POLITECNICO DI MILANO
DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING
DOCTORAL PROGRAMME IN ENVIRONMENTAL AND INFRASTRUCTURE
ENGINEERING

ON THE DEVELOPMENT OF A GENERAL UNDIFFERENCED UNCOMBINED ADJUSTMENT FOR GNSS OBSERVATIONS

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Cycle XXXIII – 2021

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Introduction

This thesis describes the development of a general processing strategy for GNSS observations and the corresponding software implementation. The strategy was developed during a doctorate program conducted at Geomatics Research & Development (GReD) s.r.l, a company spin-off of the geodetic school of Politecnico di Milano. The processing strategy has been developed for the purpose of GReD applications, geodetic monitoring of displacement and estimate of tropospheric parameters, but has a general setup in basis to serve most purposes of the use of GNSS networks of receivers. The whole strategy has been developed with a focus on flexibility, meaning that should be able to process all available GNSS signals with a general as possible paramterization. The strategy has been implemented in the framework of the goGPS project, an open source MATLAB based GNSS processing software, with an important evolution in term of proprietary software of GReD.

The strategy builds upon decades of research in the field that I hope I was able to properly reference in the bibliography. Besides specific implementation aspects, original parts are however presents. The first one is more theoretical and consists of a general way to solve integer least squares rank deficient problems. The second is more applied and consists on the use of numerical methods to solve rank deficiencies in systems of GNSS observation equation.

The document is organised as follows, after a few words on the notation used the necessary mathematical foundations are presented. They concern with least squares estimation theory and the presentation is split in two chapters. The first one (Chapter 2) describes the theory and the

corresponding problems the second (Chapter 3) regards the methods to solve such problems. Both chapters describe the cases of real and integer parameters for both full rank and rank deficient systems. Then a chapter (Chapter 4) is dedicated to the physical description of the GNSS system and the relevant physical phenomena affecting it. The Least squares theory previously presented is then applied to specific meaningful estimation cases (Chapter 5). For each problems specific behaviour of the system are highlighted. Particular attention is given to typical cases of mixed integer real rank deficiencies and to the derivation of constraints leading to integer estimates. Finally a chapter (Chapter 6) is dedicated to the software implementation and to the presentation of two cases studies.

1.1 Notation

In this section all notations used in the thesis are presented.

1.1.1 Symbols Used

In this subsection we make a list of mathematical symbol used

$\mathbb{Q}$	set of rationals numbers
$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integer numbers
$\mathcal{N}()$	null space of
$\mathcal{R}()$	range space of
$span()$	span of
$rank()$	rank of
$\perp$	orthogonal
$\cdot$	matrix product
$\in$	belonging to
$A^{\intercal}$	transpose
A^{-1}	inverse
A^{+}	pseudoinverse
$\lfloor \rceil$	round to nearest integer
$\otimes$	Kronecker product
I	Identity matrix
e	Column vector of ones
$\backslash$	Least square operator

1.1.2 Matrices, Vector, Scalars, Functions

The following conventions for variables in equations are used:

- Lower case greek letters (α, β ,...) for scalars.

- Lower case Latin letters(a, b,...) for vectors.

- Upper case Latin letters(A, B,...) for matrices.

- Upper case greek letters for functions ($\Gamma(), P(), \Phi()$,...).

The previous convection will be broken only for very specific case of constant that are almost always identify with a specific symbol in literature. Such cases occurs only in chapter 4 and are clearly indicated, the most clear examples is the use of Latin letter c for the speed of light in vacuum. Furthermore the previous convention does not apply to subscript or superscript where lower case Latin letter should be expected, similarly when speaking about the numerosity of a parameter in a sentence lower case letter will be used. For instance we might write ".... the matrx A has n columns ..." in this case the symbol "n" should be interpreted as a scalar. To denote a matrix of integers A with m rows and n columns we will say $A \in \mathbb{Z}^{m,n}$ and similar notation will be used for real and rational number.
To select specific rows and columns of matrices index of the rows and columns (separate by semicolon) are put inside brackets after the variable name. For instance $A(1, n; k, m)$ indicates the submatrix composed by rows 1 and n and columns k and m. If we want to express a range of column or rows we will use horizontal dots $\ldots$. For example sub matrix of A composed by rows from 1 to n and column from k to m will be denoted as $A(1 \ldots n; k \ldots m)$. If we want to indicate simply one element of a matrix the more succinct form A_{ij} will be used.

1.1.3 Estimates

Regarding estimates we will use the following convention:

- x the unknown variable.

- $\hat{x}$ the real/rational estimates of the variable.

- $\check{x}$ the integer estimates of the variable.

- $\acute{x}$ the true value of the unknowns

Furthemore, we will use the breve sign $(\breve{A})$ to indicate matrix or observations that have been reduced for some parameters.

2

Least Squares Adjustment Theory

Nothing is more practical than a good theory

– Kurt Lewin

This chapter deals with the least squares principle for the estimation of unknown parameters. It is fundamental to make this introduction since almost all GNSS estimations rely on such estimation principle. The chapter will first illustrate the principle, discuss the case of estimation of real variables with linear constraints and discuss the estimation in case of rank deficiencies. Then the focus is moved to the case of integer parameters, after a very brief introduction of lattice theory, estimation cases of full and rank deficient systems are discussed.

2.1 Least Squares

Consider a sampling from of a random variable vector y, that we will name observation, and a linear model A connecting them to some other deterministic variable vector x called the unknown, we can then write:

$$y \approx A \cdot x; \tag{2.1}$$

as y is extracted from a random variable the relation is not fulfilled exactly so we have to add a term e called noise to account for that. The matrix A of the system is often called "design matrix"; we will use this name in the document. Being the noise a random term we can specify a mean

$E(y) = A \cdot x$, and variance covariance matrix C_{yy}. The equation then becomes:

$$y = A \cdot x + e \tag{2.2}$$

Having an estimated value of the unknown $\hat{x}$, we can construct the following quantity called residual:

$$u = y - A \cdot \hat{x} \tag{2.3}$$

The principle of least squares consists in finding the value of $\hat{x}$ for which the quantity

$$u^\mathsf{T} \cdot C_{yy}^{-1} \cdot u \tag{2.4}$$

is minimised. This estimation principle is called in different ways in geodetic literature, "least squares adjustment by observation only" [74] or "least squares adjustment of Gauss-Markov Model" [66] and in non geodetic literature it is often called Generalized Least Squares (GLS) after the work of [3]. Trough this document we will refer to it simply as Least Squares (LS).

This estimation principle is central in our context because in case of normally distributed error (a commonly used approximation for GNSS observations, for instance see [54] Section 6.3.3) it guarantees that the estimation is of minimum variance. Furthermore if the problem is linear and not rank deficient (normality of error not required) it is also unbiased. For this reason it is called Best Linear Unbiased Estimator (BLUE). For normally distributed errors it is also the maximum likelihood estimator i.e. the so to say most probable value for the unknown. The derivation of the estimator is rather simple. Let's write the Eq 2.4 explicitly:

$$(y - A \cdot x)^\mathsf{T} \cdot W \cdot (y - A \cdot x) = \tag{2.5}$$
$$= y^\mathsf{T} \cdot W \cdot y - y^\mathsf{T} \cdot W \cdot A \cdot x - (A \cdot x)^\mathsf{T} \cdot W \cdot y + (A \cdot x)^\mathsf{T} \cdot W \cdot (A \cdot x) = \tag{2.6}$$
$$= y^\mathsf{T} \cdot W \cdot y - 2 \cdot x^\mathsf{T} \cdot A^\mathsf{T} \cdot W \cdot y + x^\mathsf{T} \cdot A^\mathsf{T} \cdot W \cdot A \cdot x \tag{2.7}$$

where we have substituted $W = C_{yy}^{-1}$, assuming that the variance covariance matrix is positive definite. A quadratic function, positive at infinity, always has a minimum. To find it we can take the derivative with respect to our unknown x and equate it to zero:

$$-2 \cdot A^\mathsf{T} \cdot W \cdot y + A^\mathsf{T} \cdot A \cdot x = 0 \tag{2.8}$$

Rearranging the terms we have:

$$A^\mathsf{T} \cdot W \cdot A \cdot x = A^\mathsf{T} \cdot W \cdot y \tag{2.9}$$

$$N \cdot x = b \tag{2.10}$$

This last equation is called "normal equation", N is called "normal matrix" , and b is called the "normal known term". We will use this terminology in the document. Now we just have to multiply the left and right terms by N^{-1} and we are left with an expression to compute the least square system.

$$\hat{x} = N^{-1} \cdot b = (A^\mathsf{T} \cdot W \cdot A)^{-1} \cdot A^\mathsf{T} \cdot W \cdot y = S \cdot y \tag{2.11}$$

Where we call S the matrix that solves the least square problem. Since the least squares solution can in practice be computed in many other way (Chapter 3) for simplicity we will denote the least square procedure as a whole with the backslash sign \. For instance $\hat{x} = A \backslash y$.
Once we have the estimates of the unknowns, we can then derive its variance covariance matrix using the covariance propagation law:

$$C_{\hat{x}\hat{x}} = S \cdot C_{yy} \cdot S^\mathsf{T} = (A^\mathsf{T} \cdot W \cdot A)^{-1} \tag{2.12}$$

We see that the variance covariance matrix of the estimates is the inverse of the normal matrix. If we now write the full expression for the residual we have:

$$u = y - A \cdot \hat{x} = (I - A \cdot S) \cdot y = (I - H) \cdot y = P \cdot y \tag{2.13}$$

The operator P is called a projector (in the general cas an oblique one), and the matrix H is sometimes called the "hat matrix"; we will use such names in the document. Like all projectors P is idempotent $P \cdot P = P$, the same applies to the hat matrix; in fact $H = I - P$ is idemepotent too and so it is a projector. Furthermore, when $W = I$, P and H are also also symmertric:

$$H^\mathsf{T} = (A \cdot (A^\mathsf{T} \cdot A)^{-1} \cdot A^\mathsf{T})^\mathsf{T} = A \cdot ((A^\mathsf{T} \cdot A)^{-1})^\mathsf{T} = A \cdot (A^\mathsf{T} \cdot A)^{-1} \cdot A^\mathsf{T}, \tag{2.14}$$

meaning that they are orthogonal projectors in the euclidean metric. This makes sense since once we are at the minimum of the quadratic expression of Eq 2.5, if we apply the minimisation starting from the minimum we will find it again.

2.1.1   Reducing the Variance Covariance Matrix to Identity

Very often it is much easier to work without the variance covarince matrix of the observation Cyy and its inverse W. To do that is very easy; in fact we can modify both the A matrix and the observation vector y in the following way:

$$A' = R \cdot A \tag{2.15}$$

$$y' = R \cdot y \tag{2.16}$$

where R is a matrix such that:

$$R^{\mathsf{T}} \cdot R = W \tag{2.17}$$

Indeed such operation is particularly simple when W is diagonal from the beginning. Often, specially in connection with the Cholesky algorithm, R is chosen to be triangular. It is easy to verify that the least square solvers will remain the same using the modified design matrix and observations.

2.1.2   Partial Parameters Elimination

In a least squares minimisation problem it is often useful to eliminate some parameters and produce a new linear system whose least squares estimates are identical to the original ones. Let's assume we have the system:

$$y = A_1 \cdot x_1 + A_2 \cdot x_2 = \begin{bmatrix} A_1 & A_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.18}$$

whose least squares estimates we call $\hat{x}_1$ and $\hat{x}_2$; we would like to have a system of the type:

$$\breve{y} = \breve{A}_1 \cdot x_1 \tag{2.19}$$

in such a way that Eq 2.19 produces the same estimate of $\hat{x}_1$ as for the original system. The system can be constructed as:

$$\breve{A}_1 = A_1 - A_2 \cdot (A_2^{\mathsf{T}} \cdot A_2)^{-1} \cdot A_2^{\mathsf{T}} \cdot A_1 = A_1 - A_2 \cdot (A_2 \backslash A_1) = (I - H_2) \cdot A_1 \tag{2.20}$$

$$\breve{y} = y - A_2 \cdot (A_2^{\mathsf{T}} \cdot A_2)^{-1} \cdot A_2^{\mathsf{T}} \cdot y = y - A_2 \cdot (A_2 \backslash y) = (I - H_2) \cdot y \tag{2.21}$$

Note that $I - A_2 \cdot (A_2^{\mathsf{T}} \cdot A_2)^{-1} \cdot A_2^{\mathsf{T}}$ is the orthogonal projector P_2 on to the range of A_2. This means that to remove the parameters x_2 one has to

project both A_1 and y on the orthogonal complement of the range of A_2. It is interesting to bring the problem to the normal form. The reduced normal matrix and normal known term are:

$$\check{N} = (A_1 - H_2 \cdot A_1)^\intercal \cdot (A_1 - H_2 \cdot A_1) \tag{2.22}$$

$$\check{b} = (A_1 - H_2 \cdot A_1)^\intercal \cdot (y - H_2 \cdot y); \tag{2.23}$$

concentrating on the first term and developing the product we get

$$\check{N} = A_1^\intercal \cdot A_1 - (H_2 \cdot A_1)^\intercal A_1 - A_1^\intercal \cdot H_2 \cdot A_1 + (H_2 \cdot A_1)^\intercal \cdot H_2 \cdot A_1 \tag{2.24}$$

Let's focus on the fourth term, bringing the transpose inside the parenthesis:

$$(H_2 \cdot A_1)^\intercal \cdot H_2 \cdot A_1 = A_1^\intercal \cdot H_2^\intercal \cdot H_2 \cdot A_1$$

From the properties of the hat matrix we know that $H^\intercal \cdot H = H^\intercal = H$; the above expression then becomes:

$$A_1^\intercal \cdot H_2^\intercal \cdot A_1 = (H_2 \cdot A_1)^\intercal \cdot A_1,$$

which is exactly the second term of equation 2.24 with the sign inverted. The two terms thus simplify and we are left with:

$$\check{N} = A_1^\intercal \cdot A_1 - A_1^\intercal \cdot H_2 \cdot A_1 \tag{2.25}$$

Substituting the expression for H_2 we get:

$$\check{N} = A_1^\intercal \cdot A_1 - A_1^\intercal \cdot A_2 \cdot (A_2^\intercal \cdot A_2)^{-1} \cdot A_2^\intercal \cdot A_1. \tag{2.26}$$

In the equation we can identify the blocks of the normal matrix, in fact

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} A_1^\intercal \cdot A_1 & A_1^\intercal \cdot A_2 \\ A_2^\intercal \cdot A_1 & A_2^\intercal \cdot A_2 \end{bmatrix}$$

So we can rewrite Eq 2.25 as:

$$\check{N} = N_{11} - N_{12} \cdot N_{22}^{-1} \cdot N_{21} \tag{2.27}$$

The same expression is also called "Schur complement" and is found in the formula of the inversion in part [83].
If we now develop the product for the known term in Eq 2.23

$$A_1^\intercal \cdot y - (H_2 \cdot A_1)^\intercal \cdot y - A_1^\intercal \cdot H_2 \cdot y + (H_2 \cdot A_1)^\intercal \cdot H_2 \cdot y \tag{2.28}$$

For the same reasoning as before the fourth term becomes equal to the second and thus cancels out. If we substitute the formula of the orthogonal projector we are left with:

$$A_1^\intercal \cdot y - A_1^\intercal \cdot A_2 \cdot (A_2^\intercal \cdot A_2)^{-1} \cdot A_2^\intercal \cdot y \qquad (2.29)$$

Considering the known term as:

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} A_1^\intercal \cdot y \\ A_2^\intercal \cdot y \end{bmatrix},$$

and the block notation used for the normal matrix, we finally have:

$$\breve{b} = b_1 - N_{21} \cdot (N_{22})^{-1} \cdot b_2 \qquad (2.30)$$

2.2 Rank Deficient Least Squares

Let's consider the linear system:

$$A \cdot x = y \qquad (2.31)$$

Where $A \in \mathbb{R}^{m,n}$.

Definition 1 *Given a matrix A we call the range space of A, $\mathcal{R}(A)$, the set of all vector y that can be generated by a linear combination of the columns of A, namely $\mathcal{R}(A) = \{y; \ y = A \cdot x, \ x \in \mathbb{R}^n\}$*

A regular basis for the range space of A, $\mathcal{R}(A)$ is a set of linear independent vector that span $\mathcal{R}(A)$.

Definition 2 *Given a matrix A we call the rank of A, $rank(A)$, the dimension of any regular basis for the range space of A*

The system is said to be rank deficient if:

$$rank(A) < n \qquad (2.32)$$

Such systems are know to have an infinite number of least squares solutions. To analyse all the possible solutions the concept of null space is introduced.

Definition 3 *Given a matrix A, we call the null space of A, $\mathcal{N}(A)$, the set of all vectors x for which $A \cdot x = 0$; $\mathcal{N}(A) = \{x; A \cdot x = 0\}$*

$\mathcal{N}(A)$ is indeed a subspace of $\mathbb{R}^n$. The null space can be used to generate all the possible solutions of a rank deficient least squares problem. In fact, calling Q a matrix with columns given by a basis of the null space X, once we have a least squares solution for the system $\hat{x}$ it is easy to show that also $\hat{x} + Q \cdot g$ is a least square solution, in fact:

$$y - A \cdot (\hat{x} + Q \cdot g) = y - A \cdot \hat{x} - A \cdot Q \cdot g = y - A \cdot \hat{x} - 0 = y - A \cdot \hat{x} \quad (2.33)$$

Identifying the symbols of the basis and the corresponding matrix, a general regular basis Q for the null space X will be represented by a matrix Q with dimensions m, d where m number of rows of A and $d = col(A) - rank(A)$. The range space of $A^\intercal$, $\mathcal{R}(A^\intercal)$, and the null space of A, $\mathcal{N}(A)$, are orthogonal complements, in fact their direct sum spans the whole $\mathbb{R}^n$. Generally speaking, to solve a rank deficient linear systems we can specify a set of linear constraints that fix the solution in the null space. The constraints can be written as:

$$K \cdot x = c \quad (2.34)$$

where $K \in \mathbb{R}^{d,n}$. The constraints do eliminate the rank deficiency if $K \cdot Q = S$ has full rank (i.e. if the intersection of the two subspace spanned by $K^\intercal$ and Q is the 0 point and not a line, plane, hyperplane).

2.2.1 Minimum Norm Solution

A very popular choice to specify the constraint for a rank deficient least squares system is to minimise the norm of the estimates,

$$Min(\|\hat{x}\|) \quad (2.35)$$

along all LS solutions. We will see that such solution can be produced using as constraint:

$$Q^\intercal \cdot x = 0^{d,1} \quad Q \in \mathcal{N}(A) \quad (2.36)$$

To show that, let's write explicitly $\|\hat{x}\|^2$. All possible solutions are given by:

$$\hat{x} = \hat{x}_1 + Q \cdot g \quad (2.37)$$

where $\hat{x}_1$ is any specific least squares solution of the system. Keeping in mind that the square of the norm can be written as:

$$(\hat{x}_1 + Q \cdot g)^\intercal \cdot (\hat{x}_1 + Q \cdot g) \quad (2.38)$$

$$\hat{x}_1^\intercal \cdot \hat{x}_1 + 2 \cdot \hat{x}_1^\intercal \cdot Q \cdot g + (Q \cdot g)^\intercal \cdot (Q \cdot g) \quad (2.39)$$

to find the minimum let's take the derivative with respect to g and equate it to 0:

$$2 \cdot \hat{x}_1^\intercal \cdot Q + 2 \cdot g^\intercal \cdot Q^\intercal \cdot Q = 0 \tag{2.40}$$

$$g = -(Q^\intercal \cdot Q)^{-1} \cdot Q \cdot \hat{x}_1 \tag{2.41}$$

if we put g in Eq 2.37 we have:

$$\hat{x}_{mn} = \hat{x}_1 - Q \cdot (Q^\intercal \cdot Q)^{-1} \cdot Q \cdot \hat{x}_1 \tag{2.42}$$

$$\hat{x}_{mn} = (I - Q \cdot (Q^\intercal \cdot Q)^{-1} \cdot Q \cdot)\hat{x}_1 \tag{2.43}$$

were we can recognise the orthogonal projector onto the span Q. So if we project any solution on the orthogonal space to Q the projection of the solution on Q will be zero proving Eq 2.36. A well known property of the minimum norm solution is that it minimises the sum of the variances of the estimates [13] [29].

2.3   Constrained Least Squares

Sometimes we would like to seek the x that minimizes:

$$(y - A \cdot x)^\intercal (y - A \cdot x) \tag{2.44}$$

subject to a linear constraints:

$$K \cdot x = c \tag{2.45}$$

It has to be noted that since it is a constraint, the system in Eq 2.45 has to be underdetermined, i.e. the number of rows must be lower or equal than $rank(K^\intercal)$. Since it has to fulfil Eq 2.45 exactly, $\hat{x}$ can move only in the null space of K. So if we have a basis X for $\mathcal{N}(K)$ and a basis Y for $\mathcal{R}(K^\intercal)$, identifying them with the matrices of their columns, we can decompose x as:

$$x = X \cdot p + Y \cdot q \tag{2.46}$$

The quadratic form in Eq 2.44 becomes:

$$(y - A \cdot X \cdot p + A \cdot Y \cdot q)^\intercal \cdot (y - A \cdot X \cdot p + A \cdot Y \cdot q) = \tag{2.47}$$

$$= y^\intercal \cdot y + (A \cdot X \cdot p)^\intercal \cdot (A \cdot X \cdot p) - 2 \cdot y^\intercal \cdot (A \cdot X \cdot p) +$$

$$+ (A \cdot Y \cdot q)^\intercal \cdot (A \cdot Y \cdot q) - 2 \cdot y^\intercal \cdot (A \cdot Y \cdot q) + \underbrace{2 \cdot (A \cdot X \cdot p)^\intercal \cdot (A \cdot Y \cdot q)}_{0}$$

$$\tag{2.48}$$

adding and subtracting $y^\mathsf{T} y$ to the previous expression one can write it as sum of quadratic forms:

$$-y^\mathsf{T} y + (y - A \cdot X \cdot p)^\mathsf{T} \cdot (y - A \cdot X \cdot p) + (y - A \cdot Y \cdot q)^\mathsf{T} \cdot (y - A \cdot Y \cdot q) \quad (2.49)$$

So our problem can be written as two separate least squares problem. The solution then becomes:

$$\hat{x} = Y \cdot ((A \cdot Y) \backslash y) + X \cdot ((A \cdot X) \backslash y) \quad (2.50)$$

Lets now apply the decomposition also to the constraint:

$$K \cdot (Y \cdot q + X \cdot p) = c \quad (2.51)$$

we can see that the second term disappears since $K \cdot X = 0$ and that the first term generates the least norm solution. So in the end we have:

$$\hat{x} = \hat{x}_{mn} + X \cdot ((A \cdot X)^+ \cdot y) \quad (2.52)$$

where $\hat{x}_{mn}$ is the least norm solution of the system $K \cdot x = c$.

2.4 Basics of Lattice Theory

In this section a very brief presentation of lattice theory is done.

Definition 4 *Given a real matrix $L \in \mathbb{R}^{m,n}$ it is called lattice $\mathcal{L}(L)$ with base L the set of all vectors generated by $L \cdot x$ where $x \in \mathbb{Z}^n$*

The lattice is called full rank lattice if $rank(L) = m$.

Definition 5 *Given a lattice $\mathcal{L}(L)$ with base L , we call fundamental parallelepiped the set of all points $p = L \cdot x, x \in [0,1]^n$*

The volume of the fundamental parallelepiped is given by $det(L)$. A full rank lattice with determinant equal to ± 1 spans the whole $\mathbb{Z}^n$. Fig 2.1 shows some examples of lattices.

Definition 6 *A matrix U is called unimodular if $U \in \mathbb{Z}^n$ and $det(U) = \pm 1$*

A lattice spanned by U, is whole $\mathbb{Z}^n$. The following three elementary operations applied to an unimodular matrix will keep its unimodular property [18] [105]:

- switch the order of the columns.

- invert the sign of a columns.

- subtract an integer number of columns to an other.

(a) Lattice of not full rank.   (b) Lattice of full rank.

(c) Lattice of full rank, spanning whole $\mathbb{Z}^2$
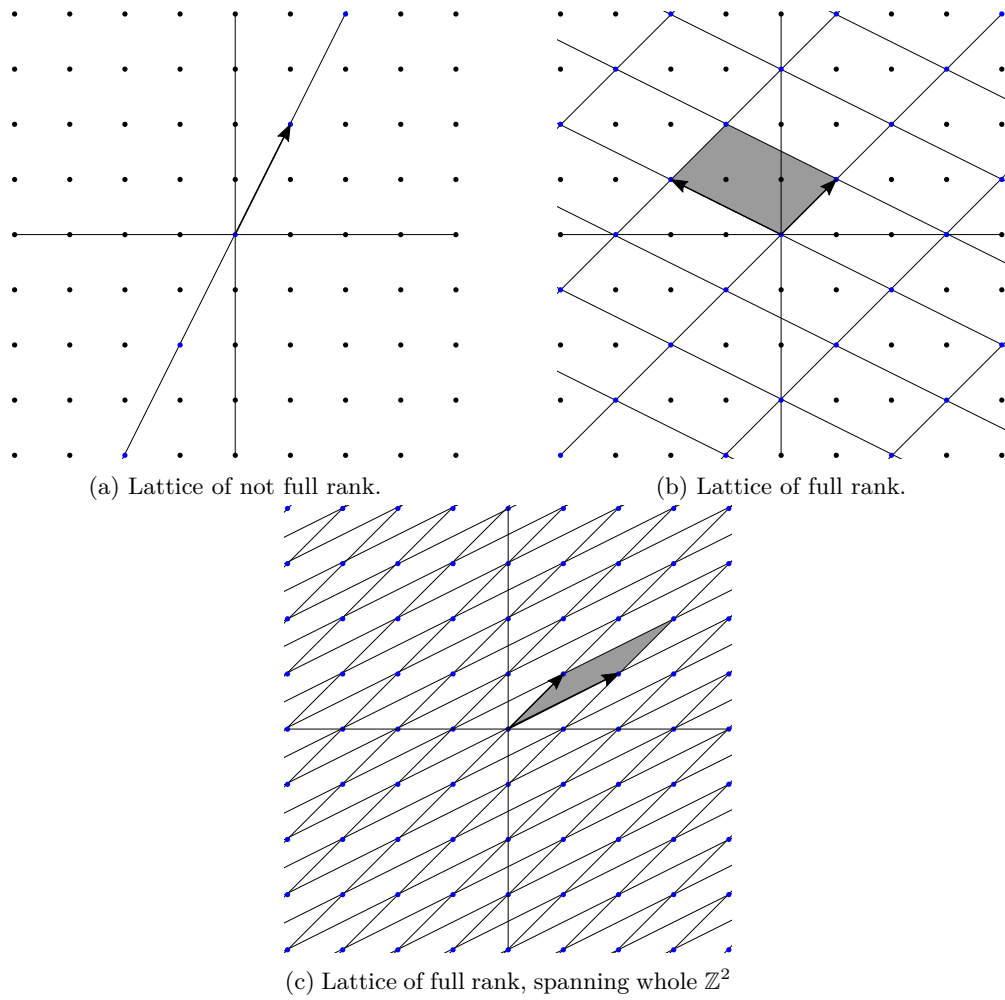
Figure 2.1: Three examples of lattices; arrows indicate $\mathbb{R}^2$ vectors that form the basis of the lattice $\mathcal{L}$.

2.4.1 Lattice Reduction

Generally, the same lattice can be generated by different lattice bases. For istance, it is easy to see that all lattice basis with determinant equal to plus minus one spans the whole $\mathbb{Z}^n$. It can be shown that a lattice basis B when multiplied by a unimodular matrix U becomes a new basis B' that spans the same lattice as B. For several reasons it is desirable to find a lattice basis for which the magnitude of the columns are as small as possible and as orthogonal as possible. Such procedure is called lattice reduction. A popular algorithm for such a task is the Lenstra–Lenstra–Lovász (LLL) [72]. The algorithm is not going to be described here since the procedure is similar to the LAMBDA decorrelation procedure [71] that will be discussed in Subsection 3.2.4.

2.5 Integer Least Squares

In this section we are going to discuss a least squares problems whose unknowns are integer. Consider a linear system of the type:

$$B \cdot z = y \tag{2.53}$$

Where $B \in \mathbb{R}^{m,n}$ and $rank(B) = n$ and $z \in \mathbb{Z}^n$, of course given B and a general y in the observable space, Eq 2.53 can not be satisfied exactly, so the problem is to find the integer vector $\check{z}$ that minimises the square of the residuals. Such integer vector is called the Integer Least Squares (ILS) estimator. Often the integer parameters are found together in a system with real ones. Consider for instance the system:

$$A \cdot x + B \cdot z = y \tag{2.54}$$

Where $x \in \mathbb{R}^{n_1,1}$ and $z \in \mathbb{Z}^{n_2,1}$. This system can be reduced to a pure integer problem by using the procedure described in sub section 2.1.2 [1] Since the constraint for a vector to have integer values can not be expressed by a set of linear equations, the previously explained constrained least square is of little use. Due to this fact, in general, to find the integer least squares different integer vectors have to be tested individually. To test the values a search space has to be defined and all the integer vectors

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[1] In literature another procedure exists which basically dicretizes also the continuous parameters and performs a search in such discretized space. This procedure is called "Ambiguity Function Method" see for instance [23] [45].

in it should be tested. A procedure to do that will be described in Sec 3.2.

2.5.1 Comparison of Integer L.S. with Real Value Estimates

Once we have the integer least squares solution we could ask our selves how it compares with real least squares solution. One way to do that is to compare the theoretical variance of the integer solution with that of the float one.

For instance let's consider a one dimensional case. In such a simple case, to recover the integer least squares estimator it is sufficient to round to the closest integer the float solution. Given the standard deviation of the estimate σ, the variance of the integer least squares for the one dimensional case can be written as:

$$\check{\sigma}^2 = \sum_{-\infty}^{\infty} i^2 \cdot \int_{i-1/2}^{i+1/2} \frac{e^{-\frac{x^2}{2 \cdot \sigma^2}}}{\sqrt{2 \cdot \pi}\sigma} dx \qquad (2.55)$$

In figure 2.2 is possible to see the difference between the standard deviation of the float and integer solutions as a function of the standard deviation of the float solution. It can be see that up to 0.5σ the variance of the integer least squares solution is smaller than the float one but after then it is bigger.

Since the integer solution is not always better than the real one we need a criterion for accepting the integer solution. One popular choice is the ratio test [33] revisited by [108]. The test checks whether the ratio between the objective function $\Omega()$ (the least squares principle) computed using the ILS ($\check{z}_1$) and computed using the second best integer vector ($\check{z}_2$) is greater than a certain threshold:

$$\frac{\Omega(\check{z}_2)}{\Omega(\check{z}_1)} > \gamma \qquad (2.56)$$

2.5.2 The Bayesian Solution

Instead of choosing one integer vector one can use a weighted linear combination of all integer vectors with significant probability. Such approach is called Bayesian approach [12] [28][42] and is also know as Best Integer Equivariant since it is the best (i.e. with minimum variance) in the
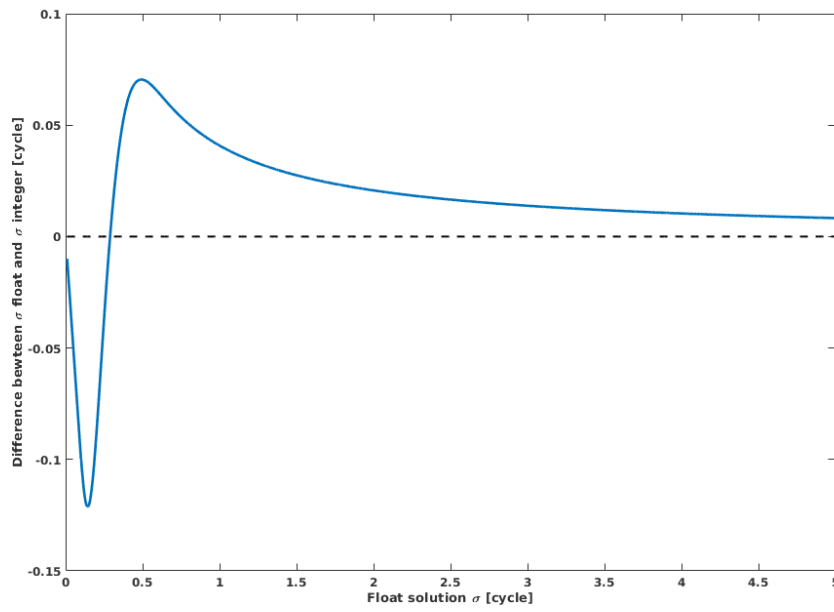
Figure 2.2: Difference between the standard deviation of the float and integer solution as a function of the standard deviation of the float solution.

broader class of Integer Equivariant Estimators [110]. In the estimator formula each integer vector is weighted by its posterior probability:

$$b = \sum_{z \in \mathbb{Z}^n} w_z \cdot z \tag{2.57}$$

where:

$$w_z = \int_{\Omega_z} \Gamma(x) dx \tag{2.58}$$

or in plain English the integral of the Probability Density Function (PDF) $\Gamma(x)$ over the so call pull in region Ω_z i.e. the region in $\mathbb{R}^n$ for which real valued estimates x are mapped to z by the integer least squares algorithm [109].

2.6   Rank Deficient Integer Least squares

Let's consider now a new linear system:

$$A \cdot z = y \tag{2.59}$$

Where $z \in \mathbb{Z}^n$. Let's assume that the system is rank deficient. We can use a basis Q for $\mathcal{N}(A)$. Taking an integer least squares solution $\check{z}_0$ where $\check{z} \in \mathbb{Z}$ we can generate a new solution $\hat{z}_1$ as $\check{z}_0 + Q \cdot g$. However, since in general $Q \in \mathbb{Q}^{m,d}$, $\hat{z}_1$ will be real too and thus not an admissible integer solution. To generate admissible solution we have to introduce the concept of integer null space.

Definition 7 *Given a matrix A we call integer null space of A, $\mathcal{I}(A) = T$, the set of all vectors $t \in \mathbb{Z}$ for which $A \cdot t = 0$, $T = \{t; \; t \in \mathbb{Z}^n, A \cdot t = 0\}$.*

A basis L for the integer null space T is a system of vectors for which the lattice $\mathcal{L}(L)$ spans the whole integer null space T. As we said before to solve the problem we have to specify a set of admissible linear constraints; in fact, not all constraints allow for the recovery of an integer least squares solution. To show this let's consider for a moment an integer least squares problem for which the observations are errorless.

$$B \cdot z = y \tag{2.60}$$

Let's assume that we know the true value of z that we will denote as $\acute{z}$. To solve the problem we use an admissible constraint of the type (2.34).

If we pick the entries of K as integers to obtain the true solution $\acute{z}$ we have to set c equal to:

$$\acute{c} = K \cdot \acute{z} \tag{2.61}$$

Since $K, \acute{z} \in \mathbb{Z}^{d,n}, \mathbb{Z}^n$ also $\acute{c} \in \mathbb{Z}$. To show that not all the constraints will result in an integer solution we now can simply keep the same integer K and choose a c whose entries are not integer. In this case obviously also the estimated solution $\hat{x}$ will not be integer.

Let's now consider all solutions equivalent to $\acute{z}$, if we have a lattice basis L for the integer null space, we can generate them as:

$$\acute{z}_e = \acute{z} + L \cdot g, \quad g \in \mathbb{Z} \tag{2.62}$$

Choosing again a K with integer entries , following Eq, 2.61, c has to be set:

$$\acute{c}_e = K \cdot (\acute{z} + L \cdot g) = \acute{c} + K \cdot L \cdot g = \acute{c} + S \cdot g \tag{2.63}$$

Since normally we do not know the true value $\acute{z}$ we can construct two strategies to find an admissible $\acute{c}_e$.

2.6.1 First Strategy

We know that if we pick a $K \in \mathbb{Z}^{d,n}$, i.e. made of integers, also $\acute{c}_e$ has to be integer. However, in the general case, not all integer vectors will satisfy Eq 2.63. Nevertheless, if we choose K wisely we can make S to be unimodular.If this is the case S will span the whole $\mathbb{Z}^d$ and thus all $g \in \mathbb{Z}^n$ will generate an admissible $\acute{c}_e$. Generally speaking there will be multiple K (infinite) that multiplied by the integer null space will give a unimodular matrix. Having an admissible constraint K_1 all other admissible constraints can be generated by $U \cdot K_1$ where U is an unimodular matrix.

2.6.2 Second Strategy

Another strategy can be used. Going back to Eq 2.63 we know that the $\acute{c}_e$ is generated form the lattice spanned by S. From lattice theory we know that given a lattice basis S the number of integers found in the fundamental parallelepiped is $n = det(S)$. So if we start from an arbitrary $c_0 \in \mathbb{Z}^d$, it is sufficient to examine the n, cs contained in the fundamental parallelepiped to find an element of $\acute{c}_e$ [2] . To better understand the

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[2]The GLONASS ambiguity resolution strategy used in the Bernese GNSS software [24] can be roughly viewed in these terms (See [43] section 4.2)

concept Fig 2.3 depicts the search space for the c values that is found in the fundamental parallelepiped of $\mathcal{L}(S)$ for a two dimensional case.
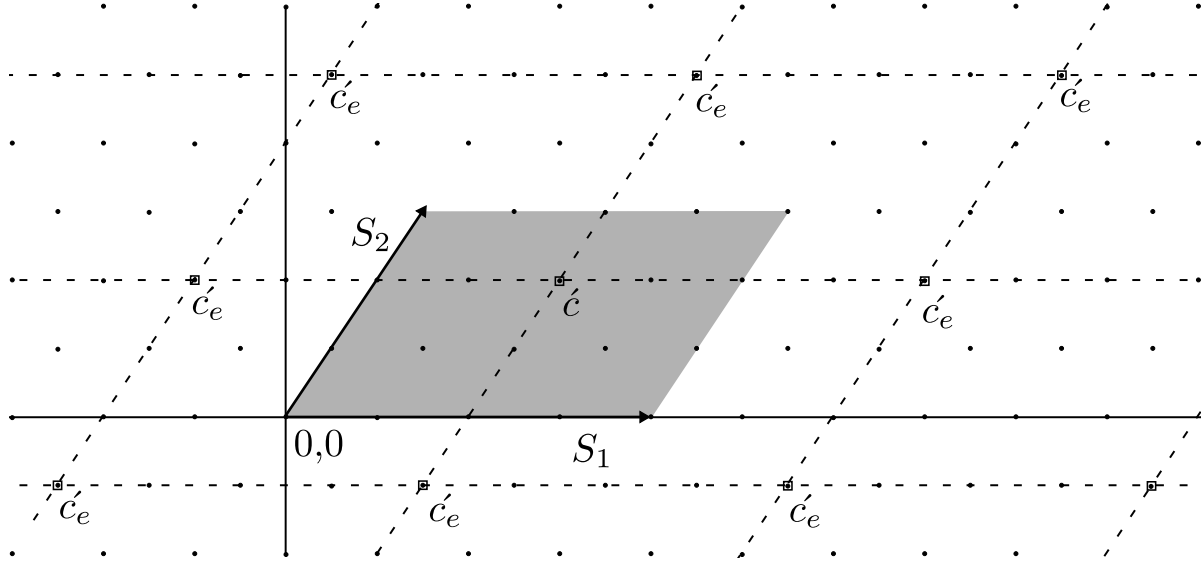
Figure 2.3: Depiction of the search space for the c part of the constraint

2.7   Unitary Integer Rank Deficient Problems

Definition 8 *An integer rank deficient problem is called unitary when an admissible constraint for the problem can be constructed by simply eliminating (constraining to zero) one or more parameters of the problem.*

It will be shown that in GNSS estimations some rank deficient integer least squares problem are of this kind and thus can be solved using common algorithms for rank deficient problems.

2.8   Mixed Integer-Rational Least Squares

Let's consider now a new linear system:

$$A \cdot x + B \cdot z = \begin{bmatrix} A & B \end{bmatrix} \cdot \begin{bmatrix} x \\ z \end{bmatrix} = C \cdot p = y, \qquad (2.64)$$

where $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^p$. Let's assume that the matrix $[AB]$ is rank deficient. Having defined the integer null space, we can then define the mixed integer null space.

Definition 9 *Given linear system $A \cdot x + B \cdot z = y$ we call mixed integer null space of $[AB]$, $\mathcal{M}([AB])$, the set of all vectors $p = \begin{bmatrix} x \\ z \end{bmatrix}, x \in \mathbb{R}^n, z \in \mathbb{Z}^p$ for which $\begin{bmatrix} A & B \end{bmatrix} \cdot \begin{bmatrix} x \\ z \end{bmatrix} = 0$*

Having a basis $G = \begin{bmatrix} Q \\ L \end{bmatrix}$ for the mixed integer linear system null space and a least squares solution for the problem $\hat{p}_0 = \begin{bmatrix} \hat{x}_0 \\ \hat{z}_0 \end{bmatrix}$ all the admissible solutions can be generated as $\hat{p}_0 + G \cdot h$ with the elements of h $h_i \in \mathbb{Z}$ if the ith column of the L matrix has non zero entries or $h_i \in \mathbb{R}$ if the ith column of the L matrix is made of zeros.

To chose a proper constraint we have to distinguish two cases for each kind of rank deficiency:

- The rank deficiency is only between the column referred to real parameters. In this case we refer to Section 2.2.

- The rank deficiency is either between the columns referred to integer parameters or between mixed columns in this case we consider only the integer part of the null space and apply one of the two strategies described in Section 2.6. This is because the constraint conditions on the integers parameters are more stringent ans so any conditions working for the integers parameters will be admissible for the real parameters.

2.9 Tykhonov Regularization

Sometimes we have some prior information on part of the parameters (for instance an approximate knowledge of its magnitude or its time/spatial variation). One could express this information using a pseudo observation. This procedure is called regularization in the sense that the estimation procedure is obliged to follow the equations of the pseudo observations.

This kind of procedure is called Tykhonov regularization[3] due to the work of Andrey Nikolayevich Tikhonov on the topic of inverse ill posed problems [114]. Let's consider the a linear system:

$$A \cdot x = y \tag{2.65}$$
$$G \cdot x = r \tag{2.66}$$

where often r is just a zero vector. Omitting the weight matrix for simplicity, if we build the hat matrix from it we have:

$$H_t = A \cdot (A^\mathsf{T} \cdot A + G^\mathsf{T} \cdot G)^{-1} \cdot A^\mathsf{T} \tag{2.67}$$

If we try to compute $H_t \cdot H_t$ we see that the new hat matrix is not idempotent anymore:

$$H_t \cdot H_t = A \cdot (A^\mathsf{T} \cdot A + G^\mathsf{T} \cdot G)^{-1} \cdot A^\mathsf{T} \cdot A \cdot (A^\mathsf{T} \cdot A + G^\mathsf{T} \cdot G)^{-1} \cdot A^\mathsf{T} \tag{2.68}$$

This means that if the regularized least squares operator is applied to the residuals of a first adjustment it will give a non zero correction to the estimation. It is therefore interesting to study the convergence at infinity of the regularised estimation. Calling as $N = A^\mathsf{T} \cdot A$ and $O = G^\mathsf{T} \cdot G$ the estimator becomes:

$$\hat{x}_1 = (N + O)^{-1} \cdot A^\mathsf{T} \cdot y \tag{2.69}$$

then the correction to such estimation, when we apply the same procedure again is:

$$\delta\hat{x} = (N + O)^{-1} \cdot A^\mathsf{T} \cdot (y - A \cdot \hat{x}_1) = \hat{x}_1 - (N + O)^{-1} \cdot N \cdot \hat{x}_1 \tag{2.70}$$

So the new value of the estimation will be

$$\hat{x}_2 = 2 \cdot \hat{x}_1 - (N + O)^{-1} \cdot N \cdot \hat{x}_1 \tag{2.71}$$

calling $M = (N + O)^{-1} \cdot N$, the estimation at iteration $n + 1$ will be:

$$\hat{x}_{n+1} = 2 \cdot \hat{x}_n - M \cdot \hat{x}_n = \hat{x}_n + (I - M) \cdot \hat{x}_n \tag{2.72}$$

and the correction at iteration $n + 1$ will be:

$$\delta\hat{x}_{n+1} = (I - M) \cdot \hat{x}_n = (I - M)^n \cdot \hat{x}_1. \tag{2.73}$$

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[3]Although Tykhonov's theory is much more general, we use it here only in the context of the use of a quadratic regularising functional. In other field is also called ridge regression or L2 regularization, see for instance [19] Subsection 2.4.1.1.

The estimate after n iterations will be:

$$\hat{x}_n = \sum_{i=0}^{n} (I - M)^n \cdot \hat{x}_1;\tag{2.74}$$

for $n \to \infty$ since $|M| < 1$ the summation will converge to M^{-1},([57] proposition 1.5.38). So in the end we have:

$$\hat{x}_\infty = M^{-1} \cdot (N+O)^{-1} \cdot A^\mathsf{T} \cdot y = N^{-1} \cdot (N+O) \cdot (N+O)^{-1} \cdot A^\mathsf{T} \cdot y = N^{-1} \cdot A^\mathsf{T} \cdot y\tag{2.75}$$

This means that the solution will converge to the non regularized solution. If we want to keep the effect of the pseudo observation equation it is just enough to iterate their residual together with those of the observations equation. In this way it will be a regular least square solution. For this reason when specifying the pseudo observation and their weight one has always to care for the initial value from which the estimation is starting.

2.10   Robust Adjustment

Sometimes the set of observations we want to adjust are contaminated at least partly by measurements that do not respect our stochastic assumption. These observations are called outliers and if not properly treated could result in biases in our estimation. An estimator that is relatively insensitive to such wrong observation is called a robust estimator. A way to make the estimator more robust than the simple least squares is to minimise a function of the residuals that is not the sum of their squares, but rather:

$$\sum_i \mathrm{P}(u_i);\tag{2.76}$$

the function $\mathrm{P}(u_i)$ is also called the loss function. These estimators are called M-Estimator [58]. For general $\mathrm{P}(u)$, no closed form for the computation of such estimator exist. To compute them one has to iterate the least squares computation adjusting the weights of the observations or the residuals of the estimation.
At each step the new weights w for the observations can be computed as:

$$w = \frac{\Phi(u)/\sigma}{u/\sigma};\tag{2.77}$$

or if one would like to use the the modified residuals, the new observations
would be:

$$u^* = \Phi(u/\sigma) \cdot \sigma \tag{2.78}$$

where $\Phi(u) = \frac{d\mathrm{P}(u)}{du}$ and σ is the standard deviation of the observations.
The method of weights has a rate of convergence that should be faster that
the method of residuals ([59] Lemma 7.8). However, since the method of
residuals can fully reuse the solver of the step before it might be faster or
more convenient in practice.

A popular M-estimator is the so called Huber-estimator. Its loss function
is defined as:

$$\mathrm{P}(u) = \begin{cases} \frac{1}{2} \cdot u^2, & if |u| < \gamma \\ \gamma \cdot (|u| - \frac{1}{2}\gamma), & if |u| > \gamma \end{cases} \tag{2.79}$$

if $\gamma = 0$ it is the least absolute estimator, if $\gamma = \infty$ it is the least squares
estimator.

3

Solving Least Squares Problems

Plurality must never be posited without necessity
                                                – William of Ockham

3.1 Algorithms for Least Squares Problems

This section will describe the different method to solve least squares problem. Four popular algorithm are presented, many more exists [56][41][101].

3.1.1 Cholesky Decomposition

The Cholesky method [11] works on the normal form. It works decomposing the normal matrix into two triangular matrix one the transpose of the other:

$$N = L \cdot L^\mathsf{T} \qquad (3.1)$$

where L is a lower triangular matrix. The element of the matrix can be computed as:

$$L = \begin{pmatrix} L_{11} & 0 & \dots & 0 \\ L_{21} & L_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & L_{nn} \end{pmatrix} = \begin{pmatrix} \sqrt{N_{11}} & 0 & \dots & 0 \\ N_{21}/L_{11} & \sqrt{N_{22} - L_{21}^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{n1}/L_{11} & N_{n2} - L_{21} \cdot L_{n1}/L_{22} & \dots & \sqrt{N_{nn} - \sum_{i=1}^{n-1} L_{ni}^2} \end{pmatrix}.$$
$$(3.2)$$

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Such a formula holds also in case the matrix N is partitioned, in this case the square root of a symmetric matrix is the cholesky decomposition of the block itself. Once the decomposition has been performed the linear system can be solved in two step as:

$$\hat{p} = L^\mathsf{T} \setminus b \tag{3.3}$$

$$\hat{x} = L \setminus \hat{p} \tag{3.4}$$

Solving such system is easy given the triangular form of L. The complexity of the Cholesky decomposition is around $\frac{1}{3} \cdot n^3$ floating point operations (See [115] Chapter 23). It is the fastest direct method to solve linear systems.

3.1.2 LDL Decomposition

The LDL decomposition is a generalisation of the cholesky decomposition. A symmetric matrix is decomposed as:

$$N = L \cdot D \cdot L^\mathsf{T} \tag{3.5}$$

where L is a lower triangular matrix with ones on the main diagonal, and D is diagonal matrix. The elements of the matrix L are:

$$L = \begin{pmatrix} 1 & 0 & \dots & 0 \\ L_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \dots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ N_{21}/D_1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_{n1}/D_1 & (N_{n2} - \sum_{i=1}^{n-1}(L_{2i} \cdot L_{ni} \cdot D_i))/D_2 & \dots & 1 \end{pmatrix} \tag{3.6}$$

and the elements of matrix D are:

$$D = \begin{pmatrix} D_1 & 0 & \dots & 0 \\ 0 & D_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_n \end{pmatrix} = \begin{pmatrix} N_{11} & 0 & \dots & 0 \\ 0 & N_{22} - L_{21}^2 \cdot D_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & N_{nn} - \sum_{i=1}^{n-1} L_{ni}^2 \cdot D_i \end{pmatrix} \tag{3.7}$$

The same decomposition can also be performed in blocks. Once the decomposition has been performed the linear system can be solved in three step as:

$$\hat{p} = L^\mathsf{T} \setminus b \tag{3.8}$$

$$\hat{q} = D \setminus \hat{p} \tag{3.9}$$

$$\hat{x} = L \setminus \hat{q} \tag{3.10}$$

3.1.3 Solving a Rank Deficient Systems Using the LDL Decomposition

Let's now consider a j columns rank deficient matrix where the jth column is a linear combination of the first $j-1$ columns.

$$A' = \begin{bmatrix} A & A \cdot f \end{bmatrix}. \tag{3.11}$$

The normal matrix can then be written as

$$N' = A'^\intercal \cdot A' = \begin{bmatrix} A^\intercal \cdot A & A^\intercal \cdot A \cdot f \\ f^\intercal \cdot A^\intercal \cdot A & f^\intercal \cdot A^\intercal \cdot A \cdot f \end{bmatrix} \tag{3.12}$$

calling $A^\intercal \cdot A = N$ the D_{jj} element of the LDL decomposition can be written in block form as:

$$D_{jj} = D_{22} = N'_{22} - L'_{21} \cdot D_{11} \cdot (L'_{21})^\intercal = N'_{22} - N'_{21} \cdot N'^{-1}_{11} \cdot N'_{11} \cdot (N'_{21} \cdot N'^{-1}_{11})^\intercal$$
$$= f^\intercal \cdot N \cdot f - f^\intercal \cdot N \cdot (f^\intercal \cdot N \cdot N^{-1})^\intercal = f^\intercal \cdot N \cdot f - f^\intercal \cdot N \cdot f = 0 \tag{3.13}$$

We see that if a particular parameter is a linear combination of the previous ones it will have a 0 at the corresponding element on the diagonal. We can use this property to identify parameters that are in rank deficiency with the previous ones. To solve the least squares problem then is simply sufficient to eliminate the parameters (setting them to 0) that are linear combination of the previous ones. The decomposition can be computed permuting the columns and rows (so called pivoting strategy), as this can improve the numerical accuracy of the factorisation (See [53] Section 11.1). This would not change the strategy discussed above, the only difference being that the elimination of parameters should be consistent with the permutation of columns and rows.

3.1.4 QR Decomposition

Another popular decomposition used to solve linear systems , is the QR decomposition[35]. The original matrix is decomposed as:

$$A = Q \cdot R \tag{3.14}$$

with Q an orthonormal matrix and R a matrix of the form:

$$R = \begin{bmatrix} U \\ 0^{m-n,n} \end{bmatrix} \tag{3.15}$$

where U is an n by n upper triangular matrix. A LS solution to the system can ten be computed as:

$$\hat{p} = Q^\intercal \cdot y \tag{3.16}$$

$$\hat{x} = R \setminus \hat{p} \tag{3.17}$$

where the first system is easily solved, being upper triangular. One has to note that the matrix R of the QR decomposition is equal to the Cholesky decomposition of the normal matrix $A^\intercal \cdot A$[40] and so it can be used to compute the variance covariance matrix of the estimates or to obtain an LS solution in a similar way to the cholesky decomposition. The QR factorization algorithm has a complexity of $2 \cdot m \cdot n^2 - \frac{2}{3} \cdot n^3$ floating point operations when computed using Householder transforms method (See [115] Chapter 10).

3.1.5 Solving a Rank Deficient Systems Using the QR Decomposition

The decomposition can be seen as a Graam Schmidt orthogonalization [1] in particular the j, j element of the R matrix can be written as:

$$R(j,j) = \|(A(\ldots,j) - Q(\ldots;1\ldots j)^\intercal \cdot A(\ldots,j)\| \tag{3.18}$$

or in plain English as norm of column j minus its projection onto the previously orthogonalized columns. It is clear that if a column is a linear combination of the previous ones such element will be 0. So we can once again detect parameters that are in rank deficiency with the previous ones, looking at the diagonal elements of R. To solve the system one can then remove the parameters from the system. Another way to solve the system would be to use the factorisation of $A^\top$ instead. Given that:

$$A^\intercal = Q' \cdot R', \tag{3.19}$$

the least norm solution to the system $A \cdot x = y$ can be computed as

$$\hat{p} = R''^\intercal \setminus y \tag{3.20}$$

$$\hat{x}_{mn} = Q' \cdot \hat{p}' \tag{3.21}$$

where R'' is the R' matrix where rows corresponding to a 0 value on the diagonal have been eliminated, and $\hat{p}'$ correspond to the values of

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[1]Although this is normally not used as a way to compute the QR decomposition due to bad numerical rounding error properties, see [101] Chapter 3.

$\hat{p}$ augmented by zeros in place of the eliminated rows of R'. Similarly to LDL decomposition a pivoting strategy can be used to improve the numerical stability of the algorithm (See [53] Section 19.4).

3.1.6 SVD decomposition

Another decomposition used to solve least squares problems is the singular value decomposition [10]. The matrix $A \in \mathbb{R}^{m,n}$ is decomposed as:

$$A = U \cdot S \cdot V \tag{3.22}$$

where U is an m by m orthonormal matrix, V is an n by n orthonormal matrix, and S is matrix of the form:

$$S = \begin{bmatrix} D \\ 0^{m-n,n} \end{bmatrix} \tag{3.23}$$

with D a diagonal matrix with non negative elements. The values on the diagonal D are called singular values of D. For convention the singular values are ordered in decreasing order. The solution of a regualar least squares system can be produced as:

$$p = U^{\intercal} \cdot y \tag{3.24}$$
$$q = [D^{-1}0^{n,m-n}] \cdot p \tag{3.25}$$
$$\hat{x} = V^{\intercal}q \tag{3.26}$$

3.1.7 Solving Rank Deficient System Using the SVD Decomposition

Similarly to the other algorithms, rank deficiencies can be identified looking at the elements of diagonal matrix D. If rank deficiencies are present the last elements of the diagonal will turn to zero and the system can simply be solved removing the corresponding rows and columns from D and putting zeros at the same place in the vector q. The resulting solution will be the least norm solution ([41] subsection 5.5.1). The complexity of the SVD computation is on the order of $2 \cdot m \cdot n^2 + 11 \cdot n^3$ floating point operations (See [115] Chapter 11), being more expensive that the Cholesky/LDL and QR decomposition. However the algorithm is also more numerically stable than the other two (See again [115]).

3.2   Algorithms for Integer Least Squares Problems

In this section we will discuss the procedure to obtain an integer solution to a non rank deficient least squares problem. All solutions start from the float solution and from their variance covariance matrix. The integer least squares estimator is the integer vector that minimises:

$$(y - A \cdot z)^\intercal \cdot (y - A \cdot z) \tag{3.27}$$

This can be seen as the point in the lattice $\mathcal{L}(A)$ closest to y [47]. In lattice theory such problem a is called the Closest Vector Problem (CVP). The same problem can also be viewed as the integer vector that minise:

$$(\hat{z} - z)^\intercal \cdot C_{\hat{z}\hat{z}}^{-1}(\hat{z} - z) \tag{3.28}$$

the previous equation can be interpreted as the square of the distance between the float solution and the integer solution, when the metric is defined by the $C_{\hat{z}\hat{z}}$ matrix.

3.2.1   Round

The simplest way to obtain an integer solution from a float one is just round the estimates to its nearest integer. However, this procedure does not guarantee that the integer solution is the integer least squares solution. Consider the simple example:

$$\hat{z} = \begin{bmatrix} 3.55 \\ 7.34 \end{bmatrix} \qquad C_{\hat{z}\hat{z}} = \begin{bmatrix} 5 & 4.9 \\ 4.9 & 5 \end{bmatrix} \tag{3.29}$$

the rounding to the next integer yields $\check{z}_r = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ wile the ILS is $\check{z} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ in fact it is easy to verify that:

$$(\hat{z} - \check{z}_r)^\intercal \cdot C_{\hat{z}\hat{z}} \cdot (\hat{z} - \check{z}_r) > (\hat{z} - \check{z})^\intercal \cdot C_{\hat{z}\hat{z}}^{-1} \cdot (\hat{z} - \check{z}) \tag{3.30}$$

$$2.09 > 0.19 \tag{3.31}$$

This is because the pull-in region of the rounding operator (Subsection 2.5.2) is different from the one of the integer least squares. The advantage of the rounding operation rely on its extremely low computational cost. For this reason it is interesting to know when it can be used safely. A lower

bound for the success rate (The probability P that the rounding operator will return the ILS) of the operator can be computed as follows[107]:

$$P(\hat{z}_r = \hat{z}) = \prod_i \left(1 - 2 \cdot \int_{-\infty}^{-\frac{1}{2}} \Psi(x; 0, \sigma_i^2) dx \right), \qquad (3.32)$$

where $\Psi(x; 0, \sigma_i^2)$ is the normal distribution centred in 0 and with variance σ_i^2, the latter being the diagonal elements of the $C_{\hat{z}\hat{z}}$ matrix. It has to be noted that this success rates depend only on $C_{\hat{z}\hat{z}}$ and thus they are sensible to errors in the stochastic model.

3.2.2 Bootstrap

In this section the bootstrap procedure is presented [1]. The procedure simply rounds the first integer parameter and conditions the rest of the integer parameters to the fixing of the first one. It then proceeds with all the other parameters. If we have the variance covariance of the integer parameters C_{zz} and we compute the LDL decomposition we can see it has a particular meaning for the integer bootstrapping. Looking at Eq 3.7 we can see how the element of the diagonal are the variances of the integer parameters given that the previous one have been fixed. Unfortunately also the bootstrap estimator does not guarantee the estimation of the ILS. For instance if we consider the example in Eq 3.29 if we apply the bootstrap estimator we have $\hat{z}_b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$ that again it is different from the ILS. Similarly to the rounding operator it has a very low computational cost. For this reason it is also interesting to know when we can use it safely. A lower bound for the success rate can be compute as [107]:

$$P(\hat{z}_r = \hat{z}) = \prod_i \left(1 - 2 \cdot \int_{-\infty}^{-\frac{1}{2}} \Psi(x, 0, \gamma_i^2) dx \right), \qquad (3.33)$$

where γ_i^2 is the variance of the conditioned integer parameters. Such variances are lower or equal to the variances of the original matrix (Eq 3.7). For this reason the success rate of the bootstrap estimator is never greater to the one of the rounding operator, and normally is much smaller. Again, like for the rounding it is very sensible to a correct stochastic model.

3.2.3 Search

To be sure to reach the integer least squares, ones has to perform a search over different candidates. In this subsection we will see how to search over such candidates. The strategy presented is taken from [117]. The strategy works for the integer least squares or for the search of the n closest integers vectors. Considering Eq 3.28 we would like firstly to list all candidates at a distance smaller than χ. To do that, one could list for all integer parameters, all integer numbers that are within χ from the float solution. This would however result in a very large search space. Another strategy could be produced by taking inspiration from the bootstrapping procedure. The fist integer parameter is taken and all z_1 for which $z_1 \cdot C_{11}^{-1} \cdot z_1 < \chi$ are listed. Then for each one of the listed z_1 one can list all the z_2 for which $z_2 \cdot C_{22}'^{-1} \cdot z_2 < \chi - z_{1i} \cdot C_{11}^{-1} \cdot z_{1i}$ where C_{22}' is the variance of the second integer parameter conditioned to the fixing of the first one. The procedure can be repeated for all ambiguities producing a tree of possible integer parameters (see Fig 3.1). Having defined the
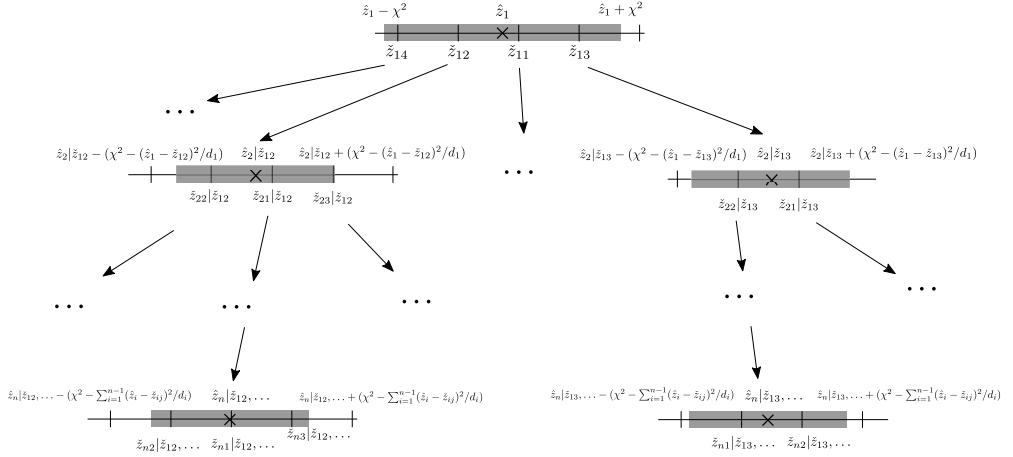
Figure 3.1: Depiction of the search tree for a given χ^2.

listing procedure one has to fix a value of χ. Instead of using a fixed value the value can be changed while doing the search. Imagine we are looking for n values one could start with the value of $\chi = \infty$ and start to evaluate the first branch of the tree. Once the algorithm has evaluated n solutions, χ is set to the maximal distance of the n solutions. Then the search continues, and each time a integer set is found whose distance is smaller then the current χ, the integer is stored in place of the current

most distant value and the χ is set to the new current most distant value. The procedure is continued till any more candidates inside χ are found. Such a procedure is called search and shrink strategy [106][20].

3.2.4 Decorrelation

As we have seen the search space depends on the conditional variances of the integer parameters. So if one finds a way to reduce such variances the search would be much faster and also the success rate of the bootstrap would increase. To try to reduce the variances we have to use a transformation that preserves the integer character of the estimates. The "integerness" should be preserved both in the in the forward and backward transformation. In Sec 2.4 we have already presented one of such transformations, performed by an unitary matrix. In particular if one is able to reduce the variances of the first considered parameter (normally the most precise) one can reduce from the beginning of the search procedure the number of branches to be evaluated itself . Since the first parameter conditional variance is just its variance, the aim of the transformation should be to reduce the variance of the parameter itself and of the firsts subsequently conditioned parameters. Consider for instance a variance covariance matrix of the form:

$$C = \begin{bmatrix} \alpha & \alpha - \epsilon \\ \alpha - \epsilon & \alpha + \epsilon \end{bmatrix} \tag{3.34}$$

where $\alpha \gg \epsilon$ if we take the unitary transformation:

$$U = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.35}$$

applying the transformation to our variance covariance matrix we have:

$$U \cdot C \cdot U^{\mathsf{T}} = \begin{bmatrix} 3 \cdot \epsilon & \epsilon \\ \epsilon & \alpha \end{bmatrix} \tag{3.36}$$

where $3 \cdot \epsilon \ll \alpha$. We have to notice that the unitary transform has also the effect to decorrelate the matrix, as one realises because the correlation coefficient in Eq 3.34 tends to 1 when $\epsilon \to 0$, while it tends to 0 in Eq 3.36, whence the name decorrelation[2]. To produce such unitary matrix

---

[2]In lattice theory the procedure is called "eduction", highlighting its computational purpose [17].

for a general case the LAMBDA decorrelation method has been proposed by [106]. To build a good decorrelating matrix matrix a chain of two operations is used, a decorrelation and a permutation step:

$$R = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.37}$$

where α has to be an integer. Consider a two by two submatrix of a variance covariance matrix:

$$C = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \tag{3.38}$$

If we apply the first operation we have:

$$R \cdot C \cdot R^\intercal = \begin{pmatrix} \sigma_1^2 & \alpha \cdot \sigma_1^2 + \sigma_{12} \\ \alpha \cdot \sigma_1^2 + \sigma_{12} & \alpha^2 \cdot \sigma_2^2 + 2 \cdot \alpha \cdot \sigma_{12} + \sigma_2^2 \end{pmatrix} \tag{3.39}$$

So if we want to minimize the bottom right term (the variance of the second component) we have to take the derivative with respect to α and put it to zero. Doing that and considering it has to be an integer, α becomes:

$$\alpha = \lfloor \frac{\sigma_{12}}{\sigma_1^2} \rceil \tag{3.40}$$

Let's compare it, for completeness, with the reduction step of the LLL algorithm [18] that works on the design matrix A. The reduction parameter there is computed as:

$$\mu_{12} = \left\lfloor \frac{A(\dots;2)^\intercal \cdot A(\dots;1)}{A(\dots;1)^\intercal \cdot A(\dots;1)} \right\rceil, \tag{3.41}$$

and if we go to the normal matrix $N = A^\intercal \cdot A$ we are left with a formula very similar to Eq 3.40, namely:

$$\mu_{12} = \left\lfloor \frac{N_{12}}{N_{11}} \right\rceil \tag{3.42}$$

The difference is that one works on the normal matrix (LLL) and the other on its inverse (LAMBDA) [71]. The decorrelation and permutation step are repeated on the various elements of the variance covariance matrix till no more reduction is possible.

3.3 Algorithms for Rank Deficient Integer Least Square Problems

Having defined the integer null space in Section 2.6, we now need a way to compute one of its basis. First of all we have to make two considerations. The first one is that any real number can be approximated with arbitrary accuracy by a rational number. The second is that any rational number can be transformed into an integer number multiplying it by its denominator. So if we have a matrix A of real entries, it can be transformed into a rational matrix and subsequently into an integer matrix $A_{\mathbb{Z}}$ by multiplying it by the least common multiple of all the single element denominators. The computed integer null space will be insensitive to such scaling, in fact $\alpha \cdot A \cdot X = 0$ for any chosen α if $A \cdot X = 0$. We will now introduce the Hermite Normal Form and then use it to compute the integer null space of a rank deficient system.

Definition 10 [3] *Given a matrix of integers $A \in \mathbb{Z}^{m,n}$, its Hermite normal form is the matrix $H = U \cdot A$, where U is an m by m unimodular matrix, and $H \in \mathbb{Z}^{m,n}$ is*

- *an upper triangular matrix*

- *its first non zero entry of each rows is always positive and to the right of the first non zero entry of the row above.*

- *in the column of each first non zero row entry all elements above the first non zero row entry are strictly smaller than it.*

The hermite normal form H exist and is unique for each $A \in \mathbb{Z}$, see [2] Theorem 2.9 and 2.13 for a proof of this statement. Effective methods to compute the normal Hermite form of a matrix A are presented e.g. in [50]. Let's now consider a rank deficient integer matrix A and compute the Hermite normal form of $A^{\intercal}$. Following [18] we will show how the Hermite normal form can be used to compute a lattice basis for the integer null space of A. Since some columns of A can be presented as linear combinations of the other columns the last rows of the Hermite form H will be zero. Let's now concentrate on the unimodular matrix U. Since we know that the last d rows of H (d being the rank deficiency) are zero

---

[3]The definition of the Hermite Normal form is not always consistent in literature, for some considerations on the matter see [102] Chapter 2.1.

we can write:

$$H(n - d \ldots n; \ldots) = \mathbf{0}^{d,m} = U(n - d \ldots n; \ldots) \cdot A^\intercal = U_r \cdot A^\intercal \quad (3.43)$$

If we transpose the formula we have:

$$A \cdot U_r^\intercal = 0 \quad (3.44)$$

Thus the last d rows are a lattice basis for the null space of A. We have now to show that the basis spans the whole integer null space, i.e. no other integer vector is in the integer null space, outside the lattice generated by $U_r^\intercal$. To show this we first have to remember that the matrix U is unimodular and thus it spans the whole $\mathbb{Z}^n$. So any integer vector not spanned by $U_r^\intercal$ has to be spanned by $U(1 \ldots s; \ldots)^\intercal = U_s^\intercal$ where $s = n - d - 1$. On the other hand if it is spanned by $U_s^\intercal$ it follows that $A \cdot U_s^\intercal \cdot g \neq \mathbf{0}, \forall g \in \mathbb{Z}^n$, and thus does not belong to the null space. From this we can conclude that, since all integers vector not spanned by $U_r^\intercal$ are not in the null space, and U spans the whole $\mathbb{Z}^n$, $U_r^\intercal$ spans the whole integer null space of A.

Once we have the Hermite normal from is it also easy to compute an admissible constraint for the integer least squares. In fact, the inverse of a unimodular matrix is unimodular. Calling V the inverse of U and V_r the last d columns of V and taking the constraint K (Eq 2.63) as $V_r^\intercal$ transposed $(K = V_r^\intercal)$we have:

$$K \cdot L = V_r^\intercal \cdot U_r^\intercal = I \quad (3.45)$$

and since the identity is clearly unimodular we have an admissible constraint.

3.3.1   Relationship Between Hermite Normal Form and Integer Estimability

It is interesting to compare this derived constraint to the concept of integer estimability introduced in [111] (Theorem 1). We prove equivalence on condition that the equation $y = A \cdot x$ contains only the integer vector x comparing with the formulation of the above theorem, this corresponds to assuming $B = 0$, $B^\perp = I_m$. in such a case the unbiased integer estimability condition for a maximal vector $F^\intercal \cdot x$ with dimension s is:

$$F = A^\intercal \cdot M \quad (Condition\ 1) \quad (3.46)$$

for some full rank M which is equivalent to $span(F) = span(A^\mathsf{T}) = \mathcal{N}(A)^\perp$, and:

$$F^\mathsf{T} \cdot Z = [I_s \; 0] \quad (Condition \; 2) \tag{3.47}$$

for some unimodular Z of dimension n.

To do so, we divide the the unimodular matrix resulting from the computation of the Hermite Normal Form U as:

$$U = \begin{bmatrix} U_s \\ U_r \end{bmatrix} = \begin{bmatrix} U(1\ldots s; \ldots) \\ U(n-d\ldots n; \ldots) \end{bmatrix}. \tag{3.48}$$

We also divide its inverse V vertically as:

$$U^{-1} = V = \begin{bmatrix} V_s & V_r \end{bmatrix} = \begin{bmatrix} V(\ldots; 1\ldots s) & V(\ldots; n-d\ldots n) \end{bmatrix}. \tag{3.49}$$

Then we can note that:

$$
\begin{array}{cccc}
& U \cdot V = I & & V \cdot U = I \\
U_s \cdot V_s = I_s & U_r \cdot V_r = I_r & V_s^\mathsf{T} \cdot U_s^\mathsf{T} = I_s & V_r^\mathsf{T} \cdot U_r^\mathsf{T} = I_r \\
U_s \cdot V_r = 0 & U_r \cdot V_s = 0 & V_r^\mathsf{T} \cdot U_s^\mathsf{T} = 0 & V_s^\mathsf{T} \cdot U_r^\mathsf{T} = 0
\end{array}
$$

Now we can change the unknown parameters using the transformation:

$$x = U^\mathsf{T} \cdot z \tag{3.50}$$

from what follows

$$z = V^\mathsf{T} \cdot x. \tag{3.51}$$

Moreover, splitting z into $z^\mathsf{T} = [z_s^\mathsf{T} z_r^\mathsf{T}]$, we can write:

$$z_s = [I_s \; 0] \cdot z = [I_s \; 0] \cdot V^\mathsf{T} \cdot x = [I_s \; 0] \cdot \begin{bmatrix} V_s^\mathsf{T} \cdot x \\ V_r^\mathsf{T} \cdot x \end{bmatrix} = V_s^\mathsf{T} \cdot x. \tag{3.52}$$

We note that $V_s^\mathsf{T} \cdot x$ is integer estimable according to [111]. In fact remembering that the span of U_r^T is $\mathcal{N}(A)$, and that $V_s^\mathsf{T} \cdot U_r^\mathsf{T} = 0$, we that $span(V_s) = \mathcal{N}(A)^\perp$ fulfilling condition 1. Furthermore, noting that:

$$V_s^\mathsf{T} \cdot U^\mathsf{T} = [I_s \; 0]. \tag{3.53}$$

with U unimodular we see that also condition 2 is fulfilled.

Moreover, observing that:

$$z_r = [0 \; I_r] \cdot z = [0 \; I_r] \cdot V^\mathsf{T} \cdot x = [0 \; I_r] \cdot \begin{bmatrix} V_s^\mathsf{T} \cdot x \\ V_r^\mathsf{T} \cdot x \end{bmatrix} = V_r^\mathsf{T} \cdot x, \tag{3.54}$$

we see that the condition $I_r \cdot z_r = z_r = 0$ is identical to the the previously introduced admissible condition:

$$V_r^\intercal \cdot x = 0 \tag{3.55}$$

which results in the estimation of:

$$z_s = V_s^\intercal \cdot x \tag{3.56}$$

that can be later on transformed back to x by:

$$x = U^\intercal \cdot z = U^\intercal \begin{pmatrix} z_s \\ 0 \end{pmatrix} = U_s^\intercal \cdot z_s \tag{3.57}$$

In other terms the condition:

$$V_r^\intercal \cdot x = 0 \tag{3.58}$$

provides the same x estimated by:

$$x = U_s^\intercal \cdot z_s \tag{3.59}$$

and this links the proposed admissible constraint to the integer estimable function $V_s^\intercal$.

3.3.2   Mixed Integer Real Case

To derive such admissible constraint for a mixed integer real rank deficient model we can follow two ways.

First Strategy

The first one is simply to reduce for all non integer parameters using Eq 2.20. Then we are left with an all integer problems for which we can compute the integer null space. However generally speaking the orthogonal projector on the subspace spanned by an integer matrix, is not integer. This latter consideration can make the passage from rationals to integers more complex and thus an other strategy could be desirable.

Second Strategy

To develop the second strategy we can have again a look at the mixed integer null space, more specifically having a matrix J whose columns spans the mixed integer null space we have:

$$0 = C \cdot J = A \cdot X + B \cdot T \tag{3.60}$$

$$A \cdot X = -B \cdot T \tag{3.61}$$

As we said before we know that $\mathcal{N}(A) \perp \mathcal{R}(A^\intercal)$. So, if we have a basis L for $\mathcal{N}(A^\intercal)$ we can write:

$$L^\intercal \cdot A \cdot X + L^\intercal \cdot B \cdot T = 0 + L^\intercal \cdot B \cdot T = 0 \tag{3.62}$$

Since L can be computed in integer form by using the Hermite normal form also, $L^\intercal \cdot B$ will be done of integers. Then we can compute T using again the Hermite normal form. Once we have T, the real part of the mixed integer null space X can be computed by inverting 3.61:

$$X = A^+ \cdot (-B \cdot T) \tag{3.63}$$

We have to highlight that this procedure will produce only the columns of J that correspond to the rank deficiencies between real and integers parameters. The ones that are either between integer parameters alone or between real parameters alone should be analysed separately with the methods discussed before.

4

GNSS Physics

Purus mathematicus, purus asinus

—

4.1 Introduction

This chapter deals with the physical processes and phenomena involved in
GNSS positioning. Its aim is not to provide and in-depth description or
understanding of the systems, rather to give a good enough introduction
in order to understand the quantities that we can model and the ones we
have to estimate.

4.2 Satellites and Signals

A GNSS system is composed mainly by two basic elements, satellites
transmitting ranging signals and receiver tracking them. Since the num-
ber of receiver types is countless we will restrict to describing the satellites.
Regarding the receivers it is enough to say that different kind of them are
able to track different satellites and signals and we will refer ideally to
a receiver capable of tracking any combination of them. A lot of differ-
ent GNSS are operative and transmitting ranging signal. They belong to
different families:

- Global Positioning System (GPS), first developed system of Ameri-
  can origin.

- GLObalnaya NAvigatsionnaya Sputnikovaya Sistema (GLONASS), its Russian counterpart.

- Galileo, the European one.

- Běidǒu Wèixīng Dǎoháng Xìtǒng, commonly known as BeiDou, the Chinese one.

- The Quasi-Zenith Satellite System (QZSS), a satellite based augmentation system that enhances and expands GPS over the Japanese and Pacific area.

- The Indian Regional Navigation Satellite System (IRNSS), an Indian regional navigation system.

- Several Satellite Augmentation Systems that also trasmit ranging Singals such as Wide Area Augmentation System (WAAS), European Geostationary Navigation Overlay Service (EGNOS), Multifunctional Satellite Augmentation System (MSAS), GPS-aided GEO augmented navigation (GAGAN) and System for Differential Correction and Monitoring, SDCM.

The working principle of these systems is the same for all of them so from a data analysis point of view they can be all treated in a similar way. Their difference lays in their orbit, signal plan, and specific systematic biases in their observation. Tab 4.1 presents a brief summary of the different GNSS orbital parameters. Each GNSS system transmits multiple ranging signals on different frequencies. To distinguish between different satellites signals from the same band most GNSS use a Code Division Multiple Access (CDMA) scheme (Sometimes with in combination with time multiplexing [51]) while GLONASS uses a Frequency Division Multiple Access (FDMA) scheme for its principal ranging signals. Most signals are either in the 1164-1350 MHz band or in the 1559-1610 MHz band, allocated by the International Telecommunication Union (ITU) for satellite radionavigation ([116] Chapter II), IRNSS transmits also a signal in the 2483.5-2500 MHz band. Tab 4.2 presents a list of the ranging signal and carrier frequency for all GNSS. Different ranging codes can be tracked in combinations, furthermore especially for encrypted signals, codeless and semicodeless tracking strategy are possible [119]. A framework to catalogue the observables resulting from different tracking techniques can be found in the definition of the Receiver INdependent EXchange (RINEX)

	N of satelites	Orbit type	Semimajor axis	Inclination	Orbital planes
GPS	33	MEO	26560	55	6 spaced by 60°
GLONASS	24	MEO	25440	64	3 spaced by 120°
Galileo	30	MEO	29600	56	3 spaced by 120°
BeiDou	27	MEO	27878	55	3 spaced by 120°
	5	GEO	42164	0	1
	3	GSO	42164	55	3
QZSS	3	GSO	42164	43	3
	1	GEO	42164	0	1
IRNSS	3	GEO	42164	5	
	4	GSO	42164	29	2

Table 4.1: Orbital parameters summary for different GNSS. The following acronyms are used: Medium Earth Orbit (MEO), GeoSynchronous Orbit (GSO), Geosynchronous Equatorial Orbit (GEO).

format ([86] Subsection 5.2.17) . From a data processing point of view all signals are processed in the same mode, the only variable part being the choice of the parameters in the observation equations and the weights of the observables.

4.3 GNSS Observables

GNSS receivers provide four types of observables:

- Phase measurement

- Pseudorange measurement

- Doppler measurement

- Signal to Noise Ratio (SNR) or Carrier to Noise Density (C/N_0)

Phase and pseudorange measurements are derived by correlating the various GNSS signal with an internal replica in the receiver. As For the pseudo range its value represents the travel time of the signal plus the desychronization between the satellite and the receiver clock. The noise of pseudorange measurement depends on the ranging signal and on the tracking algorithm implemented in the receiver but it is of the order of a

	BAND	RINEX 3.05 denomination	Frequency	Ranging code
GPS	L1	1	1574.42	C/A, L1C (Pilot and Data), P(Y), M
	L2	2	1227.60	L2 CM,L2 CL, P(Y), M
	L5	5	1176.45	L5 I, L5 Q
GLONASS	G1	1	1598.0625-16053.75	C/A,P
	G1a	4	1600.25	L1OCd, L1OCp
	G2	2	1242.9375-12486.25	C/A,P
	G2a	6	1248.06	L2OCp, L2CSI
	G3	3	1202.25	L3OCd, L3OCp
Galileo	E1	1	1575.42	E1 OS (Data and Pilot), PRS
	E6	6	1278.75	E6 CS data,E6 CS pilot, E& PRS
	E5	8	1191.795	
	E5a	5	1165.45	E5a data,E5a pilot
	E5b	7	1207.14	E5b data,E5b pilot
QZSS	L1	1	1574.42	C/A,L1C (Pilot and Data), P(Y), SAIF
	L2	2	1227.60	L2 CM, L2 CL
	L5	5	1176.45	L5 I, L5 Q
	L6	6	1278.75	LEX (Pilot and Data)
BeiDou	B1 (Old)	2	1561.098	B1I, B1Q
	B1 (New)	1	1575.42	B1C_data, B1C_pilot_a, B1C_pilot_b
	B2 (Old) / B2b	7	1207.14	B2I, B2Q
	B2a	5	1176.45	B2a (Data and Pilot)
	B2a+B2b	8	1191.795	
	B3	6	1268.52	B3I, B3Q
IRNSS	L5	5	1176.45	SPS, RS (Data and Pilot)
	S	9	2492.028	SPS, RS (Data and Pilot)
SBAS	L1	1	1574.42	C/A
	L5	5	1176.45	L5 I, L5 Q

Table 4.2: Summary table for GNSS frequencies and signals.

few decimetres to a few meters. For the phase measurement since the signal repeats after each cycle, the travel time is known with the ambiguity of an integer number of cycles. Once the first measurement is generated the Phase Lock Loop (PLL) in the receiver is able to keep track of the number of cycles from the first acquisition. The noise of the phase measurement is less variable that the pseudorange one, and it is on the order of $\frac{1}{100}$ of the wavelength. The Doppler measurement represent the derivative of the phase measuremente and thus the derivative of the travel time and clocks desyncrhonization. For a static receiver it is not very informative since the same information can be provided deriving the phase, usually with a better accuracy. The SNR and C/N_0 provides information about the strength of the received signal and can be used in the context of positioning to adjust the weights of the observables. A common format to save and share the collection of such observables is the aforementioned (RINEX) [86] format. Between these observations the pseudorange (ρ) and the phase (ϕ, in unit of length) are the ones typically used for static positioning. Their value can be expressed as:

$$\rho = \varrho + \tau^r + \tau^s + \beta^r + \beta^s + \delta_\zeta + \delta_\iota + \epsilon_\rho \tag{4.1}$$

$$\phi = \varrho + \tau^r + \tau^s + \beta^r + \beta^s + \delta_\zeta - \delta_\iota + \lambda \cdot \nu + \epsilon_\phi \tag{4.2}$$

where ϱ is the distance between the receiver and the satellite, τ^r, τ^s are the receiver and satellite clock desynchronizations in unit of length, $\beta^r + \beta^s$ are instrumental biases specific to the type of observable in unit of length, δ_ζ is the tropospheric delay in unit of length, δ_ι is the ionospheric delay in unit of length, λ and ν are the wavelength of the phase and the unknown number of cycles respectively and $\epsilon_\rho, \epsilon_\phi$ are the pseudorange and phase errors. In the following sections a way to correct or model these parameters is going to be described.

4.4 Satellite and Receivers Parameters

In these section the aspects of GNSS satellites and receivers that are more important for a GNSS data compensation, are described.

4.4.1 Satellite an Receiver Clocks

For a GNSS system to work it is fundamental to achieve the synchronisation of satellite and receiver clocks. This in order to relate the travel

time of the signal to distances. The word "clock error" is typically used to speak of the desynchronization of receivers and satellites with respect to a reference network time. In this document we will use the word desynchronization instead, highlighting the fact that the clocks do not need in principle to be correct with respect to a global time reference system but just to be synchronised one with the others [1]. The sysnchronization is achieved in two main ways. The first is trough the use of very stable frequency standards on both satellites and control stations. This guarantees that the satellite signals stay synchronised in time. Of course this synchronization is limited by the frequency stability of the satellite clocks. The other way to achieve synchronisation is to estimate such synchronisation as an unknown in the system. This is what is typically done for commercial receivers (carrying low cost unstable oscillators) and also for satellite clocks, in a network processing to estimate satellite clock drifts. International GNSS service (IGS) analysis centers routinely provide clock desynchronzation for all GNSS [69].

4.4.2   Signal Biases

Different signals are typically not emitted nor tracked synchronously. This effect has to be taken into account when processing GNSS signals. Normally such biases are slowly varying in time [91], however sub daily centimeters to decimetres variations have been observed in GPS Block II-F satellites [76] and Beidou satellites [75]. Similarly temperature dependent frequency dependent biases have been observed in the receiver equipment [85][124]. To avoid biases in the estimates.They have to be modeled as unknowns in the system of observation equations.

4.4.3   Satellite Orbits

Satellite orbits are another important factor in GNSS positioning. The various GNSS broadcast ephemerides parameters, with meter like accuracy for operational use in their navigational message (Galileo should start to broadcast an new service with decimetre accuracy in the near future). IGS and commercial enterprises provide centimetre quality orbit estimated in post processing [69]. Furthermore realtime services provide orbit with centimetre sub-decimeter quality in realtime [63]. Finally MEO

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[1]The Locata positioning system [8] makes an interesting use of this concept

GNSS orbit can be predicted with accuracy of some decimetres for a one day span [100].

4.4.4 Antenna and Multipath Effects

Antennas causes delays that are dependent on the elevation and azimuth of the exiting incoming signals. This effect has been observed for both receiver [97][39] and satellite [88] antennas. These effects are frequencies dependent and are normally modelled as a sum of a translation of the antenna center (the so called Phase Center Offset PCO) and a map of elevation and azimuth dependent terms (so called Phase Center Variations PCV). It has to be notices that the PCV could absorb also the PCO terms; the separation is kept in order to be able to model a consistent part of the signal with one translation only. Models for such delays can be computed either on the field using a particular antenna model as reference [88], or in the filed again rotating one of them in order to get absolute variations of the rotated antenna [121], or using simulated signal in an anechoic chamber [95]. The IGS provides antennas calibrations for most satellites and receivers. Another phenomenon that causes effects and very similar errors in the GNSS data is the multipath effect. In fact reflection and refraction cause advancements or delays in the correlator output of the receiver. Since the environment near the antenna is normally not changing on short times, such an effect can be estimated using observation from several days and modelled in a similar fashion as PCO and PCV [60].

4.5 Geophysical Effects on GNSS Observables and Unknowns

Several geophysical effects, affect the measurable quantities in a GNSS system. To be able to derive a correct information about our quantity of interest we have to be able either to correct these effects or to estimate them.

4.5.1 Atmosphere

The speed of a radiowave passing trough the atmosphere is not constant. It is influenced both by neutral and ionized gases. The path of the radiowave passing trough such a medium will not follow a straight line;

instead, due to the Fermat Principle, it follows the path with the least
travel time. Once the path is determined the total time of travel can be
computed integrating the velocity:

$$\tau = \frac{1}{c} \cdot \int_{\gamma} N(l) dl \tag{4.3}$$

where c is the speed of light, γ is the wave path, l is the integration variable
along the path and $N(l)$ is the refractive index of the atmosphere along
the path. If the difference between the least time path γ and the straight
unperturbed path μ is a small fraction of the total length of the path, the
same difference in time of flight can be written as:

$$\delta\tau \approx \frac{1}{c} \cdot (\int_{\mu} (N(l) - 1) dl + \cdot \int_{\gamma} dl - \int_{\mu} dl) \tag{4.4}$$

In which the first term represents the integral of the refractive index
minus one integrated along the straight line and the second and third
terms represent the difference between the real and the straight path.
This difference is often called "bending term", β, we will use the same
name in the following sections. Since the bending term is normally in the
order of few centimetres, this simplification is legitimate (comparing it to
the thickness of the atmosphere). We will now divide the delay into two
different components, the one caused by the neutral atmosphere and the
one caused by the ionized atmosphere.

Ionosphere

The atmospheric upper layers (above 80 Km) are ionized this cause the
refractive index to be significantly different from zero. The refractive in-
dex of an electromagnetic wave passing trough the ionosphere depends in
general on the frequency of the wave (for such reason is called dispersive),
the propagation direction of the wave and the local magnetic field vector.
The refractive index can be computed using Appleton-Hartree formula
([4] as cited in [62]). A simplified formula has been derived for the iono-
spheric induced delay (in meter) on a phase observation [9][62] (bending

term excluded):

$$\delta_\iota = -40.3 \frac{\int_\mu E(l) \; dl}{\varphi^2} - 1.1283 \cdot 10^{12} \frac{\int_\mu E(l) \cdot B(l) \cdot cos(\Theta(l)) \; dl}{\varphi^3}$$
$$- 812.37 \frac{\int_\mu E(l)^2 \; dl}{\varphi^4} - 1.579 \cdot 10^{22} \frac{\int_\mu E(l) \cdot B(l) \cdot (1 + cos(\Theta(l)))dl}{\varphi^4}$$

$$(4.5)$$

where φ is the signal frequency $E(l)$ is the electron density component along the path $B(l)$ is the strength of the magnetic vector along the path and $\Theta(l)$ is the angle between the magnetic vector and the unit vector of the incoming signal. The quantity $\int_\mu E(l) \; dl$ is often called Total Electron Component (TEC). Sometimes the quantity is transformed to the so called Vertical Total Electron Component (VTEC) diving it by a mapping function. For both quantity the TEC Unit (TECU) is tipically used. One TECU corresponds to $10^{16} \; electrons/m^2$. The term that depends on $1/\varphi^2$ is called first order term, is constitutes around the 99% of the delay and they are typically estimated if two or more frequencies are present for the same satellite. The term that depends on $1/\varphi^3$ is called second order term, those depending on $1/\varphi^4$ third order terms. The second and third terms are commonly referred as higher order effects and they are typically mitigated by using the expression in Eq 4.5. To do so the TEC is estimated from the observations or taken from a model. The magnetic field vector and its angle with respect to the incoming wave are typically taken constant and their value is computed at the height of approximately 350 Km, where the ionosphere normally has is peak electron density. The $\int_\mu E(l)^2 \; dl$ term is either approximated by $2.9148 \cdot 10^{-6} \cdot TEC^2$ [36] or computed more rigorously raytracing a three dimensional ionospheric model [62]. For the computation of the bending term one can again either use a raytracing approach or use the approximate formula [79][46] :

$$\beta = \frac{40.3}{4 \cdot \varphi^4} \cdot tan(\zeta)^2 \cdot 2.9148 \cdot 10^{-6} \cdot TEC^2, \qquad (4.6)$$

where ζ is the zenith angle of the satellite computed at the height of the assumed thin shell. It is important to say that while the bending term is the same for both phase and pseudoranges the others terms are not. For the pseudorange normally only the first order term is considered, it reads:

$$\delta_\iota = +40.3 \frac{\int_\mu E(l) \; dl}{\varphi^2} \qquad (4.7)$$

Quantity	Order of magnitude
VTEC	0-100 TECu
Thin shell MF at 5 degree	~ 3
Thin shell MF at 10 degree	~ 2.8
Thin shell MF at 30 degree	~ 1.7
1st order term L1/E1/B2	$\sim 0.16 \frac{m}{TECu}$
1st order term L5/E5	$\sim 0.29 \frac{m}{TECu}$
Higher order and bending terms	0.01 cm

Table 4.3: Orders of magnitude for ionospheric parameters. The thin shell Mapping Function (MF) represent a rough amplification coefficient (It is assumed that all electrons are concentrated in a thin shell at 350 Km of altitude) for the inclined satellites.

which is exactly the phase first term multiplied by -1, this is because of the group velocity (pseudorange measurement) is different from the phase velocity (phase measurement)[55]. Some orders of magnitudes for ionospheric parameters are presented in Tab 4.3.

Troposphere

There is also refraction term that is caused by the neutral part of the atmosphere; since most of the atmospheric mass is contained into the troposphere this term is normally labeled tropospheric delay. The coefficient of refraction of the neutral atmosphere does not depend on the frequency (in the GNSS frequency range), but in general it depends on the partial pressure of all gases elements in the air mix. However, for simplicity, it is normally approximated as a function of total pressure, temperature, partial water vapour pressure and partial carbon dioxide pressure [89]; the term due to carbon dioxide is usually omitted. Another characteristic of the air refraction is that in the area of the signal incoming into a GNSS station (around 100 Km horizontally) it can be approximated by a stratification of homogeneous horizontal layers. This allows the tropospheric delay to be written as a function of the elevation θ :

$$\delta(\theta) = \delta(\frac{\pi}{2}) \cdot T(\theta) \tag{4.8}$$

where $T(\theta)$ is the so called mapping function, and $\delta(\frac{\pi}{2})$ is the tropospheric delay at the zenith normally called Zenith Total Delay (ZTD) or Zenith Tropospheric Delay (ZTD), in the rest of the document we will refer to it using the symbol ζ. The delay is normally measured using meters and

so we will refer to it in this document as having unit of length. For a flat earth with no bending the mapping function would be $\frac{1}{sin(\theta)}$; since this is not the case the mapping function is typically expressed with the continued fraction form [73] normally using three parameters (a, b, c) [52]:

$$T(\theta) = \frac{1 + a/(1 + b/(1 + c))}{sin(\theta) + \frac{a}{sin(\theta) + \frac{b}{sin(\theta) + c}}} \quad (4.9)$$

Furthermore in order to partly account for anysotropy that is present in the layers, also terms that depends on the sine and cosine of the azimuth α are added: [38][7]:

$$\delta(\theta) = \zeta \cdot T(\theta) + \Gamma(\theta)cos(\alpha) \cdot \xi + \Gamma(\theta)sin(\alpha) \cdot \eta \quad (4.10)$$

where ξ and η are normally called north and east tropospheric gradient components, as they can be interpreted as a tilting of the tropospheric layers in north and east direction. The function Γ is normally called the gradient mapping function and can be written as [21] $\frac{1}{sin(\theta)tan(\theta)+C}$. Furthermore, the tropospheric delay is normally divided into the hydrostatic and wet components[2], i.e.[52]:

$$\delta(\theta) = \zeta_h \cdot T_h(\theta) + \Gamma_h(\theta)cos(\alpha) \cdot \xi_h + \Gamma_h(\theta)sin(\alpha) \cdot \eta_h + $$
$$\zeta_w \cdot T_w(\theta) + \Gamma_w(\theta)cos(\alpha) \cdot \xi_w + \Gamma_w(\theta)sin(\alpha) \cdot \eta_w \quad (4.11)$$

each one with different mapping functions and gradient mapping functions. Models for both mapping functions, zenith delays and gradients are available. Some of such models use global spatio-temporal parameters that are estimated once and then kept fixed [78][16][70], some other use local meteorological data [90][25].Finally other models use global coefficients that are periodically updated [68] [70] (typically fitting them to ray traced delays from global weather numerical prediction models). Beside that, a residual wet part is estimated from observations; this because the wet correction that can be modelled with less accuracy. Since mapping function for hydrostatic and wet delays are very similar, any mismodeling in the hydrostatic part will be absorbed easily in the wet estimates. It is worth mentioning that mapping functions that are satellite specific exists, that are not function of elevation and azimuth exist [126]. Yet they are not as common in practice. Tab 4.4 reports some order of magnitude for such effects. Other localised tropospheric phenomena (hydrometeor ,

---

[2]It has to be noted that in theory this would not be legitimate since the problem is non linear and thus the superposition of the effects is not valid. However, the deviations from linearity are small and so this is a good approximation.

Quantity	Order of magnitude
Hydrostatic ZTD at sea level	~ 2.2 m
Wet ZTD at sea level	~ 0.5 m
ZTD gradients (wet and hydrostatic summed)	~ 1 mm
Mapping function at 30 degree elevation	~ 2
Mapping function at 10 degree elevation	~ 5
Mapping function at 5 degree elevation	~ 10
Gradient mapping function at 30 degree elevation	~ 10
Gradient mapping function at 10 degree elevation	~ 90
Gradient mapping function at 5 degree elevation	~ 200

Table 4.4: Orders of magnitude for tropospheric parameters.

particulates, etc..) do affect the GNSS signals; [99] presents an overview of these effects on the observations. Such effects are commonly simply ignored.

4.5.2   Solid Earth

Receivers that are normally considered static at a local scale are not so at global one. If we want to consider them as static in the processing, we have to account for a series of geophysical phenomena that cause them to move. Once we have a model for the point movement Δx, for each pseudorange/phase ω observation we can remove this effect by projecting it along the Line Of Sight (LOS) unit vector l. The "corrected" observation ω' then will the become:

$$\omega' = \omega - l^{\intercal} \cdot \Delta x \qquad (4.12)$$

The choice about the various physical effects to be modelled depends on the signal of interest. Generally speaking it is possible to divide the movements into two broad categories: solid earth tides and loading phenomena. The solid earth tides are movements due to a time variation in the earth gravitational potential that, combined with the elasticity of the earth, generates "stretching" and deformation of the planet. Loading phenomena on the other hand, are deformations of the earth crust do to the pressure exerted by several types of mass changes; more specifically water due to sea tides, water due to hydrological cycle and atmosphere density changes due to tides and large scale weather phenomena. When applying corrections for such effects one has to check the consistency of

reference frames [15]; this is especially true if one is using epeherides or clocks from an external provider.

Tides

The displacement of a points is normally computed starting from the tidal potential. Thhis is linked to the actual displacements by the so called Love h and Shida l numbers relating the tidal displacement to vertical (υ) and horizontal (ϵ η) displacement [61].

$$\begin{bmatrix} \Delta\upsilon \\ \Delta\epsilon \\ \Delta\eta \end{bmatrix} = \begin{bmatrix} \frac{h}{|g|} \cdot \Psi \\ \frac{l}{|g|} \cdot \frac{a}{r \cdot cos(\phi)} \frac{\Psi}{\partial\lambda} \\ \frac{l}{|g|} \cdot \frac{a}{r} \frac{\Psi}{\partial\phi} \end{bmatrix} \tag{4.13}$$

where ϕ is the latitude λ the longitude, r the distance form earth centre,a is the mean earth radius, Ψ the tidal potential and $|g|$ the mean gravity force. The sun and moon contributions to the tidal potential are typically expressed as a sum of Legendre polynomials function of the angle between the celestial body and the receiver in earth fixed frame. The sum is usually stopped at degree three. Love and Shida numbers vary depending on the degree of the summation, on the latitude and on the tidal frequency (this last applies to the computation done in frequency domain). A reference procedure for the computations of such displacements can be found in [81] Subsection 7.1.1. The change in the instantaneous axis of rotation of the earth also creates a tidal potential; its effect can be approximated as:

$$\Psi(r, \lambda) = -\frac{\omega^2, r^2}{2} \cdot (\xi \cdot cos(\lambda) + \nu \cdot sin(\lambda)) \tag{4.14}$$

where ξ and ν are the cartesian coordinates of the instantaneous axis of rotation with respect to the mean pole.

Loading Effects

Other displacements of the earth surface are caused by mass loading. This loading causes the earth crust to deform. Its effect is normally computed by integrating all loading masses around the point of interest, using a specific distance dependent "transfer" (from loading mass to displacement) function[34]. The most important loading factors are the oceans. Most variation of the ocean mass is due to the ocean tides. The loading effect

Quantity	Order of magnitude
Solid earth tides	~ 0.5 m
pole variation induced tides	~ 0.01 m
Atmospheric loading	~ 0.02 m
Ocean loading	~ 0.2 m
Hydrological loading	~ 0.02 cm

Table 4.5: Orders of magnitude for solid earth effects.

of ocean tides on the coordinates of a station is typically described by a sum of harmonics:

$$\Delta r = \sum_j \alpha_j cos(\chi_j(t) - \phi_j) \qquad (4.15)$$

where α_j is the amplitude of the harmonics, ϕ_j is the phase and $\chi_j(t)$ is the astronomical argument. Online services such as [94] provide site specific parameters that allows the computation of amplitude and phase at receiver sites for the 11 main tide components [81]. Other components can be interpolated in the frequency range from the main 11 mentioned. Another source of loading is the atmosphere. To compute the loading, data from a numerical weather prediction model are used [118]. Typically the loading is divided into a tidal part (loading caused by the tides of the atmosphere) and a non tidal part. Different online services are available ([118] Section 2.1.4) they provide coefficients for the tidal part of the loading and world wide gridded maps of displacements for the non tidal part. Another source of loading is the water present in the rivers, lakes, glaciers and the soil; online services computing such loading starting from hydrological global models exist [30]. Tab 4.5 presents some order of magnitude for solid earths effects.

4.5.3 Relativity

All GNSS computations are normally done in Galilean physics. However several relativistic effects are to be taken into account to assure that the Galilean approximation does not introduce too large errors in the measurements. The biggest effects due to the behaviour of clocks in space. The constant term of time dilation due to satellite speed and gravitational potential are accounted for adjusting the frequencies of the emitted signal. However there are time varing effects that have to be taken into account. These effects could be absorbed into the computation of the clock satellite

desynchronisation and for some of them this choice is made. The first one is the time dilation $\delta\tau$ due to the variation of the speed along the orbit. Its can be computed as [67]:

$$\delta\tau = -2 \cdot \frac{v \cdot s}{c^2} \tag{4.16}$$

where s is satellite position vector, v is satellite velocity and c is the speed of light. This correction is normally not absorbed by the satellite clock estimates so it has to be applied in the stand alone positioning. There is also a non negligible effect (few centimetres) due to the oblateness of the earth gravity field, however this effect is normally absorbed in the satellite clock estimates. Another important term for precise positioning is the Shapiro signal propagation delay. This effect is due to the time dilation caused by the earth along the ray path propagation. As it is path dependent, it can not be absorbed in any of the clock desynchronization estimates. Its expression in a spherical earth approximation is ([5] Eq 96):

$$\delta\rho = \frac{2 \cdot \mu}{c^2} \cdot \ln\left(\frac{|r| + |s| + |s - r|}{|r| + |s| + |s - r|}\right) \tag{4.17}$$

where $\delta\rho$ is the Shapiro delay, r and s and the receiver and satellite vectors respectively μ is the earth gravitational constant. Another correction terms arise from the fact the GPS signals are mostly circularly polarized. This in combination with the relative rotation of the satellites GNSS antennas with respect to that of the receivers ones create a phase advancement/delay in the measurement [120]. This effect is commonly know as phase wrap-up or phase wind-up effect. It can be computed as a function of receiver and satellites attitude [48].

5

Least Squares Adjustment Theory Applied to GNSS

With three parameters, I can fit an elephant.
– William Thomson, 1st Baron Kelvin

It this chapter the GNSS stochastic model for a system of GNSS equation is presented. Selected examples are then showed to characterise the behaviour of the system in different conditions.

5.1   Observation Equation and Resulting System

The linearised observation equation (Eq 4.2) for the carrier-phase pseudodistance in unit of length of receiver r , satellite s, tracking channel c, with frequency f at epoch t reads:

$$\phi_{r,s,c,t} = -\tilde{l}_{r,s,c,t}^{\mathsf{T}} \cdot x_t^r + \tau_t^r - \mu_f \cdot \iota_{r,s,f,t} + T(\theta_{r,s}) \cdot \zeta_t + \Gamma(\theta_{r,s})cos(\alpha) \cdot \xi_t + \Gamma(\theta_{r,s})sin(\alpha) \cdot \eta_t$$
$$+\lambda_f \cdot \nu + \beta_{c,t}^r + l_{r,s,c,t} \cdot x_t^s + \tau_t^s + \beta_{f,t}^s$$
$$(5.1)$$

Similarly the linearised observation equation (Eq 4.1) for the pseudo-range measurement reads:

$$\rho_{r,s,c,t} = -\tilde{l}_{r,s,c,t}^{\mathsf{T}} \cdot x_t^r + \tau_t^r - \mu_f \cdot \iota_{r,s,f,t} + T(\theta_{r,s}) \cdot \zeta_t + G(\theta_{r,s})cos(\alpha) \cdot \xi_t + G(\theta_{r,s})sin(\alpha) \cdot \eta_t$$
$$+\beta_{c,t}^r + l_{r,s,c,t} \cdot x_t^s + \tau_t^s + \beta_{c,t}^s$$
$$(5.2)$$

Table 5.1: Unknowns. Abbreviations used: r number of receivers, s number of satellites, e number of epochs, f number of frequencies, and c number of channels per frequencies.

Parameter	Description	Number of unknowns
x^r	Coordinates of the receiver	$r \cdot 3$ (static) , $r \cdot 3 \cdot e$ (dynamic)
x^s	Coordinates of the satellites	$s \cdot 3 \cdot e$ (kinematic orbit)
τ^r	Receiver desynchronization	$r \cdot e$
τ^s	Satellite desynchronization	$s \cdot e$
ι	Slant total electron component	$r \cdot s \cdot e$
ζ	Zenith troposphere delay	$r \cdot e$
η	North gradient	$r \cdot e$
ξ	East gradient	$r \cdot e$
ν	Unknown number of cycles (Ambiguity)	$r \cdot s \cdot f \cdot c$ (without cycle slips)
β^r	Receiver electronic bias	$r \cdot f \cdot c$ (constant), $r \cdot f \cdot c \cdot e$ (time varying)
β^s	Satellite electronic bias	$s \cdot f \cdot c$ (constant), $s \cdot f \cdot c \cdot e$ (time varying)

Table 5.2: Element of the design matrix

Parameter	Description
$\tilde{l}$	Approximate unit vector from receiver to satellite
μ	Ionospheric delay coefficient $\frac{40.3}{f^2}$
T	Tropospheric mapping function
Γ	Gradient tropospheric mapping function
λ	Wavelength
θ	Satellite elevation
α	Satellite azimuth

Abbreviation	Full formula
Γ	$\Gamma(\theta)$
υ	$cos(\alpha) \cdot G(\theta)$
ψ	$sin(\alpha) \cdot G(\theta)$

The symbols used in Eq 5.1 Eq 5.2 are explained in Table 5.1 and 5.2. When speaking about "clock errors", "clock desynchronization", "electronic bias" we have to realise that while they may have different physical origin their effect on the observation is the same: a desynchronization of the measurement. To discern the different effects we will use the expressions "clock desynchronization" and "electronic biases", all desynchronization are in unit of lenght is the previous equation.

5.2 Processing Mode

Different processing modes are possible in the case of GNSS data. A very broad division could be the processing of stand alone receivers versus the processing of a network of receivers [1]. Processing of a network of receivers normally requires few auxiliary data in input since most of the parameters are going to be estimated in the network itself. Stand alone receivers processing, on the other hand, requires more data as input. Stand alone processing at a global scale using phase data precise ephemerides and most refined geophysical/physical corrections is normally called as Precise Point Positioning (PPP) [125]. Corrections to perform precise point positioning are normally routinely provided (for free) by the IGS according to a protocol of international collaboration between several analysis centres [69][77]. Looking more carefully at these corrections we can see that they are the same unknowns estimated in a network processing. So the same procedure can be applied also to non global networks, these procedure is called normally State Space Representation (SSR) [122][112], and in case of kinematics positioning Precise Point Positioning Real Time Kinematics (PPP-RTK).

5.3 Model Rank Deficiency

Several rank deficiencies are present in a system constructed from such observations [80]; we list them individually highlighting the relative null spaces. We say that a group of unknowns is "in rank deficiency" when the corresponding columns in the design matrix are linearly dependent. By applying this concept we can examine separate groups of variables,identifying the specific deficiencies that are present due to the form of observation

---

[1]Processing of stand alone GNSS satellites is also done, but very seldom and for specific application. The technique is commonly referred as reverse PPP [31]

equations.

5.3.1   Clock Desynchronization - Electronic Bias

For both transmitter and receiver electronic biases the columns of the
design matrices referring to the clock unknowns A_τ are in rank deficiency
with the columns referring to the electronic bias unknowns A_β . In fact:

$$A_\tau \cdot e - A_\beta \cdot e = 0 \tag{5.3}$$

Where e is a column matrix with elements equal to one, and 0 is a ma-
trix made of zeros and n is the number of observations. The number of
such deficiencies is one per satellite and one per receiver, included in the
adjustment.

5.3.2   Cycle Ambiguity - Phase Electronic Bias

Similarly, for both the receiver and the satellite, the ambiguity term is in
rank deficiency with the phase electronic bias. Calling A_ν the columns
referring to the ambiguities and $\lambda_1 \ldots \lambda_f$ wavelength of frequency 1 to f,
we can see how:

$$A_\beta \cdot e - A_\nu \cdot \begin{pmatrix} 1/\lambda_1 \\ \vdots \\ 1/\lambda_1 \\ \vdots \\ 1/\lambda_f \\ \vdots \\ 1/\lambda_f \end{pmatrix} = 0 \tag{5.4}$$

For each continuous set of phase data (no Cycle Slip no loss of tracking)
the number of such rank deficiencies is one per phase bias per satellite
and one per phase bias per receiver. In case we have a non repaired cycle
slip on all signals tracked by a receiver or a satellite a new rank deficiency
appears.

5.3.3 Ionosphere - Electronic Bias

Similarly we have a rank deficiency between ionospheric term and electronic bias for both the receiver and the satellite. In fact:

$$A_\beta \cdot \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \vdots \\ \mu_f \\ \vdots \\ \mu_f \end{pmatrix} - A_\iota \cdot e = 0, \tag{5.5}$$

where $\mu_1 \dots \mu_f$ are the entries for the ionospheric unknowns for observations of frequency 1 to f, and A_ι are the columns of the design matrix corresponding to the ionospheric unknown. The satellite related number of such rank deficiencies is one per satellite. Regarding the receiver for each one of them a rank deficiency is present for all the set of electronic biases that share the same ionospheric unknowns. Typically these corresponds to the different constellations, but in the unfortunate case one receiver would track different frequencies for different subsets of satellites of the same constellation there would be a rank deficiency per set of satellites. This time the number of the rank deficiencies is one per receiver and one per satellite.

5.3.4 Cycle Ambiguity - Clock Desynchronization

This term is only present in case a phase only adjustment is performed, in fact it is easy to see that:

$$A_\tau \cdot e - A_\nu \cdot \begin{pmatrix} 1/\lambda_1 \\ \vdots \\ 1/\lambda_1 \\ \vdots \\ 1/\lambda_f \\ \vdots \\ 1/\lambda_f \end{pmatrix} = 0 \tag{5.6}$$

The number of these rank deficiency is one per receiver and one per satellite. It is clear if ones introduced the pseudo-range measurements, since

they have no entry in the ambiguity term the rank deficiency disappear. However since most of the times a bias for the phase observation is added we are still left with a rank deficiency of the type described in Subsection 5.3.2.

5.3.5 Cycle Ambiguity - Ionosphere

This term is only present in case a phase only adjustment is performed, in fact it is easy to see that:

$$
A_\nu \cdot \begin{pmatrix} 1/\lambda_1 \cdot \mu_1 \\ \vdots \\ 1/\lambda_1 \cdot \mu_1 \\ \vdots \\ 1/\lambda_f \cdot \mu_f \\ \vdots \\ 1/\lambda_f \cdot \mu_f \end{pmatrix} - \underset{n \times m2}{A_\iota} \cdot e = 0 \tag{5.7}
$$

The number of this rank deficiencies is one per receiver and one per satellite.

5.3.6 Receiver Clock Desynchronization - Satellite Clock Desynchronization

Another rank deficiency exists between clock desynchronization of the receiver and clock desynchronization of the satellite. In Fact:

$$
A_{\tau^r} \cdot e - A_{\tau^s} \cdot e = 0 \tag{5.8}
$$

If all satellites are connected to all receivers the number of such rank deficiency is just one.

5.3.7 Receiver Electronics Bias - Satellite Electronic Bias

Another rank deficiency exist between the electronic bias of the receivers and those of the satellites. In Fact:

$$
A_{\beta^r} \cdot e - A_{\beta^s} \cdot e = 0 \tag{5.9}
$$

If all satellites are connected to all receivers the number of such rank deficiency is κ, i.e. the combined number of pseudoaranges and phase trackings.

5.3.8 Receiver Electronics Bias - Satellite Clock Desynchronization

Another rank deficiency exist between the electronic bias of the receiver and the clock desynchronisation of the satellite. In Fact:

$$A_{\beta^r} \cdot e - A_{\tau^s} \cdot e = 0 \tag{5.10}$$

If all satellites are connected to all receivers the number of such rank deficiencies is just one.

5.3.9 Satellite Electronics Bias - Receiver Clock Desynchronization

Another rank deficiency exists between the electronic bias of the satellites and the clock desynchronisation of the receivers. In fact:

$$A_{\beta^s} \cdot e - A_{\tau^r} \cdot e = 0 \tag{5.11}$$

If all satellites are connected to all receivers the number of such rank deficiency is one.

5.3.10 Summary Graph

To summarise all the rank deficiencies they have been put on a graph. Each node is a set of unknowns. The rank deficiencies are represented by using an arrow . The arrow is dashed in case the rank deficiency is present for phase only adjustment. On top of each arrow the dimension of the rank deficiency is reported. Same abbreviations are used as in Tab 6.1, moreover pb is used for the number of phase biases and x is used for the number of constellations.
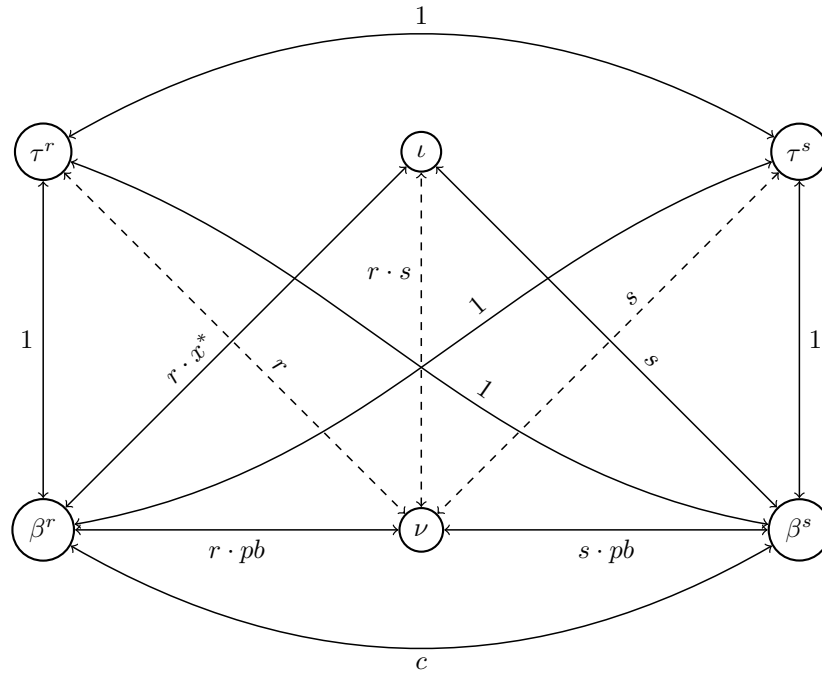
Figure 5.1: Summary graph for the rank deficiencies between GNSS unknowns. Symbols used are the same of 5.1. To help the reader we report them here: β^r and β^s are electronic biases for receivers and satellites respectively, τ^r and τ^s are clock desynchronization for receivers and satellites respectively, ι are the ionospheric unknown, and ν are the ambiguities. Furthermore r is number of receivers,s is number of satellites, e is number of epochs, f is number of frequencies, c is number of channels per frequencies, pb is the number of phase biases and x is the number of constellations.

5.4   Geometric Rank Deficiencies

Beside the rank deficiencies that we discussed in the previous sections there are also some involving the geometric parameters. In fact, when estimating both the receivers and satellites coordinates, the estimates are invariant (in term of the least squares minimisation) to both a translation and a rotation of the network. To solve the problem the so called no-net-rotation and no-net-translation conditions of the receiver coordinates can be used. Among all possible conditions this one will minimise the variance of the estimates of the receiver coordinates [65].

5.5 Solving the Rank Deficiencies

Here we report possible choices to solve the rank deficiencies described. The choice clearly depends on the type of parameter introduced. The processing strategies discussed correspond to the use of pseudorange and phase data with ionospheric parameters in the observation model. Other estimation strategies will require some variations of the procedure.

5.5.1 Clock Desynchronization - Electronic Bias

First rank deficiency might easily be solved removing one whole set of electronic bias unknowns (it might be constant or slowly time varying) per receiver and per satellite.
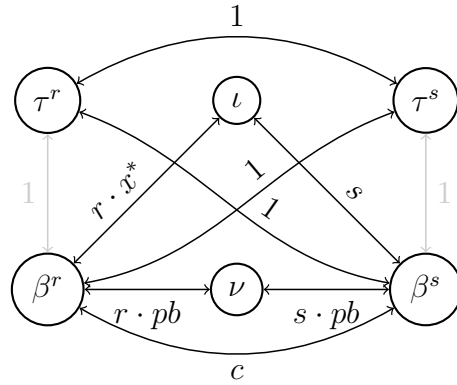
It has to be highlighted that the ionospheric bias rank deficiency is not affected by the removal of one of the biases, in fact considering the clock synchronisation also one can find:

$$A_\tau \cdot e \cdot \mu_{rem} + A_\beta \cdot \begin{pmatrix} \mu_1 - \mu_{rem} \\ \vdots \\ \mu_1 - \mu_{rem} \\ \vdots \\ \mu_f - \mu_{rem} \\ \vdots \\ \mu_f - \mu_{rem} \end{pmatrix} - \iota \cdot e = 0 \qquad (5.12)$$

where μ_{rem} is the ionospheric coefficient of the eliminated bias. Similar considerations can be made for ambiguity. To avoid unnecessary complications when dealing with electronic bias ambiguity rank deficiency, the

eliminated bias should belong to the group associated to the pseudor-
anges.

5.5.2   Electronic Bias - Ionospheric Delay

To remove this rank deficiency another electronic bias can be cancelled
for each set of biases insisting on the same ionosphere parameters (i.e.
per constellation). As before it is easier to eliminate such bias from the
pseudorange set. Care should be taken in removing such a bias, in fact it
can not be of the same frequency as one of the biases removed to eliminate
the clock desynchronization electronic bias rank deficiency.
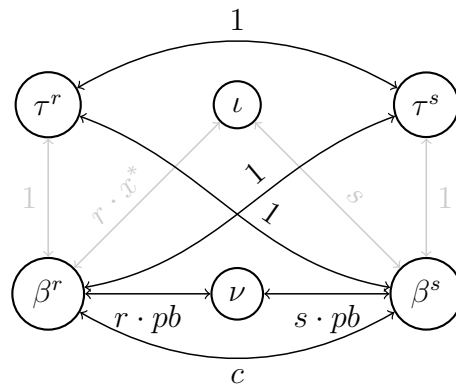
This double elimination of parameters will remove also the rank defi-
ciency between the satellite and receiver electronic bias, the rank de-
ficiency between the the receiver electronic bias and the satellite clock
desynchronization and the satellite electronic bias and the receiver clock
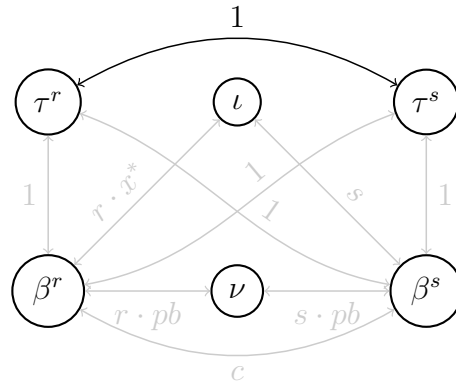desynchronization.
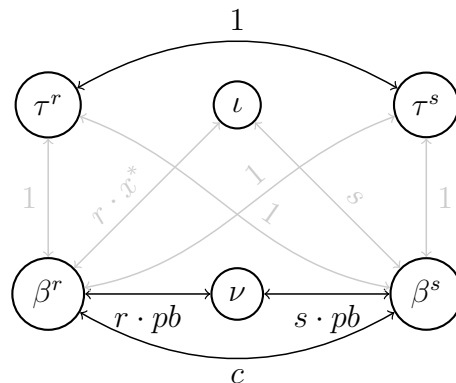
5.5.3 Receiver Electronic Bias - Satellite Electronic Bias

This rank deficiency can be eliminated removing one electronic bias from one receiver per each phase or pseudorange tracking considered. This can be already be done in the removal of the previous rank deficiency but this is not generally the case when pseudorange and phase jointly compensated. Again care should be put in the sense that if one parameter has already been removed for a tracking no additional one should be removed.
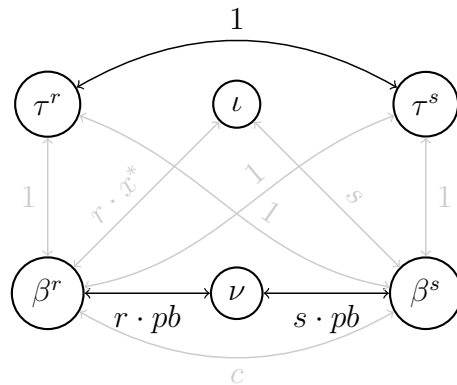
5.5.4 Cycle Ambiguity - Phase Electronic Bias

A second rank deficiency might be solved removing one ambiguity unknown per set of phase electronic bias unknowns for both the receiver and the satellite. It must be noticed that every time a set of ambiguities (insisting on the same electronic bias) experiences a cycle slip (or loss of signal) on all the satellites a new rank deficiency is introduced in the system. This means a new ambiguity unknown should be removed. The choice of the ambiguity term to be removed can not be completely arbitrary. In fact in removing the ambiguity unknown one should take care that it does not belong to the same satellite and channel of the ambiguity removed to solve the rank deficiency with respect to the satellite parameters.

5.5.5   Clock Desynchronization Rank Deficiency

The rank deficiency of clock desynchronization can be easily solved removing one of the clock unknowns. While the choice of the clock makes no difference from a mathematical point of view, the choice of a stable clock will make the variation of all the other clocks unknowns smaller.
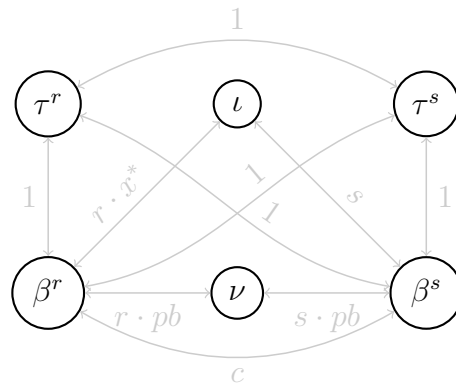
5.6   Near Rank Deficiencies

In this section some near rank deficiencies that may arise are discussed.

5.6.1   Geometrical Terms (Small Networks)

In this section we will show how unknowns whose entries in the design matrix depend on the geometry of the system (coordinates, tropospheric

parameters) can become almost rank deficient. Consider for instance a very short baseline; in such a case the line of sight vectors from the receiver to the satellites will be almost identical. Similarly also the elevations and the azimuths of the satellites will be very close one to the other. Consider then a system made up by such geometrical parameters and by satellite clock desynchronization.

$$
A = \begin{bmatrix} A' & 0 & \begin{bmatrix} I^1 \\ \vdots \\ I^n \\ I^1 \\ \vdots \\ I^n \end{bmatrix} \\ 0 & A'' & \end{bmatrix}
\tag{5.13}
$$

where A' is the design matrix corresponding to the first receiver, A'' is the design matrix corresponding to the second receiver and I^1 to I^n are the identity matrices corresponding to clocks unknowns of satellites 1 to n. As we said when the baseline is very short it is $A' \approx A''$. If this is the case we can see that a group of rank deficiencies arises. In fact, for each column i of A' we have:

$$
A \cdot X = \begin{bmatrix} A'(\ldots;i) & 0 & \begin{bmatrix} I^1 \\ \vdots \\ I^n \\ I^1 \\ \vdots \\ I^n \end{bmatrix} \\ 0 & A''(\ldots;i) & \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -A'(\ldots;i) \end{bmatrix} = 0
\tag{5.14}
$$

because $A' \approx A''$.

5.6.2 Ambiguity Clock Near Rank Deficiency

In some particular situation it is possible that we encounter near rank deficiencies arising from the same parameters we discussed above. As we said in 5.3.2 if we have a non repaired simultaneous cycle slip on all signals coming into a receiver (or exiting a satellite) a new rank deficiency arises. Let's now assume that the cycle slip is not simultaneous but occurs at near epochs. This could happen for instance in case of high ionospheric activity when scintillations cause cycle slips in phase tracking. Consider a

simple system formed by two satellites seen from one receiver. The system has $2 \cdot n$ epochs. On the first satellite we have a cycle slips between epochs n and $n+1$ on the second satellite we have a cycle slip between epochs $n+1$ and $n+2$. For simplicity we will analyse only the part relative to cycle ambiguity and receiver clock desynchronization; the design matrix becomes

$$A = \begin{bmatrix} \lambda \cdot \begin{bmatrix} 1^{n,1} & 0^{n,1} & 0^{n,1} & 0^{n,1} \\ 0^{n,1} & 1^{n,1} & 0^{n,1} & 0^{n,1} \\ 0^{n+1,1} & 0^{n+1,1} & 1^{n+1,1} & 0^{n+1,1} \\ 0^{n-1,1} & 0^{n-1,1} & 0^{n-1,1} & 1^{n-1,1} \end{bmatrix} & \begin{bmatrix} I^{2 \cdot n, 2 \cdot n} \\ \\ I^{2 \cdot n, 2 \cdot n} \end{bmatrix} \end{bmatrix} \tag{5.15}$$

To simplify the equation we multiply all the ambiguity terms by $1/\lambda$ since this does not affect the rank deficiency of the system. The normal matrix then becomes:

$$N = \begin{bmatrix} \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+1 & 0 \\ 0 & 0 & 0 & n-1 \end{bmatrix} & \begin{bmatrix} 1^{n,1} & 0^{n,1} & 1^{n,1} & 0^{n,1} \\ 0^{n,1} & 1^{n,1} & 0^{n,1} & 1^{n,1} \\ 1^{n+1,1} & 0^{n+1,1} & 1^{n+1,1} & 0^{n+1,1} \\ 0^{n-1,1} & 1^{n-1,1} & 0^{n-1,1} & 1^{n-1,1} \end{bmatrix}^{\mathsf{T}} \\ \begin{bmatrix} 1^{n,1} & 0^{n,1} & 1^{n,1} & 0^{n,1} \\ 0^{n,1} & 1^{n,1} & 0^{n,1} & 1^{n,1} \\ 1^{n+1,1} & 0^{n+1,1} & 1^{n+1,1} & 0^{n+1,1} \\ 0^{n-1,1} & 1^{n-1,1} & 0^{n-1,1} & 1^{n-1,1} \end{bmatrix} & 2 \cdot I^{2 \cdot n, 2 \cdot n} \end{bmatrix} \tag{5.16}$$

if we reduce for the clock desynchronization parameters using Eq 2.30 we have:

$$N = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+1 & 0 \\ 0 & 0 & 0 & n-1 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} n & 0 & n & 0 \\ 0 & n & 1 & n-1 \\ n & 1 & n-1 & 0 \\ 0 & n-1 & 0 & n-1 \end{bmatrix} = \begin{bmatrix} \frac{n}{2} & 0 & -\frac{n}{2} & 0 \\ 0 & \frac{n}{2} & -\frac{1}{2} & -\frac{n-1}{2} \\ -\frac{n}{2} & -\frac{1}{2} & \frac{n+3}{2} & 0 \\ 0 & -\frac{n-1}{2} & 0 & \frac{n+3}{2} \end{bmatrix} \tag{5.17}$$

We can see that for $n \gg 1$ we can approximate column three by taking the opposite of column one and column four by taking the opposite of column two. The relative error of this approximation with respect to the magnitude of the vector, expressed as the difference of third and first

column divided by the norm of the first, is:

$$o \approx \frac{\sqrt{2}}{n} \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix} ; \qquad (5.18)$$

As we see for large enough n, it may fall under the computer numerical precision.

5.7   Significant Examples

In the following sections examples of applications of the previously described theory to GNSS problems are presented.

5.7.1   On the Equivalence Between Stand Alone Processing and Baseline Processing (Small Baselines)

In this section we discuss an equivalence between stand alone processing and baseline (relative) processing. We will show that processing the data in stand alone mode and then differentiate them is equivalent to a baseline processing under certain assumptions. Consider a baseline where the two receivers are very close one to the other (few kilometres), in this case the Line of Sight (LOS) vectors from the receivers to the satellites will be very similar. So all the entries in the design matrix will be similar too. Furthermore, if we are in clear sky conditions and the satellites seen by the two receiver are the same, we will have:

$$A_1 \approx A_2 \qquad (5.19)$$

where A_1 is the design matrix for parameters of receiver 1 and the same for A_2. We can then call for both receivers the part relative to receiver parameters A and the one relative to satellite parameters C. Removing the columns relative to the first receiver parameters to solve the rank deficiency we will have:

$$\begin{pmatrix} 0 & C \\ A & C \end{pmatrix} \cdot \begin{pmatrix} x_r \\ x_s \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad (5.20)$$

If we then reduce for the the satellite parameters x_s, computing the hat matrix as $H = \begin{bmatrix} C \cdot (2 \cdot C^\intercal \cdot C)^{-1} C \\ C \cdot (2 \cdot C^\intercal \cdot C)^{-1} C \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \cdot I \\ \frac{1}{2} \cdot I \end{bmatrix}$, we have:

$$\breve{A} = \begin{bmatrix} -\frac{1}{2} \cdot A \\ \frac{1}{2} \cdot A \end{bmatrix} \tag{5.21}$$

$$\breve{y} = \begin{bmatrix} -\frac{1}{2} \cdot (y_1 - y_2) \\ \frac{1}{2} \cdot (y_2 - y_1) \end{bmatrix} \tag{5.22}$$

where the second term in both new design matrix, and new observation vector is simply the first one multiplied by -1. Adjusting the sign, and recognising that the least squares estimates are invariant with respect to scaling both design matrix and observation vector, we are left with the well known satellite differenced expression:

$$A \cdot x_r = (y_2 - y_1) \tag{5.23}$$

where x_r can be interpreted as the difference between the coordinates of the two receivers:

$$\delta x_r = x_{r2} - x_{r1}, \tag{5.24}$$

with δx_r equal to x_r of Eq 5.23. Now if we would like to compute x_{r2}, x_{r1} in a stand alone processing we would have $x_{r1} = S \cdot y_1 \; x_{r2} = S \cdot y_2$. where S is the solver for a system with A as design matrix. Putting this expressions in 5.24 we have:

$$\delta x_r = S \cdot (y_2 - y_1) \tag{5.25}$$

which is the solver for Eq 5.23. This show that for small baselines (in open sky) it is equivalent to process the data in baseline or in stand alone mode and difference them. This is of course true only for the float solution. To have an equivalence in general case one needs to fix the ambiguities in the stand alone mode. To do so one needs good corrections (global or local).

5.7.2 On the Equivalence Between Network and Baseline Processing (Small Networks)

A very similar reasoning can be made for small networks in case of network processing. Rewriting Eq 5.20 we have :

$$
\begin{bmatrix}
0 & 0 & \dots & 0 & C \\
A & 0 & \dots & 0 & C \\
0 & A & \dots & 0 & C \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \dots & A & C
\end{bmatrix}
\cdot
\begin{bmatrix}
x_{r2} \\ x_{r3} \\ \vdots \\ x_s
\end{bmatrix}
=
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n
\end{bmatrix}
\tag{5.26}
$$

reducing for C we will get:

$$
\begin{bmatrix}
-\frac{1}{n}A & -\frac{1}{n}A & \dots & -\frac{1}{n}A \\
\frac{n-1}{n}A & -\frac{1}{n}A & \dots & -\frac{1}{n}A \\
-\frac{1}{n}A & \frac{n-1}{n} & \dots & -\frac{1}{n}A \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n}A & -\frac{1}{n}A & \dots & \frac{n-1}{n}A
\end{bmatrix}
\cdot
\begin{bmatrix}
x_{r2} \\ x_{r3} \\ \vdots \\ x_{rn}
\end{bmatrix}
=
\begin{bmatrix}
y_1 - \frac{1}{n}\sum_{i=1}^{n} y_i \\
y_2 - \frac{1}{n}\sum_{i=1}^{n} y_i \\
y_3 - \frac{1}{n}\sum_{i=1}^{n} y_i \\
\vdots \\
y_n - \frac{1}{n}\sum_{i=1}^{n} y_i
\end{bmatrix}
\tag{5.27}
$$

then if we build the normal matrix from such reduced matrix we have:

$$
\check{N} =
\begin{bmatrix}
\frac{n-1}{n}A^{\mathsf{T}} \cdot A & -\frac{1}{n}A^{\mathsf{T}} \cdot A & \dots & -\frac{1}{n}A^{\mathsf{T}} \cdot A \\
-\frac{1}{n}A^{\mathsf{T}} \cdot A & \frac{n-1}{n}A^{\mathsf{T}} \cdot A & \dots & -\frac{1}{n}A^{\mathsf{T}} \cdot A \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{1}{n}A^{\mathsf{T}} \cdot A & -\frac{1}{n}A^{\mathsf{T}} \cdot A & \dots & \frac{n-1}{n}A^{\mathsf{T}} \cdot A
\end{bmatrix}
= \left(I - \frac{1}{n} \cdot e \cdot e^{\mathsf{T}}\right) \otimes A^{\mathsf{T}} \cdot A
\tag{5.28}
$$

we can now compute the inverse of such a matrix. The first term before the Kronecker product can be written as:

$$
G = (I + u^{\mathsf{T}} \cdot v), u = -\frac{1}{\sqrt{n}} \cdot e, v = \frac{1}{\sqrt{n}} \cdot e
\tag{5.29}
$$

Computing its inverse using Sherman–Morrison formula ([82] Section 2.7.1), we have:

$$
G^{-1} = (I + u^{\mathsf{T}} \cdot v) = I - \frac{-\frac{1}{n} \cdot e \cdot e^{\mathsf{T}}}{1 - \frac{1}{n} \cdot e^{\mathsf{T}} \cdot e} = I + e \cdot e^{\mathsf{T}}
\tag{5.30}
$$

where $e \in 1^{n-1,1}$. So the inverse of $\check{N}$ can be written as:

$$
\check{N}^{-1} = (I + e \cdot e^{\mathsf{T}}) \otimes (A^{\mathsf{T}} \cdot A)^{-1}.
\tag{5.31}
$$

The element corresponding to a single receiver will be simply $2(A^\intercal \cdot A)^{-1}$. Coming back to the normal equation if we would reduce for all receivers but one we would have $\frac{1}{2} A^\intercal \cdot A$. Then we can compute the know term of the normal equation:

$$\breve{b} = A^\intercal \otimes \begin{bmatrix} y_2 - \frac{1}{n} \sum_{i=1}^{n} y_i \\ y_3 - \frac{1}{n} \sum_{i=1}^{n} y_i \\ \vdots \\ y_n - \frac{1}{n} \sum_{i=1}^{n} y_i \end{bmatrix} \tag{5.32}$$

Now we can reduce it for all receiver but one (called j). To do so we need the term $\breve{N}(x_j; x_{(i\neq j)}) \cdot \breve{N}(x_{(i\neq j)}; x_{(i\neq j)})^{-1} = L$ (See Eq 2.27). We can reconstruct this term from the reduced normal equation that we have already computed. In fact, we know that $L \cdot \breve{N}(x_{(i\neq j)}; x_j) = (\frac{n-1}{n} - \frac{1}{2}) \cdot A^\intercal \cdot A$ so $L = -\frac{1}{2} \cdot e \otimes I$ where I is an identity matrix of the size of $A^\intercal \cdot A$. So computing the reduced know term, we have:

$$\breve{\breve{b}} = A^\intercal \cdot (y_j - \frac{1}{n} \sum_{i=1}^{n} y_i + \frac{1}{2} \sum_{i=2, i\neq j}^{n} y_i - \frac{n-2}{2n} \sum_{i=1}^{n} y_i) = A^\intercal \cdot (y_2 - \frac{1}{2} \sum_{i=\{1j\}} y_i) = \frac{1}{2} \cdot A^\intercal \cdot (y_j - y_1) \tag{5.33}$$

That combined with the normal matrix $\breve{N} = \frac{1}{2} A^\intercal \cdot A$ (computed after Eq 5.31) gives the same result we obtained for the baseline processing.

This means that, under the previous assumptions, adjusting all receiver together or adjusting them by baselines is equivalent. Furthermore, using the equivalence presented in the previous section, under the same assumptions, adjusting a small network of receivers, is the same (in term of estimates) than adjusting various stand alone receivers and then taking the differences. For real data evaluation of this principle see [104].

5.7.3 Analysis of the Variance of the Estimates as Function of Network Size

In this subsection we will characterise the variance of the estimates for different network sizes. The purpose of doing so is to give a general idea of the quality of the estimates we can achieve with different network sizes. An analytic solution of this problem is quite challenging, for this reason we follow a numerical strategy. To this purpose we simulated networks of different sizes picking α stations that are within β kilometres from the point with latitude and longitude equal to zero (Some example can be seen in Fig 5.2). For the position of the satellites we used the positions

of GPS satellites from IGS computed ephemerides at an arbitrary time.
In the simulation we consider the following parameters: clock desyn-
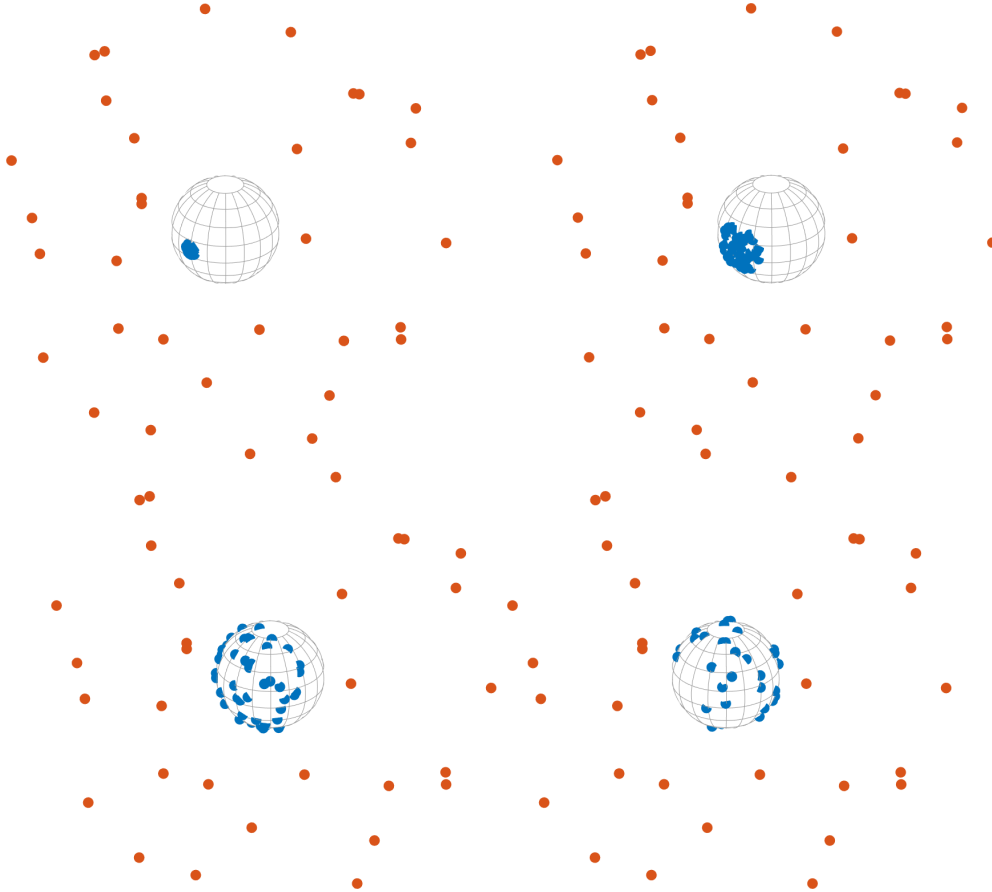
Figure 5.2: Examples of the generated random networks. Receivers in blue satellites in red.

chronization for both satellites and receivers, receiver and satellites coordinates, receiver tropospheric parameters. The simulation is done for one epoch only. We randomly extract 50 receivers from an uniform distribution within β kilometres from the origin. Then, assuming a cut-off angle of 7 degrees of elevation, all possible observations between receivers and satellites are formed. To solve the rank deficiencies of the system we set a no-net rotation and no-net translation condition on all receivers coordinates, and a no-net translation condition on receivers clock desynchronisation. For each observation we assume a standard deviation of one

centimetre. The simulation is performed for networks radius ranging from 10 to 20000 Km. We performed two type of simulations one estimating satellite coordinates and one without them. To avoid numerical problems we regularise the satellite coordinates in Tykhonov sense using pseudo observations with a standard deviation of $1000m$. In Fig 5.3 it is possible to see the the result of such analysis.

Looking at the results one can see that the variance of the satellites geometrical terms it is much higher than the one of the receiver. This is because we set the no net translation condition on the receivers. In practices this means that while the system is ill posed in general this is not too much of a problem for terrestrial positioning.

A rather obvious thing to observe is the fact that if we fix the satellite coordinates, we significantly bring down the standard deviation of the estimates. This is particularly true for receiver coordinates in a small network. Beside that, it is interesting to notice that in this case the system is so rank deficient that satellite clock desynchronization unknown could absorb also orbit mismatch too. The rank deficiency that is normally unwanted becomes a good property in such case. Lastly we see that there is a network size for which the quality of the troposphere estimates improves dramatically. This size is around few thousands kilometres for the case in which we are estimating satellites coordinates and around one thousand kilometres for the case with satellite coordinates fixed. This means that to have reliable absolute troposphere parameters estimates, we have to process networks of at least such a size.

5.7.4 Network of GNSS Ambiguities

In this section we will analyse the problem of adjusting a small network where we do estimate desynchronizations and ambiguities only. We will derive the mixed integer null space from which we will see that problem is unitary rank deficient. Then we will show how LDL or QR decomposition can be used to solve the integer rank deficiency.[2]. Consider a simple network made up of by 4 receivers and 3 satellites (See Figure 5.4). We can write the design matrix for desynchronization and ambiguity only

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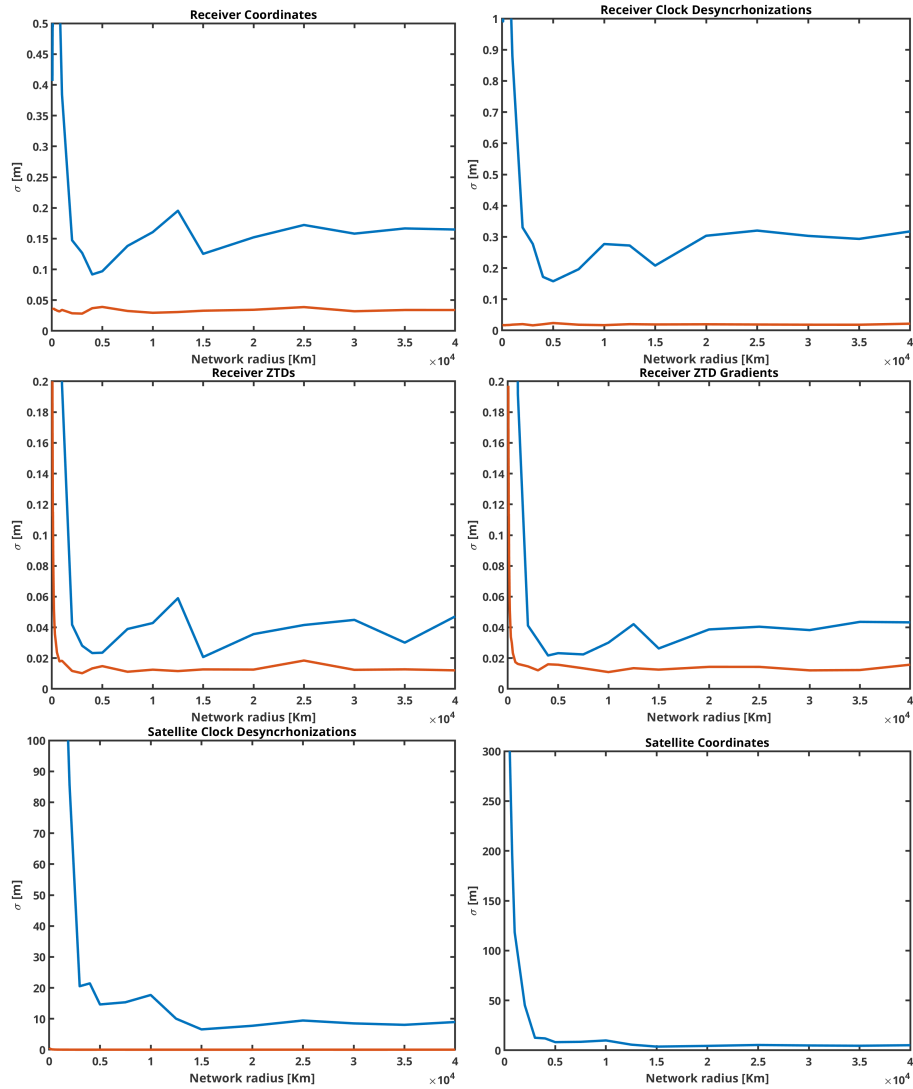[2]The same problem can be solved using graph theory [64]

Figure 5.3: Variance of the estimates for a simulated network a function of network size.
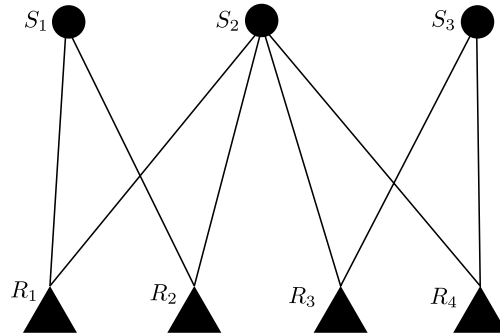
Figure 5.4: A small network GNSS network.

as:

$$
C = [AB] =
\begin{bmatrix}
\overbrace{
\begin{matrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{matrix}
}^{\tau_r}
\overbrace{
\begin{matrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{matrix}
}^{\tau_s}
&
\overbrace{
\begin{matrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{matrix}
}^{\nu}
\end{bmatrix}
$$
(5.34)

where τ_r are receiver clock desynchronization, τ_s are receiver clock desynchronization, and ν are the ambiguities. For simplicity the wavelength has been fixed to 1 since this does not affect the reasoning. To compute the mixed integer null space we can apply the strategy presented in Subsection 3.3.2. First we compute the integer matrix L of integer whose columns form a basis for $\mathcal{N}(A^\intercal)$; this can be done computing the Hermite

normal form of A:

$$U \cdot A = H \tag{5.35}$$

$$U = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tag{5.36}$$

we can see that the last two rows of U span $\mathcal{N}(A)$ so we will use them to construct L. Then we can compute $L^{\mathsf{T}} \cdot B$ and compute a basis for its null space again using the Hermite normal form. The unimodular matrix in this case will be:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}. \tag{5.37}$$

The matrix whose columns span the null space is then:

$$T = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.38}$$

and the admissible constraint computed inverting U will be:

$$
K =
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\tag{5.39}
$$

For which we can easily verify that $K \cdot T = I$. We can then try to solve the problem using LDL and QR factorization. Reducing for the desynchronization parameters B becomes:

$$
\breve{B} =
\begin{bmatrix}
0.250 & -0.250 & -0.250 & 0.250 & 0.000 & -0.000 & 0.000 & -0.000 \\
-0.250 & 0.250 & 0.250 & -0.250 & 0.000 & -0.000 & 0.000 & -0.000 \\
-0.250 & 0.250 & 0.250 & -0.250 & -0.000 & 0.000 & -0.000 & 0.000 \\
0.250 & -0.250 & -0.250 & 0.250 & -0.000 & -0.000 & -0.000 & 0.000 \\
0.000 & 0.000 & -0.000 & -0.000 & 0.250 & -0.250 & -0.250 & 0.250 \\
0.000 & 0.000 & -0.000 & -0.000 & -0.250 & 0.250 & 0.250 & -0.250 \\
0.000 & -0.000 & -0.000 & -0.000 & -0.250 & 0.250 & 0.250 & -0.250 \\
0.000 & -0.000 & -0.000 & -0.000 & 0.250 & -0.250 & -0.250 & 0.250
\end{bmatrix}
\tag{5.40}
$$

If we compute the QR factorization of $\breve{B}$ we have:

$$
Q =
\begin{bmatrix}
-0.500 & 0.274 & -0.173 & 0.382 & -0.000 & -0.593 & 0.060 & 0.378 \\
0.500 & -0.222 & 0.182 & 0.642 & -0.000 & -0.203 & -0.462 & 0.012 \\
0.500 & 0.076 & -0.826 & -0.052 & -0.000 & -0.165 & 0.178 & -0.008 \\
-0.500 & -0.420 & -0.470 & 0.208 & -0.000 & 0.225 & -0.345 & -0.374 \\
-0.000 & -0.610 & 0.115 & -0.191 & -0.500 & -0.484 & 0.282 & -0.118 \\
-0.000 & -0.563 & -0.053 & -0.076 & 0.500 & 0.120 & 0.108 & 0.631 \\
-0.000 & 0.016 & 0.047 & -0.474 & 0.500 & -0.522 & -0.425 & -0.263 \\
-0.000 & 0.063 & -0.122 & -0.359 & -0.500 & 0.082 & -0.600 & 0.486
\end{bmatrix}
$$

$$
R =
\begin{bmatrix}
-0.500 & 0.500 & 0.500 & -0.500 & 0.000 & -0.000 & 0.000 & -0.000 \\
0.000 & -0.000 & -0.000 & 0.000 & 0.000 & -0.000 & 0.000 & -0.000 \\
0.000 & 0.000 & -0.000 & 0.000 & 0.000 & -0.000 & 0.000 & -0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -0.000 & -0.000 & -0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & -0.500 & 0.500 & 0.500 & -0.500 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -0.000 & -0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & -0.000 & 0.000 \\
0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000
\end{bmatrix}
$$

we see that there are various elements on the diagonal of R whose value is equal zero, considering numerical precision. If we construct a constraint to fix them we have:

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.41}$$

which multiplied for T will give:

$$S = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.42}$$

The determinant of S is 1 showing that it is unimodular. This makes the constraint admissible. Similarly if we compute the LDL factorisation of $\breve{B}^\intercal \cdot \breve{B}$ with pivoting we have:

$$L = \begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -1.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -1.000 & 0.333 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ -0.000 & -0.000 & -0.000 & -1.000 & 1.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & -1.000 & -0.000 & 1.000 & 0.000 & 0.000 \\ -0.000 & -0.000 & -0.000 & 1.000 & -1.000 & -1.000 & 1.000 & 0.000 \\ 1.000 & -0.333 & -1.000 & -0.000 & 0.000 & -0.000 & -0.000 & 1.000 \end{bmatrix}$$

$$D = I \cdot \begin{bmatrix} 0.250 \\ 0.000 \\ 0.000 \\ 0.250 \\ -0.000 \\ 0.000 \\ 0.000 \\ -0.000 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

As we had before the element on the diagonal can be interpreted ad conditional variances. Using this interpretation the elements whose conditional variance is zero are the ones that are completely determined (with variance zero) by a combination of the previous ones i.e. the ones generating the rank deficiency. Looking at D and doing the same reasoning we did for QR we can set up a constraint matrix:

$$K = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{5.43}$$

which multiplied for T will gives:

$$S = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5.44}$$

This time the determinant is -1 and again S is unimodular. This makes the constraint admissible.

5.7.5   GLONASS Receiver Ambiguity Rank Deficiency

GLONASS satellites use a Frequency Division Multiple Access (FDMA) scheme to distinguish the different satellite signals in the receiver. For this reason the rank deficiency between the receiver clock desynchronization and the ambiguity can not simply be solved by choosing one pivot ambiguity. The wavelengths of the 14 GLONASS frequencies can be written as:

$$\lambda_{j,i} = \frac{c}{\gamma_j} \cdot \frac{1}{2840 + i} \tag{5.45}$$

where c is the speed of light, γ_j ($j \in [12]$) is a constant depending on the central frequency, and i is an index running from 1 to 14. The values of γ_j is the greatest common divisor of each frequency group. Such a value is 563500 for the first frequency and 437500 for the second frequency. Lets

pick one of the two frequencies and construct the design matrix for the single frequency receiver problem. Calling $\alpha = \frac{c}{\gamma_i}$, the design matrix for the ambiguities and the receiver clock desynchronisation can be written as:

$$A = \begin{bmatrix} \frac{\alpha}{2840+i_1} & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \frac{\alpha}{2840+i_n} & 1 \end{bmatrix}. \tag{5.46}$$

We can transform the matrix simply multiplying all ambiguity entry by $\frac{\prod_{i=i_1}^{i_n}(2840+i)}{\alpha}$. Once we do that we can find the mixed integer null space constructed by a single vector. The result is:

$$J = \begin{bmatrix} 2840 + i_1 \\ 2840 + i_2 \\ \vdots \\ 2840 + i_n \\ -\alpha \end{bmatrix} \tag{5.47}$$

If there are at least two subsequent channel the admissible constraint will be extremely simple, in fact:

$$K \cdot J = \begin{bmatrix} \dots 0 & 1 & -1 & 0 \dots \end{bmatrix} \cdot \begin{bmatrix} \vdots \\ 2840 + i \\ 2840 + i + 1 \\ \vdots \\ -\alpha \end{bmatrix} = 1 \tag{5.48}$$

or in words it is sufficient to set the difference of two subsequent channels to 0. Otherwise if no subsequent channels are present one could simply compute a generating vector for the greatest common divisor between the entry of the integer null space and use it as constraint.

Let's now consider another GLONASS single difference model where a linear bias proportional to the wavelength has been added. This can be the case if the tracking has not been set up correctly [98]. The observation equation will be:

$$A = \begin{bmatrix} \frac{\alpha}{2840+i_1} & \cdots & 0 & \frac{\alpha}{2840+i_1} & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \frac{\alpha}{2840+i_n} & \frac{\alpha}{2840+i_n} & 1 \end{bmatrix} \tag{5.49}$$

If we look for the mixed integer null space for a such system, consisting of two vectors, we get:

$$J = \begin{bmatrix} 2840 + i_1 & 1 \\ 2840 + i_2 & 1 \\ \vdots \\ 2840 + i_n & 1 \\ 0 & -1 \\ -\alpha & 0 \end{bmatrix} \tag{5.50}$$

We can then choose the first constraint similarly to the previous example and, in addition, set one the two ambiguities to zero.

5.7.6 Codeless Adjustment with Ionosphere

In case we process GNSS observations without pseudorange measurements and we include ionospheric delay in the observation model we have a rank deficiency[113]. In fact if we analyse a simple system with only the ambiguity and ionospheric unknowns we have:

$$A = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 & \gamma \cdot \lambda_1^2 \\ 0 & \lambda_2 & \ldots & 0 & \gamma \cdot \lambda_2^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_n & \gamma \cdot \lambda_n^2 \end{bmatrix} \tag{5.51}$$

Where λ are the GNSS wavelengths and $\gamma = -\frac{40.3 \cdot 10^{16}}{c^2}$. The mixed integer null space in this case is generated by the vector:

$$J = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \\ -\frac{1}{\gamma} \end{bmatrix} \tag{5.52}$$

The integer null space for all constellations and frequency combinations are presented in Tables 5.3 5.4 5.55.6 5.75.8. To generate an admissible constraint one could again compute the generating vector for the greatest common divisor of the entries in the vector spanning the integer null space.

If ones looks at the first line of the GLONASS table 5.8 one can recognise

the estimable combination discovered by [6]. It is worth mentioning that codeless processing is also useful in the case we want to estimate satellite specific code biases [49][98]; in this case the pseudorange does not fix the ionosphere and we are left with a rank deficiency similar to the one of code-less processing.

Table 5.3: GPS ambiguity-ionosphere integer null space

	L1	L2	L5
L1-L2	60	77	x
L2-L5	115	x	154
L1-L2-L5	1380	1771	1848

Table 5.4: Galileo ambiguity-ionosphere integer null space

	E1	E5a	E5b	E5	E6
E1,E5a	115	154	x	x	x
E1,E5b	59	x	77	x	x
E1,E5	233	x	x	308	x
E1,E6	125	x	x	x	154
E5a,E5b	x	118	115	x	x
E5a,E5	x	233	x	230	x
E5a,E6	x	25	x	x	23
E5b,E5	x	x	233	236	x
E5b,E6	x	x	125	x	118
E5,E6	x	x	x	250	233
E1,E5a,E5b	6785	9086	8855	x	x
E1,E5a,E5	26795	35882	x	35420	x
E1,E5a,E6	2875	3850	x	x	3542
E1,E5b,E5	13747	x	17941	18172	x
E1,E5b,E6	7375	x	9625	x	9086
E1,E5,E6	29125	x	x	38500	35882
E5a,E5b,E5	x	27494	26795	27140	x
E5a,E5b,E6	x	2950	2875	x	2714
E5a,E5,E6	x	5825	x	5750	5359
E5b,E5,E6	x	x	29125	29500	27494
E1,E5a,E5b,E5	1580905	2117038	2063215	2089780	x
E1,E5a,E5b,E6	169625	227150	221375	x	208978
E1,E5a,E5,E6	669875	897050	x	885500	825286
E1,E5b,E5,E6	1718375	x	2242625	2271500	2117038
E5a,E5b,E5,E6	x	687350	669875	678500	632362
E1,E5a,E5b,E5,E6	39522625	52925950	51580375	52244500	48691874

Table 5.5: Beidou ambiguity-ionosphere integer null space

	B1	B1C	B2a	B2b	B2ab	B3
B1,B1C	110	109	x	x	x	x
B1,B2a	575	x	763	x	x	x
B1,B2b	590	x	x	763	x	x
B1,B2ab	1165	x	x	x	1526	x
B1,B3	620	x	x	x	x	763
B1C,B2a	x	115	154	x	x	x
B1C,B2b	x	59	x	77	x	x
B1C,B2ab	x	233	x	x	308	x
B1C,B3	x	62	x	x	x	77
B2a,B2b	x	x	118	115	x	x
B2a,B2ab	x	x	233	x	230	x
B2a,B3	x	x	124	x	x	115
B2b,B2ab	x	x	x	233	236	x
B2b,B3	x	x	x	62	x	59
B2ab,B3	x	x	x	x	248	233
B1,B1C,B2a	12650	12535	16786	x	x	x
B1,B1C,B2b	6490	6431	x	8393	x	x
B1,B1C,B2ab	25630	25397	x	x	33572	x
B1,B1C,B3	6820	6758	x	x	x	8393
B1,B2a,B2b	67850	x	90034	87745	x	x
B1,B2a,B2ab	133975	x	177779	x	175490	x
B1,B2a,B3	71300	x	94612	x	x	87745
B1,B2b,B2ab	137470	x	x	177779	180068	x
B1,B2b,B3	36580	x	x	47306	x	45017
B1,B2ab,B3	144460	x	x	x	189224	177779
B1C,B2a,B2b	x	6785	9086	8855	x	x
B1C,B2a,B2ab	x	26795	35882	x	35420	x
B1C,B2a,B3	x	7130	9548	x	x	8855
B1C,B2b,B2ab	x	13747	x	17941	18172	x
B1C,B2b,B3	x	3658	x	4774	x	4543
B1C,B2ab,B3	x	14446	x	x	19096	17941
B2a,B2b,B2ab	x	x	27494	26795	27140	x
B2a,B2b,B3	x	x	7316	7130	x	6785
B2a,B2ab,B3	x	x	28892	x	28520	26795
B2b,B2ab,B3	x	x	x	14446	14632	13747
B1,B1C,B2a,B2b	746350	739565	990374	965195	x	x
B1,B1C,B2a,B2ab	2947450	2920655	3911138	x	3860780	x
B1,B1C,B2a,B3	784300	777170	1040732	x	x	965195
B1,B1C,B2b,B2ab	1512170	1498423	x	1955569	1980748	x
B1,B1C,B2b,B3	402380	398722	x	520366	x	495187
B1,B1C,B2ab,B3	1589060	1574614	x	x	2081464	1955569
B1,B2a,B2b,B2ab	15809050	x	20977922	20444585	20707820	x
B1,B2a,B2b,B3	4206700	x	5582108	5440190	x	5176955
B1,B2a,B2ab,B3	16612900	x	22044596	x	21760760	20444585
B1,B2b,B2ab,B3	8523140	x	x	11022298	11164216	10488961
B1C,B2a,B2b,B2ab	x	1580905	2117038	2063215	2089780	x
B1C,B2a,B2b,B3	x	420670	563332	549010	x	522445
B1C,B2a,B2ab,B3	x	1661290	2224684	x	2196040	2063215
B1C,B2b,B2ab,B3	x	852314	x	1112342	1126664	1058519
B2a,B2b,B2ab,B3	x	x	1704628	1661290	1682680	1580905
B1,B1C,B2a,B2b,B2ab	173899550	172318645	230757142	224890435	227786020	x
B1,B1C,B2a,B2b,B3	46273700	45853030	61403188	59842090	x	56946505
B1,B1C,B2a,B2ab,B3	182741900	181080610	242490556	x	239368360	224890435
B1,B1C,B2b,B2ab,B3	93754540	92902226	x	121245278	122806376	115378571
B1,B2a,B2b,B2ab,B3	980161100	x	1300631164	1267564270	1283884840	1206230515
B1C,B2a,B2b,B2ab,B3	x	98016110	131256356	127919330	129566360	121729685
B1,B1C,B2a,B2b,B2ab,B3	x	x	x	x	x	x

Table 5.6: IRNSS ambiguity-ionosphere integer null space

	L5	S
L5,S	1218	575

Table 5.7: QZSS ambiguity-ionosphere integer null space

	L1	L2	L5	LEX6
L1,L2	60	77	x	x
L1,L5	115	x	154	x
L1,LEX6	125	x	x	154
L2,L5	x	23	24	x
L2,LEX6	x	25	x	24
L5,LEX6	x	x	25	23
L1,L2,L5	1380	1771	1848	x
L1,L2,LEX6	1500	1925	x	1848
L1,L5,LEX6	2875	x	3850	3542
L2,L5,LEX6	x	575	600	552
L1,L2,L5,LEX6	34500	44275	46200	42504

Table 5.8: GLONASS ambiguity-ionosphere integer null space

	G1	G2	G3
G1,G2	7	9	x
$G1_1$,G3	32054	x	42615
$G1_2$,G3	16027	x	21315
$G1_3$,G3	32054	x	42645
$G1_4$,G3	16027	x	21330
$G1_5$,G3	32054	x	42675
$G1_6$,G3	16027	x	21345
$G1_7$,G3	32054	x	42705
$G1_8$,G3	16027	x	21360
$G1_9$,G3	2914	x	3885
$G1_{10}$,G3	16027	x	21375
$G1_{11}$,G3	32054	x	42765
$G1_{12}$,G3	517	x	690
$G1_{13}$,G3	32054	x	42795
$G1_{14}$,G3	16027	x	21405
G2,$G3_1$	x	32054	33145
G2,$G3_2$	x	48081	49735
G2,$G3_3$	x	96162	99505
G2,$G3_4$	x	16027	16590
G2,$G3_5$	x	96162	99575
G2,$G3_6$	x	48081	49805
G2,$G3_7$	x	32054	33215
G2,$G3_8$	x	48081	49840
G2,$G3_9$	x	8742	9065
G2,$G3_{10}$	x	16027	16625
G2,$G3_{11}$	x	96162	99785
G2,$G3_{12}$	x	1551	1610
G2,$G3_{13}$	x	32054	33285
G2,$G3_{14}$	x	48081	49945
$G1_1$,$G2_1$,G3	224378	288486	298305
$G1_2$,$G2_2$,G3	112189	144243	149205
$G1_3$,$G2_3$,G3	224378	288486	298515
$G1_4$,$G2_4$,G3	112189	144243	149310
$G1_5$,$G2_5$,G3	224378	288486	298725
$G1_6$,$G2_6$,G3	112189	144243	149415
$G1_7$,$G2_7$,G3	224378	288486	298935
$G1_8$,$G2_8$,G3	112189	144243	149520
$G1_9$,$G2_9$,G3	20398	26226	27195
$G1_{10}$,$G2_{10}$,G3	112189	144243	149625
$G1_{11}$,$G2_{11}$,G3	224378	288486	299355
$G1_{12}$,$G2_{12}$,G3	3619	4653	4830
$G1_{13}$,$G2_{13}$,G3	224378	288486	299565
$G1_{14}$,$G2_{14}$,G3	112189	144243	149835

6

Software Implementation

In theory there is no difference between theory and practice - in practice there is

– Yogi Berra

This chapter deals with the description of the implementation of a GNSS processing strategy. The strategy does not make use of differentiations between observables, for this reason it is generally called undifferenced and uncombined [1] [27][80], we will use this terminology in this document. The word undifferenced is used in the sense that observations are not differenced in between satellites and receivers, while the word uncombined refers to the fact no combination between phase/pseudorange measurements from different frequencies is made. The strategy and the resulting software have been developed starting from the open-source goGPS project [84] and it has been recently re-engineered for geodetic monitoring applications in the framework of the activity of GReD a spin-off of Politecnico of Milano. The software has been produced originally by using MATLAB programming language and it is this version that is described in this thesis. Figure 6.1 presents the main processing steps.

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[1]Sometimes the expression "raw observation approach" is used instead [103]

95
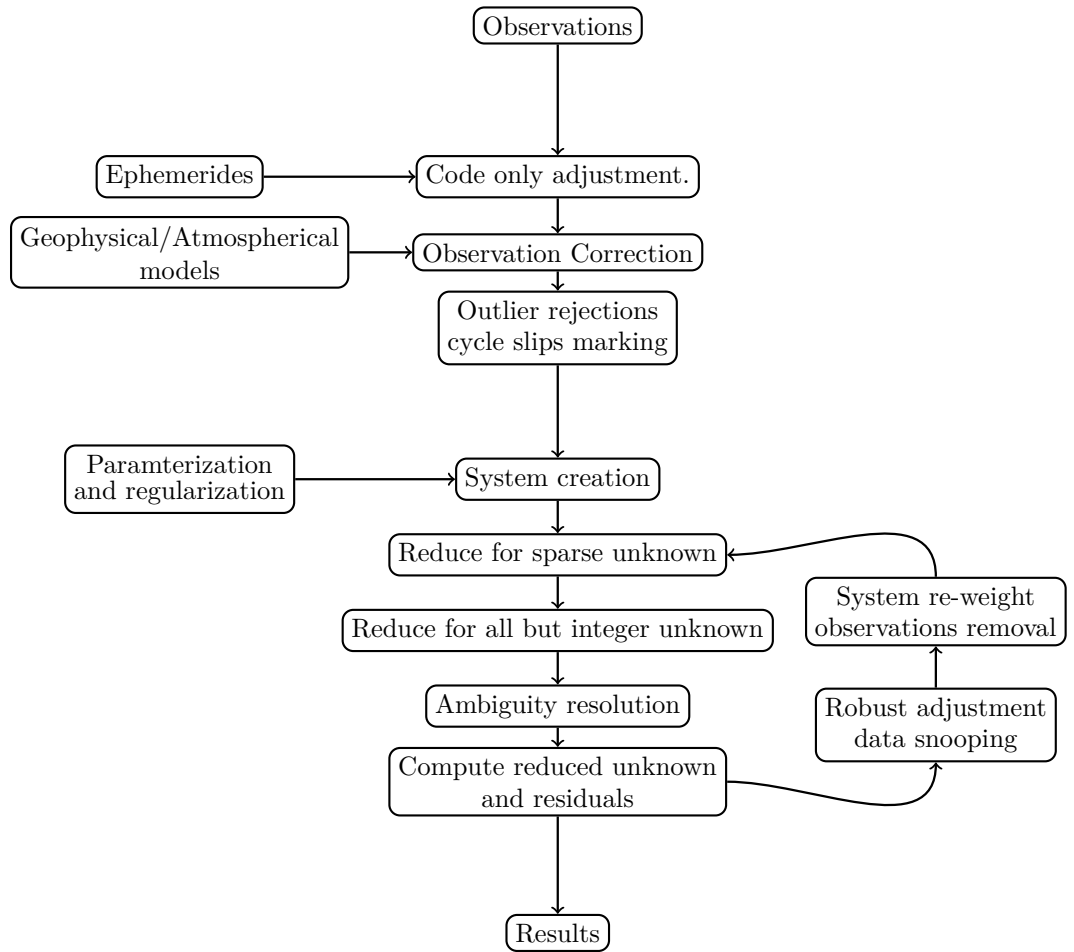
Figure 6.1: Summary graph of software processing operations

All operations before the creation of the linear system are called pre-processing. Since they were not discussed before, the following sections will discuss them. Finally some examples and results obtained from real data are presented.

6.1   Preprocessing

In this section the preprocessing of the observations is discussed. The term preprocessing is used to call all the operations that are done before

setting up the linear system of equations.

6.1.1 Code Only Adjustment

This procedure is applied to have a good linearisation point for the observation equations, computing the satellite positions at the correct time and most of the observation corrections (Chapter 4) for which an approximate position of the receiver is needed. To do that a stand alone adjustment of pseudoranges only is performed for the receiver. Particular care has to be given to the time chosen for the computation of the satellite orbits. This is because satellites move quite fast in an earth fixed frame. So it is very important to compute the satellite position precisely at the time the signal is emitted. The transmission time τ_{tx} can be computed as:

$$\tau_{tx} = \tau_{rx} - \tau_{travel} - \tau_{rec}, \tag{6.1}$$

where τ_{rx} is the nominal reception time, τ_{travel} is the travel time of the signal and τ_{rec} is the receiver desynchronization. Taking a semimajor axis of 26000 Km and assuming a circular orbit the speed of the satellite will be $\approx 26000 \cdot 2 \cdot \pi/(3600 \cdot 12) = 3.8$ Km/s; adding to that the velocity due to earth rotation, we can compute a rough upper bound of satellite velocity as $3.8 + 2 \cdot \pi/86400 \cdot 26000 \approx 5.6$ Km/s. With this quantity we can then derive an accuracy requirement for travel time. In fact if we want to compute the satellites positions with millimetre level error the transmission time has to be known with an sccuracy better than $0.001/5600 \approx 1.8 \cdot 10^{-7}$ s. This means that all effects bigger than $1.8 \cdot 10^{-7} \cdot c \approx 53$ meters have to be considered in the computation of travel time [2]. Finally one has to consider that during the flight of signals the earth rotates (Sagnac effect); for this reason the computed satellite positions have to be rotated for the angle swept during travel time. The whole procedure has to be iterated till convergence at the meter level of the receiver position. This is because the computation of both travel time and transmission time depends on the current estimate of the positions and receiver clock desynchronization.

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[2]Observation from different frequencies have different travel times, so in principle different orbit position should be computed for each frequency, fortunately even for high ionospheric activity such difference as just below 50 m.

6.1.2   Cycle Slips Detection

One of the operations needed before setting up the system of linear equation is the detection of the occurrence of cycle slips. A cycle slips occurs when the phase tracking loop is interrupted, and the continuous counting of phase cycles is broken. In such a case normally a new ambiguity parameter is set up in the linear system.

Uniqueness of Cycle Slip

Before describing our approach to the cycle slips detection we discuss the non uniqueness of cycle slips determination in an undifferenced uncombined observation processing. We will see that multiple choices of cycle slips are equivalent. Consider the case of a receiver seeing two satellites, if one of the two signals slips it would be equivalent to set a cycle slip on the first satellite or on the second. This is because the clock desyncronization could absorb the jump of the satellite that slipped and was not marked with a cycle slip. Similarly, if we have more than one satellite and one of them slips, one could put either a cycle slip on the satellite that slipped or a cycle slip on all the others. These two situations are not equivalent because the second system would be less well conditioned than the first one. There is then the very unlikely condition of multiple satellites slipping by the same number of cycles, this would create a situation of indecision similar to the one of two satellites described before. If the number of satellites that slipped for the same cycles number is bigger that the one that did not slip, an even stranger situation could occur in which putting the slip on the satellites that have not slipped would result in a better conditioned system. Having that in mind we can formulate the following criterion to choose the cycle slip in the phases:
Within all possible cycle slip sets that can explain jumps in the phase measurements the one which involves less cycle slips is used. The above discussion is indeed a reflection of the non estimability of all the ambiguities due to the rank deficiencies.

Cycle Slip Detection Procedure

In this Subsection the procedure used by us to detect cycle slips is described. Of course, there is a large literature on cycle slips detection and repairing, see for instance [14] [27] [123] and many more. The cycle slip detection procedure here presented assumes two things:

- The receiver position is fixed or slowly varying in time.

- The sampling rate of the observations is not too large.

The procedure works by differentiating observations in time. The formula for a generic phase observation differentiated in time reads:

$$\delta\phi = \delta\rho + \delta\tau^r + \delta\iota + \delta\zeta + \delta\tau^s + \gamma \cdot \lambda + \sqrt{2} \cdot \epsilon \tag{6.2}$$

where δ stands for time differentiation, ρ is the geometric distance between the satellite and the receiver, ι is the ionospheric delay of the observation, ζ is the tropospheric delay for the observable , τ^r and τ^s are the receiver and satellite clock desynchronizations respectively, γ is a variable that is equal to 1 in case of a cycle slip and 0 otherwise, λ is the amount of the cycle slip (an integer number of cycles) and ϵ is the measurement noise assumed to be uncorrelated. Electronic bias are not present in the equation since they vary slowly and thus their contribution is negligible. For each component in the observation we have normally approximate values. This approximate values can be removed from the observation in order to reduce the magnitude of the effects. Let's now analyse the various terms of the equation and their orders of magnitude.
We can start from the geometric term. Taking two subsequent epochs $t1, t2$ the geometric term can be written as:

$$\delta\rho - \delta\tilde{\rho} = (|r - s_{t1}| - |r - s_{t2}|) - (|\tilde{r} - s_{t1}| - |\tilde{r} - s_{t2}|) \tag{6.3}$$

where s is the satellite position vector, r is the receiver position vector, and $\tilde{}$ denotes an approximate value. We can then use the following approximations:

$$|r - s_{t1}| - |r - s_{t2}| \approx (e^\intercal \cdot v) \cdot \tau \tag{6.4}$$

where e is the unit vector from receiver to satellite, v is the satellite speed, τ is the time passes from epoch $t1$ to $t2$. Our expression then becomes:

$$\delta\rho - \delta\tilde{\rho} \approx ((e - \tilde{e})^\intercal \cdot v) \cdot \tau \tag{6.5}$$

Looking at Fig 6.2 we can write an approximate upper bound for this quantity. Assuming an error for the approximate position of 3 m, the magnitude of the $(e - \tilde{e})$ vector will be approximately $3/26000000 = 1.15 \cdot 10^{-7}$. Looking at Fig 6.2 and taking as earth radius the approximate value of 6700 Km the angle α between the two vectors can be at most
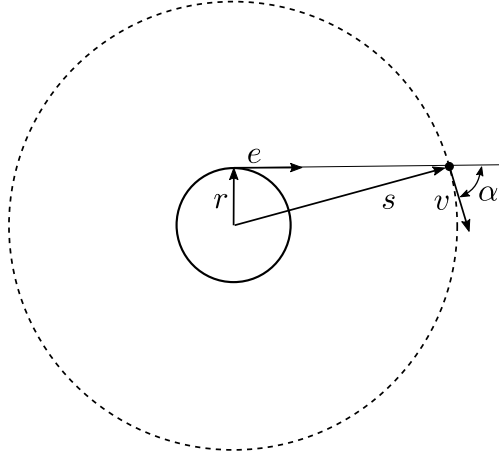
Figure 6.2: Simplified satellite receiver geometry.

$90° - atan(6700/26000) \approx 75.5°$. Taking the previously computed upper bound for satellites speed 5.6 Km/s and putting all together in the worst case the geometric term will be:

$$\delta\rho - \delta\tilde{\rho} \approx 1.15 \cdot 10^{-7} \cdot 5.6 \cdot 10^3 \cdot cos(75.5) \cdot \tau = 1.6 \cdot 10^{-4} \cdot \tau. \qquad (6.6)$$

Taking for instance a sampling rate of 30 seconds the maximum error would be around 5 mm which is well below one cycle and thus negligible. Second, we consider the tropospheric delay; its variations are typically on the order of some cm per hour [32]. If we assume a high change in tropospheric delay of 3 cm/h and a very low satellite (mapping function of 10) we have a change in tropospheric delay of around $8.3 \cdot 10^{-5}$ m/s that for typical sampling rate of 30 s gives a change of around 2 mm which again is far below one wavelength.
Last, we can consider the ionospheric delay. The ionospheric variations are normally below 10 TEC/hour [37]. Using such value, assuming a satellite at low elevation we have a mapping function of around 2.7, amd using the frequency most affected by ionosphere (GPS L5), we have a variation of ionospheric delay of around 2.3 mm/s. This multiplied by a typical sampling rate of 30 second would give a difference of ionospheric delay of 6.75 cm which is significant with respect to the wavelength of 25.48 cm. For this reason an external model for the ionospheric delay is needed in order to reduce the delay to few cm and to not mark an non existing cycle slip in high ionospheric activity cases (for istance the one provided by IGS

under the IONEX format [92] can be used). Modern GNSS satellite clocks have normally short term stability of around 10^{-12} for tenths of seconds [87]. This gives at 30 seconds a stability of the order of few centimetres. So for modern clock the term can be neglected; if one has to perform a cycle slip detection on older satellites or at lower sampling rate the satellite clock would have to be corrected (see for instance IGS precise products [69]) or to be estimated in the cycle slip detection procedure. Receiver clocks are normally too unstable so their differential effects have to be estimated.

Neglecting all irrelevant elements Eq 6.2 becomes:

$$\delta\phi = \delta\tau^r + \delta\tau^s + \gamma \cdot \lambda + \sqrt{2} \cdot \epsilon \tag{6.7}$$

We can now concentrate on the cycle slip term; in this case the cycle slip can be assimilated to an outlier. Thus it is sufficient to make a robust adjustment of the clock drifts and then mark as outliers all observations with residual greater than a certain threshold (for instance half a wavelength). In the case we are using data from modern GNSS satellites or correcting them using precise products we are left with one unknown per epoch and we can thus use a very fast robust estimator such as the sample median. At this point we have not still distinguished between outliers and cycle slips in the observations. To distinguish them is necessary to look at two subsequent time differences. In fact if we have an outlier we will see a jump in the data with similar magnitude and reverse sign in the second epoch, while this will not happen in the case of a cycle slip. This latter concept is exemplified in Fig 6.3.
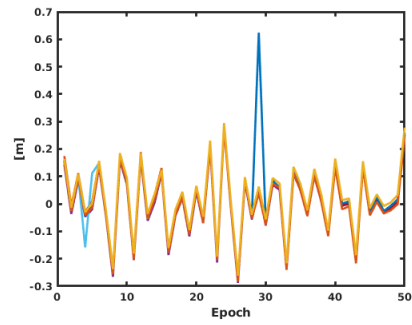
6.2 Parametrization

A lot of different parametrizations can be set up. In this section we show a framework to describe such parametrizations and to create the corresponding design matrix. The software can adjust different types of unknowns:
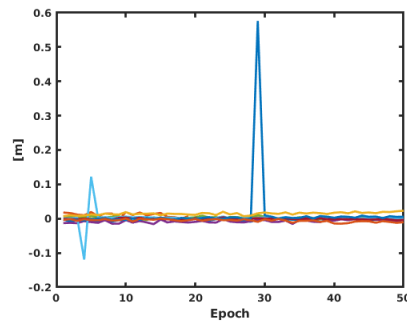
- Receiver coordinates.

- Satellite coordinates.

- ZTD.

- ZTD gradients.
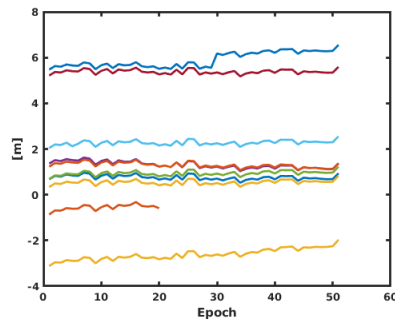
(a) Phase measurements reduced by known terms.

(b) Phase measurements reduced by known terms and differentiated in time.

(c) Phase measurements reduced by known terms, differentiated in time, median removed.

Figure 6.3: Cycle slip and outliers effect on phase measurements. Cycle slip in dark blue line, outlier in light blue line. The sampling rate is 30 seconds.

- Receiver clock desynchronization.

- Satellite clock desynchronization.

- Frequency specific biases.

- Observation specific biases.

- Biases linearly dependent on frequecies.

- Ionospheric delays.

- Cycle ambiguities terms.

In the software each unknown is catalogued in terms of 4 different group, they are:

- Receiver parametrization.

- Satellite parametrization.

- Time parametrization.

- Observation parametrization.

In term of receiver parametrization each unknown can be classified as:

- Common to all receivers.

- One per receiver.

- Common to a set of receivers.

In term of satellite parametrization each unknown can be classified as:

- Common to all satellites.

- One per satellite.

- Common to a set of satellites.

In term of time parametrization each unknown is allowed to be classified as:

- Constant in time.

- One per epoch.

- Piece wise constant, with arbitrary steps.

- Piece wise constant, with regular steps.

- Piece wise linear.

- Piece wise cubic.

In term of observation parametrization each unknown is allowed to be either:

- One per tracking.

- One per frequency.

- One per GNSS band.

- One per frequency bin.

One could argue that biases and desynchronizations are the same unknowns just paramterized in a different ways. Altough this is true we use different names for them to better organise their signature in the observations which have different behaviour in practice. Furthermore, we allow all parameters to be regularised either in their absolute values or in their time difference with arbitrary weight in the pseudo observations. A computer software has been developed such that for each parameter an arbitrary parameterization can be chosen and a corresponding design matrix constructed. The desing matrix is stored using a sparse matrix (i.e. only the non zero values and their index in the matrix are stored) in order to save memory. Since the rank deficiencies of such matrix would vary according to different parameterization we choose to solve them with the numerical methods presented in Section 3.1 to avoid the management of all possible cases.

6.3   Number of Observables

In this section the number of unknowns in the system is analysed. The number of unknowns depends in general on the chosen parametrization. We will give the number for a typical case. Table 6.1 presents the number of unknowns in the system for a typical network processing static receivers.

If we take as an example a network of 10 receivers, 60 satellites (of which one third is visible), 3 frequency per satellite with one phase and

	Number of unknown
Ionosphere	$\alpha \cdot \beta \cdot \gamma \cdot \eta$
Satellite clock desynchronization	$\alpha \cdot \gamma$
Receiver clock desynchronization	$\beta \cdot \gamma$
Ambiguities	$\alpha \cdot \beta \cdot \eta$
Satellite bias	$\alpha \cdot \eta \cdot 2$
Receiver bias	$\beta \cdot \eta \cdot 2$
Receiver coordinates	$\beta \cdot 3$
Receiver tropospheric parameters	$\beta \cdot 3 \cdot \gamma / \tau$

Table 6.1: Number of parameters for a typical processing. α is the number of visible satellites, β is the number of receivers, γ is the number of epochs, η is the number of frequencies and τ is the rate of tropospheric parametrization.

one pseudorange measurement each, 1 day of observations, 30 second sampling rate for the observations and one hour sampling for the tropospheric parameters we have the following numbers:

- Ionosphere: 1728000

- Satellite clock desynchronization: 57600

- Receiver clock desynchronization: 28800

- Ambiguity: 1200 (assuming 2 arcs per day)

- Satellite bias: 120 (assumed constant for simplicity)

- Receiver bias: 60 (assumed constant for simplicity)

- Receiver coordinate: 30

- Receiver tropospheric parameter: 720.
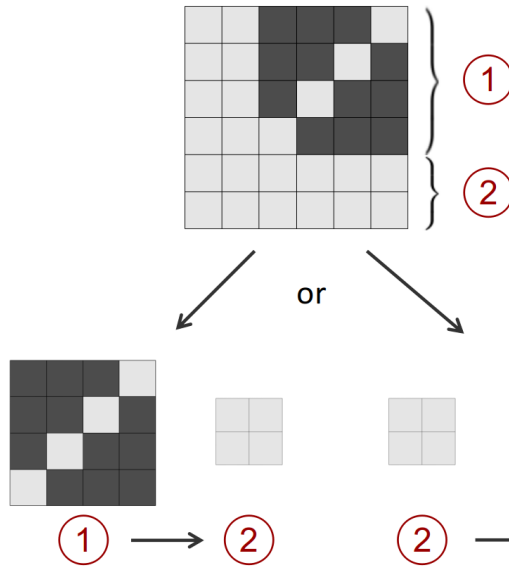
- Total: 1816530.

Figure 6.4: Example of the dependence of sparseness on the reduction order.

It is clear from the above example that the ionospheric and the desynchronization unknowns dominate in the total number and thus they require a special attention in the solving process.

6.4   Strategy to Reduce

Since the system as is so large a strategy to exploit the sparseness of the system has to be used. One possible strategy is to reduce the parameters using equation 2.27 ([93]); the reduction has to be done in a way that the reduced parameters are as sparse as possible[3]. Figure 6.4 presents a very simple example highlighting how the sparseness of the reduced matrix depends on the reduction order. In order to keep the system sparse the reduction term (the term after the minus in Eq 2.27) should have non zero elements in the same place of the part to be reduced (the term before the minus in Eq 2.27).
To understand the best strategy we have to look at the structure of the

---

[3]General solvers that use the concept of graph theory to exploit the sparseness of a generic design matrix exists (see for instance [26]). However, they have to analyse the sparseness of the matrix before solving the system. Knowing in advance the structure of the matrix it is possible to program a faster solution.

normal matrix sparseness. In particular the element N_{ij} of the normal matrix will be zero if $A(\ldots, i)^\intercal \cdot A(\ldots, j) = 0$ and this can happen only if no observations have a non zero entry for both the parameters, i.e. if there is no observation connecting the two parameters. Now let's consider the case of a reduced normal matrix $\breve{N}$ that has been reduced by another part of the normal matrix R and whose cross term (lower bottom) in the original normal matrix is X.

The $\breve{N}_{ij}$ element will be zero if the sufficient (but in general non necessary) following conditions are met:

- The non reduced N_{ij} element was zero.

- The product of the i and j column of the cross term X is zero, $X(\ldots; i)^\intercal \cdot X(\ldots; j) = 0$. This means that the two parameter i and j are not both linked to a third parameter by some observation.

- The cross element of the inverse of the matrix R corresponding to non zero elements in columns i, j of the X matrix are also zero, i.e. there is no observation connecting them directly or trough a chain of other observations.

These three conditions can be resumed by saying that the two parameters should not be connected by any observation. This connection could be either direct or by a chain of observations that connect the parameters to be reduced.

We have seen as the ionospheric parameters constitute the majority of the unknowns, altough their entries in the normal form are also diagonal. For these reason they are a good candidate in the list of parameters to be reduces. A similar reasoning can be done with respect to receiver and clock desynchronizations.

We will now discuss which is a better order to reduce such parameters. To do that, we have to list the effects of reducing each of the three parameters on the other two. The effect on the sparseness of the other parameters could be assessed too, but since they are not so numerous it is less compelling to do so. A single ionospheric parameter is connected trough observation to a single satellite clock desynchronization, and a single receiver clock desynchronization. This means that reducing for ionosphere will not decrease the sparseness of receiver and satellite clock desynchronizations. On the other hand both single receiver and clock desynchronization unknowns are connected by observations to multiple ionospheric , satellite

and receiver clock desynchronization unknowns. More specifically for each epoch each receiver desynchronization is connected by observations to all the desynchronisations of the satellites for the epoch and to all the ionospheric delays for the receiver for the epoch. The same applies for satellite desynchronisations switching the terms "satellite" and "receiver" in the previous sentence.This means that reducing one desynchronization for a receiver or a satellite, would bind all desynchronizations of the satellite and ionospheric delays for the same epoch. Again the same applies for the receiver. Taking that in mind it is clear that the first parameters to be reduced are the ionospheric ones since they are the most numerous and their reduction does not affect the receiver and satellites unknowns. The reduction of the second group of parameters depends however on the ratio between receiver and visible satellites; the more numerous should be reduced first. Normally the visible satellites are more than the receivers and thus should be reduced first. Now we have to concentrate on the rest of the parameters.

We see that the budget of remaining unknowns is dominated by ambiguities. As they are integer parameters, they have also a very nice property: if we can reliably estimate them as integer their variance will be 0 and thus we can completely eliminate them from the estimation process. It is then possible to use a subset of the network to estimate the satellite parameters and then use them to solve receivers ambiguities in stand alone mode [22]. Then the whole network can be readjusted without ambiguity terms.

6.5   Software Running Examples

In this section two examples of processing of real data are presented. One is a very small network; it represents the case of geodetic monitoring. The other one is a global network, it is presented in order to show the numerical resolution of rank deficiencies and the estimation of bias that allows ambiguity fixing in the PPP mode.

6.6   Small Network Low Cost Receivers

The first case is the adjustment of a small network of low cost geodetic receivers. In the framework of the H2020 GIMS project a network

Figure 6.5: Location of the stations of the local network.

|	
Observation used	Gps (C1C,L1C,C2L,C2L) , Galileo (C1C,L1C,C7Q,L7Q)
Clock desynchronization	Epoch-wise unconstrained
Satellite Inter Frequency Bias	Cubic spline each 2h
Receiver Inter Frequency Bias	Cubic spline each 2h
Observation Bias	Constant
Ionosphere	Epoch-wise common between receivers unconstrained.

Table 6.2: Processing settings for local network processing.

of low cost GNSS sensors were installed on a landslide in the Potoska planina site near the village of Potoki Slovenia [96]. The network consists of double frequency μblox ZED-F9P GNSS equipped receiver. Fig 6.5 shows the approximate positions of the stations on a satellite image. The data have been processed using the station POT6 as reference. The processing parameters are reported in Tab 6.2. PCO parameters have been estimated for frequencies different from GPS L1/Galileo E1 , their values are reported in Tab 6.3. For comparison the observations have been processed using Bernese software 5.2 [24], using E1 and L1 frequencies only (processing courtesy of Stefano Caldera, GReD). Figure 6.6 presents the evolution of the baseline in time. To compute the repetability of the results the movement of the landslide has been modelled as cubic spline with a spline each 28 days. The repetability is then computed as the mean of the absolute values of the residuals with respect of the computed
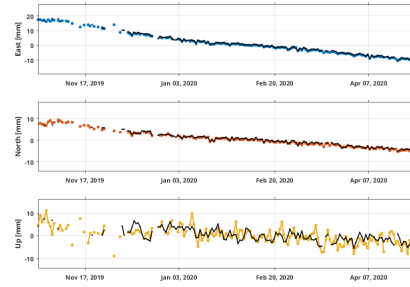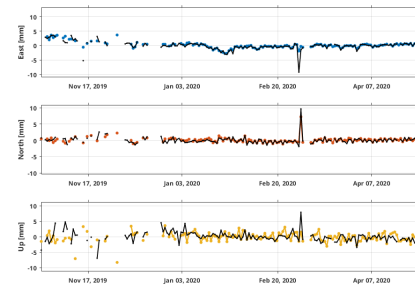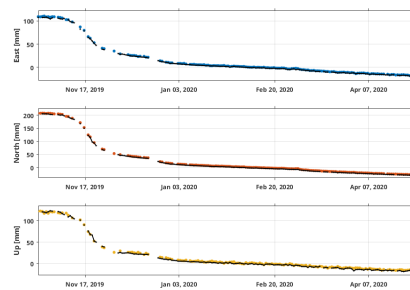
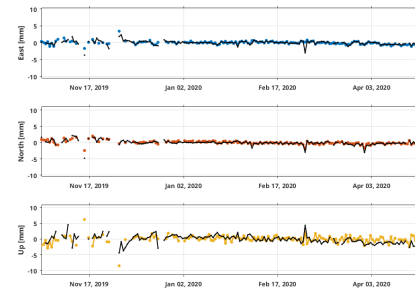(a) POT6-POT1 baseline.
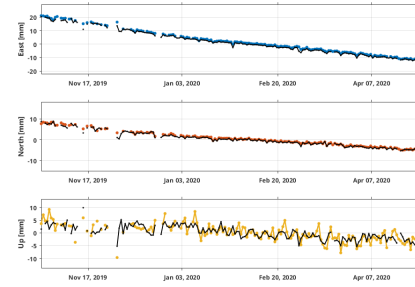
(b) POT6-POT2 baseline.
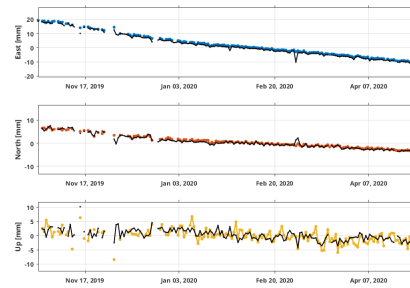
(c) POT6-POT3 baseline.

(d) POT6-POT4 baseline.

(e) POT6-POT5 baseline.

(f) POT6-POT7 baseline.

Figure 6.6: Repetability of the computed baselines. goGPS solution coloured, Bernese solution black.

| Station | G02 | | | E07 | | |
	North	East	Up	North	East	Up
POT5	-3.849	-3.041	-3.437	-6.436	-4.407	2.385
POT4	2.431	4.057	24.337	4.013	3.753	21.245
POT1	-2.100	-1.464	-1.220	-3.304	-1.836	-2.202
POT6	-5.207	0.610	-9.003	-5.817	0.044	-6.915
POT3	3.071	5.055	-3.298	4.478	6.709	-2.949
POT7	0.233	-4.392	3.512	0.377	-4.607	0.281
POT2	5.436	-0.769	-11.182	6.642	0.379	-12.056

Table 6.3: Estimated PCO, values in millimeters

| Baseline | East [mm] | | | North [mm] | | | Up [mm] | | |
	goGPS	Bernese	RMS	goGPS	Bernese	RMS	goGPS	Bernese	RMS
POT6-POT1	2.845	2.204	0.609	5.954	4.565	0.397	3.596	2.924	1.500
POT6-POT2	0.656	1.153	0.389	0.811	1.011	0.917	1.515	1.789	1.428
POT6-POT3	0.614	0.581	0.259	0.522	0.566	0.262	2.677	1.916	2.109
POT6-POT4	0.471	0.837	0.662	0.392	0.601	0.342	1.924	1.406	1.466
POT6-POT5	0.469	0.584	0.480	0.486	0.679	0.361	1.202	1.333	1.126
POT6-POT6	0.596	0.793	0.455	0.439	0.602	0.302	2.317	1.632	1.669

Table 6.4: Standard deviation of the coordinates with respect to a 28 day cubic spline for both goGPS and Bernese solution and RMS of the differences. All statistics in millimetres.

spline. In Tab 6.4 the repetabilities are presented along with the RMS of the Bernese-goGPS difference. The repeatability and RMS are on the order of half a millimetre (planar component) and few millimetres (vertical component) for most baselines confirming a good behaviour of the produced software.

6.7 Large Network of Geodetic Receivers

In this example we present the processing of a global network of stations for 1 day of observation (15 January 2020). The stations come from the IGS network, a map of their position can be seen in Fig 6.7. The stations are equipped with Trimble NETR9 receivers and are processed using GPS and Galileo observations , Tab 6.5 presents all the observations used in the adjustment for each station. Furthermore Tab 6.6 presents the processing settings for the network. Due to the various number of ob-

Station Name	Tracked Observations
DLF1	Galileo : C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS :C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
MAYG	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
NIUM	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
PNGM	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
SEYG	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
SOLO	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
GMSD	Galileo: C1X, C5X, C7X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
TLSE	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
RGDG	Galileo: C1X, C5X, C6X, C7X, C8X, L1X, L5X, L6X, L7X, L8X
	GPS: C1C, C1X, C2W, C2X, C5X, L1C, L1X, L2W, L2X, L5X
PFRR	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
LPGS	Galileo: C1X, C5X, C6X, C7X, C8X, L1X, L5X, L6X, L7X, L8X
	GPS: C1C, C1W, C2W, C2X, C5X, L1C, L1W, L2W, L2X, L5X
LMMF	Galileo: C1X, C5X, C6X, C7X, C8X, L1X, L5X, L6X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
JCTW	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2L, C2W, C5X, L1C, L2L, L2W, L5X
CAS1	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
CPVG	Galileo: C1X, C5X, C7X, C8X, L1X, L5X, L7X, L8X
	GPS: C1C, C2W, C2X, C5X, L1C, L2W, L2X, L5X
GODN	Galileo: C1X, C5X, C6X, C7X, C8X, L1X, L5X, L6X, L7X, L8X
	GPS: C1C, C1W, C2W, C2X, C5X, L1C, L1W, L2W, L2X, L5X
NYA2	Galileo: C1X, C5X, C6X, C7X, C8X, L1X, L5X, L6X, L7X, L8X
	GPS: C1C, C1W, C2W, C2X, C5X, L1C, L1W, L2W, L2X, L5X

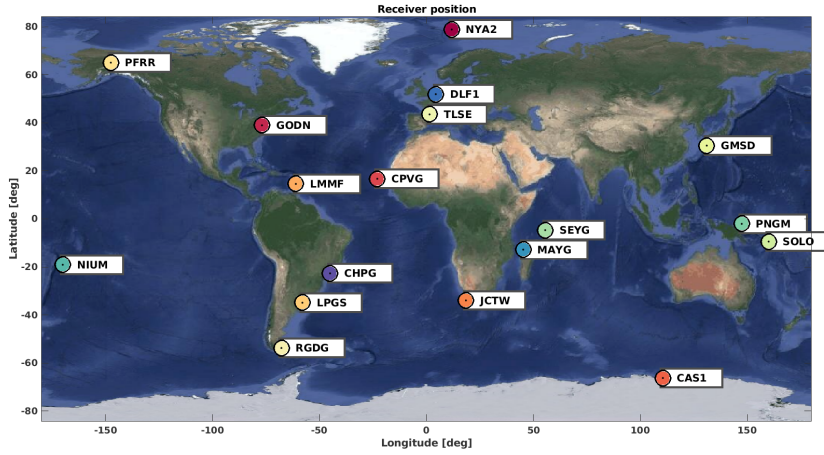Table 6.5: Observation available for GPS and Galileo.

Figure 6.7: Location of the stations of the global network.

Ephemerids	CODE final MGEX solution
Tropospheric mapping function	GMF (for compatibility with IGS solution)
Clock desynchronization	Epoch-wise unconstrained
Satellite Inter Frequency Bias	Cubic spline each 1h
Receiver Inter Frequency Bias	Cubic spline each 3h
Observation type bias	Constant
ZTD paramterization	Cubic spline 1h
ZTD regularisation	Absolute value (0.5 m), time difference (0.02 $m/\sqrt{h}$)
ZTD gradient parametrization	Cubic spline 2h
ZTD gradient regularisation	Absolute value (0.02 m), time difference (0.001 $m/\sqrt{h}$)
Ionosphere	Epoch-wise unconstrained
Coordinates	Costant

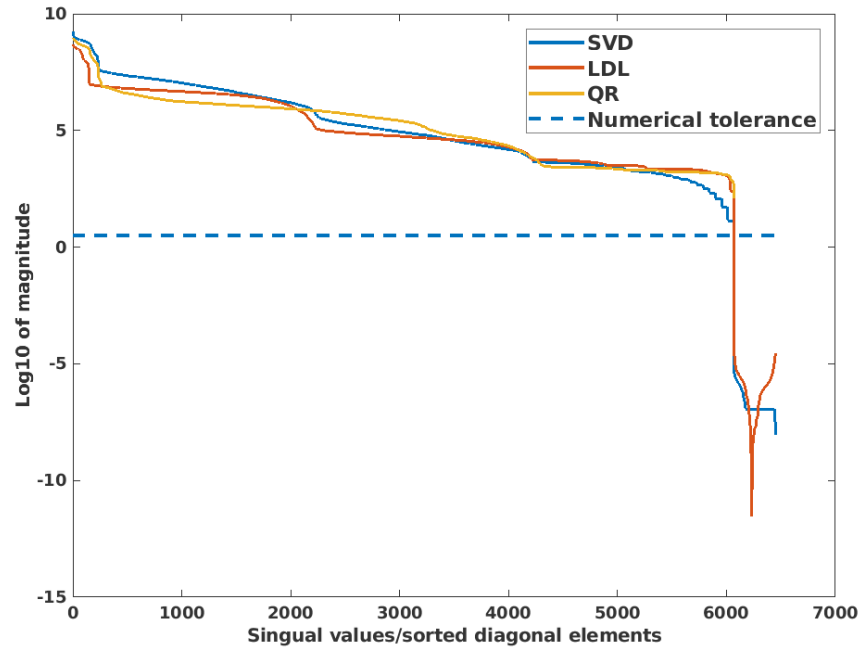Table 6.6: Processing settings for global network processing.

Figure 6.8: Singular values, elements of the diagonal computed from factorization of normal matrix corresponding to bias elements.

servations used with their corresponding biases it is worth looking at the performance of the numerical method for the resolution of both real and integer unknowns. Fig 6.8 presents the singular values and the diagonal elements of the LDL decomposition and the QR decomposition (all sorted in descending order, LDL and QR of computed with partial pivoting). It is possible to see that all three methodologies identify the same number of rank deficiencies, and that we have a clear drop of the values below the numerical tolerance after the last significant value. Similarly the analysis is performed also on the reduced normal matrix corresponding to ambiguities. In Fig 6.9 we plot the values of the diagonal elements of both LDL and QR decomposition sorted in descending order, computed with partial pivoting. Again we can see that the same number of rank deficiencies is identified and that there is a clear drop below numerical tolerance. As an indication that the constraint derived by the decomposition is correct we show the histogram of the fractional part of the decorrelated ambiguities
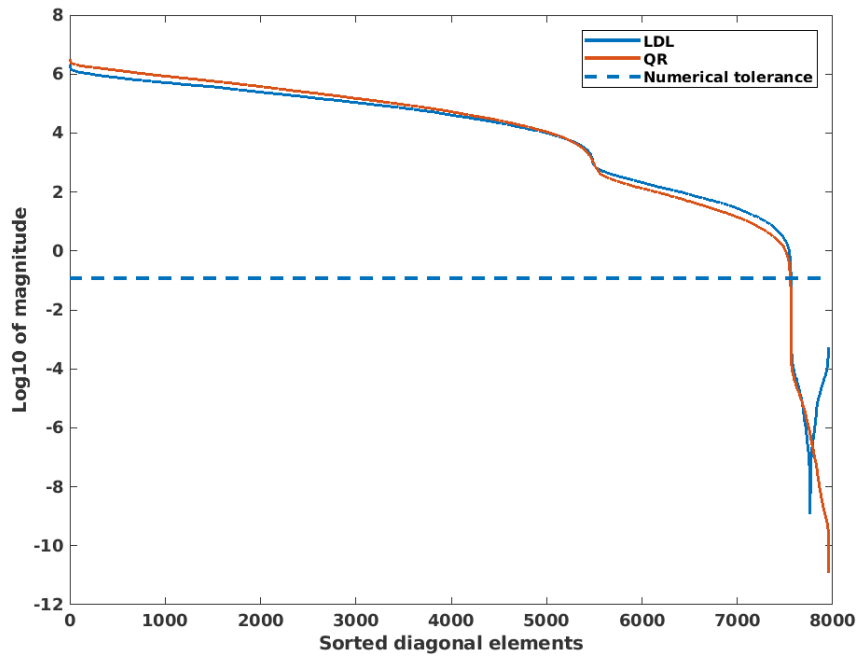
Figure 6.9: Elements of the diagonal computed from factorization of normal matrix corresponding to ambiguities.
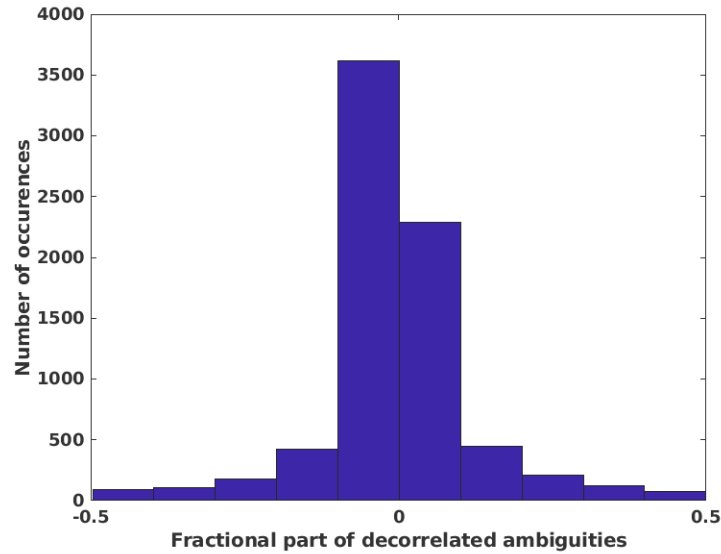
Figure 6.10: Histogram of the fractional part of the decorrelated ambiguities, for the network processing.

(Fig 6.10). The values are clearly dispersed around zero indicating a good behaviour of the processing. The network has been processed using all stations but TLSE. This last station is used to verify the PPP procedure with ambiguity fixing. After the processing of the other stations all clock and bias unknowns referring to the satellites have been stored in the program and applied to the observations. If we than apply the PPP processing to the TLSE station, we see again that the decorrelated ambiguities are clearly dispersed around zero (Fig 6.11). To validate the results we compare them with the official IGS tropospheric products [44]. In Tab 6.7 we present the mean and the standard deviation of the difference between compute and IGS ZTDs sampled at observation rate. Fig 6.12 presents the troposphere from both IGS and that computed for four stations; those with lower bias and standard deviation and those with higher. Furthermore in Tab 6.8 the difference between IGS and computed coordinates are presented. The TLSE ZTD and coordinate estimates (PPP processing) are also included in Tab 6.7 and Tab 6.8. Generally speaking we see an agreement between both the tropospheric and coordinates unknowns better than the centimetre, thus proving a good behaviour of the software.
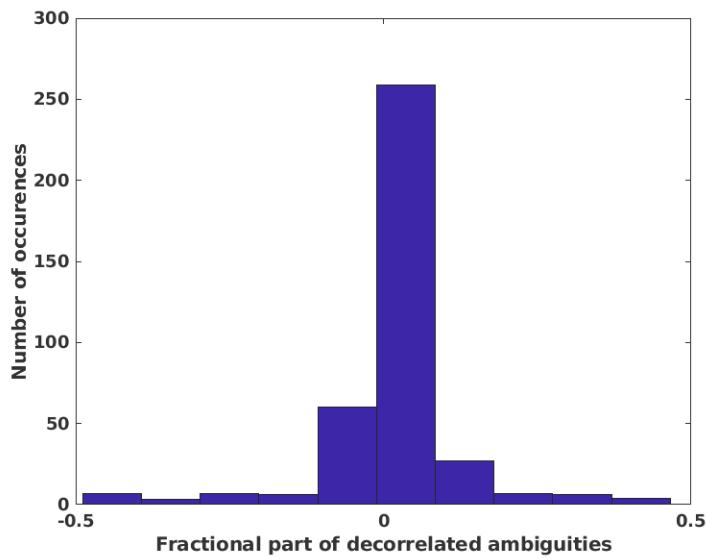
Figure 6.11: Histogram of the fractional part of the decorrelated ambiguities, for the PPP processing on TLSE station.

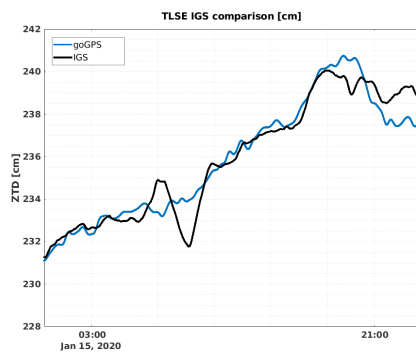| Station	Mean [cm]	Standard deviation [cm]
CHPG	-0.27	0.45
DLF1	-0.27	0.37
MAYG	-0.21	0.63
NIUM	0.26	0.79
PNGM	-0.06	0.60
SEYG	-0.14	0.54
SOLO	-0.32	0.72
GMSD	0.41	0.54
TLSE	-0.01	0.74
RGDG	0.59	0.44
LPGS	0.40	0.74
LMMF	-0.30	0.60
JCTW	0.12	0.47
CAS1	0.98	0.29
CPVG	-0.26	0.56
NYA2	-0.33	0.49
GODN	-0.93	0.88

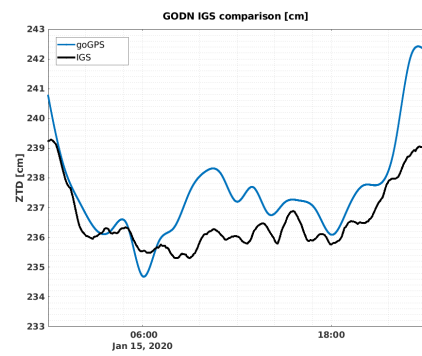Table 6.7: Mean and standard deviation of the IGS and computed ZTDS.
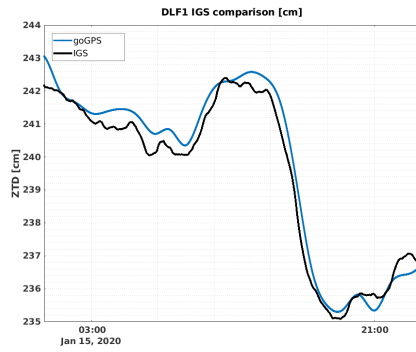
(a) DLF1 station.
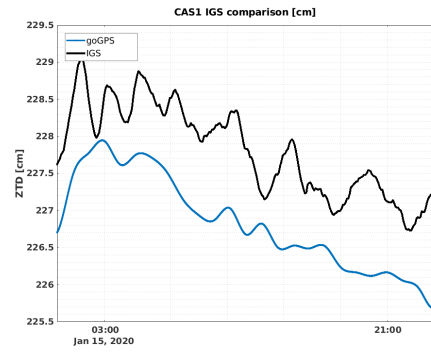
(b) CAS1 station.

(c) TLSE station.

(d) GODN station.

Figure 6.12: Computed ZTD vs IGS ones, best and worst cases.

Station	X [cm]	Y [cm]	Z [cm]
CHPG	-0.12	0.30	-0.56
DLF1	0.25	-0.36	0.61
MAYG	-0.58	-1.32	0.52
NIUM	0.15	0.51	-0.19
PNGM	0.95	-0.86	-0.64
SEYG	0.37	-0.35	-0.01
SOLO	0.40	-0.77	-0.28
GMSD	0.44	-0.23	-0.45
TLSE	0.50	-0.13	0.86
RGDG	-0.55	-0.24	-0.32
LPGS	0.60	-0.92	-0.19
LMMF	-0.92	0.37	-0.19
JCTW	0.87	0.45	-0.31
CAS1	-0.34	0.98	-2.03
CPVG	-1.13	-0.46	-0.36
GODN	-0.13	0.88	-0.31
NYA2	-0.07	0.40	1.09

Table 6.8: Difference between IGS and computed coordinates

7

Conclusions

A general strategy to adjust GNSS observables has been discussed and implemented. The rank deficiencies present in a system of GNSS observables and a list of general algorithms for their solution have been discussed. A MATLAB based software has been produced following the developed procedure. The software is very flexible and capable of processing all GNSS observables with very general user defined parametrizations. The software operations have been described and some numerical examples have been reported. The procedure contains few innovative parts namely, a general theoretical framework for the solution of integer and real integer mixed rank deficient problems, and the usage of numerical methods for the resolution of the rank deficiencies. The software could be improved and we will do that in the future. Two important future works will be the estimation of variance components of the observations variance covariance matrix, and the possibility to estimate orbital and earth rotation parameters.

Bibliography

"Now let us come to those references to authors which other books have, and you want for yours. The remedy for this is very simple:You have only to look out for some book that quotes them all, from A to Z as you say yourself, and then insert the very same alphabet in your book... ... .Besides, no one will trouble himself to verify whether you have followed them or whether you have not, being no way concerned in it"

– Miguel de Cervantes, *Don Quixote*

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