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Master of Science in Mathematical Engineering

Selfish Reciprocal Altruism: an evolutionary rule for strategy update in social networks

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Ai miei genitori,
Giovanna e Cesare

Contents

List of Figures

List of Tables

Sommario

Lo studio dell'evoluzione e della persistenza della cooperazione tra individui "egoisti", cioè che ottimizzano il proprio beneficio, è un argomento a lungo dibattuto e tutt'ora di grande interesse in vari ambiti, dalla biologia alle scienze sociali fino all'ingegneria, come testimoniato da ricerche recenti.
Il contesto standard in cui l'analisi della cooperazione è inserita è quello della *Teoria dei Giochi Evolutiva*, ossia l'applicazione dei modelli della Teoria dei Giochi a situazioni in cui l'attenzione è centrata sulla dinamica del cambio di strategia. In quest'ottica, infatti, gli individui possono esibire diversi tipi di comportamento, alcuni dei quali riescono a persistere e diffondersi all'interno della popolazione, mentre altri tendono a scomparire.
Le interazioni tra gli individui sono tipicamente modellate attraverso il *Dilemma del Prigioniero*, un gioco a due giocatori e due strategie, in cui gli individui coinvolti nell'interazione possono decidere se collaborare con l'avversario oppure sfruttarlo.
In particolare, considerando l'ambito sociale, la popolazione può essere rappresentata tramite una rete, dove i nodi sono individui e gli archi simboleggiano le possibili interazioni tra di essi. Gli individui intraprendono ripetutamente il gioco del Dilemma del Prigioniero con i loro vicini, e tra un'interazione e l'altra possono decidere se cambiare la strategia utilizzata a seconda del comportamento dei vicini. In questo modo si verifica un'evoluzione delle frazioni di cooperatori e sfruttatori all'interno della rete. Il cambiamento di strategia avviene secondo regole ben precise e, spesso, quella utilizzata è l'imitazione di un vicino che ha ottenuto risultati migliori. Questo metodo, tuttavia, non sembra essere adatto a situazioni in cui la popolazione è eterogenea e gli individui giocano quindi in condizioni molto differenti tra di loro: imitare un vicino che ha caratteristiche molto differenti non garantisce di ottenere i suoi stessi risultati. Bisogna inoltre tenere in considerazione che spesso i giocatori non hanno informazioni sulla composizione e sulla struttura del vicinato dei loro stessi vicini; sarebbe quindi sbagliato includere questo

tipo di informazione nel processo di scelta della strategia, in quanto spesso non è disponibile.
Per questi motivi proponiamo una nuova regola per l'aggiornamento di strategia, che chiamiamo *selfish reciprocal altruism*, la quale utilizza solo le informazioni locali disponibili agli individui in seguito agli incontri diretti con gli altri giocatori e si basa sul concetto di reciprocità, unito alla natura egoistica di ogni individuo.
Come risultato mostriamo che, con questa regola, la cooperazione non solo evolve all'interno della popolazione, ma addirittura riesce ad emergere anche se inizialmente presente in frazioni particolarmente ridotte.

Parole chiave: Cooperazione, evoluzione, reti sociali, reciprocità, informazioni locali, comportamento egoista.

Abstract

The emergence and persistence of cooperation among self-interested individuals is a long-standing and still debated topic in various fields of study, going from biology to social sciences and with recent attention in engineering.
The standard modeling framework for the study of cooperation is *Evolutionary Game Theory* (EGT), where individuals can exhibit different forms of behavior, some of which have the ability to persist in the population, while others have a tendency to be driven out. In particular, the paradigmatic game describing individuals' interactions is the *Prisoner's Dilemma* (PD).
When considering the social framework, a population can be represented by a network, where the nodes are individuals and links are the possible interactions among them. Individuals engage repeated PD games with their neighbors and, depending on the behavior of the neighbors, they can decide to change strategy following rules for strategy update. One of the most common rules is the imitation of a better performing neighbor. This method, however, is not suitable where neighbors might act in completely different conditions, e.g. in strongly heterogeneous networks.
We propose a new rule for strategy update, called *selfish reciprocal altruism*, where players decide to update strategy only if it is convenient for them in the next few iterations, basing their decision only on the local information gathered through direct interactions.
As a result, we show that this selfish updating does not simply favor the evolution of cooperation, but also its emergence in populations where the initial fraction of cooperators is very small.

Key-words: Cooperation, evolution, social networks, reciprocity, local information, selfish behavior.

Chapter 1

Introduction

1.1 The role of cooperation

The emergence and persistence of cooperation among self-interested individuals is a long-standing and still debated topic in various fields of study, going from biology to social sciences and with recent attention in engineering. Cooperation is often needed for evolution to construct new levels of organization. For example, genomes, cells, multicellular organisms, social insects, and human societies are all based on cooperation.

Many studies have been done about cooperation in its various scenarios, here we present a selection of these works to introduce the topic and its relevance in different applications.

One of the early works is G. Hardin's *The tragedy of the commons* (1968) [1], where the author considers the "population problem". Since the world is finite, he claims, it can only support a finite population; thus, the population growth must eventually equal zero. What we would like to reach is "the greatest good for the greatest number", but this is not possible. First of all we need to understand what we mean by "good": this changes from one person to another and different goods seem to be incommensurable. The mankind should learn from nature, where goods are compared in term of survival, but no cultural group could imitate this process so far. In fact there's no prosperous population that has had a growth rate equal zero for a certain period of time, and a positive growth rate is taken as evidence that a population is below its optimum size. Referring to Adam Smith's [2] idea that an individual who "intends only his own gain" is "lead by an invisible hand to promote...the public interest" the author introduces the concept of the *tragedy of commons*: if we imagine a pasture open to all, a rational herdsman will try to keep as many cattles as possible on the commons in order to maximize his gain. But the same reasoning will be done by every man, with the result that each of them will try to increase his herd without limit in a world that is limited. Pursuing personal interests, the mankind can thus only rush into ruin, hence proving that freedom in commons leads to tragedy.

In *The Evolution of Reciprocal Altruism* (1971) [3], R.L. Trivers presents a model to account for the natural selection of what is called *reciprocally altruistic behavior*. This model can be applied to several fields and the author discusses three different instances of this behavior: (1) behavior involved in cleaning symbioses, (2) warning cries in birds and (3) human reciprocal altruism. He defines the altruistic behavior as the one that benefits another organism, not closely related, while being apparently detrimental to the organism performing the behavior, benefit and detriment being defined in terms of contribution to fitness. A human being saving another, who is not closely related and is about to drown, is an instance of altruism. Assume that the chance of dying for the drowning man is one-half if no one leaps in to save him, but that the chance that his potential rescuer will drown if he leaps in to save him is much smaller. Assume also that the drowning man always drowns when his rescuer does and that he is always saved when the rescuer survives the rescue attempt. If the drowning man reciprocates at some future time, and if the survival chances are then exactly reversed, it is beneficial to each participant to have risked his life for the other. If we assume that the entire population is sooner or later exposed to the same risk of drowning, the two individuals who risk their lives to save each other will be selected over those who face drowning on their own. Note that the benefits of reciprocity depend on the unequal cost/benefit ratio of the altruistic act, that is, the benefit of the altruistic act to the recipient is greater than the cost of the act to the performer. Note also that the benefits and costs may differ from an individual to another depending on their age and other characteristics. A natural question is why should the rescued individual bother to reciprocate? Why not cheat? Selection will discriminate against the cheater if cheating has later adverse affects on his life which outweigh the benefit of not reciprocating. This may happen if the altruist responds to the cheating by curtailing all future possible altruistic gestures to this individual.

In human reciprocal altruism, each individual is seen as possessing both altruistic and cheating tendencies, and the model shows how selection can operate against the cheater. Sometimes cheating can pay, for example when the partner will not find it out or when he is unlikely to survive long enough to reciprocate adequately, but at the same time selection can discriminate against cheater if cheating decreases the possibility of receiving any future altruistic gesture. In human reciprocal altruism an important role is played by friendship, dislike, moralistic aggression, gratitude, sympathy, suspicion, trustworthiness, and another central aspect is the psychological influence of the cost/benefit ratio in the decision of performing an altruistic act.

In *The Evolution of Cooperation* [4] R. Axelrod and W.D. Hamilton, taking into account the manifest existence of cooperation and related group behavior in nature, introduce a model based on the assumption that interactions between pair of individuals occur on a probabilistic basis, thus being possible that two individuals interact multiple times during their life. The model is based on the Prisoner's Dilemma game and in particular on the existence of an evolutionary stable strategy in this context.

The emergence and persistence of cooperation keeps being studied, as it can be noticed for example from the recently published paper *Think global, act local: Preserving the global commons* [5], where the authors propose a method by which reciprocity can maintain cooperation in a large-scale public goods game (PGG). Here again the framework of Prisoner's Dilemma (PD) game is used: in one of the experiments performed, participants play one round of PD with their two nearest neighbors on a cyclic network after each PGG round. It's observed that participants reduce PD cooperation with neighbors who contribute little in the PGG. Instead, low PGG contributors increase their contributions if both neighbors defect in PD. In conclusion the authors claim that this "local-to-global" reciprocity seems to facilitate large-scale cooperation.

Papers [6, 11, 12] show that the same topic has become interesting also from an engineering point of view, where the attention is focused for example on sensor networks used in communications, controls and mechanics.

E. Semsar-Kazerooni and K. Khorasani in *A Game Theory Approach to Multi-Agent Team Cooperation* [6] use cooperative game theory to design a team of agents in Sensor Networks that can accomplish consensus over a common value for the agents' output. The goal is to ensure cooperation minimizing a cost function that is a combination of individual costs of each agent.

The authors of *When Structure Meets Function in Evolutionary Dynamics on Complex Networks* [11] show how the system structure is essential in the formation of collective behavior. In the specific they define the theoretical framework of evolutionary dynamics on complex networks by describing the main elements that characterize this framework and studying the impact of the network structure on the evolutionary dynamics.

Cooperation stimulation mechanisms for wireless multihop networks: A survey [12] focuses on wireless sensor networks and mobile ad hoc networks. In these types of networks nodes have to rely on their peer neighbors in transmitting packets to destinations, although a successful rate of communication is assured only if all nodes fully cooperate to relay packets for each other. This paper summarizes existing cooperation stimulation mechanisms and discusses important issues in this field.

A broad literature is available about the subject, and those presented are simply some of the examples available in the various fields. We continue our work by focusing on the Prisoner's Dilemma game and proposing a new model to account for cooperation.

1.2 About this work

In the first part of this thesis we observe that, when the PD game is played in large and well-mixed populations, defection is the dominant strategy, while cooperation cannot invade and disappears if initially present. We then consider the various mechanisms proposed in the literature to foster cooperation in this context and relax the social dilemma [14].

The work continues with the study of more recent researches, where evolutionary games have been analyzed on structured populations, represented either by regular lattices or by complex networks of contacts [21]. We notice that the locality of interactions – named *network reciprocity* after [22] – has been proposed as a new basic mechanism to explain cooperation, because, thanks to it, it is possible that groups of cooperators (C) perform better than clustered or little-connected defectors (D). Starting with the seminal works [22, 23], hundreds of influential contributions have been published on the topic, with the emerging result that heterogeneous networks of contacts, the so-called *scale-free networks* [24], benefit cooperation.

Here we focus on social networks and we question network reciprocity as a mechanism favoring cooperation. This mechanism works in biological evolution, where the strategy update is determined by a selection that acts on birth, death and competition processes, but when considering the social context the spreading of strategies is modeled in different ways, usually through imitation rules, and therefore some questions arise. Why should we imitate a better performing neighbor in a heterogeneous setting? Does it make sense to imitate a neighbor, considering that he/she potentially operates in radically different conditions?
Moreover, network reciprocity only supports the persistence of a significant level of cooperation, whereas the emergence of cooperation in a population dominated by defection remains unexplained. Cooperation needs to be present in an initial cluster to have the chance to take off.

The connectivity of the player to be imitated can be taken into account to attenuate the chances that little-connected individuals – *leaves* – imitate highly-connected ones – *hubs* – and vice-versa [25, 26], but typically the information on the size and composition of the neighbors' neighborhoods is not available to the players. Furthermore, the locality of the interaction opens the way to complex strategies requiring cognitive tasks such as remembering past interactions, foreseeing future ones, and identifying good and bad players. In this sense hubs need more memory than leaves, because they have more neighbors and consequently more interactions, but this is already intrinsic in the social structure, where central individuals have a large address book.

Therefore, we finally go back to the original idea of direct reciprocity [3] that is made practicable by network reciprocity. We implement an updating rule based on *selfish reciprocal altruism* that makes only use of local information, gathered through direct social interaction. At each update step, an individual computes his/her own expected payoff, accumulated behaving as C or D over an investment horizon of few of the next interactions and accordingly decides whether to switch strategy or not. With no mechanism discouraging defection, D will eventually dominate independently of the network structure.

However, we find that a mechanism of direct reciprocity, where Cs play less frequently with defecting neighbors, allows the invasion of cooperation in populations initially dominated by defection, its long-term persistence, and even the eventual dominance of C, provided the benefit-to-cost ratio of the interaction and the investment horizon is sufficiently large. And all this is indeed enhanced by heterogeneous network structures, provided the few initial Cs occupy the hubs of the network.

This thesis is structured as follows. In Chapter 2, we introduce the Prisoner's Dilemma game and we review some of the mechanisms proposed in the literature to favor cooperation in a population. We then illustrate some of the main properties of networks and present the network models used throughout this thesis. We discuss the Prisoner's Dilemma when played in social networks and we study the effects on the evolution of cooperators when an imitation process is taken into account. In Chapter 3, we propose a new evolutionary rule and show the results of the simulations. Finally, in Chapter 4 we give our conclusions and outline possible future works.

Chapter 2

Evolutionary Dynamics

2.1 The Prisoner's Dilemma game

The standard modeling framework for the study of cooperation is *Evolutionary Game Theory* (EGT), where individuals can exhibit different forms of behavior, some of which have the ability to persist in the population, while others have a tendency to be driven out.
Evolutionary Game Theory is the application of game theory to evolving populations. EGT originated as an application of the theory of games to biological contexts, arising from the realization that the strategic aspect plays an important role in evolution. Recently, however, EGT has become of increased interest to economists, sociologists, and anthropologists and social scientists in general.

As it can be noticed from the examples reported in the previous chapter, the paradigmatic game describing individuals' interaction is the *Prisoner's Dilemma* [13], that can be explained with the following example: two suspects of a bank robbery are caught and interrogated by the police. The police offers them separately the following deal. If a suspect testifies against his colleague (a strategy of *defection* – D), and the other does not (*cooperation* – C), his sentence will be reduced by five years. If both suspects testify, that is defect, they will get the reduction of only one year. However, if they both cooperate and do not testify, their sentence, because of the lack of a hard evidence, will be reduced by three years.

We thus obtain the payoff matrix in Table 2.1, where the strategy of the 1st player is the rows and that of the 2nd player on the columns, and the matrix contains the payoff for the 1st player.

	C	D
C	3	0
D	5	1

Table 2.1: Prisoner's Dilemma example

The Prisoner's Dilemma is a two-player-two-strategy game that can be parametrized in multiple ways: a first way to see it is to generalize the payoffs in the four different cases. As it can be seen in Table 2.2, the game payoffs become respectively: reward (R), sucker (S), temptation (T) and punishment (P) with $T > R > P > S$.

	C	D
C	R	S
D	T	P

Table 2.2: PD 4-parameter

We can normalize the above quantities by reducing the number of parameters to two. Consider the following inequalities:

$$\frac{T-P}{R-P} > \frac{R-P}{R-P} > 0 > \frac{S-P}{R-P} \tag{2.1}$$

We rename these fractions, writing T instead of $\frac{T-P}{R-P}$ and S instead of $\frac{S-P}{R-P}$. We therefore obtain the following inequalities:

$$T > 1 > 0 > S \tag{2.2}$$

The resulting payoff matrix is shown in Table 2.3.

	C	D
C	1	S
D	T	0

Table 2.3: PD 2-parameter: T and S

The two-parameter normalization can also be written in another way, as shown in Table 2.4. Cooperators (C) provide a benefit b to the other player

at a cost $c < b$, whereas defectors (D) provide no benefit at no cost.
The punishment's payoff P will thus be zero and consequently $S = -c < 0$, $T = b$ and the reward's payoff will be the difference between the benefit b and the cost c.

	C	D
C	$b - c$	$-c$
D	b	0

Table 2.4: PD 2-parameter: b and c

We can further reduce the number of parameters by introducing the benefit-to-cost ratio $r = b/c > 1$ and using the cost c as scaling unit. The game payoffs become $T = b = rc$, $R = b - c = (r - 1)c$, $P = 0$, $S = -c$. We will consider $c = 1$ in the following and this is the resulting payoff matrix:

	C	D
C	$r - 1$	-1
D	r	0

Table 2.5: PD 1-parameter

2.2 Mechanisms to favor cooperation

The one introduced in the previous section is the simplest version of the Prisoner's Dilemma game, where only two players are involved in a single round of the game. However, when studying the evolution of cooperation, we generally take into consideration an entire population of individuals playing among each others. Furthermore, the same pair of individuals will probably interact more than once during the lifetime, as described by the theory of *repeated games*.
In game theory, a *repeated game* is simply a repetition of some base game (called stage game) a certain number of times: considering an evolutionary process in a population, it will most likely happen that the same two individuals interact multiple times during their lifetimes, thus engaging the same game not once, but repeatedly. After an interaction, an individual may decide, for various reasons, to change strategy, and the same can be done by many individuals in the population. Consequently, the ratio between cooperators and defectors is not always the same and their number can vary

after each round of the game. In this way we obtain a population that is dynamic, in the sense that the strategy of its components can evolve toward a new status at every step.

In particular, let's first consider a large and well-mixed population, i.e. a population with a large number of individuals, all of which are playing in the same (or very similar) conditions. When the PD game is played this population, it turns out that defection is the dominant strategy while cooperation cannot invade and disappears if initially present in the population.

In this context, several mechanisms have been proposed to foster cooperation and relax the social dilemma [14], for example:

1. reciprocal altruism [3] (also known as *direct reciprocity*, among kin or unrelated individuals);
2. establishment of reputations [15] (also known as *indirect reciprocity*);
3. mechanisms of group rather than individual selection;
4. recognition of perceptible traits [17];
5. optional participation [18];
6. punishment of antisocial behaviors [19, 30].

Direct Reciprocity

In 1971, Trivers [3] proposes a mechanism for the evolution of cooperation that could explain the cooperation between unrelated individuals or even between members of different species, and not only among relatives or kin individuals. This mechanism, called *reciprocal altruism* or *direct reciprocity*, is based on the idea that the same two individuals can meet and interact repeatedly. In the framework of the repeated Prisoner's Dilemma game, the key point is that, if I cooperate now, you may decide to cooperate later. In every round of the game, each player can decide whether to cooperate or not. One of the many possible strategies in this game is the one known as *tit-for-tat*: cooperation is chosen on the first move, while on the following moves the player does whatever the other player did on the preceding move. This strategy, although being simple, can be the winning strategy [4], catalyzing cooperation, when in the society defection is prevailing. A wide variety of strategies is available, but one general rule seems to always be valid: direct reciprocity can lead to the evolution of cooperation only if the probability, w, of another encounter between the same two individuals exceeds the cost-to-benefit ratio of the altruistic act, that is:

$$w > \frac{c}{b} \tag{2.3}$$

Many of the human behaviors that can be seen as example of altruistic acts – such as helping in time of danger, helping the sick or sharing knowledge – meet the criterion of small cost to the giver and a great benefit to the taker, making the above rule easily satisfiable and favoring cooperation.

Indirect Reciprocity

Direct reciprocity relies on repeated encounters between the same two individuals, and both of them must be able to provide help. But often the interactions among humans are asymmetric: one person is in a position to help another, but there is no chance for a direct reciprocation. In *Evolution of indirect reciprocity by image scoring* [15], Nowak and Sigmund present a new theoretical framework which does not require the same two individuals to meet multiple times. This model, called *indirect reciprocity*, takes into account the fact that helping someone establishes a *good reputation*, which will be rewarded by others. When deciding how to act, in fact, we not only think about direct consequences of our behavior, but also consider the possible effects on our reputation. In the standard framework of indirect reciprocity, individuals of a population have the option to help one other or not. In one generation of the game, m random pairs of players are chosen, of which one is the potential donor of the altruistic act and the other is the recipient. If the donor cooperates and helps the recipient, he/she pays a cost c to himself, and the recipient receives a benefit of value b (with $b > c$). Instead, if the donor decides not to help, they will both receive a zero pay-off. Furthermore, each player has an image score, s, that is related to his/her reputation and status. If an individual, chosen for the role of donor, decides to cooperate then his/her image score increases by one unit, while if he/she doesn't cooperate it decreases by one unit. The image score of a recipient does not change. In the basic model, donors decide to help according to the image score of the recipient: a player with the strategy k, for example, provides help if and only if the image score of the potential recipient is at least k. This means that a donor provides help if the recipient is likely to help others, i.e. if the recipient has helped others in the past. Helping someone increases the chances to become the recipient of an altruistic act in the future.

In an idealized scenario, each member of the population knows the image score of all the other individuals, but in a more realistic case everyone simply have a personal perception of the image score of the other players. Calling q the probability of knowing someone's reputation, the authors state that indirect reciprocity can promote cooperation only if q exceeds the cost-to-benefit ratio of the altruistic act:

$$q > \frac{c}{b} \tag{2.4}$$

Group Selection

Selection acts not only on individuals but also on groups. A group of cooperators might be more successful than a group of defectors. A simple model of group selection [16] works as follows. Consider a population subdivided into groups. The fitness of individuals is determined by the payoff gained from an evolutionary game and interactions occur between members of the same group. In each step, a single individual in the population is chosen for reproduction with a probability proportional to its fitness and the offspring is added to the same group. If the group reaches a critical size, n, it will divide into two groups with probability q and the members will be randomly distributed over the two new groups. If the group does not divide, i.e. with probability $1-q$, a randomly selected individual is eliminated from the group. Therefore, n represents the maximum number of individuals in a single group. The total number of groups is m and it is constant; in fact, whenever a group splits into two, another one is eliminated. Considering these constraints, the population size varies from the lower bound of m to the upper bound of mn individuals. Note that selection emerges on two levels: on the one hand the evolutionary dynamics are driven by individual fitness, because only individuals are assigned payoff values and only individuals reproduce. On the other hand, groups can stay together or split when reaching a certain size and groups containing fitter individuals reach the critical size faster and separate more often. In particular, pure cooperator groups grow faster than pure defector groups, whereas in any mixed group, defectors reproduce faster than cooperators. Therefore, selection on the lower level – within groups – favors defectors, while selection on the higher level – between groups – favors cooperators.

Recognition of Perceptible Traits

The evolution of altruism can be facilitated if there exist mechanisms that allow discrimination against defectors. An example of such mechanisms is the *green beard effect* [17] regarding genes reproduction, in which altruists can recognize each other using a conspicuous tag or signal. Imagine a population where all individuals have a recognizable tag in the form of a colored beard.

Each individual plays the PD game with all its neighbors.Altruistic actions are only towards individuals with the same beard color.

Optional Participation

Another mechanism proposed to foster cooperation is the *optional participation* [18]. Differently from previous models, the optional participation considers three different strategic types instead of two: cooperators, defectors and loners. Cooperators and defectors are willing to play, while loners prefer to rely on a risk-averse behavior. In a large population, from time to time, a random sample of M individuals is offered the option to engage in a public goods game. Loners, if selected, will always refuse to play obtaining a side income that does not depend on other individuals. This model has been proposed to overcome the difficulties encountered in other settings, such as the need of repeated encounters between the same individuals or the necessity of identifying who are the defectors. In such a voluntary public good games, cooperators will not stably dominate, but neither will defectors. Their frequencies oscillate, because the public good becomes unattractive if loners abound.

Punishment

One last mechanism to favor cooperation is the punishment of antisocial behaviors, [19, 30]. Consider a large population, divided into groups of size n. Three behavioral types are coexisting in the population, contributors, defectors and punishers. Contributors incur a cost c to produce a total benefit b that is shared equally among group members. Defectors incur no costs and produce no benefits. Punishers cooperate and then punish defectors in their group, reducing each defector's payoff.

Let x be the fraction of contributors in the group, then the expected payoff for contributors is $bx - c$ and the expected payoff for defectors is bx, so the payoff disadvantage of the contributors is a constant c independent of the distribution of types in the population. Punishers act reducing each the payoff of each defector in their group by p/n at a cost k/n to themselves. If the frequency of punishers is y, the expected payoffs become $b(x+y) - c$ for contributors, $b(x+y) - py$ for defectors, and $b(x+y) - c - k(1-x-y)$ for punishers.

If the fraction of punisher is such that that the cost of being punished exceeds the cost of cooperating, that is $py > c$, then contributors have higher fitness than defectors. Punishers suffer a fitness disadvantage of $k(1-x-y)$ compared with non-punishing contributors, so it may arise the following question: why do people incur costs to punish others and provide benefits to

nonrelatives? Punishment is an altruistic act and mere contributors can be considered as "second-order free riders". Note, however, that the payoff disadvantage of punishers relative to contributors approaches zero as defectors become rare because there is no need for punishment.
Variants of this model are also possible and it is plausible that altruistic punishment is a method to pave the way to the fixation of cooperation.

2.3 Networks

Recently, evolutionary games have been studied on structured populations, represented either by regular lattices or by complex networks of contacts. In this section we introduce the concept of graph (or network) and some of the network models often used in the literature.

A graph is defined as a pair $G = (V, E)$, where V is the set of vertices, or nodes, and E is the set of edges, links or arcs. Nodes usually represent individuals, objects or subsystems, while links represent the interaction or dependencies among them.
A graph can be undirected or directed, weighted or unweighted.

An *undirected graph* is characterized by edges that have no orientation, i.e. the edge (i, j) is identical to the edge (j, i).

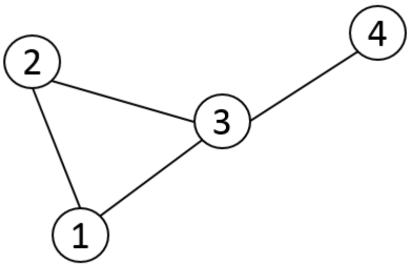


Figure 2.1: Undirected graph

A *directed graph* instead is a graph in which edges have orientations. Links are arrows, and the arrow (i, j) is considered to be directed from i to j

If a real or integer number (the weight) is assigned to each link, the graph is

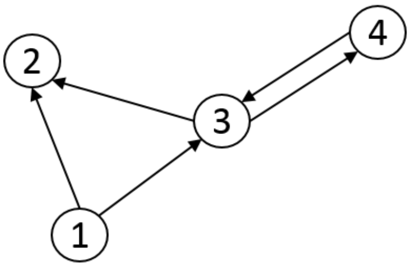
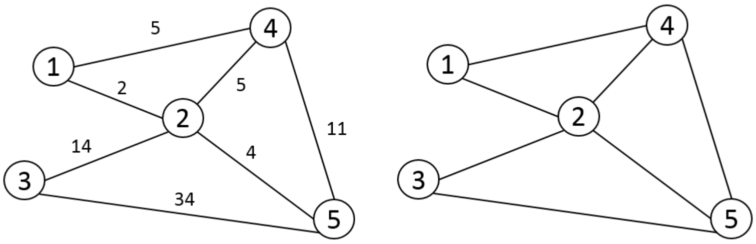


Figure 2.2: Directed graph

called *weighted*, otherwise it is *unweighted*. Such weights can represent costs, lengths, capacities or other quantities, depending on the problem at hand.



(a) *Weighted* **(b)** *Unweighted*

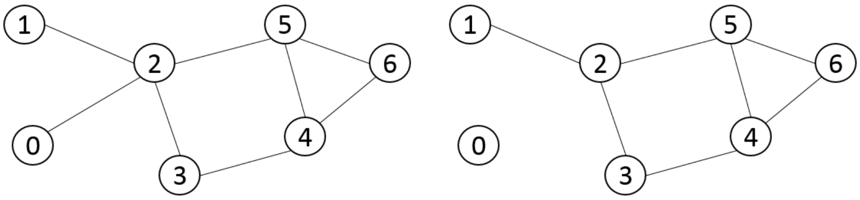
Figure 2.3: Examples of weighted and unweighted graphs

An unweighted network is completely described by the $N \times N$ *adjacency matrix* $A = [a_{ij}]$, where $N = |V|$ is the number of nodes in the network and $a_{ij} = 1$ if the link $i \to j$ exists, $a_{ij} = 0$ otherwise.

The adjacency matrix is symmetrical if the network is undirected, while it is asymmetrical if the network is directed. Usually A is a sparse matrix, i.e. the density of links ρ is small: $\rho = \frac{L}{N(N-1)}$ for directed networks and $\rho = \frac{L}{N(N-1)/2}$ for undirected ones, where $L = |E|$ and $N(N-1)$ (or respectively $N(N-1)/2$) is the maximum possible number of links in the network.

A weighted network is described also by the $N \times N$ *weight matrix* $W = [w_{ij}]$: $w_{ij} > 0$ if the link $i \to j$ exists, $w_{ij} = 0$ otherwise.

Finally, a graph is *connected* if there exists a path between every pair of vertices, ie. there are no unreachable vertices.



(a) *Connected* **(b)** *Disconnected*

Figure 2.4: Examples of connected and disconnected graphs

2.3.1 Network Properties

In this section we introduce some of the quantities used to describe the main properties that characterize networks.

Distance and Diameter

The *distance* d_{ij} is the length of the shortest path connecting i to j, measured in number of links.

For a connected network, the *diameter* D is the length of the shortest path between the two most distant nodes in the network. In other words, once the shortest path length from every node to all other nodes is calculated, the diameter is the longest of all the calculated distances:

$$D = \max_{i,j} d_{ij} \tag{2.5}$$

The *average distance* d is:

$$d = \langle d_{ij} \rangle = \frac{1}{N(N-1)} \sum_{i,j(i \neq j)} d_{ij} \tag{2.6}$$

If the network is weighted, several generalized definitions are available.

Degree and Strength

In an undirected network, the *degree* k_i of the node i is the number of links connected to i, i.e. the number of neighbors of i:

$$k_i = \sum_j a_{ij} \tag{2.7}$$

In an undirected weighted network, the *strength* s_i of node i is the total weight of the links connected to i:

$$s_i = \sum_j w_{ij} \tag{2.8}$$

If the network is directed, we differentiate among *in-*, *out-* and *total degree*, and *in-*, *out-* and *total strength* of node i.

The *degree distribution* $P(k)$ of a network specifies the fraction of nodes having exactly degree k, i.e. the probability that a randomly selected node has degree k:

$$P(k) = \frac{\#nodes\ with\ degree\ k}{N} \tag{2.9}$$

$$\sum_k P(k) = 1$$

Often, instead of the *degree distribution*, the *cumulative degree distribution* is used:

$$\bar{P}(k) = \frac{\#\ nodes\ with\ degree \geq k}{N} = \sum_{h=k}^{k_{max}} P(h) \tag{2.10}$$

$$\bar{P}(k_{min}) = 1$$

The *r-moments* of the degree distribution $P(k)$ are:

$$\langle k^r \rangle = \sum_k k^r P(k)\ ,\ r = 1, 2, ... \tag{2.11}$$

In particular, the first moment ($r = 1$) is the *average degree* of the network:

$$\langle k \rangle = \sum_k kP(k) = \frac{1}{N}\sum_i k_i = \frac{2L}{N} \tag{2.12}$$

Clustering Coefficient

The *local clustering*, or *transitivity*, *coefficient* quantifies the “local link density” by counting the triangles in the network. It counts how frequently, if we have the links $j \leftrightarrow i$ and $i \leftrightarrow l$, then we also have $j \leftrightarrow l$ and thus it exists the triangle i, j, l. Saying it in another way, in a social network it counts how frequently two friends of mine are also friends each other. The (local) *clustering coefficient* $0 \leq c_i \leq 1$ of node i is:

$$c_i = \frac{\#\ \mathit{triangles\ connected\ to\ i}}{\#\ \mathit{triples\ j, i, l\ centered\ on\ i}} = \frac{e_i}{k_i(k_i-1)/2} \tag{2.13}$$

where k_i is the degree of i, and e_i is the number of links directly connecting neighbors of i (at most $k_i(k_i-1)/2$).
The *global clustering coefficient* C is the average c_i over the whole network:

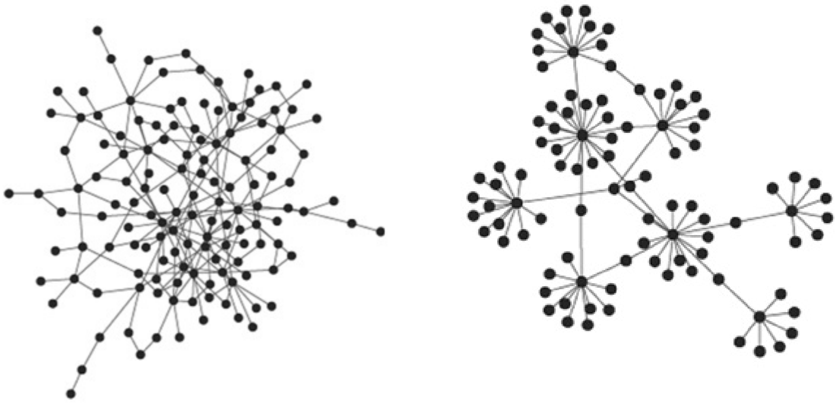
$$C = \langle c_i \rangle = \frac{1}{N}\sum_i c_i \tag{2.14}$$

Correlated Networks

A network is *correlated* if the probability $Q(h|k)$ that a node with degree k has a neighbor with degree h does depend on k. Correlations can be captured by the *average nearest neighbors degree* function:

$$k_{nn}(k) = \sum_h hQ(h|k) \tag{2.15}$$

As shown in the examples of Figure 2.5, a network is *assortative* if high-degree nodes tend to connect to high-degree nodes, on the contrary, it is *disassortative* if high-degree nodes tend to connect to low-degree ones. For example, social networks are typically assortative, while technological networks are disassortative.



(a) *Assortative* **(b)** *Disassortative*

Figure 2.5: Examples of assortative and disassortative networks

When the network is uncorrelated, the probability $Q(h|k)$ does not depend on k, thus $Q(h|k) = Q(h)$ and the degree distribution of neighbors of node i does not depend on the degree k of i. For uncorrelated network the *degree distribution of neighbors* is:

$$\begin{aligned} Q(h) &= \frac{\#\ links\ from\ nodes\ with\ degree\ h}{\#\ links\ from\ nodes\ with\ any\ degree} \\ &= \frac{hP(h)}{\sum_k kP(k)} = \frac{hP(h)}{\langle k \rangle} \end{aligned} \tag{2.16}$$

and the average nearest neighbors degree $k_{nn}(k)$ is constant and larger than the average degree $\langle k \rangle$, in fact:

$$k_{nn}(k) = \sum_h hQ(h|k) = \sum_h \frac{h^2 P(h)}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle} \tag{2.17}$$

$$k_{nn} = \frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \sigma^2}{\langle k \rangle} = \langle k \rangle + \frac{\sigma^2}{\langle k \rangle} \tag{2.18}$$

2.3.2 Network Models

Complete Network

A *complete network* is characterized by full connectivity, i.e. each node is connected to all the others, thus the network is homogeneous. Given a complete graph with N nodes, since all pairs of vertices are connected, the diameter is 1 and for every node i the degree k_i is $k_i = N - 1$, while the total number of edges is $\frac{N(N-1)}{2}$.

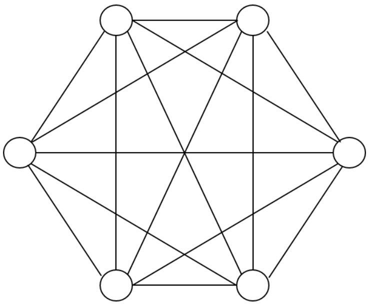


Figure 2.6: Complete network

Single-scale Network (ER Network)

The *Erdös-Rényi network* (ER) is built connecting L randomly extracted

pairs among N nodes and it is an example of a "random" network. An alternative procedure to build it is the following: starting from a graph with N nodes an no links, connect each pair i, j with a given probability p.
When N is large enough, the degree distribution of an ER network behaves as a Poisson distribution with $\langle k \rangle = \frac{2L}{N} = p(N-1)$:

$$P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \tag{2.19}$$

The network is almost homogeneous and the typical scale of node degree is $k_i = \langle k \rangle$, with small fluctuations around $\langle k \rangle$.

Scale-free Network (BA Network)

The model introduced by Barabási and Albert (BA) [24] is an example of *scale-free network* and it is obtained by adding one node at a time and connecting it according to the mechanism of preferential attachment: starting with m_0 nodes arbitrarily connected, at each step, add a new node i with $m \leq m_0$ new links and attach these m link with higher probability to nodes that have high degree. This procedure is easily understandable and follows the principle "rich get richer ": given a new node i, the probability of connecting it to an existing node j is $\frac{k_j}{\sum_h k_h}$.
When N goes to infinity, the average degree of a BA network tends to $\langle k \rangle = 2m$ and the degree distribution tends to a power-law $P(k) \approx k^{-3}$

Star Network

Star networks are characterized by one central node, called *hub*, connected to many nodes, the *leaves*, which are not connected among each other. The degree of the hub is $N - 1$, while the degree of the leaves is 1.

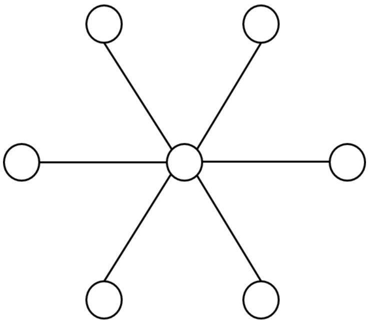


Figure 2.7: Star network

2.4 Evolutionary Dynamics in Social Networks

2.4.1 Strategy Update

As we have seen in the previous chapter, cooperation is often formulated in the framework of *Evolutionary Game Theory*. The evolutionary dynamics on networks offer an appropriate tool to study diffusion of cooperation in complex systems.

Generally, evolutionary dynamics evolve on a structured complex system, which is often characterized by a network, where nodes and edges represent respectively the individuals (or components) and their mutual interactions. Moreover, the interaction intensity on each edge can be described by a specific weight value.

The admissible states of each node are usually discrete and each state s, which corresponds to a strategy of the game, is assigned a fitness value, $f(s)$, to measure its success in a specified task performed on the evolving system. Some typical fitness functions are described as follows.

- *Random Drift*: Random drift refers to the situation in which all states are equally successful in the evolving system, i.e. $f(s) = 1$ for all states s. In other words, the fitness is not affected by the strategy adopted by the player, so that players randomly update their strategy. It is a simple approximation of a realistic evolution and it is used to study evolution under neutral selection [7].
- *Constant Selection*: In this scenario, the fitness of a state is kept unchanged in the whole evolutionary process. In other words, for each state s, $f(s)$ is a positive constant, but it may be different among states. Thus, constant selection is more general than random drift. Constant selection is often used for investigating how successful behaviors spread out over a network [8, 9].
- *Networked Game*: In a networked game, the states of nodes are usually called strategies. The fitness of an individual player (node) is acquired by playing games with its neighbors. In such a networked game, the fitness of each state depends on all the states of its neighbors, and it is

often expressed as the expected payoff of the game.

The most commonly studied case of evolutionary dynamics is the binary-state evolutionary dynamics, where only two different states are possible. This evolutionary dynamics can be used to characterize the invasion process of new ideas, behaviors and innovations [10].
In evolutionary dynamics on complex networks, every node updates its state according to the states and the fitness of its neighbors. Some typical updating rules are listed in the following, including the birth-death (BD) process, death-birth (DB) process, link dynamics (LD), and imitation (IM) process.

- In the BD process, firstly a node is selected with a probability proportional to its fitness. Then, a neighboring node of this selected node is randomly chosen. Finally, the selected node reproduces a copy and uses this copy to replace the above-chosen neighboring node, as shown in Figure 2.8. If the network is weighted, the neighboring node is chosen with a probability proportional to the weight of the corresponding edge. For example, if node i is first selected, then a neighboring node j is chosen with probability proportional to the weight w_{ij}.

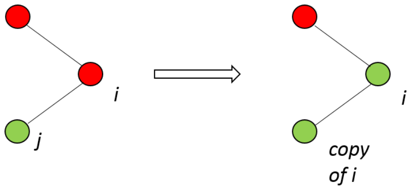


Figure 2.8: BD updating process: node i is selected with a probability proportional to its fitness. Then node j is randomly chosen among i's neighbors and it is replaced by a copy of i.

- In the DB process, firstly a node is randomly chosen. Then, a neighboring node of the above chosen node is selected with a probability proportional to its fitness. Finally, the selected neighbor reproduces a copy and uses this copy to replace the first-chosen node, as shown in Figure 2.9. For weighted networks, if node i is randomly chosen first, then a neighboring node j is selected with a probability proportional to $r_j w_{ji}$. Here, r_j denotes the fitness of node j and w_{ji} is the weight of edge (j, i).

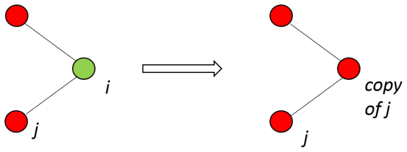


Figure 2.9: DB updating process: node i is randomly chosen. Then node j is selected among i's neighbors with a probability proportional to its fitness. Finally, node i is replaced by a copy of j.

- In the LD process, a directed edge is selected with a probability proportional to its weight at each step. If the edge (i, j) is selected, then node j adopts the state of node i at the next step, as shown in Figure 2.10.

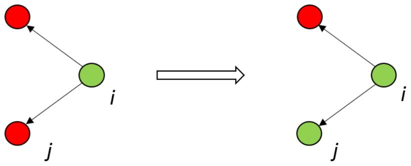


Figure 2.10: LD updating process: the directed edge (i, j) is selected with a probability proportional to its weight. Node j updates, adopting the state of node i.

- In the IM process, a node is randomly chosen firstly. Then, another node is selected, among the neighbors of the above-chosen node and the node itself, with a probability proportional to its fitness. Finally, the first node copies the state of the second node, as shown in Figure 2.11.

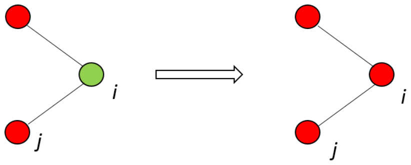


Figure 2.11: IM updating process: Node i is randomly chosen. Node j is selected with a probability proportional to its fitness. Node i imitates node j and updates its state.

2.4.2 The PD Game in Social Networks

Evolutionary Game Theory is often studied through the analysis of games such as the Prisoner's Dilemma, which is used as a metaphor describing cooperation among unrelated individuals. In the framework of evolutionary dynamics, a single round of the PD game is not enough to represent the entire process, therefore we need to make use of *(infinitely) repeated games.* When studying the evolution of the cooperation in a population, in fact, it is reasonable to assume that the same pair of players will interact more than once during the period of time considered, repeating the same game an undefined number of times. When the PD game is played in a social network, every individual engages a game with all his/her neighbors and accumulates a payoff in each of these interactions. The total payoff obtained in one round is the sum of all the payoffs gained in the single interactions. Depending on the payoff accumulated and on the behavior of their neighbors, players may decide to change strategy before the next round starts, switching from cooperation to defection, or vice versa. The update of strategy can take place according to various rules, for example the ones described in the previous section.

We now show how the evolutionary dynamic works in the context of PD game played in social networks.

As done in many contributions, we parameterize the PD game with the benefit-to-cost ratio $r = b/c > 1$ and use the cost c as scaling unit ($c = 1$ in the following). Given a population of N individuals with all-to-all connectivity (representable with a complete network), the expected payoff – calculated referring to Table 2.5 – for a C in a single round of the game is

$$\pi_{C,N_C} = (r-1)\frac{N_C - 1}{N-1} - \frac{N - N_C}{N-1} \tag{2.20}$$

while the expected payoffs for a D is

$$\pi_{D,N_C} = r\frac{N_C}{N-1} \tag{2.21}$$

where $N_C \leq N$ is the number of Cs and $\frac{N_C}{N}$ and $\frac{N-N_C}{N}$ are the fractions (or frequencies) of Cs and Ds. It is easily noticeable that π_{C,N_C} is always smaller than π_{D,N_C}, hence, in this framework, it is always convenient being a defector rather than a cooperator and thus, independently of the rule for strategy update, cooperation will disappear even if initially non zero.

This is exactly the result obtained in the classical context for infinite and well-mixed populations, where the evolution of the frequency is deterministically described by the *replicator equation* [13]. The replicator model is the limit model for large and well-mixed populations, where all individuals play in the same condition, in the sense that all individuals are equally likely to interact with each other. This setting can be interpreted in two ways: from a social point of view it represents an imitation process, where the focal individual imitates a better performing strategy, while in a biological setting it represents a situation in which the birth rates of C and D are proportional to the expected payoffs.

Consider a population of N individuals and call $x = \frac{N_C}{N}$ the fraction of cooperators. When N is large, the limits of the expected payoffs for C and D are respectively:

$$\pi_C = (r-1)x - (1-x)$$

and

$$\pi_D = rx$$

and the average payoff of the population is:

$$\langle \pi \rangle = x\pi_C + (1-x)\pi_D$$

The *replicator equation* describing the evolution of the frequency of Cs can thus be written as follows:

$$\dot{x} = x(\pi_C - \langle \pi \rangle) = x(1-x)(\pi_C - \pi_D) = -x(1-x) \tag{2.22}$$

Since $x \in (0,1)$, it will always be $\dot{x} < 0$ and it is immediately clear that the result is the same as the one seen above: the fraction of cooperators will disappear even if initially non zero.

$$\dot{x} = -x(1-x)$$

Figure 2.12: Replicator Dynamics

In a structured population, where individuals are represented by the nodes of a static network and interactions occur only between neighbors, the expected payoffs for a C and a D in a single PD round depend on the composition of their neighborhoods.

Given a node with degree k (i.e. with k neighbors), $k_C \leq k$ of which are Cs, we have

$$\pi_{C,k,k_C} = (r-1)\frac{k_C}{k} - \frac{k-k_C}{k}$$

and

$$\pi_{D,k,k_C} = r\frac{k_C}{k}$$

In a situation of Cs mainly connected to Cs and Ds mainly connected to Ds, $\pi_{C,k_i,k_{C_i}}$ can exceed $\pi_{D,k_j,k_{C_j}}$ for a C node i and a D node j and this can locally allow the diffusion of cooperation.
Consider for example the situation shown in Figure 2.13. The two central hubs have degree k and are connected to $k-1$ leaves with their same strategy. In the limit of large k, the expected payoffs for the C and D hubs in a single round of the PD game are respectively $\pi_{C,k,k-1} = r-1 > 0$ and $\pi_{D,k,1} = 0$. For the D hub it is then more convenient to imitate the better performing C strategy, allowing in this way the diffusion of cooperation.

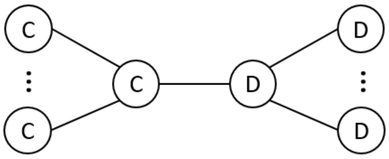


Figure 2.13: A simple heterogeneous network, with two central hubs connected to many leaves.

The locality of the interaction – *network reciprocity* [22] – can thus explain the persistence of a significant fraction of Cs in the population, but it does not support the invasion of Cs starting from a few isolated individuals. In [22] the authors found a surprisingly simple rule that partly clarifies the issue: network reciprocity favors cooperation if the benefit-to-cost ratio r exceeds the average degree $\langle k \rangle$ of the network of contacts. Under this condition, and basically independently of the network structure, the fixation probability of cooperation, i.e. the probability to go from a single C to all-C, is higher under selection for best performances than under totally random strategy update. The probability that the isolated C switches to D is however much higher than that of going from 1 to 2 Cs, i.e., cooperation does not invade, even though its fixation is favored by selection.

More recent studies, reviewed in the next section, have shown that network reciprocity is further enhanced in heterogeneous networks, typically in scale-free networks characterized by a power-law degree distribution [24]. In all studies, evolution has been described by the so-called *finite population replicator dynamics* [21], an imitation rule according to which the evolutionary dynamics converge to the deterministic replicator limit in the case of large and all-to-all networks. The results again support the persistence of cooperation, but not its emergence.

2.5 Evolution by imitation

Many studies support the importance of the role of network heterogeneity in the evolution of cooperation and they are all based on variations of the following procedure, an imitation process here described with reference to the PD game.

Evolution is modeled as a process based on discrete steps that generally coincide with generations or rounds of the game. At each step, all pairs of neighbors engage in a single round of the game and each individual accumulates the payoffs obtained playing in his/her own neighborhood. When the round of the game is over and the payoffs have been accumulated, each individual randomly selects a neighbor and updates his/her own strategy by possibly copying the one of the selected player with a probability related to the difference of their accumulated payoffs. This is a generalization of the imitation rule (IM) for strategy update seen in the previous section. More precisely, if individual i, with accumulated payoff P_i, randomly selects individual j, with accumulated payoff P_j, then i copies the strategy of j with probability:

$$prob_{i\to j} = \max\left\{0, \frac{P_j - P_i}{(r+1)\max\{k_i, k_j\}}\right\} \tag{2.23}$$

i.e. i copies j only if $P_j > P_i$ and with a probability that increases with the difference $P_j - P_i$ of the accumulated payoffs. Note that in the above rule the denominator is a normalization factor, necessary to ensure that $prob_{i\to j}$ does not exceed one and, at the same time, to reduce the chances of imitation if one or both nodes are highly connected.

The above described mechanism is simply one version of the imitation rule, but several variants are available. For example, instead of giving to every player the chance of updating his/her strategy at each step, it is possible to introduce an *asynchronous update.* After every generation of the game, only one individual or a small fraction of the population is offered the possibility to update strategy. In this case the imitation probability typically used is the smoothed imitation probability:

$$prob_{i \to j} = \frac{1}{1 + \exp(-(P_j - P_i)/K)}$$

where K is a parameter of the process. A small value of K means that individual i is highly sensitive to the expected payoff gain (the probability of imitating j sharply rises to 1 with $P_j - P_i > 0$ and drops to 0 with $P_j - P_i < 0$), whereas a large K indicates a low sensitivity.

Nonetheless, the role played by network heterogeneity seems to be unaffected by the particular choice [25, 26].

It is important to note that imitation based on the payoffs accumulated by neighbors is not a strictly local evolutionary rule. Local informations are those gathered by a player during his/her own interactions, whereas the payoffs accumulated by the player's neighbors are not necessarily available to the player and need, at best, to be estimated. The same is true for the information on our neighbor's connectivity, used to scale the imitation probability in (2.23). Thus, this type of imitation process requires informations that are not always available to the player, or, if available, demand a cognitive and memory effort to the player.

2.5.1 Evolutionary Dynamics of Social Dilemmas in Structured Heterogeneous Populations

F. C. Santos and coauthors [25, 26] were the first to show that network heterogeneity favors cooperation, by using the imitation rule (2.23) under synchronous update. They study what is the impact of moving from complete to scale-free [24] networks and document an increase of the long-term fraction of cooperation, respectively going from 0 (dominance of defection) in large complete networks to 1 (full cooperation) in large scale-free networks.
In particular they model the PD game with the 4-parameter formulation (see Table 2.2), with:

$$T > R > P > S$$

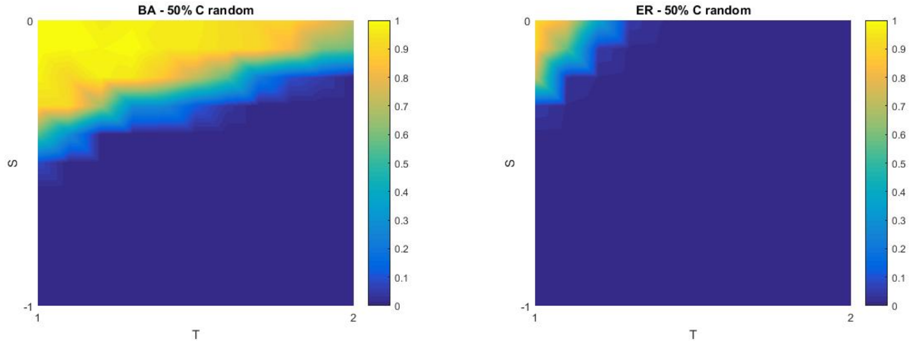
and assume $R = 1$ and $P = 0$ without loss of generality:

$$T > 1 > 0 > S$$

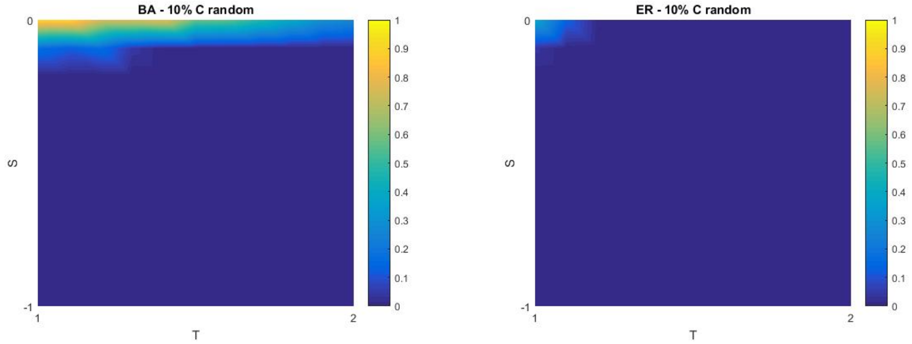
What they show is that, in heterogeneous populations, the sustainability of cooperation is simpler to achieve than in homogeneous populations. In particular they start from complete networks and move to scale-free (BA) networks, passing through the single-scale (ER) ones. Going from complete to ER populations, it clearly appears that cooperation is enhanced. When considering ER and BA graphs, the authors explain that other mechanisms should be taken into account besides the heterogeneity, but in the end, also in this case, the result is that the cooperation is enhanced.

Santos and coauthors implement a model based on an imitation rule, but their approach may be subject to a criticism: why should we imitate a better performing neighbor in a heterogeneous network? Traditionally, imitation has been used by classical *Evolutionary Game Theory* in unstructured populations as the social evolutionary driver of cultural transmission and learning. But in structured populations, where the situation is heterogeneous, a blind imitation of a better performing neighbor can seriously reduce the performance of an individual. The size and composition of the neighbors' neighborhoods do strongly matter, as exemplified in Figure 2.13, where the D hub would drastically reduce his/her own accumulated payoff by imitating the better performing C neighbor.

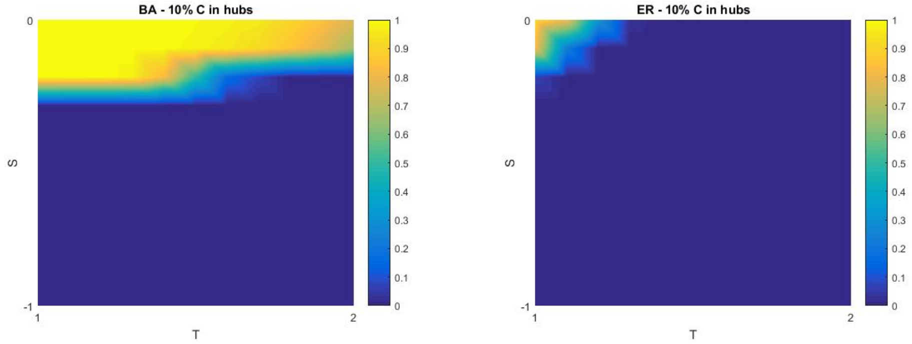
Besides the fact that imitation in a heterogeneous setting might not be the best practice, the results obtained by Santos and coauthors are based on simulations starting from random initial conditions with 50% cooperation, and thus only support the persistence of C, not its emergence.
We have reproduced their experiment using the Barabási-Albert and Erdös-Rényi network models, selecting the initial cooperators both with a random initialization and with a preferential selection for hubs. We have then repeated it with 10% initial cooperation and we have observed that defection takes over the population most of the times, even if the initial Cs occupy the hubs of scale-free networks as shown in Figure 2.14. This proves that, with this model, cooperation is hardly able to invade the population.



(a) *BA network with* 50% *initial Cs placed randomly* **(b)** *ER network with* 50% *initial Cs placed randomly*



(c) *BA network with* 10% *initial Cs placed randomly* **(d)** *ER network with* 10% *initial Cs placed randomly*



(e) *BA network with* 10% *initial Cs placed in hubs* **(f)** *ER network with* 10% *initial Cs placed in hubs*

Figure 2.14: Results of simulations based on the model implemented by Santos and coauthors [26]. For each pair (T, S) of parameters, the plots display (in color code) the eventual fraction of cooperators. In figures (a) and (b) the initial proportion of cooperators is equal to 50%, as in [26]. In all the other figures the initial proportion of cooperators is equal to 10%. Figures (a) and (c) show the results for BA networks with random initialization. Figure (e) shows the results for BA network with preferential initialization for hubs. Figures (b), (d) and (f) show the same but for ER networks

Chapter 3

Selfish Reciprocal Altruism

3.1 The model

In a heterogeneous framework, imitating a better performing neighbor might not always be the best choice. Moreover, the imitation process seen in Section 2.5 uses information that may not be available to the players. In fact, in the described model, individual i imitates individual j with a probability that depends on the payoff P_j accumulated by j and on the degree k_j of j. However, having information about these quantities is not always possible, as it is not possible in the real world to have full information about all our acquaintances. We thus introduce a new evolutionary rule, that we call *selfish reciprocal altruism*, based on strictly local information and on a selfish updating process.

Local Information is directly gathered by individuals when playing with their neighbors. No extra information is considered nor needed in our model.
Selfish Updating refers to the individuals' choice of changing their strategy only if it is convenient for themselves, in terms of accumulated payoff, in the next few generations.
For the evaluation of the accumulated payoffs we make use of an *asynchronous strategy update*: at each step, only a small portion of the population is given the chance to change strategy, so that, when an individual faces this opportunity, he/she can reasonably assume that no variations will occur in the neighborhood in the next few generations.

An asynchronous update, by itself, would not help the emergence of cooperation. In fact, the payoffs accumulated by a C and a D player with degree k and k_C cooperators neighbors are respectively:

$$P_{C,k,k_C} = (r-1)k_C - (k - k_C) \tag{3.1}$$

and

$$P_{D,k,k_C} = rk_C \tag{3.2}$$

where P_{C,k,k_C} is obviously smaller than P_{D,k,k_C}.

We therefore introduce a sort of penalization for defectors, using the idea of *direct reciprocity* [3] together with a mechanism for *optional participation* [18]. The original idea of reciprocity is maintained: when a C meets another C – that means, he/she is reciprocated in his/her cooperative behavior – he/she will be eager to play again with this individual. On the opposite, when a C meets a D, he/she is defected (or exploited) and therefore he/she reduces the probability of engaging another game with this individual.

More in detail, our evolutionary rule is defined as follows:

1. *Initialization.* Given a population of N individuals, individual i is willing to play with individual j with probability p_{ij}. Initially, for every $i, j = 1, ..., N$, $p_{ij} = 1$ if i and j are neighbors, while $p_{ij} = 0$ otherwise. (Note that, during the evolution of the game, we will have $p_{ij} \neq p_{ji}$, in general.)

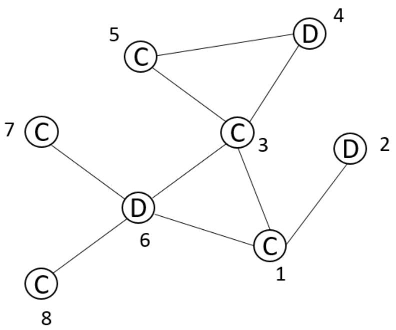
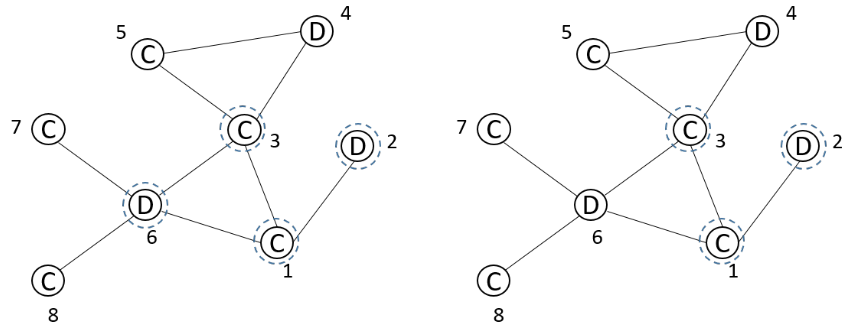


Figure 3.1: *Initialization step*: $p_{ij} = 1$ if i and j neighbors, 0 otherwise. For example, $p_{12} = 1$, while $p_{14} = 0$.

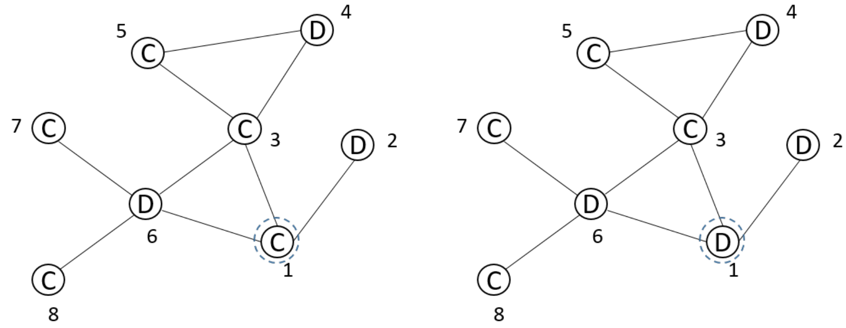
2. *Interaction.* At each round, every individual i decides the neighbors with whom he/she wants to play. Called j one of the neighbors, individual i wants to play with j with probability p_{ij}. The game between i and j takes place only if both players are willing to play, i.e. it happens with probability $p_{ij}p_{ji}$.
   If i is a C and he/she is exploited by a D neighbor j, then p_{ij} is halved.
   If i is a C and he/she is reciprocated by a C neighbor j, then p_{ij}, is restored to 1.
   If i does not engage a game with the neighbor j, then the probability p_{ij} remains unchanged.
   If i is a D, then $p_{ij} = 1$ no matter what is the strategy of j.
   Note that the information accessible to players about the strategy of neighbors depends on their own strategy. Ds are always willing to play with everyone and therefore they know the exact state of their neighbors: they either played in the previous step, and consequently learned the state, or they did not. But in the second case they are necessarily facing Cs, because Ds would never reject engaging a game. On the other hand, Cs cannot be certain of the state of their opponents and base their knowledge on the results of the latest interactions with the interested neighbor, hence being liable to miss the variation of a neighbor's strategy. In particular, the belief of the C node i on the state of the neighbor j is stored, by construction, in the probability p_{ij}. Specifically, if $p_{ij} < 1$ then i believes j to be D, otherwise it is a C.

3. *Update.* After each round, a fraction α of the population is randomly selected and given the opportunity to update strategy. When individual i is selected, he/she computes the expected payoffs accumulated in the neighborhood behaving as C or D in the next $h \geq 1$ interaction steps. The computation is done assuming that no other change will occur in the meantime in the neighborhood. Moreover, cooperators cannot know the exact state of their neighbors, hence they can only base their calculation on the information gathered in the last interactions occurred. Also note that it is plausible that a D knows the value of p_{ji}, because he/she can remember how many consecutive times he/she defected the C neighbor i. At this point, the individual compares the two expected payoffs and changes strategy only if the expected gain is positive. If the player updates strategy, the probabilities of playing with neighbors are consequently modified:
   When i is C and switches to D, p_{ij} is set to 1 for all neighbors j.
   When i is D and switches to C, p_{ij} is set to $\frac{1}{2}$ for those neighbors j that are D and remains 1 for C neighbors.



(a) *First interaction* **(b)** *Another interaction*

Figure 3.2: *Interaction step*: Consider player 1. At the first round of the game he/she will play with all the neighbors, because all the p_{ij}s are equal to 1. After the first round p_{12} and p_{16} will become $\frac{1}{2}$ because 2 and 6 are Ds while 1 is C; p_{13} will remain 1. If the interaction is not the first one, games are played according to the values of $p_{ij}p_{ji}$: for example, it may happen that player 1 engages a game with players 2 and 3, but not with 6. In this case, after the interactions, p_{12} will be halved again, while p_{13} will remain equal to 1. p_{16} will not be changed, because there is no interactions between players 1 and 6.



(a) *Selection for update* **(b)** *Updating*

Figure 3.3: *Update step*: Imagine that player 1 is selected for the update. He/she evaluates the expected payoffs over the horizon h in the two possible cases: if he/she stays C and if he/she decides to switch to D. Finally, player 1 will choose the strategy that gives him/her the higher expected payoff. If 1 decides to switch to D, then p_{12} and p_{16} are restored to 1.

As a result of this modelization, the population and its state are defined by the binary vector of status, C or D, for each individual and by the $N \times N$

matrix of the probabilities p_{ij}, that gives information both about the network structure and the reciprocity status at the same time.

Considering the description of the model, the resulting network representing the population is thus undirected, because link $i \to j$ always exists together with link $j \to i$, but directionally weighted, because $p_{ij} \neq p_{ji}$ (by construction, one of the two probabilities is always equal to 1) and adaptive, because the p_{ij}s change in time.

Note that the information used by individuals only comes from their interactions with other players. This satisfies one of the requirements of our evolutionary rule: using only local information in order to avoid the necessity of extra knowledge that might not be available to players.

Finally, it is important to observe the role of the prediction horizon h, the number of steps over which the opportunity to change strategy is evaluated. If $h = 1$ the winning strategy is always defection and everyone switches to D; only under the assumption of $h \geq 2$ it makes sense to reciprocate and cooperation can persist.
As already mentioned, when an individual evaluates whether it is convenient to change strategy or not, he/she assumes that no changes will occur in his/her neighborhood during the next h generations. In order to make this assumption reasonable, the value of α – the fraction of the population who is given the possibility to change – should be small enough, when considered together with the prediction horizon h, so that it will most likely be true that no variations occur around the selected individual during the h prediction steps. Note that αh is a gross upper-bound of the average fraction of the neighbors that might have changed state in h steps. In particular, we will choose values of the investment horizon h and of the fraction α so that $\alpha h \leq 0.25$.

3.2 Preliminary Analysis

To show that *selfish reciprocal altruism* favors the emergence and persistence of cooperation in heterogeneous social networks we have performed some preliminary analysis using an investment horizon of $h = 2$ steps.

Consider the situation at the first strategy update, just after the first round of the interaction. Keep in mind that when a C individual thinks his/her neighbor is a C, then he/she will want to play with probability one, while when the neighbor is a D, the probability is halved at every interaction. On the other hand, a D individual always wants to play with probability one. Therefore, the expected payoffs (evaluated with reference to the payoff matrix shown in Table 2.5) for a player with degree k and k_C C neighbors are those discussed below, depending on the strategy used in the first round and on the choice of behaving as C or D over the next two interaction rounds.

If at round one he/she was a C and decides to stay C in the next two rounds, then:

$$\begin{aligned} P_{CC} &= 2(r-1)k_C - (k-k_C)\Big(\frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{2}\Big) \\ &= 2(r-1)k_C - \frac{7}{8}(k-k_C) \end{aligned} \tag{3.3}$$

If at round one he/she was a C and decides to switch to D, then:

$$P_{CD} = rk_C\Big(1 + \frac{1}{2}\Big) = \frac{3}{2}rk_C \tag{3.4}$$

If at round one he/she was a D and decides to stay D in the next two rounds, then:

$$\begin{aligned} P_{DC} &= (r-1)k_C\Big(\frac{1}{2} + \frac{1}{2}1 + \frac{1}{2}\frac{1}{2}\Big) - (k-k_C)\Big(\frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{2}\Big) \\ &= \frac{5}{4}(r-1)k_C - \frac{7}{8}(k-k_C) \end{aligned} \tag{3.5}$$

If at round one he/she was a D and decides to switch to C, then:

$$P_{DD} = rk_C\Big(\frac{1}{2} + \frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{2}\Big) = \frac{7}{8}rk_C \tag{3.6}$$

Observe that expectations are computed assuming that no variations will occur in the neighborhood of the player over the considered horizon, a hypothesis that is plausible under the assumption of asynchronous update.
As an example, consider a player that behaves as C in round 1. If he/she keeps cooperating at rounds 2 and 3, the cooperation with the k_C C neighbors gives the term $2(r-1)k_C$, whereas the interaction with the $(k-k_C)$ Ds takes place with probability $\frac{1}{2}$ at round 2, because defection is for sure detected at round 1, and with probability $\frac{1}{2}\frac{1}{4} + \frac{1}{2}\frac{1}{2} = \frac{3}{8}$ at round 3, because defection occurs with probability $\frac{1}{2}$ at round 2.

The expressions for payoffs are summarized in Table 3.1, together with the conditions on parameters that gives rise to a switch of strategy.

	C	D	Condition
C	$P_{CC} = 2(r-1)k_C - \frac{7}{8}(k-k_C)$	$P_{CD} = \frac{3}{2}rk_C$	$r < 4 + \frac{7}{4}\frac{k-k_C}{k_C}$
D	$P_{DC} = \frac{5}{4}(r-1)k_C - \frac{7}{8}(k-k_C)$	$P_{DD} = \frac{7}{8}rk_C$	$r > \frac{10}{3} + \frac{7}{3}\frac{k-k_C}{k_C}$

Table 3.1: Payoff values and conditions for strategy update with investment horizon $h = 2$. Note that, if $k_C = 0$, a C player always switches to D, while a D player does not change strategy. In fact, $P_{CC} < 0$ and $P_{DC} < 0$, while $P_{CD} = 0$ and $P_{DD} = 0$: thus, if $k_C = 0$, being a D always gives higher payoffs.

From the expected payoffs written above we notice that:

- A C with only D neighbors changes to D when selected for update. Being C he/she is exploited by all neighbors and hence receives a negative payoff, while switching to D will give him/her a null payoff.
- A C with only C neighbors does not change to D provided $r \geq 4$. In this case, in fact, the payoffs are respectively $P_{CC} = 2(r-1)k_C$ and $P_{CD} = \frac{3}{2}rk_C$ and $P_{CC} \geq P_{CD}$ if and only if $r \geq 4$.
- A D with only D neighbors does not change strategy, because it would be $P_{DC} < 0$, while $P_{DD} = 0$.
- In all-to-all networks, reciprocity does not support the emergence of cooperation. Let N_C be the total number of Cs and N the size of the population, then a C decides to switch if $r < 4 + \frac{7}{4}\frac{N-N_C}{N_C-1}$, while a D switches if $r > \frac{10}{3} + \frac{7}{3}\frac{N-N_C-1}{N_C}$. If N_C is much smaller than N, then the few Cs change to D while Ds remain Ds.
- Low-degree Ds connected to a C might change to C. For example, if $k = 1$ the switch takes place provided $r > \frac{10}{3}$.

3.2.1 Preliminary analysis for star network

Recall the configuration of a star network, where a single central hub is connected to many leaves (Figure 2.7). This represents the simplest case of

heterogeneous network.
Imagine a single C is placed in the hub, while all the leaves are Ds as shown in Figure 3.4 (a): this is exactly the above mentioned situation of low-degree Ds connected to a C.

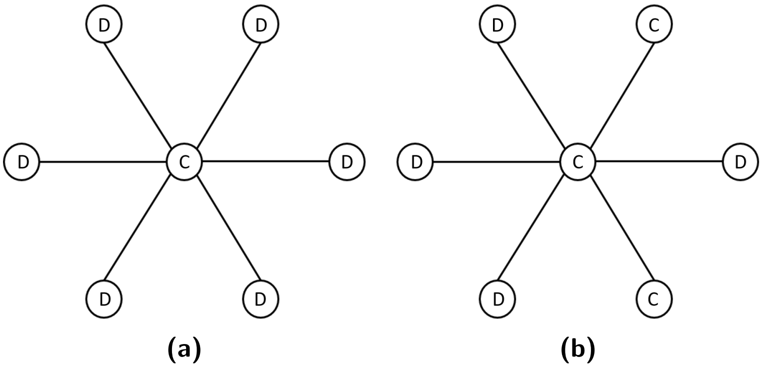


Figure 3.4: (a) Star network with a single C in the central hub. (b) Star network with C hub and some C leaves.

In this case, for a D leaf we have $P_{DC} = \frac{5}{4}(r-1)$ and $P_{DD} = \frac{7}{8}r$, so the leave changes strategy if $r > \frac{10}{3}$. It is very likely that some leaves will change strategy before the central node is selected for the update, so that the fraction of Cs will initially increase. If this fraction gets to the level above which the C hub has no incentive to change, then cooperation will eventually dominate.

We can analytically determine what is this level: let k_C be the number of C leaves and k the total number of leaves, as shown in Figure 3.4 (b).

For the C hub, we have:

$$P_{CC} = 2(r-1)k_C - \frac{7}{8}(k - k_C)$$

$$P_{CD} = \frac{3}{2}rk_C$$

so

$$\Delta P = P_{CD} - P_{CC} = \left(\frac{9}{8} - \frac{1}{2}r\right)k_C + \frac{7}{8}k$$

If we want the C hub not to change to D, then it should be $\Delta P < 0$.
If $r < \frac{9}{4}$, then ΔP is always greater than zero and the hub always switches to D.

But, if $r > \frac{9}{4}$, ΔP is smaller than zero if the number k_C of C leaves satisfies

$$k_C > \frac{\frac{7}{8}k}{\frac{1}{2}r - \frac{9}{8}}$$

So, if the value of the benefit-to-cost ratio is large enough and at least $k_C > \frac{7}{8}k/(\frac{1}{2}r - \frac{9}{8})$ leaves decide to change strategy before the C hub is selected for the update, starting from a star configuration with a single C placed in the central hub, cooperation can invade.

As we will show in the following, this result extends to complex topologies: selfish reciprocal altruism supports the emergence of cooperation in heterogeneous networks, provided the benefit-to-cost ratio r is sufficiently high and the few initial Cs occupy some of the network's hubs.

3.3 Simulations

We have performed extensive numerical simulations to confirm the intuition raised by the above analytical considerations. We test four types of networks – complete, single-scale (ER), scale-free (BA) and star – and use two different types of initialization, cooperators randomly chosen or cooperators initially in hubs.

For each of the considered typologies, we generate multiple networks, initialize them, and then perform simulations to evaluate the eventual proportion of cooperators. The parameters defining the simulation setting are listed below:

- N: Number of nodes in the network;
- $initial\ fraction\ of\ Cs$: Initial proportion of cooperators in the network;
- h: Investment horizon. It is the number of steps considered by an individual while computing the expected payoffs to decide whether to change strategy or not;
- α: Fraction of the population selected for the strategy update after each round. It is important to notice that, varying the value of α,

the dynamics of the model crucially varies too. If α is too small, the validity of the model is compromised. For example, if α is very small, it may happen that a C individual is not selected for the update for a very long time. This player, therefore, keeps being defected by a D neighbor, reducing the probability of playing with him/her every time, till the moment in which this probability becomes so small that the two players do not engage a game almost anymore. But in this case, the C individual will not realize if the D neighbor changes strategy, because the probability of playing with that neighbor has become so small that he/she will not have the chance of changing his/her belief.
On the other hand, if α is too large, the hypothesis of an asynchronous update where the neighborhood stays unchanged becomes unrealistic. A large value of α means that a greater portion of the population is selected for the update, and therefore it is more likely that some of the neighbors of a selected individual will be selected as well.
For this reasons, taking into account the horizon h, we choose a value of α such that $\alpha h \leq 0.25$;

- r: Benefit-to-cost ratio, that is, the rate between the benefit obtained by receiving an altruistic act, and the cost paid performing it. This is a very important parameter in the model. Changing it's value allows to favor the emergence of cooperation. In particular, high values of r tend to enhance cooperation;

- *number of realizations*: For each of the network typology considered (in particular, for ER and BA networks), *n_realizations* networks with the same parameters are generated to perform simulations;

- *number of runs*: Given a network and an initialization, *n_runs* simulations are performed on it, randomly assigning the opportunity of strategy update to different nodes every time.

We introduce a convergence test. For each simulation – a network and its initialization, evaluated for an investment horizon h and a specific benefit-to-cost ratio r – several iterations are performed. The simulation ends when either the convergence criteria is met or the maximum number of iteration is reached. Hence, we need some new parameters to model this rule:

- ϵ: Convergence parameter (see below);

- *maximum number of iterations*: The maximum number of iterations *n_max* allowed in a single simulation is needed to avoid the risk of never ending simulations.

Given a network with an initialization and a specified value for the investment horizon h and for the benefit-to-cost ratio r, a simulation is performed. At each iteration of the simulation, the number of cooperators in the network is computed. The code keeps track of the results of the last L iterations ($L = 20$ in the following) to check if the convergence has been reached. Specifically, the convergence criterion is based on variation of the *moving average*, i.e. the average number of cooperators in the network calculated on the last L iterations performed.

The moving average can be expressed in recursive form, as follows. Let $y(t)$ be the number of cooperators at iteration t, L the number of iterations considered and $m(t)$ the floating average at iteration t, then:

$$\begin{aligned} m(t) &= \frac{1}{L}\sum_{k=0}^{L-1} y(t-k) = \\ &= m(t-1) - \frac{1}{L}y(t-L) + \frac{1}{L}y(t) \end{aligned} \tag{3.7}$$

Once the moving average is computed, it is compared with the average at the previous step, and if the variation is smaller than ϵ we are one step closer to the convergence. If this condition is satisfied s consecutive times ($s = 10$), then the equilibrium is reached and the simulation ends.

For each of the *n_realizations* networks, *n_runs* simulations are performed. The number of cooperators is averaged on the total number of realizations and runs, and is divided by the number of nodes N to obtain the fraction of cooperators.

3.3.1 Explanatory example

To better understand how the entire process works, we present here a small example to clarify the evolution of cooperation in a network. In particular consider the network in Figure 3.5, generated according to the BA model [24].

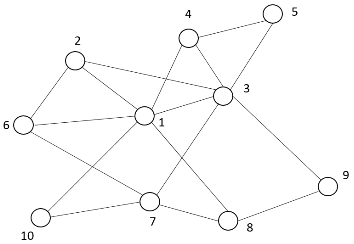


Figure 3.5: Example of a small network generated with BA algorithm [24]

The values of the parameters used in this example are: $N = 10$, $h = 4$, $\alpha = 0.05$, $r = 15$, $n\_realizations = 1$, $n\_runs = 1$, $\epsilon = 0.005$, $n\_max = 1000$.

We perform an initialization of the state of the network, so that the initial Cs are placed randomly. Here, we see that two Cs, colored in green, have been placed. Defectors appear in red.

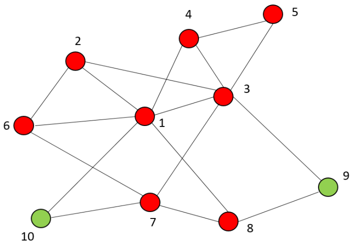
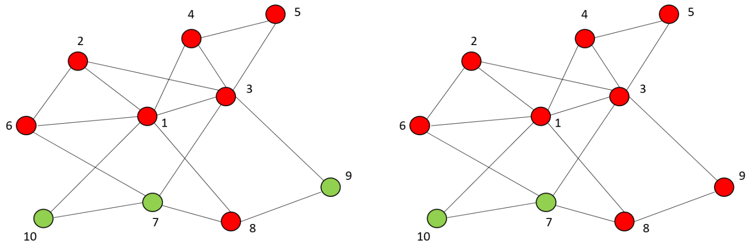


Figure 3.6: Initialization: cooperators are green, defectors are red.

At this point, the evolutionary process can start. Following the rules described at the beginning of this chapter, each node engages games with his/her neighbors, and if selected for the update, he/she can decide whether to keep playing with the actual strategy or to switch to the other one, according to his/her knowledge and belief on the status of the neighbors.

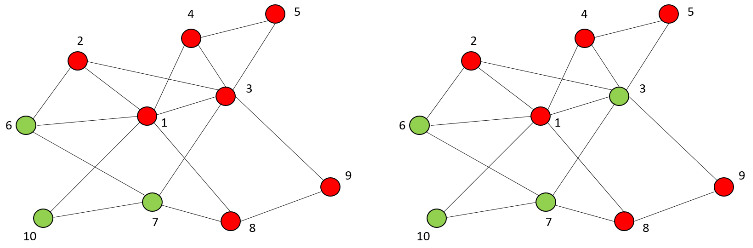
Let us describe the process of asynchronous update with an example which is purely illustrative. The first node changing strategy is node 7, that switches from D to C. However, after few iterations, node 9 realizes that it is not convenient for him/her being a C and has the chance to switch to D (see Figure 3.7).

Node 6 decides to update strategy, followed then by player 3 (see Figure 3.8).



(a) **(b)**

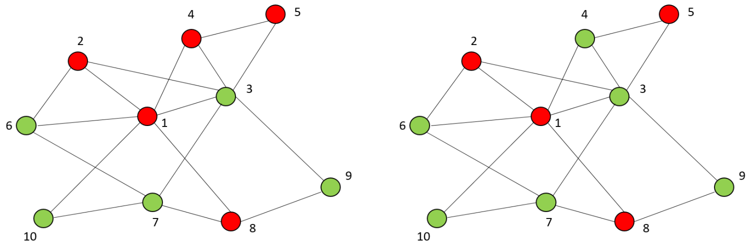
Figure 3.7



(a) **(b)**

Figure 3.8

Node 9 faces now a different neighborhood, and thus decides to go back to the original C strategy. Also node 4, evaluating the expected payoffs, chooses to move towards a cooperative attitude.

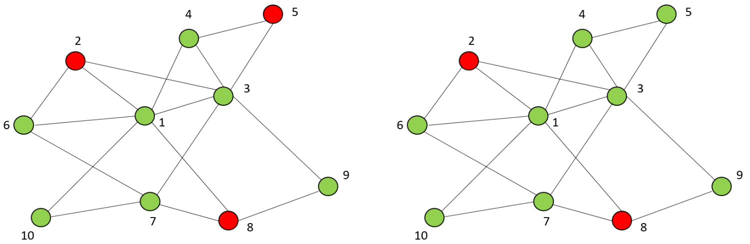


(a) **(b)**

Figure 3.9

Node 1, now surrounded by a majority of Cs, understands that is better for him/her to cooperate, and the same does node 5 because both his/her neighbors are Cs (see Figure 3.10).

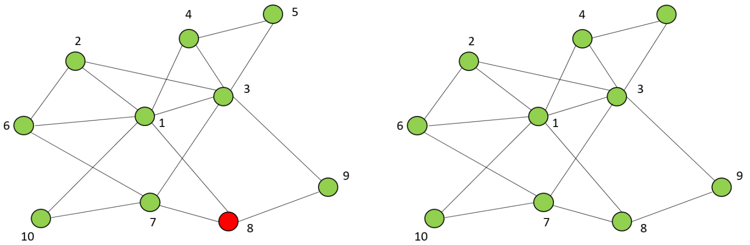
Finally, the last two defectors abandon their strategy and adopt the more



(a) **(b)**

Figure 3.10

favorable cooperation.



(a) **(b)**

Figure 3.11: The final state of the evolutionary process.

This is a very simple example of how cooperation can evolve in a network. It is possible that a player decides to change strategy, but after few iterations goes back to the other one, as it happened with node 9, because in the meantime the strategies of his/her neighbors have changed as well, and the assessment done in the past are not valid anymore. Here we started from 2 initial cooperators, and we reached a complete dominance of the cooperative behavior. We will see in the next section that, depending on the values of the parameters used in simulations, it is possible to reproduce a similar result in larger networks and also with different types of initialization.

3.3.2 Results

The values of the parameters used for the simulations are the following:

– $N = 1000$;

- $initial\ fraction\ of\ Cs \in \{0.01, 0.05, 0.10\}$
- $h \in \{2, 3, 4, 5\}$;
- $\alpha = 0.05$;
- r varying from 1 to 15;
- $n\_realizations = 15$;
- $n\_runs = 10$;
- $\epsilon = 0.005$;
- $n\_max = 1000$

The specific choice of ϵ and n_max has been made in order to ensure that the equilibrium is always reached, and thus the maximum number of iterations is only needed to handle exceptional situations.

Figures from 3.12 to 3.15 show the results of simulations with respect to the two most important parameters of our evolutionary rule, the benefit-to-cost ratio r and the investment horizon h. For each scenario, we report how the average asymptotic fraction $\langle \bar{x} \rangle$ of Cs changes with r and h.
The same type of image is shown when considering an initial fraction of cooperators of respectively 1%, 5% and 10%.
For networks generated with BA [24] and ER models, results are shown in Figure 3.12 and Figure 3.13 considering two different types of initialization: images on the left show results obtained when the initial cooperators are placed randomly, images on the right show results obtained when the initial cooperators are placed in hubs, the most connected nodes. For complete networks only the random initialization is considered, for the three different levels of the initial proportion of cooperators, as shown in Figure 3.14. Initialization in hubs does not make sense in this case, because all the nodes have the same degree, hence there are no hubs. For star networks, instead, we simply focus on one case: Figure 3.15 shows the results for a single initial cooperator placed in the only hub of the network, which is connected to all the defector leaves.

In networks generated according to the BA model [24], the selfish reciprocal altruism can promote cooperation in all cases, but the heterogeneous structure of the networks is best exploited if the initial Cs occupy the hubs of the

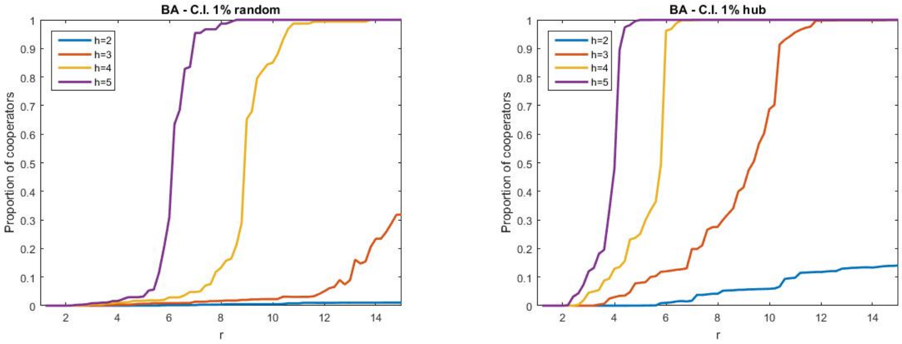
structure.
As expected, when the initial proportion of cooperators grows, cooperation can dominate even for small values of r (see figures 3.12 (a), (c) and (e) or 3.12 (b), (d) and (f)). When all the other parameters are fixed, increasing the value of h favors cooperation significantly. In fact, when the prediction horizon h is larger, cooperation tends to become more convenient than defection, in terms of expected payoff, hence players decide to switch to a cooperative behavior rather than defecting. Finally, comparing the results of the two different types of initialization with the same initial level of cooperators, it is clear that the emergence of cooperation is favored when the Cs are initially placed in hubs. Hubs are connected to many other individuals and, being C, they can influence the decision of switching to C for a higher number of individuals. When the initialization places Cs in hubs, in fact, the collaborative behavior is able to dominate also for smaller values of the benefit-to-cost ratio r and of the investment horizon h, as it can be seen comparing the left side and the right side of Figure 3.12.
Note that the results are shown in figures for r going from 1 to 15. In many cases the average asymptotic fraction $\langle \bar{x} \rangle$ of Cs reaches 1. For the cases in which the dominance of cooperation was not reached yet, we made simulations using higher values of r: if the value of r is large enough, the dominance of cooperation is reached for all the values of h and of the initial proportion of Cs (i.e. we obtain $\langle \bar{x} \rangle = 1$ in all cases).

In networks generated with the ER algorithm, as in the previous case, cooperation is enhanced when the initial proportion of Cs, the benefit-to-cost ratio r and the prediction horizon h grow. However, in this case, starting with Cs in the most connected nodes does not have any significant and beneficial effect. Comparing results obtained with a random initialization and with an initialization in hubs, the difference of the initialization does not influence the outcome (see left side and right side of Figure 3.13). This aspect is a result of the characteristic structure of ER networks: they are essentially single-scale networks, which means that the degree of each node slightly varies from the average degree of the network. Consequently there are no hubs or, in other words, the influence of the most connected nodes is not so powerful, because the degree of the other nodes is similar.
Note that also for this type of networks, increasing the benefit-to-cost ratio r to values larger than 15 results in a dominance of cooperation for all the combinations of the parameters.

In complete networks cooperation cannot invade, as shown in Figure 3.14

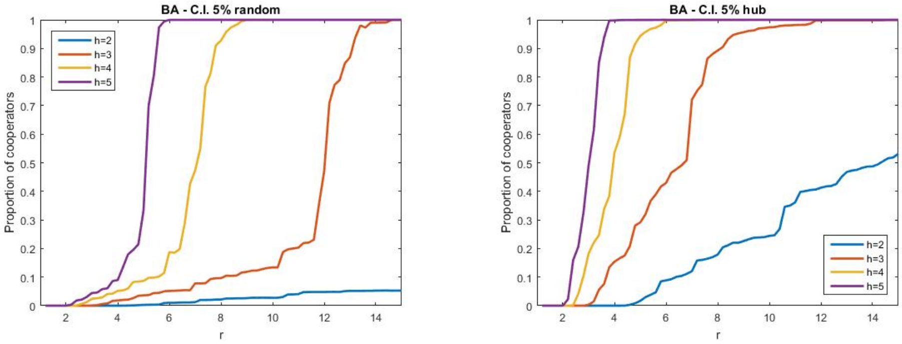
(a), and can hardly persist, as shown in Figure 3.14 (b). In such homogeneous networks, defection is often the best strategy for players, because all individuals are playing in similar conditions, as seen throughout the thesis. Although, also for this type of network, for higher values of the initial proportion of Cs it is possible to see that $\langle \bar{x} \rangle$ reaches 1.

Finally, as expected from the preliminary analysis, we can notice that in the simplest case of heterogeneous network, the star network, starting from a single cooperator placed in the central hub, cooperation is able to invade and persist, as shown in Figure 3.15.



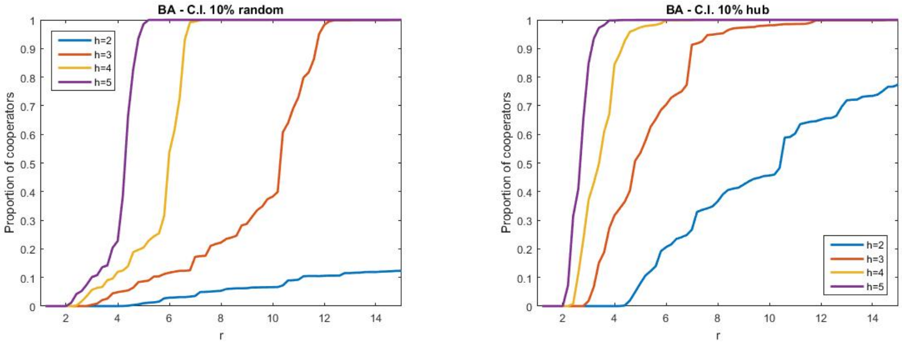
(a) *BA, random initialization with* 1% *C*

(b) *BA, hub initialization with* 1% *C*



(c) *BA, random initialization with* 5% *C*

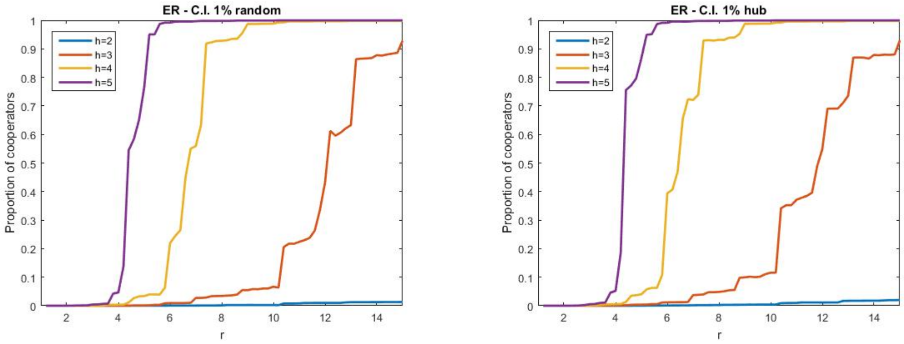
(d) *BA, hub initialization with* 5% *C*



(e) *BA, random initialization with* 10% *C*

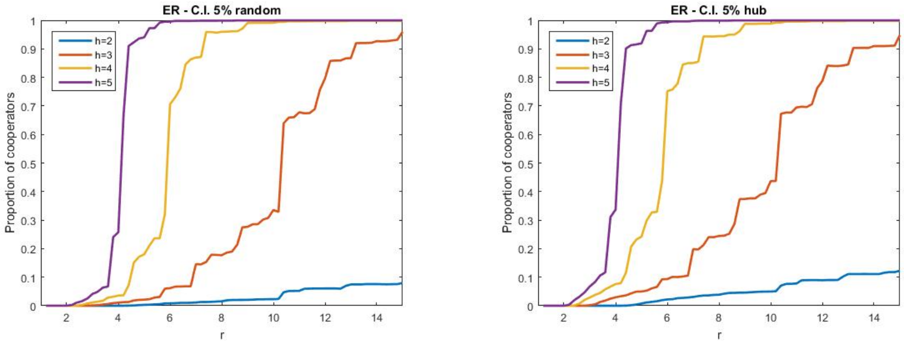
(f) *BA, hub initialization with* 10% *C*

Figure 3.12: Results obtained for scale-free Barabási-Albert networks [24]. The left side shows results for random initializations, the right side those for initializations of Cs in hubs. Images are shown for three different initial fraction of cooperators: 1%, 5% and 10%.



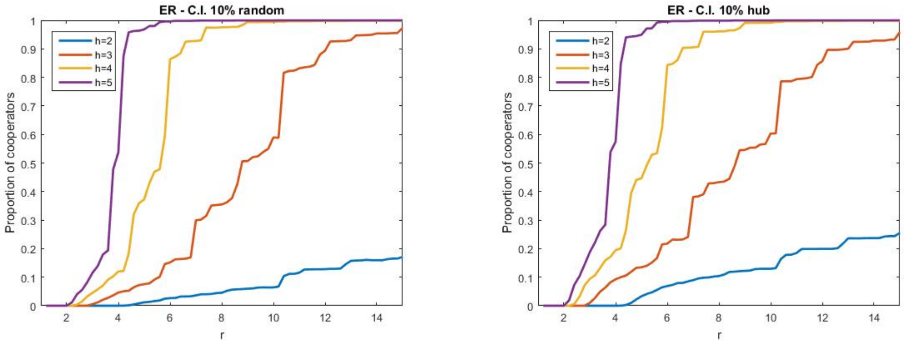
(a) *ER, random initialization with* 1% *C*

(b) *ER, hub initialization with* 1% *C*



(c) *ER, random initialization with* 5% *C*

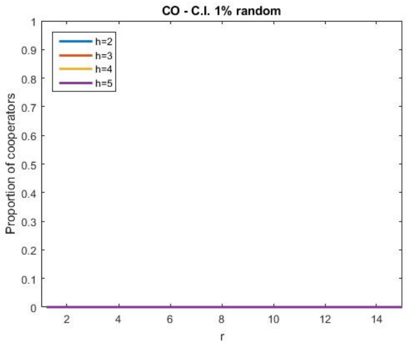
(d) *ER, hub initialization with* 5% *C*



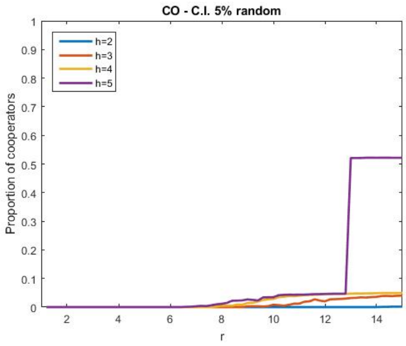
(e) *ER, random initialization with* 10% *C*

(f) *ER, hub initialization with* 10% *C*

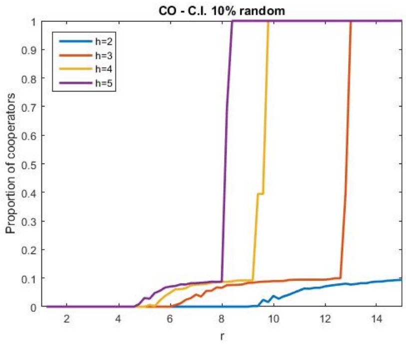
Figure 3.13: Results obtained for single-scale Erdös-Rényi networks. The left side shows results for random initializations, the right side those for initializations of Cs in hubs. Images are shown for three different initial fraction of cooperators: 1%, 5% and 10%.



(a) *Complete, random initialization with 1% C*



(b) *Complete, random initialization with 5% C*



(c) *Complete, random initialization with 10% C*

Figure 3.14: Results obtained for complete networks with random initialization. Images are shown for three different initial fraction of cooperators: 1%, 5% and 10%

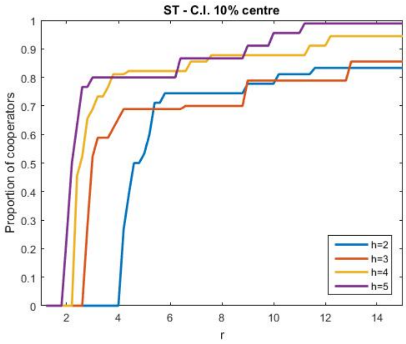


Figure 3.15: Results obtained for star networks, where a single initial cooperator is placed in the central hub, and defectors are places in all the leaves. As seen in the preliminary analysis, for the star network the dominance of cooperation can be reached.

3.4 A possible extension: Mean Field Analysis

An interesting approach for future works could be the study of the mean field model. Specifically, introducing some approximations, it is possible to simplify the above described model, so that it can represent an average behavior.

Consider discrete time steps and take into account nodes grouped according to their grade, going from 1 to the maximum grade $\bar{k}$. Call $x_k(t)$ the fraction of cooperators with degree k and $N_k = P(k)N$ the number of nodes with degree k, where $P(k)$ is the degree distribution evaluated for the degree k and N is the total number of nodes. Then, the number of cooperators with degree k at time t is $N_k x_k(t)$. The number of cooperators with degree k at time $t+1$ will be:

$$\begin{aligned} N_k x_k(t+1) = N_k x_k(t) - N_k x_k(t) Prob(C_k \to D_k) \\ + N_k(1 - x_k(t)) Prob(D_k \to C_k) \end{aligned} \tag{3.8}$$

that is the number of cooperators at the previous step, minus the number of Cs that became Ds, plus the number of Ds that switched to C.

Let's now compute these terms one by one.

The probability that a C with degree k changes to D is:

$$Prob(C_k \rightarrow D_k) = \alpha \sum_{k_C=0}^{k} Prob(C_{k,k_C} \rightarrow D_k) Prob(k_C) \tag{3.9}$$

where α is the fraction of the population selected for the strategy update at each step, $Prob(C_{k,k_C} \rightarrow D_k)$ is the probability that a C with degree k with k_C C neighbors switches to D, and $Prob(k_C)$ is the probability of having k_C C neighbors.

The probability $Prob(k_C)$ of having k_C C neighbors is:

$$Prob(k_C) = \binom{k}{k_C} P_C^{k_C} P_D^{k-k_C} \tag{3.10}$$

where P_C and P_D are respectively the probabilities that a selected neighbor is a C or a D:

$$P_C = \sum_{k'=1}^{\bar{k}} x_{k'} Prob(neighbor\ with\ degree\ k') \tag{3.11}$$

$$P_D = 1 - P_C \tag{3.12}$$

The probability of having a neighbor with degree k' can be written as follows:

$$\begin{aligned} Prob(neighbor\ with\ degree\ k') &= \frac{k' N_{k'}}{\sum_{k''=1}^{\bar{k}} k'' N_{k''}} \\ &= \frac{k' P(k')}{\langle k \rangle} \end{aligned} \tag{3.13}$$

where $P(k')$ is the degree distribution evaluated for k', and $\langle k \rangle$ is the average degree.

An individual decides to change strategy if it is convenient for him/her in the next few iterations, in terms of accumulated payoffs. We consider here an investment horizon of $h = 2$ rounds.

Let's call $\Delta Pay_{C \rightarrow D}(k_C)$ the difference between the payoff accumulated by a C player if he/she switches to D and if he/she decides to stay C. If this difference is positive, he/she will change strategy, else no variation will occur. In formulas, it can be expressed as follows:

$$Prob(C_{k,k_C} \rightarrow D_k) = \begin{cases} 1 \; if \; \Delta Pay_{C\rightarrow D}(k_C) > 0 \\ 0 \; if \; \Delta Pay_{C\rightarrow D}(k_C) > 0 \end{cases} \tag{3.14}$$

where, being π the average probability of engaging a game (that we assume being constant), the variation of the accumulated payoff is:

$$\begin{aligned} \Delta Pay_{C\rightarrow D}(k_C) = rk_C\Big[\pi + \pi\frac{\pi}{2} + (1-\pi)\pi\Big] - \Big\{(r-1)k_C\Big[\pi + \pi^2 + (1-\pi)\pi\Big] \\ - (k-k_C)\Big[\frac{\pi}{2} + \frac{\pi}{2}\frac{\pi}{4} + \Big(1-\frac{\pi}{2}\Big)\frac{\pi}{2}\Big]\Big\} \end{aligned} \tag{3.15}$$

On the other hand, the probability that a D with degree k changes to C is:

$$Prob(D_k \rightarrow C_k) = \alpha \sum_{k_C=0}^{k} Prob(D_{k,k_C} \rightarrow C_k) Prob(k_C) \tag{3.16}$$

And following the argument seen above, the probability that a D with degree k with k_C neighbors updates his/her strategy to D is:

$$Prob(D_{k,k_C} \rightarrow C_k) = \begin{cases} 1 \; if \; \Delta Pay_{D\rightarrow C}(k_C) > 0 \\ 0 \; if \; \Delta Pay_{D\rightarrow C}(k_C) > 0 \end{cases} \tag{3.17}$$

$$\begin{aligned} \Delta Pay_{D\rightarrow C}(k_C) = (r-1)k_C\Big[\pi + \pi^2 + \Big(1-\pi\Big)\pi\Big] \\ - (k-k_C)\Big[\frac{1}{2} + \frac{1}{2}\frac{1}{4} + \Big(1-\frac{1}{2}\Big)\frac{1}{2}\Big] \\ - k_C\Big[\frac{\pi}{2} + \frac{\pi}{2}\frac{\pi}{4} + \Big(1-\frac{\pi}{2}\Big)\frac{\pi}{2}\Big] \end{aligned} \tag{3.18}$$

$\Delta Pay_{D\rightarrow C}(k_C)$ being the difference between the payoff accumulated by a D player if he/she switches to C and if he/she decides to stay D.

As a future work it can be interesting to analyze the mean field model and study whether it yields the same results obtained in previous sections.

Chapter 4

Conclusions

In this thesis we have questioned the locality of interaction [22] – network reciprocity – as a mechanism favoring cooperation in social networks. Considering in specific the social context, the evolution of cooperation in structured populations has been described as an imitation process, where individuals copy the strategies of better performing neighbors [21]. However, copying a neighbor who operates in drastically different connectivity conditions does not ensure to obtain the neighbor's performance. This is evident in many real social and economic situations, where performance is intrinsically related to the connectivity of the individual or enterprise, other than to the adopted strategy.

The imitation of the best performing strategies is certainly a social driver, but in heterogeneous contexts a slightly different approach is needed. A way to deal with heterogeneous networks of contacts could be that of scaling the players' payoffs by the neighborhood size before comparison. Some sort of scaling has been implemented in the literature (as discussed in Section 2.5), but only after the payoffs comparison, thus leaving the hope to little-connected nodes of inheriting the performance of highly-connected ones. However, a proper scaling before payoff comparison would essentially kill the effect of heterogeneity in fostering cooperation. Moreover, the connectivity of neighbors is a non-local information, that is not always available.

We propose an alternative explanation for the evolution of cooperation in heterogeneous networks, based only on local information together with a selfish reciprocal altruism. With information only available through the outcome of direct pairwise interactions according to the PD game, individuals act following a selfish approach, deciding whether to switch to C or D depending on what is best for them over an investment horizon of a few interaction rounds. Instead of defecting non cooperative neighbors, we choose to allow Cs to avoid participation with D neighbors, reducing the probability of playing with them after each defection. We hence use a mix of reciprocal altruism [3] and optional participation [18].

Selfishness and reciprocity together are able to explain both the emergence and persistence of cooperation in heterogeneous networks, and eventually even its dominance, provided the benefit-to-cost ratio of the interaction is sufficiently high relative to the investment horizon. This result is strongly enhanced if the few initial cooperators occupy some of the central nodes of the network structure (hubs). Of course, the condition for the evolution of cooperation are necessarily demanding, though this is true for all proposed mechanisms. Even network reciprocity essentially works – in birth-death biological processes or according to the above criticized imitation paradigm – when cooperators are initially assumed sufficiently frequent to form clusters of protected Cs. Pure network reciprocity indeed does not explain the emergence of cooperation, whereas selfish reciprocal altruism does it. In other words, we can say that network reciprocity allows more sophisticated rules of strategy update, that in turns pave the way to cooperation.

Many variants and extensions of the present work are possible.
Different reciprocity strategies could be considered, for example nodes that behave as C with certain neighbors and as D with others according to the past history of the interaction. Another possible variant could be the introduction of a new mechanism when facing D neighbors: for example, when a player encounters a D neighbor he/she sets the probability of playing with him/her to a very low level for a certain number of rounds. After that time is passed, the probability is increased again, so that it is possible to understand whether the neighbor changed strategy in the meantime or not. This could be a method to overcome the problems raised by a very low level of α, the fraction of population updating strategy.
Group cooperation, described in terms of public goods games, could be introduced to go beyond the pairwise interaction [27]. The role of other network's aspects, such as clustering [28] and communities [29] or core-perifery structures [30], is worth to be investigated. Network dynamics and formation [31] to be possibly coupled to the game dynamics [32], are crucial to see whether the network evolves toward the structures favoring cooperation. Multiplex

and interdependent networks [33] could cope with several coupled layers of biological, social, and economic structures. All of this has so far been analyzed in the social context based on the crucial assumption that evolution proceeds by imitation, a process that should be reconsidered in real heterogeneous situations.

Finally, the mean field model seen in the previous chapter – where the evolution of cooperation is not studied looking at the behavior of single individuals, but focusing on the average behavior of nodes with degree k – could be an interesting variant when trying to reduce the complexity of calculation.

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