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Application of Gradient Dynamics in the Edgeworth-Bertrand Model for Economic Markets

Tesi di Laurea Magistrale in
Mathematical Engineering - Quantitative Finance

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Abstract

This study aims is to apply Edgewort-Bertrand Model to examine how firms do price competition under capacity constraints. This model is applied here to understand animal groups that compete for sources as biological populations. By using gradient dynamics, we examined whether the system could reach steady state by the time. Simulation shows model reaches mixed Nash Equilibrium. This study helps us to understand how groups balance competition under limited sources.In this work, I tried to replicate some results established in [3]. We see examples in economic markets as nature.

Keywords: edgeworth model, gradient dynamics, evolutionary matrix

Abstract in lingua italiana

Questo studio applica il Modello Edgeworth-Bertrand per vedere come le imprese competono sui prezzi quando hanno limitazioni di capacità. Qui, il modello viene usato per capire come gruppi di animali competono per le risorse, simili a popolazioni biologiche. Utilizzando la dinamica del gradiente, abbiamo verificato se il sistema può raggiungere uno stato stabile nel tempo. La simulazione mostra che il modello arriva a un Equilibrio di Nash misto. Questo studio ci aiuta a comprendere come i gruppi bilanciano la competizione con risorse limitate. In questo lavoro, ho cercato di replicare alcuni risultati del [3]. Vediamo esempi sia nei mercati economici che in natura.

Parole chiave: modello di edgeworth, dinamica del gradiente, matrice evolutiva

Contents

Introduction

The Edgeworth-Bertrand (EB) model plays an important role in macroeconomics. It examines how firms compete by setting prices in markets where they have limited capacity. It highlights the complexities of price competition and market outcomes when firms face production constraints. It combines Bertrand's idea of firms undercutting prices to attract consumers with Edgeworth's insight that capacity constraints can prevent firms from fully meeting demand, leading to market instability and potential price fluctuations.

This work aims to fully explore the (EB) model. We are going to study on its application.

I followed Evolving Landscapes for Population Games article by Daniel Friedman and Joel Yellin that is context of population games and foraging competition. Authors studied on gradient dynamics associated with the model lead to cyclical behavior or a mixed Nash equilibrium. Their simulation shows us asymptotic convergence to a mixed Nash equilibrium occurs, rather than cyclic outcomes. In my thesis I have replicated and extended this simulation.
To do so, we start by defining the Evolutionary Game Theory in Chapter 1, in Chapter 2 we define EB model through Bertrand model, in Chapter 3 we will dig into the key concept of population game, and finally in Chapter 4 we will see numerous applications of the Edgeworth-Bertrand model.

1 | Evolutionary Game Theory

In an evolutionary game, players choose from a range of actions or behaviors, with their payoff or fitness influenced by the choices of others. Over time, the types of behaviors in a population evolve, with more successful strategies becoming more common. The prevalence of a behavior can increase or decrease its fitness, as well as that of other behaviors, making the dynamics quite complex. We can ask which actions vanish and which actions survive over the time. We can also analyse if system approaches some stable steady-state.[1]

Evolutionary game theory, unlike classical game theory, does not assume that players make logical decisions when determining their strategic decisions. Instead, it assumes that players optimise their behaviour according to strategies that evolve over time. This theory emerged to explain situations such as biological evolution and competition in nature and has subsequently found wide application in many areas, from economics to sociology.

The basis of evolutionary game theory is the principle of "natural selection". In this context, players within a population compete to survive or succeed by following certain strategies. Effective strategies become widespread within the population because they provide a higher level of "fitness". Over time, strategic interactions between players change the distribution of strategies in the population, and eventually a stable strategy profile emerges in the population. This stable situation is called an "evolutionarily stable strategy" (ESS), and when players follow this strategy, no major changes are seen in the strategy profile without external intervention.

Evolutionary game theory offers a more dynamic approach than the static equilibrium analysis of classical game theory. That is, strategies must not only be optimal in the current situation but also be resilient to evolutionary pressures in the long term. With this theory, it becomes possible to study complex population dynamics, model how different strategies may spread in the long term, and understand how individuals adapt in a common environment.

In economics, evolutionary game theory is used to understand the evolutionary processes of markets, competition, and even price formation. In cases where firms adapt their strategies to take into account each other's pricing decisions and market shares, evolutionary game theory provides a suitable framework for analysing how firms develop strategies in a competitive environment. In this context, for example, the Bertrand and Edgeworth models are popular game theory models used to explain the evolutionary processes of price competition.

As a result of this part, evolutionary game theory allows for dynamic and multidimensional strategy analysis by presenting a world in which not only individuals but also strategies evolve. This theory contributes to our understanding of how strategic preferences evolve.

1.1. Evolutionary game model ingredients

Evolutionary game models are built using the following components: [2]

- $k \geq 1$ populations.

  Each population has its own action set. A simple game could be a 2x2 symmetric matrix format that is played by single population. Therefore, action set includes two alternative strategies. In market simulation, the population might consist of traders who sell a unit above price q and purchase a unit below price q. We might add second population, which consist of market makers that have a bid price b and ask price a.

- Dynamics

  The current state s refers to the distribution of actions chosen within each population. The dynamics explain how this distribution evolves over time.

- Payoff (or fitness)

  A function that returns real values based on the current state and the action chose by the players. Basic example is a matrix where the (i,j)th entry represents the payoff for a player who chooses the ith action while the opponent chooses for the jth action.

In a market simulation, the payoff might be the profit is determined by the trader's buy or sell orders and the prices set by the actions of all traders. The payoff $f_k(x, s)$ for an individual in group k may be nonlinear in relation to the state variable s.

1.2. Evolutionarily Stable strategies (ESS)

In 1973, John Maynard Smith and George R. Price introduced the concept of Evolutionary Stable Strategy (ESS) for the first time. [6] The authors developed the idea of an Evolutionarily Stable Strategy (ESS) as a strategy that, when it is adopted by a the population, it is hard to replace with other strategies, making it stable. ESS provides a way to study conflicts between animals and look at strategies that increase survival and reproductive fitess. By the time, these started to be used also in economy.

Let us suppose that there is single population game and action set consist of n elements. The current state s is an n-dimensional vector that can be adjusted in any direction x to transition to state $s^{'}$=ϵx+(1-ϵ)s described as a (ϵ-) minor invasion of (x-) mutants.

The current state s is an ESS if $f(s, s^{'}) > f(x, s^{'})$ for all $x \neq s$ and for all $\epsilon > 0$ sufficiently small. [2]

2 | Approach of Edgeworth-Bertrand model

2.1. Bertrand Model

In the field of economics, the Bertrand model examines how price competition occurs among rival firms. Let us consider the following hypothesis: [7]

- Rival firms produce homogeneous products and set their own price simultaneously
- No capacity constraints is assumed
- The consumers suppose that the products manufactured by various firms are similar and have the same quality; hence, they will purchase from the firm proposing a lower price

Taking these assumptions into consideration, firms try to reduce their costs in order to reach the minimum selling price so that they can be chosen by the consumers. But in doing so, the risk is to reach 0 profit, which leads to a paradox.
Firms do not have a problem producing as much product as is demanded. Equilibrium of this model is that both firms will set prices equal to marginal cost, therefore earning zero profits. But this can cause the **Bertrand Paradox**. [4] By relaxing one of the these assumptions, Bertrand Paradox is solved. One way to avoid the Bertrand Paradox is to remove the assumption that every firm does not have capacity constraints. So if firms can not supply whole market, this situation refers to Edgeworth-Bertrand model.

2.2. Edgeworth-Bertrand

According to Edgeworth, capacity should not be unlimited. The Edgeworth-Bertrand model is a Bertrand Model with a capacity constrained. Thanks to this assumption, prices increase above marginal cost as realistic. In Edgeworth's rival firms model, pure strategies are not needed to be exist , but always Nash Equilibrium can be reached in mixed strategies, where firms choose probability distributions over prices instead of an exact price [8]. As an example, for tossing coin, each side of coin has same probability distribution and for each side has 1/2 probability. However, if distribution occurs for continuous variable as price, probability distribution shows us how probabilities are spread over the entire range of prices. In this case, firms choose different prices with certain probabilities.

Considering that there are two rival firms which sell homogeneous products. In pure strategy case, they choose certain price and remain faithful on it.

In mixed strategy case,they might choose different prices with different probabilities as:

$$P(X=x)=\begin{cases} 0.30, & \text{if } x=10 \\ 0.25, & \text{if } x=12 \\ 0.40, & \text{if } x=15 \\ 0.05, & \text{if } x=20 \end{cases}$$

Since firms can not know exactly which price will be chosen by its rival firm, both firms can create their own price strategy by analyzing which strategy is more suitable for them. Through to these strategies, dynamics of market might be more stable and Nash equilibrium might be reached. Because, instead of firms change their price frequently, they follow strategies that depend on probability distributions.

2.3. Interpretations of the Edgeworth-Bertrand model

Let us start by assuming, two firms dominate the market which is duopoly. Therefore, we will focus on pairwise game which refers to a type of game involves only two players. In each time, firms can choose to participate or not participate. If they don't participate, they get zero profit.

Assumption: [3]

- The price charged by an participated firm is indicated by x $\in$ [0,1].
- If X=Z, it denotes firm is not participated.

- Firms have a positive fixed cost c which is needed to pay to start or continue to business. These cost does not change with the production quantity. It does not matter how much will be produced.
- Firms have zero marginal cost. Since marginal cost measures the additional cost of producing one more unit, if marginal cost is zero, there is no additional cost to produce one more unit.
- All consumers buy a single unit from the lowest cost firm.

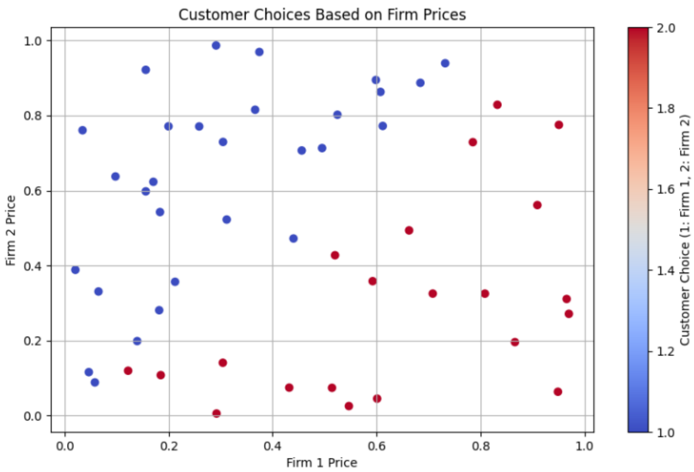

Figure 2.1: Customers choose with probability 1 whatever company offers the lowest price.

- If firms charge the same price, consumers split their total demand evenly.
- We scale maximal demand to 1.

Considering these assumptions, the pairwise payoff function, which represents profit (revenue), can be expressed as follows: [3]

$$g(Z, y) = 0 \tag{2.1}$$

$$g(x, y) = x - c, \qquad x \in [0, 1], \quad y > x, \quad y = Z \tag{2.2}$$

$$g(x, y) = 0.5x - c, \qquad y = x \in [0, 1] \tag{2.3}$$

$$g(x, y) = -c, \qquad 0 \leq y < x \leq 1 \tag{2.4}$$

In this scenario, x and y represent Firm 1 and Firm 2, respectively. If Firm 1 chooses not to participate, as shown in (2.1), it gets zero payoff because there is no need to pay the

fixed cost (c) required in order to start the business, and there is also no revenue.
In the case where Firm 2 sets a higher price than Firm 1, as in (2.2), consumers buy from Firm 1 and payoff of Firm 1 becomes difference between selling price and fixed cost.
When both firms set the same price, as illustrated in (2.3), they share the payoff equally, assuming that half of the consumers choose to buy from Firm 1 and the other half choose to buy from Firm 2.
Finally, if Firm 1's price is higher than Firm 2's, as in (2.4), consumers buy from Firm 2, resulting in Firm 1 paying the fixed cost required to start the business but earning no revenue.

I created a graph of the payoff function to illustrate how the payoff varies with all possible price combinations for Firm 1 and Firm 2, assuming that both firms have a fixed cost of c = 0.3, in accordance with the methodology described in the article I referenced.

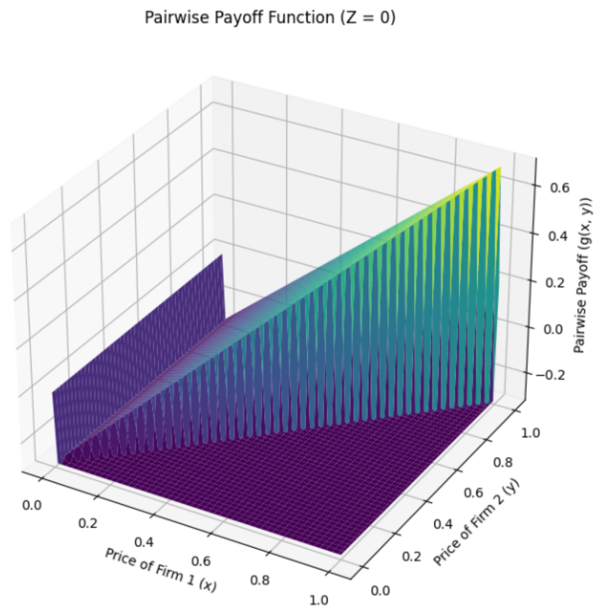

Figure 2.2: The graph of the payoff function for different combinations of prices.

Figure 1.2 clearly shows that the maximum payoff can reach 1-0.3=0.7.
Z is additional strategy representing non participation.

Remark:

If Firm 1 sets a higher price than Firm 2, consumers will choose Firm 2, and Firm 1's payoff will be -0.3.

g(x,y) is said to be a **pairwise payoff function** that represents the payoff when one player chooses x and other chooses y. (We will use in our model).

Instead of considering there are two rival firms in the market, we might work on large population of firms. Therefore, I would like to introduce to population games which involves a large or infinite population of individuals, each with their own strategies.

3 | Population Games

Population games consider how various strategies are distributed among a population of players and how these strategies change over time. The population is characterized by the distribution of strategies among individuals. Outcome of player's strategies influence not only other player's strategy, affects also prevalence of that strategy among the entire population. Population games involves evolutionary dynamics which refer to the changes and processes that occur in a population over time.

In economics and game theory, a population game can refer to a game where multiple players (in our case, rival firms) make decisions simultaneously, and the outcomes depend not only on an individual's actions but also on the distribution of actions within the entire population. This type of game considers the strategic interactions among a group of decision-makers.

There are two important mathematical objects in population games, distribution D(or density ρ) and a distributed payoff function ϕ (or pairwise payoff function g). [3]

We assume that action set which is the set of possible choices(actions) that firms can take is continuous unit interval A=[0,1]. **distributed payoff function**

Action distribution is a term used to describe how actions are distributed among a group of choices that firms have made. This distribution is indicated by its cumulative distribution function (cdf) D or by its density ρ which is provided by derivative of D with respect to x. ($\rho = D_x$)

Proposition 3.1. *Cumulative distribution function D: $\mathbb{R} \to \mathbb{R}$ on A which is a non-decreasing right continuous function such that*

$$D(x) = \begin{cases} 0, & \textit{for } x \leq 0, \\ 1, & \textit{for } x \geq 1. \end{cases}$$

Definition:
Let us assume that $\mathcal{D}$ indicates the set of all distributions [0,1]. The **distributed payoff function** ϕ:$[0,1]\times\mathcal{D}\rightarrow\mathbb{R}$ assigns the payoff $\phi(x, D)$ to any player choosing x, given that D describes the distribution of other player's actions.

Example of Population games

Let assume there is a infinite number of population. Some of them are playing first strategy and some of them are playing second strategy. Over time, perhaps the one of those strategy takes over. Technology is a good example for this kind of games.
When smart phones first introduced, there were very few people who started to use them, but over time, almost entire population started to use them.
It shows how there is competition between two strategies in a population and how the dominant strategy spreads throughout the population over time.

The Hawk and Dove game is a famous evolutionary and population game. [5]

3.1. Discrete Gradient Dynamics

Dynamics refers to the evolution, changes in the strategies, and outcomes of players in a game over time. The dynamics are controlled by a finite system (system involves finite entities.) that characterises the flow of the population or probability mass among adjacent points on a one-dimensional lattice, identified as $i = 0, 1, ...n$. These lattice points are indicating possible actions.

Distributed payoff to action k is defined by :

$$\phi_k = \sum_i g_{ki}\rho_i \tag{3.1}$$

where

g_{ki}: pairwise payoff while one player choose action k, opponent chooses action i.
ρ_i: the fraction of players choosing action i, which is defined by $\rho_i = N_i/N$. (N is large number of players; Ni is number of players who choose action i.)

Our assumption is flow of population can be only between neighboring points. (from k to k+1 or k-1). By taking into account this assumption we have a rule as below:

Mater Rule:

All flows from point k to the neighboring points k ±=1 only take place if they are "uphill" in terms of the distributed payoff. That is, mass flows from k to point k-1 if and only if the difference

$$X_k = \phi_{k-1} - \phi k \tag{3.2}$$

is positive; mass flow from k-1 to k if (2.2) is negative. When flow does occur , it is proportional to the mass ρ_k at source point k and to the payoff difference X_k.

The population dynamics are described by the real time (n+1)-dimensional discrete evolution system.

$$\frac{d\rho}{dt} = M(X) \cdot \rho \tag{3.3}$$

where the evolution matrix is

$$\mathbf{M}(\mathbf{X}) = \begin{pmatrix} X_1^- & X_1^+ & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -X_1^- & -X_1^+ + X_2 & X_2^+ & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -X_2^- & -X_2^+ + X_3 & \dots & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & -X_k^- & X_k^+ & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -X_k^+ + X_{k+1} & -X_{k+1}^- & X_{k+1}^+ & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & -X_{n-1}^+ + X_n & -X_n^- & X_n^+ \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -X_n^+ & -X_n^- \end{pmatrix}$$

where

$$X^+ = max(0, X), \qquad X^- = min(0, X) \tag{3.4}$$

Our first goal is to construct the evolutionary matrix. To achieve this evolutionary matrix, we need to calculate the distributed payoff at point k-1 and its neighbouring point k, as the difference between them is essential for determining X_k. In order to obtain the distributed payoffs, equation (2.1) must be computed. Therefore, it's necessary to construct the g function by following steps outlined in equations (2.1), (2.2), (2.3), and (2.4). In the next chapter, I will explain details about all computations and analyses.

4 | Applications

4.1. The Edgeworth-Bertrand Model as Foraging Competition

As I mentioned in previous part, I started by creating g(x,y) function.

$$\mathbf{g(x,y)} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & g_{04} & g_{05} & g_{06} \\ g_{10} & g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{20} & g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{30} & g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \\ g_{40} & g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} \\ g_{50} & g_{51} & g_{52} & g_{53} & g_{54} & g_{55} & g_{56} \\ g_{60} & g_{61} & g_{62} & g_{63} & g_{64} & g_{65} & g_{66} \end{pmatrix}$$

Firstly, I began by assuming that there are 7 strategies available for Firm 1 and Firm 2 to choose from: 0, 1, 2, ..., 6. The first element represents the choice of Firm 1, while the second element represents the choice of Firm 2. The diagonal elements of the matrix indicate the pairwise outcomes when both firms choose the same price (same strategy),such as 00, 11, ..., 66. In this case, the payoff will be 0.5x-c as mentioned in (2.3).
For other cases, the function g_{ki} represents the pairwise payoff when strategy k is chosen while the opponent has chosen strategy i.
For example, if g_{01} has a payoff of x-c as mentioned in (2.2), then g_{10} should have a payoff of -c as mentioned in (2.4).

By extending this example to all other cases and inserting it into the g(x,y) function, the pairwise function can be expressed as follows:

$$\mathbf{g}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0.5x_0 - c & x_1 - c & x_2 - c & x_3 - c & x_4 - c & x_5 - c & x_6 - c \\ -c & 0.5x_1 - c & x_2 - c & x_3 - c & x_4 - c & x_5 - c & x_6 - c \\ -c & -c & 0.5x_2 - c & x_3 - c & x_4 - c & x_5 - c & x_6 - c \\ -c & -c & -c & 0.5x_3 - c & x_4 - c & x_5 - c & x_6 - c \\ -c & -c & -c & -c & 0.5x_4 - c & x_5 - c & x_6 - c \\ -c & -c & -c & -c & -c & 0.5x_5 - c & x_6 - c \\ -c & -c & -c & -c & -c & -c & 0.5x_6 - c \end{pmatrix}$$

Assumptions in article, there is also a strategy where the firms do not enter the market, as mentioned in (2.1). The 7th column and 7th row are assigned to this strategy, called strategy Z, and the payoff for this strategy is 0. Therefore, in the pairwise payoff function, the 7th column and 7th row are completely filled with 0's as:

$$\mathbf{g}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} 0.5x_1 - c & x_2 - c & x_3 - c & x_4 - c & x_5 - c & x_6 - c & 0 \\ -c & 0.5x_2 - c & x_3 - c & x_4 - c & x_5 - c & x_6 - c & 0 \\ -c & -c & 0.5x_3 - c & x_3 - c & x_5 - c & x_6 - c & 0 \\ -c & -c & -c & 0.5x_4 - c & x_5 - c & x_6 - c & 0 \\ -c & -c & -c & -c & 0.5x_5 - c & x_6 - c & 0 \\ -c & -c & -c & -c & -c & 0.5x_6 - c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$x_k = (k-1)/5, \qquad k = 1, 2, ..., 6$$

Now, we will use the g(x,y) function to calculate the distributed payoff for action k, as described in (2.1).

We set a fixed price to start business as c=0.3 as mentioned in the article; however, we will later explore the effects of changing this fixed cost to a different value.

$$
\begin{aligned}
\Phi_0 &= -0.3\rho_0 - 0.1\rho_1 + 0.1\rho_2 + 0.3\rho_3 + 0.5\rho_4 + 0.7\rho_5 \\
\Phi_1 &= -0.3\rho_0 - 0.2\rho_1 + 0.1\rho_2 + 0.3\rho_3 + 0.5\rho_4 + 0.7\rho_5 \\
\Phi_2 &= -0.3\rho_0 - 0.3\rho_1 - 0.1\rho_2 + 0.3\rho_3 + 0.5\rho_4 + 0.7\rho_5 \\
\Phi_3 &= -0.3\rho_0 - 0.3\rho_1 - 0.3\rho_2 + 0.5\rho_4 + 0.7\rho_5 \\
\Phi_4 &= -0.3\rho_0 - 0.3\rho_1 - 0.3\rho_2 - 0.3\rho_3 + 0.1\rho_4 + 0.7\rho_5 \\
\Phi_5 &= -0.3\rho_0 - 0.3\rho_1 - 0.3\rho_2 - 0.3\rho_3 - 0.3\rho_4 + 0.2\rho_5 \\
\Phi_6 &= 0
\end{aligned}
$$

As the next step, we are now ready to find the payoff difference as described in (3.2).

$$
\begin{aligned}
X_1 &= \phi_0 - \phi 1 = 0.1\rho_1 \\
X_2 &= \phi_1 - \phi 2 = 0.1\rho_1 + 0.2\rho_2 \\
X_3 &= \phi_2 - \phi 3 = 0.2\rho_2 + 0.3\rho_3 \\
X_4 &= \phi_3 - \phi 4 = 0.3\rho_3 + 0.4\rho 4 \\
X_5 &= \phi_4 - \phi 5 = 0.4\rho_4 + 0.5\rho_5 \\
X_6 &= \phi_5 - \phi_6 = -0.3(\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4)
\end{aligned}
$$

Now, since we need the positive and negative parts of the X's in the evolutionary matrix, we have to examine them as mentioned in (3.4)

$$
\begin{aligned}
X_1^- &= min(0, X_1), & X_1^+ &= max(0, X_1) \\
X_1^- &= 0, & X_1^+ &= 0.1\rho_1 \\
X_2^- &= min(0, X_2), & X_2^+ &= max(0, X_2) \\
X_2^- &= 0, & X_2^+ &= 0.1\rho_1 + 0.2\rho_2 \\
X_3^- &= min(0, X_3), & X_3^+ &= max(0, X_3) \\
X_3^- &= 0, & X_3^+ &= 0.2\rho_2 + 0.3\rho_3 \\
X_4^- &= min(0, X_4), & X_4^+ &= max(0, X_4) \\
X_4^- &= 0, & X_4^+ &= 0.3\rho_3 + 0.4\rho_4 \\
X_5^- &= min(0, X_5), & X_5^+ &= max(0, X_5) \\
X_5^- &= 0, & X_5^+ &= 0.4\rho_4 + 0.5\rho_5 \\
X_6^- &= min(0, X_6), & X_6^+ &= max(0, X_6) \\
X_6^- &= -0.3(\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4) + 0.2\rho_5, & X_6^+ &= 0
\end{aligned}
$$

The final step in creating the evolutionary matrix is to place all the positive and negative parts of the payoff difference into M(X). Therefore, fulfilled evolutionary matrix is as below:

$$\mathbf{M(X)} = \begin{pmatrix} 0 & 0.1\rho_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1\rho_1 & 0.1\rho_1 + 0.2\rho_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1\rho_1 - 0.2\rho_2 & 0.2\rho_2 + 0.3\rho_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2\rho_2 - 0.3\rho_3 & 0.3\rho_3 + 0.4\rho_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3\rho_3 - 0.4\rho_5 & 0.4\rho_4 + 0.5\rho_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.3(\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_5) - 0.7\rho_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3(\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4) - 0.2\rho_5 & 0 \end{pmatrix}$$

The unknown variables in the game are the quantities ρ_k, which represent the proportion of the population choosing the k-th strategy. Our goal is to find these ρ_k's by using formula as mentioned in (3.3)

$$\frac{\mathbf{d}\rho}{\mathbf{dt}} = \begin{pmatrix} 0 & 0.1\rho_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1\rho_1 & 0.1\rho_1 + 0.2\rho_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1\rho_1 - 0.2\rho_2 & 0.2\rho_2 + 0.3\rho_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2\rho_2 - 0.3\rho_3 & 0.3\rho_3 + 0.4\rho_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3\rho_3 - 0.4\rho_5 & 0.4\rho_4 + 0.5\rho_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.3(\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_5) - 0.7\rho_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3(\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4) - 0.2\rho_5 & 0 \end{pmatrix} \cdot \begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \end{pmatrix}$$

This matrix differential equation is solved to calculate each element of ρ by assuming all initial values of ρ are equal to $1/7$. A graph was created to show how these ρ values change over time. The time interval selected as t=[0,2,...,20]. And solution of ρ values and graph of their change over time as below:

```
Time points (t): [ 0  2  4  6  8 10 12 14 16 18 20]
Corresponding rho values: [[0.14285714 0.14718612 0.15207194 0.15761626 0.16392    0.17106629
  0.17910549 0.18804632 0.1978539  0.20845422 0.21974284]
 [0.14285714 0.15151221 0.16120847 0.17187487 0.18321557 0.19477457
  0.206061   0.21660464 0.22604407 0.23413565 0.24075303]
 [0.14285714 0.15142017 0.16007195 0.16747491 0.17263275 0.17516086
  0.17507734 0.17277869 0.16872469 0.16339437 0.15721563]
 [0.14285714 0.14999152 0.15223184 0.14930739 0.14284549 0.13445734
  0.12540209 0.11637012 0.10779055 0.09986302 0.09265729]
 [0.14285714 0.13799922 0.12215167 0.10558335 0.0911062  0.07914962
  0.06939654 0.06146573 0.05496937 0.04959819 0.04511095]
 [0.14285714 0.07727867 0.04303137 0.02456576 0.01430085 0.00841505
  0.00503476 0.00303061 0.00183601 0.00111771 0.0006839 ]
 [0.14285714 0.18461208 0.20923277 0.22357746 0.23197915 0.23697627
  0.23992278 0.24170388 0.24278141 0.24343685 0.24383637]]
```

Figure 4.1: Solution of rho values

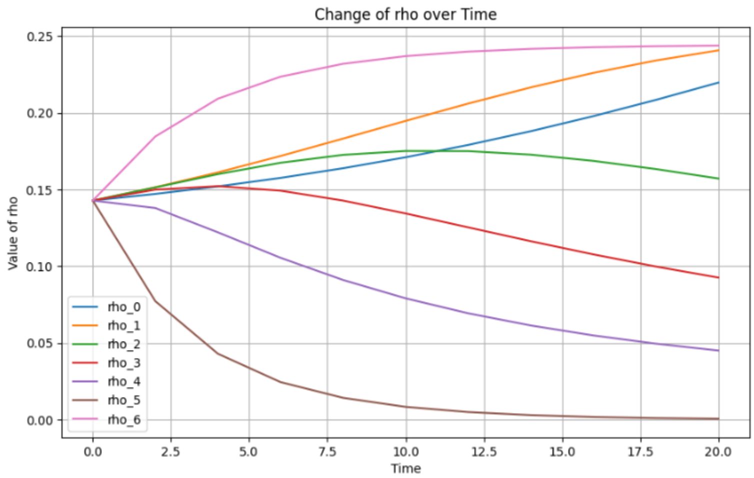

Figure 4.2: Change of rho values over time

Here, the ρ values represent the fraction of firms choosing each action(price strategy). The graph shows that at the beginning, all price strategies have the same fraction of $1/7$, since there are 7 possible prices that firms can choose to enter the market. Over time, the graph clearly illustrates us which strategies become important among firms and which ones are no longer chosen and fade out. This helps us understand which pricing strategies are successful and which are not.

If we make interpretation of elements of rho, we can see the following:

- ρ_0 shows that this strategy becomes more preferred over time, as the line goes up and its popularity increases.
- We can say the same for ρ_1, but compared to ρ_0, ρ_1 is even more preferred.
- The ρ_3 strategy increased a little at first but then went down. It might have been successful in the beginning, but it is not preferred later.
- ρ_4 strategy is decreasing over time, which means it is losing popularity.
- The ρ_5 strategy is decreasing very quickly and is not preferred anymore.
- On the other hand, ρ_6 is increasing over time and becomes the most popular one.

In order to extend the meaning of this graph as it was done in the article, I have plotted different components of ρ at different times.

I wrote a function for the asymptotic density, which is defined in the article as $\rho^*(x) = c/x^2$ for $x \geq c$. But for the case $x < c$, I defined a formula myself to achieve an exponential increase before x = c = 0.3. This function takes the last densities from the solution of the equation and calculates the asymptotic density. Therefore, we can see that the player's strategy can vary over time, but eventually, this variation tends to converge to a steady state.

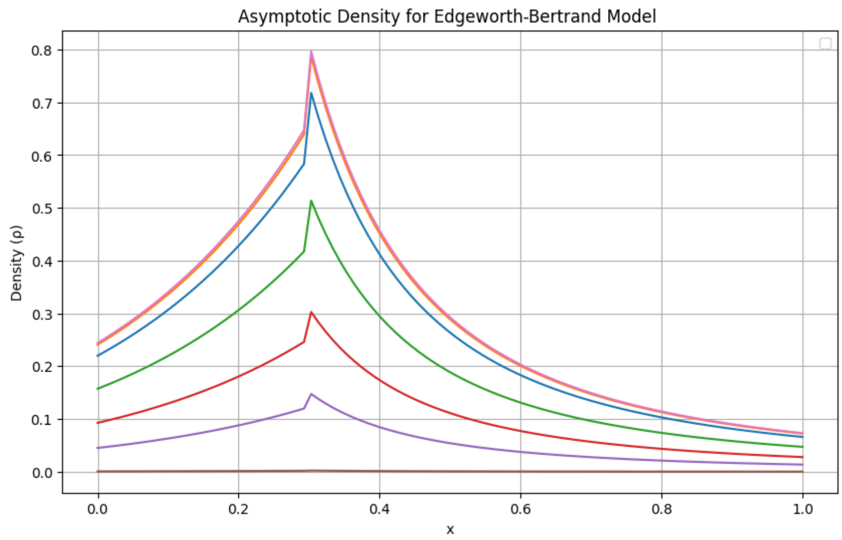

Figure 4.3: Different components of rho at different times.

This graph illustrates us how asymptotic density changes over x(price charged by an participated firm) in Edgeworth-Bertrand model.

From this graph, we can say that around x=0.3 it starts to increase, and after 0.3 it reaches its pick point. It means that in this price density is much higher. This price is chosen more by firms. It could indicate a critical price level where firms gain some advantage in the market by adopting this pricing strategy.

Prices much above c are less preferred by consumers, and firms focus less on these higher price points.

Additionally, we can say that highest or lowest prices are less preferable by time.

In Edgeworth-Bertrand model, since firms are in competition, concentration around a certain price (x>c) can be observed. This price can be Nash equilibrium between firms. When firms set this price, they can maximize their profit.

If the price is significantly above the marginal cost c, it might seem that firms could earn more. However, the Edgeworth-Bertrand model assumes that consumers will always choose the firm with the lower price, meaning that if companies set a high price, customers will simply buy from another firm. Therefore, the objective is to establish a price close to the cost.

4.1.1. Edgeworth-Bertrand Model with different initial conditions

Instead of assuming all initial conditions are equal and 1/7 as in the beginning, I also tried to make analysis by considering Edgeworth Bertrand model with different initial conditions again with same c=0.3.

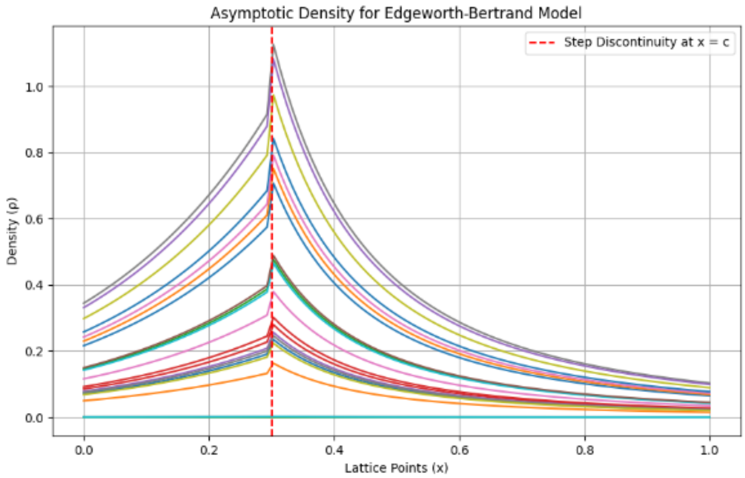

Figure 4.4: EB model with different initial conditions

We can see some similarity between Figure 4.3 and Figure 4.4. Even if in Figure 4.4, we started with different initial conditions. However, this is what we expected because both graphs represent the Edgeworth-Bertrand model, which has the same theoretical basis based on competitive pricing between firms. Since Figure 4.4 uses different initial conditions, resulting in greater variation in the density curves. But still clearly, we might see that the company's choices between prices are still bigger than marginal cost.

4.1.2. Edgeworth-Bertrand Model with different parameters

In this part, instead of considering c=0.3 as recommended in article, I would like to see what happens if we apply Edgeworth-Bertrand model with different parameters. This time, I proceeded same process with c values as 0.2, 0.3, 0.5, 0.8.

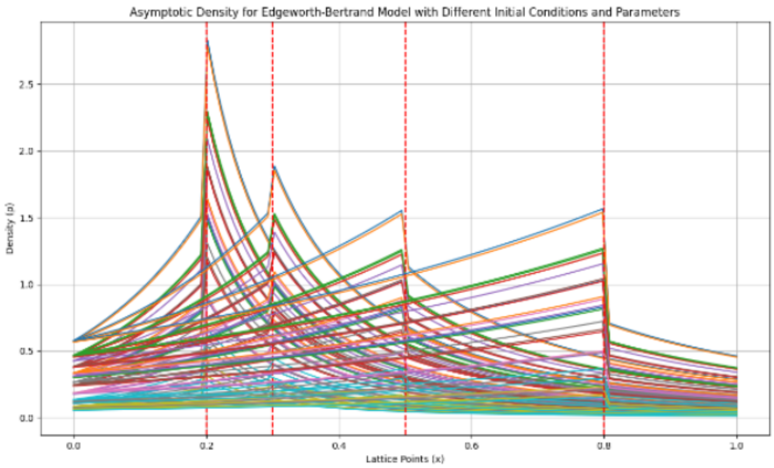

Figure 4.5: EB model with different initial conditions and different parameters

From the graph, it's evident that the company's behavior varies with each c value, and they tend to prefer prices close to the corresponding c. Given that our marginal cost is 0.8, selecting a price near 0.8 would be more advantageous for them.

I've implemented this application using 7 strategies. To explore the effects of increasing the number of strategies, I moved forward with 21 strategies. In the following section, I will discuss my results.

4.1.3. Edgeworth-Bertrand Model for more strategies

I developed a new pairwise payoff function under the assumption that there are still two rival firms. Using this updated pairwise function, I constructed a new evolutionary matrix as previously described and performed the same calculations. Finally, I have reached the graph shown below:

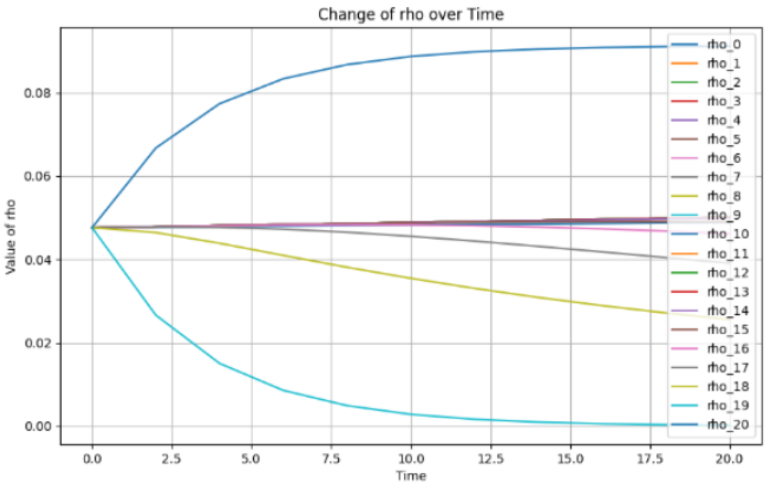

Figure 4.6: Change of rho values over time for 21 strategies

From this graph we can see that when we have more strategies there is more balance between strategies and chance of surviving in the market is increased.

If we compare Figure 4.2 and Figure 4.6, it is more clear which strategies are more preferable than others, which strategies can be dominated by others.

When there is few strategy, competition might be more fierce and it can lead that some strategies can be eliminated fast in the market. However, when there is more strategies, each strategy might competes less strongly against others. Therefore, it creates more balanced environment.

If we look at how Figure 4.4 changes for more strategies we can do same interpretation and see balance more clear as graph in below:

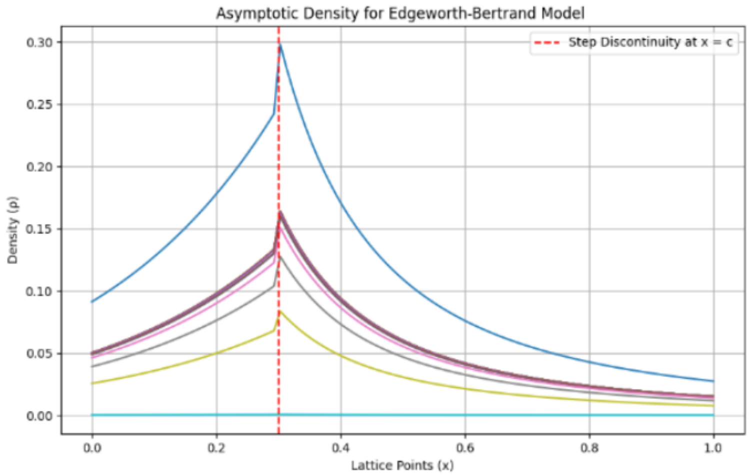

Figure 4.7: Edgeworth-Bertrand Model for 21 strategies

5 | Conclusions

As a conclusion, in this thesis I tried to reach an equilibrium point in the strategies of firms when they are choosing the possibility of price actions. Assuming there are two rival firms in the market and they produce homogenous products. If we do not assume there is a capacity constraint that firms can produce as much as demand, firms can set their price equal to marginal cost. Therefore, they get zero profit. However, in the Edgeworth-Bertrand model, we assume there is a capacity constraint and firms can not produce as much as they want. Therefore, firms can set their price above marginal cost. Even if they can set their price above marginal cost, they cannot set a high price since we assume consumers always buy from firms that have a low price.

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