order to determine the circuit equivalent we have to link the electric fields $\vec{E}_{1}$ and $\vec{E}_{2}$ to the surface currents $\vec{J}_{1}$ and $\vec{J}_{2}$ on the two sides of the metasurface. More precisely, we have to rephrase the fields variables in circuit ones, removing the magnetic field $H$ from the equations.

In section 12.4 we have demonstrated that the Boundary Conditions for a 2-layers circuit screen can be written as:

$$
\left\{\begin{align*}
\Delta V_{21} & =-s L_{0}\left(\frac{1}{2}\left(I_{2}-I_{1}\right)+\langle H\rangle \Delta z\right)  \tag{14.83}\\
\Delta H_{21} \Delta z & =-\left(I_{2}+I_{1}\right)-s C_{0}\langle V\rangle
\end{align*}\right.
$$

where:

$$
\begin{cases}V=\Delta y E & \text { voltage }  \tag{14.84}\\ I=\Delta z J_{t} & \text { net current }\end{cases}
$$

The constitutive relation for the invisibility screen can be so expressed:

$$
\begin{gather*}
\left\{\begin{array}{l}
V_{1}=0 \\
H_{1}=0
\end{array}\right.  \tag{14.85}\\
\left\{\begin{array}{l}
\langle V\rangle=\frac{1}{2} V_{2} \\
\Delta V_{21}=V_{2}
\end{array}\right. \tag{14.86}
\end{gather*}
$$

By substituting in the BCs 14.83, we find:

$$
\left\{\begin{align*}
V_{2} & =-s L_{0}\left(\frac{1}{2}\left(I_{2}-I_{1}\right)+\frac{1}{2} H_{2} \Delta z\right)  \tag{14.87}\\
H_{2} \Delta z & =-\left(I_{2}+I_{1}\right)-s C_{0} \frac{1}{2} V_{2}
\end{align*}\right.
$$

After removing the magnetic field $H_{2}$, we get a single equation:

$$
\begin{equation*}
\left(1-\frac{1}{4} s^{2} L_{0} C_{0}\right) V_{2}=s L_{0} I_{1} \tag{14.88}
\end{equation*}
$$

This equation, together with the condition $V_{1}=0$, exactly defines the circuit we were looking for. Therefore, the constitutive relations for the invisibility circuit are:

$$
\left\{\begin{array}{l}
V_{1}=0  \tag{14.89}\\
V_{2}=\frac{s L_{0}}{1-\frac{1}{4} s^{2} L_{0} C_{0}} I_{1} \quad \text { circuit constitutive relations }
\end{array}\right.
$$

In matrix form, the Thevenin equivalent will look:

$$
\left\{\begin{array}{l}
V_{1}  \tag{14.90}\\
V_{2}
\end{array}\right\}=\left[\begin{array}{ll}
0 & 0 \\
Z & 0
\end{array}\right]\left\{\begin{array}{l}
I_{1} \\
I_{2}
\end{array}\right\}
$$

where for convenience we introduced the impedance $Z$ :

$$
\begin{equation*}
Z=\frac{s L_{0}}{1-\frac{1}{4} s^{2} L_{0} C_{0}} \tag{14.91}
\end{equation*}
$$

Let us notice that impedance $Z$ is associated to a dependent voltage generator. Since in eq. 14.91 there are both the reference inductance $L_{0}$ and capacity $C_{0}$, we can check if $Z$ can be expressed as a parallel between an inductor and a capacitor:

$$
\begin{equation*}
s L / / \frac{1}{s C}=\frac{s L \cdot \frac{1}{s C}}{s L+\frac{1}{s C}}=\frac{s L}{1+s^{2} L C} \tag{14.92}
\end{equation*}
$$

Requiring that $Z=s L / /(1 /(s C))$, we find:

$$
\left\{\begin{array}{l}
L=L_{0}  \tag{14.93}\\
C=-\frac{1}{4} C_{0}
\end{array} \quad \Longrightarrow \quad Z=s L_{0} / /\left(-\frac{4}{s C_{0}}\right)\right.
$$

So for the construction of $Z$ a negative capacitor could be needed.

$$
Z=s L_{0} / /\left(-\frac{4}{s C_{0}}\right) \quad L_{0} \underbrace{8}_{\square}-\frac{1}{4} C_{0}
$$

Figure 14.12: Impedance $Z$ for the dependent source. Since the capacitor is negative, active elements could be needed.


Figure 14.13: Possible unit circuit cell for the invisibility metasurface

## Circuit analysis

Let us now analyze the invisibility circuit more in detail. On side 1 there is an ideal short-circuit, ensuring that $V_{1}=0$ and letting the current $I_{1}$ to flow without any voltage drop. Be aware that the construction of a short-circuit, i.e. an ideal null impedance, could be truly a hard task, especially at extremely high frequencies.

On side 2 there is a voltage generator $V_{2}$, which includes either the field $V_{2 \text { inc }}$ impinging from outside and the field $V_{2 i r r}$ radiated by the circuit itself:

$$
\begin{equation*}
V=V_{i n c}+V_{i r r} \tag{14.94}
\end{equation*}
$$

For the considered case, we have simply:

$$
\begin{align*}
0 & =V_{1 i n c}+V_{1 i r r}  \tag{14.95}\\
V_{2} & =V_{2 i n c}+0 \tag{14.96}
\end{align*}
$$

Finally, always on side 2 there is a dependent voltage source whose law is $V_{2}=Z I_{1}$. We can observe that the current $I_{2}$ is practically undetermined, though it can contribute to radiation. However, if you know the current $I_{\text {in }}$ coming in the circuit element from the adjacent ones, then you can calculate $I_{2}$ :

$$
\begin{align*}
I_{i n} & =I_{1}+I_{2} \quad \Longrightarrow  \tag{14.97}\\
I_{2} & =I_{i n}-I_{1}=I_{i n}-\frac{V_{2 i n c}}{Z} \tag{14.98}
\end{align*}
$$

As you can easily guess, that kind of circuit is intrinsically unstable: it has to be extremely susceptible either to the currents flowing in from other screen's region and to the assigned electric field $\vec{E}_{2 t}$.

## Thevenin equivalent

Let us see if we can conceive another circuit which still satisfy the conditions required for invisibility. We can start writing the Thevenin equivalent for a generic 2-port circuit:

$$
\left\{\begin{array}{l}
V_{1}  \tag{14.99}\\
V_{2}
\end{array}\right\}=\left[\begin{array}{rr}
Z_{1}+Z_{S} & -Z_{G}+Z_{S} \\
Z_{G}+Z_{S} & Z_{2}+Z_{S}
\end{array}\right]\left\{\begin{array}{l}
I_{1} \\
I_{2}
\end{array}\right\}
$$



Figure 14.14: Schematic circuit model for the invisibility meta-surface.
where $Z_{1}$ and $Z_{2}$ are the series impedance, $Z_{S}$ is the shunt one while $Z_{G}$ is the gyrator impedance. The structure for the general Thevenin equivalent 2-port is reported in fig. 14.15 a Requiring that the eq. 14.90 is valid, we can calculate $Z_{1}, Z_{2}, Z_{S}$ and $Z_{G}$ :

$$
\begin{align*}
Z_{1} & =Z_{2}=-\frac{1}{2} Z=-Z_{G}  \tag{14.100}\\
Z_{S} & =Z_{G}=\frac{1}{2} Z \tag{14.101}
\end{align*}
$$

The gyrator impedance can be also expressed as:

$$
\begin{equation*}
Z_{G}=\frac{1}{2} Z=\left(\frac{1}{2} s L_{0}\right) / /\left(-\frac{2}{s C_{0}}\right) \tag{14.102}
\end{equation*}
$$

The structure for the Thevenin 2-port is reported in fig. 14.15 b and 14.15 c Except for the presence of dependent generators, the circuit would be perfectly symmetric (i.e. reciprocal). Therefore, let us calculate the average fields and the discontinuities, highlighting the symmetric and anti-symmetric components for both voltages and currents. After some calculi, we find:

$$
\left\{\begin{array}{c}
\langle V\rangle  \tag{14.103}\\
\Delta V_{21}
\end{array}\right\}=Z_{G}\left[\begin{array}{cc}
1 & -\frac{1}{2} \\
2 & -1
\end{array}\right]\left\{\begin{array}{c}
\langle I\rangle \\
\Delta I_{21}
\end{array}\right\}
$$

We define the currents:

$$
\left\{\begin{align*}
I_{t o t} & =I_{2}+I_{1}=2\langle I\rangle \quad \text { net straight current }  \tag{14.104}\\
I_{\text {anti }} & =\frac{1}{2}\left(I_{2}-I_{1}\right)=\frac{1}{2} \Delta I_{21} \quad \text { ring current }
\end{align*}\right.
$$

So we get:

$$
\left\{\begin{array}{c}
\langle V\rangle  \tag{14.105}\\
\Delta V_{21}
\end{array}\right\}=Z_{G}\left[\begin{array}{cc}
\frac{1}{2} & -1 \\
1 & -2
\end{array}\right]\left\{\begin{array}{c}
I_{t o t} \\
I_{a n t i}
\end{array}\right\}
$$

That equation tell us that the symmetric and anti-symmetric components are strongly coupled (in fact the matrix is not diagonal).

Nevertheless, we can build up a circuit equivalent made by a thin straight wire and by a ring. Yet, differently from circuits with assigned permittivity $\varepsilon_{r}$

(a) General Thevenin equivalent circuit.

(b) Thevenin equivalent for "invisibility" circuit.

(c) Simplified version.

Figure 14.15: Thevenin equivalent T-circuits.


Figure 14.16: The circuit straight wire and the ring are coupled through the dependent sources, so that the symmetric components for currents and voltages are coupled to the anti-symmetric ones.


Figure 14.17: Circuit unit cell: here the straight wire and the ring are coupled by mean of a gyrator (ports 1 and 2 are indicated). The voltage generators are not shown since they does not correspond to a physical device, but just to the e.m.f. induced by propagating fields.
and permeability $\mu_{r}$, this time those two elements must interact. Actually, the straight wire can be coupled to the ring by mean of a gyrator whose characteristic impedance is $Z_{G}$. I agree that structure looks far more complicated with respect to the one analyzed in sec. 14.2 .6 , but perhaps it could be easier building the wire-ring configuration instead of the short-circuit one. A positive aspect of the wire-ring structure is that the ring current $I_{a n t i}$ is well distinct from the total one $I_{t o t}$, which flows in and out of the circuit.

Concluding, I have developed and described this circuit just for completeness: I cannot pretend it can be physically realizable at nano-scale.

### 14.2.7 Derivation by standard procedure

In the previous paragraphs the circuit equivalent for the invisibility meta-surface was directly derived from the Boundary Conditions, in few steps. Anyway, we could find the same result also performing the standard procedure described in sec. 11.5. Here we are going to verify that actually the same matrix impedance is retrieved.

## Steps for standard procedure

We start from the constitutive relation in the form:

$$
\left\{\begin{array}{c}
\Delta E_{21}  \tag{14.106}\\
\eta_{0} \Delta H_{21}
\end{array}\right\}=2\left[\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right]\left\{\begin{array}{c}
\left\langle E_{t}\right\rangle \\
\eta_{0}\left\langle H_{t}\right\rangle
\end{array}\right\}
$$

More compactly:

$$
\begin{align*}
& \Delta \mathbf{f}_{h, 21}=C_{h}\left\langle\mathbf{f}_{h}\right\rangle  \tag{14.107}\\
& C_{h}=\alpha\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]=\alpha I \quad \text { with } \alpha \rightarrow 2 \tag{14.108}
\end{align*}
$$

Then we apply the algorithm:

1. Transform the constitutive relation for electric amplitudes $\mathbf{E}$ :

$$
\begin{align*}
\Delta \mathbf{E}_{21} & =C_{E}\langle\mathbf{E}\rangle  \tag{14.109}\\
C_{E} & =T_{E} \cdot C_{h} \cdot T_{E}^{-1}  \tag{14.110}\\
C_{E} & =\alpha I \quad \text { in this case } \tag{14.111}
\end{align*}
$$

2. Calculate the transmission matrix $\overline{\bar{T}}$ :

$$
\begin{align*}
T & =\left(I-\frac{1}{2} C_{E}\right)^{-1} \cdot\left(I+\frac{1}{2} C_{E}\right)  \tag{14.112}\\
T & =\frac{1+\frac{1}{2} \alpha}{1-\frac{1}{2} \alpha} I \tag{14.113}
\end{align*}
$$

3. Calculate the scattering matrix $S_{0}$ :

$$
\begin{align*}
S & =\frac{1}{T_{22}}\left[\begin{array}{cc}
-T_{21} & 1 \\
\operatorname{det}(T) & T_{12}
\end{array}\right]  \tag{14.114}\\
S & =\left[\begin{array}{cc}
0 & G^{-1} \\
G & 0
\end{array}\right] \quad \text { where: } G=\frac{1+\frac{1}{2} \alpha}{1-\frac{1}{2} \alpha} \tag{14.115}
\end{align*}
$$

Let us notice that for $\alpha \rightarrow 2$, so for a perfect invisibility, the scattering matrix degenerates in:

$$
S=\left[\begin{array}{cc}
0 & 0  \tag{14.116}\\
\infty & 0
\end{array}\right]
$$

This result, though unusual, is consistent with the requirement for the coating material to be a perfect absorber and, at the same time, a perfect emitter (see also sec 14.2.5). We remind that:

$$
\left\{\begin{array}{l}
E_{1-}  \tag{14.117}\\
E_{2+}
\end{array}\right\}=\left[\begin{array}{ll}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{array}\right] \cdot\left\{\begin{array}{l}
E_{1+} \\
E_{2-}
\end{array}\right\}
$$

Therefore, since $E_{1-}$ and $E_{1+}$ are null because of the "darkness" condition (see 14.2.1), and since $E_{2+}$ (emitted field) can be different from zero, then that particular structure for $S$ arises.
4. Calculate the impedance matrix $Z_{E}$ :

$$
\begin{align*}
& \mathbf{E}=Z_{E} \mathbf{J}  \tag{14.118}\\
& Z_{E}=(I+S)\left(S_{0}-S\right)^{-1} Z_{i r r}  \tag{14.119}\\
& (I+S)=\left[\begin{array}{cc}
1 & G^{-1} \\
G & 1
\end{array}\right]  \tag{14.120}\\
& \left(S_{0}-S\right)=\left[\begin{array}{cc}
0 & b\left(\theta_{0}\right)-G^{-1} \\
b\left(\theta_{0}\right)-G & 0
\end{array}\right]  \tag{14.121}\\
& (I+S)\left(S_{0}-S\right)^{-1}=\left[\begin{array}{cc}
\frac{G^{-1}}{b\left(\theta_{0}\right)-G^{-1}} & \frac{1}{b\left(\theta_{0}\right)-G} \\
\overline{b\left(\theta_{0}\right)-G^{-1}} & \frac{G}{b\left(\theta_{0}\right)-G}
\end{array}\right] \tag{14.122}
\end{align*}
$$

In the limit of $\alpha \rightarrow 2$, we find simply:

$$
\begin{gather*}
\lim _{\alpha \rightarrow 2}(G)=\lim _{\alpha \rightarrow 2}\left(\frac{1+\frac{1}{2} \alpha}{1-\frac{1}{2} \alpha}\right)=\infty  \tag{14.123}\\
\lim _{\alpha \rightarrow 2}\left(G^{-1}\right)=\lim _{\alpha \rightarrow 2}\left(\frac{1-\frac{1}{2} \alpha}{1+\frac{1}{2} \alpha}\right)=0 \tag{14.124}
\end{gather*}
$$

So it follows:

$$
(I+S)\left(S_{0}-S\right)^{-1}=\frac{1}{b\left(\theta_{0}\right)}\left[\begin{array}{cc}
0 & 0  \tag{14.125}\\
1 & -b\left(\theta_{0}\right)
\end{array}\right]
$$

The radiation impedance $Z_{i r r}$ is already known (see sec 10.3.4 and eq. 10.85), so we can calculate $Z_{E}$ :

$$
\begin{align*}
& Z_{\text {irr }}=\frac{1}{2} \eta_{0}\left(I+S_{0}\right)=\frac{1}{2} \eta_{0}\left[\begin{array}{cc}
1 & b\left(\theta_{0}\right) \\
b\left(\theta_{0}\right) & 1
\end{array}\right]  \tag{14.126}\\
& Z_{E}=(I+S)\left(S_{0}-S\right)^{-1} Z_{i r r}  \tag{14.127}\\
& Z_{E}=\frac{1}{2} \eta_{0}\left(\frac{1-b\left(\theta_{0}\right)^{2}}{b\left(\theta_{0}\right)}\right)\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] \tag{14.128}
\end{align*}
$$

Explaining the factor containing $b\left(\theta_{0}\right)$, we get:

$$
\begin{align*}
& b\left(\theta_{0}\right)=\frac{1+\frac{1}{2} i \theta_{0}}{1-\frac{1}{2} i \theta_{0}}  \tag{14.129}\\
& \frac{1-b\left(\theta_{0}\right)^{2}}{b\left(\theta_{0}\right)}=\frac{-2 i \theta_{0}}{1-\frac{1}{4}\left(i \theta_{0}\right)^{2}} \tag{14.130}
\end{align*}
$$

Thus:

$$
Z_{E}=\eta_{0} \frac{-i \theta_{0}}{1-\frac{1}{4}\left(i \theta_{0}\right)^{2}}\left[\begin{array}{ll}
0 & 0  \tag{14.131}\\
1 & 0
\end{array}\right]
$$

Finally we have obtained a matrix whose structure is identical to the one in eq. 14.90 . We remind that here $Z_{E}$ links the surface currents $\mathbf{J}$ to the electric fields $\mathbf{E}$, so we have to multiply by $\Delta y / \Delta z$ in order to retrieve the circuit impedance.

$$
\begin{equation*}
Z=\frac{\Delta y}{\Delta z} \eta_{0} \frac{-i \theta_{0}}{1-\frac{1}{4}\left(i \theta_{0}\right)^{2}} \tag{14.132}
\end{equation*}
$$

Remembering that:

$$
\begin{array}{ll}
s L_{0}=\frac{\Delta x \Delta y}{\Delta z}\left(-i \theta_{0}\right) \eta_{0} & \text { reference inductance } \\
s C_{0}=\frac{\Delta z \Delta x}{\Delta y}\left(-i \theta_{0}\right) \eta_{0} & \text { reference capacity } \tag{14.134}
\end{array}
$$

after few substitutions we find again the expected result:

$$
\begin{equation*}
Z=\frac{s L_{0}}{1-\frac{1}{4} s^{2} L_{0} C_{0}} \tag{14.135}
\end{equation*}
$$

### 14.3 Techniques for cloaking device neutralization

Currently, the realization of an invisibility mantle in the visible spectrum seems to be yet technologically extremely hard, though the nano-technologies are growing more and more advanced and their costs are diminishing. During the History, sectors like nuclear physics, electronics and informatics have experienced amazing progress, faster than what had been previously foreseen, with unexpected developments.

Therefore, I cannot exclude that in a near future somebody could be able to build an effective optical cloaking device. Besides we must be notice that in the last years many research groups [64, 67, 58, 68, 69, 70] are focusing their studies on those topics: the race is open.

Now let us stop for a moment and ask again:

> Why should we construct an invisibility cloak? Which could be its applications and uses?

Honestly, I am not sure about the possible uses for a cloaking device.
One practical aim consists in shielding telecommunication antennas, in such a way that they do not interfere each other. Another application, proposed by the Tachi group [62, 71, 72] consists in making transparent the aircraft cockpits, in order to assist pilots during landing. The same concept could be applied to the car's interior, so that the driver can see better the surrounding obstacles. A similar purpose is to help medics during surgical operations, using "invisible" equipment (e.g., gloves) allowing a better view of the patient. Perhaps cloaking devices could be just an amazing attraction in a theme park or in a spectacular show of some kind: I can only guess.

For the moment, the main applications I see for an invisibility cloak are interesting for military purposes.

Probably the production of invisible dresses for a whole army (so, one uniform per each soldier) would be too expensive, but cloaking aircraft, ships and tanks on certain frequencies could turn out to be very effective in a warfare.

Even if, differently from a nuclear missile, a cloaking device would be not hurtful, though it could reveal to be a dangerous weapon if managed by the wrong person. Therefore, here I am going to show some drawbacks and limits for the proposed methods.

### 14.3.1 Echo at extremely high frequencies

A meta-surface like the one described for the invisibility cloak can function just if its thickness $\Delta x$ is much smaller than the incident wavelength $\lambda_{0}$ :

$$
\begin{equation*}
\Delta x \ll \frac{\lambda_{0}}{2 \pi} \tag{14.136}
\end{equation*}
$$

For the visible light the wavelength $\lambda_{0}$ is in the range $380 \div 750 \mathrm{~nm}$, so the metasurface could be about $\Delta x \approx 40 \mathrm{~nm}$ thick. If you desire to detect the presence of the cloak, you can hit that with an electromagnetic wave whose $\lambda_{0}$ is smaller than or at least comparable to the thickness $\Delta x$ :

$$
\begin{equation*}
\lambda_{0} \leq \Delta x \tag{14.137}
\end{equation*}
$$

In practice, it would be sufficient to illuminate the "suspect" region with UltraViolet rays, or anyway with a radiation whose frequency is higher than the one for visible light. Sensing the echo (so the reflected wave) you can thus detect the cloaking device.

### 14.3.2 Multi-frequency echo detection

Even at low frequency, such that $\lambda_{0} \gg 2 \pi \Delta x$, it could be possible to find out the cloaking device, illuminating it with an impulse or anyway with EM waves at many different frequencies. Actually the construction of a dispersionless material on a broad-band is really a hard task. In other words, the cloaking device could be invisible at certain frequencies, but not at other ones (e.g., in the Infra-Red range).

### 14.3.3 High energy EM pulse

Another technique to detect the screen consists in investing it with a high energy EM pulse, which could burn out the circuits or anyway produce a minimum measurable echo. That kind of effect is quite well known, since it was observed after solar flares or nuclear explosions. As a matter of History, in 1859 a solar storm, also known as the "Carrington event", knocked out the telegraph systems all over Europe and North America.

### 14.3.4 Acoustic or mechanical echo

Even if an object is made perfectly invisible, though it could be not transparent to other kinds of waves, like acoustic or mechanical ones. Therefore the cloaked object could be detected by a sonar echo.

### 14.3.5 Dust, rain and paints

Probably the simplest way to detect an invisible object in a room is to sprinkle or spray it with some visible powder or liquid. Excluding that the object is also incorporeal, then the powder (or liquid) will deposit on it, unveiling its presence. Moreover, powders and liquids could damage the cloaking device, making it out of order.

## Chapter 15

## Conclusions

It is only with the heart that one can see rightly; what is essential is invisible to the eye.

Antoine de Saint-Exupéry, The Little Prince, 1943

Finally we have come to the end of this work.
Starting from the Huygens' Principle we have seen how to apply it in many contexts, ranging from gravity to holography, then we focused on its electromagnetic formulation in order to design metasurfaces. By adopting the electromagnetic potentials $\rho_{A}$ and $\vec{A}$, we have derived the Boundary Conditions associated to Maxwell's Equations, hence we have compared them to the ones written in terms of electric E and magnetic B fields. As we have observed, EM potentials bring many advantages: they allow to deal also with static or low-frequency phenomena; they are convenient for numerical simulation because they do not jump to infinite values; they are suitable for mechanic and fluid-dynamic analogies; they can be easily treated in Relativity and so on. On the contrary, writing Boundary Conditions by mean of electric and magnetic fields implies many drawbacks: static phenomena like the Volta Effect can be hardly modeled, in some cases $E$ and $B$ can reach infinite values, or else they require the introduction of debated physical quantities like magnetic monopoles and currents. Thanks to EM potentials, we have verified that field discontinuities can be described in terms of electric currents $\vec{J}_{s}$ and doublets (or vortices) $\overline{\bar{D}}_{e}$ and that, in order to fully control EM fields, a finite-width metasurface should be characterized by at least 2 layers. On the basis of Boundary Conditions, we have taken into account the project of a thin screen endowed with some desired features. We have analyzed many ways to describe our system's scattering and constitutive relations, trying to adopt a quite general procedure. After that, we have proposed a circuit model for a screen whose permittivity $\varepsilon$ and permeability $\mu$ are assigned.

Finally, in the last chapters we have wondered about the possible use of active metasurfaces for some fantastic applications. Giving rise to "imagineering", we tried to describe the properties of a holographic 3D TV and investigated the conditions for an invisibility cloak, tempting to be always rigorous but also open-minded. I do not know if those wonderful devices will be ever achieved or not, but I guess it is interesting to discuss about them. I think that exploring the limits of possible through method and imagination is an effective way to
creation and invention.

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[^0]:    ${ }^{1}$ source:Scopus

[^1]:    ${ }^{1}$ more precisely, by their gravitational charge

[^2]:    ${ }^{2}$ though that's just a particular case, since the principle is valid in any number N of dimensions

