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INTEGRAL TRANSFORMS AND OPERATOR THEORY IN HYPERCOMPLEX ANALYSIS

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Abstract

In this dissertation we present extensions of the short time Fourier transform in the quaternionic and Clifford settings and we introduce and study the properties of new functional calculi based on the S-spectrum. In recent years there has been a growing interest to generalize integral transforms in the noncommutative setting. This gives the opportunity to deal with n-dimensional signals. The integral transforms have several applications in mathematical physics, precisely in signal processing, optics and time frequency analysis.
In the first part of the thesis we focus on the Bargmann and short time Fourier transforms in the slice hyperholomorphic and monogenic settings. Precisely, we study peculiar short time Fourier transforms with window functions the Gaussian and the weighted Hermite. The constructions are based on the slice hyperholomorphic Bargmann transform and the slice polyanalytic Bargmann transform, which main properties are investigated in this thesis. Moreover we focus on a short time Fourier transform, in the monogenic Clifford algebra, setting with a generic radial window function.
In the second part of the thesis we introduce new functional calculi based on the S-spectrum and the Fueter-Sce theorem. By using its integral version we define the F-functional calculus for n-tuples of commuting operators. This generates a monogenic functional calculus in the spirit of McIntosh and collaborators. The other functional calculi that we get are based on the factorization of the Fueter map, namely the Laplace operator in four real variables, in terms of the Fueter operator. The functional calculi obtained are harmonic and polyanalytic of order 2. In this thesis we focus on study-

ing the main properties of these functional calculi, such as the respective resolvent equations, the Riesz projectors and the basic algebraic rules.
In the third part, we introduce functional calculi on the S spectrum based on the factorization of the Fueter-Sce map, a suitable power of the Laplace operator in $n + 1$ variables. This suggests the introduction of new classes of functions that we plan to study more in forthcoming papers. Finally, in the last section we give an overview on possible open questions left open by this dissertation.

Summary

This thesis is divided in three parts. In the first part we study the Segal-Bargmann transform and the short-Fourier transform in the quaternionic and Clifford algebra settings. This study is motivated by a recent and increasing interest in the generalization of integral transforms to the noncommutative settings. Such kind of transforms are widely studied, in general, since they help in the analysis of vector-valued signals and images. In the non commutative settings one can deal with n-dimensional signals. Indeed, in image processing it is needed a higher-dimensional counterpart of the 1-dimensional signal. As a metter of fact, the study of hypercomplex signals can be useful in other practical fields such as optics and signal processing. The focus of the first part of the thesis is the study of the short time Fourier transform in the quaternionic and Clifford algebra settings. This integral transform can be used in several applications such as predictions of sound source position emanated by fault machine and the interpretation of ultrasonic waveforms. Furthermore, if we consider the normalized Hermite functions as window functions of the short time Fourier transform we have to deal with the theory of slice polyanalytic functions. This latter topic is a new research path that extends the theory of slice regular or slice monogenic functions to higher order.

In the second part of the thesis we focus on developing new functional calculi based on the S-spectrum and related to the Fueter-Sce construction. Firstly we study the so called F-functional calculus. The crucial object to define this functional calculus is the Fueter-Sce mapping theorem in integral form. The Fueter-Sce map can be seen as an integral transform that

maps slice hyperholomorphic functions into axially monogenic functions. Thus, the F-functional calculus is a monogenic functional calculus in the same spirit of McIntosh and collaborators. In this framework, we have been able to compute a resolvent equation, which is the appropriate tool to generate the Riesz projectors.
It is well-known that it is possible to factorize the Fueter map in terms of the Fueter operator. Therefore one can wonder the result of applying the Fueter operator or its conjugate to a slice hyperholomorphic function. In the first case we get an axially harmonic function. On the other hand, if we apply the conjugate Fueter operator to a slice hyperholomorphic function we do not get the same result, but instead an axially polyanalytic function of order 2. We are able to write an integral representation of axially harmonic functions and axially polyanalytic functions of order 2. These are crucial tools to define the respective functional calculi, both based on the S-spectrum.

In the third part we investigate the behaviour of the factorization of the Laplace operator of $n + 1$ variables elevated to an integer power depending on n, namely the Fueter-Sce map, and we apply a chosen factorization to a slice hyperholomorphic function. Due to the various factorizations of the Fueter-Sce map, the descriptions of intermediate functional calculi are much more involved. We point out that the case of dimension five, although it is a specific case, it already shows all the possible functional calculi and function spaces that can be considered in greater dimensions. It is also important to observe that all the function spaces appear also in different contexts in the literature, but, as far as we know, they are not related each other. A natural problem is to study all the function spaces that are suggested by the Fueter-Sce theorem in complex and hypercomplex setting. Moreover, it would be a challenging problem to figure out if a similar construction holds when we deal with fractional power of the Laplace operator, namely in the case of the Fueter-Sce-Qian map, in even dimension.

In this thesis we consider several problems related to the integral transforms in the hypercomplex setting and the study of the related functional calculi based on the S-spectrum. Precisely, we deal with the following topics: slice hyperholomorphic and monogenic function theories, S-functional calculus, polyanalytic function theory, Dirac operator in Clifford analysis, quaternionic Segal Bargmann-transform and Clifford Fourier transform, quaternionic short time Fourier transform with Gaussian and normalized Hermite functions as window functions, F-functional calculus and different functional calculi based on the S-spectrum. We give a brief account of the new results obtained in the three part of the thesis.

The first part of this dissertation is devoted to study hypercomplex integral transforms. All the new results obtained are summarized below.

- In [70], together with my PhD colleague K. Diki, we study a special one dimensional quaternion short-time Fourier transform (QSTFT). Its construction is based on the slice hyperholomorphic Segal-Bargmann transform. We discuss some basic properties and prove different results on the QSTFT such as Moyal formula, reconstruction formula and Lieb’s uncertainty principle. We provide also a formula for the reproducing kernel associated to the Gabor space considered in this setting.
- In [71], together with K. Diki, we show that it is possible to extend the previous results by considering a QSTFT with normalized Hermite functions as windows. It turns out that such a transform is based on the recent theory of slice polyanalytic functions on quaternions. We will use the notions of true and full slice polyanalytic Fock spaces and Segal-Bargmann transforms. Moreover, we show a closed formula for the true polyanalytic Bargmann transform in terms of the Hermite polynomials.
- The Clifford-short time Fourier transform is studied in [68]. We investigate how the short-time Fourier transform can be extended in a Clifford algebra setting. We prove some of the main properties of the Clifford short-time Fourier transform such as the orthogonality relation, the reconstruction property and the reproducing kernel formula. Moreover, we show the effects of modulating and translating the signal and the window function, respectively. The results show different features with respect to the classic case.

In the second part of the thesis we focus on the study of the new functional calculi based on the S-spectrum and on the Fueter-Sce theorem. Moreover, we give a description of the Fueter-Sce theorem in terms of the generalized CK-extension. Below we give summary of the main new results obtained.

- In [72], jointly with the PostDoc A.Guzmán Adán and with K.Diki, we provide an alternative description of the Fueter-Sce-Qian theorem in terms of the generalized CK-extension. The latter characterizes axial null solutions of the Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ in terms of their restrictions to the real line. This leads to a one-to-one correspondence between the space of axially monogenic functions in $\mathbb{R}^{n+1}$

and the space of analytic functions of one real variable. We provide explicit expressions for the Fueter-Sce-Qian map in terms of the generalized CK-extension for both cases, n even and n odd.

- Using the Cauchy formula of slice hyperholomorphic functions the Fueter-Sce-Qian theorem admits an integral representation for n odd. In [38], jointly with Professors F. Colombo, T. Qian and I. Sabadini, we show that the important relation $\Delta_{n+1}^{(n-1)/2} S_L^{-1} = F_n^L$ between the slice monogenic Cauchy kernel S_L^{-1} and the F-kernel F_n^L, that appears in the integral form of the Fueter-Sce-Qian theorem for n odd, holds also in the case we consider the fractional powers of the Laplace operator Δ_{n+1} in dimension $n+1$, i.e., for n even. Moreover, this relation is proved by computing explicitly the Fourier transform of the kernels S_L^{-1} and F_n^L as functions of the Poisson kernel. Similar results hold for the right kernels S_R^{-1} and F_n^R.

- By writing the Fueter-Sce-Qian extension theorem in integral form and it is possible to define the F-functional calculus for n-tuples of commuting operators. This functional calculus is defined on the S-spectrum and generates a monogenic functional calculus in the spirit of McIntosh and collaborators. The main goal of the papers [35, 36], joint works with Professors F. Colombo and I. Sabadini, is to show that the F-functional calculus generates the Riesz projectors. The existence of such projectors is obtained via the F-resolvent equation which was previously known only in the quaternionic setting and also its existence was under question. We prove the F-resolvent equation in the Clifford algebra setting. It is much more complicated than the one in the quaternionic case since it contains various pieces, however it still allows to nicely define the Riesz projectors.

- The results obtained in [34], jointly with Professors F. Colombo, S. Pinton and I. Sabadini, can be considered a seminal work on the introduction of an harmonic functional calculus based on the S-spectrum and on an integral representation of axially harmonic functions. This new calculus is a bridge between harmonic analysis and the spectral theory. The resolvent operator of the harmonic functional calculus is the commutative version of the pseudo S-resolvent operator. This calculus also appears, in a natural way, in the product rule for the F-functional calculus.

- In the second step of the Fueter construction an axially monogenic function is built by applying the Laplace operator Δ in four real variables to a slice hyperholomorphic function. In the papers [73, 74], jointly with Professor. S. Pinton, we use the factorization of the Laplace operator, i.e. $\Delta = \overline{\mathcal{D}}\mathcal{D}$ to split the previous procedure. From this splitting we get a class of functions that lies between the set of slice hyperholomorphic functions and the set of axially monogenic functions: the set of axially polyanalytic functions of order 2, i.e. null-solutions of $\mathcal{D}^2$. We show an integral representation formula for this kind of functions. The formula obtained is fundamental to define the associated functional calculus on the S-spectrum. Moreover we show the principal properties of this functional calculus. In particular we study a resolvent equation suitable for proving a product rule and generate the Riesz projectors.

The last part of this dissertation is devoted to present new functional calculi based on the S-spectrum and on the factorization of the Fueter-Sce map $\Delta^{\frac{n-1}{2}}$. This case is completely different from the quaternionic case presented in the second part and it will be the centre of our research for the next future.

- In [37], jointly with Professors F. Colombo, S. Pinton and I. Sabadini, we show that the extension operator from slice hyperholomorphic functions to monogenic functions admits various possible factorizations that induce different function spaces. The integral representations in such spaces allows to define the associated functional calculi based on the S-spectrum. The function spaces and the associated functional calculi define the so called *fine structure of the spectral theories on the S-spectrum*. Among the possible fine structures there are the harmonic and poly-harmonic functions and the associated harmonic and poly-harmonic functional calculi. The study of the fine structures depends on the dimension considered.

Based on the techniques developed in this thesis, we already started some new projects. We aim to tackle them in the near future. We plan to start new research investigations in the following directions

1) The F-resolvent equation for all dimensions. In this case we have to deal with fractional powers.

2) Figure out if it is possible to get a sort of commutation rule between the generalized modulation and translation operator.

3) The H^∞ calculus and the Phillips functional calculus for the F-functional calculus and the for the all *fine structures*.

4) Establish a generalized Cauchy-Kovalevskaya extension for axially harmonic functions.

5) Study the function spaces that arise from the factorization of the Fueter-Sce mapping theorem, both in the complex and hypercomplex setting.

Contents

CHAPTER 1

Introduction

In the literature there are two possible ways to extend the notion of holomorphic function of one complex variable to higher dimensions. An approach is to consider the systems of Cauchy-Riemann equations for a function of several complex variables

$$f : \Pi \subseteq \mathbb{C}^n \to \mathbb{C},$$

where Π is an open set. This gives rise to the theory of holomorphic functions in several complex variables. Another possibility is to consider quaternionic or Clifford algebra valued functions. The main advantage, in this case, is that it is still possible to work with one hypercomplex variable. This implies however that the commutativity of the product is missing. In this dissertation we are interested in the second possibility.

In the framework of quaternionic and Clifford analysis there are various notions of hyperholomorphic functions, among which two are prominent: the monogenic theory and the slice hyperholomorphic theory, both generalize the notion of holomorphic function of one complex variable. These different theories are related each other by the Fueter-Sce-Qian extension, which consists of two steps.

Step (A) extends the set of holomorphic function to the class of slice hyperholomorphic functions, by means of the so-called slice operator. This consists of replacing the complex variable $z = u + iv$ with the paravector variable $x = x_0 + \underline{x}$, where the complex unit i is replaced by $I = \frac{\underline{x}}{|\underline{x}|}$ in $\mathbb{R}^n$.

Step (B) extends slice hyperholomorphic functions to monogenic functions (or Fueter regular functions in the case of the quaternions). This extension is performed by the so called Fueter-Sce-Qian map, namely the Laplace operator in $n + 1$ variables applied $\frac{n-1}{2}$ times. If the dimension n is odd we are dealing with a pointwise differential operator, and we call this operator Fueter-Sce map. If we work in the quaternionic case, which coincides with the case $n = 3$, the operator is called Fueter map. If the dimension n is even one has to work in the setting of the fractional powers. We can summarize the Fueter-Sce construction with the following diagram

$$\mathcal{O}(D) \xrightarrow{T_F} \mathcal{SH}(\Omega_D) \xrightarrow{\Delta^{\frac{n-1}{2}}} \mathcal{AM}(\Omega_D),$$

where $\mathcal{O}(D)$ is the set of holomorphic functions defined in the symmetric open set $D \subseteq \mathbb{C}$, the set $\mathcal{SH}(\Omega_D)$ denotes the set of slice holomorphic functions defined in $\Omega_D \subset \mathbb{R}^{n+1}$, which is an open set induced by D, and $\mathcal{AM}(\Omega_D)$ is the set of axially monogenic functions.

This dissertation is divided in three parts. In the first part we focus on some specific integral transforms in the slice hyperholomorphic and monogenic settings. In the second part of this work we introduce new functional calculi based on the S-spectrum and on the Fueter-Sce theorem.

By combining the Cauchy formula for slice hyperholomorphic functions and the Fueter-Sce mapping theorem, it is possible to get an integral transform that maps slice hyperholomorphic functions to axially monogenic functions. This integral transform is called the Fueter-Sce theorem in integral form. This, together with the notion of S-spectrum, is fundamental to define a new monogenic functional calculus: the F-functional calculus. Moreover by factorizing the Fueter map we obtain a harmonic and polyanalytic functional calculus on the S-spectrum.

Finally in the last part of the dissertation, we split the Fueter-Sce map in the Clifford algebra setting. This suggests the introduction of new functional calculi, two of them are in the well known settings of polyharmonic and polyanalytic functions. This last part can be considered a seminal work on the factorization of the Fueter-Sce map and it will be the centre of our next researches.

Now, we introduce the state of the art of the first part of the thesis. We begin by introducing the definitions of the Bargmann and short time Fourier transforms in complex analysis.

In [22] the author introduced for the first time a Hilbert space of entire functions, where the creation and annihilation operators, given by

$$M_z f(z) := z f(z) \quad \text{and} \quad D f(z) := \frac{d}{dz} f(z) \tag{1.1}$$

are closed, densely defined operators that are adjoints each others. Moreover, these operators satisfy the classical commutation rule

$$[D, M_z] = \mathcal{I}$$

where $[.,.]$ and $\mathcal{I}$ are the commutator and the identity operators, respectively. Nowadays, this space is known as Fock or Segal-Bargmann space and it is given by:

$$\mathcal{F}^2(\mathbb{C}^n) := \{ f \in \mathcal{O}(\mathbb{C}^n) \mid \int_{\mathbb{C}^n} |f(z)|^2 e^{-2\pi |z|^2} d\lambda(z) < \infty \},$$

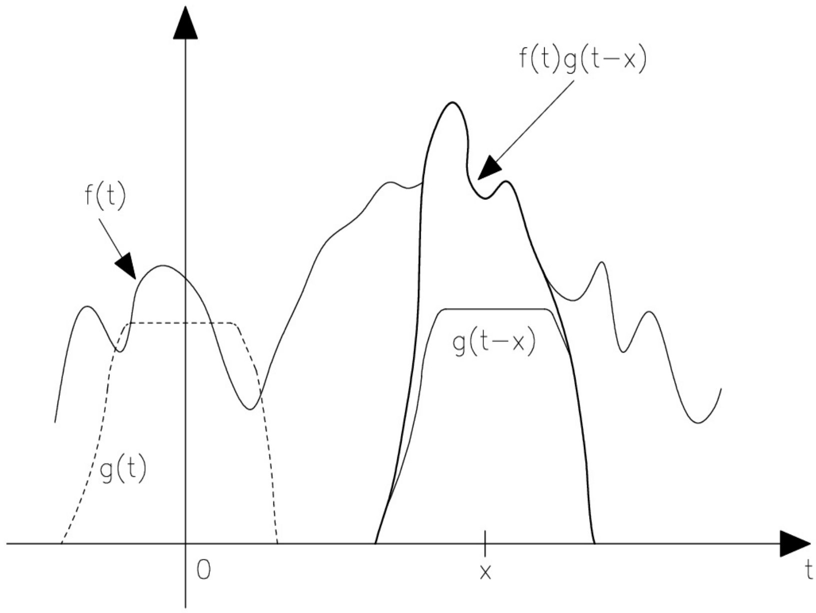
where $d\lambda(z)$ is the Lebesgue measure. This space is a reproducing kernel Hilbert space. It turns out that the operators defined in (1.1) are unitary equivalent to the position and momentum operators in quantum mechanics through the so-called Segal Bargmann transform. This is an unitary integral transform that maps functions in the space $L^2(\mathbb{C}^n)$ onto the Fock space and it is defined by

$$\mathcal{B}f(z) = \int_{\mathbb{R}^n} e^{2\pi\sqrt{2}\pi x\cdot z - \pi x\cdot x - \pi z\cdot z} f(x) dx. \tag{1.2}$$

The Segal-Bargmann transform plays also a very important role in time frequency analysis. Indeed there is a relation with the short time Fourier transform with Gaussian window, which is denoted by $\mathcal{V}_\varphi f(x, \omega)$. The relation is given by the formula

$$\mathcal{B}(f) \left(\frac{\bar{z}}{\sqrt{2}} \right) = e^{i\pi x \omega} e^{\frac{|x|^2 \pi}{2}} \mathcal{V}_\varphi f(x, \omega), \qquad \omega \in \mathbb{R}^n, \quad z = x + i\omega.$$

The idea of the short-time Fourier transform, for general window functions, is to obtain information about local properties of the signal f. To this end, the signal f is restricted to an interval and after its Fourier transform is evaluated. However, since a sharp cut-off can introduce artificial discontinuities and can create problems, it is usually chosen a smooth cut-off function φ called "window function".



Then for $f, g \in L^2(\mathbb{R}^n)$ the short-time Fourier transform is given by

$$V_g f(x,\omega) = \int_{\mathbb{R}^n} f(t)\overline{g(t-x)} e^{-2\pi i t\cdot\omega} dt.$$

For this kind of transform it is possible to show properties like the Parseval identity, the Plancherel formula, an uncertainty principle and a Paley-Wiener theorem, see [92]. The short-time Fourier transform is used in several applications such as the predictions of sound source position emanated by fault machine and the interpretation of ultrasonic waveforms.

Now we move to illustrate the state of the art related to the second part of the thesis: the functional calculi based on the S-spectrum.

The first mathematicians who understood the importance of hypercomplex analysis to define functions of noncommuting operators on Banach spaces have been A. McIntosh and his collaborators, starting from preliminary results in [102]. Using the theory of monogenic functions they developed the monogenic functional calculus and several of its applications, see [99, 101, 108, 112].

The S-spectrum is based on the slice hyperholomorphic theory of functions and their Cauchy formula. The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,x)$ is given by

$$S_L^{-1}(s,x) = \sum_{m=0} x^m s^{-1-m} = (x^2 - 2x_0 s + |s|^2)^{-1}(s - \bar{x}). \qquad (1.3)$$

The series expansions is convergent when $|x| < |s|$, and the closed expression is defined for any $x \notin [s]$. Thus, we can write any left slice hyperholomorphic function as

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S_L^{-1}(s,x)\, ds_J\, f(s),$$

where the set U is a suitable open set contained in the domain of the function f. For $x, s \in \mathbb{R}^{n+1}$, with $x \notin [s]$, we have the following identity

$$-(x^2 - 2s_0 x + |s|^2)^{-1}(x - \bar{s}) = (s^2 - 2x_0 s + |x|^2)^{-1}(s - \bar{x}).$$

This implies that it is possible to write the slice hyperholomorphic Cauchy kernel in the following equivalent way

$$S_L^{-1}(s,x) := (s^2 - 2x_0 s + |x|^2)^{-1}(s - \bar{x}). \qquad (1.4)$$

Even though $S_L^{-1}(s,x)$ written as in (1.3) is more suitable for several applications, for example for the definition of a functional calculus, it does not allow easy computations of the powers of the Laplacian in $n+1$ variables $\Delta := \partial_{x_0}^2 + \sum_{j=1}^n \partial_{x_j}^2$, with respect to the variable x applied to it. The slice hyperholomorphic Cauchy kernel written as in (1.4) is the one that allows, by iteration, the computation of $\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x)$. Indeed, for x, $s \in \mathbb{R}^{n+1}$ with $x \notin [s]$ we have

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) := F_L(s,x) = \gamma_n (s - \bar{x})(s^2 - 2x_0 s + |s|^2)^{-2}, \qquad (1.5)$$

where γ_n is a suitable constant depending on the dimension of the Clifford algebras, see (7.47). The function $F_L(s,x)$ has different type of regularities in the two variables: it is right slice hyperholomorphic the variable s while it is axially monogenic in x. By combining (1.5) and the Cauchy formula for slice hyperholomorphic functions a Fueter theorem in integral form has been proved in [54]. The left axially monogenic function $\breve{f}(x) := \Delta^{\frac{n-1}{2}} f(x)$, with f a slice hyperholomorphic function, admits the integral representation

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} F_L(s,x) ds_J f(s), \qquad (1.6)$$

where U is a suitable open set which contains the domain of the function f. The main advantage of (1.6) is that it gives the possibility to get axially monogenic functions by suitably integrating slice hyperholomorphic functions.

The Cauchy formula of slice hyperholomorphic functions generates the S-functional calculus for quaternionic linear operators or for n-tuples of not necessarily commuting operators. This calculus is based on the notion of S-spectrum. This was discovered in 2006 by F. Colombo and I. Sabadini as it is explained in the introduction of the book [59].

Let V_n be a two sided Clifford Banach module and let $T : V_n \to V_n$ be a bounded right (or left) linear operator. The S-spectrum of T is defined as

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} \quad : \quad T^2 - 2s_0T + |s|^2\mathcal{I} \quad \text{is not invertible in } \mathcal{B}(V_n)\},$$

and the S-resolvent set

$$\rho_S(T) := \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

The set $\mathcal{B}(V_n)$ denotes the set of bounded right linear operators acting on V_n.

The existence of an appropriate quaternionic spectrum was suggested by the formulation of quaternionic quantum mechanics given by G. Birkhoff and J. von Neumann [26]. A quaternionic formulation of quantum mechanics is systematically studied in the monograph [5]. However, such a formulation of quantum mechanics turned out to be more involved than expected due to a lack of an appropriate definition of spectrum in the quaternionic setting. Recently, there have been some progresses in the study of quaternionic quantum mechanics. For example, in [86] it was proved that the equivalence of real and complex quantum mechanics and the equivalence of complex and quaternionic quantum mechanics are dual problems that can be solved with the same techniques.

In accordance to (1.3), for $s \in \rho_S(T)$ we can define the left S-resolvent operator of T at s as

$$S_L^{-1}(s,T) := -(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}).$$

This is a $\mathcal{B}(V_n)$-valued slice hyperholomorphic function in the variables s and so we can define for any left slice hyperholomorphic function f defined in a suitable open set U, with $\sigma_S(U) \subset U$, the S-functional calculus:

$$f(T) = \int_{\partial(U\cap\mathbb{C}_J)} S_L^{-1}(s,T)ds_Jf(s).$$

The S-functional calculus for n-tuples of operators generalizes the Riesz-Dunford-functional calculus for holomorphic functions. It is also possible to define the S-functional calculus for right slice hyperholomorphic functions. The right S-resolvent operator of T at s is defined by

$$S_R^{-1}(s,T) := -(T-\overline{s}\mathcal{I})(T^2-2s_0T+|s|^2\mathcal{I})^{-1}.$$

Then, we can define the S functional calculus for a right slice hyperholomorphic function f, by

$$f(T)=\int_{\partial(U\cap\mathbb{C}_J)} f(s)ds_J S_R^{-1}(s,T).$$

For the S functional calculus it is possible to show a resolvent equation, that is called S-resolvent equation. One of the main differences with respect to the resolvent equation in Riesz-Dunford functional calculus is that both S-resolvent operators are involved. Precisely, we have

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T)= \quad & \big[[S_R^{-1}(s,T)-S_L^{-1}(p,T)]p-\overline{s}[S_R^{-1}(s,T)-S_L^{-1}(p,T)]\big] \\ & (p^2-2s_0p+|s|^2)^{-1}, \end{aligned}$$

for $s,p\in\rho_S(T)$.

In the sequel, we will consider bounded paravector operators T, with commuting components $T_\ell\in\mathcal{B}(V)$ for $\ell=0,1,\dots,n$, with n odd. This set of operators is denoted by $\mathcal{BC}(V_n)$, and it is a subset of $\mathcal{B}(V_n)$.

The S-functional calculus admits a commutative version. For paravector operators $T=T_0+e_1T_1+\ldots+e_nT_n$ such that $T\in\mathcal{BC}(V_n)$ the F-spectrum of T is defined as

$$\sigma_F(T)=\{s\in\mathbb{R}^{n+1}\ \ :\ \ \ s^2\mathcal{I}-s(T+\overline{T})+T\overline{T}\ \ \text{is not invertible in } \mathcal{B}(V_n)\},$$

where we have set $\overline{T}:=T_0-e_1T_1-\ldots-e_nT_n$, and the F-resolvent set

$$\rho_F(T):=\mathbb{R}^{n+1}\setminus\sigma_F(T).$$

It turns out that the F-spectrum is the commutative version of the S-spectrum, i.e., we have

$$\sigma_F(T)=\sigma_S(T),\ \ \text{for}\ \ T_0+e_1T_1+\ldots+e_nT_n\in\mathcal{BC}(V_n).$$

In the sequel also for the commutative version of the S-functional calculus we will use the symbol $\sigma_S(T)$. The definition of the commutative S-spectrum comes from the Cauchy kernel written in the second form, see

(1.4). In fact, for paravector operators $T \in \mathcal{BC}(V_n)$, the commutative version of left S-resolvent operator is given by

$$S_L^{-1}(s,T) := (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1}, \quad s \in \rho_S(T), \tag{1.7}$$

and the commutative version of the right S-resolvent operator is

$$S_R^{-1}(s,T) := (s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1}(s\mathcal{I} - \overline{T}), \quad s \in \rho_S(T). \tag{1.8}$$

For the sake of simplicity we have still denoted the commutative version of the S-resolvent operators with the same symbols as for the noncommutative ones. The operator

$$\mathcal{Q}_{c,s}(T)^{-1} := (s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1}, \quad s \in \rho_S(T),$$

is called commutative pseudo S-resolvent operator, for short, pseudo resolvent operator. In the sequel, when we mention the S-resolvent operators we intend their commutative versions.

The Fueter-Sce mapping theorem in integral form is the crucial object for the definition of the F-functional calculus. It is a mongenic functional calculus, in the same spirit of McIntosh and collaborators, based on the commutative version of the S spectrum.

We now define the F-resolvent operators. Let n be an odd number, we define the left F-resolvent operator as

$$F_n^L(s,T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-\frac{n+1}{2}}, \quad s \in \rho_S(T) \tag{1.9}$$

and the right F-resolvent operator as

$$F_n^R(s,T) := \gamma_n(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-\frac{n+1}{2}}(s\mathcal{I} - \overline{T}), \quad s \in \rho_S(T). \tag{1.10}$$

For a left slice hyperholomorphic function f and $\breve{f} = \Delta^{\frac{n-1}{2}} f$ we define the F-functional calculus as

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,T)\, ds_J\, f(s),$$

and for right slice hyperholomorphic function the F-functional calculus is given by

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} f(s)\, ds_J\, F_n^R(s,T).$$

As for the Riesz-Dunford functional calculus and the S-functional calculus, the resolvent equation for the F-functional calculus plays an important role. However, the F-resolvent equation has further differences with respect to the complex resolvent equation and with respect to the S-resolvent equation. This is a consequence of the fact that the F-functional calculus is based on an integral transform and not on a Cauchy formula.

The F-resolvent equation for $n = 3$ is known since some years and it coincides with the quaternionic F-resolvent equation, precisely it is given by

$$\begin{aligned}
&F_3^R(s,T)S_L^{-1}(x,T) + S_R^{-1}(s,T)F_3^L(x,T)\\
&-\frac{1}{4}\Big(sF_3^R(s,T)F_3^L(x,T)x - sF_3^R(s,T)TF_3^L(x,T)\\
&-F_3^R(x,T)TF_3^L(x,T)x + F_3^R(s,T)T^2F_3^L(x,T)\Big)\\
=&\big[\left(F_3^R(s,T) - F_3^L(x,T)\right)x - \bar{s}\left(F_3^R(s,T) - F_3^L(x,T)\right)\big](x^2 - 2s_0x + |s|^2)^{-1}.
\end{aligned}$$

for $T \in \mathcal{BC}(V_3)$ and for any $p, s \in \rho_S(T)$, with $s \notin [p]$.

All the constructions presented so far can be summarized in the following diagram

$$\begin{array}{ccc}
\mathcal{SH}(\Omega_D) & & \mathcal{AM}(\Omega_D)\\
\downarrow & & \\
\text{Slice Cauchy Formula} & \xrightarrow{T_{FS2}} & \text{Fueter} - \text{Sce theorem in integral from}\\
\downarrow & & \downarrow\\
\text{S}-\text{Functional calculus} & & F - \text{functional calculus}
\end{array}$$

Description of the contents

The thesis is divided in 13 chapters besides this introduction. In chapter 2 we give a brief overview of the Riesz-Dunford functional calculus for linear bounded operators. We recall the main properties like the resolvent equation, the so called Riesz projectors and the product rule. In chapter 3 we give in details the state of the art of the S-functional calculus and the F-functional calculus. Furthermore we revise the main notions for the theory of slice hyperholomorphic functions and we recall the definition and some relevant properties of the set of monogenic functions. The main contributions to the research are contained from the chapter 4 to chapter 13. In the last chapter there is an Appendix, containing computations related to the complex Hermite functions, which we did not find in literature, and refinements of the Fueter-Sce construction.

Now, we give a brief description of the contents of each chapter.

- This chapter is based on [70]. We first recall the definitions and the main properties of the Fock space and the Segal Bargmann transform in the quaternionic setting. These were originally introduced in [15, 76]. Then, we characterize the range of the Schwartz space under the Segal Bargmann transform. We also take into account relations related to the position and momentum operators in this framework. The main goal is to study a quaternionic analogue of the short time Fourier transform (QSTFT) in dimension one with Gaussian window function $\varphi(t)=2^{1/4}e^{-\pi t^2}$. To this end, we extend the formula

  $$\mathcal{B}(f)\left(\frac{\bar{z}}{\sqrt{2}}\right)=e^{i\pi x\omega}e^{\frac{|x|^2\pi}{2}}\mathcal{V}_\varphi f(x,\omega),\qquad \omega\in\mathbb{R}^n,\quad z=x+i\omega,$$

  where B is the complex Bargmann transform, to quaternionic setting. Note that we need to use the counterpart of the Segal-Bargmann transform in the quaternionic setting and the slice representation of the quaternions. For f, $g\in L^2(\mathbb{R},\mathbb{H})$ we have the following Moyal identity for the QSTFT with Gaussian window

  $$\langle\mathcal{V}_\varphi f,\mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}=2\langle f,g\rangle_{L^2(\mathbb{R},\mathbb{H})}.$$

  Moreover, we prove that is possible to reconstruct the signal f if we know its QSFT. This is the idea behind the reconstruction formula

  $$f(y)=2^{-\frac{1}{4}}\int_{\mathbb{R}^2}e^{2\pi J\omega y}\mathcal{V}_\varphi f(x,\omega)e^{-\pi(y-x)^2}dxd\omega,\ \forall y\in\mathbb{R}.$$

  Finally, we show that it is possible to write the quaternionic Fourier transform in terms of the reproducing kernel of the so-called Gabor space

  $$\mathcal{G}_{\mathbb{H}}^{\varphi}:=\{\mathcal{V}_\varphi f,\ f\in L^2(\mathbb{R},\mathbb{H})\}.$$

- This chapter is based on [71]. We generalize the previous construction of the short time Fourier transform for more general windows functions: the weighted Hermite functions, defined as

  $$\psi_n^\nu(x):=\frac{(-1)^n e^{\frac{\nu}{2}x^2}\frac{d^n}{dx^n}\left(e^{-\nu x^2}\right)}{2^{n/2}\nu^{n/2}(n!)^{1/2}\pi^{1/4}\nu^{-1/4}}.$$

If we consider $\nu = 2\pi$ and $n = 0$ in the previous formula we get the Gaussian function $\phi_0(t) = 2^{1/4}e^{-\pi t^2}$. In order to study the QSTFT with the weighted Hermite functions as windows we need the theory of slice polyanalytic functions, developed in [16, 17]. In the first section of this chapter we recall the definition and the main properties of this function theory. Later, we recall the definition of the polyanalytic Fock space, defined for a given $J \in \mathbb{S}$ and $n \geq 1$ to be

$$\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H}) := \{f \in \mathcal{SP}_{n+1}(\mathbb{H}) : \int_{\mathbb{C}_I} |f_I(q)|^2 e^{-2\pi|q|^2}\, d\lambda_I(q) < \infty\},$$

where $d\lambda_I(q)$ is the Lebesgue measure on the slice $\mathbb{C}_I$. In the polyanalytic theory it is possible to define also a different polyanalytic Fock space called *true quaternionic polyanalytci Fock space*, denoted by $\mathcal{F}_T^n(\mathbb{H})$. We show the following relation among the two different Fock spaces

$$\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H}) = \bigoplus_{j=0}^{n} \mathcal{F}_T^j(\mathbb{H}).$$

As in the previous chapter our aim is to study a polyanalytic Bargmann transform in order to get information about the a the QSTFT with weighed Hermite functions as windows. Moreover, we generalize the following identity

$$\mathcal{V}_{\psi_n}\varphi(x,\omega) = e^{-i\pi x\omega} G^{n+1}(\varphi)\left(\frac{\bar{z}}{\sqrt{2}}\right) e^{-\frac{|z|^2\pi}{2}}, \tag{1.11}$$

where G^{n+1} is the complex true polyanalytic Bargmann transform. In this context it is possible to consider a QSTFT of a vector-valued function $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ with respect to $\vec{\psi} = (\psi_0, ..., \psi_n)$. Also for this kind of signal it is possible to have a relation as (1.11). Let us consider the formula

$$\mathbf{V}_{\vec{\psi}}\vec{\varphi}(x,\omega) = e^{-\pi i x\omega} \mathbf{G}\vec{\varphi}\left(\frac{\bar{z}}{\sqrt{2}}\right) e^{-\frac{\pi|z|^2}{2}}, \tag{1.12}$$

where $\mathbf{G}$ is the full polyanalytic Bargmann transform in the complex setting. In this chapter we show that it is possible to write the true quaternionic Bargmann transform in the following way

$$B^{k+1}\varphi(q) = c_k \int_{\mathbb{R}} e^{-\pi(q^2+x^2)+2\pi\sqrt{2}qx} H_k\left(\frac{q+\overline{q}}{\sqrt{2}} - x\right)\varphi(x)dx,$$

where $c_k := 2^{\frac{3}{4}}(2^k k!(2\pi)^k)^{-\frac{1}{2}}$ and H_k are the Hermite polynomials. This expression is crucial to get a QSTFT with weighted Hermite functions as windows and to show all the main properties of this integral transform. In this chapter we show also that in the quaternionic polyanalytic theory it is possible to define a polyanalytic Bargmann transform for vector-valued signal $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ and it is defined as a sum of $n+1$ quaternionic true polyanalytic Bargmann transform. By means of this and formula (1.12) we define a QSTFT with vector-valued signal.

- This chapter is based on [68]. We provide a generalization of the short time Fourier transform in the Clifford algebra setting. The basic tool to get a good definition of this transform is to use the so-called Clifford-Fourier transform

$$\mathcal{F}_{\pm}f(y) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_{\pm}(x,y)f(x)\,dx,$$

where the kernel $K_{\pm}(x,y)$ is given by a combination of Bessel functions, see (6.7). In order to show the main properties of the Clifford short time Fourier transform we recall the definitions of the generalized translation and modulation operators, introduced in [66]. Then we define the Clifford short time Fourier transform as

$$\mathcal{V}_g f(x,\omega) = \mathcal{F}_{-}(\tau_x \bar{g} \cdot f)(\omega),$$

where τ_x is the generalized translation operator. If the function g, in the previous formula, is radial, then we get the following expression

$$\mathcal{V}_g f(x,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_{-}(t,\omega)g(t-x)f(t)\,dt.$$

Moreover, we show that it is possible to write the Clifford short time Fourier transform as a combination of the generalized modulation and translation operators. Precisely, we show the following

$$\mathcal{V}_g f(x,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{M_\omega \tau_x g(t)} f(t)\,dt.$$

We prove that all the properties that usually holds for the short-time Fourier transform like the orthogonality relation, the Moyal formula and Lieb's uncertainty principle are also valid in this context. Furthermore, we show that the behaviour of the generalized modulation and generalized translation applied to Clifford-valued signal and a radial window function is different from the classic case.

- In this chapter we give an overview of the Fueter-Sce theorem. We start by giving, with all the details, the original proof of the Fueter-Sce theorem done by M.Sce in [126]. He performed the proof in a very general and pioneering way, because it is done for a generic quadratic algebra. As particular cases we can find the algebra of quaternions and the paravectors. In these cases the operator that transforms slice hyperholomorphic functions in axially monogenic functions is $\Delta^{\frac{n-1}{2}}$, where Δ is the Laplace operator in $n+1$ variables and n is an odd number. The case $n = 3$ coincides with the quaternionic case. In the case n even we have to deal with fractional powers of the Laplacian and so we have to use the techniques of the Fourier multipliers, see [122]. In the case of n odd F. Colombo, I. Sabadini and F. Sommen, see [54], proved an integral version of the Fueter-Sce theorem. Precisely, given a slice hyperholomorphic function f they were able to get an axially monogenic function $\breve{f}$ by an integral transform whose kernel has an interesting form. In the last section we bring together the Fueter-Sce theorem and the generalized Cauchy-Kovalevskaya (CK) extension. This result asserts that axially monogenic functions are completely determined by their restriction to the real line. Conversely, any real analytic function has a unique generalized CK extension. This means that this operator is an isomorphism. Since the Fueter-Sce map is a surjective map but is not injective, it does not coincide with the generalized CK-extension. We provide a match among the Fueter-Sce map and the generalized CK-extension in the case of n odd and even. This part of the chapter is based on [72].

- This chapter is based on [38]. We recall in the previous chapter the action of the operator $\Delta^{\frac{n-1}{2}}$, in the variable x with n being an odd number, to the slice hyperholomorphic Cauchy kernel written in second form see (1.4). The expression is very simple and it is given by

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}. \qquad (1.13)$$

  In this chapter, we generalize the previous formula for any n. However, in the case of n being an even number we have to deal with the Fourier multipliers

$$(-\Delta)^{\frac{n-1}{2}} = \mathrm{F}^{-1} \left(2\pi|\cdot|\right)^{n-1} \mathrm{F} \qquad (1.14)$$

  in order to give meaning to the fractional powers of the Laplace operator. First of all, we show that the following function

$$k_L(s,x) := (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-\lambda}, \qquad \lambda \in \mathbb{R},$$

for $s^2 - 2x_0 s + |x|^2 \in \mathbb{R}^{n+1} \setminus (-\infty, 0]$, is left monogenic in the variable x if and only if $h = \frac{n-1}{2}$. In order to generalize formula (1.13) for any dimension we compute the Fourier transform of the slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,x)$ with respect to the variable x. Denoting by F the Fourier transform we get

$$\mathrm{F}[S_L^{-1}(s,\cdot)](\xi) = c_n \, \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \, e^{-is\xi_0}, \quad \xi_0 + \underline{\xi} \neq 0 \tag{1.15}$$

As further step we compute the Fourier transform of the following function

$$F_n^L(s,x) := \gamma_n (s - \bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}}.$$

The result obtained is the following

$$\mathrm{F}[F_n^L(s,\cdot)](\xi) = k_n \, \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2} \, e^{-is\xi_0}, \quad \xi_0 + \underline{\xi} \neq 0 \tag{1.16}$$

By putting together (1.15), (1.16) and (1.14) we get the desired result.

- This chapter is based on [35,36]. The main topic is the study of the so-called F-functional calculus. This arises form the Fueter-Sce theorem in integral form when we formally replace the paravector variable x with the operator T. The F-functional calculus is a monogenic functional calculus, in the same spirit of McIntosh and collaborators, based on the theory of the S-spectrum. This functional calculus was originally introduced in the paper [41]. In this chapter we solve some of the main problems left open in that paper. First of all by means of the Clifford-Appell polynomials, given by

$$P_k^n(x) := \sum_{s=0}^{k} T_s^k(n) x^{k-s} \overline{x}^s, \qquad x \in \mathbb{R}^{n+1},$$

where the coefficients $T_s^k(n)$ are constants depending on the dimension of the algebra, we write a series expansion of the F-kernels. This is obtained for paravectors x and s such that $|x| < |s|$ and it is given by

$$F_n^L(s,x) = \sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n) \, x^{m+1-n-\ell} \, \bar{x}^\ell \, s^{-1-m}, \tag{1.17}$$

where $K_\ell(m,n)$ are constants depending on the dimension of the algebra. We find also a series expansion for the resolvent operators of the F-functional calculus. It was enough to replace formally the operator T to the paravector x in the formula (1.17). We give the following definition of F-resolvent operator for a paravector operator T with commuting components

$$F_n^L(s,T) = \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - (T+\overline{T})s + T\overline{T})^{-\frac{n+1}{2}},$$

for n being an odd number. The central aim of this chapter is to get a resolvent equation for the F-functional calculus in the Clifford-setting, we denote this object F-resolvent equation. The quaternionic case has been investigated in [41], but the case of general n (odd number) has been an open problem for some years. In the case $n = 3$ (which coincides with the quaternionic case) the F-resolvent equation is written just in terms of the S-resolvent operators and the F-resolvent operators. However, this is not always possible in the general case. In order to explain how to obtain the F-resolvent equation in the general case we treat separately the cases $n = 5$ and $n = 7$. In the case $n = 5$ we show that the equation can be written in a quite reasonable way in terms of the S-resolvent operators and F-resolvent operators. The case $n = 7$ shows that it is not possible to have a simple closed form for the F-resolvent equation just in terms of the S-resolvent operators and of the F-resolvent operators. Instead, the use of the pseudo S-resolvent operators allows a reasonable structure of the resolvent equation. From the case $n = 7$ is not anymore possible to write the F-resolvent equation only in terms of F-resolvent operators, because it would lead to an equation that is too complicated. The interesting symmetries that appear in the F-resolvent equation are fundamental to study the Riesz projectors in this setting, which are defined by the following operators

$$\begin{aligned}\check{P} &= \frac{1}{\gamma_n(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)\,dp_J p^{n-1}\\ &= \frac{1}{\gamma_n(2\pi)}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{n-1}ds_J F_n^R(s,T),\end{aligned}$$

where G_1 and G_2 contain part of the S-spectrum. In the monogenic functional calculus developed by McIntosh and collaborators the resolvent equation is missing. They are able to study the Riesz projectors by using another functional calculus: the Weyl calculus. We start to study the Riesz projectors in the case $n = 5$, since in this case we have a F-resolvent equation written in terms of the F-resolvent operators. For the cases with n more than five we divide the study in the cases of the parity of $h = \frac{n-1}{2}$.

- This chapter is based on [34]. We establish a new functional calculus, that can be considered the harmonic version of the Riesz-Dunford functional calculus. The principle tools to obtain this new functional calculus are the Fueter mapping theorem and the theory of the S-spectrum. By factorizing the Laplace operator in terms of the Fueter operator , namely $\Delta = \overline{\mathcal{D}}\mathcal{D}$, we have a refinement of the Fueter construction given by

$$\mathcal{O}(D) \xrightarrow{T_{F1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D).$$

So, in the quaternionic setting, there is the class of axially harmonic functions that lies between the set of slice hyperholomorphic functions and axially monogenic functions. All the sets of functions spaces and the associated functional calculi induced by the factorization of the Fueter-Sce map are called *fine structures*. A possible fine structure of the quaternionic spectral theory on the S-spectrum is given by the following diagram

$$\begin{tikzcd}
\mathcal{SH}(\Omega_D) \arrow[d] & \mathcal{AH}(\Omega_D) & \mathcal{AM}(\Omega_D) \\
\text{Slice Cauchy Formula} \arrow[r, "\mathcal{D}"] \arrow[d] & \mathcal{AH} \text{ in integral form} \arrow[r, "\overline{\mathcal{D}}"] \arrow[d] & \text{Fueter thm. in integral form} \arrow[d] \\
S-\text{Functional calculus} & \text{Harmonic functional calculus} & F-\text{functional calculus}
\end{tikzcd}$$

In this section we show that we can write an axially harmonic function $\tilde{f}(q)$ in integral form. Precisely, given a left slice hyperholomorphic function f we have the following formula

$$\tilde{f}(q) = \mathcal{D}f(q) = -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(q)^{-1} ds_J f(s), \tag{1.18}$$

for a suitable open set U. By formally replacing the variable q with an operator T with commuting components we have

$$\tilde{f}(T) := -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s),$$

with $\tilde{f} = \mathcal{D}f$. This is the definition of the harmonic functional calculus on the S-spectrum or equivalently called Q-functional calculus. We prove several properties of this functional calculus like the basic algebraic rules and the independence from the set U and the imaginary unit $J \in \mathbb{S}$. Furthermore, we get a resolvent equation for the Q-functional calculus, which is given by

$$\begin{aligned}&\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T) + S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1} - 2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1}\\ &= [(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})p - \bar{s}(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})]\\ &\quad \mathcal{Q}_s(p)^{-1}. \end{aligned} \tag{1.19}$$

By means of this equation we show a product formula of the Q-functional calculus and the interesting symmetries of equation (1.19) allow to show that the following operators

$$\begin{aligned}\tilde{P}(T) &:= \frac{1}{2\pi}\int_{\partial(G_2\cap\mathbb{C}_J)} s\, ds_J\, \mathcal{Q}_{c,s}(T)^{-1}\\ &= \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1}\, dp_J p. \end{aligned} \tag{1.20}$$

are projectors, namely $\tilde{P}^2 = \tilde{P}$. The operator (1.20) is called Riesz projectors for the Q-functional calculus. The fine structure described in the following chapter is the suitable tool to obtain a product rule for the F-functional calculus in the quaternionic case. This is given by

$$\Delta(fg)(T) = \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T),$$

with T an operator with commuting components and the functions f and g are assumed such that the product fg is slice hyperholomorphic.

- This chapter is based on [73, 74]. We study another possible refinement of the Fueter theorem. By applying the operator $\overline{\mathcal{D}}$ to a slice hyperholomorphic function f. Then the function $\breve{f}^0 := \overline{\mathcal{D}}f$, is polyanalytic of order 2 i.e. it is in the kernel of the operator $\mathcal{D}^2$. Therefore, we have the following diagram

$$\mathcal{O}(D) \xrightarrow{T_F} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D).$$

where $\mathcal{AP}_2(\Omega_D)$ is the set of axially polyanalytic of order 2. In this chapter we describe the following fine structure

$$
\begin{array}{ccccc}
\mathcal{SH}(\Omega_D) & & \mathcal{AP}_2(\Omega_D) & & \mathcal{AM}(\Omega_D) \\
\downarrow & & & & \\
\text{Slice Cauchy Formula} & \xrightarrow{\overline{\mathcal{D}}} & \mathcal{AP}_2 \text{ integral form} & \xrightarrow{\mathcal{D}} & \text{Fueter thm. in integral form} \\
\downarrow & & \downarrow & & \downarrow \\
\text{S}-\text{Functional calculus} & & \mathcal{P}_2-\text{functionalcalculus} & & \mathcal{F}-\text{functional calculus}
\end{array}
$$

Basically we describe the central column of this digram. By applying the conjugate Fueter to the second form of the slice hyperholomorphic Cauchy kernel we get the following polyanalytic kernel

$$\overline{\mathcal{D}}S_L^{-1}(s,q) = -F_L(s,q)s + q_0F_L(s,q) = \sum_{k=0}^{1} q_0^k F_L(s,q)(-1)^{k+1}s^{1-k}.$$

This formula together with the slice hyperholomorphic Cauchy formula, imply the following integral representation for axially polyanalytic functions of order 2

$$\breve{f}^0(q) = -\frac{1}{2\pi}\sum_{k=0}^{1}(-q_0)^k \int_{\partial(U\cap\mathbb{C}_J)} F_L(s,q)s^{1-k}\, ds_J\, f(s),$$

where U is a suitable open set and the function f is a left slice hyperholomorphic. As for the other functional calculi the integral representation is the crucial object to define the functional calculus. Given an operator T with commuting components we define the left $\mathcal{P}_2$-resolvent operator as

$$\mathcal{P}_2^L(s,T) = \sum_{j=0}^{1} T_0^j(-1)^{j+1}F_L(s,T)s^{1-j}.$$

Then we can define the polyanalytic functional calculus of order 2 on the S-spectrum with the following expression

$$\breve{f}^0(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)\, ds_J\, f(s), \tag{1.21}$$

where f is a left slice hyperholomorphic function. The integral (1.21) does not depend on the set U neither on the imaginary unit $J \in \mathbb{S}$. Even for this fine structure we show a resolvent equation which is the fundamental tool to show a product rule and to generate the Riesz projectors.

- This chapter is based on [37]. We continue our investigation on the fine structures. Let $h := \frac{n-1}{2}$ be the so-called Sce exponent, and Δ be the Laplace operator in dimension $n+1$. In the Clifford framework the operator $T_{FS2} := \Delta^h$ maps the slice hyperholomorphic function $f(x)$ to the set of axially monogenic functions.

  Therefore it is possible to repeatedly apply to a slice hyperholomorphic function $f(x)$ the Dirac operator and its conjugate, until we reach the maximum power of the Laplacian, i.e., the Sce exponent. This implies the possibility to build different sets of functions which lie between the set of slice hyperholomorphic functions and the set of axially monogenic functions. In this dissertation we focus on considering dimension five. In this case the Fueter-Sce map is given by Δ^2. In dimension five there are seven spaces between the set of slice hyperholomorphic functions and axially monogenic functions, precisely: $\mathcal{ABH}(\Omega_D)$ the axially bi-harmonic functions, $\mathcal{ACH}_1(\Omega_D)$ the axially Cliffordian holomorphic functions of order 1, $\mathcal{AH}(\Omega_D)$ the axially harmonic functions, $\mathcal{AP}_2(\Omega_D)$ the axially polyanalytic of order 2, $\mathcal{ACH}_1(\Omega_D)$ the axially anti Cliffordian of order 1, $\mathcal{ACP}_{(1,2)}(\Omega_D)$ the axially polyanalytic Cliffordian of order $(1,2)$, $\mathcal{AP}_3(\Omega_D)$ the axially polyanalytic of order 3, In dimension greater than five there will be one more function space, that is not indicated in the list above, and with this addition, all the fine structures can be described using those function spaces of different orders. We are able to write any function in the previous sets in integral form. This fact is crucial to give a definition to the different functional calculi based on the S-spectrum.

- In this last chapter we present some new research directions and perspectives that are under investigations at the moment.

Part I:Integral transforms in the hypercomplex setting

In this first part we extend to the quaternionic and Clifford algebra settings the short time Fourier transform. Before we recall some basic notions of Riesz-Dunford functional calculus and hypercomplex analysis that we will need in the sequel.

In Chapter 4 and Chapter 5 we develop special one dimensional quaternion short-time Fourier transforms by using the slice hyperholomorphic and the slice polyanalytic Bargmann transforms. In these cases we consider a Gaussian function and the weighted Hermite functions as windows. Then we study the main properties of these quaternion short time Fourier transforms.

In Chapter 6, we give an extension of the short-time Fourier transform in the Clifford algebra setting. The tool to achieve this is the Clifford Fourier transform, introduced in [66]. In this case we deal with radial window functions.

CHAPTER 2

The Riesz-Dunford functional calculus

2.1 Introduction

In this chapter we recall some basic concepts of the Riesz-Dunford functional calculus. We will focus only on the functional calculus for linear bounded operators. The unbounded case is not a topic of this thesis. We will insert only some of the proofs of the results presented in this chapter. For more details see [79].

2.2 Vector-Valued functions of a complex variable

We start by recalling the definition of linear bounded operator.

Definition 2.2.1. Let us consider X and Y two complex Banach space.

- We say that the map $T : X \to Y$ is a linear operator if

$$T(\lambda x + \mu y) = \lambda Tx + \mu Ty, \qquad \text{for all} \quad x, y \in X, \quad \lambda, \mu \in \mathbb{C}.$$

- A linear operator $T : X \to Y$ is said to be bounded if there exists a constant $c \geq 0$ such that for any $x \in X$ we have

$$\|Tx\| \leq k\|x\|.$$

- The set of all bounded linear operators $T : X \to Y$ with norm defined by
$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$
is denoted by $\mathcal{B}(X, Y)$. If $X = Y$ we set $\mathcal{B}(X) := \mathcal{B}(X, X)$.

The classic notion of holomorphic function as well as the classical Cauchy theorem and Cauchy integral formula can be generalized to function with values in a normed vector space X.

Definition 2.2.2. Let X be a Banach space and $z_0 \in \mathbb{C}$. Let us consider a function $f : \mathbb{C} \to X$. The function f is holomorphic in z_0 if there exists an open disk $D(z_0, r)$, with $r > 0$, such that f admits the following power series expansion
$$f(z) = \sum_{n=0}^{\infty} T_n (z - z_0)^n, \qquad T_n \in \mathcal{B}(X), \qquad n \in \mathbb{N},$$
converging in the norm of X in $D(z_0, r)$.

Theorem 2.2.3 (Cauchy theorem)**.** *Let U be an open bounded set in $\mathbb{C}$ such that ∂U is a finite union of continuously differentiable Jordan curves. If the function $f : U \cup \partial U \to X$ is a holomorphic function, then we have*
$$\int_{\partial U} f(z) dz = 0.$$

Theorem 2.2.4 (Cauchy integral formula)**.** *Let U be an open bounded set in $\mathbb{C}$. We suppose that $V \subset U$ such that $\partial V \cup V \subseteq U$ and ∂V is a finite union of continuously differentiable Jordan curves. Then if $f : U \to X$ is holomorphic, for each $z_0 \in V$ we have*
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial V} f(z)(z - z_0)^{-1} dz.$$

2.3 The functional calculus for linear bounded operators

Let X be a complex Banach space and $T \in \mathcal{B}(X)$. We define the resolvent set of T as
$$\rho(T) := \{\lambda \in \mathbb{C} \mid (\lambda \mathcal{I} - T)^{-1} \in \mathcal{B}(X)\},$$
where $\mathcal{I}$ is the identity operator. The spectrum set of T is defined as
$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

We have the following properties

Lemma 2.3.1. *Let $T \in \mathcal{B}(X)$. Then we have*

- *the resolvent set $\rho(T)$ is open,*
- *the closed set $\sigma(T)$ is compact and nonempty*

The following function

$$R(\lambda, T) := (\lambda \mathcal{I} - T)^{-1}, \qquad \lambda \in \rho(T),$$

is called resolvent operator of T. The number

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}, \tag{2.1}$$

is called the spectral radius of T. A crucial result is to expand in series the resolvent operator.

Proposition 2.3.2. *Let $T \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$. Then for $\|T\| < |\lambda|$ we have*

$$R(\lambda, T) = \sum_{n=0}^{\infty} T^n \lambda^{-1-n}.$$

Proof. Since the geometric series $\sum_{n=0}^{\infty} u^n = \frac{1}{1-u}$ converges if $|u| < 1$ we have

$$\begin{aligned} R(\lambda, T) &= (\lambda \mathcal{I} - T)^{-1} = \lambda^{-1}(\mathcal{I} - \lambda^{-1} T)^{-1} \\ &= \lambda^{-1} \sum_{n=0}^{\infty} (\lambda^{-1} T)^n = \sum_{n=0}^{\infty} T^n \lambda^{-1-n}. \end{aligned}$$

□

Proposition 2.3.3. *Let $T \in \mathcal{B}(X)$. Then for every pair λ, $\mu \in \rho(T)$ we have*

- *The function $R(\lambda, T)$ is analytic on $\rho(T)$,*
- *$R(\lambda, T) R(\mu, T) = R(\mu, T) R(\lambda, T)$,*
- *The resolvent operator satisfies the following identity*

$$\lambda R(\lambda, T) - T R(\lambda, T) = \mathcal{I}. \tag{2.2}$$

One of the main properties that the resolvent operator enjoys is the following.

Proposition 2.3.4 (Resolvent equation)**.** *Let* $T \in \mathcal{B}(X)$*. Then for every* λ*,* $\mu \in \rho(T)$ *we have*

$$R(\lambda, T)R(\mu, T) = \frac{R(\lambda, T) - R(\mu, T)}{\mu - \lambda}. \tag{2.3}$$

Proof. By Proposition 2.3.3 we know

$$\lambda R(\lambda, T) - TR(\lambda, T) = \mathcal{I}, \tag{2.4}$$

and

$$\mu R(\mu, T) - TR(\mu, T) = \mathcal{I}. \tag{2.5}$$

Moreover, we can write

$$R(\lambda, T)R(\mu, T) = R(\mu, T)R(\lambda, T). \tag{2.6}$$

If we multiply equation (2.4) by $R(\mu, T)$ we get

$$\lambda R(\lambda, T)R(\mu, T) - TR(\lambda, T)R(\mu, T) = R(\mu, T), \tag{2.7}$$

and if we multiply the equation (2.5) by $R(\lambda, T)$ we obtain

$$\mu R(\mu, T)R(\lambda, T) - TR(\mu, T)R(\lambda, T) = R(\lambda, T). \tag{2.8}$$

By taking the difference of the equations (2.7) and (2.8) and by (2.6), we get the resolvent equation. □

Now, we list some properties of the resolvent equation that we will be useful in the sequel to show important properties for the the Riesz-Dunford functional calculus.

(I) The product of the resolvent operators $R(\lambda, T)R(\mu, T)$, at two different points $\lambda,\ \mu \in \rho(T)$, is transformed into the difference $R(\lambda, T) - R(\mu, T)$.

(II) The difference $R(\lambda, T) - R(\mu, T)$ is entangled with the Cauchy kernel $1/(\mu - \lambda)$ of holomorphic functions as follows

$$\frac{R(\lambda, T) - R(\mu, T)}{\mu - \lambda}.$$

(III) The resolvent equation preserves the holomorphicity both in λ and in $\mu \in \rho(T)$.

Definition 2.3.5. Let $T \in \mathcal{B}(X)$. We denote by $\mathcal{F}(T)$ the family of functions f which are analytic on some neighbourhood of $\sigma(T)$.

Now, we are ready to define the Riesz-Dunford functional calculus.

Definition 2.3.6. Let $f \in \mathcal{F}(T)$, and let U be an open set whose boundary ∂U is a finite union of continuously differentiable Jordan curves, oriented in the positive sense. Suppose that $\sigma(T) \subseteq U$ and that $U \cup \partial U$ is contained in the domain of analyticity of f. Then the operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi i} \int_{\partial U} R(\lambda, T) f(\lambda) d\lambda. \tag{2.9}$$

We observe that the integral in formula (2.9) does not depend on the open set U, it depends only on f. The previous definition is consistent with polynomials.

Theorem 2.3.7. *Let $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $T \in \mathcal{B}(T)$. We suppose that U is an open set whose boundary ∂U is a finite union of continuously differentiable Jordan curves oriented in the positive sense. If $\sigma(T) \subset U$ we have*

$$T^n = \frac{1}{2\pi i} \int_{\partial U} R(\lambda, T) \lambda^n d\lambda.$$

Proof. Let us denote by $B_r(0)$ a ball with radius $r > \|T\|$. Then by Proposition 2.3.2 we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial B_r(0)} R(\lambda, T) \lambda^n d\lambda &= \sum_{m=0}^{\infty} T^n \frac{1}{2\pi i} \int_{\partial B_r(0)} \lambda^{-n+m-1} d\lambda \\ &= T^n \end{aligned} \tag{2.10}$$

because

$$\int_{\partial B_r(0)} \lambda^{-n+m-1} d\lambda = \begin{cases} 1 & n = m, \\ 0 & n \neq m. \end{cases}$$

By the Cauchy theorem, the integral in (2.10) is not affected if we replace the circle $\partial B_r(0)$ by ∂U. □

As a consequence of Theorem 2.3.7 we have the following result.

Theorem 2.3.8. *Let $T \in \mathcal{B}(T)$. We suppose that U is an open set whose boundary ∂U is a finite union of continuously differentiable Jordan curves oriented in the positive sense and we assume that the set U contains $\sigma(T)$. For every $p(z) = \sum_{\ell=0}^{n} z^\ell a_\ell$ with $a_\ell \in \mathbb{C}$, we set $P(T) = \sum_{\ell=0}^{n} T^\ell a_\ell$. Then*

$$P(T) = \frac{1}{2\pi i} \int_{\partial U} R(\lambda, T) p(\lambda) d\lambda.$$

The Riesz-Dunford functional calculus enjoys the following algebraic properties.

Proposition 2.3.9. *Let f, $g \in \mathcal{F}(T)$ and α_1, $\alpha_2 \in \mathbb{C}$. Then we have*

1) $\alpha_1 f + \alpha_2 g \in \mathcal{F}(T)$ and $(\alpha_1 f + \alpha_2 g)(T) = \alpha_1 f(T) + \alpha_2 g(T)$.

2) If $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$, with $\{\alpha_n\}_{n\geq 0} \subset \mathbb{C}$, converges in a neighbourhood of $\sigma(T)$, then $f(T) = \sum_{n=0}^{\infty} \alpha_n T^n$.

Proof. 1) The first follows easily from Definition 2.3.6.

2) The second point follows from the fact that the series $\sum_{n=0}^{\infty} \alpha_n \lambda^n$ converges uniformly on the set $C_\varepsilon : \{\lambda \in \mathbb{C} \,:\, |\lambda| \leq r(T) + \varepsilon\}$ for $\varepsilon > 0$. Then by Proposition 2.3.2 we get

$$\begin{aligned} f(T) &= \frac{1}{2\pi i} \int_{C_\varepsilon} \sum_{n=0}^{\infty} \alpha_n \lambda^n R(\lambda, T) d\lambda \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \alpha_n \int_{C_\varepsilon} \lambda^n R(\lambda, T) d\lambda \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \alpha_n \int_{C_\varepsilon} \left(\sum_{k=0}^{\infty} \lambda^{-1-k} T^k \right) \lambda^n d\lambda \\ &= \sum_{n=0}^{\infty} \alpha_n T^n. \end{aligned}$$

□

One of the main important properties of the Riesz-Dunford functional calculus is the so called *product rule*. To show this property we use the resolvent equation (2.3).

Theorem 2.3.10 (product rule)**.** *Let f, $g \in \mathcal{F}(T)$ then $f \cdot g \in \mathcal{F}(T)$ and $f(T)g(T) = (f \cdot g)(T)$.*

Proof. By hypothesis we know that $f, g \in \mathcal{F}(T)$ then trivially $f \cdot g \in \mathcal{F}(T)$. Let us consider two neighbourhoods U_1, U_2 of $\sigma(T)$, whose boundary ∂U_1 and ∂U_2 are finite unions of continuously differentiable Jordan curves. Now, we assume that $U_1 \cup \partial U_1 \subseteq U_2$ and that $U_2 \cup \partial U_2$ is contained in a common region of analyticity of f and g. By Definition 2.3.6 and the resolvent equation (2.3) we get

$$\begin{aligned} f(T)g(T) &= -\frac{1}{4\pi^2}\int_{\partial U_1} f(\lambda)R(\lambda,T)d\lambda \int_{\partial U_2} g(\mu)R(\mu,T)d\mu \\ &= -\frac{1}{4\pi^2}\int_{\partial U_1}\int_{\partial U_2} f(\lambda)g(\mu)R(\lambda,T)R(\mu,T)d\mu d\lambda \\ &= -\frac{1}{4\pi^2}\int_{\partial U_1}\int_{\partial U_2} f(\lambda)g(\mu)\frac{R(\lambda,T)-R(\mu,T)}{\mu-\lambda}d\mu d\lambda. \end{aligned}$$

Now, by Fubini's theorem, Cauchy theorem (see Theorem 2.2.3) and Cauchy integral formula (see Theorem 2.2.4) we obtain

$$\begin{aligned} f(T)g(T) &= -\frac{1}{4\pi^2}\int_{\partial U_1} f(\lambda)R(\lambda,T)\left(\int_{\partial U_2}\frac{g(\mu)}{\mu-\lambda}d\mu\right)d\lambda \\ &\quad +\frac{1}{4\pi^2}\int_{\partial U_2} g(\mu)R(\mu,T)\left(\int_{\partial U_1}\frac{f(\lambda)}{\mu-\lambda}d\lambda\right)d\mu \\ &= \frac{1}{2\pi i}\int_{\partial U_1} f(\lambda)g(\lambda)R(\lambda,T)d\lambda \\ &= (f\cdot g)(T). \end{aligned}$$

□

Another important application of the resolvent equation is the study of the so called Riesz projectors.

Theorem 2.3.11 (Riesz projectors)**.** *Let $T \in \mathcal{B}(X)$ and $f \in \mathcal{F}(T)$. We suppose that $\sigma(T)=\sigma_1(T)\cup\sigma_2(T)$ with dist$(\sigma_1(T),\sigma_2(T))>0$. Then we consider two open sets Ω_1 and Ω_2 such that $\sigma(T_i)\subset\Omega_i$ with $i=1,2$ and $\overline{\Omega_1}\cap\overline{\Omega_2}=\emptyset$. Then the following operator*

$$P := \frac{1}{2\pi i}\int_{\partial\Omega_i} R(\lambda,T)d\lambda, \qquad i=1,2$$

is a projector.

Proof. We have to show that $P^2=P$. Let G_1 and G_2 tow open sets that contain the spectrum $\sigma(T)$ and such that ∂G_i is the union of a finite number of continuously differentiable Jordan curves oriented in the positive sense. Moreover we suppose that

- $\sigma_i(T)\subset G_1$, with $i=1,2$,
- $\overline{G_1}\subset G_2$,

- $\overline{G_2} \subset \Omega_i$, with $i = 1, 2$.

Now, by the Cauchy theorem we can write

$$P = \frac{1}{2\pi i} \int_{\partial G_1} R(\lambda, T) d\lambda = \frac{1}{2\pi i} \int_{\partial G_2} R(\mu, T) d\mu.$$

Therefore, by the resolvent equation (2.3) we have

$$\begin{aligned} P^2 &= \frac{1}{(2i\pi)^2} \int_{\partial G_1} \int_{\partial G_2} R(\lambda, T) R(\mu, T) d\lambda d\mu \\ &= \frac{1}{(2i\pi)^2} \int_{\partial G_1} \int_{\partial G_2} \frac{R(\lambda, T) - R(\mu, T)}{\mu - \lambda} d\lambda d\mu \\ &= \frac{1}{(2i\pi)^2} \int_{\partial G_1} R(\lambda, T) \left(\int_{\partial G_2} \frac{1}{\mu - \lambda} \right) \\ &\quad - \frac{1}{(2i\pi)^2} \int_{\partial G_2} R(\mu, T) \left(\int_{\partial G_1} \frac{1}{\mu - \lambda} \right). \end{aligned}$$

Due to the facts that

$$\int_{\partial G_2} \frac{1}{\mu - \lambda} = 2\pi i \qquad \int_{\partial G_1} \frac{1}{\mu - \lambda} = 0,$$

we obtain

$$P^2 = \frac{1}{2\pi i} \int_{\partial G_1} R(\lambda, T) d\lambda = P.$$

□

Another important result for the Riesz-Dunford functional calculus is the following.

Theorem 2.3.12 (Spectral mapping theorem)**.** *If $f \in \mathcal{F}(T)$, then $f(\sigma(T)) = \sigma[f(T)]$.*

We state a result, which is a particular case of the spectral mapping theorem, that shows how to compute the spectral radius defined in (2.1).

Proposition 2.3.13. *Let $T \in \mathcal{B}(X)$, then we have*

- $\sigma(T^n) = [\sigma(T)]^n := \{\lambda^n \, : \, \lambda \in \sigma(T)\}$.
- $r(T) = \lim_{n \to \infty} \sqrt[n]{T^n}$.

The spectral mapping theorem implies following result.

Theorem 2.3.14 (Composition rule)**.** *Let* $f \in \mathcal{F}(T)$*,* $g \in \mathcal{F}(f(T))$*, and* $F(\lambda) = g(f(\lambda))$*. Then we have*

- $F \in \mathcal{F}(T)$*,*
- $F(T) = g(f(T))$*.*

CHAPTER 3

Preliminaries on slice hypercomplex analysis

In this chapter we recall the main notions of the quaternionic and Clifford analysis. In the first section of this chapter we mention some basic results about the slice hyperholomorphic functions theory that we will need in the sequel. In the second section we revise the main concepts of the S functional calculus. Most of the results presented in this chapter are well-known, for this reason we omit all the proofs. For further information see the books [13, 28, 44, 45, 56, 59, 87, 94].

3.1 Hyperholomorphic functions

The skew-field of quaternions is defined as

$$\mathbb{H} = \{q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \ | \ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where the imaginary units satisfy the relations

$$e_1^2 = e_2^2 = e_3^2 = -1$$

and

$$e_1 e_2 = -e_2 e_1 = e_3, \ \ e_2 e_3 = -e_3 e_2 = e_1, \ \ e_3 e_1 = -e_1 e_3 = e_2.$$

Given $q \in \mathbb{H}$ we call $\mathrm{Re}(q) := q_0$ the real part of q and $\underline{q} = q_1e_1 + q_2e_2 + q_3e_3$ the imaginary part. The modulus of $q \in \mathbb{H}$ is given by $|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$, the conjugate of q is defined by $\overline{q} = q_0 - \underline{q}$ and we have $|q| = \sqrt{q\overline{q}}$. Moreover for $p, q \in \mathbb{H}$ we have

$$\overline{pq} = \overline{q}\overline{p}.$$

The symbol $\mathbb{S}$ denotes the unit sphere of purely imaginary quaternions

$$\mathbb{S} = \{\underline{q} = q_1e_1 + q_2e_2 + q_3e_3 \ \mid \ q_1^2 + q_2^2 + q_3^2 = 1\}.$$

Notice that if $J \in \mathbb{S}$, then $J^2 = -1$. Therefore J is an imaginary unit, and we denote by

$$\mathbb{C}_J = \{u + Jv \ \mid u, v \in \mathbb{R}\},$$

an isomorphic copy of the complex plane. It can be considered as a complex plane in $\mathbb{H}$ passing through 0,1 and J. It is immediate that we have

$$\mathbb{H} = \bigcup_{J \in \mathbb{S}} \mathbb{C}_J.$$

Given a non-real quaternion $q = q_0 + \underline{q} = q_0 + J_q|\underline{q}|$, we set $J_q = \underline{q}/|\underline{q}| \in \mathbb{S}$ and we associate to q the 2-sphere defined by

$$[q] := \{q_0 + J|\underline{q}| \ \mid \ J \in \mathbb{S}\}.$$

The quaternions are a particular case of real Clifford algebras.
Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \ldots, e_n$ satisfying the relations $e_\ell e_m + e_m e_\ell = 0$, $\ell \neq m$, $e_\ell^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = \{\ell_1 \ldots \ell_r\} \in \mathcal{P}\{1, 2, \ldots, n\}$, $\ell_1 < \ldots < \ell_r$ is a multi-index and $e_A = e_{\ell_1} e_{\ell_2} \ldots e_{\ell_r}$, $e_\emptyset = 1$. The Clifford algebras over two units $\mathbb{R}_2$ is the algebra of the quaternions, previously denoted by $\mathbb{H}$.

A point $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j \in \mathbb{R}_n$ called paravector and the real part x_0 of x will also be denoted by $\mathrm{Re}(x)$. The vector part of x is defined by $\underline{x} = x_1e_1 + \ldots + x_ne_n$. The conjugate of x is denoted by $\overline{x} = x_0 - \underline{x}$ and the Euclidean modulus of x is given by $|x|^2 = x_0^2 + \ldots + x_n^2$. The sphere of purely imaginary vectors with modulus 1, is defined by

$$\mathbb{S}^{n-1} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \mid x_1^2 + \ldots + x_n^2 = 1\}.$$

The element $I \in \mathbb{S}^{n-1}$ is such that $I^2 = -1$, so I is an imaginary unit, and we will denote the complex space with imaginary unit I by $\mathbb{C}_I$. Given a non-real paravector $x = x_0 + \underline{x} = x_0 + J_x|\underline{x}|$, we set $J_x := \underline{x}/|\underline{x}| \in \mathbb{S}^{n-1}$, and we associate to x the $(n-1)$- sphere defined by

$$[x] = \{x_0 + J|\underline{x}| \ \mid \ J \in \mathbb{S}^{n-1}\}.$$

In the sequel we will give the definition and the properties of slice hyperholomorphic functions only in the Clifford algebra setting, since it is always possible to recover the quaternionic case.

Definition 3.1.1. Let $U \subseteq \mathbb{R}^{n+1}$.

- We say that U is axially symmetric if, for every $u + Iv \in U$, all the elements $u + Jv$ for $J \in \mathbb{S}^{n-1}$ are contained in U.
- We say that U is a slice domain if $U \cap \mathbb{R} \neq \emptyset$ and if $U \cap \mathbb{C}_J$ is a domain in $\mathbb{C}_J$ for every $J \in \mathbb{S}^{n-1}$.

Definition 3.1.2. An axially symmetric open set $U \subset \mathbb{R}^{n+1}$ is called slice Cauchy domain if $U \cap \mathbb{C}_J$ is a Cauchy domain in $\mathbb{C}_J$ for every $J \in \mathbb{S}^{n-1}$. More precisely, U is a slice Cauchy domain if for every $J \in \mathbb{S}^{n-1}$ the boundary of $U \cap \mathbb{C}_J$ is the union of a finite number of nonintersecting piecewise continuously differentiable Jordan curves in $\mathbb{C}_J$.

On axially symmetric open sets we define the class of slice hyperholomorphic functions, in the case of Clifford algebra valued functions they are often called slice monogenic functions.

Definition 3.1.3 (Slice hyperholomorphic functions)**.** Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} := \{(u, v) \in \mathbb{R}^2 : \ u + Jv \in U \quad \forall J \in \mathbb{S}^{n-1}\}$. We say that a function $f : U \to \mathbb{R}_n$ of the form

$$f(x) = \alpha(u, v) + J\beta(u, v) \tag{3.1}$$

where $x = u + Jv$ for any $J \in \mathbb{S}^{n-1}$, is left slice hyperholomorphic if α and β are $\mathbb{R}_n$-valued differentiable functions such that

$$\alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v) \ \text{ for all } (u, v) \in \mathcal{U}, \tag{3.2}$$

and if α and β satisfy the Cauchy-Riemann system

$$\partial_u\alpha(u, v) - \partial_v\beta(u, v) = 0, \quad \partial_v\alpha(u, v) + \partial_u\beta(u, v) = 0.$$

We recall that right slice hyperholomorphic functions are of the form

$$f(x) = \alpha(u, v) + \beta(u, v)J,$$

where α, β satisfy the above conditions.

Definition 3.1.4. The set of left (resp. right) slice hyperholomorphic functions on U is denoted with the symbol $\mathcal{SH}_L(U)$ (resp. $\mathcal{SH}_R(U)$). The subset of intrinsic functions consists of those slice hyperholomorphic functions such that α, β are real-valued function and it is denoted by $\mathcal{N}(U)$.

Definition 3.1.5. Let U be an open set in $\mathbb{R}^{n+1}$. A real differentiable function $f : U \to \mathbb{R}_n$ is left monogenic if

$$\mathcal{D}f(x) = \partial_{x_0} f(x) + \sum_{i=1}^{n} e_i \partial_{x_i} f(x) = 0.$$

It is right monogenic if

$$f(x)\mathcal{D} = \partial_{x_0} f(x) + \sum_{i=1}^{n} \partial_{x_i} f(x) e_i = 0.$$

In Chapter 7 we will study a connection between the slice hyperholomorphic and monogenic functions.

In general the product of slice hyperholomorphic functions is not preserved. Indeed for example the function $f(x) = xa$, with $a \in \mathbb{R}_n \setminus \mathbb{R}$, is left slice hyperholomorphic; but the product $f(x) \cdot f(x) = xaxa$ is not slice hyperholomorphic. In order to overcome this issue we give a suitable definition of product among slice hyperholomorphic functions

Definition 3.1.6. For $f = \alpha + J\beta$, $g = \gamma + J\delta \in \mathcal{SH}_L(U)$, we define the left slice hyperholomorphic product as

$$f *_L g = (\alpha\gamma - \beta\delta) + J(\alpha\delta + \beta\gamma).$$

For $f = \alpha + \beta I$, $g = \gamma + \delta J \in \mathcal{SH}_R(U)$, we define the right slice hyperholomorphic product as

$$f *_R g = (\alpha\gamma - \beta\delta) + (\alpha\delta + \beta\gamma)J.$$

If we consider the previous example we have that $f(x) * f(x) = x^2a^2$, which is a left slice hyperholomorphic function.

The subclass of intrinsic function plays a very important rule because a pointwise multiplication with a slice hyperholomorphic function maintains the property to be slice hyperholomorphic.

Lemma 3.1.7. *Let $U \subset \mathbb{H}$ be axially symmetric. If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*

Remark 3.1.8. The slice hyperholomorphic product, in general, is not commutative. It is associative and distributive. However, if the function f is intrinsic the slice hyperholomorphic product coincides with the classical pointwise product :

$$f *_L g = fg = g *_L f.$$

Similarly, if g is intrinsic we have

$$f *_R g = fg = g *_R f.$$

For the sake of simplicity we recall the following result only for left slice hyperholomorphic functions.

Theorem 3.1.9 (Representation formula)**.** *Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain and let f be a slice hyperholomorphic function and f be a left slice hyperholomorphic on U.*

- *For any $x = u + J_x v \in U$ the following formulas hold*

$$f(x) = \frac{1}{2}[1 - J_x J]f(u + Jv) + \frac{1}{2}[1 + J_x J]f(u - Jv),$$

and

$$f(x) = \frac{1}{2}\left[f(u + Jv) + f(u - Jv) + J_x J[f(u - Jv) - f(u + Jv)]\right].$$

- *Moreover, the two quantities*

$$\alpha(u, v) := \frac{1}{2}[f(u + Jv) + f(u - Jv)],$$

and

$$\beta(u, v) = J\frac{1}{2}[f(u - Jv) - f(u + Jv)]$$

do not depend on $J \in \mathbb{S}^{n-1}$.

Definition 3.1.10. Let $U \subset \mathbb{H}$ be an axially symmetric open set. For any $f \in \mathcal{SH}_L(U)$ the slice derivative is defined by:

$$\partial_S f(x) = \lim_{\mathbb{C}_J \ni s \to x} (s - x)^{-1}(f(s) - f(x)).$$

Similarly, if $f \in \mathcal{SH}_R(U)$, then the function

$$\partial_S f(x) = \lim_{\mathbb{C}_J \ni s \to x} (f(s) - f(x))(s - x)^{-1},$$

is called the slice derivative of f.

We observe that the slice derivative of a left slice hyperholomorphic function, (respectively right), is again a left slice hyperholomorphic function, (respectively right). Furthermore, if $x \in \mathbb{R}$ then the slice derivative coincides with the partial derivative $\frac{\partial}{\partial x_0} f$.

In a suitable domain we can expand the slice hyperholomorphic into a power series.

Theorem 3.1.11. *If f is a left slice hyperholomorphic function on $B_r(a)$, a ball with radius $r > 0$ and centre $a \in \mathbb{R}$, then*

$$f(x) = \sum_{n=0}^{\infty} (x-a)^n \frac{1}{n!} (\partial_S^n f) \qquad \textit{for} \quad x \in B_r(a).$$

If f is a right slice hyperholomorphic function on $B_r(a)$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} (\partial_S^n f)(x-a)^n \qquad \textit{for} \quad x \in B_r(a).$$

In [88] Gentili and Struppa proposed another definition to extend the classical theory of holomorphic functions to the quaternionic setting. In [58] these concepts was considered in the Clifford algebra setting.

Definition 3.1.12. Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric slice domain and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $J \in \mathbb{S}^{n-1}$ and let f_J be the restriction of the function f to the complex plane $\mathbb{C}_J$. We say that f is left holomorphic if for every $J \in \mathbb{S}^{n-1}$ we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + J\frac{\partial}{\partial t}\right) f_J(x+Iy) = 0 \qquad \forall x + Jy \in U \cap \mathbb{C}_J. \tag{3.3}$$

We say that f is right holomorphic if for every $J \in \mathbb{S}^{n-1}$ we have

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}J\right) f_J(x+Iy) = 0 \qquad \forall x + Jy \in U \cap \mathbb{C}_J. \tag{3.4}$$

Remark 3.1.13. It is possible to show that any left holomorphic function (resp. right) defined in an axially symmetric slice domain satisfies the Representation formula, see Theorem 3.1.9. This implies that a left holomorphic function (resp. right) is a left (resp. right) slice function. Therefore on axially symmetric slice domain Definition 3.1.3 and Definition 3.1.12 coincide. However, in other sets like open sets that do not intersect the real axis, there exists functions that are not slice functions but satisfies (3.3) or (3.4). Thus Definition 3.1.3 is more restrictive than Definition 3.1.12, even if it is more appropriate for the operator theory, see Remark 3.2.11.

We now recall the hyperholomorphic Cauchy formulas that are the heart of the hyperholomorphic spectral theories.

Theorem 3.1.14. *Let $s,\ x \in \mathbb{R}^{n+1}$ with $|x| < |s|$, then*

$$\sum_{n=0}^{+\infty} x^n s^{-n-1} = -(x^2 - 2Re(s)x + |s|^2)^{-1}(x - \overline{s})$$

and

$$\sum_{n=0}^{+\infty} s^{-n-1} x^n = -(x - \overline{s})(x^2 - 2Re(s)x + |s|^2)^{-1}.$$

Moreover, for any $s,\ x \in \mathbb{R}^{n+1}$ with $x \notin [s]$, we have

$$-(x^2 - 2Re(s)x + |s|^2)^{-1}(x - \overline{s}) = (s - \overline{x})(s^2 - 2Re(x)s + |x|^2)^{-1}$$

and

$$-(x - \overline{s})(x^2 - 2Re(s)x + |s|^2)^{-1} = (s^2 - 2Re(x)s + |x|^2)^{-1}(s - \overline{x}).$$

In view of Theorem 3.1.14 there are two possible representations of the Cauchy kernels for left slice hyperholomorphic functions and two for right slice hyperholomorphic functions.

Definition 3.1.15. Let $s,\ x \in \mathbb{R}^{n+1}$ with $x \notin [s]$ then we define the two functions

$$\mathcal{Q}_s(x)^{-1} := (x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}, \quad \mathcal{Q}_{c,s}(x)^{-1} := (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-1},$$

that are called pseudo Cauchy kernel and commutative pseudo Cauchy kernel, respectively.

Definition 3.1.16. Let $s,\ x \in \mathbb{R}^{n+1}$ with $x \notin [s]$ then

- We say that the left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,x)$ is written in the form I if

$$S_L^{-1}(s,x) := \mathcal{Q}_s(x)^{-1}(\overline{s} - x).$$

- We say that the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,x)$ is written in the form I if

$$S_R^{-1}(s,x) := (\overline{s} - x)\mathcal{Q}_s(x)^{-1}.$$

- We say that the left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,x)$ is written in the form II if

$$S_L^{-1}(s,x) := (s-\overline{x})\mathcal{Q}_{c,s}(x)^{-1}.$$

- We say that the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,x)$ is written in the form II if

$$S_R^{-1}(s,x) := \mathcal{Q}_{c,s}(x)^{-1}(s-\overline{x}).$$

In this article, otherwise specified, we refer to $S_L^{-1}(s,x)$ and $S_R^{-1}(s,x)$ as written in the form II.

We have the following regularity for the (left and right) slice hyperholomorphic Cauchy kernels.

Lemma 3.1.17. *Let $s \notin [x]$. The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,x)$ is left slice hyperholomorphic in x and right slice hyperholomorphic in s. The right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,x)$ is left slice hyperholomorphic in s and right slice hyperholomorphic in x.*

Theorem 3.1.18 (The Cauchy formulas for slice hyperholomorphic functions)**.** *Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain, let $J \in \mathbb{S}^{n-1}$ and set $ds_J = ds(-J)$. If f is a (left) slice hyperholomorphic function on a set that contains $\overline{U}$ then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} S_L^{-1}(s,x)\, ds_J\, f(s), \qquad \textit{for any} \quad x \in U. \tag{3.5}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} f(s)\, ds_J\, S_R^{-1}(s,x), \qquad \textit{for any} \quad x \in U. \tag{3.6}$$

These integrals depend neither on U nor on the imaginary unit $J \in \mathbb{S}^{n-1}$.

A Cauchy integral theorem holds also for slice hyperholomorphic functions.

Lemma 3.1.19. *Let f and g be left slice monogenic and right slice monogenic functions, respectively, defined on an open set U. For any $J \in \mathbb{S}^{n-1}$ and any open bounded set D_J in $U \cap \mathbb{C}_J$ whose boundary is a finite union of continuously differentiable Jordan curves, we have*

$$\int_{\partial D_J} g(s) ds_I f(s) = 0.$$

3.2 S-functional calculus

The Cauchy formula of slice hyperholomorphic functions generates the S-functional calculus for Clifford linear operators or for n-tuples of not necessarily commuting operators. This calculus is based on the notion of S-spectrum, see [9, 12, 42, 43, 50–53]. This notion was discovered in 2006 by F. Colombo and I. Sabadini as it is well explained in the introduction of the book [45].

The existence of an appropriate quaternionic spectrum was suggested by the formulation of quaternionic quantum mechanics given by G. Birkhoff and J. von Neumann [26]. We note that in this framework a spectral theorem for quaternionic operators is necessary, and this was proved in [10] (see also the particular cases [11, 86]) and further generalized to fully Clifford operators in [47].

Preliminary attempts to prove the spectral theorem for quaternionic operators, without a precise notion of quaternionic spectrum, were given in [131, 135], while in [83] the spectral theorem for quaternionic matrices is treated on the right spectrum, a subset of the S-spectrum.

Now, we fix the notations. We denote by $\mathcal{B}(V_n)$ the Banach space of all bounded right linear operators acting on a the two sided Clifford Banach module $V_n = V \otimes \mathbb{R}_n$, where V is a real Banach space. In the sequel we will consider operators of the form $T = T_0 + \sum_{j=0}^{n} e_j T_j$ where $T_j \in \mathcal{B}(V_n)$ with $j = 0, 1, ..., n$. The subset of such operators in $\mathcal{B}(V_n)$ will be denoted by $\mathcal{B}^{0,1}(V_n)$.

Now, we give the appropriate definition of spectrum. This is the so-called S-spectrum, and it is defined in an unconventional way because it involves the square of the linear operator T.

Definition 3.2.1. Let V_n be a two sided Clifford Banach module and let $T \in B^{0,1}(V_n)$. For $s \in \mathbb{R}^{n+1}$ we set

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}.$$

We define the S-resolvent set $\sigma_S(T)$ of T as

$$\sigma_S(T) := \{s \in \mathbb{R}^{n+1} : \mathcal{Q}_s(T)^{-1} \quad \text{is not invertible in} \quad \mathcal{B}(V_n)\},$$

and we define the S-spectrum $\rho_S(T)$ of T as

$$\rho_S(T) := \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ is called the pseudo S-resolvent operator of T at s.

Theorem 3.2.2. *Let $T \in B^{0,1}(V_n)$ and $s \in \mathbb{R}^{n+1}$ with $\|T\| < |s|$. Then we have*

$$\sum_{n=0}^{\infty} T^n s^{-n-1} = -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I}),$$

and

$$\sum_{n=0}^{\infty} s^{-n-1} T^n = -(T - \overline{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

According to the left or right slice hyperholomorphicity, there exist two different resolvent operators.

Definition 3.2.3 (S-resolvent operators)**.** Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. Then the left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I}),$$

and the right S-resolvent operator is defined as

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

Proposition 3.2.4. *Let $T \in \mathcal{B}^{0,1}(V_n)$ and $s \in \rho_S(T)$. Then the left S-resolvent operator satisfies the equation*

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I}, \quad , \tag{3.7}$$

and the right S-resolvent operator satisfies

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{3.8}$$

The equations (3.7) and (3.8) cannot be considered the resolvent equations for the S-functional calculus because they do not satisfy the properties of the classical resolvent equation. The S-resolvent equation involves both the S-resolvent operators. Precisely, we have

Theorem 3.2.5 (S-resolvent equation)**.** *Let $T \in \mathcal{B}^{0,1}(V_n)$ then for s, $p \in \rho_S(T)$, with $p \notin [s]$, we have*

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) =& [\left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right)p+ \\ &- \overline{s}\left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right)]\mathcal{Q}_s(p)^{-1}, \end{aligned} \tag{3.9}$$

where $\mathcal{Q}_s(p) := p^2 - 2Re(s)p + |s|^2$.

Remark 3.2.6. The S-resolvent equation can be rewritten by using the left and right slice hyperholomorphic products with respect to the variables s and p, which are denoted by $*_{L,s}$, $*_{R,p}$.

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = \left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right) *_{L,s} S_L^{-1}(p,s)\mathcal{I}$$

or

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = S_L^{-1}(p,s)\mathcal{I} *_{R,p} \left(S_R^{-1}(s,T) - S_L^{-1}(p,T)\right).$$

The major differences between the resolvent equations in the S functional calculus (see (3.9)) and in the Riesz-Dunford functional calculus (see (2.3)) are listed below.

(I-S) The S-resolvent equation contains both the S-resolvent operators and an important fact to point out is that $S_L^{-1}(s,T)$ is right slice hyperholomorphic and $S_R^{-1}(s,T)$ left slice hyperholomorphic. The product $S_R^{-1}(s,T)S_L^{-1}(p,T)$ preserves the right slice hyperholomorphicity in s and the left slice hyperholomorphicity in p.

(II-S) The product $S_R^{-1}(s,T)S_L^{-1}(p,T)$ is transformed into the difference $S_R^{-1}(s,T)-S_L^{-1}(p,T)$ of the two S-resolvent operators, as in the complex case.

(III-S) The difference $S_R^{-1}(s,T)-S_L^{-1}(p,T)$ is entangled with $(p-\overline{s})(p^2-2s_0p+|s|^2)^{-1}$, which is the Cauchy kernel of slice hyperholomorphic functions, and the map

$$\begin{aligned}(s,p) \mapsto \big[&[S_R^{-1}(s,T) - S_L^{-1}(p,T)]p \\ &- \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)]\big](p^2 - 2s_0p + |s|^2)^{-1},\end{aligned}$$

for $s, p \in \mathbb{R}^{n+1} \setminus \sigma_S(T)$, preserves the right slice hyperholomorphicity in s and the left slice hyperholomorphicity in p.

Remark 3.2.7. It is important to note that the product $S_L^{-1}(p,T)S_R^{-1}(s,T)$ cannot be used in the S-resolvent equation because it destroys slice hyperholomorphicity.

In order to give the definition of the S-functional calculus we need the following classes of functions.

Notation: Let $T \in \mathcal{B}^{0,1}(V_n)$. We denote by $\mathcal{SH}_L(\sigma_S(T))$, $\mathcal{SH}_R(\sigma_S(T))$ and $N(\sigma_S(T))$ the sets of all left, right and intrinsic slice hyperholomorphic functions f, respectively, with $\sigma_S(T) \subset dom(f)$.

Definition 3.2.8 (S-functional calculus)**.** Let $T \in \mathcal{B}^{0,1}(V_n)$. Let U be a slice Cauchy domain that contains $\sigma_S(T)$ and $\overline{U}$ is contained in the domain of f. Set $ds_J = -dsJ$ for $J \in \mathbb{S}^{n-1}$, so we define for every $f \in \mathcal{SH}_L(\sigma_S(T))$

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S_L^{-1}(s,T)\, ds_J\, f(s), \tag{3.10}$$

and for every $f \in \mathcal{SH}_R(\sigma_S(T))$

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s)\, ds_J\, S_R^{-1}(s,T). \tag{3.11}$$

The definition of S-functional calculus is well posed since the integrals in (3.10) and (3.11) depend neither on U and nor on the imaginary unit $J \in \mathbb{S}^{n-1}$.

Like in the Riesz-Dunford functional calculus the resolvent equation is fundamental to show a product rule for the S-functional calculus and to study the Riesz projectors in this setting.

Theorem 3.2.9 (Product rule)**.** *Let $T \in \mathcal{B}^{0,1}(V_n)$ and assume $f \in N(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$. Then we have*

$$(fg)(T) = f(T)g(T).$$

Theorem 3.2.10 (Riesz projectors)**.** *Let $T \in \mathcal{B}^{0,1}(V_n)$ and let $\sigma_S(T) = \sigma_{1S}(T) \cup \sigma_{2S}(T)$, with $dist(\sigma_{1S}(T), \sigma_{2S}(T)) > 0$. Let U_1 and U_2 be two axially symmetric s-domains such that $\sigma_{1S}(T) \subset U_1$ and $\sigma_{2S}(T) \subset U_2$ with $U_1 \cap U_2 = \emptyset$. Set*

$$P_j := \frac{1}{2\pi} \int_{\partial(U_J \cap \mathbb{C}_J)} S_L^{-1}(s,T) ds_J, \qquad j = 1, 2,$$

$$T_j := \frac{1}{2\pi} \int_{\partial(U_J \cap \mathbb{C}_J)} S_L^{-1}(s,T) ds_J s, \qquad j = 1, 2.$$

Then P_j are projectors and $T_j = T_j P_j = P_j T_j$ for $j = 1, 2$.

Remark 3.2.11. We observe that Definition 3.1.3 is the most appropriate for the operator theory. One of the reason is that the slice structure of the function is essential for the properties of slice hyperholomorphic functions such as the Cauchy formulas, which are essential for operator theory. Moreover, if we consider the S-functional calculus restricted to functions defined on axially symmetric slice domains, this would prevent the definition of Riesz projectors via this calculus.

In [53] a commutative version of of the S-functional is established.

We will consider bounded paravector operators T, with commuting components $T_\ell \in \mathcal{B}(V)$ for $\ell = 0, 1, \dots, n$. By $\mathcal{BC}(V_n)$ we denote the subset of $\mathcal{B}(V_n)$ consisting of Clifford operators with commuting components, i.e., operators of the type $\sum_A e_A T_A$ where $A = \{\ell_1 \dots \ell_r\} \in \mathcal{P}\{1, 2, \dots, n\},\ \ell_1 < \dots < \ell_r$ is a multi-index, $T_\emptyset = T_0$, and the operators T_A commute among themselves. In this case the most appropriate definition of the S-spectrum is its commutative version, that for historical reasons is also called F-spectrum, i.e.,

$$\sigma_F(T) = \{s \in \mathbb{R}^{n+1} \ :\ s^2\mathcal{I} - (T + \overline{T})s + T\overline{T} \ \text{ is not invertible in } \mathcal{B}(V_n)\}$$

where we have set $\overline{T} := T_0 - e_1T_1 - \dots - e_nT_n$, and the F-resolvent set

$$\rho_F(T) := \mathbb{R}^{n+1} \setminus \sigma_F(T).$$

It turns out that the F-spectrum is the commutative version of the S-spectrum, i.e., we have

$$\sigma_F(T) = \sigma_S(T), \ \text{ for } \ T \in \mathcal{BC}(V_n).$$

The definition of the F-spectrum comes from the structure of the commutative S-resolvent operators. In fact, for paravector operators $T \in \mathcal{BC}(V_n)$, the commutative version of left S-resolvent operator is defined as

$$S_L^{-1}(s, T) := (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}, \quad s \in \rho_S(T), \tag{3.12}$$

and the commutative version of the right S-resolvent operator is

$$S_R^{-1}(s, T) := (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}(s\mathcal{I} - \overline{T}), \quad s \in \rho_S(T). \tag{3.13}$$

The operator

$$\mathcal{Q}_{c,s}(T)^{-1} := (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}, \quad s \in \rho_S(T), \tag{3.14}$$

is called commutative pseudo S-resolvent operator (or pseudo resolvent operator, for short). For the sake of simplicity, we have used in (3.12) and (3.13) the same symbols used for T with noncommuting components.

We can define the commutative S-functional calculus similarly as in Definition 3.2.8. Moreover, for this type of functional calculus the structure of the S-resolvent equation (see (3.9)) is maintained.

CHAPTER 4

On the quaternionic short-time Fourier and Segal-Bargmann transforms

4.1 Motivation

In this chapter we introduce an extension of the short-time Fourier transform in dimension one to the quaternionic setting .

To this end, we fix a property that relates the complex short-time Fourier transform and the complex Segal-Bargmann transform:

$$V_\varphi f(x,\omega)=e^{-\pi i x\omega}\mathcal{B}f(\bar{z})e^{\frac{-\pi|z|^2}{2}}, \qquad z=x+i\omega, \tag{4.1}$$

where V_φ is the complex short-time Fourier transform with respect to the Gaussian window φ (see [92, Def. 3.1]) and $\mathcal{B}f(z)$ denotes the complex version of the Segal-Bargmann transform, see (1.2). To achieve our aim we use the quaternionc analogue of the Segal-Bargmann transform studied in [76].

4.2 Quaternionic Segal-Bargmann transform

We briefly revise the notion of Fock space of slice hyperholomorphic functions, first introduced in [15]. Moreover, we recall the results that we need about the slice hyperholomorphic Segal- Bargmann transform, see [76].

For a given $J \in \mathbb{S}$ and $\nu > 0$ we define the slice hyperholomorphic Fock space as

$$\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H}) := \{f \in SH(\mathbb{H}); \int_{\mathbb{C}_J} |f_J(p)|^2 e^{-\nu|p|^2} d\lambda_J(p) < \infty\},$$

where $\nu > 0$, $f_J = f|_{\mathbb{C}_J}$ and $d\lambda_J(p) = dxdy$, for $p = x + yJ$, is the Lebesgue measure on $\mathbb{C}_J$. The right $\mathbb{H}$-vector space $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ is endowed with the inner product

$$\langle f, g\rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = \int_{\mathbb{C}_J} \overline{g_J(q)} f_J(q) e^{-\nu|q|^2} d\lambda_J(q), \ \forall f, g \in \mathcal{F}^{2,\nu}_{Slice}(\mathbb{H}). \quad (4.2)$$

The associated norm is given by

$$\|f\|^2_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = \int_{\mathbb{C}_J} |f_J(q)|^2 e^{-\nu|q|^2} d\lambda_J(q).$$

This quaternionic Hilbert space does not depend on the choice of the imaginary unit J. The monomial q^n, $n = 0, 1, 2, ...$, form an orthogonal basis of the slice hyperholomorphic Fock space with

$$\langle q^m, q^n\rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = \frac{m!}{\mu^m}\delta_{m,n}.$$

Furthermore, if we consider $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ in $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ we have

$$\langle f, g\rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = \sum_{n=0}^{\infty} \frac{n!}{\nu^n}\overline{b_n} a_n.$$

This implies that a given series $f(q) = \sum_{n=0}^{\infty} q^n a_n$ belongs to $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ if and only if the sequence $\{a_n\}_{n\geq 0} \subset \mathbb{H}$ satisfies the condition

$$\|f\|_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = \sum_{n=0}^{\infty} \frac{n!}{\nu^n}|a_n|^2 < \infty. \quad (4.3)$$

Now, we observe that for $q \in \mathbb{H}$ we have the following estimate

Lemma 4.2.1. *For every $f \in \mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ we have the estimate*

$$|f(q)| \leq e^{\frac{\nu}{2}|q|^2} \|f\|_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})}.$$

From the previous result we get that the evaluation map $\delta_q : \mathcal{F}^{2,\nu}_{Slice}(\mathbb{H}) \to \mathbb{H}$; $\delta_q(f) = f(q)$, is a continuous linear form. Therefore, by the Riesz's representation theorem for quaternionic Hilbert spaces, there exists a unique element $K^{\nu}_q \in \mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ such that

$$\langle f, K^{\nu}_q \rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = f(q),$$

for all $f \in \mathcal{F}^{2\nu}_{Slice}(\mathbb{H})$. The reproducing kernel function $K_\nu : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$; $(p,q) \mapsto K_\nu(p,q) = K^{\nu}_q(p)$ is given by

$$K_\nu(p,q) = \sum_{n=0}^{\infty} \frac{\nu^n p^n \overline{q}^n}{n!} = \overline{K_\nu(q,p)}.$$

Precisely, the reproducing kernel for the slice hyperholomorphic Fock space $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ is given by

$$K_\nu(p,q) = K^{\nu}_q(p) = \sum_{n=0}^{\infty} \frac{\nu^n p^n \overline{q}^n}{n!} = e_*(\nu p \bar{q}), \qquad (p,q) \in \mathbb{H} \times \mathbb{H}. \quad (4.4)$$

Now, we recall some basic result of the quaternionic Segal-Bargmann transform. It has as a domain the Hilbert space $L^2(\mathbb{R}, dx) = L^2(\mathbb{R}, \mathbb{H})$, consisting of all the square integrable $\mathbb{H}$-valued functions with respect to

$$\langle \psi, \phi \rangle := \int_{\mathbb{R}} \overline{\phi(t)} \psi(t) dt.$$

The range of the quaternionic Segal-Bargmann transform is the slice hyperholomorphic Fock space $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$. The kernel of the quaternionic Segal-Bargmann transform is given by:

$$A(q,x) = \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} e^{-\frac{\nu}{2}(q^2+x^2)+\nu\sqrt{2}qx}, \ \forall (q,x) \in \mathbb{H} \times \mathbb{R}.$$

This function can be seen as the generating function of the real weighted Hermite functions

$$h^{\nu}_n(x) := (-1)^n e^{\frac{\nu}{2}x^2} \frac{d^n}{dx^n}(e^{-\nu x^2})$$

that form an orthogonal basis of the space $L^2(\mathbb{R}; dx)$, with norm given explicitly by

$$\|h^{\nu}_n\|^2_{L^2(\mathbb{R},dx)} = 2^n \nu^n n! \left(\frac{\pi}{\nu}\right)^{\frac{1}{2}}.$$

In particular, we have the following result.

Proposition 4.2.2. *For all $q \in \mathbb{H}$ and $x \in \mathbb{R}$, we have*

$$A(q,x) = \sum_{k=0}^{\infty} f_k^\nu(q)\psi_k^\nu(x),$$

where ψ_k^ν denote the normalized weighted Hermite functions:

$$\psi_k^\nu(x) := \frac{h_k^\nu(x)}{\|h_k^\nu(x)\|_{L^2(\mathbb{R},\mathbb{H})}}, \qquad \textit{and} \qquad f_k^\nu(q) := \frac{q^k}{||q^k||_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})}}, \qquad \forall k \geq 0.$$

Then, for any quaternionic valued function φ in $L^2(\mathbb{R},\mathbb{H})$ the slice hyperholomorphic Segal-Bargmann transform is defined by

$$\mathcal{B}^S_{\mathbb{H}}(\varphi)(q) = \int_{\mathbb{R}} \mathcal{A}^S_{\mathbb{H}}(q,x)\varphi(x)dx. \tag{4.5}$$

The following result shows that the integral transform $\mathcal{B}^S_{\mathbb{H}}$ is well defined on $L^2(\mathbb{R};dx)$.

Proposition 4.2.3. *For every $q \in \mathbb{H}$ and every $\varphi \in L^2(\mathbb{R};dx)$, we have*

$$|\mathcal{B}^S_{\mathbb{H}}(\varphi)(q)| \leq \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} e^{\frac{\nu}{2}|q|^2} \|\varphi\|_{L^2(\mathbb{R};dx)}. \tag{4.6}$$

Moreover, by direct computations, we can obtain the expression of the slice hyperholomorphic Segal-Bargmann transform acting on Hermite function h_n^ν.

Lemma 4.2.4. *For every quaternion $q \in \mathbb{H}$ and the nonnegative integer n, we have*

$$\mathcal{B}^S_{\mathbb{H}}(h_n^\nu)(q) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} 2^{\frac{n}{2}}\nu^n q^n,$$

and

$$\|\mathcal{B}^S_{\mathbb{H}}(h_n^\nu)\|_{\mathcal{F}^{2,\nu}_{Slice}} = \|h_n^\nu\|_{L^2(\mathbb{R},\mathbb{H})}.$$

Now, we show an unitary property for the slice hyperholomorphic Segal-Bargmann transform, which is not found in literature in the following explicit form.

Proposition 4.2.5. *Let $f, g \in L^2(\mathbb{R},\mathbb{H})$. Then, we have*

$$\langle \mathcal{B}^S_{\mathbb{H}}(f), \mathcal{B}^S_{\mathbb{H}}(g)\rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} = \langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{4.7}$$

Proof. Any $f, g \in L^2(\mathbb{R}, \mathbb{H})$ can be expanded as

$$f(x) = \sum_{k=0}^{\infty} h_k^{\nu}(x)\alpha_k,$$

$$g(x) = \sum_{k=0}^{\infty} h_k^{\nu}(x)\beta_k,$$

where $\{\alpha_k\}_{k\geq 0}$, $\{\beta_k\}_{k\geq 0} \subset \mathbb{H}$.

$$\begin{aligned} \langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{g(x)} f(x)\, dx = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \overline{h_k^{\nu}(x)\beta_k} h_k^{\nu}(x)\alpha_k\, dx \\ &= \sum_{k=0}^{\infty} \overline{\beta_k} \left(\int_{\mathbb{R}} \overline{h_k^{\nu}(x)} h_k^{\nu}(x)\, dx \right) \alpha_k \\ &= \sum_{k=0}^{\infty} \|h_k^{\nu}(x)\|^2_{L^2(\mathbb{R},\mathbb{H})} \overline{\beta_k}\alpha_k. \end{aligned} \tag{4.8}$$

On the other way, since

$$\langle f, h_k^{\nu}\rangle_{L^2(\mathbb{R},\mathbb{H})} = \sum_{j=0}^{\infty} \left(\int_{\mathbb{R}} \overline{h_k^{\nu}(x)} h_j^{\nu}(x)\, dx \right) \alpha_j = \|h_k^{\nu}(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\alpha_k.$$

By [76] we have that

$$\begin{aligned} \mathcal{B}^S_{\mathbb{H}}(f)(q) &= \sum_{k=0}^{\infty} q^k \frac{\langle f, h_k^{\nu}\rangle_{L^2(\mathbb{R},\mathbb{H})}}{\|h_k^{\nu}(x)\|_{L^2(\mathbb{R},\mathbb{H})}\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}} \\ &= \sum_{k=0}^{\infty} q^k \frac{\|h_k^{\nu}(x)\|_2^2}{\|h_k^{\nu}(x)\|_{L^2(\mathbb{R},\mathbb{H})}\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\alpha_k \\ &= \sum_{k=0}^{\infty} q^k \frac{\|h_k^{\nu}(x)\|_{L^2(\mathbb{R},\mathbb{H})}}{\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\alpha_k. \end{aligned} \tag{4.9}$$

Using the similar computations we obtain

$$\overline{\mathcal{B}^S_{\mathbb{H}}(g)(q)} = \sum_{k=0}^{\infty} \frac{\|h_k^{\nu}(x)\|_{L^2(\mathbb{R},\mathbb{H})}}{\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}} \overline{q^k\beta_k}. \tag{4.10}$$

By putting together (4.9) and (4.10) we obtain

$$\begin{aligned}
\langle \mathcal{B}_{\mathbb{H}}^S(f), \mathcal{B}_{\mathbb{H}}^S(g)\rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} &= \sum_{k=0}^{\infty}\int_{\mathbb{C}_J} \|h_k^\nu(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\frac{\overline{q}^k}{\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\cdot \\
&\quad \cdot\frac{q^k}{\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\alpha_k e^{-\nu|q|^2}\, d\lambda_J(q) \\
&= \sum_{k=0}^{\infty}\|h_k^\nu(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\left(\int_{\mathbb{C}_J}\frac{\overline{q}^k}{\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\cdot\right. \\
&\quad \left.\cdot\frac{q^K}{\|q^k\|_{\mathcal{F}^{2,\nu}_{Slice}}}e^{-\nu|q|^2}\, d\lambda_J(q)\right)\alpha_k \\
&= \sum_{k=0}^{\infty}\|h_k^\nu(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\frac{1}{\|q^k\|^2_{\mathcal{F}^{2,\nu}_{Slice}}}\cdot \\
&\quad \cdot\left(\int_{\mathbb{C}_J}\overline{q^k}q^k e^{-\nu|q|^2}\, d\lambda_J(q)\right)\alpha_k \\
&= \sum_{k=0}^{\infty}\|h_k^\nu(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\frac{1}{\|q^k\|^2_{\mathcal{F}^{2,\nu}_{Slice}}}\|q^k\|^2_{\mathcal{F}^{2,\nu}_{Slice}}\alpha_k \\
&= \sum_{k=0}^{\infty}\|h_k^\nu(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\alpha_k
\end{aligned}$$

Finally, since (4.8) and (4.11) are equal we obtain the thesis. □

If we consider $f = g$ in Proposition 4.2.5 we have the following result.

Theorem 4.2.6. *The quaternionic Segal-Bargmann transform realizes a surjective isometry from $L^2(\mathbb{R},\mathbb{H})$ onto the slice hyperholomorphic Fock space $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$.*

4.3 Range of the Schwartz space and some operators

We characterize the range of the Schwartz space under the Segal-Bargmann transform with parameter $\nu = 1$ in the slice hyperholomorphic setting of quaternions. We consider also some equivalence relations related to the position and momentum operators in this setting. The quaternionic Schwartz space on the real line, that we are considering in this framework, is defined by

$$\mathcal{S}_{\mathbb{H}}(\mathbb{R}) := \left\{ \psi : \mathbb{R} \longrightarrow \mathbb{H} \ : \ \sup_{x\in\mathbb{R}} \left| x^\alpha \frac{d^\beta}{dx^\beta}(\psi)(x) \right| < \infty \right\}, \ \forall \alpha, \beta \in \mathbb{N}\}.$$

For $J \in \mathbb{S}$, the classical Schwartz space is given by

$$\mathcal{S}_{\mathbb{C}_J}(\mathbb{R}) := \left\{ \varphi : \mathbb{R} \longrightarrow \mathbb{C}_J; \ : \ \sup_{x\in\mathbb{R}} \left| x^\alpha \frac{d^\beta}{dx^\beta}(\varphi)(x) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N} \right\}\}.$$

Clearly, we have that

$$\mathcal{S}_{\mathbb{C}_J}(\mathbb{R}) \subset \mathcal{S}_{\mathbb{H}}(\mathbb{R}) \subset L^2_{\mathbb{H}}(\mathbb{R}).$$

Moreover, we prove the following.

Lemma 4.3.1. *Let $\psi : x \longmapsto \psi(x)$ be a quaternionic valued function. Let $I, J \in \mathbb{S}$ be such that $I \perp J$. Then, $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$ if and only if there exist $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}_J}(\mathbb{R})$ such that we have*

$$\psi(x) = \varphi_1(x) + \varphi_2(x)I, \ \forall x \in \mathbb{R}.$$

Proof. Let $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$. Then, we can write

$$\psi(x) = \varphi_1(x) + \varphi_2(x)I,$$

where φ_1 and φ_2 are $\mathbb{C}_J-$valued functions. Note that for all $\alpha, \beta \in \mathbb{N}$ we have

$$\left| x^\alpha \frac{d^\beta}{dx^\beta}(\psi)(x) \right|^2 = \left| x^\alpha \frac{d^\beta}{dx^\beta}(\varphi_1)(x) \right|^2 + \left| x^\alpha \frac{d^\beta}{dx^\beta}(\varphi_2)(x) \right|^2.$$

In particular, this implies that $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$ if and only if $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}_J}(\mathbb{R})$. □

Let us now denote by $\mathcal{SF}(\mathbb{H})$ the range of $\mathcal{S}_{\mathbb{H}}(\mathbb{R})$ under the quaternionic Segal-Bargmann transform $\mathcal{B}^S_{\mathbb{H}}$. Therefore, we have the following characterization of $\mathcal{SF}(\mathbb{H})$.

Theorem 4.3.2. *A function $f(q) = \displaystyle\sum_{k=0}^{\infty} q^k c_k$ belongs to $\mathcal{SF}(\mathbb{H})$ if and only if*

$$\sup_{k\in\mathbb{N}} |c_k| k^p \sqrt{k!} < \infty, \forall p > 0.$$

i.e,

$$\mathcal{SF}(\mathbb{H}) = \{\sum_{k=0}^{\infty} q^k c_k, \ c_k \in \mathbb{H} \text{ and } \sup_{k\in\mathbb{N}} |c_k| k^p \sqrt{k!} < \infty, \forall p > 0\}.$$

Proof. Let $f \in \mathcal{SF}(\mathbb{H})$, then by definition $f = \mathcal{B}^S_{\mathbb{H}}\psi$ where $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$. Let $I, J \in \mathbb{S}$, be such that $I \perp J$. Thus, Lemma 4.3.1 implies that

$$\psi(x) = \varphi_1(x) + \varphi_2(x)I,$$

where $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}_J}(\mathbb{R})$. Therefore, we have

$$\mathcal{B}^S_{\mathbb{H}}(\psi)(q) = \mathcal{B}^S_{\mathbb{H}}(\varphi_1)(q) + \mathcal{B}^S_{\mathbb{H}}(\varphi_2)(q)I.$$

Then, we take the restriction to the complex plane $\mathbb{C}_J$ and get:

$$\mathcal{B}^S_{\mathbb{H}}(\psi)(z) = \mathcal{B}_{\mathbb{C}_J}(\varphi_1)(z) + \mathcal{B}_{\mathbb{C}_J}(\varphi_2)(z)I, \quad \forall z \in \mathbb{C}_J,$$

where the complex Bargmann transform is given by

$$\mathcal{B}_{\mathbb{C}_J}(\varphi_l)(z) = \frac{1}{\pi^{\frac{3}{4}}} \int_{\mathbb{R}} e^{-\frac{1}{2}(z^2+x^2)+\sqrt{2}zx} \varphi_l(x)dx, \ l = 1, 2.$$

In particular, we set $f_J := \mathcal{B}^S_{\mathbb{H}}(\psi)$, $f_1 := \mathcal{B}_{\mathbb{C}_J}(\varphi_1)$ and $f_2 := \mathcal{B}_{\mathbb{C}_J}(\varphi_2)$. Then, we have $f_1, f_2 \in \mathcal{SF}(\mathbb{C}_J)$. Thus, by applying the classical result in complex analysis, see [115] we have

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } f_2(z) = \sum_{n=0}^{\infty} b_n z^n, \ \forall z \in \mathbb{C}_J.$$

Moreover, for all $p > 0$ the following conditions hold

$$\sup_{n\in\mathbb{N}} |a_n| n^p \sqrt{n!} < \infty \text{ and } \sup_{n\in\mathbb{N}} |b_n| n^p \sqrt{n!} < \infty.$$

In particular, we have then

$$f_J(z) = \sum_{n=0}^{\infty} a_n z^n + (\sum_{n=0}^{\infty} a_n z^n)I, \ \forall z \in \mathbb{C}_J.$$

Therefore,

$$f_J(z) = \sum_{n=0}^{\infty} z^n c_n \text{ with } c_n = a_n + b_n I, \text{ for all } z \in \mathbb{C}_J.$$

Thus, by taking the slice hyperholomorphic extension we get

$$f(q) = \sum_{n=0}^{\infty} q^n c_n, \ \forall q \in \mathbb{H}.$$

Moreover, note that $c_n = a_n + b_n I, n \in \mathbb{N}$. Then, $|c_n| \leq |a_n| + |b_n|, \forall n \in \mathbb{N}$. Thus, for all $p > 0$, we have

$$\sup_{n\in\mathbb{N}} |c_n| n^p \sqrt{n!} \leq \sup_{n\in\mathbb{N}} |a_n| n^p \sqrt{n!} + \sup_{n\in\mathbb{N}} |b_n| n^p \sqrt{n!} < \infty.$$

Finally, we conclude that

$$\mathcal{SF}(\mathbb{H}) = \{f(q) = \sum_{k=0}^{\infty} q^k c_k,\ c_k \in \mathbb{H} \text{ and } \sup_{k\in\mathbb{N}} |c_k| k^p \sqrt{k!} < \infty, \forall p > 0\}.$$

□

Now, let us consider on $L^2(\mathbb{R}, \mathbb{H}) = L^2_{\mathbb{H}}(\mathbb{R})$ the position and momentum operators defined by

$$X : \varphi \mapsto X\varphi(x) = x\varphi(x) \text{ and } D : \varphi \mapsto D\varphi(x) = \frac{d}{dx}\varphi(x).$$

Their domains are given respectively by

$$\mathcal{D}(X) := \{\varphi \in L^2_{\mathbb{H}}(\mathbb{R});\ X\varphi \in L^2_{\mathbb{H}}(\mathbb{R})\} \text{ and } \mathcal{D}(D) := \{\varphi \in L^2_{\mathbb{H}}(\mathbb{R});\ D\varphi \in L^2_{\mathbb{H}}(\mathbb{R})\}.$$

First, let us prove the following

Lemma 4.3.3. *For all* $(q, x) \in \mathbb{H} \times \mathbb{R}$, *we have*

$$\partial_S \mathcal{A}^S_{\mathbb{H}}(q, x) = (-q + \sqrt{2}x)\mathcal{A}^S_{\mathbb{H}}(q, x).$$

Proof. Let $(q, x) \in \mathbb{H} \times \mathbb{R}$. Then, by definition of the quaternionic Segal-Bargmann kernel we can write

$$\mathcal{A}^S_{\mathbb{H}}(q, x) := \pi^{-\frac{3}{4}} e^{-\frac{x^2}{2}} e^{-\frac{q^2}{2}} e^{\sqrt{2}qx}.$$

In this case, we can apply the Leibnitz rule with respect to the slice derivative and get

$$\partial_S \mathcal{A}^S_{\mathbb{H}}(q, x) = \pi^{-\frac{3}{4}} e^{-\frac{x^2}{2}} \left(e^{-\frac{q^2}{2}} \partial_S (e^{\sqrt{2}xq}) + \partial_S (e^{-\frac{q^2}{2}}) e^{\sqrt{2}xq} \right).$$

However, using the series expansion of the exponential function and applying the slice derivative we know that

$$\partial_S (e^{-\frac{q^2}{2}}) = -q e^{-\frac{q^2}{2}} \text{ and } \partial_S (e^{\sqrt{2}xq}) = \sqrt{2}x e^{\sqrt{2}xq}.$$

Therefore, we obtain

$$\partial_S \mathcal{A}^S_{\mathbb{H}}(q, x) = (-q + \sqrt{2}x) A^S_{\mathbb{H}}(q, x).$$

□

Theorem 4.3.4. *Let $\varphi \in \mathcal{D}(X)$. Then, we have*

$$(\partial_S + q)\, \mathcal{B}_{\mathbb{H}}^S(\varphi)(q) = \sqrt{2}\mathcal{B}_{\mathbb{H}}^S(x\varphi)(q), \ \forall q \in \mathbb{H}.$$

Proof. Let $\varphi \in \mathcal{D}(X)$ and $q \in \mathbb{H}$. Then, we have

$$\partial_S \mathcal{B}_{\mathbb{H}}^S(\varphi)(q) = \int_{\mathbb{R}} \partial_S \mathcal{A}_{\mathbb{H}}^S(q,x)\varphi(x)dx.$$

Therefore, using Lemma 4.3.3 we obtain

$$\partial_S \mathcal{B}_{\mathbb{H}}^S(\varphi)(q) = \sqrt{2}\mathcal{B}_{\mathbb{H}}^S(x\varphi)(q) - q\mathcal{B}_{\mathbb{H}}^S(\varphi)(q).$$

Finally, we get

$$(\partial_S + q)\, \mathcal{B}_{\mathbb{H}}^S(\varphi)(q) = \sqrt{2}\mathcal{B}_{\mathbb{H}}^S(x\varphi)(q), \ \forall q \in \mathbb{H}.$$

□

As a quick consequence, we have

Corollary 4.3.5. *The position operator X on $L^2_{\mathbb{H}}(\mathbb{R})$ is equivalent to the operator $\dfrac{1}{\sqrt{2}}(\partial_S + q)$ on the space $\mathcal{F}^{2,1}_{Slice}(\mathbb{H})$ via the quaternionic Segal-Bargmann transform $\mathcal{B}_{\mathbb{H}}^S$. In other words, for all $\varphi \in \mathcal{D}(X)$ we have*

$$X(\varphi) = (\mathcal{B}_{\mathbb{H}}^S)^{-1}\frac{(\partial_S + q)}{\sqrt{2}}\mathcal{B}_{\mathbb{H}}^S(\varphi).$$

On the other hand, we have also the following

Theorem 4.3.6. *We denote by $M_q : \varphi \longmapsto M_q\varphi(q) = q\varphi(q)$ the creation operator on $\mathcal{F}^{2,1}_{Slice}(\mathbb{H})$. Then, we have*

$$(\mathcal{B}_{\mathbb{H}}^S)^{-1} M_q \mathcal{B}_{\mathbb{H}}^S = \frac{1}{\sqrt{2}}(X - D) \text{ on } \mathcal{D}(X) \cap \mathcal{D}(D).$$

Proof. Let $\varphi \in \mathcal{D}(X) \cap \mathcal{D}(D)$. Then, we have

$$\begin{aligned}\mathcal{B}_{\mathbb{H}}^S(D\varphi)(q) &= \int_{\mathbb{R}} \mathcal{A}_{\mathbb{H}}^S(q,x)\frac{d}{dx}\varphi(x)dx \\ &= -\int_{\mathbb{R}} \frac{d}{dx}\mathcal{A}_{\mathbb{H}}^S(q,x)\varphi(x)dx.\end{aligned}$$

However, note that for all $(q,x) \in \mathbb{H} \times \mathbb{R}$, we have

$$\frac{d}{dx}\mathcal{A}_{\mathbb{H}}^S(q,x) = (-x + \sqrt{2}q)\mathcal{A}_{\mathbb{H}}^S(q,x).$$

Therefore,

$$\mathcal{B}_{\mathbb{H}}^{S}(D\varphi)(q)=\mathcal{B}_{\mathbb{H}}^{S}(x\varphi)(q)-\sqrt{2}q\mathcal{B}_{\mathbb{H}}^{S}(\varphi)(q).$$

Thus, we obtain

$$M_q\mathcal{B}_{\mathbb{H}}^{S}(\varphi)=\mathcal{B}_{\mathbb{H}}^{S}\left(\frac{1}{\sqrt{2}}(X-D)\right)(\varphi).$$

Finally, we just need to apply $(\mathcal{B}_{\mathbb{H}}^{S})^{-1}$ to complete the proof. □

Corollary 4.3.7. *The position operator X on $L^2_{\mathbb{H}}(\mathbb{R})$ is equivalent to the operator $\frac{1}{\sqrt{2}}(\partial_S+q)$ on the space $\mathcal{F}^{2,1}_{Slice}(\mathbb{H})$ via the quaternionic Segal-Bargmann transform $\mathcal{B}_{\mathbb{H}}^{S}$. In other words, for all $\varphi\in\mathcal{D}(X)$ we have*

$$X(\varphi)=(\mathcal{B}_{\mathbb{H}}^{S})^{-1}\frac{(\partial_S+q)}{\sqrt{2}}\mathcal{B}_{\mathbb{H}}^{S}(\varphi).$$

On the other hand, we have also the following

Theorem 4.3.8. *We denote by $M_q:\varphi\longmapsto M_q\varphi(q)=q\varphi(q)$ the creation operator on $\mathcal{F}^{2,1}_{Slice}(\mathbb{H})$. Then, we have*

$$(\mathcal{B}_{\mathbb{H}}^{S})^{-1}M_q\mathcal{B}_{\mathbb{H}}^{S}=\frac{1}{\sqrt{2}}(X-D)\text{ on }\mathcal{D}(X)\cap\mathcal{D}(D).$$

Proof. Let $\varphi\in\mathcal{D}(X)\cap\mathcal{D}(D)$. Then, we have

$$\begin{aligned}\mathcal{B}_{\mathbb{H}}^{S}(D\varphi)(q)&=\int_{\mathbb{R}}\mathcal{A}_{\mathbb{H}}^{S}(q,x)\frac{d}{dx}\varphi(x)dx\\&=-\int_{\mathbb{R}}\frac{d}{dx}\mathcal{A}_{\mathbb{H}}^{S}(q,x)\varphi(x)dx.\end{aligned}$$

However, note that for all $(q,x)\in\mathbb{H}\times\mathbb{R}$, we have

$$\frac{d}{dx}\mathcal{A}_{\mathbb{H}}^{S}(q,x)=(-x+\sqrt{2}q)\mathcal{A}_{\mathbb{H}}^{S}(q,x).$$

Therefore,

$$\mathcal{B}_{\mathbb{H}}^{S}(D\varphi)(q)=\mathcal{B}_{\mathbb{H}}^{S}(x\varphi)(q)-\sqrt{2}q\mathcal{B}_{\mathbb{H}}^{S}(\varphi)(q).$$

Thus, we obtain

$$M_q\mathcal{B}_{\mathbb{H}}^{S}(\varphi)=\mathcal{B}_{\mathbb{H}}^{S}\left(\frac{1}{\sqrt{2}}(X-D)\right)(\varphi).$$

Finally, we just need to apply $(\mathcal{B}_{\mathbb{H}}^{S})^{-1}$ to complete the proof. □

4.4 The 1D quaternion Fourier transform

In this section, we study the one dimensional quaternion Fourier transform (QFT). Namely, we are considering here the 1D left sided QFT studied in chapter 3 of the book [25]. In order to have less problems with computations we add -2π to the exponential. The QFT is different from the one considered for example in [30, 96, 97]. Firs of all, we are making the integration over the real line, so we are working with a 1-dimensional Fourier transform. Whereas, Hitzer and collaborators usually integrate in $\mathbb{R}^2$ and they sometimes consider two roots of the unit. They take into consideration a definition of quaternion Fourier transform where the signal lies between two exponential functions. Moreover, we have the possibility to have a kind of convolution theorem, see Remark (4.4.4). This type of result is harder to obtain in the theory developed by Hitzer and collaborators.

Definition 4.4.1. The left sided 1D quaternionic Fourier transform of a quaternion valued signal $\psi : \mathbb{R} \longrightarrow \mathbb{H}$ is defined on $L^1(\mathbb{R}; dx) = L^1(\mathbb{R}; \mathbb{H})$ by

$$\mathcal{F}_J(\psi)(\omega) = \int_{\mathbb{R}} e^{-2\pi J\omega t}\psi(t)dt$$

for a given $J \in \mathbb{S}$. Its inverse is defined by

$$\widetilde{\mathcal{F}}_J(\phi)(t) = \int_{\mathbb{R}} e^{2\pi J\omega t}\phi(\omega)d\omega.$$

Let $I \in \mathbb{S}$ be such that $J \perp I$. We can split the signal ψ via symplectic decomposition into simplex and perplex parts with respect to J such that we have:

$$\psi(t) = \psi_1(t) + \psi_2(t)I$$

where $\psi_1(t), \psi_2(t) \in \mathbb{C}_I$. The left sided 1D QFT of ψ becomes

$$\mathcal{F}_J(\psi)(\omega) = \int_{\mathbb{R}} e^{-2\pi J\omega t}\psi_1(t)dt + \int_{\mathbb{R}} e^{-2\pi J\omega t}\psi_2(t)dtJ$$

so that

$$\mathcal{F}_J(\psi)(\omega) = \mathcal{F}_J(\psi_1)(\omega) + \mathcal{F}_J(\psi_2)(\omega)J.$$

According to [25], most of the properties may be inherited from the classical complex case thanks to the equivalence between $\mathbb{C}_J$ and the standard complex plane and the fact that QFT can be decomposed into a sum of complex subfield functions.

Now, we define two fundamental operators for time-frequency analysis.

Translation

$$\tau_x\psi(t) := \psi(t-x) \qquad x \in \mathbb{R}.$$

Modulation

$$M_\omega\psi(t) = e^{2\pi J\omega t}\psi(t), \qquad \omega \in \mathbb{R}.$$

As in the classical case we have a commutative relation between the two operators.

Lemma 4.4.2. *Let ψ be a function in $L^2(\mathbb{R},\mathbb{H})$ then we have*

$$\tau_x M_\omega\psi(t) = e^{-2\pi J\omega x} M_\omega \tau_x \psi(t), \qquad \omega, x \in \mathbb{R}. \tag{4.11}$$

Proof. It is just a matter of computations

$$\begin{aligned} \tau_x M_\omega\psi(t) &= M_\omega\psi(t-x) = e^{2\pi J\omega(t-x)}\psi(t-x) \\ &= e^{2\pi J\omega t}e^{-2\pi J\omega x}\psi(t-x) \\ &= e^{-2\pi J\omega x}e^{2\pi J\omega t}\psi(t-x) \\ &= e^{-2\pi J\omega x}M_\omega\tau_x\psi(t). \end{aligned}$$

□

From [25, Table 3.2] we have the following properties

$$\mathcal{F}_J(\tau_x\psi) = M_{-x}\mathcal{F}_J(\psi), \tag{4.12}$$

$$\mathcal{F}_J(M_\omega\psi) = \tau_\omega\mathcal{F}_J(\psi). \tag{4.13}$$

From (4.12) and (4.13) follow easily that

$$\mathcal{F}_J(M_\omega\tau_x\psi) = \tau_\omega M_{-x}\mathcal{F}_J(\psi). \tag{4.14}$$

Then, we prove a version of the Plancherel theorem for 1D QFT.

Theorem 4.4.3. *Let $\phi, \psi \in L^2(\mathbb{R},\mathbb{H})$. Then, we have*

$$\langle \mathcal{F}_J(\phi), \mathcal{F}_J(\psi)\rangle_{L^2(\mathbb{R},\mathbb{H})} = \langle \phi, \psi\rangle_{L^2(\mathbb{R},\mathbb{H})}.$$

In particular, for any $\phi \in L^2(\mathbb{R},\mathbb{H})$ we have

$$||\mathcal{F}_J(\phi)||_{L^2(\mathbb{R},\mathbb{H})} = ||\phi||_{L^2(\mathbb{R},\mathbb{H})}.$$

Proof. Let $\phi, \psi \in L^2(\mathbb{R}, \mathbb{H})$. By inversion formula for the 1D QFT, see [25], we have

$$\phi(\omega) = \widetilde{\mathcal{F}}_J(\mathcal{F}_J(\phi))(\omega), \ \forall \omega \in \mathbb{R}.$$

Thus, direct computations using Fubini's theorem lead to

$$\begin{aligned}
\langle \phi, \psi \rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{\psi(\omega)} \left(\int_{\mathbb{R}} e^{2\pi I \omega t} \mathcal{F}_J(\phi)(t) dt \right) d\omega \\
&= \int_{\mathbb{R}} \overline{\left(\int_{\mathbb{R}} e^{-2\pi I \omega t} \psi(\omega) d\omega \right)} \mathcal{F}_J(\phi)(t) dt \\
&= \int_{\mathbb{R}} \overline{\mathcal{F}_J(\psi)(t)} \mathcal{F}_J(\phi)(t) dt \\
&= \langle \mathcal{F}_J(\phi), \mathcal{F}_J(\psi) \rangle_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

As a direct consequence, we have for any $\phi \in L^2(\mathbb{R}, \mathbb{H})$

$$\begin{aligned}
||\mathcal{F}_J(\phi)||^2_{L^2(\mathbb{R},\mathbb{H})} &= \langle \mathcal{F}_J(\phi), \mathcal{F}_J(\phi) \rangle_{L^2(\mathbb{R},\mathbb{H})} \\
&= \langle \phi, \phi \rangle_{L^2(\mathbb{R},\mathbb{H})} \\
&= ||\phi||^2_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

□

The following remark may be of interest in some other contexts.

Remark 4.4.4. The formal convolution of two given signals $\phi, \psi : \mathbb{R} \longrightarrow \mathbb{H}$ when it exists is defined by

$$(\phi * \psi)(t) := \int_{\mathbb{R}} \phi(\tau) \psi(t - \tau) d\tau.$$

In particular, if the window function ϕ is real valued the 1D QFT satisfies the classical property

$$\mathcal{F}_J(\phi * \psi) = \mathcal{F}_J(\phi) \mathcal{F}_J(\psi).$$

4.5 Quaternion short time Fourier transform with a Gaussian window

The idea of the short-time Fourier transform is to obtain information about local properties of the signal f. In order to achieve this aim the signal f is restricted to an interval and after its Fourier transform is evaluated. However, since a sharp cut-off can introduce artificial discontinuities and can create problems, it is usually chosen a smooth cut-off function φ called "window function".

The aim of this section is to propose a quaternionic analogue of the short-time Fourier transform in dimension one with a Gaussian window function $\varphi(t) = 2^{1/4}e^{-\pi t^2}$. For this, we consider the following formula [92, Prop. 3.4.1]

$$V_\varphi f(x,\omega) = e^{-\pi ix\omega}\mathcal{B}f(\bar{z})e^{\frac{-\pi|z|^2}{2}}, \tag{4.15}$$

where the variables $(x,\omega) \in \mathbb{R}^2$ have been converted into a complex vector $z = x + i\omega$, and $\mathcal{B}f(z)$ is the complex version of the Segal-Bargmann transform according to [92]. Therefore, we want to extend (4.15) to the quaternionic setting. To this end, we use the quaternionic analogue of the Segal-Bargmann transform and the slicing representation of the quaternions $q = x + J\omega$, where $J \in \mathbb{S}$.

If the signal is complex we denote the short-time Fourier transform as V_φ, while if the signal is $\mathbb{H}$-valued we identify the short-time Fourier transform as $\mathcal{V}_\varphi$.

Definition 4.5.1. Let $f : \mathbb{R} \to \mathbb{H}$ be a function in $L^2(\mathbb{R},\mathbb{H})$. We define the 1D quaternion short time Fourier transform of f with respect to $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ as

$$\mathcal{V}_\varphi f(x,\omega) = e^{-J\pi x\omega}\mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}, \tag{4.16}$$

where $q = x + J\omega$ and $\mathcal{B}^S_{\mathbb{H}}(f)(q)$ is the quaternionic Segal-Bargmann transform defined in (4.5).

Using (4.5) with $\nu = 2\pi$, we can write (4.16) in the following way

$$\mathcal{V}_\varphi f(x,\omega) = 2^{\frac{3}{4}}\int_{\mathbb{R}} e^{-\pi\left(\frac{\bar{q}^2}{2}+t^2\right)+2\pi\bar{q}t-J\pi x\omega-\frac{|q|^2\pi}{2}} f(t)\,dt. \tag{4.17}$$

From this formula we are able to put in relation the 1D quaternion short-time Fourier transform and the 1D quaternion Fourier transform defined in section 3.

Lemma 4.5.2. *Let f be a function in $L^2(\mathbb{R},\mathbb{H})$ and $\varphi(t) = 2^{1/4}e^{-\pi t^2}$, recalling the 1D quaternion Fourier transform we have*

$$\mathcal{V}_\varphi f(x,\omega) = \sqrt{2}\mathcal{F}_I(f\cdot\tau_x\varphi)(\omega). \tag{4.18}$$

Proof. By putting $q = x + J\omega$ in (4.17) we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= 2^{\frac{3}{4}} e^{-J\pi x\omega} e^{-\frac{x^2\pi}{2}} e^{-\frac{\omega^2\pi}{2}} \int_{\mathbb{R}} e^{-\pi t^2} e^{-\frac{\pi}{2}\left(x^2-\omega^2-2x\omega J\right)} \cdot \\
&\quad \cdot e^{2\pi(x-J\omega)t} f(t)\,dt \\
&= 2^{\frac{3}{4}} \int_{\mathbb{R}} e^{-\pi t^2-\pi x^2+2\pi xt} e^{-2\pi J\omega t} f(t)\,dt \\
&= \sqrt{2} \int_{\mathbb{R}} e^{-2\pi J\omega t} f(t) 2^{\frac{1}{4}} e^{-\pi(t-x)^2}\,dt \\
&= \sqrt{2} \int_{\mathbb{R}} e^{-2\pi J\omega t} f(t)\varphi(t-x)\,dt = \sqrt{2}\mathcal{F}_I(f\cdot\tau_x\varphi)(\omega).
\end{aligned}$$

□

Now, we prove a formula which relates the 1D quaternion Fourier transform and its signal through the 1D short-time Fourier transform.

Proposition 4.5.3. *If φ is a Gaussian function $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ and $f \in L^2(\mathbb{R},\mathbb{H})$ then*

$$\mathcal{V}_\varphi f(x,\omega) = \sqrt{2}e^{-2\pi J\omega x}\mathcal{V}_\varphi\mathcal{F}_J(f)(\omega,-x). \tag{4.19}$$

Proof. Recalling the definition of modulation and of inner product on $L^2(\mathbb{R},\mathbb{H})$, by Lemma 4.5.2 we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= \sqrt{2}\int_{\mathbb{R}} \overline{e^{2\pi J\omega t}\varphi(t-x)} f(t)\,dt \qquad (4.20)\\
&= \sqrt{2}\int_{\mathbb{R}} \overline{M_\omega\tau_x\varphi(t)} f(t)\,dt = \sqrt{2}\langle f, M_\omega\tau_x\varphi\rangle.
\end{aligned}$$

Using the Plancherel theorem for the 1D quaternion Fourier transform, the property (4.14) and the fact that $\mathcal{F}_J(\varphi) = \varphi$ we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= \sqrt{2}\langle\mathcal{F}_J(f), \mathcal{F}_J(M_\omega\tau_x\varphi)\rangle \\
&= \sqrt{2}\langle\mathcal{F}_J(f), \tau_\omega M_{-x}\mathcal{F}_J(\varphi)\rangle \\
&= \sqrt{2}\langle\mathcal{F}_J(f), \tau_\omega M_{-x}\varphi\rangle.
\end{aligned}$$

Finally, from (4.11) and (4.20) we get

$$\mathcal{V}_\varphi f(x,\omega) = \sqrt{2}e^{-2\pi J\omega x}\langle\mathcal{F}_J(f), M_{-x}\tau_\omega\varphi\rangle = \sqrt{2}e^{-2\pi J\omega x}\mathcal{V}_\varphi\mathcal{F}_J(f)(\omega,-x).$$

□

4.5.1 Moyal fromula

Now, we prove the Moyal formula and an isometric relation for the 1D quaternion short-time Fourier transform. This will be showed in two different ways. In the first one we use the properties of the quaternionic Segal-Bargmann transform, whereas in the second way we use Lemma 4.5.2 and some basic properties of 1D quaternion Fourier transform.

Proposition 4.5.4. *For any $f \in L^2(\mathbb{R}, \mathbb{H})$*

$$\|\mathcal{V}_\varphi f\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \sqrt{2}\|f\|_{L^2(\mathbb{R},\mathbb{H})}. \tag{4.21}$$

Proof. We use the slicing representation of the quaternions $q = x + J\omega$ and formula (4.16) to get

$$\begin{aligned} \|\mathcal{V}_\varphi f\|^2_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^2 \, d\omega \, dx \\ &= \int_{\mathbb{R}} |e^{-J\pi x\omega}|^2 \left| \mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)\right|^2 e^{-|q|^2\pi} \, d\omega \, dx \\ &= \int_{\mathbb{R}} \left| \mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)\right|^2 e^{-|q|^2\pi} \, d\omega \, dx. \end{aligned}$$

Now, using the change of variable $p = \frac{\bar{q}}{\sqrt{2}}$ we have that $dA(p) = \frac{1}{2}\, d\omega \, dx$, hence by Theorem 4.2.6 we have

$$\begin{aligned} \|\mathcal{V}_\varphi f\|^2_{L^2(\mathbb{R},\mathbb{H})} &= 2\int_{\mathbb{R}^2} |\mathcal{B}^S_{\mathbb{H}}(f)(p)|^2 e^{-2\pi|q|^2} \, dA(p) \\ &= 2\|\mathcal{B}^S_{\mathbb{H}}(f)\|^2_{\mathcal{F}^{2,2\pi}_{Slice}} = 2\|f\|^2_{L^2(\mathbb{R},\mathbb{H})}. \end{aligned}$$

Therefore

$$\|\mathcal{V}_\varphi f\|_{L^2(\mathbb{R},\mathbb{H})} = \sqrt{2}\|f\|_{L^2(\mathbb{R},\mathbb{H})}.$$

$\square$

Thus, the 1D quaternionic short-time Fourier transform is an isometry from $L^2(\mathbb{R}, \mathbb{H})$ into $L^2(\mathbb{R}^2, \mathbb{H})$.

Proposition 4.5.5 (Moyal formula)**.** *Let f, g be functions in $L^2(\mathbb{R}, \mathbb{H})$. Then we have*

$$\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{4.22}$$

Proof. From (4.16) we get

$$\begin{aligned}
\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \int_{\mathbb{R}^2} \overline{\mathcal{V}_\varphi g(x,\omega)}\mathcal{V}_\varphi f(x,\omega)\, d\omega\, dx \\
&= \int_{\mathbb{R}^2} \overline{e^{-J\pi x\omega}\mathcal{B}^S_{\mathbb{H}}(g)\left(\frac{\bar{q}}{\sqrt{2}}\right) e^{-\frac{|q|^2\pi}{2}} e^{-J\pi x\omega}} \cdot \\
&\quad \cdot \mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right) e^{-\frac{|q|^2\pi}{2}}\, d\omega\, dx \\
&= \int_{\mathbb{R}^2} \overline{\mathcal{B}^S_{\mathbb{H}}(g)\left(\frac{\bar{q}}{\sqrt{2}}\right)} e^{J\pi x\omega} e^{-J\pi x\omega} \cdot \\
&\quad \cdot \mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right) e^{-|q|^2\pi}\, d\omega\, dx \\
&= \int_{\mathbb{R}^2} \overline{\mathcal{B}^S_{\mathbb{H}}(g)\left(\frac{\bar{q}}{\sqrt{2}}\right)} \mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right) e^{-|q|^2\pi}\, d\omega\, dx.
\end{aligned}$$

Using the same change of variables as before $p = \frac{\bar{q}}{\sqrt{2}}$ and from (4.7) we obtain

$$\begin{aligned}
\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= 2\int_{\mathbb{R}^2} \overline{\mathcal{B}^S_{\mathbb{H}}(g)(p)}\mathcal{B}^S_{\mathbb{H}}(f)(p) e^{-2|q|^2\pi}\, d\omega\, dx \\
&= 2\langle \mathcal{B}^S_{\mathbb{H}}(f), \mathcal{B}^S_{\mathbb{H}}(g)\rangle_{\mathcal{F}^{2,2\pi}_{Slice}(\mathbb{H})} = 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

□

Remark 4.5.6. If we put $f = \frac{h_k^{2\pi}(t)}{\|h_k^{2\pi}(t)\|_2^2}$ by Lemma 4.2.4 we have

$$\mathcal{V}_\varphi f(x,\omega) = e^{-J\pi x\omega} e^{-\frac{\pi}{2}|q|^2} \frac{2^{3/4}}{2^k k!}\bar{q}^k.$$

Remark 4.5.7. From (4.18) we can prove (4.22) in a different way. This proof may be of interest in some other contexts.

Let us assume $f, g \in L^2(\mathbb{R},\mathbb{H})$ and recall $\varphi(t) = 2^{1/4}e^{-\pi t^2}$, by Lemma 4.5.2 and Plancherel theorem for the 1D quaternion Fourier transform we have

$$\begin{aligned}
\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \int_{\mathbb{R}^2} \overline{\mathcal{V}_\varphi g(x,\omega)}\mathcal{V}_\varphi f(x,\omega)\, d\omega\, dx \\
&= 2\int_{\mathbb{R}^2} \overline{\mathcal{F}_I(g\cdot\tau_x\varphi)(\omega)}\mathcal{F}_I(f\cdot\tau_x\varphi)(\omega)\, d\omega\, dx \\
&= 2\int_{\mathbb{R}^2} \overline{g(\omega)\cdot\tau_x\varphi(\omega)} f(\omega)\cdot\tau_x\varphi(\omega)\, d\omega\, dx.
\end{aligned}$$

Now, by Fubini's theorem and the fact that $\|\varphi\|_2^2 = 1$ we get

$$\begin{aligned}
\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= 2\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \overline{g(\omega)\cdot \tau_x\varphi(\omega)} f(\omega)\cdot \tau_x\varphi(\omega)\,dx \right) d\omega \\
&= 2\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \overline{g(\omega)} f(\omega)\varphi^2(x-\omega)\,dx \right) d\omega \\
&= 2\int_{\mathbb{R}} \overline{g(\omega)} f(\omega) \left(\int_{\mathbb{R}} \varphi^2(x-\omega)\,dx \right) d\omega \\
&= 2\int_{\mathbb{R}} \overline{g(\omega)} f(\omega) \|\varphi\|_2^2\,d\omega = 2\int_{\mathbb{R}} \overline{g(\omega)} f(\omega)\,d\omega \\
&= 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

Hence

$$\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{4.23}$$

If we put $f = g$ in (4.23) we obtain (4.21).

4.5.2 Inversion formula and adjoint of QSTFT

The 1D QSTFT with Gaussian window φ satisfies a reconstruction formula that we prove in the following.

Theorem 4.5.8. *Let $f \in L^2(\mathbb{R},\mathbb{H})$. Then, we have*

$$f(y) = 2^{-\frac{1}{4}} \int_{\mathbb{R}^2} e^{2\pi J\omega y} \mathcal{V}_\varphi f(x,\omega) e^{-\pi(y-x)^2}\,dx d\omega,\ \forall y \in \mathbb{R}.$$

Proof. For all $y \in \mathbb{R}$, we set

$$g(y) = 2^{-\frac{1}{4}} \int_{\mathbb{R}^2} e^{2\pi J\omega y} \mathcal{V}_\varphi f(x,\omega) e^{-\pi(y-x)^2}\,dx d\omega.$$

Let $h \in L^2(\mathbb{R},\mathbb{H})$. Fubini's theorem combined with Moyal formula for

QSTFT leads to

$$\begin{aligned}\langle g,h\rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{h(y)}g(y)dy\\ &= 2^{-\frac{1}{4}}\int_{\mathbb{R}^3}\overline{h(y)}e^{2\pi J\omega y}\mathcal{V}_\varphi f(x,\omega)e^{-\pi(y-x)^2}dxd\omega dy\\ &= 2^{-1}\sqrt{2}\int_{\mathbb{R}^2}\left(\overline{\int_{\mathbb{R}} e^{-2\pi J\omega y}2^{\frac{1}{4}}e^{-\pi(y-x)^2}h(y)dy}\right)\mathcal{V}_\varphi f(x,\omega)dxd\omega\\ &= 2^{-1}\int_{\mathbb{R}^2}\overline{\mathcal{V}_\varphi h(x,\omega)}\mathcal{V}_\varphi f(x,\omega)dxd\omega\\ &= 2^{-1}\langle\mathcal{V}_\varphi f,\mathcal{V}_\varphi h\rangle_{L^2(\mathbb{R}^2)}\\ &= \langle f,h\rangle_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

Hence, we have

$$f(y)=g(y)=2^{-\frac{1}{4}}\int_{\mathbb{R}^2}e^{2\pi J\omega y}\mathcal{V}_\varphi f(x,\omega)e^{-\pi(y-x)^2}dxd\omega.$$

This ends the proof. □

We note that the QSTFT admits a left side inverse that we can compute as follows

Theorem 4.5.9. *Let φ denote the Gaussian window $\varphi(t)=2^{1/4}e^{-\pi t^2}$ and let us consider the operator $\mathcal{A}_\varphi : L^2(\mathbb{R}^2,\mathbb{H})\longrightarrow L^2(\mathbb{R},\mathbb{H})$ defined for any $F\in L^2(\mathbb{R}^2,\mathbb{H})$ by*

$$\mathcal{A}_\varphi(F)(y)=2^{\frac{3}{4}}\int_{\mathbb{R}^2}e^{2\pi J\omega y}F(x,\omega)e^{-\pi(y-x)^2}dxd\omega,\ \forall y\in\mathbb{R}.$$

Then, $\mathcal{A}_\varphi$ is the adjoint of $\mathcal{V}_\varphi$. Moreover, the following identity holds

$$\mathcal{V}_\varphi^*\mathcal{V}_\varphi=2Id. \tag{4.24}$$

Proof. Let $F\in L^2(\mathbb{R}^2,\mathbb{H})$ and $h\in L^2(\mathbb{R},\mathbb{H})$. We use some calculations similar to the previous result and get

$$\begin{aligned}
\langle \mathcal{A}_\varphi(F), h\rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{h(y)}\mathcal{A}_\varphi(F)(y)dy \\
&= 2^{\frac{3}{4}} \int_{\mathbb{R}^3} \overline{h(y)} e^{2\pi J\omega y} F(x,\omega) e^{-\pi(y-x)^2} dx d\omega dy \\
&= \int_{\mathbb{R}^2} \sqrt{2} \left(\overline{\int_{\mathbb{R}} e^{-2\pi J\omega y} 2^{\frac{1}{4}} e^{-\pi(y-x)^2} h(y) dy} \right) F(x,\omega) dx d\omega \\
&= \int_{\mathbb{R}^2} \overline{\mathcal{V}_\varphi h(x,\omega)} F(x,\omega) dx d\omega \\
&= \langle F, \mathcal{V}_\varphi h\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned}$$

In particular, this shows that

$$\mathcal{A}(\varphi)(F) = \mathcal{V}_\varphi^*(F), \ \forall F \in L^2(\mathbb{R}^2, \mathbb{H}).$$

From reconstruction formula we obtain (4.24).

□

Remark 4.5.10. We note that the identity $\mathcal{V}_\varphi^*\mathcal{V}_\varphi = 2Id$ provides another proof for the fact that QSTFT is an isometric operator and the adjoint $\mathcal{V}_\varphi^*$ defines a left inverse.

4.5.3 The eigenfunctions of the 1D quaternion Fourier transform

Through the 1D QSTFT we can prove in another way that the eigenfunctions of the 1D quaternion Fourier transform are given by the Hermite functions.

Proposition 4.5.11. *The Hermite functions $h_k^{2\pi}(t)$ are eigenfunctions of the 1D quaternion Fourier transform :*

$$\mathcal{F}_J(h_k^{2\pi})(t) = 2^{-1/2}(-I)^k h_k^{2\pi}(t), \qquad t \in \mathbb{R}.$$

Proof. By the first identity of Lemma 4.2.4 we have

$$\begin{aligned}
\mathcal{V}_\varphi(h_k^{2\pi})(x,-\omega) &= e^{J\pi x\omega} \mathcal{B}_{\mathbb{H}}^S(h_k^{2\pi}) \left(\frac{q}{\sqrt{2}}\right) e^{-\frac{\pi|q|^2}{2}} \qquad (4.25) \\
&= e^{J\pi x\omega} 2^{1/4} 2^{k/2} (2\pi)^k 2^{-k/2} q^k e^{-\frac{\pi|q|^2}{2}} \\
&= e^{J\pi x\omega} 2^{1/4} (2\pi)^k q^k e^{-\frac{\pi|q|^2}{2}}.
\end{aligned}$$

Recalling that $q = x + J\omega$ and using (4.19) we obtain

$$\begin{aligned} \mathcal{V}_\varphi \mathcal{F}_J(h_k^{2\pi})(x,-\omega) &= 2^{-1/2} e^{2\pi J\omega x} \mathcal{V}_\varphi h_k^{2\pi}(\omega, x) \\ &= 2^{-1/2} e^{2\pi J\omega x} e^{-J\pi\omega x} \mathcal{B}_{\mathbb{H}}^S(h_k^{2\pi}) \left(\frac{\omega - Ix}{\sqrt{2}} \right) e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{-1/2} e^{\pi J\omega x} \mathcal{B}_{\mathbb{H}}^S(h_k^{2\pi}) \left(\frac{-Iq}{\sqrt{2}} \right) e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{-1/2} e^{\pi J\omega x} 2^{1/4} 2^{k/2} (2\pi)^k (-I)^k 2^{-k/2} q^k e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{-1/2} (-J)^k e^{J\pi\omega x} 2^{1/4} (2\pi)^k q^k e^{-\frac{|q|^2\pi}{2}}. \end{aligned}$$

Combining with (4.25) we get

$$\mathcal{V}_\varphi \mathcal{F}_J(h_k^{2\pi})(x,-\omega) = 2^{-1/2}(-I)^k \mathcal{V}_\varphi h_k^{2\pi}(x,-\omega).$$

From (4.24) we know that $\mathcal{V}_\varphi$ is injective, hence we have the thesis. □

4.5.4 Reproducing kernel property

The inversion formula gives us the possibility to write the 1D QSTFT using the reproducing kernel associated to the quaternion Gabor space, introduced in [4], with a Gaussian window that is defined by

$$\mathcal{G}_{\mathbb{H}}^{\varphi} := \{\mathcal{V}_\varphi f,\ f \in L^2(\mathbb{R}, \mathbb{H})\}.$$

Theorem 4.5.12. *Let f be in $L^2(\mathbb{R}, \mathbb{H})$ and $\varphi(t) = 2^{1/4} e^{-\pi t^2}$. If*

$$\mathbb{K}_\varphi(\omega, x; \omega', x') = \int_{\mathbb{R}} e^{-2\pi J\omega' t} \varphi(t - x') \overline{e^{-2\pi J\omega t} \varphi(t - x)}\, dt,$$

then $\mathbb{K}_\varphi(\omega, x; \omega', x')$ is the reproducing kernel of the space $\mathcal{G}_{\mathbb{H}}^{\varphi}$, i.e.

$$\mathcal{V}_\varphi f(x', \omega') = \int_{\mathbb{R}^2} \mathbb{K}_\varphi(\omega, x; \omega', x') \mathcal{V}_\varphi f(x, \omega)\, dx d\omega.$$

Proof. By Lemma 4.5.2 and the reconstruction formula we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x',\omega') &= 2^{3/4}\int_{\mathbb{R}} e^{-2\pi J\omega' t} f(t) e^{-\pi(t-x')^2}\, dt \\
&= 2^{3/4}\int_{\mathbb{R}} e^{-2\pi J\omega' t} e^{-\pi(t-x')^2} 2^{-\frac{1}{4}} \cdot \\
&\quad \cdot \left(\int_{\mathbb{R}^2} e^{2\pi J\omega t} e^{-\pi(t-x)^2} \mathcal{V}_\varphi f(x,\omega)\, dx\, d\omega\right) dt \\
&= \sqrt{2}\int_{\mathbb{R}^3} e^{-2\pi I(\omega'-\omega)t} e^{-\pi(t-x')^2} e^{-\pi(t-x)^2} \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x,\omega)\, dx\, d\omega\, dt.
\end{aligned}$$

Using Fubini's theorem we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x',\omega') &= \sqrt{2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-2\pi J(\omega'-\omega)t} e^{-\pi(t-x')^2} e^{-\pi(t-x)^2}\, dt\right) \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x,\omega)\, dx\, d\omega \\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-2\pi J\omega' t} 2^{1/4} e^{-\pi(t-x')^2} \overline{2^{1/4} e^{-2\pi J\omega t} e^{-\pi(t-x)^2}}\, dt\right) \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x,\omega)\, dx\, d\omega \\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-2\pi J\omega' t} \varphi(t-x') \overline{e^{-2\pi J\omega t}\varphi(t-x)}\, dt\right) \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x,\omega)\, dx\, d\omega \\
&= \int_{\mathbb{R}^2} \mathbb{K}_\varphi(\omega, x; \omega', x') \mathcal{V}_\varphi f(x,\omega)\, dx d\omega.
\end{aligned}$$

□

4.5.5 Lieb's uncertainty principle for QSTFT

The QSTFT follows the Lieb's uncertainty principle as the classical complex case. Indeed, we first study the weak uncertainty principle which is the subject of this result

Theorem 4.5.13 (Weak uncertainty principle)**.** *Let $f \in L^2(\mathbb{R},\mathbb{H})$ be a unit vector (i.e $||f|| = 1$), U an open set of $\mathbb{R}^2$ and $\varepsilon \geq 0$ such that*

$$\int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega \geq 1-\varepsilon.$$

Then, we have

$$|U| \geq \frac{1-\varepsilon}{2},$$

where $|U|$ denotes the Lebesgue measure of U.

Proof. We note that using Definition of QSTFT and Proposition 4.2.3 we obtain

$$\begin{aligned} |\mathcal{V}_\varphi f(x,\omega)| &= |\mathcal{B}_{\mathbb{H}}^S f(\bar{q}/\sqrt{2})| e^{-\frac{|q|^2}{2}\pi} \\ &= |\mathcal{B}_{\mathbb{H}}^S f(p)| e^{-\pi|p|^2};\ p=\bar{q}/\sqrt{2} \\ &\leq \sqrt{2}||f||_{L^2(\mathbb{R})}. \end{aligned}$$

Thus, by hypothesis we get

$$1-\varepsilon \leq \int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega \leq ||\mathcal{V}_\varphi f||_\infty^2 |U| \leq 2|U|.$$

Hence, we have

$$|U| \geq \frac{1-\varepsilon}{2}.$$

□

Theorem 4.5.14 (Lieb's inequality)**.** *Let $f \in L^2(\mathbb{R},\mathbb{H})$ and $2 \leq p < \infty$. Then, we have*

$$\int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^p dx d\omega \leq \frac{2^{p+1}}{p} ||f||_{L^2(\mathbb{R},\mathbb{H})}^p$$

Proof. Let $I, J \in \mathbb{S}$ be such that I is orthogonal to J. Then, for $f \in L^2(\mathbb{R},\mathbb{H})$, there exist $f_1, f_2 \in L^2(\mathbb{R},\mathbb{C}_I)$ such that

$$f(t) = f_1(t) + f_2(t)I,\ \forall t \in \mathbb{R}$$

and for which the classical Lieb's inequality [110] holds , i.e:

$$\int_{\mathbb{R}^2} |V_\varphi f_l(x,\omega)|^p dx d\omega \leq \frac{2}{p} ||f_l||_{L^2(\mathbb{R},\mathbb{C}_I)}^p;\ l=1,2.$$

In particular, by definition of QSTFT we have

$$\mathcal{V}_\varphi f(x,\omega) = V_\varphi f_1(x,\omega) + V_\varphi f_2(x,\omega) I,\ \forall (x,\omega) \in \mathbb{R}^2.$$

Thus,

$$\begin{aligned} |\mathcal{V}_\varphi f(x,\omega)|^p &\leq (|V_\varphi f_1(x,\omega)| + |V_\varphi f_2(x,\omega)|)^p \\ &\leq 2^{p-1} \left(|V_\varphi f_1(x,\omega)|^p + |V_\varphi f_2(x,\omega)|^p \right). \end{aligned}$$

We use the classical Lieb's inequality on each component combined with the fact that $||f_l||_p \leq ||f||_p$ for $l = 1, 2$ and get

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^p dx d\omega &\leq \frac{2^p}{p}\left(||f_1||^p_{L^2(\mathbb{R})} + ||f_2||^p_{L^2(\mathbb{R})}\right)\\ &\leq \frac{2^{p+1}}{p}||f||^p_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

This ends the proof. □

The next result improves the weak uncertainty principle in the sense that it gives a best sharper estimate for $|U|$.

Theorem 4.5.15. *Let $f \in L^2(\mathbb{R}, \mathbb{H})$ be a unit vector, U an open set of $\mathbb{R}^2$ and $\varepsilon \geq 0$ such that*

$$\int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

Then, we have

$$|U| \geq c_p(1-\varepsilon)^{\frac{p}{p-2}},$$

where $|U|$ denotes the Lebesgue measure of U and $c_p = \left(\frac{2^{p+1}}{p}\right)^{-\frac{2}{p-2}}$.

Proof. Let $f \in L^2(\mathbb{R}, \mathbb{H})$ be such that $||f||_{L^2(\mathbb{R},\mathbb{H})} = 1$. We first apply Holder inequality with exponents $q = \dfrac{p}{2}$ and $q' = \dfrac{p}{p-2}$. Then, using Lieb's inequality for QSTFT we get

$$\begin{aligned}\int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega &= \int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^2 \chi_U(x,\omega) dx d\omega\\ &\leq \left(\int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^p dx d\omega\right)^{\frac{2}{p}} |U|^{\frac{p-2}{p}}\\ &\leq \left(\frac{2^{p+1}}{p}\right)^{\frac{2}{p}} |U|^{\frac{p-2}{p}}.\end{aligned}$$

Hence, by hypothesis we obtain

$$|U| \geq c_p(1-\varepsilon)^{\frac{p}{p-2}}$$

where $c_p = \left(\frac{2^{p+1}}{p}\right)^{-\frac{2}{p-2}}$. □

CHAPTER 5

On the polyanalytic short-time Fourier transform in the quaternionic setting

5.1 Motivation

In this chapter we build a short-time Fourier transform for more generic window functions: the weighted Hermite functions, which are defined as follows

$$\psi_n^\nu(x) := \frac{(-1)^n e^{\frac{\nu}{2}x^2} \frac{d^n}{dx^n}\left(e^{-\nu x^2}\right)}{2^{n/2}\nu^{n/2}(n!)^{1/2}\pi^{1/4}\nu^{-1/4}}. \tag{5.1}$$

In our case we will consider the parameter $\nu = 2\pi$. We note that for $n = 0$ we have $\psi_0(t) = 2^{1/4}e^{-\pi t^2}$, which is exactly the window function that we have took into accaount in the previous chapther. Therefore, this chapther can be considered a generalization of the previous one.
The study of the QSTFT with respect to the weighted Hermite functions as windows is related to the theory of slice polyanalytic functions of a quaternionic variable. Recently, this topic has been intensively investigated, see [6, 7, 16–18]. We will recall some basic notions in the next section. The

main idea of the results of this chapter is to fix the following property

$$V_{\psi_n}\varphi(x,\omega) = e^{-\pi i x\omega}\mathcal{G}^{n+1}\varphi(\bar{z})e^{-\frac{\pi|z|^2}{2}}, \tag{5.2}$$

where V_{ψ_n} is the complex short-time Fourier transform with respect to the weighted Hermite functions ψ_n (see [2, Prop.1]) and $G^{n+1}\varphi$ denotes the complex true polyanalytic Segal-Bargmann transform. We extend (5.2) in the quaternionic setting. To reach this aim we need the slice version of the quaternions.
It is possible to introduce a short-time Fourier transform of a vector-valued function $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$

$$\mathbf{V}_{\vec{\psi}}\vec{\varphi}(x,\omega) = e^{-\pi i x\omega}\mathbf{G}\vec{\varphi}(\bar{z})e^{-\frac{\pi|z|^2}{2}},$$

where $\mathbf{V}_{\vec{\psi}}\vec{\varphi}$ denotes the complex short-time Fourier transform with respect to the vector-valued window $\vec{\psi} = (\psi_0, ..., \psi_n)$ (see [4, Formula 20]), and $\mathbf{G}\vec{\varphi}$ is the complex polyanalytic Bargmann transform (full-poly Bargmann). Also in this case we extend the formula to the quaternions.
Based on the properties of the true quaternionic polyanalytic Bargmann transform and the full-poly one (see also [24]) we prove the main results of the QSTFT.

5.2 Preliminaries on slice polyanalytic functions

In this section we recall briefly the main notations and concepts of the theory of slice polyanalytic functions. This extends to higher order the theory of slice hyperholomoprhic functions summarized in chapter 2.

Example The function $F(q) = 1 - \bar{q}qe_2$, for $q = x + Jy \in \mathbb{H}$ is not slice hyperholomorphic, indeed

$$\left(\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}\right)F(q) = -(x+Jy)e_2, \qquad \forall J \in \mathbb{S}$$

However,

$$\left(\frac{\partial}{\partial x} + J\frac{\partial}{\partial y}\right)^2 F(q) = 0, \qquad \forall J \in \mathbb{S}.$$

Then we say that the function F is slice polyanalytic of order 2 on $\mathbb{H}$.

Definition 5.2.1 (Slice polyanalytic functions)**.** Let $n \in \mathbb{N}$ and denote by $\mathcal{C}^n(U)$ the set of continuously differentiable functions with all their derivatives up to order n on an axially symmetric open set $U \subseteq \mathbb{H}$. We set

$\mathcal{U} := \{(x,y) \in \mathbb{R}^2 \,:\, x + Jy \subset U\}$. A function $f : U \to \mathbb{H}$ is called left slice function, if it is of the form

$$f(q) = \alpha(x,y) + J\beta(x,y) \quad \text{for} \quad q = x + Jy \in U$$

with the two functions α, $\beta : \mathcal{U} \to \mathbb{H}$ that satisfy the compatibility conditions $\alpha(x,-y) = \alpha(x,y)$, $\beta(x,-y) = -\beta(x,y)$. If in addition α and β are in $\mathcal{C}^n(U)$ and satisfy the poly Cauchy-Riemann equations of order $n \in \mathbb{N}$

$$\frac{1}{2^n}(\partial_x + J\partial_y)^n(\alpha(x,y) + J\beta(x,y)) = 0, \quad \text{for all } \ J \in \mathbb{S}, \tag{5.3}$$

Then the function f is called left slice polyanalytic function of order $M \in \mathbb{N}$. The set of such kind of functions will be denoted by $\mathcal{SP}_n(\mathbb{H})$.

The definition is easily adapted in the case of right slice polyanalytic functions. Moreover, if we consider $n = 1$ in the previous definition we get the set of slice hyperholomorphic functions.

Now, we list a series of basic results about slice polyanalytic functions, see [16, 17], that will be useful in the sequel.

Theorem 5.2.2 (Representation formula)**.** *Let f be a slice polyanalytic function of order $n{+}1$ defined on an axially symmetric slice domain $\Omega \subset \mathbb{H}$. Let $I \in \mathbb{S}$, then for any $q = x + Jy \in \Omega$ the following equality holds*

$$f(q) = \frac{1 - JI}{2} f(x + Iy) + \frac{1 + JI}{2} f(x - Iy).$$

Lemma 5.2.3 (Splitting Lemma)**.** *Let f be a slice polyanalytic function of order n on a domain $\Omega \subseteq \mathbb{H}$. Then, for any imaginary units I and J with $I \perp J$ there exist $F, G : \Omega_J \longrightarrow \mathbb{C}_J$ polyanalytic functions of order n such that for all $z = x + Jy \in \Omega_J$, we have*

$$f_J(z) = F(z) + G(z)I.$$

Remark 5.2.4. For $n = 1$ in Lemma 5.2.3 we obtain the classic Splitting Lemma (see [87, Lemma 1.3]).

Proposition 5.2.5. *(Poly-decomposition) A function $f : \Omega \to \mathbb{H}$ defined on an axially symmetric slice domain is slice polyanalytic of order n if and only if there exist unique slice regular functions $f_0, ..., f_{n-1}$ on Ω such that we have the following decomposition*

$$f(q) := \sum_{k=0}^{n-1} \bar{q}^k f_k(q); \quad \forall q \in \Omega.$$

Proposition 5.2.6. *(Identity Principle) Let f and g be two slice polyanalytic functions of order n on a slice domain $\Omega \subset \mathbb{H}$. If, for some $J \in \mathbb{S}$, f and g coincide on U a subdomain of Ω_J, then $f \equiv g$ everywhere in Ω.*

Remark 5.2.7. For $n = 1$ we obtain the classical Identity Principle.

The Leibniz rule will be useful for our calculations in the next section. The proof is based on direct computations using the definition of slice derivative, see (3.1.10).

Proposition 5.2.8. *Let $f : \mathbb{H} \to \mathbb{H}$ be an intrinsic function and $g : \mathbb{H} \to \mathbb{H}$ be a slice regular function. Then, we have*

$$\partial_s(fg) = f(\partial_s g) + (\partial_s f)g. \tag{5.4}$$

It is possible to generalize the previous result by applying more time the slice derivatives.

Proposition 5.2.9. *Let $f : \mathbb{H} \to \mathbb{H}$ be an intrinsic function and $g : \mathbb{H} \to \mathbb{H}$ be a slice regular function. Then, for any $k \in \mathbb{N}$, we have*

$$\partial_s^k(fg) = \sum_{m=0}^{k} \binom{k}{m} (\partial_s^m f)(\partial_s^{k-m} g). \tag{5.5}$$

In [17] the authors introduced the quaternionic polyanalytic Fock space defined for a given $J \in \mathbb{S}$ and $n \geq 1$ to be

$$\widetilde{\mathcal{F}}_J^{n+1}(\mathbb{H}) := \{f \in \mathcal{SP}_{n+1}(\mathbb{H}) : \int_{\mathbb{C}_J} |f_J(q)|^2 e^{-2\pi|q|^2}\, d\lambda_J(q) < \infty\}, \tag{5.6}$$

where $d\lambda_J(q)$ is the Lebesgue measure on the slice $\mathbb{C}_J$. It is given by $d\lambda_J(q) = dxdy$ for $q = x + Jy$. Moreover, the space is endowed with the following inner product

$$\langle f, g\rangle_{\widetilde{\mathcal{F}}_J^{n+1}(\mathbb{H})} = \int_{\mathbb{C}_J} \overline{g_J(q)} f_J(q) e^{-2\pi|q|^2}\, d\lambda_J(q).$$

In [17, Prop. 4.1] and [17, Prop. 4.2] it is showed that the polyanalytic Fock space is a quaternionic reproducing kernel Hilbert space which does not depend on the choice of $J \in \mathbb{S}$. Thus, from now we will denote the quaternionic polyanalytic Fock space by $\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H})$.
Now, we give the definition of the quaternionic true polyanalytic Fock space.

Definition 5.2.10. Let $I \in \mathbb{S}$. A slice function $f : \mathbb{H} \to \mathbb{H}$ belongs to the quaternionic true polyanalytic Fock space $\mathcal{F}_T^n(\mathbb{H})$, if and only if

i) $\displaystyle\int_{\mathbb{C}_J} |f_J(q)|^2 e^{-2\pi|q|^2}\, d\lambda_J(q) < \infty.$

ii) There exists a slice regular function H such that

$$f(q) = (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} e^{2\pi|q|^2} \partial_s^n (e^{-2\pi|q|^2} H(q)).$$

Remark 5.2.11. By using similar arguments used in [17, Prop. 4.2] and [15, 17, 76] one can show that the Definition 5.2.10 do not depend on the choice of $J \in \mathbb{S}$.

We observe that for $n = 0$ the two different Fock spaces are the same.

5.3 Polyanalytic Bargmann transform

In this section we show the relation between the quaternionic Fock space and the quaternionic true polyanalytic Fock space for very $n \geq 0$. In order to prove this we show two preliminary results.

Lemma 5.3.1. *Let $k \geq 1$. Then, for all $q \in \mathbb{H}$ we have*

$$\partial_s^k e^{-2\pi|q|^2} = (-2\pi)^k \bar{q}^k e^{-2\pi|q|^2}. \tag{5.7}$$

Proof. We prove the formula by induction. Let us start with $k = 1$, we observe that

$$e^{-2\pi|q|^2} = e^{-2\pi q\bar{q}} = \sum_{n=0}^{\infty} \frac{(-2\pi)^n}{n!} q^n \bar{q}^n.$$

Now, we evaluate the slice derivative and get

$$\begin{aligned}
\partial_s e^{-2\pi|q|^2} &= \sum_{n=1}^{\infty} \frac{(-2\pi)^n}{n!} n q^{n-1} \bar{q}^n \\
&= \sum_{h=0}^{\infty} \frac{(-2\pi)^{h+1}}{(h+1)!} (h+1) q^h \bar{q}^{h+1} \\
&= -2\pi \left(\sum_{h=0}^{\infty} \frac{(-2\pi)^h}{h!} q^h \bar{q}^h \right) \bar{q} \\
&= -2\pi e^{-2\pi|q|^2} \bar{q}.
\end{aligned}$$

Let us assume that the formula holds for k. We have to prove that it holds for $k+1$

$$\begin{aligned}\partial_s^{k+1}e^{-2\pi|q|^2} &= \partial_s(\partial_s^k e^{-2\pi|q|^2})\\ &= (-2\pi)^k\bar{q}^k\partial_s e^{-2\pi|q|^2}\\ &= (-2\pi)^{k+1}\bar{q}^{k+1}e^{-2\pi|q|^2}.\end{aligned}$$

□

Remark 5.3.2. We use similar arguments to justify that for any $k\geq 1$, we have

$$\overline{\partial_J}^k e^{-2\pi|q|^2} = (-2\pi)^k q^k e^{-2\pi|q|^2}. \tag{5.8}$$

Proposition 5.3.3. *Let g be a slice regular function on $\mathbb{H}$. We consider the following function*

$$u(q)=\sum_{k=0}^{n}(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|q|^2}\partial_s^k(e^{-2\pi|q|^2}g(q)).$$

Then u is a slice polyanalytic function of order $n+1$ on $\mathbb{H}$.

Proof. By the generalized Leibniz formula (5.5) we have

$$\begin{aligned}u(q) &= \sum_{k=0}^{n}(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|q|^2}\partial_s^k(e^{-2\pi|q|^2}g(q))\\ &= \sum_{k=0}^{n}(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|q|^2}\sum_{m=0}^{k}\binom{k}{m}\partial_s^m e^{-2\pi|q|^2}\partial_s^{k-m}g(q)\\ &:= \sum_{k=0}^{n}c_k\underline{g}_k(q),\end{aligned} \tag{5.9}$$

where $c_k:=(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}$ and $\underline{g}_k(q):=e^{2\pi|q|^2}\sum_{m=0}^{k}\binom{k}{m}\partial_s^m e^{-2\pi|q|^2}\partial_s^{k-m}g(q)$. By Lemma 5.3.1 we get

$$\begin{aligned}\underline{g}_k(q) &= e^{2\pi|q|^2}\sum_{m=0}^{k}\binom{k}{m}(-2\pi)^m\bar{q}^m e^{-2\pi|q|^2}\partial_s^{k-m}g(q)\\ &= \sum_{m=0}^{k}\binom{k}{m}(-2\pi)^m\bar{q}^m\partial_s^{k-m}g(q)\\ &:= \sum_{m=0}^{k}\bar{q}^m\beta_m(q),\end{aligned}$$

where $\beta_m(q) = \binom{k}{m}(-2\pi)^m \partial_s^{k-m} g(q)$.
Since g is a slice regular function and the iteration of slice derivatives is slice regular too, we get that β_m is slice regular. This implies by Proposition 5.2.5 that $\underline{g}_k$ is a slice polyanalytic function of order $k+1$. Finally by (5.9) we get that u is a slice polyanalytic function of order $n+1$. $\square$

For our future purpose we recall the so-called quaternionic Hermite polynomials (for more details see [80, 132]).

$$H^{2\pi}_{m,p}(q,\bar{q}) := (-1)^{m+p} e^{2\pi|q|^2} \partial_s^p \overline{\partial_J}^m e^{-2\pi|q|^2}, \qquad m,p \in \mathbb{N}. \tag{5.10}$$

Remark 5.3.4. Using Remark 5.3.2 it is possible to write the quaternionic Hermite polynomials in another way:

$$\begin{aligned} H^{2\pi}_{m,p}(q,\bar{q}) &= (-1)^{m+p} e^{2\pi|q|^2} \partial_s^p [(-2\pi)^m q^m e^{-2\pi|q|^2}] \\ &= (-1)^{m+p} e^{2\pi|q|^2} (-2\pi)^m \partial_s^p (q^m e^{-2\pi|q|^2}) \\ &= (2\pi)^m (-1)^p e^{2\pi|q|^2} \partial_s^p (q^m e^{-2\pi|q|^2}). \end{aligned}$$

Therefore

$$H^{2\pi}_{m,p}(q,\bar{q}) = (2\pi)^m (-1)^p e^{2\pi|q|^2} \partial_s^p (q^m e^{-2\pi|q|^2}). \tag{5.11}$$

The following orthogonality relation holds for the quaternionic Hermite polynomials (for the proof see the Appendix A (Thm. 14.0.2))

$$\int_{\mathbb{C}_J} \overline{H^{2\pi}_{m,p}(q,\bar{q})} H^{2\pi}_{m',p'}(q,\bar{q}) e^{-2\pi|q|^2} \, d\lambda_J(q) = \frac{m!p!(2\pi)^{p+m}}{2} \delta_{m,m'} \delta_{p,p'}. \tag{5.12}$$

In the next result we will show that two different quaternionic true polyanalytic Fock spaces are orthogonal to each other.

Lemma 5.3.5. *Let $f \in \mathcal{F}_T^j(\mathbb{H})$ and $g \in \mathcal{F}_T^m(\mathbb{H})$ with $j \neq m$. Then we have*

$$\langle f, g \rangle_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})} = 0.$$

Proof. By Definition 5.2.10 there exist two slice regular functions H and L such that

$$f(q) = (-1)^j \sqrt{\frac{1}{(2\pi)^j j!}} e^{2\pi|q|^2} \partial_s^j (e^{-2\pi|q|^2} H(q))$$

and

$$g(q) = (-1)^m \sqrt{\frac{1}{(2\pi)^m m!}} e^{2\pi|q|^2} \partial_s^m (e^{-2\pi|q|^2} L(q)).$$

Thus, we can use the series expansion theorem for slice regular functions to write

$$H(q)=\sum_{k=0}^{\infty}q^k\alpha_k \quad \text{and} \quad L(q)=\sum_{p=0}^{\infty}q^p\beta_p, \quad \{\alpha_k\}_{k\geq 0},\{\beta_p\}_{p\geq 0}\subset \mathbb{H}.$$

Therefore, using the quaternionic Hermite polynomials and developing a bit the calculations we easily get that

$$f(q)=\sqrt{\frac{1}{(2\pi)^j j!}}\sum_{k=0}^{\infty}\frac{H_{k,j}^{2\pi}(q,\bar{q})}{(2\pi)^k}\alpha_k$$

and

$$g(q)=\sqrt{\frac{1}{(2\pi)^m m!}}\sum_{p=0}^{\infty}\frac{H_{p,m}^{2\pi}(q,\bar{q})}{(2\pi)^p}\beta_p.$$

Hence, using the orthogonality of the quaternionic Hermite polynomials (5.12) combined with the condition $j\neq m$ we obtain

$$\begin{aligned}\langle f,g\rangle_{\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H})} &=\int_{\mathbb{C}_J}\overline{g(q)}f(q)e^{-2\pi|q|^2}d\lambda_J(q)\\ &=\sqrt{\frac{1}{(2\pi)^j j!}}\sqrt{\frac{1}{(2\pi)^m m!}}\sum_{k,p=0}^{\infty}\frac{\overline{\beta_k}}{(2\pi)^{k+p}}\cdot\\ &\quad\cdot\int_{\mathbb{C}_J}\left(\overline{H_{p,m}^{2\pi}(q,\bar{q})}H_{k,j}^{2\pi}(q,\bar{q})e^{-2\pi|q|^2}d\lambda_J(q)\right)\alpha_p\\ &=0.\end{aligned}$$

□

Now, we are ready to prove the relation between the quaternionic polyanalytic Fock space (see (5.6)) and the quaternionic true polyanalytic Fock space (see Definition 5.2.10).

Theorem 5.3.6. *The quaternionic polyanalytic Fock space $\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H})$ is the direct sum of true polyanalytic Fock spaces $\mathcal{F}_T^j(\mathbb{H})$, $j=0,...,n$, i.e.*

$$\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H})=\bigoplus_{j=0}^{n}\mathcal{F}_T^j(\mathbb{H}). \tag{5.13}$$

Proof. We prove the equality by double inclusion. Let $f \in \widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})$. Let $J \in \mathbb{S}$. We choose $I \in \mathbb{S}$ be such that $I \perp J$. Since f, in particular, is slice polyanalytic of order $n+1$ by the Splitting Lemma (see Lemma 5.2.3) there exist $F, G : \mathbb{C}_J \to \mathbb{C}_J$ polyanalytic functions of order $n+1$ in the complex polyanalytic Fock space $\widetilde{\mathcal{F}}^{n+1}(\mathbb{C}_J)$ (see [4]), such that

$$f_J(z) = F(z) + G(z)I.$$

By [21, 135] we know that

$$\widetilde{\mathcal{F}}^{n+1}(\mathbb{C}_J) = \bigoplus_{j=0}^{n} \mathcal{F}^j_T(\mathbb{C}_J).$$

Therefore there exist unique $f_k, p_k \in \mathcal{F}^j_T(\mathbb{C}_J)$ such that

$$F(z) = \sum_{k=0}^{n} f_k(z),$$

$$G(z) = \sum_{k=0}^{n} p_k(z).$$

By definition of the complex true polyanalytic Fock space we have that both f_k and p_k satisfy the following integrability conditions

$$\int_{\mathbb{C}_J} |f_k(z)|^2 e^{-2\pi|z|^2} d\lambda_J(z) < \infty, \tag{5.14}$$

$$\int_{\mathbb{C}_J} |p_k(z)|^2 e^{-2\pi|z|^2} d\lambda_J(z) < \infty. \tag{5.15}$$

Moreover, they can be written as

$$f_k(z) = (-1)^k \sqrt{\frac{1}{(2\pi)^k k!}} e^{2\pi|z|^2} \partial_z^k (e^{-2\pi|z|^2} s_k(z)), \tag{5.16}$$

$$p_k(z) = (-1)^k \sqrt{\frac{1}{(2\pi)^k k!}} e^{2\pi|z|^2} \partial_z^k (e^{-2\pi|z|^2} h_k(z)), \tag{5.17}$$

where s_k, h_k are entire functions. Thus, we have

$$\begin{aligned} f_J(z) &= \sum_{k=0}^{n}(f_k(z)+p_k(z)I) \\ &= \sum_{k=0}^{n}(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|z|^2}\partial_z^k\left(e^{-2\pi|z|^2}(s_k(z)+h_k(z)I)\right) \\ &:= \sum_{k=0}^{n}(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|z|^2}\partial_z^k(e^{-2\pi|z|^2}g(z)). \end{aligned} \tag{5.18}$$

By hypothesis we know that f is a slice polyanalytic function of order $n+1$. Moreover, by Proposition 5.3.3 we know that the following function

$$u(q) = \sum_{k=0}^{n}(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|q|^2}\partial_s^k(e^{-2\pi|q|^2}g(q))$$

is slice polyanalytic of order $n+1$. Since the functions f and u coincide on the slice $\mathbb{C}_J$, by the Identity Principle (see Proposition 5.2.6) we have that $f(q)=u(q)$. Now, we call

$$u_k(q) := (-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|q|^2}\partial_s^k(e^{-2\pi|q|^2}g(q)), \qquad 0\leq k\leq n.$$

In order to finish this first part we have to prove

$$\int_{\mathbb{C}_J}|u_{k,I}(q)|^2e^{-2\pi|q|^2}d\lambda_J(q)<\infty.$$

Since $g_J(z)=s_k(z)+h_k(z)I$, by (5.16) and (5.17) we have

$$\begin{aligned} u_{k,I}(z) &= (-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|z|^2}\partial_z^k(e^{-2\pi|z|^2}s_k(z)) \\ &\quad +(-1)^k\sqrt{\frac{1}{(2\pi)^k k!}}e^{2\pi|z|^2}\partial_z^k(e^{-2\pi|z|^2}h_k(z))I \\ &= f_k(z)+p_k(z)I. \end{aligned}$$

Thus by (5.14) and (5.15) we get

$$\begin{aligned} \int_{\mathbb{C}_J}|u_{k,J}(q)|^2e^{-2\pi|q|^2}d\lambda_J(q) &= \int_{\mathbb{C}_J}|f_{k,J}(z)|^2e^{-2\pi|z|^2}d\lambda_J(z) \\ &\quad +\int_{\mathbb{C}_J}|p_{k,J}(z)|^2e^{-2\pi|z|^2}d\lambda_J(z)<\infty. \end{aligned}$$

Now, we move on the other inclusion. Let $f \in \bigoplus_{j=0}^{n} \mathcal{F}_T^j(\mathbb{H})$. This means that there exist unique functions $f_k \in \mathcal{F}_T^j(\mathbb{H})$, $k = 0, ..., n$, such that

$$f(q) = \sum_{k=0}^{n} f_k(q). \tag{5.19}$$

By definition of quaternionic true polyanalytic Fock space we have that

$$f_k(q) = (-1)^k \sqrt{\frac{1}{(2\pi)^k k!}} e^{2\pi|q|^2} \partial_s^k \left(e^{-2\pi|q|^2} H(q) \right),$$

where H is a slice regular function. Thus, by Proposition 5.3.3 f is a slice polyanalytic function of order $n+1$. Finally, we have to prove that

$$\int_{\mathbb{C}_J} |f_J(q)|^2 e^{-2\pi|q|^2} \, d\lambda_J(q) < \infty.$$

By equality (5.19) and the triangle inequality we have that

$$|f_J(q)| \leq \sum_{k=0}^{n} |f_{k,J}(q)|.$$

Therefore,

$$|f_J(q)|^2 \leq \left(\sum_{k=0}^{n} |f_{k,J}(q)| \right)^2 \leq (n+1) \sum_{k=0}^{n} |f_{k,J}(q)|^2.$$

Now we multiply by $e^{-2\pi|q|^2}$ and integrate.

$$\begin{aligned}
&\int_{\mathbb{C}_J} |f_J(q)|^2 e^{-2\pi|q|^2} \, d\lambda_J(q) \leq (n+1) \int_{\mathbb{C}_J} \sum_{k=0}^{n} |f_{k,J}(q)|^2 e^{-2\pi|q|^2} \, d\lambda_J(q) \\
&\leq (n+1) \left[\int_{\mathbb{C}_J} |f_{1,I}(q)|^2 e^{-2\pi|q|^2} \, d\lambda_J(q) + ... + |f_{n,I}(q)|^2 e^{-2\pi|q|^2} \, d\lambda_J(q) \right] \\
&< \infty.
\end{aligned}$$

The previous conclusion holds because $f_k \in \mathcal{F}_T^j(\mathbb{H})$, for $k = 0, ..., n$, by hypothesis.

The sum in formula 5.13 is a direct sum. Indeed, if we consider $g \in \mathcal{F}_T^i(\mathbb{H}) \cap \mathcal{F}_T^j(\mathbb{H})$, with $i \neq j$, by Lemma 5.3.5 we get that

$$\langle g, g \rangle_{\widetilde{\mathcal{F}}_{Slice}^{n+1}} = 0.$$

Since $\widetilde{\mathcal{F}}^{n+1}_{Slice}$ is a quaternionic Hilbert space we get that $g = 0$. We have proved that $\mathcal{F}^i_T(\mathbb{H}) \cap \mathcal{F}^j_T(\mathbb{H}) = \{0\}$. Finally, by Lemma 5.3.5 we get that the direct sum in formula (5.13) is orthogonal. □

Remark 5.3.7. A similar result was proved in [24, Thm. 3.4] following a different method.

Now, we give the definition of the true quaternionic polyanalytic Bargmann transform for $\varphi \in L^2(\mathbb{R}, \mathbb{H})$ (inspired from [84])

$$B^{n+1}\varphi(q) := (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} \sum_{j=0}^{n} \binom{n}{j} (-2\pi\bar{q})^j \partial_s^{n-j} \mathcal{B}\varphi(q), \quad (5.20)$$

where $\mathcal{B}\varphi(q)$ is the quaternionic analogue of the Segal-Bargmann transform (see (4.5) with $\nu = 2\pi$). Using the Leibniz rule and Lemma 5.3.1 we get the following definition.

Definition 5.3.8. The true quaternionic polyanalytic Bargmann transform of order $n+1$ of a function $\varphi \in L^2(\mathbb{R}, \mathbb{H})$ is defined by the formula

$$B^{n+1}\varphi(q) = (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} e^{2\pi|q|^2} \partial_s^n [e^{-2\pi|q|^2} \mathcal{B}\varphi(q)]. \quad (5.21)$$

Remark 5.3.9. For $n = 0$ we obtain the quaternionic Segal-Bargmann transform $B^1\varphi(q) = \mathcal{B}\varphi(q)$.

Theorem 5.3.10. *The true quaternionic polyanalytic Bargmann transform* $B^{n+1} : L^2(\mathbb{R}, \mathbb{H}) \to \mathcal{F}^n_T(\mathbb{H})$ *is an isometric isomorphism.*

Proof. Firstly, we remark that by Theorem 5.13 the norm of the true quaternionic polyanalytic Fock space $\mathcal{F}^n_T(\mathbb{H})$ is induced by the norm of the space $\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})$. Thus we get

$$\|B^{n+1}(\varphi)\|_{\mathcal{F}^n_T(\mathbb{H})} = \|B^{n+1}(\varphi)\|_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})}.$$

Therefore, we have to prove that

$$\|B^{n+1}(\varphi)\|_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})} = \|\varphi\|_{L^2(\mathbb{R},\mathbb{H})}. \quad (5.22)$$

Let $\varphi \in L^2(\mathbb{R}, \mathbb{H})$. We expand it in the following way

$$\varphi(x) = \sum_{k=0}^{\infty} \psi_k^{2\pi}(x) \alpha_k,$$

where $\psi_k^{2\pi}(x)$ are the normalized weighted Hermite functions (see (5.1)) and $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathbb{H}$. By Lemma 4.2.4 we have

$$\begin{aligned}
\mathcal{B}\varphi(q) &= \sum_{k=0}^{\infty} \mathcal{B}\left(\psi_k^{2\pi}(x)\right)\alpha_k \\
&= \sum_{k=0}^{\infty} \frac{2^{1/4}2^{k/2}(2\pi)^k q^k}{2^{k/2}(2\pi)^{k/2}\sqrt{k!}2^{-1/4}}\alpha_k \\
&= \sqrt{2}\sum_{k=0}^{\infty} \frac{(2\pi)^{k/2}}{\sqrt{k!}} q^k \alpha_k. \qquad (5.23)
\end{aligned}$$

Now, we insert this in (5.21) and using (5.11) we obtain

$$\begin{aligned}
B^{n+1}\varphi(q) &= \sqrt{2}\sum_{k=0}^{\infty}(-1)^n\sqrt{\frac{1}{(2\pi)^n n!}}e^{2\pi|q|^2}\partial_s^n\left[e^{-2\pi|q|^2}\frac{(2\pi)^{k/2}}{\sqrt{k!}}q^k\right]\alpha_k \\
&= \sqrt{2}\sqrt{\frac{1}{(2\pi)^n n!}}\sum_{k=0}^{\infty}\frac{(2\pi)^{k/2}}{\sqrt{k!}}(-1)^n e^{2\pi|q|^2}\partial_s^n\left[e^{-2\pi|q|^2}q^k\right]\alpha_k \\
&= \sqrt{2}\sqrt{\frac{1}{(2\pi)^n n!}}\sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}(2\pi)^{k/2}}(-1)^n(2\pi)^k e^{2\pi|q|^2}\partial_s^n\left[e^{-2\pi|q|^2}q^k\right]\alpha_k \\
&= \sqrt{2}\sqrt{\frac{1}{(2\pi)^n n!}}\sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}(2\pi)^{k/2}}H_{k,n}^{2\pi}(q,\bar{q})\alpha_k.
\end{aligned}$$

Therefore we get

$$B^{n+1}\varphi(q) = \sqrt{2}\sqrt{\frac{1}{(2\pi)^n n!}}\sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}(2\pi)^{k/2}}H_{k,n}^{2\pi}(q,\bar{q})\alpha_k. \qquad (5.24)$$

Now, we evaluate the $\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})$ norm of $B^{n+1}\varphi$.

$$\begin{aligned}
\|B^{n+1}\varphi(q)\|^2_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})} &= \frac{2}{n!(2\pi)^n}\int_{\mathbb{C}_J}\left(\sum_{k=0}^{\infty}\frac{1}{\sqrt{k!}(2\pi)^{k/2}}\overline{\alpha_k}\overline{H^{2\pi}_{k,n}(q,\bar{q})}\right)\cdot\\
&\quad\cdot\left(\sum_{\ell=0}^{\infty}\frac{1}{\sqrt{\ell!}(2\pi)^{\ell/2}}H^{2\pi}_{\ell,n}(q,\bar{q})\alpha_k\right)e^{-2\pi|q|^2}\,d\lambda_J(q)\\
&= \frac{2}{n!(2\pi)^n}\sum_{k,\ell=0}^{\infty}\frac{1}{\sqrt{k!}(2\pi)^{k/2}}\frac{1}{\sqrt{\ell!}(2\pi)^{\ell/2}}\cdot\\
&\quad\cdot\overline{\alpha_k}\left(\int_{\mathbb{C}_J}\overline{H^{2\pi}_{k,n}(q,\bar{q})}H^{2\pi}_{\ell,n}(q,\bar{q})e^{-2\pi|q|^2}\,d\lambda_J(q)\right)\alpha_k.
\end{aligned}$$

Due to the orthogonality relation of the quaternionic Hermite polynomials (5.12) we obtain

$$\begin{aligned}
\|B^{n+1}\varphi(q)\|^2_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})} &= \frac{2}{n!(2\pi)^n}\sum_{k=0}^{\infty}\frac{1}{k!(2\pi)^k}\overline{\alpha_k}\left(\int_{\mathbb{C}_J}\overline{H^{2\pi}_{k,n}(q,\bar{q})}H^{2\pi}_{k,n}(q,\bar{q})\right.\\
&\quad\left.\cdot e^{-2\pi|q|^2}\,d\lambda_J(q)\alpha_k\right)\\
&= \frac{2}{n!(2\pi)^n}\sum_{k=0}^{\infty}\frac{1}{k!(2\pi)^k}k!n!(2\pi)^{k+n}\frac{1}{2}|\alpha_k|^2=\sum_{k=0}^{\infty}|\alpha_k|^2.
\end{aligned}$$

Thus we have

$$\|B^{n+1}\varphi(q)\|^2_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})}=\sum_{k=0}^{\infty}|\alpha_k|^2. \tag{5.25}$$

On the other hand

$$\begin{aligned}
\|\varphi\|^2_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}}\left(\sum_{k=0}^{\infty}\overline{\alpha_k}\overline{\psi^{2\pi}_k(x)}\right)\left(\sum_{k=0}^{\infty}\psi^{2\pi}_k(x)\alpha_k\right)\,dx\\
&= \sum_{k=0}^{\infty}\overline{\alpha_k}\left(\int_{\mathbb{R}}\overline{\psi^{2\pi}_k(x)}\psi^{2\pi}_k(x)\,dx\right)\alpha_k\\
&= \sum_{k=0}^{\infty}|\alpha_k|^2.
\end{aligned}$$

Therefore we get

$$\|\varphi\|^2_{L^2(\mathbb{R},\mathbb{H})}=\sum_{k=0}^{\infty}|\alpha_k|^2. \tag{5.26}$$

Since (5.25) and (5.26) are equal we obtain (5.22). Finally, we have to prove that B^{n+1} is surjective. This means that for a function $h \in \mathcal{F}_T^n(\mathbb{H})$ we have to find a function $\psi \in L^2(\mathbb{R}, \mathbb{H})$ such that

$$B^{n+1}\psi(q) = h(q).$$

By the definition of the quaternionic true polyanalytic Fock space $\mathcal{F}_T^n(\mathbb{H})$ (see Definition 5.2.10) we know that there exists a slice regular function H such that

$$h(q) = (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} e^{2\pi|q|^2} \partial_s^n (e^{-2\pi|q|^2} H(q)).$$

From the Splitting Lemma (see Remark 5.2.4) for slice regular functions we can write H on the slice $\mathbb{C}_J$ as

$$H_I(z) = F(z) + G(z)i, \quad z = x + jy \in \mathbb{C}_J,$$

where $F(z)$ and $G(z)$ are holomorphic functions. Thus

$$\begin{aligned} h_I(z) &= (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} e^{2\pi|z|^2} \partial_z^n \left(e^{-2\pi|z|^2} (F(z) + G(z)I) \right) \\ &= (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} e^{2\pi|z|^2} \partial_z^n \left(e^{-2\pi|z|^2} F(z) \right) \\ &\quad + (-1)^n \sqrt{\frac{1}{(2\pi)^n n!}} e^{2\pi|z|^2} \partial_z^n \left(e^{-2\pi|z|^2} G(z) \right) I \\ &:= P(z) + Q(z)I. \end{aligned}$$

By hypothesis $h \in \mathcal{F}_T^n(\mathbb{H})$, this implies that

$$\int_{\mathbb{C}_J} |P(z)|^2 e^{-2\pi|z|^2} d\lambda_J(z) < \infty,$$

and

$$\int_{\mathbb{C}_J} |Q(z)|^2 e^{-2\pi|z|^2} d\lambda_J(z) < \infty.$$

Moreover, since the functions F and G are holomorphic we obtain that $P(z)$ and $Q(z)$ belong to the space $\mathcal{F}_T^n(\mathbb{C}_J)$. Now, since the complex polyanalytic Bargmann $B_{\mathbb{C}}^{n+1}$ is an isometric isomorphism from $L^2(\mathbb{R}, \mathbb{C}_J) \to \mathcal{F}_T^n(\mathbb{C}_J)$ (see [2, Thm. 1]), and in particular is surjective, we can find two functions $\psi_1(s)$ and $\psi_2(s)$, with $s \in \mathbb{R}$, which belong to the space $L^2(\mathbb{R}, \mathbb{C}_J)$ such that

$$B^{n+1}_{\mathbb{C}_J}\psi_1(z) = P(z), \qquad B^{n+1}_{\mathbb{C}_J}\psi_2(z) = Q(z).$$

Hence

$$\begin{aligned} h_J(z) &= P(z) + Q(z)I = B^{n+1}_{\mathbb{C}_J}\psi_1(z) + B^{n+1}_{\mathbb{C}_J}\psi_2(z)I = \\ &= B^{n+1}_{\mathbb{C}_J}(\psi_1(s) + \psi_2(s)I) := B^{n+1}_{\mathbb{C}_J}\psi_I(z). \end{aligned}$$

Finally, we get thesis by the classical Identity Principle, see Remark 5.2.7. □

Now, we give the proof of the following corollary for the sake of completeness.

Corollary 5.3.11. *Let* $\varphi, \phi \in L^2(\mathbb{R}, \mathbb{H})$. *Then*

$$\langle B^{n+1}(\varphi), B^{n+1}(\phi)\rangle_{\mathcal{F}^n_T(\mathbb{H})} = \langle \varphi, \phi\rangle_{L^2(\mathbb{R},\mathbb{H})}.$$

Proof. It is known that any $\varphi, \phi \in L^2(\mathbb{R}, \mathbb{H})$ can be expanded as

$$\varphi(x) = \sum_{k=0}^{\infty} \psi_k^{2\pi}(x)\alpha_k,$$

$$\phi(x) = \sum_{k=0}^{\infty} \psi_k^{2\pi}(x)\beta_k,$$

where $\{\alpha_k\}_{k\geq 0}$, $\{\beta_k\}_{k\geq 0} \subset \mathbb{H}$. Since $\psi_k^{2\pi}(x)$ are normalized Hermite functions we have

$$\langle \varphi, \phi\rangle_{L^2(\mathbb{R},\mathbb{H})} = \sum_{k=0}^{\infty} \overline{\beta_k}\alpha_k. \tag{5.27}$$

On the other hand, by (5.24) we have

$$B^{n+1}(\varphi)(q) = \sqrt{2}\sqrt{\frac{1}{(2\pi)^n n!}}\sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}(2\pi)^{k/2}} H^{2\pi}_{k,n}(q,\bar{q})\alpha_k, \tag{5.28}$$

$$B^{n+1}(\phi)(q) = \sqrt{2}\sqrt{\frac{1}{(2\pi)^n n!}}\sum_{\ell=0}^{\infty} \frac{1}{\sqrt{\ell!}(2\pi)^{\ell/2}} H^{2\pi}_{\ell,n}(q,\bar{q})\beta_k. \tag{5.29}$$

Now, we put together (5.28) and (5.29) and by the orthogonality relation of the quaternionic Hermite polynomials (5.12) we get

$$\begin{aligned}\langle B^{n+1}(\varphi), B^{n+1}(\phi)\rangle_{\mathcal{F}^n_T(\mathbb{H})} &= \int_{\mathbb{C}_J} \overline{B^{n+1}(\phi)(q)} B^{n+1}(\varphi)(q) e^{-2\pi|q|^2}\, d\lambda_J(q) \\ &= \frac{2}{n!(2\pi)^n} \sum_{k=0}^{\infty} \frac{1}{k!(2\pi)^k} \overline{\beta_k} \left(\int_{\mathbb{C}_J} \overline{H^{2\pi}_{k,n}(q,\bar{q})} H^{2\pi}_{k,n}(q,\bar{q}) \right. \\ &\left. \cdot e^{-2\pi|q|^2}\, d\lambda_J(q) \right) \alpha_k \\ &= \sum_{k=0}^{\infty} \overline{\beta_k} \alpha_k.\end{aligned}$$

Therefore

$$\langle B^{n+1}(\varphi), B^{n+1}(\phi)\rangle_{\mathcal{F}^n_T(\mathbb{H})} = \sum_{k=0}^{\infty} \overline{\beta_k} \alpha_k. \tag{5.30}$$

ince (5.27) and (5.30) are equal we obtain the thesis. □

This notation will be very useful in the sequel. A function $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ is in the space $L^2(\mathbb{R}, \mathbb{H}^{n+1})$ if

$$||\vec{\varphi}||^2_{L^2(\mathbb{R},\mathbb{H}^{n+1})} := \sum_{j=0}^{n} ||\varphi_j||^2_{L^2(\mathbb{R},\mathbb{H})} < \infty. \tag{5.31}$$

Moreover, it is also possible to consider an inner product for vector-valued functions $\vec{f} = (f_0, ..., f_n)$ and $\vec{g} = (g_0, ..., g_n)$ as

$$\langle \vec{f}, \vec{g}\rangle_{L^2(\mathbb{R}^2,\mathbb{H}^{n+1})} = \sum_{j=0}^{n} \langle f_j, g_j\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{5.32}$$

See ([1]) for more details.
Now, we define the quaternionic full-polyanalytic Bargmann transform.

Definition 5.3.12. Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector-valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. The quaternionic full-polyanalytic Bargmann transform is defined as

$$\mathfrak{B}\vec{\varphi}(q) = \sum_{j=0}^{n} B^{j+1}\varphi_j(q), \tag{5.33}$$

where $B^{j+1}\varphi_j(q)$ is the true quaternionic polyanalytic Bargmann transform, defined in (5.21).

Remark 5.3.13. For $n = 0$ in (5.33) we obtain the quaternionic Segal-Bargmann transform.

Theorem 5.3.14. *The quaternionic full-polyanalytic Bargmann transform* $\mathfrak{B} : L^2(\mathbb{R}, \mathbb{H}^{n+1}) \to \widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})$ *is an isometric isomorphism.*

Proof. Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$ such that each component belongs to $L^2(\mathbb{R}, \mathbb{H})$. Then, we have

$$\begin{aligned} ||\mathfrak{B}\vec{\varphi}(q)||^2_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})} &= \int_{\mathbb{C}_J} \overline{\mathfrak{B}\vec{\varphi}(q)} \mathfrak{B}\vec{\varphi}(q) e^{-2\pi|q|^2}\, d\lambda_J(q) = \\ &= \sum_{j,m=0}^{n} \int_{\mathbb{C}_J} \overline{B^{j+1}(\varphi_j)(q)} B^{m+1}(\varphi_m)(q) e^{-2\pi|q|^2} d\lambda_J(q). \end{aligned}$$

From Lemma 5.3.5 everything is zero when $j \neq m$, so we focus only on the case $j = m$. Then, by Theorem 5.3.10 we obtain

$$||\mathfrak{B}\vec{\varphi}(q)||^2_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})} = \sum_{j=0}^{n} ||B^{j+1}\varphi_j||^2_{\mathcal{F}^n_T(\mathbb{H})} = \sum_{j=0}^{n} ||\varphi_j||^2_{L^2(\mathbb{R},\mathbb{H})} = ||\vec{\varphi}||^2_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.$$

Finally, the quaternionic full-polyanalytic Bargmann transform $\mathfrak{B} : L^2(\mathbb{R}, \mathbb{H}^{n+1}) \longrightarrow \widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})$ is surjective because is the sum of true quaternionic polyanalytic Bargmann transforms, which are surjective (see Theorem 5.3.10). □

5.4 Reproducing kernel of the true polyanalytic Fock space

In this section we give an explicit expression of the reproducing kernel of the quaternionic true polyanalytic Fock space. It is obtained by extending suitably the kernel of the complex case, see [2]. In order to prove our result we follow similar arguments of [46]. Before we need the following easy result, see [6].

Lemma 5.4.1. *If* $f \in \mathcal{SP}_n(\mathbb{H})$ *and* $g \in \mathcal{SP}_m(\mathbb{H})$*, then we have*

$$f * g \in \mathcal{SP}_{n+m-1}(\mathbb{H}).$$

In the following result we use this notation

$$(g)^{k*} = \underbrace{g * ... * g}_{k-times}.$$

Proposition 5.4.2. *The reproducing kernel of the quaternionic true poly Fock space $\mathcal{F}_T^n(\mathbb{H})$ is given by*

$$K_{n+1}(q,r)=2e_*(2\pi q\bar{r})*\left(\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}\frac{1}{k!}(2\pi(\bar{q}q-q\bar{r}-\bar{q}r+\bar{r}r))^{k*}\right).$$

Proof. Let us start by proving that $K_{n+1}(q,r)$ is a slice polyanalytic function of order $n+1$ with respct to the variable q.
We set

$$\varphi_{n+1}(q,.):=\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}\frac{1}{k!}(2\pi(\bar{q}q-q\bar{r}-\bar{q}r+\bar{r}r))^{k*}.$$

Now, we know that the function $q\longmapsto 2e_*(2\pi q\bar{r})$ is slice regular on $\mathbb{H}$ with respect to the variable q. Moreover, $\varphi_{n+1}(q,r)$ is a slice polyanalytic function of order $n+1$ on $\mathbb{H}$ with respect to q, by Remark (5.4.1). Thus, the function $K_{n+1}(q,r)$ is a slice polyanalytic function of order $n+1$ on $\mathbb{H}$ with respect to the variable q. Now, by Theorem 5.2.2 for $q=x+Iy$ we can write

$$K_{n+1}(q,r)=\frac{(1-JI)}{2}K_{n+1}^{\mathbb{C}_J}(x+yI,r)+\frac{(1+JI)}{2}K_{n+1}^{\mathbb{C}_J}(x-yI,r),$$

where $K_{n+1}^{\mathbb{C}_J}$ is the reproducing kernel of the complex true polyanalytic Fock space. From this formula it is clear that $K_{n+1}(q,r)\in\mathcal{F}_T^n(\mathbb{H})$. Now, by applying another time Theorem 5.2.2 we obtain

$$\begin{aligned}\langle f(.),\overline{K_{n+1}(q,.)}\rangle_{\widetilde{\mathcal{F}}_{Slice}^{n+1}(\mathbb{H})} &= \int_{\mathbb{C}_J}K_{n+1}(q,r)f(q)d\lambda_J(q)\\ &= \frac{(1-JI)}{2}\int_{\mathbb{C}_J}K_{n+1}^{\mathbb{C}_J}(x+yI,r)f(q)d\lambda_J(q)+\\ &\quad +\frac{(1+JI)}{2}\int_{\mathbb{C}_J}K_{n+1}^{\mathbb{C}_J}(x-yI,r)f(q)d\lambda_J(q)\\ &= \frac{(1-JI)}{2}f(x+Iy)+\frac{(1+JI)}{2}f(x-Iy)\\ &= f(q).\end{aligned}$$

This ends the proof. □

Remark 5.4.3. If $n=0$ in Proposition 5.4.2 we obtain (4.4) with $\nu=2\pi$.

Now, we are ready to show an estimate for the quaternionic-full polyanalytic Bargmann transform and the true quaternionic one.

Proposition 5.4.4. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. For every $q \in \mathbb{H}$ and every $\vec{\varphi} \in L^2(\mathbb{R}, \mathbb{H}^{n+1})$, we have*

$$|\mathfrak{B}\vec{\varphi}(q)| \leq \sqrt{2(n+1)}e^{\pi|q|^2}||\vec{\varphi}||_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.$$

Proof. From [17, Prop 4.5] we know that for any $f \in \widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})$ and $q \in \mathbb{H}$ we have

$$|f(q)| \leq \sqrt{2(n+1)}e^{\pi|q|^2}||f||_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})}.$$

Now, we specify this inequality for the quaternionic full-polyanalytic Bargmann transform by setting $f(q) := \mathfrak{B}\vec{\varphi}(q)$ and get

$$|\mathfrak{B}\vec{\varphi}(q)| \leq \sqrt{2(n+1)}e^{\pi|q|^2}||\mathfrak{B}\vec{\varphi}||_{\widetilde{\mathcal{F}}^{n+1}_{Slice}(\mathbb{H})}.$$

Thus, by the isometry property proved in Theorem 5.3.14 we obtain

$$|\mathfrak{B}\vec{\varphi}(q)| \leq \sqrt{2(n+1)}e^{\pi|q|^2}||\vec{\varphi}||_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.$$

□

Proposition 5.4.5. *For every $f \in \mathcal{F}^n_T(\mathbb{H})$, we have the following estimate*

$$|f(q)| \leq \sqrt{2}e^{\pi|q|^2}||f||_{\mathcal{F}^n_T(\mathbb{H})}.$$

Proof. From the reproducing kernel property of the space $\mathcal{F}^n_T(\mathbb{H})$ and the Cauchy-Schwartz inequality we have

$$|f(q)| = \left|\langle f, K_{n+1}\rangle_{\mathcal{F}^n_T(\mathbb{H})}\right| \leq ||f||_{\mathcal{F}^n_T(\mathbb{H})}||K_{n+1}||_{\mathcal{F}^n_T(\mathbb{H})}.$$

In particular, using Proposition 5.4.2 we have

$$||K_{n+1}||^2_{\mathcal{F}^n_T(\mathbb{H})} = K_{n+1}(q,q) = 2e^{2\pi|q|^2}.$$

Thus, we have

$$||K_{n+1}||_{\mathcal{F}^n_T(\mathbb{H})} = \sqrt{2}e^{\pi|q|^2}.$$

Finally, we obtain

$$|f(q)| \leq \sqrt{2}e^{\pi|q|^2}||f||_{\mathcal{F}^n_T(\mathbb{H})}.$$

□

Proposition 5.4.6. *For every $q \in \mathbb{H}$ and $\varphi \in L^2(\mathbb{R}, \mathbb{H})$, we have*

$$|B^{n+1}\varphi(q)| \leq \sqrt{2}e^{\pi|q|^2}||\varphi||_{L^2(\mathbb{R},\mathbb{H})}.$$

Proof. We follow a similar reasoning of Proposition 5.4.4, then we apply Theorem 3.9. □

Remark 5.4.7. For $n = 0$ in Proposition 5.4.4 and Proposition 5.4.6 we get the same estimate of Proposition 4.2.3 with $\nu = 2\pi$.

5.5 Closed formula of the quaternionic polyanalytic Bargmann transform

Inspired from Vasilevski paper [134] we write the following closed integral transform of the true polyanalytic Bargmann in the complex case, for $\varphi \in L^2(\mathbb{C})$

$$\widehat{B}^{k+1}\varphi(z) = 2^{\frac{3}{4}}(2^k k!(2\pi)^k)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} H_k\left(\frac{z+\bar{z}}{\sqrt{2}} - x\right)\varphi(x)dx,$$

where H_k are the weighted Hermite polynomials defined as

$$H_j(y) = (-1)^j e^{2\pi y^2}\frac{d^j}{dy^j}e^{-2\pi y^2} = j!\sum_{m=0}^{[\frac{j}{2}]}\frac{(-1)^m(4\pi y)^{j-2m}}{m!(j-2m)!}, \tag{5.34}$$

and $[.]$ denotes the integer part. We want to prove the equality between $\widehat{B}^{k+1}\varphi(z)$ and $B^{k+1}\varphi(z)$, which is defined by

$$B^{k+1}\varphi(z) = \sqrt{\frac{1}{(2\pi)^k k!}}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-2\pi\overline{z})^j\partial_z^{k-j}(B\varphi)(z).$$

Thus, by definition of the Segal-Bargmann transform we have

$$\begin{aligned} B^{k+1}\varphi(z) &= 2^{\frac{3}{4}}\sqrt{\frac{1}{(2\pi)^k k!}}(-1)^k\int_{\mathbb{R}}\sum_{j=0}^{k}\binom{k}{j}(-2\pi\overline{z})^j\partial_z^{k-j}\left(e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}\right) \\ &\quad\cdot\varphi(x)dx \\ &= 2^{\frac{3}{4}}\sqrt{\frac{1}{(2\pi)^k k!}}(-1)^k\int_{\mathbb{R}}\left[(\partial_z - 2\pi\overline{z})^k e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}\right]\varphi(x)dx. \end{aligned}$$

Then, in order to prove the equality between $\widehat{B}^{k+1}$ and B^{k+1} we need the following result.

Proposition 5.5.1. *For any $k \geq 0$, $z \in \mathbb{C}$ and $x \in \mathbb{R}$, we have*

$$(\partial_z - 2\pi\overline{z})^k e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} = (-1)^k 2^{-\frac{k}{2}} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} H_k\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right).$$

Proof. We prove the statement by induction. For $k=0$ we have $H_0\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right) = 1$, thus the result holds in this case. Let us assume that the equality is true for k. We will prove the result for $k+1$ thanks to the induction hypothesis and Leibniz rule. Indeed, we have

$$\begin{aligned}
&(\partial_z - 2\pi\overline{z})^{k+1} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} \\
&= (\partial_z - 2\pi\overline{z})(\partial_z - 2\pi\overline{z})^k \left(e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}\right) \\
&= (\partial_z - 2\pi\overline{z}) \left[(-1)^k 2^{-\frac{k}{2}} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} H_k\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right)\right] \\
&= (-1)^k 2^{-\frac{k}{2}} \left[\partial_z (e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}) H_k\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right)\right. \\
&\quad + e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} \partial_z H_k\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right) \\
&\quad \left. -2\pi\overline{z} H_k\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right) e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}\right].
\end{aligned}$$

We write $z = u + iv$ and develop the computations using formula (14.5) in Appendix B (with $\nu = 2\pi$) to get

$$\begin{aligned}
\partial_z H_k\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right) &= \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) H_k(\sqrt{2}u - x) \\
&= \frac{\sqrt{2}}{2}\frac{d}{du} H_k(\sqrt{2}u - x) \\
&= 2\sqrt{2}\pi k H_{k-1}\left(\frac{z+\overline{z}}{\sqrt{2}} - x\right)
\end{aligned}$$

and

$$\partial_z \left(e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}\right) = (-2\pi z + 2\pi\sqrt{2}x) e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx}.$$

Thus, if we set $y = \dfrac{z+\overline{z}}{\sqrt{2}} - x$ and by using formula (14.4) in Appendix

B (with $\nu = 2\pi$) we obtain

$$\begin{aligned}
&(\partial_z - 2\pi\overline{z})^{k+1} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} \\
&= (-1)^k 2^{-\frac{k}{2}} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} \Big[(-2\pi z + 2\pi\sqrt{2}x)H_k(y) \\
&\quad +2\sqrt{2}\pi k H_{k-1}(y) - 2\pi\overline{z}H_k(y)\Big] \\
&= (-1)^{k+1} \frac{2^{-\frac{k}{2}}}{\sqrt{2}} \sqrt{2} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} \Big[2\pi\sqrt{2}yH_k(y) - 2\sqrt{2}\pi k H_{k-1}(y)\Big] \\
&= (-1)^{k+1} 2^{-\frac{k+1}{2}} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} \left(4\pi y H_k(y) - 4\pi k H_{k-1}(y)\right) \\
&= (-1)^{k+1} 2^{-\frac{k+1}{2}} e^{-\pi(z^2+x^2)+2\pi\sqrt{2}zx} H_{k+1}(y).
\end{aligned}$$

Thus replacing y by $\frac{z+\overline{z}}{\sqrt{2}} - x$ we have the result for $k+1$. □

Due to Proposition 5.5.1 we have that

$$\widehat{B}^{k+1}\varphi(z) = B^{k+1}\varphi(z).$$

Lemma 5.5.2. *The weighted Hermite polynomials $H_k\left(\frac{q+\overline{q}}{\sqrt{2}} - x\right)$, with $q \in \mathbb{H}$ and $x \in \mathbb{R}$, are slice polyanalytic of order $k+1$ on $\mathbb{H}$.*

Proof. We know that

$$H_k\left(\frac{q+\overline{q}}{\sqrt{2}} - x\right) = k! \sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m \left(4\pi(\frac{q+\overline{q}}{\sqrt{2}} - x)\right)^{k-2m}}{m!(k-2m)!}.$$

We note that $H_k\left(\frac{q+\overline{q}}{\sqrt{2}} - x\right)$ is a slice function since it is the sum of slice functions. To justify that it is slice polyanalytic of order $k+1$, we proceed by induction on k. In order, to get the thesis it is enough to prove that

$$\overline{\partial_J}^{k+1}\left(4\pi\left(\frac{q+\overline{q}}{\sqrt{2}} - x\right)^{k-2m}\right) = 0, \qquad 0 \leq m \leq \left[\frac{k}{2}\right].$$

Let us begin the induction: the case $k = 1$ is trivial. Now, we assume that the statement holds for k and we prove it for $k+1$. We have by inductive hypothesis

$$\begin{aligned}\overline{\partial_J}^{k+2}\left(4\pi\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)\right)^{k+1-2m} &= \overline{\partial_J}^{k+1}\overline{\partial_J}\left(4\pi\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)\right)^{k+1-2m}\\ &= (k+1-2m)2\sqrt{2}\pi\\ &\quad \cdot\overline{\partial_J}^{k+1}\left(4\pi\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)\right)^{k-2m}\\ &= 0.\end{aligned}$$

This means that when we apply $\overline{\partial_J}^{k+1}$ to $H_k\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)$ each member of the sum is zero. Thus,

$$\overline{\partial_J}^{k+1}H_k\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)=0.$$

Let us recall $\widehat{B}^{k+1}\varphi$ expression in the quaternionic setting

$$\widehat{B}^{k+1}\varphi(q)=2^{\frac{3}{4}}(2^k k!(2\pi)^k)^{-\frac{1}{2}}\int_{\mathbb{R}}e^{-\pi(q^2+x^2)+2\pi\sqrt{2}qx}H_k\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)\varphi(x)dx.$$

Since the function $G(q):=e^{-\pi(q^2+x^2)+2\pi\sqrt{2}x}$ is slice regular and $H_k\left(\frac{q+\bar{q}}{\sqrt{2}}-x\right)$ is intrinsic and we proved that it is slice polyanalytic of order $k+1$. Then, by [17, Prop 3.3] we have that the function $e^{-\pi(q^2+x^2)+2\pi\sqrt{2}q}H_k\left(\frac{q+\overline{q}}{\sqrt{2}}-x\right)$ is slice polyanalytic of order $k+1$. This means that $\widehat{B}^{k+1}\varphi$ is slice polyanalytic of order $k+1$. □

Proposition 5.5.3. *The two true quaternionic full-polyanalytic Bargmann transforms B^{k+1} and $\widehat{B}^{k+1}$ are equal.*

Proof. Since $B^{k+1}\varphi(z)=\widehat{B}^{k+1}\varphi(z)$ and $\widehat{B}^{k+1}\varphi, B^{k+1}\varphi$ are slice polyanalytic of order $k+1$ by the Identity Principle for slice polyanalytic functions we get that

$$\widehat{B}^{k+1}\varphi(q)=B^{k+1}\varphi(q).$$

□

Remark 5.5.4. The formula for $\widehat{B}^{k+1}\varphi$ is a closed formula for the polyanalytic Bargmann transform. Moreover, for $k=0$ it turns out that $\widehat{B}^{k+1}\varphi$ reduces to the quaternionic analogue of the Segal-Bargmann transform (see (4.5) with $\nu=2\pi$).

5.6 Quaternion short-time Fourier transform with normalized Hermite functions as windows

The short-time Fourier transform provides a simultaneous description of the temporal and spectral behaviour of a signal, which varies over the time. In order to find the frequency spectrum of a signal φ at a specific time x, one can localize the signal φ to neighbourhood of x and after evaluates the Fourier transform of the restriction. This procedure of localization is made by choosing a cut-off function, called "*window function*".
The aim of this section is to introduce a quaternionic analogue of the short-time Fourier transform in dimension one with normalized weighted Hermite functions as windows, $\psi_n(t) = \frac{h_n^{2\pi}(t)}{\|h_n^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}}$, see (5.1). To develop this concept we need the theory of slice polyanalytic functions, see Section 2. We start by considering this formula [4, Prop.1]

$$V_{\psi_n}\varphi(x,\omega) = e^{-\pi i x\omega} G^{n+1}\varphi\left(\frac{\bar{z}}{\sqrt{2}}\right) e^{-\frac{\pi|z|^2}{2}}, \tag{5.35}$$

where the variables $(x,\omega) \in \mathbb{R}^2$ have been converted into a complex vector $z = x + i\omega$, and $G^{n+1}\varphi(z)$ is the complex true polyanalytic version of the Segal-Bargmann transform.
In this context it is possible to consider a quaternion short-time Fourier transform of a vector-valued function $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ with respect to $\vec{\psi} = (\psi_0, ..., \psi_n)$. Also for this kind of signal it is possible to have a relation as (5.35). Let us consider the following formula [4, Formula 20]

$$\mathbf{V}_{\vec{\psi}}\vec{\varphi}(x,\omega) = e^{-\pi i x\omega} \mathbf{G}\vec{\varphi}\left(\frac{\bar{z}}{\sqrt{2}}\right) e^{-\frac{\pi|z|^2}{2}}, \tag{5.36}$$

where $\mathbf{G}$ is the complex full-polyanalytic Segal-Bargmann transform. We want to extend (5.35) and (5.36) to the quaternionic setting.

Definition 5.6.1. Let $\varphi : \mathbb{R} \to \mathbb{H}$ be a function in $L^2(\mathbb{R},\mathbb{H})$. We define the 1D-true polyanalytic quaternion short time Fourier transform (true-poly QSTFT) with respect to $\psi_n(t) = \frac{h_n^{2\pi}(t)}{\|h_n^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}}$ as

$$\mathcal{V}_{\psi_n}\varphi(x,\omega) = e^{-J\pi x\omega} B^{n+1}(\varphi)\left(\frac{\bar{q}}{\sqrt{2}}\right) e^{-\frac{|q|^2\pi}{2}}, \tag{5.37}$$

where $q = x + J\omega$ and B^{n+1} is the true quaternionic polyanalytic Bargmann transform, defined in (5.21).

It is possible to define a vector-valued quaternionic short-time Fourier transform.

Definition 5.6.2. Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector-valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. We define the 1D-polyanalytic quaternion short-time Fourier transform (full-poly QSTFT) with respect to $\vec{\psi} = (\psi_0, ..., \psi_n)$ as

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x, \omega) = e^{-J\pi x\omega}\mathfrak{B}(\vec{\varphi})\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}, \tag{5.38}$$

where $q = x + J\omega$ and $\mathfrak{B}$ is the quaternionic full-polyanalytic Bargmann transform, defined in (5.33).

Remark 5.6.3. For $n = 0$ in (5.37) and (5.38) we obtain the definition of the 1D-quaternion short-time Fourier transform with respect to the Gaussian window $g(t) = 2^{1/4}e^{-\pi t^2}$, see Definition 4.5.1. Indeed, by (5.1) we get

$$\psi_0(t) = \frac{h_0^{2\pi}(t)}{\|h_0^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}} = \frac{e^{-\pi t^2}}{\left(\frac{\pi}{2\pi}\right)^{\frac{1}{4}}} = 2^{1/4}e^{-\pi t^2}.$$

Moreover, we have already observed that $B^1\varphi = \mathcal{B}\varphi$, which is the quaternionic analogue of the Bargmann transform. Therefore, with formulas (5.37) and (5.38) we are working in a more general setting than Chapter 4.

It is possible to put in relation the true-poly QSTFT and the full-poly one. Indeed, we have the following result.

Proposition 5.6.4. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. The sum of the true-poly QSTFTs with respect to ψ_j of φ_j, with $0 \leq j \leq n$, is the full-poly QSTFT with respect to $\vec{\psi}$ of $\vec{\varphi}$, i.e.*

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x, \omega) = \sum_{j=0}^{n} \mathcal{V}_{\psi_j}\varphi_j(x, \omega). \tag{5.39}$$

Proof. From (5.38) and (5.33) turns out that

$$\begin{aligned}
\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x, \omega) &= e^{-J\pi x\omega}\mathfrak{B}(\vec{\varphi})\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}} \\
&= e^{-J\pi x\omega}\left(\sum_{j=0}^{n} B^{j+1}\varphi_j\left(\frac{\bar{q}}{\sqrt{2}}\right)\right)e^{-\frac{|q|^2\pi}{2}} \\
&= \sum_{j=0}^{n} e^{-J\pi x\omega}B^{j+1}\varphi_j\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}.
\end{aligned}$$

Thus, by (5.37) we have

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega) = \sum_{j=0}^{n} \mathcal{V}_{\psi_j}\varphi_j(x,\omega).$$

□

5.6.1 Moyal formulas

Here, we show that a Moyal formula and an isometric relation hold both for the true-poly QSTFT and the full-poly one.

Theorem 5.6.5. *Let φ, ϕ be functions in $L^2(\mathbb{R},\mathbb{H})$. Then we have*

$$\langle \mathcal{V}_{\psi_n}\varphi, \mathcal{V}_{\psi_n}\phi\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = 2\langle \varphi, \phi\rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{5.40}$$

Proof. By Definition 5.6.1 we have

$$\begin{aligned}
\langle \mathcal{V}_{\psi_n}\varphi, \mathcal{V}_{\psi_n}\phi\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \int_{\mathbb{R}^2} \overline{\mathcal{V}_{\psi_n}\phi(x,\omega)}\mathcal{V}_{\psi_n}\varphi(x,\omega)dxd\omega \\
&= \int_{\mathbb{R}^2} \overline{e^{-J\pi x\omega}B^{n+1}(\phi)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}}\cdot \\
&\quad \cdot e^{-J\pi x\omega}B^{n+1}(\varphi)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}dxd\omega \\
&= \int_{\mathbb{R}^2} \overline{B^{n+1}(\phi)\left(\frac{\bar{q}}{\sqrt{2}}\right)}e^{J\pi x\omega}e^{-J\pi x\omega}B^{n+1}(\varphi)\left(\frac{\bar{q}}{\sqrt{2}}\right) \\
&\quad \cdot e^{-|q|^2\pi}dxd\omega \\
&= \int_{\mathbb{R}^2} \overline{B^{n+1}(\phi)\left(\frac{\bar{q}}{\sqrt{2}}\right)}B^{n+1}(\varphi)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-|q|^2\pi}dxd\omega
\end{aligned}$$

We put $p = \frac{\bar{q}}{\sqrt{2}}$ and by Corollary 5.3.11 we get

$$\begin{aligned}
\langle \mathcal{V}_{\psi_n}\varphi, \mathcal{V}_{\psi_n}\phi\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= 2\int_{\mathbb{R}^2} \overline{B^{n+1}(\phi(p))}B^{n+1}(\varphi(p))e^{-2|p|^2\pi}dp \\
&= 2\langle B^{n+1}\varphi, B^{n+1}\phi\rangle_{\mathcal{F}_T^n(\mathbb{H})}. \\
&= 2\langle \varphi, \phi\rangle_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

□

Corollary 5.6.6. *For any $\sigma \in L^2(\mathbb{R},\mathbb{H})$ we have*

$$\|\mathcal{V}_{\psi_n}\sigma\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \sqrt{2}\|\sigma\|_{L^2(\mathbb{R},\mathbb{H})}.$$

Proof. It follows trivially by Theorem 5.6.5 by putting $\phi = \varphi := \sigma$. □

We prove an isometry property for the full-poly QSTFT.

Theorem 5.6.7. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$, then*

$$\|\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \sqrt{2}\|\vec{\varphi}\|_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.$$

Proof. First of all we note that

$$\|\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} = \sum_{j=0}^{n} \|\mathcal{V}_{\psi_j}\varphi_j\|^2_{L^2(\mathbb{R}^2,\mathbb{H})},$$

where $\mathcal{V}_{\psi_j}$ are the true-poly QSTFTs. Therefore by Corollary 5.6.6 we have

$$\begin{aligned} \|\mathbb{V}_{\vec{\psi}_j}\vec{\varphi}\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} &= \|\mathcal{V}_{\psi_0}\varphi_0\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} + ... + \|\mathcal{V}_{\psi_n}\varphi_n\|^2_{L^2(\mathbb{R}^2,\mathbb{H})} \\ &= 2\|\varphi_0\|^2_{L^2(\mathbb{R},\mathbb{H})} + ... + 2\|\varphi_n\|^2_{L^2(\mathbb{R},\mathbb{H})} \\ &= 2\sum_{j=0}^{n} \|\varphi_j\|^2_{L^2(\mathbb{R},\mathbb{H})} = 2\|\vec{\varphi}\|^2_{L^2(\mathbb{R},\mathbb{H}^{n+1})}. \end{aligned}$$

□

Now, we prove a Moyal formula for the full-poly QSTFT. In order to do this we need the following polarization identity (see [89, Formula 2.4]) for $u, v \in \mathbb{H}$

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right) + \frac{1}{4}\sum_{\boldsymbol{\tau}=\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}} \left(\|u\boldsymbol{\tau}+v\|^2 - \|u\boldsymbol{\tau}-v\|^2\right)\boldsymbol{\tau}.$$

Proposition 5.6.8. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ and $\vec{\phi} = (\phi_0, ..., \phi_n)$ be vector valued functions in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. Then, we have*

$$\langle \mathbb{V}_{\vec{\psi}}\vec{\phi}, \mathbb{V}_{\vec{\psi}}\vec{\varphi}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = 2\langle\vec{\phi}, \vec{\varphi}\rangle_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.$$

Proof. By the polarization identity with $u := \mathbb{V}_{\vec{\psi}}\vec{\phi}$ and $v := \mathbb{V}_{\vec{\psi}}\vec{\varphi}$ and the linearity of the full-poly QSTFT, which comes from the linearity of the quaternionic full-polyanalytic Bargmann, we get

$$\begin{aligned}\langle \mathbb{V}_{\vec{\psi}}\vec{\phi}, \mathbb{V}_{\vec{\psi}}\vec{\varphi}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \frac{1}{4}\left(\|\mathbb{V}_{\vec{\psi}}\vec{\phi}+\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|^2-\|\mathbb{V}_{\vec{\psi}}\vec{\phi}-\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|^2\right)+\\ &+\frac{1}{4}\sum_{\boldsymbol{\tau}=\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}}\left(\|\mathbb{V}_{\vec{\psi}}\vec{\phi}\cdot\boldsymbol{\tau}+\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|^2-\|\mathbb{V}_{\vec{\psi}}\vec{\phi}\cdot\boldsymbol{\tau}-\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|^2\right)\boldsymbol{\tau}\\ &= \frac{1}{4}\left(\|\mathbb{V}_{\vec{\psi}}(\vec{\phi}+\vec{\varphi})\|^2-\|\mathbb{V}_{\vec{\psi}}(\vec{\phi}-\vec{\varphi})\|^2\right)+\\ &+\frac{1}{4}\sum_{\boldsymbol{\tau}=\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}}\left(\|\mathbb{V}_{\vec{\psi}}(\vec{\phi}\cdot\boldsymbol{\tau}+\vec{\varphi})\|^2-\|\mathbb{V}_{\vec{\psi}}(\vec{\phi}\cdot\boldsymbol{\tau}-\vec{\varphi})\|^2\right)\boldsymbol{\tau}.\end{aligned}$$

Since the space $L^2(\mathbb{R},\mathbb{H}^{n+1})$ is a right vector quaternionic space we have that $\vec{\phi}\pm\vec{\varphi}$ and $\vec{\phi}\boldsymbol{\tau}\pm\vec{\varphi}$ stay in $L^2(\mathbb{R},\mathbb{H}^{n+1})$. Therefore by Theorem 5.6.7 we have

$$\begin{aligned}\langle \mathbb{V}_{\vec{\psi}}\vec{\phi}, \mathbb{V}_{\vec{\psi}}\vec{\varphi}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \frac{1}{2}\left(\|\vec{\phi}+\vec{\varphi}\|^2-\|\vec{\phi}-\vec{\varphi}\|^2\right)+\\ &+\frac{1}{2}\sum_{\boldsymbol{\tau}=\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}}\left(\|\vec{\phi}\cdot\boldsymbol{\tau}+\vec{\varphi}\|^2-\|\vec{\phi}\cdot\boldsymbol{\tau}-\vec{\varphi}\|^2\right)\boldsymbol{\tau}.\end{aligned} \tag{5.41}$$

Using another time the polarization identity with $u:=\vec{\phi}$ and $v:=\vec{\varphi}$ in (5.41) we obtain

$$\langle \mathbb{V}_{\vec{\psi}}\vec{\phi}, \mathbb{V}_{\vec{\psi}}\vec{\varphi}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = 2\langle\vec{\phi},\vec{\varphi}\rangle_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.$$

□

5.6.2 Reconstruction formula

It is possible to recover the value of the signal if we know its true-poly QSTFT. In order to show a reconstruction formula we recall the following formula for the true quaternionic polyanalytic Bargmann (see Section 5). Let $\varphi\in L^2(\mathbb{R},\mathbb{H})$, thus we have

$$(B^{n+1}\varphi)(q) = c_n\int_{\mathbb{R}} e^{-\pi(q^2+t^2)+2\pi\sqrt{2}qt}H_n\left(\frac{q+\bar{q}}{\sqrt{2}}-t\right)\varphi(t)dt, \tag{5.42}$$

where $c_n := 2^{\frac{3}{4}}(2^n n!(2\pi)^n)^{-\frac{1}{2}}$ and H_n are the weighted Hermite polynomials, see (5.34).

We have the following relation between the weighted Hermite functions and the weighted Hermite polynomials

$$h_n^{2\pi}(u) = H_n(u)e^{-\pi u^2}. \tag{5.43}$$

Before to prove the reconstruction formula we need the following auxiliary result.

Lemma 5.6.9. *Let φ be a function in $L^2(\mathbb{R},\mathbb{H})$ and $\psi_n(t) = \frac{h_n^{2\pi}(t)}{\|h_n^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}}$. Then we have*

$$\mathcal{V}_{\psi_n}\varphi(x,\omega) = \sqrt{2}\int_{\mathbb{R}} e^{-2\pi J\omega t}\psi_n(x-t)\varphi(t)dt. \tag{5.44}$$

Proof. From (5.37) and (5.42) we get

$$\begin{aligned}\mathcal{V}_{\psi_n}\varphi(x,\omega) &= e^{-J\pi x\omega}\left(B^{n+1}\varphi\right)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{\frac{3}{4}}(2^n n!(2\pi)^n)^{-\frac{1}{2}}e^{-J\pi x\omega}\int_{\mathbb{R}} e^{-\pi\left(\frac{\bar{q}^2}{2}+t^2\right)+2\pi\bar{q}t} \\ &\quad \cdot H_n\left(\frac{\frac{\bar{q}}{\sqrt{2}}+\frac{q}{\sqrt{2}}}{\sqrt{2}}-t\right)\varphi(t)\,dt\; e^{-\frac{|q|^2\pi}{2}}.\end{aligned}$$

Now, since $q = x + J\omega$ we get

$$\begin{aligned}\mathcal{V}_{\psi_n}\varphi(x,\omega) &= 2^{\frac{3}{4}}(2^n n!(2\pi)^n)^{-\frac{1}{2}}e^{-J\pi x\omega}e^{-\frac{\pi x^2}{2}}e^{J\pi x\omega}e^{\frac{\omega^2\pi}{2}}e^{-\frac{\pi x^2}{2}}e^{-\frac{\omega^2\pi}{2}}\,\cdot \\ &\quad \cdot\int_{\mathbb{R}} e^{-\pi t^2+2\pi xt-2\pi J\omega t}H_n\left(x-t\right)\varphi(t)dt \\ &= 2^{\frac{3}{4}}(2^n n!(2\pi)^n)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-2\pi J\omega t}e^{-\pi t^2+2\pi xt-\pi x^2}H_n(x-t)\varphi(t)dt \\ &= 2^{\frac{3}{4}}(2^n n!(2\pi)^n)^{-\frac{1}{2}}\frac{\|h_n^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}}{\|h_n^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}}\int_{\mathbb{R}} e^{-2\pi J\omega t}e^{-\pi(x-t)^2}H_n(x-t)\varphi(t)dt.\end{aligned}$$

Hence by (5.43) we get

$$\begin{aligned}\mathcal{V}_{\psi_n}\varphi(x,\omega) &= 2^{\frac{3}{4}}(2^n n!(2\pi)^n)^{-\frac{1}{2}}(2^n(2\pi)^n n!2^{-\frac{1}{2}})^{\frac{1}{2}}\int_{\mathbb{R}} e^{-2\pi J\omega t}\frac{h_n^{2\pi}(x-t)}{\|h_n^{2\pi}\|_{L^2(\mathbb{R},\mathbb{H})}}\varphi(t)dt \\ &= \sqrt{2}\int_{\mathbb{R}} e^{-2\pi J\omega t}\psi_n(x-t)\varphi(t)dt.\end{aligned}$$

This concludes the proof. □

It is possible to have something similar also for the full-poly QSFT.

Corollary 5.6.10. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector-valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. Then for $\vec{\psi} = (\psi_0, ..., \psi_n)$ we have*

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x, \omega) = \sqrt{2}\int_{\mathbb{R}} e^{-2\pi J\omega t}\sum_{j=0}^{n}\psi_j(x-t)\varphi_j(t)dt.$$

Proof. By Proposition 5.6.4 we know that

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x, \omega) = \sum_{j=0}^{n}\mathcal{V}_{\psi_j}\varphi_j(x, \omega).$$

For each member of the sum we know that the equality (5.44) holds. So we have

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x.\omega) = \sqrt{2}\int_{\mathbb{R}} e^{-2\pi J\omega t}\sum_{j=0}^{n}\psi_j(x-t)\varphi_j(t)dt.$$

□

Now, we are ready to prove the reconstruction formula for the true-poly QSTFT.

Theorem 5.6.11. *Let $\varphi \in L^2(\mathbb{R}, \mathbb{H})$. Then for all $y \in \mathbb{R}$ we have*

$$\varphi(y) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}^2} e^{2\pi J\omega y}[\mathcal{V}_{\psi_n}\varphi(x, \omega)]\psi_n(x-y)dxd\omega. \tag{5.45}$$

Proof. Let us set

$$\phi(y) := \frac{1}{\sqrt{2}}\int_{\mathbb{R}^2} e^{2\pi J\omega y}[\mathcal{V}_{\psi_n}\varphi(x, \omega)]\psi_n(x-y)dxd\omega, \qquad \forall y \in \mathbb{R}.$$

Let $\Theta \in L^2(\mathbb{R}, \mathbb{H})$. By Fubini's theorem, Lemma 5.6.9 and the Moyal formula (5.40) we get

$$\begin{aligned}
\langle \phi, \Theta\rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}}\overline{\Theta(y)}\phi(y)dy \\
&= \frac{1}{\sqrt{2}}\int_{\mathbb{R}^3}\overline{\Theta(y)}e^{2\pi J\omega y}[\mathcal{V}_{\psi_n}\varphi(x, \omega)]\psi_n(x-y)dxd\omega dy \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\left(\sqrt{2}\int_{\mathbb{R}}\overline{e^{-2\pi J\omega y}\psi_n(x-y)\Theta(y)dy}\right)\mathcal{V}_{\psi_n}\varphi(x, \omega)dxd\omega \\
&= \frac{1}{2}\int_{\mathbb{R}^2}\overline{\mathcal{V}_{\psi_n}\Theta(x, \omega)}\mathcal{V}_{\psi_n}\varphi(x, \omega)dxd\omega \\
&= \frac{1}{2}\langle\mathcal{V}_{\psi_n}\varphi, \mathcal{V}_{\psi_n}\Theta\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \langle\varphi, \Theta\rangle_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

Therefore, since for all $\Theta \in L^2(\mathbb{R}, \mathbb{H})$ we have $\langle \phi, \Theta \rangle_{L^2(\mathbb{R},\mathbb{H})} = \langle \varphi, \Theta \rangle_{L^2(\mathbb{R},\mathbb{H})}$ we conclude that

$$\varphi(y) = \phi(y) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^2} e^{2\pi J\omega y} [\mathcal{V}_{\psi_n} \varphi(x, \omega)] \psi_n(x - y) dx d\omega.$$

$\square$

Remark 5.6.12. It is possible to have a kind of reconstruction formula also for the full-poly QSTFT. Basically, we use the reconstruction formula (5.45) for each component of the vector-valued function $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$. Thus for $0 \leq j \leq n$ we have

$$\varphi_j(y) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}^2} e^{2\pi J\omega y} [\mathcal{V}_{\psi_j} \varphi_j(x, \omega)] \psi_j(x - y) dx d\omega \qquad \forall y \in \mathbb{R}.$$

Remark 5.6.13. If $n = 0$ in (5.45) we obtain the formula proved Theorem 4.5.8.

Both the true-poly QTSFT and the full poly one admit a left-side inverse, which is possible to compute.

Theorem 5.6.14. *Let us consider the operator $\mathcal{A}_{\psi_n} : L^2(\mathbb{R}^2, \mathbb{H}) \to L^2(\mathbb{R}, \mathbb{H})$ defined for any $\Lambda \in L^2(\mathbb{R}^2, \mathbb{H})$ by*

$$\mathcal{A}_{\psi_n}(\Lambda)(y) = \sqrt{2} \int_{\mathbb{R}^2} e^{2\pi J\omega y} \Lambda(x, \omega) \psi_n(x - y) dx d\omega, \qquad \forall y \in \mathbb{R}. \quad (5.46)$$

Then $\mathcal{A}_{\psi_n}$ is the adjoint of $\mathcal{V}_{\psi_n}$. Moreover, we have the following identity

$$\mathcal{V}_{\psi_n}^* \mathcal{V}_{\psi_n} = 2Id. \quad (5.47)$$

Proof. Firstly we show that $\mathcal{A}_{\psi_n}$ is the adjoint operator of $\mathcal{V}_{\psi_n}$. Let $h \in L^2(\mathbb{R}, \mathbb{H})$. The application of Fubini's theorem and formula (5.44) lead to

$$\begin{aligned}
\langle \mathcal{A}_{\psi_n}(\Lambda), h \rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{h(y)} \mathcal{A}_{\psi_n}(\Lambda)(y) dy \\
&= \sqrt{2} \int_{\mathbb{R}^3} \overline{h(y)} e^{2\pi J\omega y} \Lambda(x, \omega) \psi_n(x - y) dx d\omega dy \\
&= \int_{\mathbb{R}^2} \sqrt{2} \left(\int_{\mathbb{R}} \overline{e^{-2\pi J\omega y} \psi_n(x - y) h(y)} dy \right) \Lambda(x, \omega) dx d\omega \\
&= \int_{\mathbb{R}^2} \overline{\mathcal{V}_{\psi_n} h(x, \omega)} \Lambda(x, \omega) dx d\omega = \langle \Lambda, \mathcal{V}_{\psi_n} h \rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned}$$

Now, we prove (5.47). It follows by formula (5.45)

$$\begin{aligned}\mathcal{V}^*_{\psi_n}(\mathcal{V}_{\psi_n}\varphi)(y) &= \sqrt{2}\int_{\mathbb{R}^2} e^{2\pi J\omega y}[\mathcal{V}_{\psi_n}\varphi(x,\omega)]\psi_n(x-y)dxd\omega \\ &= 2\varphi(y).\end{aligned}$$

□

Remark 5.6.15. The formula (5.46) is the left-side inverse of the true-poly QSTFT.

We can prove a similar result for the full-poly QSTFT.

Theorem 5.6.16. *Let us consider* $\mathbb{A}^*_{\vec{\psi}} : L^2(\mathbb{R}^2,\mathbb{H}) \to L^2(\mathbb{R},\mathbb{H}^{n+1})$ *defined for any* $\Lambda \in L^2(\mathbb{R}^2,\mathbb{H})$ *by*

$$\mathbb{A}^*_{\vec{\psi}}\Lambda = \left(\mathcal{A}^*_{\vec{\psi}}\Lambda, ..., \mathcal{A}^*_{\vec{\psi}}\Lambda\right), \tag{5.48}$$

where $\vec{\psi} = (\psi_0, ..., \psi_n)$ *and* $\mathcal{A}^*_{\vec{\psi}}\Lambda$ *are defined as follow*

$$\mathcal{A}^*_{\vec{\psi}}\Lambda = \sqrt{2}\int_{\mathbb{R}^2} e^{2\pi J\omega y}\Lambda(x,\omega)\psi_j(x-y)dxd\omega \qquad 0 \le j \le n.$$

Then $\mathbb{A}^*_{\vec{\psi}}$ *is full-poly adjoint operator of* $\mathbb{V}_{\vec{\psi}}$*. Moreover,*

$$\mathbb{V}^*_{\vec{\psi}}\mathbb{V}_{\vec{\psi}} = 2Id_{L^2(\mathbb{R},\mathbb{H}^{n+1})}. \tag{5.49}$$

Proof. The construction of the vector (5.48) follows from the definition of the full-poly adjoint operator and Theorem 5.6.14. Finally, (5.49) follows from the fact that the full-poly QSTFT is an isometric operator, see Theorem 5.6.7. □

Remark 5.6.17. The vectorial operator proposed in formula (5.48) can be considered a left-side inverse of the full-poly QSTFT.

5.6.3 Reproducing kernel property

Now, we will write the reproducing kernel of the quaternionic Gabor space associated to the true-poly QSTFT. We define this space as

$$\mathcal{G}^{\psi_n}_{\mathbb{H}} := \{\mathcal{V}_{\psi_n}\varphi,\ \varphi \in L^2(\mathbb{R},\mathbb{H})\}.$$

We also consider a vector-valued version of the previous space which is given by

$$\mathbb{G}_{\mathbb{H}}^{\vec{\psi}} = \{\mathbb{V}_{\vec{\psi}}\vec{\varphi},\ \vec{\varphi} \in L^2(\mathbb{R}, \mathbb{H}^{n+1})\},$$

where $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ and $\vec{\psi} = (\psi_0, ..., \psi_n)$.

Theorem 5.6.18. *Let φ be in $L^2(\mathbb{R}, \mathbb{H})$ and $\psi_n(t) = \frac{h_n^{2\pi}(t)}{\|h_n^{2\pi}\|_{L^2(\mathbb{R}^2,\mathbb{H})}}$. If*

$$K_{\psi_n}(x, \omega; x', \omega') = \int_{\mathbb{R}} e^{2\pi I(\omega'-\omega)t}\psi_n(x'-t)\psi_n(x-t)dt.$$

Then, K_{ψ_n} is the reproducing kernel of the space $\mathcal{G}_{\mathbb{H}}^{\psi_n}$, i.e

$$\mathcal{V}_{\psi_n}\varphi(x', \omega') = \langle \mathcal{V}_{\psi_n}\varphi, K_{\psi_n}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.$$

Proof. We use Lemma 5.6.9, the inversion formula (5.45) and Fubini's theorem to get

$$\begin{aligned}
\mathcal{V}_{\psi_n}\varphi(x', \omega') &= \sqrt{2}\int_{\mathbb{R}} e^{-2\pi J\omega' t}\varphi(t)\psi_n(x'-t)dt \\
&= \int_{\mathbb{R}} e^{-2\pi J\omega' t}\psi_n(x'-t)\left(\int_{\mathbb{R}^2} e^{2\pi J\omega t}[\mathcal{V}_{\psi_n}\varphi(x,\omega)]\psi_n(x-t)dxd\omega\right)dt \\
&= \int_{\mathbb{R}^3} e^{-2\pi J\omega' t}\psi_n(x'-t)e^{2\pi J\omega t}\psi_n(x-t)\mathcal{V}_{\psi_n}\varphi(x,\omega)dxd\omega dt \\
&= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} e^{-2\pi Jt(\omega'-\omega)}\psi_n(x'-t)\psi_n(x-t)dt\right)\mathcal{V}_{\psi_n}\varphi(x,\omega)dxd\omega \\
&= \int_{\mathbb{R}^2}\overline{K_{\psi_n}(x,\omega;x',\omega')}\mathcal{V}_{\psi_n}\varphi(x,\omega)dxd\omega \\
&= \langle \mathcal{V}_{\psi_n}\varphi, K_{\psi_n}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned}$$

□

A similar theorem holds for the full-poly QSTFT, that we can state as follows.

Theorem 5.6.19. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n) \in L^2(\mathbb{R}, \mathbb{H}^{n+1})$. The following functions*

$$K_{\vec{\psi}}(x, \omega; x', \omega') = \sum_{j=0}^{n}\int_{\mathbb{R}} e^{2\pi I(\omega'-\omega)t}\psi_j(x'-t)\psi_j(x-t)dt$$

are the reproducing kernel of the space $\mathbb{G}_{\mathbb{H}}^{\vec{\psi}}$, i.e:

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x', \omega') = \langle \mathbb{V}_{\psi_N}\vec{\varphi}, K_{\vec{\psi}}\rangle_{L^2(\mathbb{R}^2,\mathbb{H}^{n+1})}.$$

Proof. It follows from the previous theorem and the fact that we can write the full-poly QSTFT as sums of true-poly QSTFTs (Proposition 5.6.4) and the definition of inner product of the space $L^2(\mathbb{R}^2, \mathbb{H}^{n+1})$ (see (5.32)). In particular, we have

$$\begin{aligned}\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x', \omega') &= \sum_{j=0}^{n} \mathcal{V}_{\psi_j}\varphi_j(x', \omega')\\ &= \sum_{j=0}^{n} \langle \mathcal{V}_{\psi_j}\varphi_j, K_{\psi_j}\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}\\ &= \langle \mathbb{V}_{\vec{\psi}}\vec{\varphi}, K_{\vec{\psi}}\rangle_{L^2(\mathbb{R}^2,\mathbb{H}^{n+1})}.\end{aligned}$$

□

Remark 5.6.20. If $n = 0$ in Theorem 5.6.18 and Theorem 5.6.19 we recover the same result of Theorem 4.5.12.

5.6.4 Lieb's uncertainty principle

In this section we want to extend to the quaternionic polyanalytic theory the Lieb's uncertainty principle. Let us recall that the uncertainty principles state that a signal cannot be simultaneously sharply located both in time and frequency domains. This is emphasised by the following generic principle [92]:

"A function cannot be concentrated on small sets in the time-frequency plane, no matter which time-frequency representation is used."

Theorem 5.6.21. *(Weak uncertainty principle) Let $\varphi \in L^2(\mathbb{R}, \mathbb{H})$ be an unit vector, U an open set of $\mathbb{R}^2$ and $\varepsilon \geq 0$ such that*

$$\int_U |\mathcal{V}_{\psi_n}\varphi(x, \omega)|^2 dx d\omega \geq 1 - \varepsilon, \tag{5.50}$$

then $|U| \geq \frac{1-\varepsilon}{2}$.

Proof. From the definition of true-poly QSTFT and Proposition 5.4.6 we have

$$\begin{aligned}|\mathcal{V}_{\psi_n}\varphi(x, \omega)| &= |e^{-J\pi x\omega}| \left| B^{n+1}\varphi\left(\frac{\bar{q}}{\sqrt{2}}\right)\right| e^{-\frac{|q|^2\pi}{2}}\\ &\leq \sqrt{2}e^{\frac{|q|^2\pi}{2}} \|\varphi\|_{L^2(\mathbb{R},\mathbb{H})} e^{-\frac{|q|^2\pi}{2}}\\ &= \sqrt{2}.\end{aligned} \tag{5.51}$$

Thus by (5.50) we get

$$1-\varepsilon \leq \int_U |\mathcal{V}_{\psi_n}\varphi(x,\omega)|^2 dxd\omega \leq \|\mathcal{V}_{\psi_n}\varphi\|_\infty^2 |U| \leq 2|U|.$$

Hence

$$|U| \geq \frac{1-\varepsilon}{2}.$$

□

A weak uncertainty principle holds also for the full-poly QSTFT.

Theorem 5.6.22. *Let $\vec{\varphi}=(\varphi_0,...,\varphi_n)$ be a vector valued function in $L^2(\mathbb{R},\mathbb{H}^{n+1})$ with $\|\varphi_j\|_{L^2(\mathbb{R},\mathbb{H})}=1$, for all $1\leq j\leq n$, and $\vec{\psi}=(\psi_0,...,\psi_n)$ be a vector-valued window function. If U is an open set of $\mathbb{R}^2$ and $\varepsilon\geq 0$ we suppose*

$$\int_U |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^2 dxd\omega \geq 1-\varepsilon, \tag{5.52}$$

then $|U|\geq \frac{1-\varepsilon}{2(n+1)^2}$.

Proof. By Proposition 5.6.4 we know that

$$\mathbb{V}_{\vec{\psi}}\vec{\varphi}=\sum_{j=0}^n \mathcal{V}_{\psi_j}\varphi_j(x,\omega).$$

Thus by the estimate (5.51) applied at each members of the sum turns out that

$$\left|\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)\right| \leq \sum_{j=0}^n \left|\mathcal{V}_{\psi_j}\varphi_j(x,\omega)\right| \leq \sqrt{2}(n+1). \tag{5.53}$$

Therefore by (5.52) and (5.53) we obtain

$$1-\varepsilon \leq \int_U |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^2 dxd\omega \leq \|\mathbb{V}_{\vec{\psi}}\vec{\varphi}\|_\infty^2 |U| \leq 2|U|(n+1)^2.$$

Then

$$|U| \geq \frac{1-\varepsilon}{2(n+1)^2}.$$

□

In order to improve the estimates of the weak uncertainty principles we need the following L^p- estimate of the true-poly QSTFT and the full-poly one. For the first one we omit the proof since it can be shown with exactly the same computations of Theorem 4.5.14.

Proposition 5.6.23. *Let $\varphi \in L^2(\mathbb{R}, \mathbb{H})$ and $p \in [2, \infty)$ then we have*

$$\int_{\mathbb{R}^2} |\mathcal{V}_{\psi_n}\varphi(x,\omega)|^p dx d\omega \leq \frac{2^{p+1}}{p} \|\varphi\|^p_{L^2(\mathbb{R},\mathbb{H})}. \tag{5.54}$$

Proposition 5.6.24. *Let $\vec{\varphi} = (\varphi_0, ..., \varphi_n)$ be a vector valued function in $L^2(\mathbb{R}, \mathbb{H}^{n+1})$. For $p \in [2, \infty)$ we have*

$$\int_{\mathbb{R}^2} |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^p dx d\omega \leq \frac{2^{p+1}}{p}(n+1)^{p-1} \|\vec{\varphi}\|^p_{L^2(\mathbb{R},\mathbb{H}^{n+1})}. \tag{5.55}$$

Proof. From (5.39) we obtain

$$|\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^p \leq \left(\sum_{j=0}^{n} |\mathcal{V}_{\psi_j}\varphi_j(x,\omega)| \right)^p \leq (n+1)^{p-1} \sum_{j=0}^{n} |\mathcal{V}_{\psi_j}\varphi_j(x.\omega)|^p.$$

Now, we integrate and apply (5.54) at each member of the sum and we get

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^p dx d\omega &\leq (n+1)^{p-1} \sum_{j=0}^{n} \int_{\mathbb{R}^2} |\mathcal{V}_{\psi_j}\varphi_j(x,\omega)|^p dx d\omega \\ &\leq \frac{2^{p+1}}{p}(n+1)^{p-1} \sum_{j=0}^{n} \|\varphi_j\|^p_{L^2(\mathbb{R},\mathbb{H})} \\ &\leq \frac{2^{p+1}}{p}(n+1)^{p-1} \left(\sum_{j=0}^{n} \|\varphi_j\|^2_{L^2(\mathbb{R},\mathbb{H})} \right)^{\frac{p}{2}} \\ &= \frac{2^{p+1}}{p}(n+1)^{p-1} \|\vec{\varphi}\|^p_{L^2(\mathbb{R},\mathbb{H}^{n+1})}.\end{aligned}$$

□

Next, we show that the inequalities (5.54) and (5.55) yield a sharper estimate for Theorem 5.6.21 and Theorem 5.6.22, respectively.

Theorem 5.6.25. *Let us consider $\vec{\varphi} = (\varphi_0, ..., \varphi_n) \in L^2(\mathbb{R}, \mathbb{H}^{n+1})$, such that $\|\varphi_j\|_{L^2(\mathbb{R},\mathbb{H})} = 1$, for any $0 \leq j \leq n$. Let U be an open set of $\mathbb{R}^2$, $\varepsilon \geq 0$ and*

$$\int_U |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^2 dx d\omega \geq 1 - \varepsilon, \tag{5.56}$$

then

$$|U| \geq \left(\frac{2^{p+1}}{p} \right)^{-\frac{2}{p-2}} (1-\varepsilon)^{\frac{p}{p-2}} (n+1)^{\frac{2-3p}{p-2}}, \qquad \textit{for} \quad p > 2.$$

Proof. The estimate (5.55) implies that

$$\begin{aligned}1-\varepsilon &\leq \int_U |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^2 dx d\omega = \int_{\mathbb{R}^2} |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^2 \chi_U(x,\omega) dx d\omega \\ &\leq \left(\int_{\mathbb{R}^2} |\mathbb{V}_{\vec{\psi}}\vec{\varphi}(x,\omega)|^p dx d\omega\right)^{\frac{2}{p}} |U|^{\frac{p-2}{p}} \\ &\leq \left(\frac{2^{p+1}}{p}\right)^{\frac{2}{p}} (n+1)^{\frac{3p-2}{p}} |U|^{\frac{p-2}{p}}.\end{aligned}$$

Therefore

$$|U| \geq \left(\frac{2^{p+1}}{p}\right)^{-\frac{2}{p-2}} (1-\varepsilon)^{\frac{p}{p-2}} (n+1)^{\frac{2-3p}{p-2}}, \qquad \text{for} \quad p > 2.$$

□

Theorem 5.6.26. *Let φ be in $L^2(\mathbb{R},\mathbb{H})$. If we assume that U is an open set of $\mathbb{R}^2$, $\varepsilon \geq 0$ and $\|\varphi\|_{L^2(\mathbb{R},\mathbb{H})} = 1$ such that*

$$\int_U |\mathcal{V}_{\psi_n}\varphi(x,\omega)|^2 dx d\omega \geq 1-\varepsilon.$$

Then, we have

$$|U| \geq \left(\frac{2^{p+1}}{p}\right)^{-\frac{2}{p-2}} (1-\varepsilon)^{\frac{p}{p-2}}, \qquad \textit{for} \quad p > 2.$$

Proof. It follows by using similar techniques of Theorem 5.6.25. □

Remark 5.6.27. If $n = 0$ in Theorem 5.6.25 and Theorem 5.6.26, these results coincide with Theorem 4.5.15.

CHAPTER 6

On the Clifford short-time Fourier transform and its properties

6.1 Motivation

In this chapter we investigate how the short-time Fourier transform can be extended in a Clifford algebra setting. It is crucial to use a suitable generalization of the Fourier transform in the Clifford algebra setting. H.De Bie explains in [62] that nowadays the emphasis of the research on hypercomplex integral transform lies on three different methods: the eigenfunction approach, the generalized roots of -1 approach and the spin group approach.

In the first one the Clifford-Fourier transform is defined as the following integral transform

$$\mathcal{F}_{\pm}f(y) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_{\pm}(x,y) f(x)\, dx, \tag{6.1}$$

where the kernel $K_{\pm}(x,y)$ is given by an explicit expression. For this kind of Clifford-Fourier transform many generalizations were found, see [29,61, 63–65] and some important properties such as the uncertainty principle and Riemann-Lebesgue lemma were proved [81, 82].

In the second approach the definition of Fourier transform is given in the following way.

Definition 6.1.1. Denote by $\mathcal{I}_n$ the set $\{i \in \mathbb{R}_n | \, i^2 = -1\}$ of geometric square roots of minus one. Let $F_1 := \{i_1, ..., i_\mu\}$, $F_2 = \{i_{\mu+1}, ..., i_n\}$ be two ordered finite sets of such square roots, $i_k \in \mathcal{I}_n$, for all $k = 1, ..., n$. The geometric Fourier transform $\mathcal{F}_{F_1,F_2}$ of a function $f : \mathbb{R}^n \to \mathbb{R}_n$ takes the form

$$\mathcal{F}_{F_1,F_2}(f)(u) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left(\prod_{k=1}^{\mu} e^{-i_k x_k u_k} \right) f(x) \left(\prod_{k=\mu+1}^{n} e^{-i_k x_k u_k} \right) dx.$$

Finally, in the third approach the Fourier transform is defined as follows

$$\hat{f}(\phi) := \int_{\mathbb{R}^2} \phi(x, y) f(x, y) \phi(-x, -y) \, dx \, dy,$$

where ϕ is a group of morphism.

The aim of this chapter is to generalize the short-time Fourier transform on the Clifford algebra setting using the first approach.

Now, we recall briefly some notions that we will use in this chapter. The product of two 1-vectors $x = \sum_{j=0}^{n} x_j e_j$ and $y = \sum_{j=0}^{n} y_j e_j$ splits into a scalar part and a 2-vector part

$$\underline{x}\underline{y} = -\langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y},$$

with

$$\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^{n} x_j y_j = -\frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}),$$

and

$$\underline{x} \wedge \underline{y} = \sum_{j<k} e_j e_k (x_j y_k - x_k y_j) = \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}).$$

For any $\underline{x}, \underline{y} \in \mathbb{R}_n$, we have $|\underline{x}\underline{y}| \leq 2^n |\underline{x}||\underline{y}|$ and $|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$.

In the sequel, we consider functions defined on $\mathbb{R}^n$ and taking values in the real Clifford algebra $\mathbb{R}_n$. Such functions can be expressed as

$$f(x) = \sum_A e_A f_A(x),$$

where $e_A = e_{i_1} e_{i_2} ... e_{i_k}$, $1 \leq i_1 < i_2 < ... < i_k \leq n$ and f_A are real valued functions. We define the modulus $|f|$ of any function f which takes values in $\mathbb{R}_n$ as

$$|f| := \sqrt{\sum_A |f_A|^2}. \tag{6.2}$$

Now, we indicate the Dirac operator in $\mathbb{R}^n$ as

$$\partial_{\underline{x}} = \sum_{i=1}^{n} e_i \partial_{x_i}.$$

We denote by $\mathcal{P}$ the space of polynomials taking values in $\mathbb{R}_n$, i.e

$$\mathcal{P} := \mathbb{R}[x_1, ..., x_n] \otimes \mathbb{R}_n.$$

The set of homogeneous polynomials of degree k is then denoted by $\mathcal{P}_k$, while the space $\mathcal{M}_k := \ker\partial_{\underline{x}} \cap \mathcal{P}_k$ is called the space of spherical monogenics of degree k.

We denote by $L^p(\mathbb{R}^n) \otimes \mathbb{R}_n$ as the module of all Clifford-valued functions $f : \mathbb{R}^n \to \mathbb{R}_n$ with finite norm

$$\|f\|_p = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|, & p = \infty, \end{cases}$$

where $dx = dx_1...dx_n$ represents the usual Lebesgue measure in $\mathbb{R}^n$. We consider the following inner product for two functions $f, g : \mathbb{R}^n \to \mathbb{R}_n$

$$\langle f, g \rangle = \int_{\mathbb{R}^n} \overline{f(x)} g(x)\, dx.$$

Let us denote the Schwartz space as $\mathcal{S}(\mathbb{R}^n)$. In the rest of the chapter we will consider functions in this space which are radial, so real valued functions, or which takes value in the Clifford algebra $\mathbb{R}_n$. In this last case we denote the Schwartz space as $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$. For this kind of space it is possible to introduce a basis $\{\psi_{j,k,l}\}$ (see [128]), which is defined by

$$\psi_{2j,k,l}(x) := L_j^{\frac{n}{2}+k-1}(|\underline{x}|^2) M_k^{(l)} e^{-\frac{|\underline{x}|^2}{2}}, \tag{6.3}$$

$$\psi_{2j+1,k,l}(x) := L_j^{\frac{n}{2}+k}(|\underline{x}|^2) \underline{x} M_k^{(l)} e^{-\frac{|\underline{x}|^2}{2}}, \tag{6.4}$$

where $j, k \in \mathbb{N}$, $\{M_k^{(l)} \in \mathcal{M}_k : l = 1, ..., \dim\mathcal{M}_k\}$ is a basis for $\mathcal{M}_k$, and L_j^{α} are the Laguerre polynomials.

In the sequel we define the commutator of the Clifford operators A, B in the following way

$$[A, B] = AB - BA. \tag{6.5}$$

6.2 Clifford-Fourier transform

In this section, we recall the definition and some properties of the Clifford-Fourier transform introduced by H.De Bie and Y.Xu in [66].

Definition 6.2.1 (Clifford-Fourier Transform)**.** On the Schwartz class of functions $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$, we define the Clifford-Fourier transform as

$$\mathcal{F}_{\pm} f(y) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_{\pm}(x, y) f(x)\, dx, \tag{6.6}$$

and their inverses as

$$\mathcal{F}_{\pm}^{-1} f(y) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widetilde{K_{\pm}}(x, y) f(x)\, dx,$$

where

$$K_{\pm}(x, y) := e^{\mp i \frac{\pi}{2} \Gamma_{\underline{y}}} e^{-i \langle \underline{x}, \underline{y} \rangle},$$

and

$$\widetilde{K_{\pm}}(x, y) := e^{\pm i \frac{\pi}{2} \Gamma_{\underline{y}}} e^{i \langle \underline{x}, \underline{y} \rangle},$$

$\Gamma_{\underline{y}} := \frac{\partial_y y - y \partial_y}{2} + \frac{n}{2} = -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j})$ is the Gamma operator.

In the paper [66] the authors write an explicit formula of the kernel by using the Gegenbauer polynomials $C_k^{\lambda}(\omega)$ and the Bessel functions $J_{\alpha}(t)$:

$$K_{-}(x, y) = A_{\lambda} + B_{\lambda} + (\underline{x} \wedge \underline{y}) C_{\lambda}, \tag{6.7}$$

with

$$A_{\lambda} = 2^{\lambda-1} \Gamma(\lambda+1) \sum_{k=0}^{\infty} (i^n + (-1)^k)(|\underline{x}||\underline{y}|)^{-\lambda} J_{k+\lambda}(|\underline{x}||\underline{y}|) C_k^{\lambda}(\langle \xi, \eta \rangle),$$

$$B_{\lambda} = -2^{\lambda-1} \Gamma(\lambda) \sum_{k=0}^{\infty} (k+\lambda)(i^n - (-1)^k)(|\underline{x}||\underline{y}|)^{-\lambda} J_{k+\lambda}(|\underline{x}||\underline{y}|) C_k^{\lambda}(\langle \xi, \eta \rangle),$$

$$C_{\lambda} = -(2\lambda) 2^{\lambda-1} \Gamma(\lambda) \sum_{k=0}^{\infty} (i^n + (-1)^k)(|\underline{x}||\underline{y}|)^{-\lambda-1} J_{k+\lambda}(|\underline{x}||\underline{y}|) C_{k-1}^{\lambda+1}(\langle \xi, \eta \rangle),$$

where $\xi = \frac{x}{|\underline{x}|}$, $\eta = \frac{y}{|\underline{y}|}$ and $\lambda = \frac{n-2}{2}$.

It is important to observe that the kernel is not symmetric, in the sense that $K_{-}(x, y) \neq K_{-}(y, x)$. Hence, we adopt the convention that we always integrate over the first variable in the kernels. Throughout the paper, we only focus on the kernel $K_{-} = e^{i \frac{\pi}{2} \Gamma_{\underline{y}}} e^{-i \langle \underline{x}, \underline{y} \rangle}$, from the following proposition (see [66, Prop. 3.4]) it is possible to recover $K_{+}(x, y) = e^{-i \frac{\pi}{2} \Gamma_{\underline{y}}} e^{-i \langle \underline{x}, \underline{y} \rangle}$.

Remark 6.2.2. The main difference between this Fourier transform and the ones considered in [30, 96, 97] is that the kernel of the Clifford-Fourier transform is developed by considering that it is the eigenvalue of a basis of the Clifford-space $L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$. On the other hand, the approach considered by Hitzer and collaborators consists of replacing the classic complex imaginary unit i with generalized roots of -1.

Proposition 6.2.3. *For $x, y \in \mathbb{R}^n$,*

$$K_+(x,y) = (K_-(x,-y))^c,$$

$$\widetilde{K_\pm}(x,y) = (K_\pm(x,y))^c,$$

where ^{c} stands for the complex conjugate.

It is important to remark that in the case d is even A_λ, B_λ, C_λ are real valued so the complex conjugate in $K_-(x,y)$ can be omitted.

From the explicit expression of the kernel it is possible to derive some easy properties [66, Prop. 5.1].

Proposition 6.2.4. *Let $n = 2$. Then the kernel of Clifford-Fourier transform satisfies*

$$K_-(x,z)K_-(y,z) = K_-(x+y,z).$$

If the dimension d is even and $n > 2$ then

$$K_-(x,z)K_-(y,z) \neq K_-(x+y,z).$$

If we consider the Clifford conjugate and even dimension (see [109, Prop. 3.5]) we have

$$K_-(y,x) = \overline{K_-(x,y)}. \tag{6.8}$$

Furthermore, the following trivial formula holds

$$K_-(-x,y) = K_-(x,-y). \tag{6.9}$$

Moreover, in [66] for even dimensions the kernel is rewritten as a finite sum of Bessel functions (see [66, Thm. 4.3])

$$K_-(x,y) = (-1)^{\lambda+1}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left(A_\lambda^*(s,t) + B_\lambda^*(s,t) + (\underline{x} \wedge \underline{y})C_\lambda^*(s,t)\right),$$

where $s = \langle \underline{x}, \underline{y} \rangle$ and $t = |\underline{x} \wedge \underline{y}| = \sqrt{|\underline{x}|^2|\underline{y}|^2 - s^2}$ and

$$A^*_\lambda(s,t) = \sum_{l=0}^{\lfloor \frac{\lambda+1}{2} - \frac{3}{4} \rfloor} s^{\lambda-1-2l} \frac{1}{2^l l!} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-2l)} \tilde{J}_{(2\lambda-2l-1)/2}(t),$$

$$B^*_\lambda(s,t) = -\sum_{l=0}^{\lfloor \frac{\lambda+1}{2} - \frac{1}{2} \rfloor} s^{\lambda-2l} \frac{1}{2^l l!} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-2l+1)} \tilde{J}_{(2\lambda-2l-1)/2}(t),$$

$$C^*_\lambda(s,t) = -\sum_{l=0}^{\lfloor \frac{\lambda+1}{2} - \frac{1}{2} \rfloor} s^{\lambda-2l} \frac{1}{2^l l!} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-2l+1)} \tilde{J}_{(2\lambda-2l+1)/2}(t)$$

with $\tilde{J}_\alpha(t) = t^{-\alpha} J_\alpha(t)$.

The above formula helps H.De Bie and Y.Xu to prove a very important estimate of the kernel, [66, Thm. 5.3].

Lemma 6.2.5. *Let n be even. For $x, y \in \mathbb{R}^n$, one has*

$$|K_-(x,y)| \leq c(1+|\underline{x}|)^\lambda (1+|\underline{y}|)^\lambda. \tag{6.10}$$

Some problems for the Clifford-Fourier transform arise when we consider odd dimensions. Indeed in this case it is not known if it is possible to write $K_-(x,y)$ as sums of Bessel functions. Moreover, it seems not possible to obtain an upper bound of the kernel as in (6.10). So in the rest of the chapter we focus only on the even dimensions more than two.

Let us define the following space of functions

$$B(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) : \|f\|_B := \int_{\mathbb{R}^n} (1+|\underline{y}|)^\lambda |f(y)|\, dy < \infty \right\}. \tag{6.11}$$

Due to the boundedness of the kernel we have the following important theorem (see [66, Thm 6.1]).

Theorem 6.2.6. *Let d be an even integer. The Clifford-Fourier transform is well defined on $B(\mathbb{R}^n) \otimes \mathbb{R}_n$. In particular, for $f \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$, $\mathcal{F}_\pm f$ is a continuous function.*

Now, we list some important properties of the Clifford-Fourier transform.

Proposition 6.2.7. *[109, Prop. 3.6][Plancherel theorem] If $f, g \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ then*

$$\int_{\mathbb{R}^n} \overline{f(y)} g(y)\, dy = \int_{\mathbb{R}^n} \overline{\mathcal{F}_-(f)(x)} \mathcal{F}_-(g)(x)\, dx. \tag{6.12}$$

Proposition 6.2.8. *[109, Prop. 3.7][Parseval's identity] If $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$, then*

$$\|f\|_2 = \|\mathcal{F}_-(f)\|_2. \tag{6.13}$$

Theorem 6.2.9. *[66, Thm. 6.6] For the basis $\{\psi_{j,k,l}\}$ of $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ (see* (6.3) *and* (6.4)*), one has*

$$\mathcal{F}_\pm(\psi_{2j,k,l}) = (-1)^{j+k}(\mp 1)^k \psi_{2j,k,l},$$
$$\mathcal{F}_\pm(\psi_{2j+1,k,l}) = i^n(-1)^{j+1}(\mp 1)^{k+n-1}\psi_{2j+1,k,l}.$$

When we restrict to the basis $\{\psi_{j,k,l}\}$ we have

$$\mathcal{F}_\pm^{-1}\mathcal{F}_\pm = Id. \tag{6.14}$$

Moreover, when n is even, (6.14) *holds for all $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$.*

Now we prove a property, which was proved in a more general setting in [65, Thm. 6.3].

Theorem 6.2.10. *When n is even for the basis $\{\psi_{j,k,l}\}$ of $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$, one has*

$$\mathcal{F}_\pm\left(\mathcal{F}_\pm(\psi_{j,k,l})\right) = \psi_{j,k,l}. \tag{6.15}$$

Moreover, the formula (6.15) *holds for all $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$.*

Proof. We distinguish two cases depending on j.
If j is even we apply two times Theorem 6.2.9 and we get

$$\mathcal{F}_\pm\left(\mathcal{F}_\pm(\psi_{j,k,l})\right) = (-1)^{j+k}(-1)^{j+k}(\mp 1)^k(\mp 1)^k\psi_{2j,k,l} = \psi_{2j,k,l}.$$

If j is odd, as before, we apply two times Theorem 6.2.9 and we obtain

$$\begin{aligned} \mathcal{F}_\pm\left(\mathcal{F}_\pm(\psi_{j,k,l})\right) &= i^n(-1)^{j+1}(\mp 1)^{k+n-1} i^n(-1)^{j+1}(\mp 1)^{k+n-1}\psi_{2j+1,k,l} \\ &= (i^2)^n\psi_{2j+1,k,l} = (-1)^n\psi_{2j+1,k,l} = \psi_{2j+1,k,l}. \end{aligned}$$

This proves formula (6.15).

Finally, since $\mathcal{F}_\pm$ is continuous on $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ [66, Thm. 6.3] and $\{\psi_{j,k,l}\}$ is a dense subset of $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ we obtain that for all $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$

$$\mathcal{F}_\pm\left(\mathcal{F}_\pm(f)\right)(x) = f(x). \tag{6.16}$$

□

Remark 6.2.11. Due to the density of $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ in $L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ the Plancherel's theorem, Parseval's identity, and the equations (6.14) and (6.16) can be extended to functions in $L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$.

6.3 Generalized translation and modulation operators

In this section we introduce the generalized translation and the generalized modulation, which are fundamental tools for developing a time-frequency analysis in a Clifford algebra setting. The generalized translation was introduced for the first time in the paper [66], where the authors "fixed" the well-known property of the translation of the classical Fourier transform and derived the following definition.

Definition 6.3.1. Let $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$. For $y \in \mathbb{R}^n$ the generalized translation operator $f \mapsto \tau_y f$ is defined by

$$\mathcal{F}_-\tau_y f(x) = K_-(y,x)\mathcal{F}_- f(x), \qquad x \in \mathbb{R}^n.$$

By the inversion formula of $\mathcal{F}_-$, the translation can be expressed as an integral operator

$$\tau_y f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} (K_-(\xi,x))^c K_-(y,\xi)\mathcal{F}_- f(\xi)\, d\xi.$$

Since we are working with even dimensions $K_-(\xi,x)$ is a real-valued function so we can omit the complex conjugate

$$\tau_y f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(\xi,x) K_-(y,\xi)\mathcal{F}_- f(\xi)\, d\xi. \tag{6.17}$$

We note that the generalized translation has the following properties, see [66, Prop. 7.2].

Proposition 6.3.2. *If $n=2$ for all functions $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ one has*

$$\tau_y f(x) = f(x-y).$$

If the dimension n is even and $n>2$ then in general

$$\tau_y f(x) \neq f(x-y).$$

However, τ_y coincides with the classical translation operator if f is a radial function [66, Thm. 7.3].

Proposition 6.3.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a radial function on $\mathbb{R}^n$, $f(x) = f_0(|\underline{x}|)$, with $f_0 : \mathbb{R}_+ \mapsto \mathbb{R}$, then $\tau_y f(x) = f(x-y)$.*

Using the generalized translation, it is possible to define a convolution operation for functions with values in Clifford algebra.

Definition 6.3.4. For $f, g \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$, the generalized convolution is defined by

$$(f *_{Cl} g)(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \tau_y f(x) g(y)\, dy, \qquad x \in \mathbb{R}^n.$$

We remark that if f and g take value in the Clifford algebra, then $f *_{Cl} g$ is not commutative in general. Moreover, in the case when one of the two functions is radial we have the following well-known property of the Fourier transform [66, Thm. 8.2].

Theorem 6.3.5. *If $g \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$, and $f \in \mathcal{S}(\mathbb{R}^n)$ is a radial function, then $f *_{Cl} g$ satisfies*

$$\mathcal{F}_-(f *_{Cl} g)(x) = \mathcal{F}_- f(x) \mathcal{F}_- g(x).$$

In particular, since f is a scalar function we have the commutativity of the convolution, i.e.

$$f *_{Cl} g = g *_{Cl} f.$$

Now we have all the tools to build the generalized modulation. Let f be in $\mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ and like the generalized translation we fix the following property:

$$\mathcal{F}_-(M_y f)(\xi) = \tau_y \mathcal{F}_-(f)(\xi), \qquad \xi, y \in \mathbb{R}^n, \tag{6.18}$$

where M_y is the generalized modulation operator. By Definition 6.3.1 we have

$$\mathcal{F}_-(M_y f)(\xi) = \mathcal{F}_-^{-1}\left(K_-(y,x)\mathcal{F}_-\left(\mathcal{F}_-(f)\right)(x)\right)(\xi).$$

Now we apply $\mathcal{F}_-$ and use Theorem 6.2.10, so we are able to give the following definition.

Definition 6.3.6. Let $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$. For $y \in \mathbb{R}^n$ the generalized modulation is defined by

$$M_y f(x) = K_-(y,x) f(x), \qquad x \in \mathbb{R}^n.$$

Remark 6.3.7. As in the classical Fourier analysis we can relate the Clifford-Fourier transform of the translation with the modulation of the Clifford-Fourier transform:

$$\mathcal{F}_-(\tau_y f(x)) = K_-(y,x)\mathcal{F}_- f(x) = M_y(\mathcal{F}_- f(x)). \tag{6.19}$$

Remark 6.3.8. In the theory that we are going to develop for even dimensions $n > 2$ there is not a commutative relationship between the modulation operator and the translation operator, as it happens in the classical case. Indeed, let $f \in \mathcal{S}(\mathbb{R}^n)$ be a radial function, by Definition 6.3.6 and Proposition 6.3.3 we get

$$M_\omega \tau_y f(x) = K_-(\omega, x) f(x-y), \quad \omega, y \in \mathbb{R}^n. \tag{6.20}$$

Now, if we exchange the roles between the translation and modulation operators by formulas (6.17) and (6.18) we have

$$\begin{aligned} \tau_y M_\omega f(x) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(\xi, x) K_-(y, \xi) \mathcal{F}_-(M_\omega f)(\xi)\, d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(\xi, x) K_-(y, \xi) \tau_\omega \mathcal{F}_-(f)(\xi)\, d\xi, \quad \omega, y \in \mathbb{R}^n. \end{aligned}$$

Therefore

$$\tau_y M_\omega f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(\xi, x) K_-(y, \xi) \tau_\omega \mathcal{F}_-(f)(\xi)\, d\xi. \tag{6.21}$$

Since (6.20) and (6.21) are very different we do not have any commutative relations between the generalised translation and modulation operators.

Remark 6.3.9. One my wonder if for dimension $d = 2$ there exists a commutative formula between the generalized translation and modulation operator. One can verify by Proposition 6.3.2 that for $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$

$$M_\omega \tau_y f(x) = K_-(\omega, x) f(x-y),$$
$$\tau_y M_\omega f(x) = K_-(\omega, x-y) f(x-y).$$

Now, due to Proposition 6.2.4 and the fact that K_- is not symmetric we deduce that it is not possible to make this computation $K_-(\omega, x-y) = K_-(\omega, x)K_-(\omega, -y)$. Thus, we conclude that never exists a commutative formula between the generalized translation and modulation operator.

We end this section proving some easy but important formulas of time-frequency analysis.

Lemma 6.3.10. *If $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $y, \omega \in \mathbb{R}^n$ then*

$$\mathcal{F}_-(\tau_y M_\omega f)(x) = M_y \tau_\omega \mathcal{F}_- f(x), \qquad x \in \mathbb{R}^n, \tag{6.22}$$

$$\mathcal{F}_-(M_\omega \tau_y f)(x) = \tau_\omega M_y \mathcal{F}_- f(x) \qquad x \in \mathbb{R}^n. \tag{6.23}$$

Proof. Formula (6.22) follows by applying (6.19) and (6.18)

$$\mathcal{F}_-(\tau_y M_\omega f)(x) = M_y \mathcal{F}_-(M_\omega f)(x) = M_y \tau_\omega \mathcal{F}_- f(x).$$

Formula (6.23) follows by applying (6.18) and (6.19)

$$\mathcal{F}_-(M_\omega \tau_y f)(x) = \tau_\omega \mathcal{F}_-(\tau_y f)(x) = \tau_\omega M_y \mathcal{F}_-(f)(x).$$

□

6.4 The Clifford short-time Fourier transform

In this section we generalize the short-time Fourier transform, using the Clifford-Fourier transform. As signal we consider a function f Clifford-valued and firstly we assume the same hypothesis also for the window function g.

Definition 6.4.1. Let $f, g \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$. The Clifford short-time Fourier transform of a function f with respect to g is defined as

$$\mathcal{V}_g f(x, \omega) = \mathcal{F}_-(\tau_x \bar{g} \cdot f)(\omega), \qquad \text{for} \quad x, \omega \in \mathbb{R}^n. \tag{6.24}$$

We want to manipulate formula (6.24) in order to write it as an integral. From the definition of the Clifford-Fourier transform (6.6) and the formula of the generalized translation operator (6.17) we get

$$\begin{aligned}
\mathcal{V}_g f(x, \omega) &= \mathcal{F}_-(\tau_x \bar{g} \cdot f)(\omega) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t, \omega) \tau_x \bar{g}(t) f(t)\, dt \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} K_-(t, \omega) K_-(\xi, t) K_-(x, \xi) \mathcal{F}_-(\bar{g})(\xi) f(t)\, d\xi\, dt \\
&= (2\pi)^{-\frac{3}{2}n} \int_{\mathbb{R}^{3n}} K_-(t, \omega) K_-(\xi, t) K_-(x, \xi) K_-(z, \xi) \\
&\quad \bar{g}(z) f(t)\, dz\, d\xi\, dt.
\end{aligned}$$

Since it is difficult to work with this amount of non commuting kernels we choose to work with a radial window function. So if $g \in \mathcal{S}(\mathbb{R}^n)$ is a radial function by definition of the Clifford-Fourier transform and Proposition 6.3.3 we get

$$\begin{aligned}
\mathcal{V}_g f(x, \omega) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t, \omega) \tau_x g(t) f(t)\, dt \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t, \omega) g(t - x) f(t)\, dt.
\end{aligned}$$

Thus we have the following definition.

Definition 6.4.2. Let $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in \mathcal{S}(\mathbb{R}^n)$ be a radial function. The Clifford short-time Fourier transform of a function with respect to g is defined, for $x, \omega \in \mathbb{R}^n$, as

$$\begin{aligned}
\mathcal{V}_g f(x, \omega) &= \mathcal{F}_-(\tau_x \bar{g} \cdot f)(\omega) \qquad (6.25) \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t, \omega) g(t - x) f(t)\, dt.
\end{aligned}$$

In the sequel we will use this integral formula for proving all the properties of the Clifford short-time Fourier transform.

Now we are going to show that the Clifford short-time Fourier transform as the Clifford-Fourier transform is well-defined on $B(\mathbb{R}^n) \otimes \mathbb{R}_n$. First of all we recall the following notion

$$B(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) : \|f\|_B := \int_{\mathbb{R}^n} (1 + |\underline{y}|)^\lambda |f(y)| \, dy < \infty \right\}$$

where $\lambda = \frac{d-2}{2}$ and d is even more that two. Now, we introduce the following spaces of real valued functions for $1 \leq p < \infty$

$$B^p(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{B^p} := \left(\int_{\mathbb{R}^n} (1 + |\underline{y}|)^\lambda |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty \right\},$$

$$W_{p\lambda}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{W_{p\lambda}} ; = \left(\int_{\mathbb{R}^n} (1 + |\underline{y}|)^{\lambda p} |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty \right\}.$$

Remark 6.4.3. If $p = 1$ all the spaces introduced coincide.

For the spaces $B^p(\mathbb{R}^n)$ and $W_{p\lambda}(\mathbb{R}^n)$ we have the following inclusion.

Lemma 6.4.4. *Let n be even more than two. For $p \geq 1$ we have*

$$W_{p\lambda}(\mathbb{R}^n) \subseteq B^p(\mathbb{R}^n).$$

Proof. It is enough to prove that $\|.\|_{B^p} \leq \|.\|_{W_{p\lambda}}$. Firstly we observe that since $\lambda = \frac{n-2}{2} \geq 1$ and $p \geq 1$ we have

$$(1 + |\underline{y}|)^\lambda \leq (1 + |\underline{y}|)^{\lambda p}.$$

Then

$$(1 + |\underline{y}|)^\lambda |f(y)|^p \leq (1 + |\underline{y}|)^{\lambda p} |f(y)|^p.$$

Therefore, by the monotonicity of the integral we have

$$\int_{\mathbb{R}^n} (1 + |\underline{y}|)^\lambda |f(y)|^p \, dy \leq \int_{\mathbb{R}^n} (1 + |\underline{y}|)^{\lambda p} |f(y)|^p \, dy.$$

Thus

$$\|f\|_{B^p} \leq \|f\|_{W_{p\lambda}}.$$

$\square$

Remark 6.4.5. From the properties of the L^p-spaces we do not have any relations of inclusion between the spaces $B(\mathbb{R}^n)$ and $B^p(\mathbb{R}^n)$. For the same reason there is not any inclusion between $B(\mathbb{R}^n)$ and $W_{p\lambda}(\mathbb{R}^n)$.

Now we prove two inequalities which will be fundamental for defining the domain of the Clifford short-time Fourier transform.

Lemma 6.4.6. *Let d be even more than two. If $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in W_{2\lambda}(\mathbb{R}^n)$ is a radial function, then $f \cdot g \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$. In particular we have,*

$$\|f \cdot g\|_B \leq c\|g\|_{W_{2\lambda}}\|f\|_2, \tag{6.26}$$

where c is a positive constant.

Proof. From the Hölder inequality we get

$$\begin{aligned}
\|f \cdot g\|_B &\leq c\int_{\mathbb{R}^n} (1+|\underline{y}|)^\lambda |f(y)||g(y)|\, dy \\
&= c\int_{\mathbb{R}^n} (1+|\underline{y}|)^\lambda |g(y)||f(y)|\, dy \\
&\leq c\left(\int_{\mathbb{R}^n} (1+|\underline{y}|)^{2\lambda}|g(y)|^2\, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |f(y)|^2\, dy\right)^{\frac{1}{2}} \\
&= c\|g\|_{W_{2\lambda}}\|f\|_2.
\end{aligned}$$

□

Proposition 6.4.7. *Let d be even more than two. If $f \in B^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in B^2(\mathbb{R}^n)$ is a radial function, then $f \cdot g \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$. In particular, we have*

$$\|f \cdot g\|_B \leq c\|f\|_{B^2}\|g\|_{B^2}, \tag{6.27}$$

where c is a positive constant.

Proof. From the Hölder inequality we obtain

$$\begin{aligned}
\|f \cdot g\|_B &\leq c\int_{\mathbb{R}^n} (1+|\underline{y}|)^\lambda |f(y)||g(y)|\, dy \\
&= c\int_{\mathbb{R}^n} (1+|\underline{y}|)^{\frac{\lambda}{2}} |f(y)|(1+|\underline{y}|)^{\frac{\lambda}{2}}|g(y)|\, dy \\
&\leq c\left(\int_{\mathbb{R}^n} (1+|\underline{y}|)^{\lambda}|f(y)|^2\, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (1+|\underline{y}|)^{\lambda}|g(y)|^2 dy\right)^{\frac{1}{2}} \\
&= c\|f\|_{B^2}\|g\|_{B^2}.
\end{aligned}$$

□

Now we show that the space $W_{p\lambda}(\mathbb{R}^n)$ is invariant under translation of radial functions.

Lemma 6.4.8. *Let d be even more than two and $p \geq 1$. If g is a radial function in $W_{p\lambda}(\mathbb{R}^n)$ then $\tau_x g \in W_{p\lambda}(\mathbb{R}^n)$, i.e*

$$\|\tau_x g\|_{W_{p\lambda}} \leq (1+|\underline{x}|)^{\lambda} \|g\|_{W_{p\lambda}}, \qquad x \in \mathbb{R}^n. \tag{6.28}$$

Proof. Since g is a radial function by Proposition 6.3.3 we have

$$\|\tau_x g\|^p_{W_{p\lambda}} = \int_{\mathbb{R}^n} (1+|\underline{t}|)^{p\lambda} |\tau_x g(t)|^p \, dt = \int_{\mathbb{R}^n} (1+|\underline{t}|)^{p\lambda} |g(t-x)|^p \, dt.$$

Now, we put $t - x = z$ and by the triangle inequality we obtain

$$\begin{aligned}
\|\tau_x g\|^p_{W_{p\lambda}} &= \int_{\mathbb{R}^n} (1+|\underline{z}+\underline{x}|)^{p\lambda} |g(z)|^p \, dt \\
&\leq \int_{\mathbb{R}^n} (1+|\underline{z}|+|\underline{x}|)^{p\lambda} |g(z)|^p \, dz \\
&\leq \int_{\mathbb{R}^n} (1+|\underline{z}|)^{p\lambda} (1+|\underline{x}|)^{p\lambda} |g(z)|^p \, dz \\
&= (1+|\underline{x}|)^{p\lambda} \int_{\mathbb{R}^n} (1+|\underline{z}|)^{p\lambda} |g(z)|^p \, dz \\
&= (1+|\underline{x}|)^{p\lambda} \|g\|^p_{W_{p\lambda}}.
\end{aligned}$$

So we gain the thesis. □

Remark 6.4.9. Using the same techniques and the hypothesis of Lemma 6.4.8 it is possible to prove that also the space $B^p(\mathbb{R}^n)$ is invariant under translation. Indeed

$$\|\tau_x g\|_{B^p} \leq (1+|\underline{x}|)^{\frac{\lambda}{p}} \|g\|_{B^p}, \qquad x \in \mathbb{R}^n. \tag{6.29}$$

Now, we have all the tools for proving that $\mathcal{V}_g f \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$.

Theorem 6.4.10. *Let $n > 2$ and even. If $f \in B^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in B^2(\mathbb{R}^n)$ is a radial function, then the Clifford short-time Fourier transform is well defined.*

Proof. If we prove that $\tau_x g \cdot f \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$, then by Theorem 6.2.6 and Definition 6.4.2 we have the thesis.
So, we focus on proving that $\tau_x g \cdot f \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$. By inequalities (6.27) and (6.29) we have

$$\|\tau_x g \cdot f\|_B \leq c \|\tau_x g\|_{B^2} \|f\|_{B^2} \leq c(1+|\underline{x}|)^{\frac{\lambda}{2}} \|g\|_{B^2} \|f\|_{B^2} < \infty,$$

where c is a positive constant. □

It is possible to prove the well-posedness of the Clfford short-time Fourier transform choosing different spaces for the signal f and the window function g.

Theorem 6.4.11. *Let $n > 2$ and even. If $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in W_{2\lambda}(\mathbb{R}^n)$ is a radial function, then the Clifford short-time Fourier transform is well defined.*

Proof. As before it is an application of Theorem 6.2.6. So we focus on proving that $\tau_x g \cdot f \in B(\mathbb{R}^n) \otimes \mathbb{R}_n$. By inequalities (6.26) and (6.28) we have

$$\|\tau_x g \cdot f\|_B \leq c\|\tau_x g\|_{W_{2\lambda}} \|f\|_2 \leq c(1+|\underline{x}|)^\lambda \|g\|_{W_{2\lambda}} \|f\|_2 < \infty,$$

where c is positive constant. $\square$

In the rest of the chapter we consider (except some cases) the signal $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and the window function $g \in W_{2\lambda}(\mathbb{R}^n)$ a radial function.

6.5 Elementary properties of the Clifford short-time Fourier transform

In this section we prove some basic properties of the Clifford short-time Fourier transform.

Proposition 6.5.1. *Let $n > 2$ and even. If $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in W_{2\lambda}(\mathbb{R}^n)$ is a radial function then*

1 (Right linearity) If $\lambda, \mu \in \mathbb{R}_n$ and $h \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ then

$$[\mathcal{V}_g(f\lambda + h\mu)](\omega, x) = \mathcal{V}_g(f)(x,\omega)\lambda + \mathcal{V}_g(h)(x,\omega)\mu.$$

2 (Parity)

$$\mathcal{V}_g f(x,\omega) = \mathcal{V}_g f(-x,\omega).$$

Proof. The first one follows from the Definition 6.4.2. The second one follows from the hypothesis of radiality of g. $\square$

In the next proposition we list some equivalent forms of the Clifford short-time Fourier transform.

Proposition 6.5.2. *Let $n > 2$ and even. We suppose that $g \in W_{2\lambda}(\mathbb{R}^n)$ is a radial function and $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$. Then*

1.

$$\mathcal{V}_g f(x,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{M_\omega \tau_x g(t)} f(t)\, dt, \tag{6.30}$$

2.

$$\mathcal{V}_g f(x,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{\tau_\omega M_x (\mathcal{F}_- g)(t)} (\mathcal{F}_- f)(t)\, dt, \tag{6.31}$$

3.

$$\begin{aligned} \mathcal{V}_g f(x,\omega) &= \mathcal{V}_{\mathcal{F}_-(g)} \mathcal{F}_-(f)(\omega, x) \\ &\quad -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{[\tau_x, M_\omega] g(t)} f(t)\, dt, \end{aligned} \tag{6.32}$$

4.

$$\begin{aligned} \mathcal{V}_g f(x,\omega) &= \mathcal{F}_- \left(\mathcal{F}_-(f) \cdot \tau_\omega \mathcal{F}_-(g) \right)(x) \\ &\quad -(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{[\tau_x, M_\omega] g(t)} f(t)\, dt, \end{aligned} \tag{6.33}$$

where $[.,.]$ *is the commutator defined in* (6.5).

Proof. 1. To prove the equality (6.30) we use Definition 6.4.2, the relation (6.8) and the radiality of the function g

$$\begin{aligned} \mathcal{V}_g f(x,\omega) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t,\omega) \tau_x g(t) f(t)\, dt \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{K_-(\omega,t) \tau_x g(t)} f(t)\, dt \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{M_\omega \tau_x g(t)} f(t)\, dt. \end{aligned}$$

2. We prove (6.31) using (6.30), Plancherel's theorem (see Proposition 6.2.7) and the formula (6.23)

$$\begin{aligned} \mathcal{V}_g f(x,\omega) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{(\mathcal{F}_- \left(M_\omega \tau_x g \right)(y)} \mathcal{F}_-(f)(y)\, dy \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \overline{\tau_\omega M_x \mathcal{F}_-(g)(y)} \mathcal{F}_-(f)(y)\, dy. \end{aligned}$$

3. To show (6.32) we observe that since g is a radial function also its Clifford-Fourier transform is radial. Thus by relation (6.8) we have

$$\begin{aligned}\mathcal{V}_{\mathcal{F}_-(g)}\mathcal{F}_-(f)(\omega,x) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} K_-(t,x)\tau_\omega\mathcal{F}_-g(t)\mathcal{F}_-f(t)\,dt \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{K_-(x,t)\tau_\omega\mathcal{F}_-g(t)}\mathcal{F}_-f(t)\,dt \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{M_x\tau_\omega\mathcal{F}_-g(t)}\mathcal{F}_-f(t)\,dt.\end{aligned}$$

Finally, using Plancherel's theorem, Theorem 6.2.10 and the relation (6.23) we obtain

$$\begin{aligned}\mathcal{V}_{\mathcal{F}_-(g)}\mathcal{F}_-(f)(\omega,x) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{\mathcal{F}_-\left(M_x\tau_\omega\mathcal{F}_-g(t)\right)}f(t)\,dt \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{\tau_x M_\omega g(t)}f(t)\,dt. \qquad (6.34)\end{aligned}$$

Since the generalized translation and generalized modulation does not commute (see Remark 6.3.8) we cannot exchange the rules of τ_x and M_ω. To change the order we use the commutator defined in (6.5), so we can relate this formula with the Clifford short-time Fourier transform of f with respect to g by using (6.30):

$$\begin{aligned}\mathcal{V}_{\mathcal{F}_-(g)}\mathcal{F}_-(f)(\omega,x) &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{[\tau_x, M_\omega]g(t)}f(t)\,dt + \\ &\quad +(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{M_\omega\tau_x g(t)}f(t)\,dt \\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} \overline{[\tau_x, M_\omega]g(t)}f(t)\,dt + \mathcal{V}_g f(x,\omega).\end{aligned}$$

4. Finally, formula (6.33) follows from the relations (6.8) and (6.30)

$$\begin{aligned}
\mathcal{F}_-\left(\left(\mathcal{F}_-(f)\cdot\tau_\omega\mathcal{F}_-(g)\right)(x)\right. &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} K_-(y,x)\mathcal{F}_-(f)(y)\\
&\quad\cdot\tau_\omega\mathcal{F}_-(g)(y)\,dy\\
&= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\overline{K_-(x,y)\tau_\omega\mathcal{F}_-(g)(y)}\\
&\quad\cdot\mathcal{F}_-(f)(y)\,dy\\
&= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\overline{M_x\tau_\omega\mathcal{F}_-(g)(y)}\\
&\quad\cdot\mathcal{F}_-(f)(y)\,dy\\
&= \mathcal{V}_{\mathcal{F}_-(g)}\mathcal{F}_-(\omega,x).
\end{aligned}$$

Using formula (6.32) we obtain the thesis.

□

Remark 6.5.3. The formulas proved in Proposition 6.5.2 are similar to the classical case (see [92, Lemma 3.1.1]). The main difference is the presence of the following integral

$$\int_{\mathbb{R}^n}\overline{[\tau_x,M_\omega]g(t)}f(t)\,dt.$$

This is due to the lack of commutativity.

Remark 6.5.4. Another difference with respect to the classical case is that it is not possible to write the Clifford short-Fourier transform as a convolution of the Clifford-Fourier transfrom of the signal and the Clifford-Fourier transform of the window function. For example it is not possible to prove a formula like this

$$\mathcal{V}_g f(x,\omega) = (\overline{M_x\mathcal{F}_-(g)} *_{Cl} \mathcal{F}_-(f))(\omega).$$

Indeed, by the definition of convolution (see Definition 6.3.4) we have

$$(\overline{M_x\mathcal{F}_-(g)} *_{Cl} \mathcal{F}_-(f))(\omega) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\overline{\tau_y M_x\mathcal{F}_-(g)(\omega)\mathcal{F}_-(f)(y)}\,dy.$$

Now, it is not possible to compute $\tau_y M_x\mathcal{F}_-(g)$ using the classic formula of the translation because $M_x\mathcal{F}_-(g)$ is no longer radial and so it is not possible to derive a relation with the Clifford short-time Fourier transform.

Now we study the continuity of the Clifford short-time Fourier transform.

Theorem 6.5.5. *Let $n > 2$ and even. If $g \in W_{2\lambda}(\mathbb{R}^n)$ is a radial function and $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ then the Clifford short-time Fourier transform is a continuous operator on $L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$.*

Proof. We remark that $g \in W_{2\lambda}(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n)$, hence by the formula (6.30) it is enough to prove the continuity of the operators τ_x and M_ω. The continuity of the translation operator follows from the following fact

$$\lim_{x \to 0} \|\tau_x g - g\|_2 = 0.$$

On the other side the continuity of the modulation operator follows from the formula (6.18) and the Parseval's identity (see Proposition 6.2.8)

$$\lim_{\omega \to 0} \|M_\omega g - g\|_2 = \lim_{\omega \to 0} \|\tau_\omega \mathcal{F}_-(g) - \mathcal{F}_-(g)\|_2 = 0.$$

$\square$

Before to state the next result, we have to introduce some notations. We call $f \otimes g$ the tensor product between f and g and it acts in the following way

$$(f \otimes g)(x,t) = f(x)g(t).$$

Let $\mathcal{T}$ be the asymmetric coordinate transform of a function f on $\mathbb{R}^{2n}$

$$\mathcal{T} f(x,t) = f(t, t-x). \tag{6.35}$$

Definition 6.5.6 (Partial Clifford-Fourier transform)**.** Let $f \in \mathcal{S}(\mathbb{R}^{2n}) \otimes \mathbb{R}_n$. We define the partial Clifford-Fourier transform in the following way

$$\mathcal{F}_{2\text{-}} f(x,\omega) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t,\omega) f(x,t)\, dt. \tag{6.36}$$

Remark 6.5.7. The partial Clifford-Fourier transform is the Cliffrod Fourier transform defined in (6.6) with respect to the second variable, hence the variable x is considered as a parameter.

Remark 6.5.8. All the properties which hold for the Clifford-Fourier transform as the Plancherel's theorem and the Parseval's identity hold also for the partial Clifford-Fourier transform.

Now, we show another way to write the Clifford short-time Fourier transform using the tensor product and the partial Clifford-Fourier transform, as in the classical case [92].

Lemma 6.5.9. *If $g \in W_{2\lambda}(\mathbb{R}^n)$ is a radial function and $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ then*

$$\mathcal{V}_g f(x,\omega) = \mathcal{F}_{2\text{-}} \left(\mathcal{T}(f \otimes g)(x,\omega) \right).$$

Proof. From the definition of the partial Clifford-Fourier transform and the definition of $\mathcal{T}$ (see formula (6.35)) we obtain

$$\begin{aligned}\mathcal{F}_{2^-}\left(\mathcal{T}(f\otimes g)(x,\omega)\right) &= \int_{\mathbb{R}^n} K_-(t,\omega)\mathcal{T}\left(f\otimes g\right)(x,t)\,dt \\ &= \int_{\mathbb{R}^n} K_-(t,\omega)\mathcal{T}\left(f(x)g(t)\right)\,dt \\ &= \int_{\mathbb{R}^n} K_-(t,\omega)f(t)g(t-x)\,dt.\end{aligned}$$

Now, using the fact that the window function g is real-valued, thus can commute, and Definition 6.4.2 we get

$$\begin{aligned}\mathcal{F}_{2^-}\left(\mathcal{T}(f\otimes g)(x,\omega)\right) &= \int_{\mathbb{R}^n} K_-(t,\omega)f(t)g(t-x) \\ &= \int_{\mathbb{R}^n} K_-(t,\omega)g(t-x)f(t)\,dt \\ &= \mathcal{V}_g f(x,\omega).\end{aligned}$$

□

Now, we prove a sort of "covariance" property. The main difference from the classical case (see [92, Lemma 3.1.3]) is that we do not have an equality, this is due to the lack of commutativity.

Lemma 6.5.10. *Let $n>2$ and even. We suppose that $g\in W_{2\lambda}(\mathbb{R}^n)$ is a radial function and $f\in L^2(\mathbb{R}^n)\otimes\mathbb{R}_n$, thus for $x,\mu,\omega,\eta\in\mathbb{R}^n$ we have*

$$|\mathcal{V}_g f(x-\mu,\omega-\eta)|\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{\eta}|)^\lambda(1+|\underline{x}|)^\lambda(1+|\underline{\mu}|)^\lambda\|f\|_2\|g\|_{W_{2\lambda}},$$

where c is a positive constant.

Proof. Firstly we use Lemma 6.2.5

$$\begin{aligned}|\mathcal{V}_g f(x-\mu,\omega-\eta)| &\leq c\int_{\mathbb{R}^n}|K_-(t,\omega-\eta)||f(t)g(t-x+\mu)|\,dt \\ &\leq c\int_{\mathbb{R}^n}(1+|\underline{t}|)^\lambda(1+|\underline{\omega}-\underline{\eta}|)^\lambda|f(t)g(t-x+\mu)|\,dt \\ &\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{\eta}|)^\lambda\int_{\mathbb{R}^n}(1+|\underline{t}|)^\lambda|f(t)||g(t-x+\mu)|\,dt,\end{aligned}$$

where c is a positive constant. Now, by Hölder inequality we get

$$|\mathcal{V}_g f(x-\mu,\omega-\eta)|\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{\eta}|)^\lambda\|f\|_2\left(\int_{\mathbb{R}^n}(1+|\underline{t}|)^{2\lambda}|g(t-x+\mu)|^2\,dt\right)^{\frac{1}{2}}.$$

Putting $t - x + \mu = z$ in the integral above we obtain

$$\begin{aligned}
|\mathcal{V}_g f(x-\mu,\omega-\eta)| &\leq c(1+|\underline{\omega}|)^{\lambda}(1+|\underline{\eta}|)^{\lambda}\|f\|_2 \\
&\qquad \left(\int_{\mathbb{R}^n}(1+|\underline{z}+\underline{x}-\underline{\mu}|)^{2\lambda}|g(z)|^2\,dz\right)^{\frac{1}{2}} \\
&\leq c(1+|\underline{\omega}|)^{\lambda}(1+|\underline{\eta}|)^{\lambda}\|f\|_2 \\
&\qquad \cdot\left(\int_{\mathbb{R}^n}(1+|\underline{z}|+|\underline{x}|+|\underline{\mu}|)^{2\lambda}|g(z)|^2\,dz\right)^{\frac{1}{2}} \\
&\leq c(1+|\underline{\omega}|)^{\lambda}(1+|\underline{\eta}|)^{\lambda}(1+|\underline{x}|)^{\lambda}(1+|\underline{\mu}|)^{\lambda}\|f\|_2 \\
&\qquad \cdot\left(\int_{\mathbb{R}^n}(1+|\underline{z}|)^{2\lambda}|g(z)|^2\,dz\right)^{\frac{1}{2}} \\
&= c(1+|\underline{\omega}|)^{\lambda}(1+|\underline{\eta}|)^{\lambda}(1+|\underline{x}|)^{\lambda}(1+|\underline{\mu}|)^{\lambda}\|f\|_2\|g\|_{W_{2\lambda}}.
\end{aligned}$$

□

6.6 Modulation and translation of the signal and of the window function

In this section we discuss what happens if we modulate and translate the signal and the window function, respectively. Surprisingly, there are some differences with respect to the classical Fourier analysis: the estimates depends on the Clifford-Fourier transform of f and the convolution between a function and g.

Proposition 6.6.1. *Let $n > 2$ and even. If $f \in \mathcal{S}(\mathbb{R}^n) \otimes \mathbb{R}_n$ and g is a radial function in $\mathcal{S}(\mathbb{R}^n)$, then for $\theta, \eta, \mu, x, \omega \in \mathbb{R}^n$ we have*

$$\begin{aligned}
|\mathcal{V}_{\tau_\theta g}\tau_\mu M_\eta f(x,\omega)| \;\leq\; & c(1+|\underline{\omega}|)^{\lambda}(1+|\underline{\mu}|)^{\lambda}(1+|\underline{\theta}|)^{2\lambda}\left((1+|.|)^{2\lambda}*|g|\right)(x) \\
& \cdot\int_{\mathbb{R}^n}(1+|.|)^{2\lambda}|\tau_\eta\mathcal{F}_- f(.)|d\cdot
\end{aligned}$$

where . is a fixed variable and c is a positive constant.

Proof. By Definition 6.4.2 we get

$$|\mathcal{V}_{\tau_\theta g}\tau_\mu M_\eta f(x,\omega)| \leq c\int_{\mathbb{R}^n}|K_-(t,\omega)||\tau_\theta g(t-x)||\tau_\mu M_\eta f(t)|\,dt. \quad (6.37)$$

Since by hypothesis the function g is radial to compute the translation of the function $g(t-x)$ we can use the ordinary formula of the translation (see

Proposition 6.3.3), thus

$$\tau_\theta g(t-x) = g(t-x-\theta). \tag{6.38}$$

On the other hand, in order to compute $\tau_\mu M_\eta f(t)$, since we are not translating a radial function, we have to use the formula (6.17) and the relation (6.18)

$$\begin{aligned}\tau_\mu M_\eta f(t) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} K_-(\xi,t) K_-(\mu,\xi) \mathcal{F}_-(M_\eta f)(\xi)\, d\xi \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} K_-(\xi,t) K_-(\mu,\xi) \tau_\eta(\mathcal{F}_- f)(\xi)\, d\xi.\end{aligned}$$

Therefore

$$\tau_\mu M_\eta f(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} K_-(\xi,t) K_-(\mu,\xi) \tau_\eta(\mathcal{F}_- f)(\xi)\, d\xi. \tag{6.39}$$

Putting (6.38) and (6.39) in (6.37) we obtain

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g} \tau_\mu M_\eta f(x,\omega)| &\leq c \int_{\mathbb{R}^{2n}} |K_-(t,\omega)||g(t-x-\theta)||K_-(\xi,t)||K_-(\mu,\xi)| \\ &\quad \cdot |\tau_\eta \mathcal{F}_- f(\xi)|\, d\xi\, dt.\end{aligned}$$

Using the upper bound of the kernel K_- (see Lemma 6.2.5) we get

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g} \tau_\mu M_\eta f(x,\omega)| &\leq c(1+|\underline{\omega}|)^\lambda (1+|\underline{\mu}|)^\lambda \int_{\mathbb{R}^{2n}} (1+|\underline{t}|)^{2\lambda} (1+|\underline{\xi}|)^{2\lambda} |g(t-x-\theta)| \\ &\quad |\tau_\eta \mathcal{F}_- f(\xi)|\, d\xi\, dt.\end{aligned}$$

Now, we put $z = t-\theta$

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g} \tau_\mu M_\eta f(x,\omega)| &\leq c(1+|\underline{\omega}|)^\lambda (1+|\underline{\mu}|)^\lambda \int_{\mathbb{R}^{2n}} (1+|\underline{z}+\underline{\theta}|)^{2\lambda} (1+|\underline{\xi}|)^{2\lambda} |g(z-x)| \\ &\quad \cdot |\tau_\eta \mathcal{F}_- f(\xi)| d\xi dz \\ &\leq c(1+|\underline{\omega}|)^\lambda (1+|\underline{\mu}|)^\lambda (1+|\underline{\theta}|)^{2\lambda} \int_{\mathbb{R}^{2n}} (1+|\underline{z}|)^{2\lambda} |g(z-x)| \cdot \\ &\quad \cdot (1+|\underline{\xi}|)^{2\lambda} |\tau_\eta \mathcal{F}_- f(\xi)|\, d\xi\, dz.\end{aligned}$$

Now, since g is radial we have the following equality

$$g(z-x) = g(x-z) = \tau_z g(x). \tag{6.40}$$

By Fubini' theorem and the definition of convolution we get

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g}\tau_\mu M_\eta f(x,\omega)| &\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{\mu}|)^\lambda(1+|\underline{\theta}|)^{2\lambda}\left((1+|z|)^{2\lambda}*|g|\right)(x)\\ &\quad\cdot\int_{\mathbb{R}^n}(1+|\underline{\xi}|)^{2\lambda}|\tau_\eta\mathcal{F}_-f(\xi)|\,d\xi.\end{aligned}$$

□

We observed that in Clifford-Fourier analysis there is not a commutative relations between the modulation and the translation operator (see Remark 6.3.8). This is confirmed by the following estimate where we exchange the order of the modulation and translation in the signal.

Proposition 6.6.2. *Let $n > 2$ and even. If $f \in \mathcal{S}(\mathbb{R}^n)\otimes\mathbb{R}_n$ and g is a radial function in $\mathcal{S}(\mathbb{R}^n)$, then for $\theta,\eta,\mu,x,\omega\in\mathbb{R}^n$ we have*

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g}M_\eta\tau_\mu f(x,\omega)| &\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{\eta}|)^\lambda(1+|\underline{\mu}|)^\lambda(1+|\underline{\theta}|)^{3\lambda}\\ &\quad\big((1+|.|)^{3\lambda}*|g|\big)(x)\int_{\mathbb{R}^n}(1+|.|)^{2\lambda}|\mathcal{F}_-f(.)|d\cdot\end{aligned}$$

where . is a fixed variable and c is a positive constant.

Proof. By Definition 6.4.2 and the radiality of g we get

$$|\mathcal{V}_{\tau_\theta g}M_\eta\tau_\mu f(x,\omega)| \leq c\int_{\mathbb{R}^n}|K_-(t,\omega)||g(t-x-\theta)||M_\eta\tau_\mu f(t)|\,dt. \quad (6.41)$$

To translate the function f we need the formula (6.17) (because f is not radial), thus

$$\begin{aligned}M_\eta\tau_\mu f(t) &= K_-(\eta,t)\tau_\mu f(t)\\ &= K_-(\eta,t)\int_{\mathbb{R}^n}K_-(\xi,t)K_-(\mu,\xi)\mathcal{F}_-f(\xi)\,d\xi. \quad (6.42)\end{aligned}$$

Putting (6.42) in (6.41) we obtain

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g}M_\eta\tau_\mu f(x,\omega)| &\leq c\int_{\mathbb{R}^{2n}}|K_-(t,\omega)||g(t-x-\theta)||K_-(\eta,t)|\\ &\quad\cdot|K_-(\xi,t)||K_-(\mu,\xi)||\mathcal{F}_-f(\xi)|\,d\xi dt.\end{aligned}$$

Now by the upper bound of the kernel (see Lemma 6.2.5) we get

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g}M_\eta\tau_\mu f(x,\omega)| &\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{\eta}|)^\lambda(1+|\underline{\mu}|)^\lambda\int_{\mathbb{R}^{2n}}(1+|\underline{t}|)^{3\lambda}(1+|\underline{\xi}|)^{2\lambda}\\ &\quad\cdot|g(t-x-\theta)||\mathcal{F}_-f(\xi)|\,d\xi dt.\end{aligned}$$

Putting $z = t - \theta$, using the equality (6.40) and the Fubini's theorem we obtain

$$\begin{aligned}|\mathcal{V}_{\tau_\theta g} M_\eta \tau_\mu f(x,\omega)| &\leq c(1+|\underline{\omega}|)^\lambda (1+|\underline{\eta}|)^\lambda (1+|\underline{\mu}|)^\lambda \int_{\mathbb{R}^{2n}} (1+|\underline{z}+\underline{\theta}|)^{3\lambda} \\ &\quad \cdot (1+|\underline{\xi}|)^{2\lambda} |g(z-x)||\mathcal{F}_- f(\xi)|\, d\xi dz \\ &= c(1+|\underline{\omega}|)^\lambda (1+|\underline{\eta}|)^\lambda (1+|\underline{\mu}|)^\lambda (1+|\underline{\theta}|)^{3\lambda} \\ &\quad \cdot \left((1+|z|)^{3\lambda} * |g|\right)(x) \int_{\mathbb{R}^n} (1+|\xi|)^{2\lambda} |\mathcal{F}_- f(\xi)|\, d\xi.\end{aligned}$$

□

Remark 6.6.3. One can wonder if it is possible to have an estimate for

$$|\mathcal{V}_{M_q \tau_\theta g} M_\eta \tau_\mu f(x,\omega)|, \qquad q,\theta,\eta,\mu,x,\omega \in \mathbb{R}^n. \tag{6.43}$$

Basically with respect to Proposition 6.6.2 we make the modulation of the window function. However, it is not possible to have an estimate for (6.43). From Definition 6.4.2 we have

$$|\mathcal{V}_{M_q \tau_\theta g} M_\eta \tau_\mu f(x,\omega)| \leq c \int_{\mathbb{R}^n} |K_-(t,\omega)||\tau_x M_q \tau_\theta g(t)||M_\eta \tau_\mu f(t)|\, dt. \tag{6.44}$$

It is not possible to use the ordinary formula of translation for computing $\tau_x M_q \tau_\theta g$, because we are not translating a radial function. Thus by formula (6.17) and the property (6.23) we obtain

$$\begin{aligned}\tau_x M_q \tau_\theta g(t) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} K_-(z,t) K_-(x,z) \mathcal{F}_-(M_q \tau_\theta g)(z)\, dz \\ &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} K_-(z,t) K_-(x,z) \tau_q M_\theta \mathcal{F}_-(g)(z)\, dz.\end{aligned}$$

Therefore

$$\tau_x M_q \tau_\theta g(t) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^n} K_-(z,t) K_-(x,z) \tau_q M_\theta \mathcal{F}_-(g)(z)\, dz. \tag{6.45}$$

Putting (6.42) and (6.45) in (6.44) we get

$$\begin{aligned}|\mathcal{V}_{M_q \tau_\theta g} M_\eta \tau_\mu f(x,\omega)| &\leq c \int_{\mathbb{R}^{3d}} |K_-(t,\omega)||K_-(z,t)||K_-(x,z)||K_-(\eta,t)| \\ &\quad \cdot |K_-(\xi,t)||K_-(\mu,\xi)||\tau_q M_\theta \mathcal{F}_-(g)(z)||\mathcal{F}_- f(\xi)|\, dz d\xi dt.\end{aligned}$$

Using the upper bound of the kernel (see Lemma (6.2.5)) and the Fubini's theorem we obtain

$$\begin{aligned}|\mathcal{V}_{M_q\tau_\theta g}M_\eta\tau_\mu f(x,\omega)| \quad \leq \quad & c(1+|\underline{\omega}|)^\lambda(1+|\underline{x}|)^\lambda(1+|\underline{\eta}|)^\lambda(1+|\underline{\mu}|)^\lambda \cdot \\ & \int_{\mathbb{R}}(1+|\underline{t}|)^{4\lambda}\,dt \cdot \int_{\mathbb{R}^n}(1+|\underline{z}|)^{2\lambda}|\tau_q M_\theta \mathcal{F}_-(g)(z)|\,dz \\ & \cdot \int_{\mathbb{R}^n}(1+|\underline{\xi}|)^{2\lambda}|\mathcal{F}_-(f)(\xi)|\,d\xi.\end{aligned}$$

Since $\lambda = \frac{d-2}{2}$ and $n > 2$ is even we have that the integral $\int_{\mathbb{R}}(1+|\underline{t}|)^{4\lambda}\,dt$ is not convergent. Therefore it is not possible to have an estimate for $|\mathcal{V}_{M_q\tau_\theta g}M_\eta\tau_\mu f(x,\omega)|$.

A similar reasoning proves that we cannot obtain estimates also for $|\mathcal{V}_{\tau_\theta M_q g}M_\eta\tau_\mu f(x,\omega)|$, $|\mathcal{V}_{\tau_\theta M_q g}\tau_\mu M_\eta f(x,\omega)|$, $|\mathcal{V}_{M_q\tau_\theta g}\tau_\mu M_\eta f(x,\omega)|$.

Remark 6.6.4. It is not possible to repeat the same estimates of Proposition 6.6.1 and Proposition 6.6.2 using as a window function the modulation of g. Below, we perform the computation to show where the problem arises. Let $\theta, \mu, \eta, x, \omega \in \mathbb{R}^n$, we want to make an estimate of $|\mathcal{V}_{M_\theta g}\tau_\mu M_\eta f(x,\omega)|$. From the Definition 6.4.2 we have

$$|\mathcal{V}_{M_\theta g}\tau_\mu M_\eta f(x,\omega)| \leq c\int_{\mathbb{R}^n}|K_-(t,\omega)||\tau_x M_\theta g(t)||\tau_\mu M_\eta f(t)|\,dt. \tag{6.46}$$

Since $M_\theta g$ is no longer radial we have to use the formula (6.17) for computing the translation. Thus by formula (6.18) we get

$$\begin{aligned}\tau_x M_\theta g(t) &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^n}K_-(z,t)K_-(x,z)\mathcal{F}_-(M_\theta g)(z)\,dz \\ &= (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^n}K_-(z,t)K_-(x,z)\tau_\theta\mathcal{F}_-(g)(z)\,dz.\end{aligned}$$

Therefore

$$\tau_x M_\theta g(t) = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^n}K_-(z,t)K_-(x,z)\tau_\theta\mathcal{F}_-(g)(z)\,dz. \tag{6.47}$$

Putting (6.47) and (6.39) in (6.46) and using the upper bound of the kernel we obtain

$$\begin{aligned}|\mathcal{V}_{M_\theta g}\tau_\mu M_\eta f(x,\omega)| &\leq c\int_{\mathbb{R}^n}|K_-(t,\omega)||K_-(z,t)||K_-(x,z)||\tau_\theta\mathcal{F}_-(g)(z)|\cdot\\ &\quad\cdot|K_-(\xi,t)||K_-(\mu,\xi)||\tau_\eta\mathcal{F}_-(f)(\xi)|\,dzd\xi dt\\ &\leq c(1+|\underline{\omega}|)^\lambda(1+|\underline{x}|)^\lambda(1+|\underline{\mu}|)^\lambda\int_{\mathbb{R}^n}(1+|\underline{t}|)^{3\lambda}dt\cdot\\ &\quad\cdot\int_{\mathbb{R}^n}(1+|\underline{z}|)^{2\lambda}|\tau_\theta\mathcal{F}_-(g)(z)|dz\int_{\mathbb{R}^n}(1+|\underline{\xi}|)^{2\lambda}\\ &\quad|\tau_\eta\mathcal{F}_-(f)(\xi)|d\xi.\end{aligned}$$

As in the previous remark the integral $\int_{\mathbb{R}^n}(1+|\underline{t}|)^{3\lambda}dt$ does not converge, so it not possible to have an estimate. The same considerations hold also for $|\mathcal{V}_{M_\theta g}M_\eta\tau_\mu f(x,\omega)|$.

6.7 Further properties of the Clifford short-time Fourier transform

There are some properties for the Clifford short-time Fourier transform which hold also for the complex case and quaternionic case (see [111]).

Theorem 6.7.1 (Orthogonality relation)**.** *Let $n>2$ and even. Let $g_1, g_2\in\mathcal{S}(\mathbb{R}^n)$ be radial functions and $f_1,f_2\in L^2(\mathbb{R}^n)\otimes\mathbb{R}_n$. Then we have*

$$\int_{\mathbb{R}^{2n}}\overline{\mathcal{V}_{g_1}f_1(x,\omega)}\mathcal{V}_{g_2}f_2(x,\omega)\,d\omega\,dx=\langle f_1,f_2\rangle\left(\int_{\mathbb{R}^n}g_1(z)g_2(z)\,dz\right). \tag{6.48}$$

Proof. From Definition 6.4.2 and the Plancherel's theorem (see Proposition 6.2.7) we have

$$\begin{aligned}\int_{\mathbb{R}^{2n}}\overline{\mathcal{V}_{g_1}f_1(x,\omega)}\mathcal{V}_{g_2}f_2(x,\omega)\,d\omega\,dx &= \int_{\mathbb{R}^{2n}}\overline{\mathcal{F}_-(\tau_x\bar{g}_1\cdot f_1)(\omega)}\mathcal{F}_-(\tau_x\bar{g}_2\cdot f_2)(\omega)\\ &\quad d\omega\,dx\\ &= \int_{\mathbb{R}^{2n}}\overline{(\tau_x\bar{g}_1\cdot f_1)}(t)(\tau_x\bar{g}_2\cdot f_2)(t)\,dt\,dx.\end{aligned}$$

Now, since g_1 and g_2 are real valued functions we can omit the conjugate. Moreover, the window functions g_1, g_2 can commute. Thus

$$\int_{\mathbb{R}^{2n}}\overline{\mathcal{V}_{g_1}f_1(x,\omega)}\mathcal{V}_{g_2}f_2(x,\omega)\,d\omega\,dx=\int_{\mathbb{R}^{2n}}\overline{f_1}(t)f_2(t)g_1(t-x)g_2(t-x)\,dt\,dx.$$

By hypothesis we can use Fubini's theorem for changing the order of integration

$$\int_{\mathbb{R}^{2n}} \overline{\mathcal{V}_{g_1} f_1(x,\omega)} \mathcal{V}_{g_2} f_2(x,\omega)\, d\omega\, dx = \int_{\mathbb{R}^n} \overline{f_1}(t) f_2(t) \left(\int_{\mathbb{R}^n} g_1(t-x) g_2(t-x)\, dx \right) dt.$$

Finally, by a change of variable we get

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \overline{\mathcal{V}_{g_1} f_1(x,\omega)} \mathcal{V}_{g_2} f_2(x,\omega)\, d\omega\, dx &= \int_{\mathbb{R}^n} \overline{f_1}(t) f_2(t) \left(\int_{\mathbb{R}^n} g_1(z) g_2(z)\, dz \right) dt \\ &= \langle f_1, f_2 \rangle \left(\int_{\mathbb{R}^n} g_1(z) g_2(z)\, dz \right).\end{aligned}$$

□

Remark 6.7.2. We can prove the above theorem also using Lemma 6.5.9, and this proof may be of interest in some other contexts. Supposing the same hypothesis of Theorem 6.7.1, by Plancherel's theorem and the fact that g_1 and g_2 are real valued, then we have

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \overline{\mathcal{V}_{g_1} f_1(x,\omega)} \mathcal{V}_{g_2} f_2(x,\omega)\, d\omega\, dx &= \int_{\mathbb{R}^{2n}} \overline{\mathcal{F}_2\text{-}\mathcal{T}(f_1 \otimes g_1)(x,\omega)} \\ &\quad \cdot \mathcal{F}_2\text{-}\mathcal{T}(f_2 \otimes g_2)(x,\omega)\, d\omega\, dx \\ &= \int_{\mathbb{R}^{2n}} \overline{\mathcal{T}(f_1 \otimes g_1)(x,t)} \\ &\quad \cdot \mathcal{T}(f_2 \otimes g_2)(x,t)\, dt\, dx \\ &= \int_{\mathbb{R}^{2n}} \overline{f_1}(t) f_2(t) g_1(t-x) g_2(t-x)\, dt\, dx.\end{aligned}$$

Using the same arguments of Theorem 6.7.1 we obtain the equality (6.48).

Corollary 6.7.3. *Let $n > 2$ and even. If $g \in \mathcal{S}(\mathbb{R}^n)$ is a radial function and $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ then*

$$\|\mathcal{V}_g f(x,\omega)\|_2^2 = \|f\|_2^2 \int_{\mathbb{R}^n} g^2(z)\, dz.$$

Proof. If we put $f_1 = f_2 := f$ and $g_1 = g_2 := g$ in the equality (6.48) we obtain the thesis. □

Theorem 6.7.4 (Reconstruction formula)**.** *Let us assume that* $f \in L^2(\mathbb{R}^n)\otimes \mathbb{R}_n$, $g \in \mathcal{S}(\mathbb{R}^n)$ *is a radial function and* $\int_{\mathbb{R}^n} g^2(z)\,dz \neq 0$. *Then for all* $f \in L^2(\mathbb{R}^n)\otimes\mathbb{R}_n$ *we have*

$$f(y) = \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\int_{\mathbb{R}^{2n}} M_\omega \tau_x g(y)\mathcal{V}_g f(x,\omega)\,d\omega\,dx.$$

Proof. Let us assume

$$\widetilde{f}(y) := \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\int_{\mathbb{R}^{2n}} M_\omega \tau_x g(y)\mathcal{V}_g f(x,\omega)\,d\omega\,dx.$$

Let $h \in L^2(\mathbb{R}^n)\otimes\mathbb{R}_n$. By the equality (6.30), the orthogonality relation (see Theorem 6.7.1) and Fubini's theorem we obtain

$$\begin{aligned}
\langle \tilde{f}, h\rangle &= \int_{\mathbb{R}^n} \overline{\widetilde{f}(y)} h(y)\,dy = \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\left(\int_{\mathbb{R}^{3d}} \overline{M_\omega\tau_x g(y)\mathcal{V}_g f(x,\omega)}\,d\omega\,dx\right) \\
&\quad h(y)\,dy \\
&= \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\int_{\mathbb{R}^{3d}} \overline{M_\omega\tau_x g(y)\mathcal{V}_g f(x,\omega)} h(y)\,d\omega\,dx\,dy \\
&= \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\int_{\mathbb{R}^{3d}} \overline{\mathcal{V}_g f(x,\omega)}\;\overline{M_\omega\tau_x g(y)}\,h(y)\,dy\,d\omega\,dx \\
&= \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\int_{\mathbb{R}^{2n}} \overline{\mathcal{V}_g f(x,\omega)}\left(\int_{\mathbb{R}^n} \overline{M_\omega\tau_x g(y)}\,h(y)\,dy\right)\,d\omega\,dx \\
&= \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\int_{\mathbb{R}^{2n}} \overline{\mathcal{V}_g f(x,\omega)}\mathcal{V}_g h(x,\omega)\,d\omega\,dx \\
&= \frac{1}{\int_{\mathbb{R}^n} g^2(z)\,dz}\langle f,h\rangle\int_{\mathbb{R}^n} g^2(z)\,dz = \langle f,h\rangle.
\end{aligned}$$

We proved that for all $h \in L^2(\mathbb{R}^n)\otimes\mathbb{R}_n$

$$\langle \widetilde{f}, h\rangle = \langle f, h\rangle.$$

This implies $\widetilde{f}(y) = f(y)$. □

The inversion formula gives us the possibility to write the Clifford short-time Fourier transform using the reproducing kernel associated to the Clifford Gabor space, introduced in [4], defined by

$$\mathcal{G}^g_{\mathbb{R}_n} := \{\mathcal{V}_g f, \quad f \in L^2(\mathbb{R}^n)\otimes\mathbb{R}_n\},$$

where g is a radial window function.

Theorem 6.7.5 (Reproducing kernel)**.** *Let $n > 2$ and even. If $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and g is a radial function in $\mathcal{S}(\mathbb{R}^n)$, we have that*

$$\mathbb{K}_g(\omega, x; \omega', x') = \frac{1}{(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^n} g^2(z)\, dz} \int_{\mathbb{R}^n} K_-(t, \omega') g(t-x') \overline{K_-(t, \omega) g(t-x)}\, dt$$

is the reproducing kernel of the space $\mathcal{G}^g_{\mathbb{R}_n}$, i.e

$$\mathcal{V}_g f(x', \omega') = \int_{\mathbb{R}^{2n}} \mathbb{K}_g(\omega, x; \omega', x') \mathcal{V}_g f(x, \omega)\, d\omega\, dx.$$

Proof. By the reconstruction formula (see Theorem 6.7.4), Fubini's theorem and since g is a real valued function we have

$$\begin{aligned}
\mathcal{V}_g f(x', \omega') &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t, \omega') g(t - x') f(t)\, dt \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} K_-(t, \omega') g(t - x') \frac{1}{\int_{\mathbb{R}^n} g^2(z)\, dz} \\
&\quad \left(\int_{\mathbb{R}^{2n}} M_\omega \tau_x g(t) \mathcal{V}_g f(x, \omega)\, d\omega\, dx \right) dt \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{3d}} K_-(t, \omega') g(t - x') \frac{1}{\int_{\mathbb{R}^n} g^2(z)\, dz} \\
&\quad M_\omega \tau_x g(t) \mathcal{V}_g f(x, \omega) d\omega dx dt \\
&= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{3d}} K_-(t, \omega') g(t - x') \frac{1}{\int_{\mathbb{R}^n} g^2(z)\, dz} K_-(\omega, t) \\
&\quad g(t - x) \mathcal{V}_g f(x, \omega)\, dt\, d\omega\, dx \\
&= \int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^n} K_-(t, \omega') g(t - x') \frac{1}{(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^n} g^2(z)\, dz} K_-(\omega, t) g(t - x) dt \right) \\
&\quad \mathcal{V}_g f(x, \omega) d\omega dx \\
&= \int_{\mathbb{R}^{2n}} \mathbb{K}_g(\omega, x; \omega', x') \mathcal{V}_g f(x, \omega)\, d\omega\, dx.
\end{aligned}$$

□

For this reproducing kernel it is possible to have the following bound.

Corollary 6.7.6. *Let $n > 2$ and even. Let us assume that $f \in L^2(\mathbb{R}^n) \otimes \mathbb{R}_n$ and $g \in \mathcal{S}(\mathbb{R}^n)$ is a radial function, then we have that*

$$|\mathbb{K}_g(\omega, x; \omega', x')| \leq C(1 + |\underline{\omega}|)^\lambda (1 + |\underline{\omega'}|)^\lambda (1 + |\underline{x}|)^{2\lambda} (1 + |\underline{x'}|)^{2\lambda},$$

where C is a constant.

Proof. By the relation (6.8) and the fact that g is a real valued function we have that

$$\mathbb{K}_g(\omega, x; \omega', x') = \frac{1}{(2\pi)^{\frac{d}{2}} \int_{\mathbb{R}^n} g^2(z)\, dz} \int_{\mathbb{R}^n} K_-(t,\omega')K_-(\omega,t)g(t-x')g(t-x)\, dt.$$

Using the upper bound of the kernel K_- (see Lemma 6.2.5) we get

$$\begin{aligned}
|\mathbb{K}_g(\omega, x; \omega', x')| &\leq \frac{c}{(2\pi)^{\frac{d}{2}} \left|\int_{\mathbb{R}^n} g^2(z)\, dz\right|} \int_{\mathbb{R}^n} |K_-(t,\omega')||K_-(\omega,t)||g(t-x')| \\
&\quad |g(t-x)|\, dt \\
&\leq \frac{c}{(2\pi)^{\frac{d}{2}} \left|\int_{\mathbb{R}^n} g^2(z)\, dz\right|} (1+|\underline{\omega}|)^{\lambda}(1+|\underline{\omega'}|)^{\lambda} \\
&\quad \cdot \int_{\mathbb{R}^n} (1+|\underline{t}|)^{2\lambda} |g(t-x')||g(t-x)|\, dt,
\end{aligned}$$

where c is a positive constant. If we put $t = z + x + x'$ we get

$$\begin{aligned}
|\mathbb{K}_g(\omega, x; \omega', x')| &\leq \frac{c}{(2\pi)^{\frac{d}{2}} \left|\int_{\mathbb{R}^n} g^2(z)\, dz\right|} (1+|\underline{\omega}|)^{\lambda}(1+|\underline{\omega'}|)^{\lambda} \cdot \\
&\quad \cdot \int_{\mathbb{R}^n} (1+|\underline{z}|+|\underline{x}|+|\underline{x'}|)^{2\lambda} |g(z+x)||g(z+x')|\, dz \\
&\leq \frac{c}{(2\pi)^{\frac{d}{2}} \left|\int_{\mathbb{R}^n} g^2(z)\, dz\right|} (1+|\underline{\omega}|)^{\lambda}(1+|\underline{\omega'}|)^{\lambda}(1+|\underline{x}|)^{2\lambda} \\
&\quad \cdot (1+|\underline{x'}|)^{2\lambda} \int_{\mathbb{R}^n} (1+|\underline{z}|)^{2\lambda} |g(z+x)||g(z+x')|\, dz.
\end{aligned}$$

We denote as $C := \frac{c \int_{\mathbb{R}^n} (1+|\underline{z}|)^{2\lambda} |g(z+x)||g(z+x')|\, dz}{(2\pi)^{\frac{d}{2}} \left|\int_{\mathbb{R}^n} g^2(z)\, dz\right|}$. Thus we get

$$|\mathbb{K}_g(\omega, x; \omega', x')| \leq C(1+|\underline{\omega}|)^{\lambda}(1+|\underline{\omega'}|)^{\lambda}(1+|\underline{x}|)^{2\lambda}(1+|\underline{x'}|)^{2\lambda}.$$

□

6.8 Lieb's Uncertainty principle

In this section we want to extend the Lieb's Uncertainty Principle [110] in the Clifford algebra setting. Firstly let us recall that in general the uncertainty principles state that a function and its Fourier transform cannot be simultaneously sharply localized.

Before to prove a weak uncertainty principle for the Clifford short-time Fourier transform we go through the following important estimate.

Proposition 6.8.1. *Let $n > 2$ and even. If g is a radial function in $W_{p\lambda}(\mathbb{R}^n)$, with $p \geq 1$, then*

$$\|M_\omega \tau_x g\|_p \leq c(1 + |\underline{\omega}|)^\lambda (1 + |\underline{x}|)^\lambda \|g\|_{W_{p\lambda}},$$

where c is a positive constant.

Proof. Since g is a radial function by Proposition 6.3.3 we get

$$\|M_\omega \tau_x g\|_p^p = \int_{\mathbb{R}^n} |K_-(\omega, t) g(t - x)|^p \, dt.$$

Now, if we put $s = t - x$ by the upper bound of Lemma 6.2.5 we obtain

$$\begin{aligned}
\|M_\omega \tau_x g\|_p^p &\leq c \int_{\mathbb{R}^n} |K_-(\omega, s + x)|^p |g(s)|^p \, ds \\
&\leq c \int_{\mathbb{R}^n} (1 + |\underline{\omega}|)^{\lambda p} (1 + |\underline{s} + \underline{x}|)^{\lambda p} |g(s)|^p \, ds \\
&\leq c(1 + |\underline{\omega}|)^{\lambda p} (1 + |\underline{x}|)^{\lambda p} \int_{\mathbb{R}^n} (1 + |\underline{s}|)^{\lambda p} |g(s)|^p \, ds \\
&= c(1 + |\underline{\omega}|)^{\lambda p} (1 + |\underline{x}|)^{\lambda p} \|g\|_{W_{p\lambda}}^p.
\end{aligned}$$

□

Remark 6.8.2. This result it is very different from the classical result of the Fourier analysis, in which we have $\|M_\omega \tau_x f\|_p = \|f\|_p$.

Proposition 6.8.3 (Weak uncertainty principle)**.** *Let $n > 2$ and even. Suppose that $\|f\|_2 = \|g\|_{W_{2\lambda}} = 1$, with g a radial function. Let $U \subseteq \mathbb{R}^{2n}$ be an open set and $\varepsilon \geq 0$ such that*

$$\int\int_U (1 + |\underline{x}|)^{-2\lambda} (1 + |\underline{\omega}|)^{-2\lambda} |\mathcal{V}_g f(x, \omega)|^2 \, dx \, d\omega \geq 1 - \varepsilon. \tag{6.49}$$

Then $|U| \geq (1 - \varepsilon)\frac{1}{c}$, where c is positive constant different from zero.

Proof. Formula (6.30), Hölder inequality and Proposition 6.8.1 imply that

$$\begin{aligned}
|\mathcal{V}_g f(x, \omega)| &= \int_{\mathbb{R}^n} |\overline{M_\omega \tau_x g(t)}| |f(t)| \, dt \leq \|M_\omega \tau_x g\|_2 \|f\|_2 \\
&\leq c(1 + |\underline{x}|)^\lambda (1 + |\underline{\omega}|)^\lambda \|g\|_{W_{2\lambda}} \|f\|_2 = c(1 + |\underline{x}|)^\lambda (1 + |\underline{\omega}|)^\lambda.
\end{aligned}$$

Therefore, using this calculus we get

$$\begin{aligned} 1-\varepsilon &\leq \int\int_U (1+|\underline{x}|)^{-2\lambda}(1+|\underline{\omega}|)^{-2\lambda}|\mathcal{V}_g f(x,\omega)|^2\,dx\,d\omega \\ &\leq c\int\int_U (1+|\underline{x}|)^{-2\lambda}(1+|\underline{\omega}|)^{-2\lambda}(1+|\underline{x}|)^{2\lambda}(1+|\underline{\omega}|)^{2\lambda}\,dx\,d\omega \\ &= c|U|. \end{aligned}$$

hence $|U| \geq (1-\varepsilon)\frac{1}{c}$. □

Remark 6.8.4. The Lieb's uncertainty principle in this context is quite different from the classical one, since we have the presence of polynomials in (6.49), but these are crucial for the convergence of the integrals.

Remark 6.8.5. In order to improve the above proposition, in the classical Fourier analysis, Lieb [110] estimated the L^p- norm of the short-time Fourier transform using the Hausdorff-Young's inequality. However, as remarked in [109], it is not possible to extend this inequality in the Clifford algebra setting since we cannot use the Riesz interpolation theorem.

Part II: Functional calculi based on the S-spectrum and the Fueter-Sce theorem

In this second part of the dissertation we present new functional calculi based on the S-spectrum. For our construction it is essential the Fueter-Sce theorem.

In chapter 7, we give an overview of the proof of the Fueter-Sce theorem done by M. Sce in a very general and pioneering way. Then we recall from [54] that even the Fueter-Sce map can be considered as an integral transform, that maps slice hyperholomorphic functions into monogenic functions. This result is achieved by applying the Fueter-Sce map, namely a suitable integer power of the Laplace operator in $n + 1$ variables, to the slice hyperholomorphic Cauchy kernel.

In Chapter 8 we compute the Fourier transform of the slice Cauchy kernel. This is the correct tool to compute the Fueter-Sce-Qian map applied to the slice hyperholomorphic Cauchy kernel. The Fueter-Sce-Qian map is the fractional version of the Laplace operator, therefore we have to deal with Fourier multipliers.

Based on the integral representation of the Fueter-Sce theorem, in Chapter 9 we provide a new functional calculus: the F-functional calculus. This a monogenic functional calculus in the same spirit of McIntosh and collaborators but it is based on the commutative version of the S-spectrum. Therefore the F-functional calculus can be considered as a bridge between the spectral theory on the S-spectrum and the monogenic spectral theory.

In Chapter 10 and Chapter 11 we provide the definitions and the main properties of the harmonic and polyanalytic functional calculi based on the S-spectrum. These are based on integral representations of axially harmonic and polyanalytic functions.

CHAPTER 7

Fueter-Sce theorem

7.1 Motivation

Holomorphic functions of one complex variable $f : \Omega \subseteq \mathbb{C} \to \mathbb{C}$ can be extended to quaternionic-valued functions or, more in general, to Clifford algebra-valued functions using the Fueter-Sce-Qian extension theorem. This theorem is due to R.Fueter [85] for the quaternionic setting, it was generalized by M.Sce [126], to Clifford algebra $\mathbb{R}_n$ for n odd while the case of even dimension was proved by T.Qian in [122] (see also the recent monograph [125]). The method of T.Qian requires the use of the Fourier transform in the space of distributions and is deeply different from the method of R. Fueter and M. Sce.

This extension theorem is also called the Fueter-Sce-Qian construction and gives two different notions of hyperholomorphic functions. Consider functions defined on an open set U in the quaternions $\mathbb{H}$ or in $\mathbb{R}^{n+1}$ for Clifford algebra-valued functions, then the Fueter-Sce-Qian extension consists of two steps.

Step (A) extends holomorphic functions to the class of slice hyperholomorphic functions. These functions are also called slice monogenic for Clifford algebra-valued functions and slice regular in the quaternionic case.

See Chapter 3.

Step (B) extends slice hyperholomorphic functions to monogenic functions or Fueter regular functions in the case of the quaternions. The theory of monogenic functions is widely studied and the literature is rich, see e.g. the books [28, 55, 67, 94] and references therein.

Both the classes of hyperholomorphic functions have a Cauchy formula that can be used to define functions of quaternionic operators or of n-tuples of operators.

The Cauchy formula of slice hyperholomorphic functions generates the S-functional calculus for quaternionic linear operators or for n-tuples of not necessarily commuting operators. This calculus is based on the notion of S-spectrum, see Chapter 3.

Step (A) has generated the following research directions: the foundation of the quaternionic spectral theory on the S-spectrum, see the books [44, 45] and, for paravector operators, [59]; quaternionic evolution operators; Phillips functional calculus; H^∞-functional calculus, see [44]; the characteristic operator functions and applications to linear system theory [14]; quaternionic perturbation theory and invariant subspaces [33]; Schur analysis in this setting [13]. For some new classes of fractional diffusion problems based on fractional powers of quaternionic linear operators, see the book [44] and the more recent contributions [39, 40, 48].

Step (B) generates the monogenic functional calculus based on the monogenic spectrum. Some of the research directions in this area are: monogenic spectral theory and applications [99]; singular integrals and Fourier transform, see the recent book [125].

The first mathematicians who understood the importance of hypercomplex analysis to define functions of noncommuting operators on Banach spaces have been A. McIntosh and his collaborators. Using the theory of monogenic functions they developed the monogenic functional calculus and several of its applications, see [99, 101, 108, 112].

In this chapter we give a detailed proof of the Fueter-Sce theorem in a very general setting, see [57]. Moreover, we show that it is possible to write the Fueter-Sce mapping theorem in integral form. To prove this we follow the arguments of [54].

7.2 Futer-Sce theorem in quadratic algebras

In this section we recall the proof of the Fueter-Sce theorem, see [57, 126]. The proof was made in a very general setting. We consider modules with units which are quadratic. This means that each element of the quadratic modules satisfies a quadratic equation with respect to the multiplicative structure induced in the module by their tensor algebra. This quadratic modules generalizes the quaternionic and Clifford algebras.

Let M be a module on a field $\mathbf{F}$ with characteristic not equal 2 and let $1 = i_0, i_1, \dots, i_n$ be a basis. After identifying the unit of $\mathbf{F}$ with the unit of M, we can write the elements in M in the form

$$x = x_0 + x_1 i_1 + \cdots + x_n i_n = x_0 + \mathbf{x} \qquad (x_i \in \mathbf{F}).$$

Let $\mathbf{T}$ be the tensor algebra over M and let us assume that for the elements x^2 in $\mathbf{T}$ one has

$$\mathbf{x}^2 = q(\mathbf{x}) = \sum_{j,k=1}^{n} a_{jk} x_j x_k, \tag{7.1}$$

where $q(\mathbf{x})$ denotes a quadratic form on $\mathbf{F}$; it follows that x^2 (which belongs to $\mathbf{T}$) is in M.

We observe that M is closed with respect to the operation that to the pair x, y associates $\dfrac{xy + yx}{2}$. This gives a Jordan algebra M^+.

Definition 7.2.1 (Quadratic module)**.** If we consider the module M in $\mathbf{T}$ equipped with the multiplicative structure of M^+, we will say that M is a quadratic module and we denote it by M_q.

Remark 7.2.2. By reducing $q(\mathbf{x})$ to a canonical form, it is possible to note that M^+ is a Jordan algebra, central, simple, of degree 2, then M_q can be embedded only in algebras $\mathbf{A}$ such that $\mathbf{A}^+$ contains such a Jordan algebra. Among these algebras we find those obtainable with the Cayley-Dickson process; these algebras are themselves quadratic modules.

Remark 7.2.3. If, in addition, $\mathbf{A} \supset \mathrm{M}_q$ is associative, it contains the algebra quotient of $\mathbf{T}$ and of the ideal generated by (7.1); thus the smallest associative algebra containing a quadratic module is a Clifford algebra.

Now, we give the definition of conjugate element of $x \in \mathrm{M}_q$.

Definition 7.2.4. We call conjugate of an element $x = x_0 + \mathbf{x}$ in M_q the element $\bar{x} = x_0 - \mathbf{x}$.

It is immediate that

$$x + \bar{x} = 2x_0 = t(x) \qquad (\text{trace of } x)$$

$$x\bar{x} = x_0^2 - q(\mathbf{x}) = n(x) \qquad (\text{norm of } x)$$

are in $\mathbf{F}$ and that the elements x in $\mathbf{M}_q$ satisfy the equation in $\mathbf{F}$

$$x^2 - t(x)x + n(x) = 0. \tag{7.2}$$

Now, we define the inverse element in $\mathbf{M}_q$.

Definition 7.2.5. If x is an element in $\mathbf{M}_q$ with nonzero norm, then the inverse element of x is defined by

$$x^{-1} := \frac{\bar{x}}{n(x)}. \tag{7.3}$$

Remark 7.2.6. It is obvious that the element in (7.3) is a solution to the equation $x \cdot y = 1$ in the variable y.

Now, we set

$$y^2 := \frac{1}{\varepsilon} q(\mathbf{x}),$$

and so

$$n(x) = x_0^2 - \varepsilon y^2,$$

where y, $\varepsilon \in \mathbf{F}$ or to one of its extensions $\mathbf{F}^o$.

In the sequel, we shall consider $\mathbf{M}_q$ on $\mathbf{F}^o$ and we will focus on the case $y \neq 0$.

Definition 7.2.7. We say that a function $w(x)$ in $\mathbf{M}_q$ is *biholomorphic* if

$$w(x) = u(x_0, y) + \frac{1}{y} v(x_0, y)\mathbf{x} \tag{7.4}$$

where $u(x_0, y)$ and $v(x_0, y)$ are functions of x_0 and y, satisfying

$$\frac{\partial u}{\partial x_0} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \varepsilon \frac{\partial v}{\partial x_0}. \tag{7.5}$$

Remark 7.2.8. The derivations have the usual formal properties.

Remark 7.2.9. If we consider $\varepsilon = -1$ in Definition 7.2.7 we get the Definition of a slice hyperholomorhic function, see Definition 3.1.3.

The biholomorphic functions enjoy the following fundamental property.

Theorem 7.2.10. *The powers of a biholomorphic function are still biholomorphic functions.*

Proof. Let us consider w a biholomorphic function of the form (7.4). Then by the binomial theorem we have

$$w^m = \left(u + \frac{1}{y} v\mathbf{x}\right)^m = \sum_{k=0}^{m} \binom{m}{k} u^{m-k} \frac{1}{y^k} v^k \mathbf{x}^k.$$

Splitting the sum and considering that $\mathbf{x}^{2k} = (q(\mathbf{x}))^k = \varepsilon^k y^{2k}$, we have

$$\begin{aligned} w^m &= \sum_{k=0}^{m} \binom{m}{k} u^{m-k} \frac{1}{y^k} v^k \mathbf{x}^k \\ &= \sum_{k=0}^{[m/2]} \binom{m}{2k} u^{m-2k} \frac{1}{y^{2k}} v^{2k} \mathbf{x}^{2k} + \sum_{k=0}^{[m/2]} \binom{m}{2k+1} u^{m-2k-1} \frac{1}{y^{2k+1}} v^{2k+1} \mathbf{x}^{2k+1} \\ &= \sum_{k=0}^{[m/2]} \binom{m}{2k} u^{m-2k} \frac{1}{y^{2k}} v^{2k} \varepsilon^k y^{2k} + \sum_{k=0}^{[m/2]} \binom{m}{2k+1} u^{m-2k-1} \frac{1}{y^{2k+1}} v^{2k+1} \varepsilon^k y^{2k} \mathbf{x} \\ &= \left(\sum_{k=0}^{[m/2]} (\varepsilon)^k \binom{m}{2k} u^{m-2k} v^{2k}\right) + \frac{1}{y} \left(\sum_{k=0}^{[m/2]} (\varepsilon)^k \binom{m}{2k+1} u^{m-2k-1} v^{2k+1}\right) \mathbf{x} \end{aligned}$$

where $[m/2]$ is the integer part of $m/2$. We get the thesis since x and x^{-1} are biholomorphic functions. $\square$

We denote by ∂ the operator

$$i_1 \frac{\partial}{\partial x_1} + \cdots + i_n \frac{\partial}{\partial x_n}.$$

We set the quadratic form inverse of $q(\mathbf{x})$ as

$$q^{-1}(\mathbf{x}) = \sum_{j,k=1}^{n} \alpha_{jk} x_j x_k.$$

This is helpful to give the definition of the following operator

$$\square w = \frac{\partial^2 w}{\partial x_0^2} - q^{-1}(\partial) w, \tag{7.6}$$

Remark 7.2.11. The operator defined in (7.6) in the Clifford algebras setting is the Laplace operator.

Lemma 7.2.12. *Let $s \in \mathbb{N}$. Then we set*

$$u_s := \frac{\partial u_{s-1}}{\partial y}\frac{1}{y}, \qquad v_s := \frac{\partial v_{s-1}}{\partial y}\frac{1}{y} - \frac{v_{s-1}}{y^2} = \frac{\partial}{\partial y}\frac{v_{s-1}}{y} \tag{7.7}$$

$$w_s = u_s + \frac{1}{y}v_s\mathbf{x}.$$

We assume that the function w_0 is holomorphic. Then the functions u_s, v_s satisfy the relations

$$\frac{\partial u_s}{\partial x_0} = \frac{\partial v_s}{\partial y} + 2s\frac{v_s}{y}, \qquad \frac{\partial u_s}{\partial y} = \varepsilon\frac{\partial v_s}{\partial x_0}. \tag{7.8}$$

Proof. We show the result by induction on s. For $s = 0$, formula (7.8) reduces to (7.5), which holds because w_0 is biholomorphic by hypothesis. So let us suppose that (7.8) hold for $s-1$; then

$$\begin{aligned}
\frac{\partial u_s}{\partial x_0} &= \frac{1}{y}\frac{\partial^2 u_{s-1}}{\partial x_0 \partial y} = \frac{1}{y}\frac{\partial}{\partial y}\left[\frac{\partial v_{s-1}}{\partial y} + 2(s-1)\frac{v_{s-1}}{y}\right] \\
&= \frac{1}{y}\frac{\partial}{\partial y}\left[yv_s + \frac{v_{s-1}}{y} + 2(s-1)\frac{v_{s-1}}{y}\right] \\
&= \frac{1}{y}\frac{\partial}{\partial y}\left[yv_s + (2s-1)\frac{v_{s-1}}{y}\right] \\
&= \frac{1}{y}\frac{\partial}{\partial y}\left[yv_s - \frac{v_{s-1}}{y}\right] + \frac{2s}{y}\frac{\partial}{\partial y}\frac{v_{s-1}}{y} \\
&= \frac{1}{y}\frac{\partial}{\partial y}yv_s - \frac{1}{y}\frac{\partial}{\partial y}\frac{v_{s-1}}{y} + 2s\frac{v_s}{y} \\
&= \frac{1}{y}\frac{\partial}{\partial y}yv_s - \frac{v_s}{y} + 2s\frac{v_s}{y} \\
&= \frac{\partial}{\partial y}v_s + \frac{v_s}{y} - \frac{v_s}{y} + 2s\frac{v_s}{y} \\
&= \frac{\partial}{\partial y}v_s + 2s\frac{v_s}{y}.
\end{aligned}$$

This prove the first equality of (7.7).

$$\begin{aligned}
\frac{\partial u_s}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial u_{s-1}}{\partial y}\right) = \varepsilon \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial v_{s-1}}{\partial x_0}\right) \\
&= \varepsilon\left(-\frac{1}{y^2}\frac{\partial v_{s-1}}{\partial x_0} + \frac{1}{y}\frac{\partial}{\partial y}\frac{\partial}{\partial x_0}v_{s-1}\right) \\
&= \varepsilon \frac{\partial}{\partial x_0}\left(\frac{1}{y}\frac{\partial v_{s-1}}{\partial y} - \frac{v_{s-1}}{y^2}\right) \\
&= \varepsilon \frac{\partial v_s}{\partial x_0}.
\end{aligned}$$

$\square$

Theorem 7.2.13. *If $w_0 = u_0 + \frac{1}{y}v_0\mathbf{x}$ is a biholomorphic function and n is odd, then:*

$$\square^{(n-1)/2} w_0 = 0. \tag{7.9}$$

Before to show the previous theorem we need some preliminary results.

Proof. We use the same notations of Lemma 7.2.12. By applying the equations in (7.7). We get

$$\begin{aligned}
\frac{\partial^2 u_s}{\partial x_0^2} &= \frac{\partial}{\partial x_0}\left(\frac{\partial v_s}{\partial y} + 2s\frac{v_s}{y}\right) \qquad (7.10) \\
&= \frac{\partial}{\partial y}\frac{\partial v_s}{\partial x_0} + 2s\frac{\partial}{\partial x_0}\left(\frac{v_s}{y}\right) \\
&= \frac{1}{\varepsilon}\frac{\partial^2 u_s}{\partial y^2} + \frac{2s}{\varepsilon}\frac{\partial u_s}{\partial y}\frac{1}{y} = \frac{1}{\varepsilon}\frac{\partial (y u_{s+1})}{\partial y} + \frac{2s}{\varepsilon}u_{s+1} \\
&= \frac{1}{\varepsilon}\left(u_{s+1} + y\frac{\partial u_{s+1}}{\partial y} + 2s u_{s+1}\right) \\
&= \frac{1}{\varepsilon}\left[(2s+1)u_{s+1} + y\frac{\partial u_{s+1}}{\partial y}\right].
\end{aligned}$$

Similarly we have

$$\frac{\partial^2 v_s}{\partial x_0^2} = \frac{\partial}{\partial x_0}\left(\frac{\partial v_s}{\partial x_0}\right) = \frac{1}{\varepsilon}\frac{\partial}{\partial x_0}\left(\frac{\partial u_s}{\partial y}\right) = \frac{1}{\varepsilon}\frac{\partial}{\partial y}\left(\frac{\partial v_s}{\partial y} + 2s\frac{v_s}{y}\right). \tag{7.11}$$

Hence (7.11) and (7.11) imply

$$
\begin{aligned}
\frac{\partial^2 w_s}{\partial x_0^2} &= \frac{\partial^2 u_s}{\partial x_0^2} + \frac{1}{y}\frac{\partial^2 v_s}{\partial x_0^2}\mathbf{x} \\
&= \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{y\varepsilon}\frac{\partial}{\partial y}\left(\frac{\partial v_s}{\partial y} + 2s\frac{v_s}{y}\right)\mathbf{x} \\
&= \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{y\varepsilon}\frac{\partial}{\partial y}\left[yv_{s+1} + \frac{1}{y}(2s+1)v_s\right]\mathbf{x} \\
&= \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{y\varepsilon}\left[v_{s+1} + y\frac{\partial v_{s+1}}{\partial y} - \frac{1}{y^2}(2s+1)v_s + \right. \\
&\quad \left. + \frac{(2s+1)}{y}\frac{\partial v_s}{\partial y}\right]\mathbf{x} \\
&= \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{y\varepsilon}\left[v_{s+1} + y\frac{\partial v_{s+1}}{\partial y} + v_{s+1}(2s+1)\right]\mathbf{x} \\
&= \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{y\varepsilon}\left[y\frac{\partial v_{s+1}}{\partial y} + (2s+2)v_{s+1}\right]\mathbf{x} \\
&= \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{\varepsilon}\left[\frac{\partial v_{s+1}}{\partial y} + (2s+2)\frac{v_{s+1}}{y}\right]\mathbf{x}.
\end{aligned}
$$

Therefore

$$
\frac{\partial^2 w_s}{\partial x_0^2} = \frac{1}{\varepsilon}\left[y\frac{\partial u_{s+1}}{\partial y} + (2s+1)u_{s+1}\right] + \frac{1}{\varepsilon}\left[\frac{\partial v_{s+1}}{\partial y} + (2s+2)\frac{v_{s+1}}{y}\right]\mathbf{x}.
$$

Since $y^2 = \frac{1}{\varepsilon}q(\mathbf{x})$ we get

$$
\frac{\partial}{\partial q}y^2 = \frac{1}{\varepsilon}\frac{\partial}{\partial q}q.
$$

This implies

$$
\frac{\partial y}{\partial q} = \frac{1}{2y\varepsilon}.
$$

Hence, we have

$$
\frac{\partial u_s}{\partial x_j} = \frac{\partial u_s}{\partial y}\frac{\partial y}{\partial q}\frac{\partial q}{\partial x_j} = \frac{\partial u_s}{\partial y}\frac{1}{2\varepsilon y}\frac{\partial q}{\partial x_j} = \frac{1}{2\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j}. \tag{7.12}
$$

and

$$\begin{aligned}
\frac{\partial}{\partial x_j}\left(\frac{1}{y}v_s\mathbf{x}\right) &= \left(\frac{\partial}{\partial x_j}\frac{1}{y}\right)v_s\mathbf{x} + \frac{1}{y}\left(\frac{\partial}{\partial x_j}v_s\right)\mathbf{x} + \frac{1}{y}v_s\frac{\partial \mathbf{x}}{\partial x_j} \\
&= \frac{\partial y^{-1}}{\partial y}\frac{\partial y}{\partial q}\frac{\partial q}{\partial x_j}v_s\mathbf{x} + \frac{1}{y}\frac{\partial v_s}{\partial y}\frac{\partial y}{\partial q}\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{y}v_s i_j \\
&= -\frac{1}{y^2}\frac{1}{2\varepsilon y}\frac{\partial q}{\partial x_j}v_s\mathbf{x} + \frac{1}{y}\frac{1}{2\varepsilon y}\frac{\partial v_s}{\partial y}\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{y}v_s i_j \\
&= -\frac{1}{2\varepsilon y^3}v_s\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{2\varepsilon y^2}\frac{\partial v_s}{\partial y}\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{y}v_s i_j \\
&= \frac{1}{2\varepsilon y}\left[\frac{1}{y}\frac{\partial v_s}{\partial y} - \frac{1}{y^2}v_s\right]\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{y}v_s i_j \\
&= \frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{y}v_s i_j. \qquad (7.13)
\end{aligned}$$

Therefore we have

$$\frac{\partial w_s}{\partial x_j} = \frac{1}{2\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j} + \frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_j}\mathbf{x} + \frac{1}{y}v_s i_j. \qquad (7.14)$$

From (7.12) and the definition of v_s (see (7.7)) we have

$$\begin{aligned}
\frac{\partial}{\partial x_k}\left(\frac{1}{2\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j}\right) &= \frac{1}{2\varepsilon}\left(\frac{\partial u_{s+1}}{\partial x_k}\right)\frac{\partial q}{\partial x_j} + \frac{1}{2\varepsilon}u_{s+1}\frac{\partial^2 q}{\partial x_k \partial x_j} \\
&= \frac{1}{4\varepsilon^2}u_{s+2}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j} + \frac{1}{2\varepsilon}u_{s+1}2a_{jk} \\
&= \frac{1}{4\varepsilon^2 y}\frac{\partial u_{s+1}}{\partial y}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j} + \frac{1}{\varepsilon}u_{s+1}a_{jk}.
\end{aligned}$$

Now, from (7.13) we have

$$\begin{aligned}
\frac{\partial}{\partial x_k}\left(\frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_j}\mathbf{x}\right) &= \frac{1}{2\varepsilon}\frac{\partial}{\partial x_k}\left(\frac{v_{s+1}}{y}\right)\frac{\partial q}{\partial x_j}\mathbf{x}+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}\frac{\partial^2 q}{\partial x_k\partial x_j}\mathbf{x}+\\
&+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}\frac{\partial q}{\partial x_j}\frac{\partial \mathbf{x}}{\partial x_k}\\
&=\frac{1}{4\varepsilon^2 y}\left(\frac{1}{y}\frac{\partial v_{s+1}}{\partial y}-\frac{1}{y^2}v_{s+1}\right)\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}\mathbf{x}+\\
&+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}2a_{jk}\mathbf{x}+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}\frac{\partial q}{\partial x_j}i_k\\
&=\frac{1}{4\varepsilon^2 y^2}\frac{\partial v_{s+1}}{\partial y}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}\mathbf{x}-\frac{1}{4\varepsilon^2 y^3}v_{s+1}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}\mathbf{x}\\
&+\frac{1}{\varepsilon}\frac{v_{s+1}}{y}a_{jk}\mathbf{x}+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}\frac{\partial q}{\partial x_j}i_k.
\end{aligned}$$

Therefore from (7.13) we get

$$\frac{\partial}{\partial x_k}\left(\frac{v_s}{y}\right)i_j=\frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_k}i_j.$$

Hence, we have

$$\begin{aligned}
\frac{\partial^2 w_s}{\partial x_k\partial x_j} \quad = \quad & \frac{1}{4\varepsilon^2 y}\frac{\partial u_{s+1}}{\partial y}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}+\frac{1}{\varepsilon}u_{s+1}a_{jk}-\frac{1}{4\varepsilon^2 y^3}v_{s+1}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}\mathbf{x}\\
&+\frac{1}{4\varepsilon^2 y^2}\frac{\partial v_{s+1}}{\partial y}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}\mathbf{x}+\frac{v_{s+1}}{\varepsilon y}\left[a_{jk}\mathbf{x}+\frac{1}{2}\frac{\partial q}{\partial x_j}i_k+\frac{1}{2}\frac{\partial q}{\partial x_k}i_j\right].
\end{aligned}$$

Since

$$\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial q}{\partial x_k}\frac{\partial q}{\partial x_j}=4q(x)=4\varepsilon y^2 \qquad \sum_{j,k=1}^{n}\alpha_{jk}a_{jk}=n,$$

and

$$\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial q}{\partial x_j}i_k=\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial q}{\partial x_k}i_j=2\mathbf{x},$$

we get

$$\begin{aligned}
\sum_{j,k=1}^{n}\alpha_{jk}\frac{\partial^2 w_s}{\partial x_k\partial x_j} &= \frac{1}{\varepsilon}\left[\frac{\partial u_{s+1}}{\partial y}y+nu_{s+1}+\frac{\partial v_{s+1}}{\partial y}\mathbf{x}-\frac{v_{s+1}}{y}\mathbf{x}+\frac{v_{s+1}}{y}n\mathbf{x}+2\frac{v_{s+1}}{y}\mathbf{x}\right]\\
&=\frac{1}{\varepsilon}\left[\frac{\partial u_{s+1}}{\partial y}y+nu_{s+1}+\frac{\partial v_{s+1}}{\partial y}\mathbf{x}+(n+1)\frac{v_{s+1}}{y}\mathbf{x}\right].
\end{aligned}$$

Therefore we have

$$\sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2 w_s}{\partial x_k \partial x_j} = \frac{1}{\varepsilon}\left[\frac{\partial u_{s+1}}{\partial y} y + n u_{s+1} + \frac{\partial v_{s+1}}{\partial y}\mathbf{x} + (n+1)\frac{v_{s+1}}{y}\mathbf{x}\right].$$

Finally, by taking into account (7.6),we obtain

$$\varepsilon \Box w_s = -(n-2s-1) w_{s+1}. \tag{7.15}$$

Thus, if n is odd, for $s = (n-1)/2$ one has

$$\varepsilon \Box w_{(n-1)/2} = 0,$$

namely (7.9). □

Definition 7.2.14. A function w with values the quadratic module $\mathbf{M}_q$ is called JB-*monogenic* (Jordan B-monogenic) if, for a symmetric matrix B with entries b_{ij} and with determinant different from zero ($|B| \neq 0$) one has

$$\frac{\partial w}{\partial x_0} - \frac{1}{2\varepsilon} \sum_{j,k=1}^{n} b_{jk} \left[\frac{\partial w}{\partial x_j} i_k + i_k \frac{\partial w}{\partial x_j}\right] = 0.$$

Theorem 7.2.15. *Let us suppose that w_0 is biholomorphic. Then we set*

$$S := \frac{1}{2\varepsilon} \sum_{j,k=1}^{n} b_{jk} \frac{\partial q}{\partial x_j} i_k.$$

If

$$\mathbf{x} = S.$$

in $\mathbf{M}_q$, the $(n-1)/2$ power of $\Box$ of all biholomorphic functions is JB-monogenic.

Proof. Firstly, we show the following identity.

$$2 - \frac{1}{\varepsilon y^2}(S\mathbf{x} + \mathbf{x}S) = \frac{1}{\varepsilon y^2}[\mathbf{x}(\mathbf{x} - S) + (\mathbf{x} - S)\mathbf{x}]. \tag{7.16}$$

We can get (7.16) with some manipulations:

$$\begin{aligned}
\frac{1}{\varepsilon y^2}[\mathbf{x}(\mathbf{x} - S) + (\mathbf{x} - S)\mathbf{x}] &= \frac{1}{\varepsilon y^2}[2\mathbf{x}^2 - \mathbf{x}S - S\mathbf{x}] \\
&= \frac{1}{\varepsilon y^2} 2\mathbf{x}^2 - \frac{1}{\varepsilon y^2}[S\mathbf{x} + \mathbf{x}S] \\
&= \frac{2}{\varepsilon y^2}\varepsilon y^2 - \frac{1}{\varepsilon y^2}[S\mathbf{x} + \mathbf{x}S] \\
&= 2 - \frac{1}{\varepsilon y^2}(S\mathbf{x} + \mathbf{x}S).
\end{aligned}$$

Now from the relations (7.8) we have

$$\begin{aligned}\frac{\partial w_s}{\partial x_0} &= \frac{\partial u_s}{\partial x_0}+\frac{1}{y}\frac{\partial v_s}{\partial x_0}\mathbf{x} \\ &= \frac{\partial v_s}{\partial y}+2s\frac{v_s}{y}+\frac{1}{y\varepsilon}\frac{\partial u_s}{\partial y}\mathbf{x} \\ &= \frac{\partial v_s}{\partial y}+2s\frac{v_s}{y}+\frac{u_{s+1}}{\varepsilon}\mathbf{x}.\end{aligned} \tag{7.17}$$

By (7.17) we get

$$\frac{\partial w_s}{\partial x_j}=\frac{1}{2\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j}+\frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_j}\mathbf{x}+\frac{1}{y}v_s i_j.$$

Hence we have

$$\begin{aligned}&\frac{1}{2\varepsilon}\sum_{j,k=1}^{n} b_{jk}\left[\frac{\partial w_s}{\partial x_j}i_k+i_k\frac{\partial w_s}{\partial x_j}\right]=\frac{1}{2\varepsilon}\left(\sum_{j,k=1}^{n} b_{jk}\left[\frac{1}{2\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j}+\right.\right.\\ &\left.+\frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_j}\mathbf{x}+\frac{1}{y}v_s i_j\right]i_k+b_{jk}i_k\left[\frac{1}{2\varepsilon}u_{s+1}\frac{\partial q}{\partial x_j}+\frac{1}{2\varepsilon y}v_{s+1}\frac{\partial q}{\partial x_j}\mathbf{x}\right.\\ &\left.\left.+\frac{1}{y}v_s i_j\right]\right).\end{aligned}$$

Moreover, by recalling the definition of S we have

$$\begin{aligned}\frac{1}{4\varepsilon^2}\sum_{j,k=1}^{n} b_{jk}u_{s+1}\frac{\partial q}{\partial x_j}i_k+\frac{1}{4\varepsilon^2}\sum_{j,k=1}^{n} b_{jk}u_{s+1}\frac{\partial q}{\partial x_j}i_k &= \frac{1}{\varepsilon}\frac{u_{s+1}}{2\varepsilon}\sum_{j,k=1}^{n} b_{jk}u_{s+1}\frac{\partial q}{\partial x_j}i_k \\ &= \frac{1}{\varepsilon}u_{s+1}S,\end{aligned} \tag{7.18}$$

and

$$\begin{aligned}&\frac{1}{4\varepsilon^2}\sum_{j,k=1}^{n} b_{jk}\frac{v_{s+1}}{y}\frac{\partial q}{\partial x_j}\mathbf{x}i_k+\frac{1}{4\varepsilon^2}\sum_{j,k=1}^{n} b_{jk}i_k\frac{v_{s+1}}{y}\frac{\partial q}{\partial x_j}\mathbf{x}\\ &=\frac{1}{4\varepsilon^2}\frac{v_{s+1}}{y}\mathbf{x}\sum_{j,k=1}^{n} b_{jk}\frac{\partial q}{\partial x_j}i_k+\frac{1}{4\varepsilon^2}\frac{v_{s+1}}{y}\sum_{j,k=1}^{n} b_{jk}\frac{\partial q}{\partial x_j}i_k\mathbf{x}\\ &+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}\mathbf{x}S+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}S\mathbf{x}.\end{aligned} \tag{7.19}$$

Since $v_{s+1}=\frac{1}{y}\frac{\partial v_s}{\partial y}-\frac{1}{y^2}v_s$ we get

$$\begin{aligned}\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}\mathbf{x}S+\frac{1}{2\varepsilon}\frac{v_{s+1}}{y}S\mathbf{x} &= \frac{1}{2\varepsilon y^2}\frac{\partial v_s}{\partial y}\mathbf{x}S-\frac{1}{2\varepsilon y^3}v_s\mathbf{x}S+\frac{1}{2\varepsilon y^2}\frac{\partial v_s}{\partial y}S\mathbf{x}\\ &\quad -\frac{1}{2\varepsilon y^3}v_s S\mathbf{x} \qquad (7.20)\\ &= \frac{1}{2\varepsilon y^2}\frac{\partial v_s}{\partial y}(\mathbf{x}S+S\mathbf{x})-\frac{v_s}{y}\left[\frac{1}{2\varepsilon y^2}(\mathbf{x}S+S\mathbf{x})\right].\end{aligned}$$

From the polar form of a quadratic form we get

$$\begin{aligned}\frac{1}{2\varepsilon}\left(\sum_{j,k=1}^{n} b_{jk}\frac{v_s}{y}i_ji_k+\sum_{j,k=1}^{n} b_{jk}\frac{v_s}{y}i_ki_j\right) &= \frac{1}{2\varepsilon}\frac{v_s}{y}\left(\sum_{j,k=1}^{n} b_{jk}i_ji_k+\sum_{j,k=1}^{n} b_{jk}i_ki_j\right)\\ &= \frac{1}{\varepsilon}\frac{v_s}{y}\sum_{j,k=1}^{n} b_{jk}a_{jk}. \qquad (7.21)\end{aligned}$$

By putting together (7.17), (7.18), (7.19), (7.21) and (7.21) we get

$$\begin{aligned}0 &= \frac{1}{\varepsilon}u_{s+1}(\mathbf{x}-S)+\frac{\partial v_s}{\partial y}\left[1-\frac{1}{2\varepsilon^2y^2}(\mathbf{x}S+S\mathbf{x})\right]+\\ &\quad +\frac{v_s}{y}\left[2s+\frac{1}{2\varepsilon^2y^2}(\mathbf{x}S+S\mathbf{x})\right]-\frac{v_s}{y}\frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk}a_{jk}\\ &= \frac{1}{\varepsilon}u_{s+1}(\mathbf{x}-S)+\frac{\partial v_s}{\partial y}\left[1-\frac{1}{2\varepsilon^2y^2}(\mathbf{x}S+S\mathbf{x})\right]+\\ &\quad +\frac{v_s}{y}\left[2s+1-1+\frac{1}{2\varepsilon^2y^2}(\mathbf{x}S+S\mathbf{x})\right]-\frac{v_s}{y}\frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk}a_{jk}\end{aligned}$$

Moreover, we know that

$$w_{s+1}=u_{s+1}+\frac{1}{y}v_{s+1}\mathbf{x}=u_{s+1}+\frac{1}{y^2}\frac{\partial v_s}{\partial y}\mathbf{x}-\frac{1}{y^3}v_s\mathbf{x}.$$

Finally, by (7.16) we obtain

$$\begin{aligned}0 &= \frac{1}{2\varepsilon}u_{s+1}(\mathbf{x}-S)+\frac{1}{2\varepsilon}u_{s+1}(\mathbf{x}-S)+\frac{1}{2\varepsilon y^2}\frac{\partial v_s}{\partial y}[\mathbf{x}(\mathbf{x}-S)+(\mathbf{x}-S)\mathbf{x}]+\\ &\quad +\frac{v_s}{y}\left[2s+1-\frac{1}{2\varepsilon y^2}[\mathbf{x}(\mathbf{x}-S)+(\mathbf{x}-S)\mathbf{x}]\right]-\frac{v_s}{y}\frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk}a_{jk}\\ &= \frac{1}{2\varepsilon}[w_{s+1}(\mathbf{x}-S)+(\mathbf{x}-S)w_{s+1}]+\frac{v_s}{y}(2s+1)-\frac{v_s}{y}\frac{1}{\varepsilon}\sum_{j,k=1}^{n} b_{jk}a_{jk}\end{aligned}$$

and this is satisfied if $s = \dfrac{n-1}{2}$ and if

$$\mathbf{x} = S. \tag{7.22}$$

□

If in Definition 7.2.14 we consider that $\mathbf{M}_q$, or B, is a scalar and w is such that the jacobian matrix $\partial \mathbf{w}/\partial \mathbf{x}$ is symmetric, then we have the following.

Definition 7.2.16. A function w is said *B-monogenic on the left* if it satisfies the following equation

$$\frac{\partial w}{\partial x_0} - \frac{1}{\varepsilon} \sum_{j,k=1}^{n} b_{jk} i_j \frac{\partial w}{\partial x_k} = D_B w = 0. \tag{7.23}$$

In a similar way one defines functions B-monogenic on the right.

Remark 7.2.17. With computations similar to the one done in Theorem 7.2.15 one can see that if (7.22) holds, the $(n-1)/2$ power of $\Box$ of biholomorphic functions is B-monogenic on the left and on the right.

Remark 7.2.18. If $\mathbf{M}_q$ is alternative, multiplying on the left (7.23) by $\bar{D}_B$, the conjugate operator of D_B, one finds that

$$\bar{D}_B D_B w = \left[\frac{\partial^2}{\partial x_0^2} - \frac{1}{\varepsilon} g(\delta) \right] w = 0, \tag{7.24}$$

where $g(\mathbf{x})$ is the quadratic form associated with the matrix BAB^t. Thus if B satisfies the relation

$$BAB^t = \varepsilon^2 A^{-1}, \tag{7.25}$$

then (7.24) coincides with (7.6) and we can say that: B-monogenic functions are solutions of the equation $\Box w = 0$.

7.2.1 Comments

The Fueter mapping theorem was proved by R.Fueter in the Mid Thirties, see [85] and it can be considered a special case of the previous construction, in which $\mathbf{F} = \mathbb{R}$ and $\mathbf{M} = \mathbb{H}$. Basically, it provides a way to generate Fueter regular functions starting from holomorphic functions of one complex variable. The Fueter's theorem can be stated as follows

Theorem 7.2.19 (Fueter mapping theorem). *Let $f_0(z)=\alpha(u,v)+i\beta(u,v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D=\{q=q_0+\underline{q} \mid (q_0,|\underline{q}|)\in D\}$$

be the open set induced by D in $\mathbb{H}$. Then the operator T_{F1} defined by

$$f(q)=T_{F1}(f_0):=\alpha(q_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(q_0,|\underline{q}|)$$

maps the set of holomorphic functions in the set of intrinsic slice hyper-holomorphic functions. Moreover, the function

$$\breve{f}(q):=T_{F2}\left(\alpha(q_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(q_0,|\underline{q}|)\right),$$

where $T_{F2}=\Delta$ and Δ is the Laplacian in four real variables q_ℓ, $\ell=0,1,2,3$, is in the kernel of the Fueter operator i.e.

$$\mathcal{D}\breve{f}=0 \quad on \quad \Omega_D.$$

Another special case, of the construction presented in the previous section, is to take $\mathbf{F}=\mathbb{R}$ and $\mathbf{M}=\mathbb{R}^{n+1}$, identified with the set of paravectors. In this case by applying $\Delta^{\frac{n-1}{2}}$ (Δ is the Laplace operator in $n+1$ variables) to a function induced on the set of paravectors by a holomorphic function, one obtains a monogenic function with values in the real Clifford algebra $\mathbb{R}_n$ over an odd number n of imaginary units. Now, we state a version of the Fueter-Sce theorem that is commonly known in the recent literature.

Theorem 7.2.20 (Fueter-Sce mapping theorem). *Let $n\geq 3$ be an odd number. Let $f(z)=\alpha(u,v)+i\beta(u,v)$ be a holomoprhic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D:=\{x=x_0+\underline{x} \mid (x_0,|\underline{x}|)\in D\},$$

be the open set induced by D in $\mathbb{R}^{n+1}$. The operator T_{FS1} defined by

$$T_{FS1}(f)=\alpha(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}\beta(x_0,|\underline{x}|) \tag{7.26}$$

maps the holomorphic function $f(z)$ in the set of intrinsic slice hyperholomorphic function. Then the function

$$\breve{f}(x):=T_{FS2}\left(\alpha(x_0,|\underline{x}|)+\frac{\underline{x}}{|\underline{x}|}\beta(x_0,|\underline{x}|)\right)$$

where $T_{FS2} := \Delta^{\frac{n-1}{2}}$ *is in the kernel of the Dirac operator, i.e.,*

$$\mathcal{D}\breve{f}(x) = 0, \qquad \textit{on} \qquad \Omega_D.$$

We observe that in the previous theorem the operator T_{FS1} maps holomorphic functions into the set of intrinsic slice hyperholomorphic functions, denoted by $\mathcal{N}(\Omega_D)$. Similarly, when we apply the operator T_{FS1} to the set of intrinsic slice hyperholomorphic functions, we obtain a subclass of the monogenic functions that is called axially monogenic functions.

Definition 7.2.21 (Axially monogenic function)**.** Let $D \subset \mathbb{C}$. Let Ω_D be an axially symmetric open set in $\mathbb{R}^{n+1}$, and let $x = x_0 + \underline{x} = x_0 + \underline{\omega} r \in \Omega_D$ with $r := |\underline{x}|$ and $\underline{\omega} := \frac{\underline{x}}{|\underline{x}|}$. A function $f : \Omega_D \to \mathbb{R}_n$ is an axially monogenic function if it is monogenic, i.e, $\mathcal{D}f = 0$, and it has the form

$$f(x_0 + \underline{x}) = A(x_0, |\underline{x}|) + \underline{\omega} B(x_0, |\underline{x}|), \tag{7.27}$$

where the functions A and B satisfy the even-odd conditions, see (3.2). We denote this set of functions as $\mathcal{AM}(\Omega_D)$.

We can visualize the Fueter-Sce construction by the following diagram

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{N}(\Omega_D) \xrightarrow{T_{FS2}} \mathcal{AM}(\Omega_D), \tag{7.28}$$

where $\mathcal{O}(D)$ is the set of holomorphic function.

Remark 7.2.22. We observe that the map T_{FS1} can naturally be defined on $\mathbb{R}$-valued functions of one real variable. This is due to the equivalence between the space of intrinsic holomorphic functions in $\mathbb{C}$, and the space $\mathcal{A}(\mathbb{R})$ of $\mathbb{R}$-valued real-analytic functions on $\mathbb{R}$ that can be holomorphically extended to the entire complex plane $\mathbb{C}$. In this way, we obtain a map $T_{FS1} : \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_n \to \mathcal{SH}(\mathbb{R}^{n+1})$ which factorizes the Fueter-Sce extension as follows

$$\begin{tikzcd} \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_n \arrow[r, "T_{FS1}"] & \mathcal{SH}(\mathbb{R}^{n+1}) \arrow[d, "T_{FS2}"] \\ & \mathcal{AM}(\mathbb{R}^{n+1}) \end{tikzcd}$$

Sce's results are broader than Theorem 7.2.20 from two different points of view:

- the algebra in which the results are proved

- the type of functions obtained.

The quadratic module M_q can be embedded in algebras of specific form, like for example in the Cayley-Dickson algebras, in particular the octonions and case of associative algebras and in all Clifford algebras.

By Theorem 7.2.15 we know that if we consider a function $f_0(z) = f_0(u+iv) = \alpha(u,v) + i\beta(u,v)$ holomorphic, then

$$\Delta^{\frac{n-1}{2}} f(x) = \Delta^{\frac{n-1}{2}} \left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|) \right)$$

is a JB-monogenic function, i.e.,

$$\frac{\partial f}{\partial x_0} - \frac{1}{2\varepsilon} \sum_{j,k=1}^{n} b_{j,k} \left[\frac{\partial f}{\partial x_j} i_k + i_k \frac{\partial f}{x_j} \right],$$

where the matrix $B = [b_{jk}]$ is symmetric and nondegenerate.

The case in which monogenic functions are obtained occurs when $B = I$, with I being the identity matrix, and M_q is the set of paravectors in a Clifford algebra. Operators of Cauchy-Fueter type in which there are coefficients b_{jk} such that the matrix $B = [b_{jk}]$ is orthogonal have been considered in [127].

About 40 years after the results of Sce, T. Qian in [122] showed that the Futer-Sce theorem holds in the case of a Clifford algebra over an even number n of imaginary units. He showed that also in this case, Sce's construction gives a monogenic function. He makes use of the Fourier multiplier

$$(-\Delta)^{\frac{n-1}{2}} = F^{-1} (2\pi|\cdot|)^{n-1} F, \tag{7.29}$$

in order to give meaning to the fractional powers of the Laplacian corresponding to the cases where n is even. Here F and F^{-1} are the Fourier and inverse Fourier transform in $\mathbb{R}^{n+1}$.

Qian's extension makes use of the following more constructive approach. Consider a meromorphic intrinsic function $f(z)$ on $\mathbb{C}$, together with its Laurent expansion around $z = 0$

$$f(z) = \sum_{j \in \mathbb{Z}} a_j z^j.$$

It is clearly seen that all the coefficients a_j in the above expansion must be real numbers and therefore, they play no role when applying the slice extension map T_{FS1}. Thus, to determine the action of the Fueter-Sce-Qian mapping $\tau_n := \operatorname{sgn}(-x_0)^{n-1}\Delta^{\frac{n-1}{2}}$, with n being even, on the slice extension $f(x_0+\underline{x}) := T_{FS1}[f](x)$ of f, it is enough to consider only the actions $\tau_n\left[x^j\right],\ j\in\mathbb{Z}$.

A direct computation of these actions, using the Fourier multiplier definition (7.29), is possible for all dimensions $n\in\mathbb{N}$ only when we consider negative powers of x, i.e. $x^{-\ell}$ with $\ell\in\mathbb{N}$. Observe that x^ℓ ($\ell\in\mathbb{N}$) is not in the Schwartz class of rapidly decreasing functions and therefore, $\Delta^{\frac{n-1}{2}}\left[x^\ell\right]$ is not well-defined as a function if n is even. However, it is possible to define suitable actions of the Fueter-Sce-Qian mapping on positive powers of the paravector x by means of the Kelvin inversion

$$I[f](x) = \frac{\bar{x}}{|x|^{n+1}} f\left(\frac{\bar{x}}{|x|^2}\right), \tag{7.30}$$

which maps monogenic functions into monogenic functions.

In general, the following relations have been established for the action of the Fueter-Sce-Qian mapping on integer powers of x for all dimensions $n\in\mathbb{N}$,

$$\tau_n\left[x^\ell\right] = \begin{cases} \operatorname{sgn}(-x_0)^{n-1}\Delta^{\frac{n-1}{2}}\left[x^\ell\right], & \ell<0,\\ 0, & 0\le\ell\le n-2,\\ I\left[\Delta^{\frac{n-1}{2}}\left[x^{-\ell+n-2}\right]\right], & n-1\le\ell. \end{cases} \tag{7.31}$$

In [122], it was shown that (7.31) indeed provides a suitable extension of the pointwise differential operator $\Delta^{\frac{n-1}{2}}\left[x^\ell\right]$ when n is odd since, in this case, both approaches coincide.

Remark 7.2.23. It is worth noticing that the above definition of the Fueter-Sce-Qian mapping is slightly different form the customary definition used in the literature, see e.g. [103, 122]. Our modification consists in the introduction of the (almost constant) factor $\operatorname{sgn}(-x_0)^{n-1}$ after the action $\Delta^{\frac{n-1}{2}}\left[x^\ell\right]$ when $\ell<0$. As we will show in the following section, this small difference allows for a better description of τ_n in terms of the GCK operator. Observe also that this change still preserves the property that τ_n extends the pointwise differential operator $\Delta^{\frac{n-1}{2}}$ from n odd to $n\in\mathbb{N}$. Indeed, it is obvious that $\operatorname{sgn}(-x_0)^{n-1}\equiv 1$ if n is odd.

More explicit expressions for the actions (7.31) will be provided in section 7.4, see also [78, 103, 122].

Nowadays there are several generalizations of the Futer-Sce-Qian mapping theorem, see [49, 119–121, 129] and the survey [124]. Moreover, in [123] T. Qian proved that the Fueter-Sce's theorem is useful to show boundedness of singular integrals.

7.3 Fueter-Sce-Qian theorem and generalized CK-extension

The aim of this section is to bring together two essential results in Clifford analysis: the Fueter-Sce-Qian theorem and the generalized Cauchy-Kovalevskaya extension [67, 75, 95] (CK-extension for short). There is a vast literature on both results, which essentially provide two different ways of transforming analytic functions of one (real or complex) variable into monogenic functions

In general, every monogenic function defined at $\mathbb{R}^n$ is determined by its restriction to the hyperplane $x_0 = 0$. On the other hand, any real-analytic function $f(\underline{x})$ defined in a region of $\mathbb{R}^n$, has a unique monogenic extension $f(x_0, \underline{x})$ called Cauchy-Kovalevskaya extension (CK-extension). However, it is possible to deal with restrictions to submanifolds not only of codimension one but of arbitrary codimension. This kind of expansion is called generalized CK-extension (GCK-extension). It was obtained in [67] for monogenic functions, defined in $\Omega \subset \mathbb{R}^{n+p}$, by considering their restrictions to $\mathbb{R}^p$. The set Ω is a $SO(n)$-invariant $(n+p)$-dimensional neighbourhood of $\Omega_1 := \Omega \cap \mathbb{R}^p$. By this a Taylor series for monogenic functions $f(\underline{x}, \underline{y})$ is obtained, where $\underline{y} \in \mathbb{R}^p$ is considered as a parameter. Thus, we get a power series in the variable $\underline{x} \in \mathbb{R}^n$. The following is a particular case of such a result for codimension $p = 1$, see [67].

Theorem 7.3.1 (Generalized CK-extension)**.** *Let $f_0(x_0)$ be a Clifford-valued analytic function in a real domain $\Omega_1 \subset \mathbb{R}$. Then there exists a unique sequence $\{f_j(x_0)\}$ of analytic functions such that the series*

$$f(x_0, \underline{x}) = \sum_{j=0}^{\infty} \underline{x}^j f_j(x_0)$$

converges in an axially symmetric slice $(n+1)$-diemensional neighbourhood $\Omega \subset \mathbb{R}^{n+1}$ of Ω_1 and its sum is monogenic. Moreover,

$$\begin{aligned} f(x_0, \underline{x}) &= \Gamma\left(\frac{n}{2}\right)\left(\frac{|\underline{x}|\partial_{x_0}}{2}\right)^{-\frac{n}{2}}\left[\frac{|\underline{x}|\partial_{x_0}}{2} J_{\frac{n}{2}-1}(|\underline{x}|\partial_{x_0}) + \right. \\ &\quad \left. + \frac{\underline{x}\partial_{x_0}}{2} J_{\frac{n}{2}}(|\underline{x}|\partial_{x_0})\right] f_0(x_0), \qquad (7.32) \end{aligned}$$

where J_ν is the Bessel function of the first kind of order ν.

Formula (7.32) is known as the generalized CK-extension of f_0, and it is denoted it by GCK$[f_0](x_0, \underline{x})$. This extension operator defines an isomoprhism between right modules:

$$\mathrm{GCK} : \mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \to \mathcal{AM}(\Omega),$$

whose inverse is given by the restriction operator to the real line, i.e. GCK$[f_0](x_0, 0) = f_0(x_0)$.

The actions of the Fueter-Sce-Qian, the GCK and the slice monogenic extension maps can be summarized in the following diagram:

$$\begin{tikzcd} \mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \arrow[r, "T_{FS1}"] \arrow[d, dashed, "?"] & \mathcal{SH}(\Omega) \arrow[d, "\tau_n"] \\ \mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \arrow[r, "\mathrm{GCK}"] & \mathcal{AM}(\Omega) \end{tikzcd} \tag{7.33}$$

Obviously, the map GCK does not coincide with $\tau_n \circ T_{FS1}$ since GCK is an isomorphism between right modules while $\tau_n \circ T_{FS1}$ is not. Nevertheless, the above diagram can be completed (and made commutative) by adding the missing left vertical arrow. We address this problem in the following section.

7.3.1 The odd dimensional case

In this case, the action of the pointwise differential operator $\Delta^{\frac{n-1}{2}}$ on slice monogenic functions has been explicitly computed, see e.g. [118, Lem.3.2] and [93, Thm. 11.33]. This result will be extremely useful when deriving the connection between the Fueter-Sce-Qian's theorem and the generalized CK-extension.

Lemma 7.3.2 ([93, 118])**.** *If $n \in \mathbb{N}$ is odd, then the action of the pointwise differential operator $\Delta^{\frac{n-1}{2}}$ on a slice hyperholomorphic function*

$$f(x_0 + \underline{x}) = \alpha(x_0, r) + \underline{\omega}\, \beta(x_0, r),$$

is given by

$$\Delta^{\frac{n-1}{2}} [f(x_0 + \underline{x})] = A(x_0, r) + \underline{\omega}\, B(x_0, r),$$

with

$$A(x_0, r) = (n-1)!! \left(\frac{1}{r}\partial_r\right)^{\frac{n-1}{2}} [\alpha](x_0, r), \tag{7.34}$$

and

$$B(x_0,r) = (n-1)!! \left(\partial_r \frac{1}{r}\right)^{\frac{n-1}{2}} [\beta](x_0,r). \tag{7.35}$$

We can now formulate the main result of this section.

Theorem 7.3.3. *Let $f(u+iv) = \alpha(u,v) + i\beta(u,v)$ be an intrinsic holomorphic function defined on an intrinsic complex domain $\Omega_2 \subset \mathbb{C}$. Then for n odd and $r = |\underline{x}|$we have*

$$\begin{aligned} \Delta^{\frac{n-1}{2}} [f(x_0+\underline{x})] &= (n-1)!!\,\mathrm{GCK}\left[\left(\frac{1}{r}\partial_r\right)^{\frac{n-1}{2}} [\alpha](x_0,0)\right] \\ &= (-1)^{\frac{n-1}{2}} \frac{(n-1)!!}{(n-2)!!}\,\mathrm{GCK}\left[f^{(n-1)}(x_0)\right]. \end{aligned}$$

Setting $\gamma_n = (-1)^{\frac{n-1}{2}} \frac{(n-1)!!}{(n-2)!!}$ and $\Omega_1 = \Omega_2 \cap \mathbb{R}$, we obtain the following commutative diagram.

$$\begin{tikzcd} \mathcal{A}(\Omega_1)\otimes\mathbb{R}_n \arrow[r, "T_{FS1}"] \arrow[d, "\gamma_m \partial_{x_0}^{n-1}"'] & \mathcal{SM}(\Omega) \arrow[d, "T_{FS2}"] \\ \mathcal{A}(\Omega_1)\otimes\mathbb{R}_n \arrow[r, "\mathrm{GCK}"] & \mathcal{AM}(\Omega) \end{tikzcd} \tag{7.36}$$

Proof. The fact that f is an holomorphic intrinsic function means that $\alpha(u,v)$ is an even analytic function in v while $\beta(u,v)$ is an odd analytic function in v. Thus, after setting $u = x_0$ and $v = r$, we easily see that the functions

$$A(x_0,r) = (n-1)!! \left(\frac{1}{r}\partial_r\right)^{\frac{n-1}{2}} [\alpha](x_0,r),$$

and

$$B(x_0,r) = (n-1)!! \left(\partial_r \frac{1}{r}\right)^{\frac{n-1}{2}} [\beta](x_0,r).$$

also are even and odd analytic functions in the variable r, respectively. Using Lemma 7.3.2 we identify these functions as the components of the axial monogenic function that results from the action of the Fueter-Sce-Qian map on f, i.e.

$$\Delta^{\frac{n-1}{2}} f(x_0+\underline{x}) = A(x_0,r) + \underline{\omega} B(x_0,r).$$

By virtue of the generalized CK-extension Theorem 7.3.1, the above axially monogenic function is completely determined by its restriction to the real line

$$\Delta^{\frac{n-1}{2}} f(x_0 + \underline{x})\big|_{\underline{x}=0}.$$

Since $B(x_0, r)$ is odd in the variable r, we easily obtain that

$$\Delta^{\frac{n-1}{2}} f(x_0 + \underline{x})\Big|_{\underline{x}=0} = A(x_0, 0) = (n-1)!! \left(\frac{1}{r}\partial_r\right)^{\frac{n-1}{2}} [\alpha](x_0, 0).$$

Therefore,

$$\Delta^{\frac{n-1}{2}} f(x_0 + \underline{x}) = (n-1)!! \text{ GCK} \left[\left(\frac{1}{r}\partial_r\right)^{\frac{n-1}{2}} [\alpha](x_0, 0)\right].$$

Since the function f is an intrinsic holomorphic function we have

$$\alpha(x_0, r) = \sum_{j=0}^{\infty} \frac{(-1)^j r^{2j}}{(2j)!} \partial_{x_0}^{2j}[\alpha](x_0, 0),$$

which for any $\ell \in \mathbb{N}$ yields

$$\begin{aligned} \left(\frac{1}{r}\partial_r\right)^{\ell} [\alpha](x_0, r) &= \sum_{j=\ell}^{\infty} (-1)^j \frac{(2j)(2j-2)\cdots(2j-2\ell+2)}{(2j)!} r^{2j-2\ell} \cdot \\ &\quad \partial_{x_0}^{2j}[\alpha](x_0, 0) \\ &= \sum_{j=0}^{\infty} (-1)^{j+\ell} 2^{\ell} \frac{(j+\ell)!}{j!(2j+2\ell)!} r^{2j} \partial_{x_0}^{2j+2\ell}[\alpha](x_0, 0). \end{aligned}$$

Taking $r = 0$, we obtain

$$\left(\frac{1}{r}\partial_r\right)^{\ell} [\alpha](x_0, 0) = \frac{(-1)^{\ell}}{(2\ell-1)!!} \partial_{x_0}^{2\ell}[\alpha](x_0, 0) = \frac{(-1)^{\ell}}{(2\ell-1)!!} f^{(2\ell)}(x_0),$$

which proves the desired result when substituting $\ell = \frac{n-1}{2}$. □

7.3.2 The even case

We now extend Theorem 7.3.3 to any dimension $n \in \mathbb{N}$ regardless of the parity of n. To that end, we make use of the more constructive approach developed by Qian and his collaborators, see e.g. [103, 122].

As we discussed prevously, this approach focusses on the basic actions $\tau_n\left[x^j\right]$ defined in (7.31) for all $j \in \mathbb{Z}$. These actions on integer powers of the paravector x have been explicitly computed for all dimensions $n \in \mathbb{N}$ and all powers $j \in \mathbb{Z}$, in terms of the so-called monogenic monomials, see e.g. [78, 103, 122].

Definition 7.3.4 (Monogenic monomials)**.** Let $n, k \in \mathbb{N}$, we define the monogenic monomials $P^{(-k)}$ and $P^{(k-1)}$ respectively by

$$P^{(-k)} := \frac{(-1)^{k-1}\sigma_{n+1}\lambda_n}{(k-1)!}\,\partial_{x_0}^{k-1}[E], \qquad P^{(k-1)} := I[P^{(-k)}].$$

Here $\lambda_n = 2^{n-1}\left(\Gamma\left(\frac{n+1}{2}\right)\right)^2$, $\sigma_{n+1} = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$ is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^{n+1}$, $E(x)$ is the so-called Cauchy kernel, i.e. the fundamental solution of $\mathcal{D} = \partial_{x_0} + \partial_{\underline{x}}$ given by

$$E(x) = \frac{1}{\sigma_{n+1}}\frac{\bar{x}}{|x|^{n+1}},$$

while I is the Kelvin inversion defined in (7.30), i.e.

$$I[f](x) = \sigma_{n+1}\,E(x)\,f\left(\frac{\bar{x}}{|x|^2}\right).$$

Using the Fourier multiplier definition of the fractional Laplacian (7.29), it has been shown that (see for example [78, 103, 122])

$$(-\Delta)^{\frac{n-1}{2}}\left[x^{-k}\right] := F^{-1}\left[(2\pi|\cdot|)^{n-1}\,F[(\cdot)^{-k}]\right](x) = P^{(-k)}(x)$$

Combining this identity with the definition of τ_m given in (7.31) one easily obtains the following result.

Theorem 7.3.5 ([103, 122])**.** *Let $n \in \mathbb{N}$, $\ell \in \mathbb{Z}$. The actions of the Fueter-Sce-Qian mapping $\tau_n[x^\ell]$ defined in (7.31) are given by the expressions*

$$\tau_n[x^\ell] = (-1)^{\frac{1-n}{2}}\begin{cases} \operatorname{sgn}(-x_0)^{n-1}P^{(\ell)}(x), & \ell < 0,\\ 0, & 0 \le \ell \le n-2,\\ P^{(\ell+1-n)}(x), & n-1 \le \ell. \end{cases}$$

From the above theorem, it is clear that our goal now reduces to find how the monogenic monomials can be expressed in terms of the generalized CK-extension map. To that end, we first note that the Cauchy kernel $E(x)$ is an axial monogenic function on $\mathbb{R}^{n+1} \setminus \{0\}$ since

$$E(x_0, \underline{x}) = x_0(x_0^2+r^2)^{-\frac{n+1}{2}} - \underline{\omega}\,r(x_0^2+r^2)^{-\frac{n+1}{2}}, \qquad \text{with} \quad r = |\underline{x}| \;\text{ and }\; \underline{\omega} = \frac{\underline{x}}{r}.$$

It is thus clear that the monomials with negative indices $P^{(-k)}$, $k \in \mathbb{N}$, are also axial monogenic. The axial monogenicity of the rest of the monomials $P^{(k)}$, with $k \in \mathbb{N}$, can be shown as a consequence of the following result.

Lemma 7.3.6. *The following statements hold:*

$i)$ The Kelvin inversion I preserves axial monogenicity.

$ii)$ Given a domain $\Omega_1 \subset \mathbb{R}$ and $f_0 \in \mathcal{A}(\Omega_1) \otimes \mathbb{R}_n$, we have

$$I \circ \mathrm{GCK}[f_0] = \mathrm{sgn}(x_0)^{n+1}\, \mathrm{GCK}\left[x_0^{-n} f_0(x_0^{-1})\right].$$

Proof. In [67, Chpt. II], it was proved that the Kelvin inversion I preserves monogenicity. Thus, to prove $i)$, it suffices to show that I preserves the axial form (7.27).
We first observe that $|x_0+\underline{x}|^\gamma = (x_0^2+|\underline{x}|^2)^{\frac{\gamma}{2}}$ ($\gamma \in \mathbb{R}$) is a scalar function of x_0 and $|\underline{x}|$. Thus for any axial monogenic function $f(x_0,\underline{x}) = A(x_0,|\underline{x}|) + \underline{x}\, B(x_0,|\underline{x}|)$ we have that

$$\begin{aligned} I[f](x_0,\underline{x}) &= \frac{x_0-\underline{x}}{|x_0+\underline{x}|^{n+1}} \left[A\left(\frac{x_0}{|x_0+\underline{x}|^2}, \frac{|\underline{x}|}{|x_0+\underline{x}|^2}\right) \right. \\ &\quad \left. - \frac{\underline{x}}{|x_0+\underline{x}|^2}\, B\left(\frac{x_0}{|x_0+\underline{x}|^2}, \frac{|\underline{x}|}{|x_0+\underline{x}|^2}\right)\right] \\ &= A_1(x_0,|\underline{x}|) + \underline{x}\, B_1(x_0,|\underline{x}|), \end{aligned}$$

for a suitable pair of functions A_1 and B_1. The second statement $ii)$ easily follows from the properties of the generalized CK-extension (see Theorem 7.3.1). Indeed,

$$I \circ \mathrm{GCK}[f_0](x_0,\underline{x}) = \frac{x_0-\underline{x}}{|x_0+\underline{x}|^{n+1}}\, \mathrm{GCK}[f_0]\left(\frac{x_0}{|x_0+\underline{x}|^2}, -\frac{\underline{x}}{|x_0+\underline{x}|^2}\right).$$

Hence, the restriction of this axial monogenic function to the real line is given by

$$\begin{aligned} I \circ \mathrm{GCK}[f_0](x_0,0) &= \frac{x_0}{|x_0|^{n+1}}\, \mathrm{GCK}[f_0]\left(x_0^{-1},0\right) \\ &= \mathrm{sgn}(x_0)^{n+1} x_0^{-n} f_0(x_0^{-1}), \end{aligned}$$

which proves the result. □

We can now write the monogenic monomials $P^{(k)}$ as generalized CK-extensions of suitable initial analytic functions of one real variable.

Proposition 7.3.7. *For all $n, k \in \mathbb{N}$ we have that*

$$P^{(-k)} = \frac{\lambda_n (n+k-2)!}{(k-1)!(n-1)!} \operatorname{sgn}(x_0)^{n-1} \operatorname{GCK}[x_0^{-k-n+1}], \tag{7.37}$$

$$P^{(k-1)} = \frac{\lambda_n (n+k-2)!}{(k-1)!(n-1)!} \operatorname{GCK}[x_0^{k-1}], \tag{7.38}$$

Or equivalently,

$$P^{(-k)} = \frac{\lambda_n}{(n-1)!} \operatorname{sgn}(-x_0)^{n-1} \operatorname{GCK} \circ \partial_{x_0}^{n-1}[x_0^{-k}], \tag{7.39}$$

$$P^{(k-1)} = \frac{\lambda_n}{(n-1)!} \operatorname{GCK} \circ \partial_{x_0}^{n-1}[x_0^{n+k-2}]. \tag{7.40}$$

Proof. From $P^{(-k)} = \frac{(-1)^{k-1}\sigma_{n+1}\lambda_n}{(k-1)!} \, \partial_{x_0}^{k-1}[E]$ we obtain

$$\begin{aligned} P^{(-k)}\big|_{\underline{x}=0} &= \frac{(-1)^{k-1}\lambda_n}{(k-1)!} \, \partial_{x_0}^{k-1}\left[\frac{x_0}{|x_0|^{n+1}}\right] \\ &= \frac{(-1)^{k-1}\lambda_n}{(k-1)!} \operatorname{sgn}(x_0)^{n+1} \, \partial_{x_0}^{k-1}\left[x_0^{-n}\right] \\ &= \frac{\lambda_n (n+k-2)!}{(k-1)!(n-1)!} \operatorname{sgn}(x_0)^{n-1} \, x_0^{-k-n+1}. \end{aligned}$$

On the other hand, substituting the indentity $x_0^{-k-n+1} = (-1)^{n-1}\frac{(k-1)!}{(k+n-2)!}\partial_{x_0}^{n-1}[x_0^{-k}]$ in the above equality yields

$$P^{(-k)}\big|_{\underline{x}=0} = \frac{\lambda_n}{(n-1)!} \operatorname{sgn}(-x_0)^{n-1} \, \partial_{x_0}^{n-1}[x_0^{-k}].$$

Using the fact that axially monogenic functions are completely determined by their restrictions to the real line, we obtain from the two last equalities that (7.37) and (7.39) hold.
For $P^{(k-1)}$ with $k \in \mathbb{N}$ we have, by virtue of Lemma 7.3.6 and (7.37), that

$$\begin{aligned} P^{(k-1)} &= I[P^{(-k)}] \\ &= \frac{\lambda_n (n+k-2)!}{(k-1)!(n-1)!} \operatorname{sgn}(x_0)^{n-1} \, I \circ \operatorname{GCK}[x_0^{-k-n+1}] \\ &= \frac{\lambda_n (n+k-2)!}{(k-1)!(n-1)!} \operatorname{GCK}[x_0^{k-1}]. \end{aligned}$$

Substituting the identity $x_0^{k-1} = \frac{(k-1)!}{(n+k-2)!}\partial_{x_0}^{n-1}[x_0^{n+k-2}]$ in the above equality we obtain,

$$P^{(k-1)} = \frac{\lambda_n}{(n-1)!}\, \mathrm{GCK} \circ \partial_{x_0}^{n-1}[x_0^{n+k-2}],$$

which completes the proof. □

We can now extend the above relation between the Fueter-Sce-Qian map and the generalized CK-extension when acting on the basic monomials x_0^k, $k \in \mathbb{Z}$, to general analytic functions by considering their Laurent expansions.

Theorem 7.3.8. *Let $\Omega_2 \subset \mathbb{C}$ be an intrinsic complex domain and let $f : \Omega_2 \to \mathbb{C}$ be a holomorphic intrinsic function. Then for all dimensions $m \in \mathbb{N}$ we have*

$$\tau_n\left[f(x_0 + \underline{x})\right] = \frac{(-1)^{\frac{1-n}{2}}2^{n-1}}{(n-1)!}\Gamma\left(\frac{n+1}{2}\right)^2 \mathrm{GCK} \circ \partial_{x_0}^{n-1}[f(x_0)]. \quad (7.41)$$

Setting $\gamma_n = \frac{(-1)^{\frac{1-n}{2}}2^{n-1}}{(n-1)!}\Gamma\left(\frac{n+1}{2}\right)^2$ and $\Omega_1 = \Omega_2 \cap \mathbb{R}$, we obtain the following extension of the commutative diagram (7.36) to all dimensions $n \in \mathbb{N}$

$$\begin{tikzcd}
\mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \arrow[r, "TFS1"] \arrow[d, "\gamma_n \partial_{x_0}^{n-1}"'] & \mathcal{SM}(\Omega) \arrow[d, "\tau_n"] \\
\mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \arrow[r, "\mathrm{GCK}"] & \mathcal{AM}(\Omega)
\end{tikzcd}$$

Remark 7.3.9. The previously defined constant $\gamma_n = \frac{(-1)^{\frac{1-n}{2}}2^{n-1}}{(n-1)!}\Gamma\left(\frac{n+1}{2}\right)^2$ is an extension to all dimensions $n \in \mathbb{N}$ of the constant $\gamma_n = (-1)^{\frac{n-1}{2}}\frac{(n-1)!!}{(n-2)!!}$ introduced in Theorem 7.34 for odd values of n. Indeed, if n is odd, then

$$\Gamma\left(\frac{n+1}{2}\right)^2 = \left[\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)\cdots\left(\frac{2}{2}\right)\right]^2 = \frac{\left((n-1)!!\right)^2}{2^{n-1}},$$

and $(-1)^{\frac{1-n}{2}} = (-1)^{\frac{n-1}{2}}$ since the power $\frac{n-1}{2}$ is integer. The combination of these two facts easily yields

$$\frac{(-1)^{\frac{1-n}{2}}2^{n-1}}{(n-1)!}\Gamma\left(\frac{n+1}{2}\right)^2 = (-1)^{\frac{n-1}{2}}\frac{(n-1)!!}{(n-2)!!}, \quad \text{for } n \text{ odd.}$$

Proof. We can assume without loosing generality that f is holomorphic around the origin. If that is not the case, we can always obtain such a function by applying a translation argument. Let us consider the Laurent expansion of f at $z = 0$, i.e.

$$f(z) = \sum_{j\in\mathbb{Z}} a_j\, z^j, \quad a_j \in \mathbb{R}.$$

The action of the Fueter mapping on this function is thus given by

$$\tau_n f(x_0 + \underline{x}) = \sum_{j\in\mathbb{Z}} a_j\ \tau_n\left[x^j\right], \quad a_j \in \mathbb{R}.$$

We recall that the action of τ_n does not affect the convergence of the above series, see [103]. Hence, combining Theorem 7.3.5 and Proposition 7.3.7, we obtain

$$\begin{aligned}
\tau_n f(x_0+\underline{x}) &= \sum_{k=1}^{\infty} a_{-k}\ \tau_n\left[x^{-k}\right] + \sum_{k=1}^{\infty} a_k\ \tau_n\left[x^k\right] \\
&= (-1)^{\frac{1-n}{2}} \sum_{k=1}^{\infty} a_{-k}\ \mathrm{sgn}(-x_0)^{n-1}\ P^{(-k)} \\
&\quad +(-1)^{\frac{1-n}{2}} \sum_{k=n-1}^{\infty} a_k\ P^{(k+1-n)} \\
&= (-1)^{\frac{1-n}{2}} \sum_{k=1}^{\infty} a_{-k}\ \mathrm{sgn}(-x_0)^{n-1}\ P^{(-k)} + (-1)^{\frac{1-n}{2}} \sum_{k=0}^{\infty} a_{k+n-1}\ P^{(k)} \\
&= (-1)^{\frac{1-n}{2}} \sum_{k=1}^{\infty} a_{-k}\ \frac{\lambda_n}{(n-1)!}\,\mathrm{GCK}\circ\partial_{x_0}^{n-1}[x_0^{-k}] \\
&\quad +(-1)^{\frac{1-n}{2}} \sum_{k=0}^{\infty} a_{k+n-1}\ \frac{\lambda_n}{(n-1)!}\ \mathrm{GCK}\circ\partial_{x_0}^{n-1}[x_0^{n+k-1}] \\
&= (-1)^{\frac{1-m}{2}} \frac{\lambda_n}{(n-1)!}\ \mathrm{GCK}\circ\partial_{x_0}^{n-1}\left[\sum_{k=1}^{\infty} a_{-k}\ x_0^{-k} + \sum_{k=0}^{\infty} a_{k+n-1}\ x_0^{n+k-1}\right] \\
&= (-1)^{\frac{1-n}{2}} \frac{\lambda_n}{(n-1)!}\ \mathrm{GCK}\circ\partial_{x_0}^{n-1}\left[\sum_{k=1}^{\infty} a_{-k}\ x_0^{-k} + \sum_{k=0}^{\infty} a_k\ x_0^{k}\right] \\
&= (-1)^{\frac{1-n}{2}} \frac{\lambda_n}{(n-1)!}\ \mathrm{GCK}\circ\partial_{x_0}^{n-1}[f(x_0)],
\end{aligned}$$

which completes the proof. □

7.4 Fueter-Sce theorem in integral form

In this section we recall how to show a version of the Fueter-Sce mapping theorem in integral form, see [41, 54]. This theorem will be fundamental in the sequel. Precisely given a holomorphic function f we will see how to generate a monogenic function $\breve{f}$ by an integral transform whose kernel has an interesting form.

We begin by computing the powers of the Laplacian applied to the slice Cauchy kernels. A key point is to use the second form of the slice Cauchy kernels, see Definition 3.1.16. Even if the first form of the slice Cauchy kernels is more suitable for several applications, for example, for the definition of the functional calculus, the explicit computations of The Laplacian applied to the first form of the slice Cauchy kernels does not yield a simple closed formula.

Theorem 7.4.1. *Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$ and let*

$$\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}$$

be the Laplace operator in the variables x.

- *If we consider the left slice monogenic Cauchy kernel $S_L^{-1}(s,x)$ written in form II, i.e,*

  $$S_L^{-1}(s,x) = (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-1}.$$

  Then for $h \geq 1$ we have

  $$\Delta^h S_L^{-1}(s,x) = c(h)(s-\bar{x})(s^2 - 2x_0(x)s + |x|^2)^{-(h+1)}. \qquad (7.42)$$

- *If we consider the right slice monogenic Cauchy kernel $S_L^{-1}(s,x)$ written in form II, i.e,*

  $$S_R^{-1}(s,x) = (s^2 - 2x_0 s + |x|^2)^{-1}(s-\bar{x}).$$

  Then for $h \geq 1$ we have

  $$\Delta^h S_R^{-1}(s,x) = c(h)(s^2 - 2x_0 s + |x|^2)^{-(h+1)}(s-\bar{x}). \qquad (7.43)$$

where $c(h) := (-1)^h \prod_{\ell=1}^{h}(2\ell) \prod_{\ell=1}^{h}(n-2\ell+1)$.

Proof. We shall only consider the left slice monogenic case. The other one follows by similar computations. We prove the result by induction on h. We start with the case $h=1$. We have

$$\frac{\partial}{\partial x_0} S_L^{-1}(s,x) = -(s-\bar{x})(-2s+2x_0)(s^2-2x_0s+|x|^2)^{-2}-(s^2-2x_0s+|x|^2)^{-1},$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x_0^2} S_L^{-1}(s,x) &= 2(-2s+2x_0)(s^2-2x_0+|x|^2)^{-2}-2(s-\bar{x})\cdot \\ &\quad (s^2-2x_0s+|x|^2)^{-3}+2(s-\bar{x})(-2s+2x_0)^2 \\ &\quad \cdot(s^2-2x_0s+|x|^2)^{-3}.\end{aligned}$$

Furthermore, for $1 \leq i \leq n$ we have

$$\frac{\partial}{\partial x_i} S_L^{-1}(s,x) = e_i(s^2-2x_0s+|x|^2)^{-1}-2(s-\bar{x})x_i(s^2-2x_0s+|x|^2)^{-2}.$$

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2} S_L^{-1}(s,x) &= -4x_ie_i(s^2-2x_0s+|x|^2)^{-2}-2(s-\bar{x})\cdot \\ &\quad (s^2-2x_0s+|x|^2)^{-2}+8(s-\bar{x})x_i^2(s^2-2x_0s+|x|^2)^{-3}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta S_L^{-1}(s,x) &= 2(-2s+2x_0)(s^2-2x_0+|x|^2)^{-2}+2(s-\bar{x})(-2s+2x_0)^2 \\ &\quad \cdot(s^2-2x_0s+|x|^2)^{-3}-4\sum_{i=1}^n x_ie_i(s^2-2x_0s+|x|^2)^{-2} \\ &\quad +8\sum_{i=1}^n x_i^2(s-\bar{x})(s^2-2x_0s+|x|^2)^{-3}-2(n+1)(s-\bar{x})\cdot \\ &\quad (s^2-2x_0+|x|^2)^{-2} \\ &= -4\left(s-x_0+\sum_{i=1}^n e_ix_i\right)(s^2-2x_0s+|x|^2)^{-2}+2(s-\bar{x})\cdot \\ &\quad \left[(-2s+2x_0)^2+\sum_{i=1}^n 4x_i^2\right](s^2-2x_0s+|x|^2)^{-3} \\ &\quad -2(n+1)(s-\bar{x})(s^2-2x_0s+|x|^2)^{-2} \\ &= -4(s-\bar{x})(s^2-2x_0s+|x|^2)^{-2}+8(s-\bar{x})\cdot \\ &\quad (s^2-2x_0s+|x|^2)^{-2}-2(n+1)(s-\bar{x})(s^2-2x_0s+|x|^2)^{-2} \\ &= -2(n-1)(s-\bar{x})(s^2-2x_0s+|x|^2)^{-2}.\end{aligned}$$

This proves formula (7.42) for $h = 1$.

Now we assume that formula (7.42) holds for $h \in \mathbb{N}$ and let us show that it holds for $h + 1$. In order to avoid the constants during the computations we consider the function

$$G_h(s, x) = (s - \bar{x})(s^2 - 2x_0 s + |x|^2)^{-(h+1)}. \tag{7.44}$$

By similar computations done before we have

$$\begin{aligned} \frac{\partial^2}{\partial x_0^2} G_h(s,x) &= 2(h+1)(-2s+2x_0)(s^2-2x_0s+|x|^2)^{-h-1} + (h+1)(h+2) \cdot \\ &\quad (s-\bar{x})(-2s+2x_0)^2(s^2-2x_0s+|x|^2)^{-h-3} - 2(h+1) \cdot \\ &\quad (s-\bar{x})(s^2-2x_0s+|x|^2)^{-h-2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x_i^2} G_h(s,x) &= -4(h+1)e_i x_i (s^2-2x_0s+|x|^2)^{-h-2} + 4(h+1)(h+2) \cdot \\ &\quad x_i^2 (s-\bar{x})(s^2-2x_0s+|x|^2)^{-h-3} - 2(h+1)(s-\bar{x}) \cdot \\ &\quad (s^2-2x_0s+|x|^2)^{-h-2}, \end{aligned}$$

we obtain

$$\Delta G_h(s,x) = -2(h+1)(n-2h-1)(s-\bar{x})(s^2-2x_0s+|x|^2)^{-h-2}.$$

Finally by taking into account the constants we get

$$\Delta^{h+1} S_L^{-1}(s,x) = (-1)^{h+1} \prod_{\ell=1}^{h+1} (2\ell) \prod_{\ell=1}^{h+1} (n-2\ell+1)(s-\bar{x})(s^2-2x_0s+|x|^2)^{-h-2}.$$

This ends the proof. $\square$

Now, we study the properties of regularity of the function $\Delta^h S_L^{-1}(s,x)$ in both variables.

Proposition 7.4.2. *Let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. The function $\Delta^h S_L^{-1}(s,x)$ is right slice hyperholomorphic in the variables s for all $h \geq 0$. The function $\Delta^h S_R^{-1}(s,x)$ is left slice hyperholomorphic in the variable s for all $h \geq 0$.*

Proof. By Remark 3.1.13 we can prove the slice hyperholomorphicity by using Definition 3.1.12. We observe that for $h = 0$ it follows by Lemma

3.1.17. When $h \geq 1$ and $s = u + Jv$, $J \in \mathbb{S}^{n-1}$ and by using the function introduced in (7.44) we have

$$\begin{aligned} \frac{\partial}{\partial u} G_h(u+Jv,x) &= (u^2 - v^2 + 2Juv - 2x_0 u - 2x_0 Jv + |x|^2)^{-(h+1)} \\ &\quad -(h+1)(u + Jv - \bar{x})(u^2 - v^2 + 2Juv - 2x_0 u \\ &\quad -2x_0 Jv + |x|^2)^{-(h+2)}(2u + 2Jv - 2x_0) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial v} G_h(u+Jv,x) &= J(u^2 - v^2 + 2Juv - 2x_0 u - 2x_0 Jv + |x|^2)^{-(h+1)} \\ &\quad -(h+1)(u + Jv - \bar{x})(u^2 - v^2 + 2Juv - 2x_0 u \\ &\quad -2x_0 Jv + |x|^2)^{-(h+2)}(-2v + 2Ju - 2x_0 J). \end{aligned}$$

Therefore, this implies that

$$\frac{\partial}{\partial u} G_h(u+Jv,x) + \frac{\partial}{\partial v} G_h(u+Jv,x) J = 0.$$

By similar computations it is possible to show that $\Delta^h S_R^{-1}(s,x)$ is left slice hyperholomorphic in the variable s. □

Remark 7.4.3. The function $\Delta^h S_L^{-1}(s,x)$ is not slice hyperholomorphic in the variable x. Indeed for every $h \geq 1$ we have

$$\frac{\partial}{\partial u} G_h(s,u+Jv) + J \frac{\partial}{\partial v} G_h(s,u+Jv) = 2h(s^2 - 2us + u^2 + v^2)^{-(h+1)}.$$

We cannot get zero for any values of $h \geq 1$.

For $h = \frac{n-1}{2}$ the function $\Delta^h S_L^{-1}(s,x)$ is monogenic in the variable x as proved in the following result.

Proposition 7.4.4. *Let n be an odd number and x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. The function $\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x)$ is a left monogenic in the variables x. The function $\Delta^{\frac{n-1}{2}} S_R^{-1}(s,x)$ is a right monogenic in the variable x.*

Proof. It is enough to show that the function $G_{\frac{n-1}{2}}(s,x)$ is monogenic in the variable x.

$$\begin{aligned} \frac{\partial}{\partial x_0} G_{\frac{n-1}{2}}(s,x) &= -(s^2 - 2x_0 s + |x|^2)^{-\frac{n-1}{2}-1} + 2\left(\frac{n-1}{2} + 1\right) \cdot \\ &\quad (s - \bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n-1}{2}-2}(s - x_0), \end{aligned}$$

and for $1 \leq i \leq n$ we have

$$\begin{aligned}\frac{\partial}{\partial x_i} G_{\frac{n-1}{2}}(s,x) &= e_i(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-1} - 2\left(\frac{n-1}{2}+1\right)\cdot \\ &\quad x_i(s-\bar{x})(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-2}.\end{aligned}$$

Hence, we get

$$\begin{aligned}&\left(\frac{\partial}{\partial x_0}+\sum_{j=1}^n e_j\frac{\partial}{\partial x_j}\right)G_{\frac{n-1}{2}}(s,x) = -(n+1)(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-1}\\
&+2\left(\frac{n-1}{2}+1\right)(s-\bar{x})(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-2}(s-x_0)\\
&-2\sum_{i=1}^n\left(\frac{n-1}{2}+1\right)x_ie_i(s-\bar{x})(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-2}\\
&= -(n+1)(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-1}+2\left(\frac{n-1}{2}+1\right)(s-\bar{x})s\\
&\cdot(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-2}-2\left(\frac{n-1}{2}+1\right)x(s-\bar{x})(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-2}\\
&= -(n+1)(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-1}+(n+1)(s^2-\bar{x}s-xs+|x|^2)\cdot\\
&(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-2}\\
&= -(n+1)(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-1}+(n+1)(s^2-2x_0s+|x|^2)^{-\frac{n-1}{2}-1}\\
&= 0.\end{aligned}$$

□

We observe that

$$c(h) = (-1)^h\prod_{\ell=1}^h(2\ell)\prod_{\ell=1}^h(n-2\ell+1) = (-1)^h 4^h[h!]^2.$$

Definition 7.4.5 (F_n-kernels)**.** Let n be an odd number. Let $x,\ s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$ the F_n^L kernel as

$$F_n^L(s,x) = \gamma_n(s-\bar{x})Q_{c,s}(x)^{-\frac{n+1}{2}}, \tag{7.45}$$

and F_n^R kernel as

$$F_n^R(s,x) = \gamma_n Q_{c,s}(x)^{-\frac{n+1}{2}}(s-\bar{x}), \tag{7.46}$$

where

$$\gamma_n := (-1)^{\frac{n-1}{2}} 2^{n-1} [(\frac{1}{2}(n-1))!]^2. \tag{7.47}$$

Later we will make use of the following formulas. For $s \notin [x]$, the F_n-kernels satisfy the following equations

$$F_n^L(s,x)s - xF_n^L(s,x) = \gamma_n \mathcal{Q}_{c,s}^{-\frac{n-1}{2}}(x) \tag{7.48}$$

and

$$sF_n^R(s,x) - F_n^R(s,x)x = \gamma_n \mathcal{Q}_{c,s}^{-\frac{n-1}{2}}(x). \tag{7.49}$$

Theorem 7.4.6 (Fueter-Sce mapping theorem in integral form)**.** *Let n be an odd number. Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain, let $J \in \mathbb{S}$ and set $ds_J = ds(-J)$.*

- *If f is a (left) slice hyperholomorphic function on a set W, such that $\overline{U} \subset W$, then the left monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,x) ds_J f(s). \tag{7.50}$$

- *If f is a right slice hyperholomorphic function on a set W, such that $\overline{U} \subset W$, then the right monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} f(s) ds_J F_n^R(s,x). \tag{7.51}$$

The integrals depend neither on U and nor on the imaginary unit $J \in \mathbb{S}$.

Proof. By Theorem 3.1.18 and Theorem 7.4.1 we get

$$\begin{aligned}
\breve{f}(x) &= \Delta^{\frac{n-1}{2}} f(x) \\
&= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} \Delta^{\frac{n-1}{2}} S_L^{-1}(s.x) ds_J f(s) \\
&= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,x) ds_J f(s).
\end{aligned}$$

By Proposition 7.4.4 it follows that$\breve{f}(x)$ is a monogenic function. □

The main advantage to have a Fueter-Sce theorem as an integral transform is that there is no need to compute the powers of the Laplacian applied to a slice hyperholomorphic function to obtain a monogenic function. Moreover, we will see in Chapter 9 that the integral version of the Fueter-Sce theorem is crucial to define a new monogenic functional calculus.

CHAPTER 8

The Poisson kernel and the Fourier transform of the slice monogenic Cauchy kernels

8.1 Motivation

In the previous chapter we have obtained a very simple expression by applying the operator $\Delta^{\frac{n-1}{2}}$ (for n odd), in the variable in x, to the function $S_L^{-1}(s,x)$, written in the second form. The expression is given by

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}. \tag{8.1}$$

In this section we show that formula (8.1) holds also when we consider that n is even. However, in this case $\Delta^{\frac{n-1}{2}}$ is not a differential operator but it is a fractional operator and we have to deal with the Fourier multipliers. Basically in this chapter we study the following problems.

Problem 8.1.1. *(A) Determine the type of hyperholomorphicity of the map*

$$(s,x) \mapsto (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-h},$$

for $h \in \mathbb{R}$, with respect to s and x for $s \notin [x]$.

(B) Compute explicitly the Fourier transform of the slice monogenic Cauchy kernels, written in the second form, and of the F_n-kernels as functions of the Poisson kernel.

(C) Show that the relation $\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}$ is true for all dimensions n replacing suitably the constants γ_n.

8.2 Monogenicity of the Futer-Sce kernel in even dimension

We start by recalling that given a paravector $y = u + J_y v \in \mathbb{R}^{n+1} \setminus (-\infty, 0]$, for $\alpha \in \mathbb{R}$, we can define the fractional powers as

$$y^{\alpha} := e^{\alpha \log y} = e^{\alpha(\ln|y| + J \arg(y))}, \tag{8.2}$$

where $\arg(y) = \arccos \frac{u}{|y|}$. The definition is analogue for the quaternions and the fractional powers so defined are slice monogenic functions, see [44].

In the following result we give an answer to the first point of Problem 8.1.1.

Theorem 8.2.1. *Let λ be a real number and $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let us define:*

$$k_L(s,x) := (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-\lambda}, \qquad \lambda \in \mathbb{R},$$

and

$$k_R(s,x) := (s^2 - 2x_0 s + |x|^2)^{-\lambda}(s-\bar{x}), \qquad \lambda \in \mathbb{R},$$

for

$$s^2 - 2x_0 s + |x|^2 \in \mathbb{R}^{n+1} \setminus (-\infty, 0].$$

Then, the function $k_L(s,x)$ is left monogenic function in the variable x and $k_R(s,x)$ right monogenic in the variable x if and only if $\lambda = \frac{n+1}{2}$.

Proof. We give the details for $k_L(s,x)$, similarly we proceed for $k_R(s,x)$. For simplicity, in the proof, we write $k(s,x)$ for $k_L(s,x)$. We have to compute $(\partial_{x_0} + \partial_{\underline{x}})k(s,x)$, where $\partial_{\underline{x}} = \sum_{j=1}^{n} e_j \partial_{x_j}$. First, we put $s = u + Jv$, thus

$$s^2 - 2x_0 s + |x|^2 = (u^2 - v^2 - 2x_0 u + |x|^2) + J(2uv - 2x_0 v).$$

Using the formula of fractional powers, in (8.2), we get

$$k(s,x) = (s-\bar{x}) e^{\alpha(u,v)}, \tag{8.3}$$

where

$$\begin{aligned}\alpha(u,v) \;&:=\; -\frac{\lambda}{2}\ln[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]+\\ &\quad -\lambda J\arccos\frac{u^2-v^2-2x_0u+|x|^2}{\sqrt{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}}.\end{aligned}$$

Let us denote

$$\beta(u,v) := \frac{u^2-v^2-2x_0u+|x|^2}{\sqrt{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}}.$$

So we have

$$k(s,x)=(s-\bar{x})e^{-\frac{\lambda}{2}\ln[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]-\lambda J\arccos\beta(u,v)}.$$

To compute $\partial_{x_0}k(s,x)$ we calculate the derivative of $\beta(u,v)$ with respect to x_0

$$\frac{\partial\beta(u,v)}{\partial x_0}=\frac{(-2u+2x_0)(2uv-2x_0v)^2+(u^2-v^2-2x_0u+|x|^2)(2uv-2x_0v)2v}{[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]^{3/2}}.$$

Thus

$$\begin{aligned}\frac{\partial\arccos\beta(u,v)}{\partial x_0} \;&=\; -\frac{\partial_{x_0}\beta(u,v)}{\sqrt{1-\beta^2(u,v)}}\\ &=\; -\frac{(-2u+2x_0)(2uv-2x_0v)+(u^2-v^2-2x_0u+|x|^2)2v}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}.\end{aligned}$$

So we have

$$\begin{aligned}\frac{\partial k(s,x)}{\partial x_0} \;&=\; -e^{\alpha(u,v)}+\frac{(u+Jv-\bar{x})2\lambda\,[(u^2-v^2-2x_0u+|x|^2)(u-x_0+Jv)}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\\ &\quad -\frac{J(2uv-2x_0v)(Jv+u-x_0)]]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}e^{\alpha(u,v)}.\end{aligned} \tag{8.4}$$

Now, we compute the derivative of $k(s,x)$ with respect to x_j, $1\leq j\leq n$. As before we start from the derivative of $\beta(u,v)$

$$\frac{\partial\beta(u,v)}{\partial x_j}=\frac{2x_j(2uv-2x_0v)^2}{[(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2]^{3/2}}.$$

Thus

$$\frac{\partial\arccos\beta(u,v)}{\partial x_j}=-\frac{2x_j(2uv-2x_0v)}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}.$$

Therefore

$$\begin{aligned}\frac{\partial k(s,x)}{\partial x_j} &= e_j e^{\alpha(u,v)} - \frac{(u+Jv-\bar{x})2\lambda}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\cdot \\ &\quad \left[(u^2-v^2-2x_0u+|x|^2)-J(2uv-2x_0v)\right] x_j e^{\alpha(u,v)}.\end{aligned}$$

Now, we are ready to compute $\partial_{\underline{x}} k(s,x)$

$$\begin{aligned}\partial_{\underline{x}} k(s,x) &= \sum_{j=1}^{n} e_j \frac{\partial k(s,x)}{\partial x_j} \\ &= -ne^{\alpha(u,v)} - \frac{(u+Jv-\bar{x})2\lambda}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\cdot \\ &\quad \left[(u^2-v^2-2x_0u+|x|^2)-J(2uv-2x_0v)\right]\cdot\left(\sum_{j=1}^{n} e_j x_j\right) e^{\alpha(u,v)}\end{aligned}$$

This implies that

$$\begin{aligned}\partial_{\underline{x}} k(s,x) = -ne^{\alpha(u,v)} - \frac{(u+Jv-\bar{x})2\lambda}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2}\cdot \\ \left[(u^2-v^2-2x_0u+|x|^2)-J(2uv-2x_0v)\right] \underline{x} e^{\alpha(u,v)}.\end{aligned} \tag{8.5}$$

Hence form (8.4) and (8.5) we get

$$\begin{aligned}(\partial_{x_0}+\partial_{\underline{x}})k(s,x) = &-(n+1)e^{\alpha(u,v)} + \frac{2\lambda(u+Jv-\bar{x})(u+Jv-x)}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2} \\ &-\frac{[(u^2-v^2-2x_0u+|x|^2)J(2uv-2x_0v)]}{(u^2-v^2-2x_0u+|x|^2)^2+(2uv-2x_0v)^2} e^{\alpha(u,v)}.\end{aligned}$$

Now, we observe that

$$(u+Jv-\bar{x})(u+Jv-x) = (u^2-v^2-2ux_0+|x|^2)+J(2uv-2x_0v).$$

Setting

$$\gamma(u,v) := (u^2-v^2-2ux_0+|x|^2)+J(2uv-2x_0v)$$

we therefore obtain

$$\begin{aligned}(\partial_{x_0}+\partial_{\underline{x}})k(s,x) &= \left[-(n+1)+2\lambda\frac{\gamma(u,v)\cdot\overline{\gamma(u,v)}}{|\gamma(u,v)|^2}\right] e^{\alpha(u,v)} \\ &= \left[-(n+1)+2\lambda\right] e^{\alpha(u,v)},\end{aligned}$$

so we finally get

$$(\partial_{x_0}+\partial_{\underline{x}})k(s,x) = 0$$

if and only if $\lambda = \frac{n+1}{2}$. □

8.3 The Fourier transform of the slice monogenic Cauchy kernels

The main result of this section is the explicit computation of the Fourier transform of the slice monogenic Cauchy kernels $S_L^{-1}(s,x)$ and $S_R^{-1}(s,x)$ with respect to x when s is a real number. Then by extension we get the Fourier transform when s is a paravector. Firstly, let us introduce the definition of Fourier transform that we will use. This solves the first of the second point of Problem 8.1.1.

Definition 8.3.1. Let $f \in \mathcal{S}(\mathbb{R}^{n+1})$. The Fourier transform of the function f is

$$\hat{f}(\xi) := \mathrm{F}[f(x)](\xi) = \int_{\mathbb{R}^{n+1}} f(x) e^{-i(x,\xi)} dx,$$

where

$$(x,\xi) = \sum_{j=0}^{n} x_j \xi_j.$$

Definition 8.3.2. We define the inverse Fourier transform of the function f in the following way

$$F^{-1}[f(\xi)](x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} f(\xi) e^{i(x,\xi)} d\xi.$$

In this paper we will use the following important result

Theorem 8.3.3 (Plancherel's Theorem)**.** *If $f, g \in \mathcal{S}(\mathbb{R}^{n+1})$ then*

$$\int_{\mathbb{R}^{n+1}} f(x)\overline{g(x)} dx = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \mathrm{F}(f)(\xi)\overline{\mathrm{F}(g(\xi))} d\xi. \tag{8.6}$$

Remark 8.3.4. Using the Plancherel's theorem it is possible to obtain

$$\int_{\mathbb{R}^{n+1}} F^{-1}(f)(x)\overline{F^{-1}(g(x))} dx = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} f(\xi)\overline{g(\xi)} d\xi.. \tag{8.7}$$

In the sequel we will need this result.

Theorem 8.3.5. *[91, Sect. B.5] Let $f(|\underline{x}|)$ be a radial function in $\mathcal{S}(\mathbb{R}^n)$ with $n \geq 2$. Then the Fourier transform of f is also radial and has the form*

$$\hat{f}(|\underline{\xi}|) = (2\pi)^{\frac{n}{2}} |\underline{\xi}|^{-\frac{n-2}{2}} \int_0^\infty J_{\frac{n-2}{2}}(|\underline{\xi}|r) r^{\frac{n}{2}} f(r)\, dr,$$

where $r = |\underline{x}|$ and $J_{\frac{n-2}{2}}$ are the Bessel functions.

Remark 8.3.6. (See [117]) The formula in Theorem 8.3.5 is also valid for all functions
$$f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Now, we start our computations.

Theorem 8.3.7. *Let us assume $s_0 \in \mathbb{R}$ and $x \in \mathbb{R}^{n+1}$. If we consider the slice monogenic Cauchy kernels $S_L^{-1}(s_0, x)$ and $S_R^{-1}(s_0, x)$ written in form II (see Definition 3.1.16) then their Fourier transforms with respect to x are equal and given by*

$$\mathrm{F}[S_L^{-1}(s,\cdot)](\xi) = \mathrm{F}[S_R^{-1}(s_0,\cdot)](\xi) = c_n \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is_0\xi_0}, \quad \xi_0 + \underline{\xi} \neq 0$$

where
$$c_n := i2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

Moreover, if $s = s_0 + \underline{s} \in \mathbb{R}^{n+1}$ is a paravector the term $e^{-is_0\xi_0}$ extends to the intrinsic entire slice monogenic function $e^{-is\xi_0}$ and we have the Fourier transforms of the Cauchy kernels:

$$\mathrm{F}[S_L^{-1}(s,\cdot)](\xi) = c_n \, \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \, e^{-is\xi_0}, \quad \xi_0 + \underline{\xi} \neq 0 \tag{8.8}$$

and

$$\mathrm{F}[S_R^{-1}(s,\cdot)](\xi) = c_n \, e^{-is\xi_0} \, \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}}, \quad \xi_0 + \underline{\xi} \neq 0. \tag{8.9}$$

The extension $\mathrm{F}[S_L^{-1}(s,\cdot)](\xi)$ is right slice monogenic in s, while $\mathrm{F}[S_R^{-1}(s,\cdot)](\xi)$ is right slice monogenic in s.

Proof. In the following proof we always work with $s = s_0 \in \mathbb{R}$ since we have

$$\begin{aligned} S_L^{-1}(s,x) &= (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-1} \\ &= (s^2 - 2x_0 s + |x|^2)^{-1}(s-\bar{x}) = S_R^{-1}(s,x). \end{aligned}$$

The extension from $s = s_0$ to $s = s_0 + \underline{s}$ is immediate. So in our computations, we set

$$S^{-1}(s,x) := S_L^{-1}(s,x) = S_R^{-1}(s,x) = \frac{s - x_0 + \underline{x}}{(s-x_0)^2 + |\underline{x}|^2}, \quad s \in \mathbb{R}.$$

We put $x = x_0 + \underline{x}$ and we recall the identification of the paravectors with $(x_0, ..., x_n)$. Since the function $S^{-1}(s, x)$ is not in $L^1(\mathbb{R}^{n+1})$ we have to perform the computations in the distributional sense. Firstly, we consider the following function

$$f_{\underline{x}}(x_0) := \frac{s - x_0}{(s - x_0)^2 + |\underline{x}|^2}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx &= \int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}_0\left(\mathrm{F}_n\varphi_{\underline{x}}\right)(x_0)}dx_0 d\underline{x} \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}} \left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)\overline{\left(\mathrm{F}_n\varphi_{\underline{x}}\right)(\xi_0)}d\xi_0 d\underline{x}, \end{aligned}$$

where $d\underline{x} = dx_1...dx_n$, F_0 is the Fourier transform with respect to the variable x_0 and F_n is the Fourier transform with respect to the other variables. Now, we compute $\mathrm{F}_0 f_{\underline{x}}(\xi_0)$. First of all we make the following change of variables $s + y = x_0$, thus by basic properties of the Fourier transform we have

$$\begin{aligned} \mathrm{F}_0 f_{\underline{x}}(\xi_0) &= -\mathrm{F}_y[y(y^2 + |\underline{x}|^2)^{-1}](\xi_0)e^{-is\xi_0} \\ &= -i\frac{d}{d\xi_0}\mathrm{F}_y\left(\frac{1}{|\underline{x}|^2 + y^2}\right)(\xi_0)e^{-is\xi_0} \\ &= -i\frac{\pi}{|\underline{x}|}\left(\frac{d}{d\xi_0}e^{-|\underline{x}||\xi_0|}\right)e^{-is\xi_0} \\ &= i\frac{\pi\xi_0}{|\xi_0|}e^{-|\underline{x}||\xi_0|}e^{-is\xi_0}. \end{aligned} \tag{8.10}$$

Since $\varphi_{\underline{x}}(x_0) = \varphi(x)$ and by Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} \mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)\overline{\varphi(\xi)}d\underline{\xi}d\xi_0 \\ &= \int_{\mathbb{R}^{n+1}} \mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)\overline{\varphi(\xi)}d\xi. \end{aligned}$$

We finish this first part by computing $\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)$. By Theorem 8.3.5 with $r = |\underline{x}|$ we have

$$\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0) = (2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}i\frac{\pi\xi_0}{|\xi_0|}e^{-is\xi_0}\int_0^\infty J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}e^{-r|\xi_0|}dr.$$

Now, we make another change of variables $t=|\underline{\xi}|r$.

$$\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0)=(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-n} i\frac{\pi\xi_0}{|\xi_0|}e^{-is\xi_0}\int_0^\infty J_{\frac{n-2}{2}}(t)t^{\frac{n}{2}}e^{-\frac{t|\xi_0|}{|\underline{\xi}|}}dt.$$

From [90, formula 6.623(2)] we know that

$$\int_0^\infty e^{-at}t^{\nu+1}J_\nu(bt)dt=\frac{2a(2b)^\nu\Gamma\left(\nu+\frac{3}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{\nu+\frac{3}{2}}},\qquad \nu>-1,\quad a>0,\quad b>0.$$

In our case $a:=\frac{|\xi_0|}{|\underline{\xi}|}$, $\nu:=\frac{n}{2}-1$, $b:=1$. Since $n\geq 2$ all conditions on the parameters are satisfied. Thus, we have

$$\begin{aligned}\mathrm{F}_n\left(\mathrm{F}_0 f_{\underline{x}}\right)(\xi_0) &=\frac{i}{2}(2\pi)^{\frac{n}{2}+1}\frac{\xi_0}{|\xi_0|}|\underline{\xi}|^{-n}e^{-is\xi_0}2\frac{|\xi_0|}{|\underline{\xi}|}2^{\frac{n}{2}-1}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\left(\frac{\xi_0^2}{|\underline{\xi}|^2}+1\right)^{\frac{n+1}{2}}}\\ &= \frac{i2^n\pi^{\frac{n+1}{2}}\xi_0|\underline{\xi}|^{-n-1}e^{-is\xi_0}|\underline{\xi}|^{n+1}\Gamma\left(\frac{n+1}{2}\right)}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}\\ &= \frac{i2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)\xi_0 e^{-is\xi_0}}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}.\end{aligned}$$

Hence

$$\int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx=c_n\int_{\mathbb{R}^{n+1}}\frac{\xi_0}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}e^{-is\xi_0}\overline{\varphi}(\xi)d\xi,\quad (8.11)$$

where $c_n:=i2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)$. Now, we compute the Fourier transform of

$$h_{\underline{x}}(x_0):=\frac{\underline{x}}{(s-x_0)^2+|\underline{x}|^2}=\sum_{j=1}^n e_jx_ju_{\underline{x}}(x_0),$$

where we have set

$$u_{\underline{x}}(x_0):=\frac{1}{(s-x_0)^2+|\underline{x}|^2}.$$

Let $\varphi\in\mathcal{S}(\mathbb{R}^{n+1})$

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx &= \int_{\mathbb{R}^{n+1}}\sum_{j=1}^n e_jx_ju_{\underline{x}}(x_0)\overline{\mathrm{F}_0\left(\mathrm{F}_n(\varphi_{\underline{x}})\right)(x_0)}dx_0d\underline{x}\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}}\sum_{j=1}^n e_jx_j\mathrm{F}_0(u_{\underline{x}})(\xi_0)\overline{\mathrm{F}_n\varphi_{\underline{x}}(\xi_0)}d\xi_0d\underline{x}.\end{aligned}$$

Now, we compute $\mathrm{F}_0(u_{\underline{x}})(\xi_0)$ using the following change of variable $s + y = x_0$

$$\mathrm{F}_0(u_{\underline{x}})(\xi_0) = \mathrm{F}_y\left(\frac{1}{y^2+|\underline{x}|^2}\right)(\xi_0)e^{-is\xi_0} = \frac{\pi}{|\underline{x}|}e^{-|\underline{x}||\xi_0|}e^{-is\xi_0}.$$

By Fubini's theorem and the fact that $\varphi_{\underline{x}}(x_0) = \varphi(x)$ we have

$$\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx = \int_{\mathbb{R}}\int_{\mathbb{R}^n}\sum_{j=1}^n e_j\mathrm{F}_n\left(x_j\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0)\overline{\varphi(\xi)}d\underline{\xi}d\xi_0.$$

From the following basic property of the Fourier transform

$$\mathrm{F}[xf(x)](\xi) = i\frac{d}{d\xi}(\mathrm{F}f(x))(\xi), \quad x,\xi\in\mathbb{R}$$

we get

$$\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx = i\int_{\mathbb{R}}\int_{\mathbb{R}^n}\sum_{j=1}^n e_j\frac{\partial}{\partial\xi_j}\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0)\overline{\varphi(\xi)}d\underline{\xi}d\xi_0.$$

We complete the proof of this theorem by computing $\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0)$. By Theorem 8.3.5, with $r = |\underline{x}|$, we get

$$\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0) = (2\pi)^{\frac{n}{2}}\pi e^{-is\xi_0}|\underline{\xi}|^{-\frac{n-2}{2}}\int_0^\infty J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}-1}e^{-r|\xi_0|}dr.$$

Now we put $t = r|\underline{\xi}|$

$$\mathrm{F}_n(\mathrm{F}_0(u_{\underline{x}}))(\xi_0) = \frac{1}{2}(2\pi)^{\frac{n}{2}+1}e^{-is\xi_0}|\underline{\xi}|^{1-n}\int_0^\infty J_{\frac{n-2}{2}}(t)t^{\frac{n}{2}-1}e^{-t\frac{|\xi_0|}{|\underline{\xi}|}}dt.$$

From [90, formula 6.623 (1)] we know that

$$\int_0^\infty e^{-at}t^\nu J_\nu(bt)dt = \frac{(2b)^\nu\Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}(a^2+b^2)^{\nu+\frac{1}{2}}}, \qquad \nu > -\frac{1}{2}, \quad a>0, \quad b>0.$$

Thus by putting $b := 1$, $\nu := \frac{n}{2} - 1$ and $a := \frac{|\xi_0|}{|\underline{\xi}|}$ we obtain

$$\begin{aligned}
\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0) &= \frac{1}{2}(2\pi)^{\frac{n}{2}+1}e^{-is\xi_0}\left(\frac{|\underline{\xi}|^{1-n}2^{\frac{n}{2}-1}\Gamma\left(\frac{n-1}{2}\right)}{\sqrt{\pi}\left(\frac{\xi_0^2}{|\underline{\xi}|^2}+1\right)^{\frac{n-1}{2}}}\right) \\
&= 2^{n-1}\pi^{\frac{n+1}{2}}e^{-is\xi_0}\Gamma\left(\frac{n-1}{2}\right)\left(\frac{|\underline{\xi}|^{1-n}|\underline{\xi}|^{n-1}}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n-1}{2}}}\right) \\
&= 2^{n-1}\pi^{\frac{n+1}{2}}e^{-is\xi_0}\Gamma\left(\frac{n-1}{2}\right)\left(\frac{1}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n-1}{2}}}\right).
\end{aligned}$$

We compute the derivative

$$\begin{aligned}
\frac{\partial}{\partial\xi_j}\left(\frac{1}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n-1}{2}}}\right) &= -\frac{\left(\frac{n-1}{2}\right)(\xi^2+|\underline{\xi}|^2)^{\frac{n-3}{2}}2\xi_j}{(\xi_0^2+|\underline{\xi}|^2)^{n-1}} \\
&= -\frac{2\xi_j\left(\frac{n-1}{2}\right)}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}.
\end{aligned} \tag{8.12}$$

Now, by using the following property of the Gamma function $\Gamma(x+1) = x\Gamma(x)$, for $x > 0$, we obtain

$$\begin{aligned}
\sum_{j=1}^{n} e_j\frac{\partial}{\partial\xi_j}\mathrm{F}_n\left(\mathrm{F}_0(u_{\underline{x}})\right)(\xi_0) &= -\frac{2^{n-1}\pi^{\frac{n+1}{2}}e^{-is\xi_0}\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)2\sum_{j=1}^{n}e_j\xi_j}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}} \\
&= -\frac{2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}\underline{\xi}e^{-is\xi_0}.
\end{aligned}$$

Hence we get

$$\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx = -c_n\int_{\mathbb{R}^{n+1}}\frac{\underline{\xi}}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}e^{-is\xi_0}\overline{\varphi(\xi)}d\xi, \tag{8.13}$$

where $c_n := i2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)$. Finally from (8.11) and (8.13) we get

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}} S^{-1}(s,x)\overline{\mathrm{F}(\varphi)(x)}dx &= \int_{\mathbb{R}^{n+1}} f_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx \\
&\quad -\int_{\mathbb{R}^{n+1}} h_{\underline{x}}(x_0)\overline{\mathrm{F}(\varphi)(x)}dx \\
&= c_n\int_{\mathbb{R}^{n+1}}\frac{\bar{\xi}}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}}e^{-is\xi_0}\overline{\varphi(\xi)}d\xi.
\end{aligned}$$

This proves (8.8) and (8.9), respectively.

We are now in the position to observe that the term $e^{-is\xi_0}$, for $s = s_0$ extends to the entire intrinsic slice monogenic function $e^{-i(s_0+\underline{s})\xi_0}$. So the function

$$s_0 \mapsto c_n \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} e^{-is_0\xi_0}$$

has the right slice monogenic extension in $s \in \mathbb{R}^{n+1}$

$$\mathrm{F}[S_L^{-1}(s, \cdot)](\xi) = c_n \ \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}} \ e^{-is\xi_0}$$

while the right slice monogenic extension in $s \in \mathbb{R}^{n+1}$ is

$$\mathrm{F}[S_R^{-1}(s, \cdot)](\xi) = c_n \ e^{-is\xi_0} \ \frac{\bar{\xi}}{(\xi_0^2 + |\underline{\xi}|^2)^{\frac{n+1}{2}}}$$

and this conclude the proof. □

8.4 The Fourier transform of the F_n-kernels

Thanks to Theorem 8.2.1 the F_n-kernels are meaningful also with n odd where we interpret the fractional powers of paravectors are slice monogenic functions.

As we have shown when n be an odd number and x, $s \in \mathbb{R}^{n+1}$, for $s \notin [x]$, the relations

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s, x) = \gamma_n (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and

$$\Delta^{\frac{n-1}{2}} S_R^{-1}(s, x) = \gamma_n (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}(s - \bar{x}),$$

where γ_n are given by (7.47). We now let n be any natural number and when n is even we interpret the terms $(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}$ as fractional power for x, $s \in \mathbb{R}^{n+1}$. We recall, for $s \notin [x]$, the definition of the left F_n^L-kernel as

$$F_n^L(s, x) := \gamma_n (s - \bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}},$$

and the right F_n^R-kernel as

$$F_n^R(s, x) := \gamma_n (s^2 - 2x_0)s + |x|^2)^{-\frac{n+1}{2}}(s - \bar{x}),$$

where γ_n, are given by (7.47), are now interpreted in terms of the Euler's Gamma function

$$\gamma_n := (-1)^{\frac{n-1}{2}} 2^{n-1} \left[\Gamma\left(\frac{n+1}{2} \right) \right]^2 .$$

We will compute the Fourier transforms of $F_n^L(s,x)$ and $F_n^R(s,x)$ with respect to x. This solves the second part of the second point of Problem 8.1.1.

Remark 8.4.1. Observe that when $s = s_0 \in \mathbb{R}$, for $s \notin [x]$, then we have

$$\begin{aligned} F_n^L(s_0,x) &= \gamma_n (s_0 - \bar{x})(s_0^2 - 2x_0 s_0 + |x|^2)^{-\frac{n+1}{2}} \\ &= \gamma_n (s_0^2 - 2x_0 s_0 + |x|^2)^{-\frac{n+1}{2}} (s_0 - \bar{x}) \\ &= F_n^R(s_0,x). \end{aligned}$$

So for simplicity in the following when $s = s_0 \in \mathbb{R}$ we use the notation

$$F_n(s_0,x) := F_n^L(s_0,x) = F_n^R(s_0,x).$$

Theorem 8.4.2. *Let us assume $x \in \mathbb{R}^{n+1}$ and s a real number. The Fourier transform of $F_n(s,x)$ with respect to x is*

$$\widehat{F_n}(s,\xi) = k_n \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2} e^{-is\xi_0},$$

where

$$k_n := i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2} \right).$$

Moreover, if $s = s_0 + \underline{s} \in \mathbb{R}^{n+1}$ is a paravector the term $e^{-is_0\xi_0}$ extends to the intrinsic entire slice monogenic function $e^{-is\xi_0}$ and we have the Fourier transforms of the kernels F_n^L and F_n^R:

$$\mathrm{F}[F_n^L(s,\cdot)](\xi) = k_n \, \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2} \, e^{-is\xi_0}, \quad \xi_0 + \underline{\xi} \neq 0$$

and

$$\mathrm{F}[F_n^R(s,\cdot)](\xi) = k_n \, e^{-is\xi_0} \, \frac{\bar{\xi}}{\xi_0^2 + |\underline{\xi}|^2}, \quad \xi_0 + \underline{\xi} \neq 0.$$

The extension $\mathrm{F}[F_n^L(s,\cdot)](\xi)$ is right slice monogenic in s, while $\mathrm{F}[F_n^R(s,\cdot)](\xi)$ is right slice monogenic in s.

Proof. We observe that the Fourier transform of the kernel $F_n(s,\cdot)$ is meaningful and from similar computations at the beginning of Theorem 8.3.7 we obtain

$$\begin{aligned}
\widehat{F_n}(s,\xi) &= \gamma_n s \int_{\mathbb{R}} e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}} \\
&\quad J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0 \\
&\quad -\gamma_n \int_{\mathbb{R}} x_0 e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}}\cdot \\
&\quad J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0 \\
&\quad +\gamma_n i\sum_{j=1}^n e_j \int_{\mathbb{R}} e^{-ix_0\xi_0}\frac{\partial}{\partial \xi_j}\Big((2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\cdot \\
&\quad \int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}}J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0\Big) \\
&:= \gamma_n\left(\widehat{F_{n,1}}(s,\xi)+\widehat{F_{n,2}}(s,\xi)+\widehat{F_{n,3}}(s,\xi)\right).
\end{aligned}$$

Now, we focus on the first two members

$$\begin{aligned}
\widehat{F_{n,1}}(s,\xi) &+ \widehat{F_{n,2}}(s,\xi) \\
&= s\int_{\mathbb{R}} e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}}\cdot \\
&\quad J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0 \\
&\quad - \int_{\mathbb{R}} x_0 e^{-ix_0\xi_0}(2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\int_0^\infty (s^2-2x_0s+x_0^2+r^2)^{-\frac{n+1}{2}}\cdot \\
&\quad J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0 \\
&= (2\pi)^{\frac{n}{2}}|\underline{\xi}|^{-\frac{n-2}{2}}\int_{\mathbb{R}}(s-x_0)e^{-ix_0\xi_0}\int_0^\infty [(s-x_0)^2+r^2]^{-\frac{n+1}{2}}\cdot \\
&\quad J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0.
\end{aligned}$$

Firstly, we solve the integral in the variable r. From [90, formula 6.565 (3)] we know that for $b>0$, $\nu>-1$ we have

$$\int_0^\infty x^{\nu+1}(x^2+a^2)^{-\nu-\frac{3}{2}}J_\nu(bx)dx = \frac{b^\nu\sqrt{\pi}}{2^{\nu+1}|a|e^{|a|b}\Gamma\left(\nu+\frac{3}{2}\right)}. \tag{8.14}$$

In our case $b := |\underline{\xi}|$, $\nu = \frac{n}{2} - 1$ and $a = s - x_0$. Then

$$\begin{aligned}\widehat{F_{n,1}}(s,\xi) + \widehat{F_{n,2}}(s,\xi) &= \frac{(2\pi)^{\frac{n}{2}} 2^{-\frac{n}{2}} \sqrt{\pi} |\underline{\xi}|^{-\frac{n}{2}+1} |\underline{\xi}|^{\frac{n}{2}-1}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{\mathbb{R}} e^{-ix_0\xi_0} \frac{(s-x_0)}{|s-x_0|} \\ &\quad e^{-|\underline{\xi}||s-x_0|} dx_0 \\ &= \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} \int_{\mathbb{R}} e^{-ix_0\xi_0} \frac{(s-x_0)}{|s-x_0|} e^{-|\underline{\xi}||s-x_0|} dx_0.\end{aligned}$$

Now, we put $s + y = x_0$. Thus we have

$$\widehat{F_{n,1}}(s,\xi) + \widehat{F_{n,2}}(s,\xi) = -\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)} e^{-is\xi_0} \int_{\mathbb{R}} e^{-iy\xi_0} \frac{y}{|y|} e^{-|\underline{\xi}||y|} dy.$$

From [116, formula 3.2 pag 11] we know that

$$\int_0^\infty \cos(xu) \frac{e^{-ax}}{x} dx = -\frac{\log(a^2 + u^2)}{2}.$$

Thus by the Euler's formula we get

$$\begin{aligned}\mathrm{F}\left(\frac{1}{|y|} e^{-|\underline{\xi}||y|}\right)(\xi_0) &= 2\int_0^\infty \cos(y\xi_0) \frac{1}{y} e^{-|\underline{\xi}|y} dy \\ &= -\log(\xi_0^2 + |\underline{\xi}|^2).\end{aligned}$$

Using basic properties of the Fourier transform we obtain

$$\begin{aligned}\int_{\mathbb{R}} e^{-iy\xi_0} \frac{y}{|y|} e^{-|\underline{\xi}||y|} dy &= \mathrm{F}\left(\frac{y}{|y|} e^{-|\underline{\xi}||y|}\right)(\xi_0) \\ &= i\frac{d}{d\xi_0} \mathrm{F}\left(\frac{1}{|y|} e^{-|\underline{\xi}||y|}\right)(\xi_0) \\ &= -i\frac{d}{d\xi_0} \left(\log(\xi_0^2 + |\underline{\xi}|^2)\right) \\ &= -\frac{2i\xi_0}{\xi_0^2 + |\underline{\xi}|^2}.\end{aligned}$$

Therefore

$$\widehat{F_{n,1}}(s,\xi) + \widehat{F_{n,2}}(s,\xi) = \frac{2i\pi^{\frac{n+1}{2}} e^{-is\xi_0} \xi_0}{\Gamma\left(\frac{n+1}{2}\right)(\xi_0^2 + |\underline{\xi}|^2)}.$$

Finally, we multiply by $\gamma_n := (-1)^{\frac{n-1}{2}} 2^{n-1} \left[\Gamma\left(\frac{n+1}{2}\right)\right]^2$

$$\gamma_n(\widehat{F_{n,1}}(s,\xi) + \widehat{F_{n,2}}(s,\xi)) = \frac{i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\xi_0^2 + |\underline{\xi}|^2} \xi_0 e^{-is\xi_0}. \quad (8.15)$$

Now we compute $\widehat{F_{n,3}}(s,\xi)$.

$$\begin{aligned}\widehat{F_{n,3}}(s,\xi) &= i(2\pi)^{\frac{n}{2}}\sum_{j=1}^{n} e_j\frac{\partial}{\partial \xi_j}\left(|\underline{\xi}|^{-\frac{n-2}{2}}\int_{\mathbb{R}} e^{-ix_0\xi_0}\int_0^{\infty}[(s-x_0)^2+r^2]^{-\frac{n+1}{2}}\right.\\ &\quad \left. J_{\frac{n-2}{2}}(|\underline{\xi}|r)r^{\frac{n}{2}}drdx_0\right).\end{aligned}$$

As before we compute the integral in the variable r using (8.14) with $b := |\underline{\xi}|$, $\nu = \frac{n}{2}-1$ and $a = s-x_0$. Thus we have

$$\begin{aligned}\widehat{F_{n,3}}(s,\xi) &= \frac{i(2\pi)^{\frac{n}{2}}2^{-\frac{n}{2}}\sqrt{\pi}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^{n} e_j\frac{\partial}{\partial \xi_j}\left(|\underline{\xi}|^{-\frac{n}{2}+1}|\underline{\xi}|^{\frac{n}{2}-1}\int_{\mathbb{R}} e^{-ix_0\xi_0}|s-x_0|^{-1}\cdot\right.\\ &\quad \left. e^{-|\underline{\xi}||s-x_0|}dx_0\right)\\ &= \frac{i\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^{n} e_j\frac{\partial}{\partial \xi_j}\left(\int_{\mathbb{R}} e^{-ix_0\xi_0}|s-x_0|^{-1}e^{-|\underline{\xi}||s-x_0|}dx_0\right).\end{aligned}$$

We put $s+y=x_0$.

$$\widehat{F_{n,3}}(s,\xi) = \frac{i\pi^{\frac{n+1}{2}}e^{-is\xi_0}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^{n} e_j\frac{\partial}{\partial \xi_j}\left(\int_{\mathbb{R}} e^{-iy\xi_0}|y|^{-1}e^{-|\underline{\xi}||y|}dy\right).$$

From (8.15) we know that

$$\int_{\mathbb{R}} e^{-iy\xi_0}|y|^{-1}e^{-|\underline{\xi}||y|}dy = -\log(\xi_0^2+|\underline{\xi}|^2).$$

Therefore

$$\begin{aligned}\widehat{F_{n,3}}(s,\xi) &= -\frac{i\pi^{\frac{n+1}{2}}e^{-is\xi_0}}{\Gamma\left(\frac{n+1}{2}\right)}\sum_{j=1}^{n} e_j\frac{\partial}{\partial \xi_j}\left(\log(|\underline{\xi}|^2+\xi_0^2)\right)\\ &= -\frac{2i\pi^{\frac{n+1}{2}}e^{-is\xi_0}}{\Gamma\left(\frac{n+1}{2}\right)(\xi_0^2+|\underline{\xi}|^2)}\sum_{j=1}^{n} e_j\xi_j\\ &= -\frac{2i\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)(\xi_0^2+|\underline{\xi}|^2)}\,\underline{\xi}e^{-is\xi_0}.\end{aligned}$$

Finally, multiplying by $\gamma_n := (-1)^{\frac{n-1}{2}}2^{n-1}\left[\Gamma\left(\frac{n+1}{2}\right)\right]^2$ we have

$$\gamma_n\widehat{F_{n,3}}(s,\xi) = -\frac{i(-1)^{\frac{n-1}{2}}2^n\pi^{\frac{n+1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\xi_0^2+|\underline{\xi}|^2}\,\underline{\xi}e^{-is\xi_0}. \tag{8.16}$$

Putting together (8.15) and (8.16) we get

$$\widehat{F_n}(s,\xi) = k_n \frac{\bar{\xi}}{\xi^2 + |\underline{\xi}|^2} e^{-is\xi_0},$$

where $k_n := i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right)$. Finally, the slice monogenic extensions are obtained reasoning as in the case of the Cauchy kernel. □

8.5 The relation of the kernels S^{-1} and F_n via the Fourier transform

In this section we show the last point of Problem 8.1.1. Before to prove a fundamental result we recall that when n is even the operator $\Delta^{\frac{n-1}{2}}$ is defined by the Fourier multipliers

$$\Delta^{\frac{n-1}{2}} f(x) = F^{-1}[(i|\xi|)^{n-1} \mathrm{F}(f(x))(\xi)](x), \tag{8.17}$$

where F and F^{-1}, are respectively, the Fourier and the inverse Fourier transformations, given, respectively in Definition 8.3.1 and Definition 8.3.2.

Theorem 8.5.1. *For $x \in \mathbb{R}^{n+1}$ and $s \in \mathbb{R}$ we have that*

$$\Delta^{\frac{n-1}{2}} S^{-1}(s,x) = \gamma_n (s - \bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}} \tag{8.18}$$

Proof. If n is odd, this can be proved through pointwise differential computation, see Theorem 7.4.1. While for the case n even the result will be showed for any $\varphi \in \mathcal{S}(\mathbb{R}^{n+1})$. Firstly we prove the equality for $s \in \mathbb{R}$. The formula (8.7) and Theorem 8.3.3 imply that we can pass the factional Laplacian to the test function, so we have

$$\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x) \overline{\varphi(x)} dx = \int_{\mathbb{R}^{n+1}} S^{-1}(s,x) \overline{\Delta^{\frac{n-1}{2}} \varphi(x)} dx.$$

Using another time Theorem 8.3.3 we get

$$\begin{aligned}\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x) \overline{\varphi(x)} dx &= \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \mathrm{F}\left(S^{-1}(s,.)\right)(\xi) \cdot \\ &\quad \overline{\mathrm{F}\left(\Delta^{\frac{n-1}{2}} \varphi(x)\right)(\xi)} d\xi.\end{aligned}$$

From Theorem 8.3.7 and Theorem 8.4.2 we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x)\overline{\varphi(x)}dx \\
&\quad = \frac{1}{(2\pi)^{n+1}} i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^{n+1}} \frac{\bar{\xi} e^{-is\xi_0}}{(\xi_0^2+|\underline{\xi}|^2)^{\frac{n+1}{2}}} \cdot \\
&\quad |\xi|^{n-1}\overline{\hat{\varphi}(\xi)}d\xi \\
&\quad = \frac{1}{(2\pi)^{n+1}} i(-1)^{\frac{n-1}{2}} 2^n \pi^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^{n+1}} \frac{\bar{\xi} e^{-is\xi_0}}{\xi_0^2+|\underline{\xi}|^2} \overline{\hat{\varphi}(\xi)}d\xi \\
&\quad = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} \widehat{F_n}(s,\xi)\overline{\hat{\varphi}(\xi)}d\xi.
\end{aligned}$$

Finally by applying another time the Theorem 8.3.3 we get

$$\int_{\mathbb{R}^{n+1}} \Delta^{\frac{n-1}{2}} S^{-1}(s,x)\overline{\varphi(x)}dx = \int_{\mathbb{R}^{n+1}} F_n(s,x)\overline{\varphi(x)}dx.$$

□

Corollary 8.5.2. *The relation* (8.18) *extends to* $s \in \mathbb{R}^{n+1}$*, considering the left and the right slice monogenic extensions.*

Proof. The extension of the equation (8.18) to $s \in \mathbb{R}^{n+1}$ follows from the Identity principle, because the function $e^{-is\xi_0}$ is trivially intrinsic slice monogenic. □

CHAPTER 9

The F-functional calculus for bounded operators

9.1 Motivation

The Fueter-Sce mapping theorem in integral form, introduced in Theorem 7.4.6, provides an integral transform that turns slice hyperholomorphic functions into monogenic ones. If we formally replace the variable x of this integral transform by an operator T, we get a functional calculus for monogenic functions based on the theory of slice hyperholomorphic functions: the F-functional calculus. This functional calculus is defined on the S-spectrum and generates a monogenic functional calculus in the spirit of McIntosh and collaborators, see [99, 101, 108, 112]. We can summarize the construction that leads to the F-functional calculus by the following diagram

$$
\begin{array}{ccc}
\mathcal{SH}(U) & & \mathcal{AM}(U) \\
\downarrow & & \\
\text{Slice Cauchy Formula} & \xrightarrow{T_{FS2}} & \text{Fueter} - \text{Sce theorem in integral form} \\
\downarrow & & \downarrow \\
\text{S}-\text{Functional calculus} & & F - \text{functional calculus}
\end{array}
\tag{9.1}
$$

Remark 9.1.1. Observe that in the above diagram the arrow from the space of axially monogenic function $\mathcal{AM}(U)$ is missing because the F-functional calculus is deduced from the slice hyperholomorphic Cauchy formula.

9.2 The F-resolvent operators and the F-functional calculus

In this subsection we will show that the action of the Fueter-Sce map to the paravector monomial x^k leads to the Clifford-Appell polynomials, given by [31, 32]. The family of $\{P_k\}_{k\in\mathbb{N}}$ of Clifford-Appell polynomials is defined as

$$P_k^n(x) := \sum_{s=0}^{k} T_s^k(n) x^{k-s}\overline{x}^s, \qquad x \in \mathbb{R}^{n+1}, \tag{9.2}$$

where $T_s^k(n)$ is defined as

$$T_s^k(n) := \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}, \qquad n \geq 1, \tag{9.3}$$

where $(.)_k$ stands for the Pochhammer symbol defined by $(a)_s = \frac{\Gamma(a+s)}{\Gamma(a)}$ or $(a)_s = a(a+1)(a+2)...(a+s-1)$, for $s > 0$ and $(a)_0 = 1$ for $s = 0$. It is proved in [32] that

$$\sum_{s=0}^{k} T_s^k(n) = 1. \tag{9.4}$$

The polynomials $P_k^n(x)$ satisfy the Appell property

$$\frac{1}{2}\overline{\mathcal{D}} P_k^n(x_0 + \underline{x}) = k P_{k-1}^n(x_0 + \underline{x}), \tag{9.5}$$

where $\overline{\mathcal{D}} = \partial_{x_0} - \sum_{\ell=1}^{n} e_\ell \partial_{x_\ell}$. Moreover the Clifford-Appell polynomials are monogenic i.e.

$$\mathcal{D} P_k^n(x) = 0.$$

Since the Clifford-Appell polynomials are axially monogenic, see [32], and by the fact that

$$P_k^n(x_0+\underline{0}) = x_0^k \sum_{s=0}^{k} T_k^s(n) = x_0^k,$$

it is clear that

$$P_k^n(x) = GCK[x_0^k]. \tag{9.6}$$

Combining this result with Theorem 7.3.3 we obtain the following result.

Theorem 9.2.1. *Let $n \geq 1$ be a fixed odd number and $k \geq 0$. Then, for any $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ it holds that*

$$\Delta^{\frac{n-1}{2}}(x^{n+k-1}) = \gamma_n \frac{(n+k-1)!}{(n-1)!k!} P_k^n(x), \tag{9.7}$$

where γ_n are defined in (7.47).

Proof. By the identity $\partial_{x_0}^{n-1}[x_0^{n-1+k}] = \frac{(n+k-1)!}{k!} x_0^k$ and the fact that $P_k^n(x)$ is the generalized CK-extension of the monomial x_0^k we get

$$\begin{aligned} P_k^n(x) &= GCK[x_0^k] \\ &= \frac{k!}{(n+k-1)!} GCK \circ \partial_{x_0}^{n-1}[x_0^{n-1+k}]. \end{aligned}$$

Finally, using Theorem 7.3.3 we obtain

$$\Delta^{\frac{n-1}{2}}(x^{n+k-1}) = \gamma_n \frac{(n+k-1)!}{(n-1)!k!} P_k^n(x).$$

□

Remark 9.2.2. The above result extends to arbitrary dimension those obtained for the quaternionic setting in [77]. Indeed, if we consider $n = 3$ in (9.7) we get

$$\begin{aligned} \Delta(x^{k+2}) &= -\frac{2(k+2)!}{k!} P_k^3(x) \\ &= -2(k+2)(k+1) P_k^3(x), \end{aligned}$$

which is exactly the identity obtained in [77, Rem. 3.9].

Corollary 9.2.3. *Let $x \in \mathbb{R}^{n+1}$ and $n \geq 3$ be a fixed odd number. Then*

$$\Delta^{\frac{n-1}{2}}(x^m) = \begin{cases} \gamma_n \frac{m!}{(n-1)!(m-n+1)!} P^n_{m+1-n}(x) & \textit{if} \quad m > n-1, \\ \\ \gamma_n & \textit{if} \quad m = n-1, \\ \\ 0 & \textit{if} \quad m < n-1. \end{cases}$$

where γ_n are defined in (7.47).

Proof. For the case $m > n-1$, it is enough to set $m := n+k-1$ in formula (9.7).
For the second case we substitute $m = n-1$ in the first case and we obtain

$$\Delta^{\frac{n-1}{2}}(x^{n-1}) = 4^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right) \left(\frac{n-1}{2}\right)! \left(\frac{n-3}{2}\right)! = \gamma_n.$$

Finally, the case $m < n-1$ is trivial because the number of derivatives to perform is more than the degree of the monomial. □

We can now give the following

Definition 9.2.4 (F-kernel series)**.** Let $s, x \in \mathbb{R}^{n+1}$. We define the left F-kernel series as

$$\sum_{m=n-1}^{+\infty} \Delta^{\frac{n-1}{2}} x^m s^{-1-m},$$

and the right F-kernel series as

$$\sum_{m=n-1}^{+\infty} s^{-1-m} \Delta^{\frac{n-1}{2}} x^m.$$

Proposition 9.2.5. *For $s, x \in \mathbb{R}^{n+1}$ with $|x| < |s|$, the F-kernel series converge.*

Proof. By Corollary 9.2.3 for $m \geq n-1$ and by formula (9.2) we get

$$\begin{aligned} |\Delta^{\frac{n-1}{2}} x^m| &= |\gamma_n| \left| \frac{m!}{(n-1)!(m-n+1)!} P^n_{m+1-n}(x) \right| \qquad (9.8) \\ &\leq |\gamma_n| \frac{m!}{(n-1)!(m-n+1)!} \sum_{\ell=0}^{m+1-n} T^{m+1-n}_{\ell}(n) |x|^{m+1-n} \end{aligned}$$

From formula 9.4 we know that

$$\sum_{\ell=0}^{m+1-n} T_\ell^{m+1-n}(n) = 1.$$

This implies that

$$\sum_{m=n-1}^{+\infty} |\Delta^{\frac{n-1}{2}} x^m s^{-1-m}| \leq \frac{|\gamma_n|}{(n-1)!} \sum_{m=n-1}^{+\infty} \frac{m!}{(m-n+1)!} |x|^{m-n+1} |s|^{-1-m}.$$

The last series converge, by the ratio test since $|x| < |s|$. Indeed

$$\begin{aligned} \lim_{m \to +\infty} \frac{(m+1)!}{(m-n+2)!} \frac{(m-n+1)!}{m!} \frac{|x|^{m-n+2}|s|^{-2-m}}{|x|^{m-n+1}|s|^{-1-m}} &= \lim_{m \to +\infty} \frac{(m+1)}{(m-n+2)} |x||s|^{-1} \\ &= |x||s|^{-1} < 1. \end{aligned}$$

The convergence of the right F-kernel series can be proved with similar computations. □

Lemma 9.2.6. *Let $x, s \in \mathbb{R}^{n+1}$. For $|x| < |s|$, we have*

$$F_n^L(s,x) = \sum_{m=0}^{+\infty} \Delta^{\frac{n-1}{2}} x^m s^{-1-m},$$

and

$$F_n^R(s,x) = \sum_{m=0}^{+\infty} s^{-1-m} \Delta^{\frac{n-1}{2}} x^m.$$

Proof. We rewrite the left Cauchy kernel using the Taylor expansion

$$S_L^{-1}(s,x) = \sum_{m=0}^{+\infty} x^m s^{-1-m},$$

and by applying the Fueter-Sce map we get

$$F_n^L(s,x) = \Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \sum_{m=0}^{+\infty} \left(\Delta^{\frac{n-1}{2}} x^m \right) s^{-1-m},$$

where we can exchange the sum and the Laplacian by Proposition 9.2.5. A similar reasoning holds for the right F-kernel series. □

Proposition 9.2.7. *Let $x, s \in \mathbb{R}^{n+1}$. Then, for $|x| < |s|$, we have*

$$\sum_{m=n-1}^{+\infty} \Delta^{\frac{n-1}{2}} x^m s^{-1-m} = \gamma_n (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}}, \tag{9.9}$$

and

$$\sum_{m=n-1}^{+\infty} s^{-1-m} \Delta^{\frac{n-1}{2}} x^m = \gamma_n (s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}} (s-\bar{x}). \tag{9.10}$$

Proof. By Proposition 9.2.5 and Theorem 9.2 we have that

$$\begin{aligned}\sum_{m=n-1}^{+\infty} \Delta^{\frac{n-1}{2}} x^m s^{-1-m} &= \Delta^{\frac{n-1}{2}} \sum_{m=0}^{+\infty} x^m s^{-1-m} \qquad (9.11)\\ &= \Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}},\end{aligned}$$

and similarly we can prove (9.10). □

In order to state the following results we need to introduce the following notation

$$K_\ell(m,n) := \gamma_n \frac{\left(\frac{n+1}{2}\right)_{m+1-n-\ell} \left(\frac{n-1}{2}\right)_\ell}{\ell!(m-n+1-\ell)!}. \tag{9.12}$$

Definition 9.2.8 (Series expansions of the F-kernels)**.** Let $x, s \in \mathbb{R}^{n+1}$. For $|x| < |s|$, we have

$$F_n^L(s,x) = \sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n)\, x^{m+1-n-\ell}\, \bar{x}^\ell\, s^{-1-m},$$

and

$$F_n^R(s,x) = \sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n) s^{-1-m}\, x^{m+1-n-\ell}\, \bar{x}^\ell,$$

where $K_\ell(m,n)$ is defined in (9.12).

Now, we define the F-kernel operators by formally replacing the variable x with the operator T with commuting components.

Definition 9.2.9 (F-kernel operators)**.** Let $s \in \mathbb{R}^{n+1}$. For $\|T\| < |s|$, we define the left F-kernel operators as

$$\sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n)\, T^{m+1-n-\ell}\, \bar{T}^\ell\, s^{-1-m},$$

and and the right F-kernel operators as

$$\sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n) s^{-1-m}\, T^{m+1-n-\ell}\, \bar{T}^\ell$$

where $K_\ell(m,n)$ is as in (9.12).

Theorem 9.2.10. *Let n be a fixed odd number. Then for $s \in \mathbb{R}^{n+1}$ and $\|T\| < |s|$, we have*

$$\sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n)\, T^{m+1-n-\ell}\, \bar{T}^\ell\, s^{-1-m} = \gamma_n (s\mathcal{I}-\overline{T})(s^2\mathcal{I}-(T+\overline{T})s+T\overline{T})^{-\frac{n+1}{2}},$$

and

$$\sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n) s^{-1-m}\, T^{m+1-n-\ell}\, \bar{T}^\ell = \gamma_n (s^2\mathcal{I}-(T+\overline{T})s+T\overline{T})^{-\frac{n+1}{2}} (s\mathcal{I}-\overline{T}),$$

where $K_\ell(m,n)$ is defined in (9.12) and the constants γ_n are defined as in (7.47).

Proof. It follows by the previous results by replacing x by the paravector operator T. □

Remark 9.2.11. In the second form of the slice Cauchy kernels is involved the term $|x|^2 = x\bar{x} = \bar{x}x$. For this identity it is necessary that the components of x commute. This has implications when we formally replace the paravector x with the operator T. These are the reasons why we require that in the F-functional calculus the components of T have to commute.

Definition 9.2.12 (F-resolvent operators)**.** Let $T \in \mathcal{BC}(X)$. For $s \in \rho_S(T)$, we define the left F-resolvent operator as

$$F_n^L(s,T) = \gamma_n (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - (T+\overline{T})s + T\overline{T})^{-\frac{n+1}{2}},$$

and the right F-resolvent operator as

$$F_n^R(s,T) = \gamma_n (s^2\mathcal{I} - (T+\overline{T})s + T\overline{T})^{-\frac{n+1}{2}} (s\mathcal{I} - \overline{T}).$$

By Proposition 7.4.2 we get the following result.

Lemma 9.2.13. *Let $T \in \mathcal{BC}(V_n)$ we have*

- *The left F-resolvent operator is a $\mathcal{B}(V_n)$-valued right slice hyperholomorphic of the variable s on $\rho_S(T)$.*

- *The right F-resolvent operator is a $\mathcal{B}(V_n)$-valued left slice hyperholomorphic of the variable s on $\rho_S(T)$.*

Now, we have all the tools to give the definition for the F-functional calculus.

Definition 9.2.14 (The F-functional calculus for bounded operators)**.** Let n be an odd number, let $T \in \mathcal{BC}(V_n)$, assume that the operators T_ℓ, $\ell = 1, .., n$ have real spectrum and set $ds_J = ds/J$. For any function $f \in \mathcal{SR}_L(\sigma_S(T))$, we define

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,T)\, ds_J\, f(s). \tag{9.13}$$

For any $f \in \mathcal{SR}_R(\sigma_S(T))$, we define

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} f(s)\, ds_J\, F_n^R(s,T), \tag{9.14}$$

where $J \in \mathbb{S}^{n-1}$ and U is a slice Cauchy domain U.

Theorem 9.2.15. *The F-functional calculus is well defined, that is, the integrals in* (9.13) *and* (9.14) *depend neither on the imaginary unit $J \in \mathbb{S}^{n-1}$ nor on the slice Cauchy domain U.*

Proof. We will show only the case $\breve{f} = \Delta^{\frac{n-1}{2}} f$, with $f \in \mathcal{SH}_L(\sigma_S(T))$. The other case follows by similar computations.

The independence from the set U follows by the Cauchy integral theorem (see Theorem 3.1.19) and from the facts that the function f is left slice hyperholomorphic and $F_n^L(s,T)$ is right slice hyperholomorphic in the variable s.

To show the independence from the imaginary unit, we consider two imaginary units J, $I \in \mathbb{S}^{n-1}$ with $J \neq I$ and two bounded slice Caichy domains U_p, U_s with $\sigma_S(U) \subset U_p$, $\overline{U}_p \subset U_s$ and $\overline{U}_s \subset \operatorname{dom}(f)$. Then every $s \in \partial(U \cap \mathbb{C}_J)$ belongs to the unbounded slice Cauchy domain $\mathbb{R}^{n+1} \setminus U_p$. Since $\lim_{p\to\infty} F_n^L(s,p) = 0$, by the slice hyperholomorphic Cauchy kernel we have

$$\begin{aligned} F_n^L(s,T) &= \frac{1}{2\pi} \int_{\partial((\mathbb{R}^{n+1}\setminus U_p)\cap\mathbb{C}_I)} F_n^L(s,T) dp_I S_R^{-1}(p,s) \\ &= \frac{1}{2\pi} \int_{\partial(U_p\cap\mathbb{C}_I)} F_n^L(s,T) dp_I S_L^{-1}(s,p), \end{aligned}$$

where the last identity holds because $\partial((\mathbb{R}^{n+1}\setminus U_p)\cap\mathbb{C}_I)=-\partial(U_p\cap\mathbb{C}_I)$ and $S_R^{-1}(p,s)=-S_L^{-1}(s,p)$. Thus

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)} F_n^L(s,T)ds_J f(s)\\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)}\left(\frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_I)} F_n^L(s,T)dp_I S_L^{-1}(s,p)\right) ds_J f(s).\end{aligned}$$

Since the integrand is continuous and the path of integration is bounded, we can exchange the order of integration by Fubini's theorem. Thus we get

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_I)} F_n^L(p,T)\left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)} S_L^{-1}(s,p)ds_J f(s)\right)\\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_I)} F_n^L(p,T)dp_I f(p).\end{aligned}$$

□

Now, we define the kernel of the Fueter-Sce map as

$$\ker(\Delta^{\frac{n-1}{2}}):=\left\{f\in\mathcal{SH}_L(\Omega),\quad \Delta^{\frac{n-1}{2}}f(x)=0\right\},$$

where the set Ω is an axially symmetric slice domain in $\mathbb{R}^{n+1}$. A similar definition for right slice hyperholomorphic functions is possible.

Lemma 9.2.16. *Let n be a fixed odd number. Let Ω be an axially symmetric slice domain on $\mathbb{R}^{n+1}$. Then, a slice hyperholomorphic function f belongs to $\ker(\Delta^{\frac{n-1}{2}})$ if and only if f it is a polynomial $\mathbb{R}_n$-valued of degree $n-2$ in the variable x.*

$$f(x)=\sum_{k=0}^{n-2}x^k\alpha_k,\quad \forall x\in\Omega. \tag{9.15}$$

Proof. Let us assume that the function f is left slice hyperholomorphic. By Theorem 7.3.3 we have that $\Delta^{\frac{n-1}{2}}f=0$ if and only if

$$GCK[f^{(n-1)}(x_0)]=0.$$

This holds if and only if

$$f^{(n-1)}(x_0)=0.$$

Finally, since f is defined in a piecewise connected open set $\mathbb{R}$, it can be uniquely extended to a holomorphic function to a connected set in $\mathbb{R}^2$. Thus

f is a polynomial of degree at most $n-2$, i.e.

$$f(x)=\sum_{k=0}^{n-2} x^k \alpha_k, \quad \{\alpha_k\}_{0\leq k\leq n-2}\subset \mathbb{R}_n.$$

□

Theorem 9.2.17. *Let U be an arbitrary bounded connected slice Cauchy domain. Let us suppose that f, $g\in \mathcal{SH}_L(U)$ and $\breve{f}=\Delta^{\frac{n-1}{2}}f=\Delta^{\frac{n-1}{2}}g=\breve{g}$ with $f\neq g$. Then $\breve{f}(T)=\breve{g}(T)$.*

Proof. The function $f-g$ is left slice hyperholomorphic and belongs to the kernel of the Fueter-Sce map. Thus by Lemma 9.2.16 we have

$$f(s)-g(s)=\sum_{k=0}^{n-2} s^k\alpha_k, \qquad \alpha_k\in\mathbb{R}_n.$$

From the definition of the F-functional calculus we have that

$$\begin{aligned}\breve{f}(T)-\breve{g}(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,T)ds_J(f(s)-g(s))\\ &= \frac{1}{2\pi}\sum_{k=0}^{n-2}\int_{\partial(B_r(0)\cap\mathbb{C}_J)} F_n^L(s,T)ds_J s^k\alpha_k,\end{aligned}$$

we change the domain of integration to the ball $B_r(0)$ with $\|T\|<r$ by the Cauchy's integral theorem and the slice hyperholomorphicity of $F_n^L(s,T)$ in s. By Theorem 9.2.10 and by the Cauchy's integral theorem we get

$$\begin{aligned}\breve{f}(T)-\breve{g}(T) &= \frac{1}{2\pi}\sum_{k=0}^{n-2}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}\sum_{m=n-1}^{+\infty}\sum_{\ell=0}^{m-n+1}K_\ell(m,n)T^{m+1-n-\ell}\bar{T}^\ell\\ &\quad ds_J s^{k-1-m}\alpha_k\\ &= \frac{1}{2\pi}\sum_{k=0}^{n-2}\sum_{m=n-1}^{+\infty}\sum_{\ell=0}^{m-n+1}K_\ell(m,n)T^{m+1-n-\ell}\bar{T}^\ell\\ &\quad \int_{\partial(B_r(0)\cap\mathbb{C}_J)} s^{k-1-m}ds_J\alpha_k\\ &= 0.\end{aligned}$$

□

Now we want to show Theorem 9.2.17 for the case in which the set U is disconnected. In order to show this result we need the following result, which is based on the monogenic functional calculus of McIntosh and collaborators, see [99, 101, 108, 112].

Lemma 9.2.18. *Let $T \in \mathcal{BC}(V_n)$. Suppose that G contains just some points of the S-spectrum of T and assume that the closed smooth curve $\partial(G\cap\mathbb{C}_J)$ belongs to the F-resolvent set of T, for every $I \in \mathbb{S}^{n-1}$. Then*

$$\int_{\partial(G\cap\mathbb{C}_J)} s^m ds_J F_n^R(s,T) = 0,$$

$$\int_{\partial(G\cap\mathbb{C}_J)} F_n^L(p,T) dp_I p^m = 0,$$

for all $m \leq n-2$.

Proof. Since $\Delta^{\frac{n-1}{2}} x^m = 0$, if $m \leq n-2$, we have that

$$\int_{\partial(G\cap\mathbb{C}_J)} s^m ds_J F_n^R(s,x) = 0,$$

$$\int_{\partial(G\cap\mathbb{C}_J)} F_n^L(p,x) dp_I p^m = 0,$$

for $m \leq n-2$ and for all x such that $x \notin [s]$ if $s \in \partial(G\cap\mathbb{C}_J)$ (respectively, for all x such that $x \notin [p]$ if $p \in \partial(G\cap\mathbb{C}_J)$). We recall that $F_n^L(p,x)$ is left monogenic in x for every p, such that $x \notin [p]$. Therefore we can use the definition of the monogenic functional calculus

$$F_n^L(p,T) = \int_{\partial\Omega} \mathcal{G}_\omega(T)\mathbf{n}(\omega)F_n^L(p,\omega)d\mu(\omega),$$

where the open set Ω contains the monogenic spectrum of T, $\mathcal{G}_\omega(T)$ is the monogenic resolvent operator, $\mathbf{n}(\omega)$ is the unit normal vector to $\partial\Omega$ and $d\mu(\omega)$ is the surface element. From the Fubini's theorem it follows that

$$\begin{aligned}\int_{\partial(G\cap\mathbb{C}_J)} F_n^L(p,T)dp_I p^n &= \int_{\partial(G\cap\mathbb{C}_J)} \int_{\partial\Omega} \left(\mathcal{G}_\omega(T)\mathbf{n}(\omega)F_n^L(p,\omega)d\mu(\omega)\right) p^n dp_J \\ &= \int_{\partial\Omega} \mathcal{G}_\omega(T)\mathbf{n}(\omega) \left(\int_{\partial(G\cap\mathbb{C}_J)} F_n^L(s,x)p^n dp_J\right) d\mu(\omega) \\ &= 0,\end{aligned}$$

which concludes the proof. $\square$

Theorem 9.2.19. *Let U be an arbitrary bounded slice Cauchy domain.Let $T = T_1e_1 + T_2e_2 + T_3e_3$ Let us suppose that $f,\ g \in \mathcal{SH}_L(U)$ and $\breve{f} = \Delta^{\frac{n-1}{2}} f = \Delta^{\frac{n-1}{2}} g = \breve{g}$ with $f \neq g$. Then $\breve{f}(T) = \breve{g}(T)$.*

Proof. If the set U is connected the result follows by Theorem 9.2.17. If U is not connected we can write

$$f(s) - g(s) = \sum_{k=0}^{n-2} \sum_{r=1}^{n} \chi_{U_r}(s) s^k \alpha_k,$$

where U_r, $r = 1, ..., n$ are the connected components of U and χ_{U_r} is the characteristic function of U_r. Hence we have

$$\breve{f}(T) - \breve{g}(T) = \frac{1}{2\pi} \sum_{k=0}^{n-2} \sum_{r=1}^{n} \int_{\partial(U_r \cap \mathbb{C}_J)} F_n^L(s,T) ds_J s^k \alpha_k.$$

By Lemma 9.2.18 we obtain that

$$\breve{f}(T) - \breve{g}(T) = 0.$$

□

Remark 9.2.20. In the above proof we cannot perform computations as in Theorem 9.2.17 because the set $\mathbb{R}^{n+1} \setminus U_r$ contains part of the S-spectrum of T and thus $F_L^n(s,T)$ is not slice hyperholomorphic in that set.

Now, we show some algebraic properties of the F-functional calculus.

Proposition 9.2.21. *Let n be a fixed odd number. Let $T \in \mathcal{BC}^{0,1}(V_n)$ be such that $T = \sum_{i=1}^{n} T_i e_i$, and assume that the operators T_i, $1 \leq i \leq n$ have real spectrum.*

- *If $\breve{f} = \Delta^{\frac{n-1}{2}} f$ and $\breve{g} = \Delta^{\frac{n-1}{2}} g$ with $f,\ g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{R}_n$, then*
$$(\breve{f}a + \breve{g})(T) = \breve{f}(T)a + \breve{g}(T).$$
- *If $\breve{f} = \Delta^{\frac{n-1}{2}} f$ and $\breve{g} = \Delta^{\frac{n-1}{2}} g$ with $f,\ g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{R}_n$, then*
$$(a\breve{f} + \breve{g})(T) = \breve{f}(T)a + \breve{g}(T).$$

Proof. It follows by the linearity of the integrals in Definition 9.2.14. □

Proposition 9.2.22. *Let n be a fixed odd number. Let $T \in \mathcal{BC}^{0,1}(V_n)$ be such that $T = \sum_{i=1}^{n} T_i e_i$, and assume that the operators T_i, $1 \leq i \leq n$ have real spectrum.*

1) Let $\breve{f} = \Delta^{\frac{n-1}{2}} f$ *with* $f \in \mathcal{SH}_L(\sigma_S(T))$ *and assume that* $f(x) = \sum_{\ell=0}^{\infty} x^\ell a_\ell$ *with* $a_\ell \in \mathbb{R}_n$, *where the series converges on a ball* $B_r(0)$ *with* $\sigma_S(T) \subset B_r(0)$. *Then*

$$\breve{f}(T) = \left(\sum_{r=0}^{m-n+1} K_r(m,n)(-1)^r \right) \sum_{\ell=n-1}^{\infty} T^{\ell+1-n} a_\ell,$$

where $K_r(m,n)$ *is defined in* (9.12).

2) Let $\breve{f} = \Delta^{\frac{n-1}{2}} f$ *with* $f \in \mathcal{SH}_R(\sigma_S(T))$ *and assume that* $f(x) = \sum_{\ell=0}^{\infty} x^\ell a_\ell$ *with* $a_\ell \in \mathbb{R}_n$, *where the series converges on a ball* $B_r(0)$ *with* $\sigma_S(T) \subset B_r(0)$. *Then*

$$\breve{f}(T) = \left(\sum_{r=0}^{m-n+1} K_r(m,n)(-1)^r \right) \sum_{\ell=n-1}^{\infty} a_\ell T^{\ell+1-n}.$$

Proof. We will show only the first point because the second point can be proved by similar arguments. We consider an imaginary unit $J \in \mathbb{S}^{n-1}$ and a radius $0 < R < r$ such that $\sigma_S(T) \subset B_R(0)$. Since the expansion in series of f converges uniformly on $\partial(B_R(0) \cap \mathbb{C}_J)$ we have

$$\begin{aligned} \breve{f}(T) &= \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} F_n^L(s,T) ds_J \sum_{\ell=0}^{\infty} s^\ell a_\ell \\ &= \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} F_n^L(s,T) ds_J s^\ell a_\ell. \end{aligned}$$

By Theorem 9.2.10 and from the fact that $T_0 = 0$ we have

$$F_n^L(s,T) = \left(\sum_{\ell=0}^{m-n+1} K_r(m,n)(-1)^r \right) \sum_{m=n-1}^{+\infty} T^{m+1-n} s^{-1-m}.$$

Therefore we get

$$
\begin{aligned}
\breve{f}(T) &= \frac{\left(\sum_{\ell=0}^{m-n+1} K_r(m,n)(-1)^r\right)}{2\pi} \sum_{\ell=0}^{\infty} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} \sum_{m=n-1}^{\infty} T^{m+1-n} s^{-1-m} \\
&\quad ds_J s^{\ell} a_{\ell} \\
&= \frac{\left(\sum_{\ell=0}^{m-n+1} K_r(m,n)(-1)^r\right)}{2\pi} \sum_{\ell=0}^{\infty} \sum_{m=n-1}^{\infty} T^{m+1-n} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} s^{-1-m} \\
&\quad ds_J s^{\ell} a_{\ell} \\
&= \left(\sum_{r=0}^{m-n+1} K_r(m,n)(-1)^r\right) \sum_{\ell=n-1}^{\infty} T^{\ell+1-n} a_{\ell}.
\end{aligned}
$$

□

To state the next result we need the following notations

$$\mathcal{M}_m^L(T,\bar{T}) := \sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n)\, T^{m+1-n-\ell}\, \bar{T}^{\ell}$$

and

$$\mathcal{M}_m^R(T,\bar{T}) := \sum_{m=n-1}^{+\infty} \sum_{\ell=0}^{m-n+1} K_\ell(m,n) T^{m+1-n-\ell}\, \bar{T}^{\ell},$$

where $K_\ell(m,n)$ is defined in (9.12).

Theorem 9.2.23. *Let n be a fixed odd number. Let $T \in \mathcal{BC}^{0,1}(V_n)$ and $k \geq n-1$. Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain with $\sigma_S(T) \subset U$. For every imaginary unit $J \in \mathbb{S}^{n-1}$, we have*

$$\mathcal{M}_k^L(T,\bar{T}) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,T) ds_J s^k, \tag{9.16}$$

and

$$\mathcal{M}_k^R(T,\bar{T}) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_J)} s^k ds_J F_n^R(s,T). \tag{9.17}$$

Proof. We will prove only formula (9.16), since the equality (9.17) can be proved with similar arguments. By Theorem 9.2.10 and by the Cauchy integral theorem we get

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} F_n^L(s,T)\,ds_J s^m \\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} \sum_{m=n-1}^{+\infty}\sum_{\ell=0}^{m-n+1} K_\ell(m,n)\,T^{m+1-n-\ell}\,\bar{T}^\ell s^{-1-m} ds_J s^k \\
&= \sum_{m=n-1}^{+\infty}\sum_{\ell=0}^{m-n+1} K_\ell(m,n)\,T^{m+1-n-\ell}\,\bar{T}^\ell \left(\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)} s^{-1-m+k} ds_J\right) \\
&= \mathcal{M}_k^L(T,\bar{T}).
\end{aligned}
$$

□

9.3 The F-resolvent equation for $n = 5$ and for $n = 7$

The F-resolvent equation has further differences with respect to the complex resolvent equation and with respect to the S-resolvent equation. This is a consequence of the fact that the F-functional calculus is based on an integral transform and not on a Cauchy formula.

The F-resolvent equation for $n = 3$ is known since some years and it coincides with the quaternionic F-resolvent equation. Precisely it is (see [41])

$$
\begin{aligned}
&F_3^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_3^L(p,T) \\
&-\frac{1}{4}\big(sF_3^R(s,T)F_3^L(p,T)p - sF_3^R(s,T)TF_3^L(p,T) - F_3^R(s,T)TF_3^L(p,T)p \\
&+F_3^R(s,T)T^2F_3^L(p,T)\big) = \big[\big(F_3^R(s,T) - F_3^L(p,T)\big)p - \bar{s}\big(F_3^R(s,T) - F_3^L(p,T)\big)\big]\cdot \\
&(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}
$$

for $T \in \mathcal{BC}(V_3)$ and for any $p, s \in \rho_S(T)$, with $s \notin [p]$. The equation is written just in terms of S-resolvent operators and of F-resolvent operators. The case of general n (odd number) is not so simple, and its existence has been an open problem for some years.

In order to explain how to obtain it in the general case we treat separately the cases $n = 5$ and $n = 7$. In the case $n = 5$ it is clear that the equation can be written in a quite reasonable way in terms of the F-resolvent operators. But starting from $n = 7$ this choice cannot be made anymore because it leads to an equation that is ways too complicated. This is the reason for which we cannot replace the pseudo S-resolvent operators. In the sequel will be fundamental the following result.

Theorem 9.3.1 (The left and right F-resolvent equations)**.** *Let n be an odd number and let $T \in \mathcal{B}^{0,1}(V_n)$. Let $s \in \rho_S(T)$. Then the F-resolvent operators satisfy the equations*

$$F_n^L(s,T)s - TF_n^L(s,T) = \gamma_n \mathcal{Q}_{c,s}(T)^{-\frac{n-1}{2}} \tag{9.18}$$

and

$$sF_n^R(s,T) - F_n^R(s,T)T = \gamma_n \mathcal{Q}_{c,s}(T)^{-\frac{n-1}{2}}, \tag{9.19}$$

where the constants γ_n are given by (7.47).

Proof. We prove the relation (9.18), since (9.19) follows by similar arguments. Thus we have

$$F_n^L(s,T)s = \gamma_n(s\mathcal{I} - \bar{T})s\mathcal{Q}_{c,s}^{-\frac{n+1}{2}}(T),$$

and

$$TF_n^L(s,T) = \gamma_n(Ts - T\bar{T})\mathcal{Q}_{c,s}^{-\frac{n+1}{2}}(T).$$

By making the difference, we obtain

$$\begin{aligned} F_n^L(s,T)s - TF_n^L(s,T) &= \gamma_n(s^2\mathcal{I} - s(T+\bar{T}) + T\bar{T})\mathcal{Q}_{c,s}^{-\frac{n+1}{2}}(T) \\ &= \gamma_n \mathcal{Q}_{c,s}(T)^{-\frac{n-1}{2}}. \end{aligned}$$

□

9.3.1 The F-resolvent equation for $n = 5$

In this case we show the F- resolvent equation establishes a link between the difference $F_5^R(s,T) - F_5^L(p,T)$, the slice Cauchy kernel and suitable bounded operators.

To prove the F resolvent equation we need the following technical result involving the pseudo S-resolvent operators.

Lemma 9.3.2 (The F- resolvent equation for $n = 5$, with the pseudo S-resolvent operators)**.** *Let $T \in \mathcal{BC}^{0,1}(V_5)$. Then for $p, s \in \rho_S(T)$ the following equation holds*

$$\begin{aligned} &F_5^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_5^L(p,T) \\ &+\gamma_5 \mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) \\ &+\gamma_5[\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_s^{-1}(T)\mathcal{Q}_{c,p}^{-2}(T)] \\ &= \left\{[F_5^R(s,T) - F_5^L(p,T)]p - \bar{s}[F_5^R(s,T) - F_5^L(p,T)]\right\}(p^2 - 2s_0p + |s|^2)^{-1}. \end{aligned} \tag{9.20}$$

where γ_5 is given by (7.47) for $n = 5$.

Proof. Let us start by left multiplying the S-resolvent equation (3.9) by $\gamma_5\mathcal{Q}_{c,s}^{-2}(T)$ so that we get

$$\begin{aligned} F_5^R(s,T)S_L^{-1}(p,T) &= \big\{[F_5^R(s,T)-\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)]p- \\ &\quad \bar{s}[F_5^R(s,T)-\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)]\big\}\cdot \\ &\quad (p^2-2s_0p+|s|^2)^{-1}. \end{aligned} \tag{9.21}$$

Now, we multiply the S-resolvent equation on the right by $\gamma_5\mathcal{Q}_{c,p}^{-2}(T)$ and we obtain

$$\begin{aligned} S_R^{-1}(s,T)F_5^L(p,T) &= \big\{[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)-F_5^L(p,T)]p \\ &\quad -\bar{s}[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)-F_5^L(p,T)]\big\} \\ &\quad \cdot(p^2-2s_0p+|s|^2)^{-1}. \end{aligned} \tag{9.22}$$

We multiply the S-resolvent equation on the left by $\mathcal{Q}_{c,s}^{-1}(T)$ and on the right by $\mathcal{Q}_{c,p}^{-1}(T)$, we get

$$\begin{aligned} &\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) = \big\{[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T) \\ &-\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)]p-\bar{s}[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T) \\ &-\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)]\big\}(p^2-2s_0p+|s|^2)^{-1}. \end{aligned} \tag{9.23}$$

Now, we sum (9.21), (9.22) and (9.23) multiplied by γ_5 to get

$$\begin{gathered} F_5^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_5^L(p,T)+ \\ \gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)=\big\{[F_5^R(s,T)-F_5^L(p,T)]p \\ -\bar{s}[F_5^R(s,T)-F_5^L(p,T)]\big\}(p^2-2s_0p+|s|^2)^{-1}+\big\{[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) \\ -\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)-\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)+\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)]p \\ -\bar{s}[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)-\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)-\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T) \\ +\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)]\big\}(p^2-2s_0p+|s|^2)^{-1}. \end{gathered}$$

Finally, we verify that

$$\begin{aligned} &\big\{[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)-\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)-\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T) \\ &+\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)]p-\bar{s}[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) \\ &-\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)-\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T) \\ &+\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)]\big\}\cdot(p^2-2s_0p+|s|^2)^{-1} \end{aligned}$$

$$= -\gamma_5[\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-2}(T)].$$

Now observe that by the definitions of the S-resolvent operators we have

$$\gamma_5\left(S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) - \mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) - \mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\right.$$
$$\left.+\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)\right) = \gamma_5\left(\mathcal{Q}_{c,s}^{-1}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-2}(T)\right.$$
$$-\mathcal{Q}_{c,s}^{-1}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-2}(T) - \mathcal{Q}_{c,s}^{-2}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T)+$$
$$\left.+\mathcal{Q}_{c,s}^{-2}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T)\right) = \gamma_5[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)$$
$$+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)],$$

and this implies that

$$\left\{[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) - \gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1} - \gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\right.$$
$$+\gamma_5\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T) + \gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)]p$$
$$-\bar{s}[\gamma_5 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) - \gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)$$
$$\left.+\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)]\right\}(p^2-2s_0p+|s|^2)^{-1}$$
$$= \gamma_5\left\{[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T) + \mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)]p\right.$$
$$\left.-\bar{s}[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T) + \mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)]\right\}(p^2-2s_0p+|s|^2)^{-1}$$
$$= \gamma_5\left\{[\mathcal{Q}_{c,s}^{-1}(T)(sp-p^2)\mathcal{Q}_{c,p}^{-2}(T) + \mathcal{Q}_{c,s}^{-2}(T)(sp-p^2)\mathcal{Q}_{c,p}^{-1}(T)]\right.$$
$$\left.-[\mathcal{Q}_{c,s}^{-1}(T)(\bar{s}s-\bar{s}p)\mathcal{Q}_{c,p}^{-2}(T) + \mathcal{Q}_{c,s}^{-2}(T)(\bar{s}s-\bar{s}p)\mathcal{Q}_{c,p}^{-1}(T)]\right\}(p^2-2s_0p+|s|^2)^{-1}$$
$$= \gamma_5[\mathcal{Q}_{c,s}^{-2}(T)(sp-p^2-\bar{s}s+\bar{s}p)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T)\cdot$$
$$(sp-p^2-s\bar{s}+\bar{s}p)\mathcal{Q}_{c,p}^{-2}(T)](p^2-2s_0p+|s|^2)^{-1}$$
$$= -\gamma_5[\mathcal{Q}_{c,s}^{-2}(T)(p^2-2s_0p+|s|^2)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T)\cdot$$
$$(p^2-2s_0p+|s|^2)\mathcal{Q}_{c,p}^{-2}(T)](p^2-2s_0p+|s|^2)^{-1}$$
$$= -\gamma_5[\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-2}(T)].$$

□

Using the above preliminary lemma and the relations between the pseudo S-resolvent operators and the F-resolvent operators, we obtain for $n=5$ the F-resolvent equation. This equation has strong similarities with the case $n=3$.

Theorem 9.3.3 (The F- resolvent equation for $n=5$)**.** *Let $T \in \mathcal{BC}^{0,1}(V_5)$. Then, for $p, s \in \rho_S(T)$, the following equation holds*

$$F_5^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_5^L(p,T) + \gamma_5^{-1}\left(s^2F_5^R(s,T)F_5^L(p,T)p^2\right.$$

$$
\begin{aligned}
&-3s^2F_5^R(s,T)TF_5^L(p,T)p - 3sF_5^R(s,T)TF_5^L(p,T)p^2 \\
&+3sF_5^R(s,T)T^2F_5^L(p,T)p - 2sF_5^R(s,T)|T|^2TF_5^L(p,T) \\
&+2sF_5^R(s,T)|T|^2F_5^L(p,T)p - 2F_5^R(s,T)|T|^2TF_5^L(p,T)p + \\
&sF_5^R(s,T)\overline{T}^2F_5^L(p,T)p - sF_5^R(s,T)|T|^2\overline{T}F_5^L(p,T) \\
&-F_5^R(s,T)|T|^2\overline{T}F_5^L(p,T)p + F_5^R(s,T)|T|^4F_5^L(p,T) \\
&+sF_5^R(s,T)F_5^L(p,T)p^3 - F_5^R(s,T)TF_5^L(p,T)p^3 \\
&+2F_5^R(s,T)T^2F_5^L(p,T)p^2 - F_5^R(s,T)T^3F_5^L(p,T)p \\
&+2F_5^R(s,T)T^2|T|^2F_5^L(p,T) + s^3F_5^R(s,T)F_5^L(p,T)p \\
&-s^3F_5^R(s,T)TF_5^L(p,T) + 2s^2F_5^R(s,T)T^2F_5^L(p,T) - sF_5^R(s,T)T^3F_5^L(p,T)\big) \\
&= \left\{[F_5^R(s,T) - F_5^L(p,T)]p - \bar{s}[F_5^R(s,T) - F_5^L(p,T)]\right\}(p^2 - 2s_0p + |s|^2)^{-1},
\end{aligned}
$$

where we have set for the sake of simplicity

$$|T|^2 = \overline{T}T.$$

Proof. Firstly we remark that

$$
\begin{aligned}
&\gamma_5\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) \\
&= \gamma_5\mathcal{Q}_{c,s}^{-2}(T)(s\mathcal{I} - \overline{T})(p\mathcal{I} - \overline{T})\mathcal{Q}_{c,p}^{-2}(T) \\
&= \gamma_5\mathcal{Q}_{c,s}^{-2}(T)\left(sp\mathcal{I} - s\overline{T} - \overline{T}p + \overline{T}^2\right)\mathcal{Q}_{c,p}^{-2}(T) \\
&= \gamma_5\left[s\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-2}(T)p - s\mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T) - \mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T)p\right. \\
&\left.+\mathcal{Q}_{c,s}^{-2}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-2}(T)\right].
\end{aligned}
$$

Putting this in (9.20) we deduce that

$$
\begin{aligned}
&F_5^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_5^L(p,T) + \gamma_5\left[s\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-2}(T)p+\right. \\
&\left.-s\mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T) - \mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T)p + \mathcal{Q}_{c,s}^{-2}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-2}(T)\right] \\
&+\gamma_5[\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-2}(T)] = \left\{[F_5^R(s,T) - F_5^L(p,T)]p\right. \\
&\left.-\bar{s}[F_5^R(s,T) - F_5^L(p,T)]\right\}(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}
$$

Now, we use the right and left F-resolvent equation for $n = 5$ (see Theorem 9.3.1) namely

$$F_5^L(p,T)p - TF_5^L(p,T) = \gamma_5\mathcal{Q}_{c,p}^{-2}(T), \tag{9.24}$$

and

$$sF_5^R(s,T) - F_5^R(s,T)T = \gamma_5 \mathcal{Q}_{c,s}^{-2}(T). \tag{9.25}$$

We go through the computations terms by terms

$$\begin{aligned} s\mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T) &= \gamma_5^{-2} s\left(sF_5^R(s,T) - F_5^R(s,T)T\right)\overline{T} \\ &\quad \left(F_5^L(p,T)p - TF_5^L(p,T)\right) \\ &= \gamma_5^{-2}\left(s^2 F_5^R(s,T)\overline{T}F_5^L(p,T)p\right. \\ &\quad -s^2F_5^R(s,T)|T|^2F_5^L(p,T) - sF_5^R(s,T)|T|^2F_5^L(p,T)p \\ &\quad \left.+sF_5^R(s,T)|T|^2TF_5^L(p,T)\right), \end{aligned} \tag{9.26}$$

$$\begin{aligned} \mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T)p &= \gamma_5^{-2}\left(sF_5^R(s,T)\overline{T}F_5^L(p,T)p^2\right. \\ &\quad -sF_5^R(s,T)|T|^2F_5^L(p,T)p - F_5^R(s,T)|T|^2F_5^L(p,T)p^2 \\ &\quad \left.+F_5^R(s,T)|T|^2TF_5^L(p,T)p\right), \end{aligned} \tag{9.27}$$

$$\begin{aligned} \mathcal{Q}_{c,s}^{-2}(T)\overline{T}\mathcal{Q}_{c,p}^{-2}(T)p &= \gamma_5^{-2}\left(sF_5^R(s,T)\overline{T}F_5^L(p,T)p^2\right. \\ &\quad -sF_5^R(s,T)|T|^2F_5^L(p,T)p - F_5^R(s,T)|T|^2F_5^L(p,T)p^2 \\ &\quad \left.+F_5^R(s,T)|T|^2TF_5^L(p,T)p\right), \end{aligned} \tag{9.28}$$

$$\begin{aligned} \mathcal{Q}_{c,s}^{-2}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-2}(T) &= \gamma_5^{-2}\Big(sF_5^R(s,T)\overline{T}^2F_5^L(p,T)p \\ &\quad -sF_5^R(s,T)|T|^2\overline{T}F_5^L(p,T) - F_5^R(s,T)|T|^2\overline{T}F_5^L(p,T)p \\ &\quad +F_5^R(s,T)|T|^4F_5^L(p,T)\Big). \end{aligned} \tag{9.29}$$

Moreover by (9.24) and (9.25) we have

$$\begin{aligned} \mathcal{Q}_{c,p}^{-1}(T) &= \mathcal{Q}_{c,p}^{-2}(T)\mathcal{Q}_{c,p}(T) \\ &= \gamma_5^{-1}\left(F_5^L(p,T)p - TF_5^L(p,T)\right)\left(p^2 - (T+\overline{T})p + |T|^2\right) \\ &= \gamma_5^{-1}\left(F_5^L(p,T)p^3 - 2TF_5^L(p,T)p^2 - \overline{T}F_5^L(p,T)p^2 + T^2F_5^L(p,T)p\right. \\ &\quad \left.+2|T|^2F_5^L(p,T)p - T|T|^2F_5^L(p,T)\right), \end{aligned} \tag{9.30}$$

$$\mathcal{Q}_{c,s}(T)^{-1} = \mathcal{Q}_{c,s}(T)\mathcal{Q}_{c,s}(T)^{-2} \tag{9.31}$$

$$\begin{aligned}
&= \gamma_5^{-1}\left(s^2 - s(T+\overline{T}) + |T|^2\right)\left(sF_5^R(s,T) - F_5^R(s,T)T\right)\\
&= \gamma_5^{-1}\left(s^3F_5^R(s,T) - 2s^2F_5^R(s,T)T - s^2F_5^R(s,T)\overline{T}\right.\\
&\quad \left. + sF_5^R(s,T)T^2 + 2sF_5^R(s,T)|T|^2 - F_5^R(s,T)T|T|^2\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-1}(T) &= \gamma_5^{-2}\left(sF_5^R(s,T) - F_5^R(s,T)T\right)\left(F_5^L(p,T)p^3\right.\\
&\quad -2TF_5^L(p,T)p^2 - \overline{T}F_5^L(p,T)p^2\\
&\quad \left.+T^2F_5^L(p,T)p + 2|T|^2F_5^L(p,T)p - T|T|^2F_5^L(p,T)\right)\\
&= \gamma_5^{-2}\left(sF_5^R(s,T)F_5^L(p,T)p^3 - 2sF_5^R(s,T)TF_5^L(p,T)p^2\right.\\
&\quad -sF_5^R(s,T)\overline{T}F_5^L(p,T)p^2 + sF_5^R(s,T)T^2F_5^L(p,T)p+\\
&\quad 2sF_5^R(s,T)|T|^2F_5^L(p,T)p - sF_5^R(s,T)T|T|^2F_5^L(p,T)\\
&\quad -F_5^R(s,T)TF_5^L(p,T)p^3 + 2F_5^R(s,T)T^2F_5^L(p,T)p^2\\
&\quad +F_5^R(s,T)|T|^2F_5^L(p,T)p^2 - F_5^R(s,T)T^3F_5^L(p,T)p\\
&\quad \left.-2F_5^R(s,T)|T|^2TF_5^L(p,T)p + F_5^R(s,T)T^2|T|^2F_5^L(p,T)\right).
\end{aligned} \tag{9.32}$$

and

$$\begin{aligned}
\mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-2}(T) &= \gamma_5^{-2}\left(s^3F_5^R(s,T) - 2s^2F_5^R(s,T)T\right.\\
&\quad -s^2F_5^R(s,T)\overline{T} + F_5^R(s,T)T^2 + 2sF_5^R(s,T)|T|^2\\
&\quad \left.-F_5^R(s,T)T|T|^2\right)\left(F_5^L(p,T)p - TF_5^L(p,T)\right)\\
&= \gamma_5^{-2}\left(s^3F_5^R(s,T)F_5^L(p,T)p - 2s^2F_5^R(s,T)TF_5^L(p,T)p\right.\\
&\quad -s^2F_5^R(s,T)\overline{T}F_5^L(p,T)p + sF_5^R(s,T)T^2F_5^L(p,T)p\\
&\quad +2sF_5^R(s,T)|T|^2F_5^L(p,T)p - F_5^R(s,T)T|T|^2F_5^L(p,T)p\\
&\quad -s^3F_5^R(s,T)TF_5^L(p,T) + 2s^2F_5^R(s,T)T^2F_5^L(p,T)\\
&\quad +s^2F_5^R(s,T)|T|^2F_5^L(p,T) - sF_5^R(s,T)T^3F_5^L(p,T)\\
&\quad \left.-2sF_5^R(s,T)|T|^2TF_5^L(p,T) + F_5^R(s,T)T^2|T|^2F_5^L(p,T)\right).
\end{aligned} \tag{9.33}$$

The statement is obtained by taking the sum of (9.26), (9.27), (9.28), (9.29), (9.30), (9.32), (9.33). □

9.3.2 The F-resolvent equation for $n = 7$

As it is clearly visible from the case $n = 5$, that there are intrinsic complications in the structure of the F-resolvent equation. The case $n = 7$ shows that it is not possible to have a reasonable closed form for the F-resolvent equation just in terms of the S-resolvent operators and of the F-resolvent operators. Instead, the use of the pseudo S-resolvent operators allows a reasonable structure of the resolvent equation. The point of the matter is that in this case it is not possible to obtain a F-resolvent equation as a relation between $F_7^R(s,T)F_7^L(p,T)$, and $F_7^R(s,T) - F_7^L(p,T)$. However, it is possible to prove a form of the F- resolvent equation which will be fundamental in the section on the Riesz projects. As before, we begin with a technical result.

Lemma 9.3.4 (The F- resolvent equation for $n = 7$ with the pseudo S-resolvent operators)**.** *Let $T \in \mathcal{BC}^{0,1}(V_7)$. Then for $p, s \in \rho_S(T)$ the following equation holds*

$$
\begin{aligned}
&F_7^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_7^L(p,T) \\
&+\gamma_7[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) + \mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) \\
&+\mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-3}(T)\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-2}(T)] \\
&= \left\{\left[F_7^R(s,T) - F_7^L(p,T)\right]p - \bar{s}\left[F_7^R(s,T) - F_7^L(p,T)\right]\right\}(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}
\tag{9.34}
$$

Proof. First of all, we left multiply the S-resolvent equation (3.9) by $\gamma_7\mathcal{Q}_{c,s}^{-3}(T)$, so that we get

$$
\begin{aligned}
F_7^R(s,T)S_L^{-1}(p,T) \;=\; &\left\{[F_7^R(s,T) - \gamma_7\mathcal{Q}_{c,s}^{-3}(T)S_L^{-1}(p,T)]p\right. \\
&\left.-\bar{s}[F_7^R(s,T) - \gamma_7\mathcal{Q}_{c,s}^{-3}(T)S_L^{-1}(p,T)]\right\}\cdot \\
&\cdot(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}
\tag{9.35}
$$

Now, we right multiply the S-resolvent equation (3.9) by $\gamma_7\mathcal{Q}_{c,p}^{-3}(T)$

$$
\begin{aligned}
S_R^{-1}(s,T)F_7^L(p,T) \;=\; &\left\{[\gamma_7 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-3}(T) - F_7^L(p,T)]p\right. \\
&\left.-\bar{s}[\gamma_7 S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-3}(T) - F_7^L(p,T)]\right\} \\
&\cdot(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}
\tag{9.36}
$$

Then we multiply the S-resolvent equation on the left by $\mathcal{Q}_{c,s}^{-1}(T)$ and on the right by $\mathcal{Q}_{c,p}^{-2}(T)$

$$
\begin{aligned}
&\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) \\
&= \left\{[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) - \mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)]p\right. \\
&\left.-\bar{s}[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T) - \mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)]\right\} \\
&\cdot(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}
\tag{9.37}
$$

We now multiply the S-resolvent equation on the left by $\mathcal{Q}_{c,s}^{-2}(T)$ and on the right by $\mathcal{Q}_{c,p}^{-1}(T)$

$$
\begin{aligned}
&\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) && (9.38)\\
&=\big\{[\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)-\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)]p\\
&-\bar{s}[\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)-\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)]\big\}\\
&(p^2-2s_0p+|s|^2)^{-1}. && (9.39)
\end{aligned}
$$

We sum (9.35), (9.36) and (9.37), (9.38) multiplied by γ_7, and we obtain

$$
\begin{aligned}
&F_7^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_7^L(p,T)+\gamma_7\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&+\gamma_7\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)=\{\big[F_7^R(s,T)-F_7^L(p,T)\\
&+\gamma_7S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-3}(T)-\gamma_7\mathcal{Q}_{c,s}^{-3}(T)S_L^{-1}(p,T)+\gamma_7\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&-\gamma_7\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)+\gamma_7\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)\\
&-\gamma_7\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)\big]\,p-\bar{s}\,\big[F_7^R(s,T)-F_7^L(p,T)\\
&+\gamma_7S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-3}(T)-\gamma_7\mathcal{Q}_{c,s}^{-3}(T)S_L^{-1}(p,T)+\gamma_7\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&-\gamma_7\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)+\gamma_7\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)\\
&-\gamma_7\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)\big]\,(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}
$$

By the definition of S- resolvent operators we have

$$
\begin{aligned}
&S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-3}(T)-\mathcal{Q}_{c,s}^{-3}(T)S_L^{-1}(p,T)+\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&-\mathcal{Q}_{c,s}^{-1}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)+\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-1}(T)-\mathcal{Q}_{c,s}^{-2}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)\\
&=\mathcal{Q}_{c,s}^{-1}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-3}(T)-\mathcal{Q}_{c,s}^{-3}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-2}(T)\\
&-\mathcal{Q}_{c,s}^{-1}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T)-\mathcal{Q}_{c,s}^{-2}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-2}(T)\\
&=\mathcal{Q}_{c,s}^{-1}(T)s\mathcal{Q}_{c,p}^{-3}(T)-\mathcal{Q}_{c,s}^{-3}(T)p\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)s\mathcal{Q}_{c,p}^{-2}(T)-\mathcal{Q}_{c,s}^{-1}(T)p\mathcal{Q}_{c,p}^{-3}(T)+\\
&\mathcal{Q}_{c,s}^{-3}(T)s\mathcal{Q}_{c,p}^{-1}(T)-\mathcal{Q}_{c,s}^{-2}(T)p\mathcal{Q}_{c,p}^{-2}(T)\\
&=\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&F_7^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_7^L(p,T)+\gamma_7\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&+\gamma_7\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)=\big\{\big[F_7^R(s,T)-F_7^L(p,T)\big]\,p\\
&-\bar{s}\,\big[F_7^R(s,T)-F_7^L(p,T)\big]\big\}\,(p^2-2s_0p+|s|^2)^{-1}+\gamma_7\,\big\{\big[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)\\
&+\mathcal{Q}_{c,s}^{-3}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)\big]\,p
\end{aligned}
$$

$$-\bar{s}\left[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)\right]\Big\}\cdot (p^2-2s_0p+|s|^2)^{-1}.$$

Finally we have to verify that

$$\begin{aligned}&\gamma_7\Big\{\left[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)\right.\\&\left.+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)\right]p-\bar{s}\left[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)\right.\\&\left.+\mathcal{Q}_{c,s}^{3}(T)(s-p)\mathcal{Q}_{c,p}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)\right]\Big\}(p^2-2s_0p+|s|^2)^{-1}\\&=-\gamma_7[\mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-2}(T)].\end{aligned}$$

This follows from

$$\begin{aligned}&\gamma_7\Big\{\left[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)\right]p\\&-\bar{s}\left[\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)(s-p)\mathcal{Q}_{c,p}^{-2}(T)\right]\Big\}\cdot\\&(p^2-2s_0p+|s|^2)^{-1}\\&=\gamma_7\left[\mathcal{Q}_{c,s}^{-1}(T)(sp-p^2)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(sp-p^2)\mathcal{Q}_{c,p}^{-1}(T)\right.\\&+\mathcal{Q}_{c,s}^{-2}(T)(sp-p^2)\mathcal{Q}_{c,p}^{-2}(T)-\mathcal{Q}_{c,s}^{-1}(T)(\bar{s}s-\bar{s}p)\mathcal{Q}_{c,p}^{-3}(T)\\&\left.-\mathcal{Q}_{c,s}^{-3}(T)(\bar{s}s-\bar{s}p)\mathcal{Q}_{c,p}^{-1}(T)-\mathcal{Q}_{c,s}^{-2}(T)(\bar{s}s-\bar{s}p)\mathcal{Q}_{c,p}^{-2}(T)\right](p^2-2s_0p+|s|^2)^{-1}\\&=\gamma_7\left[\mathcal{Q}_{c,s}^{-1}(T)(sp-p^2-\bar{s}s+\bar{s}p)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)(sp-p^2-\bar{s}s+\bar{s}p)\mathcal{Q}_{c,p}^{-1}(T)\right.\\&\left.+\mathcal{Q}_{c,s}^{-2}(T)(sp-p^2-\bar{s}s+\bar{s}p)\mathcal{Q}_{c,p}^{-2}(T)\right](p^2-2s_0p+|s|^2)^{-1}\\&=-\gamma_7[\mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-3}(T)+\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-2}(T)].\end{aligned}$$

□

The results of the previous lemma allows us to obtain the so-called pseudo F-resolvent equation for $n=7$.

Theorem 9.3.5 (The pseudo F-resolvent equation for $n=7$)**.** *Let $T\in\mathcal{BC}^{0,1}(V_7)$. Then for $p,s\in\rho_S(T)$ the following equation holds*

$$\begin{aligned}&F_7^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_7^L(p,T)\\&+\gamma_7^{-1}\left[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)F_7^L(p,T)p^2-\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)TF_7^L(p,T)p\right.\\&-\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\overline{T}F_7^L(p,T)p+\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)|T|^2F_7^L(p,T)\\&+s^2F_7^R(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)-sF_7^R(s,T)TS_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)\\&-sF_7^R(s,T)\overline{T}S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)+F_7^R(s,T)|T|^2S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)\\&+\mathcal{Q}_{c,s}^{-1}(T)F_7^L(p,T)p-\mathcal{Q}_{c,s}^{-1}(T)TF_7^L(p,T)\end{aligned}\tag{9.40}$$

$$
\begin{aligned}
&+sF_7^R(s,T)\mathcal{Q}_{c,p}^{-1}(T)-F_7^R(s,T)T\mathcal{Q}_{c,p}^{-1}(T)\big]+\gamma_7^{-2}[s^3F_7^R(s,T)F_7^L(p,T)p^3\\
&-s^3F_7^R(s,T)TF_7^L(p,T)p^2-s^2F_7^R(s,T)TF_7^L(p,T)p^3\\
&+s^2F_7^R(s,T)T^2F_7^L(p,T)p^2+s^3F_7^R(s,T)F_7^L(p,T)p[|T|^2-p(T+\overline{T})]\\
&-s^3F_7^R(s,T)TF_7^L(p,T)[|T|^2-p(T+\overline{T})]-s^2F_7^R(s,T)TF_7^L(p,T)p\cdot\\
&[|T|^2-p(T+\overline{T})]+s^2F_7^R(s,T)T^2F_7^L(p,T)[|T|^2-p(T+\overline{T})]\\
&+\big[|T|^2-s(T+\overline{T})\big]\,sF_7^R(s,T)F_7^L(p,T)p^3-\big[|T|^2-s(T+\overline{T})\big]\,sF_7^R(s,T)T\cdot\\
&F_7^L(p,T)p^2-\big[|T|^2-s(T+\overline{T})\big]\,F_7^R(s,T)TF_7^L(p,T)p^3+\big[|T|^2-s(T+\overline{T})\big]\cdot\\
&F_7^R(s,T)T^2F_7^L(p,T)p^2+\big[|T|^2-s(T+\overline{T})\big]\big(sF_7^R(s,T)F_7^L(p,T)p\\
&-sF_7^R(s,T)TF_7^L(p,T)-F_7^R(s,T)TF_7^L(p,T)p+F_7^R(s,T)T^2F_7^L(p,T)\big)\cdot\\
&\big[|T|^2-p(T+\overline{T})\big]\\
&=\big\{\big[F_7^R(s,T)-F_7^L(p,T)\big]p-\bar{s}\big[F_7^R(s,T)-F_7^L(p,T)\big]\big\}(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}
$$

Proof. We use the F-resolvent equations (9.18) and (9.19) with $n=7$

$$F_7^L(p,T)p-TF_7^L(p,T)=\gamma_7\mathcal{Q}_{c,p}^{-3}(T), \tag{9.41}$$

$$sF_7^R(s,T)-F_7^R(s,T)T=\gamma_7\mathcal{Q}_{c,s}^{-3}(T). \tag{9.42}$$

Now we substitute (9.41) and (9.42) in the equation of Lemma 9.3.4. We go term by term

$$
\begin{aligned}
&\mathcal{Q}_{c,s}(T)^{-1}S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&=\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-3}(T)\\
&=\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-3}(T)p-\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\overline{T}\mathcal{Q}_{c,p}^{-3}(T)\\
&=\gamma_7^{-1}\big\{\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\big[F_7^L(p,T)p-TF_7^L(p,T)\big]p\\
&-\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\overline{T}\big[F_7^L(p,T)p-TF_7^L(p,T)\big]\big\}\\
&=\gamma_7^{-1}\big[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)F_7^L(p,T)p^2-\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)TF_7^L(p,T)p\\
&-\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)\overline{T}F_7^L(p,T)p+\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)|T|^2F_7^L(p,T)\big],
\end{aligned}
\tag{9.43}
$$

$$
\begin{aligned}
&\mathcal{Q}_{c,s}^{-2}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T)\\
&=\mathcal{Q}_{c,s}^{-3}(T)(s\mathcal{I}-\overline{T})S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)\\
&=s\mathcal{Q}_{c,s}^{-3}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)-\mathcal{Q}_{c,s}^{-3}(T)\overline{T}S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T)
\end{aligned}
\tag{9.44}
$$

$$
\begin{aligned}
&= \gamma_7^{-1} \left\{ s \left[sF_7^R(s,T) - F_7^R(s,T)T \right] S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) \right. \\
&\left. - \left[sF_7^R(s,T) - F_7^R(s,T)T \right] \overline{T} S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) \right\} \\
&= \gamma_7^{-1} \left[s^2F_7^R(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) - sF_7^R(s,T)TS_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) \right. \\
&\left. -sF_7^R(s,T)\overline{T}S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) + F_7^R(s,T)|T|^2 S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-2}(T) \right],
\end{aligned}
$$

$$
\mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-3}(T) = \gamma_7^{-1} \left(\mathcal{Q}_{c,s}^{-1}(T)F_7^L(p,T)p - \mathcal{Q}_{c,s}^{-1}(T)TF_7^L(p,T) \right) \tag{9.45}
$$

$$
\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-1}(T) = \gamma_7^{-1} \left(sF_7^R(s,T)\mathcal{Q}_{c,p}(T) - F_7^R(s,T)T\mathcal{Q}_{c,p}(T) \right). \tag{9.46}
$$

Now, since

$$
\begin{aligned}
\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T) \quad = \quad & \gamma_7^{-2} \left[sF_7^R(s,T)F_7^L(p,T)p - sF_7^R(s,T)TF_7^L(p,T) \right. \\
& \left. -F_7^R(s,T)TF_7^L(p,T)p + F_7^R(s,T)T^2F_7^L(p,T) \right],
\end{aligned}
$$

we get

$$
\begin{aligned}
&\mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-2}(T) = \mathcal{Q}_{c,s}(T)\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T)\mathcal{Q}_{c,p}(T) \qquad (9.47) \\
&= (s^2 - s(T+\overline{T}) + |T|^2)\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T)(p^2 - p(T+\overline{T}) + |T|^2) \\
&= s^2\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T)p^2 + s^2\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T)\left(|T|^2 - p(T+\overline{T})\right) \\
&+ \left(|T|^2 - s(T+\overline{T})\right)\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T)p^2 + \left(|T|^2 - s(T+\overline{T})\right) \\
&\mathcal{Q}_{c,s}^{-3}(T)\mathcal{Q}_{c,p}^{-3}(T)\left(|T|^2 - p(T+\overline{T})\right) \\
&= \gamma_7^{-2} \left\{ s^3F_7^R(s,T)F_7^L(p,T)p^3 - s^3F_7^R(s,T)TF_7^L(p,T)p^2 \right. \\
&-s^2F_7^R(s,T)TF_7^L(p,T)p^3 + s^2F_7^R(s,T)T^2F_7^L(p,T)p^2 \\
&+s^3F_7^R(s,T)F_7^L(p,T)p[|T|^2 - p(T+\overline{T})] - s^3F_7^R(s,T)TF_7^L(p,T) \\
&[|T|^2 - p(T+\overline{T})] - s^2F_7^R(s,T)TF_7^L(p,T)p[|T|^2 - p(T+\overline{T})] \\
&+s^2F_7^R(s,T)T^2F_7^L(p,T)[|T|^2 - p(T+\overline{T})] + \left[|T|^2 - s(T+\overline{T})\right] \\
&sF_7^R(s,T)F_7^L(p,T)p^3 - \left[|T|^2 - s(T+\overline{T})\right] sF_7^R(s,T)TF_7^L(p,T)p^2 \\
&- \left[|T|^2 - s(T+\overline{T})\right] F_7^R(s,T)TF_7^L(p,T)p^3 \left[|T|^2 - s(T+\overline{T})\right]. \\
&+F_7^R(s,T)T^2F_7^L(p,T)p^2 + \left[|T|^2 - s(T+\overline{T})\right] \left(sF_7^R(s,T)F_7^L(p,T)p \right. \\
&\left. -sF_7^R(s,T)TF_7^L(p,T) - F_7^R(s,T)TF_7^L(p,T)p + F_7^R(s,T)T^2F_7^L(p,T) \right) \cdot \\
&\left. \cdot \left[|T|^2 - p(T+\overline{T})\right] \right\}.
\end{aligned}
$$

Taking the sum of (9.43), (9.44), (9.45), (9.46) and (9.47) we get the statement. □

9.4 The Riesz projectors for the F-functional calculus for $n = 5$

In this section we study the Riesz projectors for the F-functional calculus for $n = 5$. This case is contained in the general result, see next section, but the explicit computation that can be done in this particular case shows the path for the general case and why we have introduced the pseudo F-resolvent equation.

We recall two preliminary lemmas that will be useful in the sequel and in the next section.

Lemma 9.4.1. *Let $B \in \mathcal{B}(V_n)$. Let G be a bounded slice Cauchy domain and let f be an intrinsic slice hyperholomorphic function whose domain contains G. Then for $p \in G$, and for any $J \in \mathbb{S}^{n-1}$ we have*

$$\frac{1}{2\pi}\int_{\partial(G\cap\mathbb{C}_J)} f(s)ds_J(\bar{s}B - Bp)(p^2 - 2s_0p + |s|^2)^{-1} = Bf(p).$$

In this section we give an answer to [41, Rem. 5.5]. Indeed, we prove that due to the equations proved in the previous section we can generate the Riesz projectors for the F-functional calculus. We begin by recalling the definition of projectors.

Definition 9.4.2. Let V_n be a Banach module and let $P : V_n \to V_n$ be a linear operator. If $P^2 = P$ we say that P is a projector.

Theorem 9.4.3. *Let $T \in \mathcal{BC}^{0.1}(V_5)$ be such that $T = \sum_{\ell=1}^{5} e_\ell T_\ell$. Let $\sigma_S(T) = \sigma_{S,1}(T) \cup \sigma_{S,2}(T)$ with*

$$\mathit{dist}\,(\sigma_{S,1}(T), \sigma_{S,2}(T)) > 0$$

and with

$$\sigma(T_\ell) \subset \mathbb{R} \ \text{ for all } \ell = 1, ..., 5.$$

Let G_1, G_2 be two admissible sets for T such that $\sigma_{S,1}(T) \subset G_1$ and $\bar{G}_1 \subset G_2$ and such that $dist\,(G_2, \sigma_{S,2}(T)) > 0$. Then the operator

$$\check{P} = \frac{1}{\gamma_5(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_J)} F_5^L(p,T)dp_J p^4 = \frac{1}{\gamma_5(2\pi)}\int_{\partial(G_2\cap\mathbb{C}_J)} s^4 ds_J F_5^R(s,T)$$

is a projector.

Proof. If we multiply the F- resolvent equation in Theorem 9.3.3 by s^2 on the left and p^2 on the right we get

$$
\begin{aligned}
&s^2F_5^R(s,T)S_L^{-1}(p,T)p^2+s^2S_R^{-1}(s,T)F_5^L(p,T)p^2\\
&+\gamma_5^{-1}\big(s^4F_5^R(s,T)F_5^L(p,T)p^4-3s^4F_5^R(s,T)TF_5^L(p,T)p^3\\
&-3s^3F_5^R(s,T)TF_5^L(p,T)p^4+3s^3F_5^R(s,T)T^2F_5^L(p,T)p^3\\
&-2s^3F_5^R(s,T)|T|^2TF_5^L(p,T)p^2+2s^3F_5^R(s,T)|T|^2F_5^L(p,T)p^3\\
&-2s^2F_5^R(s,T)|T|^2TF_5^L(p,T)p^3+s^3F_5^R(s,T)T^2F_5^L(p,T)p^3\\
&++s^3F_5^R(s,T)|T|^2TF_5^L(p,T)p^2+s^2F_5^R(s,T)|T|^2TF_5^L(p,T)p^3\\
&+s^2F_5^R(s,T)|T|^4F_5^L(p,T)p^2+s^3F_5^R(s,T)F_5^L(p,T)p^5\\
&-s^2F_5^R(s,T)TF_5^L(p,T)p^5+2s^2F_5^R(s,T)T^2F_5^L(p,T)p^4\\
&-s^2F_5^R(s,T)T^3F_5^L(p,T)p^3+2s^2F_5^R(s,T)T^2|T|^2F_5^L(p,T)p^2\\
&+s^5F_5^R(s,T)F_5^L(p,T)p^3-s^5F_5^R(s,T)TF_5^L(p,T)p^2\\
&+2s^4F_5^R(s,T)T^2F_5^L(p,T)p^2-s^3F_5^R(s,T)T^3F_5^L(p,T)p^2\big)\\
&=s^2\left\{[F_5^R(s,T)-F_5^L(p,T)]p.-\bar{s}[F_5^R(s,T)-F_5^L(p,T)]\right\}\cdot\\
&(p^2-2s_0p+|s|^2)^{-1}p^2.
\end{aligned}
$$

Now, we multiply the equation by ds_J on the left, integrate it over $\partial(G_2\cap\mathbb{C}_J)$ with respect to ds_J and then we multiply it by dp_J on the right and integrate over $\partial(G_1\cap\mathbb{C}_J)$ with respect to dp_J, we obtain

$$
\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)dp_Jp^2+\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JS_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2\\
&+\gamma_5^{-1}\Bigg(\int_{\partial(G_2\cap\mathbb{C}_J)}s^4ds_JF_5^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^4-3\int_{\partial(G_2\cap\mathbb{C}_J)}s^4ds_JF_5^R(s,T)T\int_{\partial(G_2\cap\mathbb{C}J)}F_5^L(p,T)p^3\\
&-3\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^4+3\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3\\
&-2\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)|T|^2T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2+2\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)|T|^2\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3\\
&-2\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)|T|^2T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3+\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)|T|^2T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2+\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)|T|^2T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)|T|^4\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2+\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^5\\
&-\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^5+2\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^4\\
&-\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)T^3\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3+2\int_{\partial(G_2\cap\mathbb{C}_J)}s^2ds_JF_5^R(s,T)T^2|T|^2\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^5ds_JF_5^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^3-\int_{\partial(G_2\cap\mathbb{C}_J)}s^5ds_JF_5^R(s,T)T\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2\\
&+2\int_{\partial(G_2\cap\mathbb{C}_J)}s^4ds_JF_5^R(s,T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2-\int_{\partial(G_2\cap\mathbb{C}_J)}s^3ds_JF_5^R(s,T)T^3\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jp^2\Bigg)
\end{aligned}
$$

$$= \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^2 \left\{[F_5^R(s,T)-F_5^L(p,T)]p - \bar{s}[F_5^R(s,T)-F_5^L(p,T)]\right\}(p^2-2s_0p+|s|^2)^{-1}dp_Jp^2.$$

From Lemma 9.2.18 the expression simplifies to

$$\gamma_5(2\pi)^2\frac{1}{\gamma_5}\left(\frac{1}{2\pi}\int_{G_2\cap\mathbb{C}_J} s^4 ds_J F_5^R(s,T)\right)\frac{1}{\gamma_5}\left(\frac{1}{2\pi}\int_{G_1\cap\mathbb{C}_J} F_5^R(p,T)dp_Jp^4\right)$$
$$= \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^2 \left\{[F_5^R(s,T)-F_5^L(p,T)]p - \bar{s}[F_5^R(s,T)-F_5^L(p,T)]\right\}\cdot$$
$$(p^2-2s_0p+|s|^2)^{-1}p^2dp_J.$$

By definition of projectors we have

$$\frac{(2\pi)^2}{\gamma_5^{-1}}\check{P}^2 = \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^2 \left\{[F_5^R(s,T)-F_5^L(p,T)]p\right.$$
$$\left.-\bar{s}[F_5^R(s,T)-F_5^L(p,T)]\right\}(p^2-2s_0p+|s|^2)^{-1}p^2dp_J.$$

Now, we work on the integral on the right hand side. As $\bar{G}_1 \subset G_2$, for any $s\in\partial(G_2\cap\mathbb{C}_J)$ the functions

$$p\mapsto p(p^2-2s_0p+|s|^2)^{-1}p^2,$$
$$p\mapsto (p^2-2s_0p+|s|^2)^{-1}p^2$$

are slice hyperholomorphic on $\bar{G}_1$. By Lemma 3.1.19 we have

$$\int_{\partial(G_1\cap\mathbb{C}_J)} p(p^2-2s_0p+|s|^2)^{-1}dp_Jp^2 = 0,$$
$$\int_{\partial(G_1\cap\mathbb{C}_J)} (p^2-2s_0p+|s|^2)^{-1}p^2dp_J = 0.$$

This implies that

$$\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^2F_5^R(s,T)p(p^2-2s_0p+|s|^2)^{-1}dp_Jp^2 = 0$$

and

$$\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^2\bar{s}F_5^R(s,T)(p^2-2s_0p+|s|^2)^{-1}dp_Jp^2 = 0,$$

from which we deduce

$$\frac{(2\pi)^2}{\gamma_5^{-1}}\check{P}^2 = \int_{\partial(G_2\cap\mathbb{C}_J)} s^2ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} [\bar{s}F_5^L(p,T)-F_5^L(p,T)p]$$
$$(p^2-2s_0p+|s|^2)^{-1}dp_Jp^2.$$

From Lemma 9.4.1 with $B =: F_5^L(p,T)$ and $f(s) := s^2$ we get

$$\check{P}^2 = \frac{1}{(2\pi)\gamma_5} \int_{\partial(G_1\cap\mathbb{C}_J)} F_5^L(p,T)dp_J p^4 = \check{P}.$$

□

The above computations can be generalized to study the general case where, however, we have to consider separately the case in which the Sce exponent is even or odd.

9.5 The F-resolvent equation for n odd

Following the strategy used in the cases $n = 5, 7$ we can now write the F-resolvent equations in the general case, involving the pseudo S-resolvent operators when it is not necessary (or possible) to replace the F-resolvent operators. Using the pseudo S-resolvent operators there are interesting symmetries that allows to use the F-resolvent equation for further applications, among which to compute the Riesz projectors. We start by stating a technical result which generalizes Lemma 9.3.2 and Lemma 9.3.4.

Lemma 9.5.1 (The general structure of the F-resolvent equation with the pseudo S-resolvent operators)**.** *Let $n > 3$ be an odd number, and let $h = \frac{n-1}{2}$ be the Sce exponent. Let us consider $T \in \mathcal{BC}^{0,1}(V_n)$. Then for $p, s \in \rho_S(T)$ the following equation holds*

$$\begin{aligned} &F_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_n^L(p,T) \qquad (9.48)\\ &+\gamma_n \left[\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i+1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T) + \sum_{i=0}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\\ &= \left\{\left[F_n^R(s,T) - F_n^L(p,T)\right]p - \bar{s}\left[F_n^R(s,T) - F_n^L(p,T)\right]\right\}(p^2 - 2s_0p + |s|^2)^{-1}. \end{aligned}$$

Proof. We left multiply the S-resolvent equation (3.9) by $\gamma_n \mathcal{Q}_{c,s}^{-h}(T)$

$$\begin{aligned} F_n^R(s,T)S_L^{-1}(p,T) &= \left\{[F_n^R(s,T) - \gamma_n \mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T)]p \right. \qquad (9.49)\\ &\left. -\bar{s}[F_n^R(s,T) - \gamma_n \mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T)]\right\} \cdot \\ &\cdot (p^2 - 2s_0p + |s|^2)^{-1} \end{aligned}$$

and we right multiply it by $\gamma_n \mathcal{Q}_{c,p}^{-h}(T)$, so we get

$$S_R^{-1}(s,T)F_n^L(p,T) = \left\{[\gamma_n S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T) - F_n^L(p,T)]p \right. \qquad (9.50)$$

$$-\bar{s}[\gamma_n S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T) - F_n^L(p,T)]\big\} \cdot \\ \cdot(p^2-2s_0p+|s|^2)^{-1}. \tag{9.51}$$

We now multiply S-resolvent equation on the left and on the right by $\mathcal{Q}_{c,s}^{-h+1+i}(T)$ and $\mathcal{Q}_{c,p}^{-i-1}(T)$, respectively. Then, we sum on the index $0 \leq i \leq h-2$ and we obtain

$$\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_{c,p}^{-i-1}(T) = \tag{9.52}$$
$$\left\{ \left[\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T) S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T) - \mathcal{Q}_{c,s}^{-h+1+i}(T) S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\right] p \right.$$
$$\left. -\bar{s}\left[\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T) S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T) - \mathcal{Q}_{c,s}^{-h+1+i}(T) S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\right\}$$
$$(p^2-2s_0p+|s|^2)^{-1}.$$

Now we sum (9.49), (9.50) and (9.52) multiplied by γ_n, and we get

$$F_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_n^L(p,T) \tag{9.53}$$
$$= +\gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i+1}(T) S_R^{-1}(s,T) S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)$$
$$= \big\{ \big[F_n^R(s,T) - \gamma_n \mathcal{Q}_{c,s}^{-h}(T) S_L^{-1}(p,T)$$
$$+\gamma_n S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T) - F_n^L(p,T) + \gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T) S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T)$$
$$-\mathcal{Q}_{c,s}^{-h+1+i}(T) S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\big]\, p$$
$$-\bar{s}\left[F_n^R(s,T) - \gamma_n \mathcal{Q}_{c,s}^{-h}(T) S_L^{-1}(p,T) + \gamma_n S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T) - F_n^L(p,T)\right.$$
$$+\gamma_n \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T) S_R^{-1}(s,T)\mathcal{Q}_{c,s}^{-i-1}(T)$$
$$\left.-\mathcal{Q}_{c,s}^{-h+1+i}(T) S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\big\}\,(p^2-2s_0p+|s|^2)^{-1}.$$

Putting in order the terms in the right hand side of the previous equation we get

$$
\begin{aligned}
&F_n^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_n^L(p,T) \\
&+\gamma_n\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i+1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T) \\
&=\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1} \\
&+\gamma_n\left\{\left[S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T)-\mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T)+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\cdot\right.\right. \\
&\mathcal{Q}_{c,p}^{-i-1}(T)-\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\Big]p-\bar{s}\left[S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T)\right. \\
&-\mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T)+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T) \\
&\left.\left.-\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}
\tag{9.54}
$$

Now, using the definition of left and right S-resolvent operators we get

$$
\begin{aligned}
&S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T)-\mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T) \\
&+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T)-\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T) \\
&=\mathcal{Q}_{c,s}^{-1}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-h}(T)-\mathcal{Q}_{c,s}^{-h}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T) \\
&+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-1}(T)-\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-2}(T) \\
&=\mathcal{Q}_{c,s}^{-1}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-h}(T)-\mathcal{Q}_{c,s}^{-h}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T)+\mathcal{Q}_{c,s}^{-h}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-1}(T) \\
&+\sum_{i=1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-1}(T)-\mathcal{Q}_{c,s}^{-1}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-h}(T) \\
&-\sum_{i=0}^{h-3}\mathcal{Q}_{c,s}^{-h+1+i}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-2}(T) \\
&=\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-h}(T)+\mathcal{Q}_{c,s}^{-h}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T) \\
&+\sum_{i=1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-1}(T)-\sum_{i=0}^{h-3}\mathcal{Q}_{c,s}^{-h+1+i}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-2}(T) \\
&=\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-h}(T)+\mathcal{Q}_{c,s}^{-h}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)
\end{aligned}
\tag{9.55}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-1}(T)-\sum_{i=1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-1}(T)\\
&=\mathcal{Q}_{c,s}^{-1}(T)(s-p)\mathcal{Q}_{c,p}^{-h}(T)+\mathcal{Q}_{c,s}^{-h}(T)(s-p)\mathcal{Q}_{c,p}^{-1}(T)\\
&+\sum_{i=1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s-p)\mathcal{Q}_{c,p}^{-i-1}(T)\\
&=\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)(s-p)\mathcal{Q}_{c,p}^{-i-1}(T).
\end{aligned}
$$

Then we compute

$$
\begin{aligned}
&\gamma_n\Big\{\Big[S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T)-\mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T)\\
&+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T)-\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\Big]p\\
&-\bar{s}\Big[S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T)-\mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T)+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\cdot\\
&\mathcal{Q}_{c,p}^{-i-1}(T)-\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\Big]\Big\}(p^2-2s_0p+|s|^2)^{-1}\\
&=\gamma_n\left\{\left[\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)(s-p)\mathcal{Q}_{c,p}^{-i-1}(T)\right]p\right.\\
&\left.-\bar{s}\left[\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)(s-p)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1}\\
&=\gamma_n\left[\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)(sp-p^2)\mathcal{Q}_{c,p}^{-i-1}(T)\right.\\
&\left.-\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)(|s|^2-\bar{s}p)\mathcal{Q}_{c,p}^{-i-1}(T)\right](p^2-2s_0p+|s|^2)^{-1}\\
&=\gamma_n\left[\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)(sp-p^2-|s|^2+\bar{s}p)\mathcal{Q}_{c,p}^{-i-1}(T)\right](p^2-2s_0p+|s|^2)^{-1}\\
&=-\gamma_n\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T)(p^2-2s_0p+|s|^2)(p^2-2s_0p+|s|^2)^{-1}
\end{aligned}
$$

$$= -\gamma_n \sum_{i=0}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T).$$

Hence

$$\begin{aligned}
&\gamma_n \Big\{ \big[S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T) - \mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T) \\
&+ \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T) \\
&-\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\big] p - \bar{s}\left[S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-h}(T) \right.\\
&-\mathcal{Q}_{c,s}^{-h}(T)S_L^{-1}(p,T) + \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+1+i}(T)S_R^{-1}(s,T)\mathcal{Q}_{c,p}^{-i-1}(T) \\
&\left.-\mathcal{Q}_{c,s}^{-h+1+i}(T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\Big\} (p^2 - 2s_0 p + |s|^2)^{-1} \\
&= -\gamma_n \sum_{i=0}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T).
\end{aligned} \tag{9.56}$$

Finally, by substituting (9.56) in (9.54) we get (9.48). □

Remark 9.5.2. Equation (9.48) generalizes (9.20) and (9.34). Indeed if we put $n = 5$, then $h = 2$ and we get

$$\begin{aligned}
&F_5^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_5^L(p,T) + \\
&\gamma_5 \left[\sum_{i=0}^{0} \mathcal{Q}_{c,s}^{-1+i}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T) + \sum_{i=0}^{1} \mathcal{Q}_{c,s}^{-2+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T) \right] \\
&= \left\{ \left[F_5^R(s,T) - F_5^L(p,T) \right] p - \bar{s}\left[F_5^R(s,T) - F_5^L(p,T) \right] \right\} (p^2 - 2s_0 p + |s|^2)^{-1}.
\end{aligned}$$

By developing the computations we obtain

$$\begin{aligned}
&F_5^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_5^L(p,T) \\
&+\gamma_5 \left[\mathcal{Q}_{c,s}^{-1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-2}(T)\mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T)\mathcal{Q}_{c,p}^{-2}(T) \right] \\
&= \left\{ [F_5^R(s,T) - F_5^L(p,T)]p - \bar{s}[F_5^R(s,T) - F_5^L(p,T)] \right\} (p^2 - 2s_0 p + |s|^2)^{-1},
\end{aligned}$$

which is exactly (9.20).

Now if we put $n = 7$ in (9.48), then $h = 3$ and we obtain

$$\begin{aligned}
&F_7^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_7^L(p,T) \\
&+\gamma_7 \left[\sum_{i=0}^{1} \mathcal{Q}_{c,s}^{-2+i}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T) + \sum_{i=0}^{2} \mathcal{Q}_{c,s}^{-3+i}(T)\mathcal{Q}_{c,s}^{-i-1}(T) \right]
\end{aligned}$$

$$= \left\{ \left[F_7^R(s,T) - F_7^L(p,T)\right] p - \bar{s} \left[F_7^R(s,T) - F_7^L(p,T)\right] \right\} (p^2 - 2s_0 p + |s|^2)^{-1}.$$

By developing the computations we get

$$\begin{aligned}
& F_7^R(s,T) S_L^{-1}(p,T) + S_R^{-1}(s,T) F_7^L(p,T) \\
& + \gamma_7 [\mathcal{Q}_{c,s}^{-2}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-1}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_{c,p}^{-2}(T) \\
& + \mathcal{Q}_{c,s}^{-3}(T) \mathcal{Q}_{c,p}^{-1}(T) + \mathcal{Q}_{c,s}^{-2}(T) \mathcal{Q}_{c,p}^{-2}(T) + \mathcal{Q}_{c,s}^{-1}(T) \mathcal{Q}_{c,p}^{-3}(T)] \\
& = \left\{ \left[F_7^R(s,T) - F_7^L(p,T)\right] p - \bar{s} \left[F_7^R(s,T) - F_7^L(p,T)\right] \right\} (p^2 - 2s_0 p + |s|^2)^{-1},
\end{aligned}$$

which is exactly (9.34).

Remark 9.5.3. The proof of the previous lemma shows that the structure of the resolvent equations of the hyperholomorphic functional calculi is crucial. In fact the term

$$\left\{ \left[F_n^R(s,T) - F_n^L(p,T)\right] p - \bar{s} \left[F_n^R(s,T) - F_n^L(p,T)\right] \right\} (p^2 - 2s_0 p + |s|^2)^{-1}$$

involves the difference of the F-resolvent operators entangled with the Cauchy kernel of slice hyperholomorphic functions. This term is equal to a function involving the products of the F-resolvent operators and of the S-resolvent operators that appear in the term

$$F_n^R(s,T) S_L^{-1}(p,T) + S_R^{-1}(s,T) F_n^L(p,T)$$

and of a more complicated part that involves the S-resolvent operators and the pseudo S-resolvent operators, namely

$$\gamma_n \left[\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i+1}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_{c,p}^{-i-1}(T) + \sum_{i=0}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T) \mathcal{Q}_{c,p}^{-i-1}(T) \right].$$

In order to find a pseudo F-resolvent equation we divide into two cases according to the parity of the Sce exponent $h = \dfrac{n-1}{2}$. To state the following result we introduce this notations

$$\begin{aligned}
\mathcal{A}_0(s,p,T) := & -s^h F_n^R(s,T) T F_n^L(p,T) p^{h-1} - s^{h-1} F_n^R(s,T) T F_n^L(p,T) p^h \\
& + s^{h-1} F_n^R(s,T) T^2 F_n^L(p,T) p^{h-1} \\
& + s^h F_n^R(s,T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k F_n^L(p,T) p^{h-2k} \\
& - s^h F_n^R(s,T) \sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} (|T|^2 - 2T_0 p)^k T F_n^L(p,T) p^{h-1-2k}
\end{aligned}$$

$$
\begin{aligned}
&- s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}T(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\\
&+ s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}T(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}_0(s,p,T) :=& \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right)\\
&- \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right)\\
&- \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right)\\
&+ \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{C}_0(s,p,T) :=& \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h\\
&- \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^kT\right)F_n^L(p,T)p^{h-1}\\
&- \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h\\
&+ \left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^kT\right)F_n^L(p,T)p^{h-1}.
\end{aligned}
$$

9.5.1 The general structure of the pseudo F-resolvent equation for h odd

The main result of this subsection is the following theorem.

Theorem 9.5.4 (The general structure of the pseudo F-resolvent equation for h odd number)**.** *Let $n > 3$ be an odd number as well as h. Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then for $p, s \in \rho_S(T)$ the following equation holds*

$$
\begin{aligned}
&F_n^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_n^L(p,T)\\
&+\gamma_n\left[s\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T)p-s\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)\right.\\
&-\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)p+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-i-2}(T)\\
&\left.+\sum_{i=0,i\neq\frac{h-1}{2}}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\\
&+\gamma_n^{-1}\left[\,s^hF_n^R(s,T)F_n^L(p,T)p^h+\mathcal{A}_0(s,p,T)+\mathcal{B}_0(s,p,T)+\mathcal{C}_0(s,p,T)\right]\\
&=\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1},
\end{aligned}
\tag{9.57}
$$

where the three terms $\mathcal{A}_0(s,p,T)$, $\mathcal{B}_0(s,p,T)$ and $\mathcal{C}_0(s,p,T)$ are defined above.

Proof. We start by rewriting formula (9.48) as

$$
\begin{aligned}
&F_n^R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_n^L(p,T)\\
&+\gamma_n\left[\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i+1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)+\mathcal{Q}_{c,s}^{-\frac{h+1}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+1}{2}}(T)\right.\\
&\left.+\sum_{i=0,i\neq\frac{h-1}{2}}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T)\right]\\
&=\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1}.
\end{aligned}
$$

Now, we focus on the term $\mathcal{Q}_{c,s}^{-\frac{h+1}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+1}{2}}(T)$ and with some manipulations we obtain

$$
\begin{aligned}
\mathcal{Q}_{c,s}^{-\frac{h+1}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+1}{2}}(T) &= \mathcal{Q}_{c,s}^{-\frac{h+1}{2}}(T)\mathcal{Q}_{c,s}s^{-\frac{h-1}{2}}(T)\mathcal{Q}_{c,s}^{-\frac{1-h}{2}}(T)\cdot\\
&\quad\mathcal{Q}_{c,p}^{-\frac{1-h}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h-1}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+1}{2}}(T)\\
&= \mathcal{Q}_{c,s}^{-h}(T)\mathcal{Q}_{c,s}^{-\frac{1-h}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{1-h}{2}}(T)\mathcal{Q}_{c,p}^{-h}(T).
\end{aligned}
$$

By the binomial formula we get

$$
\begin{aligned}
&\mathcal{Q}_{c,s}^{-\frac{h+1}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+1}{2}}(T) = \mathcal{Q}_{c,s}^{-h}(T)(s^2\mathcal{I} - 2sT_0 + T\overline{T})^{\frac{h-1}{2}} \cdot \\
&\quad (p^2\mathcal{I} - 2pT_0 + T\overline{T})^{\frac{h-1}{2}}\mathcal{Q}_{c,p}^{-h}(T) \\
&= \mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=0}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0 s)^k\right) \\
&\left(\sum_{k=0}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0 p)^k p^{h-1-2k}\right)\mathcal{Q}_{c,p}^{-h}(T) \\
&= \mathcal{Q}_{c,s}^{-h}(T)\left(s^{h-1} + \sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0 s)^k\right) \\
&\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0 p)^k p^{h-1-2k} + p^{h-1}\right)\mathcal{Q}_{c,p}^{-h}(T) \\
&= s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1} + s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0 p)^k p^{h-1-2k}\right) \cdot \\
&\mathcal{Q}_{c,p}^{-h}(T) + \mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0 s)^k\right) \\
&\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2 - 2T_0 p)^k p^{h-1-2k}\right) \cdot \\
&\mathcal{Q}_{c,p}^{-h}(T) + \mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2 - 2T_0 s)^k\right)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1}.
\end{aligned}
$$

Now, we use the left and right F-resolvent equations, (see Theorem 9.3.1)

$$F_n^L(p,T)p - TF_n^L(p,T) = \gamma_n \mathcal{Q}_{c,p}^{-h}(T)$$

and

$$sF_n^R(s,T) - F_n^R(s,T)T = \gamma_n \mathcal{Q}_{c,s}^{-h}(T).$$

We go through the computations term by term

$$
\begin{aligned}
&s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1} \qquad (9.58)\\
&=\gamma_n^{-2}\left[s^hF_n^R(s,T)F_n^L(p,T)p^h-s^hF_n^R(s,T)TF_n^L(p,T)p^{h-1}\right.\\
&\left.-s^{h-1}F_n^R(s,T)TF_n^L(p,T)p^h+s^{h-1}F_n^R(s,T)T^2F_n^L(p,T)p^{h-1}\right].
\end{aligned}
$$

Then we consider

$$
\begin{aligned}
&s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^k\mathcal{Q}_{c,p}^{-h}(T)p^{h-1-2k}\right)\\
&=\gamma_n^{-2}\left[s^hF_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right.\\
&-s^hF_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\\
&-s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}T(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\\
&\left.+s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}T(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right].
\end{aligned}
$$

Then we compute the term

$$
\begin{aligned}
&\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}\mathcal{Q}_{c,s}^{-h}(T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^k\mathcal{Q}_{c,p}^{-h}(T)p^{h-1-2k}\right)\\
&=\gamma_n^{-2}\left[\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right)\right.\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right)\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right)\\
&\left.+\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right)\right].
\end{aligned}
$$

We have also

$$
\begin{aligned}
&\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k}\mathcal{Q}_{c,s}^{-h}(T)(|T|^2-2T_0s)^k\right)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1}\\
&=\gamma_n^{-2}\left[\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h\right.\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^kT\right)F_n^L(p,T)p^{h-1}\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h\\
&\left.+\left(\sum_{k=1}^{\frac{h-1}{2}} \binom{\frac{h-1}{2}}{k} s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^kT\right)F_n^L(p,T)p^{h-1}\right].
\end{aligned}
$$

Finally by using the definition of left and right S-resolvent operators we get

$$
\begin{aligned}
&\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i+1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)\\
&=\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s\mathcal{I}-\overline{T})(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-2}(T)\\
&=s\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T)p-s\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)\\
&-\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)p+\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-i-2}(T),
\end{aligned}
$$

and this concludes the proof. □

9.5.2 The general structure of the pseudo F-resolvent equation for h even number

In this last subsection we consider the case in which $h=(n-1)/2$ is an even number. To state the following result we need these notations

$$\begin{aligned}\mathcal{A}_1(s,p,T) :&= -s^h F_n^R(s,T)TF_n^L(p,T)p^{h-1} - s^{h-1}F_n^R(s,T)TF_n^L(p,T)p^h \\ &\quad + s^{h-1}F_n^R(s,T)T^2F_n^L(p,T)p^{h-1}\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_1(s,p,T) :&= s^h F_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k F_n^L(p,T)p^{h-2k} \\ &\quad - s^h F_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k TF_n^L(p,T)p^{h-1-2k} \\ &\quad - s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}T(|T|^2-2T_0p)^k F_n^L(p,T)p^{h-2k} \\ &\quad + s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}T(|T|^2-2T_0p)^k TF_n^L(p,T)p^{h-1-2k}\end{aligned}$$

and

$$\begin{aligned}\mathcal{C}_1(s,p,T) :=& \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k F_n^L(p,T)p^{h-2k}\right) \\ &- \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k TF_n^L(p,T)p^{h-1-2k}\right) \\ &- \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k F_n^L(p,T)p^{h-2k}\right) \\ &+ \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k TF_n^L(p,T)p^{h-1-2k}\right) \\ &+ \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h \\ &- \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k T\right)F_n^L(p,T)p^{h-1} \\ &- \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h \\ &+ \left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^k T\right)F_n^L(p,T)p^{h-1}.\end{aligned}$$

Theorem 9.5.5 (The general structure of the pseudo F-resolvent equation for h even number)**.** *Let $n > 3$ be an odd number and h be even. Let $T \in \mathcal{BC}^{0,1}(V_n)$. Then for $p, s \in \rho_S(T)$ the following equation holds*

$$\begin{aligned}
&F_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_n^L(p,T) \\
&+\gamma_n \Big[-s\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\overline{T}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T) - \mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\overline{T}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p \\
&+\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T) + \sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T) \\
&+s\sum_{i=0, i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T)p - s\sum_{i=0, i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T) \\
&-\sum_{i=0, i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)p + \sum_{i=0, i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-i-2}(T)\Big] \\
&+\gamma_n^{-1}\left[\mathcal{A}_1(s,p,T) + \mathcal{B}_1(s,p,T) + \mathcal{C}_1(s,p,T) + s^hF_n^R(s,T)F_n^L(p,T)p^h\right] \\
&= \left\{\left[F_n^R(s,T) - F_n^L(p,T)\right]p - \bar{s}\left[F_n^R(s,T) - F_n^L(p,T)\right]\right\}(p^2 - 2s_0p + |s|^2)^{-1},
\end{aligned} \tag{9.59}$$

where the three terms $\mathcal{A}_1(s,p,T)$, $\mathcal{B}_1(s,p,T)$ and $\mathcal{C}_1(s,p,T)$ are defined above.

Proof. Let us begin by writing formula (9.48) as

$$\begin{aligned}
&F_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_n^L(p,T) \\
&+\gamma_n\Bigg[\mathcal{Q}_{c,s}^{-\frac{h}{2}}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-\frac{h}{2}}(T) + \sum_{i=0, i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i+1}(T)S_R^{-1}(s,T)\cdot \\
&S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T) + \sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T)\Bigg] \\
&= \left\{\left[F_n^R(s,T) - F_n^L(p,T)\right]p - \bar{s}\left[F_n^R(s,T) - F_n^L(p,T)\right]\right\}(p^2 - 2s_0p + |s|^2)^{-1}.
\end{aligned}$$

Now, we focus on the $\mathcal{Q}_{c,s}^{-\frac{h}{2}}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-\frac{h}{2}}(T)$. By definition of left and right S-resolvent operators we get

$$\mathcal{Q}_{c,s}^{-\frac{h}{2}}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-\frac{h}{2}}(T)$$

$$= \mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)(s\mathcal{I}-\overline{T})(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)$$
$$= s\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p - s\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\overline{T}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T) - \mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\overline{T}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p$$
$$+\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T).$$

We continue the calculations only on the term $s\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p$. By the binomial formula we get

$$s\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p = s\mathcal{Q}_{c,s}^{-h}(T)(s^2\mathcal{I}-2sT_0+T\overline{T})^{\frac{h-2}{2}}\cdot$$
$$(p^2\mathcal{I}-2pT_0+T\overline{T})^{\frac{h-2}{2}}\mathcal{Q}_{c,p}^{-h}(T)p$$
$$= s\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=0}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2-2k}(|T|^2-2T_0s)^k\right)\cdot$$
$$\left(\sum_{k=0}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kp^{h-2-2k}\right)\mathcal{Q}_{c,p}^{-h}(T)p$$
$$= s\mathcal{Q}_{c,s}^{-h}(T)\left(s^{h-2}+\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2-2k}(|T|^2-2T_0s)^k\right)\cdot$$
$$\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}(|T|^2-2T_0p)^kp^{h-2-2k}+p^{h-2}\right)\mathcal{Q}_{c,p}^{-h}(T)p$$
$$= s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\mathcal{Q}_{c,p}^{h}(T)p^{h-1} + s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kp^{h-1-2k}\right)\cdot$$
$$\mathcal{Q}_{c,p}^{-h}(T)+\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-1}{2}}{k}s^{h-1-2k}(|T|^2-2T_0s)^k\right)\cdot$$
$$\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kp^{h-1-2k}\right)\mathcal{Q}_{c,p}^{-h}(T)$$
$$+\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}(|T|^2-2T_0s)^k\right)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1}.$$

Now, we use the left and right F-resolvent equations in Theorem 9.3.1

and we go through the computations term by term

$$s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1}=\gamma_n^{-2}\left[s^hF_n^R(s,T)F_n^L(p,T)p^h\right.$$
$$-s^hF_n^R(s,T)TF_n^L(p,T)p^{h-1}-s^{h-1}F_n^R(s,T)TF_n^L(p,T)p^h$$
$$\left.+s^{h-1}F_n^R(s,T)T^2F_n^L(p,T)p^{h-1}\right],$$

and

$$s^{h-1}\mathcal{Q}_{c,s}^{-h}(T)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k\mathcal{Q}_{c,p}^{-h}(T)p^{h-1-2k}\right)$$
$$=\gamma_n^{-2}\left[s^hF_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right.$$
$$-s^hF_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}$$
$$-s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}T(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}$$
$$\left.+s^{h-1}F_n^R(s,T)\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}T(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right],$$

then we consider the term

$$\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}\mathcal{Q}_{c,s}^{-h}(T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^k\mathcal{Q}_{c,p}^{-h}(T)p^{h-1-2k}\right)$$
$$=\gamma_n^{-2}\left[\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right)\right.$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-1}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right)$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kF_n^L(p,T)p^{h-2k}\right)$$
$$\left.+\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k-1}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}(|T|^2-2T_0p)^kTF_n^L(p,T)p^{h-1-2k}\right)\right],$$

and the other term

$$\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}\mathcal{Q}_{c,s}^{-h}(T)(|T|^2-2T_0s)^k\right)\mathcal{Q}_{c,p}^{-h}(T)p^{h-1}$$
$$=\gamma_n^{-2}\left[\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h\right.$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-2k}F_n^R(s,T)(|T|^2-2T_0s)^kT\right)F_n^L(p,T)p^{h-1}$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^k\right)F_n^L(p,T)p^h$$
$$\left.+\left(\sum_{k=1}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}s^{h-1-2k}F_n^R(s,T)T(|T|^2-2T_0s)^kT\right)F_n^L(p,T)p^{h-1}\right].$$

Finally by using the definition of left and right S-resolvent operators we obtain

$$\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i+1}(T)S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}^{-i-1}(T)$$
$$=\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)(s\mathcal{I}-\overline{T})(p\mathcal{I}-\overline{T})\mathcal{Q}_{c,p}^{-i-2}(T)$$
$$=s\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{i+2}(T)p-s\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)$$
$$-\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}\mathcal{Q}_{c,p}^{-i-2}(T)p+\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\overline{T}^2\mathcal{Q}_{c,p}^{-i-2}(T)$$

and this concludes the proof. □

9.5.3 Comments

The general structure of the F-resolvent equation involving the pseudo S-resolvent operators is obtained in Lemma 9.5.1: for $n>3$, n odd, and Sce exponent $h = \frac{n-1}{2}$ the equation is

$$\begin{aligned} & F_n^R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_n^L(p,T) \\ & + \gamma_n \left[\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i+1}(T) S_R^{-1}(s,T) S_L^{-1}(p,T) \mathcal{Q}_{c,p}^{-i-1}(T) + \sum_{i=0}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T) \mathcal{Q}_{c,p}^{-i-1}(T) \right] \\ & = \left\{ \left[F_n^R(s,T) - F_n^L(p,T) \right] p - \bar{s} \left[F_n^R(s,T) - F_n^L(p,T) \right] \right\} (p^2 - 2s_0 p + |s|^2)^{-1} \end{aligned} \tag{9.60}$$

for paravector operators $T \in \mathcal{BC}(V_n)$ and for $p, s \in \rho_S(T)$.

This general structure is useful to study the Riesz projectors of the F-functional calculus (see next section). This equation has two different expressions according to the fact that the Sce exponent $h := (n-1)/2$ is an even or odd number. Precisely, for h odd is obtained in Theorem 9.5.4 while the case in which h is even is obtained in Theorem 9.5.5.

Let us summarize, for the F-resolvent equation, the major similarities and the main differences with respect to S-resolvent equation and the resolvent equation (see (3.9)) of the Riesz-Dunford functional calculus (see (2.3)).

(I-Fa) There are two different F-resolvent operators $F_n^L(s,T)$ and $F_n^R(p,T)$ which are right slice hyperholomorphic in s and left slice hyperholomorphic in p, respectively.

(I-Fb) There are additional terms containing:

(i) the operator $F_n^R(s,T)BF_n^L(s,T)$ which, for any bounded operator B, preserves the right slice hyperholomorphicity in s and the left slice hyperholomorphicity in p .

(ii) the commutative version of the S-resolvent operators which appears in the terms:

$$F_n^R(s,T)S_L^{-1}(p,T) \quad \text{and} \quad S_R^{-1}(s,T)F_n^L(p,T).$$

(iii) the commutative pseudo S-resolvent operators $\mathcal{Q}_{c,}(T)$ and $\mathcal{Q}_{c,p}(T)$.

(II-F) The difference $F_n^L(s,T) - F_n^R(s,T)$ is entangled with of the Cauchy kernel of slice hyperholomorphic functions, in the same way as the S-resolvent equation, i.e.,

$$(s,p) \mapsto \left[[F_n^L(s,T) - F_n^R(s,T)]p - \bar{s}[F_n^L(s,T) - F_n^R(s,T)] \right] (p^2 - 2s_0 p + |s|^2)^{-1}$$

and it preserves the slice hyperholomorphicity on the right in s and on the left in p.

(III-F) The term

$$\Big[[F_n^L(s,T)-F_n^R(s,T)]p-\overline{s}[F_n^L(s,T)-F_n^R(s,T)]\Big](p^2-2s_0p+|s|^2)^{-1}$$

equals not only terms that involve suitable product of the F-resolvent operators, but also the terms described in the above items (i), (ii) and (iii).

Remark 9.5.6. Similarly to the case of the S-resolvent equation, the products of the form

$$F_n^L(s,T)BF_n^R(s,T),$$

where B is a linear operator, cannot be used in the F-resolvent equation because they destroy slice hyperholomorphicity.

9.6 The Riesz Projectors for the F-functional calculus: the general case of n odd

In the monogenic functional calculus developed by McIntosh and collaborators, [99, 101, 108, 112], the resolvent equation is missing. They are able to study the Riesz projectors by using another functional calculus: the Weyl calculus, see [100]. For the F-functional calculus, which is a monogenic functional calculus, the interesting symmetries that appear in the equations of Theorem 9.5.4 and Theorem 9.5.5 allow to study the Riesz projectors.

Theorem 9.6.1. *Let $n>3$ be an odd number and let $T=\sum_{i=1}^n e_\ell T_\ell\in\mathcal{BC}^{0,1}(V_n)$. Let $\sigma_S(T)=\sigma_{S,1}(T)\cup\sigma_{S,2}(T)$ with*

$$\mathit{dist}\,(\sigma_{S,1}(T),\sigma_{S,2}(T))>0,$$

and

$$\sigma(T_\ell)\subset\mathbb{R}\ \text{ for all } \ell=1,...,n.$$

Let G_1, G_2 be two admissible sets for T such that $\sigma_{S,1}(T)\subset G_1$ and $\bar{G}_1\subset G_2$ and such that $dist\,(G_2,\sigma_{S,2}(T))>0$. Then the operator

$$\begin{aligned}\check{P} &= \frac{1}{\gamma_n(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{n-1} \\ &= \frac{1}{\gamma_n(2\pi)}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{n-1}ds_JF_n^R(s,T).\end{aligned}\tag{9.61}$$

is a projector.

Proof. We divide the proof in two cases, according to the parity of $h = \frac{n-1}{2}$.

CASE I: The Sce exponent h is odd.

We start by multiplying the equation of Theorem 9.5.4 by s^h on the left and p^h on the right, and since $T_0 = 0$ we get

$$\begin{aligned}
&s^h F_n^R(s,T) S_L^{-1}(p,T) p^h + s^h S_R^{-1}(s,T) F_n^L(p,T) p^h \\
&+\gamma_n \left[s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) \mathcal{Q}_{c,p}^{-i-2}(T) p^{h+1} + s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) T \mathcal{Q}_{c,p}^{-i-2}(T) p^h \right. \\
&+ s^h \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) T \mathcal{Q}_{c,p}^{-i-2}(T) p^{h+1} + s^h \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) T^2 \mathcal{Q}_{c,p}^{-i-2}(T) p^h + \\
&\left. + s^h \sum_{i=0, i \neq \frac{h-1}{2}}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T) \mathcal{Q}_{c,p}^{i+1}(T) p^h \right] + \gamma_n^{-1} \left[s^{2h} F_n^R(s,T) F_n^L(p,T) p^{2h} \right. \\
&\left. + s^h \mathcal{A}_0(s,p,T) p^h + s^h \mathcal{B}_0(s,p,T) p^h + s^h \mathcal{C}_0(s,p,T) p^h \right] \\
&= s^h \left\{ \left[F_n^R(s,T) - F_n^L(p,T) \right] p - \bar{s} \left[F_n^R(s,T) - F_n^L(p,T) \right] \right\} \cdot \\
&(p^2 - 2s_0 p + |s|^2)^{-1} p^h.
\end{aligned} \tag{9.62}$$

Now, we multiply equation (9.62) by ds_J on the left, integrate it over $\partial(G_2 \cap \mathbb{C}_J)$ with respect to ds_J and then we multiply it by dp_J on the right and integrate over $\partial(G_1 \cap \mathbb{C}_J)$ with respect to dp_J. We obtain

$$\begin{aligned}
&\int_{\partial(G_2 \cap \mathbb{C}_J)} s^h ds_J F_n^R(s,T) \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p,T) dp_J p^h \\
&+ \int_{\partial(G_2 \cap \mathbb{C}_J)} s^h ds_J S_R^{-1}(s,T) \int_{\partial(G_1 \cap \mathbb{C}_J)} F_n^L(p,T) dp_J p^h \\
&+\gamma_n \left[\int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) ds_J \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-2}(T) dp_J p^{h+1} \right. \\
&+ \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1} \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) ds_J T \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-2}(T) dp_J p^h \\
&+ \int_{\partial(G_2 \cap \mathbb{C}_J)} s^h \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) ds_J T \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-2}(T) dp_J p^{h+1} \\
&+ \int_{\partial(G_2 \cap \mathbb{C}_J)} s^h \sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) ds_J T^2 \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-2}(T) dp_J p^h
\end{aligned} \tag{9.63}$$

$$
\begin{aligned}
&+\int_{\partial(G_2\cap\mathbb{C}_J)} s^h \sum_{i=0, i\neq\frac{h-1}{2}}^{h-1} \mathcal{Q}_{c,s}^{-h+i}(T)ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-1}(T)dp_Jp^h\Bigg]\\
&+\gamma_n^{-1}\Bigg[\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h}ds_JF_n^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h}+\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J} s^hds_J\mathcal{A}_0(s,p,T)dp_Jp^h+s^hds_J\mathcal{B}_0(s,p,T)dp_Jp^h\\
&+s^hds_J\mathcal{C}_0(s,p,T)dp_Jp^h\Bigg]\\
&=\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J\int_{\partial(G_1\cap\mathbb{C}_J)} s^h\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}\\
&(p^2-2s_0p+|s|^2)^{-1}dp_Jp^h.
\end{aligned}
$$

Recalling the definition of $\mathcal{A}_0$, $\mathcal{B}_0$, $\mathcal{C}_0$ and the fact that $T_0=0$ we have

$$
\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} s^hds_J\mathcal{A}_0(s,p,T)dp_Jp^h+s^hds_J\mathcal{B}_0(s,p,T)dp_Jp^h+s^hds_J\mathcal{C}_0(s,p,T)dp_Jp^h\\
&=-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h}F_n^R(s,T)ds_JT\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-1}\\
&-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1}F_n^R(s,T)ds_JT\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h}\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1}ds_JF_n^R(s,T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-1}\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h}ds_JF_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}|T|^{2k}\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-2k}\\
&-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h}ds_JF_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}|T|^{2k}T\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)p^{2h-1-2k}\\
&-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1}ds_JF_n^R(s,T)\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}T|T|^{2k}\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-2k}\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1}F_n^R(s,T)ds_J\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}T|T|^{2k}T\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-1-2k}\\
&+\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k}ds_JF_n^R(s,T)|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}|T|^{2k}\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-2k}\right)\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k}ds_JF_n^R(s,T)|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}|T|^{k}T\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h-1-2k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h-2k-1}ds_JF_n^R(s,T)T|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}|T|^{2k}\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h-2k}\right)\\
&+\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h-2k-1}ds_JF_n^R(s,T)T|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}|T|^{2k}T\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h-1-2k}\right)\\
&+\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h-2k}ds_JF_n^R(s,T)|T|^{2k}\right)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h}\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h-2k}ds_JF_n^R(s,T)|T|^{2k}T\right)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h-1}\\
&-\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h-1-2k}ds_JF_n^R(s,T)T|T|^{2k}\right)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h}\\
&+\left(\sum_{k=1}^{\frac{h-1}{2}}\binom{\frac{h-1}{2}}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h-1-2k}ds_JF_n^R(s,T)T^2|T|^{2k}\right)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h-1}.
\end{aligned}
$$

Now, since $h\leq 2h-1$ by Lemma 9.2.18 we get

$$
\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_JF_n^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)dp_Jp^h\\
&=\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_JS_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^h\\
&=0.
\end{aligned}
$$

Moreover, since $2h-2k\leq 2h-1$ and $2h-1-2k\leq 2h-1$ we obtain

$$
\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}s^hds_J\mathcal{A}_0(s,p,T)dp_Jp^h+s^hds_J\mathcal{B}_0(s,p,T)dp_Jp^h\\
&+s^hds_J\mathcal{C}_0(s,p,T)dp_Jp^h=0.
\end{aligned}
$$

Now, we focus on the term

$$
\int_{\partial(G_2\cap\mathbb{C}_J)}s^{h+1}\sum_{i=0}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h+1}.
$$

First of all we split the sum in two parts and write

$$\sum_{i=0}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T) = \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T) + \sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T),$$

where $\lfloor . \rfloor$ is the floor of a number. In the first sum the powers of $\mathcal{Q}_{c,s}^{-1}(T)$ are more than the powers of $\mathcal{Q}_{c,p}^{-1}(T)$, and conversely in the second sum.

Since $T_0 = 0$, by the binomial formula we get

$$\sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T) = \mathcal{Q}_{c,s}^{-h}(T) \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} s^{2k} |T|^{2(i-k)} \mathcal{Q}_{c,p}^{-i-2}(T) +$$
$$+ \sum_{i=\lfloor \frac{h-2}{2} \rfloor+1}^{h-2} \mathcal{Q}_{c,s}^{-h+i}(T) \sum_{k=0}^{h-2-i} \binom{h-2-i}{k} p^{2k} |T|^{2(h-2-i-k)} \mathcal{Q}_{c,p}^{-h}(T).$$

Consider the first sum. By the F- resolvent equation, see (9.19) we get

$$\begin{aligned} &\sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} s^{2k} \mathcal{Q}_{c,s}^{-h}(T) |T|^{2(i-k)} \mathcal{Q}_{c,p}^{-i-2}(T) \\ &= \gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} s^{2k} \left(s F_n^R(s,T) - F_n^R(s,T) T \right) |T|^{2(i-k)} \mathcal{Q}_{c,p}^{-i-2}(T). \end{aligned}$$

Hence we have to compute the following integrals

$$\gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+2+2k} ds_J F_n^R(s,T) |T|^{2(i-k)} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-2}(T) dp_J p^{h+1},$$

$$\gamma_n^{-1} \sum_{i=0}^{\lfloor \frac{h-2}{2} \rfloor} \sum_{k=0}^{i} \binom{i}{k} \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1+2k} ds_J F_n^R(s,T) T |T|^{2(i-k)} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-i-2}(T) dp_J p^{h+1}.$$

Now, since h is odd then we can write $h = 2N+1$, with $N \in \mathbb{N}$. This implies that

$$\begin{aligned} h+2+2k &\leq 2i+2+2N+1 \leq 2\lfloor \frac{h-2}{2} \rfloor + 2 + 2N + 1 \\ &= 2(N-1)+2+2N+1 = 4N+1 = 2h-1. \end{aligned}$$

Similarly we get

$$h+1+2k \leq 2h-1.$$

Therefore by Lemma 9.2.18 we get

$$\gamma_n^{-1}\sum_{i=0}^{\lfloor\frac{h-2}{2}\rfloor}\sum_{k=0}^{i}\binom{i}{k}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+2+2k}ds_J F_n^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_J p^{h+1}=0$$
$$\gamma_n^{-1}\sum_{i=0}^{\lfloor\frac{h-2}{2}\rfloor}\sum_{k=0}^{i}\binom{i}{k}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1+2k}ds_J F_n^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_J p^{h+1}=0.$$

Now, we focus on the second sum. By the F- resolvent equation, see (9.18), we get

$$\sum_{i=\lfloor\frac{h-2}{2}\rfloor+1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\sum_{k=0}^{h-2-i}\binom{h-2-i}{k}|T|^{2(h-2-i-k)}\mathcal{Q}_{c,p}^{-h}(T)p^{2k}$$
$$=\gamma_n^{-1}\sum_{i=\lfloor\frac{h-2}{2}\rfloor+1}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\sum_{k=0}^{h-2-i}\binom{h-2-i}{k}|T|^{2(h-2-i-k)}\left(F_n^R(p,T)p\right.$$
$$\left.-TF_n^R(p,T)\right)p^{2k}.$$

Hence we have to compute the following integrals

$$\gamma_n^{-1}\sum_{i=\lfloor\frac{h-2}{2}\rfloor+1}^{h-2}\sum_{k=0}^{2+i-h}\binom{h-2-i}{k}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1}ds_J\mathcal{Q}_{c,s}^{-h+i}(T)|T|^{2(h-2-i-k)}$$
$$\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^R(p,T)dp_Jp^{h+2k+2},$$

$$\gamma_n^{-1}\sum_{i=\lfloor\frac{h-2}{2}\rfloor+1}^{h-2}\sum_{k=0}^{h-2-i}\binom{h-2-i}{k}\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1}ds_J\mathcal{Q}_{c,s}^{-h+i}(T)|T|^{2(h-2-i-k)}T$$
$$\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^R(p,T)dp_Jp^{h+2k+1}.$$

Since $h=2N+1$, with $N\in\mathbb{N}$ we get

$$\begin{aligned}2k+2+h &\leq 2(h-2-i)+2+h=2h-4-2i+2+h+2+2N+1\\ &\leq 4N+2-4-2\left(\lfloor\frac{h-2}{2}\rfloor+1\right)=4N+1=2h-1,\end{aligned}$$

and similarly

$$2k+1+h\leq 2h-1,$$

together with Lemma 9.2.18 we get

$$\gamma_n^{-1}\sum_{i=\lfloor\frac{h-2}{2}\rfloor+1}^{h-2}\sum_{k=0}^{h-2-i}\binom{h-2-i}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{h+1}ds_J\mathcal{Q}_{c,s}^{-h+i}(T)|T|^{2(h-2-i-k)}\int_{\partial(G_1\cap\mathbb{C}_J)}$$
$$F_n^R(p,T)dp_Jp^{h+2k+2}=0$$
$$\gamma_n^{-1}\sum_{i=\lfloor\frac{h-2}{2}\rfloor+1}^{h-2}\sum_{k=0}^{h-2-i}\binom{h-2-i}{k}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{h+1}ds_J\mathcal{Q}_{c,s}^{-h+i}(T)|T|^{2(h-2-i-k)}T\int_{\partial(G_1\cap\mathbb{C}_J)}$$
$$F_n^R(p,T)dp_Jp^{h+2k+1}=0.$$

Similar arguments applied to the other members of (9.63) lead to

$$\gamma_n^{-1}\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h}ds_JF_n^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h}$$
$$=\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}s^h\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}$$
$$(p^2-2s_0p+|s|^2)^{-1}dp_Jp^h.$$

Since $h=\frac{n-1}{2}$, by formula (9.61) we get

$$\begin{aligned}\frac{(2\pi)^2}{\gamma_n^{-1}}\check{P}^2=&\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}s^h\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p\right.\\ &\left.-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1}p^hdp_J.\end{aligned}$$

Now, we work on the integral on the right hand side. As $\bar{G}_1\subset G_2$, for any $s\in\partial(G_2\cap\mathbb{C}_J)$ the functions

$$p\mapsto p(p^2-2s_0p+|s|^2)^{-1}p^h,$$
$$p\mapsto(p^2-2s_0p+|s|^2)^{-1}p^h$$

are slice hyperholomorphic on $\bar{G}_1$. By Lemma 3.1.19 we have

$$\int_{\partial(G_1\cap\mathbb{C}_J)} p(p^2-2s_0p+|s|^2)^{-1}dp_Jp^h=0,$$

$$\int_{\partial(G_1\cap\mathbb{C}_J)} (p^2-2s_0p+|s|^2)^{-1}p^hdp_J=0.$$

This implies that

$$\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J\int_{\partial(G_1\cap\mathbb{C}_J)} s^hF_n^R(s,T)p(p^2-2s_0p+|s|^2)^{-1}dp_Jp^h=0$$

and

$$\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J\int_{\partial(G_1\cap\mathbb{C}_J)} s^h\bar{s}F_n^R(s,T)(p^2-2s_0p+|s|^2)^{-1}dp_Jp^h=0.$$

Then we have

$$\frac{(2\pi)^2}{\gamma_n^{-1}}\check{P}^2=\int_{\partial(G_2\cap\mathbb{C}_J)} s^hds_J\int_{\partial(G_1\cap\mathbb{C}_J)} [\bar{s}F_n^L(p,T)-F_7^L(p,T)p](p^2-2s_0p+|s|^2)^{-1}dp_Jp^h.$$

From Lemma 9.4.1 with $B=:F_n^L(p,T)$ and $f(s):=s^h$ we get

$$\check{P}^2=\frac{1}{(2\pi)\gamma_n}\int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T)dp_Jp^{2h}=\check{P}.$$

CASE II: The Sce exponent h is even.

We multiply the equation of Theorem 9.5.5 by s^h left and p^h on the right, and since $T_0=0$ we get

$$\begin{aligned}
&s^hF_n^R(s,T)S_L^{-1}(p,T)p^h+s^hS_R^{-1}(s,T)F_n^L(p,T)p^h \qquad (9.64)\\
&+\gamma_n\Big[s^{h+1}\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p^h+s^h\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)p^{h+1}\\
&+s^h\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T^2\mathcal{Q}_{c,p}^{\frac{h+2}{2}}(T)p^h+s^h\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-1}(T)p^h\\
&+s^{h+1}\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T)p^{h+1}+s^{h+1}\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)T\mathcal{Q}_{c,p}^{-i-2}(T)p^h\\
&s^h\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\mathcal{Q}_{c,p}^{-i-2}(T)p^{h+1}+s^h\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)T^2\mathcal{Q}_{c,p}^{-i-2}(T)p^h\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\gamma_n^{-1}\left[s^h\mathcal{A}_1(s,p,T)p^h+s^h\mathcal{B}_1(s,p,T)p^h+s^h\mathcal{C}_1(s,p,T)p^h+s^{2h}F_n^R(s,T)F_n^L(p,T)p^{2h}\right]\\
&=s^h\left\{\left[F_n^R(s,T)-F_n^L(p,T)\right]p-\bar{s}\left[F_n^R(s,T)-F_n^L(p,T)\right]\right\}(p^2-2s_0p+|s|^2)^{-1}p^h.
\end{aligned}
$$

Now, we multiply by ds_J on the left, integrate it over $\partial(G_2\cap\mathbb{C}_J)$ with respect to ds_J and then we multiply it by dp_J on the right and integrate over $\partial(G_1\cap\mathbb{C}_J)$ with respect to dp_J, and we obtain

$$
\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_J)}s^hF_n^R(s,T)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)dp_Jp^h \qquad (9.65)\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_JS_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)p^h+\\
&+\gamma_n\left[\int_{\partial(G_2\cap\mathbb{C}_J)}s^{h+1}ds_J\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)dp_Jp^h\right.\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_J\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)dp_Jp^{h+1}\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_J\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)dp_Jp^h\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_J\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-1}(T)dp_Jp^h\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^{h+1}ds_J\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\int_{\partial(G_2\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h+1}\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^{h+1}\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^h\\
&+\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_J\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h+1}\\
&\left.+\int_{\partial(G_2\cap\mathbb{C}_J)}s^hds_J\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^h\right]\\
&+\gamma_n^{-1}\left[\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}s^hds_J\mathcal{A}_1(s,p,T)dp_Jp^h+s^hds_J\mathcal{B}_1(s,p,T)dp_Jp^h\right.\\
&\left.+\int_{\partial(G_2\cap\mathbb{C}_J)}s^{2h}ds_JF_n^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2h}\right]
\end{aligned}
$$

$$+ s^h ds_J \mathcal{C}_1(s,p,T) dp_J p^h = \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^h \left\{ \left[F_n^R(s,T) - F_n^L(p,T)\right] p \right.$$
$$\left. -\bar{s}\left[F_n^R(s,T) - F_n^L(p,T)\right]\right\} (p^2 - 2s_0 p + |s|^2)^{-1} dp_J p^h.$$

From the definition of $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1$ and recalling that $T_0 = 0$ we have

$$\int_{\partial(G_2\cap\mathbb{C}_J)} \int_{\partial(G_1\cap\mathbb{C}_J)} s^h ds_J \mathcal{A}_1(s,p,T) dp_J p^h + s^h ds_J \mathcal{B}_1(s,p,T) dp_J p^h + s^h ds_J \mathcal{C}_1(s,p,T) dp_J p^h$$
$$= -\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h} F_n^R(s,T) ds_J T \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1}$$
$$-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1} F_n^R(s,T) ds_J T \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h}$$
$$+\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1} ds_J F_n^R(s,T) T^2 \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1}$$
$$+\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h} ds_J F_n^R(s,T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-2k}$$
$$-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h} ds_J F_n^R(s,T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T|T|^{2k} \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) p^{2h-1-2k}$$
$$-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1} ds_J F_n^R(s,T) \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T|T|^{2k} \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-2k}$$
$$+\int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-1} F_n^R(s,T) ds_J \sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} T^2|T|^{2k} \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1-2k}$$
$$+\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k} ds_J F_n^R(s,T)|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-2k}\right)$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k} ds_J F_n^R(s,T)|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{k} T \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1-2k}\right)$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k-1} ds_J F_n^R(s,T) T|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-2k}\right)$$
$$+\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k-1} ds_J F_n^R(s,T) T|T|^{2k}\right)\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2k} T \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1-2k}\right)$$
$$+\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k} ds_J F_n^R(s,T)|T|^{2k}\right) \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h}$$
$$-\left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2\cap\mathbb{C}_J)} s^{2h-2k} ds_J F_n^R(s,T)|T|^{2k} T\right) \int_{\partial(G_1\cap\mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1}$$

$$- \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{2h-1-2k} ds_J F_n^R(s,T) T |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h}$$
$$+ \left(\sum_{k=1}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{2h-1-2k} ds_J F_n^R(s,T) T^2 |T|^{2k} \right) \int_{\partial(G_1 \cap \mathbb{C}_J)} F_n^L(p,T) dp_J p^{2h-1}.$$

Now we observe that since $h \leq 2h-1$ by Lemma 9.2.18 we have

$$\begin{aligned}
&\int_{\partial(G_2 \cap \mathbb{C}_J)} s^h ds_J F_n^R(s,T) \int_{\partial(G_1 \cap \mathbb{C}_J)} S_L^{-1}(p,T) dp_J p^h \\
&= \int_{\partial(G_2 \cap \mathbb{C}_J)} s^h ds_J S_R^{-1}(s,T) \int_{\partial(G_1 \cap \mathbb{C}_J)} F_n^L(p,T) dp_J p^h \\
&= 0.
\end{aligned}$$

Moreover, since $2h-2k \leq 2h-1$ and $2h-1-2k \leq 2h-1$ we get

$$\begin{aligned}
&\int_{\partial(G_2 \cap \mathbb{C}_J)} \int_{\partial(G_1 \cap \mathbb{C}_J)} s^h ds_J \mathcal{A}_1(s,p,T) dp_J p^h + s^h ds_J \mathcal{B}_1(s,p,T) dp_J p^h \\
&+ s^h ds_J \mathcal{C}_1(s,p,T) dp_J p^h = 0.
\end{aligned}$$

Now, we focus on computing the integral

$$\int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1} ds_J \mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T) dp_J p^h.$$

By the binomial formula and recalling that $T_0 = 0$ we get

$$\begin{aligned}
&\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T) = \mathcal{Q}_{c,p}^{-\frac{2-h}{2}}(T) \mathcal{Q}_{c,p}^{-\frac{h-2}{2}}(T) \mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T) = (p^2 + |T|^2)^{\frac{h-2}{2}} \mathcal{Q}_{c,p}^{-h}(T) \\
&= \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \mathcal{Q}_{c,p}^{-h}(T) p^{2k}.
\end{aligned}$$

By the F-resolvent, see (9.18), we deduce that

$$\begin{aligned}
&\int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1} ds_J \mathcal{Q}_{c,s}^{\frac{h+2}{2}}(T) T \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{\frac{h+2}{2}}(T) dp_J p^h \qquad (9.66) \\
&= \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1} ds_J \mathcal{Q}_{c,s}^{\frac{h+2}{2}}(T) T \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \int_{\partial(G_1 \cap \mathbb{C}_J)} \mathcal{Q}_{c,p}^{h}(T) dp_J p^{2k+h} \\
&= \int_{\partial(G_2 \cap \mathbb{C}_J)} s^{h+1} ds_J \mathcal{Q}_{c,s}^{\frac{h+2}{2}}(T) T \sum_{k=0}^{\frac{h-2}{2}} \binom{\frac{h-2}{2}}{k} |T|^{2\left(\frac{h-2}{2}-k\right)} \int_{\partial(G_1 \cap \mathbb{C}_J)} F_n^L(p,T) dp_J p^{2k+h+1}
\end{aligned}$$

$$-\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1}ds_J\mathcal{Q}_{c,s}^{\frac{h+2}{2}}(T)T^2\sum_{k=0}^{\frac{h-2}{2}}\binom{\frac{h-2}{2}}{k}|T|^{2\left(\frac{h-2}{2}-k\right)}\int_{\partial(G_1\cap\mathbb{C}_J)}F_n^L(p,T)dp_Jp^{2k+h}.$$

We observe that

$$h+1+2k\leq h+1+h-2=2h-1,$$

similarly we have $h+2k\leq 2h-1$, so formula (9.66) together with Lemma 9.2.18 imply that

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1}ds_J\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)dp_Jp^h=0.$$

Using similar arguments, we obtain

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h}ds_J\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)dp_Jp^{h+1}=0$$
$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h}ds_J\mathcal{Q}_{c,s}^{-\frac{h+2}{2}}(T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-\frac{h+2}{2}}(T)dp_Jp^{h}=0.$$

By similar computations made when h is odd we get

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h}ds_J\sum_{i=0}^{h-1}\mathcal{Q}_{c,s}^{-h+i}(T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-1}(T)dp_Jp^h=0,$$

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1}ds_J\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\int_{\partial(G_2\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h+1}=0,$$

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h+1}\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h}=0,$$

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h}ds_J\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h+1}=0,$$

$$\int_{\partial(G_2\cap\mathbb{C}_J)} s^{h}ds_J\sum_{i=0,i\neq\frac{h-2}{2}}^{h-2}\mathcal{Q}_{c,s}^{-h+i}(T)T^2\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}^{-i-2}(T)dp_Jp^{h}=0.$$

By formula (9.61) we get

$$\frac{(2\pi)^2}{\gamma_n^{-1}}\check{P}^2 = \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} s^h \left\{ \left[F_n^R(s,T) - F_n^L(p,T)\right] p \right.$$
$$\left. -\bar{s}\left[F_n^R(s,T) - F_n^L(p,T)\right]\right\} (p^2 - 2s_0p + |s|^2)^{-1} p^h dp_J.$$

Finally, by following exactly the same steps done when h is odd we get

$$\check{P}^2 = \check{P}.$$

□

CHAPTER 10

Axially harmonic functions and the harmonic functional calculus on the S-spectrum

10.1 The fine structure of hyperholomorphic spectral theory

The main purpose of this chapter is to show that by using the Fueter mapping theorem and the spectral theory on the S-spectrum we can define a functional calculus for harmonic functions in four variables. This new calculus can be seen as the harmonic version of the Riesz-Dunford functional calculus.

In this chapter we further refine the diagram (9.1), observing that, in the case of the quaternions, the map T_{F2} can be factorized as $T_{F2} = \Delta = \overline{\mathcal{D}}\mathcal{D}$, so there is an intermediate step between slice hyperholomorphic functions and Fueter regular functions, and the intermediate class of functions that appears is the one of axially harmonic functions $\mathcal{AH}(\Omega_D)$, see Definition 10.2.2. Thus the diagram (7.28) can be written as:

$$\mathcal{O}(D) \xrightarrow{T_{F1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D).$$

It is important to define precisely what we mean by intermediate functional calculus between the S-functional calculus and the F-functional calculus, both from the points of view of the function theory and of the operator theory. The notions of *fine structures of the spectral theory on the S-spectrum* arise naturally from the Fueter extension theorem.

Definition 10.1.1 (Fine structure of the spectral theory on the S-spectrum)**.** We will call *fine structure of the spectral theory on the S-spectrum* the set of functions spaces and the associated functional calculi induced by a factorization of the operator T_{F2}, in the Fueter extension theorem.

Remark 10.1.2. In the Clifford algebra setting the map T_{F2} becomes the Fueter-Sce operator given by $T_{FS2} = \Delta_{n+1}^{\frac{n-1}{2}}$ and its splitting is more involved. We are investigating it in general, when n is odd, and in the case $n = 5$ we have a complete description of all the possible fine structures, see Chapter 12.

The fine structure of the quaternionic spectral theory on the S-spectrum is illustrated in the following diagram

$$\begin{array}{ccccc}
\mathcal{SH}(\Omega_D) & & \mathcal{AH}(\Omega_D) & & \mathcal{AM}(\Omega_D) \\
\downarrow & & & & \\
\text{Slice Cauchy Formula} & \xrightarrow{\mathcal{D}} & \mathcal{AH} \text{ in integral form} & \xrightarrow{\overline{\mathcal{D}}} & \text{Fueter theorem in integral form} \\
\downarrow & & \downarrow & & \downarrow \\
S - \text{Functional calculus} & & \text{Harmonic functional calculus} & & F - \text{functional calculus}
\end{array}$$

where the description of the central part of the diagram, i.e., the fine structure, is the main topic of this paper.

Remark 10.1.3. As for the space of axially monogenic functions, the arrow from the space of axially harmonic functions is missing. In fact, like the F-functional calculus, also the harmonic functional calculus is deduced from the slice hyperholomorphic Cauchy formula.

To sum up, the main problems addressed in this chapter are:

Problem 10.1.4. *In the Fueter extension theorem consider the factorization*

$$\mathcal{SH}(U) \xrightarrow{\mathcal{D}} X(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D),$$

and give an integral representation of the functions in the space $X(\Omega_D) := \mathcal{D}(\mathcal{SH}(\Omega_D))$ and, using this integral transform, define its functional calculus.

Problem 10.1.5. *Determine a product rule formula for the F- functional calculus.*

As we will see in the sequel, the above problems are related. In fact, the product rule of the F-functional calculus is based on the functional calculus in Problem 10.1.4.

10.2 Axially harmonic functions

In this chapter we solve the first part of Problem 10.1.4. We begin by rewriting the Fueter mapping theorem (see Theorem 7.2.19) in a more refined way, considering the factorization of the Laplace operator Δ in terms of the Fueter operator $\mathcal{D}$ and its conjugate $\overline{\mathcal{D}}$.

Theorem 10.2.1 (Fueter mapping theorem (refined))**.** *Let $f_0(z) = \alpha(u, v) + i\beta(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D = \{q = q_0 + \underline{q} \mid (q_0, |\underline{q}|) \in D\} \tag{10.1}$$

be the open set induced by D in $\mathbb{H}$. The operator T_{F1} defined by

$$f(q) = T_{F1}(f_0) := \alpha(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}\beta(q_0, |\underline{q}|)$$

maps the set of holomorphic functions in the set of intrinsic slice hyperholomorphic functions. Then the function

$$\tilde{f}(q) := T'_{F2}\left(\alpha(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|}\beta(q_0, |\underline{q}|)\right),$$

where $T'_{F2} := \mathcal{D}$ is the Fueter operator, is in the kernel of the Laplace operator, i.e.,

$$\Delta\tilde{f} = 0 \quad on \quad \Omega_D.$$

Moreover,

$$\breve{f}(q) := T''_{F2}\tilde{f},$$

where $T''_{F2} = \overline{\mathcal{D}}$ and $\overline{\mathcal{D}} = \partial_{q_0} - \sum_{i=1}^{3} e_i\partial_{q_i}$, is the kernel of the Fueter operator, i.e.,

$$\mathcal{D}\breve{f} = 0 \quad on \quad \Omega_D.$$

In Theorem 10.2.1 we have applied to the slice hyperholomorphic function f firstly the Fueter operator and then the operator $\overline{\mathcal{D}}$, while in Theorem

7.2.19 we apply directly the Laplacian. Therefore, there is a class of functions that lies between the class of slice hyperholomorphic functions and the class of axially monogenic functions: it is the so-called class of axially harmonic functions that we introduce below.

Definition 10.2.2 (Axially harmonic function)**.** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set not intersecting the real line, and let

$$\mathcal{U} = \{(u, v) \in \mathbb{R} \times \mathbb{R}^+ \mid u + \mathbb{S}v \in U\}.$$

Let $f : U \to \mathbb{H}$ be a function, of class $\mathcal{C}^3$, of the form

$$f(q) = \alpha(u, v) + J\beta(u, v), \quad q = u + Jv, \quad J \in \mathbb{S},$$

where α and β are $\mathbb{H}$-valued functions. More in general, let f as above and let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let

$$\mathcal{U} = \{(u, v) \in \mathbb{R}^2 \mid u + \mathbb{S}v \in U\},$$

and assume that

$$\alpha(u, v) = \alpha(u, -v), \quad \beta(u, v) = -\beta(u, -v) \text{ for all } (u, v) \in \mathcal{U}. \tag{10.2}$$

Let us set

$$\tilde{f}(q) := \mathcal{D}f(q), \text{ for } q \in U.$$

If

$$\Delta\tilde{f}(q) = 0, \text{ for } q \in U$$

we say that $\tilde{f}$ is axially harmonic on U.

The axially monogenic functions satisfy a system of differential equations called Vekua system, see [60]. In the case of axially harmonic functions, the functions $A(q_0, r)$ and $B(q_0, r)$ satisfy a second order system of differential equations.

Theorem 10.2.3. *Let U be an axially symmetric open set in $\mathbb{H}$, not intersecting the real line, and let $\tilde{f}(q) = A(q_0, r) + \underline{\omega}B(q_0, r)$ be an axially harmonic function on U, $r > 0$ and $\underline{\omega} \in \mathbb{S}$. Then the functions $A(q_0, r)$ and $B(q_0, r)$ satisfy the following system*

$$\begin{cases} \partial_{q_0}^2 A(q_0, r) + \partial_r^2 A(q_0, r) + \dfrac{2}{r}\partial_r A(q_0, r) = 0 \\ \partial_{q_0}^2 B(q_0, r) + \partial_r^2 B(q_0, r) + \dfrac{2r\partial_r B(q_0, r) - 2B(q_0, r)}{r^2} = 0. \end{cases}$$

Proof. An axially harmonic function is written as

$$\tilde{f}(q) = A(q_0, r) + \underline{\omega} B(q_0, r), \quad q = q_0 + r\underline{\omega} \in U$$

and it is in the kernel of the operator $\Delta = \mathcal{D}\overline{\mathcal{D}}$. We denote the Fueter operator as $\mathcal{D} = \partial_{q_0} + \partial_{\underline{q}}$ and $\overline{\mathcal{D}} = \partial_{q_0} - \partial_{\underline{q}}$, where $\partial_{\underline{q}} = e_1\partial_{q_1} + e_2\partial_{q_2} + e_3\partial_{q_3}$. We know that (see [118])

$$\partial_{\underline{q}}(A(q_0, r) + \underline{\omega} B(q_0, r)) = \underline{\omega}\partial_r A(q_0, r) - \partial_r B(q_0, r) - \frac{2}{r} B(q_0, r). \quad (10.3)$$

This implies that

$$\begin{aligned} \overline{\mathcal{D}} f(q) &= (\partial_{q_0} - \partial_{\underline{q}})(A(q_0, r) + \underline{\omega} B(q_0, r)) \\ &= \left(\partial_{q_0} A(q_0, r) + \partial_r B(q_0, r) + \frac{2}{r} B(q_0, r) \right) + \underline{\omega} \left(\partial_{q_0} B(q_0, r) - \partial_r A(q_0, r) \right). \end{aligned}$$

By setting

$$A'(q_0, r) := \partial_{q_0} A(q_0, r) + \partial_r B(q_0, r) + \frac{2}{r} B(q_0, r)$$

and

$$B'(q_0, r) := \partial_{q_0} B(q_0, r) - \partial_r A(q_0, r),$$

we get

$$\overline{\mathcal{D}} f(q_0, r) = A'(q_0, r) + \underline{\omega} B'(q_0, r).$$

Now, by applying another time formula (10.3) we obtain

$$\partial_{\underline{q}} \overline{D} f(q) = \underline{\omega}\partial_r A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r} B'(q_0, r).$$

Therefore we have

$$\begin{aligned} \Delta f(q) &= \mathcal{D}\overline{\mathcal{D}} f(q) \\ &= (\partial_{q_0} + \partial_{\underline{q}}) \overline{D} f(q_0, r) \\ &= \left(\partial_{q_0} A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r} B'(q_0, r) \right) \underline{\omega} \left(\partial_{q_0} B'(q_0, r) + \partial_r A'(q_0, r) \right). \end{aligned}$$

Since the function f is axially harmonic we have $\Delta f(q) = 0$, thus we get

$$\begin{cases} \partial_{q_0} A'(q_0, r) - \partial_r B'(q_0, r) - \frac{2}{r} B'(q_0, r) = 0 \\ \partial_{q_0} B'(q_0, r) + \partial_r A'(q_0, r) = 0. \end{cases} \quad (10.4)$$

Now, we write the system (10.4) in terms of A and B by substituting A' and B'

$$\begin{aligned}\partial_{q_0}A'(q_0,r)-\partial_r B'(q_0,r)-\frac{2}{r}B'(q_0,r) &= \partial^2_{q_0}A(q_0,r)+\partial_{q_0}\partial_r B(q_0,r)+\frac{2}{r}\partial_{q_0}B(q_0,r)\\ &\quad -\partial_r\partial_{q_0}B(q_0,r)+\partial^2_r A(q_0,r)-\frac{2}{r}\partial_{q_0}B(q_0,r)\\ &\quad +\frac{2}{r}\partial_r A(q_0,r) \qquad (10.5)\\ &= \partial^2_{q_0}A(q_0,r)+\partial^2_r A(q_0,r)+\frac{2}{r}\partial_r A(q_0,r).\end{aligned}$$

$$\begin{aligned}\partial_{q_0}B'(q_0,r)+\partial_r A'(q_0,r) &= \partial^2_{q_0}B(q_0,r)-\partial_{q_0}\partial_r A(q_0,r)+\partial_r\partial_{q_0}A(q_0,r)\\ &\quad +\partial^2_r B(q_0,r)+\frac{2}{r}\partial_r B(q_0,r)-\frac{2}{r^2}B(q_0,r) \qquad (10.6)\\ &= \partial^2_{q_0}B(q_0,r)+\partial^2_r B(q_0,r)+\frac{2r\partial_r B(q_0,r)-2B(q_0,r)}{r^2}.\end{aligned}$$

By putting (10.5) and (10.6) in (10.4) we get the statement. □

Remark 10.2.4. If we suppose that a function f is harmonic over the ball $B_r(p)$ of radius r and center p, and continuous in the closure of the ball we can write

$$f(q)=\frac{r^2-|q-p|^2}{|\partial B_r(p)|r}\int_{\partial B_r(p)}\frac{f(y)}{|y-q|^4}dy,$$

where $|\partial B_r(p)|$ is the measure of the sphere and dy is the surface element.

10.3 Integral representation of axially harmonic functions

In this section we show how to write an axially harmonic function in integral form. The main advantage of this approach is that it is enough to compute an integral of slice hyperholomorphic functions in order to get an axially harmonic function. The crucial point to get the integral representation is to apply the Fueter operator $\mathcal{D}$ to the slice hyperholomorphic Cauchy kernels written in second form, see Definition 3.1.16.

Theorem 10.3.1. *Let $s,\ q\in\mathbb{H}$ be such that $q\notin[s]$ then*

$$\mathcal{D}S^{-1}_L(s,q)=-2\mathcal{Q}_{c,s}(q)^{-1}$$

and

$$S^{-1}_R(s,q)\mathcal{D}=-2\mathcal{Q}_{c,s}(q)^{-1}.$$

Proof. We prove only the first equality since the second one follows with similar computations. First we apply ∂_{q_0} and ∂_{q_i} for $i = 1, 2, 3$ to the left slice hyperholomorphic Cauchy kernel

$$S_L^{-1}(s, q) = (s - \bar{q})\mathcal{Q}_{c,s}(q)^{-1}.$$

Thus, we have

$$\begin{aligned}
\partial_{q_0} S_L^{-1}(s, q) &= -\mathcal{Q}_{c,s}(q)^{-1} - (s - \overline{q})\mathcal{Q}_{c,s}(q)^{-2}(-2s + 2q_0) \\
&= -\mathcal{Q}_{c,s}(q)^{-1} - 2q_0(s - \overline{q})\mathcal{Q}_{c,s}(q)^{-2} + 2(s - \overline{q})\mathcal{Q}_{c,s}(q)^{-2}s \\
&= -\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_0}{2}F_L(s, q) - \frac{1}{2}F_L(s, q)s.
\end{aligned}$$

Then for $i = 1, 2, 3$ we get

$$\begin{aligned}
\partial_{q_i} S_L^{-1}(s, q) &= e_i\mathcal{Q}_{c,s}(q)^{-1} - 2q_i(s - \overline{q})\mathcal{Q}_{c,s}(q)^{-2} \\
&= e_i\mathcal{Q}_{c,s}(q)^{-1} + \frac{1}{2}q_iF_L(s, q).
\end{aligned}$$

Thus, by Theorem 9.3.1 with $n = 3$, we obtain

$$\begin{aligned}
\mathcal{D}S_L^{-1}(s, q) &= \partial_{q_0} S_L^{-1}(s, q) + \sum_{i=1}^{3} e_i\partial_{q_i} S_L^{-1}(s, q) \\
&= -\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_0}{2}F_L(s, q) - \frac{1}{2}F_L(s, q)s - 3\mathcal{Q}_{c,s}(q)^{-1} + \frac{q}{2}F_L(s, q) \\
&= -4\mathcal{Q}_{c,s}(q)^{-1} - \frac{1}{2}\left(F_L(s, q)s - qF_L(s, q)\right) \\
&= -2\mathcal{Q}_{c,s}(q)^{-1}.
\end{aligned}$$

□

Remark 10.3.2. Although the slice hyperholomorphic Cauchy kernel written in form I is more suitable in various cases, like for the definition of S-functional calculus, it does not allow easy computations of $\mathcal{D}S_L^{-1}(s, q)$.

We observe that when we apply the Laplace operator to a monomial q^n we get a polynomial in terms of q and $\bar{q}$, see [85, page 316 formula (12)], [77, Thm. 3.2]. The same feature happens when the Fueter operator is applied to the monomial q^n, see [23, Lemma 1].

Lemma 10.3.3. *For all $n \geq 1$ we have*

$$\mathcal{D}q^n = q^n\mathcal{D} = -2\sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1}.$$

Remark 10.3.4. Since

$$\overline{\sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1}} = \sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1}$$

we deduce that $\mathcal{D}q^n$ is real.

Definition 10.3.5. Let $s, q \in \mathbb{H}$, we define the commutative Q-series as

$$-2\sum_{m=1}^{+\infty}\sum_{k=1}^{m} q^{m-k}\overline{q}^{k-1}s^{-1-m} \qquad \text{and} \qquad -2\sum_{m=1}^{+\infty}\sum_{k=1}^{m} s^{-1-m}q^{m-k}\overline{q}^{k-1}.$$

Remark 10.3.6. The two series in Definition 10.3.5 coincide, where they converge, since $\sum_{k=1}^{m} q^{m-k}\overline{q}^{k-1}$ is real.

Proposition 10.3.7. *For $s,\ q \in \mathbb{H}$ with $|q| < |s|$, the commutative Q-series converges.*

Proof. To prove the convergence, it is sufficient to prove the convergence of the modulus of the series, i.e., we consider

$$\sum_{m=1}^{+\infty} 2m|q|^{m-1}|s|^{-1-m}.$$

The last series converges by the ratio test. Indeed, since $|q| < |s|$, we have

$$\lim_{m\to\infty} \frac{(m+1)|q|^m|s|^{-2-m}}{m|q|^{m-1}|s|^{-1-m}} = \lim_{m\to\infty} \frac{m+1}{m}|q||s|^{-1} < 1.$$

□

Lemma 10.3.8. *For $q,\ s \in \mathbb{H}$ such that $|q| < |s|$, we have*

$$\mathcal{Q}_{c,s}(q)^{-1} = \sum_{m=1}^{+\infty}\sum_{k=1}^{m} q^{m-k}\overline{q}^{k-1}s^{-1-m} = \sum_{m=1}^{+\infty}\sum_{k=1}^{m} s^{-1-m}q^{m-k}\overline{q}^{k-1}.$$

Proof. We prove the first equality since the second one can be proved in a similar way. By Theorem 3.1.14, we can expand the left Cauchy kernel as

$$S_L^{-1}(s,q) = \sum_{m=0}^{\infty} q^m s^{-1-m}.$$

By Theorem 10.3.1 and Proposition 10.3.7, which allows to exchange the series with the Fueter operator, we have

$$-2\mathcal{Q}_{c,s}(q)^{-1} = \mathcal{D}S_L^{-1}(s,q) = \sum_{m=0}^{\infty} (\mathcal{D}q^m)\, s^{-1-m}.$$

We get the statement by applying Lemma 10.3.3. □

Remark 10.3.9. Using the well-known equality

$$(a^n - b^n) = (a-b)\sum_{k=1}^{n} a^{n-k} b^{k-1}$$

for $a = q$ and $b = \overline{q}$, and by Lemma 10.3.3 we have

$$\mathcal{D}q^n = \begin{cases} -2nq^{n-1} & \text{if } \mathrm{Im}(q) = 0, \\ -(\underline{q})^{-1}(q^n - \overline{q}^n) & \text{if } \mathrm{Im}(q) \neq 0. \end{cases}$$

With this result, we can prove Theorem 10.3.1 by using the series expansion of the kernel in the following way: if $|q| < |s|$ and $\underline{q} \neq 0$ then

$$\begin{aligned} \mathcal{D}S_L^{-1}(s,q) &= \sum_{m=0}^{\infty} (\mathcal{D}q^m)\, s^{-1-m} \\ &= -(\underline{q})^{-1} \left(\sum_{m=1}^{\infty} q^m s^{-1-m} - \sum_{m=1}^{\infty} \overline{q}^m s^{-1-m} \right) \\ &= -(\underline{q})^{-1}(S_L^{-1}(s,q) - S_L^{-1}(s,\overline{q})) \\ &= -(\underline{q})^{-1}(2\underline{q}\mathcal{Q}_{c,s}(q)^{-1}) \\ &= -2\mathcal{Q}_{c,s}(q)^{-1}, \end{aligned}$$

if $|q| < |s|$ and $\underline{q} = 0$, we have

$$\begin{aligned}
\mathcal{Q}_{c,s}(q)\mathcal{D}S_L^{-1}(s,q) &= (s^2 - 2qs + q^2)\left(-2\sum_{m=1}^{\infty} mq^{m-1}s^{-1-m}\right) \\
&= -2\sum_{m=1}^{\infty} mq^{m-1}s^{1-m} + 4\sum_{m=1}^{\infty} mq^m s^{-m} - 2\sum_{m=1}^{\infty} mq^{m+1}s^{-m-1} \\
&= -2\sum_{m=0}^{\infty} q^m s^{-m} + 2\sum_{m=1}^{\infty} mq^m s^{-m} - 2\sum_{m=2}^{\infty} mq^m s^{-m} \\
&\quad + 2\sum_{m=2}^{\infty} q^m s^{-m} \\
&= -2.
\end{aligned}$$

Now, we study the regularity of the function $\mathcal{D}S_L^{-1}(s,q)$ in both variables.

Proposition 10.3.10. *Let s, $q \in \mathbb{H}$ be such that $q \notin [s]$. The function $\mathcal{D}S_L^{-1}(s,q)$ is an intrinsic slice hyperholomorphic function in s.*

Proof. This follows by Theorem 10.3.1 and the shape of the commutative pseudo Cauchy kernel. □

Remark 10.3.11. The function $\mathcal{D}(S_L^{-1}(s,q))$ is not left slice hyperholomorphic in the variable q. Indeed, let $q = u + Jv$ for an arbitrary $J \in \mathbb{S}$ then $\mathcal{Q}_{c,s}(q)^{-1} = (s^2 - 2us + u^2 + v^2)^{-1}$ and we have the following two relations

$$\frac{\partial}{\partial u}\mathcal{Q}_{c,s}(u+Jv)^{-1} = -(-2s+2u)\mathcal{Q}_{c,s}(u+Jv)^{-2}$$

and

$$\frac{\partial}{\partial v}\mathcal{Q}_{c,s}(u+Jv)^{-1} = -2v\mathcal{Q}_{c,s}(u+Jv)^{-2},$$

which yield

$$\begin{aligned}
\left(\frac{\partial}{\partial u} + J\frac{\partial}{\partial v}\right)\mathcal{Q}_{c,s}(u+Jv)^{-1} &= -(-2s+2u+2Jv)\mathcal{Q}_{c,s}(u+Jv)^{-2} \\
&= 2(s-q)\mathcal{Q}_{c,s}(u+Jv)^{-2} = -\frac{1}{2}F_L(s,\overline{q}).
\end{aligned}$$

The function $\mathcal{D}S_L^{-1}(s,q)$ turns out to be harmonic in q, as proved in the following result.

Proposition 10.3.12. *Let s, $q \in \mathbb{H}$ be such that $q \notin [s]$. Then the function $\mathcal{D}S_L^{-1}(s,q)$ is harmonic in the real components of q.*

Proof. Since the left slice hyperholomorphic Cauchy kernel is a C^∞ function for any $q \notin [s]$, we can apply to it a differential operator of any order. The result follows by the facts that the Laplacian is a real operator, thus it commutes with $\mathcal{D}$, and by Proposition 7.4.4 with $n = 3$. Indeed

$$\Delta \mathcal{D} S_L^{-1}(s,q) = \mathcal{D} \Delta S_L^{-1}(s,q) = \mathcal{D} F_L(s,q) = 0.$$

□

Finally as a consequence of the definition of the F-kernel we have this result.

Lemma 10.3.13. *Let s, $q \in \mathbb{H}$ be such that $q \notin [s]$, then*

$$\mathcal{D}^2 S_L^{-1}(s,q) = F_L(s, \bar{q}).$$

Proof. By Theorem 10.3.1 we have

$$\mathcal{D}^2 S_L^{-1}(s,q) = -2\mathcal{D}\mathcal{Q}_{c,s}(q)^{-1}. \tag{10.7}$$

Firstly, we apply the derivatives with respect to q_0 and q_i, with $i = 1, 2, 3$ to the commutative pseudo Cauchy kernel

$$\frac{\partial}{\partial q_0} \mathcal{Q}_{c,s}(q)^{-1} = -2(-s + q_0)(s^2 - 2q_0 s + |q|^2)^{-2},$$

and for $i = 1, 2, 3$ we get

$$\frac{\partial}{\partial q_i} \mathcal{Q}_{c,s}(q)^{-1} = -2q_i(s^2 - 2q_0 s + |q|^2)^{-2}.$$

Thus we obtain

$$\begin{aligned} \mathcal{D}\mathcal{Q}_{c,s}(q)^{-1} &= \frac{\partial}{\partial q_0} \mathcal{Q}_{c,s}(q)^{-1} + \sum_{i=1}^{3} e_i \frac{\partial}{\partial q_i} \mathcal{Q}_{c,s}(q)^{-1} \\ &= -2(-s + q_0 + \underline{q})(s^2 - 2q_0 s + |q|^2)^{-2} \\ &= 2(s - q)(s^2 - 2q_0 s + |q|^2)^{-2}. \end{aligned}$$

Therefore by (10.7) we get

$$\mathcal{D}^2 S_L^{-1}(s,q) = -4(s - q)(s^2 - 2q_0 s + |q|^2)^{-2} = F_L(s, \bar{q}).$$

□

Remark 10.3.14. By Proposition 7.4.4 it is clear that the function $F_L(s, \bar{q})$ is axially anti-monogenic. This observation together with Lemma 6.35 implies the construction of the following diagram

$$\mathcal{O}(D) \xrightarrow{T_{F1}} SH(\Omega_D) \xrightarrow{\mathcal{D}} AH(\Omega_D) \xrightarrow{\mathcal{D}} \overline{AM(\Omega_D)}, \tag{10.8}$$

where Ω_D is defined as in (10.1) and $\overline{AM(\Omega_D)}$ is the set of axially anti-monogenic functions. In order to avoid this set of functions in the constructions like the one in (10.8) we impose that the composition of the operators appearing in (10.8) must be the Fueter map (in the case of this chapter Δ). This is very important when we increase the dimension of the algebra, see Chapter 12.

We observe that if we set $q = u + Jv$ and we apply the 2-dimensional Laplacian

$$\Delta_2 := \overline{\partial_J}\partial_J = \left(\frac{\partial}{\partial u} - J\frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + J\frac{\partial}{\partial v}\right),$$

to the commutative pseudo Cauchy kernel we get its square.

Lemma 10.3.15. *Let s, $q = u + Jv \in \mathbb{H}$ be such that $q \notin [s]$, then*

$$\Delta_2 \mathcal{Q}_{c,s}(q)^{-1} = 4\mathcal{Q}_{c,s}(q)^{-2}.$$

Proof. We set $q = u + Jv$, $I \in \mathbb{S}$. By Remark 10.3.11 we know that

$$\partial_J \mathcal{Q}_{c,s}(u + Jv)^{-1} = 2(s - u - Jv)\mathcal{Q}_{c,s}(q)^{-2}.$$

Now, we have

$$\frac{\partial}{\partial u}\partial_J \mathcal{Q}_{c,s}(u+Jv)^{-1} = -2\mathcal{Q}_{c,s}(u+Jv)^{-2} - 4(s-u-Jv)\mathcal{Q}_{c,s}(u+Jv)^{-3}(-2s+2u),$$

and

$$\frac{\partial}{\partial v}\partial_J \mathcal{Q}_{c,s}(u+Jv)^{-1} = -2I\mathcal{Q}_{c,s}(u+Jv)^{-2} - 8(s-u-Jv)\mathcal{Q}_{c,s}(u+Jv)^{-3}v.$$

By definition of the 2-dimensional Laplacian and since the variable s commute with $\mathcal{Q}_{c,s}(u+Jv)$, we get

$$\begin{aligned}
\Delta_2 \mathcal{Q}_{c,s}(q)^{-1} &= \left(\frac{\partial}{\partial u} - J\frac{\partial}{\partial v}\right)\partial_I \mathcal{Q}_{c,s}(u+Jv)^{-1} \\
&= -4(s-u-Jv)\mathcal{Q}_{c,s}(u+Jv)^{-3}(-2s+2u) \\
&\quad +8J(s-u-Jv)\mathcal{Q}_{c,s}(u+Jv)^{-3}v - 4\mathcal{Q}_{c,s}(u+Jv)^{-2} \\
&= 8(s-u-Jv)(s-u)\mathcal{Q}_{c,s}(u+Jv)^{-3} \\
&\quad +8J(s-u-Jv)v\mathcal{Q}_{c,s}(u+Jv)^{-2} - 4\mathcal{Q}_{c,s}(u+Jv)^{-2} \\
&= 8(s^2-su-us+u^2-Jsv+Juv+Jsv-Juv+v^2)\mathcal{Q}_{c,s}(u+Jv)^{-3} \\
&\quad -4\mathcal{Q}_{c,s}(u+Jv)^{-2} \\
&= 8\mathcal{Q}_{c,s}(u+Jv)\mathcal{Q}_{c,s}(u+Jv)^{-3} - 4\mathcal{Q}_{c,s}(u+Jv)^{-2} \\
&= 8\mathcal{Q}_{c,s}(u+Jv)^{-2} - 4\mathcal{Q}_{c,s}(u+Jv)^{-2} = 4\mathcal{Q}_{c,s}(u+Jv)^{-2}.
\end{aligned}$$

□

We conclude this section with an integral representation of axially harmonic functions that will allow us to define the harmonic functional calculus based on the S-spectrum.

Theorem 10.3.16 (Integral representation of axially harmonic functions)**.** *Let $W \subset \mathbb{H}$ be an open set. Let U be a slice Cauchy domain such that $\overline{U} \subset W$. Then for $J \in \mathbb{S}$ and $ds_J = ds(-J)$ we have:*

1) If $f \in \mathcal{SH}_L(W)$, then the function $\tilde{f}(q) = \mathcal{D}f(q)$ is harmonic and it admits the following integral representation

$$\tilde{f}(q) = -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(q)^{-1} ds_J f(s), \quad q \in U. \tag{10.9}$$

2) If $f \in \mathcal{SH}_R(W)$, then the function $\tilde{f}(q) = f(q)\mathcal{D}$ is harmonic and it admits the following integral representation

$$\tilde{f}(q) = -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} f(s) ds_J \mathcal{Q}_{c,s}(q)^{-1}, \quad q \in U. \tag{10.10}$$

The integrals depend neither on U nor on the imaginary unit $J \in \mathbb{S}$.

Proof. We prove only the first statement because the other proof is similar. We can write the function f by using the Cauchy formula for slice hyperholomorphic functions, see Theorem 3.1.18. Now, by applying the left Fueter operator to $f(q)$ and by Theorem 10.3.1 we get

$$\begin{aligned}\tilde{f}(q) &= \mathcal{D}f(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{D}S_L^{-1}(s,q)ds_J f(s)\\ &= -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(q)^{-1}ds_J f(s).\end{aligned}$$

Since $\tilde{f}(q) = \mathcal{D}f(q)$ and by Proposition 10.3.12, it is immediately verified that $\tilde{f}(q)$ is a harmonic function. The independence of integral in (10.9) from the set U and the imaginary unit $J \in \mathbb{S}$ follows by the Cauchy formula. □

In this section we have described the central part of the following diagram

$$\mathcal{O}(D) \xrightarrow{T_{F1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D). \quad (10.11)$$

Remark 10.3.17. In the quaternionic setting it is possible to obtain another digram besides in (10.11). This comes from the factorization $\Delta = \mathcal{D}\overline{\mathcal{D}}$ and is called second fine structure in the quaternionic setting, see Definition 10.1.1. The set of functions that lies between the set of slice hyperholomorphic functions and the set of axially monogenic functions is the set of axially polyanalytic functions of order 2, for more details see Chapter 11 .

10.4 The harmonic functional calculus on he S spectrum

In this section we introduce the harmonic functional calculus on the S-spectrum, which is based on the integral representation of axially harmonic functions. Recall that X denotes a two-sided quaternionic Banach space.

We give meaning to the substitution of the variable q with the operator T in the power series introduced in Definition 10.3.5.

Definition 10.4.1. Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$, we formally define the commutative pseudo S-resolvent series as

$$-2\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m} \quad \text{and} \quad -2\sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}T^{m-k}\overline{T}^{k-1}.$$

Remark 10.4.2. The two series in Definition 10.4.1 coincide, where they converge.

Proposition 10.4.3. *Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$ and $\|T\| < |s|$, the series in the Definition 10.4.1 converges. Moreover, we have*

$$\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m} = \sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}T^{m-k}\overline{T}^{k-1} = \mathcal{Q}_{c,s}(T)^{-1}. \tag{10.12}$$

Proof. For the convergence of the series it is sufficient to prove the convergence of the series of the operator norm:

$$\sum_{m=1}^{\infty} m\|T\|^{m-1}|s|^{-1-m}. \tag{10.13}$$

Since

$$\lim_{m\to\infty} \frac{(m+1)\|T\|^m|s|^{-2-m}}{m\|T\|^{m-1}|s|^{-1-m}} = \lim_{m\to\infty}\frac{m+1}{m}\|T\||s|^{-1} < 1,$$

by the ratio test the series (10.13) is convergent. To prove the equality (10.12), we show that

$$\begin{aligned}\mathcal{Q}_{c,s}(T)\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right) &= \left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T)\\ &= \mathcal{I}.\end{aligned} \tag{10.14}$$

The first equality in (10.14) is a consequence of the following facts: for any positive integer m the operator $\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}$ does not contain any imaginary units, so it is real and then it commutes with any power of s. Secondly, the components of T are commuting among them and the operator $\mathcal{Q}_{c,s}(T)$, see formula (3.14), can be written as: $s^2\mathcal{I} - 2sT_0 + \sum_{i=0}^{3} T_i^2$.

Now we prove the second equality in (10.14). First we observe that

$$\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T) = \left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\cdot$$
$$(s^2 - s(T+\overline{T}) + T\overline{T})$$
$$\begin{aligned}&= \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{1-m} - \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m+1-k}\overline{T}^{k-1}s^{-m} - \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k}s^{-m}\\ &\quad + \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k+1}\overline{T}^{k}s^{-1-m}.\end{aligned}$$

Making the change of index $m' = 1 + m$ in the second and fourth series, we have

$$\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T) =$$

$$= \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{1-m} - \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k-1}s^{1-m'} - \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k}s^{-m}$$

$$+ \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k}s^{-m'}$$

$$= \mathcal{I} + \sum_{m=2}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{1-m} - \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k-1}s^{1-m'} +$$

$$- \overline{T}s^{-1} - \sum_{m=2}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k}s^{-m} + \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k}s^{-m'}.$$

Simplifying the opposite terms in the first and second series and in the third and fourth series, we finally get

$$\begin{aligned}\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T) &= \mathcal{I} + \sum_{m=2}^{\infty}\overline{T}^{m-1}s^{1-m} - \sum_{m=2}^{\infty}\overline{T}^{m-1}s^{1-m} \\ &= \mathcal{I}.\end{aligned}$$

□

Lemma 10.4.4. *Let $T \in \mathcal{BC}(X)$. The commutative pseudo S-resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$ is a $\mathcal{B}(X)$-valued right and left slice hyperholomorphic function of the variable s in $\rho_S(T)$.*

Proof. It follows by Proposition 10.3.10. □

Remark 10.4.5. We point out an important difference between the commutative and the noncommutative pseudo S-resolvent operator. For $T \in \mathcal{B}(X)$ with noncommuting components the operator $\mathcal{Q}_{c,s}(T)$ is not well defined because $T\overline{T} \neq \overline{T}T$. But in the case $T \in \mathcal{BC}(X)$ then it turns out to be well defined and the inverse is $\mathcal{B}(X)$-valued slice hyperholomorphic function for $s \in \rho_S(T)$.

The noncommutative pseudo S-resolvent operator $\mathcal{Q}_s(T)^{-1}$ turns out to be well defined for operators $T \in \mathcal{B}(X)$ with noncommuting components, but it is not a $\mathcal{B}(X)$-valued slice hyperholomorphic function.

Remark 10.4.6. The functional calculus based on axially harmonic functions in integral form will be called *harmonic functional calculus (on the S-spectrum)* or, since it is based on the commutative pseudo S-resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$, Q-functional calculus.

Definition 10.4.7 (Harmonic functional calculus on the S-spectrum)**.** Let $T \in \mathcal{BC}(X)$ and set $ds_J = ds(-J)$ for $J \in \mathbb{S}$. For every function $\tilde{f} = \mathcal{D}f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J f(s), \tag{10.15}$$

where U is an arbitrary bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset dom(f)$ and $J \in \mathbb{S}$ is an arbitrary imaginary unit.
For every function $\tilde{f} = f\mathcal{D}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\tilde{f}(T) := -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J \mathcal{Q}_{c,s}(T)^{-1}, \tag{10.16}$$

where U and J are as above.

We note the the presence of $-\frac{1}{\pi}$ in front of (10.15) and (10.16) is justified by Theorem 10.3.16.

Theorem 10.4.8. *The harmonic functional calculus on the S-spectrum is well-defined, i.e., the integrals in* (10.15) *and* (10.16) *depend neither on the imaginary unit $J \in \mathbb{S}$ nor on the slice Cauchy domain U.*

Proof. Here we show only the case $\tilde{f} = \mathcal{D}f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, since the other one follows by analogous arguments.
Since $\mathcal{Q}_{c,s}(T)^{-1}$ is a right slice hyperholomorphic function in s and f is left slice hyperholomorphic, the independence from the set U follows by the Cauchy integral formula, see Theorem 3.1.18 and Theorem 3.1.19.
Now, we want to show the independence from the imaginary unit. Let us consider two imaginary units $J, I \in \mathbb{S}$ with $J \neq I$ and two bounded slice Cauchy domains U_q, U_s with $\sigma_s(T) \subset U_q$, $\overline{U}_q \subset U_s$ and $\overline{U}_s \subset dom(f)$. Then every $s \in \partial(U_s \cap \mathbb{C}_J)$ belongs to the unbounded slice Cauchy domain $\mathbb{H} \setminus U_q$. Recall that $\mathcal{Q}_{c,q}(T)^{-1}$ is right slice hyperholomorphic on $\rho_S(T)$, also at infinity, since $\lim_{q \to +\infty} \mathcal{Q}_{c,q}(T)^{-1} = 0$. Thus the Cauchy formula implies

$$\begin{aligned} \mathcal{Q}_{c,s}(T)^{-1} &= \frac{1}{2\pi} \int_{\partial((\mathbb{H} \setminus U_q) \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I S_R^{-1}(q,s) \\ &= \frac{1}{2\pi} \int_{\partial(U_q \cap \mathbb{C}_I)} \mathcal{Q}_{c,q}(T)^{-1} dq_I S_L^{-1}(s,q). \end{aligned} \tag{10.17}$$

The last equality is due to the fact that $\partial\left((\mathbb{H}\setminus U_q)\cap\mathbb{C}_I\right)=-\partial(U_q\cap\mathbb{C}_I)$ and $S_R^{-1}(q,s)=-S_L^{-1}(s,q)$. Combining (10.15) and (10.17) we get

$$\begin{aligned}\tilde{f}(T) &= -\frac{1}{\pi}\int_{\partial(U_s\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_J f(s)\\ &= -\frac{1}{\pi}\int_{\partial(U_s\cap\mathbb{C}_J)}\left(\frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_I)}\mathcal{Q}_{c,q}(T)^{-1}dq_I S_L^{-1}(s,q)\right)ds_J f(s).\end{aligned}$$

Due to Fubini's theorem we can exchange the order of integration and by the Cauchy formula we obtain

$$\begin{aligned}\tilde{f}(T) &= -\frac{1}{\pi}\int_{\partial(U_q\cap\mathbb{C}_I)}\mathcal{Q}_{c.q}(T)^{-1}dq_I\left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)}S_L^{-1}(s,q)ds_J f(s)\right)\\ &= -\frac{1}{\pi}\int_{\partial(U_q\cap\mathbb{C}_I)}\mathcal{Q}_{c,q}(T)^{-1}dq_I f(q).\end{aligned}$$

This proves the statement. □

Problem 10.4.9. *Let Ω be a slice Cauchy domain. It might happen that $f,g\in\mathcal{SH}_L(U)$ (resp. $f,g\in\mathcal{SH}_R(U)$) and $\mathcal{D}f=\mathcal{D}g$ (resp. $f\mathcal{D}=g\mathcal{D}$). Is it possible to show that for any $T\in\mathcal{BC}(X)$, with $\sigma_S(T)\subset U$, we have $\tilde{f}(T)=\tilde{g}(T)$?*

We start to address this problem by observing that $\mathcal{D}(f-g)=0$ (resp. $(f-g)\mathcal{D}=0$). Therefore it is necessary to study the sets

$$(\ker\mathcal{D})_{\mathcal{SH}_L(U)}:=\{f\in\mathcal{SH}_L(U):\mathcal{D}(f)=0\}$$

and

$$(\ker\mathcal{D})_{\mathcal{SH}_R(U)}:=\{f\in\mathcal{SH}_R(U):(f)\mathcal{D}=0\}.$$

Theorem 10.4.10. *Let U be a connected slice Cauchy domain of $\mathbb{H}$, then*

$$\begin{aligned}(\ker\mathcal{D})_{\mathcal{SH}_L(U)} &= \{f\in\mathcal{SH}_L(U):f\equiv\alpha\quad\textit{for some }\alpha\in\mathbb{H}\}\\ &= \{f\in\mathcal{SH}_R(U):f\equiv\alpha\quad\textit{for some }\alpha\in\mathbb{H}\}=(\ker D)_{\mathcal{SH}_R(U)}.\end{aligned}$$

Proof. We prove the result in the case $f\in\mathcal{SH}_L(U)$ since the case $f\in\mathcal{SH}_R(U)$ follows with similar arguments. We proceed by double inclusion. The fact that

$$(\ker D)_{\mathcal{SH}_L(U)}\supseteq\{f\in\mathcal{SH}_L(U):f\equiv\alpha\quad\text{for some }\alpha\in\mathbb{H}\}$$

is obvious. The other inclusion can be proved observing that if $f \in (\ker \mathcal{D})_{\mathcal{SH}_L(U)}$, after a change of variable if needed, there exists $r > 0$ such that the function f can be expanded in a convergent series at the origin

$$f(q) = \sum_{k=0}^{\infty} q^k \alpha_k \quad \text{for } \{\alpha_k\}_{k \in \mathbb{N}_0} \subset \mathbb{H} \text{ and for any } q \in B_r(0)$$

where $B_r(0)$ is the ball centered at 0 and of radius r. By Lemma 10.3.3, we have

$$0 = \mathcal{D}f(q) \equiv \sum_{k=1}^{\infty} \mathcal{D}(q^k)\alpha_k = -2 \sum_{k=1}^{\infty} \sum_{s=1}^{k} q^{k-s}\overline{q}^{s-1}\alpha_k, \quad \forall q \in B_r(0).$$

If we restrict the previous series in a neighbourhood Ω of 0 of the real line we get

$$\sum_{k=1}^{\infty} q_0^{k-1}\alpha_k, \qquad \forall q_0 \in \Omega.$$

and this implies that

$$\alpha_k = 0, \quad \forall k \geq 1,$$

which yields $f(q) \equiv \alpha_0$ in Ω and since U is connected $f(q) \equiv \alpha_0$ for any $q \in U$. □

We solve the problem 10.4.9 in the case U connected.

Proposition 10.4.11. *Let $T \in \mathcal{BC}(X)$ and let U be a connected slice Cauchy domain with $\sigma_S(T) \subset U$. If $f, g \in \mathcal{SH}_L(U)$ (resp. $f, g \in \mathcal{SH}_R(U)$) satisfy the property $\mathcal{D}f = \mathcal{D}g$ (resp. $f\mathcal{D} = g\mathcal{D}$) then $\tilde{f}(T) = \tilde{g}(T)$.*

Proof. We prove the theorem in the case $f, g \in \mathcal{SH}_L(U)$ since the case $f, g \in \mathcal{SH}_R(U)$ follows by similar arguments. By definition of the harmonic functional calculus on the S-spectrum, see Definition 10.4.7, we have

$$\tilde{f}(T) - \tilde{g}(T) = -\frac{1}{\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J (f(s) - g(s)).$$

Since $\mathcal{Q}_{c,s}(T)^{-1}$ is slice hyperholomorphic in the variable s by Theorem 3.1.18, we can change the domain of integration to $B_r(0) \cap \mathbb{C}_J$ for some

$r > 0$ with $\|T\| < r$. Moreover, by hypothesis we have that $f(s) - g(s) \in (\ker D)_{\mathcal{SH}_L(U)}$, thus by Theorem 10.4.10 and Proposition 10.4.3 we get

$$\begin{aligned}\tilde{f}(T) - \tilde{g}(T) &= -\frac{1}{\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J(f(s) - g(s)) \\ &= \int_{\partial(B_r(0)\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J\alpha \\ &= \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}\int_{\partial(B_r(0)\cap\mathbb{C}_J)} s^{-1-m} ds_J\alpha = 0.\end{aligned}$$

□

In order to solve Problem 10.4.9, in the case U not connected, we need the following lemma, which is based on the monogenic functional calculus developed by McIntosh and collaborators, see [99, 101, 108, 112].

Lemma 10.4.12. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_0 + T_1e_1 + T_2e_2$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum. Let G be a bounded slice Cauchy domain such that $(\partial G) \cap \sigma_S(T) = \emptyset$. For every $J \in \mathbb{S}$ we have*

$$\int_{\partial(G\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J = 0. \tag{10.18}$$

Proof. Since $\Delta(1) = 0$ and $\Delta(q) = 0$, by Theorem 7.4.6 we also have

$$\int_{\partial(G\cap\mathbb{C}_J)} F_L(s, q) ds_J = \Delta(1) = 0, \tag{10.19}$$

and

$$\int_{\partial(G\cap\mathbb{C}_J)} F_L(s, q) ds_J s = \Delta(q) = 0, \tag{10.20}$$

for all $q \notin \partial G$ and $J \in \mathbb{S}$. By the monogenic functional calculus [99, 101] we have

$$F_L(s, T) = \int_{\partial\Omega} G(\omega, T)\mathbf{D}\omega F_L(s, \omega),$$

where $\mathbf{D}\omega$ is a suitable differential form, the open set Ω contains the left spectrum of T and $G(\omega, T)$ is the Fueter resolvent operator. By Theorem 9.3.1 we can write

$$\mathcal{Q}_{c,s}(T)^{-1} = -\frac{1}{4}(F_L(s, T)s - TF_L(s, T)),$$

thus we have

$$\begin{aligned}
&\int_{\partial(G\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J = -\frac{1}{4}\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,T)s - TF_L(s,T) ds_J \\
&= -\frac{1}{4}\left(\int_{\partial(G\cap\mathbb{C}_J)}\int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega F_L(s,\omega)s\, ds_J\right. \\
&\quad \left. -T\int_{\partial(G\cap\mathbb{C}_J)}\int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega F_L(s,\omega) ds_J\right) \\
&= -\frac{1}{4}\left(\int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega\left(\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,\omega)ds_J s\right)\right. \\
&\quad \left. -T\int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega\left(\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,\omega)ds_J\right)\right) \\
&= 0,
\end{aligned}$$

where the second equality is a consequence of the Fubini's Theorem and the last equality is a consequence of formulas (10.19) and (10.20). □

Remark 10.4.13. To define a monogenic functional McIntosh and collaborators, see [99, 101, 108, 112], had as hypothesis that the component T_0 of the operator $T = T_0 + T_1e_1 + T_2e_2 + T_3e_3$ is zero. However, it is possible to set zero a different component of the operator T.

Finally in the following result we give an answer to Problem 10.4.9.

Proposition 10.4.14. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_0 + T_1e_1 + T_2e_2$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum. Let U be a slice Cauchy domain with $\sigma_S(T) \subset U$. If $f, g \in \mathcal{SH}_L(U)$ (resp. $f, g \in \mathcal{SH}_R(U)$) satisfy the property $\mathcal{D}f = \mathcal{D}g$ (resp $f\mathcal{D} = g\mathcal{D}$) then $\tilde{f}(T) = \tilde{g}(T)$.*

Proof. If U is connected we can use Proposition 10.4.11. If U is not connected then $U = \cup_{l=1}^n U_l$ where the U_l are the connected components of U. Hence, we have $f(s) - g(s) = \sum_{l=1}^n \chi_{U_l}(s)\alpha_l$ and we can write

$$\tilde{f}(T) - \tilde{g}(T) = -\sum_{l=1}^n \frac{1}{\pi}\int_{\partial(U_l\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} ds_J \alpha_l.$$

The last summation is zero by Lemma 10.4.12. □

We conclude this section with some algebraic properties of the harmonic functional calculus.

Proposition 10.4.15. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_0 + T_1 e_1 + T_2 e_2$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum.*

- *If $\tilde{f} = \mathcal{D} f$ and $\tilde{g} = \mathcal{D} g$ with $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*
$$(\tilde{f} a + \tilde{g})(T) = \tilde{f}(T) a + \tilde{g}(T).$$
- *If $\tilde{f} = f \mathcal{D}$ and $\tilde{g} = g \mathcal{D}$ with $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*
$$(a \tilde{f} + \tilde{g})(T) = a \tilde{f}(T) + \tilde{g}(T).$$

Proof. The above identities follow immediately from the linearity of the integrals in (10.15), resp. (10.16). □

Proposition 10.4.16. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_0 + T_1 e_1 + T_2 e_2$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum.*

- *If $\tilde{f} = \mathcal{D} f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$ and assume that $f(q) = \sum_{m=0}^{\infty} q^m a_m$ with $a_m \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*
$$\tilde{f}(T) = -2 \sum_{m=1}^{\infty} \sum_{k=1}^{m} T^{m-k} \overline{T}^{k-1} a_m.$$
- *If $\tilde{f} = f \mathcal{D}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$ and assume that $f(q) = \sum_{m=0}^{\infty} a_m q^m$ with $a_m \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*
$$\tilde{f}(T) = -2 \sum_{m=1}^{\infty} \sum_{k=1}^{m} a_m T^{m-k} \overline{T}^{k-1}.$$

Proof. We prove the first assertion since the second one can be proven similarly. We choose an imaginary unit $J \in \mathbb{S}$ and a radius $0 < R < r$ such that $\sigma_S(T) \subset B_R(0)$. Then the series expansion of f converges uniformly on $\partial(B_R(0) \cap \mathbb{C}_J)$, and so

$$\begin{aligned} \tilde{f}(T) &= -\frac{1}{\pi} \int_{\partial(B_R(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} \, ds_J \, \sum_{l=0}^{\infty} s^l a_l \\ &= -\frac{1}{\pi} \sum_{l=0}^{\infty} \int_{\partial(B_R(0) \cap \mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1} \, ds_J s^l a_l. \end{aligned}$$

By replacing $\mathcal{Q}_{c,s}(T)^{-1}$ with its series expansion, see Proposition 10.4.3, we further obtain

$$\begin{aligned}\tilde{f}(T) &= -\frac{1}{\pi}\int_{\partial(B_R(0)\cap\mathbb{C}_J)}\sum_{m=1}^{\infty}\sum_{k=1}^{m}T^{m-k}\overline{T}^{k-1}s^{-1-m}\,ds_J\sum_{l=0}^{\infty}s^l a_l\\ &= -\frac{1}{\pi}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{l=0}^{\infty}T^{m-k}\overline{T}^{k-1}\int_{\partial(B_R(0)\cap\mathbb{C}_J)}s^{-1-m}\,ds_J\,s^l a_l\\ &= -2\sum_{m=1}^{\infty}\sum_{k=1}^{m}T^{m-k}\overline{T}^{k-1}a_m.\end{aligned}$$

The last equality is due to the fact that $\int_{\partial(B_R(0)\cap\mathbb{C}_J)} s^{-1-m}\,ds_J\,s^l$ is equal to 2π if $l=m$, and 0 otherwise. $\square$

Now, we prove other important properties of the harmonic functional calculus.

Theorem 10.4.17. *Let $T\in\mathcal{BC}(X)$. Let $m\in\mathbb{N}_0$, and let $U\subset\mathbb{H}$ be a bounded slice Cauchy domain with $\sigma_S(T)\subset U$. For every $J\in\mathbb{S}$ we have*

$$H_m(T)=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1}, \tag{10.21}$$

where

$$H_m(T):=\sum_{k=0}^{m}T^{m-k}\overline{T}^{k}.$$

Proof. We start by considering U to be the ball $B_r(0)$ with $\|T\|<r$. We know that

$$\mathcal{Q}_{c,s}(T)^{-1}=\sum_{n=1}^{+\infty}\sum_{k=1}^{n}T^{n-k}\overline{T}^{k-1}s^{-1-n}$$

for every $s\in\partial B_r(0)$. By Proposition 10.4.3 we know that the series converges on $\partial B_r(0)$. Thus we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1} &= \frac{1}{2\pi}\sum_{n=1}^{+\infty}\sum_{k=1}^{n}T^{n-k}\overline{T}^{k-1}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}s^{-n+m}ds_J\\ &= \sum_{k=1}^{m+1}T^{m+1-k}\overline{T}^{k-1}\\ &= \sum_{k=0}^{m}T^{m-k}\overline{T}^{k}\\ &= H_m(T)\end{aligned}$$

since

$$\int_{\partial(B_r(0)\cap\mathbb{C}_J)}s^{-n+m}ds_J = \begin{cases}0 & \text{if } n\neq m+1\\ 2\pi & \text{if } n=m+1.\end{cases}$$

This proves the result for the case $U = B_r(0)$. Now we get the result for an arbitrary bounded Cauchy domain U that contains $\sigma_S(T)$. Then there exists a radius r such that $\overline{U}\subset B_r(0)$. The operator $\mathcal{Q}_{c,s}(T)^{-1}$ is right slice hyperholomorphic and the monomial s^{m+1} is left slice hyperholomorphic on the bounded slice Cauchy domain $B_r(0)\setminus U$. By the Cauchy's integral formula we get

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1}-\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1}\\ &=\frac{1}{2\pi}\int_{\partial((B_r(0)\setminus U)\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1}=0.\end{aligned}$$

Finally we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1} &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_Js^{m+1}\\ &= H_m(T),\end{aligned}$$

and this concludes the proof. □

Remark 10.4.18. Unlike what happens in the S-functional calculus (see [45, Thm. 3.2.2]) we do not have a left slice hyperholomorphic polynomial on the left hand side of equality (10.21), but we have harmonic polynomials. Another difference with respect to [45, Thm. 3.2.2] is that in Theorem

10.4.17 we do not have a difference between right and left part, because by Proposition 10.4.3

$$\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m} = \sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}T^{m-k}\overline{T}^{k-1} = \mathcal{Q}_{c,s}(T)^{-1}.$$

For the intrinsic functions we have the following result.

Lemma 10.4.19. *Let $T \in \mathcal{BC}(X)$. If $f \in \mathcal{N}(\sigma_S(T))$ and U is a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset dom(f)$, then we have*

$$\tilde{f}(T) = -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1}ds_J f(s) = -\frac{1}{\pi}\int_{\partial(U\cap\mathbb{C}_J)} f(s)ds_J\mathcal{Q}_{c,s}(T)^{-1}.$$

Proof. It follows by the definitions of intrinsic functions, of the Q-functional calculus and Runge's theorem. □

10.5 The resolvent equation for the harmonic functional calculus

In this section we prove various resolvent equations for the pseudo S-resolvent operator $\mathcal{Q}_{c,s}(T)^{-1}$. The first version of this equation is written in terms of $\mathcal{Q}_{c,s}(T)^{-1}$ and of the S-resolvent operators.

Theorem 10.5.1 (The Q-resolvent equation with S-resolvent operators)**.** *Let $T \in \mathcal{BC}(X)$. Then, for p, $s \in \rho_S(T)$ with $s \notin [p]$, the following equalities hold*

$$\begin{aligned}\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} \;=\;& \left\{\left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T) - S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right]p\right.\\ &\left.-\bar{s}\left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T) - S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right]\right\}\cdot\\ &(p^2-2s_0p+|s|^2)^{-1}, \qquad (10.22)\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} =\;& (p^2-2s_0p+|s|^2)^{-1}\left\{s\left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right.\right.\\ &\left.-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right] - \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right.\\ &\left.\left.-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right]\bar{p}\right\}. \qquad (10.23)\end{aligned}$$

Proof. By the definition of left S-resolvent operator we have

$$\mathcal{Q}_{c,p}(T)^{-1}p = \overline{T}\mathcal{Q}_{c,p}(T)^{-1} + S_L^{-1}(p,T). \qquad (10.24)$$

By iterating (10.24) we get

$$\begin{aligned}
&\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}(p^2-2s_0p+|s|^2)=\\
&=\mathcal{Q}_{c,s}(T)^{-1}[\mathcal{Q}_{c,p}(T)^{-1}p]p-2s_0\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}p\\
&\quad+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&=\mathcal{Q}_{c,s}(T)^{-1}[\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+S_L^{-1}(p,T)]p-2s_0\mathcal{Q}_{c,s}(T)^{-1}\\
&\quad[\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+S_L^{-1}(p,T)]\\
&\quad+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&=\mathcal{Q}_{c,s}(T)^{-1}\overline{T}[\mathcal{Q}_{c,p}(T)^{-1}p]+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p\\
&\quad-2s_0\mathcal{Q}_{c,s}(T)^{-1}[\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+S_L^{-1}(p,T)]+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&=\mathcal{Q}_{c,s}(T)^{-1}\overline{T}[\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+S_L^{-1}(p,T)]+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p\\
&\quad-2s_0\mathcal{Q}_{c,s}(T)^{-1}[\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+S_L^{-1}(p,T)]+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}.
\end{aligned}$$

Now, by the definition of the right S-resolvent operator we have

$$\mathcal{Q}_{c,s}(T)^{-1}\overline{T}=s\mathcal{Q}_{c,s}(T)^{-1}-S_R^{-1}(s,T). \tag{10.25}$$

This equality implies

$$\begin{aligned}
&\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}(p^2-2s_0p+|s|^2)\\
&=[\mathcal{Q}_{c,s}(T)^{-1}\overline{T}]\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+[\mathcal{Q}_{c,s}(T)^{-1}\overline{T}]S_L^{-1}(p,T)+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p\\
&\quad-2s_0[\mathcal{Q}_{c,s}(T)^{-1}\overline{T}]\mathcal{Q}_{c,p}(T)^{-1}-2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&=[s\mathcal{Q}_{c,s}(T)^{-1}-S_R^{-1}(s,T)]\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+[s\mathcal{Q}_{c,s}(T)^{-1}-S_R^{-1}(s,T)]S_L^{-1}(p,T)\\
&\quad+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p-2s_0[s\mathcal{Q}_{c,s}(T)^{-1}-S_R^{-1}(s,T)]\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad-2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&=s[\mathcal{Q}_{c,s}(T)^{-1}\overline{T}]\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\\
&\quad-S_R^{-1}(s,T)S_L^{-1}(p,T)+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p-2s_0s\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad+2s_0S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}-2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&=s[s\mathcal{Q}_{c,s}(T)^{-1}-S_R^{-1}(s,T)]\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\overline{T}\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)S_L^{-1}(p,T)+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p\\
&\quad-2s_0s\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}+2s_0S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\\
&-2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)+|s|^2\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}.
\end{aligned}$$

Now, since $s^2-2s_0s+|s|^2=0$ we get

$$\begin{aligned}
&\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}(p^2-2s_0p+|s|^2)\\
&=(s^2-2s_0s+|s|^2)\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}-sS_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad-S_R^{-1}(s,T)\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)S_L^{-1}(p,T)
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p+2s_0S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}-2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\\
&=-sS_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\overline{T}\mathcal{Q}_{c,p}(T)^{-1}+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\\
&\quad -S_R^{-1}(s,T)S_L^{-1}(p,T)+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p+2s_0S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad -2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\\
&=-sS_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)[\overline{T}\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad +S_L^{-1}(p,T)]+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p+2s_0S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad -2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T).
\end{aligned}
$$

Finally, by using another time formula (10.24) and the fact that $2s_0-s=\bar{s}$ we obtain

$$
\begin{aligned}
&\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}(p^2-2s_0p+|s|^2)\\
&=-sS_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p\\
&\quad +\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p+2s_0S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}-2s_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\\
&=\left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right]p\\
&\quad -\bar{s}\left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right].
\end{aligned}
$$

It is possible to obtain formula (10.23) with similar computations. □

Remark 10.5.2. We can rewrite the equations obtained in Theorem 10.5.1 by using the left or right slice hyperholomorphic products, see Definition 3.1.6, in the variables $s,p\in\rho_S(T)$ with $s\notin[p]$,

$$
\begin{aligned}
\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} &= \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right]\\
&\quad *_{s,L}(p-\bar{s})(p^2-2s_0p+|s|^2)^{-1}\mathcal{I},
\end{aligned}
$$

or

$$
\begin{aligned}
\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1} &= (p-\bar{s})(p^2-2s_0p+|s|^2)^{-1}\mathcal{I}*_{p,R}\\
&\quad \left[\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}\right].
\end{aligned}
$$

Theorem 10.5.3 (Left and right generalized Q-resolvent equations)**.** *Let $T\in\mathcal{BC}(X)$ with $s\in\rho_S(T)$ and set*

$$
\mathcal{M}_m^L(s,T):=\sum_{i=0}^{m-1}\overline{T}^iS_L^{-1}(s,T)s^{m-i-1}
$$

and

$$
\mathcal{M}_m^R(s,T):=\sum_{i=0}^{m-1}s^{m-i-1}S_R^{-1}(s,T)\overline{T}^i.
$$

Then for $m \geq 1$ and $s \in \rho_S(T)$, the following equations hold

$$\mathcal{Q}_{c,s}(T)^{-1}s^m - \overline{T}^m \mathcal{Q}_{c,s}(T)^{-1} = \mathcal{M}_m^L(s,T) \tag{10.26}$$

and

$$s^m \mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}\overline{T}^m = \mathcal{M}_m^R(s,T).$$

Proof. We prove the result by induction on m. We will prove only (10.26) since the other equality is proven with similar techniques. The case $m = 1$ is trivial because

$$\mathcal{M}_1^L(s,t) = S_L^{-1}(s,t) = \mathcal{Q}_{c,s}^{-1}(T)s - \overline{T}\mathcal{Q}_{c,s}^{-1}(T).$$

We assume that the equation holds for $m-1$ and we will prove it for m. By inductive hypothesis, we have

$$\begin{aligned}\overline{T}^m \mathcal{Q}_{c,s}(T)^{-1} &= \overline{T}\overline{T}^{m-1}\mathcal{Q}_{c,s}(T)^{-1} = \overline{T}(\mathcal{Q}_{c,s}(T)^{-1}s^{m-1} - \mathcal{M}_{m-1}^L(s,T)) \\ &= \overline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-1} - \overline{T}\mathcal{M}_{m-1}^L(s,T).\end{aligned}$$

Since

$$\overline{T}\mathcal{M}_{m-1}^L(s,T) = \sum_{i=0}^{m-2} \overline{T}^{i+1} S_L^{-1}(s,T)s^{m-i-2} = \sum_{i=1}^{m-1} \overline{T}^i S^{-1}(s,T)s^{m-i-1}$$

and

$$\overline{T}\mathcal{Q}_{c,s}(T)^{-1} = \mathcal{Q}_{c,s}(T)^{-1}s - S_L^{-1}(s,t),$$

we have

$$\begin{aligned}\overline{T}^m \mathcal{Q}_{c,s}(T)^{-1} &= \mathcal{Q}_{c,s}(T)^{-1}s^m - S_L^{-1}(s,T)s^{m-1} - \sum_{i=1}^{m-1} \overline{T}^i S^{-1}(s,T)s^{m-i-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1}s^m - \sum_{i=0}^{m-1} \overline{T}^i S^{-1}(s,T)s^{m-i-1} \\ &= \mathcal{Q}_{c,s}(T)^{-1}s^m - \mathcal{M}_m^L(s,T).\end{aligned}$$

□

Now, our goal is to obtain a resolvent equation for the Q-functional calculus in which a term of the following form

$$[\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1}] *_{s,left} S_L^{-1}(s,q)$$

is transformed into the product of $\mathcal{Q}_{c,s}(T)^{-1}$ and $\mathcal{Q}_{c,p}(T)^{-1}$ and other terms involving the S-resolvent operators. Moreover, we want to maintain the

slice hyperholomorphicity.
In order to achieve this aim we need to recall a suitable modification of the classic S-resolvent equation, see [6, Thm. 6.7].

Theorem 10.5.4. *Let $T \in \mathcal{BC}(X)$ and $B \in \mathcal{B}(X)$ such that it commutes with T, then we have*

$$\begin{aligned} S_R^{-1}(s,T)BS_L^{-1}(p,T) &= [\left(S_R^{-1}(s,T)B - BS_L^{-1}(p,T)\right)p + \qquad (10.27)\\ &\quad -\bar{s}\left(S_R^{-1}(s,T)B - BS_L^{-1}(p,T)\right)]\mathcal{Q}_s(p)^{-1}, \end{aligned}$$

where $\mathcal{Q}_s(p) := p^2 - 2s_0p + |s|^2$.

Remark 10.5.5. If we consider $B = \mathcal{I}$ in (10.27) we get (3.9).

Now, we have all the tools to obtain a new resolvent equation for the Q-functional calculus.

Theorem 10.5.6. *Let $T \in \mathcal{BC}(X)$. For s, $p \in \rho_S(T)$ with $s \notin [p]$ we have the following equation*

$$\begin{aligned} &\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T) + S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1} - 2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1} =\\ &= [(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})p - \bar{s}(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})]\\ &\quad \mathcal{Q}_s(p)^{-1}, \qquad (10.28) \end{aligned}$$

where $\underline{T} = T_1e_1 + T_2e_2 + T_3e_3$.

Proof. We will show this result in seven steps.

Step I. We consider $B = T$ in (10.27) and we multiply it on the right by $4\mathcal{Q}_{c,p}(T)^{-1}$, then we get

$$\begin{aligned} -S_R^{-1}(s,T)TF_L(p,T) &= [\left(4S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1} + TF_L(p,T)\right)p \qquad (10.29)\\ &\quad -\bar{s}\left(4S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1} + TF_L(p,T)\right)]\mathcal{Q}_s(p)^{-1}. \end{aligned}$$

Step II. We consider $B = \mathcal{I}$ in (10.27) and we multiply it on the right by $-4\mathcal{Q}_{c,p}(T)^{-1}p$, then we obtain

$$\begin{aligned} S_R^{-1}(s,T)F_L(s,T)p &= [\left(-4S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p - F_L(p,T)p\right)p \qquad (10.30)\\ &\quad -\bar{s}\left(-4S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p - F_L(p,T)p\right)]\mathcal{Q}_s(p)^{-1}. \end{aligned}$$

Step III. We substitute $B = T$ in (10.27) and we multiply it on the left by $4\mathcal{Q}_{c,s}(T)^{-1}$, then we get

$$\begin{aligned} -F_R(s,T)TS_L^{-1}(s,T) &= [\left(-F_R(s,T)T - 4\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right)p \qquad (10.31)\\ &\quad -\bar{s}\left(-F_R(s,T)T - 4\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right)]\mathcal{Q}_s(p)^{-1}. \end{aligned}$$

Step IV. We substitute $B=\mathcal{I}$ in (10.27) and we multiply it on the left by $-4s\mathcal{Q}_{c,s}(T)^{-1}$, then we obtain

$$\begin{aligned} sF_R(s,T)S_L^{-1}(s,T) &= [\left(sF_R(s,T)+4s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right)p \\ &\quad -\bar{s}\left(sF_R(s,T)+4s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right)]\mathcal{Q}_s(p)^{-1}. \end{aligned} \tag{10.32}$$

Step V. We make the sum of formulas (10.29), (10.30), (10.31), (10.32) and by Theorem 9.3.1 we get

$$\begin{aligned} &-4S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}-4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(s,T) \\ &=[\left(4\mathcal{Q}_{c,p}(T)^{-1}-4\mathcal{Q}_{c,s}(T)^{-1}\right)p-\bar{s}\left(4\mathcal{Q}_{c,p}(T)^{-1}-4\mathcal{Q}_{c,s}(T)^{-1}\right)]\mathcal{Q}_s(p)^{-1} \\ &+4\left[\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right.\right. \\ &\left.+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right)p-\bar{s}\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p\right. \\ &\left.\left.-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right)\right]\mathcal{Q}_s(p)^{-1}. \end{aligned} \tag{10.33}$$

Step VI. We show that

$$\begin{aligned} &\left[\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right)p+\right. \\ &\left.-\bar{s}\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right)\right]\mathcal{Q}_s(p)^{-1} \\ &=-\mathcal{Q}_{c,s}(T)^{-1}T\mathcal{Q}_{c,p}(T)^{-1}. \end{aligned} \tag{10.34}$$

We focus on proving formula (10.34). First of all, we observe that by the definition of the S-resolvent operators we have

$$\begin{aligned} &S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T) \\ &=\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I}-\bar{T})T\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}T(p\mathcal{I}-\bar{T})\mathcal{Q}_{c,p}(T)^{-1} \\ &=\mathcal{Q}_{c,s}(T)^{-1}sT\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}Tp\mathcal{Q}_{c,p}(T)^{-1}. \end{aligned}$$

Thus we get

$$\begin{aligned} &\left[\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right)p\right. \\ &\quad\left.-\bar{s}\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right)\right]\mathcal{Q}_s(p)^{-1} \\ &=\left(\mathcal{Q}_{c,s}(T)^{-1}sTp\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}Tp^2\mathcal{Q}_{c,p}(T)^{-1}+\right. \\ &\quad\left.-\mathcal{Q}_{c,s}(T)^{-1}|s|^2T\mathcal{Q}_{c,p}(T)^{-1}+\mathcal{Q}_{c,s}(T)^{-1}\bar{s}Tp\mathcal{Q}_{c,p}(T)^{-1}\right)\mathcal{Q}_s(p)^{-1} \\ &=\left(\mathcal{Q}_{c,s}(T)^{-1}(s+\bar{s})Tp\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}Tp^2\mathcal{Q}_{c,p}(T)^{-1}\right. \\ &\left.-\mathcal{Q}_{c,s}(T)^{-1}|s|^2T\mathcal{Q}_{c,p}(T)^{-1}\right)\mathcal{Q}_s(p)^{-1} \\ &=-\mathcal{Q}_{c,s}(T)^{-1}T\mathcal{Q}_s(p)\mathcal{Q}_{c,p}(T)^{-1}\mathcal{Q}_s(p)^{-1} \\ &=-\mathcal{Q}_{c,s}(T)^{-1}T\mathcal{Q}_{c,p}(T)^{-1}. \end{aligned}$$

Step VII. By similar computations we have the following equality

$$\begin{aligned}&\left[\left(s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p\right)p+\right.\\&\left.-\bar{s}\left(s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p\right)\right]\mathcal{Q}_s(p)^{-1}\\&=\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1}.\end{aligned}\tag{10.35}$$

Step VIII. We put together (10.34) and (10.35) to obtain

$$\begin{aligned}&\left[\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)\right.\right.\\&\quad\left.+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right)p-\bar{s}\left(S_R^{-1}(s,T)T\mathcal{Q}_{c,p}(T)^{-1}-S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}p\right.\\&\quad\left.\left.-\mathcal{Q}_{c,s}(T)^{-1}TS_L^{-1}(p,T)+s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)\right)\right]\mathcal{Q}_s(p)^{-1}\\&=-\mathcal{Q}_{c,s}(T)^{-1}T\mathcal{Q}_{c,p}(T)^{-1}+\mathcal{Q}_{c,s}(T)\bar{T}\mathcal{Q}_{c,p}(T)^{-1}\\&=-2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1}.\end{aligned}\tag{10.36}$$

Finally by putting formula (10.36) in (10.33) we get

$$\begin{aligned}&S_R^{-1}(s,T)\mathcal{Q}_{c,p}(T)^{-1}+\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)=\\&[\left(\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1}\right)p-\bar{s}\left(\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1}\right)]\mathcal{Q}_s(p)^{-1}\\&+2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1}.\end{aligned}$$

This proves the statement. □

By using formula (10.28), it is possible to obtain an interesting and nice formula for the product rule of the Q-functional calculus.

Theorem 10.5.7. *Let $T\in\mathcal{BC}(X)$. We assume that $f\in\mathcal{N}(\sigma_S(T))$ and $g\in\mathcal{SH}_L(\sigma_S(T))$ then we have*

$$\mathcal{D}(fg)(T)=f(T)(\mathcal{D}g)(T)+(\mathcal{D}f)(T)g(T)+(\mathcal{D}f)(T)\underline{T}(\mathcal{D}g)(T).$$

If $g\in\mathcal{SH}_R(\sigma_S(T))$ then we have

$$\mathcal{D}(gf)(T)=f(T)(\mathcal{D}g)(T)+(\mathcal{D}f)(T)g(T)+(\mathcal{D}f)(T)\underline{T}(\mathcal{D}g)(T).\tag{10.37}$$

Proof. Let G_1 and G_2 be two bounded slice Cauchy domains as in the proof of Theorem 5.22. Let us consider $p\in\partial(G_1\cap\mathbb{C}_J)$ and $s\in\partial(G_1\cap\mathbb{C}_J)$. By the definitions of the S-functional calculus and the Q-functional calculus

we get

$$
\begin{aligned}
& f(T)(\mathcal{D}g)(T)+(\mathcal{D}f)(T)g(T)+(\mathcal{D}f)(T)\underline{T}(\mathcal{D}g)(T)\\
&=-\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} S^{-1}_L(s,T)ds_J f(s)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_J g(p)\\
&-\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_J f(s)\int_{\partial(G_1\cap\mathbb{C}_J)} S^{-1}_L(p,T)dp_J g(p)\\
&+\frac{1}{\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}ds_J f(s)\underline{T}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_J g(p).
\end{aligned}
$$

By the fact that the function f is intrinsic and by Theorem 10.5.6 we get

$$
\begin{aligned}
& f(T)(\mathcal{D}g)(T)+(\mathcal{D}f)(T)g(T)+(\mathcal{D}f)(T)\underline{T}(\mathcal{D}g)(T)\\
&=\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} f(s)ds_J[-S^{-1}_R(s,T)\mathcal{Q}_{c,p}(T)^{-1}\\
&\quad -\mathcal{Q}_{c,s}(T)^{-1}S^{-1}_L(p,T)+2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1}]dp_J g(p)\\
&=-\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} f(s)ds_J[(\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1})p\\
&\quad -\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1})]\mathcal{Q}_s(p)^{-1}dp_J g(p)\\
&=-\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,s}(T)^{-1}p\mathcal{Q}_s(p)^{-1}dp_J g(p)\\
&\quad +\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}p\mathcal{Q}_s(p)^{-1}dp_J g(p)\\
&\quad +\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\bar{s}\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_s(p)^{-1}dp_J g(p)\\
&\quad -\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\bar{s}\mathcal{Q}_{c,p}(T)^{-1}\mathcal{Q}_s(p)^{-1}dp_J g(p).
\end{aligned}
$$

Since the maps $p\mapsto p\mathcal{Q}_s(p)^{-1}$ and $p\mapsto\mathcal{Q}_s(p)$ are intrinsic slice hyperholomorphic on $\bar{G}_1$, by the Cauchy integral formula we get that the first and the third integrals in the above formula are zero. By Lemma 9.5 and the definition of Q-functional calculus we get

$$f(T)(\mathcal{D}g)(T)+(\mathcal{D}f)(T)g(T)+(\mathcal{D}f)(T)\underline{T}(\mathcal{D}g)(T)=$$
$$=-\frac{1}{2\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}[\bar{s}\mathcal{Q}_{c,p}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1}p]$$
$$\mathcal{Q}_s(p)\,dp_J\,g(p)$$
$$=-\frac{1}{\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_Jf(p)g(p)$$
$$=-\frac{1}{\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_J(fg)(p)$$
$$=\mathcal{D}(fg)(T).$$

Formula (10.37) follows by similar computations. □

10.6 The Riesz projectors for harmonic functional calculus

We now take advantage of the Q-resolvent equation in Theorem 10.5.6 to study the Riesz projectors for the harmonic functional calculus.

Theorem 10.6.1 (The Riesz projectors)**.** *Let $T=T_0+T_1e_1+T_2e_2$ and assume that the operators T_l, $l=0,\,1,\,2$, have real spectrum. Let $\sigma_S(T)=\sigma_1\cup\sigma_2$ with $\operatorname{dist}(\sigma_1,\sigma_2)>0$.*

Let $G_1,\,G_2\subset\mathbb{H}$ be two bounded slice Cauchy domains such that $\sigma_1\subset G_1$, $\overline{G}_1\subset G_2$ and $\operatorname{dist}(G_2,\sigma_2)>0$. Then the operator

$$\tilde{P}(T):=\frac{1}{2\pi}\int_{\partial(G_2\cap\mathbb{C}_J)}s\,ds_J\mathcal{Q}_{c,s}(T)^{-1}=\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}\,dp_Jp$$

is a projection, i.e.,

$$\tilde{P}^2=\tilde{P}.$$

Moreover, the operator $\tilde{P}$ commutes with T, i.e. we have

$$T\tilde{P}=\tilde{P}T. \tag{10.38}$$

Proof. From the definition of right S-resolvent operator we have

$$S_R^{-1}(s,T)=s\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}\bar{T}. \tag{10.39}$$

By inserting formula (10.39) in the equation (10.28) and by multiplying on the right by p we get

$$\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(p,T)p+s\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}p-\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,p}(T)^{-1}p$$
$$-2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1}p=[(\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1})p$$
$$-\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}-\mathcal{Q}_{c,p}(T)^{-1})]\mathcal{Q}_s(p)^{-1}p. \tag{10.40}$$

Now, we multiply formula (10.40) by ds_J and we integrate it on $\partial(G_2 \cap \mathbb{C}_J)$ with respect to ds_J and if we multiply on the right by dp_J and we integrate on $\partial(G_1 \cap \mathbb{C}_J)$ with respect to dp_J. Therefore, we get

$$\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \mathcal{Q}_{c,s}(T)^{-1} \int_{\partial(G_1\cap\mathbb{C}_J)} S_L^{-1}(p,T) dp_J p + \int_{\partial(G_2\cap\mathbb{C}_J)} s ds_J \mathcal{Q}_{c,s}(T)^{-1} \cdot \int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p - 2\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \mathcal{Q}_{c,s}(T)^{-1} \underline{T} \int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p = \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} [(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})p - \bar{s}(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})] \cdot \mathcal{Q}_s(p)^{-1} dp_J p.$$

By Lemma 10.3.12 we obtain

$$(2\pi)^2 \tilde{P}^2 = \int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} [(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})p - \bar{s}(\mathcal{Q}_{c,s}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1})] \mathcal{Q}_s(p)^{-1} dp_J p.$$

Now, since the functions $p \mapsto \mathcal{Q}_s(p)^{-1}$ and $p \mapsto \mathcal{Q}_s(p)^{-1}$ are slice hyperholomorphic and do not have singularities inside $\partial(G_1 \cap \mathbb{C}_J)$ by the Cauchy theorem we get

$$\int_{\partial(G_1\cap\mathbb{C}_J)} p\mathcal{Q}_s(p)^{-1} dp_J p^2 = \int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_s(p) dp_J p = 0. \tag{10.41}$$

Therefore, we have

$$\tilde{P}^2 = \frac{1}{(2\pi)^2} \int_{\partial(G_2\cap\mathbb{C}_J)} \int_{\partial(G_1\cap\mathbb{C}_J)} ds_J [\bar{s}\mathcal{Q}_{c,p}(T)^{-1} - \mathcal{Q}_{c,p}(T)^{-1} p] \mathcal{Q}_s(p)^{-1} dp_J p.$$

By exchanging the role of the integrals and Lemma 9.5 with $B := \mathcal{Q}_{c,p}(T)^{-1}$ we get

$$\tilde{P}^2 = \frac{1}{2\pi} \int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1} dp_J p = \tilde{P}.$$

Now, we prove (10.38). By the following formula

$$\overline{T}\mathcal{Q}_{c,p}(T)^{-1} = \mathcal{Q}_{c,p}(T)^{-1} p - S_L^{-1}(p,T),$$

we get

$$\begin{aligned} T\tilde{P} &= -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\overline{T}\mathcal{Q}_{c,p}(T)^{-1}dp_Jp \\ &= -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}(\mathcal{Q}_p(T)^{-1}p - S_L^{-1}(p,T))dp_Jp \\ &= -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_jp^2 + \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)dp_Jp. \end{aligned}$$

Thus we obtain

$$T\tilde{P} = -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_Jp^2 + \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)dp_Jp. \tag{10.42}$$

On the other side, since

$$\mathcal{Q}_{c,p}(T)^{-1}\overline{T} = \mathcal{Q}_{c,p}(T)^{-1}p - S_R^{-1}(p,T),$$

we get

$$\begin{aligned} \tilde{P}T &= -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}pdp_J\mathcal{Q}_{c,p}(T)^{-1}\overline{T} \\ &= -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_Jp^2 + \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}pdp_JS_R^{-1}(p,T). \end{aligned}$$

From the fact that $p\chi_{G_1}(p)$ is intrinsic slice hyperholomorphic in G_1, it follows by [45, Thm. 3.2.11] that

$$\tilde{P}T = -\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_Jp^2 + \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)pdp_J. \tag{10.43}$$

Since (10.42) and (10.43) are equal we get the statement. □

Remark 10.6.2. The Q-resolvent equation stated in Theorem 10.5.1 preserves the slice hyperholomorphicity, however it is not useful to prove Theorem 10.6.1.

Remark 10.6.3. Theorem 10.6.1 can be proved using directly the F-functional Calculus. Indeed, in the same hypothesis of the theorem, it is proved in [45, Thm. 7.4.2] that

$$\check{P}_1^2 = \check{P}_1 \quad \text{and} \quad T\check{P}_1 = \check{P}_1T$$

for

$$\check{P}_1 := -\frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}F_L(p,T)\,dp_Jp^2.$$

Now, using Theorem 9.3.1 and [45, Lemma 7.4.1], we have

$$\begin{aligned}\tilde{P} &= \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1}\, dp_J p\\ &= -\frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)p - TF_L(p,T)\, dp_J p\\ &= -\frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)\, dp_J p^2 = \check{P}.\end{aligned}$$

10.7 The product rule for the F-functional calculus

By means of the harmonic functional calculus we can show a product rule for the F-functional calculus.

Theorem 10.7.1. *Let $T \in \mathcal{BC}(X)$ and assume $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ then we have*

$$\Delta(fg)(T) = \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T). \tag{10.44}$$

Proof. Let G_1 and G_2 be two bounded slice Cauchy domains such that contain $\sigma_S(T)$ and $\overline{G}_1 \subset G_2$, with $\overline{G}_2 \subset dom(f) \cap dom(g)$. We choose $p \in \partial(G_1\cap\mathbb{C}_J)$ and $s \in \partial(G_2\cap\mathbb{C}_J)$. For every $J \in \mathbb{S}$, from the definitions of F-functional calculus, S-functional calculus and Q- functional calculus, we get

$$\begin{aligned}&\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) =\\ &= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J F_R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} S^{-1}_L(p,T)dp_J g(p)\\ &+\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J S^{-1}_R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J g(p)\\ &-\frac{1}{(\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1}ds_J f(s)\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1}dp_J g(p).\end{aligned}$$

Since the function f is intrinsic by Lemma 10.4.19 we have

$$\begin{aligned}&\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) =\\ &= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J F_R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} S^{-1}_L(p,T)dp_J g(p)\\ &+\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J S^{-1}_R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J g(p)\\ &-\frac{1}{(\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{Q}_{c,s}(T)^{-1}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1}dp_J g(p).\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\Delta f(T)g(T)+f(T)\Delta g(T)-\mathcal{D}f(T)\mathcal{D}g(T)\\
&\qquad=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}f(s)ds_J\left[F_R(s,T)S_L^{-1}(p,T)+\right.\\
&\qquad\left.+S_R^{-1}(s,T)F_L(p,T)-4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\right]dp_Jg(p).
\end{aligned}$$

By the following equation (see [45, Lemma 7.3.2])

$$\begin{aligned}
&F_R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_L(p,T)-4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\
&\qquad=[(F_R(s,T)-F_L(p,T))p-\overline{s}(F_R(s,T)-F_L(p,T))]\mathcal{Q}_s(p)^{-1},
\end{aligned}$$

where $\mathcal{Q}_s(p)=p^2-2s_0p+|s|^2$, we obtain

$$\begin{aligned}
&\Delta f(T)g(T)+f(T)\Delta g(T)-\mathcal{D}f(T)\mathcal{D}g(T)\\
&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}f(s)\left[(F_R(s,T)-F_L(p,T))\,p\right.\\
&\left.-\bar{s}\left(F_R(s,T)-F_L(p,T)\right)\right]\mathcal{Q}_s(p)^{-1}dp_Jg(p).
\end{aligned}$$

By the linearity of the integrals follows that

$$\begin{aligned}
&\Delta f(T)g(T)+f(T)\Delta g(T)-\mathcal{D}f(T)\mathcal{D}g(T)=\\
&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}F_R(s,T)p\mathcal{Q}_s(p)^{-1}dp_Jg(p)\\
&-\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}F_L(p,T)p\mathcal{Q}_s(p)^{-1}dp_Jg(p)\\
&-\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\bar{s}F_R(s,T)\mathcal{Q}_s(p)^{-1}dp_Jg(p)\\
&+\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\bar{s}F_L(p,T)\mathcal{Q}_s(p)^{-1}dp_Jg(p).
\end{aligned}$$

Since the functions $p\mapsto p\mathcal{Q}_s(p)^{-1}$, $p\mapsto\mathcal{Q}_s(p)^{-1}$ are intrinsic slice hyperholomorphic on $\bar{G}_1$, by the Cauchy integral formula, see Theorem 3.1.19 we have

$$\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}F_R(s,T)p\mathcal{Q}_s(p)^{-1}dp_Jg(p)=0,$$

$$\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\bar{s}F_R(s,T)\mathcal{Q}_s(p)^{-1}dp_Jg(p)=0.$$

Thus, we get

$$
\begin{aligned}
&\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) = \\
&-\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)p\mathcal{Q}_s(p)^{-1}dp_J g(p) \\
&+\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} \bar{s}F_L(p,T)\mathcal{Q}_s(p)^{-1}dp_J g(p) \\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} \left[\bar{s}F_L(p,T) - F_L(p,T)p\right]\mathcal{Q}_s(p)^{-1}dp_J g(p).
\end{aligned}
$$

By applying Lemma 9.4.1 with $B := F_L(p,T)$ and by the definition of the F-functional calculus we obtain

$$
\begin{aligned}
&\Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T) \\
&= \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J f(p)g(p) \\
&= \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J (fg)(p) \\
&= \Delta(fg)(T).
\end{aligned}
$$

□

Corollary 10.7.2. *Let $T \in \mathcal{BC}(X)$ and assume $g \in \mathcal{N}(\sigma_S(T))$ and $f \in \mathcal{SH}_R(\sigma_S(T))$ then we have*

$$\Delta(fg)(T) = \Delta f(T)g(T) + f(T)\Delta g(T) - \mathcal{D}f(T)\mathcal{D}g(T). \tag{10.45}$$

Remark 10.7.3. The product fg in Theorem 10.7.1 and Corollary 10.7.2 is respectively slice hyperholomorphic left or right slice hyperholomorphic.

Remark 10.7.4. The classical formula

$$\Delta(fg) = \Delta(f)\cdot g + f\cdot\Delta(g) + 2\langle\nabla f, \nabla g\rangle, \tag{10.46}$$

is true for any C^2 quaternionic valued functions and it inspires formula (10.44). However, formula (10.44) is true only for slice hyperholomorphic functions. Indeed its proof relies heavily on the slice hyperholomorphic Cauchy integral representation formula. Thus (10.44) is not applicable in the case when f and g are real valued functions and, in this sense, it is not a generalization of the formula (10.46).

Remark 10.7.5. Formula (10.44) is a general case of the well-known formula $\Delta(qg(q)) = q\Delta(g(q)) + 2\mathcal{D}(g(q))$. Indeed, it is enough to replace the operator T by q and to take $f(q) := q$ in formula (10.44).

CHAPTER 11

A polyanalytic functional calculus and its properties on the S-spectrum

11.1 Motivation

In the previous chapter we studied a possible splitting of the diagram (9.1) and have showed that between the set of slice hyperholomorphic functions and the set of axially monogenic functions lies the set of of axially harmonic functions. Moreover, by means of this splitting, we developed an harmonic functional calculus.
The main goal of this chapter is to understand another, different splitting of (9.1).

By rearranging the maps in the Fueter theorem it is possible to get the set of axially polyanalytic functions of order 2, i.e. functions in the kernel of $\mathcal{D}^2$. In this paper we study the splitting

$$\mathcal{O}(D) \xrightarrow{T_F} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D), \tag{11.1}$$

where $\mathcal{AP}_2(\Omega_D)$ is the set of axially polyanalytic functions of order 2. The goal of this chapter is to describe the central part of this diagram

$$
\begin{array}{ccccc}
\mathcal{SH}(\Omega_D) & & \mathcal{AP}_2(\Omega_D) & & \mathcal{AM}(\Omega_D) \\
\downarrow & & & & \\
\text{Slice Cauchy Formula} & \xrightarrow{\overline{\mathcal{D}}} & \mathcal{AP}_2 \text{ integral form} & \xrightarrow{\mathcal{D}} & \text{Fueter thm. in integral form} \\
\downarrow & & \downarrow & & \downarrow \\
\text{S}-\text{Functional calculus} & & \mathcal{P}_2-\text{functionalcalculus} & & F-\text{functional calculus}
\end{array}
$$

Axially polyanalytic functions play an important role in the study of elasticity problems, see [113, 114]. The theory of polyanalytic functions is also used to investigate problems in time-frequency analysis, see for example [3] and to study well-known Hilbert spaces, see for instance [2,21,133]. For more information about polyanalytic functions see [4, 27].

11.2 Functions spaces of axial type in the quaternionic setting

We start by recalling the definition of fine structure.

Definition 11.2.1 (Fine structure of slice hyperholomorphic spectral theory)**.** A fine structure of a slice hyperholomorphic spectral theory is the set of functions spaces and the associated functional calculi induced by a factorization of the operator Δ.

In the quaternionic case only two fine structures are possible. One of them is studied in the previous chapter, and the other one is the main topic of this chapter.
The first fine structure studied corresponds to the factorization $\Delta = \mathcal{D}\overline{\mathcal{D}}$. In that case, we have the following diagram

$$\mathcal{O}(D) \xrightarrow{T_F} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D), \tag{11.2}$$

where the $\mathcal{AH}(\Omega_D)$ is the set of axially harmonic functions and Ω_D is defined as in Theorem 11.2.2.
The aim of this chapter is to study the fine structure which corresponds to the other possible factorization of the Laplacian, $\Delta = \overline{\mathcal{D}}\mathcal{D}$. To this end, we need the following splitting of the Fueter theorem (see [85]).

Theorem 11.2.2. *Let $f_0(z) = \alpha(u, v) + i\beta(u, v)$ be a holomorphic function defined in a domain (open and connected) D in the upper-half complex plane and let*

$$\Omega_D = \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 \mid (q_0, |\underline{q}|) \in D\} \tag{11.3}$$

be the open set induced by D in $\mathbb{H}$. The map

$$f(q)=T_F(f_0):=\alpha(q_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(q_0,|\underline{q}|)$$

takes the holomorphic function $f_0(z)$ and gives the intrinsic slice hyperholomorphic function f induced by f_0. Then the function

$$\breve{f}^0(q):=\overline{\mathcal{D}}\left(\alpha(q_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(q_0,|\underline{q}|)\right),$$

is in the kernel of $\mathcal{D}^2$, i.e.

$$\mathcal{D}^2\breve{f}^0=0 \qquad \text{on} \quad \Omega_D.$$

Moreover,

$$\breve{f}(q)=\mathcal{D}\breve{f}^0(q),$$

is axially monogenic.

From the previous theorem we have the following diagram

$$\mathcal{O}(D)\xrightarrow{T_{F1}}\mathcal{SH}(\Omega_D)\xrightarrow{\overline{\mathcal{D}}}\mathcal{AP}_2(\Omega_D)\xrightarrow{\mathcal{D}}\mathcal{AM}(\Omega_D), \tag{11.4}$$

where $\mathcal{AP}_2(\Omega_D)$ is the set of axially polyanalytic functions of order 2. Now, we give a rigorous definition of this set. First of all we assume that an axial function is of the form

$$f(q)=\alpha(q_0,|\underline{q}|)+\frac{\underline{q}}{|\underline{q}|}\beta(q_0,|\underline{q}|), \tag{11.5}$$

where α and β satisfy the conditions (6.38), with $u:=q_0$ and $v:=|\underline{q}|$.

Definition 11.2.3 (Axially polyanalytic function of order 2)**.** Let $f:\Omega_D\subseteq\mathbb{H}\to\mathbb{H}$ be of axial type and of class $\mathcal{C}^3(\Omega_D)$, where the set Ω_D is defined in (11.3). Then the function

$$\breve{f}^0(q)=\overline{\mathcal{D}}f(q) \qquad \text{on} \quad \Omega_D$$

is called an axially polyanalytic function of order 2 if

$$\mathcal{D}^2\breve{f}^0(q)=0 \qquad \text{on} \quad \Omega_D.$$

It is possible to write a polyanalytic function as a sum of axially monogenic functions, see [27]. Specifically, we can write the so called polyanalytic decomposition as

$$\breve{f}^0(q)=\breve{f}_0(q)+q_0\breve{f}_1(q), \tag{11.6}$$

where the $\breve{f}_0(q)$ and $\breve{f}_1(q)$ are axially monogenic functions. As well as the monogenic functions satisfy a system of differential equations, called Vekua systems also the axially polyanalytic functions of order 2 satisfy a system of differential equations, but of order two.

Theorem 11.2.4. *Let U be an axially symmetric open set in $\mathbb{H}$, not intersecting the real line, and let $\breve{f}^0(q) = A(q_0, r) + \underline{\omega} B(q_0, r)$ be an axially polyanalytic function of order 2 on U, $r > 0$ and $\underline{\omega} \in \mathbb{S}$. Then the functions $A(q_0, r)$ and $B(q_0, r)$ satisfy the following system*

$$\begin{cases} \partial_{x_0}^2 A(q_0,r) - 2\partial_{x_0}\partial_r B(q_0,r) - \frac{4}{r}\partial_{x_0} B(q_0,r) - \partial_r^2 A(q_0,r) - \frac{2}{r}\partial_r A(q_0,r) = 0 \\ \partial_{x_0}^2 B(q_0,r) + 2\partial_{x_0}\partial_r A(q_0,r) - \partial_r^2 B(q_0,r) - 2\frac{r\partial_r B(q_0,r) - B(q_0,r)}{r^2} = 0. \end{cases}$$

Proof. If we consider a function of axial type (11.5), by [118] we know that

$$\mathcal{D}(f) := \left(\partial_{q_0} A(q_0,r) - \partial_r B(q_0,r) - \frac{2}{r} B(q_0,r)\right) + \underline{\omega}\left(\partial_{q_0} B(q_0,r) + \partial_r A(q_0,r)\right). \tag{11.7}$$

Now, we set

$$A'(q_0,r) := \partial_{q_0} A(q_0,r) - \partial_r B(q_0,r) - \frac{2}{r} B(q_0,r),$$
$$B'(q_0,r) := \partial_{q_0} B(q_0,r) + \partial_r A(q_0,r).$$

By applying another time (11.7) we get

$$\mathcal{D}^2 f(q) = \left(\partial_{q_0} A'(q_0,r) - \partial_r B'(q_0,r) - \frac{2}{r} B'(q_0,r)\right) + \underline{\omega}\left(\partial_{q_0} B'(q_0,r) + \partial_r A'(q_0,r)\right).$$

Now, we develop further the computations

$$\begin{aligned} \mathcal{D}^2 f(q) &= \Big(\partial_{q_0}^2 A(q_0,r) - \partial_{q_0}\partial_r B(q_0,r) - \frac{2}{r}\partial_{q_0} B(q_0,r) - \partial_{q_0}\partial_r B(q_0,r) - \partial_r^2 A(q_0,r). \\ &\quad -\frac{2}{r}\left(\partial_{q_0} B(q_0,r) + \partial_r A(q_0,r)\right)\Big) + \underline{\omega}\Big(\partial_{q_0}^2 B(q_0,r) + \partial_{q_0}\partial_r A(q_0,r) \\ &\quad +\partial_r\partial_{q_0} A(q_0,r) - \partial_r^2 B(q_0,r) - 2\partial_r\left(\frac{B(q_0,r)}{r}\right)\Big) \qquad (11.8) \\ &= \left(\partial_{q_0}^2 A(q_0,r) - 2\partial_{q_0}\partial_r B(q_0,r) - \frac{4}{r}\partial_{x_0} B(q_0,r) - \partial_r^2 A(q_0,r) - \frac{2}{r}\partial_r A(q_0,r)\right) \\ &\quad +\underline{\omega}\left(\partial_{q_0}^2 B(q_0,r) + 2\partial_{q_0}\partial_r A(q_0,r) - \partial_r^2 B(q_0,r) - 2\frac{r\partial_r B(q_0,r) - B(q_0,r)}{r^2}\right). \end{aligned}$$

If we consider the function f polyanalytic of order 2, i.e. $\mathcal{D}^2(f) = 0$, by (11.8) we get the statement. □

In conclusion, even if $\Delta = \mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D}$ the application of $\mathcal{D}$ or the operator $\overline{\mathcal{D}}$ to the set of slice hyperholomorphic functions gives arise to two completely different fine structures.

11.3 Integral representation of polyanalytic functions of order 2

In this section we show how to write a polyanalytic function of order 2 in integral form. Basically, we provide an integral transform that turns slice hyperholomorphic functions into axially polyanalytic functions of order 2. The crucial point to show the integral representation is to apply the operator $\overline{\mathcal{D}}$ to the second form of the slice hyperholomorphic Cauchy kernels.

Theorem 11.3.1. *Let $s, q \in \mathbb{H}$, be such that $s \notin [q]$ then*

$$\overline{\mathcal{D}}S_L^{-1}(s,q) = -F_L(s,q)s + q_0F_L(s,q) = \sum_{k=0}^{1} q_0^k F_L(s,q)(-1)^{k+1}s^{1-k}, \tag{11.9}$$

and

$$S_R^{-1}(s,q)\overline{\mathcal{D}} = -sF_R(s,q) + q_0F_R(s,q) = \sum_{k=0}^{1} q_0^k s^{1-k}F_R(s,q)(-1)^{k+1}. \tag{11.10}$$

Proof. We start by applying the derivative with respect to ∂_{q_0} to the left slice hyperholomorphic Cauchy kernel

$$\partial_{q_0}S_L^{-1}(s,q) = -\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_0}{2}F_L(s,q) - \frac{1}{2}F_L(s,q)s. \tag{11.11}$$

Now, we make the derivative with respect to ∂_{q_i},

$$\partial_{q_i}S_L^{-1}(s,q) = e_i\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_i}{2}F_L(s,q), \qquad i=1,2,3. \tag{11.12}$$

Formula (11.11) and (11.12) imply that

$$\begin{aligned}\overline{\mathcal{D}}S_L^{-1}(s,q) &= \left(\partial_{q_0} - \sum_{i=1}^{3} e_i\partial_{q_i}\right)S_L^{-1}(s,q) = 2\mathcal{Q}_{c,s}(q)^{-1} + \frac{q_0}{2}F_L(s,q)\\ &\quad -\frac{1}{2}F_L(s,q)s - \frac{\underline{q}}{2}F_L(s,q)\\ &= 2\mathcal{Q}_{c,s}(q)^{-1} + \frac{\bar{q}}{2}F_L(s,q) - \frac{1}{2}F_L(s,q)s.\end{aligned}$$

From the equality $F_L(s,q)s - qF_L(s,q) = -4\mathcal{Q}_{c,s}(q)^{-1}$ it follows the thesis. By similar computations we obtain formula (11.10). $\square$

Now, we study the regularities of $\overline{\mathcal{D}}S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)\overline{\mathcal{D}}$ both in s and in q.

Proposition 11.3.2. *Let $s,\ q \in \mathbb{H}$, be such that $s \notin [q]$. The function $\overline{\mathcal{D}}S_L^{-1}(s,q)$ is a right slice hyperholomorphic function in the variable s, while $S_R^{-1}(s,q)\overline{\mathcal{D}}$ is left slice hyperholomorphic in the variable s.*

Proof. By Theorem 11.3.1 we know that $\overline{\mathcal{D}}S_L^{-1}(s,q)$ is a sum of right slice hyperholomorphic functions in the variable s. Indeed $\mathcal{Q}_{c,s}(q)^{-1}$ is a right slice hyperholomorphic function as well as $\bar{q}F_L(s,q)$ and $F_L(s,q)s$. The left slice hyperholomorphicity of the function $S_R^{-1}(s,T)\overline{\mathcal{D}}$ follows by similar arguments. $\square$

Proposition 11.3.3. *Let $s,\ q \in \mathbb{H}$, be such that $s \notin [q]$. The function $\overline{\mathcal{D}}S_L^{-1}(s,q)$ is left polyanalytic of order 2 and $S_R^{-1}(s,q)\overline{\mathcal{D}}$ is right polyanalytic of order 2 with respect to the variable q.*

Proof. It follows from the fact that the function $F_L(s,q)$ is axially monogenic in the variable q and the Laplace operator is a real operator, thus it can commute with other operators. Therefore, we get

$$\mathcal{D}^2\left(\overline{\mathcal{D}}S_L^{-1}(s,q)\right) = \mathcal{D}\Delta S_L^{-1}(s,q) = \mathcal{D}F_L(s,q) = 0.$$

The right polyanalyticity of $S_R^{-1}(s,T)\overline{\mathcal{D}}$ follows similarly. $\square$

The expressions obtained in Theorem 11.3.1 can be considered a polyanalytic decomposition of $\overline{\mathcal{D}}S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)\overline{\mathcal{D}}$, respectively, see formula (11.6). Indeed the functions $-F_L(s,q)s$ and $F_L(s,q)$ are left axially monogenic in the variable q. Similarly, the functions $-sF_R(s,q)$ and $F_R(s,q)$ are right axially monogenic in the variable q.

Now, we have all what we need to write an axially polyanalytic function of order 2 as an integral formula. This will be fundamental to define the polyanalytic functional calculus of order 2 based on the S-spectrum.

Theorem 11.3.4 (Integral representation of axially polyanalytic functions of order 2)**.** *Let $W \subset \mathbb{H}$ be an open set. Let U be a slice Cauchy domain such that $\overline{U} \subset W$. Then for $J \in \mathbb{S}$ and $ds_J = ds(-J)$ we have*

1. *if $f \in \mathcal{SH}_L(W)$, then the function $\breve{f}^0(q) = \overline{\mathcal{D}} f(q)$ is polyanalytic of order 2 and it admits the following integral representation*
$$\breve{f}^0(q) = -\frac{1}{2\pi} \sum_{k=0}^{1} (-q_0)^k \int_{\partial(U \cap \mathbb{C}_J)} F_L(s,q) s^{1-k} \, ds_J \, f(s) \quad \forall q \in U; \tag{11.13}$$
2. *if $f \in \mathcal{SH}_R(W)$, then the function $\breve{f}^0(q) = f(q)\overline{\mathcal{D}}$ is polyanalytic of order 2 and it admits the following integral representation*
$$\breve{f}^0(q) = -\frac{1}{2\pi} \sum_{k=0}^{1} (-q_0)^k \int_{\partial(U \cap \mathbb{C}_J)} f(s) \, ds_J \, s^{1-k} F_R(s,q) \quad \forall q \in U. \tag{11.14}$$

The integrals depend neither on U nor on the imaginary unit $J \in U$.

Proof. We get the thesis by applying the conjugate Fueter operator $\overline{\mathcal{D}}$ to the Cauchy formulas, see Theorem 3.1.18. By Theorem 11.3.1 it follows (11.13) and (11.14). Finally, the function $\breve{f}^0(q)$ is polyanalytic of order 2 by Proposition 11.3.3. □

In this section we have described the second central row of the diagram (11.1). It is clear the reason of the lack of the arrow that connects the set of axially polyanalytic functions and their integral representation. Indeed, we obtain it by means of the slice Cauchy formula.

11.4 Series expansion of the kernel of the fine structure spaces

Now, we address the following.

Problem Is it possible to write a series expansion of $\overline{\mathcal{D}} S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)\overline{\mathcal{D}}$ in terms of q and $\bar{q}$?

In order to answer this question we recall the following series expansion of the slice hyperholomorphic Cauchy kernels. For q, $s \in \mathbb{H}$ with $|q| < |s|$ we have
$$S_L^{-1}(s,q) = \sum_{n=0}^{\infty} q^n s^{-1-n}, \qquad S_R^{-1}(s,q) = \sum_{n=0}^{\infty} s^{-1-n} q^n. \tag{11.15}$$

Therefore it is clear that in order to find the series expansions of $\overline{\mathcal{D}} S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)\overline{\mathcal{D}}$ it is fundamental to understand the action of the conjugate Fueter operator $\overline{\mathcal{D}}$ over the monomial q^n.

Lemma 11.4.1. *For $n \geq 1$ we have*

$$\overline{\mathcal{D}}q^n = 2\left(nq^{n-1} + \sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1}\right). \tag{11.16}$$

Moreover,

$$q^n\overline{\mathcal{D}} = \overline{\mathcal{D}}q^n. \tag{11.17}$$

Proof. By Lemma 10.3.3 we know that

$$\mathcal{D}q^n = (\partial_{q_0} + \partial_{\underline{q}})q^n = -2\sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1},$$

where $\partial_{\underline{q}} := \sum_{i=1}^{3} e_i\partial_{q_i}$. Then we have

$$\partial_{\underline{q}}q^n = -2\sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1} - nq^{n-1}. \tag{11.18}$$

Therefore

$$\overline{\mathcal{D}}q^n = \left(\partial_{q_0} - \partial_{\underline{q}}\right)q^n = 2\left(nq^{n-1} + \sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1}\right). \tag{11.19}$$

Finally formula (11.17) follows with similar computations. □

It is possible to write polynomials $\overline{\mathcal{D}}q^n$ in terms of the Clifford-Appell polynomials in the quaternionic setting, see [32] and formula (9.2) with $n = 3$. This family of axially monogenic homogeneous polynomials is defined, for any $\ell \geq 0$, as

$$Q_\ell(q, \bar{q}) = \frac{2}{(\ell+1)(\ell+2)}\sum_{j=0}^{\ell}(\ell - j + 1)q^{\ell-j}\bar{q}^j. \tag{11.20}$$

Proposition 11.4.2. *Let $n \geq 2$, then for $q \in \mathbb{H}$, we have*

$$\overline{\mathcal{D}}q^n = 2n\sum_{k=0}^{1} q_0^k(-1)^k(n+1-2k)Q_{n-1-k}(q, \bar{q}). \tag{11.21}$$

Proof. We write

$$\overline{\mathcal{D}}q^n = \left(\overline{\mathcal{D}}q^n - q_0\Delta q^n\right) + q_0\Delta q^n = g_0(q) + g_1(q), \tag{11.22}$$

and we consider $g_0(q)$. From the fact that

$$\Delta q^n = -4\sum_{k=1}^{n-1}(n-k)q^{n-k-1}\bar{q}^{k-1}, \tag{11.23}$$

see Corollary 9.2.3, with $n = 3$, and by Lemma 11.3.1 we can write

$$g_0(q) = \overline{\mathcal{D}}q^n - q_0\Delta q^n = 2nq^{n-1} + 2\sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1} + 4q_0\sum_{k=1}^{n-1}(n-k)q^{n-k-1}\bar{q}^{k-1}.$$

Since $2q_0 = q + \bar{q}$ we obtain

$$\begin{aligned} g_0(q) &= 2\left(nq^{n-1} + \sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1} + \sum_{k=1}^{n-1}(n-k)q^{n-k}\bar{q}^{k-1} + \sum_{k=1}^{n-1}(n-k)q^{n-k-1}\bar{q}^{k}\right) \\ &= 2\left(\sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1} + \sum_{k=1}^{n-1}(n-k)q^{n-k}\bar{q}^{k-1} + \sum_{k=0}^{n-1}(n-k)q^{n-k-1}\bar{q}^{k}\right) \\ &= 2\left(\sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1} + \sum_{k=1}^{n}(n-k)q^{n-k}\bar{q}^{k-1} + \sum_{k=1}^{n}(n-k+1)q^{n-k}\bar{q}^{k-1}\right) \\ &= 4\sum_{k=1}^{n}(n-k+1)q^{n-k}\bar{q}^{k-1}. \end{aligned}$$

By formula (11.23) we get

$$g_0(q) = -\Delta(q^{n+1}).$$

This implies that

$$\overline{\mathcal{D}}q^n = -\Delta(q^{n+1}) + q_0\Delta q^n. \tag{11.24}$$

By [77, Rem. 3.9] we know that for $n \geq 2$ we have

$$\Delta(q^n) = -2n(n-1)Q_{n-2}(q,\bar{q}) \tag{11.25}$$

where the homogenous polynomials $Q_n(q,\bar{q})$ are defined in (11.20). Finally by combining formula (11.24) and formula (11.25) we get

$$\begin{aligned} \overline{\mathcal{D}}q^n &= 2n\left[(n+1)Q_{n-1}(q,\bar{q}) - 2q_0(n-1)Q_{n-2}(q.\bar{q})\right] \\ &= 2n\sum_{k=0}^{1} q_0^k(-1)^k(n+1-2k)Q_{n-1-k}(q,\bar{q}). \end{aligned}$$

□

Formula (11.21) can be considered as the polyanalytic decomposition of the polynomials $\overline{\mathcal{D}}q^n$ since the functions $(n+1)Q_{n-1}(q,\bar{q})$ and $(n-1)Q_{n-2}(q.\bar{q})$ are left and right axially monogenic.

Remark 11.4.3. The polynomials $\overline{\mathcal{D}}q^n$ were also obtained in [69], by means of other tools, see [18].

Now, we have all the instruments to introduce the following.

Definition 11.4.4. Let s, $q \in \mathbb{H}$, we define the left $\overline{\mathcal{D}}$-kernel series as

$$2\sum_{n=1}^{\infty}\left(nq^{n-1}+\sum_{k=1}^{n}q^{n-k}\bar{q}^{k-1}\right)s^{-1-n}, \tag{11.26}$$

and the right $\overline{\mathcal{D}}$-kernel series as

$$2\sum_{n=1}^{\infty}s^{-1-n}\left(nq^{n-1}+\sum_{k=1}^{n}q^{n-k}\bar{q}^{k-1}\right). \tag{11.27}$$

Proposition 11.4.5. *Let s, $q \in \mathbb{H}$ with $|q|<|s|$, the left and right $\overline{\mathcal{D}}$-kernel series are convergent.*

Proof. We show only the convergence of the left $\overline{\mathcal{D}}$-kernel series. The convergence of the right one follows by similar computations.
In order to show the convergence it is enough to prove that the series of moduli is convergent, i.e.

$$4\sum_{n=1}^{+\infty}n|q|^{n-1}s^{-1-n}<+\infty.$$

The series converges by the ratio test, indeed

$$\lim_{n\to+\infty}\frac{(n+1)|q|^{n}|s|^{-2-n}}{n|q|^{n-1}|s|^{-1-n}}=|q||s|^{-1}<1. \tag{11.28}$$

□

The following result contains the solution to the problem stated at the beginning of this section.

Lemma 11.4.6. *For q, $s \in \mathbb{H}$ such that $|q|<|s|$, we have*

$$\begin{aligned}\sum_{k=0}^{1}q_0^kF_L(s,q)(-1)^{k+1}s^{1-k} &= 2\sum_{n=1}^{\infty}\left(nq^{n-1}+\sum_{j=1}^{n}q^{n-j}\bar{q}^{j-1}\right)s^{-1-n}\\ &= 2\sum_{n=2}^{\infty}\sum_{j=0}^{1}nq_0^j(-1)^j(n+1-2j)Q_{n-1-j}(q,\bar{q})s^{-1-n},\end{aligned}$$

and

$$\sum_{k=0}^{1} s^{1-k}(-1)^{k+1}F_R(s,q)q_0^k = 2\sum_{n=1}^{\infty} s^{-1-n}\left(nq^{n-1}+\sum_{k=1}^{n} q^{n-k}\bar{q}^{k-1}\right) \tag{11.29}$$
$$= 2\sum_{n=2}^{\infty}\sum_{j=0}^{1} ns^{-1-n}q_0^j(-1)^j(n+1-2j)Q_{n-1-j}(q,\bar{q}).$$

Proof. By formulas (11.15) we know that we can expand the left Cauchy kernel as

$$S_L^{-1}(s,q)=\sum_{n=0}^{\infty} q^n s^{-1-n}.$$

Thus by Proposition 11.4.5 (which allows to exchange the operator $\overline{\mathcal{D}}$ with the sum) and by Theorem 11.3.1 we get

$$\begin{aligned}\sum_{k=0}^{1} q_0^k F_L(s,q)(-1)^{k+1}s^{1-k} &= \overline{\mathcal{D}}S_L^{-1}(s,q)\\ &= \sum_{n=0}^{\infty}(\overline{\mathcal{D}}q^n)s^{-1-n}\\ &= 2\left(\sum_{n=1}^{\infty} nq^{n-1}+\sum_{j=1}^{n} q^{n-j}\bar{q}^{j-1}\right)s^{-1-n}.\end{aligned}$$

The second equality of the statement follows by applying Proposition 11.4.2 in the last equality of the previous computations.
By similar arguments it is possible to prove the equalities (11.29).

□

Basically, we have given two possible answers to the initial problem. Indeed, we get two possible expansions of $\overline{\mathcal{D}}S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)\overline{\mathcal{D}}$, respectively. These will be fundamental in the next section.

11.5 The polyanalytic functional calculus of order 2 on the S-spectrum and its properties

In this section we will analyse the central third row of the diagram (11.1). From the shape of the slice hyperholomorphic Cauchy kernel, that we use to prove the integral representation (see Theorem 11.3.4), we have to restricted to the case of commuting operators.

Definition 11.5.1. Let $T = T_0 + \sum_{i=1}^3 e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$, we formally define the right $\overline{\mathcal{D}}$-kernel operator as

$$2\sum_{n=1}^{\infty}\left(nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1}\right) s^{-1-n}$$

and the left $\overline{\mathcal{D}}$-kernel operator as

$$2\sum_{n=1}^{\infty} s^{-1-n}\left(nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1}\right).$$

Now, we recall the expansion in series of F-resolvent operators in terms of T and $\bar{T}$, see Theorem 9.2.10 with $n = 3$. Let $T = T_0 + \sum_{i=1}^3 e_i T_i \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$ with $\|T\| < |s|$ we have

$$F_L(s,T) = -4\sum_{n=2}^{\infty}\sum_{\ell=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k-1}s^{-1-n} \tag{11.30}$$

$$F_R(s,T) = -4\sum_{n=2}^{\infty}\sum_{\ell=1}^{n-1}(n-k)s^{-1-n}T^{n-k-1}\bar{T}^{k-1}.$$

This is fundamental for the following result.

Proposition 11.5.2. *Let $T = T_0 + \sum_{i=1}^3 e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$ and $\|T\| < |s|$, the series in Definition 11.5.1 converges. Moreover, we have*

$$\sum_{j=0}^{1} T_0^j (-1)^{j+1} F_L(s,T) s^{1-j} = 2\sum_{n=1}^{\infty}\left(nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1}\right) s^{-1-n} \tag{11.31}$$

and

$$\sum_{j=0}^{1} s^{1-j}(-1)^{j+1} F_R(s,T) T_0^j = 2\sum_{n=1}^{\infty} s^{-1-n}\left(nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1}\right). \tag{11.32}$$

Proof. First of all, we show the convergence of the series. It is sufficient to prove that the series of the operator norm:

$$4\sum_{n=1}^{\infty} n\|T\|^{n-1}s^{-1-n}.$$

is convergent. This follows from computations similar to those in the proof of Proposition 11.4.5.
Now we prove equality (11.31). By formulas (11.30) we know how to expand in series $F_L(s,T)$, thus we have

$$\sum_{j=0}^{1} T_0^j(-1)^{j+1}F_L(s,T)s^{1-j} = 4\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k-1}s^{-n} -4T_0\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k-1}s^{-1-n}.$$

Now, to show equality (11.31) it is enough to prove the following equality

$$\begin{aligned}
&4T_0\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k-1}s^{-1-n}\\
&= 4\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k-1}s^{-n}\\
&-2\sum_{n=1}^{\infty}nT^{n-1}s^{-1-n} - 2\sum_{n=1}^{\infty}\sum_{k=1}^{n}T^{n-k}\bar{T}^{k-1}s^{-1-n}.
\end{aligned}$$

At this point, we are going to manipulate the series in the left hand side of the previous equality in order to obtain the terms in the right hand side. By using the relation: $2T_0 = T+\bar{T}$, we obtain

$$\begin{aligned}
&4T_0\textstyle\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k-1}s^{-1-n}\\
&= 2\textstyle\sum_{n=2}^{\infty}\sum_{k=1}^{n}(n-k)T^{n-k}\bar{T}^{k-1}s^{-1-n} + 2\sum_{n=2}^{\infty}\sum_{k=1}^{n-1}(n-k)T^{n-k-1}\bar{T}^{k}s^{-1-n}\\
&= 2\textstyle\sum_{\ell=3}^{\infty}\sum_{k=1}^{\ell-1}(\ell-k-1)T^{\ell-1-k}\bar{T}^{k-1}s^{-\ell} + 2\sum_{\ell=3}^{\infty}\sum_{\alpha=2}^{\ell-1}(\ell-\alpha)T^{\ell-\alpha-1}\bar{T}^{\alpha-1}s^{-\ell}\\
&= 2\textstyle\sum_{\ell=3}^{\infty}\sum_{k=1}^{\ell-1}(\ell-k)T^{\ell-1-k}\bar{T}^{k-1}s^{-\ell} - 2\sum_{\ell=3}^{\infty}\sum_{k=1}^{\ell-1}T^{\ell-1-k}\bar{T}^{k-1}s^{-\ell}\\
&+2\textstyle\sum_{\ell=3}^{\infty}\sum_{\alpha=1}^{\ell-1}(\ell-\alpha)T^{\ell-\alpha-1}\bar{T}^{\alpha-1}s^{-\ell} - 2\sum_{\ell=3}^{\infty}(\ell-1)T^{\ell-2}s^{-\ell},
\end{aligned}$$

where in the second equality we change indexes in the first sum with $\ell = n+1$, as well as, in the second sum with $\ell = n+1$ and $k = \alpha - 1$. Now, starting the first and the third series from $\ell = 2$ we get

$$
\begin{aligned}
&4T_0 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (n-k) T^{n-k-1} \bar{T}^{k-1} s^{-1-n} \\
&= 2 \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} (\ell-k) T^{\ell-1-k} \bar{T}^{k-1} s^{-\ell} - 2s^{-2} - 2 \sum_{\ell=3}^{\infty} \sum_{k=1}^{\ell-1} T^{\ell-1-k} \bar{T}^{k-1} s^{-\ell} \\
&+ 2 \sum_{\ell=2}^{\infty} \sum_{\alpha=1}^{\ell-1} (\ell-\alpha) T^{\ell-\alpha-1} \bar{T}^{\alpha-1} s^{-\ell} - 2s^{-2} - 2 \sum_{\ell=3}^{\infty} (\ell-1) T^{\ell-2} s^{-\ell} \\
&= 4 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (n-k) T^{nl-k-1} \bar{T}^{k-1} s^{-n} - 2 \sum_{\ell=2}^{\infty} (\ell-1) T^{\ell-2} s^{-\ell} \\
&- 2 \sum_{\ell=2}^{\infty} \sum_{k=1}^{\ell-1} T^{\ell-1-k} \bar{T}^{k-1} s^{-\ell} \\
&= 4 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} (n-k) T^{n-k-1} \bar{T}^{k-1} s^{n} - 2 \sum_{\ell=1}^{\infty} n T^{n-1} s^{-n-1} \\
&- 2 \sum_{n=1}^{\infty} \sum_{k=1}^{n} T^{n-k} \bar{T}^{k-1} s^{-n-1},
\end{aligned}
$$

where the last equality is obtained by the change of indexes in the second and in the third series with $n = \ell - 1$. By similar arguments it is possible to prove (11.32). $\square$

Corollary 11.5.3. *Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in \mathcal{BC}(X)$, $s \in \mathbb{H}$ and $\|T\| < |s|$, then*

$$
\sum_{j=0}^{1} T_0^j (-1)^{j+1} F_L(s,T) s^{1-j} = 2n \sum_{n=1}^{\infty} \left(\sum_{k=0}^{1} T_0^k (-1)^k (n+1-2k) Q_{n-1-k}(T, \bar{T}) \right) s^{-1-n}
$$

and

$$
\sum_{j=0}^{1} s^{1-j} (-1)^{j+1} F_R(s,T) T_0^j = \sum_{n=1}^{\infty} s^{-1-n} \left(\sum_{k=0}^{1} T_0^k (-1)^k (n+1-2k) Q_{n-1-k}(T, \bar{T}) \right).
$$

Proof. This result follows by Proposition 11.5.2 and from the fact that we can write the right $\overline{\mathcal{D}}$-kernel operator in terms of $Q_\ell(q,\bar{q}) = \frac{2}{(\ell+1)(\ell+2)} \sum_{j=0}^{\ell} (\ell - j + 1) T^{\ell-j} \bar{T}^j$, see Proposition 11.21. $\square$

Now, we can give the following

Definition 11.5.4 ($\mathcal{P}_2$-resolvent operators)**.** Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in \mathcal{BC}(X)$. For $s \in \rho_S(T)$, we define the left $\mathcal{P}_2$-resolvent operator as

$$\mathcal{P}_2^L(s,T) = \sum_{j=0}^{1} T_0^j (-1)^{j+1} F_L(s,T) s^{1-j},$$

and the right $\mathcal{P}_2$-resolvent operator as

$$\mathcal{P}_2^R(s,T) = \sum_{j=0}^{1} s^{1-j} (-1)^{j+1} F_R(s,T) T_0^j.$$

Lemma 11.5.5. *Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in \mathcal{BC}(X)$. Then*

- *the left $\mathcal{P}_2$-resolvent operator is a $\mathcal{B}(X)$-valued right slice hyperholomorphic function of the variable s in $\rho_S(T)$;*
- *the right $\mathcal{P}_2$-resolvent operator is a $\mathcal{B}(X)$-valued left slice hyperholomorphic function of the variable s in $\rho_S(T)$.*

Proof. It follows by similar arguments of Proposition 11.3.2. □

Definition 11.5.6 (Polyanalytic functional calculus of order 2 on the S-spectrum)**.** Let $T = T_0 + \sum_{i=1}^{3} e_i T_i \in \mathcal{BC}(X)$ and set $ds_J = ds(-J)$ for $J \in \mathbb{S}$. For every function $\breve{f}^0 = \overline{\mathcal{D}} f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$\breve{f}^0(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} \mathcal{P}_2^L(s,T)\, ds_J\, f(s), \tag{11.33}$$

where $\overline{U} \subset \mathrm{dom}(f)$ and $J \in \mathbb{S}$ is an arbitrary imaginary unit.
For every $\breve{f}^0 = f\overline{\mathcal{D}}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\breve{f}^0(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s)\, ds_J\, \mathcal{P}_2^R(s,T), \tag{11.34}$$

where U and J are as above.

Theorem 11.5.7. *The polyanalytic functional calculus of order 2 on the S-spectrum is well defined, i.e., the integrals* (11.33) *and* (11.34) *depend neither on the imaginary unit $J \in \mathbb{S}$ nor on the slice Cauchy domain U.*

Proof. Here we show only the case $\breve{f}^\circ = \overline{\mathcal{D}}f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, since the other one follows by analogous arguments.
Since $\mathcal{P}_2^L(s,T)$ is a $\mathcal{B}(X)$-valued right slice hyperholomorphic function in s and f is left slice hyperholomorphic, the independence from the set U follows by the Cauchy integral formula, see Theorem 3.1.18 and Theorem 3.1.19.
Now, we want to show the independence from the imaginary unit. Let us consider two imaginary units $J, I \in \mathbb{S}$ with $J \neq I$ and two bounded slice Cauchy domains U_q, U_s with $\sigma_S(T) \subset U_q$, $\overline{U}_q \subset U_s$ and $\overline{U}_s \subset dom(f)$. Then every $s \in \partial(U_s \cap \mathbb{C}_J)$ belongs to the unbounded slice Cauchy domain $\mathbb{H} \setminus U_q$. Recall that $\mathcal{P}_2^L(q,T)$ is right slice hyperholomorphic on $\rho_S(T)$, also at infinity, since $\lim_{q\to+\infty} \mathcal{P}_2^L(q,T) = 0$. Thus the Cauchy formula implies

$$\begin{aligned}\mathcal{P}_2^L(s,T) &= \frac{1}{2\pi}\int_{\partial((\mathbb{H}\setminus U_q)\cap\mathbb{C}_I)} \mathcal{P}_2^L(q,T)dq_I S_R^{-1}(q,s)\\ &= \frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_I)} \mathcal{P}_2^L(q,T)dq_I S_L^{-1}(s,q).\end{aligned}$$

The last equality is due to the fact that $\partial\left((\mathbb{H}\setminus U_q)\cap\mathbb{C}_I\right) = -\partial(U_q\cap\mathbb{C}_I)$ and $S_R^{-1}(q,s) = -S_L^{-1}(s,q)$. By Definition 10.4.7 and (10.17) we get

$$\begin{aligned}\breve{f}^\circ(T) &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)ds_J f(s)\\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)} \left(\frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_I)} \mathcal{P}_2^L(q,T)dq_I S_L^{-1}(s,q)\right) ds_J f(s).\end{aligned}$$

Due to Fubini's theorem we can exchange the order of integration and by the Cauchy formula we obtain

$$\begin{aligned}\breve{f}^\circ(T) &= \frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_I)} \mathcal{P}_2^L(q,T)dq_I \left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)} S_L^{-1}(s,q)ds_J f(s)\right)\\ &= \frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_I)} \mathcal{P}_2^L(q,T)dq_I f(q).\end{aligned}$$

This proves the statement. □

The following result is also important to have a well posed functional calculus.

Theorem 11.5.8. *Let U be a slice Cauchy domain. If $f, g \in \mathcal{SH}_L(U)$ (resp. $f, g \in \mathcal{SH}_R(U)$) and $\overline{\mathcal{D}}f = \overline{\mathcal{D}}g$ (resp. $f\overline{\mathcal{D}} = g\overline{\mathcal{D}}$) then for any $T \in \mathcal{BC}(X)$*

such that $T = T_0e_0 + T_1e_1 + T_2e_2$*, and assuming that the operators* T_ℓ*,* $\ell = 0, 1, 2$*, have real spectrum, we have*

$$\breve{f}^0(T) = \breve{g}^0(T).$$

In order to prove the previous theorem we need some auxiliary results. First of all, we have to study the following sets

$$(\ker \overline{\mathcal{D}})_{\mathcal{SH}_L(U)} := \{f \in \mathcal{SH}_L(U) : \overline{\mathcal{D}}(f) = 0\}$$

and

$$(\ker \overline{\mathcal{D}})_{\mathcal{SH}_R(U)} := \{f \in \mathcal{SH}_R(U) : (f)\overline{\mathcal{D}} = 0\}.$$

It is necessary to study these sets because in the hypothesis of Theorem 11.5.8 we have $\overline{\mathcal{D}}(f - g) = 0$ (resp. $(f - g)\overline{\mathcal{D}} = 0$).

Theorem 11.5.9. *Let* U *be a connected slice Cauchy domain of* $\mathbb{H}$*, then*

$$\begin{aligned}(\ker \overline{\mathcal{D}})_{\mathcal{SH}_L(U)} &= \{f \in \mathcal{SH}_L(U) : f \equiv \alpha \quad \textit{for some } \alpha \in \mathbb{H}\} \\ &= \{f \in \mathcal{SH}_R(U) : f \equiv \alpha \quad \textit{for some } \alpha \in \mathbb{H}\} = (\ker \overline{\mathcal{D}})_{\mathcal{SH}_R(U)}.\end{aligned}$$

Proof. We prove the result in the case $f \in \mathcal{SH}_L(U)$ since the case $f \in \mathcal{SH}_R(U)$ follows by similar arguments. We proceed by double inclusion. The fact that

$$(\ker \overline{\mathcal{D}})_{\mathcal{SH}_L(U)} \supseteq \{f \in \mathcal{SH}_L(U) : f \equiv \alpha \quad \text{for some } \alpha \in \mathbb{H}\}$$

is obvious. The other inclusion can be proved observing that if $f \in (\ker \overline{\mathcal{D}})_{\mathcal{SH}_L(U)}$, after a change of variable if needed, there exists $r > 0$ such that the function f can be expanded in a convergent series at the origin

$$f(q) = \sum_{k=0}^{\infty} q^k \alpha_k \quad \text{for } \{\alpha_k\}_{k\in\mathbb{N}_0} \subset \mathbb{H} \text{ and for any } q \in B_r(0)$$

where $B_r(0)$ is the ball centred at 0 and of radius r. By Lemma 11.4.1, we have

$$\begin{aligned} 0 &= \overline{\mathcal{D}} f(q) \equiv \sum_{k=1}^{\infty} \overline{\mathcal{D}}(q^k)\alpha_k \qquad (11.35) \\ &= 2\sum_{k=1}^{\infty}\left(kq^{k-1} + \sum_{s=1}^{k} q^{k-s}\overline{q}^{s-1}\right)\alpha_k, \quad \forall q \in B_r(0).\end{aligned}$$

If we restrict the previous series in (11.36) in a neighbourhood Ω of 0 of the real line we get

$$0=\sum_{k=1}^{\infty} q_0^{k-1}\alpha_k \quad \forall\, q_0\in\Omega$$

and this implies

$$\alpha_k=0, \quad \forall k\geq 1.$$

Thus $f(q)\equiv\alpha_0$ in $B_r(0)$ and since U is connected $f(q)\equiv\alpha_0$ for any $q\in U$. □

To define a monogenic functional McIntosh and collaborators, see [99, 101, 108, 112], had as hypothesis that the component T_0 of the operator $T=T_0+T_1e_1+T_2e_2+T_3e_3$ is zero. However, it is possible to set zero a different component of the operator T. In a polyanalytic functional calculus is not convenient to have $T_0=0$, due to the left and right structure of the $\overline{\mathcal{D}}$-kernel (see Definition 11.5.1). For this reason, in the present work we impose the last component of the operator T to be zero, i.e., $T_3=0$.

Lemma 11.5.10. *Let $T\in\mathcal{BC}(X)$ be such that $T=T_0e_0+T_1e_1+T_2e_2$, and assume that the operators T_ℓ, $\ell=0,1,2$, have real spectrum. Let G be a bounded slice Cauchy domain such that $(\partial G)\cap\sigma_S(T)=\emptyset$. For every $J\in\mathbb{S}$ we have*

$$\int_{\partial(G\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_J=0 \quad \textit{and} \quad \int_{\partial(G\cap\mathbb{C}_J)} ds_J\mathcal{P}_2^R(s,T)=0. \quad (11.36)$$

Proof. We prove only the first equality of (11.36), since the other one follows by similar computations. Since $\Delta(1)=0$ and $\Delta(q)=0$, by Theorem 7.4.6 we have

$$\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,q)ds_J=\Delta(1)=0, \quad (11.37)$$

and

$$\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,q)ds_Js=\Delta(q)=0, \quad (11.38)$$

for all $q\notin\partial G$ and $J\in\mathbb{S}$. By the monogenic functional calculus of McIntosh and collaborators we have

$$F_L(s,T)=\int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega F_L(s,\omega),$$

where $\mathbf{D}\omega$ is a suitable differential form, the open set Ω contains the left spectrum of T and $G(\omega, T)$ is the Fueter resolvent operator. By Definition 11.5.1 we have

$$\begin{aligned}
&\int_{\partial(G\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)ds_J = \int_{\partial(G\cap\mathbb{C}_J)} (-F_L(s,T)s + T_0 F_L(s,T))\, ds_J \\
&= -\Bigg(\int_{\partial(G\cap\mathbb{C}_J)} \int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega F_L(s,\omega)s\, ds_J \\
&\quad -T_0 \int_{\partial(G\cap\mathbb{C}_J)} \int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega F_L(s,\omega) ds_J\Bigg) \\
&= -\frac{1}{4}\Bigg(\int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega \left(\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,\omega)ds_J s\right) \\
&\quad -T_0 \int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega \left(\int_{\partial(G\cap\mathbb{C}_J)} F_L(s,\omega)ds_J\right)\Bigg) \\
&= 0,
\end{aligned}$$

where the second equality is a consequence of the Fubini's Theorem and the last equality is a consequence of formulas (11.37) and (11.38). □

Proof of Theorem 11.5.8. We prove the theorem when $f, g \in \mathcal{SH}_L(U)$. The case of $f, g \in \mathcal{SH}_R(U)$ follows by similar arguments. We divide the proof in two cases.

$\boxed{U \text{ is connected}}$

By definition of the P_2-functional calculus on the S-spectrum, see Definition 10.4.7, we have

$$\breve{f}^0(T) - \breve{g}^0(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)ds_J(f(s)-g(s)).$$

Since $\mathcal{P}_2^L(s,T)$ is slice hyperholomorphic in the variable s by Theorem 3.1.18, we can change the domain of integration to $B_r(0)\cap\mathbb{C}_J$ for some $r > 0$ with $\|T\| < r$. Moreover, by hypothesis we have that $f(s) - g(s) \in$

$(\ker \overline{\mathcal{D}})_{\mathcal{SH}_L(\Omega)}$, thus by Theorem 11.5.9 and Proposition 10.4.3 we get

$$\begin{aligned}\breve{f}^0(T) - \breve{g}^0(T) &= \frac{1}{2\pi} \int_{\partial(B_r(0)\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T) ds_J (f(s) - g(s)) \\ &= \frac{1}{2\pi} \int_{\partial(B_r(0)\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T) ds_J \alpha \\ &= \frac{1}{\pi} \sum_{m=1}^{\infty} \left(mT^{m-1} + \sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1} \right) \\ &\quad \cdot \int_{\partial(B_r(0)\cap\mathbb{C}_J)} s^{-1-m} ds_J \alpha = 0.\end{aligned}$$

$\boxed{U \text{ is not connected}}$

In this case we write the set U in the following way $U = \cup_{\ell=1}^n U_\ell$ where the U_ℓ are the connected components of U. Hence, there exist constants $\alpha_\ell \in \mathbb{H}$ for $\ell = 1, \dots, n$, such that $f(s) - g(s) = \sum_{\ell=1}^n \chi_{U_\ell}(s)\alpha_\ell$. Thus we can write

$$\breve{f}^\circ(T) - \breve{g}^\circ(T) = \sum_{\ell=1}^{n} \frac{1}{2\pi} \int_{\partial(U_\ell\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T) ds_J \alpha_\ell.$$

Finally, by Lemma 11.5.10, we get $\breve{f}^\circ(T) - \breve{g}^\circ(T) = 0$. □

Remark 11.5.11. If the set U in Theorem 11.5.8 is connected we can show the result for operators of the following form $T = T_0e_0 + T_1e_1 + T_2e_2 + T_3e_3$. However, in order to have a well defined functional calculus also for not connected sets, as it happens for the monogenic functional calculus of McIntosh, we need to annihilate a component of the operator T.

We conclude this section by proving some algebraic properties of the P_2-functional calculus.

Proposition 11.5.12. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_0e_0 + T_1e_1 + T_2e_2$, and assume that the operators T_ℓ, $\ell = 0, 1, 2$, have real spectrum.*

- *If $\breve{f}^\circ = \overline{\mathcal{D}}f$ and $\breve{f}^\circ = \overline{\mathcal{D}}g$ with $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*
$$(\breve{f}^\circ a + \breve{g}^\circ)(T) = \breve{f}^\circ(T)a + \breve{g}^\circ(T).$$
- *If $\breve{f}^\circ = f\overline{\mathcal{D}}$ and $\breve{g}^\circ = g\overline{\mathcal{D}}$ with $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*
$$(a\breve{f}^\circ + \breve{g}^\circ)(T) = a\breve{f}^\circ(T) + \breve{g}^\circ(T).$$

Proof. The above identities follow immediately from the linearity of the integrals in (10.15), resp. (8.15). □

Proposition 11.5.13. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_0e_0 + T_1e_1 + T_2e_2$, and assume that the operators T_ℓ, $\ell = 0, 1, 2$, have real spectrum.*

- *If $\breve{f}^\circ = \overline{\mathcal{D}}f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$ and assume that $f(q) = \sum_{m=0}^{\infty} q^m a_m$ with $a_m \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$\breve{f}^\circ(T) = \sum_{m=1}^{\infty} \left(mT^{m-1} + \sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1} \right) a_m.$$

- *If $\breve{f}^\circ = f\mathcal{D}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$ and assume that $f(q) = \sum_{m=0}^{\infty} a_m q^m$ with $a_m \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$\breve{f}^\circ(T) = \sum_{m=1}^{\infty} a_m \left(mT^{m-1} + \sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1} \right).$$

Proof. We prove the first assertion since the second one can be proven by following similar arguments. We choose an imaginary unit $J \in \mathbb{S}$ and a radius $0 < R < r$ such that $\sigma_S(T) \subset B_R(0)$. Then the series expansion of f converges uniformly on $\partial(B_R(0) \cap \mathbb{C}_J)$, and so

$$\begin{aligned} \breve{f}^0(T) &= \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)\, ds_J \sum_{\ell=0}^{\infty} s^\ell a_\ell \\ &= \frac{1}{2\pi} \sum_{\ell=0}^{\infty} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)\, ds_J s^\ell a_\ell. \end{aligned}$$

By Proposition 10.4.3, we further obtain

$$\begin{aligned} \breve{f}^0(T) &= \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_J)} \sum_{m=1}^{\infty} \left(mT^{m-1} + \sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1} \right) s^{-1-m}\, ds_J \sum_{\ell=0}^{\infty} s^\ell a_\ell \\ &= \frac{1}{2\pi} \sum_{m=1}^{\infty} \sum_{\ell=0}^{\infty} \left(mT^{m-1} + \sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1} \right) \int_{\partial(B_R(0)\cap\mathbb{C}_J)} s^{-1-m+\ell}\, ds_J\, a_\ell \\ &= \sum_{m=1}^{\infty} \left(mT^{m-1} + \sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1} \right) a_m. \end{aligned}$$

The last equality is due to the fact that $\int_{\partial(B_R(0)\cap\mathbb{C}_J)} s^{-1-m+\ell}\, ds_J$ is equal to 2π if $\ell = m$, and 0 otherwise. $\square$

Theorem 11.5.14. *Let $T \in \mathcal{BC}(X)$. Let $m \in \mathbb{N}_0$, and let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain with $\sigma_S(T) \subset U$. For every $J \in \mathbb{S}$ we have*

$$P_m^2(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)ds_J s^{m+1}, \tag{11.39}$$

where

$$P_m^2(T) := (m+1)T^m + \sum_{k=0}^{m} T^{m-k}\bar{T}^k.$$

Proof. We start by considering U to be the ball $B_r(0)$ with $\|T\| < r$. By Proposition 10.4.3 We know that we can expand the left $\overline{\mathcal{D}}$-kernel operator as

$$\mathcal{P}_2^L(s,T) = \sum_{n=1}^{+\infty}\left(nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1}\right)s^{-1-n}$$

for every $s \in \partial B_r(0)$. Since the series converges on $\partial B_r(0)$, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)ds_J s^{m+1} &= \frac{1}{2\pi}\sum_{n=1}^{+\infty}\left(nT^{n-1} + \sum_{k=1}^{n} T^{n-k}\bar{T}^{k-1}\right)\cdot \\ &\quad \cdot \int_{\partial(B_r(0)\cap\mathbb{C}_J)} s^{-n+m} ds_J \\ &= (m+1)T^m + \sum_{k=1}^{m+1} T^{m+1-k}\bar{T}^{k-1} \\ &= (m+1)T^m + \sum_{k=0}^{m} T^{m-k}\overline{T}^k = P_m^2(T),\end{aligned}$$

where we have used

$$\int_{\partial(B_r(0)\cap\mathbb{C}_J)} s^{-n+m} ds_J = \begin{cases} 0 & \text{if } n \neq m+1 \\ 2\pi & \text{if } n = m+1. \end{cases}$$

This proves the result for the case $U = B_r(0)$. Now we get the result for an arbitrary bounded Cauchy domain U that contains $\sigma_S(T)$. The operator $\mathcal{P}_2^L(s,T)$ is right slice hyperholomorphic and the monomial s^{m+1} is left slice hyperholomorphic on the bounded slice Cauchy domain $B_r(0) \setminus U$. By the Cauchy's integral theorem (see Theorem 3.1.19) we get

$$\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_Js^{m+1}-\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_Js^{m+1}$$
$$=\frac{1}{2\pi}\int_{\partial((B_r(0)\setminus U)\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_Js^{m+1}=0.$$

Finally we have

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_Js^{m+1} &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_Js^{m+1}\\ &= P_m^2(T).\end{aligned}$$

and this concludes the proof. □

Finally, by using the same methodology developed in [45, Thm. 3.2.11] we have the following result

Lemma 11.5.15. *Let $T\in\mathcal{BC}(X)$. If $f\in N(\sigma_S(T))$ and U is a bounded slice Cauchy domain such that $\sigma_S(T)\subset U$ and $\overline{U}\subset dom(f)$, then we have*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}\mathcal{P}_2^L(s,T)ds_Jf(s)=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}f(s)ds_J\mathcal{P}_2^R(s,T).$$

11.6 Resolvent equation and product rule for the polyanalytic functional calculus

In this section we want to address the following problem.

Problem 11.6.1. *Is it possible to show a resolvent equation for the P_2-functional calculus on the S-spectrum that enjoys similar properties of the holomorphic resolvent equation?*

In order to answer to this question it is essential the following result.

Theorem 11.6.2. *Let $T\in\mathcal{BC}(X)$. For $q,s\in\rho_S(T)$, with $s\notin[q]$ the following equation holds*

$$\begin{aligned}&S_R^{-1}(s,T)\mathcal{P}_2^L(q,T)+\mathcal{P}_2^R(s,T)S_L^{-1}(q,T)-4\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,q}(T)^{-1}\\ &\qquad=[(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(q,T))q\\ &\qquad-\bar{s}(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(q,T))]\mathcal{Q}_s(q)^{-1},\end{aligned}$$

where $\mathcal{Q}_s(q):=q^2-2s_0q+|q|^2$ and $\underline{T}=T_1e_1+T_2e_2+T_3e_3$.

Proof. We divide the proof in nine steps.

Step I. We multiply the S-resolvent equation (see (3.9)) on the right by $4\mathcal{Q}_{c,q}^{-1}(T)q$ and we get

$$\begin{aligned} -S_R^{-1}(s,T)F_L(q,T)q &= [(4S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1}q+F_L(q,T)q)q \\ &\quad -\bar{s}(4S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1}q+F_L(q,T)q)]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.40}$$

Step II. We multiply the S-resolvent equation on the right by $-4T_0\mathcal{Q}_{c,q}^{-1}(T)$ and we obtain

$$\begin{aligned} S_R^{-1}(s,T)T_0F_L(q,T) &= [(-4S_R^{-1}(s,T)T_0\mathcal{Q}_{c,q}(T)^{-1}-T_0F_L(q,T))q \\ &\quad -\bar{s}(-4S_R^{-1}(s,T)T_0\mathcal{Q}_{c,q}(T)^{-1}-T_0F_L(q,T))]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.41}$$

Step III. We sum the equations (11.41) and (11.42), we get

$$\begin{aligned} S_R^{-1}(s,T)\mathcal{P}_2^L(q,T) &= [-\mathcal{P}_2^L(q,T)q+\bar{s}\mathcal{P}_2^L(q,T)]\mathcal{Q}_s(q)^{-1} \\ &\quad +4[-S_R^{-1}(s,T)(T_0-q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1}q \\ &\quad +\bar{s}S_R^{-1}(s,T)(T_0-q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1}]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.42}$$

Step IV. We multiply the S-resolvent equation on the left by $4\mathcal{Q}_{c,s}(T)^{-1}s$ and we get

$$\begin{aligned} -sF_R(s,T)S_L^{-1}(q,T) &= [(-sF_R(s,T)-4s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T))q \\ &\quad -\bar{s}(-sF_R(s,T)-4s\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T))]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.43}$$

Step V. We multiply the S-resolvent equation on the left by $-4T_0\mathcal{Q}_{c,s}(T)^{-1}$ and we get

$$\begin{aligned} T_0F_R(s,T)S_L^{-1}(q,T) &= [(T_0F_R(s,T)+4T_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T))q \\ &\quad -\bar{s}(T_0F_R(s,T)+4T_0\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T))]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.44}$$

Step VI. We sum the equations (11.44) and (11.45), we obtain

$$\begin{aligned} \mathcal{P}_2^R(s,T)S_L^{-1}(q,T) &= [\mathcal{P}_2^R(s,T)q-\bar{s}\mathcal{P}_2^R(s,T)]\mathcal{Q}_s(q)^{-1} \\ &\quad +4[\mathcal{Q}_{c,s}(T)^{-1}(T_0-s\mathcal{I})S_L^{-1}(q,T)q \\ &\quad -\bar{s}\mathcal{Q}_{c,s}(T)^{-1}(T_0-s\mathcal{I})S_L^{-1}(q,T)]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.45}$$

Step VII. We sum the equations (11.42) and (11.38), we get

$$\begin{aligned} &S_R^{-1}(s,T)\mathcal{P}_2^L(q,T)+\mathcal{P}_2^R(s,T)S_L^{-1}(q,T) \\ &=[(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(q,T))q-\bar{s}(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(q,T))]\mathcal{Q}_s(q)^{-1} \\ &+4[(\mathcal{Q}_{c,s}(T)^{-1}(T_0-s\mathcal{I})S_L^{-1}(q,T)-S_R^{-1}(s,T)(T_0-q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1})q \\ &-\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}(T_0-s\mathcal{I})S_L^{-1}(q,T)-S_R^{-1}(s,T)(T_0-q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1}. \end{aligned} \tag{11.46}$$

Step VIII. We manipulate the term

$$4[(\mathcal{Q}_{c,s}(T)^{-1}(T_0-s\mathcal{I})S_L^{-1}(q,T)-S_R^{-1}(s,T)(T_0-q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1})q \\ -\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}(T_0-s\mathcal{I})S_L^{-1}(q,T)-S_R^{-1}(s,T)(T_0-q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1}, \tag{11.47}$$

which is in the right hand side of the equation (11.46). This term is the sum of the following two terms

$$4T_0[(\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1})q \\ -\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1}, \tag{11.48}$$

and

$$4[(S_R^{-1}(s,T)q\mathcal{Q}_{c,q}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}sS_L^{-1}(q,T))q \\ -\bar{s}(S_R^{-1}(s,T)q\mathcal{Q}_{c,q}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}sS_L^{-1}(q,T))]\mathcal{Q}_s(q)^{-1}. \tag{11.49}$$

Firstly, we focus on the term (11.48). By the definitions of the left and the right S-resolvent operators, we have

$$\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1} \\ =\mathcal{Q}_{c,s}(T)^{-1}(q\mathcal{I}-\bar{T})\mathcal{Q}_{c,q}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I}-\bar{T})\mathcal{Q}_{c,q}(T)^{-1} \\ =\mathcal{Q}_{c,s}(T)^{-1}(q-s)\mathcal{Q}_{c,q}(T)^{-1}.$$

Thus the term (11.48) can be rewritten in the following way

$$4T_0[(\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1})q \\ -\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)-S_R^{-1}(s,T)\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1} \\ =4T_0[(\mathcal{Q}_{c,s}(T)^{-1}(q-s)\mathcal{Q}_{c,q}(T)^{-1})q-\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}(q-s)\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1} \\ =4T_0[\mathcal{Q}_{c,s}(T)^{-1}(-sq+q^2+|s|^2-\bar{s}q)\mathcal{Q}_{c,q}(T)^{-1}]\mathcal{Q}_s(q)^{-1} \\ =4T_0[\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_s(q)\mathcal{Q}_{c,q}(T)^{-1}]\mathcal{Q}_s(q)^{-1} \\ =4T_0\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1}. \tag{11.50}$$

Now we focus on the term (11.49). By the definitions of the left and the right S-resolvent operators, we have

$$S_R^{-1}(s,T)q\mathcal{Q}_{c,q}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}sS_L^{-1}(q,T) \\ =\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I}-\bar{T})q\mathcal{Q}_{c,q}(T)^{-1}-\mathcal{Q}_{c,s}(T)^{-1}s(q\mathcal{I}-\bar{T})\mathcal{Q}_{c,q}(T)^{-1} \\ =-\mathcal{Q}_{c,s}(T)^{-1}(\bar{T}q-s\bar{T})\mathcal{Q}_{c,q}(T)^{-1}.$$

Thus the term (11.49) can be rewritten in the following way

$$\begin{aligned}
&4[(S_R^{-1}(s,T)q\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}sS_L^{-1}(q,T))q \\
&-\bar{s}(S_R^{-1}(s,T)q\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}sS_L^{-1}(q,T))]\mathcal{Q}_s(q)^{-1} \\
&= -4[(\mathcal{Q}_{c,s}(T)^{-1}(\bar{T}q - s\bar{T})\mathcal{Q}_{c,q}(T)^{-1})q \\
&-\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}(\bar{T}q - s\bar{T})\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1} \\
&= -4\mathcal{Q}_{c,s}(T)^{-1}(\bar{T}q^2 - s\bar{T}q - \bar{s}\bar{T}q + |s|^2\bar{T})\mathcal{Q}_{c,q}(T)^{-1}\mathcal{Q}_s(q)^{-1} \\
&= -4\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_s(q)\mathcal{Q}_{c,q}(T)^{-1}\mathcal{Q}_s(q)^{-1} \\
&= -4\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,q}(T)^{-1}.
\end{aligned} \tag{11.51}$$

In conclusion by (11.50) and (11.51) we can write

$$\begin{aligned}
&4[(\mathcal{Q}_{c,s}(T)^{-1}(T_0 - s\mathcal{I})S_L^{-1}(q,T) - S_R^{-1}(s,T)(T_0 - q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1})q \\
&-\bar{s}(\mathcal{Q}_{c,s}(T)^{-1}(T_0 - s\mathcal{I})S_L^{-1}(q,T) \\
&-S_R^{-1}(s,T)(T_0 - q\mathcal{I})\mathcal{Q}_{c,q}(T)^{-1})]\mathcal{Q}_s(q)^{-1} \\
&= 4T_0\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1} - 4\mathcal{Q}_{c,s}(T)^{-1}\bar{T}\mathcal{Q}_{c,q}(T)^{-1} \\
&= 4\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,q}(T)^{-1}.
\end{aligned} \tag{11.52}$$

Step IX. Finally, by (11.52) and (11.46) we get

$$\begin{aligned}
&S_R^{-1}(s,T)\mathcal{P}_2^L(q,T) + \mathcal{P}_2^R(s,T)S_L^{-1}(q,T) - 4\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,q}(T)^{-1} \\
&= [(\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T))q - \bar{s}(\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T))]\mathcal{Q}_s(q)^{-1}.
\end{aligned}$$

□

Lemma 11.6.3. *Let $T \in \mathcal{BC}(X)$ and let $s \in \rho_S(T)$. The commutative pseudo S-resolvent operator satisfies the equations*

$$\mathcal{Q}_{c,s}(T)^{-1} = \frac{1}{4}(\mathcal{P}_2^L(s,T) + \underline{T}F_L(s,T)) \tag{11.53}$$

and

$$\mathcal{Q}_{c,s}(T)^{-1} = \frac{1}{4}(\mathcal{P}_2^R(s,T) + F_R(s,T)\underline{T}). \tag{11.54}$$

Proof. By Theorem 9.3.1, with $n = 3$, we have

$$\begin{aligned}
4\mathcal{Q}_{c,s}(T)^{-1} &= -F_L(s,T)s + TF_L(s,T) \\
&= -F_L(s,T)s + T_0F_L(s,T) + \underline{T}F_L(s,T) \\
&= \mathcal{P}_2^L(s,T) + \underline{T}F_L(s,T).
\end{aligned}$$

To prove the other equality in the statement we can proceed in a similar way. □

By means of the previous result we can write a resolvent equation for the P_2-functional calculus.

Theorem 11.6.4. *Let $T \in \mathcal{BC}(X)$. For $q,\ s \in \rho_S(T)$, with $s \notin [q]$ the following equation holds*

$$\begin{aligned}
&S_R^{-1}(s,T)\mathcal{P}_2^L(q,T) + \mathcal{P}_2^R(s,T)S_L^{-1}(q,T)\\
&-\frac{1}{4}\left(\mathcal{P}_2^R(s,T)\underline{T}\mathcal{P}_2^L(q,T) + \mathcal{P}_2^R(s,T)\underline{T}^2F_L(q,T) + F_R(s,T)\underline{T}^2\mathcal{P}_2^L(q,T)\right.\\
&\left.+F_R(s,T)\underline{T}^3F_L(q,T)\right) = [(\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T))q\\
&-\bar{s}(\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T))]\mathcal{Q}_s(q)^{-1}. \qquad (11.55)
\end{aligned}$$

Proof. From the identities (11.53) and (11.54) we obtain

$$\begin{aligned}
4^2\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,q}(T)^{-1} &= \left(\mathcal{P}_2^R(s,T) + F_R(s,T)\underline{T}\right)\underline{T}\left(\mathcal{P}_2^L(q,T) + \underline{T}F_L(q,T)\right)\\
&= \mathcal{P}_2^R(s,T)\underline{T}\mathcal{P}_2^L(q,T) + \mathcal{P}_2^R(s,T)\underline{T}^2F_L(q,T) +\\
&\quad +F_R(s,T)\underline{T}^2\mathcal{P}_2^L(q,T) + F_R(s,T)\underline{T}^3F_L(q,T).
\end{aligned}$$

Replacing this identity in (11.40) we obtain the thesis $\square$

We can write the equation (11.55) in the following way

$$\begin{aligned}
&S_R^{-1}(s,T)\mathcal{P}_2^L(q,T) + \mathcal{P}_2^R(s,T)S_L^{-1}(q,T)\\
&-\frac{1}{4}\left(\mathcal{P}_2^R(s,T)\underline{T}\mathcal{P}_2^L(q,T) + \mathcal{P}_2^R(s,T)\underline{T}^2F_L(q,T) + F_R(s,T)\underline{T}^2\mathcal{P}_2^L(q,T)\right.\\
&\left.+F_R(s,T)\underline{T}^3F_L(q,T)\right) = [\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T)] *_{s,left} S_L^{-1}(q,s).
\end{aligned}$$

This equation can be considered a resolvent equation for the P_2-functional calculus. The main differences and the major similarities with the holomorphic resolvent equation are listed below.

- Due to the noncommutative setting there are two different $\overline{\mathcal{D}}$-kernel operators $\mathcal{P}_2^L(q,T)$ and $\mathcal{P}_2^R(s,T)$, which are right slice hyperholomorphic in q and left slice hyperholomorphic in s, respectively.
- The difference $\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T)$ is suitably multiplied by the Cauchy kernel of the slice hyperholomorphic functions
- The term
$$[\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(q,T)] *_{s,left} S_L^{-1}(q,s)$$
is equal not only to the product of the $\mathcal{P}$-resolvent operators but also to other terms:

- the S-resolvent operators,
- the F-resolvent operators.

- The resolvent equation preserves the slice hyperholomorphicity on the right in s and on the left in q.

As it happens for the holomorphic functional calculus the resolvent equation is crucial to obtain a product formula.

Theorem 11.6.5. *Let $T \in \mathcal{BC}(X)$ and assume $f \in \mathcal{N}(\sigma_S(T))$. If $g \in \mathcal{SH}_L(\sigma_S(T))$, then we have*

$$\overline{\mathcal{D}}(fg)(T) = f(T)(\overline{\mathcal{D}}g)(T) + (\overline{\mathcal{D}}f)(T)g(T) - \mathcal{D}(f)(T)\underline{T}\mathcal{D}(g)(T). \tag{11.56}$$

If $g \in SH_R(\sigma_S(T))$, then we have

$$\overline{\mathcal{D}}(gf)(T) = g(T)(\overline{\mathcal{D}}f)(T) + (\overline{\mathcal{D}}g)(T)f(T) - \mathcal{D}(g)(T)\underline{T}\mathcal{D}(f)(T). \tag{11.57}$$

Proof. Let G_1 and G_2 be two bounded slice Cauchy domains such that they contain the S-spectrum and $\overline{G}_1 \subset G_2$ and $\overline{G}_2 \subset \mathrm{dom}(f) \cap \mathrm{dom}(g)$. We choose $p \in \partial(G_1 \cap \mathbb{C}_J)$ and $s \in \partial(G_2 \cap \mathbb{C}_J)$. For every $J \in \mathbb{S}$, from the definitions of the P_2-functional calculus, the S-functional calculus and the Q-functional calculus we get

$$\begin{aligned}
&f(T)(\overline{\mathcal{D}}g)(T) + (\overline{\mathcal{D}}f)(T)g(T) - \mathcal{D}(f)(T)\underline{T}\mathcal{D}(g)(T)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J S_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(p,T)dp_J g(p)\\
&+ \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{P}_2^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} S_L^{-1}(p,T)dp_J g(p)\\
&- \frac{1}{\pi^2}\left(\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{Q}_{c,s}(T)^{-1}\right)\underline{T}\left(\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1}dp_J g(p)\right)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} f(s)ds_J[S_R^{-1}(s,T)\mathcal{P}_2^L(p,T) + \mathcal{P}_2^R(s,T)S_L^{-1}(p,T)\\
&\qquad - 4\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)^{-1}]dp_J g(p).
\end{aligned}$$

Now from equation (11.40) we obtain

$$\begin{aligned}
&f(T)(\overline{\mathcal{D}}g)(T) + (\overline{\mathcal{D}}f)(T)g(T) - \mathcal{D}(f)(T)\underline{T}\mathcal{D}(g)(T)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} f(s)ds_J[(\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(p,T))p\\
&\qquad - \bar{s}(\mathcal{P}_2^R(s,T) - \mathcal{P}_2^L(p,T))]\mathcal{Q}_s(p)^{-1}dp_J g(p).
\end{aligned}$$

Since $p\mathcal{Q}_s(p)^{-1}$ and $\mathcal{Q}_s(p)^{-1}$ are intrinsic slice hyperholomorphic on $\overline{G}_1$, by the Cauchy integral formula we get

$$\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\mathcal{P}_2^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} p\mathcal{Q}_s(p)^{-1}dp_Jg(p)=0$$

and

$$\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\bar{s}\mathcal{P}_2^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_s(p)^{-1}dp_Jg(p)=0.$$

Therefore, we obtain

$$\begin{aligned}
&\overline{\mathcal{D}}(fg)(T)=f(T)(\overline{\mathcal{D}}g)(T)+(\overline{\mathcal{D}}f)(T)g(T)-\mathcal{D}(f)(T)\underline{T}\mathcal{D}(g)(T)\\
&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}[\bar{s}\mathcal{P}_2^L(p,T)-\mathcal{P}_2^L(p,T)p]\mathcal{Q}_s(p)^{-1}dp_Jg(p).
\end{aligned}$$

By Fubini's theorem, Lemma 9.4.1 with $B:=\mathcal{P}_2^L(p,T)$, and the definition of the P_2-functional calculus we get

$$\begin{aligned}
&f(T)(\overline{\mathcal{D}}g)(T)+(\overline{\mathcal{D}}f)(T)g(T)-\mathcal{D}(f)(T)\underline{T}\mathcal{D}(g)(T)\\
&=\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{P}_2^L(p,T)dp_Jf(p)g(p)\\
&=\overline{\mathcal{D}}(fg)(T).
\end{aligned}$$

□

In the previous chapter a product rule for the F- functional calculus is proved, see Theorem 10.7.1. The formula is obtained in terms of the Q-functional calculus, i.e. the operator $\mathcal{D}$ is involved. In the following result we show a product rule for the F-functional calculus in which the P_2-functional calculus is involved, namely the operator $\overline{\mathcal{D}}$ plays a role.

Theorem 11.6.6. *Let $T\in\mathcal{BC}(X)$ and assume $f\in\mathcal{N}(\sigma_S(T))$ and $g\in\mathcal{SH}_L(\sigma_S(T))$. Then we have*

$$\begin{aligned}
\Delta(fg)(T) &= (\Delta f)(T)g(T)+f(T)(\Delta g)(T)-\frac{1}{4}(\overline{\mathcal{D}}f)(T)(\overline{\mathcal{D}}g)(T)\\
&\quad-\frac{1}{4}(\overline{\mathcal{D}}f)(T)\underline{T}(\Delta g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}(\overline{\mathcal{D}}g)(T)\\
&\quad-\frac{1}{4}(\Delta f)(T)\underline{T}^2(\Delta g)(T). \qquad (11.58)
\end{aligned}$$

Proof. Let G_1 and G_2 be two bounded slice Cauchy domains like in the proof of Theorem 11.6.5. Let us consider $p \in \partial(G_1 \cap \mathbb{C}_J)$ and $s \in \partial(G_2 \cap \mathbb{C}_J)$. Then by the definitions of the F-functional calculus, the S-functional calculus, the P_2-functional calculus and from the fact that f is intrinsic, we get

$$
\begin{aligned}
&(\Delta f)(T)g(T) + f(T)(\Delta g)(T) - \frac{1}{4}(\overline{\mathcal{D}}f)(T)(\overline{\mathcal{D}}g)(T) \\
&- \frac{1}{4}(\overline{\mathcal{D}}f)(T)\underline{T}(\Delta g)(T) - \frac{1}{4}(\Delta f)(T)\underline{T}(\overline{\mathcal{D}}g)(T) - \frac{1}{4}(\Delta f)(T)\underline{T}^2(\Delta g)(T) \\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J F_R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} S_L^{-1}(p,T)dp_J g(p) \\
&+ \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J S_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J g(p) \\
&- \frac{1}{4(2\pi)^2}\left[\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{P}_2^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(p,T)dp_J g(p)\right. \\
&+ \int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{P}_2^R(s,T)\underline{T}\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J g(p) \\
&+ \int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J F_R(s,T)\underline{T}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(p,T)dp_J g(p) \\
&\left.+ \int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J F_R(s,T)\underline{T}^2\int_{\partial(G_1\cap\mathbb{C}_J)} F_L(p,T)dp_J g(p)\right] \\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} f(s)ds_J[F_R(s,T)S_L^{-1}(p,T) + S_R^{-1}(s,T)F_L(p,T) \\
&- \frac{1}{4}\mathcal{P}_2^R(s,T)\mathcal{P}_2^L(p,T) - \frac{1}{4}\mathcal{P}_2^R(s,T)\underline{T}F_L(p,T) - \frac{1}{4}F_R(s,T)\underline{T}\mathcal{P}_2^L(p,T) \\
&- \frac{1}{4}F_R(s,T)\underline{T}^2F_L(p,T)]dp_J g(p).
\end{aligned}
$$

From Lemma 11.6.3 we get

$$
\begin{aligned}
&\frac{1}{4}\left(P_2^R(s,T)\mathcal{P}_2^L(p,T) + \mathcal{P}_2^R(s,T)\underline{T}F_L(p,T) + F_R(s,T)\underline{T}\mathcal{P}_2^L(p,T)\right. \\
&\left.+F_R(s,T)\underline{T}^2F_L(p,T)\right) = 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}.
\end{aligned}
$$

Therefore, we get

$$\begin{aligned}&(\Delta f)(T)g(T)+f(T)(\Delta g)(T)-\frac{1}{4}(\overline{\mathcal{D}}f)(T)(\overline{\mathcal{D}}g)(T)\\&-\frac{1}{4}(\overline{\mathcal{D}}f)(T)\underline{T}(\Delta g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}(\overline{\mathcal{D}}g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}^2(\Delta g)(T)\\&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}f(s)ds_J[F_R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_L(p,T)\\&-4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}]dp_Jg(p).\end{aligned}$$

Now, by [45, Lemma 7.3.2], we know that

$$\begin{aligned}&F_R(s,T)S_L^{-1}(p,T)+S_R^{-1}(s,T)F_L(p,T)-4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-1}\\&\quad=[(F_R(s,T)-F_L(s,T))p-\bar{s}(F_R(s,T)-F_L(p,T))]\mathcal{Q}_s(p)^{-1}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&(\Delta f)(T)g(T)+f(T)(\Delta g)(T)-\frac{1}{4}(\overline{\mathcal{D}}f)(T)(\overline{\mathcal{D}}g)(T)\\&-\frac{1}{4}(\overline{\mathcal{D}}f)(T)\underline{T}(\Delta g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}(\overline{\mathcal{D}}g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}^2(\Delta g)(T)\\&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}f(s)ds_J[(F_R(s,T)-F_L(s,T))p\\&-\bar{s}(F_R(s,T)-F_L(p,T))]dp_Jg(p).\end{aligned}$$

By using similar arguments of the proof of Theorem 11.6.5 i.e., the linearity of the integrals, the Cauchy integral formula and Lemma 9.5, we obtain

$$\begin{aligned}&(\Delta f)(T)g(T)+f(T)(\Delta g)(T)-\frac{1}{4}(\overline{\mathcal{D}}f)(T)(\overline{\mathcal{D}}g)(T)\\&-\frac{1}{4}(\overline{\mathcal{D}}f)(T)\underline{T}(\Delta g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}(\overline{\mathcal{D}}g)(T)-\frac{1}{4}(\Delta f)(T)\underline{T}^2(\Delta g)(T)\\&=\frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)}F_L(p,T)dp_Jf(p)g(p)\\&=\Delta(fg)(T).\end{aligned}$$

□

Remark 11.6.7. If in formula (11.58) we replace the operator T with a generic quaternion $q\in\mathbb{H}$ and by considering $f(q)=q^n$ and $g(q)=q$, we get $\Delta(qg(q))=q\Delta(g(q))+2\mathcal{D}(g(q))$.

11.7 Riesz projectors for the polyanalytic functional calculus

The aim of this section is to investigate the Riesz projectors for the P_2-functional calculus. Before, we need some auxiliary results.

Theorem 11.7.1. *Let $T \in \mathcal{BC}(X)$ with $s \in \rho_S(T)$ then we have*

$$\mathcal{P}_2^L(s,T)s - T\mathcal{P}_2^L(s,T) = 4(S_L^{-1}(s,T) - \underline{T}\mathcal{Q}_{c,s}(T)^{-1}) \tag{11.59}$$

and

$$s\mathcal{P}_2^R(s,T) - \mathcal{P}_2^R(s,T)T = 4(S_R^{-1}(s,T) - \mathcal{Q}_{c,s}(T)^{-1}\underline{T}). \tag{11.60}$$

Proof. From the definition of the left $\overline{\mathcal{D}}$-kernel operator and formula (9.18) we get

$$\begin{aligned}
\mathcal{P}_2^L(s,T)s - T\mathcal{P}_2^L(s,T) &= (-F_L(s,T)s + T_0F_L(s,T))s \\
&\quad - T(-F_L(s,T)s + T_0F_L(s,T)) \\
&= (-F_L(s,T)s + TF_L(s,T))s \\
&\quad + T_0(F_L(s,T)s - TF_L(s,T)) \\
&= 4\mathcal{Q}_{c,s}(T)^{-1}s - 4T_0\mathcal{Q}_{c,s}(T)^{-1} \\
&= 4(s - T_0 + \underline{T})\mathcal{Q}_{c,s}(T)^{-1} - 4\underline{T}\mathcal{Q}_{c,s}(T)^{-1} \\
&= 4S_L^{-1}(s,T) - 4\underline{T}\mathcal{Q}_{c,s}(T)^{-1}.
\end{aligned}$$

The equation (11.60) follows by similar arguments. □

In the following result we provide a suitable generalization of the previous result.

Theorem 11.7.2. *Let $T \in \mathcal{BC}(X)$ with $s \in \rho_S(T)$ and set*

$$\mathcal{A}_m^L(s,T) := 4\sum_{i=0}^{m-1} T^iS_L^{-1}(s,T)s^{m-i-1}, \qquad \mathcal{A}_m^R(s,T) := 4\sum_{i=0}^{m-1} s^{m-i-1}S_R^{-1}(s,T)T^i$$

and

$$\mathcal{B}_m^L(s,T) := 4\sum_{i=0}^{m-1} T^i\underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-i-1}, \qquad \mathcal{B}_m^R(s,T) := 4\sum_{i=0}^{m-1} s^{m-i-1}\mathcal{Q}_{c,s}(T)^{-1}\underline{T}T^i.$$

Then for $m \in \mathbb{N}$, we have the following equation

$$\mathcal{P}_2^L(s,T)s^m - T^m\mathcal{P}_2^L(s,T) = \mathcal{A}_m^L(s,T) - \mathcal{B}_m^L(s,T). \tag{11.61}$$

Similarly

$$s^m\mathcal{P}_2^R(s,T) - \mathcal{P}_2^R(s,T)T^m = \mathcal{A}_m^R(s,T) - \mathcal{B}_m^R(s,T). \tag{11.62}$$

Proof. We will show only formula (11.61) because formula (11.62) follows by similar computations. We prove the result by induction on m. If $m = 1$ we have by formula (11.59)

$$\begin{aligned}\mathcal{P}_2^L(s,T)s - T\mathcal{P}_2^L(s,T) &= 4(S_L^{-1}(s,T) - \underline{T}\mathcal{Q}_{c,s}(T)^{-1})\\ &= \mathcal{A}_1^L(s,T) - \mathcal{B}_1^L(s,T).\end{aligned}$$

Now, we assume that the equation holds for $m-1$ and we will prove it for m. By inductive hypothesis we have

$$\begin{aligned} T^m\mathcal{P}_2^L(s,T) &= TT^{m-1}\mathcal{P}_2^L(s,T) \\ &= T(\mathcal{P}_2^L(s,T)s^{m-1} - \mathcal{A}_{m-1}^L(s,T) + \mathcal{B}_{m-1}^L(s,T)) \\ &= T\mathcal{P}_2^L(s,T)s^{m-1} - T\mathcal{A}_{m-1}^L(s,T) + T\mathcal{B}_{m-1}^L(s,T). \end{aligned} \tag{11.63}$$

By using formula (11.59), we obtain

$$T\mathcal{P}_2^L(s,T)s^{m-1} = \mathcal{P}_2^L(s,T)s^m - 4S_L^{-1}(s,T)s^{m-1} + 4\underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-1}. \tag{11.64}$$

Moreover, we have

$$\begin{aligned} T\mathcal{A}_{m-1}^L(s,T) &= 4\sum_{i=0}^{m-2} T^{i+1}S_L^{-1}(s,T)s^{m-i-2} \\ &= 4\sum_{\ell=1}^{m-1} T^{\ell}s_L^{-1}(s,T)S^{m-\ell-1} \end{aligned} \tag{11.65}$$

and

$$\begin{aligned} T\mathcal{B}_{m-1}^L(s,T) &= 4\sum_{i=0}^{m-2} T^{i+1}\underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-i-2} \\ &= 4\sum_{\ell=1}^{m-1} T^{\ell}\underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-\ell-1}. \end{aligned} \tag{11.66}$$

Eventually, by inserting formulas (11.64), (11.65) and (11.66) in (11.63),

we get

$$
\begin{aligned}
T^m\mathcal{P}_2^L(s,T) &= \mathcal{P}_2^L(s,T)s^m - 4S_L^{-1}(s,T)s^{m-1} + 4\underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-1} \\
&\quad - 4\sum_{\ell=1}^{m-1} T^\ell S_L^{-1}(s,T)s^{m-\ell-1} + 4\sum_{\ell=1}^{m-1} T^\ell \underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-\ell-1} \\
&= \mathcal{P}_2^L(s,T)s^m - 4\sum_{\ell=0}^{m-1} T^\ell S_L^{-1}(s,T)s^{m-\ell-1} \\
&\qquad + 4\sum_{\ell=0}^{m-1} T^\ell \underline{T}\mathcal{Q}_{c,s}(T)^{-1}s^{m-\ell-1} \\
&= \mathcal{P}_2^L(s,T)s^m - \mathcal{A}_m^L(s,T) + \mathcal{B}_m^L(s,T).
\end{aligned}
$$

□

Remark 11.7.3. Using the relation $2\underline{T} = T - \bar{T}$, we can also write the term $\mathcal{B}_m^L(s,T)$ of Theorem 11.7.2 in the following way

$$
\mathcal{B}_m^L(s,T) = 2\sum_{i=0}^{m-1} T^{i+1}\mathcal{Q}_{c,s}(T)^{-1}s^{m-i-1} - 2\bar{T}\sum_{i=0}^{m-1} T^i\mathcal{Q}_{c,s}(T)^{-1}s^{m-i-1}.
$$

The interesting symmetries that appear in the equation (11.40) allow to study the Riesz projectors.

Theorem 11.7.4 (Riesz projectors)**.** *Let $T = T_0 + T_1e_1 + T_2e_2$ and assume that the operators T_ℓ, with $\ell = 0,1,2$ have real spectrum. Let $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with $dist(\sigma_1,\sigma_2) > 0$. Let G_1, $G_2 \subset \mathbb{H}$ be two bounded slice Cauchy domains such that $\sigma_1 \subset G_1$, $\bar{G}_1 \subset G_2$ and $dist(G_2,\sigma_2) > 0$. Then the operator*

$$
\breve{P}_0 := \frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)\,ds_J\,s = \frac{1}{8\pi}\int_{\partial(G_2\cap\mathbb{C}_J)} p\,dp_J\,\mathcal{P}_2^R(p,T)
$$

is a projection i.e.

$$
\breve{P}_0^2 = \breve{P}_0.
$$

Moreover, we have the following commutative relation with respect the operator T

$$
T\breve{P}_0 = \breve{P}_0T. \tag{11.67}
$$

Proof. By Theorem 11.7.1 we know that

$$
S_R^{-1}(s,T) = \frac{1}{4}\left(s\mathcal{P}_2^R(s,T) - \mathcal{P}_2^R(s,T)T\right) + \mathcal{Q}_{c,s}(T)^{-1}. \tag{11.68}
$$

Now, by substituting (11.68) in (11.40) we get

$$\begin{aligned}
&\frac{1}{4}s\mathcal{P}_2^R(s,T)\mathcal{P}_2^L(p,T)-\frac{1}{4}\mathcal{P}_2^R(s,T)T\mathcal{P}_2^L(p,T)+\mathcal{Q}_{c,s}(T)^{-1}\mathcal{P}_2^L(p,T)\\
&+\mathcal{P}_2^R(s,T)S_L^{-1}(p,T)-4\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T) \qquad (11.69)\\
&=[\left(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\right)p-\bar{s}\left(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\right)]\mathcal{Q}_s(p)^{-1}.
\end{aligned}$$

Now, we multiply the equation (11.69) on the right by p, we get

$$\begin{aligned}
&\frac{1}{4}s\mathcal{P}_2^R(s,T)\mathcal{P}_2^L(p,T)p-\frac{1}{4}\mathcal{P}_2^R(s,T)T\mathcal{P}_2^L(p,T)p+\mathcal{Q}_{c,s}(T)^{-1}\mathcal{P}_2^L(p,T)p\\
&+\mathcal{P}_2^R(s,T)S_L^{-1}(p,T)p-4\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\mathcal{Q}_{c,p}(T)p \qquad (11.70)\\
&=[\left(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\right)p-\bar{s}\left(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\right)]\mathcal{Q}_s(p)^{-1}p.
\end{aligned}$$

Now, we multiply formula (11.71) by ds_J on the left and we integrate it on $\partial(G_2\cap\mathbb{C}_J)$ with respect to ds_J. Similarly, if we multiply formula (11.71) on the right by dp_J and we integrate it on $\partial(G_1\cap\mathbb{C}_J)$ with respect to dp_J. Thus we obtain

$$\begin{aligned}
&\frac{1}{4}\int_{\partial(G_2\cap\mathbb{C}_J)}sds_J\mathcal{P}_2^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{P}_2^L(p,T)dp_Jp-\frac{1}{4}\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\mathcal{P}_2^R(s,T)T\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{P}_2^L(p,T)dp_Jp\\
&-\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\mathcal{Q}_{c,s}(T)^{-1}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{P}_2^L(p,T)dp_Jp+\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\mathcal{P}_2^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)}S_L^{-1}(p,T)dp_Jp+\\
&-4\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\mathcal{Q}_{c,s}(T)^{-1}\underline{T}\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_{c,p}(T)^{-1}dp_J=\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}[\Big(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\Big)p\\
&-\bar{s}\Big(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\Big)]\mathcal{Q}_s(p)^{-1}dp_Jp.
\end{aligned}$$

By Lemma 11.5.10 and Lemma 10.3.12 we have

$$\begin{aligned}
4(2\pi)^2\breve{P}_0^2=&\int_{\partial(G_2\cap\mathbb{C}_J)}ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}[\left(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\right)p\\
&-\bar{s}\left(\mathcal{P}_2^R(s,T)-\mathcal{P}_2^L(p,T)\right)]\mathcal{Q}_s(p)^{-1}dp_Jp. \qquad (11.71)
\end{aligned}$$

Now, since the functions $p\mapsto\mathcal{Q}_s(p)^{-1}$ and $p\mapsto\mathcal{Q}_s(p)^{-1}$ are slice hyperholomorphic and do not have singularities inside $\partial(G_1\cap\mathbb{C}_J)$ by the Cauchy theorem we get

$$\int_{\partial(G_1\cap\mathbb{C}_J)}p\mathcal{Q}_s(p)^{-1}dp_Jp^2=\int_{\partial(G_1\cap\mathbb{C}_J)}\mathcal{Q}_s(p)dp_Jp=0. \qquad (11.72)$$

This implies that formula (11.71) can be written as

$$\begin{aligned}(\breve{P}_0)^2 &= -\frac{1}{4(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(p,T)\mathcal{Q}_s(p)^{-1}pdp_Jp \\ &\quad +\frac{1}{4(2\pi)^2}\int_{\partial(G_1\cap\mathbb{C}_J)} ds_J \int_{\partial(G_1\cap\mathbb{C}_J)} \bar{s}\mathcal{P}_2^L(p,T)\mathcal{Q}_s(p)^{-1}pdp_J \\ &= \frac{1}{4(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} ds_J[\bar{s}\mathcal{P}_2^L(p,T)-\mathcal{P}_2^L(p,T)p]\mathcal{Q}_s(p)^{-1}dp_Jp.\end{aligned}$$

By Fubini's theorem and Lemma 9.5 with $B := \mathcal{P}_2^L(p,T)$ we get

$$\breve{P}_0^2 = \frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(p,T)dp_Jp = \breve{P}_0.$$

Now, we want to show the commutativity relation (11.67). By (11.59) we know that

$$T\mathcal{P}_2^L(p,T) = \mathcal{P}_2^L(s,T)s - 4\left(S_L^{-1}(s,T) - \underline{T}\mathcal{Q}_{c,s}(T)^{-1}\right).$$

From the definition of Riesz projector we get

$$\begin{aligned}T\breve{P}_0 &= \frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{P}_2^L(s,T)ds_Js^2 - \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} S_L^{-1}(s,T)ds_Js \\ &\quad +\frac{\underline{T}}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} \mathcal{Q}_{c,s}(T)^{-1}ds_Js.\end{aligned}$$

On the other side, by (11.60) we obtain

$$\mathcal{P}_2^R(s,T)T = s\mathcal{P}_2^R(s,T) - 4\left(S_R^{-1}(s,T) - \underline{T}\mathcal{Q}_{c,s}(T)^{-1}\right).$$

This togetehr with the definition of Riesz projectors we get

$$\begin{aligned}\breve{P}_0T &= \frac{1}{8\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} s^2ds_J\mathcal{P}_2^R(s,T) - \frac{1}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_J)} sds_JS_R^{-1}(s,T) \\ &\quad +\frac{\underline{T}}{2\pi}\int_{G_1\cap\mathbb{C}_J} sds_J\mathcal{Q}_{c,s}(T)^{-1}.\end{aligned}$$

Thus, we have the statement. □

Part III: Further functional calculi on the S-spectrum based on the Fueter-Sce theorem and conclusions

In this last part of the dissertation we start to study the functional calculi based on the S-spectrum and rising from the factorization of the Fueter-Sce map. Due to the various factorizations the arguments to get new functional calculi are more involved than the quaternionic case. We focus on the dimension five, because there are all the functional calculi and function spaces that can be considered in greater dimensions.

Finally, we conclude the thesis by describing some new research directions.

CHAPTER 12

The fine sructure of the spectral theory on the S-spectrum in dimension five

12.1 Motivation

This chapter belongs to a new research direction that is related to the Fueter-Sce-Qian mapping theorem.

We recall the fine structure of the spectral theories on the S-spectrum taking advantage of the following observation. Let $h := \frac{n-1}{2}$ be the so-called Sce exponent, and Δ be the Laplace operator in dimension $n+1$: the operator $T_{FS2} := \Delta^h$ maps the slice hyperholomorphic function $f(x)$ to the set of axially monogenic functions. The powers of the Laplace operator Δ^h can be factorized in terms of the Dirac operator $\mathcal{D}$ and its conjugate $\overline{\mathcal{D}}$ because

$$\mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D} = \Delta.$$

So it is possible to repeatedly apply to a slice hyperholomorphic function $f(x)$ the Dirac operator and its conjugate, until we reach the maximum power of the Laplacian, i.e., the Sce exponent. This implies the possibility to build different sets of functions which lie between the set of slice hyperholomorphic functions and the set of axially monogenic functions.

We will call *fine structure of the spectral theory on the S-spectrum* the set of the functions spaces and the associated functional calculi induced by a factorization of the operator T_{FS2} in the Fueter-Sce extension theorem.

One of the most important factorizations leads to the so-called Dirac fine structure that corresponds to an alternating sequence of $\mathcal{D}$ and $\overline{\mathcal{D}}$, h times, for example

$$T_{FS2} = \Delta^h = \mathcal{D}\overline{\mathcal{D}} \ldots \mathcal{D}\overline{\mathcal{D}}.$$

In Chapter 10 and Chapter 11 we studied the Dirac fine structure when $n = 3$, i.e., the quaternionic case. Let $D \subset \mathbb{C}$ and let $\Omega_D \subseteq \mathbb{R}^{n+1}$ be the set induced by D, see Theorem 7.2.20. In particular we have studied the fine structure associated with the factorization:

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D), \tag{12.1}$$

where $\mathcal{AH}(\Omega_D)$ is the set of axially harmonic functions and their integral representation give rise to the harmonic functional calculus on the S-spectrum. This structure also allows to obtain a product formula for the F-functional calculus, see Theorem 10.7.1.

However, since $\Delta = \mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D}$, we can interchange the order of the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ in (12.1). This gives rise to the factorization:

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D). \tag{12.2}$$

where $\mathcal{AP}_2(\Omega_D)$ is a space of polyanalytic functions.

Clearly, as the dimension of the Clifford algebra increases there are more possibilities and we denote by $\mathcal{FS}(\Omega_D)$, the set of function spaces associated with the fine structures. These functions spaces lie between the set of slice hyperholomorphic functions and axially monogenic functions, in dimension five there are seven such spaces, precisely: $\mathcal{ABH}(\Omega_D)$ the axially bi-harmonic functions, $\mathcal{ACH}_1(\Omega_D)$ the axially Cliffordian holomorphic functions of order 1, (which is a short cut for order $(1,1)$), $\mathcal{AH}(\Omega_D)$ the axially harmonic functions, $\mathcal{AP}_2(\Omega_D)$ the axially polyanalytic of order 2, $\overline{\mathcal{ACH}_1(\Omega_D)}$ the axially anti Cliffordian of order 1, $\mathcal{ACP}_{(1,2)}(\Omega_D)$ the axially polyanalytic Cliffordian of order $(1,2)$, $\mathcal{AP}_3(\Omega_D)$ the axially polyanalytic of order 3. Now, we give the precise definitions of the previous functions spaces.

Definition 12.1.1 (holomorphic Cliffordian of order k)**.** Let U be an open set. A function $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$ of class $\mathcal{C}^{2k+1}(U)$ is said to be (left) holomorphic Cliffordian of order k if

$$\Delta^k \mathcal{D} f(x) = 0 \quad \forall x \in U,$$

where $0 \leq k \leq \frac{n-1}{2}$.

Remark 12.1.2. For $k := \frac{n-1}{2}$ in Definition 12.1.1 we get the class of functions studied in [104–106].

Remark 12.1.3. Every holomorphic Cliffordian function of order k is holomorphic Cliffordian of order $k+1$. If $k = 0$ in Definition 12.1.1 we get the set of (left) monogenic functions.

Definition 12.1.4 (anti-holomorphic Cliffordian of order k)**.** Let U be an open set. A function $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$ of class $\mathcal{C}^{2k+1}(U)$ is said to be (left) anti-holomorphic Cliffordian of order k if

$$\Delta^k \overline{\mathcal{D}} f(x) = 0 \quad \forall x \in U,$$

where $0 \leq k \leq \frac{n-1}{2}$.

Definition 12.1.5 (polyharmonic of degree k)**.** Let $k \geq 1$. A function $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$ of class $\mathcal{C}^{2k}(U)$ is called polyharmonic of degree k in the open set $U \subset \mathbb{R}^{n+1}$ if

$$\Delta^k f(x) = 0, \quad \forall x \in U.$$

For $p = 1$ the function is called harmonic and for $p = 2$ the function is called bi-harmonic. The polyharmonic functions are studied in [20].

Definition 12.1.6 (polyanalytic of order m)**.** Let $m \geq 1$. Let $U \subset \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}_n$ be a function of class $\mathcal{C}^m(U)$. We say that f is (left) polyanalytic of order m on U if

$$\mathcal{D}^m f(x) = 0, \qquad \forall x \in U.$$

It is very important to point out that these function spaces appear in different contexts in the literature and they seem to be unrelated. In this paper we show that they all appear as fine structures in the Fueter-Sce construction. In dimension greater than five there will be one more function space, that is not indicated in the list above, and with this addition, all the fines structures can be described using those function spaces of different orders.

12.2 Function spaces generated by the Fueter-Sce mapping theorem

Working in the Clifford algebra with five imaginary units, i.e., $n = 5$ the second Fueter-Sce map is Δ^2 where the Laplace operator Δ is in dimension 6.

We recall that there exist different possible factorizations of Δ^2 in terms of the Dirac operator $\mathcal{D}$ and its conjugate $\overline{\mathcal{D}}$ choosing different configurations of products of $\mathcal{D}$ and $\overline{\mathcal{D}}$.

The case of dimension five is different from what happens in the quaternionic case, in which the Fueter map can be factorized only as $\mathcal{D}\overline{\mathcal{D}}$ and $\overline{\mathcal{D}}\mathcal{D}$: here we obtain a reacher structure, see Chapter 10 and Chapter 11.

In the setting of slice hyperholomorphic functions, functions of the form (3.1) together with the even-odd conditions are called slice functions. In the monogenic setting such functions, often considered only in the upper half space, are called function of axial type. We will use both terminology according to the setting. Now, we assume that the axial functions (or slice functions) of the form

$$f(x) = \alpha(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\beta(x_0, |\underline{x}|)$$

are of class $\mathcal{C}^5(\Omega_D)$, where Ω_D is as in the Fueter-Sce mapping theorem. We will consider functions $f : \Omega_D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ with values in the Clifford algebra $\mathbb{R}_n$ where we consider the case $n = 5$.

Definition 12.2.1 ($\mathcal{ABH}(\Omega_D)$ axially bi-harmonic function)**.** Let $f : \Omega_D \subseteq \mathbb{R}^6 \to \mathbb{R}_5$ be of axial type and of class $\mathcal{C}^5(\Omega_D)$. Then, the function

$$\tilde{f}_1(x) := \mathcal{D}f(x) \qquad \text{on} \quad \Omega_D$$

is called an axially bi-harmonic function, since by the Fueter-Sce mapping theorem, it satisfies

$$\Delta^2 \tilde{f}_1(x) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this set of functions by $\mathcal{ABH}(\Omega_D)$.

Definition 12.2.2 ($\mathcal{ACH}_1(\Omega_D)$ axially Cliffordian functions of order one)**.** Consider the function $\tilde{f}_1(x) := \mathcal{D}f(x) \in \mathcal{ABH}(\Omega_D)$ and apply the conjugate Dirac operator $\overline{\mathcal{D}}$ to $\tilde{f}_1(x)$. Then we get

$$f^\circ(x) := \overline{\mathcal{D}}\tilde{f}_1(x) = \Delta f(x) \qquad \text{on} \quad \Omega_D, \tag{12.3}$$

which is an axially Cliffordian functions of order one (which is the short cut for order $(1, 1)$) by the Fueter-Sce mapping theorem, i.e.,

$$\Delta\mathcal{D}f^\circ(x) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this set of functions by $\mathcal{ACH}_1(\Omega_D)$.

Definition 12.2.3 ($\overline{\mathcal{ACH}_1(\Omega_D)}$ axially anti-Cliffordian functions of order one)**.** Consider the function $\tilde{f}_1(x) := \mathcal{D}f(x) \in \mathcal{ABH}(\Omega_D)$ and apply the Dirac operator $\mathcal{D}$ to $\tilde{f}_1(x)$ we obtain

$$f_\circ(x) = \mathcal{D}\tilde{f}_1(x) = \mathcal{D}^2 f(x), \qquad \text{on} \quad \Omega_D,$$

which is axially anti-Cliffordian functions of order one by the Fueter-Sce mapping theorem, i.e.,

$$\Delta\overline{\mathcal{D}}\left(f_\circ(x)\right) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this set of functions by $\overline{\mathcal{ACH}_1(\Omega_D)}$.

Definition 12.2.4 ($\mathcal{AH}(\Omega_D)$ axially harmonic functions)**.** Consider the function $f^\circ(x) := \overline{\mathcal{D}}\tilde{f}_1(x) = \Delta f(x) \in \mathcal{ACH}_1(\Omega_D)$ and apply the Dirac operator $\mathcal{D}$ to $f^\circ(x)$. We get,

$$\tilde{f}_0(x) = \mathcal{D}f^\circ(x) = \Delta\mathcal{D}f(x),$$

which is an axially harmonic functions, by the Fueter-Sce mapping theorem, i.e.,

$$\Delta\tilde{f}_0(x) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this set of functions as $\mathcal{AH}(\Omega_D)$.

Definition 12.2.5 ($\mathcal{AP}_2(\Omega_D)$ axially polyanalytic functions of order two)**.** If we apply the operator $\overline{\mathcal{D}}$ to $f^\circ(x) := \overline{\mathcal{D}}\tilde{f}_1(x) = \Delta f(x) \in \mathcal{ACH}_1(\Omega_D)$ we obtain

$$\breve{f}_1^\circ(x) = \overline{\mathcal{D}}f^\circ(x) = \Delta\overline{\mathcal{D}}f(x),$$

which is an axially polyanalytic functions of order two, by the Fueter-Sce mapping theorem, i.e.,

$$\mathcal{D}^2\breve{f}_1^\circ(x) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this set of functions by $\mathcal{AP}_2(\Omega_D)$.

Definition 12.2.6 ($\mathcal{APC}_{(1,2)}(\Omega_D)$ axially Cliffordian polyanalytic functions of order $(1,2)$)**.** Let $f : \Omega_D \subseteq \mathbb{R}^6 \to \mathbb{R}_5$ be of axial type and of class $\mathcal{C}^5(\Omega_D)$. Apply to (7.26) the conjugate of the Dirac operator. In this case we obtain

$$\breve{f}^\circ(x) = \overline{\mathcal{D}}f(x) \qquad \text{on} \quad \Omega_D, \tag{12.4}$$

which is an axially Cliffordian polyanalytic functions of order $(1,2)$, by the Fueter-Sce mapping theorem, i.e.,

$$\Delta\mathcal{D}^2\breve{f}^\circ(x) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this class of functions as $\mathcal{APC}_{(1,2)}(\Omega_D)$.

Definition 12.2.7 ($\mathcal{AP}_3(\Omega_D)$ axially polyanalytic function of order three)**.** Let $\breve{f}^\circ(x) = \overline{\mathcal{D}}f(x) \in \mathcal{APC}_{(1,2)}(\Omega_D)$. Applying the operator conjugate Dirac operator $\overline{\mathcal{D}}$ to $\breve{f}^\circ(x)$, we get

$$\breve{f}_0^\circ(x) = \overline{\mathcal{D}}^2 f(x) \qquad \text{on} \quad \Omega_D,$$

which is an axially polyanalytic function of order three, i.e.,

$$\mathcal{D}^3 \breve{f}_0^\circ(x) = 0 \qquad \text{on} \quad \Omega_D.$$

We denote this class of functions as $\mathcal{AP}_3(\Omega_D)$.

Remark 12.2.8. Keeping in mind the above notations we have

$$\breve{f}(x) = \Delta\overline{\mathcal{D}}\tilde{f}_0(x) = \mathcal{D}\breve{f}_1^\circ(x) = \overline{\mathcal{D}}^2 f_\circ(x) = \Delta f^\circ(x) = \overline{\mathcal{D}}\tilde{f}_1(x),$$

where $\breve{f}$ is axially monogenic and also

$$\breve{f}(x) = \mathcal{D}^2\breve{f}_0^\circ(x) = \Delta\mathcal{D}\breve{f}^\circ(x) \qquad \text{on} \quad \Omega_D.$$

Remark 12.2.9. In the general case appears the same classes of functions but with different orders.

Taking advantage of the function spaces of axial functions defined in this section we can now define the fine structure associated with this spaces that appear in the Clifford algebra $\mathbb{R}_5$.

12.3 Function space of axial type in dimension five

By applying the Fueter-Sce map $T_{FS2} := \Delta^h$, where $h := \frac{n-1}{2}$ and is the Sce exponent, to a slice hyperholomorphic function $f(x)$ we get the monogenic function $\breve{f}(x) = \Delta^h f(x)$.
Due to the factorization of the Laplace operator in terms of $\mathcal{D}$ and $\overline{\mathcal{D}}$ it is possible to apply these two operators to a slice hyperholomorphic function $f(x)$ a number of times, until we reach the maximum power of the Laplacian, i.e., the Sce exponent.
This implies the possibility to build different sets of functions between the set of slice hyperholomorphic functions and the set of axially monogenic functions, (see the previous section). This fact leads to the definition of fine structure of slice hyperholomorphic spectral theory.

Definition 12.3.1 (Fine structure of slice hyperholomorphic spectral theory)**.** A fine structure of slice hyperholomorphic spectral theory is the set of functions spaces and the associated functional calculi induced by a factorization of the operator T_{FS2}, in the Fueter-Sce extension theorem.

The factorization $T_{FS2} = \Delta^h = \mathcal{D}\overline{\mathcal{D}}...\mathcal{D}\overline{\mathcal{D}}$ is of particular interest.

Definition 12.3.2 (Dirac fine structure)**.** The Dirac fine structure corresponds to an alternating sequence of products of the Dirac operator $\mathcal{D}$ and of its conjugate $\overline{\mathcal{D}}$ until we obtain $\Delta^{\frac{n-1}{2}}$.

In Chapter 10 we studied the Dirac fine structure when $n = 3$ this is also the quaternionic case. In particular we studied the sequence represented by the following diagram:

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D). \tag{12.5}$$

The fine structure in (12.5) allows to obtain a product rule for the F-functional calculus.
However, since $\Delta = \mathcal{D}\overline{\mathcal{D}} = \overline{\mathcal{D}}\mathcal{D}$, we can exchange the roles of the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ in (12.5). This gives rise to the sequence represented by the following diagram:

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D), \tag{12.6}$$

which is investigated in Chapter 11 . Even if the diagrams (12.5) and (12.6) come from the Fueter mapping theorem and the factorization of the Fueter operator $T_{F2} = \Delta$, we get two different fine structures.

In each fine structure above and in all the fine structures we consider in the sequel the final set of function spaces is always the set of axially monogenic functions.

In the Clifford algebra setting the splitting of the second Fueter-Sce mapping is more complicated, due to the fact that we are dealing with integer powers of the Laplacian. Moreover, when n is even the Laplace operator has a fractional power and so we have to work in the space of distributions using the Fourier multipliers, see [122].

Due to the fact that for $n = 5$ we deal with the operator Δ^2, we get more Dirac fine structures, which are all different and important at the same time. In order to label all the fine structures, we will denote every fine structures, with an ordered sequence of the applied operators. For example, in the quaternionic case, we call (12.5) the Dirac fine structure of the kind $(\mathcal{D}, \overline{\mathcal{D}})$ and (12.6) the Dirac fine structure of the kind $(\overline{\mathcal{D}}, \mathcal{D})$. Also in the case $n = 5$ we have a structure in which we apply alternately the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ until we reach the second Fueter-Sce mapping.

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{ABH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AHC}_1(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D). \tag{12.7}$$

We call (12.7) the Dirac fine structure of the kind $(\mathcal{D}, \overline{\mathcal{D}}, \mathcal{D}, \overline{\mathcal{D}})$.

Remark 12.3.3. Even when $n = 5$ the Dirac fine structure (12.7) is of fundamental importance to obtain a product formula for the F-functional calculus (see Theorem 12.8.1).

Remark 12.3.4. In order to avoid, at the end of the sequence of spaces the set of axially-anti monogenic function, we impose the condition that the composition of all the operators between spaces Clifford valued functions, must be equal to the operator $T_{FS2} = \Delta^{(n-1)/2}$ in the Fueter-Sce mapping theorem.

However, by rearranging the sequence of $\mathcal{D}$ and $\overline{\mathcal{D}}$ it is possible to obtain other fine structures, in which other sets of functions are involved. Thus, we have the Dirac fine structures $(\mathcal{D}, \overline{\mathcal{D}}, \overline{\mathcal{D}}, \mathcal{D})$

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{ABH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AHC}_1(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D),$$

and the Dirac fine structure $(\mathcal{D}, \mathcal{D}, \overline{\mathcal{D}}, \overline{\mathcal{D}})$

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{ABH}(\Omega_D) \xrightarrow{\mathcal{D}} \overline{\mathcal{AHC}_1(\Omega_D)} \xrightarrow{\overline{\mathcal{D}}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D).$$

All the previous Dirac fine structures are obtained by applying first the Dirac operator. Nevertheless, it is possible to apply the operator $\overline{\mathcal{D}}$ as first operator. In this case other three Dirac fine structures arise. We have the Dirac fine structure of the kind $(\overline{\mathcal{D}}, \mathcal{D}, \overline{\mathcal{D}}, \mathcal{D})$

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{APC}_{(1,2)}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AHC}_1(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D),$$

the Dirac fine structure $(\overline{\mathcal{D}}, \mathcal{D}, \mathcal{D}, \overline{\mathcal{D}})$

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{APC}_{(1,2)}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AHC}_1(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AM}(\Omega_D),$$

and the Dirac fine structure $(\overline{\mathcal{D}}, \overline{\mathcal{D}}, \mathcal{D}, \mathcal{D})$

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{APC}_{(1,2)}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}} \mathcal{AP}_3(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D).$$

The following diagram summarizes all the Dirac fine structures with their function spaces:

$$\begin{tikzcd}
 & & & \mathcal{AP}_3(\Omega_D) \arrow[dr, "\mathcal{D}"] & & \\
 & & \mathcal{APC}_{(1,2)}(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"] \arrow[dr, "\mathcal{D}"] & & \mathcal{AP}_2(\Omega_D) \arrow[dr, "\mathcal{D}"] & \\
\mathcal{O}(D) \arrow[r, "T_{FS1}"] & \mathcal{SH}(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"] \arrow[dr, "\mathcal{D}"] & & \mathcal{ACH}_1(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"] \arrow[dr, "\mathcal{D}"] & & \mathcal{AM}(\Omega_D) \\
 & & \mathcal{ABH}(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"] \arrow[dr, "\mathcal{D}"] & & \mathcal{AH}(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"] & \\
 & & & \overline{\mathcal{ACH}_1(\Omega_D)} \arrow[ur, "\overline{\mathcal{D}}"] & &
\end{tikzcd}$$

Remark 12.3.5. In all the previous Dirac fine structures it is possible to combine the Dirac operator and its conjugate. In this way we get a fine structure which is "weaker" then the previous ones; in the sense that we are skipping some classes of functions. We call these kind of fine structures coarser. Up to now we have mentioned just some of them and in the Appendix we show the complete landscape of the fine structures in dimension five.

The Laplace fine structure is of the kind (Δ, Δ), which is a coarser fine structure with respect to the Dirac one, is given by

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\Delta} \mathcal{ACH}_1(\Omega_D) \xrightarrow{\Delta} \mathcal{AM}(\Omega_D).$$

Other interesting coarser fine structures are the harmonic ones, in which appear only the harmonic and bi-harmonic sets of functions

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{ABH}(\Omega_D) \xrightarrow{\Delta} \mathcal{AH}(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D),$$

and the polyanalytic one, in which there appear only the polyanalytic functions of orders three and two

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} \mathcal{SH}(\Omega_D) \xrightarrow{\overline{\mathcal{D}}^2} \mathcal{AP}_3(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AP}_2(\Omega_D) \xrightarrow{\mathcal{D}} \mathcal{AM}(\Omega_D).$$

We observe that it is not possible to have coarser fine structure in the quaternionic case. This is due to the fact that we are dealing with the Laplacian at power 1.

12.4 System of differential equations for fine structure spaces of axial type

In analogy with the Vekua-type system of differential equations for axially monogenic functions in this section we give all the systems of differential equations for the fine structure spaces in dimension five. Let D be a domain in the upper-half complex plane. Let Ω_D be an axially symmetric open set in $\mathbb{R}^6$ and let $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in \Omega_D$. A function $f : \Omega_D \to \mathbb{R}_5$ is of axial type if there exist two functions $A = A(x_0, r)$ and $B = B(x_0, r)$, independent of $\underline{\omega} \in \mathbb{S}^4$ and with values in $\mathbb{R}_5$, such that

$$f(x) = A(x_0, r) + \underline{\omega} B(x_0, r), \quad \text{where } \ r > 0.$$

So we characterize the class of functions that lies between the set of slice hyperholomorphic and the set of axially monogenic functions, that we have denoted as axially functions. We recall by [118], that if $f(x) = A(x_0, r) + \underline{\omega} B(x_0, r)$ then

$$\begin{aligned} \mathcal{D}f \ &= \ \left(\partial_{x_0} A(x_0, r) - \partial_r B(x_0, r) - \frac{4}{r} B(x_0, r)\right) \\ &\quad + \underline{\omega} \left(\partial_{x_0} B(x_0, r) + \partial_r A(x_0, r)\right), \end{aligned} \tag{12.8}$$

$$\begin{aligned} \overline{\mathcal{D}}f \ &= \ \left(\partial_{x_0} A(x_0, r) + \partial_r B(x_0, r) + \frac{4}{r} B(x_0, r)\right) \\ &\quad + \underline{\omega} \left(\partial_{x_0} B(x_0, r) - \partial_r A(x_0, r)\right). \end{aligned} \tag{12.9}$$

Theorem 12.4.1. *Let $D \subseteq \mathbb{C}$. Let Ω_D be an axially symmetric open set in $\mathbb{R}^6$ and let $f_\circ(x) = A(x_0, r) + \underline{\omega} B(x_0, r)$ be an axially anti cliffordian holomorphic function of order 1. Then $A = A(x_0, r)$ and $B = B(x_0, r)$ satisfy the following system*

$$\begin{cases} \partial_{x_0}^3 A + \partial_{x_0} \partial_r^2 A + \frac{4}{r} \partial_{x_0} \partial_r A + \partial_r \partial_{x_0}^2 B + \partial_r^3 B + 8\frac{\partial_r^2 B}{r} + 8\frac{\partial_r B}{r^2} - 8\frac{B}{r^3} + \frac{4}{r} \partial_{x_0}^2 B = 0 \\ \\ \partial_{x_0}^3 B + \partial_{x_0} \partial_r^2 B - 4\partial_r \left(\frac{\partial_{x_0} B}{r}\right) - \partial_r \partial_{x_0}^2 A - \partial_r^3 A - 4\partial_r \left(\frac{\partial_r A_1}{r}\right) = 0. \end{cases}$$

Proof. Let us consider $f_\circ(x) = A + \underline{\omega} B$. By similar computations done in Theorem 10.2.3 we have

$$\begin{aligned} \Delta(f_\circ(x)) \ &= \ \left(\partial_{x_0}^2 A + \partial_r^2 A + \frac{4}{r} \partial_r A\right) \\ &\quad + \underline{\omega} \left(\partial_{x_0}^2 B + \partial_r^2 B + 4\partial_r \left(\frac{B}{r}\right)\right). \end{aligned} \tag{12.10}$$

Now, we set

$$A' := \partial_{x_0}^2 A + \partial_r^2 A + \frac{4}{r} \partial_r A \quad \text{and} \quad B' := \partial_{x_0}^2 B + \partial_r^2 B + 4\frac{\partial_r}{r} B - \frac{4}{r^2} B.$$

Then by formula (12.9) we have

$$\begin{aligned}\Delta\overline{\mathcal{D}}(f_\circ(x)) &= \overline{\mathcal{D}}(A'+\underline{\omega}B') = (\partial_{x_0}A'+\partial_r B'+\frac{4}{r}B')+\underline{\omega}(\partial_{x_0}A'-\partial_r B')\\ &= \partial_{x_0}^3 A+\partial_{x_0}\partial_r^2 A+\frac{4}{r}\partial_{x_0}\partial_r A+\partial_r^3 B+\frac{4}{r}\partial_r^2 B-\frac{4}{r^2}\partial_r B\\ &+ \frac{8}{r^3}B-\frac{4}{r^2}\partial_r B+\partial_r\partial_{x_0}^2 B+\frac{4}{r}\partial_r^2 B+\frac{16}{r^2}\partial_r B-\frac{16}{r^3}B+\frac{4}{r}\partial_{x_0}^2 B\\ &+ \underline{\omega}(\partial_{x_0}\partial_r^2 B+\frac{4}{r}\partial_{x_0}\partial_r B-\frac{4}{r^2}\partial_{x_0}B+\partial_{x_0}^3 B-\partial_r\partial_{x_0}^2 A-\partial_r^3 A\\ &+ \frac{4}{r^2}\partial_r A-\frac{4}{r}\partial_r^2 A),\end{aligned} \tag{12.11}$$

so we finally have

$$\begin{aligned}\Delta\overline{\mathcal{D}}(f_\circ(x)) &= \partial_{x_0}^3 A+\partial_{x_0}\partial_r^2 A+\frac{4}{r}\partial_{x_0}\partial_r A+\partial_r^3 B+\frac{8}{r}\partial_r^2 B+\frac{8}{r^2}\partial_r B-\frac{8}{r^3}B\\ &+ \partial_r\partial_{x_0}^2 B+\frac{4}{r}\partial_{x_0}^2 B+\underline{\omega}(\partial_{x_0}\partial_r^2 B+\frac{4}{r}\partial_{x_0}\partial_r B-\frac{4}{r^2}\partial_{x_0}B\\ &+ \partial_{x_0}^3 B-\partial_r\partial_{x_0}^2 A-\partial_r^3 A+\frac{4}{r^2}\partial_r A-\frac{4}{r}\partial_r^2 A).\end{aligned} \tag{12.12}$$

Since $(^\circ f(x))$ is anti Cliffordian holomorphic of order one we have that $\Delta\overline{\mathcal{D}}(f_\circ(x)) = 0$. □

Theorem 12.4.2. *Let $D \subseteq \mathbb{C}$. Let Ω_D be an axially symmetric open set in $\mathbb{R}^6$ and let $\tilde{f}_1(x) = A(x_0,r)+\underline{\omega}B(x_0,r)$ be an axially bi-harmonic function. Then $A := A(x_0,r)$ and $B := B(x_0,r)$ satisfy the following system*

$$\begin{cases}\partial_{x_0}^4 A+2\partial_{x_0}^2\partial_r^2 A+\partial_r^4 A-\frac{8}{r^3}\partial_r A+\frac{8}{r^2}\partial_r^2 A+\frac{8}{r}\partial_r^3 A+\frac{4}{r}\partial_r\partial_{x_0}^2 A=0\\ \\ \partial_r^4 B+\frac{8}{r}\partial_r^3 B-\frac{2}{4}r^3\partial_r B+\frac{12}{r^4}B+2\partial_r^2\partial_{x_0}^2 B-\frac{8}{r^2}\partial_{x_0}^2 B+\frac{8}{r}\partial_{x_0}^2\partial_r B+\partial_{x_0}^4 B=0.\end{cases}$$

Proof. By formula (12.12) we have

$$C := \partial_{x_0}^3 A+\partial_{x_0}\partial_r^2 A+\frac{4}{r}\partial_{x_0}\partial_r A+\partial_r^3 B+\frac{8}{r}\partial_r^2 B+\frac{8}{r^2}\partial_r B-\frac{8}{r^3}B+\partial_r\partial_{x_0}^2 B+\frac{4}{r}\partial_{x_0}^2 B$$

and

$$D := \partial_{x_0}\partial_r^2 B+\frac{4}{r}\partial_{x_0}\partial_r B-\frac{4}{r^2}\partial_{x_0}B+\partial_{x_0}^3 B-\partial_r\partial_{x_0}^2 A-\partial_r^3 A+\frac{4}{r^2}\partial_r A-\frac{4}{r}\partial_r^2 A,$$

Therefore, by formula (12.8) we have

$$
\begin{aligned}
&\Delta^2(f(x)) = \mathcal{D}(\overline{\mathcal{D}}\Delta f(x)) = (\partial_{x_0}C - \partial_r D - \frac{4}{r}D) + \underline{\omega}(\partial_{x_0}D + \partial_r C)\\
&= \partial_{x_0}^4 A + \partial_{x_0}^2\partial_r^2 A + \frac{4}{r}\partial_{x_0}^2\partial_r A + \partial_{x_0}\partial_r^3 B + \frac{8}{r}\partial_{x_0}\partial_r^2 B + \frac{8}{r^2}\partial_{x_0}\partial_r B - \frac{8}{r^3}\partial_{x_0}B + \partial_r\partial_{x_0}^3 B\\
&+\frac{4}{r}\partial_{x_0}^3 B - \partial_{x_0}\partial_r^3 B - \frac{4}{r}\partial_{x_0}\partial_r^2 B + \frac{8}{r^2}\partial_{x_0}\partial_r B - \frac{8}{r^3}\partial_{x_0}B - \partial_r\partial_{x_0}^3 B + \partial_r^2\partial_{x_0}^2 A + \partial_r^4 A\\
&+\frac{8}{r^3}\partial_r A - \frac{4}{r^2}\partial_r^2 A + \frac{4}{r}\partial_r^3 A - \frac{4}{r^2}\partial_r^2 A - \frac{4}{r}\partial_{x_0}\partial_r^2 B - \frac{16}{r^2}\partial_{x_0}\partial_r B + \frac{16}{r^3}\partial_{x_0}B - \frac{4}{r}\partial_{x_0}^3 B\\
&+\frac{4}{r}\partial_r\partial_{x_0}^2 A + \frac{4}{r}\partial_r^3 A - \frac{16}{r^3}\partial_r A + \frac{16}{r^2}\partial_r^2 A + \underline{\omega}(\partial_r\partial_{x_0}^3 A + \partial_{x_0}\partial_r^3 A + \frac{4}{r}\partial_{x_0}\partial_r^2 A + \partial_r^4 B\\
&-\frac{8}{r^2}\partial_r^2 B + \frac{8}{r}\partial_r^3 B - \frac{16}{r^3}\partial_r B + \frac{8}{r^2}\partial_r^2 B + \frac{24}{r^4}B - \frac{4}{r^2}\partial_{x_0}\partial_r A - \frac{8}{r^3}\partial_r B\\
&+\partial_r^2\partial_{x_0}^2 B - \frac{4}{r^2}\partial_{x_0}^2 B + \frac{4}{r}\partial_r\partial_{x_0}^2 B + \partial_{x_0}^2\partial_r^2 B + \frac{4}{r}\partial_{x_0}^2\partial_r B - \frac{4}{r^2}\partial_{x_0}^2 B + \partial_{x_0}^4 B\\
&-\partial_r\partial_{x_0}^3 A - \partial_r^3\partial_{x_0}A + \frac{4}{r^2}\partial_{x_0}\partial_r A - \frac{4}{r}\partial_r^2\partial_{x_0}A)\\
&= \partial_{x_0}^4 A + 2\partial_{x_0}^2\partial_r^2 A + \partial_r^4 A - \frac{8}{r^3}\partial_r A + \frac{8}{r^2}\partial_r^2 A + \frac{8}{r}\partial_r^3 A + \frac{4}{r}\partial_r\partial_{x_0}^2 A + \underline{\omega}(\partial_r^4 B\\
&+\frac{8}{r}\partial_r^3 B - \frac{1}{2}r^3\partial_r B + \frac{12}{r^4}B + 2\partial_r^2\partial_{x_0}^2 B - \frac{8}{r^2}\partial_{x_0}^2 B + \frac{8}{r}\partial_{x_0}^2\partial_r B + \partial_{x_0}^4 B).
\end{aligned}
$$

Since the function $\tilde{f}_1(x)$ is bi-harmonic, i.e. $\Delta^2\tilde{f}_1(x) = 0$, we have the thesis. □

Theorem 12.4.3. *Let $D \subseteq \mathbb{C}$. Let Ω_D be an axially symmetric open set in $\mathbb{R}^6$ and let $\breve{f}_0^\circ(x) = A(x_0, r) + \underline{\omega}B(x_0, r)$ be an axially polyanalytic function of order three. Then $A := A(x_0, r)$ and $B := B(x_0, r)$ satisfy the following system*

$$
\begin{cases}
\partial_{x_0}^3 A + \partial_r^3 B - 3\partial_{x_0}^2\partial_r B - 3\partial_{x_0}\partial_r^2 A - \frac{12}{r}\partial_{x_0}^2 B - \frac{12}{r}\partial_{x_0}\partial_r A\\
+8\frac{\partial_r^2 B}{r} + 8\frac{\partial_r B}{r^2} - 8\frac{B}{r^3} = 0\\
\\
\partial_{x_0}^3 B - \partial_r^3 A + 3\partial_{x_0}^2\partial_r A - 3\partial_{x_0}\partial_r^2 B - 12\frac{\partial_{x_0}\partial_r B}{r} + 12\frac{\partial_{x_0}B}{r^2} - 4\frac{\partial_r^2 A}{r} + \frac{4}{r^2}\partial_r A = 0.
\end{cases}
$$

Proof. First of all we start by computing $\mathcal{D}^2\breve{f}_0^\circ(x)$.

$$\mathcal{D}^2\breve{f}_0^\circ(x) = \mathcal{D}(\mathcal{D}\breve{f}_0^\circ(x)) = \mathcal{D}(A' + \underline{\omega}B'),$$

where

$$A' := \partial_{x_0}A - \partial_r B - \frac{4}{r}B \quad \text{and} \quad B' := \partial_{x_0}B + \partial_r A.$$

By formula (12.8) we get

$$\begin{aligned}\mathcal{D}^2\breve{f}_0^\circ(x) &= \left(\partial_{x_0}A'(x_0,r)-\partial_r B'(x_0,r)-\frac{4}{r}B'(x_0,r)\right) \\ &\quad +\underline{\omega}\left(\partial_{x_0}B'(x_0,r)+\partial_r A'(x_0,r)\right) \\ &= \left(\partial_{x_0}^2 A-2\partial_{x_0}\partial_r B-\frac{8}{r}\partial_{x_0}B-\partial_r^2 A-\frac{4}{r}\partial_r A\right) \\ &\quad +\underline{\omega}\left(\partial_{x_0}^2 B+2\partial_{x_0}\partial A-\partial_r^2 B+-4\partial_r\frac{\partial_r B}{r}+4\frac{B}{r^2}\right) \\ &= A''+\underline{\omega}B''.\end{aligned} \tag{12.13}$$

where

$$A'':=\partial_{x_0}^2 A-2\partial_{x_0}\partial_r B-\frac{8}{r}\partial_{x_0}B-\partial_r^2 A-\frac{4}{r}\partial_r A$$

and

$$B'':=\partial_{x_0}^2 B+2\partial_{x_0}\partial A-\partial_r^2 B-4\partial_r\frac{\partial_r B}{r}+4\frac{B}{r^2}.$$

Finally by applying another time formula (12.8) we get

$$\begin{aligned}\mathcal{D}^3\breve{f}_0^\circ(x) &= \mathcal{D}(\mathcal{D}^2\breve{f}_0^\circ(x))=\mathcal{D}(A''+\underline{\omega}B'') \\ &= \left(\partial_{x_0}A''-\partial_r B''-\frac{4}{r}B''\right)+\underline{\omega}(\partial_{x_0}B''+\partial_r A'') \\ &= \Bigg(\partial_{x_0}^3 A+\partial_r^3 B-3\partial_{x_0}^2\partial_r B-3\partial_{x_0}\partial_r^2 A-\frac{12}{r}\partial_{x_0}^2 B-\frac{12}{r}\partial_{x_0}\partial_r A \\ &\quad +8\frac{\partial_r^2 B}{r}+8\frac{\partial_r B}{r^2}-8\frac{B}{r^3}\Bigg)+\underline{\omega}\Big(\partial_{x_0}^3 B-\partial_r^3 A+3\partial_{x_0}^2\partial_r A-3\partial_{x_0}\partial_r^2 B \\ &\quad -12\frac{\partial_{x_0}\partial_r B}{r}+12\frac{\partial_{x_0}B}{r^2}-4\frac{\partial_r^2 A}{r}+\frac{4}{r^2}\partial_r A\Big).\end{aligned}$$

We get the statement from the fact that the function $\breve{f}_0^\circ(x)$ is polyanalytic of order three, i.e. $\mathcal{D}^3\breve{f}_0^\circ(x)=0$. □

Theorem 12.4.4. *Let $D\subseteq\mathbb{C}$. Let Ω_D be an axially symmetric open set in $\mathbb{R}^6$. Then*

- *$f^\circ(x)=A(x_0,r)+\underline{\omega}B(x_0,r)$ is axially Cliffordian of order one if and only if $A:=A(x_0,r)$ and $B:=B(x_0,r)$ satisfy the following* *system*

$$\begin{cases} \partial_{x_0}A+\partial_{x_0}\partial_r^2A+\frac{4}{r}\partial_{x_0}\partial_rA-\partial_r\partial_{x_0}^2B-\partial_r^3B-8\frac{\partial_rB}{r^2}+8\frac{B}{r^3}-4\frac{\partial_{x_0}^2B}{r}=0\\ \\ \partial_{x_0}^3B+\partial_{x_0}\partial_r^2B+4\partial_r\left(\frac{\partial_{x_0}B}{r}\right)+\partial_r\partial_{x_0}^2A+\partial_r^3A+4\partial_r^2\left(\frac{A}{r}\right)=0. \end{cases}$$

- $\tilde{f}_0(x)=A(x_0,r)+\underline{\omega}B(x_0,r)$ *is axially harmonic if and only if* $A:=A(x_0,r)$ *and* $B:=B(x_0,r)$ *satisfy the following system*

$$\begin{cases} \partial_{x_0}^2A+\partial_r^2A+\frac{4}{r}\partial_rA=0\\ \\ \partial_{x_0}^2B+\partial_r^2B+4\partial_r\left(\frac{B}{r}\right)=0. \end{cases}$$

- $\breve{f}_1^\circ(x)=A(x_0,r)+\underline{\omega}B(x_0,r)$ *is axially polyanalytic of order two if and only if* $A:=A(x_0,r)$ *and* $B:=B(x_0,r)$ *satisfy the following system*

$$\begin{cases} \partial_{x_0}^2A-2\partial_{x_0}\partial_rB-\frac{8}{r}\partial_{x_0}B-\partial_r^2A-\frac{4}{r}\partial_rA=0\\ \\ \partial_{x_0}^2B+2\partial_{x_0}\partial_rA-\partial_r^2B-4\partial_r\left(\frac{B}{r}\right)=0. \end{cases}$$

- $\breve{f}^\circ(x)=A(x_0,r)+\underline{\omega}B(x_0,r)$ *is axially Cliffordian polyanalytic of order* $(1,2)$ *if and only if* $A:=A(x_0,r)$ *and* $B:=B(x_0,r)$ *satisfy the following system*

$$\begin{cases} \partial_{x_0}^4A-2\partial_r\partial_{x_0}^3B-2\partial_{x_0}\partial_r^3B-8\frac{\partial_{x_0}^3B}{r}-8\frac{\partial_{x_0}\partial_r^2B}{r}-\partial_r^4A-8\frac{\partial_r^3A}{r}-8\frac{\partial_rA}{r^3}\\ -4\frac{\partial_r^2A}{r^2}-8\frac{A}{r^4}-16\frac{\partial_{x_0}\partial_rB}{r^2}=0\\ \\ \partial_{x_0}^4B+2\partial_r\partial_{x_0}^3A+2\partial_{x_0}\partial_r^3A+8\frac{\partial_r^2\partial_{x_0}A}{r}-12\frac{\partial_r\partial_{x_0}A}{r^2}-4\frac{\partial_r\partial_{x_0}B}{r^2}-\partial_r^4B\\ +8\frac{\partial_{x_0}A}{r^3}-8\frac{\partial_r^2B}{r^2}+24\frac{\partial_rB}{r^3}-24\frac{B}{r^4}+4\frac{\partial_{x_0}^2B}{r^2}=0. \end{cases}$$

Proof. We do not give all the details of the proof because they are tedious computations we just mentions that: the first system follows by applying the Dirac operator $\mathcal{D}$ to (12.10). The computations are similar to that ones done in Theorem 12.4.1. The second system follows by formula (12.10).

The third system follows by formula (12.13). The fourth system follows by applying the operator $\mathcal{D}$ to $\Delta\mathcal{D}\breve{f}^\circ(x)$, which formula is possible to get by the first point of this theorem. $\square$

In the next section we will give the integral representation of the functions belonging to the function spaces associated with the fine structure. These spaces are called, for short, fine structure spaces.

12.5 Integral representation of the functions of the fine structure spaces

In this section we construct the kernels associated with the spaces of the fine structures. The strategy follows the construction of the Fueter-Sce mapping theorem in integral form. Precisely, we proceed by applying to the left (and to the right) slice hyperholomorphic Cauchy kernels some suitable operators that define the required kernels. These new kernels will be used to give these functions the appropriate integral representations. In the proofs of the following theorems we consider just the left hyperholomorphic Cauchy kernel since for the right hyperholomorphic Cauchy kernel computations are similar.

Theorem 12.5.1 (Structure of the slice $\mathcal{D}$-kernels $S^{-1}_{\mathcal{D},L}$ and $S^{-1}_{\mathcal{D},R}$)**.** *Let $s, x \in \mathbb{R}^6$ be such that $x \notin [s]$, then*

$$S^{-1}_{\mathcal{D},L}(s,x) := \mathcal{D}\left(S^{-1}_L(s,x)\right) = -4\mathcal{Q}_{c,s}(x)^{-1}, \tag{12.14}$$

and

$$S^{-1}_{\mathcal{D},R}(s,x) := \left(S^{-1}_R(s,x)\right)\mathcal{D} = -4\mathcal{Q}_{c,s}(x)^{-1}. \tag{12.15}$$

We denote by $S^{-1}_{\mathcal{D},L}$ and $S^{-1}_{\mathcal{D},R}$ the left and the right slice $\mathcal{D}$-kernels.

Proof. We compute the following derivatives

$$\frac{\partial}{\partial x_0} S^{-1}_L(s,x) = -\mathcal{Q}_{c,s}(x)^{-1} + 2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}(s-x_0), \tag{12.16}$$

and, for $1 \le i \le 5$, we get

$$\frac{\partial}{\partial x_i} S^{-1}_L(s,x) = e_i\mathcal{Q}_{c,s}(x)^{-1} - 2x_i(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}. \tag{12.17}$$

Finally, we have

$$\begin{aligned}
\mathcal{D}\left(S_L^{-1}(s,x)\right) &= \frac{\partial}{\partial x_0}S_L^{-1}(s,x)+\sum_{i=1}^{5}e_i\frac{\partial}{\partial x_i}S_L^{-1}(s,x)\\
&= -\mathcal{Q}_{c,s}(x)^{-1}-2x_0(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}+2(s-\bar{x})s\mathcal{Q}_{c,s}(x)^{-2}\\
&\quad -5\mathcal{Q}_{c,s}(x)^{-1}-2\underline{x}(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}\\
&= -6\mathcal{Q}_{c,s}(x)^{-1}-2x(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}+2(s-\bar{x})s\mathcal{Q}_{c,s}(x)^{-2}\\
&= -6\mathcal{Q}_{c,s}(x)^{-1}+2(s^2-\bar{x}s-xs+|x|^2)\mathcal{Q}_{c,s}(x)^{-2}\\
&= -4\mathcal{Q}_{c,s}(x)^{-1}.
\end{aligned}$$

□

Now, we apply the Laplacian of $\mathbb{R}^6$, i.e.,

$$\Delta := \sum_{i=0}^{5}\frac{\partial^2}{\partial x_i^2},$$

to the slice hyperholomorphic Cauchy kernel

Theorem 12.5.2 (Structure of the slice Δ-kernels $S^{-1}_{\Delta,L}$ and $S^{-1}_{\Delta,R}$)**.** *Let $s,x\in\mathbb{R}^6$ be such that $x\notin[s]$, then*

$$S^{-1}_{\Delta,L}(s,x) := \Delta S_L^{-1}(s,x) = -8S_L^{-1}(s,x)\mathcal{Q}_{c,s}(x)^{-1}, \tag{12.18}$$

and

$$S^{-1}_{\Delta,R}(s,x) := \Delta S_R^{-1}(s,x) = -8\mathcal{Q}_{c,s}(x)^{-1}S_R^{-1}(s,x). \tag{12.19}$$

We denote by $S^{-1}_{\Delta,L}$ and $S^{-1}_{\Delta,R}$ the left and the right slice Δ-kernels.

Proof. By formula (12.16) we get

$$\begin{aligned}
\frac{\partial^2}{\partial x_0^2}S_L^{-1}(s,x) &= (-2s+2x_0)\mathcal{Q}_{c,s}(x)^{-2}+(2x_0-2s)\mathcal{Q}_{c,s}(x)^{-2}\\
&\quad -2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}+8(s-\bar{x})(x_0-s)^2\mathcal{Q}_{c,s}(x)^{-3}.
\end{aligned}$$

By formula (12.17), for $1\leq i\leq 5$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial x_i^2}S_L^{-1}(s,x) &= -2x_ie_i\mathcal{Q}_{c,s}(x)^{-2}-2x_ie_i\mathcal{Q}_{c,s}(x)^{-2}\\
&\quad -2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}+8(s-\bar{x})x_i^2\mathcal{Q}_{c,s}(x)^{-3}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}\Delta S_L^{-1}(s,x) &= \frac{\partial^2}{\partial x_0^2} S_L^{-1}(s,x) + \sum_{i=1}^{5} \frac{\partial^2}{\partial x_i^2} S_L^{-1}(s,x)\\ &= 4(x_0-s)\mathcal{Q}_{c,s}(x)^{-2} - 2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} + 8(s-\bar{x})(x_0-s)^2\mathcal{Q}_{c,s}(x)^{-3}\\ &\quad -4\underline{x}\mathcal{Q}_{c,s}(x)^{-2} - 10(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} + 8|\underline{x}|^2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}\\ &= -4(s-x_0+\underline{x})\mathcal{Q}_{c,s}(x)^{-2} - 12(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} +\\ &\quad +8(s-\bar{x})[(x_0-s)^2+|\underline{x}|^2]\mathcal{Q}_{c,s}(x)^{-3}\\ &= -16(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} + 8(s-\bar{x})(x_0^2+s^2-2x_0s+|\underline{x}|^2)\mathcal{Q}_{c,s}(x)^{-3}\\ &= -8(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}\\ &= -8S_L^{-1}(s,x)\mathcal{Q}_{c,s}(x)^{-1}.\end{aligned}$$

□

Theorem 12.5.3 (Structure of the slice $\Delta\mathcal{D}$-kernels $S^{-1}_{\Delta\mathcal{D},L}$ and $S^{-1}_{\Delta\mathcal{D},R}$). *Let $x,\, s \in \mathbb{R}^6$ be such that $x \notin [s]$, then*

$$S^{-1}_{\Delta\mathcal{D},L}(s,x) := \Delta\mathcal{D}\left(S_L^{-1}(s,x)\right) = 16\mathcal{Q}_{c,s}(x)^{-2}, \tag{12.20}$$

and

$$S^{-1}_{\Delta\mathcal{D},R}(s,x) := \left(S_R^{-1}(s,x)\right)\mathcal{D}\Delta = 16\mathcal{Q}_{c,s}(x)^{-2}. \tag{12.21}$$

We denote by $S^{-1}_{\Delta\mathcal{D},L}$ and $S^{-1}_{\Delta\mathcal{D},R}$ the left and the right slice $\Delta\mathcal{D}$-kernels.

Proof. In order to show formula (12.20) it is enough to apply the Dirac operator to (12.18). Thus, we have

$$\frac{\partial}{\partial x_0}\left(\Delta S_L^{-1}(s,x)\right) = -8[-\mathcal{Q}_{c,s}(x)^{-2} - 2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(2x_0-2s)].$$

For $1 \leq i \leq 5$, we have

$$\frac{\partial}{\partial x_i}\left(\Delta S_L^{-1}(s,x)\right) = -8[e_i\mathcal{Q}_{c,s}(x)^{-2} - 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}x_i].$$

Finally by formula (7.48), with $n = 5$, we have

$$\begin{aligned}
\mathcal{D}\Delta S_L^{-1}(s,x) &= \frac{\partial}{\partial x_0}\left(\Delta S_L^{-1}(s,x)\right) + \sum_{i=1}^{5} e_i \frac{\partial}{\partial x_i}\left(\Delta S_L^{-1}(s,x)\right) \\
&= -8[-6\mathcal{Q}_{c,s}(x)^{-2} + \frac{4}{\gamma_5}F_L^5(s,x)(s-x_0) - \frac{4}{\gamma_5}\underline{x}F_L^5(s,x)] \\
&= -8\left[-6\mathcal{Q}_{c,s}(x)^{-2} + \frac{4}{\gamma_5}\left(F_L^5(s,x)s - xF_L^5(s,x)\right)\right] \\
&= -8\left(-6\mathcal{Q}_{c,s}(x)^{-2} + 4\mathcal{Q}_{c,s}(x)^{-2}\right) \\
&= 16\mathcal{Q}_{c,s}(x)^{-2}.
\end{aligned}$$

Formula (12.21) follows with similar reasoning.

□

Theorem 12.5.4 (Structure of the slice $\overline{\mathcal{D}}$-kernels $S^{-1}_{\overline{\mathcal{D}},L}$ and $S^{-1}_{\overline{\mathcal{D}},R}$)**.** *Let x, $s \in \mathbb{R}^6$ be such that $x \notin [s]$, then*

$$\begin{aligned}
S^{-1}_{\overline{\mathcal{D}},L}(s,x) &:= \overline{\mathcal{D}}(S_L^{-1}(s,x)) \\
&= 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}(s-x_0) + 2\mathcal{Q}_{c,s}(x)^{-1}
\end{aligned} \tag{12.22}$$

and

$$\begin{aligned}
S^{-1}_{\overline{\mathcal{D}},R}(s,x) &:= (S_R^{-1}(s,x))\overline{\mathcal{D}} \\
&= 4(s-x_0)\mathcal{Q}_{c,s}(x)^{-2}(s-\bar{x}) + 2\mathcal{Q}_{c,s}(x)^{-1}.
\end{aligned} \tag{12.23}$$

We denote by $S^{-1}_{\overline{\mathcal{D}},L}$ and $S^{-1}_{\overline{\mathcal{D}},R}$ the left and the right slice $\overline{\mathcal{D}}$-kernels.

Proof. By the relations (12.16) and (12.17) and using the fact that $2x_0 =$

$x+\bar{x}$, we have that

$$\begin{aligned}
\overline{\mathcal{D}}(S_L^{-1}(s,x)) &= \frac{\partial}{\partial x_0}(S_L^{-1}(s,x)) - \sum_{i=1}^{5} e_i \frac{\partial}{\partial x_i}(S_L^{-1}(s,x)) \\
&= -\mathcal{Q}_{c,s}(x)(x)^{-1} + 5\mathcal{Q}_{c,s}(x)(x)^{-1} + 2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}(s-x_0) \\
&\quad + 2\underline{x}(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} \\
&= 4\mathcal{Q}_{c,s}(x)^{-1} + 2((s-\bar{x})s - \bar{x}(s-\bar{x}))\mathcal{Q}_{c,s}(x)^{-2} \\
&= 4\mathcal{Q}_{c,s}(x)\mathcal{Q}_{c,s}(x)^{-2} + 2((s-\bar{x})s - \bar{x}(s-\bar{x}))\mathcal{Q}_{c,s}(x)^{-2} \\
&= (6s^2 - 8sx_0 + 4|x|^2 - 4\bar{x}s + 2\bar{x}^2)\mathcal{Q}_{c,s}(x)^{-2} \\
&= (6s^2 - 4\bar{x}s - 4xs + 4|x|^2 - 4\bar{x}s + 2\bar{x}^2)\mathcal{Q}_{c,s}(x)^{-2} \\
&= [4(s^2 - \bar{x}s) - 2(\bar{x}s + xs - |x|^2 - \bar{x}^2) \\
&\quad + 2(s^2 - \bar{x}s + |x|^2 - xs)]\mathcal{Q}_{c,s}(x)^{-2} \\
&= 4(s-\bar{x})s\mathcal{Q}_{c,s}(x)^{-2} - 4x_0(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} \\
&\quad + 2((s-\bar{x})s + x(\bar{x}-s))\mathcal{Q}_{c,s}(x)^{-2} \\
&= 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}(s-x_0) + 2\mathcal{Q}_{c,s}(x)^{-1}.
\end{aligned}$$

□

Theorem 12.5.5 (Structure of the slice $\overline{\mathcal{D}}^2$-kernels $S^{-1}_{\overline{\mathcal{D}}^2,L}$ and $S^{-1}_{\overline{\mathcal{D}}^2,R}$)**.** *Let x, $s \in \mathbb{R}^6$ be such that $x \notin [s]$, then*

$$S^{-1}_{\overline{\mathcal{D}}^2,L}(s,x) := \overline{\mathcal{D}}^2(S_L^{-1}(s,x)) = 32(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(s-x_0)^2 \quad (12.24)$$

and

$$S^{-1}_{\overline{\mathcal{D}}^2,R}(s,x) := (S_R^{-1}(s,x))\overline{\mathcal{D}}^2 = 32(s-x_0)^2\mathcal{Q}_{c,s}(x)^{-3}(s-\bar{x}). \quad (12.25)$$

We denote by $S^{-1}_{\overline{\mathcal{D}}^2,L}$ and $S^{-1}_{\overline{\mathcal{D}}^2,R}$ the left and the right slice $\overline{\mathcal{D}}^2$-kernels.

Proof. It is useful to compute $\overline{\mathcal{D}}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2})$. By the relations:

$$\frac{\partial}{\partial x_0}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}) = -\mathcal{Q}_{c,s}(x)^{-2} - 2(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(-2s+2x_0)$$

and

$$\frac{\partial}{\partial x_i}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}) = e_i\mathcal{Q}_{c,s}(x)^{-2} - 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(x_i),$$

we have

$$\begin{aligned}\overline{\mathcal{D}}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}) &= \frac{\partial}{\partial_{x_0}}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}) - \sum_{i=1}^{5} e_i \frac{\partial}{\partial_{x_i}}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}) \\ &= 4\mathcal{Q}_{c,s}(x)^{-2} + 4((s-\bar{x})s - \bar{x}(s-\bar{x}))\mathcal{Q}_{c,s}(x)^{-3} \\ &= 4\mathcal{Q}_{c,s}(x)\mathcal{Q}_{c,s}(x)^{-3} + 4((s-\bar{x})s - \bar{x}(s-\bar{x}))\mathcal{Q}_{c,s}(x)^{-3} \\ &= (8s^2 - 8sx_0 + 4|x|^2 - 8\bar{x}s + 4\bar{x}^2)\mathcal{Q}_{c,s}(x)^{-3} \\ &= (8s^2 - 8\bar{x}s)\mathcal{Q}_{c,s}(x)^{-3} - (8x_0 s - 8x_0\bar{x})\mathcal{Q}_{c,s}(x)^{-3} \\ &= 8(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(s-x_0),\end{aligned} \tag{12.26}$$

where in the fourth equality we used $|x|^2 + \bar{x}^2 = 2x_0\bar{x}$. Now, using formula (12.22) and Leibnitz formula for $\overline{\mathcal{D}}$, we can compute $\overline{\mathcal{D}}^2(S_L^{-1}(s,x))$:

$$\begin{aligned}&\overline{\mathcal{D}}^2(S_L^{-1}(s,x)) = \overline{\mathcal{D}}[4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}(s-x_0) + 2\mathcal{Q}_{c,s}(x)^{-1}] \\ &= 4\overline{\mathcal{D}}[(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}](s-x_0) - 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} + 2\overline{\mathcal{D}}[\mathcal{Q}_{c,s}(x)^{-1}] \\ &= 4(8(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}s - 8(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}x_0)(s-x_0) \\ &\quad - 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} + 4(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2} \\ &= 32(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(s-x_0)^2.\end{aligned}$$

□

Theorem 12.5.6 (Structure of the slice $\mathcal{D}^2$-kernels $S^{-1}_{\mathcal{D}^2,L}$ and $S^{-1}_{\mathcal{D}^2,R}$)**.** *Let x, $s \in \mathbb{R}^6$ be such that $x \notin [s]$, then*

$$S^{-1}_{\mathcal{D}^2,L}(s,x) := \mathcal{D}^2(S_L^{-1}(s,x)) = 8S_L^{-1}(s,\bar{x})\mathcal{Q}_{c,s}(x)^{-1} \tag{12.27}$$

and

$$S^{-1}_{\mathcal{D}^2,R}(s,x) := (S_R^{-1}(s,x))\mathcal{D}^2 = 8\mathcal{Q}_{c,s}(x)^{-1}S_R^{-1}(s,\bar{x}). \tag{12.28}$$

We denote by $S^{-1}_{\mathcal{D}^2,L}$ and $S^{-1}_{\mathcal{D}^2,R}$ the left and the right slice $\mathcal{D}^2$-kernels.

Proof. By (12.14) and relations (12.16) and (12.17), we have that

$$\begin{aligned}\mathcal{D}^2(S_L^{-1}(s,x)) &= -4\mathcal{D}(\mathcal{Q}_{c,s}(x)^{-1}) \\ &= -4(2s - 2x_0 - 2\sum_{i=1}^{5} e_i x_i)\mathcal{Q}_{c,s}(x)^{-2} \\ &= 8(x-s)\mathcal{Q}_{c,s}(x)^{-2}.\end{aligned}$$

□

Theorem 12.5.7 (Structure of the slice $\Delta\overline{\mathcal{D}}$-kernels $S^{-1}_{\Delta\overline{\mathcal{D}},L}$ and $S^{-1}_{\Delta\overline{\mathcal{D}},R}$). *Let $x,\, s \in \mathbb{R}^6$ be such that $x \notin [s]$, then*

$$S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,x) := \Delta\overline{\mathcal{D}}(S^{-1}_L(s,x)) = -64(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(s-x_0) \quad (12.29)$$

and

$$S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,x) := (S^{-1}_R(s,x))\Delta\overline{\mathcal{D}} = -64(s-x_0)\mathcal{Q}_{c,s}(x)^{-3}(s-\bar{x}). \quad (12.30)$$

We denote by $S^{-1}_{\Delta\overline{\mathcal{D}},L}$ and $S^{-1}_{\Delta\overline{\mathcal{D}},R}$ the left and the right slice $\Delta\overline{\mathcal{D}}$-kernels.

Proof. We show only formula (12.29) because it is possible to prove formula (12.30) with similar arguments. By formulas (12.18) and (12.26), we have that

$$\Delta\overline{\mathcal{D}}(S^{-1}_L(s,x)) = -8\overline{\mathcal{D}}((s-\bar{x})\mathcal{Q}_{c,s}(x)^{-2}) = -64(s-\bar{x})\mathcal{Q}_{c,s}(x)^{-3}(s-x_0).$$

□

Now, we study the regularity of the previous kernels.

Proposition 12.5.8. *Let $x,\, s \in \mathbb{R}^6$ be such that $x \notin [s]$. We have that*

1. *$S^{-1}_{\Delta^{1-\ell}\mathcal{D},L}(s,x)$ (resp. $S^{-1}_{\Delta^{1-\ell}\mathcal{D},R}(s,x)$) is slice right (resp. left) hyperholomorphic in s and it is $\ell+1$-harmonic in x for $0 \leq \ell \leq 1$;*

2. *$S^{-1}_{\Delta,L}(s,x)$ (resp. $S^{-1}_{\Delta,R}(s,x)$) is slice right (resp. left) hyperholomorphic in s and it is left (resp. right) holomorphic Cliffordian of order 1 in x;*

3. *$S^{-1}_{\overline{\mathcal{D}},L}(s,x)$ (resp. $S^{-1}_{\overline{\mathcal{D}},R}(s,x)$) is slice right (resp. left) hyperholomorphic in s and it is left (resp. right) polyanalytic Cliffordian of order $(1,2)$ in x;*

4. *$S^{-1}_{\Delta^{\ell}\overline{\mathcal{D}}^{2-l},L}(s,x)$ (resp. $S^{-1}_{\Delta^{\ell}\overline{\mathcal{D}}^{2-l},R}(s,x)$) is slice right (resp. left) hyperholomorphic in s and it is left (resp. right) polyanalytic of order $3-\ell$ in x for $0 \leq \ell \leq 1$;*

5. *$S^{-1}_{\mathcal{D}^2,L}(s,x)$ (resp. $S^{-1}_{\mathcal{D}^2,R}(s,x)$) is slice right (resp. left) hyperholomorphic in s and it is left (resp. right) anti-holomorphic Cliffordian of order 1 in x.*

Proof. We prove the regularity for the left kernels since for the right kernels the arguments are similar. The right slice hyperholomorphicity in s of the left kernels is due to the fact that each kernel can be written as a left combination of intrinsic slice hyperholomorphic functions in s: $\mathcal{Q}_{c,s}(x)^{-m}$, $s\mathcal{Q}_{c,s}(x)^{-m}$, $s^2\mathcal{Q}_{c,s}(x)^{-m}$ and $s^3\mathcal{Q}_{c,s}(x)^{-m}$ for $1 \leq m \leq 3$ (see Theorems from 12.5.1 to 12.5.7).

As the operators $\mathcal{D}$, $\overline{\mathcal{D}}$ and Δ can be commuted, we have:

1. $\Delta^{\ell+1}(S^{-1}_{\Delta^{1-\ell}\mathcal{D}}(s,x)) = \Delta^{\ell+1}(\Delta^{1-\ell}\mathcal{D}(S^{-1}_L(s,x))) = \mathcal{D}\Delta^2(S^{-1}_L(s,x)) = \mathcal{D}F^5_L(s,x) = 0$, wich implies $S^{-1}_{\Delta^{1-\ell}\mathcal{D},L}(s,x)$ is $\ell+1$-harmonic in x for $0 \leq \ell \leq 1$;

2. $\Delta\mathcal{D}(S^{-1}_{\Delta,L}(s,x)) = \Delta\mathcal{D}(\Delta(S^{-1}_L(s,x))) = \mathcal{D}\Delta^2(S^{-1}_L(s,x)) = \mathcal{D}F^5_L(s,x) = 0$, wich implies $S^{-1}_{\Delta,L}(s,x)$ is holomorphic Cliffordian of order 1 in x;

3. $\Delta\mathcal{D}^2(S^{-1}_{\overline{\mathcal{D}},L}(s,x)) = \Delta\mathcal{D}^2(\overline{\mathcal{D}}(S^{-1}_L(s,x))) = \mathcal{D}\Delta^2(S^{-1}_L(s,x)) = \mathcal{D}F^5_L(s,x) = 0$, wich implies $S^{-1}_{\overline{\mathcal{D}},L}(s,x)$ is polyanalytic Cliffordian of order $(1,2)$ in x;

4. $\mathcal{D}^{3-\ell}(S^{-1}_{\Delta^\ell\overline{\mathcal{D}}^{2-l},L}(s,x)) = \mathcal{D}^{3-\ell}(\Delta^\ell\overline{\mathcal{D}}^{2-l}(S^{-1}_L(s,x))) = \mathcal{D}\Delta^2(S^{-1}_L(s,x)) = \mathcal{D}F^5_L(s,x) = 0$, wich implies $S^{-1}_{\Delta^\ell\overline{\mathcal{D}}^{2-l},L}(s,x)$ is polyanalytic of order $3-\ell$ in x for $0 \leq \ell \leq 1$;

5. $\Delta\overline{\mathcal{D}}(S^{-1}_{\mathcal{D}^2,L}(s,x)) = \Delta\overline{\mathcal{D}}(\mathcal{D}^2(S^{-1}_L(s,x))) = \mathcal{D}\Delta^2(S^{-1}_L(s,x)) = \mathcal{D}F^5_L(s,x) = 0$, wich implies $S^{-1}_{\mathcal{D}^2,L}(s,x)$ is anti-holomorphic Cliffordian of order 1 in x.

□

Remark 12.5.9. We can write the formulas of left and right slice kernels in Theorem 12.5.1 up to Theorem 12.5.7 in terms of the F-kernels. By using formula (7.45) and (7.46) we have

$$\begin{aligned}
S^{-1}_{\mathcal{D},L}(s,x) &= -\frac{1}{16}\left[F^L_5(s,x)s^3 - (x+2x_0)F^L_5(s,x)s^2 + (2x_0x+|x|^2)F^L_5(s,x)s\right.\\
&\quad \left. -x|x|^2F^L_5(s,x)\right],\\
S^{-1}_{\mathcal{D},R}(s,x) &= -\frac{1}{16}\left[s^3F^R_5(s,x) - s^2F^R_5(s,x)(x+2x_0) + sF^R_5(s,x)(2x_0x+|x|^2)\right.\\
&\quad \left. -F^R_5(s,x)x|x|^2\right],
\end{aligned}$$

$$\begin{aligned} S^{-1}_{\Delta,L}(s,x) &= -\frac{1}{8}\left[F_5^L(s,x)s^2-2x_0F_5^L(s,x)s+|x|^2F_5^L(s,x)\right], \\ S^{-1}_{\Delta,R}(s,x) &= -\frac{1}{8}\left[s^2F_5^R(s,x)-2sF_5^R(s,x)x_0+F_5^R(s,x)|x|^2\right], \end{aligned}$$

$$\begin{aligned} S^{-1}_{\Delta\mathcal{D},L}(s,x) &= \frac{1}{4}\left[F_5^L(s,x)s-xF_5^L(s,x)\right], \\ S^{-1}_{\Delta\mathcal{D},R}(s,x) &= \frac{1}{4}\left[sF_5^R(s,x)-F_5^R(s,x)x\right], \end{aligned}$$

$$\begin{aligned} S^{-1}_{\overline{\mathcal{D}},L}(s,x) &= \frac{1}{32}\left[3F_5^L(s,x)s^3-(8x_0+x)F_5^L(s,x)s^2+(4x_0^2+2x_0x+3|x|^2)F_5^L(s,x)s\right. \\ &\quad \left.-(x|x|^2+2x_0|x|^2)F_5^L(s,x)\right], \\ S^{-1}_{\overline{\mathcal{D}},R}(s,x) &= \frac{1}{32}\left[3s^3F_5^R(s,x)-s^2F_5^R(s,x)(8x_0+x)+sF_5^R(s,x)(4x_0^2+2x_0x+3|x|^2)\right. \\ &\quad \left.-F_5^R(s,x)(x|x|^2+2x_0|x|^2)\right], \end{aligned}$$

$$\begin{aligned} S^{-1}_{\mathcal{D}^2,L}(s,x) &= -\frac{1}{8}\left[F_5^L(s,x)s^2-2xF_5^L(s,x)s+x^2F_5^L(s,x)\right], \\ S^{-1}_{\mathcal{D}^2,R}(s,x) &= -\frac{1}{8}\left[s^2F_5^R(s,x)-2sF_5^R(s,x)x+F_5^R(s,x)x^2\right], \end{aligned}$$

$$\begin{aligned} S^{-1}_{\overline{\mathcal{D}}^2,L}(s,x) &= \frac{1}{2}[F_5^L(s,x)s^2-2x_0F_5^L(s,x)s+x_0^2F_5^L(s,x)], \\ S^{-1}_{\overline{\mathcal{D}}^2,R}(s,x) &= \frac{1}{2}[s^2F_5^R(s,x)-2sF_5^R(s,x)x_0+F_5^R(s,x)x_0^2], \end{aligned}$$

$$\begin{aligned} S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,x) &= -F_5^L(s,x)s+x_0F_5^L(s,x), \\ S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,x) &= -sF_5^R(s,x)+F_5^R(s,x)x_0. \end{aligned}$$

The kernels $S^{-1}_{\Delta^\ell\overline{\mathcal{D}}^2,L}(s,x)$ and $S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,x)$, and the right counterparts, are written in terms of their polyanalytic decomposition.

12.5.1 The integral representation for the fine structure functions spaces

Now, we can give the integral representation for the functions of the fine structure spaces.

Theorem 12.5.10. *Let $W \subset \mathbb{R}^6$ be an open set. Let U be a slice Cauchy domain such that $\bar{U} \subset W$. Then for $J \in \mathbb{S}^4$ and $ds_J = ds(-J)$ we have the following integral representation.*

- **(Integral representation of ℓ-harmonic functions, $0 \leq \ell \leq 1$)** *If $f \in \mathcal{SH}_L(W)$, the function $\tilde{f}_\ell(x) := \Delta^{1-\ell}\mathcal{D}f(x)$ is $\ell+1$-harmonic for $0 \leq \ell \leq 1$ and it admits the following integral representation*

$$\tilde{f}_\ell(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\Delta^{1-\ell}\mathcal{D},L}(s,x) ds_J f(s).$$

  If $f \in \mathcal{SH}_R(W)$, the function $\tilde{f}_\ell(x) := f(x)\Delta^{1-\ell}\mathcal{D}$ is $\ell+1$-harmonic for $0 \leq \ell \leq 1$ and it admits the following integral representation

$$\tilde{f}_\ell(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\Delta^{1-\ell}\mathcal{D},R}(s,x).$$

- **(Integral representation of holomorphic Cliffordian functions of order 1)** *If $f \in \mathcal{SH}_L(W)$, the function $f^\circ(x) := \Delta f(x)$ is left holomorphic Cliffordian of order 1 and it admits the following integral representation*

$$f^\circ(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\Delta,L}(s,x) ds_J f(s).$$

  If $f \in \mathcal{SH}_R(W)$, the function $f^\circ(x) := \Delta f(x)$ is right holomorphic Cliffordian of order 1 and it admits the following integral representation

$$f^\circ(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\Delta,R}(s,x).$$

- **(Integral representation of polyanalytic Cliffordian functions of order $(1,2)$)** *If $f \in \mathcal{SH}_L(W)$, the function $\breve{f}^\circ(x) := \overline{\mathcal{D}}f(x)$ is left polyanalytic Cliffordian of order $(1,2)$ and it admits the following integral representation*

$$\breve{f}^\circ(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\overline{\mathcal{D}},L}(s,x) ds_J f(s).$$

If $f \in \mathcal{SH}_R(W)$, the function $\breve{f}^{\circ}(x) := \Delta f(x)$ is right polyanalytic Cliffordian of order $(1, 2)$ and it admits the following integral representation

$$\breve{f}^{\circ}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\overline{\mathcal{D}},R}(s, x).$$

- **(Integral representation of polyanalytic functions of order $3 - \ell$, $0 \leq \ell \leq 1$)** *If $f \in \mathcal{SH}_L(W)$, the function $\breve{f}_\ell(x) := \Delta^\ell \overline{\mathcal{D}}^{2-\ell} f(x)$ is left polyanalytic of order $3 - \ell$ for $0 \leq \ell \leq 1$ and it admits the following integral representation*

  $$\breve{f}_\ell(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\Delta^\ell \overline{\mathcal{D}}^{2-\ell},L}(s, x) ds_J f(s).$$

  If $f \in \mathcal{SH}_R(W)$, the function $\breve{f}_\ell(x) := f(x)\Delta^\ell \overline{\mathcal{D}}^{2-l}$ is right polyanalytic of order $3 - \ell$ for $0 \leq \ell \leq 1$ and it admits the following integral representation

  $$\breve{f}_\ell(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\Delta^\ell \overline{\mathcal{D}}^{2-l},R}(s, x).$$

- **(Integral representation of anti-holomorphic Cliffordian functions of order** 1) *If $f \in \mathcal{SH}_L(W)$, the function $f_0(x) := \mathcal{D}^2 f(x)$ is left anti-holomorphic Cliffordian of order 1 and it admits the following integral representation*

  $$f_\circ(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\mathcal{D}^2,L}(s, x) ds_J f(s).$$

  If $f \in \mathcal{SH}_R(W)$, the function $f_0(x) := f(x)\mathcal{D}^2$ is right anti-holomorphic Cliffordian of order 1 and it admits the following integral representation

  $$f_\circ(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\mathcal{D}^2,R}(s, x).$$

Moreover, the integrals do not depend on U nor on the imaginary unit $J \in \mathbb{S}^4$.

Proof. We prove the integral representation for the $\ell + 1$-harmonic functions starting from a left slice hyperholomorphic functions since the other cases can be proved with similar arguments. We start by using the Cauchy formula. By Theorem 12.5.1 and Theorem 12.5.3, we have for $0 \leq \ell \leq 1$

$$\begin{aligned}\tilde{f}_\ell(x) &= \Delta^{1-\ell}\mathcal{D}f(x)\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} \Delta^{1-\ell}\mathcal{D}S_L^{-1}(s,x)ds_I f(s)\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)} S^{-1}_{\Delta^{1-\ell}\mathcal{D},L}(s,x)ds_I f(s).\end{aligned}$$

By Proposition 7.4.4 the function $\tilde{f}_\ell(x)$ is $\ell+1$-harmonic. □

12.6 Series expansion of the kernels of the fine structures spaces

In this section our aim is to write the kernel of the previous integral theorem in terms of convergent series of x and $\bar{x}$. In order to do this we need to investigate the application of the operators $\mathcal{D}$, Δ, $\Delta\mathcal{D}$, $\overline{\mathcal{D}}$, $\overline{\mathcal{D}}^2$, $\mathcal{D}^2$ and $\Delta\overline{\mathcal{D}}$ to the monomial x^m, with $m \in \mathbb{N}$. We already know that:

Lemma 12.6.1. *[23, Lemma 1] For $m \geq 1$ we have*

$$\mathcal{D}(x^m) = (x^m)\mathcal{D} = -4\sum_{k=0}^{m-1}\overline{x}^{m-k-1}x^k = -4\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1}. \qquad (12.31)$$

We will use the following well-known equality

$$\Delta(xf(x)) = x\Delta f(x) + 2\mathcal{D}f(x), \qquad (12.32)$$

for any $x \in \mathbb{R}^6$ and for any $\mathcal{C}^2$ function f.

Proposition 12.6.2. *Let $x \in \mathbb{R}^6$ and $m \geq 2$. Then we have*

$$\Delta x^m = -8\sum_{k=1}^{m-1}(m-k)x^{m-k-1}\overline{x}^{k-1} = -8\sum_{k=1}^{m-1} kx^{k-1}\bar{x}^{m-k-1}. \qquad (12.33)$$

Proof. The proof is by induction on m. For $m=2$ and $x = x_0+\sum_{i=1}^5 e_i x_i = x_0 + \underline{x}$ we have

$$\Delta x^2 = \Delta(x_0^2 + 2x_0\underline{x} - |\underline{x}|^2) = -8.$$

Let us suppose that the statement is true for m, we want to prove it for

$m+1$. Then, we have

$$\begin{aligned}\Delta x^{m+1} &= \Delta(x^m x) = 2\mathcal{D}(x^m) + x\Delta x^m \\ &= -8\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1} - 8\sum_{k=1}^{m-1}(m-k)x^{m-k}\overline{x}^{k-1} \\ &= -8\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1} - 8\sum_{k=1}^{m}(m-k)x^{m-k}\overline{x}^{k-1} \\ &= -8\sum_{k=1}^{m}(m-k+1)x^{m-k}\overline{x}^{k-1},\end{aligned}$$

where the first equality is an application of formula (12.32) and the second equality is consequence of the inductive hypothesis and formula (12.31). The second equality follows by rearranging the indexes. $\square$

Proposition 12.6.3. *Let $x \in \mathbb{R}^6$, for $m \geq 2$ we have*

$$\mathcal{D}^2(x^m) = (x^m)\mathcal{D}^2 = -8\sum_{k=1}^{m-1} kx^{m-k-1}\overline{x}^{k-1}.$$

Proof. We show the result by induction on m. For $m = 2$, we have by formula (12.31) that

$$\mathcal{D}^2(x^2) = \mathcal{D}^2(x_0^2 + 2x_0\underline{x} - |\underline{x}|^2) = -8\mathcal{D}(x_0) = -8.$$

We suppose the statement is true for m and we prove it for $m+1$. First we observe that

$$x^{m+1} = x^m(\overline{x} + x) - x^{m-1}|x|^2.$$

Thus, by the Leibniz formula for the Dirac operator and the fact that $\mathcal{D}|x|^2 = 2x$ we get

$$\begin{aligned}\mathcal{D}(x^{m+1}) &= \mathcal{D}(x^m(\overline{x}+x)) - \mathcal{D}(x^{m-1}|x|^2) \\ &= \mathcal{D}(x^m)(\overline{x}+x) + 2x^m - \mathcal{D}(x^{m-1})|x|^2 - 2x^m \\ &= \mathcal{D}(x^m)(\overline{x}+x) - \mathcal{D}(x^{m-1})|x|^2.\end{aligned}$$

By using another time the Leibniz formula and the inductive hypothesis we get

$$\begin{aligned}
&\mathcal{D}^2(x^{m+1}) = \mathcal{D}\left(\mathcal{D}(x^m)(\overline{x}+x) - \mathcal{D}(x^{m-1})|x|^2\right)\\
&= (\mathcal{D}^2 x^m)(x+\overline{x}) + 2\mathcal{D}x^m - \mathcal{D}^2(x^{m-1})|x|^2 - 2x\mathcal{D}(x^{m-1})\\
&= -8\sum_{k=1}^{m-1} kx^{m-k}\overline{x}^{k-1} - 8\sum_{k=1}^{m-1} kx^{m-k-1}\overline{x}^{k} - 8\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1}\\
&\quad + 8\sum_{k=1}^{m-2} kx^{m-k-1}\overline{x}^{k} + 8\sum_{k=1}^{m-1} x^{m-k}\overline{x}^{k-1}\\
&= -8\sum_{k=1}^{m-1} kx^{m-k}\overline{x}^{k-1} - 8(m-1)\bar{x}^{n-1} - 8\bar{x}^{m-1}\\
&= -8\sum_{k=1}^{m} kx^{m-k}\overline{x}^{k-1}.
\end{aligned}$$

Finally since $\mathcal{D}(x^m) = (x^m)\mathcal{D}$ we get

$$(x^m)\mathcal{D}^2 = \left((x^m)\mathcal{D}\right)\mathcal{D} = \mathcal{D}(x^m)\mathcal{D} = \mathcal{D}^2(x^m).$$

$\square$

Now, we need some preliminaries results to get a formula for $\Delta\mathcal{D}x^m$.

Lemma 12.6.4. *Let $f : \mathbb{R}^{n+1} \to \mathbb{R}_n$ then*

$$\overline{\Delta f(x)} = \Delta\overline{f}(x).$$

Proof. We know that we can write $f(x) = \sum_{A\subset\{1,\dots,n\}} e_A f_A$, thus we have

$$\begin{aligned}
\overline{\Delta f(x)} &= \sum_{A\subset\{1,\dots,n\}} e_A \Delta f_A(x_A)\\
&= \sum_{A\subset\{1,\dots,n\}} \overline{e}_A \Delta f_A(x_A)\\
&= \Delta\left(\sum_{A\subset\{1,\dots,n\}} \overline{e}_A f_A(x_A)\right)\\
&= \Delta\overline{f}(x).
\end{aligned}$$

$\square$

Corollary 12.6.5. *Let $m \geq 2$. Then for $x \in \mathbb{R}^6$ we have*

$$\overline{\Delta x^m} = \Delta \overline{x}^m. \tag{12.34}$$

Proof. If we consider $n = 5$ and $f(x) = x^m$ in Lemma 12.6.4, we get the statement. □

Corollary 12.6.6. *Let $m \geq 2$. Then for any $x \in \mathbb{R}^6$ we have*

$$\Delta \bar{x}^m = -8 \sum_{k=1}^{m-1} k x^{m-k-1} \overline{x}^{k-1}.$$

Proof. It follows by Corollary 12.6.5 and Proposition 12.32. □

Lemma 12.6.7. *Let $m \geq 3$, for any $x \in \mathbb{R}^6$ we have*

$$\sum_{k=1}^{m} \Delta(x^{m-k}\overline{x}^{k-1}) = -4 \sum_{k=1}^{m-2} (m-k-1) k x^{m-k-2} \overline{x}^{k-1}.$$

Proof. We shall prove this formula by induction on m. For $m = 3$ we have

$$\Delta(x^2 + x\overline{x} + \overline{x}^2) = \Delta(3x_0^2 - |\underline{x}|^2) = -4.$$

Let us suppose the statement is true for m, we want to prove it for $m+1$. Now, formula (12.32) and Corollary 12.6.6 imply that

$$\begin{aligned}\sum_{k=1}^{m+1} \Delta(x^{m+1-k}\overline{x}^{k-1}) &= \sum_{k=1}^{m} \Delta(x^{m+1-k}\overline{x}^{k-1}) + \Delta \overline{x}^m \\ &= 2\sum_{k=1}^{m} \mathcal{D}(x^{m-k}\overline{x}^{k-1}) + x\sum_{k=1}^{m} \Delta(x^{m-k}\overline{x}^{k-1}) - 8\sum_{k=1}^{m-1} k x^{m-k-1}\overline{x}^{k-1}.\end{aligned}$$

Finally, by formula (12.31), the inductive hypothesis and Lemma 12.6.3 we get

$$\begin{aligned}\sum_{k=1}^{m+1} \Delta(x^{m+1-k}\overline{x}^{k-1}) &= -\frac{1}{2}\mathcal{D}^2 x^m + x\sum_{k=1}^{m}\Delta(x^{m-k}\overline{x}^{k-1}) - 8\sum_{k=1}^{m-1} kx^{m-k-1}\overline{x}^{k-1} \\ &= -\frac{1}{2}\mathcal{D}^2 x^m - 4\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-1}\overline{x}^{k-1} \\ &\quad -8\sum_{k=1}^{m-1} kx^{m-k-1}\overline{x}^{k-1} \\ &= 4\sum_{k=1}^{m-1} kx^{m-k-1}\overline{x}^{k-1} - 4\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-1}\overline{x}^{k-1} \\ &\quad -8\sum_{k=1}^{m-1} kx^{m-k-1}\overline{x}^{k-1} \\ &= -4\sum_{k=1}^{m-1}(m-k)kx^{m-k-1}\overline{x}^{k-1}.\end{aligned}$$

□

Proposition 12.6.8. *Let $m \geq 3$, for any $x \in \mathbb{R}^6$ we have*

$$\Delta\mathcal{D}x^m = 16\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-2}\overline{x}^{k-1}.$$

Proof. It follows by formula (12.31) and Lemma 12.6.7. □

Let us recall the following fact

$$\partial_{\underline{x}}(\underline{x}^m) = \begin{cases} -m\underline{x}^{m-1} & m \quad \text{even}, \\ -(m+4)\underline{x}^{m-1} & m \quad \text{odd}, \end{cases} \tag{12.35}$$

where $\partial_{\underline{x}} = \sum_{j=1}^{5} e_j \frac{\partial}{\partial x_j}$.

Proposition 12.6.9. *Let $m \geq 1$. For any $x \in \mathbb{R}^6$, if $\underline{x} \neq 0$ we have*

$$\overline{\mathcal{D}}x^m = 2\left[mx^{m-1} + 2\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1}\right]. \tag{12.36}$$

On the other hand if $\underline{x} = 0$ we have

$$\overline{\mathcal{D}}x^m = 6mx^{m-1}.$$

Moreover,

$$\overline{\mathcal{D}}(x^m) = (x^m)\overline{\mathcal{D}}.$$

Proof. We will perform a direct computations. By the Binomial theorem and formula (12.35) we get

$$\begin{aligned}
\overline{\mathcal{D}}(x^m) &= \left(\frac{\partial}{\partial x_0} - \partial_{\underline{x}}\right)(x_0 + \underline{x})^m = \left(\frac{\partial}{\partial x_0} - \partial_{\underline{x}}\right)\left(\sum_{k=0}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^k\right) \\
&= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)x_0^{m-k-1}\underline{x}^k - \sum_{k=0}^{m}\binom{m}{k}x_0^{m-k}\partial_{\underline{x}}(\underline{x}^k) \\
&= m\sum_{k=0}^{m-1}\frac{(m-1)!}{k!(m-k-1)!}x_0^{m-k-1}\underline{x}^k + m\sum_{k=1}^{m}\frac{(m-1)!}{(k-1)!(m-k)!}x_0^{m-k}\underline{x}^{k-1} + \\
&\quad +4\sum_{k=1,k \text{ odd}}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^{k-1}.
\end{aligned}$$

By rearranging the indices of the sum and by using another time the binomial theorem we get

$$\overline{\mathcal{D}}(x^m) = 2mx^{m-1} + 4\sum_{k=1,k \text{ odd}}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^{k-1}. \tag{12.37}$$

If $\underline{x} = 0$ we get

$$\overline{\mathcal{D}}x^m = 6mx^{m-1}.$$

On the other side, if $\underline{x} \neq 0$ by the Binomial theorem we have

$$\begin{aligned}
2\sum_{k=1,k \text{ odd}}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^{k-1} &= \sum_{k=1}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^{k-1} + \sum_{k=1}^{m}\binom{m}{k}x_0^{m-k}(-\underline{x})^{k-1} \\
&= (\underline{x})^{-1}\left[\sum_{k=1}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^k - \sum_{k=1}^{m}\binom{m}{k}x_0^{m-k}(-\underline{x})^k\right] \\
&= (\underline{x})^{-1}(x^m - \bar{x}^m).
\end{aligned}$$

Now, since $x^m - \bar{x}^m = 2\underline{x}\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}$ we get

$$2\sum_{k=1}^{m}\binom{m}{k}x_0^{m-k}\underline{x}^{k-1} = 2\sum_{k=1}^{m}x^{m-k}\bar{x}^{k-1}. \tag{12.38}$$

By putting formula (12.38) in (12.37) we obtain formula (12.36).
Finally, since $\partial_{\underline{x}}(\underline{x}^m) = (\underline{x}^m)\partial_{\underline{x}}$, we can repeat the previous computations, thus we get

$$\overline{\mathcal{D}}(x^m) = (x^m)\overline{\mathcal{D}}.$$

□

To compute $\overline{\mathcal{D}}^2 x^m$ we need the following result.

Lemma 12.6.10. *Let $m \geq 2$. For any $x \in \mathbb{R}^6$ we have*

$$\partial_{\underline{x}}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) = \sum_{k=1}^{m-1}(2k-m)x^{m-k-1}\bar{x}^{k-1}.$$

Proof. By Proposition 12.6.3 and formula (12.31) we have

$$\begin{aligned} -8\sum_{k=1}^{m-1} kx^{m-k-1}\bar{x}^{k-1} &= \mathcal{D}(\mathcal{D}x^m) = -4\mathcal{D}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) \\ &= -4\left[\frac{\partial}{\partial x_0}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) + \partial_{\underline{x}}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right)\right]. \end{aligned} \tag{12.39}$$

Now, since $\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1} = (2\underline{x})^{-1}(x^m - \bar{x}^n)$ we get

$$\begin{aligned} \frac{\partial}{\partial x_0}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) &= \frac{\partial}{\partial x_0}\left[(2\underline{x})^{-1}(x^m - \bar{x}^m)\right] \\ &= (2\underline{x})^{-1}m(x^{m-1} - \overline{x}^{m-1}) \\ &= m\sum_{k=1}^{m-1} x^{m-1-k}\bar{x}^{k-1}. \end{aligned} \tag{12.40}$$

Therefore by formula (12.39) and formula (12.40) we get

$$\begin{aligned} \partial_{\underline{x}}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) &= 2\sum_{k=1}^{m-1} kx^{m-k-1}\bar{x}^{k-1} - \frac{\partial}{\partial x_0}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) \\ &= 2\sum_{k=1}^{m-1} kx^{m-k-1}\bar{x}^{k-1} - m\sum_{k=1}^{m-1} x^{m-1-k}\bar{x}^{k-1} \\ &= \sum_{k=1}^{m-1}(2k-m)x^{m-k-1}\bar{x}^{k-1}. \end{aligned}$$

□

Proposition 12.6.11. *Let $m \geq 2$. Then for any $x \in \mathbb{R}^6$ we have*

$$\begin{aligned} \overline{\mathcal{D}}^2(x^m) &= (x^m)\overline{\mathcal{D}}^2 \\ &= 4\left[m(m-1)x^{m-2} + 2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\bar{x}^{k-1}\right]. \end{aligned} \tag{12.41}$$

Proof. By applying two times Proposition 12.6.9 and the fact that $\overline{\mathcal{D}} = \frac{\partial}{\partial x_0} - \partial_{\underline{x}}$ we have

$$\begin{aligned} \overline{\mathcal{D}}^2(x^m) &= \overline{\mathcal{D}}(\overline{\mathcal{D}}x^m) \\ &= 2m\overline{\mathcal{D}}(x^{m-1}) + 4\overline{\mathcal{D}}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) \\ &= 2m\left(2(m-1)x^{m-2} + 4\sum_{k=1}^{m-1} x^{m-1-k}\bar{x}^{k-1}\right) + \\ &\quad +4\left[\frac{\partial}{\partial x_0}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right) - \partial_{\underline{x}}\left(\sum_{k=1}^{m} x^{m-k}\bar{x}^{k-1}\right)\right]. \end{aligned}$$

By formula (12.40) and Lemma 12.6.10 we get

$$\begin{aligned} \overline{\mathcal{D}}^2 x^m &= 4\Bigg[m(m-1)x^{m-2} + 2m\sum_{k=1}^{m-1} x^{m-1-k}\bar{x}^{k-1} \\ &\quad +m\sum_{k=1}^{m-1} x^{m-1-k}\bar{x}^{k-1} - \sum_{k=1}^{m-1}(2k-m)x^{m-k-1}\bar{x}^{k-1}\Bigg] \\ &= 4\left[m(m-1)x^{m-2} + 2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\bar{x}^{k-1}\right]. \end{aligned}$$

Finally, since $\overline{\mathcal{D}}(x^m) = (x^m)\overline{\mathcal{D}}$ we get

$$\overline{\mathcal{D}}^2(x^m) = (x^m)\overline{\mathcal{D}}^2.$$

□

Proposition 12.6.12. *Let $m \geq 3$. For any $x \in \mathbb{R}^6$ we have*

$$\Delta\overline{\mathcal{D}}(x^m) = (x^m)\Delta\overline{\mathcal{D}} = -16\sum_{k=1}^{m-2}(m-k-1)(m+k)x^{m-k-2}\bar{x}^{k-1}. \tag{12.42}$$

Proof. By applying the operator Δ to formula (12.6.10) we get

$$\Delta\overline{\mathcal{D}}x^m = 2m\Delta(x^{m-1}) + 4\sum_{k=1}^{m}\Delta(x^{m-k}\bar{x}^{k-1}).$$

Therefore, by Proposition 12.6.2 and Lemma 12.6.7 we obtain

$$\begin{aligned}\Delta\overline{\mathcal{D}}(x^m) &= -16m\sum_{k=1}^{m-2}(m-1-k)x^{m-2-k}\bar{x}^{k-1} + \\ &\quad -16\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-2}\bar{x}^{k-1} \\ &= -16\sum_{k=1}^{m-2}(m-k-1)(m+k)x^{m-2-k}\bar{x}^{k-1}.\end{aligned}$$

Finally, since the laplacian is a real operator and $\overline{\mathcal{D}}(x^m) = (x^m)\overline{\mathcal{D}}$ we get

$$\Delta\overline{\mathcal{D}}(x^m) = (x^m)\Delta\overline{\mathcal{D}}.$$

□

Definition 12.6.13 (left and right kernel series)**.** Let s, $x \in \mathbb{R}^6$, then we define

- the left $\mathcal{D}$-kernel series as

$$-4\sum_{m=1}^{\infty}\sum_{k=1}^{m}x^{m-k}\overline{x}^{k-1}s^{-1-m}, \tag{12.43}$$

  and the right $\mathcal{D}$-kernel series as

$$-4\sum_{m=1}^{\infty}\sum_{k=1}^{m}s^{-1-m}x^{m-k}\overline{x}^{k-1},$$

- the left Δ-kernel series as

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)x^{m-k-1}\overline{x}^{k-1}s^{-1-m}, \tag{12.44}$$

  the right Δ-kernel series as

$$-8\sum_{m=2}^{\infty}\sum_{k=2}^{m-1}(m-k)s^{-1-m}x^{m-k-1}\overline{x}^{k-1},$$

- the left $\Delta\mathcal{D}$-kernel series as

$$16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-2}\overline{x}^{k-1}s^{-1-m}, \tag{12.45}$$

the right $\Delta\mathcal{D}$-kernel series as

$$16\sum_{m=3}^{\infty}\sum_{k=3}^{m-2}(m-k-1)ks^{-1-m}x^{m-k-2}\overline{x}^{k-1},$$

- the left $\mathcal{D}^2$-kernel series as

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}kx^{m-k-1}\overline{x}^{k-1}s^{-1-m}, \tag{12.46}$$

the right $\mathcal{D}^2$-kernel series as

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}ks^{-1-m}x^{m-k-1}\overline{x}^{k-1},$$

- the left $\overline{\mathcal{D}}$-kernel series as

$$2\left[\sum_{m=1}^{\infty}\left(mx^{m-1}+2\sum_{k=1}^{m}x^{m-k}\overline{x}^{k-1}\right)\right]s^{-1-m}, \tag{12.47}$$

the right $\overline{\mathcal{D}}$-kernel series as

$$2\left[\sum_{m=1}^{\infty}s^{-1-m}\left(mx^{m-1}+2\sum_{k=1}^{m}x^{m-k}\overline{x}^{k-1}\right)\right],$$

- the left $\overline{\mathcal{D}}^2$-kernel series as

$$4\left[\sum_{m=2}^{\infty}\left(m(m-1)x^{m-2}+2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\overline{x}^{k-1}\right)s^{-1-m}\right], \tag{12.48}$$

the right $\overline{\mathcal{D}}^2$-kernel series as

$$4\left[\sum_{m=2}^{\infty}s^{-1-m}\left(m(m-1)x^{m-2}+2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\overline{x}^{k-1}\right)\right],$$

- the left $\Delta\overline{\mathcal{D}}$-kernel series as

$$-16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)x^{m-k-2}\bar{x}^{k-1}s^{-1-m}. \quad (12.49)$$

the right $\Delta\overline{\mathcal{D}}$-kernel series as

$$-16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)s^{-1-m}x^{m-k-2}\bar{x}^{k-1}.$$

Remark 12.6.14. The left and the right $\mathcal{D}$-kernel series are equal, where they converge, as well as the left and the right $\Delta\mathcal{D}$-kernel series.

We collect some technical lemmas that we have used in the proofs of some of the following results.

Lemma 12.6.15. *For $m \geq 3$ we have*

$$\sum_{k=1}^{m-2}(m-k-1)k = \frac{m(m-1)(m-2)}{6}.$$

Proof. We know that $\sum_{k=1}^{m-2} k = \frac{(m-1)(m-2)}{2}$ and $\sum_{k=1}^{m-2} k^2 = \frac{(m-2)(m-1)(2m-3)}{6}$. Thus we have

$$\begin{aligned}\sum_{k=1}^{m-2}(m-k-1)k &= (m-1)\sum_{k=1}^{m-2} k - \sum_{k=1}^{m-2} k^2\\ &= \frac{(m-1)^2(m-2)}{2} - \frac{(m-2)(m-1)(2m-3)}{6}\\ &= \frac{(m-1)(m-2)}{2}\left((m-1) - \frac{(2m-3)}{3}\right)\\ &= \frac{m(m-1)(m-2)}{6}.\end{aligned}$$

□

Lemma 12.6.16. *For $m \geq 3$ we have*

$$\sum_{k=1}^{m-2}(m-k-1)(m+k) = \frac{2m(m-1)(m-2)}{3}.$$

Proof. Since $\sum_{k=1}^{m-2} k = \frac{(m-1)(m-2)}{2}$ we get

$$\begin{aligned} m\sum_{k=1}^{m-2}(m-1-k) &= m^2(m-2) - m(m-2) - \frac{m(m-1)(m-2)}{2} \\ &= \frac{m(m-1)(m-2)}{2}. \end{aligned} \tag{12.50}$$

Finally, by formula (12.50) and Lemma 12.6.15 we get

$$\begin{aligned} \sum_{k=1}^{m-2}(m-k-1)(m+k) &= \sum_{k=1}^{m-2}(m-k-1)k + m\sum_{k=1}^{m-2}(m-k-1) \\ &= \frac{m(m-1)(m-2)}{6} + \frac{m(m-1)(m-2)}{2} \\ &= \frac{2m(m-1)(m-2)}{3}. \end{aligned}$$

□

Proposition 12.6.17. *For $s,x \in \mathbb{R}^6$ with $|x| < |s|$, all the left kernel series are convergent.*

Proof. In order to show the convergence of the series it is enough to show the convergence of the series of the moduli.

- In order to prove that the left $\mathcal{D}$-kernel series is convergent it is sufficient to show that the following series is convergent.

$$\sum_{m=1}^{\infty} m|x|^{m-1}|s|^{-1-m}. \tag{12.51}$$

  This series converges by the ratio test. Indeed, since $|x| < |s|$, we have

$$\lim_{m\to\infty} \frac{(m+1)|x|^m|s|^{-2-m}}{m|x|^{m-1}|s|^{-1-m}} = \lim_{m\to\infty} \frac{m+1}{m}|x||s|^{-1} < 1.$$

- To prove the convergence of the Δ-kernel series, we have to show that the following series convergences

$$\sum_{m=2}^{\infty}\left(\sum_{k=1}^{m-1}(m-k)\right)|x|^{m-2}|s|^{-1-m}. \tag{12.52}$$

Since $\sum_{k=1}^{m-1}(m-k)=\frac{m(m-1)}{2}$, we have

$$\lim_{m\to\infty}\frac{(m+1)m|x|^{m-1}|s|^{-2-m}}{m(m-1)|x|^{m-2}|s|^{-1-m}}=|x||s|^{-1}<1. \tag{12.53}$$

Therefore, by the ratio test the series is convergent.

- In order to prove the convergence of the $\Delta\mathcal{D}$-kernel series, we have to show the convergence of the following series

$$\sum_{m=1}^{\infty}\left(\sum_{k=1}^{m-2}(m-k-1)k\right)|x|^{m-3}|s|^{-1-m}. \tag{12.54}$$

Now, since $\sum_{k=1}^{m-2}(m-k-1)k=\frac{m(m-1)(m-2)}{6}$ (see Lemma 12.6.15), by applying the ratio test we get

$$\lim_{m\to\infty}\frac{m(m+1)(m-1)|x|^{m-2}|s|^{-2-m}}{m(m-1)(m-2)|x|^{m-3}|s|^{-1-m}}=|x||s|^{-1}<1.$$

Therefore the series is convergent.

- To show the convergence of the $\mathcal{D}^2$- kernel series we need to show the convergence of the following series

$$\sum_{m=2}^{\infty}\left(\sum_{k=1}^{m-1}k\right)|x|^{m-2}|s|^{-1-m}.$$

Since $\sum_{k=1}^{m-1}k=\frac{(m-1)m}{2}$, by applying the ratio test we reobtain the same limit of (12.53), and so the series is convergent.

- In order to prove the convergence of the $\overline{\mathcal{D}}$-kernel series we put the modulus to the series in (12.47) and after some manipulations we get that

$$\left|\sum_{m=1}^{\infty}\left(mx^{m-1}+2\sum_{k=1}^{m}x^{m-k}\overline{x}^{k-1}\right)\right||s|^{-1-m}\leq\sum_{m=1}^{\infty}3m|x|^{m-1}|s|^{-1-m}.$$

The convergence of the series $\sum_{m=1}^{\infty}3m|x|^{m-1}|s|^{-1-m}$ follows by similar arguments for the convregence of the series in (12.51).

- As done in the previous point, we insert the modulus in the series

(12.48) and after some computations we get

$$\begin{aligned}
&\left| \sum_{m=2}^{\infty} \left(m(m-1)x^{m-2} + 2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\overline{x}^{k-1} \right) \right| |s|^{-1-m} \\
&\leq \sum_{m=2}^{\infty} \left(m(m-1) + \sum_{k=1}^{m-1}(2m-k) \right) |x|^{m-2}|s|^{-1-m} \\
&= \sum_{m=2}^{\infty} \frac{5m(m-1)}{2}|x|^{m-2}|s|^{-1-m}.
\end{aligned}$$

The convergence of the series $\sum_{m=2}^{\infty} \frac{5m(m-1)}{2}|x|^{m-2}|s|^{-1-m}$ follows similarly as the series in (12.52).

- Finally, in order to show the convergence of the left $\Delta\overline{\mathcal{D}}$ it is enough to show the convergence of the following series

$$\sum_{m=3}^{\infty} \left(\sum_{k=1}^{m-2}(m-k-1)(m+k) \right) |x|^{m-3}|s|^{-1-m}.$$

Since $\sum_{k=1}^{m-2}(m-k-1)(m+k) = \frac{2m(m-1)(m-2)}{3}$ (see Lemma 12.6.16), the convergence follows similarly as done for the series in (12.54).

□

Remark 12.6.18. Similarly, all the right kernel series are convergent.

Now, we can expand in series the left and the right kernels, computed in the previous section.

Lemma 12.6.19. *For s, $x \in \mathbb{R}^6$ such that $|x| < |s|$ we can expand*

- *the left slice $\mathcal{D}$-kernel as*

$$S^{-1}_{\mathcal{D},L}(s,x) = -4\sum_{m=1}^{\infty}\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1}s^{-1-m},$$

and the right slice $\mathcal{D}$-kernel as

$$S^{-1}_{\mathcal{D},R}(s,x) = -4\sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}x^{m-k}\overline{x}^{k-1},$$

- *the left slice Δ-kernel as*

$$S^{-1}_{\Delta,L}(s,x) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)x^{m-k-1}\overline{x}^{k-1}s^{-1-m},$$

and the right slice Δ-kernel as

$$S^{-1}_{\Delta,R}(s,x) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)s^{-1-m}x^{m-k-1}\overline{x}^{k-1},$$

- *the left slice $\Delta\mathcal{D}$-kernel as*

$$S^{-1}_{\Delta\mathcal{D},L}(s,x) = 16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-2}\overline{x}^{k-1}s^{-1-m},$$

and the right slice $\Delta\mathcal{D}$-kernel as

$$S^{-1}_{\Delta\mathcal{D},R}(s,x) = 16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)ks^{-1-m}x^{m-k-2}\overline{x}^{k-1},$$

- *we can expand the left slice $\mathcal{D}^2$-kernel as*

$$S^{-1}_{\mathcal{D}^2,L}(s,x) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}kx^{m-k-1}\bar{x}^{k-1}s^{-1-m},$$

and the right slice $\mathcal{D}^2$-kernel as

$$S^{-1}_{\mathcal{D}^2,R}(s,x) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}ks^{-1-m}x^{m-k-1}\bar{x}^{k-1},$$

- *the left slice $\overline{\mathcal{D}}$-kernel as*

$$S^{-1}_{\overline{\mathcal{D}},L}(s,x) = 2\left[\sum_{m=1}^{\infty}\left(mx^{m-1}+2\sum_{k=1}^{m}x^{m-k}\bar{x}^{k-1}\right)\right]s^{-1-m},$$

and the right slice $\overline{\mathcal{D}}$-kernel as

$$S^{-1}_{\overline{\mathcal{D}},R}(s,x) = 2\left[\sum_{m=1}^{\infty}s^{-1-m}\left(mx^{m-1}+2\sum_{k=1}^{m}x^{m-k}\bar{x}^{k-1}\right)\right],$$

- *the left slice $\mathcal{D}^2$-kernel as*

$$S^{-1}_{\overline{\mathcal{D}}^2,L}(s,x)=4\left[\sum_{m=2}^{\infty}\left(m(m-1)x^{m-2}+2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\bar{x}^{k-1}\right)s^{-1-m}\right],$$

and the right slice $\overline{\mathcal{D}}^2$-kernel as

$$S^{-1}_{\overline{\mathcal{D}}^2,R}(s,x)=4\left[\sum_{m=2}^{\infty}s^{-1-m}\left(m(m-1)x^{m-2}+2\sum_{k=1}^{m-1}(2m-k)x^{m-k-1}\bar{x}^{k-1}\right)\right],$$

- *the left slice $\Delta\overline{\mathcal{D}}$-kernel as*

$$S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,x)=-16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)x^{m-k-2}\bar{x}^{k-1}s^{-1-m},$$

and the right slice $\Delta\overline{\mathcal{D}}$-kernel as

$$S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,x)=-16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)s^{-1-m}x^{m-k-2}\bar{x}^{k-1}.$$

Proof. We know that we can expand the left Cauchy kernel in the following way

$$S^{-1}_L(s,x)=\sum_{m=0}^{\infty}x^m s^{-1-m}. \tag{12.55}$$

In order to obtain all the expansions it is enough to apply to formula (12.55) the operators $\mathcal{D}$, Δ, $\Delta\mathcal{D}$, $\mathcal{D}^2$, $\overline{\mathcal{D}}$, $\overline{D}^2$ and $\Delta\overline{\mathcal{D}}$. Due to Proposition 12.6.17 we can exchange the roles of the operators with the sum. Finally, in order to get the expansions written in terms of x and $\bar{x}$ we apply formula (12.31), Proposition 12.6.2, Proposition 12.6.8, Proposition 12.6.3, Proposition 12.6.9, Proposition 12.6.11 and Proposition 12.6.12. By similar arguments we have the result for the right kernels. □

Remark 12.6.20. By Lemma 12.6.19 together with Theorem 12.5.1, Theorem 12.5.2, Theorem 12.5.3 and Theorem 12.5.6, we deduce the following equalities

$$\begin{aligned}\mathcal{Q}_{c,s}(x)^{-2} &= \sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)kx^{m-k-2}\overline{x}^{k-1}s^{-1-m}\\ &= \sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)ks^{-1-m}x^{m-k-2}\overline{x}^{k-1},\end{aligned}$$

$$S_L^{-1}(s,x)\mathcal{Q}_{c,s}(x)^{-1} = \sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)x^{m-k-1}\overline{x}^{k-1}s^{-1-m},$$

$$Q_{c,s}(x)^{-1} = \sum_{m=1}^{\infty}\sum_{k=1}^{m} x^{m-k}\overline{x}^{k-1}s^{-1-m} = \sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}x^{m-k}\overline{x}^{k-1},$$

$$S_L^{-1}(s,\bar{x})\mathcal{Q}_{c,s}(x)^{-1} = \sum_{m=2}^{\infty}\sum_{k=1}^{m-1} kx^{m-k-1}\bar{x}^{k-1}s^{-1-m}.$$

It is possible to obtain similar equalities with the right kernels.

12.7 The functional calculi of the fine structures

Using the expressions of the kernels written in terms of x and $\bar{x}$ and the fine Fueter-Sce integral theorems, we can define the fine Fueter-Sce functional calculi.

Definition 12.7.1. Let $T = T_0 + \sum_{i=1}^{5} e_iT_i \in \mathcal{BC}^{0,1}(V_5)$, $s \in \mathbb{R}^6$. We formally define

- the left and the right $\mathcal{D}$-kernel operator series as

$$-4\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\bar{T}^{k-1}s^{-1-m},$$

  and

$$-4\sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}T^{m-k}\bar{T}^{k-1};$$

- the left and the right Δ-kernel operator series as

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)T^{m-k-1}\bar{T}^{k-1}s^{-1-m},$$

  and

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)s^{-1-m}T^{m-k-1}\bar{T}^{k-1};$$

- the left and the right $\Delta\mathcal{D}$-kernel operator series as

$$16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)kT^{m-k-2}\bar{T}^{k-1}s^{-1-m},$$

and

$$16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)ks^{-1-m}T^{m-k-2}\bar{T}^{k-1}.$$

- the left and the right $\mathcal{D}^2$-kernel operator series as

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}kT^{m-k-1}\bar{T}^{k-1}s^{-1-m},$$

and

$$-8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}ks^{-1-m}T^{m-k-1}\bar{T}^{k-1};$$

- the left and the right $\overline{\mathcal{D}}$-kernel operator series as

$$2\left[\sum_{m=1}^{\infty}\left(mT^{m-1}+2\sum_{k=1}^{m}T^{m-k}\bar{T}^{k-1}\right)s^{-1-m}\right],$$

and

$$2\left[\sum_{m=1}^{\infty}s^{-1-m}\left(mT^{m-1}+2\sum_{k=1}^{m}T^{m-k}\bar{T}^{k-1}\right)\right];$$

- the left and the right $\overline{\mathcal{D}}^2$-kernel operator series as

$$4\left[\sum_{m=2}^{\infty}\left(m(m-1)T^{m-2}+2\sum_{k=1}^{m-1}(2m-k)T^{m-k-1}\bar{T}^{k-1}\right)s^{-1-m}\right],$$

and

$$4\left[\sum_{m=2}^{\infty}s^{-1-m}\left(m(m-1)T^{m-2}+2\sum_{k=1}^{m-1}(2m-k)T^{m-k-1}\bar{T}^{k-1}\right)\right];$$

- the left and the right $\Delta\overline{\mathcal{D}}$-kernel operator series as

$$-16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)T^{m-k-2}\bar{T}^{k-1}s^{-1-m},$$

and

$$-16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)s^{-1-m}T^{m-k-2}\bar{T}^{k-1}.$$

Proposition 12.7.2. *Let us consider $T \in \mathcal{BC}^{0,1}(V_5)$, $s \in \mathbb{R}^6$ and $\|T\| < |s|$ then the series previously defined are convergent and, in particular, we can expand*

- *the left and right $\mathcal{D}$-resolvent operator as*

$$\begin{aligned} S^{-1}_{\mathcal{D},L}(s,T) &= S^{-1}_{\mathcal{D},R}(s,T) = -4\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\bar{T}^{k-1}s^{-1-m} \\ &= -4\sum_{m=1}^{\infty}\sum_{k=1}^{m} s^{-1-m}T^{m-k}\bar{T}^{k-1}; \end{aligned}$$

- *the left and right Δ-resolvent operator as*

$$S^{-1}_{\Delta,L}(s,T) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)T^{m-k-1}\bar{T}^{k-1}s^{-1-m},$$

and

$$S^{-1}_{\Delta,R}(s,T) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}(m-k)s^{-1-m}T^{m-k-1}\bar{T}^{k-1};$$

- *the left and right $\Delta\mathcal{D}$-resolvent operator as*

$$\begin{aligned} S^{-1}_{\Delta\mathcal{D},L}(s,T) &= S^{-1}_{\Delta\mathcal{D},R}(s,T) \\ &= 16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)kT^{m-k-2}\bar{T}^{k-1}s^{-1-m} \\ &= 16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)ks^{-1-m}T^{m-k-2}\bar{T}^{k-1}; \end{aligned}$$

- *the left and right $\mathcal{D}^2$-resolvent operator as*

$$S^{-1}_{\mathcal{D}^2,L}(s,T) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}kT^{m-k-1}\bar{T}^{k-1}s^{-1-m},$$

and

$$S^{-1}_{\mathcal{D}^2,R}(s,T) = -8\sum_{m=2}^{\infty}\sum_{k=1}^{m-1}ks^{-1-m}T^{m-k-1}\bar{T}^{k-1};$$

- *the left and right $\overline{\mathcal{D}}$-resolvent operator as*

$$S^{-1}_{\overline{\mathcal{D}},L}(s,T) = 2\left[\sum_{m=1}^{\infty}\left(mT^{m-1} + 2\sum_{k=1}^{m} T^{m-k}\bar{T}^{k-1}\right)s^{-1-m}\right],$$

and

$$S^{-1}_{\overline{\mathcal{D}},R}(s,T) = 2\left[\sum_{m=1}^{\infty} s^{-1-m}\left(mT^{m-1} + 2\sum_{k=1}^{m} T^{m-k}\bar{T}^{k-1}\right)\right];$$

- *the left and right $\overline{\mathcal{D}}^2$-resolvent operator as*

$$S^{-1}_{\overline{\mathcal{D}}^2,L}(s,T) = 4\left[\sum_{m=2}^{\infty}\left(m(m-1)T^{m-2} + 2\sum_{k=1}^{m-1}(2m-k)T^{m-k-1}\bar{T}^{k-1}\right)s^{-1-m}\right],$$

and

$$S^{-1}_{\overline{\mathcal{D}}^2,R}(s,T) = 4\left[\sum_{m=2}^{\infty} s^{-1-m}\left(m(m-1)T^{m-2} + 2\sum_{k=1}^{m-1}(2m-k)T^{m-k-1}\bar{T}^{k-1}\right)\right];$$

- *the left and right $\Delta\overline{\mathcal{D}}$-resolvent operator as*

$$S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,T) = -16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)T^{m-k-1}\bar{T}^{k-1}s^{-1-m},$$

and

$$S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,T) = -16\sum_{m=3}^{\infty}\sum_{k=1}^{m-2}(m-k-1)(m+k)s^{-1-m}T^{m-k-1}\bar{T}^{k-1}.$$

Proof. The convergences of the series for $\|T\| < |s|$ can be proved considering the series of the operator norms and reasoning as in Proposition 12.6.17. We prove only the first equality between the kernels and the series because the other equalities follow by similar arguments. Since

$$S^{-1}_{\mathcal{D},L}(s,T) = S^{-1}_{\mathcal{D},R}(s,T) = -4\mathcal{Q}_{c,s}(T)^{-1},$$

it is sufficient to prove

$$\begin{aligned}\mathcal{Q}_{c,s}(T)\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right) &= \left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T) \\ &= \mathcal{I}. \end{aligned} \tag{12.56}$$

The first equality in (12.56) is a consequence of the following facts: for any positive integer m the sum $\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}$ is an operator of the components of T with real coefficients which then commute with any power of s, the components of T are commuting among each other and the operator $\mathcal{Q}_{c,s}(T)$ can be written in the following form: $s^2\mathcal{I} - 2sT_0 + \sum_{i=0}^{5} T_i^2$. Now we want to prove the second equality in (12.56). First we observe that

$$\begin{aligned}
&\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T)\\
&=\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)(s^2 - s(T+\overline{T}) + T\overline{T})\\
&=\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{1-m} - \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m+1-k}\overline{T}^{k-1}s^{-m} - \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k}s^{-m}\\
&\quad+\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k+1}\bar{T}^{k}s^{-1-m}.
\end{aligned}$$

Making the change of index $m' = 1 + m$ in the second and fourth series, we have

$$\begin{aligned}
&\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T) =\\
&=\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{1-m} - \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k-1}s^{1-m'} - \sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k}s^{-m}\\
&\quad+\sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k}s^{-m'}\\
&=\mathcal{I}+\sum_{m=2}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{1-m} - \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k-1}s^{1-m'}+\\
&\quad-\bar{T}s^{-1} - \sum_{m=2}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k}s^{-m} + \sum_{m'=2}^{\infty}\sum_{k=1}^{m'-1} T^{m'-k}\overline{T}^{k}s^{-m'}.
\end{aligned}$$

Simplifying the opposite terms in the first and second series and in the third and fourth series, in the end we get

$$\begin{aligned}\left(\sum_{m=1}^{\infty}\sum_{k=1}^{m} T^{m-k}\overline{T}^{k-1}s^{-1-m}\right)\mathcal{Q}_{c,s}(T) &= \mathcal{I}+\sum_{m=2}^{\infty}\overline{T}^{m-1}s^{1-m}-\sum_{m=2}^{\infty}\overline{T}^{m-1}s^{1-m}\\ &= \mathcal{I}.\end{aligned}$$

□

We can now define the resolvent operators of the fine structure and study their regularity. Based on the previous series expansions and the structure of the kernels of the function spaces we can now define the resolvent operators associated to the fine structure spaces. Using these resolvent operators associated with the S-spectrum we will define the functional calculi associated with the respective functions spaces.

Definition 12.7.3 (The resolvent operators associated with the fine structure)**.** Let $T \in \mathcal{BC}^{0,1}(V_5)$ and $s \in \rho_S(T)$, we recall that

$$\mathcal{Q}_{c,s}(T)^{-1} := (s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-1}.$$

- The left and the right $\mathcal{D}$**-resolvent operators** $S^{-1}_{\mathcal{D},L}(s,T)$ and $S^{-1}_{\mathcal{D},R}(s,T)$ are defined as
$$S^{-1}_{\mathcal{D},L}(s,T) := -4\mathcal{Q}_{c,s}(T)^{-1}, \tag{12.57}$$
and
$$S^{-1}_{\mathcal{D},R}(s,T) := -4\mathcal{Q}_{c,s}(T)^{-1}. \tag{12.58}$$

- The left and the right Δ**-resolvent operators** $S^{-1}_{\Delta,L}(s,T)$ and $S^{-1}_{\Delta,R}(s,T)$ are defined as
$$S^{-1}_{\Delta,L}(s,T) := -8S^{-1}_{L}(s,T)\mathcal{Q}_{c,s}(T)^{-1}, \tag{12.59}$$
and
$$S^{-1}_{\Delta,R}(s,T) := -8\mathcal{Q}_{c,s}(T)^{-1}S^{-1}_{R}(s,T). \tag{12.60}$$

- The left and the right $\Delta\mathcal{D}$**-resolvent operators** $S^{-1}_{\Delta\mathcal{D},L}(s,T)$ and $S^{-1}_{\Delta\mathcal{D},R}(s,T)$ are defined as
$$S^{-1}_{\Delta\mathcal{D},L}(s,T) := 16\mathcal{Q}_{c,s}(T)^{-2}, \tag{12.61}$$
and
$$S^{-1}_{\Delta\mathcal{D},R}(s,T) := 16\mathcal{Q}_{c,s}(T)^{-2}. \tag{12.62}$$

- The left and the right $\overline{\mathcal{D}}$**-resolvent operators** $S^{-1}_{\overline{\mathcal{D}},L}(s,T)$ and $S^{-1}_{\overline{\mathcal{D}},R}(s,T)$ are defined as

$$S^{-1}_{\overline{\mathcal{D}},L}(s,T) := 4(s\mathcal{I}-\bar{T})\mathcal{Q}_{c,s}(T)^{-2}(s\mathcal{I}-T_0)+2\mathcal{Q}_{c,s}(T)^{-1} \quad (12.63)$$

and

$$S^{-1}_{\overline{\mathcal{D}},R}(s,T) := 4(s\mathcal{I}-T_0)\mathcal{Q}_{c,s}(T)^{-2}(s\mathcal{I}-\bar{T})+2\mathcal{Q}_{c,s}(T)^{-1}. \quad (12.64)$$

- The left and the right $\overline{\mathcal{D}}^2$**-resolvent operators** $S^{-1}_{\overline{\mathcal{D}}^2,L}(s,T)$ and $S^{-1}_{\overline{\mathcal{D}}^2,R}(s,T)$ are defined as

$$S^{-1}_{\overline{\mathcal{D}}^2,L}(s,T) := 32(s\mathcal{I}-\bar{T})\mathcal{Q}_{c,s}(T)^{-3}(s\mathcal{I}-T_0)^2 \quad (12.65)$$

and

$$S^{-1}_{\overline{\mathcal{D}}^2,R}(s,T) := 32(s\mathcal{I}-T_0)^2\mathcal{Q}_{c,s}(T)^{-3}(s\mathcal{I}-\bar{T}). \quad (12.66)$$

- The left and the right $\mathcal{D}^2$**-resolvent operators** $S^{-1}_{\mathcal{D}^2,L}(s,T)$ and $S^{-1}_{\mathcal{D}^2,R}(s,T)$ are defined as

$$S^{-1}_{\mathcal{D}^2,L}(s,T) := 8S^{-1}_L(s,\bar{T})\mathcal{Q}_{c,s}(T)^{-1} \quad (12.67)$$

and

$$S^{-1}_{\mathcal{D}^2,R}(s,T) := 8\mathcal{Q}_{c,s}(T)^{-1}S^{-1}_R(s,\bar{T}). \quad (12.68)$$

- The left and the right $\Delta\overline{\mathcal{D}}$**-resolvent operators** $S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,T)$ and $S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,T)$ are defined as

$$S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,T) := -64(s\mathcal{I}-\bar{T})\mathcal{Q}_{c,s}(T)^{-3}(s\mathcal{I}-T_0) \quad (12.69)$$

and

$$S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,x) := -64(s\mathcal{I}-T_0)\mathcal{Q}_{c,s}(T)^{-3}(s\mathcal{I}-\bar{T}). \quad (12.70)$$

Now, we study the regularity of the previous kernels.

Proposition 12.7.4. *Let $T \in \mathcal{BC}^{0,1}(V_5)$. Then the resolvent operators associated with the fine structure in Definition 12.7.3 are slice hyperholomorphic operators valued functions for $s \in \rho_S(T)$.*

Proof. It follows by a direct computations and in the case of the S-resolvent operators or the F-resolvent operators. □

Now, we can to define the functional calculi associated with the fine structure.

Definition 12.7.5 (The functional calculi of the fine structure in dimension five)**.** Let $T \in \mathcal{BC}^{0,1}(V_5)$ and set $ds_J = ds(-J)$ for $J \in \mathbb{S}^4$. Let f be a function that belongs to $\mathcal{SH}_L(\sigma_S(T))$ or belongs to $\mathcal{SH}_R(\sigma_S(T))$. Let U be a bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathrm{dom}(f)$.

Keeping in mind the resolvent operators associated with the fine structure in Definition 12.7.3 we define functional calculi associated of the fine structure as:

- (**The** $\ell + 1$**-harmonic functional calculus for** $0 \leq \ell \leq 1$) For every function $\tilde{f}_\ell = \Delta^{1-\ell}\mathcal{D}f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$ and $0 \leq \ell \leq 1$, we set

$$\tilde{f}_\ell(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\Delta^{1-\ell}\mathcal{D},L}(s,T) ds_J f(s), \tag{12.71}$$

and, for every function $\tilde{f}_\ell = f\Delta^{1-\ell}\mathcal{D}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\tilde{f}_\ell(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\Delta^{1-\ell}\mathcal{D},R}(s,T). \tag{12.72}$$

- (**The holomorphic Cliffordian functional calculus of order 1**) For every function $f^\circ = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$f^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\Delta,L}(s,T) ds_J f(s), \tag{12.73}$$

and, for every function $f^\circ = \Delta f$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$f^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\Delta,R}(s,T). \tag{12.74}$$

- (**The polyanalytic Cliffordian functional calculus of order** $(1,2)$) For every function $\breve{f}^\circ = \overline{\mathcal{D}}f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$\breve{f}^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\overline{\mathcal{D}},L}(s,T) ds_J f(s), \tag{12.75}$$

and, for every function $\breve{f}^\circ = f\overline{\mathcal{D}}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\breve{f}^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\overline{\mathcal{D}},R}(s,T). \tag{12.76}$$

- (**The polyanalytic functional calculus of order** $3-\ell$ **for** $0 \leq \ell \leq 1$)
For every function $\breve{f}_\ell = \Delta^\ell \overline{\mathcal{D}}^{2-\ell} f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$\breve{f}_\ell(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\Delta^\ell \overline{\mathcal{D}}^{2-\ell}, L}(s,T) ds_J f(s), \tag{12.77}$$

and, for every function $\breve{f}_\ell = f\overline{\mathcal{D}}$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\breve{f}^\circ(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\Delta^\ell \overline{\mathcal{D}}^{2-\ell}, R}(s,T). \tag{12.78}$$

- (**The anti-holomorphic Cliffordian functional calculus of order** 1)
For every function $f_0 = \mathcal{D}^2 f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$f_0(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\mathcal{D}^2, L}(s,T) ds_J f(s), \tag{12.79}$$

and, for every function $f_0 = f\mathcal{D}^2$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$f_0(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} f(s) ds_J S^{-1}_{\mathcal{D}^2, R}(s,T). \tag{12.80}$$

Theorem 12.7.6. *The previous functional calculi are well defined, in the sense that the integrals in Definition 12.7.5 depend neither on the imaginary unit $J \in \mathbb{S}^4$ and nor on the slice Cauchy domain U.*

Proof. We prove the result for the functional calculi defined using the left slice hyperholomorphic functions since the right counterpart can be proved with similar computations. The only property of the kernels that we shall use to prove the theorem is that they are all right slice hyperholomoprhic in the variable s (see Proposition 12.5.8). For this reason, in what follows, we can refer to an arbitrary left kernel described in Proposition 12.7.2 with the symbol $K_L(s,T)$.
Since $K_L(s,T)$ is a right-slice hyperholomorphic function in s and f is left slice hyperholomorphic, the independence from the set U follows by the Cauchy integral formula (see Theorem 3.1.18).
Now, we want to show the independence from the imaginary unit, let us consider two imaginary units $J, I \in \mathbb{S}^4$ with $J \neq I$ and two bounded slice Cauchy domains U_x, U_s with $\sigma_S(T) \subset U_x$, $\overline{U}_x \subset U_s$ and $\overline{U}_s \subset dom(f)$. Then every $s \in \partial(U_s \cap \mathbb{C}_J)$ belongs to the unbounded slice Cauchy domain $\mathbb{R}^6 \setminus U_x$.

Since $\lim_{x\to+\infty} K_L(x,T)=0$, the slice hyperholomorphic Cauchy formula implies

$$\begin{aligned} K_L(s,T) &= \frac{1}{2\pi}\int_{\partial(\mathbb{R}^6\setminus(U_x\cap\mathbb{C}_I))} K_L(x,T)dx_I S_R^{-1}(x,s) \\ &= \frac{1}{2\pi}\int_{\partial(U_x\cap\mathbb{C}_I)} K_L(x,T)dx_I S_L^{-1}(s,x). \qquad (12.81) \end{aligned}$$

The last equality is due to the facts that $\partial(\mathbb{R}^6\setminus(U_x\cap\mathbb{C}_I))=-\partial(U_x\cap\mathbb{C}_I)$ and $S_R^{-1}(x,s)=-S_L^{-1}(s,x)$. Thus, by formula (12.81) we get

$$\begin{aligned} \tilde{f}(T) &= \int_{\partial(U_s\cap\mathbb{C}_J)} K_L(s,T)ds_J f(s) \\ &= \int_{\partial(U_s\cap\mathbb{C}_J)} \left(\frac{1}{2\pi}\int_{\partial(U_x\cap\mathbb{C}_I)} K_L(x,T)dx_I S_L^{-1}(s,x)\right) ds_J f(s). \end{aligned}$$

Due to Fubini's theorem we can exchange the order of integration and by the Cauchy formula we obtain

$$\begin{aligned} \tilde{f}(T) &= \int_{\partial(U_x\cap\mathbb{C}_I)} K_L(x,T)dx_I \left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_J)} S_L^{-1}(s,x)ds_J f(s)\right) \\ &= \int_{\partial(U_x\cap\mathbb{C}_I)} K_L(x,T)dx_I f(x). \end{aligned}$$

This proves the statement. □

In what follows we denote by P one of the operators: $\Delta^{1-\ell}\mathcal{D}$, Δ, $\overline{\mathcal{D}}$, $\Delta^{\ell}\overline{\mathcal{D}}^{2-\ell}$ and $\mathcal{D}^2$ for $\ell=0,\,1$.

Problem 12.7.7. *Let Ω be a slice Cauchy domain. It might happen that $f,g\in\mathcal{SH}_L(\Omega)$ (resp. $f,g\in\mathcal{SH}_R(\Omega)$) and $\mathsf{P}f=\mathsf{P}g$ (resp. $f\mathsf{P}=g\mathsf{P}$). Is it possible to show that for any $T\in\mathcal{BC}^{0,1}(V_5)$, with $\sigma_S(T)\subset\Omega$, we have $(\mathsf{P}f)(T)=(\mathsf{P}g)(T)$ (resp. $(f\mathsf{P})(T)=(g\mathsf{P})(T)$)?*

In order to address the problem we need an auxiliary result. We start by observing that by hypothesis we have $\mathsf{P}(f-g)=0$ (resp. $(f-g)\mathsf{P}=0$). Therefore it is necessary to study the set

$$(\ker\mathsf{P})_{\mathcal{SH}_L(\Omega)} := \{f\in\mathcal{SH}_L(\Omega) : \mathsf{P}(f)=0\}$$

and

$$(\ker\mathsf{P})_{\mathcal{SH}_R(\Omega)} := \{f\in\mathcal{SH}_R(\Omega) : (f)\mathsf{P}=0\}.$$

Theorem 12.7.8. *Let Ω be a connected slice Cauchy domain of $\mathbb{R}^6$, then*

$$(\ker \mathsf{P})_{\mathcal{SH}_L(\Omega)} = \{f \in \mathcal{SH}_L(\Omega) : f \equiv \alpha_0 + \cdots + x^t \alpha_t \quad \text{for some } \alpha_0, \ldots, \alpha_t \in \mathbb{R}_5\}$$
$$= \{f \in \mathcal{SH}_R(\Omega) : f \equiv \alpha_0 + \cdots + \alpha_t x^t \quad \text{for some } \alpha_0, \ldots, \alpha_t \in \mathbb{R}_5\}$$
$$= (\ker \mathsf{P})_{\mathcal{SH}_R(\Omega)},$$

where t is equal to the degree of P minus 1.

Proof. We prove the result in the case $f \in \mathcal{SH}_L(\Omega)$ since the case $f \in \mathcal{SH}_R(\Omega)$ follows by similar arguments. We proceed by double inclusion. The fact that

$$(\ker \mathsf{P})_{\mathcal{SH}_L(\Omega)} \supseteq \{f \in \mathcal{SH}_L(\Omega) : f \equiv \alpha_0 + \cdots + x^t \alpha_t \quad \text{for some } \alpha_0, \ldots, \alpha_t \in \mathbb{R}_5\}$$

is obvious. The other inclusion can be proved observing that if $f \in (\ker \mathsf{P})_{\mathcal{SH}_L(\Omega)}$, after a change of variable if needed, there exists $r > 0$ such that the function f can be expanded in a convergent series at the origin

$$f(x) = \sum_{k=0}^{\infty} x^k \alpha_k \quad \text{for } \{\alpha_k\}_{k \in \mathbb{N}_0} \subset \mathbb{R}_5 \text{ and for any } x \in B_r(0)$$

where $B_r(0)$ is the ball centred at 0 and of radius r. We have

$$0 = \mathsf{P} f(x) \equiv \sum_{k=\deg(\mathsf{P})}^{\infty} \mathsf{P}(x^k) \alpha_k, \quad \forall x \in B_r(0).$$

By Lemma 12.6.1, Proposition 12.6.2, Proposition 12.6.3, Proposition 12.6.8, Proposition 12.6.9, Proposition 12.6.11 and Proposition 12.6.12 we can compute explicitly the expressions $\mathsf{P}(x^k)$ and, when they are restricted to a neighborhood of zero of the real axis, they coincide up to a constant to $x^{k-\deg(\mathsf{P})}$. Since the power series is identically zero, its coefficients α_k's must be zero for any $k \geq \deg(\mathsf{P})$. This yields $f(x) \equiv \alpha_0 + \cdots + x^t \alpha_t$ in $B_r(0)$ and, since Ω is connected, we also have $f(x) \equiv \alpha_0 + \cdots + x^t \alpha_t$ for any $x \in \Omega$. □

We solve the problem 12.7.7 in the case in which Ω is connected.

Proposition 12.7.9. *Let $T \in \mathcal{BC}^{0,1}(V_5)$ and let U be a connected slice Cauchy domain with $\sigma_S(T) \subset U$. If $f, g \in \mathcal{SH}_L(U)$ (resp. $f, g \in \mathcal{SH}_R(U)$) satisfy the property $\mathsf{P} f = \mathsf{P} g$ (resp. $f \mathsf{P} = g \mathsf{P}$) then $(\mathsf{P} f)(T) = (\mathsf{P} g)(T)$ (resp. $(f \mathsf{P})(T) = (g \mathsf{P})(T)$).*

Proof. We prove the theorem in the case $f, g \in \mathcal{SH}_L(\Omega)$ since the case $f, g \in \mathcal{SH}_R(\Omega)$ follows by similar arguments. By Definition 12.7.5, we have

$$(\mathsf{P}f)(T) - (\mathsf{P}g)(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_J)} S^{-1}_{\mathsf{P},L}(s,T) ds_J (f(s) - g(s)).$$

Since $S^{-1}_{\mathsf{P},L}(s,T)$ is slice hyperholomorphic in the variable s by Proposition 12.7.4, we can change the domain of integration to $B_r(0) \cap \mathbb{C}_J$ for some $r > 0$ with $\|T\| < r$. Moreover, by hypothesis we have that $f(s) - g(s) \in (\ker \mathsf{P})_{\mathcal{SH}_L(\Omega)}$, thus by Theorem 12.7.8 and Proposition 12.7.2 we get

$$\begin{aligned}
&(\mathsf{P}f)(T) - (\mathsf{P}g)(T) = \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} S^{-1}_{\mathsf{P},L}(s,T) ds_J (f(s) - g(s)) \\
&= \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_J)} S^{-1}_{\mathsf{P},L}(s,T) ds_J (\alpha_0 + \dots + s^t \alpha_t) \\
&= \frac{1}{2\pi} \sum_{m=\deg(\mathsf{P})}^{\infty} \left(g_{\mathsf{P},m}(T, \bar{T}) \right) \int_{\partial(B_r(0) \cap \mathbb{C}_J)} s^{-1-m} ds_J (\alpha_0 + \dots + s^t \alpha_t) = 0.
\end{aligned}$$

where $g_{\mathsf{P},m}(T, \bar{T})$ is a polynomial in T and $\bar{T}$ (see Proposition 12.7.2) and $t := \deg(\mathsf{P}) - 1$. $\square$

Now, we write the resolvent operators associated with the fine structures in terms of the F- resolvent operators.

Proposition 12.7.10. *Let $T \in \mathcal{BC}^{0,1}(V_5)$ and $s \in \rho_S(T)$. Then, we have*

$$\begin{aligned}
S^{-1}_{\mathcal{D},L}(s,T) &= -\frac{1}{16} \big[F_5^L(s,T) s^3 - (T + 2T_0) F_5^L(s,T) s^2 \\
&\quad + (2T_0 x + |x|^2) F_5^L(s,T) s - T|T|^2 F_5^L(s,T) \big],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\mathcal{D},R}(s,T) &= -\frac{1}{16} \big[s^3 F_5^R(s,T) - s^2 F_5^R(s,T)(T + 2T_0) \\
&\quad + s F_5^R(s,T)(2T_0 T + |T|^2) - F_5^R(s,T) T|T|^2 \big],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\Delta,L}(s,T) &= -\frac{1}{8} \left[F_5^L(s,T) s^2 - 2T_0 F_5^L(s,T) s + |T|^2 F_5^L(s,x) \right], \\
S^{-1}_{\Delta,R}(s,T) &= -\frac{1}{8} \left[F_5^R(s,T) T^2 - 2s F_5^R(s,T) T_0 + F_5^R(s,x) |T|^2 \right],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\Delta\mathcal{D},L}(s,T) &= \frac{1}{4}\left[F_5^L(s,T)s - TF_5^L(s,T)\right],\\
S^{-1}_{\Delta\mathcal{D},R}(s,T) &= \frac{1}{4}\left[sF_5^R(s,T) - F_5^R(s,T)T\right],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\overline{\mathcal{D}},L}(s,T) &= \frac{1}{32}\left[3F_5^L(s,T)s^3 - (8T_0+T)F_5^L(s,T)s^2\right.\\
&\quad \left.+(4T_0^2+2T_0T+3|T|^2)F_5^L(s,T)s - (T|T|^2+2T_0|T|^2)F_5^L(s,T)\right],\\
S^{-1}_{\overline{\mathcal{D}},R}(s,T) &= \frac{1}{32}\left[3s^3F_5^R(s,T) - s^2F_5^R(s,T)(8T_0+T)\right.\\
&\quad \left.+sF_5^R(s,T)(4T_0^2+2T_0T+3|T|^2) - F_5^R(s,x)(T|T|^2+2T_0|T|^2)\right],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\mathcal{D}^2,L}(s,T) &= -\frac{1}{8}\left[F_5^L(s,T)s^2 - 2TF_5^L(s,T)s + T^2F_5^L(s,T)\right],\\
S^{-1}_{\mathcal{D}^2,R}(s,T) &= -\frac{1}{8}\left[s^2F_5^R(s,T) - 2sF_5^R(s,T)T + F_5^R(s,T)T^2\right],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\overline{\mathcal{D}}^2,L}(s,T) &= \frac{1}{2}[F_5^L(s,T)s^2 - 2T_0F_5^L(s,T)s + x_0^2F_5^L(s,T)],\\
S^{-1}_{\overline{\mathcal{D}}^2,R}(s,T) &= \frac{1}{2}[s^2F_5^R(s,T) - 2sF_5^R(s,T)T_0 + F_5^R(s,T)T_0^2],
\end{aligned}$$

$$\begin{aligned}
S^{-1}_{\Delta\overline{\mathcal{D}},L}(s,T) &= -F_5^L(s,x)s + T_0F_5^L(s,T),\\
S^{-1}_{\Delta\overline{\mathcal{D}},R}(s,T) &= -sF_5^R(s,T) + F_5^R(s,T)T_0.
\end{aligned}$$

Proof. it follows by formally replacing the variable x of Remark 12.5.9 by the paravector operator T. □

In order to solve Problem 12.7.7, in the case Ω not connected, we need the following lemma, which is based on the monogenic functional calculus developed in [99, 101, 108, 112]. We chose to annihilate the last component of the operator T, namely $T_4 = 0$. In the monogenic functional calculus McIntosh and collaborators consider zero the first component $T_0 = 0$.

However, in our case this is a drawback due to the structure of the polyanalytic resolvent operators, see Proposition 12.7.10.

Lemma 12.7.11. *Let $T \in \mathcal{BC}^{0,1}(V_5)$ be such that $T = T_0e_0 + T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 0, 1, 2, 3$, have real spectrum. Let G be a bounded slice Cauchy domain such that $(\partial G) \cap \sigma_S(T) = \emptyset$. For every $J \in \mathbb{S}^4$ we have*

$$\int_{\partial(G\cap\mathbb{C}_J)} S^{-1}_{\mathsf{P},L}(s,T)ds_J(\alpha_0 + \cdots + s^t\alpha_t) = 0, \tag{12.82}$$

and

$$\int_{\partial(G\cap\mathbb{C}_J)} (\alpha_0 + \cdots + s^t\alpha_t)ds_J S^{-1}_{\mathsf{P},R}(s,T) = 0, \tag{12.83}$$

where $t = \deg(\mathsf{P}) - 1$ and $\alpha_j \in \mathbb{R}_5$ for any $0 \leq j \leq t$.

Proof. Since $\Delta^2(1) = 0$, $\Delta^2(x) = 0$, $\Delta^2(x^2) = 0$ and $\Delta^2(x^3) = 0$ by Theorem 7.4.6 we also have

$$\int_{\partial(G\cap\mathbb{C}_J)} F^L_5(s,x)ds_J = \Delta^2(1) = 0, \tag{12.84}$$

and

$$\int_{\partial(G\cap\mathbb{C}_J)} F^L_5(s,x)ds_J s = \Delta^2(x) = 0,$$

and

$$\int_{\partial(G\cap\mathbb{C}_J)} F^L_5(s,x)ds_J s^2 = \Delta^2(x^2) = 0,$$

and

$$\int_{\partial(G\cap\mathbb{C}_J)} F^L_5(s,x)ds_J s^3 = \Delta^2(x^3) = 0 \tag{12.85}$$

for all $x \notin \partial G$ and $J \in \mathbb{S}^4$. By the monogenic functional we have

$$F^L_5(s,T) = \int_{\partial\Omega} G(\omega,T)\mathbf{D}\omega F^L_5(s,\omega),$$

where $\mathbf{D}\omega$ is a suitable differential form, the open set Ω contains the left spectrum of T and $G(\omega,T)$ is the Fueter resolvent operator. By Proposition 12.7.10 we can write

$$S^{-1}_{\mathsf{P},L}(s,T) = \sum_{\ell=0}^{4-\deg(\mathsf{P})} g_{\mathsf{P},\ell}(T,\bar{T})F^L_5(s,T)s^\ell,$$

and thus we have

$$S^{-1}_{\mathsf{P},L}(s,T)(\alpha_0+\cdots+s^t\alpha_t)=\sum_{\ell=0}^{3}g'_{\mathsf{P},\ell}(T,\bar{T})F_5^L(s,T)s^\ell\beta_\ell$$

for appropriate polynomial in T, $\bar{T}$: $g'_{\mathsf{P},\ell}(T,\bar{T})$, and $\beta_\ell\in\mathbb{R}_5$. We can conclude the proof of the theorem observing that for any $\ell=0,1,2,3$ we have

$$\begin{aligned}&g_{\mathsf{P},\ell}(T,\bar{T})\int_{\partial(G\cap\mathbb{C}_J)}F_5^L(s,T)ds_Js^\ell\beta_\ell\\ &=-\left(g_{\mathsf{P},\ell}(T,\bar{T})\int_{\partial(G\cap\mathbb{C}_J)}\int_{\partial\Omega}G(\omega,T)\mathbf{D}\omega F_L(s,\omega)s^\ell\,ds_J\right)\beta_\ell\\ &=\left(g_{\mathsf{P},\ell}(T,\bar{T})\int_{\partial\Omega}G(\omega,T)\mathbf{D}\omega\left(\int_{\partial(G\cap\mathbb{C}_J)}F_L(s,\omega)ds_Js^\ell\right)\right)\beta_\ell=0\end{aligned}$$

where the second equality is a consequence of the Fubini's Theorem and the last equality is a consequence of formulas (12.84)-(12.85). Therefore, we get

$$\int_{\partial(G\cap\mathbb{C}_J)}S^{-1}_{\mathsf{P},L}(s,T)ds_J(\alpha_0+\cdots+s^t\alpha_t)=0.$$

By similar computations it is possible to show (12.83). □

Finally in the following result we give an answer to the question in Problem 12.7.7.

Proposition 12.7.12. *Let $T\in\mathcal{BC}^{0,1}(V_5)$ be such that $T=T_0e_0+T_1e_1+T_2e_2+T_3e_3$, and assume that the operators T_ℓ, $\ell=0,1,2,3$, have real spectrum. Let U be a slice Cauchy domain with $\sigma_S(T)\subset U$. If $f,g\in\mathcal{SH}_L(U)$ (resp. $f,g\in\mathcal{SH}_R(U)$) satisfy the property $\mathsf{P}f=\mathsf{P}g$ (resp $f\mathsf{P}=g\mathsf{P}$) then $(\mathsf{P}f)(T)=(\mathsf{P}g)(T)$ (resp. $(f\mathsf{P})(T)=(g\mathsf{P})(T)$).*

Proof. If U is connected we can use Proposition 12.7.9. If U is not connected then $U=\cup_{\ell=1}^m U_\ell$ where the U_ℓ are the connected components of U. Hence, there exist constants $\alpha_{\ell,i}\in\mathbb{R}_5$ for $\ell=1,\dots,m$ and $i=0,1,2,3$ such that $f(s)-g(s)=\sum_{\ell=1}^m\sum_{i=0}^t\chi_{U_\ell}(s)s^i\alpha_{\ell,i}$ where $t=\deg(\mathsf{P})-1$. Thus we can write

$$(\mathsf{P}f)(T)-(\mathsf{P}g)(T)=\sum_{\ell=1}^m\frac{1}{2\pi}\int_{\partial(U_\ell\cap\mathbb{C}_J)}S^{-1}_{\mathsf{P},L}(s,T)ds_J(\alpha_{\ell,0}+\cdots+s^t\alpha_{\ell,t}).$$

the last summation is zero by Lemma 12.7.11. □

Remark 12.7.13. It is possible to prove some other important properties for these functional calculi, this will be investigated in a forthcoming work.

12.8 The product rule for the F-functional calculus in dimension five

In order to obtain a product rule for the F-functional calculus in dimension five, it is crucial the Dirac fine structure of the kind $(\mathcal{D}, \overline{\mathcal{D}}, \mathcal{D}, \overline{\mathcal{D}})$, see (12.7).

Theorem 12.8.1 (Product rule for the F-functional calculus for $n = 5$)**.** *Let $T \in \mathcal{BC}(V_5)$. We assume $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$, then we have*

$$\begin{aligned}\Delta^2(fg)(T) &= \Delta^2(f)(T)g(T) + f(T)\Delta^2(g)(T) + \Delta(f)(T)\Delta(g)(T)\\ &\quad - \mathcal{D}\Delta(f)(T)\mathcal{D}(g)(T) - \mathcal{D}(f)(T)\Delta\mathcal{D}(g)(T).\end{aligned}$$

Proof. Let G_1 and G_2 be two bounded slice Cauchy domain such that contain $\sigma_S(T)$ and $\bar{G}_1 \subset G_2$, with $\bar{G}_2 \subset dom(f) \cap dom(g)$. We choose $p \in \partial(G_1 \cap \mathbb{C}_J)$ and $s \in \partial(G_2 \cap \mathbb{C}_J)$.
For every $I \in \mathbb{S}^4$, from the definitions of F-functional calculus, SC-functional calculus, holomorphic Cliffordian functional calculus of order 1 and $\ell+1$-harmonic functional calculus ($0 \leq \ell \leq 1$) we get

$$\begin{aligned}&\Delta^2(f)(T)g(T) + f(T)\Delta^2(g)(T) + \Delta(f)(T)\Delta(g)(T)\\
&- \mathcal{D}\Delta(f)(T)\mathcal{D}(g)(T) - \mathcal{D}(f)(T)\Delta\mathcal{D}(g)(T)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J F_5^R(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} S_L^{-1}(p,T)dp_J g(p)\\
&+ \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J S_R^{-1}(s,T)\int_{\partial(G_1\cap\mathbb{C}_J)} F_5^L(p,T)dp_J g(p)\\
&+ \frac{16}{\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{Q}_{c,s}(T)^{-1}S_R^{-1}(s,T)\int_{\partial(G_2\cap\mathbb{C}_J)} S_L^{-1}(p,T)\mathcal{Q}_{c,p}(T)^{-1}dp_J g(p)\\
&+ \frac{16}{\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{Q}_{c,s}(T)^{-2}\int_{\partial(G_2\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-1}dp_J g(p)\\
&+ \frac{16}{\pi^2}\int_{\partial(G_2\cap\mathbb{C}_J)} f(s)ds_J \mathcal{Q}_{c,s}(T)^{-1}\int_{\partial(G_2\cap\mathbb{C}_J)} \mathcal{Q}_{c,p}(T)^{-2}dp_J g(p)\\
&= \frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)} f(s)ds_J\{F_5^R(s,T)S_L^{-1}(s,T) + S_R^{-1}(s,T)F_5^L(p,T)\\
&+ 2^6[\mathcal{Q}_{c,s}(T)^{-1}S_R^{-1}(s,T)S_L^{-1}(p,T)\mathcal{Q}_{c,p}(T)^{-1} + \mathcal{Q}_{c,s}(T)^{-2}\mathcal{Q}_{c,p}(T)^{-1}\\
&+ \mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,p}(T)^{-2}]\}dp_J g(p)\end{aligned}$$

By Lemma 9.3.2 we get

$$\begin{aligned}
&\Delta^2(f)(T)g(T)+f(T)\Delta^2(g)(T)+\Delta(f)(T)\Delta(g)(T)\\
&-\mathcal{D}\Delta(f)(T)\mathcal{D}(g)(T)-\mathcal{D}(g)(T)\Delta\mathcal{D}g(T)\\
&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}\int_{\partial(G_1\cap\mathbb{C}_J)}f(s))ds_J\{[F_5^R(s,T)-F_5^L(p,T)]p\\
&-\bar{s}[F_5^R(s,T)-F_5^L(p,T)]\}\mathcal{Q}_s(p)^{-1}\}dp_Jg(p).
\end{aligned}$$

Now, since the functions $p\mapsto p\mathcal{Q}_s(p)^{-1}$, $p\mapsto \mathcal{Q}_s(p)^{-1}$ are intrinsic slice hyperholomoprhic on $\bar{G}_1$ by the Cauchy integral formula we have

$$\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^R(s,T)p\mathcal{Q}_s(p)^{-1}dp_Jg(p)=0,$$

$$\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}\bar{s}F_5^R(s,T)\mathcal{Q}_s(p)^{-1}dp_Jg(p)=0.$$

Therefore we obtain

$$\begin{aligned}
&\Delta^2(f)(T)g(T)+f(T)\Delta^2(g)(T)+\Delta(f)(T)\Delta(g)(T)\\
&-\mathcal{D}\Delta(f)(T)\mathcal{D}(g)(T)-\mathcal{D}(f)(T)\Delta\mathcal{D}(g)(T)\\
&=\frac{1}{(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_J)}f(s)ds_J\int_{\partial(G_1\cap\mathbb{C}_J)}[\bar{s}F_5(p,T)^L-F_5^L(p,T)p]\mathcal{Q}_s(p)^{-1}dp_Jg(p).
\end{aligned}$$

From Lemma 9.4.1 with $B:=F_5^L(p,T)$ we get

$$\begin{aligned}
&\Delta^2(f)(T)g(T)+f(T)\Delta^2(g)(T)+\Delta(f)(T)\Delta(g)(T)\\
&-\mathcal{D}\Delta(f)(T)\mathcal{D}(g)(T)-\mathcal{D}(f)(T)\Delta\mathcal{D}(g)(T)\\
&=\frac{1}{(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_Jf(p)g(p)\\
&=\frac{1}{(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_J)}F_5^L(p,T)dp_J(fg)(p)\\
&=\Delta^2(fg)(T).
\end{aligned}$$

□

CHAPTER 13

Conclusion and further research in progress

In this dissertation we get results on integral transforms in the hypercomplex setting and functional calculi on the S-spectrum. We studied more in details the Segal-Bargmann transform and the short-time Fourier transform in the quaternionic and Cliffod algebra settings. The methods from the Fock space and Segal-Bargman theories can be used to show several results on the Gaussian RBF kernel in complex and hypercomplex analysis. The latter is one of the most used kernels in modern machine learning kernel methods, and in support vector machines (SVMs) classification algorithms, see [130]. These methods have been recently investigated in the paper [8].

Furthermore, we study in detail a monogenic functional calculus, a harmonic functional calculus and polyanalytic functional calculus on the S-spectrum. Nowadays, the S-functional calculus has generated the following research directions: the Phillips functional calculus; H^{∞}-functional calculus, see [44]; Schur analysis [13] and new classes of fractional diffusion problems based on fractional powers of quaternionic linear operators, see [39, 40, 44].

For perspectives and further researches, we aim to figure out if it is possible to have results, like the ones listed before, for the F-functional calculus, the Q-functional calculus and the P_2-functional calculus. We observe that the H^∞-functional calculus in the monogenic setting was obtained by McIntosh and collaborators, see [99]. However, even if the F-functional calculus is in the same spirit McIntosh, the approach is different since it is based on the S-spectrum and the Fueter-Sce theorem.

We started working on some different problems that are still under progress. Such problems are related to the following topics

1) The F-resolvent equation for all dimensions. In this case we have to deal with fractional powers.

2) Figure out if it is possible to get a sort of commutation rule between the generalized modulation and translation operator.

3) Establish a generalized Cauchy-Kovalevskaya extension for axially harmonic functions.

4) Generalize to all odd dimensions the *fine structures*.

5) Study the function spaces that arise from the splitting of the Fueter-Sce mapping theorem, both in the complex and hypercomplex settings.

In the next page we explain some research problems that we are considering now.

13.0.1 The F-resolvent equation for all dimensions

The monogenic functional calculus developed by McIntosh and collaborators holds for all dimensions. In this dissertation we studied the F-functional calculus only when the dimension is odd.

In order to have a F-resolvent equation also for dimensions even we need to develop a F-functional calculus when n is even. To achieve this we need to show a Fueter-Sce-Qian mapping theorem in integral form for any n.

As for the case n odd, the first step is to compute the action of the Fueter-Sce-Qian map on the second form of the slice hyperholomorphic Cauchy kernel. This was done in Chapter 8, where we show that for all n we have

$$\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = F_n^L(s,x),$$

where

$$F_n^L(s,x) := \gamma_n (s-\bar{x})(s^2 - 2x_0 s + |x|^2)^{-\frac{n+1}{2}}.$$

In this case the main issue to face is that the function $F_n^L(s,x)$ is monogenic in x if $(s^2-2x_0s+|x|^2)\in\mathbb{R}^{n+1}\setminus(-\infty,0]$. Therefore in the statement of the Fueter-Sce-Qian mapping theorem in integral form we have to take into account this feature.

Theorem 13.0.1 (Fueter-Sce-Qian theorem in integral version for even dimensions)**.** *Let n be an even number. Let f be a slice hyperholomorphic function defined in an open set that contains $\bar{U}$, where U is a bounded axially symmetric slice open set. We suppose that $\partial(U\cap\mathbb{C}_J)$ is union of a finite number of rectifiable Jordan curves for any $J\in\mathbb{S}^{n-1}$. Then if $x\in U$ and $U\cap\mathbb{R}^{n+1}\setminus(-\infty,0]=\emptyset$ the function $\breve{f}(x)$, given by*

$$\breve{f}(x)=\Delta^{\frac{n-1}{2}}f(x),$$

is monogenic and it admits the following integral representation

$$\breve{f}(x)=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}F_n^L(s,x)ds_Jf(s),\qquad ds_J=ds/J,$$

and

$$\breve{f}(x)=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_J)}f(s)ds_JF_n^R(s,x),\qquad ds_J=ds/J,$$

where the integrals depends neither on U nor on the imaginary unit $J\in\mathbb{S}^{n-1}$.

The idea to show Theorem 13.0.1 is similar to the one in Theorem 7.4.6, but in this case we have to pay attention to correctly exchange the integral and the Fueter-Sce-Qian mapping theorem.

Once proved a Fueter-Sce-Qian theorem in integral version, it is possible to define the F-functional calculus. Now we aim to solve the following problems

Problem 13.0.2. *Is it possible to have a resolvent equation for the F-functional calculus in even dimensions?*

If it is possible

Problem 13.0.3. *Does the F-resolvent equation allow to study the Riesz projectors?*

Moreover,

Problem 13.0.4. *Is it possible to develop the theory of* fine structures *in even dimensions?*

13.0.2 A new monogenic product between axially monogenic functions

In the monogenic theory of functions the pointwise product of monogenic functions is not anymore monogenic. For this reason it was introduced the so called CK-product. For a pair of monogenic functions $A(x_0, \underline{x})$ and $B(x_0, \underline{x})$ the CK-product is defined as

$$A(x_0, \underline{x}) \odot_{CK} B(x_0, \underline{x}) = CK[A(0, \underline{x}) \cdot B(0, \underline{x})].$$

The building blocks for the expansion in series of a generic monogenic function are the so-called Fueter variables, defined by

$$\xi_\ell = x_\ell - e_\ell x_0, \qquad \ell = 1, ...n.$$

Now, let us consider the Clifford-Appel polynomials, defined Chapter 9 as

$$P_k^n(x) := \sum_{s=0}^{k} T_s^k(n) x^{k-s} \overline{x}^s, \qquad x \in \mathbb{R}^{n+1},$$

where $T_s^k(n)$ is defined as

$$T_s^k(n) := \binom{k}{s} \frac{\left(\frac{n+1}{2}\right)_{k-s} \left(\frac{n-1}{2}\right)_s}{(n)_k}, \qquad n \geq 1,$$

Now, one can wonder if the CK-product works well also for this kind of polynomials. In [19, Prop. 3.7] (one can easily extend in $\mathbb{R}^{n+1}$) the CK-product between Clifford-Appel polynomials is computed and it is given by

$$(P_k^n \odot_{CK} P_s^n)(x) = \frac{c_k c_s}{c_{k+s}} P_{k+s}^n. \tag{13.1}$$

The drawback of this formula is the presence of unsuitable constants c_k, depending on the dimension, and the degree k, which turn out to be unsuitable for certain computations.

Therefore, we aim to define a new kind of product between the axially monogenic functions. This will make use of the generalized CK-extension and of the following fact

$$P_k^n(x) = GCK[x_0^k].$$

Definition 13.0.5. Let $A(x_0, \underline{x})$ and $B(x_0, \underline{x})$ be axially monogenic functions then

$$A(x_0, \underline{x}) \odot_{GCK} B(x_0, \underline{x}) = GCK[A(x_0, 0) \cdot B(x_0, 0)]. \tag{13.2}$$

In these cases we restrict the functions to a submanifold of dimension one. We will see that this gives arise to a natural properties of the product between Clifford-Appel polynomials.

Lemma 13.0.6. *Let $n \in \mathbb{N}$ and $\ell, k \geq 0$. Then for any $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ we have*

$$P_\ell^n(x) \odot_{GCK} P_k^n(x) = P_{k+\ell}^n(x). \tag{13.3}$$

By means of the Clifford-Appel polynomials and the generalized CK product we can consider an axially monogenic rational function defined by

$$\begin{aligned} \breve{r}(q) &= D + C \odot_{GCK} (I - P_1^n(q)A)^{-\odot_{GCK}} \odot_{GCK} (P_1^n(q)B) \\ &= \sum_{n=0}^{\infty} P_1^n(q) C A^n B, \end{aligned} \tag{13.4}$$

where A, B , C and D are matrices of suitable sizes. Formula (13.4) can be also useful to introduce the counterpart of Shur analysis for axially monogenic functions.

13.0.3 A generalized Cauchy-Kovalevskaya extension for axially harmonic functions

In general, every monogenic function defined at $\mathbb{R}^n$ is determined by its restriction to the hyperplane $x_0 = 0$. On the other hand, any real-analytic function $f(\underline{x})$ defined in a region of $\mathbb{R}^n$, has a unique monogenic extension $f(x_0, \underline{x})$ called Cauchy-Kovalevskaya extension (CK-extension). However, it is possible to deal with restrictions to submanifolds not only of codimension one but of arbitrary codimension. This kind of expansion is called generalized CK-extension (GCK-extension).

Our aim is to construct a similar tool for the axially harmonic functions in the Clifford algebra setting. This will be helpful to find a splitting of diagram (7.36). Now we state the generalized CK-extension for axially harmonic functions that we are investigating at the moment.

Theorem 13.0.7. *Let A_0 and A_1 be two Clifford-valued functions of one real variable x_0, defined in an open subset Ω_1 of the real line. Then there exist a unique sequence of functions $\{A_j\}_{j \in \mathbb{N}_0}$ such that the series*

$$f(x_0, \underline{x}) = \sum_{j=0}^{\infty} \underline{x}^j A_j(x_0) \tag{13.5}$$

converges in an axially symmetric $(n+1)$-dimensional neighborhood Ω of Ω_1 and such that $f(x_0, \underline{x})$ is harmonic (i.e. $\Delta_{\mathbb{R}^{n+1}} f(x_0, \underline{x}) = 0$). Moreover,

$$\begin{aligned} f(x_0, \underline{x}) &= \Gamma\left(\frac{n}{2}\right)\left(\frac{|\underline{x}|\partial_{x_0}}{2}\right)\left[\left(\frac{|\underline{x}|\partial_{x_0}}{2}\right)^{-\frac{n}{2}} J_{\frac{n}{2}-1}(|\underline{x}|\partial_{x_0})[A_0(x_0)] \right. \\ &\quad \left. + \frac{n\underline{x}}{2} J_{\frac{n}{2}}\left(|\underline{x}|\partial_{x_0}\right)\left[A_1(x_0)\right]\right], \end{aligned} \tag{13.6}$$

where J_ν is the Bessel function of the first kind of order ν. Formula 13.6 is known as the generalized CK-extension of A_0 and A_1, and it is denoted by $HGCK[(A_0, A_1)](x_0, \underline{x})$. The initial functions A_0 and A_1 can be recovered by

$$f(x_0, \underline{x})|_{\underline{x}=0} = A_0(x_0),$$

$$-\frac{1}{n}\partial_{\underline{x}}[f(x_0, \underline{x})]|_{\underline{x}=0} = A_1(x_0).$$

One of the main difference with respect the monogenic generalized CK extension is the presence of two initial functions, namely A_0 and A_1.

Now, we have all the tools to split diagram (7.36).

Theorem 13.0.8. *Let $f(z) = \alpha(u, v) + i\beta(u, v)$ be an intrinsic holomorphic function defined on an intrinsic complex domain $\Omega_2 \subset \mathbb{C}$. Then for $n \geq 3$ and odd we have*

$$\Delta_{\mathbb{R}^{n+1}}^{\frac{n-3}{2}} \mathcal{D}[f(x_0 + \underline{x})] = \gamma_n HGCK[(f^{(n-2)}(x_0), 0)]. \tag{13.7}$$

Setting $\Omega_1 = \Omega_2 \cap \mathbb{R}$, we obtain the following diagram

$$\begin{tikzcd} \mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \arrow[r, "S"] \arrow[d, "\gamma_n(\partial_{x_0}^{n-2}{,}0)"'] & \mathcal{SH}(\Omega) \arrow[d, "\Delta_{\mathbb{R}^{n+1}}^{\frac{n-3}{2}}\mathcal{D}"] \\ (\mathcal{A}(\Omega_1) \otimes \mathbb{R}_n)^2 \arrow[r, "HGCK"] \arrow[d, "\partial_{x_0} P_1"'] & \mathcal{AH}(\Omega) \arrow[d, "\overline{\mathcal{D}}"] \\ \mathcal{A}(\Omega_1) \otimes \mathbb{R}_n \arrow[r, "GCK"] & \mathcal{AM}(\Omega) \end{tikzcd} \tag{13.8}$$

where S is the slice operator and P_1 is the projection of the first component.

The previous theorem sheds light on the nature of Fueter operator.

Corollary 13.0.9. *Let* $f(z) = \alpha(u, v) + i\beta(u, v)$ *an intrinsic holomorphic function. Then*

$$\mathcal{D}[f(x_0 + \underline{x})] = -2HGCK[(f^{(1)}(x_0), 0)].$$

Our next aim is to solve the following problem

Problem 13.0.10. *Develop generalized CK extensions for polyharmonic, polyanalytic, holomorphic Cliffordian and polyanalytic Cliffordian functions. Moreover we would like to figure out if these possible extensions are related to the Fueter-Sce map.*

CHAPTER 14

Appendices

We add the appendices A and B due to the lack of references. In the appendix C it is possible to visualize all the possible *fine structures* explained in Chapter 12.

14.0.1 Appendix A: Complex Hermite polynomials

We prove an orthogonality relation for the complex Hermite polynomials, with a general parameter $\alpha > 0$. Then, we show some basic properties of the Hermite polynomials, for a general parameter $\nu > 0$. We consider the complex Hermite polynomials defined by

$$H^{\alpha}_{m,p}(z,\overline{z}) = (-1)^{m+p} e^{\alpha|z|^2} \frac{\partial^{m+p}}{\partial z^m \partial \overline{z}^p}\left(e^{-\alpha|z|^2}\right), \qquad \alpha > 0.$$

First, using some direct calculations we observe that we have

$$H^{\alpha}_{0,p}(z,\overline{z}) = \alpha^p z^p$$

and

$$H^{\alpha}_{1,p}(z,\overline{z}) = \alpha^{p+1}\overline{z}z^p - \alpha^p p z^{p-1}.$$

In order to revise the calculations of the complex Hermite polynomials norm in $L^{2,\alpha}(\mathbb{C}) := L^2(\mathbb{C}, e^{-\alpha|z|^2} d\lambda_i(z))$, where $d\lambda_i(z)$ denotes the Lebesgue measure of the complex plane $\mathbb{C}$. We follow the ideas of [98]. Let us consider the operator given by

$$A := -\frac{\partial}{\partial z} + \alpha\overline{z}.$$

Then, we can prove the following result.

Lemma 14.0.1. *For all $m \in \mathbb{N}$, we have*

$$A^m\left((\alpha z)^p\right) = H^\alpha_{m,p}(z,\overline{z}).$$

Proof. We use an induction process to prove this result. Firstly, for $m = 1$ we have

$$\begin{aligned} A\left((\alpha z)^p\right) &= -\alpha^p p z^{p-1} + \alpha^{p+1}\overline{z}z^p \\ &= H^\alpha_{1,p}(z,\overline{z}). \end{aligned}$$

Now, let us suppose that this relation holds for m and prove it for $m+1$. Indeed, we use the induction hypothesis combined with the Leibniz rule to get

$$\begin{aligned} A^{m+1}\left((\alpha z)^p\right) &= A\left(H^\alpha_{m,p}(z,\overline{z})\right) \\ &= -(-1)^{m+p}\left(\alpha\overline{z}e^{\alpha|z|^2}\frac{\partial^{m+p}}{\partial z^m\partial\overline{z}^p}e^{-\alpha|z|^2} + e^{\alpha|z|^2}\frac{\partial^{m+p+1}}{\partial z^{m+1}\partial\overline{z}^p}e^{-\alpha|z|^2}\right) \\ &+ (-1)^{m+p}\alpha\overline{z}e^{\alpha|z|^2}\frac{\partial^{m+p}}{\partial z^m\partial\overline{z}^p}e^{-\alpha|z|^2} \\ &= H^\alpha_{m+1,p}(z,\overline{z}). \end{aligned}$$

Thus, the result holds by induction, this ends the proof. ☐

Theorem 14.0.2. *Let $\alpha > 0$ and $m, p \in \mathbb{N}$. Then, we have*

$$||H^\alpha_{m,p}(z,\overline{z})||_{L^{2,\alpha}(\mathbb{C})} = \alpha^{p+m-1}\pi m!p!\,.$$

Proof. We set $\varphi_p(z) = (\alpha z)^p$, then using direct computations we obtain

$$\begin{aligned} H^\alpha_{m,p}(z,\overline{z}) &= A^m\left(\varphi_p(z)\right) \\ &= \left(-\frac{\partial}{\partial z} + \alpha\overline{z}\right)^m\left(\varphi_p(z)\right) \\ &= \alpha^p\sum_{j=0}^m(-1)^j\binom{m}{j}\left(\frac{\partial^j}{\partial z^j}M^{m-j}_{\overline{z}}(z^p)\right)\alpha^{m-j}, \end{aligned}$$

where the conjugate multiplication operator is given by $M_{\overline{z}}f = \overline{z}f$. Then, using the fact that

$$\left(\frac{\partial}{\partial z}\right)^j z^p = \frac{\Gamma(p+1)}{\Gamma(p-j+1)} z^{p-j}, \quad j = 0, 1, 2, ...$$

we obtain

$$H^{\alpha}_{m,p}(z,\overline{z}) = \alpha^p m! \sum_{j=0}^{m} (-1)^j \frac{p!}{j!(m-j)!(p-j)!} \alpha^{m-j} z^{p-j} \overline{z}^{m-j}.$$

Then, we pass to the polar coordinates $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in [0, 2\pi]$ and get

$$H^{\alpha}_{m,p}(re^{i\theta}, re^{-i\theta}) = \alpha^p m! e^{i\theta(p-m)} \sum_{j=0}^{m} \frac{p!(-1)^j}{j!(m-j)!(p-j)!} \alpha^{m-j} r^{p+m-2j}.$$

We change the summation index to $k = m - j$, so we get

$$\begin{aligned} H^{\alpha}_{m,p}(re^{i\theta}, re^{-i\theta}) &= \alpha^p m! e^{i\theta(p-m)} \\ &\quad \cdot \sum_{k=0}^{m} \frac{p!(-1)^{m-k}}{(m-k)!k!(p-m+k)!} \alpha^k r^{p-m+2k}. \end{aligned} \tag{14.1}$$

After that we use the classical formula for the generalized Laguerre polynomials given by

$$L^{\beta}_m(x) := \sum_{k=0}^{m} (-1)^k \binom{m+\beta}{m-k} \frac{x^k}{k!}.$$

Thus, if $\beta := p - m$, with $p > m$ we have

$$L^{p-m}_m(x) := \sum_{k=0}^{m} (-1)^k \frac{p!}{(m-k)!(p-m+k)!} \frac{x^k}{k!}.$$

In particular, from the formula (14.1) we get

$$H^{\alpha}_{m,p}(re^{i\theta}, re^{-i\theta}) = \alpha^p m! (-1)^m e^{i\theta(p-m)} r^{p-m} L^{p-m}_m(\alpha r^2).$$

Now, we compute the orthogonality relation using the Fubini's theorem. Let $m', p' \in \mathbb{N}$ and $d\lambda_i(z)$ be the Lebesgue measure in the complex plane, then we have

$$\langle H^\alpha_{m,p}(z,\bar z), H^\alpha_{m',p'}(z,\bar z)\rangle_{L^{2,\alpha}(\mathbb{C})} = \int_{\mathbb{C}} H_{m,p}(z,\overline{z})\overline{H_{m',p'}(z,\overline{z})}e^{-\alpha|z|^2}d\lambda_i(z)$$
$$= \alpha^{2p}(m!)(m'!)(-1)^m(-1)^{m'}\left(\int_0^{2\pi} e^{i\theta(p-m)}e^{-i\theta(p'-m')}d\theta\right)$$
$$\cdot\left(\int_0^\infty r^{p-m}r^{p'-m'}rL_m^{p-m}(\alpha r^2)L_{m'}^{p'-m'}(\alpha r^2)e^{-\alpha r^2}dr\right).$$

We set $\ell := p-m$ and $\ell' = p'-m'$. Thus we get

$$\langle H^\alpha_{m,p}(z,\bar z), H^\alpha_{m',p'}(z,\bar z)\rangle_{L^{2,\alpha}(\mathbb{C})} = 2\pi\alpha^{2p}(m!)(m'!)\cdot$$
$$\cdot(-1)^m(-1)^{m'}\delta_{\ell,\ell'}\left(\int_0^\infty r^{p-m+1}r^{p'-m'}L_m^{p-m}(\alpha r^2)L_{m'}^{p'-m'}(\alpha r^2)e^{-\alpha r^2}dr\right).$$

Since $\ell = p-m = p'-m' = \ell'$ we derive that $p = m+\ell$ and $p' = m'+\ell$. Therefore

$$\langle H^\alpha_{m,p}(z,\bar z), H^\alpha_{m',p'}(z,\bar z)\rangle_{L^{2,\alpha}(\mathbb{C})} = \langle H^\alpha_{m,m+\ell}(z,\bar z), H^\alpha_{m',m'+\ell}(z,\bar z)\rangle_{L^{2,\alpha}(\mathbb{C})}$$
$$= 2\pi\alpha^{2p}(m!)(m'!)(-1)^m(-1)^{m'}\left(\int_0^\infty r^{\ell+1}r^\ell L_m^\ell(\alpha r^2)L_{m'}^\ell(\alpha r^2)e^{-\alpha r^2}dr\right)$$
$$= 2\pi\alpha^{2p}(m!)(m'!)(-1)^m(-1)^{m'}\left(\int_0^\infty r^{2\ell+1}L_m^\ell(\alpha r^2)L_{m'}^\ell(\alpha r^2)e^{-\alpha r^2}dr\right).$$

We know that (see [90] pag. 809 paragraph 7.414 formula 3)

$$\int_0^\infty L_k^\gamma(t)L_j^\gamma(t)t^\gamma e^{-t}dt = \frac{\Gamma(\gamma+k+1)}{k!}\delta_{k,j}.$$

Then, we use the following change of variables $s = \alpha r^2$ and get

$$\begin{aligned}\langle H^\alpha_{m,p}(z,\bar z), H^\alpha_{m',p'}(z,\bar z)\rangle_{L^{2,\alpha}(\mathbb{C})} =& \pi\alpha^{2p-1}(m!)(m'!)(-1)^m(-1)^{m'}\cdot\\ &\cdot\int_0^\infty\left(\frac{s}{\alpha}\right)^\ell L_m^\ell(s)L_{m'}^\ell(s)e^{-s}ds\\ =&\pi\alpha^{2p-1-\ell}(m!)^2\frac{\Gamma(m+\ell+1)}{m!}\delta_{m,m'}.\end{aligned}$$

Since $\ell = p-m$ and $p-m = p'-m'$ we get

$$\begin{aligned}\langle H^\alpha_{m,p}(z,\bar z), H^\alpha_{m',p'}(z,\bar z)\rangle_{L^{2,\alpha}(\mathbb{C})} &= \pi\alpha^{p+m-1}m!\Gamma(p+1)\delta_{p,p'}\delta_{m,m'}\\ &= \pi\alpha^{p+m-1}m!p!\delta_{m,m'}\delta_{p,p'}.\end{aligned}$$

Therefore

$$\|H^{\alpha}_{m,p}(z,\bar{z})\|^2_{L^{2,\alpha}(\mathbb{C})} = \pi\alpha^{p+m-1}m!p! \quad .$$

□

Remark 14.0.3. In particular, for $\alpha = 2\pi$ we obtain

$$\langle H^{2\pi}_{m,p}(z,\bar{z}), H^{2\pi}_{m',p'}(z,\bar{z})\rangle_{L^{2,\alpha}(\mathbb{C})} = \frac{m!p!(2\pi)^{p+m}}{2}\delta_{m,m'}\delta_{p,p'}.$$

14.0.2 Appendix B

Let us consider the following function for $\nu > 0$

$$W(x,t) = e^{-\frac{\nu}{2}t^2+\nu\sqrt{2}xt} = \sum_{n=0}^{\infty}\frac{H^{\nu}_n(x)}{n!}\frac{t^n}{2^{\frac{n}{2}}},$$

where H^{ν}_n are the weighted Hermite polynomials defined by

$$H^{\nu}_n(x) = (-1)^n e^{\nu x^2}\left(\frac{d}{dx}\right)^n e^{-\nu x^2} = n!\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^n(2x\nu)^{n-2m}}{m!(n-2m)!}.$$

Putting $t = \sqrt{2}\lambda$ we get

$$W(x,\lambda) = e^{-\nu\lambda^2+2\nu x\lambda} = \sum_{n=0}^{\infty}\frac{H^{\nu}_n(x)}{n!}\lambda^n. \tag{14.2}$$

Relabelling λ with t we call the function $W(x,t)$ in (14.2) as the generating function of the weighted Hermite polynomials. In order to obtain a recurrence relation, which relates the weighted Hermite polynomials with their consecutive indices, we derive the equation (14.2) with respect to t:

$$\frac{\partial W(x,t)}{\partial t} = (-2\nu t + 2\nu x)e^{-\nu t^2+2\nu xt} = \sum_{n=1}^{\infty} nH^{\nu}_n(x)\frac{t^{n-1}}{n!}.$$

Using another time the generating function (14.2) we obtain

$$-2\nu\sum_{n=0}^{\infty}H^{\nu}_n(x)\frac{t^{n+1}}{n!} + 2\nu x\sum_{n=0}^{\infty}H^{\nu}_n(x)\frac{t^n}{n!} = \sum_{n=1}^{\infty}H^{\nu}_n(x)\frac{t^{n-1}}{(n-1)!}. \tag{14.3}$$

By a change of indices in the first sum we get

$$-2\nu \sum_{n=1}^{\infty} H_{n-1}^{\nu}(x) \frac{t^n}{(n-1)!} = -2\nu \sum_{n=1}^{\infty} n H_{n-1}^{\nu}(x) \frac{t^n}{n!}.$$

Thus, by another change of indices in (14.3) we obtain

$$-2\nu \sum_{n=1}^{\infty} n H_{n-1}^{\nu}(x) \frac{t^n}{n!} + 2\nu x \sum_{n=0}^{\infty} H_n^{\nu}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_{n+1}^{\nu}(x) \frac{t^n}{n!}.$$

By identifying, the coefficients of t^n we get

$$H_{n+1}^{\nu}(x) = 2\nu x H_n^{\nu}(x) - 2n\nu H_{n-1}^{\nu}(x). \tag{14.4}$$

It is possible to derive another recurrence relation satisfied by the weighted Hermite polynomials. We set $t = \sqrt{2}\lambda$ as in (14.2) and after we differentiate $W(x,t)$ with respect to x

$$\frac{\partial}{\partial x} W(x,t) = 2\nu t e^{-\nu t^2 + 2\nu x t} = \sum_{n=0}^{\infty} \left(\frac{d}{dx} H_n^{\nu}(x) \right) \frac{t^n}{n!}.$$

Using the generating function we obtain

$$2\nu \sum_{n=0}^{\infty} H_n^{\nu}(x) \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} \left(\frac{d}{dx} H_n^{\nu}(x) \right) \frac{t^n}{n!}.$$

By a change of variables we can identify the coefficients of t^n to get

$$\frac{d}{dx} H_n^{\nu}(x) = 2\nu n H_{n-1}^{\nu}(x). \tag{14.5}$$

Remark 14.0.4. If we put $\nu = 1$ in the formulas (14.4) and (14.5) we recover the classical formulas that can be found in [107].

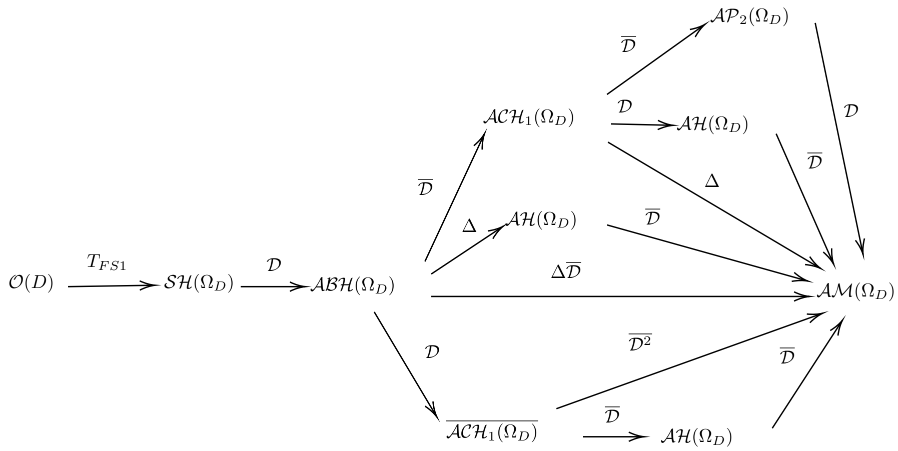
14.1 Appendix C: visualization of all possible fine structures in dimension five

In this appendix we show all the possible fine structures in dimension five. Firstly, we recall the symbols of the classes of functions involved

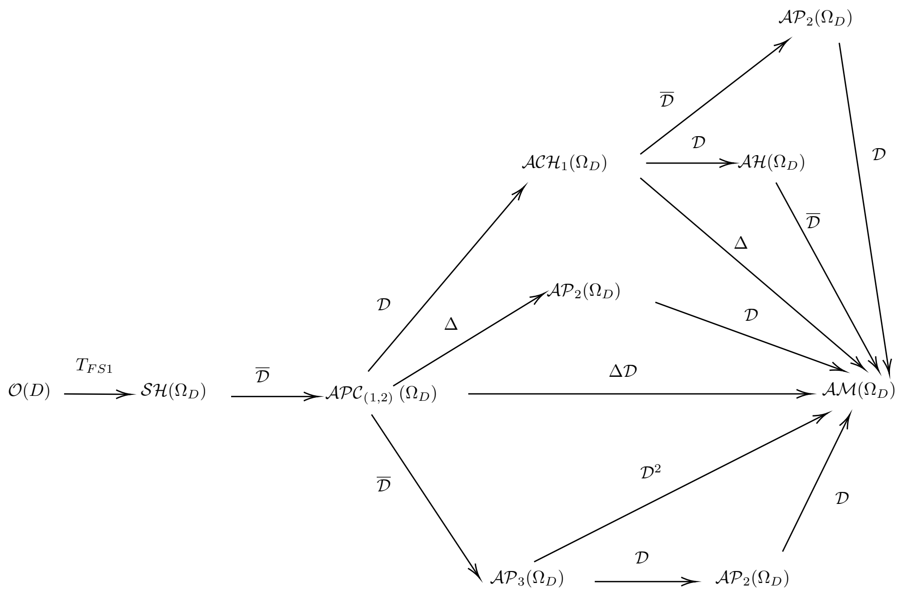
$\mathcal{ABH}(\Omega_D)$: axially bi-harmonic functions,
$\mathcal{ACH}_1(\Omega_D)$: axially Cliffordian holomorphic functions of order 1
$\mathcal{AH}(\Omega_D)$: axially harmonic functions,

$\mathcal{AP}_2(\Omega_D)$: axially polyanalytic of order 2,
$\overline{\mathcal{ACH}_1(\Omega_D)}$: axially anti Cliffordian of order 1,
$\mathcal{ACP}_{(1,2)}$: axially polyanalytic Cliffordian of order $(1, 2)$,
$\mathcal{AP}_3(\Omega_D)$: axially polyanalytic of order 3.

If we apply first the Dirac operator we have the following diagram



If we apply fist the conjugate of the Dirac operator we get



Finally, all the other possible fine structures are given by the diagram:

$$
\begin{tikzcd}
& & & \mathcal{AP}_2\left(\Omega_D\right) \arrow[dr, "\mathcal{D}"] & \\
\mathcal{O}(D) \arrow[r, "T_{FS1}"] & \mathcal{SH}(\Omega_D) \arrow[r, "\Delta"] \arrow[urr, "\Delta\overline{\mathcal{D}}"] \arrow[drr, "\Delta\mathcal{D}"'] & \mathcal{ACH}_1(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"] \arrow[rr, "\Delta"] \arrow[dr, "\mathcal{D}"] & & \mathcal{AM}(\Omega_D) \\
& & & \mathcal{AH}(\Omega_D) \arrow[ur, "\overline{\mathcal{D}}"'] &
\end{tikzcd}
$$

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