POLITECNICO DI MILANO
DEPARTMENT OF MATHEMATICS "F. BRIOSCHI"
DOCTORAL PROGRAMME IN MATHEMATICAL MODELS AND METHODS IN ENGINEERING

ANALYSIS AND APPROXIMATION OF MOMENT EQUATIONS FOR PDES WITH STOCHASTIC DATA

Doctoral Dissertation of:
Francesca Bonizzoni

Advisor:
Prof. Fabio Nobile

Tutor:
Prof. Fabio Nobile

The Chair of the Doctoral Program:
Prof. Roberto Lucchetti

2013 – XXV

Acknowledgements

I want to express my grateful thanks to my advisor, Fabio Nobile, for supporting and guiding me during these three years, and for helping me to take the first steps in the research environment. The long discussions with him in front of the blackboard have enriched my cultural baggage, and his accurate corrections to each page of the thesis have contributed to my scientific growth. He has given me the possibility of participating to many conferences all around the world, and the opportunity to follow him in Lausanne.

I would also thank Annalisa Buffa, Monica Riva, Daniel Kressner and his group for the interesting collaborations established.

I'm grateful to my Phd mates in MOX. Thank you for each laugh during coffee breaks and all the time spent together. It's always a great pleasure to come back to Milan!

Thank you to all my new friends who have welcomed me at the EPFL. You have made me discover Swiss fondue and raclette. We had fun together on sledges and during apéros and barbecues along the lake.

Thank you to my friends going back a long way: you are always present in my life!

A great reward is for my family, who supported me and sustained all my choices. Although the distance, you are always my reference point and my home.

Finally, the biggest thanks are for Giuseppe: you walked with me along my path, which is now our path, you are the partner of my life and my new family.

Abstract

This thesis concerns the study of partial differential equations (PDE) with uncertain input data described as random variables or random fields with known probability laws. Such mathematical models are known as *stochastic PDEs* (SPDEs). The solution u of an SPDE is itself stochastic. Given complete statistical information on the input data, the aim of the present thesis is to infer on the statistical moments of u.

The approach proposed consists in deriving the so called *moment equations*, that is the deterministic equations solved by the statistical moments of the stochastic solution u. We consider PDEs with randomness arising either in the loading term or in the coefficient.

Concerning the first class, the stochastic counterpart of the Hodge Laplace problem in mixed formulation is considered, which finds applications in electrostatic/magnetostatic problems with uncertain current/charge, as well as in the fluid flow in porous media with uncertain sources or sinks. The moment equations are exactly derived, and their well-posedness is proved. The stability of both full and sparse tensor product finite element discretizations is obtained.

We then consider the boundary value problem modeling the single phase flow (Darcy law) in a heterogeneous porous medium, where the permeability is described as a lognormal random field, i.e. $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, $Y(\omega, x)$ being a Gaussian random field. Under the assumption of small variability of Y, we perform a *perturbation analysis*, which consists in expanding the solution of the SPDE in Taylor series. We analyze the approximation properties of the K-th order Taylor polynomial $T^K u$, predict the divergence of the Taylor series, and provide an estimate of the optimal order K_{opt} such that adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it.

We approximate the statistical moments of the stochastic solution with the statistical moments of its Taylor polynomial. We derive the *recursive* problem solved by the expected value of $T^K u$ and show its well-posedness. An algorithm to solve the first moment problem at a prescribed order K is proposed. All the computations are performed in low-rank format, the tensor train (TT) format.

On a model problem with only few random input parameters we show that the solution obtained with our TT-algorithm compares well with the one obtained by a Stochastic Collocation method. However, our algorithm is capable of dealing also with a very large number of random variables and even infinite-dimensional random fields, and providing a valid solution, whereas the Stochastic Collocation method is unfeasible.

The dependence of the complexity of the algorithm on the prescribed tolerance tol in the TT-computations is studied via numerical tests. We numerically predict the existence of an optimal tol depending both on the order of approximation K and the standard deviation of the field Y. If the optimal tolerance is chosen, the performance of the moment equations is far superior to a standard Monte Carlo method.

Keywords: Uncertainty Quantification, moment equations, sparse tensor product approximations, Hodge Laplacian, Darcy law, lognormal permeability, perturbation technique, low-rank format.

Contents

Introduction

Motivations

Mathematical models are helpful instruments for qualitative and quantitative investigation in many different disciplines. They provide an interpretation and predict the behavior of phenomena arising in natural science, biology, engineering, economy, finance, etc. Numerical analysis develops and studies algorithms for numerically solving problems coming from mathematical modeling.

In many applications, the parameters of the model are not precisely known. This may be due to several factors.

- Measurement errors or errors due to the accuracy of the measuring instruments.
- Incomplete knowledge of quantities which can be measured only point-wise, like the permeability field of a heterogenous porous medium.
- Intrinsic uncertainty of certain phenomena, like winds, earthquake sources, responses of biological tissues, ...

Recently, many efforts have been made in trying to treat and include this uncertainty in the model. A very convenient framework is offered by the probability theory. In particular, starting from a suitable partial differential equation (PDE) model, the uncertain input data are described as random variables or random fields with known probability laws. This kind of mathematical models are known as stochastic PDE (SPDE).

The solution of an SPDE is itself stochastic: it is a function of space and time as well as the random realizations. The goal of *Uncertainty Quantification* is to infer on the solution, i.e. to understand how the uncertainty in the input data of the model reflects onto the solution. The quantities of interest may be the statistics of the solution itself, like mean and variance, or statistics of functionals of the solution.

In this thesis we focus on second order linear PDEs with randomness arising either in the forcing terms or in the coefficients. In the first case, the mixed formulation of the SPDE is considered. Interesting applications may be the electrostatic/magnetostatic problem with uncertain current/charge and the fluid flow in a heterogeneous (uncertain) porous medium with uncertain sources or sinks.

The sampling and polynomial approaches

The *Monte Carlo method* is the most straightforward approach to tackle the uncertainty quantification problem. It consists in generating a sample of M independent realizations, solve the PDE corresponding to each realization, and then combine the set of solutions to obtain an approximation of the quantity of interest. We refer to [22, 87]. Very weak assumptions on the SPDE are needed for the convergence of the Monte Carlo method. Moreover, it features a rate of convergence independent of the dimension of the problem (i.e. number of random variables in the equation) of the order of $M^{-1/2}$. This rate of convergence is very slow and a very large sample size is usually needed to achieve an acceptable accuracy. This represents the main limitation in the use of Monte Carlo methods to SPDEs where each evaluation can be very costly. It should be noted that Monte Carlo method does not exploit any possible regularity that the solution of the SPDE might have with respect to the random parameters.

A number of improvements have been proposed in recent years. Suppose that, for each realization of the input parameters, the solution of the SPDE belongs to a function space V, defined on the physical domain D. If V is endowed with a hierarchic structure $V_0 \subset V_1 \ldots \subset V_L \subset V$, $\{V_l\}_{l=o}^{L}$ being a set of nested finite dimensional subspaces, one can construct an approximation of the solution on each subspace (for instance a finite element approximation) using a certain number of realizations.

The *Multilevel Monte Carlo method* is based on the idea that a large number of realizations is needed for small l, whereas only few realizations are needed for large l. We refer to [13, 45, 46, 58, 61] for a deeper discussion on the Multilevel Monte Carlo method, and in particular to [13, 32, 94] for the application of this idea to SPDEs.

The Monte Carlo method has been improved also in the direction of the distribution of the sampling points. The distribution of points of an independent randomly generated sample is affected by clumping. The *Quasi Monte Carlo method* (see [22, 77]) exploits quasi random samples, which are deterministically generated so as to ensure a better uniformity (low-discrepancy sequences). Application of Quasi Monte Carlo methods to SPDEs have been proposed in [48, 68].

The solution u of an SPDE can be viewed as a mapping defined on the space of random parameters $\{Y = (Y_1, \ldots, Y_N) \in \mathbb{R}^N\}$, with values onto a space/time function space. A *generalized Polynomial Chaos Expansion* (gPC expansion) consists in approximating the function $u(Y)$ by multivariate polynomials in Y. Such an approximation can be obtained via a Galerkin projection, leading to the *Stochastic Galerkin method* [11,40,43,47,75,91,95], or via an interpolation strategy, leading to the *Stochastic Collocation method* [10, 42, 78, 79, 98]. The Stochastic Galerkin method entails the solution of $N_h \times M$ coupled linear systems, where N_h is the dimension of the discrete subspace in the physical space and M is the dimension of the polynomial space in Y. On the other hand, the Stochastic Collocation method consists in collocating the problem in a set of points $\{Y_i\}_{i=1}^{Q}$ (sparse grid) and entails the solution of Q decoupled linear systems, with, however, $Q \gg M$ in general. On the other hand, the Stochastic Collocation method is fully parallelizable and can exploit pre-existing deterministic solvers.

A gPC technique exploits the regularity of the solution as a function of the random parameters, and exhibits rates of convergence superior to the Monte Carlo method, at least when the dimension of the problem is not too high.

Moment equations and perturbation analysis

The Monte Carlo method requires the generation of a large number of realizations, and the evaluation of the PDE at each realization, so that it turns to be very expensive. On the other hand the gPC can successfully treat problems with moderately small dimension in the space of parameters, whereas, for large dimensions, it is unfeasible.

Given complete statistical information on the input data, the aim of the present thesis is to analyze both theoretically and numerically an alternative method to both Monte Carlo like approaches and gPC techniques, namely the so called *moment equations*. The moment equations are suitable deterministic equations solved by the statistical moments of the stochastic solution u of the SPDE. We focus, in particular, on linear PDEs with randomness arising either in the loading terms or in the coefficients.

For what concerns the first class, a family of stochastic saddle-point problems is considered. More precisely, we study the stochastic counterpart of the mixed formulation of the Hodge Laplace problem, where the Hodge Laplace is a second order differential operator acting on differential forms. See [6, 7]. In this case, we derive exactly the moment equations, and state their well-posedness.

Concerning the second class, we study the Darcy boundary value problem modeling the fluid flow in a heterogeneous porous medium. The uncertain permeability is described as a lognormal random field: $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, $Y(\omega, x)$ being a Gaussian random field. Under the assumption of small variability of Y, we perform a *perturbation analysis* based on the Taylor expansion of the solution $u(Y)$. We refer to [9,29,33] and to the geophysical literature [51, 52, 86, 93]. In [56] the perturbation technique has been applied to SPDE defined on random domains.

We investigate the approximation properties of the Taylor polynomial $T^K u$ and predict the divergence of the Taylor series and the existence of an optimal order K_{opt} of the Taylor polynomial such that adding new terms will deteriorate the accuracy instead of improving it. The divergence of the Taylor series is a result limited to the lognormal model. Indeed, in [9] the authors prove the convergence of the Taylor series in the case where the permeability is described as a linear combination of bounded random variables.

In the approach adopted in this thesis, the entire field and not a finite dimensional approximation of it is considered. Hence, the Taylor polynomial involves the Gateaux derivatives of u with respect to Y, and can not be directly computed.

Deterministic equations solved by the statistical moments of the Taylor polynomial are derived, and the statistical moments of u are approximated by the statistical moments of $T^K u$. We mainly focus our attention on the recursive problem solved by the expected value of $T^K u$, studying its structure, stating its well-posedness and proving Hölder-type regularity results.

The moment equation approach entails the solution of high dimensional problems defined on tensor product domains. The most straightforward discretization technique is the *full tensor product discretization* (FTP), which suffers from the *curse of dimensionality*, meaning that the number of degrees of freedom grows exponentially in the dimension d of the problem. To overcome the curse of dimensionality, we explore either *sparse tensor product discretizations* (STP) or *low-rank techniques*.

We discretize the moment equations derived from the mixed formulation of the stochastic Hodge Laplacian with a STP technique. If a STP is adopted, the number of degrees of freedom is no more exponential in d, but linear up to logarithmic terms. However, almost the same rate of accuracy as with a FTP discretization is obtained. See [21, 97]. We show, in particular, the stability of the STP discretization and provide a convergence result. To our knowledge this is one of the very few results available on the stability of sparse approximations of tensorized mixed problems.

On the other hand, full tensor products finite element spaces are used to discretize the recursive problem solved by the expected value of $T^K u$, u being the solution of the Darcy problem with stochastic coefficient. We tackle the curse of dimensionality of the FTP approximation by representing the high dimensional tensors in a *low-rank* or *data-sparse format*. Low-rank formats come from the Numerical Linear Algebra framework, and aim at representing approximated high dimensional tensors using a dramatically reduced number of parameters. We refer to [23, 57, 96] for a deeper introduction on classical low-rank formats. In recent years, two formats both based on the singular value decomposition have been introduced: the Hierarchical Tucker (HT) format [49, 53, 54] and the Tensor Train (TT) format [53, 81]. A tensor in TT-format is expressed using three dimensional tensors, called TT-cores, whose sizes are called TT-ranks. The TT-format presents a linear structure and can be easily manipulated. Moreover, many linear algebra operations together with the rounding operation, which consists in approximating the original tensor with a one with lower TT-rank up to a prescribed accuracy tol, are implemented in the Matlab TT-Toolbox available at `http://spring.inm.ras.ru/osel/?page_id=24`. For these reasons we have chosen to develop a code which employs only TT-format representations of tensors.

If the random field Y is parametrized by a finite number of random variables, then the Taylor polynomial $T^K u$ can be directly computed by solving the recursive problem for the derivatives of u. This finite-dimensional situation can be achieved expanding the field in series (Karhunen-Loève expansion [43, 70, 72, 73] or Fourier expansion [50]) and then truncating the series. We compare the complexity of this method with our TT-algorithm, and highlight the superiority of the latter.

Performing the computations in TT-format allows us to handle the entire field Y. Precisely, we take into account the complete Karhunen-Loève expansion of Y (up to machine precision). We compare the expected value of u computed via our TT-algorithm with a stochastic collocation solution, and highlights that the same level of accuracy can be obtained. On the other hand, when a stochastic collocation method is unfeasible, i.e. when Y is parametrized by a large number of random variables, the TT-algorithm still provides a valid solution.

A fundamental question we have tried to answer is how the complexity of the TT-algorithm depends on the precision tol achieved in the TT-computations. We have performed some numerical tests which point out the existence of an optimal tol depending both on the order of approximation, (the order K of the Taylor polynomial), and the standard deviation of Y. If the optimal tolerance is chosen, the performance of the moment equations is far superior to a standard Monte Carlo method. The question of how to determine a priori the optimal tolerance is still open and under investigation.

The main limitations of our TT-algorithm come from the storage point of view, and prevent us to grow significantly in K (order of Taylor approximation). We believe that a great improvement will follow from the implementation of sparse tensors toolboxes, which are still missing in Matlab (or other programming languages).

Outline

The outline of the thesis is the following.

Chapter 1 contains the overview of the thesis, focusing on the main results obtained and pointing to the corresponding chapters for a deeper discussion.

Chapter 2 studies a saddle-point problem with stochastic loading terms, namely the mixed form of the stochastic Hodge Laplacian, where the Hodge Laplace operator is a second order differential operator acting on differential forms. The moment equations are derived, and their well-posedness is proved. Both full and sparse tensor product finite element discretizations are provided and analyzed. In particular we prove the stability of both discretizations, and show that a sparse approximation provides almost the same rate of accuracy as a full approximation, with a drastic reduction in the number of degrees of freedom.

Chapter 3 focuses on the Darcy problem modeling the fluid flow in a heterogeneous porous medium. The permeability of the physical domain is described as a lognormal random field. Under the assumption of small variability of the field, a perturbation analysis based on the Taylor expansion of the solution is performed. The approximation properties of the Taylor polynomial are explored, and the divergence of the Taylor series is predicted. We also predict the existence of an optimal degree K_{opt} of the Taylor polynomial and provide an a priori estimate for it.

Chapter 4 is dedicated to the derivation of the moment equations for the stochastic Darcy problem with lognormal permeability introduced in Chapter 3. The recursive structure of the first moment problem is highlighted, and its well-posedness is proved. Hölder-type regularity results are also obtained.

Chapter 5 proposes an algorithm to numerically solve the recursive first moment problem derived in Chapter 4. All the correlations involved are represented in a low-rank format (Tensor Train format). The complexity of the algorithm is studied and some numerical tests in the one dimensional setting are performed.

Parts of the material contained in this thesis have been already submitted for publication or are ready to be submitted. In particular

Chapter 2 is based on: F. Bonizzoni, A. Buffa, F. Nobile, *Moment equations for the mixed formulation of the Hodge Laplacian with stochastic data*, available as MOX Report 31/2012 - Department of Mathematics, Politecnico di Milano.

Chapter 3 is based on a paper in preparation: F. Bonizzoni, F. Nobile, *Perturbation analysis for the Darcy problem with lognormal permeability*. A short version can be found in F. Bonizzoni, F. Nobile, *Perturbation analysis for the stochastic Darcy problem*. Proceeding in the European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2012).

The material in Chapters 4 and 5 is instead still unpublished.

The work of this thesis has been carried out for one half at MOX laboratory, Department of Mathematics, Politecnico di Milano, and for the other half at CSQI - MATHICSE, École Polythecnique Fédérale de Lausanne.

This work has been supported by the italian grant *Fondo per gli Investimenti della Ricerca di Base FIRB-IDEAS (Project n. RBID08223Z)* “Advanced numerical techniques for uncertainty quantification in engineering and life science problems”.

CHAPTER 1

Thesis overview

This chapter briefly introduces the main results obtained in this thesis and highlights the crucial points of each chapter, pointing to it for a deeper discussion.

1.1 Problem setting and notations

Let D be a bounded domain in $\mathbb{R}^n$ for $n \geq 1$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, that is Ω is the set of outcomes, $\mathcal{F}$ is the σ-algebra of events and $\mathbb{P} : \Omega \to [0, 1]$ is the probability measure. In this thesis we deal with boundary value problems of the form

$$T(a(\omega, x))u(\omega, x) = f(\omega, x), \text{ a.e. in } D, \text{ a.s. in } \Omega \tag{1.1}$$

where T is a partial differential operator linear with respect to u, and invertible from V to V' for any fixed $\omega \in \Omega$, V being a suitable Hilbert space. In problem (1.1), the uncertainty affects either the forcing term $f(\omega, x)$ or the coefficient $a(\omega, x)$ of the differential operator $T(a)$. As soon as the input parameters randomly vary, so does the solution. We model both the uncertain input data and the solution u as random variables or random fields.

A random variable Y is a measurable mapping defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values onto $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$. We define the expected value $\mathbb{E}[Y]$ as

$$\mathbb{E}[Y] := \int_\Omega Y(\omega) d\mathbb{P}(\omega)$$

The space of random variables with finite expected value is denoted as $L^1(\Omega, \mathbb{P})$. Similarly, for any positive integer k, $L^k(\Omega, \mathbb{P})$ denotes the space of random variables with finite k-th statistical moment

$$\mathbb{E}\left[Y^k\right] := \int_\Omega Y^k(\omega) d\mathbb{P}(\omega).$$

The space $L^k(\Omega, \mathbb{P})$ is a Banach space and, for $k = 2$, it is a Hilbert space with the natural inner product

$$(X, Y) := \int_\Omega X(\omega) Y(\omega) d\mathbb{P}(\omega).$$

All the previous definitions generalize to the case of a random field $Y : \Omega \to V$, where V is a Hilbert or Banach space of functions defined on the physical domain D. See [1, 16, 50] for more on random fields and stochastic processes. In this setting, $L^k(\Omega; V)$ denotes the Bochner space of random fields such that

$$\|Y\|_{L^k(\Omega;V)} := \left(\int_\Omega \|Y(\omega)\|_V^k \, d\mathbb{P}(\omega) \right)^{1/k} < +\infty.$$

We introduce the k-points correlation function of Y

$$\mathbb{E}\left[Y^{\otimes k}\right](x_1, \dots, x_k) := \mathbb{E}\left[Y(\omega, x_1) \otimes \cdots \otimes Y(\omega, x_k)\right].$$

If $Y(\omega, x)$ is a random field with values into the functional space V, then

$$\mathbb{E}\left[Y^{\otimes k}\right] \in V^{\otimes k} := \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}.$$

Note that, if V is a Hilbert space, the definition of the tensor product space $V \otimes V$ is quite natural since it exploits the scalar product in V. (See e.g. [85]). On the other hand, in the case V Banach space, $V \otimes V$ can be endowed with different non-equivalent norms. (See e.g. [53,88]). In Chapter 2, V will be a Sobolev space of differential forms, and, in Chapters 3, 4 and 5, the classical Sobolev space $H^1(D)$.

The aim of the work is to develop suitable techniques to approximate the statistical moments of $u(\omega, x)$, the stochastic solution of problem (1.1). In particular, we derive the deterministic equations solved by the statistical moments of $u(\omega, x)$, known as *moment equations*. If the uncertainty concerns only the loading term $f(\omega, x)$, the k-th moment equation is obtained tensorizing the stochastic problem with itself k times, and then taking the expectation. On the other hand, if the uncertainty affects the coefficient $a(\omega, x)$ of the differential operator, we adopt a *perturbation approach*, develop u in Taylor series and approximate the statistical moments of u using the statistical moments of its Taylor polynomial. In both cases, the solution of high dimensional problems is needed. High dimensional problems are affected by the *curse of dimensionality*, that is the exponential growth of the problem complexity in its dimension. To overcome this obstacle, we propose to use either sparse discretizations or low-rank techniques.

1.2 Results on linear problems in mixed formulation with stochastic forcing term

Given a bounded domain $D \subset \mathbb{R}^n$, $n \geq 1$, in Chapter 2 we consider a problem of the form

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} u(\omega, x) \\ p(\omega, x) \end{bmatrix} = \begin{bmatrix} f_1(\omega, x) \\ f_2(\omega, x) \end{bmatrix}, \quad \text{a.e. in } D, \quad \text{a.s. in } \Omega \qquad (1.2)$$

Table 1.1: *Proxy fields correspondences in the case* $n = 3$.

k	d	$H\Lambda^k(D)$
0	∇	$H^1(D)$
1	curl	$H(\text{curl}, D)$
2	div	$H(\text{div}, D)$
3	0	$L^2(D)$

where the uncertainty affects only the loading terms $f_1(\omega, x)$, $f_2(\omega, x)$, whereas $T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$ is a deterministic partial differential operator. Problems of this type arise when mixed formulations of elliptic problems are considered. Note that elliptic problems (not in mixed form) with uncertain loading terms are well studied in literature. See [28, 91, 92, 97].

In Chapter 2, T is the mixed form of the Hodge Laplace operator, which is a second order differential operator acting on differential forms. The *finite element exterior calculus* is a theoretical approach aimed at better understanding finite element schemes, and possibly developing new ones with desirable properties. See [5–7]. To explain the results obtained, we first briefly recall the problem setting and main notations.

1.2.1 Introduction on finite element exterior calculus

A differential k-form on a domain $D \subset \mathbb{R}^n$ is a map u which associates to each $x \in D$ an alternating k-form on $\mathbb{R}^n$, $u_x \in \mathit{Alt}^k\mathbb{R}^n$. The space of all smooth differential k-forms is denoted with $\Lambda^k(D)$. The exterior derivative d maps $\Lambda^k(D)$ into $\Lambda^{k+1}(D)$ for each $k \geq 0$ and is defined as

$$\mathrm{d}u_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} u_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}), \quad u \in \Lambda^k(D),$$

$v_1, \ldots, v_{k+1} \in \mathbb{R}^n$, where the hat is used to indicate a suppressed argument. The coderivative operator δ, which maps $\Lambda^k(D)$ onto $\Lambda^{k-1}(D)$, is the formal adjoint of d. The space of all square integrable differential forms $L^2\Lambda^k(D)$ is the completition of $\Lambda^k(D)$ with respect to the norm induced by the inner product

$$(u, w)_{L^2\Lambda^k(D)} := \int_D (u_x, w_x)_{\mathit{Alt}^k\mathbb{R}^n}\, vol, \tag{1.3}$$

where vol is the volume form in $\Lambda^n(D)$. Finally, the space of differential forms in $L^2\Lambda^k(D)$ with exterior derivative in $L^2\Lambda^{k+1}(D)$ is denoted with $H\Lambda^k(D)$. The space $H\Lambda^k(D)$ can be naturally endowed with boundary conditions.

Since $\mathit{Alt}^0\mathbb{R}^n = \mathbb{R}$, $\mathit{Alt}^n\mathbb{R}^n = \mathbb{R}$ and both $\mathit{Alt}^1\mathbb{R}^n$ and $\mathit{Alt}^{n-1}\mathbb{R}^n$ can be identified with $\mathbb{R}^n$, it is natural to establish correspondences between the exterior derivative and the classical differential operators, and between $H\Lambda^k(D)$ and classical Hilbert function spaces. These correspondences, known as *proxy fields*, are summarized in Table 1.1 for $n = 3$.

A feature of the exterior calculus is the rearrangement of the spaces $H\Lambda^k(D)$ in cochain complexes, known as de Rham complexes:

$$0 \to H\Lambda^0(D) \xrightarrow{\mathrm{d}} H\Lambda^1(D) \xrightarrow{\mathrm{d}} \cdots H\Lambda^n(D) \xrightarrow{\mathrm{d}} 0 \tag{1.4}$$

The cochain (1.4) is indeed a complex since the exterior derivative satisfies the important property $\mathrm{d} \circ \mathrm{d} = 0$. In terms of proxy fields, this means that $\mathrm{curl} \circ \nabla = 0$ and $\mathrm{div} \circ \mathrm{curl} = 0$.

1.2.2 The stochastic Hodge Laplacian

The Hodge Laplacian is the differential operator $\delta\mathrm{d} + \mathrm{d}\delta$, mapping k-forms into k-forms, and the Hodge Laplace problem is the boundary value problem for the Hodge Laplacian. It unifies some problems important in applications, such as the Darcy problem, modeling the fluid flow in porous media, and the magnetostatic/electrostatic problem.

Let us introduce the following bilinear operators

$$\begin{aligned} A : H\Lambda^k(D) &\to (H\Lambda^k(D))' \quad , & B : H\Lambda^k(D) &\to (H\Lambda^{k-1}(D))' \\ \langle Av, w\rangle &:= (\mathrm{d}v, \mathrm{d}w) & \langle Bv, q\rangle &:= (v, \mathrm{d}q) \end{aligned}$$

Given $\alpha \geq 0$, the weak mixed form of the Hodge Laplacian is

$$T := \begin{bmatrix} A & B^* \\ B & -\alpha \mathrm{Id} \end{bmatrix} : V_k \to V_k',$$

where B^* is the adjoint of B, $V_k := \begin{bmatrix} H\Lambda^k(D) \\ H\Lambda^{k-1}(D) \end{bmatrix}$, and V_k' is the dual of V_k. The *stochastic Hodge Laplacian* problem is the boundary value problem associated with the deterministic differential operator T and stochastic forcing term $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$, $m \geq 1$ integer: find $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m(\Omega; V_k)$ s.t.

$$\boxed{T \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \quad \text{a.s. in } V_k'.} \tag{1.5}$$

An easy extension of the well-posedness result for the deterministic problem gives the well-posedness of problem (1.5), as it will be shown in Chapter 2.

For simplicity, here we have considered homogeneous Neumann boundary conditions on the entire boundary ∂D, whereas, in Chapter 2, we have imposed homogeneous Dirichlet boundary conditions on Γ_D, and homogeneous Neumann boundary conditions on Γ_N, where $\{\Gamma_D, \Gamma_N\}$ is a partition of ∂D. It seems not straightforward to extend the results stated in Chapter 2 to non-homogeneous boundary conditions. The main difficulty is the characterization of the space $\mathrm{Tr}\left(H\Lambda^k(D)\right)$, where Tr is the trace operator.

1.2.3 Moment equations for the stochastic Hodge Laplacian

To derive the equation for the first statistical moment we start from the stochastic problem (1.5) and take the expectation on both sides of the equation. Using the linearity of the Hodge Laplace operator, we end up with the following deterministic saddle-point problem: find $E_s \in V_k$ s.t.

$$T(E_s) = \mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{in } V_k'.$$

This problem is well-posed provided that $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^1(\Omega; V_k')$, and its unique solution is given by the expected value of $\begin{bmatrix} u \\ p \end{bmatrix}$.

To obtain the m-th moment equation (m positive integer), we tensorize the stochastic problem with itself m times and then take the expectation. Hence, it has the form: find $M_s^{\otimes m} \in V_k^{\otimes m}$ s.t.

$$\boxed{T^{\otimes m} M_s^{\otimes m} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{in } (V_k')^{\otimes m}} \tag{1.6}$$

where $\mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} := \mathbb{E}\left[\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}^{\otimes m}\right]$. Similarly to what observed for the first statistical moment problem, the m-th moment problem is well-posed provided that $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$, and its unique solution is given by the m-th statistical moment of $\begin{bmatrix} u \\ p \end{bmatrix}$, $\mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$.

In Chapter 2 we prove the well-posedness of problem (1.6) with two different approaches.

- The deterministic Hodge Laplacian is well-posed, so that the inverse operator T^{-1} exists and is bounded. From a tensor product argument, it follows that $(T^{-1})^{\otimes m}$ is bounded, and is the inverse of $T^{\otimes m}$.
- The m-th moment problem is composed of m nested saddle-point problems. As alternative proof, we show a *tensorial inf-sup condition* for the tensor product space $V_k^{\otimes m}$.

The proof of the tensorial inf-sup condition is an original and relevant contribution of this thesis. The main difficulty comes from the fact that a tensor product of an inf-sup operator is not straightforwardly an inf-sup operator. A big effort was done in the construction of the minimal inf-sup operator P for the deterministic mixed Hodge Laplacian operator T such that $P^{\otimes m}$ is an inf-sup operator for $T^{\otimes m}$.

The tensorial inf-sup operator becomes crucial in the discrete setting. Indeed, when considering a finite dimensional subspace $V_{k,h} \subset V_k$, the tensor product argument applies only if the finite dimensional subspace of $V_k^{\otimes m}$ is the *full tensor product* $V_{k,h}^{\otimes m}$. If a *sparse tensor product* discretization is considered instead, an inf-sup condition is needed to prove the stability of the discretization.

1.2.4 Full and sparse finite element discretizations

The most straightforward finite element discretization of problem (1.6) is obtained considering the tensor product space $V_{k,h}^{\otimes m} := \underbrace{V_{k,h} \otimes \cdots \otimes V_{k,h}}_{m \text{ times}}$, where $V_{k,h}$ is a finite element discretization of V_k. Being $N_h = \dim(V_{k,h})$, then $\dim\left(V_{k,h}^{\otimes m}\right) = N_h^m$, which is impractical for m moderately large. This exponential growth in the dimensionality of the problem is known as curse of dimensionality. To overcome this problem, we propose the following sparse tensor product discretization. Let $\left\{V_{k,l_j}\right\}_{j\geq 1}$ be a sequence of nested finite dimensional subspaces of V_k whose limit for $j \to +\infty$ is dense in V_k. We define the sparse tensor product subspace of $V_k^{\otimes m}$ as

$$V_{k,L}^{(m)} := \bigoplus_{|\mathbf{l}|\leq L} Z_{k,l_1} \otimes \cdots \otimes Z_{k,l_m}, \tag{1.7}$$

where Z_{k,l_j} is the orthogonal complement of $V_{k,l_{j-1}}$ in V_{k,l_j}. See [21] and the references therein for more on sparse discretizations.

The sparse tensor product finite element (STP-FE) discretization of problem (1.6) has the form: find $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$ such that

$$\boxed{T^{\otimes m} M_{s,L}^{(m)} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad in \ \left(V_{k,L}^{(m)}\right)'} \tag{1.8}$$

The following stability result is the main result obtained in Chapter 2.

Theorem 1.2.1 (Stability of the STP-FE discretization)**.** *For every $\alpha \geq 0$ there exists $\bar{h}_0 > 0$ such that for all $h_0 \leq \bar{h}_0$ problem* (1.8) *is a stable discretization for the m-th moment problem* (1.6)*. In particular, for every $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$, there exists a test function $M_{t,L}^{(m)} \in V_{k,L}^{(m)}$ and positive constants $C_{m,disc}, C'_{m,disc}$ s.t.*

$$\begin{aligned} \left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)} \right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} &\geq C_{m,disc} \| M_{s,L}^{(m)} \|^2_{V_k^{\otimes m}}, \\ \| M_{t,L}^{(m)} \|_{V_k^{\otimes m}} &\leq C'_{m,disc} \| M_{s,L}^{(m)} \|_{V_k^{\otimes m}}. \end{aligned} \tag{1.9}$$

To prove the inf-sup condition (1.9), for each fixed trial function $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$, we choose the corresponding test function as $M_{t,L}^{(m)} = \Pi_L^{(m)}\left(P^{\otimes m} M_{s,L}^{(m)}\right)$, where $\Pi_L^{(m)} : V_k^{\otimes m} \to V_{k,L}^{(m)}$ is a projection and $P^{\otimes m}$ is the tensorial inf-sup operator constructed earlier. We then observe that

$$\begin{aligned} &\left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)} \right\rangle \\ &= \left\langle T^{\otimes m} M_{s,L}^{(m)}, \Pi_L^{(m)} \ P^{\otimes m} \ M_{s,L}^{(m)} \right\rangle \\ &= \left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)} \right\rangle - \left\langle T^{\otimes m} M_{s,L}^{(m)}, \left(\mathrm{Id}^{\otimes m} - \Pi_L^{(m)}\right) \ P^{\otimes m} \ M_{s,L}^{(m)} \right\rangle. \end{aligned}$$

Now, the main ingredients are:

- the continuous tensorial inf-sup condition, proved in Chapter 2, Section 2.4.4

- a sparse tensor product version of the *GAP property* (see [20]), which, roughly speaking, gives a measure of the distance between the spaces $V_k^{\otimes m}$ and $V_{k,L}^{(m)}$.

Finally, in Chapter 2 we derive an error estimate for the approximated solution of (1.8), inspired by the arguments in [21].

Theorem 1.2.2 (Order of convergence of the STP-FE discretization)**.**

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} = \mathcal{O}(h_L^{r(1-\lambda)}),$$

$0 < \lambda < 1$, *provided that*

$$\begin{bmatrix} u \\ p \end{bmatrix} \in L^m\left(\Omega; \begin{bmatrix} H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^r\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix}\right)$$
$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}p \end{bmatrix} \in L^m\left(\Omega; \begin{bmatrix} H^r\Lambda^{k+1}(D) \cap H_{\Gamma_D}\Lambda^{k+1}(D) \\ H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \end{bmatrix}\right).$$

We believe that this contribution is highly original, and one of the few proofs of stability of sparse approximations of tensorized mixed problems. Only after finishing and submitting the work [17], we became aware of the work [60], which treats the electromagnetic problem using very similar techniques.

1.3 Results on the perturbation approach for the Darcy problem with lognormal permeability tensor

Chapters 3, 4 and 5 focus on three aspects of the same problem: the stochastic Darcy equation with permeability coefficient described as a lognormal random field. In Chapter 3 we introduce the perturbation technique and study its effectiveness. In Chapter 4 we derive and analyze the moment equations for the stochastic Darcy problem. Finally, in Chapter 5 we propose a low-rank algorithm to solve them.

1.3.1 Problem setting

Let D be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) and $f \in L^2(D)$. We are interested in the following stochastic linear elliptic boundary value problem, that is the stochastic Darcy problem: find a random field $u : \Omega \times \bar{D} \to \mathbb{R}$ such that

$$\begin{cases} -\mathrm{div}_x\left(a(\omega, x)\nabla_x u(\omega, x)\right) = f(x) & x \in D,\ \omega \in \Omega \\ u(\omega, x) = g(x) & x \in \Gamma_D,\ \omega \in \Omega \\ a(\omega, x)\nabla_x u(\omega, x) \cdot \mathbf{n} = 0 & x \in \Gamma_N,\ \omega \in \Omega \end{cases} \tag{1.10}$$

Problem (1.10) models the single-phase fluid flow in a heterogeneous porous medium, whose permeability is described by the random field $a(\omega, x)$.

The weak formulation is: find $u \in L^p\left(\Omega; H^1(D)\right)$ $(p > 0)$ such that $u|_{\Gamma_D} = g$ a.s., and

$$\boxed{\int_D a(\omega, x)\nabla_x u(\omega, x) \cdot \nabla_x v(x)\, dx = \int_D f(x)v(x)\, dx \quad \forall v \in H^1_{\Gamma_D}(D),\ \text{a.s. in } \Omega.} \tag{1.11}$$

We make the following assumptions:

A1 : *The permeability field* $a \in L^p\left(\Omega; C^0(\bar{D})\right)$ *for every* $p \in (0,\infty)$.
Then the quantities

$$a_{min}(\omega) := \min_{x\in\bar{D}} a(\omega,x), \quad a_{max}(\omega) := \max_{x\in\bar{D}} a(\omega,x)$$

are well-defined, and $a_{max} \in L^p(\Omega)$ for every $p \in (0,\infty)$.

A2 : $a_{min}(\omega) > 0$ *a.s.,* $\dfrac{1}{a_{min}(\omega)} \in L^p(\Omega)$ *for every* $p \in (0,\infty)$.

These assumptions guarantee the well-posedness of problem (1.11), which follows by applying the Lax-Milgram lemma for any fixed $\omega \in \Omega$, and then taking the $L^p(\Omega,\mathbb{P})$-norm.

A frequently used model in geophysical applications describes the permeability as a lognormal random field: $a(\omega,x) = \mathrm{e}^{Y(\omega,x)}$, where $Y(\omega,x)$ is a Gaussian random field. See [14,36,51,52,93]. In recent years, the lognormal model has appeared and has been analyzed also in the mathematical literature: see [24,25,41,47].

The lognormal model is adopted in Chapters 3, 4, 5. We assume $Y(\omega,x)$ centered to lighten the notations, and introduce $\sigma^2 = \dfrac{1}{|D|}\displaystyle\int_D \mathbb{V}ar\,[Y]\ dx$. If $Y(\omega,x)$ is a stationary random field, then its variance is independent of $x \in D$ and coincides with σ^2. By a little abuse of notation, we refer to σ as the standard deviation of Y also in the case of a non-stationary random field. By an application of the Kolmogorov continuity theorem, in Chapter 3 we show that if the covariance function Cov_Y of the Gaussian field $Y(\omega,x)$ is Hölder regular with exponent $0 < t \leq 1$, then there exists a version of $Y(\omega,x)$ Hölder continuous with exponent $0 < \alpha < t/2$, which we still denote with $Y(\omega,x)$. Hence $\|Y\|_{L^\infty(D)}$, a_{max} and a_{min} are well-defined random variables. Moreover, assumptions **A1**, **A2** turn to be fulfilled, so that problem (1.11) is well-posed.

The first result we have in Chapter 3 concerns a bound on $\|Y\|_{L^\infty(D)}$ as a function of σ, obtained exploiting two different techniques. The obtained estimates hold under different assumptions, however they are quite similar and predict almost the same behavior for $\mathbb{E}\left[\|Y\|^k_{L^\infty(D)}\right]$, that is

$$\mathbb{E}\left[\|Y\|^k_{L^\infty(D)}\right] \leq C\ \sigma^k (k-1)!!, \tag{1.12}$$

with $C > 0$.

1.3.2 Perturbation analysis in the infinite dimensional setting

Thanks to the Doob-Dynkin lemma, the unique solution of problem (1.11) is a function of the random field Y: $u = u(Y,x)$. Under the assumption of small variability, that is $0 < \sigma < 1$, the aim of Chapter 3 is to perform a *perturbation analysis* based on the Taylor expansion of the solution u in a neighborhood of 0, and study the approximation properties of the Taylor polynomial. The work presented in Chapter 3 can be seen as both an extension and theoretical analysis of [51,52,93,99], and an extension of [33] to the case of a lognormal permeability field. The perturbation technique has also been applied to the case of randomly varying domains in [56].

The K-th order Taylor polynomial is

$$T^K u(Y,x) := \sum_{k=0}^{K} \frac{D^k u(0)[Y]^k}{k!},$$

where $D^k u(0)[Y]^k$ is the k-th Gateaux derivative of u in 0 evaluated along the vector $\underbrace{(Y,\ldots,Y)}_{k\text{ times}}$, and $D^0 u(0)[Y]^0 := u^0(x)$ is independent of the random field Y. u^0 solves the deterministic Laplace equation with loading term f and boundary conditions as in (1.10), whereas $D^k u(0)[Y]^k$ is the solution of the following problem: given $D^l u(0)[Y]^l \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ for all $l < k$, find $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ such that

$$\boxed{\int_D \nabla_x D^k u(0)[Y]^k \cdot \nabla_x v \, dx = -\sum_{l=1}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) \int_D Y^l \nabla_x D^{k-l} u(0)[Y]^{k-l} \cdot \nabla_x v \, dx} \tag{1.13}$$

$\forall\, v \in H^1_{\Gamma_D}(D)$, a.s. in Ω. By the Lax Milgram lemma, and a recursion argument, in Chapter 3 we prove the following theorem.

Theorem 1.3.3. *Problem* (1.13) *is well-posed, that is it admits a unique solution that depends continuously on the data. Moreover, it holds*

$$\|D^k u(0)[Y]^k\|_{H^1(D)} \leq C \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k k! < +\infty, \quad \forall k \geq 1 \quad \textit{a.s. in } \Omega \tag{1.14}$$

where $C = C(C_P, \|u^0\|_{H^1(D)})$, C_P *being the Poincaré constant. Moreover,* $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ *for any* $p > 0$.

From estimate (1.14) (and a similar one for $\|D^k u(tY)[Y]^k\|_{H^1(D)}$, $0 \leq t \leq 1$), we are able to derive an upper bound on the norm of the Taylor polynomial as well as the residual

$$R^K u(Y,x) := \frac{1}{K!} \int_0^1 (1-t)^K D^{K+1} u(tY)[Y]^{K+1} dt.$$

Theorem 1.3.4. *If the upper bound* (1.12) *holds, then*

$$\left\|T^K u\right\|_{L^1(\Omega;H^1(D))} \leq \sum_{k=0}^{K} C_1 \left(\frac{\sigma}{\log 2} \right)^k (k-1)!! \tag{1.15}$$

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq \frac{(K+1)!}{(\log 2)^{K+1}} \sum_{j=K+1}^{+\infty} C_2 \, \frac{\sigma^j}{j!!}. \tag{1.16}$$

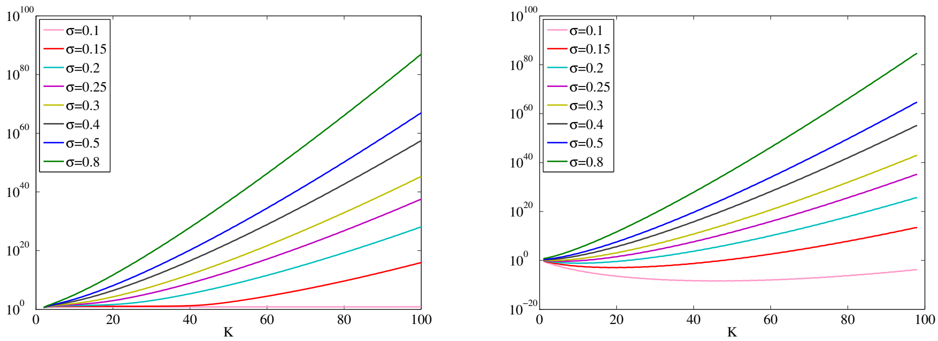
with $C_1, C_2 > 0$.

The upper bounds on the $L^1\left(\Omega; H^1(D)\right)$-norm of the Taylor polynomial and Taylor residual are the original contributions of Chapter 3. They lead us to predict the *divergence of the Taylor series for every positive* σ. See Figure 1.1, where the upper bounds (1.15) and (1.16) are plotted as a function of K, for different values of σ.

We also predict the existence of an optimal order K^{σ}_{opt} depending on σ such that adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it. In Chapter 3 we provide an explicit estimate $\bar{K}^{\sigma}$ for K^{σ}_{opt}:

$$\bar{K}^{\sigma} = \left\lfloor \left(\frac{\log 2}{\sigma} \right)^{2} \right\rfloor - 4.$$

From the practical point of view, the algorithm proposed in Chapter 5 to approximate the expected value $\mathbb{E}\left[u\right]$ entails the computation of the k-th order correction $\mathbb{E}\left[D^{k}u(0)[Y]^{k}\right]$, for each $k = 0, \dots, \bar{K}^{\sigma}$.



(a) (b)

Figure 1.1: *1.1(a): Semilogarithmic plot of the upper bound* (1.15) *on the* $L^1\left(\Omega; H^1(D)\right)$*-norm of* $T^K u$ *for different values of the standard deviation* σ*. 1.1(b): Semilogarithmic plot of the upper bound* (1.16) *on the* $L^1\left(\Omega; H^1(D)\right)$*-norm of* $R^K u$ *for different values of the standard deviation* σ*.*

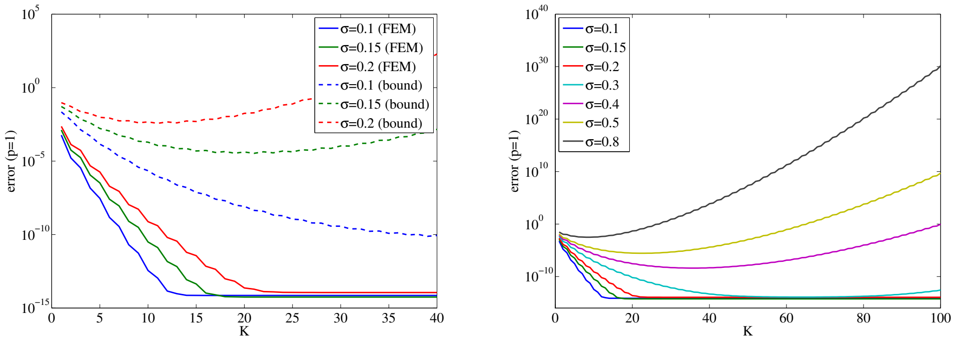
1.3.3 Finite number of independent Gaussian random variables

If the uncertainty is modeled as a vector of i.i.d centered Gaussian random variables $\mathbf{Y} = (Y_1(\omega), \dots, Y_N(\omega))$, the Taylor polynomial can be explicitly computed. In Chapter 3 we realize some numerical tests in the simple case $N = 1$ which confirm the divergence of the Taylor series. We compute the error $\left\|u_h - T^K u_h\right\|_{L^1(\Omega;L^2(D))}$ by linear FEM in space and high order Hermite quadrature formula. Figure 1.2(a) compares the error $\left\|u_h - T^K u_h\right\|_{L^1(\Omega;L^2(D))}$ with the upper bound (1.16). Figure 1.2(b) represents the semilogarithmic plot of the computed error $\left\|u_h - T^K u_h\right\|_{L^1(\Omega;L^2(D))}$ for different values of the standard deviation σ. Figure 1.3 is the logarithmic plot of the computed error $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ as a function of σ: the behavior $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)} = O(\sigma^{K+1})$ is numerically observed.

1.3.4 Finite number of independent *bounded* random variables

A completely different result is obtained if the permeability field is described as linear combination of bounded random variables:

$$a(\omega, x) = \mathbb{E}[a](x) + \sum_{n=1}^{N} \phi_n(x) Y_n(\omega),$$



(a) (b)

Figure 1.2: *1.2(a): Comparison between the computed error* $\left\|u_h - T^K u_h\right\|_{L^1(\Omega;L^2(D))}$ *and its upper bound* (1.16)*. 1.2(b): Semilogarithmic plot of the computed error* $\left\|u_h - T^K u_h\right\|_{L^1(\Omega;L^2(D))}$ *for different values of the standard deviation* σ*.*

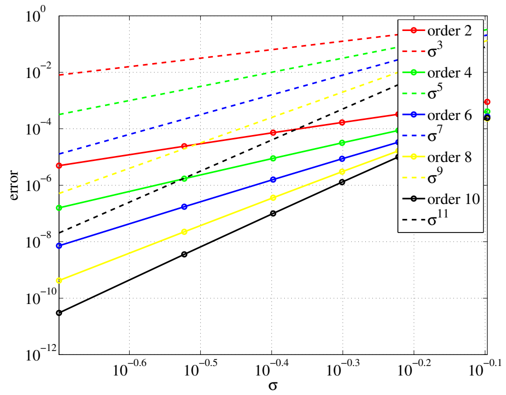


Figure 1.3: *Error* $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ *as a function of* σ*, where the reference solution* $\mathbb{E}\left[u_h\right]$ *is computed with a high order Hermite quadrature formula.*

where $Y_n(\Omega) \subset [-\gamma_n, \gamma_n]$, $0 < \gamma_n < +\infty \; \forall n$. Indeed, under the assumption of small variability, in [9] the authors prove the convergence of the Taylor series. Our conclusions are therefore specific to the case of a lognormal model for the coefficient $a(\omega, x)$.

1.4 Results on the derivation and analysis of the moment equations

Chapter 4 still focuses on problem (1.11) with lognormal permeability field $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, $Y(\omega, x)$ being a centered Gaussian random field with small standard deviation $0 < \sigma < 1$. We want to infer on the stochastic solution u using its Taylor polynomial $T^K u$.

If the random field $Y(\omega, x)$ is parametrized by a finite number of random variables $\mathbf{Y} = (Y_1(\omega), \ldots, Y_N(\omega))$, then $T^K u$ can be explicitly computed. This situation can be achieved using a truncated spectral decomposition of $Y(\omega)$ (Karhunen-Loève expansion, see [50] or Fourier expansion, see [43, 70, 72, 73]). In the approach adopted in Chapter 4 the entire field $Y(\omega, x)$ is considered, and it is not approximated in a finite dimensional probability space. In this case, the Taylor polynomial is not directly computable, but deterministic equations solved by its statistical moments can be derived.

The major part of the chapter is dedicated to the approximation of the first statistical moment:

$$\mathbb{E}\left[u(Y, x)\right] \approx \mathbb{E}\left[T^K u(Y, x)\right] = \sum_{k=0}^{K} \frac{\mathbb{E}\left[u^k\right]}{k!},$$

where u^k is a compact notation to denote the k-th Gateaux derivative $D^k u(0)[Y]^k$.

1.4.1 The structure of the problem

Given a function $v(x_1, \ldots, x_s, \ldots, x_n) : D^{\times n} \to \mathbb{R}$, $1 \le s \le n$ integers, we introduce the following notation to denote the evaluation of v on $\mathrm{diag}(D^{\times s}) \times D^{\times(n-s)}$:

$$\mathrm{Tr}_{|1:s} v(x_1, x_{s+1}, \ldots, x_n) := v(\underbrace{x_1, \ldots, x_1}_{s \text{ times}}, x_{s+1}, \ldots, x_n).$$

To obtain the K-th order approximation $\mathbb{E}\left[T^K u\right]$ we need the k-th order correction $\mathbb{E}\left[u^k\right]$, for $k = 0, \ldots, K$. $\mathbb{E}\left[u^k\right]$ solves the following problem:

$$\int_D \nabla \mathbb{E}\left[u^k\right](x) \cdot \nabla v(x)\, dx = -\sum_{l=1}^{k} \binom{k}{l} \int_D \mathrm{Tr}_{|1:l+1} \mathbb{E}\left[\nabla u^{k-l} \otimes Y^{\otimes l}\right](x) \cdot \nabla v(x)\, dx, \tag{1.17}$$

$\forall v \in H^1_{\Gamma_D}(D)$. Each element of the loading term $\mathrm{Tr}_{|1:l+1} \mathbb{E}\left[\nabla u^{k-l} \otimes Y^{\otimes l}\right]$ can be computed only after solving the following problem for the $(l+1)$-points correlation $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$: given all the lower order terms $\mathbb{E}\left[u^{k-l-s} \otimes Y^{\otimes(s+l)}\right] \in H^1_{\Gamma_D}(D) \otimes$

Table 1.2: *Recursive structure of the K-th approximation problem of the mean.*

$\mathbb{E}\left[u^0\right] = u^0$	$\mathbb{E}\left[u^1\right] = 0$	$\mathbb{E}\left[u^2\right]$	$\mathbb{E}\left[u^3\right] = 0$	$\cdot^{\cdot^{\cdot}}$
$\mathbb{E}\left[u^0 \otimes Y\right] = u^0 \otimes \mathbb{E}\left[Y\right] = 0$	$\mathbb{E}\left[u^1 \otimes Y\right]$	$\mathbb{E}\left[u^2 \otimes Y\right] = 0$	$\cdot^{\cdot^{\cdot}}$	$\cdot^{\cdot^{\cdot}}$
$\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right] = u^0 \otimes \mathbb{E}\left[Y^{\otimes 2}\right]$	$\mathbb{E}\left[u^1 \otimes Y^{\otimes 2}\right] = 0$	$\cdot^{\cdot^{\cdot}}$	$\cdot^{\cdot^{\cdot}}$	$\cdot^{\cdot^{\cdot}}$
$\mathbb{E}\left[u^0 \otimes Y^{\otimes 3}\right] = u^0 \otimes \mathbb{E}\left[Y^{\otimes 3}\right] = 0$	$\cdot^{\cdot^{\cdot}}$	$\cdot^{\cdot^{\cdot}}$	$\cdot^{\cdot^{\cdot}}$	$\cdot^{\cdot^{\cdot}}$

$(L^2(D))^{\otimes(s+l)}$, find $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \in H^1_{\Gamma_D}(D) \otimes (L^2(D))^{\otimes l}$ such that

$$\boxed{\begin{aligned} &\int_{D^{\times(l+1)}} \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) \mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \cdot \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) v \, dx_1 \dots dx_{l+1} \\ &= -\sum_{s=1}^{k-l} \begin{pmatrix} k-l \\ s \end{pmatrix} \\ &\quad \int_{D^{\times(l+1)}} \mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l-s} \otimes Y^{\otimes(s+l)}\right] \cdot \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) v \, dx_1 \dots dx_{l+1} \end{aligned}} \tag{1.18}$$

$\forall\, v \in H^1_{\Gamma_D}(D) \otimes (L^2(D))^{\otimes l}$. Problem (1.17) is a particular case of problem (1.18) with $l = 0$.

Table 1.2 summarizes the *recursive structure* of the first moment problem. The first column contains the input terms of the recursion, whereas the first row contains the increasing order corrections of the mean, that is the output terms of the recursion. Each non-zero term $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$, can be obtained only once we have computed all the previous terms in the k-th diagonal, that is $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right]$, $\mathbb{E}\left[u^1 \otimes Y^{\otimes(k-1)}\right]$, ..., $\mathbb{E}\left[u^{k-l-1} \otimes Y^{\otimes(l+1)}\right]$. To compute $\mathbb{E}\left[T^K u(Y,x)\right]$, we need all the elements in the upper triangular part of the table, that is all the elements in the k-th diagonals with $k = 0, \dots, K$. Since we assumed $\mathbb{E}\left[Y\right](x) = 0$ w.l.o.g., all the $(2k+1)$-points correlations of Y vanish. As a consequence, all the terms in the odd diagonals vanish.

Thanks to the Lax Milgram lemma, in Chapter 4 we prove the following theorem.

Theorem 1.4.5. *Let Y be a centered Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. For every $k \geq 0$ and $l = 0, \dots, k-1$ integers, problem* (1.18) *is well-posed.*

1.4.2 Regularity results

We end Chapter 4 with some regularity results on the correlations involved in the recursion described in Table 1.2. If the covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$ $0 < t \leq 1$, we prove that the k-points correlation $\mathbb{E}\left[Y^{\otimes k}\right]$ has a mixed Hölder regularity of exponent $t/2$, that is $\mathbb{E}\left[Y^{\otimes k}\right]$ is $t/2$ Hölder continuous separately in each variable $x_1, \dots, x_k \in D$. *This regularity is preserved in all the steps of the recursive first moment problem*, and ensures, together with an elliptic regularity result, that $\mathbb{E}\left[T^K u\right] \in \mathcal{C}^{1,t/2}(\bar{D})$.

Theorem 1.4.6. *Let Y be a centered Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. Moreover, suppose that the domain is convex and $\mathcal{C}^{1,t/2}$, and $u^0 \in \mathcal{C}^{1,t/2}(\bar{D})$. Then, for every positive integers k and s,*

$$\mathbb{E}\left[u^k \otimes Y^{\otimes s}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{1,t/2}(\bar{D})\right).$$

This result can be generalized to more regular covariances Cov_Y as follows. Suppose $u^0 \in \mathcal{C}^{r_1,t/2}(\bar{D})$. $Y \in L^p\left(\Omega; \mathcal{C}^{r_2,t/2}(\bar{D})\right)$, $r_1, r_2 \geq 0$, $\forall\ p$, and D in $\mathcal{C}^{r+1,t/2}$. Then

$$\mathbb{E}\left[u^k \otimes Y^{\otimes s}\right] \in \mathcal{C}^{r_2,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{r+1,t/2}(\bar{D})\right),$$

where $r := \min\{r_1 - 1, r_2\}$.

1.5 Low rank approximation of the moment equations

Chapter 5 concerns the numerical solution of the first moment problem and the recursion (1.18).

1.5.1 Finite element discretization of the first moment equation

Given a triangulation of the domain D, let us introduce the Lagrangian piecewise linear (or polynomial) FE basis, as well as the piecewise constant basis

$$\begin{aligned} V_h &= span\,\{\phi_n\}_{n=1}^{N_v} \subset H^1_{\Gamma_D}(D) \\ W_h &= span\,\{\psi_i\}_{i=1}^{N_e} \subset L^2(D) \end{aligned}$$

where N_v and N_e are the number of vertices and elements of the triangulation respectively. In Chapter 5 we discretize problem (1.18) adopting a *full tensor product* approach, so that the finite element approximation of $H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$ is given by

$$V_h \otimes (W_h)^{\otimes l} = span\,\{\phi_n \otimes \psi_{i_1} \otimes \ldots \otimes \psi_{i_l},\ n = 1, \ldots, N_v,\ i_1, \ldots, i_l = 1, \ldots, N_e\}.$$

In this setting, each $(l+1)$-points correlation $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$ is represented by a tensor denoted as $\mathcal{C}_{u^{k-l} \otimes Y^{\otimes l}}$ of order $(l+1)$ and size $N_v \times \underbrace{N_e \times \ldots \times N_e}_{l \text{ times}}$.

Let us introduce the following notation. Given two tensors $\mathcal{Y}$ and $\mathcal{X}$ of order d and $r+1$ respectively and an integer s such that $d \geq s + r$, then $\mathcal{Z} := \mathcal{X} \times_{s,r} \mathcal{Y}$ denotes the tensor of order $d - r + 1$ with entries:

$$\begin{aligned} &\mathcal{Z}(k_1, \ldots, k_{s-1}, j, k_{s+r}, \ldots, k_d) \\ &= \sum_{i_s} \cdots \sum_{i_{s+r-1}} \mathcal{X}(i_s, \ldots, i_{s+r-1}, j)\mathcal{Y}(k_1, \ldots, k_{s-1}, i_s, \ldots, i_{s+r-1}, k_{s+r}, \ldots, k_d). \end{aligned}$$

In Chapter 5 we derive the *full tensor product finite element formulation of problem* (1.18) that reads

$$\boxed{A \times_{1,1} \mathcal{C}_{u^{k-l} \otimes Y^{\otimes l}} = -\sum_{s=1}^{k-l} \begin{pmatrix} k-l \\ s \end{pmatrix} \mathcal{B}^s \times_{1,s+1} \mathcal{C}_{u^{k-l-s} \otimes Y^{\otimes (s+l)}}} \tag{1.19}$$

where A denotes the stiffness matrix with respect to the Lagrangian piecewise linear basis. For each s, $\mathcal{B}^s$ is a sparse tensor of order $s+2$ and size $N_v \times \underbrace{N_e \times \ldots \times N_e}_{s \text{ times}} \times N_v$ defined as

$$\mathcal{B}^s(n, i_1, \ldots, i_s, m) := \int_D \psi_{i_1}(x) \ldots \psi_{i_s}(x)\, \nabla\phi_n(x) \cdot \nabla\phi_m(x)\, dx.$$

With the aim of developing a black-box solver for the k-th order problem, we would need as input for the algorithm the stiffness matrix and the tensors $\mathcal{B}^s$.

1.5.2 Tensor Train format

Problem (1.19) involves high dimensional tensors. Since the number of entries of a tensor grows exponentially in its order d (curse of dimensionality), it is possible to explicitly store only tensors of small order d. For large d it is necessary to use data-sparse or low-rank formats. Between the classical low-rank formats, we mention the Canonical Polyadic (CP) and the Tucker format. We refer to the Matlab Tensor Toolbox [12] for a Matlab implementation of tensors in CP and Tucker format. We refer to [23, 57] and [96] for a deeper introduction on CP and Tucker formats respectively.

The research in the field of low-rank approximations is very active and in recent years various formats have been proposed ([49, 53, 66]). Among them, a new format based on the singular value decomposition (SVD) has been introduced: the *Tensor-Train format*. Given a d dimensional tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times \cdots \times n_d}$, its tensor train (TT) representation is given by

$$\mathcal{X}(i_1, \ldots, i_d) = \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{d-1}=1}^{r_{d-1}} G_1(i_1, \alpha_1) G_2(\alpha_1, i_2, \alpha_2) \ldots G_d(\alpha_{d-1}, i_d),$$

where $G_j \in \mathbb{R}^{r_{j-1} \times n_j \times r_j}$, $j = 1, \ldots, d$, are three dimensional arrays called *cores* of the TT-decomposition ($r_0 = r_d = 1$), and the set $(r_1, \ldots, r_{d-1})$ is known as *TT-rank*. The storage complexity is $O\left((d-2)nr^2 + 2rn\right)$ where $n = \max\{n_1, \ldots, n_d\}$, $r = \max\{r_1, \ldots, r_d\}$, so that the curse of dimensionality is broken (provided that r does not increase with d). It presents a linear structure (see Figure 1.4), which makes the TT-format easy to handle with, from the algorithmic point of view. We refer to [81, 82] and to the Matlab TT-Toolbox available at
`http://spring.inm.ras.ru/osel/?page_id=24`, where the main algebraic operations between TT-tensors are implemented, together with compression algorithms. For these reasons, we have decided to develop a code where all the tensors are represented in TT-format.

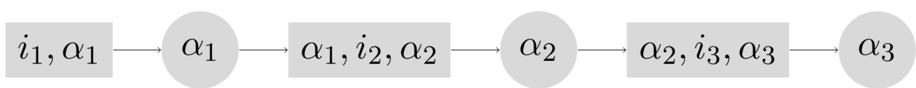


Figure 1.4: *Representation of the TT-format of a tensor.*

1.5.3 The recursive algorithm

The main achievement of Chapter 5 is the algorithm we have developed to solve problem (1.19) for $k = 0, \dots, K$ and $l = 0, \dots, k$, that is to compute all the correlations in Table 1.2. *This algorithm employs only TT-format representations of tensors.*

The inputs of the algorithm are the order of approximation K we aim to compute, the number of elements in the triangulation of D, the covariance function Cov_Y and the load function f, whereas the output is the K-th order approximation of $\mathbb{E}[u]$.

The first tool consists in the computation in TT-format of $\mathbb{E}\left[Y^{\otimes k}\right]$ for $k = 0, \dots, K$, that is the first column in Table 1.2. We refer to [67], where the authors propose an algorithm which, starting from the Karhunen-Loève (KL) expansion of the Gaussian field $Y(\omega, x)$, computes an approximation of $\mathbb{E}\left[Y^{\otimes k}\right]$ denoted with $\mathcal{C}^{TT}_{Y^{\otimes k}}$, with a prescribed accuracy tol. For more on the KL-expansion, see e.g. [43, 70, 72, 73]. $\mathcal{C}^{TT}_{Y^{\otimes k}}$ is constructed in such a way to preserve the following symmetry of $\mathbb{E}\left[Y^{\otimes k}\right]$:

$$\mathbb{E}\left[Y^{\otimes k}\right](x_1, \dots, x_k) = \mathbb{E}\left[Y^{\otimes k}\right](x_k, \dots, x_1).$$

As a consequence, its TT-rank $(r_1, \dots, r_{k-1})$ satisfies $r_p = r_{k-p}$ for $p = 1, \dots, k/2$.

The algorithm to solve problem (1.19) has a *recursive structure* which reflects the structure of the first moment problem:

for $k = 0, \dots, K$

> **for** $l = k-1, k-2, \dots, 0$
>
> Solve the boundary value problem (1.19) in TT-format to compute the $(l+1)$-points correlation function $\mathcal{C}^{TT}_{u^{k-l} \otimes Y^{\otimes l}}$
>
> **end**

The solution $\mathcal{C}^{TT}_{u^k}$ for $l = 0$ represents the k-th correction to the mean $\mathbb{E}[u]$

end

1.5.4 The storage requirements of the algorithm

In Chapter 5 we investigate the storage requirements of both the TT-format correlations $\mathcal{C}^{TT}_{Y^{\otimes k}}$, that is the input of the algorithm, and the TT-correlations involved in the recursion. The storage requirements of a tensor in TT-format highly depends on its TT-rank. The storage requirements is a limiting aspect of our algorithm since it prevents us to grows significantly in the order of the Taylor polynomial K. We believe that a great improvement will follow from the implementation of sparse tensor toolboxes, which are missing in Matlab.

In [67] the authors show that the TT-ranks of the exact TT-representation of $\mathbb{E}\left[Y^{\otimes k}\right]$ satisfy:

$$r_p = \begin{pmatrix} N + p - 1 \\ p \end{pmatrix} \tag{1.20}$$

for $p = 1, \dots, k/2$, where N is the number of independent random variables which parametrize the random field $Y(\omega, x)$. Hence, (1.20) is an upper bound for each approximated $\mathcal{C}^{TT}_{Y^{\otimes k}}$ computed with a prescribed accuracy tol. See Figure 1.5, obtained in

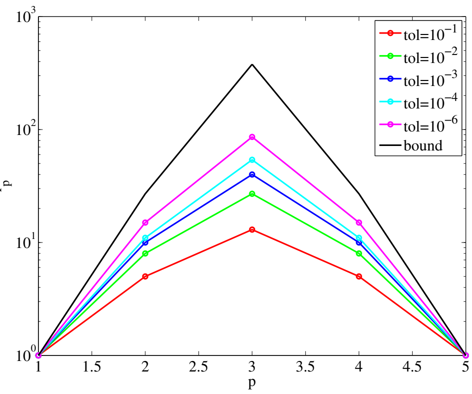


Figure 1.5: *Semilogarithmic plot of the upper bound for the TT-ranks in* (1.20) *(black line) compared with the TT-ranks of the approximated* $\mathcal{C}^{TT}_{Y^{\otimes 4}}$ *computed for different tolerances tol.*

the case of a Gaussian covariance function $Cov_Y(x_1, x_2) = \mathrm{e}^{-\frac{\|x_1 - x_2\|^2}{L^2}}$ with correlation length $L = 0.2$. As expected, the smaller tol is, the higher the TT-ranks.

We numerically verify that the upper bound (1.20) is satisfied also by all the correlations in Table 1.2. See Figure 1.6.

The TT-ranks strongly affect also the computational cost of the recursive algorithm. Indeed, they correspond to the number of linear systems to be solved to compute the K-th order approximation of $\mathbb{E}[u]$. In Chapter 5 we show that the computational cost of the TT-algorithm can be considered proportional to

$$M_2' = \sum_{n=2:2:K} \sum_{p=0}^{n-1} r_p, \tag{1.21}$$

under the assumption that the dominant cost is the one of solving a "deterministic problem", where $(r_1, \ldots, r_{n-1})$ are the TT-ranks of the n-points correlation $\mathcal{C}^{TT}_{Y^{\otimes n}}$. By a comparison between computational costs, we state that the moment equations is convenient with respect to the direct computation of the multivariate Taylor polynomial from a truncated KL-expansion.

1.5.5 Numerical results

In Chapter 5 we perform some numerical tests and solve the stochastic Darcy problem with deterministic loading term $f(x) = x$ in the one dimensional domain $D = [0, 1]$, both for a Gaussian and exponential covariance function of the Gaussian random field $Y(\omega, x)$.

Given a "reference" solution denoted with $\mathbb{E}[u]$, the error $\left\|\mathbb{E}[u] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ comes from different contributions: the truncation of the KL-expansion, the TT approximation, the truncation of the Taylor series and the FEM approximation. In our numerical tests we compute a "reference" solution on the same grid as the moment equations, so that we don't see the FEM contribution to the error.

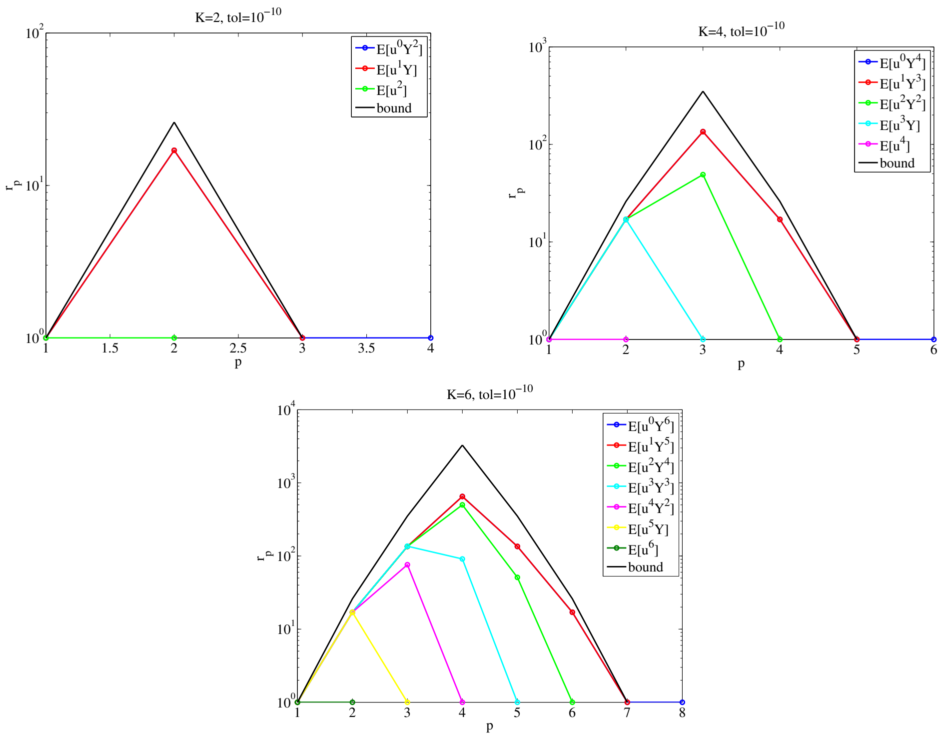


Figure 1.6: *Semilogarithmic plot of the TT-ranks of the correlations needed to solve the K-th order problem, for $K = 2, 4, 6$.*

Truncated KL

Let $Y(\omega, x)$ be a stationary Gaussian random field with Gaussian covariance function

$$Cov_Y(x_1, x_2) = \sigma^2 e^{-\frac{\|x_1 - x_2\|^2}{(0.2)^2}}, \quad 0 < \sigma < 1.$$

Let us take a uniform discretization of the spatial domain $D = [0, 1]$ in $N_h = 100$ intervals ($h = 1/N_h$). The first numerical experiments in Chapter 5 are realized starting from the same truncated KL-expansion both to compute the reference solution, that is the collocation solution (see Figure 1.7), and the TT solution. In particular, $N = 11$ random variables are considered, which capture the 99% of variance of the field. The TT computations are done at machine precision. Hence, we observe only the error due to the truncation of the Taylor series.

In Figure 1.8(a) we plot in logarithmic scale the error $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ as a function of σ: we numerically observe the behavior $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)} = O(\sigma^{K+1})$ predicted in Chapter 3. Figure 1.8(b) represents the computed error as a function of K (at least up to $K = 6$) for different values of σ. It turns out that for $\sigma < 1$ it is always useful to take into account higher order corrections.

Considering the case of $Y(\omega, x)$ conditioned to N_{oss} available point-wise observations is very relevant in applications. Indeed, from the practical point of view it is possible to measure the permeability of a heterogeneous porous medium only in a small number of fixed points, so that the natural model considered in the geophysical literature describes the permeability as a conditioned lognormal random field. See

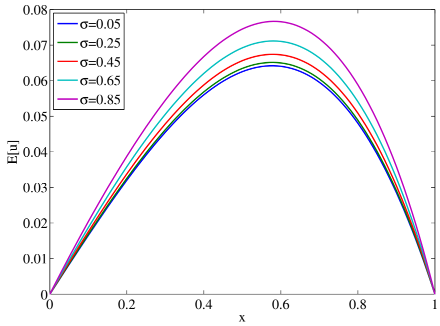
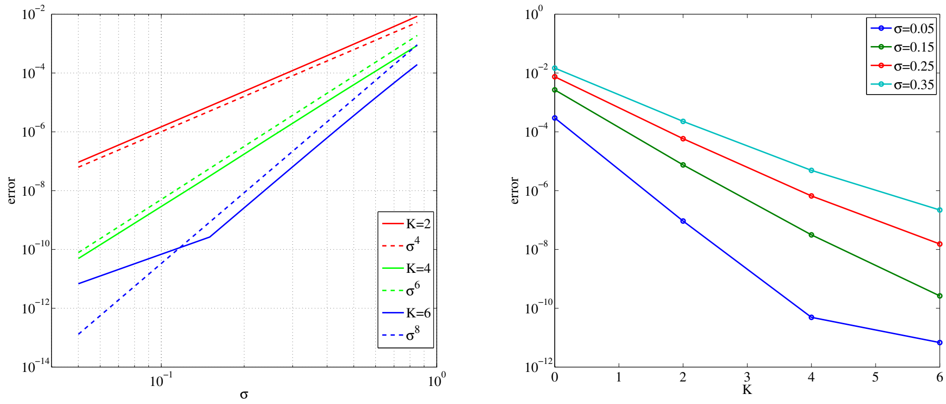


Figure 1.7: *Reference solution $\mathbb{E}[u]$ computed via the collocation method, for different values of σ.*



(a) (b)

Figure 1.8: *1.8(a) Logarithmic plot of the computed error $\left\|\mathbb{E}[u] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ as a function of σ. 1.8(b) Semilogarithmic plot of the computed error $\left\|\mathbb{E}[u] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ as a function of K for different σ.*

e.g. [51, 52, 86]. The more observations are available, the smaller the total variance of the field will be. This, actually, favors the use of perturbation methods.

The conditioned covariance function Cov_Y is non-stationary, but still Hölder continuous, so that it is included in the setting described in Chapter 3 and 4. In Chapter 5 we perform some numerical tests and conclude that the error $\left\|\mathbb{E}[u] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ is about 1 order of magnitude smaller for $N_{oss} = 3$ (compared to $N_{oss} = 0$) and 2 orders of magnitude smaller for $N_{oss} = 5$.

Complete KL

We study the case where $Y(\omega, x)$ is a stationary centered Gaussian random field with exponential covariance function

$$Cov_Y(x_1, x_2) = \sigma^2\, \mathrm{e}^{-\frac{\|x_1 - x_2\|}{0.2}}, \quad 0 < \sigma < 1.$$

The truncated KL-expansion is computed and $N = N_h = 100$ random variables are considered, so that the 100% of variance of the field is captured. Since $N = 100$, a collocation method becomes unfeasible. By a qualitative comparison with $\mathbb{E}[u]$ computed via the Monte Carlo method with $M = 10000$ samples, we show that our algorithm is effective and provides a valid solution also in this framework. In Figure 1.9, the TT-solution is always contained in the confidence interval of the Monte Carlo solution.

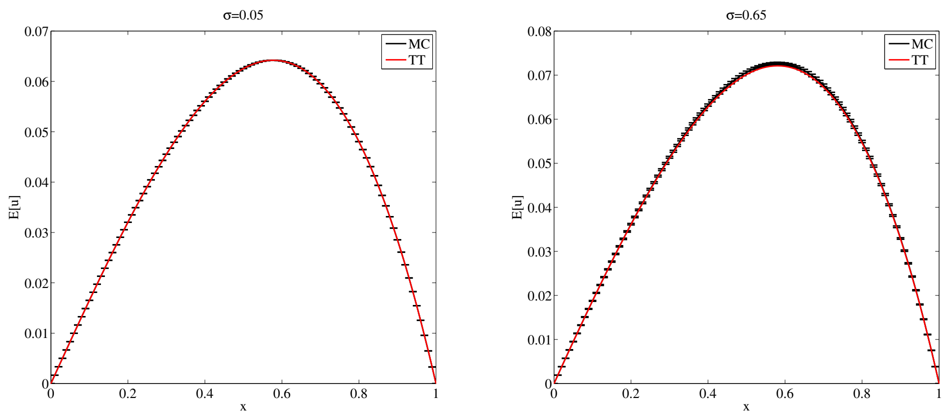


Figure 1.9: *Comparison between the second order correction computed via our TT-code, and $\mathbb{E}[u]$ computed via the Monte Carlo method ($M = 10000$ samples) for $\sigma = 0.05$ (left) and $\sigma = 0.65$ (right).*

Let us now consider a stationary Gaussian random field $Y(\omega, x)$ with Gaussian covariance function of correlation length $L = 0.2$, and its complete KL-expansion ($N = 26$ random variables have to be taken to capture the 100% of variance).

We run our TT-code imposing different tolerances in the computation of the TT-correlations $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes k}}$. In Figure 1.10 we plot the error as a function of K, for different tolerances, with $\sigma = 0.05$ (left) and $\sigma = 0.55$ (right).

We investigate the dependence of the error $\left\| \mathbb{E}[u] - \mathbb{E}\left[T^K u\right] \right\|_{L^2(D)}$ on the complexity of the code under the assumption that the complexity of the recursive algorithm is mainly due to the number of linear systems we have to solve in the recursion, that is M_2' in (1.21). Figure 1.11 represents the logarithmic plot of the error as a function of M_2' for different tolerances, with $\sigma = 0.05, 0.25, 0.85$. We compare it with the quantity $\frac{\sigma_{MC}}{\sqrt{M_2'}}$ (black line), which gives an idea of the behavior a the Monte Carlo estimate, where σ_{MC} is the estimated standard deviation of the Monte Carlo estimator. Note that, for small σ (e.g. $\sigma = 0.05$), the smaller the tolerance imposed is, the higher the accuracy reached. This is not the case if we let σ grow. Indeed, the TT-error is no more the most influencing component of the error, which is dominated, instead, by the truncation error. For a fixed truncation level, there is therefore an optimal choice of the tolerance tol_{opt}. Figure 1.11 shows that, if the optimal tolerance is chosen, the performance of the moment equations is far superior to a standard Monte Carlo method. The question of how to determine a priori the optimal tolerance as a function of K and σ is still open and under investigation.

In conclusions, the moment equation method coupled with a TT-approximation is a competitive method. It reaches the same level of accuracy as a collocation method and provides a valid solution also in the case where the collocation method is unfeasible. Moreover, for a given computational cost, the error committed by our approach is much

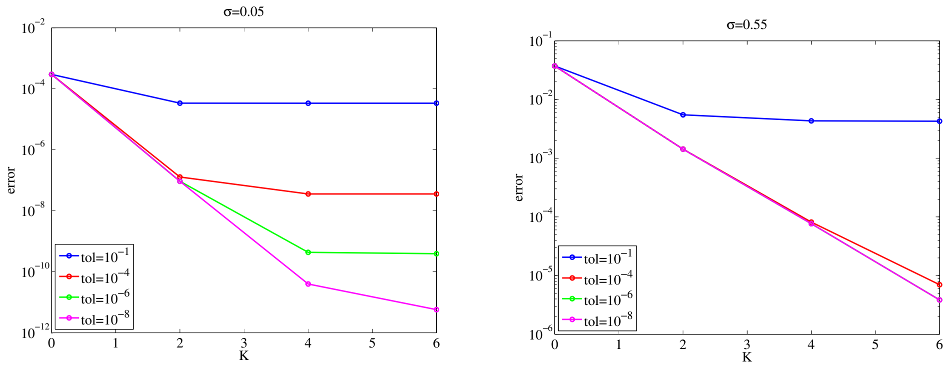


Figure 1.10: *Semilogarithmic plot of the computed error* $\left\|\mathbb{E}\left[u(Y,x) - T^K u(Y,x)\right]\right\|_{L^2(D)}$ *as a function of* K*, for different tolerances*

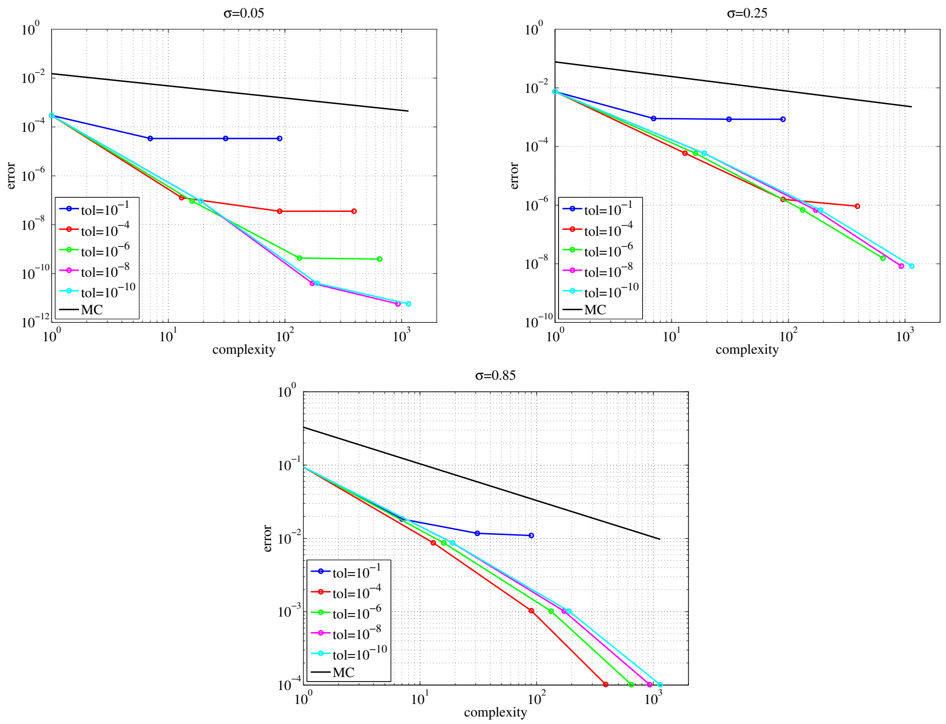


Figure 1.11: *Logarithmic plot of the computed error* $\left\|\mathbb{E}\left[u(Y,x) - T^K u(Y,x)\right]\right\|_{L^2(D)}$ *versus its computational cost* M_2' (1.21)*, for different values of* σ*. In each plot, the black line represents the behavior of the Monte Carlo method.*

smaller than the one of the Monte Carlo method.

CHAPTER 2

Moment equations for the mixed formulation of the Hodge Laplacian with stochastic data

This chapter consists of the paper *F. Bonizzoni, A. Buffa, F. Nobile*, "Moment equations for the mixed formulation of the Hodge Laplacian with stochastic data", available as MOX Report 31/2012 - Department of Mathematics, Politecnico di Milano.

2.1 Introduction

Many engineering applications are affected by uncertainty. This uncertainty may be due to the incomplete knowledge on the input data or some intrinsic variability of them. For example, if we model the two-phase flow in a porous medium, randomness arises in the permeability tensor, due to impossibility of a full characterization of conductivity properties of subsurface media, but also in the source term, typically pressure gradients or impervious boundaries. See for example [10, 39, 51, 52, 86, 93, 99, 100]. Similar situations appear in many other applications, such as combustion flows, earthquake engineering, biomedical engineering and finance. Probability theory provides an effective tool to include uncertainty in the model. We refer to [1,16,69] for probability measures on Banach spaces, and to [37,62,63,83] and the references therein for stochastic partial differential equations. We notice that the SPDEs that we consider in this work differ from those in [37,62,63,83] since we are taking L^m-intregrable processes.

In this work we focus on the linear Hodge-Laplace problem in mixed formulation, with stochastic forcing term and homogeneous boundary conditions. This problem includes the magnetostatic and electrostatic equations as well as the Darcy problem for mono-phase flows in saturated media.

The exterior calculus is a theoretical approach that, using tools from differential geometry, allows to simultaneously treat many different problems. In particular, the

Hodge Laplacian $\mathrm{d}\delta + \delta\mathrm{d}$, where δ is the formal adjoint of the exterior derivative d, maps differential k-forms to differential k-forms, and unifies some important second-order differential operators, such as the Laplacian and curl $-$ curl problems arising in electromagnetics. For more details, see [6, 7, 30].

The solution of the mixed formulation of the stochastic Hodge-Laplace problem is a couple (u, p) of random fields taking values in a suitable space of differential forms. The description of these random fields requires the knowledge of their moments. A possible approach is to compute the moments by the Monte-Carlo method in which, after sampling the probability space, the deterministic PDE is solved for each sample and the results are combined to obtain statistical information about the random field. This is a widely used technique, but it features a very slow convergence rate. Improvements can be achieved by several techniques. We mention for instance the Multilevel Monte-Carlo method appeared in recent years in literature, and applied to both stochastic ODEs and PDEs: see [13, 26, 45, 58, 61] and the references therein.

An alternative strategy is to directly calculate the moments of interest of the stochastic solution without doing any sampling. Indeed, the aim of the present work is to derive the moment equations, that is the deterministic equations solved by the m-points correlation function of the stochastic solution, show their well-posedness and propose a stable sparse finite element approximation. The stochastic problem has the form

$$T \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ a.e. in } D,$$

where T is a second order linear differential operator, D is a domain in $\mathbb{R}^n$, and the forcing terms $f_1(\omega, x)$, $f_2(\omega, x)$ are random fields, with $x \in D$, $\omega \in \Omega$ and Ω indicating the set of possible outcomes. The m-th moment equation involves the tensor product operator $T^{\otimes m} := \underbrace{T \otimes \cdots \otimes T}_{m \text{ times}}$ and the forcing term is given by the m-points correlation function of the couple $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$.

We start proving the well-posedness of the m-th moment equation. Although this comes easily from a tensorial argument, we also present a direct proof of the inf-sup condition for the tensor operator $T^{\otimes m}$. This proof will be a key tool to show the stability of a sparse finite element approximation.

Concerning the numerical approximation of the m-th moment equation, a tensorized FE approach for the numerical approximation of the moment equations is viable only for small m, as the number of degrees of freedom increases exponentially in m. For large m one should consider instead sparse approximations (see e.g. [21, 78, 79, 91, 92] and the references therein). We consider both full tensor product and sparse tensor product finite element approximations, and prove their stability using the tools from the finite element exterior calculus. See [5–7, 31]. In particular, the stability of a full tensor product approximation is a simple consequence of a tensor product argument. On the contrary, a tensor product argument does not apply if sparse tensor product approximations are considered and a direct proof of the inf-sup condition is needed, and will be proved in Section 2.6. We also provide optimal order of convergence estimates both for the full and the sparse approximations.

The analysis on well-posedness and stable discretization for the m-points correlation problem developed in this work will be necessary to analyze more complex situations with randomness appearing in the operator itself instead of simply in the right hand side. This case can be treated for small randomness by a perturbation approach (Taylor or Neumann expansions, see e.g. [9, 51, 93] and the references therein) and is currently under investigation.

The outline of the paper is the following: in Section 2.2 after recalling the definitions of the classical Sobolev spaces, we generalize them to the Sobolev spaces of differential forms. We then recall the main results on the mixed formulation of the Hodge-Laplace problem in the deterministic setting, stating the well-posedness of the problem and translating it to the language of partial differential equations using the proxy fields. In Section 2.3 we consider the stochastic counterpart of the mixed Hodge Laplacian problem, and we prove the well-posedness of its weak formulation. Section 2.4 is dedicated to the analysis of the moment equations. In particular, we provide the constructive proof of the inf-sup condition for the tensor product operator $T^{\otimes m}$. In Section 2.5 we focus on two problems of particular interest from the point of view of applications: the stochastic magnetostatic equations and the stochastic Darcy problem. Finally, in Section 2.6, we provide both full and sparse finite element discretizations for the deterministic m-th moment problem, we prove their stability and optimal order of convergence estimates.

2.2 Sobolev spaces of differential forms and the deterministic Hodge-Laplace problem

In this section we first recall the main concepts and definitions concerning the finite element exterior calculus and the Sobolev spaces of differential forms, which generalize the classical Sobolev spaces. We prove the inf-sup condition for the mixed formulation of the Hodge-Laplace problem providing a choice of test functions different from the classical one proposed in [6]. This will be needed later on to prove the equivalent inf-sup condition for the m-points correlation problem. Finally, we use the proxy fields correspondences to translate the Hodge-Laplace problem in the three dimensional case to the language of partial differential equations with the aim of showing that this general setting includes some important problems of practical interest. For more details we refer to [6, 7, 30].

2.2.1 Classical Sobolev spaces

Let $D \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We denote with $L^m(D)$ the Lebesgue space of index m with $1 \leq m < \infty$. $L^m(D)$ is a Banach space endowed with the standard norm

$$\|f\|_{L^m(D)} := \left(\int_D |f(x)|^m dx \right)^{1/m}. \tag{2.1}$$

When $p = 2$ we obtain the only Hilbert space of this class, with inner product given by

$$(f, g)_{L^2(D)} := \int_D f(x) g(x) dx, \quad f, g \in L^2(D).$$

We denote with $H^s(D)$ the Sobolev space defined as:

$$H^s(D) := \left\{ f \in L^2(D) | \ D^\alpha f \in L^2(D) \text{ for all } |\alpha| \leq s \right\}. \tag{2.2}$$

$H^s(D)$ is a Hilbert space with the natural inner product

$$(f,g)_{H^s(D)} := \sum_{|\alpha|\leq s} \langle D^\alpha f, D^\alpha g\rangle_{L^2(D)}, \qquad \text{for } f,g \in H^s(D).$$

For more on the Lebesgue spaces $L^m(D)$ and the Sobolev spaces $H^s(D)$ see for example [64]. As it will be useful later on, we also recall the following Sobolev spaces constrained by boundary conditions on $\Gamma_D \subset \partial D$:

$$\begin{aligned} H^1_{\Gamma_D}(D) &= \left\{ v \in L^2(D) \mid \nabla v \in L^2(D),\ v|_{\Gamma_D} = 0 \right\}, \\ H_{\Gamma_D}(\text{curl}, D) &= \left\{ v \in (L^2(D))^n \mid \text{curl} v \in (L^2(D))^n,\ v \times \nu|_{\Gamma_D} = 0 \right\}, \\ H_{\Gamma_D}(\text{div}, D) &= \left\{ v \in (L^2(D))^n \mid \text{div} v \in L^2(D),\ v \cdot \nu|_{\Gamma_D} = 0 \right\}, \end{aligned}$$

where ν is the outer-pointing normal versor. These spaces are Hilbert spaces with respect to the graph norm.

Considering now a probability space $(\Omega, \mathrm{d}\mathbb{P})$, the definition of L^m generalizes immediately. In this case we will use the notation $\left(L^m(\Omega, d\mathbb{P}), \|\cdot\|_{L^m(\Omega,d\mathbb{P})}\right)$ to denote the Banach space of real random variables on Ω with finite m-th moment. If $m = 2$, $\left(L^2(\Omega, d\mathbb{P}), \|\cdot\|_{L^2(\Omega,d\mathbb{P})}\right)$ is the Hilbert space of all real random variables on Ω with finite second moment, equipped with the usual inner product

$$(f(\omega), g(\omega))_{L^2(\Omega,d\mathbb{P})} := \int_\Omega f(\omega)g(\omega)d\mathbb{P}(\omega), \text{ for } f,g \in L^2(\Omega, d\mathbb{P}).$$

2.2.2 Sobolev spaces of differential forms

In order to generalize the definitions of the Sobolev spaces $H^s(D)$ to differential forms, we need to briefly recall the basic objects and results of exterior algebra and exterior calculus, inspired by [6]. The natural setting is a sufficiently smooth finite dimensional manifold D with or without boundary. For our purposes, we can restrict ourselves to the particular case of a n-dimensional bounded domain $D \subset \mathbb{R}^n$ with boundary denoted by $\partial D \subset \mathbb{R}^{n-1}$. In this way, at each point $x \in D$ the tangent space is naturally identified with $\mathbb{R}^n$ and we make this assumption throughout the paper. We denote by $Alt^k\mathbb{R}^n$, $0 \leq k \leq n$, the space of alternating k-linear maps on $\mathbb{R}^n$. Clearly, $Alt^0\mathbb{R}^n = \mathbb{R}$ and $Alt^n\mathbb{R}^n = \mathbb{R}$, and the unique element in $Alt^n\mathbb{R}^n$ is a volume form vol_n. We recall the wedge product $\wedge : Alt^k\mathbb{R}^n \times Alt^l\mathbb{R}^n \to Alt^{k+l}\mathbb{R}^n$ and the inner product $(\cdot,\cdot)_{Alt^k\mathbb{R}^n} : Alt^k\mathbb{R}^n \times Alt^k\mathbb{R}^n \to \mathbb{R}$ for $k + l \leq n$. Starting from this inner product, the Hodge star operator $\star : Alt^k\mathbb{R}^n \to Alt^{n-k}\mathbb{R}^n$ is defined: $u \wedge \star w = (u,w)_{Alt^k\mathbb{R}^n}\, vol_n$ (see e.g. [6]).

A differential k-form on D is a map u which associates to each $x \in D$ an element $u_x \in Alt^k\mathbb{R}^n$. We denote by $\Lambda^k(D)$ the space of all smooth differential k-forms on D. The wedge product of alternating k-forms may be applied point-wise to define the wedge product of differential forms: $(u \wedge w)_x = u_x \wedge w_x$. The exterior derivative d^k maps $\Lambda^k(D)$ into $\Lambda^{k+1}(D)$ for each $k \geq 0$ and is defined as

$$\mathrm{d}^k u_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \partial_{v_j} u_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}), \qquad u \in \Lambda^k(D),$$

$v_1, \ldots, v_{k+1} \in \mathbb{R}^n$, where the hat is used to indicate a suppressed argument. The exterior derivative satisfies the key property $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0, \forall\, k$. The coderivative operator $\delta^k : \Lambda^k(D) \to \Lambda^{k-1}(D)$ is the formal adjoint of the exterior derivative and it is defined by

$$\star\, \delta^k u = (-1)^k \mathrm{d}^{n-k} \star u, \quad u \in \Lambda^k(D). \tag{2.3}$$

To lighten the notation, in the following we omit the apex k when no ambiguity arises. The trace operator $\mathrm{Tr} : \Lambda^k(D) \to \Lambda^k(\partial D)$ is defined as the pullback of the inclusion $\partial D \hookrightarrow D$. We denote with *vol* the unique volume form in $\Lambda^n(D)$ such that at each $x \in D$, vol_n is the unique form associated with $Alt^n\mathbb{R}^n$. Given two differential k-forms on D it is possible to define their L^2-inner product as the integral of their point-wise inner product in $Alt^k\mathbb{R}^n$:

$$(u, w) := \int_D (u_x, w_x)_{Alt^k\mathbb{R}^n}\, vol = \int_D u \wedge \star w,\ u, w \in \Lambda^k(D). \tag{2.4}$$

In the following we will denote with $\|\cdot\|$ the norm induced by the L^2-inner product $(\cdot,\cdot)$. The following integration by parts formula holds:

$$(\mathrm{d}u, v) = (u, \delta v) + \int_{\partial D} \mathrm{Tr}(u) \wedge \mathrm{Tr}(\star v),\ u \in \Lambda^k(D),\ v \in \Lambda^{k+1}(D). \tag{2.5}$$

The completion of $\Lambda^k(D)$ in the norm induced by the scalar product (2.4) defines the Hilbert space $L^2\Lambda^k(D)$. The Sobolev space of square integrable k-forms whose exterior derivative is also square integrable is given by

$$H\Lambda^k(D) = \left\{u \in L^2\Lambda^k(D) |\ \mathrm{d}u \in L^2\Lambda^{k+1}(D)\right\}. \tag{2.6}$$

It is a Hilbert space equipped with the inner product

$$(u, w)_{H\Lambda^k} := (u, w) + (\mathrm{d}u, \mathrm{d}w).$$

In analogy with $H\Lambda^k(D)$, it is possible to define the Hilbert space

$$H^*\Lambda^k(D) := \left\{u \in L^2\Lambda^k(D) |\ \delta u \in L^2\Lambda^{k-1}(D)\right\}. \tag{2.7}$$

Let $\partial D = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$. As it is standard ([6]), the spaces (2.6) and (2.7) can be endowed with boundary conditions:

$$H_{\Gamma_D}\Lambda^k(D) := \left\{u \in H\Lambda^k(D) |\ \mathrm{Tr}(u)|_{\Gamma_D} = 0\right\}. \tag{2.8}$$

$$H^*_{\Gamma_N}\Lambda^k(D) := \left\{u \in H^*\Lambda^k(D) |\ \mathrm{Tr}(\star u)|_{\Gamma_N} = 0\right\}.$$

With the spaces defined in (2.8) and the exterior derivative operator, we can construct the L^2 de Rham complex:

$$0 \to H_{\Gamma_D}\Lambda^0(D) \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} H_{\Gamma_D}\Lambda^n(D) \to 0. \tag{2.9}$$

Since $\mathrm{d} \circ \mathrm{d} = 0$, we have

$$\mathfrak{B}_k \subseteq \mathfrak{Z}_k, \tag{2.10}$$

where $\mathfrak{B}_k$ is the image of d in $H_{\Gamma_D}\Lambda^k(D)$ while $\mathfrak{Z}_k$ is the kernel of d in $H_{\Gamma_D}\Lambda^k(D)$.

The following orthogonal decomposition of $L^2\Lambda^k(D)$, known as Hodge decomposition, holds:

$$L^2\Lambda^k(D) = \mathfrak{B}_k \oplus \mathfrak{B}_k^\perp \tag{2.11}$$

where $\mathfrak{B}_k^\perp$ is the L^2-complement of $\mathfrak{B}_k$.
We define two projection operators $\pi^\perp$ and π° as follows:

$$\begin{aligned} \pi^\perp : \mathfrak{B}_k \oplus \mathfrak{B}_k^\perp &\to \mathfrak{B}_k^\perp \\ v = \mathrm{d}v^\circ + v^\perp &\mapsto v^\perp \end{aligned} \tag{2.12}$$

$$\begin{aligned} \pi^\circ : \mathfrak{B}_k \oplus \mathfrak{B}_k^\perp &\to \mathfrak{B}_{k-1}^\perp \\ v = \mathrm{d}v^\circ + v^\perp &\mapsto v^\circ. \end{aligned} \tag{2.13}$$

Hence, given $v \in L^2\Lambda^k(D)$, it can be uniquely expressed as $v = \mathrm{d}\pi^\circ\, v + \pi^\perp\, v$. We recall a classical result in the theory of Sobolev spaces:

Lemma 2.2.1 (Poincaré inequality)**.** *There exists a positive constant C_P that depends only on the domain D such that*

$$\|v\| \leq C_P \|\mathrm{d}v\| \quad \forall v \in \mathfrak{Z}_k^\perp \tag{2.14}$$

where $\mathfrak{Z}_k^\perp$ is the orthogonal complement of $\mathfrak{Z}_k$ in $H_{\Gamma_D}\Lambda^k(D)$.

For the sake of simplicity, we consider only the case of geometries which are trivial from the topological point of view. More precisely, from now on, we make the following

Assumption A1. *The domain $D \subset \mathbb{R}^n$ is bounded, Lipschitz and contractible. Its boundary ∂D is given by the disjoint union of two open sets Γ_D and Γ_N, with $\Gamma_D, \Gamma_N \neq \emptyset$, Γ_D contractible as well and with boundary sufficiently regular (at least piecewise C^1).*

Under Assumption A1, $\mathfrak{B}_k^\perp = \mathfrak{B}_k^*$, where $\mathfrak{B}_k^*$ is the image of δ in $H^*_{\Gamma_N}\Lambda^k(D)$. This relation is proved in the three dimensional case in [38], and generalizes to the n dimensional case (see e.g. [74]).

From now on we make the following regularity assumption on the domain D, which will be needed to prove the stability of the numerical schemes we propose in this paper.

Assumption A2. *For every $0 \leq k \leq n$, there exists $0 < s \leq 1$ such that*

$$H_{\Gamma_D}\Lambda^k(D) \cap H^*_{\Gamma_N}\Lambda^k(D) \subseteq H^s\Lambda^k(D). \tag{2.15}$$

Inclusion (2.15) is verified for an s-regular domain s.t. $\Gamma_D = \partial D$ and $\Gamma_N = \emptyset$. In particular, if ∂D is smooth, then D is 1-regular, and if ∂D is Lipschitz, then D is $1/2$-regular. See [6] and the references therein. We assume the second inclusion to be verified in our more general setting where $\Gamma_N \neq \emptyset$ and $\Gamma_D \subsetneq \partial D$.

Remark 2.2.2. *The case of non-trivial topology can likely be treated following [7], but it would make the exposition of our results much more difficult.*

Remark 2.2.3. *We assume $\Gamma_D, \Gamma_N \neq \emptyset$, but the two limit cases treated in [6] can be considered with suitable modifications of our argument.*

We end the section by introducing the following notations for two Hilbert spaces we will use later on:

$$W_k := \begin{bmatrix} L^2\Lambda^k(D) \\ L^2\Lambda^{k-1}(D) \end{bmatrix}, \quad V_k := \begin{bmatrix} H_{\Gamma_D}\Lambda^k(D) \\ H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix}, \tag{2.16}$$

with the inner products $(\cdot,\cdot)_{W_k}$, $(\cdot,\cdot)_{V_k}$, and the norms $\|\cdot\|_{W_k}$, $\|\cdot\|_{V_k}$.

2.2.3 Mixed formulation of the Hodge-Laplace problem

The Hodge Laplacian is the differential operator $\delta\mathrm{d}+\mathrm{d}\delta$ mapping k-forms into k-forms, and the Hodge-Laplace problem is the boundary value problem for the Hodge Laplacian. We consider a particular case of the mixed formulation of the Hodge-Laplace problem with variable coefficients described in [6, 7, 30], which allows to include the Darcy problem (see Section 2.2.3). Given a non negative coefficient $\alpha \in \mathbb{R}_+$ and source terms $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in W_k$, find $\begin{bmatrix} u \\ p \end{bmatrix}$ such that

$$\begin{cases} \delta\mathrm{d}u + \mathrm{d}p = f_1 \text{ in } D \\ \delta u - \alpha p = f_2 \text{ in } D \\ \begin{cases} \mathrm{Tr}(u) = 0 \text{ on } \Gamma_D \\ \mathrm{Tr}(p) = 0 \text{ on } \Gamma_D \end{cases} \quad \begin{cases} \mathrm{Tr}(\star u) = 0 \text{ on } \Gamma_N \\ \mathrm{Tr}(\star \mathrm{d}u) = 0 \text{ on } \Gamma_N \end{cases} \end{cases} \tag{2.17}$$

We introduce $T : V_k \to V_k'$, the linear operator of order two represented by the matrix:

$$T := \begin{bmatrix} \delta\mathrm{d} & \mathrm{d} \\ \delta & -\alpha\mathrm{Id} \end{bmatrix} = \begin{bmatrix} A & B^* \\ B & -\alpha\mathrm{Id} \end{bmatrix}, \tag{2.18}$$

where $V_k' = \begin{bmatrix} (H_{\Gamma_D}\Lambda^k(D))' \\ (H_{\Gamma_D}\Lambda^{k-1}(D))' \end{bmatrix}$ is the dual space of V_k defined in (2.16), the operators A and B are defined as:

$$\begin{gathered} A : H_{\Gamma_D}\Lambda^k(D) \to (H_{\Gamma_D}\Lambda^k(D))' \\ \langle Av, w\rangle := (\mathrm{d}v, \mathrm{d}w) \end{gathered} \tag{2.19}$$

$$\begin{gathered} B : H_{\Gamma_D}\Lambda^k(D) \to (H_{\Gamma_D}\Lambda^{k-1}(D))' \\ \langle Bv, q\rangle := (v, \mathrm{d}q) \end{gathered} \tag{2.20}$$

and B^* is the adjoint of B. Moreover we introduce the linear operators

$$F_1 \in (H_{\Gamma_D}\Lambda^k(D))', \quad F_2 \in (H_{\Gamma_D}\Lambda^{k-1}(D))'$$

defined as:

$$\begin{gathered} F_1 : H_{\Gamma_D}\Lambda^k(D) \to \mathbb{R} \\ F_1(v) := (f_1, v) \end{gathered} \tag{2.21}$$

$$F_2 : H_{\Gamma_D}\Lambda^{k-1}(D) \to \mathbb{R} \qquad F_2(q) := (f_2, q) \tag{2.22}$$

The weak formulation of the deterministic mixed Hodge Laplacian with homogeneous essential boundary conditions on Γ_D and homogeneous natural boundary conditions on Γ_N is

Deterministic Problem:

$$\boxed{\begin{array}{c} \textit{Given } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in V_k', \textit{ find } \begin{bmatrix} u \\ p \end{bmatrix} \in V_k \textit{ s.t.} \\[2em] T \begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \textit{ in } V_k', \end{array}} \tag{2.23}$$

Theorem 2.2.4. *For every $\alpha > 0$, problem* (2.23) *is well-posed, so that there exists a unique solution that depends continuously on the data. In particular, there exist positive constants C_1, C_1' that depend only on the Poincaré constant C_P and on the parameter α, such that for any $\begin{bmatrix} u \\ p \end{bmatrix} \in V_k$ there exists $\begin{bmatrix} v \\ q \end{bmatrix} \in V_k$ with*

$$\left\langle T \begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right\rangle_{V_k', V_k} \geq C_1 \left\| \begin{bmatrix} u \\ p \end{bmatrix} \right\|_{V_k}^2 = C_1 \left(\|u\|_{H\Lambda^k}^2 + \|p\|_{H\Lambda^{k-1}}^2 \right), \tag{2.24}$$

$$\left\| \begin{bmatrix} v \\ q \end{bmatrix} \right\|_{V_k} \leq C_1' \left\| \begin{bmatrix} u \\ p \end{bmatrix} \right\|_{V_k}. \tag{2.25}$$

The same result holds with $\alpha = 0$ provided that F_2 corresponds to $f_2 \in \delta H_{\Gamma_D}\Lambda^k(D)$.

The well-posedness of problem (2.23) is proved in [6] by showing the equivalent inf-sup condition for the bounded bilinear and symmetric form $\langle T\cdot, \cdot\rangle : V_k \times V_k \to \mathbb{R}$ (2.24), (2.25) (see [8, 19]). However, we report it entirely (with a slightly different choice of test functions) as a preparatory step for the proofs we will propose later on.

Proof. We need to show (2.24) and (2.25). Let us start considering $\alpha > 0$. For a given $\begin{bmatrix} u \\ p \end{bmatrix}$ we use the Hodge decomposition (2.11):

$$\begin{bmatrix} u \\ p \end{bmatrix} = \begin{bmatrix} \mathrm{d}u^\circ + u^\perp \\ \mathrm{d}p^\circ + p^\perp \end{bmatrix}, \tag{2.26}$$

with $\mathrm{d}u^\circ \in \mathfrak{B}_k$, $\mathrm{d}p^\circ \in \mathfrak{B}_{k-1}$, $u^\perp \in \mathfrak{B}_k^\perp$ and $p^\perp \in \mathfrak{B}_{k-1}^\perp$. We choose as test functions

$$\begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} u^\perp + \mathrm{d}p^\perp \\ \gamma u^\circ - \mathrm{d}p^\circ \end{bmatrix}, \tag{2.27}$$

where γ is a positive parameter to be set later. Relation (2.27) can also be written in a compact form as

$$\begin{bmatrix} v \\ q \end{bmatrix} = P \begin{bmatrix} u \\ p \end{bmatrix}, \tag{2.28}$$

where

$$P = \begin{bmatrix} \pi^\perp & \mathrm{d}\pi^\perp \\ \gamma\pi^\circ & -\mathrm{d}\pi^\circ \end{bmatrix} \tag{2.29}$$

and the operators $\pi^\perp$, π° are defined in (2.12) and (2.13) respectively. Substituting (2.27) into (2.24), using the property $\mathrm{d}\circ\mathrm{d} = 0$, the Hodge decomposition (2.11) and the Poincaré inequality (2.14) we find

$$\begin{aligned}
\left\langle T\begin{bmatrix} u \\ p \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix}\right\rangle_{V_k',V_k} &= (\mathrm{d}u,\mathrm{d}v) + (v,\mathrm{d}p) + (u,\mathrm{d}q) - \alpha\,(p,q) \\
&= \|\mathrm{d}u^\perp\|^2 + \|\mathrm{d}p^\perp\|^2 + \gamma\|\mathrm{d}u^\circ\|^2 + \alpha\|\mathrm{d}p^\circ\|^2 - \alpha\gamma\left(p^\perp, u^\circ\right) \\
&\geq \|\mathrm{d}u^\perp\|^2 + \|\mathrm{d}p^\perp\|^2 + \gamma\|\mathrm{d}u^\circ\|^2 + \alpha\|\mathrm{d}p^\circ\|^2 \\
&\quad - \frac{\alpha\gamma^{1/2}}{2}\left(C_P^2\|\mathrm{d}p^\perp\|^2 + \gamma C_P^2\|\mathrm{d}u^\circ\|^2\right) \\
&\geq \|\mathrm{d}u^\perp\|^2 + \left(1 - \frac{\alpha}{2}\gamma^{1/2}C_P^2\right)\|\mathrm{d}p^\perp\|^2 + \\
&\quad \gamma\left(1 - \frac{\alpha\gamma^{1/2}C_P^2}{2}\right)\|\mathrm{d}u^\circ\|^2 + \alpha\|\mathrm{d}p^\circ\|^2.
\end{aligned}$$

It is possible to choose γ in order to make (2.24) true with $C_1 = C_1(C_P,\alpha)$. The inequality (2.25) with $C_1 = C_1'(C_P,\alpha)$ follows from the Hodge decomposition (2.11) and Poincaré inequality (2.14).

The proof in the case $\alpha = 0$ is very similar. Suppose $f_2 \in \delta H_{\Gamma_D}\Lambda^k(D)$. In order to have a unique solution, we need to look for $p \in \mathfrak{B}_{k-1}^\perp$. Fixed $u = \mathrm{d}u^\circ + u^\perp \in H_{\Gamma_D}\Lambda^k(D)$ we again choose the test functions as in (2.28): $v = \mathrm{d}p + u^\perp \in H_{\Gamma_D}\Lambda^k(D)$ and $q = u^\circ \in \mathfrak{B}_{k-1}^\perp$. Using the Poincaré inequality (2.14) and the orthogonal decomposition (2.11) we are able to prove the relations (2.24) and (2.25). □

A simple consequence of Theorem 2.2.4 (see [19]) is that there exists a positive constant $K = K(C_P,\alpha)$ such that

$$\left\|\begin{bmatrix} u \\ p \end{bmatrix}\right\|_{V_k} \leq K\left\|\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\right\|_{V_k'}. \tag{2.30}$$

Another way to express the result given in Theorem 2.2.4 is: $\forall \begin{bmatrix} u \\ p \end{bmatrix} \in V_k$ it holds

$$\left\langle T\begin{bmatrix} u \\ p \end{bmatrix}, P\begin{bmatrix} u \\ p \end{bmatrix}\right\rangle_{V_k',V_k} \geq C_1\left\|\begin{bmatrix} u \\ p \end{bmatrix}\right\|_{V_k}^2 \tag{2.31}$$

$$\|P\|_{\mathcal{L}(V_k,V_k)} \leq C_1'. \tag{2.32}$$

Translation to the language of partial differential equations

Let us consider the case $D \subset \mathbb{R}^3$, naturally identifying the tangent space at each point $x \in D$ with $\mathbb{R}^3$. Thanks to the identification of $Alt^0\mathbb{R}^3$ and $Alt^3\mathbb{R}^3$ with $\mathbb{R}$, and of $Alt^1\mathbb{R}^3$ and $Alt^2\mathbb{R}^3$ with $\mathbb{R}^3$, we can establish correspondences between the spaces of

	$H_{\Gamma_D}\Lambda^k(D)$	d	$Tr\|_{\Gamma_D}u$
$k=0$	$H^1_{\Gamma_D}(D)$	∇	$u\|_{\Gamma_D}$
$k=1$	$H_{\Gamma_D}(\text{curl}, D)$	curl	$u \times n\|_{\Gamma_D}$
$k=2$	$H_{\Gamma_D}(\text{div}, D)$	div	$u \cdot n\|_{\Gamma_D}$
$k=3$	$L^2(D)$	0	0

Table 2.1: *Correspondences in terms of proxy fields between the space of differential forms $H_{\Gamma_D}\Lambda^k(D)$ and the classical spaces of functions and vector fields, in the case $n = 3$.*

differential forms and scalar or vector fields. These fields are called proxy fields. In particular, we can identify each 0-form and 3-form with a scalar-valued function, and each 1-form and 2-form with a vector-valued function. Table 2.1 summarizes the correspondences in terms of proxy fields for the spaces of differential forms $H_{\Gamma_D}\Lambda^k(D)$, the exterior derivative operators and the trace operators. Based on the identifications in Table 2.1 we can reinterpret the de Rham complex (2.9) as follows:

$$0 \to H^1_{\Gamma_D}(D) \xrightarrow{\nabla} H_{\Gamma_D}(\text{curl}, D) \xrightarrow{\text{curl}} H_{\Gamma_D}(\text{div}, D) \xrightarrow{\text{div}} L^2(D) \to 0 \tag{2.33}$$

In this section we will use the symbol $(\cdot,\cdot)$ to denote the inner product in $L^2(D)$, that corresponds by proxy identifications to the inner product in $L^2\Lambda^k(D)$.

- Let us start with $k = 0$. In this case $H_{\Gamma_D}\Lambda^{-1}(D) = 0$, so $p = 0$. Then $u \in H^1_{\Gamma_D}(D)$ satisfies

$$(\nabla u, \nabla v) = (f_1, v) \quad \forall v \in H^1_{\Gamma_D}(D). \tag{2.34}$$

  We obtain the usual weak formulation of the Poisson equation equipped with homogeneous Dirichlet boundary conditions on Γ_D and homogeneous Neumann boundary conditions on Γ_N.

- For $k = 1$ and $\alpha = 0$, the linear operator T of order two defined in (2.18) is represented by the matrix

$$T = \begin{bmatrix} \text{curl}^2 & \nabla \\ -\text{div} & 0 \end{bmatrix}. \tag{2.35}$$

  Problem (2.23) is the weak formulation of the magnetostatic/electrostatic equations (see for example [18, 59, 76]). Indeed, $V_1 = \begin{bmatrix} H_{\Gamma_D}(\text{curl}, D) \\ H^1_{\Gamma_D}(D) \end{bmatrix}$ and $\begin{bmatrix} u \\ p \end{bmatrix} \in V_1$ satisfies

$$\begin{cases} (\text{curl}u, \text{curl}v) + (\nabla p, v) = (f_1, v) \\ (u, \nabla q) = (f_2, q)\,. \end{cases} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_1. \tag{2.36}$$

- When $k = 2$,

$$T = \begin{bmatrix} -\nabla\text{div} & \text{curl} \\ \text{curl} & -\alpha\text{Id} \end{bmatrix}.$$

  Problem (2.23) is the mixed formulation of the vectorial Poisson equation: find

$$\left[\begin{array}{c} u \\ p \end{array}\right] \in V_2 = \left[\begin{array}{c} H_{\Gamma_D}(\mathrm{div}, D) \\ H_{\Gamma_D}(\mathrm{curl}, D) \end{array}\right] \text{ s.t.}$$

$$\left\{\begin{array}{l} (\mathrm{div}u, \mathrm{div}v) + (\mathrm{curl}p, v) = (f_1, v) \\ (u, \mathrm{curl}q) - \alpha\,(p, q) = (f_2, q) \end{array}\right. \quad \forall \left[\begin{array}{c} v \\ q \end{array}\right] \in V_2. \tag{2.37}$$

- Finally, for $k = 3$, problem (2.23) models the fluid flow in porous media. We can reinterpret the linear tensor operator of order two T as

$$T = \left[\begin{array}{cc} 0 & \mathrm{div} \\ -\nabla & -\alpha\mathrm{Id} \end{array}\right], \tag{2.38}$$

  where $\alpha > 0$ is linked to the inverse of the permeability. Hence, (2.23) is the Darcy problem: find $\left[\begin{array}{c} u \\ p \end{array}\right] \in V_3 = \left[\begin{array}{c} L^2(D) \\ H_{\Gamma_D}(\mathrm{div}, D) \end{array}\right]$ s.t.

$$\left\{\begin{array}{l} (\mathrm{div}p, v) = (f_1, v) \\ (u, \mathrm{div}q) - \alpha\,(p, q) = 0 \end{array}\right. \quad \forall \left[\begin{array}{c} v \\ q \end{array}\right] \in V_3. \tag{2.39}$$

2.3 Stochastic Sobolev spaces of differential forms and stochastic Hodge Laplacian

2.3.1 Stochastic Sobolev spaces of differential forms

Let $v_1 \in V_1$ and $v_2 \in V_2$, where V_1, V_2 are Hilbert spaces. Let $v_1 \otimes v_2 : V_1 \times V_2 \to \mathbb{R}$ denote the symmetric bilinear form which acts on each couple $(w_1, w_2) \in V_1 \times V_2$ by

$$v_1 \otimes v_2(w_1, w_2) = (v_1, w_1)_{V_1} (v_2, w_2)_{V_2},$$

where $(\cdot,\cdot)_{V_1}$ denotes the inner product in V_1 and $(\cdot,\cdot)_{V_2}$ the inner product in V_2. Let us define an inner product $(\cdot,\cdot)_{V_1 \otimes V_2}$ on the set of such symmetric bilinear forms as

$$(v_1 \otimes v_2, v_1' \otimes v_2')_{V_1 \otimes V_2} = (v_1, v_1')_{V_1} (v_2, v_2')_{V_2}, \tag{2.40}$$

and extend it by linearity to the set

$$\mathrm{span}\,\{v_1 \otimes v_2 \;:\; v_1 \in V_1, v_2 \in V_2\} \tag{2.41}$$

composed of finite linear combinations of such symmetric bilinear forms.

Definition 2.3.5. *Given V_1 and V_2 Hilbert spaces, the tensor product $V_1 \otimes V_2$ is the Hilbert space defined as the completition of the set* (2.41) *under the inner product $(\cdot,\cdot)_{V_1 \otimes V_2}$ in* (2.40).

In the following we will denote with $\|\cdot\|_{V_1 \otimes V_2}$ the norm induced by the inner product $(\cdot,\cdot)_{V_1 \otimes V_2}$. Definition 2.3.5 naturally generalizes to the tensor product of m Hilbert spaces, with $m \geq 2$ integer. For more details on tensor product spaces and on norms on tensor product spaces see for example [71, 85] and the references therein.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space and V a separable Hilbert space. The stochastic counterpart of V is the Hilbert space given by the tensor product $V \otimes$

$L^2(\Omega, d\mathbb{P})$, where $L^2(\Omega, d\mathbb{P})$ is the Hilbert space defined in Section 2.2.1. Let $L^2(\Omega; V)$ be the Bochner space composed of functions u such that $\omega \mapsto \|u(\omega)\|_V^2$ is measurable and integrable, so that

$$\|u\|_{L^2(\Omega;V)} := \left(\int_\Omega \|u(\omega)\|_V^2 d\mathbb{P}(\omega)\right)^{1/2}$$

is finite. We observe that there is a unique isomorphism from $V \otimes L^2(\Omega, d\mathbb{P})$ to $L^2(\Omega; V)$ which maps $\psi \otimes \mu \in V \otimes L^2(\Omega, d\mathbb{P})$ onto the function $\omega \mapsto \mu(\omega)\psi \in V$.

The definition of the Hilbert space $L^2(\Omega; V)$ easily generalizes to the space $L^m(\Omega; V)$ with $m \geq 1$ integer. We say that a random field $u : \Omega \to V$ is in the Bochner space $L^m(\Omega; V)$ if $\omega \mapsto \|u(\omega)\|_V^m$ is measurable and integrable, so that

$$\|u\|_{L^m(\Omega;V)} := \left(\int_\Omega \|u(\omega)\|_V^m d\mathbb{P}(\omega)\right)^{1/m}$$

is finite.

In the following we focus on two stochastic Sobolev spaces of differential forms, namely $L^m(\Omega; W_k)$ and $L^m(\Omega; V_k)$ with $m \geq 1$ integer, where W_k and V_k are the Sobolev spaces of differential forms defined in (2.16).

2.3.2 Stochastic mixed Hodge-Laplace problem

Let be given $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$, with $m \geq 1$, defined as the stochastic version of (2.21) and (2.22):

$$\begin{aligned} F_1(\omega) &: H_{\Gamma_D}\Lambda^k(D) \to \mathbb{R} \\ &F_1(\omega)(v) := (f_1(\omega), v) \end{aligned}$$

$$\begin{aligned} F_2(\omega) &: H_{\Gamma_D}\Lambda^{k-1}(D) \to \mathbb{R} \\ &F_2(\omega)(q) := (f_2(\omega), q) \end{aligned}$$

where $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^m(\Omega; V_k)$ is given. The stochastic counterpart of problem (2.23) is:

Stochastic Problem:

$$\boxed{\begin{array}{c} \textit{Given } m \geq 1 \textit{ and } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')\textit{, find } \begin{bmatrix} u \\ p \end{bmatrix} \in L^m(\Omega; V_k) \textit{ s.t.} \\ \\ T\begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \textit{ in } V_k', \textit{ a.e. in } \Omega. \end{array}} \tag{2.42}$$

Theorem 2.3.6 (Well-posedness of the stochastic Hodge Laplacian)**.** *For every $\alpha > 0$ problem* (2.42) *is well-posed, so that there exists a unique solution that depends continuously on the data. The same result holds with $\alpha = 0$ provided that F_2 corresponds to $f_2 \in L^m\left(\Omega; \delta H_{\Gamma_D}\Lambda^k(D)\right)$.*

Proof. Thanks to Theorem 2.2.4, for almost all $\omega \in \Omega$, problem (2.42) admits a unique solution $\begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} \in V_k$, the mapping $\omega \mapsto \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix}$ is measurable and we have:

$$\left\| \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} \right\|_{V_k} \leq K \left\| \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \right\|_{V_k'} \quad \text{a.e. in } \Omega \tag{2.43}$$

with $K = K(C_P, \alpha)$ independent of ω (see (2.30)). For any $m \geq 1$,

$$\left(\mathbb{E} \left[\left\| \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} \right\|_{V_k}^m \right] \right)^{1/m} \leq K \left(\mathbb{E} \left[\left\| \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix} \right\|_{V_k'}^m \right] \right)^{1/m}.$$

By hypothesis $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$, hence we conclude that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m(\Omega; V_k)$. □

2.4 Deterministic problems for the statistics of u and p

We are interested in the statistical moments of the unique stochastic solution $\begin{bmatrix} u \\ p \end{bmatrix}$ of the stochastic problem (2.42). We exploit the linearity of the system $T \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}$ to derive the moment equations, that is the deterministic equations solved by the statistical moments of the unique stochastic solution $\begin{bmatrix} u \\ p \end{bmatrix}$. At the beginning we focus on the first moment equation. Then, after recalling the definition of the m-th statistical moment ($m \geq 2$ integer) and the main concepts about the tensor product of operators defined on Hilbert spaces, we establish the well-posedness of the m-th moment problem. The main achievement is the constructive proof of the inf-sup condition for the tensor product operator $T^{\otimes m}$ stated in Theorem 2.4.13. Indeed, this proof extends to the case of sparse tensor product approximations (see Section 2.6.4).

2.4.1 Equation for the mean

Following [92, 97], we provide a way to compute the first statistical moment of the unique stochastic solution of the stochastic Hodge Laplace problem (2.42).

Given a random field $v \in L^1(\Omega; V)$, where V in a Hilbert space, its first statistical moment $\mathbb{E}[v] \in V$ is well defined, and is given by:

$$\mathbb{E}[v](x) := \int_\Omega v(\omega, x) d\mathbb{P}, \quad x \in D. \tag{2.44}$$

Definition (2.44) easily applies to the vector case ($V = V_k, V = W_k$).

Suppose that $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^1(\Omega; V_k')$, so that the unique solution of the stochastic problem is such that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^1(\Omega; V_k)$. To derive the first moment equation we simply apply the mean operator to the stochastic problem (2.42). We exploit the commutativity between the operators T defined in (2.18) and $\mathbb{E}$ defined in (2.44), so that $\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}$ is a solution of:

Mean Problem

$$\boxed{\begin{array}{c} \textit{Given } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^1\left(\Omega; V_k'\right), \textit{ find } E_s \in V_k \textit{ s.t.} \\[2ex] T(E_s) = \mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \textit{ in } V_k', \end{array}} \tag{2.45}$$

where $\mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in V_k'$ is defined as:

$$\mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\left(\begin{bmatrix} v \\ q \end{bmatrix}\right) := \left(\mathbb{E}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix}\right)_{W_k} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_k.$$

Theorem 2.2.4 states the well-posedness of problem (2.45), hence $\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}$ is its unique solution. We notice that problem (2.45) has exactly the same structure as problem (2.42) with loading term given by the mean of the loading term in (2.42).

2.4.2 Statistical moments of a random function

Let $u \in L^m\left(\Omega; V\right)$, where V is a Hilbert space and $L^m\left(\Omega; V\right)$ is defined as in Section 2.3.1. Then $u^{\otimes m} := \underbrace{u \otimes \cdots \otimes u}_{m \text{ times}} \in L^1(\Omega, V^{\otimes m})$, where from now on $V^{\otimes m}$ denotes the tensor product space $\underbrace{V \otimes \cdots \otimes V}_{m \text{ times}}$. Hence we can give the following definition:

Definition 2.4.7. *Given $u \in L^m\left(\Omega; V\right)$, $m \geq 2$ integer, the m-th moment of $u(\omega)$ is defined by*

$$\mathcal{M}^m\left[u\right] := \mathbb{E}\left[u \otimes \cdots \otimes u\right] = \int_\Omega u(\omega) \otimes \cdots \otimes u(\omega) d\mathbb{P}(\omega) \in V^{\otimes m}. \tag{2.46}$$

It clearly holds $\|\mathcal{M}^m\left[u\right]\|_{V^{\otimes m}} \leq \|u\|^m_{L^m(\Omega;V)}$. Definition 2.4.7 with $m = 1$ is (2.44). Moreover, Definition 2.4.7 easily generalizes to the vector case.

2.4.3 Tensor product of operators on Hilbert spaces

We will see that the deterministic equation for the m-th moment involves the tensor product of the operator T. Hence, we need to describe some aspects of the theory of tensor product operators on Hilbert spaces. For more details see for example [85] and the references therein.

Suppose that $T_1 : V_1 \to V_1'$, $T_2 : V_2 \to V_2'$ are continuous operators on the Hilbert spaces V_1 and V_2 respectively. $T_1 \otimes T_2$ is defined on functions of the type $\phi \otimes \psi$, with $\phi \in V_1$, $\psi \in V_2$ as:

$$(T_1 \otimes T_2)(\phi \otimes \psi) = T_1\phi \otimes T_2\psi \in V_1' \otimes V_2'.$$

This definition extends to $V_1 \otimes V_2$ by linearity and density. The tensor product of two bounded operators on Hilbert space is still a bounded operator, as stated by the following

Proposition 2.4.8. *Let $T_1 : V_1 \to V_1'$, $T_2 : V_2 \to V_2'$ be bounded operators on Hilbert spaces V_1 and V_2 respectively. Then*

$$\|T_1 \otimes T_2\|_{\mathcal{L}(V_1\otimes V_2, V_1'\otimes V_2')} = \|T_1\|_{\mathcal{L}(V_1,V_1')}\|T_2\|_{\mathcal{L}(V_2,V_2')}.$$

Proof. See [85]. $\square$

The definition of the tensor product of two operators on Hilbert spaces and Proposition 2.4.8 generalize to tensor product of any finite number of operators defined on Hilbert spaces.

We detail now the vector case, since it will be useful in the next section. Let $V_1 = V_2 = V_k$, where V_k is defined in (2.16), and $T_1 = T_2 = T$, where $T = (T)_{i,j=1,2} : V_k \to V_k'$ is the linear operator of order two defined in (2.18). The tensor product operator $T^{\otimes m} := \underbrace{T \otimes \cdots \otimes T}_{m}$, ($m \geq 1$ integer), is the operator of order $2m$ that maps tensors in $V_k^{\otimes m}$ to tensors in $(V_k')^{\otimes m}$ defined as

$$(T^{\otimes m})_{i_1 \dots i_{2m}} = T_{i_1 i_2} \otimes \cdots \otimes T_{i_{2m-1} i_{2m}}. \tag{2.47}$$

Given $X \in V_k^{\otimes m}$, $T^{\otimes m}X$ is a tensor of order m in $(V_k')^{\otimes m}$ given by

$$(T^{\otimes m}X)_{i_1 \dots i_m} = \sum_{j_1,\dots,j_m=1}^{2} (T_{i_1 j_1} \otimes \cdots \otimes T_{i_m j_m}) X_{j_1 \dots j_m}, \quad i_1, \dots, i_m = 1, 2. \tag{2.48}$$

Definition 2.4.9. *Let T and V_k be as before and let $X \in V_k^{\otimes m}$ and $Y \in V_k^{\otimes m}$. We define*

$$\left\langle T^{\otimes m}X, Y \right\rangle = \sum_{i_1,\dots,i_m=1}^{2} \sum_{j_1,\dots,j_m=1}^{2} \left\langle T_{i_1,j_1} \cdots T_{i_m,j_m} X_{j_1,\dots,j_m}, Y_{i_1,\dots,i_m} \right\rangle. \tag{2.49}$$

2.4.4 Equation for the m-th moment

Following [97], we analyze the m-th moment equation for $m \geq 2$. Suppose $\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m(\Omega; V_k')$ so that $\begin{bmatrix} u \\ p \end{bmatrix} \in L^m(\Omega; V_k)$. To derive the deterministic m-th moment problem we tensorize the stochastic problem (2.42) with itself m times:

$$\underbrace{T \otimes \ldots \otimes T}_{m \text{ times}} \begin{bmatrix} u(\omega) \\ p(\omega) \end{bmatrix}^{\otimes m} = \begin{bmatrix} F_1(\omega) \\ F_2(\omega) \end{bmatrix}^{\otimes m} \quad \text{in } (V_k')^{\otimes m}, \text{ for a.e. } \omega \in \Omega.$$

We take the expectation on both sides and we exploit the commutativity between the operators T and $\mathbb{E}$. By definition, $\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}^{\otimes m} = \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$. Thus, $\mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$ is a solution of

m-Points Correlation Problem:

$$\boxed{\begin{gathered} \textit{Given } m \geq 2 \textit{ integer and } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m\left(\Omega; V_k'\right), \textit{ find } M_s^{\otimes m} \in V_k^{\otimes m} \textit{ s.t.} \\ \\ T^{\otimes m} M_s^{\otimes m} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \textit{ in } (V_k')^{\otimes m}, \end{gathered}} \tag{2.50}$$

where $\mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in (V_k')^{\otimes m}$ is defined as:

$$\mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \left(\begin{bmatrix} v \\ q \end{bmatrix} \right) := \left(\mathcal{M}^m \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} v \\ q \end{bmatrix} \right)_{W_k^{\otimes m}} \quad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_k^{\otimes m}.$$

We notice that in the right-hand side of (2.50) we have the m-points correlation of the loading terms of problem (2.42).

Remark 2.4.10. *Note that problem* (2.45) *is a saddle-point problem, and* (2.50) *is composed of m "nested" saddle-point problems. Indeed, if for example $m = 2$, $T \otimes T$ can be represented by the matrix*

$$T \otimes T = \left[\begin{array}{cc|cc} \delta \mathrm{d} \otimes \delta \mathrm{d} & \delta \mathrm{d} \otimes \mathrm{d} & \mathrm{d} \otimes \delta \mathrm{d} & \mathrm{d} \otimes \mathrm{d} \\ \delta \mathrm{d} \otimes \delta & \delta \mathrm{d} \otimes -\alpha \mathrm{Id} & \mathrm{d} \otimes \delta & \mathrm{d} \otimes -\alpha \mathrm{Id} \\ \hline \delta \otimes \delta \mathrm{d} & \delta \otimes \mathrm{d} & -\alpha \mathrm{Id} \otimes \delta \mathrm{d} & -\alpha \mathrm{Id} \otimes \mathrm{d} \\ \delta \otimes \delta & \delta \otimes -\alpha \mathrm{Id} & -\alpha \mathrm{Id} \otimes \delta & -\alpha \mathrm{Id} \otimes -\alpha \mathrm{Id} \end{array} \right]. \tag{2.51}$$

Theorem 2.4.11 (Well-posedness of the *m*-th problem)**.** *For every $\alpha > 0$, problem* (2.50) *is well-posed, so that there exists a unique solution that depends continuously on the data. The same result holds with $\alpha = 0$ provided that F_2 corresponds to $f_2 \in L^m\big(\Omega; \delta H_{\Gamma_D} \Lambda^k(D)\big)$.*

Proof. Theorem 2.4.11 can be proved by a simple tensor product argument, as follows. Since problem (2.23) is well-posed, the inverse operator T^{-1} exists and is linear and bounded. Now we take into account the tensor operator $(T^{-1})^{\otimes m} = \underbrace{T^{-1} \otimes \ldots \otimes T^{-1}}_{m \text{ times}}$. It is the inverse operator of $T^{\otimes m}$. Moreover, it is linear and bounded (Proposition 2.4.8). Hence we can immediately conclude the well-posedness of problem (2.50). $\square$

Remark 2.4.12. *The approach presented in the proof is not completely satisfactory in view of a finite dimensional approximation. Indeed, when considering a finite dimensional version of the operator, $T_h := T|_{V_{k,h}} : V_{k,h} \rightarrow V_{k,h}'$, where $V_{k,h}$ is a finite dimensional subspace of V_k, and aiming at proving the well-posedness of the tensor* *operator* $(T_h)^{\otimes m} = \underbrace{T_h \otimes \ldots \otimes T_h}_{m\ times}$*, this tensor product argument applies only if the finite dimensional subspace of* $V_k^{\otimes m}$ *is a tensor product space* $V_{k,h}^{\otimes m}$*. It will not apply straightforwardly if sparse tensor product spaces are considered instead.*

Constructive proof of inf-sup condition for the tensorized problem

Here we propose an alternative proof of Theorem 2.4.11 that consists in showing the inf-sup condition for $T^{\otimes m}$. This proof will be used later on to prove the stability of a sparse tensor product finite element discretization, which is of practical interest for moderately large m as it reduces considerably the curse of dimensionality with respect to a full tensor product approximation.
A result equivalent to Theorem 2.4.11 is the following

Theorem 2.4.13 (Tensorial inf-sup condition)**.** *For every* $M_s^{\otimes m} \in V_k^{\otimes m}$*, there exist a test function* $M_t^{\otimes m} \in V_k^{\otimes m}$ *and positive constants*

$$\begin{aligned} C_m &= C_m(\alpha, C_{P,1}, \|T\|_{\mathcal{L}(V_k,V_k')}, \|P\|_{\mathcal{L}(V_k,V_k)}), \\ C_m' &= C_m'(\alpha, C_{P,1}, \|T\|_{\mathcal{L}(V_k,V_k')}, \|P\|_{\mathcal{L}(V_k,V_k)})) \end{aligned}$$

s.t.

$$\left\langle T^{\otimes m} M_s^{\otimes m}, M_t^{\otimes m} \right\rangle_{(V_k')^{\otimes m}, V_k^{\otimes m}} \geq C_m \|M_s^{\otimes m}\|^2_{V_k^{\otimes m}}, \tag{2.52}$$

$$\|M_t^{\otimes m}\|_{V_k^{\otimes m}} \leq C_m' \|M_s^{\otimes m}\|_{V_k^{\otimes m}}, \tag{2.53}$$

where $C_{P,1}$ *will be introduced in* (2.60) *and* P *is defined in* (2.29)*.*

Before presenting the proof we state the tensorized versions of the Hodge decomposition and the Poincaré inequality, which are two keys ingredients in the proof of the inf-sup condition for the deterministic problem (2.23).

Let us write the space $V_k^{\otimes m}$ as

$$V_k^{\otimes m} = V_k \otimes V_k^{\otimes(m-1)} = \begin{bmatrix} H_{\Gamma_D}\Lambda^k(D) \\ H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix} \otimes V_k^{\otimes(m-1)} = \begin{bmatrix} U_k^m \\ U_{k-1}^m \end{bmatrix} \tag{2.54}$$

where we defined

$$U_k^m := H_{\Gamma_D}\Lambda^k(D) \otimes V_k^{\otimes(m-1)}, \tag{2.55}$$

$$U_{k-1}^m := H_{\Gamma_D}\Lambda^{k-1}(D) \otimes V_k^{\otimes(m-1)}. \tag{2.56}$$

We obtain the tensorial Hodge decomposition following the idea of the one dimensional Hodge decomposition (2.11). Indeed, for every integer $m \geq 2$, we split U_k^m (U_{k-1}^m is analogous) as:

Tensorial Hodge Decomposition:

$$U_k^m = \mathfrak{B}_k^m \oplus \mathfrak{B}_k^{m,\perp} \tag{2.57}$$

where

$$\begin{aligned} \mathfrak{B}_k^m &:= \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)}\, U_{k-1}^m = \mathfrak{B}_k \otimes V_k^{\otimes(m-1)} \\ \mathfrak{B}_k^{m,\perp} &:= \mathfrak{B}_k^\perp \otimes V_k^{\otimes(m-1)} \end{aligned}$$

and $\mathfrak{B}_k$, $\mathfrak{B}_k^{\perp}$ are defined in Section 2.2. The tensor operators $\pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)}$ and $\pi^{\circ} \otimes \mathrm{Id}^{\otimes(m-1)}$, where $\pi^{\perp}$ and π° are defined in (2.12) and (2.13) respectively, act on U_k^m (U_{k-1}^m is analogous) as:

$$\begin{aligned} \pi^{\perp} \otimes \mathrm{Id}^{\otimes(m-1)} : U_k^m = \mathfrak{B}_k^m \oplus \mathfrak{B}_k^{m,\perp} &\to \mathfrak{B}_k^{m,\perp} \\ v = \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} v^{\circ} + v^{\perp} &\mapsto v^{\perp} \end{aligned} \tag{2.58}$$

$$\begin{aligned} \pi^{\circ} \otimes \mathrm{Id}^{\otimes(m-1)} : U_k^m = \mathfrak{B}_k^m \oplus \mathfrak{B}_k^{m,\perp} &\to \mathfrak{B}_{k-1}^{m,\perp} \\ v = \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} v^{\circ} + v^{\perp} &\mapsto v^{\circ}. \end{aligned} \tag{2.59}$$

The tensorial Poincaré inequality is proved in the following lemma.

Lemma 2.4.14 (Tensorial Poincaré inequality). *For every integer $m \geq 2$, there exists a positive constant $C_{P,1}$ such that*

$$\|v\|_{(L^2\Lambda^k)^{\otimes m}} \leq C_{P,1} \|\mathrm{Id} \otimes \ldots \otimes \underbrace{\mathrm{d}}_{i} \otimes \ldots \otimes \mathrm{Id}\, v\|_{L^2\Lambda^k \otimes \ldots \otimes \underbrace{L^2\Lambda^{k+1}}_{i} \otimes \ldots \otimes L^2\Lambda^k}, \tag{2.60}$$

$\forall v \in L^2\Lambda^k(D) \otimes \ldots \otimes \underbrace{(\mathfrak{Z}_k^{\perp})}_{i} \otimes \ldots \otimes L^2\Lambda^k(D)$, *where $\mathfrak{Z}_k^{\perp}$ is defined in Section 2.2.2.*

Proof. We know that $H\Lambda^k(D)$ is a Hilbert space with the inner product $(u, v)_{H\Lambda^k}$ and $(u, u)_{H\Lambda^k} = \|u\|^2_{H\Lambda^k}$. Besides, we know that $\mathfrak{Z}_k^{\perp}$ is a Hilbert space with the equivalent inner product $(\mathrm{d}u, \mathrm{d}v)$ and norm $\|\mathrm{d}u\| = \sqrt{(\mathrm{d}u, \mathrm{d}u)}$. A consequence of the Open Mapping Theorem states that given m Hilbert spaces $H_1, \ldots, H_m$, the topology of $H_1 \otimes \ldots \otimes H_m$ depends only on the topology and not on the choice of the inner products of $H_1, \ldots, H_m$. If we apply this statement with $H_i = \mathfrak{Z}_k^{\perp}$ and $H_j = H\Lambda^k(D)$, $i \neq j$, we can conclude the inequality (2.60). $\square$

A simple consequence of the previous lemma is:

$$\|v\|_{(L^2\Lambda^k)^{\otimes m}} \leq C_{P,m} \|\mathrm{d}^{\otimes m} v\|_{(L^2\Lambda^{k+1})^{\otimes m}} \quad \forall v \in \left(\mathfrak{Z}_k^{\perp}\right)^{\otimes m}, \tag{2.61}$$

where $C_{P,m} > 0$ depends only on the domain D and on m.

Proof of Theorem 2.4.13. As shown before, $\mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$ is a solution of (2.50). The uniqueness of the solution of problem (2.50) is related to the global inf-sup condition (2.52), (2.53) (see [8, 19]). Suppose $\alpha > 0$ (the case $\alpha = 0$ is analogous). To lighten the notations, in the proof we use the brackets $\langle \cdot, \cdot \rangle$ without specifying the spaces we consider, when no ambiguity arises. We use the tensorial Hodge decomposition (2.57) and the tensorial Poicaré inequality (Lemma 2.4.14). We prove (2.52) by induction. In Theorem 2.2.4 we already proved the inf-sup condition with $m = 1$. Now suppose $m = 2$. We fix $M_s^{\otimes 2} = \begin{bmatrix} (M_s^{\otimes 2})_{1:} \\ (M_s^{\otimes 2})_{2:} \end{bmatrix}$ where $(M_s^{\otimes 2})_{1:}$ ($(M_s^{\otimes 2})_{2:}$ respectively) means that in the tensor of order two $M_s^{\otimes 2} = (M_s^{\otimes 2})_{ij=1,2}$ we fix $i = 1$ ($i = 2$ respectively) and let j vary. Using (2.54) and (2.57) with $m = 2$ we decompose

$$M_s^{\otimes 2} = \begin{bmatrix} \mathrm{d} \otimes \mathrm{Id}(M_s^{\circ})_{1:} + (M_s^{\perp})_{1:} \\ \mathrm{d} \otimes \mathrm{Id}(M_s^{\circ})_{2:} + (M_s^{\perp})_{2:} \end{bmatrix} \in \begin{bmatrix} U_k^2 \\ U_{k-1}^2 \end{bmatrix},$$

where

$$\begin{aligned}(M_s^{\perp})_{1:} &= \pi^{\perp} \otimes \mathrm{Id}(M_s^{\otimes 2})_{1:} \in \mathfrak{B}_k^{2,\perp}\\ (M_s^{\perp})_{2:} &= \pi^{\perp} \otimes \mathrm{Id}(M_s^{\otimes 2})_{2:} \in \mathfrak{B}_{k-1}^{2,\perp}\\ (M_s^{\circ})_{1:} &= \pi^{\circ} \otimes \mathrm{Id}(M_s^{\otimes 2})_{1:} \in \mathfrak{B}_{k-1}^{2,\perp}\\ (M_s^{\circ})_{2:} &= \pi^{\circ} \otimes \mathrm{Id}(M_s^{\otimes 2})_{2:} \in \mathfrak{B}_{k-2}^{2,\perp}.\end{aligned}$$

We choose $M_t^{\otimes 2} = P \otimes PM_s^{\otimes 2}$, where P is defined in (2.29), so that:

$$\begin{aligned}\left\langle T \otimes TM_s^{\otimes 2}, M_t^{\otimes 2}\right\rangle &= \left\langle T \otimes TM_s^{\otimes 2}, P \otimes PM_s^{\otimes 2}\right\rangle\\ &= \sum_{i,j=1}^{2} \left\langle T_{ij} \otimes T(M_s^{\otimes 2})_{j:}, (P \otimes PM_s^{\otimes 2})_{i:}\right\rangle. \qquad (2.62)\end{aligned}$$

Let $\langle T_{ij} \otimes T(M_s^{\otimes 2})_{j:}, (P \otimes PM_s^{\otimes 2})_{i:}\rangle = \mathcal{I}_{ij}$. We will bound each term $\mathcal{I}_{ij}$ for $i, j = 1, 2$.

Using (2.48) we explicit the term $(P \otimes PM_s^{\otimes 2})_{i:}$:

$$(P \otimes PM_s^{\otimes 2})_{i:} = P_{i1} \otimes P(M_s^{\otimes 2})_{1:} + P_{i2} \otimes P(M_s^{\otimes 2})_{2:}. \qquad (2.63)$$

Let us start from the case $i = j = 1$.

$$\mathcal{I}_{11} = \left\langle A \otimes T(M_s^{\otimes 2})_{1:}, (\pi^{\perp} \otimes P(M_s^{\otimes 2})_{1:} + \mathrm{d}\pi^{\perp} \otimes P(M_s^{\otimes 2})_{2:})\right\rangle. \qquad (2.64)$$

Since $\mathrm{d} \circ \mathrm{d} = 0$, $\left\langle A \otimes T(M_s^{\otimes 2})_{1:}, \mathrm{d}\pi^{\perp} \otimes P(M_s^{\otimes 2})_{2:})\right\rangle = 0$ and $A \otimes T(\mathrm{d} \otimes \mathrm{Id} M_s^{\circ})_{1:} \equiv 0$. Hence,

$$\begin{aligned}\mathcal{I}_{11} &= \left\langle A \otimes T(M_s^{\perp})_{1:}, \mathrm{Id} \otimes P(M_s^{\perp})_{1:}\right\rangle\\ &= \left\langle \mathrm{d} \otimes T(M_s^{\perp})_{1:}, \mathrm{d} \otimes P(M_s^{\perp})_{1:}\right\rangle\\ &\geq C_1 \|\mathrm{d} \otimes \mathrm{Id}(M_s^{\perp})_{1:}\|^2_{L^2\Lambda^{k+1} \otimes V_k}.\end{aligned}$$

The last step follows from (2.31). If $i = 1$ and $j = 2$ we find

$$\mathcal{I}_{12} = \left\langle B^* \otimes T(M_s^{\otimes 2})_{2:}, \pi^{\perp} \otimes P(M_s^{\otimes 2})_{1:} + \mathrm{d}\pi^{\perp} \otimes P(M_s^{\otimes 2})_{2:}\right\rangle. \qquad (2.65)$$

Since $\pi^{\perp} \otimes P(M_s^{\otimes 2})_{1:} \in \mathfrak{B}_k^{2,\perp}$, $\left\langle B^* \otimes T(M_s^{\otimes 2})_{2:}, \pi^{\perp} \otimes P(M_s^{\otimes 2})_{1:}\right\rangle = 0$. Hence,

$$\begin{aligned}\mathcal{I}_{12} &= \left\langle B^* \otimes T(M_s^{\perp})_{2:}, \mathrm{d} \otimes P(M_s^{\perp})_{2:}\right\rangle\\ &= \left\langle \mathrm{d} \otimes T(M_s^{\perp})_{2:}, \mathrm{d} \otimes P(M_s^{\perp})_{2:}\right\rangle\\ &\geq C_1 \|\mathrm{d} \otimes \mathrm{Id}(M_s^{\perp})_{2:}\|^2_{L^2\Lambda^{k} \otimes V_k}.\end{aligned}$$

If $i = 2$ and $j = 1$ we find

$$\mathcal{I}_{21} = \left\langle B \otimes T(M_s^{\otimes 2})_{1:}, \gamma\pi^{\circ} \otimes P(M_s^{\otimes 2})_{1:} - \mathrm{d}\pi^{\circ} \otimes P(M_s^{\otimes 2})_{2:}\right\rangle. \qquad (2.66)$$

Since $\langle B \otimes T(M_s^{\otimes 2})_{1:}, \mathrm{d}\pi^{\circ} \otimes P(M_s^{\otimes 2})_{2:}\rangle = 0$, and $\left\langle B \otimes T(M_s^{\perp})_{1:}, \mathrm{Id} \otimes P(M_s^{\circ})_{1:}\right\rangle = 0$, we have:

$$\begin{aligned}\mathcal{I}_{21} &= \gamma \left\langle B \otimes T(\mathrm{d} \otimes \mathrm{Id}(M_s^{\circ})_{1:}), \mathrm{Id} \otimes P(M_s^{\circ})_{1:}\right\rangle\\ &= \gamma \left\langle \mathrm{d} \otimes T(M_s^{\circ})_{1:}, \mathrm{d} \otimes P(M_s^{\circ})_{1:}\right\rangle\\ &\geq \gamma C_1 \|\mathrm{d} \otimes \mathrm{Id}(M_s^{\circ})_{1:}\|^2_{L^2\Lambda^{k} \otimes V_k}.\end{aligned}$$

If $i = j = 2$

$$
\begin{aligned}
\mathcal{I}_{22} &= -\alpha \left\langle \mathrm{Id} \otimes T(M_s^{\otimes 2})_{2:}, \gamma\pi^\circ \otimes P(M_s^{\otimes 2})_{1:} - \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle \\
&= \alpha \left\langle \mathrm{Id} \otimes T(M_s^{\otimes 2})_{2:}, \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle \qquad (2.67)\\
&\quad - \alpha \left\langle \mathrm{Id} \otimes T(M_s^{\otimes 2})_{2:}, \gamma\pi^\circ \otimes P(M_s^{\otimes 2})_{1:} \right\rangle . \qquad (2.68)
\end{aligned}
$$

Since $\left\langle \mathrm{Id} \otimes T(M_s^{\perp})_{2:}, \mathrm{d}\pi^\circ \otimes P(M_s^{\otimes 2})_{2:} \right\rangle = 0$, we find

$$
\begin{aligned}
(2.67) &= \alpha \left\langle \mathrm{d} \otimes T(M_s^\circ)_{2:}, \mathrm{d} \otimes P(M_s^\circ)_{2:} \right\rangle \\
&\geq \alpha C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{2:} \|^2_{L^2\Lambda^{k-1} \otimes V_k}.
\end{aligned}
$$

Moreover, since $\langle \mathrm{Id} \otimes T(\mathrm{d}\pi^\circ \otimes \mathrm{Id}(M_s^{\otimes 2})_{2:}), \pi^\circ \otimes P(M_s^{\otimes 2})_{1:} \rangle = 0$, we find

$$
\begin{aligned}
(2.68) &= -\alpha\gamma \left\langle \mathrm{Id} \otimes T(M_s^{\perp})_{2:}, \mathrm{Id} \otimes P(M_s^\circ)_{1:} \right\rangle \\
&\geq -\frac{\alpha}{2}\gamma^{1/2} \left(\| \mathrm{Id} \otimes T(M_s^{\perp}) \|^2_{L^2\Lambda^{k-1} \otimes V_k'} + \gamma \| \mathrm{Id} \otimes P(M_s^\circ)_{1:} \|^2_{L^2\Lambda^{k-1} \otimes V_k} \right) \\
&\geq -\frac{\alpha}{2}\gamma^{1/2} \Big(C_{P,1}^2 \|T\|^2_{\mathcal{L}(V_k, V_k')} \| \mathrm{d} \otimes \mathrm{Id}(M_s^{\perp})_{2:} \|^2_{L^2\Lambda^{k} \otimes V_k} \\
&\quad + \gamma C_{P,1}^2 \|P\|^2_{\mathcal{L}(V_k, V_k)} \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:} \|^2_{L^2\Lambda^{k} \otimes V_k} \Big) ,
\end{aligned}
$$

where we used Proposition 2.4.8 and Lemma 2.4.14. Using the lower bounds on $\mathcal{I}_{11}$, $\mathcal{I}_{12}$, $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$, we can now conclude that:

$$
\begin{aligned}
(2.62) &\geq C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^{\perp})_{1:} \|^2_{L^2\Lambda^{k+1} \otimes V_k} \\
&\quad + \left(C_1 - \frac{\alpha}{2}\gamma^{1/2} C_{P,1}^2 \|T\|^2_{\mathcal{L}(V_k, V_k')} \right) \| \mathrm{d} \otimes \mathrm{Id}(M_s^{\perp})_{2:} \|^2_{L^2\Lambda^{k} \otimes V_k} \\
&\quad + \gamma \left(C_1 - \frac{\alpha}{2}\gamma^{1/2} C_{P,1}^2 \|P\|^2_{\mathcal{L}(V_k, V_k)} \right) \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{1:} \|^2_{L^2\Lambda^{k} \otimes V_k} \\
&\quad + \alpha C_1 \| \mathrm{d} \otimes \mathrm{Id}(M_s^\circ)_{2:} \|^2_{L^2\Lambda^{k-1} \otimes V_k}.
\end{aligned}
$$

Hence, if we choose γ sufficiently small, condition (2.52) is satisfied for $m = 2$. Now suppose that the problem for the $(m-1)$-th moment is well-posed, and in particular that the inf-sup condition is verified with the test function $M_t^{\otimes(m-1)} = P^{\otimes(m-1)} M_s^{\otimes(m-1)}$:

$$
\left\langle T^{\otimes(m-1)} M_s^{\otimes(m-1)}, P^{\otimes(m-1)} M_s^{\otimes(m-1)} \right\rangle \geq C_{m-1} \| M_s^{\otimes(m-1)} \|^2_{V_k^{\otimes(m-1)}}, \qquad (2.69)
$$

where $C_{m-1} = C_{m-1}(C_{P,1}, \alpha, \|T\|_{\mathcal{L}(V_k, V_k')}, \|P\|_{\mathcal{L}(V_k, V_k)}) > 0$. We want to prove (2.52). As before, we fix $M_s^{\otimes m} = \begin{bmatrix} (M_s^{\otimes m})_{1:} \\ (M_s^{\otimes m})_{2:} \end{bmatrix}$ where $(M_s^{\otimes m})_{1:}$ ($(M_s^{\otimes m})_{2:}$ respectively) means that in the tensor of order m, $M_s^{\otimes m} = (M_s^{\otimes m})_{i_1 \ldots i_m = 1,2}$, we fix $i_1 = 1$ ($i_1 = 2$ respectively) and let $i_2, \ldots, i_m$ vary. Using (2.54) and (2.57) we decompose

$$
M_s^{\otimes m} = \begin{bmatrix} (M_s^{\perp})_{1:} + \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)}(M_s^\circ)_{1:} \\ (M_s^{\perp})_{2:} + \mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)}(M_s^\circ)_{2:} \end{bmatrix} \in \begin{bmatrix} U_k^m \\ U_{k-1}^m \end{bmatrix},
$$

where now

$$
\begin{aligned}
(M_s^{\perp})_{1:} &= \pi^\perp \otimes \mathrm{Id}^{\otimes(m-1)}(M_s^{\otimes m})_{1:} \in \mathfrak{B}_k^{m,\perp} \\
(M_s^{\perp})_{2:} &= \pi^\perp \otimes \mathrm{Id}^{\otimes(m-1)}(M_s^{\otimes m})_{1:} \in \mathfrak{B}_{k-1}^{m,\perp} \\
(M_s^{\circ})_{1:} &= \pi^\circ \otimes \mathrm{Id}^{\otimes(m-1)}(M_s^{\otimes m})_{1:} \in \mathfrak{B}_{k-1}^{m,\perp} \\
(M_s^{\circ})_{2:} &= \pi^\circ \otimes \mathrm{Id}^{\otimes(m-1)}(M_s^{\otimes m})_{1:} \in \mathfrak{B}_{k-2}^{m,\perp}.
\end{aligned}
$$

We choose $M_t^{\otimes m} = P^{\otimes m} M_s^{\otimes m}$, so that:

$$\begin{aligned}\left\langle T^{\otimes m} M_s^{\otimes m}, M_t^{\otimes m}\right\rangle &= \left\langle T^{\otimes m} M_s^{\otimes m}, P^{\otimes m} M_s^{\otimes m}\right\rangle \\ &= \sum_{i,j=1}^{2} \left\langle T_{i,j} \otimes T^{m-1} (M_s^{\otimes m})_{j:}, (P^{\otimes m} M_s^{\otimes m})_{i:}\right\rangle . \end{aligned} \tag{2.70}$$

Let $\mathcal{J}_{ij} = \langle T_{i,j} \otimes T^{m-1} (M_s^{\otimes m})_{j:}, (P^{\otimes m} M_s^{\otimes m})_{i:}\rangle$. We follow a completely similar reasoning as before, and we apply (2.69). If $i = j = 1$,

$$\begin{aligned}\mathcal{J}_{11} &= \left\langle A \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{1:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{1:}\right\rangle \\ &\geq C_{m-1} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\perp})_{1:}\|^2_{L^2\Lambda^{k+1}\otimes V_k^{\otimes(m-1)}} .\end{aligned}$$

If $i = 1$ and $j = 2$,

$$\begin{aligned}\mathcal{J}_{12} &= \left\langle B^* \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{2:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{1:}\right\rangle \\ &\geq C_{m-1} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\perp})_{2:}\|^2_{L^2\Lambda^{k}\otimes V_k^{\otimes(m-1)}} .\end{aligned}$$

If $i = 2$ and $j = 1$,

$$\begin{aligned}\mathcal{J}_{21} &= \left\langle B \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{1:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{2:}\right\rangle \\ &\geq \gamma C_{m-1} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{1:}\|^2_{L^2\Lambda^{k}\otimes V_k^{\otimes(m-1)}} .\end{aligned}$$

If $i = j = 2$,

$$\begin{aligned}\mathcal{J}_{22} &= -\alpha \left\langle \mathrm{Id} \otimes T^{\otimes(m-1)} (M_s^{\otimes m})_{2:}, (P \otimes P^{\otimes(m-1)} M_s^{\otimes m})_{2:}\right\rangle \\ &\geq \alpha C_{m-1} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{2:}\|^2_{L^2\Lambda^{k-1}\otimes V_k^{\otimes(m-1)}} + \\ &\quad - \frac{\alpha}{2}\gamma^{1/2} \Big(C_{P,1}^2 \|T\|^{2(m-1)}_{\mathcal{L}(V_k,V_k')} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\perp})_{2:}\|^2_{L^2\Lambda^{k}\otimes V_k^{\otimes(m-1)}} + \\ &\quad + \gamma C_{P,1}^2 \|P\|^{2(m-1)}_{\mathcal{L}(V_k,V_k)} \|\mathrm{d} \otimes \mathrm{Id}^{\otimes(m-1)} (M_s^{\circ})_{1:}\|^2_{L^2\Lambda^{k}\otimes V_k^{\otimes(m-1)}} \Big) .\end{aligned}$$

Hence, if we choose γ sufficiently small, condition (2.52) is satisfied. Relation (2.53) follows from the orthogonal decomposition (2.57) and the tensorial Poincaré inequality in Lemma 2.4.14. □

Another way to express the result given in Theorem 2.4.13 is the following: $\forall M_s^{\otimes m}$ it holds

$$\left\langle T^{\otimes m} M_s^{\otimes m}, P^{\otimes m} M_s^{\otimes m}\right\rangle_{(V_k')^{\otimes m}, V_k^{\otimes m}} \geq C_m \left\| M_s^{\otimes m}\right\|^2_{V_k^{\otimes m}} .$$

As a simple consequence of Proposition 2.4.8 we have also the bound on $P^{\otimes m}$.

Remark 2.4.15. *We underline that the operator P is not the classical one presented in [6] to prove the well-posedness of the deterministic Hodge-Laplace problem. Indeed it is the minimal one such that the inf-sup condition for $\langle T^{\otimes m}\cdot, \cdot\rangle : V_k^{\otimes m} \times V_k^{\otimes m} \to \mathbb{R}$ (for every finite $m \geq 1$) is satisfied. With the classical operator, the inf-sup condition for $m \geq 2$ is not automatically satisfied.*

2.5 Some three-dimensional problems important in applications

In Section 2.2.3 we have reinterpreted the deterministic Hodge-Laplace problem in $n = 3$ dimensions in terms of PDEs. Here we translate in terms of partial differential equations the stochastic Hodge-Laplace problem. In particular, we focus on the two problems obtained for $k = 1$ and $k = 3$: the stochastic magnetostatic/electrostatic equations and the stochastic Darcy equations, and we explicitly write the systems solved by the mean and the two-points correlation of the unique stochastic solution of the stochastic problem.

2.5.1 The stochastic magnetostatic/electrostatic equations

Take $k = 1$ and $\alpha = 0$. Let $f_1 \in L^m(\Omega; L^2\Lambda^1(D))$, $f_2 \in L^m(\Omega; L^2\Lambda^0(D))$ be stochastic functions, $m \geq 1$ integer, representing an uncertain current and an uncertain charge respectively. The stochastic magnetostatic/electrostatic problem is the stochastic counterpart of problem (2.36). Thanks to Theorem 2.3.6, the stochastic magnetostatic/electrostatic problem admits a unique stochastic solution that depends continuously on the data. If $m \geq 1$, the first statistical moment $\mathcal{M}^1 \begin{bmatrix} u \\ p \end{bmatrix} = \mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (2.45)): find $E_s = \begin{bmatrix} E_{s,1} \\ E_{s,2} \end{bmatrix} \in V_1$ such that

$$\begin{cases} (\mathrm{curl} E_{s,1}, \mathrm{curl} v) + (\nabla E_{s,2}, v) = (\mathbb{E}[f_1], v) \\ (E_{s,1}, \nabla q) = (\mathbb{E}[f_2], q). \end{cases} \qquad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_1, \tag{2.71}$$

where the parenthesis in (2.71) mean the L^2-inner product. In the case $m \geq 2$, the second statistical moment $\mathcal{M}^2 \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (2.50) with $m = 2$): find

$$M_s^{\otimes 2} \in V_1 \otimes V_1 = \begin{bmatrix} H_{\Gamma_D}(\mathrm{curl}, D) \otimes H_{\Gamma_D}(\mathrm{curl}, D) & H_{\Gamma_D}(\mathrm{curl}, D) \otimes H^1_{\Gamma_D}(D) \\ H^1_{\Gamma_D}(D) \otimes H_{\Gamma_D}(\mathrm{curl}, D) & H^1_{\Gamma_D}(D) \otimes H^1_{\Gamma_D}(D) \end{bmatrix}$$

such that

$$\begin{cases} \big(\mathrm{curl} \otimes \mathrm{curl}(M_s^{\otimes 2})_{11}, \mathrm{curl} \otimes \mathrm{curl}(M_t^{\otimes 2})_{11}\big) + \big(\mathrm{curl} \otimes \nabla(M_s^{\otimes 2})_{12}, \mathrm{curl} \otimes \mathrm{Id}(M_t^{\otimes 2})_{11}\big) \\ \quad + \big(\nabla \otimes \mathrm{curl}(M_s^{\otimes 2})_{21}, \mathrm{Id} \otimes \mathrm{curl}(M_t^{\otimes 2})_{11}\big) + \big(\nabla \otimes \nabla(M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{11}\big) \\ \quad = \big(\mathcal{M}^2[f_1], (M_t^{\otimes 2})_{11}\big) \\ \\ -\big(\mathrm{curl} \otimes \mathrm{Id}(M_s^{\otimes 2})_{11}, \mathrm{curl} \otimes \nabla(M_t^{\otimes 2})_{12}\big) - \big(\nabla \otimes \mathrm{Id}(M_s^{\otimes 2})_{12}, \mathrm{Id} \otimes \nabla(M_t^{\otimes 2})_{12}\big) \\ \quad = \big(\mathbb{E}[f_1 f_2], (M_t^{\otimes 2})_{12}\big) \\ \\ -\big(\mathrm{Id} \otimes \mathrm{curl}(M_s^{\otimes 2})_{12}, \nabla \otimes \mathrm{curl}(M_t^{\otimes 2})_{21}\big) - \big(\mathrm{Id} \otimes \nabla M_s^{\otimes 2})_{21}, \nabla \otimes \mathrm{Id}(M_t^{\otimes 2})_{21}\big) \\ \quad = \big(\mathbb{E}[f_2 f_1], (M_t^{\otimes 2})_{21}\big) \\ \\ \big((M_s^{\otimes 2})_{11}, \nabla \otimes \nabla(M_t^{\otimes 2})_{22}\big) = \big(\mathcal{M}^2[f_2], (M_t^{\otimes 2})_{22}\big) \end{cases} \tag{2.72}$$

$\forall M_t^{\otimes 2} \in V_1 \otimes V_1$, where the parenthesis in (2.72) have to be intended as inner product in $(L^2(D))^3 \otimes (L^2(D))^3$.

2.5.2 The stochastic Darcy problem

Let $k = 3$, $f_2 \equiv 0$ and $f_1 \in L^m(\Omega; L^2\Lambda^3(D))$, $m \geq 1$ integer, representing an uncertain source in porous media flow. The stochastic Darcy problem is the stochastic counterpart of problem (2.39). Thanks to Theorem 2.3.6, the stochastic Darcy problem admits a unique stochastic solution that depends continuously on the data. If $m \geq 1$, the first statistical moment $\mathcal{M}^1 \begin{bmatrix} u \\ p \end{bmatrix} = \mathbb{E} \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (2.45)): find $E_s = \begin{bmatrix} E_{s,1} \\ E_{s,2} \end{bmatrix} \in V_3$ such that

$$\begin{cases} (\mathrm{div} E_{s,2}, v) = (\mathbb{E}[f_1], v) \\ (E_{s,1}, \mathrm{div} q) - \alpha (E_{s,2}, q) = 0 \end{cases} \qquad \forall \begin{bmatrix} v \\ q \end{bmatrix} \in V_3. \tag{2.73}$$

where the parenthesis in (2.73) mean the L^2-inner product. In the case $m \geq 2$, the second statistical moment $\mathcal{M}^2 \begin{bmatrix} u \\ p \end{bmatrix}$ is well-defined, and is the unique solution of (see (2.50) with $m = 2$): find

$$M_s^{\otimes 2} \in V_3 \otimes V_3 = \begin{bmatrix} L^2(D) \otimes L^2(D) & L^2(D) \otimes H_{\Gamma_D}(\mathrm{div}, D) \\ H_{\Gamma_D}(\mathrm{div}, D) \otimes L^2(D) & H_{\Gamma_D}(\mathrm{div}; D) \otimes H_{\Gamma_D}(\mathrm{div}; D) \end{bmatrix}$$

such that

$$\begin{cases} (\mathrm{div} \otimes \mathrm{div}(M_s^{\otimes 2})_{22}, (M_t)_{11}) = (\mathcal{M}^2[f_1], (M_t)_{11}) \\[2ex] \left(\mathrm{div} \otimes \mathrm{Id}(M_s^{\otimes 2})_{21}, \mathrm{Id} \otimes \mathrm{div}(M_t^{\otimes 2})_{12}\right) - \alpha \left(\mathrm{div} \otimes \mathrm{Id}(M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{12}\right) = 0 \\[2ex] \left(\mathrm{Id} \otimes \mathrm{div}(M_s^{\otimes 2})_{12}, \mathrm{div} \otimes \mathrm{Id}(M_t^{\otimes 2})_{21}\right) - \alpha \left(\mathrm{Id} \otimes \mathrm{div}(M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{21}\right) = 0 \\[2ex] \left((M_s^{\otimes 2})_{11}, \mathrm{div} \otimes \mathrm{div}(M_t^{\otimes 2})_{22}\right) - \alpha \left((M_s^{\otimes 2})_{12}, \mathrm{div} \otimes \mathrm{Id}(M_t^{\otimes 2})_{22}\right) \\ \quad -\alpha \left((M_s^{\otimes 2})_{21}, \mathrm{Id} \otimes \mathrm{div}(M_t^{\otimes 2})_{22}\right) + \alpha^2 \left((M_s^{\otimes 2})_{22}, (M_t^{\otimes 2})_{22}\right) = 0 \end{cases} \tag{2.74}$$

$\forall M_t^{\otimes 2} \in V_3 \otimes V_3$, where the parenthesis in (2.72) have to be intended as inner product in $L^2(D) \otimes L^2(D)$.

2.6 Finite element discretization of the moment equations

In this section we aim at deriving a stable discretization for the moment equations, i.e. the deterministic problems solved by the statistics of the unique stochastic solution $\begin{bmatrix} u \\ p \end{bmatrix}$. First we recall the main concepts concerning the finite element differential forms and the existence of a stable finite element discretization for the mean problem

$k = 0$	$\mathcal{P}_r^-\Lambda^0(\mathcal{T}_h)$	Lagrangian elements of degree $\leq r$
$k = 1$	$\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$	Nédélec 1-nd kind $H(\text{curl})$ elements of order $r-1$
$k = 2$	$\mathcal{P}_r^-\Lambda^2(\mathcal{T}_h)$	Nédélec 1-nd kind $H(\text{div})$ elements of order $r-1$
$k = 3$	$\mathcal{P}_r^-\Lambda^3(\mathcal{T}_h)$	Discontinuous elements of degree $\leq r-1$

Table 2.2: *Proxy fields correspondences between finite element differential forms $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ and the classical finite element spaces for $n = 3$.*

(2.45). Then we construct both full and sparse tensor product finite element discretizations for the m-th problem, with $m \geq 2$ integer, we prove their stability and provide optimal order of convergence estimates.

2.6.1 Finite element differential forms

Following [6], throughout this section we assume that the domain $D \subset \mathbb{R}^n$ is a polyhedral domain in $\mathbb{R}^n$ partitioned into a finite set of n-simplices. These simplices are such that their union is the closure of D and the intersection of any two of them, if non-empty, is a common sub simplex. We denote the partition with $\mathcal{T}_h$ and the discretization parameter with h. To discretize the moment equations we use the finite element differential forms

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) = \left\{ v \in H\Lambda^k(D) |\, v|_T \in \mathcal{P}_r^-\Lambda^k(T)\; \forall\, T \in \mathcal{T}_h \right\}, \tag{2.75}$$

where the space $\mathcal{P}_r^-\Lambda^k(T)$ and the de Rham subcomplex

$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}_h) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \mathcal{P}_r^-\Lambda^n(\mathcal{T}_h) \to 0$$

are treated in [6, 59]. Since we are particularly interested in the $n = 3$ case, we resume in Table 2.2 the correspondences between the finite element differential forms (2.75) and the classical finite element spaces of scalar and vector functions. The spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ are not the only possible choice. Indeed, in [6, 7, 30, 59] the authors present other finite element differential forms to discretize the deterministic Hodge Laplacian.

In [7] the authors propose the construction of a projector

$$\Pi_{k,h} : H\Lambda^k(D) \to \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$$

which is a cochain map, that is it commutes with the exterior derivative, and such that the following approximation property holds:

$$\|v - \Pi_{k,h}v\|_{L^2\Lambda^k} \leq Ch^s \|v\|_{H^s\Lambda^k}, \quad \forall\, v \in H^s\Lambda^k(D),\; 0 \leq s \leq r, \tag{2.76}$$

where $H^s\Lambda^k(D)$ is the space of differential k-forms with square integrable partial derivatives of order at most s, and C is independent of h. Note that the inequality (2.76) for $s = 0$ implies the stability of the projector in L^2. Moreover, from (2.76) it follows the boundedness of the projector $\Pi_{k,h}$ in the $H\Lambda^k(D)$-norm. Since we are dealing with Dirichlet boundary conditions on Γ_D, we need the existence of cochain projectors which also respect the boundary conditions. To this aim, we make the following assumption:

Assumption A3. *There exists a bounded cochain projector, that by abuse of notation we denote still by* $\Pi_{k,h}$*,*

$$\Pi_{k,h} : H_{\Gamma_D}\Lambda^k(D) \to \mathcal{P}^-_{r,\Gamma_D}\Lambda^k(\mathcal{T}_h) := \mathcal{P}^-_r\Lambda^k(\mathcal{T}_h) \cap H_{\Gamma_D}\Lambda^k(D), \tag{2.77}$$

such that (2.76) *is satisfied for every* $v \in H^s\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D)$, $0 \leq s \leq r$.

Assumption A3 is satisfied in the two and three dimensional case: see [90]. The n dimensional case is still a topic of current research, whereas if natural boundary conditions are imposed on ∂D, the existence of such an operator is proved in [6], and if essential boundary conditions are imposed on ∂D, the existence of such an operator is proved in [31].

2.6.2 Discrete mean problem

The problem solved by the mean of the unique stochastic solution of the stochastic Hodge Laplacian turns out to be the deterministic Hodge Laplacian. In [6] the authors study the finite element formulation of the deterministic Hodge Laplacian with natural boundary conditions on ∂D ($\Gamma_D = \emptyset$). In [7] all the results obtained in [6] for $\Gamma_D = \emptyset$ are extended to include the case of essential boundary conditions on ∂D ($\Gamma_N = \emptyset$). Under Assumption A3, all the results in [6, 7] apply to the general case $\Gamma_D,\ \Gamma_N \neq \emptyset$.

Let $\left(\mathcal{P}^-_{r,\Gamma_D}\Lambda^k(\mathcal{T}_h), \mathrm{d}\right)$ be the finite element de Rham subcomplex, h the discretization parameter, and $V_{k,h} = \begin{bmatrix} \mathcal{P}^-_{r,\Gamma_D}\Lambda^k(\mathcal{T}_h) \\ \mathcal{P}^-_{r,\Gamma_D}\Lambda^{k-1}(\mathcal{T}_h) \end{bmatrix}$. The finite element formulation of problem (2.45) is:

Mean Problem - FE Formulation

$$\boxed{\begin{array}{c} \textit{Given} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^1(\Omega; V_k'),\ \textit{find } E_{s,h} \in V_{k,h} \textit{ s.t.} \\ \\ T(E_{s,h}) = \mathbb{E}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \textit{ in } V_{k,h}'. \end{array}} \tag{2.78}$$

In [6] the authors show the stability of (2.78) by proving the inf-sup condition for the bounded bilinear and symmetric form $\langle T\cdot, \cdot\rangle$ restricted to the finite element spaces. Moreover, using a quasi-optimal error estimate and the interpolation property (2.76), the authors deduce the following order of convergence estimate:

$$\left\| \mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix} - E_{s,h} \right\|_{V_k} = \mathcal{O}(h^r) \tag{2.79}$$

for

$$\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix} \in \begin{bmatrix} H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^r\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix}$$

such that

$$\mathrm{d}\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix} \in \begin{bmatrix} H^r\Lambda^{k+1}(D) \cap H_{\Gamma_D}\Lambda^{k+1}(D) \\ H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \end{bmatrix},$$

where $\mathbb{E}\begin{bmatrix} u \\ p \end{bmatrix}$ and $E_{s,h}$ are the unique solutions of problems (2.45) and (2.78) respectively.

2.6.3 Discrete m-th moment problem: full tensor product approximation

The full tensor product finite element formulation (FTP-FE) of problem (2.50) is:

m-Points Correlation Problem (FTP-FE):

$$\boxed{\begin{array}{c} \textit{Given } m \geq 2 \textit{ integer and } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m\left(\Omega; V_k'\right), \textit{ find } M_{s,h}^{\otimes m} \in V_{k,h}^{\otimes m} \textit{ s.t.} \\ \\ T^{\otimes m} M_{s,h}^{\otimes m} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \textit{ in } (V_{k,h}')^{\otimes m} \end{array}} \tag{2.80}$$

Theorem 2.4.11 applies to problem (2.80), as a consequence of a tensor product structure (see Remark 2.4.12). We conclude therefore the stability of the full tensor product finite element discretization $V_{k,h}^{\otimes m}$.

Let $M_s^{\otimes m} = \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix}$ be the unique solution of problem (2.50) and $M_{s,h}^{\otimes m}$ be the unique solution of problem (2.80). Exploiting the Galerkin orthogonality and the stability of the discretization, we can obtain the following quasi-optimal convergence estimate:

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,h}^{\otimes m} \right\|_{V_k^{\otimes m}} \leq C \inf_{M_h^{\otimes m} \in V_{k,h}^{\otimes m}} \left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_h^{\otimes m} \right\|_{V_k^{\otimes m}}. \tag{2.81}$$

To study the approximation properties of the space $V_{k,h}^{\otimes m}$ we construct the tensorial projection operator $\Pi_{k,h}^{\otimes m}$ as follows.

Definition 2.6.16. *Let $\Pi_{k,h} : H_{\Gamma_D}\Lambda^k(D) \to \mathcal{P}_{r,\Gamma_D}^{-}\Lambda^k(\mathcal{T}_h)$ be a bounded cochain projector satisfying Assumption A3. Given $m \geq 2$ integer, we define*

$$\Pi_{k,h}^{\otimes m} := \underbrace{\Pi_{k,h} \otimes \ldots \otimes \Pi_{k,h}}_{m \ times} : \left(H_{\Gamma_D}\Lambda^k(D)\right)^{\otimes m} \to \left(\mathcal{P}_{r,\Gamma_D}^{-}\Lambda^k(\mathcal{T}_h)\right)^{\otimes m}. \tag{2.82}$$

Since $\Pi_{k,h}$ is bounded in $H\Lambda^k$-norm by a constant which we denote as C_π, $\Pi_{k,h}^{\otimes m}$ is bounded in $(H\Lambda^k)^{\otimes m}$-norm by $(C_\pi)^m$ (Proposition 2.4.8). Moreover, since it is the tensor product of cochain projectors, it is itself a cochain projector. We state the approximation properties of $\Pi_{k,h}^{\otimes m}$ in the following

Proposition 2.6.17. *The projector $\Pi_{k,h}^{\otimes m}$ introduced in Definition 2.6.16 is such that*

$$\|v - \Pi_{k,h}^{\otimes m} v\|_{(L^2\Lambda^k)^{\otimes m}} \leq C h^s \|v\|_{(H^s\Lambda^k)^{\otimes m}} \tag{2.83}$$

$v \in (H^s\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D))^{\otimes m}$, $0 \leq s \leq r$, where C is independent of h.

Proof. We already know the result for $m = 1$ (see (2.76)). Let $m = 2$. By triangle inequality,

$$
\begin{aligned}
&\|v - \Pi_{k,h}^{\otimes 2} v\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\quad \leq \|v - \Pi_{k,h} \otimes \mathrm{Id}\, v\|_{L^2\Lambda^k \otimes L^2\Lambda^k} + \|\Pi_{k,h} \otimes (\mathrm{Id} - \Pi_{k,h})\, v\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\quad \leq C h^s \|v\|_{H^s\Lambda^k \otimes L^2\Lambda^k} + C_\pi \|v - \mathrm{Id} \otimes \Pi_{k,h} v\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\quad \leq C h^s \|v\|_{H^s\Lambda^k \otimes L^2\Lambda^k} + C\, C_\pi h^s \|v\|_{L^2\Lambda^k \otimes H^s\Lambda^k} \\
&\quad \leq C h^s (1 + C_\pi) \|v\|_{H^s\Lambda^k \otimes H^s\Lambda^k},
\end{aligned}
$$

where we used (2.76). By induction on m, we conclude (2.83). □

From the approximation properties of the projector $\Pi_{k,h}^{\otimes m}$ (2.83), it follows

Theorem 2.6.18 (Order of convergence of the FTP-FE discretization)**.**

$$
\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,h}^{\otimes m} \right\|_{V_k^{\otimes m}} = \mathcal{O}(h^r), \tag{2.84}
$$

provided that

$$
\begin{bmatrix} u \\ p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^r\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix} \right)
$$
$$
\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^r\Lambda^{k+1}(D) \cap H_{\Gamma_D}\Lambda^{k+1}(D) \\ H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \end{bmatrix} \right).
$$

2.6.4 Discrete m-th moment problem: sparse tensor product approximation

In Section 2.6.3 we proved the stability of the full tensor product finite element discretization $V_{k,h}^{\otimes m} = \underbrace{V_{k,h} \otimes \ldots \otimes V_{k,h}}_{m \text{ times}}$. The main problem of this approach is that it is strongly affected by the curse of dimensionality. Indeed, if $\dim(V_{k,h}) = N_h$, the space $V_{k,h}^{\otimes m}$ has dimension $(N_h)^m$ which is impractical for m moderately large. A reduction in the dimensionality of the problem is possible if we consider a sparse tensor product finite element (STP-FE) approximation instead (see e.g. [21, 56, 91, 92, 97] and the references therein).

Let $\mathcal{T}_0$ be a regular mesh of the physical domain $D \subset \mathbb{R}^n$, and $\{\mathcal{T}_l\}_{l=0}^\infty$ be a sequence of partitions obtained by uniform mesh refinement, that is $h_l = h_{l-1}/2$, where h_l is the discretization parameter of $\mathcal{T}_l$. We have a sequence $\left\{\mathcal{P}_r^-\Lambda^k(\mathcal{T}_l)\right\}_{l=0}^\infty$ of finite dimensional subspaces of the space V_k, which are nested and dense in V_k. Let us define the orthogonal complement of $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_{l-1})$ in $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_l)$: $S_{k,l} = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_l) \setminus \mathcal{P}_r^-\Lambda^k(\mathcal{T}_{l-1})$, and set $Z_{k,l} = \begin{bmatrix} S_{k,l} \\ S_{k-1,l} \end{bmatrix}$. For every integer $m \geq 2$, we define the sparse tensor product finite element space of level $L > 0$, $V_{k,L}^{(m)}$, as:

$$
V_{k,L}^{(m)} := \bigoplus_{|\mathbf{l}| \leq L} \left(Z_{k,l_1} \otimes \ldots \otimes Z_{k,l_m} \right), \tag{2.85}
$$

where $\underline{l}$ is a multi index in $\mathbb{N}_0^m$ and $|\mathbf{l}|$ is its length $l_1 + \ldots + l_m$. At the numerical level it may not be needed to explicitly build a basis for $Z_{k,l}$. In [55] the authors propose to use a redundant basis for the space (2.85) and an algorithm to solve the m-th moment problem in the sparse tensor product framework.

The sparse tensor product finite element (STP-FE) approximation of problem (2.50) is:

m-Points Correlation Problem (STP-FE):

$$\boxed{\begin{array}{c} \text{Given } m \geq 2 \text{ integer and } \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \in L^m\left(\Omega; V_k'\right), \text{ find } M_{s,L}^{(m)} \in V_{k,L}^{(m)} \text{ s.t.} \\[2ex] T^{\otimes m} M_{s,L}^{(m)} = \mathcal{M}^m \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \text{in } \left(V_{k,L}^{(m)}\right)' \end{array}} \tag{2.86}$$

To prove the stability of (2.86) we can not use a tensor product argument as we did to prove the stability of the FTP-FE discretization. We need to explicitly prove the inf-sup condition for the tensor product operator $T^{\otimes m}$ restricted to the STP-FE space $V_{k,L}^{(m)}$. The proof of this sparse inf-sup condition rests on two key ingredients. On one hand, we make use of the continuous inf-sup operator $P^{\otimes m}$ introduced in the proof of Theorem 2.4.13. On the other hand, we use a reasoning similar to the one proposed in [20] which defines and uses the so-called *GAP property*: we seek for its analogue in the case of STP-FE space, which will be called in what follows *STP-GAP property*. The main ingredient of the STP-GAP property is the sparse tensorial projection operator $\Pi_{kL}^{(m)}$.

Definition 2.6.19. *Let $\Pi_{k,h} : H_{\Gamma_D}\Lambda^k(D) \to \mathcal{P}_{r,\Gamma_D}^{-}\Lambda^k(\mathcal{T}_h)$ be a bounded cochain projector satisfying Assumption A3. Given $m \geq 2$ integer, we define*

$$\Pi_{\mathbf{k},L}^{(m)} := \sum_{|\underline{l}| \leq L} \otimes \Delta_{k_j, l_j}, \tag{2.87}$$

where $\Delta_{k,l} := \Pi_{k,h_l} - \Pi_{k,h_{l-1}}$ and $\mathbf{k} = (k_1, \ldots, k_m)$.

It is easy to verify that $\Pi_{\mathbf{k},L}^{(m)}$ is a bounded cochain projector. Moreover,

$$\Pi_{\mathbf{k},L}^{(m)} \left(H_{\Gamma_D}\Lambda^k(D)\right)^{\otimes m} = \bigoplus_{|\underline{l}| \leq L} \left(S_{k,l_1} \otimes \ldots \otimes S_{k,l_m}\right), \quad \mathbf{k} = (k, \ldots, k).$$

To lighten the notation, in what follows $\Pi_{\mathbf{k}L}^{(m)}$ is denoted with $\Pi_L^{(m)}$ when no ambiguity arises.

We state the STP-GAP property for $m = 2$, but its generalization to $m \geq 2$ is straightforward.

Lemma 2.6.20 (STP-GAP property)**.** *For every $v_h \in \Pi_L^{(2)}\left(H_{\Gamma_D}\Lambda^k(D) \otimes H_{\Gamma_D}\Lambda^k(D)\right)$ there exist $0 < s \leq 1$ and positive constants $C^{(1)}$, $C^{(2)}$, $C^{(3)}$, $C^{(4)}$ independent of h_0*

such that

$$\left\|\mathrm{d}\pi^{\circ} \otimes \mathrm{d}\pi^{\circ} v_h - \Pi_L^{(2)}\left(\mathrm{d}\pi^{\circ} \otimes \mathrm{d}\pi^{\circ} v_h\right)\right\|_{H\Lambda^k \otimes H\Lambda^k} \leq C^{(1)} h_0^s \left\|v_h\right\|_{H\Lambda^k \otimes H\Lambda^k}, \tag{2.88}$$

$$\left\|\mathrm{d}\pi^{\circ} \otimes \pi^{\perp} v_h - \Pi_L^{(2)}\left(\mathrm{d}\pi^{\circ} \otimes \pi^{\perp} v_h\right)\right\|_{H\Lambda^k \otimes H\Lambda^k} \leq C^{(2)} h_0^s \left\|v_h\right\|_{H\Lambda^k \otimes H\Lambda}, \tag{2.89}$$

$$\left\|\pi^{\perp} \otimes \mathrm{d}\pi^{\circ} v_h - \Pi_L^{(2)}\left(\pi^{\perp} \otimes \mathrm{d}\pi^{\circ} v_h\right)\right\|_{H\Lambda^k \otimes H\Lambda^k} \leq C^{(3)} h_0^s \left\|v_h\right\|_{H\Lambda^k \otimes H\Lambda^k}, \tag{2.90}$$

$$\left\|\pi^{\perp} \otimes \pi^{\perp} v_h - \Pi_L^{(2)}\left(\pi^{\perp} \otimes \pi^{\perp} v_h\right)\right\|_{H\Lambda^k \otimes H\Lambda^k} \leq C^{(4)} h_0^s \left\|v_h\right\|_{H\Lambda^k \otimes H\Lambda^k}, \tag{2.91}$$

where $\pi^{\perp}$, π° *are defined in* (2.12) *and* (2.13)*, respectively. Note that* v_h *is uniquely expressed as* $v_h = \mathrm{d}\pi^{\circ} \otimes \mathrm{d}\pi^{\circ} v_h + \mathrm{d}\pi^{\circ} \otimes \pi^{\perp} v_h + \pi^{\perp} \otimes \mathrm{d}\pi^{\circ} v_h + \pi^{\perp} \otimes \pi^{\perp} v_h$ *thanks to the continuous Hodge decomposition* (2.57).

Proof. Let $v_h \in \Pi_L^{(2)}\left(H_{\Gamma_D}\Lambda^k(D) \otimes H_{\Gamma_D}\Lambda^k(D)\right)$, so that $\Pi_L^{(2)} v_h = v_h$. Since $\Pi_L^{(2)}$ is a cochain map, it holds:

$$\mathrm{d} \otimes \mathrm{d}\, v_h = \mathrm{d} \otimes \mathrm{d}\Pi_L^{(2)}\, v_h = \Pi_L^{(2)} \mathrm{d} \otimes \mathrm{d}\, v_h, \tag{2.92}$$

$$\mathrm{d} \otimes \mathrm{Id}\, v_h = \mathrm{d} \otimes \mathrm{Id}\Pi_L^{(2)}\, v_h = \Pi_L^{(2)} \mathrm{d} \otimes \mathrm{Id}\, v_h, \tag{2.93}$$

$$\mathrm{Id} \otimes \mathrm{d}\, v_h = \mathrm{Id} \otimes \mathrm{d}\Pi_L^{(2)}\, v_h = \Pi_L^{(2)} \mathrm{Id} \otimes \mathrm{d}\, v_h. \tag{2.94}$$

By definition of $\mathfrak{B}_k^{\perp}$ and Assumption A1, $\mathfrak{B}_k^{\perp} \subset H_{\Gamma_D}\Lambda^k \cap H^*_{\Gamma_N}\Lambda^k$, so that, thanks to Assumption A2,

$$\left\|\Delta_{k,l} w\right\|_{L^2\Lambda^k} \leq C\, h_{l-1}^s \left\|w\right\|_{H^s\Lambda^k} \leq \tilde{C}\, h_{l-1}^s \left\|w\right\|_{H\Lambda^k} \quad \forall w \in \mathfrak{B}_k^{\perp}. \tag{2.95}$$

- Let us start proving inequality (2.91). To this end, we need to bound four quantities:

$$\left\|\pi^{\perp} \otimes \pi^{\perp} v_h - \Pi_L^{(2)}\left(\pi^{\perp} \otimes \pi^{\perp} v_h\right)\right\|_{L^2\Lambda^k \otimes L^2\Lambda^k}, \tag{2.96}$$

$$\left\|\mathrm{d}\pi^{\perp} \otimes \pi^{\perp} v_h - \Pi_L^{(2)}\left(\mathrm{d}\pi^{\perp} \otimes \pi^{\perp} v_h\right)\right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k}, \tag{2.97}$$

$$\left\|\pi^{\perp} \otimes \mathrm{d}\pi^{\perp} v_h - \Pi_L^{(2)}\left(\pi^{\perp} \otimes \mathrm{d}\pi^{\perp} v_h\right)\right\|_{L^2\Lambda^k \otimes L^2\Lambda^{k+1}}, \tag{2.98}$$

$$\left\|\mathrm{d}\pi^{\perp} \otimes \mathrm{d}\pi^{\perp} v_h - \Pi_L^{(2)}\left(\mathrm{d}\pi^{\perp} \otimes \mathrm{d}\pi^{\perp} v_h\right)\right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^{k+1}}. \tag{2.99}$$

Using that $v_h = \sum_{L=0}^{+\infty} \sum_{|\mathbf{l}|=L} \Delta_{k,l_1} \otimes \Delta_{k,l_2} v_h$, the triangular inequality and (2.95),

$$\begin{aligned}
(2.96) &\leq \sum_{|\mathbf{l}|>L} \left\|\left(\Delta_{k,l_1} \otimes \Delta_{k,l_2}\right)\left(\pi^{\perp} \otimes \pi^{\perp}\right) v_h\right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&= \sum_{|\mathbf{l}|>L} \left\|\left(\Delta_{k,l_1}\pi^{\perp} \otimes \mathrm{Id}\right)\left(\mathrm{Id} \otimes \Delta_{k,l_2}\pi^{\perp}\right) v_h\right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\leq \sum_{|\mathbf{l}|>L} C\, h_{l_1-1}^s \left\|\left(\mathrm{Id} \otimes \Delta_{k,l_2}\pi^{\perp}\right) v_h\right\|_{H\Lambda^k \otimes L^2\Lambda^k} \\
&\leq \sum_{|\mathbf{l}|>L} C\, h_{l_1-1}^s \;\; h_{l_2-1}^s \left\|v_h\right\|_{H\Lambda^k \otimes H\Lambda^k}
\end{aligned} \tag{2.100}$$

where $C > 0$ is independent on h_l $\forall l$. Observing that

$$(\mathrm{d} \otimes \mathrm{Id}) \left(\pi^\perp \otimes \pi^\perp v_h\right) = \mathrm{d} \otimes \pi^\perp v_h \in \Pi_{k,L} \left(H_{\Gamma_D}\Lambda^k(D)\right) \otimes \mathfrak{B}_k^\perp$$

so that $(\Delta_{k+1,l_1} \otimes \mathrm{Id}) \left(\mathrm{d} \otimes \pi^\perp v_h\right) = 0$ if $l_1 > L$, we can bound (2.97):

$$\begin{aligned}
(2.97) &= \left\| \mathrm{d} \otimes \pi^\perp v_h - \Pi_L^{(2)} \left(\mathrm{d} \otimes \pi^\perp v_h\right) \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\leq \sum_{l_1=0}^{L} \sum_{l_2=L-l_1+1}^{+\infty} \left\| \left(\Delta_{k+1,l_1} \otimes \Delta_{k,l_2}\right) \left(\mathrm{d} \otimes \pi^\perp\right) v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} \\
&\leq \sum_{l_1=0}^{L} \sum_{l_2=L-l_1+1}^{+\infty} \left\| \Delta_{k+1,l_1} \right\|_{\mathcal{L}(L^2\Lambda^{k+1}, L^2\Lambda^{k+1})} \left\| \left(\mathrm{Id} \otimes \Delta_{k,l_2}\right) \left(\mathrm{d} \otimes \pi^\perp\right) v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} \\
&\leq C \sum_{l_1=0}^{L} \sum_{l_2=L-l_1+1}^{+\infty} h_{l_2-1}^s \left\| \mathrm{d} \otimes \mathrm{Id}\, v_h \right\|_{L^2\Lambda^{k+1} \otimes H\Lambda^k} \\
&\leq C(L+1) \sum_{l_2=1}^{+\infty} h_{l_2-1}^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq C h_0^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \qquad (2.101)
\end{aligned}$$

where we have used that $\|\Delta_{k+1,l_1}\|_{\mathcal{L}(L^2\Lambda^{k+1}, L^2\Lambda^{k+1})}$ is bounded by a constant independent of h_{l_1}. By symmetry, we can obtain that

$$(2.98) \leq C h_0^s \left\| v_h \right\|_{H\Lambda^k \otimes H\Lambda^k}. \qquad (2.102)$$

Finally, using (2.92), we have

$$(\mathrm{d} \otimes \mathrm{d}) \left(\pi^\perp \otimes \pi^\perp\right) v_h = \mathrm{d} \otimes \mathrm{d}\, v_h = \mathrm{d} \otimes \mathrm{d}\, \Pi_L^{(2)}\, v_h = \Pi_L^{(2)}\, (\mathrm{d} \otimes \mathrm{d}) \left(\pi^\perp \otimes \pi^\perp\right) v_h,$$

so that the quantity in (2.99) vanishes. Thus, putting together (2.100), (2.101), (2.102), we conclude (2.91).

- Let us prove inequality (2.90). We need to bound two quantities:

$$\left\| \pi^\perp \otimes \mathrm{d}\pi^\circ\, v_h - \Pi_L^{(2)} \left(\pi^\perp \otimes \mathrm{d}\pi^\circ v_h\right) \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k}, \qquad (2.103)$$

$$\left\| \mathrm{d}\pi^\perp \otimes \mathrm{d}\pi^\circ\, v_h - \Pi_L^{(2)} \left(\mathrm{d}\pi^\perp \otimes \mathrm{d}\pi^\circ v_h\right) \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k}. \qquad (2.104)$$

Since $\pi^\perp \otimes \mathrm{d}\pi^\circ\, v_h = \pi^\perp \otimes \mathrm{Id}\, v_h - \pi^\perp \otimes \pi^\perp\, v_h$ and $\pi^\perp \otimes \mathrm{Id}\, v_h \in \mathfrak{B}_k^\perp \otimes$

$\Pi_{k,L}(H_{\Gamma_D}\Lambda^k(D))$, and using (2.91),

$$
\begin{aligned}
(2.103) &\leq \left\| \pi^\perp \otimes \mathrm{Id}\ v_h - \Pi_L^{(2)}\ \pi^\perp \otimes \mathrm{Id}\ v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\quad + \left\| \pi^\perp \otimes \pi^\perp\ v_h - \Pi_L^{(2)}\ \pi^\perp \otimes \pi^\perp\ v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} \\
&\leq \sum_{l_2=0}^{L} \sum_{l_1=L+1-l_2}^{+\infty} \left\| (\Delta_{k,l_1} \otimes \Delta_{k,l_2}) \left(\pi^\perp \otimes \mathrm{Id}\right)\ v_h \right\|_{L^2\Lambda^k \otimes L^2\Lambda^k} + C\, h_0^s\ \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq \sum_{l_2=0}^{L} \sum_{l_1=L+1-l_2}^{+\infty} \|\Delta_{k,l_2}\|_{\mathcal{L}(L^2\Lambda^k, L^2\Lambda^k)}\ h_{l_1-1}^s\ \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} + C\, h_0^s\ \|v_h\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq C\, h_0^s\ \|v_h\|_{H\Lambda^k \otimes H\Lambda^k}\,.
\end{aligned} \tag{2.105}
$$

Moreover, using (2.91)

$$
\begin{aligned}
(2.104) &\leq \left\| \mathrm{d}\pi^\perp \otimes \mathrm{Id}\ v_h - \Pi_L^{(2)}\ \mathrm{d}\pi^\perp \otimes \mathrm{Id}\ v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} \\
&\quad + \left\| \mathrm{d}\pi^\perp \otimes \pi^\perp\ v_h - \Pi_L^{(2)}\ \mathrm{d}\pi^\perp \otimes \pi^\perp\ v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} \\
&\leq C\, h_0^s\ \|v_h\|_{H\Lambda^k \otimes H\Lambda^k}\,.
\end{aligned} \tag{2.106}
$$

In the last inequality we exploited (2.93), which implies that $\mathrm{d}\pi^\perp \otimes \mathrm{Id}\ v_h = \mathrm{d} \otimes \mathrm{Id}\ v_h = \mathrm{d} \otimes \mathrm{Id}\ \Pi_L^{(2)}\ v_h = \Pi_L^{(2)}\ \mathrm{d}\pi^\perp \otimes \mathrm{Id}\ v_h$, so that

$$
\left\| \mathrm{d}\pi^\perp \otimes \mathrm{Id}\ v_h - \Pi_L^{(2)}\ \mathrm{d}\pi^\perp \otimes \mathrm{Id}\ v_h \right\|_{L^2\Lambda^{k+1} \otimes L^2\Lambda^k} = 0.
$$

Using (2.105) and (2.106) we conclude (2.90).

- To show (2.89), we write v_h as $v_h = \mathrm{Id} \otimes \mathrm{d}\pi^\circ\ v_h + \mathrm{Id} \otimes \pi^\perp\ v_h$ and proceed as in the proof of (2.90).
- To show (2.88) we observe that

$$
\begin{aligned}
&\left\| \mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h - \Pi_L^{(2)}\left(\mathrm{d}\pi^\circ \otimes \mathrm{d}\pi^\circ v_h\right) \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&= \left\| \left(\mathrm{Id} \otimes \mathrm{Id} - \Pi_L^{(2)}\right)\left(\mathrm{Id} \otimes \mathrm{Id} - \mathrm{d}\pi^\circ \otimes \pi^\perp - \pi^\perp \otimes \mathrm{d}\pi^\circ - \pi^\perp \otimes \pi^\perp\right) v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\leq \left\| v_h - \Pi_L^{(2)}\ v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} + \left\| \mathrm{d}\pi^\circ \otimes \pi^\perp\ v_h - \Pi_L^{(2)} \mathrm{d}\pi^\circ \otimes \pi^\perp\ v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\quad + \left\| \pi^\perp \otimes \mathrm{d}\pi^\circ\ v_h - \Pi_L^{(2)} \pi^\perp \otimes \mathrm{d}\pi^\circ\ v_h \right\|_{H\Lambda^k \otimes H\Lambda^k} \\
&\quad + \left\| \pi^\perp \otimes \pi^\perp\ v_h - \Pi_L^{(2)} \pi^\perp \otimes \pi^\perp\ v_h \right\|_{H\Lambda^k \otimes H\Lambda^k}
\end{aligned}
$$

and we conclude (2.88) using that $v_h = \Pi_L^{(2)}\ v_h$, and (2.89), (2.90), (2.91).

□

We are now ready to prove the main result of this section. It deals with vector quantities in V_k. In this context, $\Pi_L^{(m)}$ denotes the projector from $V_k^{\otimes m}$ onto $V_{k,L}^{(m)}$.

Theorem 2.6.21 (Stability of the STP-FE discretization)**.** *For every $\alpha \geq 0$ there exists $\bar{h}_0 > 0$ such that for all $h_0 \leq \bar{h}_0$ problem* (2.86) *is a stable discretization for the m-th moment problem* (2.50)*. In particular, for every $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$, there exists a test function $M_{t,L}^{(m)} \in V_{k,L}^{(m)}$ and positive constants $C_{m,disc} = C_{m,disc}\left(C_m\right)$ (C_m is introduced in* (2.52)*), $C'_{m,disc} = C'_{m,disc}\left(\alpha, \|P\|_{\mathcal{L}(V_k,V_k)}, \left\|\Pi_L^{(m)}\right\|_{\mathcal{L}(V_k^{\otimes m},V_{k,L}^{(m)})}, \|T\|_{\mathcal{L}(V_k,V_k')}\right)$ s.t.*

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)} \right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} \geq C_{m,disc} \|M_{s,L}^{(m)}\|^2_{V_k^{\otimes m}}, \tag{2.107}$$

$$\|M_{t,L}^{(m)}\|_{V_k^{\otimes m}} \leq C'_{m,disc} \|M_{s,L}^{(m)}\|_{V_k^{\otimes m}}. \tag{2.108}$$

Proof. Suppose $\alpha > 0$ (the case $\alpha = 0$ is analogous). We fix $M_{s,L}^{(m)} \in V_{k,L}^{(m)}$ and look for a sparse test function $M_{t,L}^{(m)} \in V_{k,L}^{(m)}$ such that (2.107) and (2.108) are satisfied. We choose $M_{t,L}^{(m)} = \Pi_L^{(m)} P^{\otimes m} M_{s,L}^{(m)}$. Thanks to Proposition 2.4.8 and the boundness of the operators P and $\Pi_L^{(m)}$, we immediately conclude (2.108). In the proof of (2.107), we use brackets $\langle \cdot, \cdot \rangle$ without specifying the spaces taken into account, when no ambiguity arises.

$$\begin{aligned}
&\left\langle T^{\otimes m} M_{s,L}^{(m)}, M_{t,L}^{(m)} \right\rangle \\
&\quad = \left\langle T^{\otimes m} M_{s,L}^{(m)}, \Pi_L^{(m)} \; P^{\otimes m} \; M_{s,L}^{(m)} \right\rangle \\
&\quad = \left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)} \right\rangle - \left\langle T^{\otimes m} M_{s,L}^{(m)}, \left(\mathrm{Id}^{\otimes m} - \Pi_L^{(m)}\right) \; P^{\otimes m} \; M_{s,L}^{(m)} \right\rangle .
\end{aligned}$$

We observe that, thanks to the continuous inf-sup condition (2.52),

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, P^{\otimes m} M_{s,L}^{(m)} \right\rangle \geq C_m \; \left\| M_{s,L}^{(m)} \right\|^2_{V_k^{\otimes m}}, \tag{2.109}$$

and, from Lemma 2.6.20,

$$\begin{aligned}
&\left\langle T^{\otimes m} M_{s,L}^{(m)}, \left(\mathrm{Id}^{\otimes m} - \Pi_L^{(m)}\right) \; P^{\otimes m} \; M_{s,L}^{(m)} \right\rangle \\
&\quad \leq \|T\|^m_{\mathcal{L}(V_k,V_k')} \left\| M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} \left\| \left(\mathrm{Id}^{\otimes m} - \Pi_L^{(m)}\right) \; P^{\otimes m} \; M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} \\
&\quad \leq C \; h_0^s \; \|T\|^m_{\mathcal{L}(V_k,V_k')} \left\| M_{s,L}^{(m)} \right\|^2_{V_k^{\otimes m}} .
\end{aligned}$$

Therefore, for h_0 sufficiently small, (2.107) follows. □

Another way to express the result given in Theorem 2.6.21 is the following: $\forall M_{s,L}^{(m)}$ it holds

$$\left\langle T^{\otimes m} M_{s,L}^{(m)}, \Pi_L^{(m)} P^{\otimes m} M_{s,L}^{(m)} \right\rangle_{\left(V_{k,L}^{(m)}\right)', V_{k,L}^{(m)}} \geq C_{m,disc} \left\| M_{s,L}^{(m)} \right\|^2_{V_k^{\otimes m}} .$$

Remark 2.6.22. *Note that the choice of the set of multi-indexes $\mathcal{I} := \{\mathbf{l} \in \mathbb{N}^m \colon \; |\mathbf{l}| \leq L\}$ is not the only possible in* (2.85)*. Indeed, with the same technique showed in the proof* *of Theorem 2.6.21 it is possible to prove the stability of any*

$$V_{k,L}^{(m)} := \bigoplus_{\mathbf{l}\in\Lambda(L)} Z_{k,l_1} \otimes \cdots \otimes Z_{k,l_m},$$

provided that $V_{k,L}^{(m)}$ contains $V_{k,l_0}^{\otimes m}$ for a sufficiently small h_{l_0}. However, since the m-th moment problem has a tensor product structure, we believe that the set $\mathcal{I}$ is the recommended one.

Let $\mathcal{M}^m \left[\begin{array}{c} u \\ p \end{array}\right]$ be the unique solution of problem (2.50) and $M_{s,L}^{(m)}$ be the unique solution of problem (2.86). Exploiting the Galerkin orthogonality and the stability of the discretization, we can obtain the following quasi-optimal convergence estimate:

$$\left\|\mathcal{M}^m \left[\begin{array}{c} u \\ p \end{array}\right] - M_{s,L}^{(m)}\right\|_{V_k^{\otimes m}} \leq C \inf_{M_{t,L}^{(m)}\in V_{k,L}^{(m)}} \left\|\mathcal{M}^m \left[\begin{array}{c} u \\ p \end{array}\right] - M_{t,L}^{(m)}\right\|_{V_k^{\otimes m}}. \qquad (2.110)$$

To state the approximation properties of the sparse projector $\Pi_L^{(m)}$ and, as a consequence, of the sparse space $V_{k,L}^{(m)}$ we need the following technical lemma. See Appendix for the proof.

Lemma 2.6.23. *It holds:*

$$\sum_{|\mathbf{l}|>L} 2^{-\gamma|\mathbf{l}|} = \sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma - 1}\right)^{m-i} \left(\begin{array}{c} L+m \\ i \end{array}\right) 2^{-\gamma L} \leq \left(\frac{1}{1-2^{-\lambda\gamma}}\right)^m 2^{-L\gamma(1-\lambda)} \qquad (2.111)$$

for every real $\gamma > 0$ and integer $L > 0$, with $0 < \lambda < 1$.

Proposition 2.6.24. *The projector $\Pi_L^{(m)}$ introduced in Definition 2.6.19 is such that*

$$\|v - \Pi_L^{(m)} v\|_{(L^2\Lambda^k)^{\otimes m}} \leq C h_L^{s(1-\lambda)} \|v\|_{(H^s\Lambda^k)^{\otimes m}}, \qquad (2.112)$$

$0 < \lambda < 1$, for all $v \in (H^s_{\Gamma_D}\Lambda^k(D))^{\otimes m}$, $0 < s \leq r$, where $C = C(m, \lambda, s)$ is independent of h_L.

Proof. Following [21], we proceed in three steps. We start considering the approximation properties of $\Delta_{k,l}$. Using the triangular inequality and (2.76) we have:

$$\left\|\Delta_{k,l} \otimes \mathrm{Id}^{\otimes(m-1)} v\right\|_{(L^2\Lambda^k)^{\otimes m}} \leq C h_{l-1}^s \|v\|_{H^s\Lambda^k \otimes (L^2\Lambda^k)^{\otimes(m-1)}},$$

for every $0 < s \leq r$. Now, we consider the tensor product $\otimes_{j=1}^m \Delta_{k,l_j}$. By recursion,

$$\left\|\otimes_{j=1}^m \Delta_{k,l_j} v\right\|_{(L^2\Lambda^k)^{\otimes m}} \leq C h_{\mathbf{l}-1}^s \|v\|_{(H^s\Lambda^k)^{\otimes m}},$$

where $h^s_{\mathbf{l}-1} = h^s_{l_1-1} \dots h^s_{l_m-1}$. Finally, using (2.6.23):

$$\begin{aligned}
\left\| v - \Pi_L^{(m)} v \right\|_{(L^2\Lambda^k)^{\otimes m}} &= \left\| \sum_{|\mathbf{l}|>L} \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(L^2\Lambda^k)^{\otimes m}} \leq \sum_{|\mathbf{l}|>L} \left\| \otimes_{j=1}^m \Delta_{k,l_j} v \right\|_{(L^2\Lambda^k)^{\otimes m}} \\
&\leq \sum_{|\mathbf{l}|>L} C h^s_{\mathbf{l}-1} \|v\|_{(H^s\Lambda^k)^{\otimes m}} \\
&= C \|v\|_{(H^s\Lambda^k)^{\otimes m}} h_0^{sm} \sum_{|\mathbf{l}|>L} 2^{-s|\mathbf{l}-1|} \\
&= C \|v\|_{(H^s\Lambda^k)^{\otimes m}} h_0^{sm} 2^{sm} \sum_{|\mathbf{l}|>L} 2^{-s|\mathbf{l}|} \\
&\leq C \|v\|_{(H^s\Lambda^k)^{\otimes m}} h_0^{sm} 2^{sm} 2^{-Ls(1-\lambda)} \left(\frac{1}{1-2^{-s\lambda}} \right)^m \\
&= C \|v\|_{(H^s\Lambda^k)^{\otimes m}} \left(\frac{2^s h_0^s}{1-2^{-s\lambda}} \right)^m 2^{-Ls(1-\lambda)}
\end{aligned}$$

for every $0 < s \leq r$. □

It follows

Theorem 2.6.25 (Order of convergence of the STP-FE discretization)**.**

$$\left\| \mathcal{M}^m \begin{bmatrix} u \\ p \end{bmatrix} - M_{s,L}^{(m)} \right\|_{V_k^{\otimes m}} = \mathcal{O}(h_L^{r(1-\lambda)}), \tag{2.113}$$

$0 < \lambda < 1$, *provided that*

$$\begin{bmatrix} u \\ p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \\ H^r\Lambda^{k-1}(D) \cap H_{\Gamma_D}\Lambda^{k-1}(D) \end{bmatrix} \right)$$
$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}p \end{bmatrix} \in L^m \left(\Omega; \begin{bmatrix} H^r\Lambda^{k+1}(D) \cap H_{\Gamma_D}\Lambda^{k+1}(D) \\ H^r\Lambda^k(D) \cap H_{\Gamma_D}\Lambda^k(D) \end{bmatrix} \right).$$

The previous theorem states that the STP - FE approximation has almost the same rate of convergence as the FTP - FE. On the other hand, the great advantage of the sparse approximation with respect to the full one is represented by a drastic reduction of the dimensionality of the sparse finite element space.

2.7 Conclusions

The present work addresses the mixed formulation of the Hodge Laplacian defined on a n-dimensional domain $D \subseteq \mathbb{R}^n$, $(n \geq 1)$, with stochastic forcing terms. The well-posedness of this problem is equivalent to the inf-sup condition of a suitable bounded bilinear and symmetric form $\langle T\cdot, \cdot \rangle$ coming from the weak formulation of the mixed Hodge Laplacian.

We have studied the moment equations, i.e. the deterministic equations solved by the statistical moments of the unique stochastic solution. In particular, if T is the (deterministic) operator that defines the starting problem, we show that the m-th moment equation involves the tensor product operator $T^{\otimes m} := \underbrace{T \otimes \cdots \otimes T}_{m \text{ times}}$. The main achievement of the paper has been to characterize an operator P and its tensorial version $P^{\otimes m}$ that allows us to construct suitable test functions to prove the inf-sup condition for the tensor problem $\langle T^{\otimes m} \cdot, \cdot \rangle$ both at the continuous level and at the discrete level with full or sparse FE discretizations. By this tool we have been able to show that known stable FE approximations for the deterministic problem are also stable and optimally convergent for the tensorial problem both in the full and sparse versions.

Appendix

Proof of Lemma 2.6.23. To prove the equality, we observe that

$$\sum_{|\mathbf{l}|>L} 2^{-\gamma|\mathbf{l}|} = \sum_{j=L+1}^{\infty} \sum_{|l|=j} 2^{-\gamma j} = \sum_{j=L+1}^{\infty} \binom{j+m-1}{m-1} 2^{-\gamma j}.$$

It is sufficient to show that

$$\sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} 2^{-\gamma j} = \sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma - 1}\right)^{m-i} \binom{L+K}{i} 2^{-\gamma L}. \qquad (2.114)$$

for every integer K. We prove (2.114) by induction on m. If $m = 1$,

$$\sum_{j=L+1}^{\infty} \binom{j+K-1}{0} 2^{-\gamma j} = \frac{2^{-\gamma(L+1)}}{1-2^{-\gamma}} = \frac{1}{2^\gamma - 1} 2^{-\gamma L}.$$

Let us assume the result true for $m-1$. Then

$$\begin{aligned}
&\sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} 2^{-\gamma j} \\
&= \sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} \frac{2^\gamma - 1}{2^\gamma - 1} 2^{-\gamma j} \\
&= \frac{1}{2^\gamma - 1} \left(\sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} \left(2^{-\gamma(j-1)} - 2^{-\gamma j}\right) \right) \\
&= \frac{1}{2^\gamma - 1} \left(\sum_{i=L}^{\infty} \binom{i+K}{m-1} 2^{-\gamma i} - \sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} 2^{-\gamma j} \right) \\
&= \frac{1}{2^\gamma - 1} \left(\binom{L+K}{m-1} 2^{-\gamma L} + \sum_{j=L+1}^{\infty} \left(\binom{j+K}{m-1} - \binom{j+K-1}{m-1} \right) 2^{-\gamma j} \right) \\
&= \frac{1}{2^\gamma - 1} \left(\binom{L+K}{m-1} 2^{-\gamma L} + \sum_{j=L+1}^{\infty} \binom{j+K-1}{m-2} 2^{-\gamma j} \right).
\end{aligned}$$

Applying recursively the previous equality we get:

$$\sum_{j=L+1}^{\infty} \binom{j+K-1}{m-1} 2^{-\gamma j} = \sum_{i=0}^{m-1} \left(\frac{1}{2^\gamma - 1}\right)^{m-i} \binom{L+K}{i} 2^{-\gamma L}.$$

Let us now show the inequality. Let $0 < \lambda < 1$.

$$\begin{aligned}
\sum_{i=0}^{m-1}\left(\frac{1}{2^{\gamma}-1}\right)^{m-i}\binom{L+m}{i}2^{-\gamma L} &\leq \sum_{i=0}^{m-1}\left(\frac{1}{2^{\lambda\gamma}-1}\right)^{m-i}\binom{L+m}{i}2^{-\gamma L} \\
&= \frac{2^{-\gamma L}}{(2^{\lambda\gamma}-1)^m}\sum_{i=0}^{m-1}(2^{\lambda\gamma}-1)^i\binom{L+m}{i} \\
&\leq \frac{2^{-\gamma L}}{(2^{\lambda\gamma}-1)^m}\left(2^{\lambda\gamma}\right)^{L+m} \\
&= \left(\frac{1}{1-2^{-\lambda\gamma}}\right)^m 2^{-L\gamma(1-\lambda)}.
\end{aligned}$$

□

CHAPTER 3

Perturbation analysis for the Darcy problem with lognormal permeability tensor

This chapter is based on a paper in preparation: F. Bonizzoni, F. Nobile, *Perturbation analysis for the Darcy problem with lognormal permeability*. A short version can be found in F. Bonizzoni, F. Nobile, *Perturbation analysis for the stochastic Darcy problem*. Proceeding in the European Congress on Computational Methods in Applied Sciences and Engineering (ECCOMAS 2012).

3.1 Introduction

The situation we are interested in is the steady single-phase flow of a fluid in a randomly heterogeneous saturated porous medium. Here randomness typically arises in the forcing terms, as for instance pressure gradients, as well as in the permeability tensor, due to the impossibility of a full characterization of conductivity properties of subsurface media. See for example [14, 36, 51, 52, 93, 99, 100]. The case of stochastic forcing term has been treated in Chapter 2, whereas in this chapter we focus on the following linear elliptic SPDE

$$- \operatorname{div}_x(a(\omega, x)\nabla_x u(\omega, x)) = f(x), \tag{3.1}$$

where the forcing term is deterministic and the permeability tensor is modeled as a lognormal random field, i.e. $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$ with $Y(\omega, x)$ a Gaussian random field. We treat this problem for small randomness by a perturbation approach, expanding the solution in Taylor series. The goal of the work is to infer on the solution of the Darcy problem (3.1) using its Taylor polynomial.

The Taylor polynomial is directly computable if the permeability field is parametrized by a finite number of independent random variables, i.e.

$$a(\omega, x) = a(Y_1(\omega), \ldots, Y_N(\omega), x).$$

See for example [9], where the permeability tensor is described as a linear combination of bounded random variables. On the other hand, if the permeability field is an infinite-dimensional random field, it becomes necessary to derive the equations solved by the moments of the k-th derivative of the stochastic solution with respect to the Gaussian field Y. See [56], where the authors consider a domain with random boundary perturbations, [51,52,93,99] from the engineering literature, where a lognormal permeability field is considered, and [33], where the permeability field is a linear combination of countably many bounded random variables.

In the literature, when an infinite-dimensional random field is taken into account, the majority of the authors computes only a second order correction. The aim of the present work is both to understand if it is useful to compute higher order corrections and to determine the appropriate order of the Taylor polynomial to achieve a prescribed accuracy. This work can been seen on the one hand as an extension and theoretical analysis of [51,52,93,99], and, on the other hand, as an extension of [33] to the case of a lognormal random field. In particular, we predict the divergence of the Taylor series in the case of lognormal permeability.

The outline of the chapter is the following. In order to understand how the Taylor series of an illustrative analytic function $u : \mathbb{R} \to \mathbb{R}$ of a Gaussian random variable behaves, in Section 3.2 we present two examples. Section 3.3 introduces the problem at hand and states some results on the statistical moments of the extrema of a Gaussian random field. In Section 3.4 we expand the stochastic solution in Taylor series, provide bounds on the $L^1(\Omega; H^1(D))$ and $L^2(\Omega; H^1(D))$ norms of the Taylor polynomial and predict the divergence of the Taylor series. Moreover, we state the existence and provide a formula to compute the optimal order of the Taylor polynomial such that, adding new terms to the Taylor polynomial will deteriorate the accuracy instead of improving it. Section 3.5 is dedicated to the case of a finite-dimensional permeability field. Finally, in Section 3.6 we perform some numerical tests in a one-dimensional case which confirm the divergence of the Taylor series predicted in Section 3.4.

3.2 Taylor expansion: preliminary examples

Let us consider problem (3.1), where the permeability is modeled as a lognormal random field, i.e. $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, with $Y(\omega, x)$ a centered Gaussian random field. Thanks to the Doob-Dynkin Lemma (see e.g. [84]), $u(\omega, x) = u(Y(\omega, x), x)$. The aim of the chapter is to understand the approximation properties of the Taylor polynomial of u centered in $Y = 0$ with respect to the $L^1(\Omega; H^1(D))$ and $L^2(\Omega; H^1(D))$ norms.

We start with an illustrative section in which Y is a random variable and $u : \mathbb{R} \to \mathbb{R}$ is an analytic function of $y = Y(\omega)$ and is independent of the spatial variable x. The Taylor series of u centered in $y_0 = 0$ converges in $I \subseteq \mathbb{R}$. Considering a centered Gaussian measure on $\mathbb{R}$, if $I = \mathbb{R}$, we do expect that the expected value of the K-th order Taylor polynomial converges to the expected value of u as $K \to +\infty$. This fact is illustrated in the first example, which can be obtained from problem (3.1) with $a(y) = \mathrm{e}^{-y}$. The second example shows that this result doesn't hold anymore if $I \subset \mathbb{R}$.

3.2.1 Exponential function

Let us denote with $L^1_\rho(\mathbb{R})$ the space of integrable functions with respect to the centered Gaussian measure with standard deviation $\sigma > 0$, $\rho(y) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{y^2}{2\sigma^2}}$, that is, the space of functions v with finite expected value

$$\mathbb{E}[v] := \int_{-\infty}^{+\infty} v(y)\, d\rho(y) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} v(y)\, \mathrm{e}^{-\frac{y^2}{2\sigma^2}} dy < +\infty.$$

Let $u(y) = \mathrm{e}^y$. The K-th order Taylor polynomial of u centered in $y_0 = 0$ is:

$$T^K u(y) := \sum_{k=0}^{K} \frac{u^{(k)}(0)}{k!}\, y^k = \sum_{k=0}^{K} \frac{y^k}{k!} \quad \forall y \in \mathbb{R}.$$

It is easy to verify that the Taylor series is absolutely convergent everywhere in $\mathbb{R}$ and converges exactly to the function u:

$$u(y) = \lim_{K\to+\infty} T^K u(y) = \sum_{k=0}^{+\infty} \frac{y^k}{k!} \quad \forall y \in \mathbb{R}.$$

The expected value of u is:

$$\mathbb{E}[u] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \mathrm{e}^y\, \mathrm{e}^{-\frac{y^2}{2\sigma^2}} = \mathrm{e}^{\frac{\sigma^2}{2}} < +\infty,$$

so that $u \in L^1_\rho(\mathbb{R})$. Since $\left|T^K u(y)\right| \le \sum_{k=0}^{K} \frac{|y|^k}{k!} \le \mathrm{e}^{|y|} \in L^1_\rho(\mathbb{R})$, thanks to the dominated convergence theorem we conclude that, for every $\sigma > 0$,

$$\mathbb{E}[u] = \lim_{K\to\infty} \mathbb{E}\left[T^K u\right]. \tag{3.2}$$

For completeness, we present also the direct computation. We recall that the moments of a Gaussian distribution are

$$\mathbb{E}\left[y^k\right] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} y^k\, \mathrm{e}^{-\frac{y^2}{2\sigma^2}} dy = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sigma^k (k-1)!! & \text{if } k \text{ is even, } k \ge 1. \end{cases}$$

Hence,

$$\begin{aligned} \lim_{K\to+\infty} \mathbb{E}\left[T^K u(y)\right] &= \lim_{K\to+\infty} \sum_{l=0}^{[K/2]} \frac{\mathbb{E}\left[y^{2l}\right]}{(2l)!} = \lim_{K\to+\infty} \sum_{l=0}^{[K/2]} \frac{\sigma^{2l}(2l-1)!!}{(2l)!} \\ &= \lim_{K\to+\infty} \sum_{l=0}^{[K/2]} \frac{\sigma^{2l}}{2^l (l)!} = \mathrm{e}^{\frac{\sigma^2}{2}}, \end{aligned}$$

where we denoted with $\left[\frac{K}{2}\right]$ the integer part of $\frac{K}{2}$, and we used the convention $(-1)!! = 1$. We conclude that (3.2) is verified for every $\sigma > 0$.

3.2.2 Rational function

Let $u(y) = \dfrac{1}{1+y^2}$. The K-th order Taylor polynomial of u in $y_0 = 0$ is:

$$T^K u(y) = \sum_{k=0}^{K} (-1)^k y^{2k}.$$

It is easy to verify that the Taylor series is point-wise convergent to u in the open interval $I = (-1, 1)$, that is:

$$u(y) = \lim_{K \to +\infty} T^K u(y) = \sum_{k=0}^{+\infty} (-1)^k y^{2k} \quad \forall\, y \in I.$$

As before, we want to study the behavior of the expected value of the Taylor polynomial with respect to the Gaussian measure $\rho(y) = \dfrac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{y^2}{2\sigma^2}}$, with $\sigma > 0$. The expected value of u is:

$$\mathbb{E}\left[u\right] = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \frac{1}{1+y^2}\, \mathrm{e}^{-\frac{y^2}{2\sigma^2}} \leq \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{y^2}{2\sigma^2}} = 1 < +\infty,$$

so that $u \in L^1_\rho(\mathbb{R})$. The expected value of the Taylor polynomial is

$$\mathbb{E}\left[T^K u(y)\right] = \sum_{k=0}^{K} \mathbb{E}\left[(-1)^k\, y^{2k}\right] = \sum_{k=0}^{K} (-1)^k \mathbb{E}\left[y^{2k}\right] = \sum_{k=0}^{K} (-1)^k \sigma^{2k} (2k-1)!!.$$

Observe that

$$\begin{aligned}
\sum_{k=1}^{K} (-1)^k \sigma^{2k} (2k-1)!! &= \sum_{l=1}^{[K/2]} \left(\sigma^{4l}(4l-1)!! - \sigma^{4l-2}(4l-3)!!\right) \\
&= \sum_{l=1}^{[K/2]} \left(\sigma^2(4l-1) - 1\right) \sigma^{4l-2}(4l-3)!!,
\end{aligned}$$

where $\sigma^2(4l-1) - 1 > 0$ iff $l > \dfrac{1}{4}\left(\dfrac{1}{\sigma^2} + 1\right)$. Using Stirling's formula:

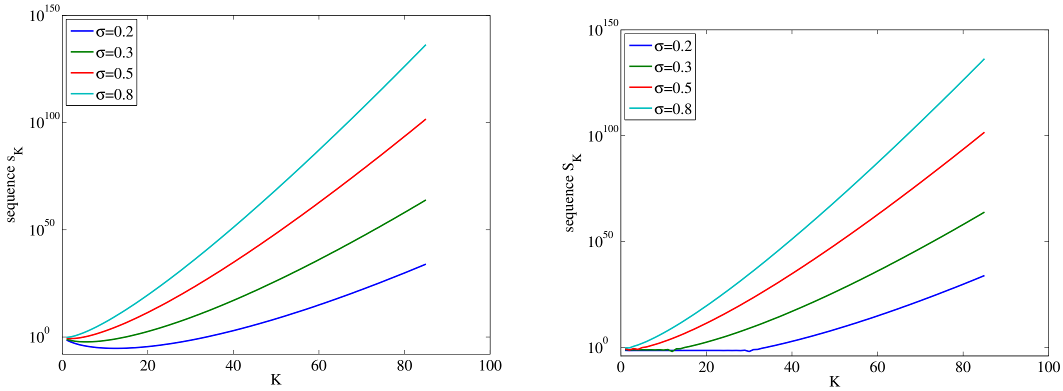
$$\sigma^{4l-2}(4l-3)!! = \sigma^{4l-2} \frac{(4l-2)!}{2^{2l-1}(2l-1)!} \sim \sqrt{2} \left(\frac{2\sigma^2(2l-1)}{e}\right)^{2l-1} \longrightarrow +\infty,$$

so that $\mathbb{E}\left[T^K u(y)\right] \longrightarrow +\infty$.

In Figure 3.1(a) we plot in semilogarithmic scale the sequence

$$\{s_k\}_k := \left\{\sigma^{2k}(2k-1)!!\right\}_k.$$

We notice that there exists k_σ depending on σ such that, for $k \leq k_\sigma$ the behavior of the sequence is dominated by the exponential factor σ^{2k} and the sequence in decreasing,



(a) (b)

Figure 3.1: *3.1(a): First 100 elements of the sequence* $\left\{\sigma^{2k}(2k-1)!!\right\}$ *as a function of* k *in semilogarithmic scale for different values of the standard deviation* σ. *3.1(b): First 100 elements of the sequence* $\left\{\left|\sum_{k=1}^{K}(-1)^k\sigma^{2k}(2k-1)!!\right|\right\}$ *as a function of the order of the Taylor polynomial* K *for different value of* σ.

whereas, for $k > k_\sigma$, the bifactorial term prevails and the sequence starts increasing. This fact reflects on the behavior of the sequence

$$\{S_K\}_K := \left\{\left|\sum_{k=1}^{K}(-1)^k\sigma^{2k}(2k-1)!!\right|\right\}_K$$

which starts diverging for K such that $s_K \geq 1$ (see Figure 3.1(b)).

3.3 Problem setting

3.3.1 Well-posedness of the stochastic Darcy problem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, where Ω is the set of outcomes, $\mathcal{F}$ the σ-algebra of events and $\mathbb{P} : \Omega \to [0,1]$ a probability measure. Let D be an open bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ and $f : D \to \mathbb{R}$ be a square integrable function defined on D. We are interested in the following stochastic linear elliptic boundary value problem, that is the stochastic Darcy problem: find a random field $u : \Omega \times \bar{D} \to \mathbb{R}$ such that

$$\begin{cases} -\text{div}_x\left(a(\omega,x)\nabla_x u(\omega,x)\right) = f(x) & x \in D,\ \omega \in \Omega \\ u(\omega,x) = g(x) & x \in \Gamma_D,\ \omega \in \Omega \\ a(\omega,x)\nabla_x u(\omega,x)\cdot \mathbf{n} = 0 & x \in \Gamma_N,\ \omega \in \Omega \end{cases} \tag{3.3}$$

where $\{\Gamma_D,\ \Gamma_N\}$ is a partition of the boundary of the domain ∂D, div_x, ∇_x are differential operators with respect to the spatial variable x and $\mathbf{n}$ is the outward normal unit vector to $\partial\Omega$. We require equation (3.3) to be satisfied $\mathbb{P}$-almost everywhere in Ω or, in other words, almost surely (a.s.). Here $a : \Omega \times \bar{D} \to \mathbb{R}$ is a random field which represents the uncertain permeability in the heterogeneous porous medium D, u represents the hydraulic head, $\mathbf{v} = -a\nabla u$ the velocity field and Γ_N an impervious boundary. We look for the solution in the Bochner space $L^p\left(\Omega; H^1(D)\right)$, $p > 0$, that is the space of functions $v(\omega, x)$ such that $\|v\|_{L^p(\Omega;H^1)} := \left(\int_\Omega \|v(\omega)\|_{H^1}^p \, \mathrm{d}\mathbb{P}(\omega)\right)^{1/p} < \infty$. See e.g. [35]. If $p = 2$, this space is a Hilbert space with the natural inner product

$$(v(\omega, x), w(\omega, x))_{L^2(\Omega;H^1(D))}$$
$$:= \int_\Omega \left(\int_D v(\omega, x) w(\omega, x)\, dx\right) d\mathbb{P}(\omega) + \int_\Omega \left(\int_D \nabla_x v(\omega, x) \cdot \nabla_x w(\omega, x)\, dx\right) d\mathbb{P}(\omega),$$

and is canonically isomorphic to the tensor product space $L^2(\Omega, \mathbb{P}) \otimes H^1(D)$. (See e.g. [85]).

The weak formulation of (3.3) is:

Stochastic Darcy problem - weak formulation

$$\boxed{\begin{array}{c}\text{Given } f \in L^2(D) \text{ and } g \in H^{1/2}(\Gamma_D),\\ \text{find } u \in L^p\left(\Omega; H^1(D)\right) \text{ s.t. } u|_{\Gamma_D} = g \text{ a.s., and}\\[1em] \displaystyle\int_D a(\omega, x)\nabla_x u(\omega, x) \cdot \nabla_x v(x)\, dx = \int_D f(x) v(x)\, dx \\[1em] \forall v \in H^1_{\Gamma_D}(D), \text{ a.s. in } \Omega.\end{array}} \tag{3.4}$$

We denoted with $H^1_{\Gamma_D}(D)$ the subspace of $H^1(D)$ of functions whose trace vanishes on Γ_D. Let us assume

A1 : *The permeability field* $a \in L^p\left(\Omega; C^0(\bar{D})\right)$ *for every* $p \in (0, \infty)$.

Then, the quantities

$$a_{min}(\omega) := \min_{x \in \bar{D}} a(\omega, x) \tag{3.5}$$
$$a_{max}(\omega) := \max_{x \in \bar{D}} a(\omega, x) \tag{3.6}$$

are well defined, and $a_{max} \in L^p(\Omega)$ for every $p \in (0, +\infty)$. Moreover, we assume

A2 : $a_{min}(\omega) > 0$ *a.s.,* $\dfrac{1}{a_{min}(\omega)} \in L^p(\Omega)$ *for every* $p \in (0, \infty)$.

We recall here the well posedness result of problem (3.4), based on the Lax Milgram Lemma. See [24, 41, 47].

Theorem 3.3.1. *If the permeability field $a(\omega, x)$ satisfies **A1**, **A2**, then problem* (3.4) *is well-posed for every $p \in (0, \infty)$, that is it admits a unique solution that depends continuously on the data.*

Proof. By an application of the Lax Milgram Lemma, for every fixed $\omega \in \Omega$ we conclude the existence of a unique $u(\omega, \cdot) \in H^1_{\Gamma_D}(D)$ a.s. such that

$$\|u(\omega, \cdot)\|_{H^1(D)} \leq \frac{\sqrt{C_P^2 + 1}}{a_{min}(\omega)} \|f\|_{L^2(D)},$$

where C_P is the Poincaré constant. By applying the L^p-norm in probability and using **A2**, we conclude

$$\mathbb{E}\left[\|u\|_{H^1(D)}^p\right]^{1/p} \leq \sqrt{C_P^2+1}\,\mathbb{E}\left[\frac{1}{(a_{min}(\omega))^p}\right]^{1/p}\|f\|_{L^2(D)} < \infty.$$

□

3.3.2 The lognormal model

A frequently used model in geophysical applications describes the permeability field $a(\omega,x)$ as a lognormal random field, that is $a(\omega,x) = \mathrm{e}^{Y(\omega,x)}$, where $Y:\Omega\times\bar{D}\to\mathbb{R}$ is a Gaussian random field. See for example [14, 36, 51, 52, 93, 99, 100]. Let us define the mean-free Gaussian random field $Y'(\omega,x) := Y(\omega,x) - \mathbb{E}\left[Y\right](x)$, and assume that its covariance kernel, $Cov_{Y'}: D\times D\to\mathbb{R}$, is Hölder continuous with exponent t for some $0 < t \leq 1$. Then, the following result holds, which extends the result in [24] obtained only for centered stationary second order random fields Y with covariance function:

$$Cov_Y(x_1,x_2) = \nu(\|x_1-x_2\|)$$

for some $\nu \in C^{0,1}(\mathbb{R}^+)$.

Proposition 3.3.2. *Let $Y:\Omega\times D\to\mathbb{R}$ be a Gaussian random field, and $Y'(\omega,x) := Y(\omega,x)-\mathbb{E}\left[Y\right](x)$ with covariance function $Cov_{Y'}\in\mathcal{C}^{0,t}(\overline{D\times D})$ for some $0<t\leq 1$. Suppose $\mathbb{E}\left[Y\right]\in\mathcal{C}^{0,t/2}(\bar{D})$. Then it holds*

$$\sup_{x_1,x_2}\mathbb{E}\left[\frac{|Y'(\omega,x_1)-Y'(\omega,x_2)|^{2p}}{\|x_1-x_2\|^{tp}}\right]^{1/2p} < +\infty, \quad \forall p>0. \tag{3.7}$$

Moreover, there exists a version of Y whose trajectories belong to $\mathcal{C}^{0,\alpha}(\bar{D})$ a.s. for $0<\alpha<t/2$.

Proof. We have:

$$\begin{aligned}
&\mathbb{E}\left[|Y'(\omega,x_1)-Y'(\omega,x_2)|^2\right]\\
&= \mathbb{E}\left[(Y')^2(\omega,x_1)\right] + \mathbb{E}\left[(Y')^2(\omega,x_2)\right] - 2\mathbb{E}\left[Y'(\omega,x_1)Y'(\omega,x_2)\right]\\
&= Cov_{Y'}(x_1,x_1) + Cov_{Y'}(x_2,x_2) - 2Cov_{Y'}(x_1,x_2)\\
&= \frac{Cov_{Y'}(x_1,x_1)-Cov_{Y'}(x_1,x_2)}{\|(x_1,x_1)-(x_1,x_2)\|^t}\|(x_1,x_1)-(x_1,x_2)\|^t\\
&\quad + \frac{Cov_{Y'}(x_2,x_2)-Cov_{Y'}(x_1,x_2)}{\|(x_2,x_2)-(x_1,x_2)\|^t}\|(x_2,x_2)-(x_1,x_2)\|^t\\
&\leq 2\,C_H\,\|x_1-x_2\|^t
\end{aligned}$$

where C_H is the Hölder continuity constant of $Cov_{Y'}$. Since $Y'(\omega,x_1)-Y'(\omega,x_2)$ is a mean free Gaussian random variable,

$$\mathbb{E}\left[|Y'(\omega,x_1)-Y'(\omega,x_2)|^{2p}\right] \leq C_p\,\mathbb{E}\left[|Y'(\omega,x_1)-Y'(\omega,x_2)|^2\right]^p$$

for every positive integer p, where $C_p = (2p-1)!!$. Hence,

$$\mathbb{E}\left[|Y'(\omega,x_1) - Y'(\omega,x_2)|^{2p}\right] \leq C_p(2\,C_H)^p \|x_1 - x_2\|^{tp},$$

so that (3.7) is verified. Using the Kolmogorov continuity theorem, we deduce the existence of a version of Y Hölder continuous with exponent $\alpha < (tp-d)/2p$. Letting $p \to +\infty$, we obtain $\alpha < t/2$. $\square$

In what follows, we identify the Hölder regular version of the field with $Y(\omega,x)$, so that $\|Y(\omega)\|_{L^\infty(D)}$, $a_{min}(\omega)$ and $a_{max}(\omega)$ are well-defined random variables. Using the Fernique's theorem (see e.g. [34]), in [24] the author shows that

$$\frac{1}{a_{min}(\omega)} \in L^p(\Omega,\mathbb{P})\;\forall\,p>0, \quad a_{max}(\omega) \in L^p(\Omega,\mathbb{P})\;\forall\,p>0.$$

Hence, **A1** and **A2** are satisfied, and problem (3.4) is well posed. In the rest of the chapter we assume the permeability field of the heterogeneous porous medium D to be lognormal random field as described before.

3.3.3 Conditioned Gaussian fields

From the point of view of applications it is very interesting to study also the case of a random field conditioned to available observations. Take for example the fluid flow in a heterogeneous porous medium: the permeability varies randomly, and can be measured only in a certain number of spatial points. Assuming that N_{oss} point-wise measurements of the permeability have been collected (e.g. by exploratory wells), one can construct a conditioned random field Y whose covariance function is non-stationary, but still Hölder continuous.

To define a conditioned field, we start from the following classical result of multivariate statistical analysis. See e.g. [4].

Proposition 3.3.3. *Let $\mathbf{X}(\omega) = (X_1(\omega),\ldots,X_N(\omega))$ be a random vector composed of two sub vectors $\mathbf{X}^{(1)} = (X_1^{(1)},\ldots,X_{N_1}^{(1)})$, $\mathbf{X}^{(2)} = (X_1^{(2)},\ldots,X_{N_2}^{(2)})$, with $N_2 = N - N_1$. Suppose that $\mathbf{X} \sim \mathcal{N}(\mu,\Sigma)$, where*

$$\mu = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},$$

$\mu^{(i)}$ being the mean of $\mathbf{X}^{(i)}$, $i = 1,2$, and $\Sigma_{i,j}$ being the covariance matrix of $\mathbf{X}^{(i)}$ and $\mathbf{X}^{(j)}$, $i,j = 1,2$. Then, the conditioned random vector $\mathbf{X}_{cond}$, defined as $\mathbf{X}^{(1)}$ given $\mathbf{X}^{(2)} = \mathbf{x}^{(2)}$, is Gaussian with mean μ_{cond} and covariance matrix Σ_{cond} given respectively by

$$\mu_{cond} = \mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}^{(2)} - \mu^{(2)}) \tag{3.8}$$

$$\Sigma_{cond} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \tag{3.9}$$

Suppose, for simplicity, the Gaussian random field $Y(\omega,x)$ to be stationary, with expected value μ and covariance function Cov_Y. Let N_{oss} be the number of point-wise available measurements whose coordinates and values are $\{x_i^o\}_{i=1}^{N_{oss}}$ and $\{y^o\}_{i=1}^{N_{oss}}$ respectively, and define the Gaussian random vector $\mathbf{X}^{(2)} := (Y(\omega, x_1^o), \dots, Y(\omega, x_{N_{oss}}^o))$. Proposition 3.3.3 states that

$$Y_{cond}(\omega, x) := \left(Y(\omega, x) | \, X^{(2)}(\omega) = (y_1^o, \dots, y_{N_{oss}}^o)\right)$$

is a Gaussian random field with

$$\mathbb{E}\left[Y_{cond}\right](x) = \mu + \sum_{i,j} (\Sigma^{-1})_{i,j} (y_j^o - \mu) Cov_Y(x, x_i^o), \tag{3.10}$$

$$Cov_{Y_{cond}}(x_1, x_2) = Cov_Y(x_1, x_2) - \sum_{i,j} (\Sigma^{-1})_{i,j} Cov_Y(x_1, x_i^o) Cov_Y(x_j^o, x_2), \tag{3.11}$$

where Σ is the covariance matrix of $\mathbf{X}^{(2)}$. Therefore, if $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, then $\mathbb{E}\left[Y_{cond}\right] \in \mathcal{C}^{0,t}(\bar{D})$ and $Cov_{Y_{cond}} \in \mathcal{C}^{0,t}(\overline{D \times D})$, so that Proposition 3.3.2 holds.

3.3.4 Upper bounds for the statistical moments of $\|Y'\|_{L^\infty(D)}$

We derive two upper bounds for the statistical moments of the well-defined random variable $\|Y'\|_{L^\infty(D)}$. The first estimate is inspired from [24] and is obtained by performing a spectral decomposition of the non-stationary random field Y. On the other hand, the second one holds only for smooth fields and exploits the *Euler characteristic heuristic* method, which consists in the approximation of the upper tail probability of the maximum of a random field using the Euler characteristic of the excursion set (see [1]).

We denote with $\sigma^2 := \frac{1}{|D|} \int_D \mathbb{V}ar\left[Y(\omega, x)\right] \, dx$. If $Y(\omega, x)$ is a stationary field, then its variance is independent of $x \in D$ and coincides with σ^2. By a little abuse of notation, in what follows we will refer to σ as the standard deviation of Y also in the case of a non-stationary random field.

We start recalling the well known Karhunen-Loève expansion of a random field. [43, 70, 72, 73].

Proposition 3.3.4 (Karhunen-Loève expansion)**.** *Let $Y(\omega, x)$ be a random field, with continuous covariance function $Cov_Y(x_1, x_2)$. Let T be the linear, symmetric and compact operator defined as*

$$\begin{aligned} T : L^2(D) &\to L^2(D), \\ \phi &\mapsto \int_D Cov_Y(x_1, x_2) \phi(x_2) \, dx_2. \end{aligned}$$

Then,

$$Y(\omega, x) = \mathbb{E}\left[Y\right](x) + \sum_{j=1}^{+\infty} \sqrt{\lambda_j} \, \phi_j(x) \, \xi_j(\omega), \quad (\omega, x) \in \Omega \times D, \tag{3.12}$$

where $\{\lambda_j\}_{j \geq 1}$ is the decreasing sequence of non-negative eigenvalues of T, $\{\phi_j(x)\}_{j \geq 1}$ are the corresponding eigenfunctions, which form an orthonormal basis for $L^2(D)$, and

$\{\xi_j(\omega)\}_{j\geq 1}$ are the centered uncorrelated random variables with unit variance defined as

$$\xi_j(\omega) = \frac{1}{\sqrt{\lambda_j}} \int_D \left(Y(\omega,x) - \mathbb{E}\left[Y\right](x)\right) \phi_j(x)\, dx.$$

Moreover, it holds

$$\int_D \mathbb{V}ar\left[Y(\cdot,x)\right]\, dx = \sum_{j=1}^{+\infty} \lambda_j. \tag{3.13}$$

Recall that we assumed $Y(\omega,x)$ to be a Gaussian random field. Hence, $\xi_j(\omega) = \frac{1}{\sqrt{\lambda_j}} \int_D Y'(\omega,x)\, \phi_j(x)\, dx$ is a Gaussian random variable for every $j \geq 1$, $\{\xi_j\}_{j\geq 1}$ are independent and λ_j are proportional to σ^2: $\lambda_j = \sigma^2 \widetilde{\lambda}_j\ \forall\, j \geq 1$.

We make the following

Assumption A1.

1. *ϕ_j is Hölder continuous with exponent $0 < \gamma \leq 1$ for every $j \geq 1$.*
2. *$R_\gamma := \sum_{j=1}^{+\infty} \widetilde{\lambda}_j \, \|\phi_j\|^2_{\mathcal{C}^{0,\gamma}(\bar{D})} < +\infty.$*

Assumption A1 is fulfilled in many practical models such as an exponential or Gaussian covariance function. Following [24], it is possible to show that, for any γ as in Assumption A1 and any $\beta < \gamma$,

$$\|Y'\|_{L^k\left(\Omega; C^{0,\beta}(\bar{D})\right)} \leq C^{1/k} \sqrt{R_\gamma}\, \sigma \left((k-1)!!\right)^{1/k}, \quad \forall k > 0$$

where C is a positive constant independent of k and σ. In particular,

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq C\, R_\gamma^{k/2} \sigma^k \, (k-1)!!, \quad \forall k > 0. \tag{3.14}$$

An estimate of the type (3.14) can also be obtained with a different approach which, however, is valid only for smooth fields. In [1] the authors propose to approximate the upper tail probability of the maximum of a smooth random field using the Euler characteristic of the excursion set. This method is known as *Euler characteristic heuristic*.

Assumption A2. *The domain is a d-dimensional rectangle $D = [0,T]^d$. The centered Gaussian field $Y'(\omega,x)$ is stationary and satisfies the following regularity assumptions:*

- *Y' is $\mathcal{C}^2$ on an open neighborhood of D.*
- *Y' does not present degenerate critical points on the n-th dimensional boundary $\partial_n D$ of D, for each $n = 0, \ldots, d$.*
- *$Y'|_{\partial_n D}$ does not present critical points on $\cup_{j=0}^{n-1} \partial_{j-i} D$ for each $n = 0, \ldots, d$.*

Lemma 3.3.5. *Suppose that the physical domain D and the random field Y satisfy Assumption A2. Then,*

$$\mathbb{E}\left[\|Y'\|^k_{L^\infty(D)}\right] \leq C\, \sigma^{k-2}\, k\, (k-1)!!, \quad \forall k \tag{3.15}$$

where $C = C(D, |\Lambda|)$, Λ *being the variance matrix of* ∇Y

$$\Lambda_{i,j} = \mathbb{E}\left[\frac{\partial Y}{\partial x_i}(\omega, x)\frac{\partial Y}{\partial x_j}(\omega, x)\right].$$

Proof. See Section 3.8. □

Lemma 3.3.5 is based on the following result, proved in [1]:

$$|\mathbb{P}\left\{\sup Y'(x,\omega) \geq y\right\} - \mathbb{E}\left\{\varphi\left(A_y\left(Y', D\right)\right)\right\}| < O\left(\mathrm{e}^{\frac{-\delta y^2}{2\sigma^2}}\right), \tag{3.16}$$

where $\delta > 1$, φ is the Euler characteristic and $A_y\left(Y', D\right) := \{x \in D|\ Y'(\omega, x) \geq y\}$ is the excursion set of Y' over D. It holds under weaker assumptions than in Assumption A2 both on the domain D and the Gaussian field Y', so that a generalized version of Lemma 3.3.5 may be deduced. However, Y' has to be smooth with constant variance. In [27] the authors obtain the same type of result as (3.16) under the weaker assumption of Y' Gaussian random field with stationary increments.

The bound (3.15) is weaker than (3.14) as it predicts a scaling σ^{k-2} instead of σ^k for the k-th moment of the random variable $\|Y'\|_{L^\infty(D)}$. On the other hand, the bound (3.14) involves the exponential term $R_\gamma^{k/2}$ where R_γ depends on the covariance function of the random field. In what follows we will choose either bound depending on the context and the application.

For simplicity, in the rest of the chapter, we assume the Gaussian random field $Y(\omega, x)$ *to be centered.*

3.4 Perturbation analysis in the infinite dimensional case

Thanks to the Doob-Dynkin Lemma [84], the solution u of problem (3.4) is a function of Y: $u = u(Y, x)$. In this section, under the assumption of small standard deviation of Y, we perform a perturbation analysis based on the Taylor expansion of the solution u in a neighborhood of the zero-mean of Y and we study the approximation properties of the Taylor polynomial of u. We exhibit upper bounds for the norms of the Taylor polynomial $\left\|T^K u(Y,x)\right\|_{L^1(\Omega;H^1(D))}$, $\left\|T^K u(Y,x)\right\|_{L^2(\Omega;H^1(D))}$, and for the errors $\left\|u - T^K u\right\|_{L^1(\Omega;H^1(D))}$, $\left\|u - T^K u\right\|_{L^2(\Omega;H^1(D))}$. We predict the existence of an optimal order of the Taylor polynomial $\bar{K}^\sigma$ depending on σ after which the error starts increasing and eventually diverges to infinity. Finally, we provide a formula to compute both $\bar{K}^\sigma$ and the minimum error achievable with a perturbation technique based on a $\bar{K}^\sigma$-th order Taylor polynomial.

3.4.1 Taylor expansion

Let $0 < \sigma < 1$ be the standard deviation of the centered Gaussian random field $Y(\omega, x)$. Given a function $u(Y) : L^\infty(D) \to H^1(D)$ which is $(K+1)$-times Gateaux differentiable, we denote its k-th ($0 \leq k \leq K+1$) Gateaux derivative in $\bar{Y} \in L^\infty(D)$ evaluated in the vector $\underbrace{(Y, \ldots, Y)}_{k \text{ times}}$ as $D^k u(\bar{Y})[Y]^k$. The K-th order ($K \geq 1$) Taylor polynomial of u centered in 0 is:

$$T^K u(Y,x) := \sum_{k=0}^{K} \frac{D^k u(0)[Y]^k}{k!}, \tag{3.17}$$

where $D^0 u(0)[Y]^0 := u^0(x)$ is independent of the random field Y. The K-th order residual of the Taylor expansion $R^K u(Y,x) := u(Y,x) - T^K u(Y,x)$ can be expressed as

$$R^K u(Y,x) := \frac{1}{K!} \int_0^1 (1-t)^K D^{K+1} u(tY)[Y]^{K+1} dt. \tag{3.18}$$

See for example [2, 3].

Since the solution u of the stochastic Darcy problem is infinitely many times differentiable, its Taylor polynomial can then be used to approximate the statistical moments of u, e.g. $\mathbb{E}[u](x) \approx \sum_{k=0}^{K} \frac{1}{k!} \mathbb{E}\left[D^k u(0)[Y]^k\right]$, leading to the so called *"moment equations"*. (see e.g. [9, 51, 56, 86, 92, 93, 97, 100]). We don't detail here with the derivation and algorithmic implementation of the moment equations, which will be the topic of Chapters 4 and 5. Rather, we investigate the accuracy of the Taylor expansion for the problem at hand.

3.4.2 Upper bound on the norm of the Taylor polynomial

The problem solved by u^0 is the deterministic Laplacian problem: given $f \in L^2(D)$ and $g \in H^{1/2}(\Gamma_D)$, find $u^0 \in H^1(D)$ such that $u|_{\Gamma_D} = g$ and

$$\int_D \nabla u^0(x) \cdot \nabla v(x)\, dx = \int_D f(x) v(x)\, dx \quad \forall v \in H^1_{\Gamma_D}(D). \tag{3.19}$$

The problem solved by the k-th Gateaux derivative of u, $D^k u(0)[Y]^k$ ($k \geq 1$) is (see e.g. [10, 51, 93])

k-th derivative problem - lognormal random field

Given $f \in L^2(D),\ g \in H^{1/2}(\Gamma_D),$ *and all lower order derivatives* $D^l u(0)[Y]^l \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right),\ l < k,$ *find* $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ *s.t.*

$$\int_D \nabla_x D^k u(0)[Y]^k \cdot \nabla_x v\, dx = -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l \nabla_x D^{k-l} u(0)[Y]^{k-l} \cdot \nabla_x v\, dx$$

$$\forall\, v \in H^1_{\Gamma_D}(D), \quad \text{a.s. in } \Omega. \tag{3.20}$$

By the Lax Milgram lemma, the boundness of $\|Y\|_{L^\infty(D)}$ and a recursion argument, we can state the following result.

Theorem 3.4.6. *Problem* (3.20) *is well-posed, that is it admits a unique solution that depends continuously on the data. Moreover, it holds*

$$\|D^k u(0)[Y]^k\|_{H^1(D)} \leq C \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k k! < +\infty, \quad \forall k \geq 1 \quad \textit{a.s. in } \Omega \tag{3.21}$$

where $C = C(C_P, \|u^0\|_{H^1(D)})$, C_P *being the Poincaré constant. Moreover,* $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ *for any* $p > 0$.

Proof. For every fixed $\omega \in \Omega$, problem (3.20) is of the form: find $w \in H^1_{\Gamma_D}(D)$ such that

$$\mathscr{A}(w, v) = \mathscr{L}(v) \quad \forall v \in H^1_{\Gamma_D}(D),$$

where the bilinear form $\mathscr{A}$ and the linear form $\mathscr{L}$ are respectively defined as

$$\begin{aligned} \mathscr{A} : H^1_{\Gamma_D}(D) \times H^1_{\Gamma_D}(D) &\to \mathbb{R} \\ (w, v) &\mapsto \int_D \nabla w(x) \cdot \nabla v(x) \, dx \end{aligned}$$

$$\begin{aligned} \mathscr{L} : H^1_{\Gamma_D}(D) &\to \mathbb{R} \\ v &\mapsto -\sum_{l=1}^{k} \binom{k}{l} \int_D Y^l \nabla_x D^{k-l} u(0)[Y]^{k-l} \cdot \nabla_x v \, dx. \end{aligned}$$

It is easy to verify that $\mathscr{A}$ is continuous and coercive. Moreover, $\mathscr{L}$ is continuous:

$$\begin{aligned} |\mathscr{L}(v)| &\leq \sum_{l=1}^{k} \binom{k}{l} \left| \int_D Y^l \nabla_x D^{k-l} u(0)[Y]^{k-l} \cdot \nabla_x v \, dx \right| \\ &\leq \sum_{l=1}^{k} \binom{k}{l} \|Y(\omega)\|^l_{L^\infty(D)} \left\| D^{k-l} u(0)[Y]^{k-l} \right\|_{H^1} \|v\|_{H^1}. \end{aligned}$$

Thanks to the Lax Milgram Lemma we conclude the well-posedness of problem (3.20) for every fixed $\omega \in \Omega$. To prove (3.21), we follow [10]. Let us take $v = D^k u(0)[Y]^k$ in (3.20). By the Cauchy-Schwarz inequality

$$\begin{aligned} \int_D \left| \nabla D^k u(0)[Y]^k \right|^2 \mathrm{dx} &\leq \sum_{l=1}^{k} \binom{k}{l} \left| \int_D Y^l \nabla D^{k-l} u(0)[Y]^{k-l} \cdot \nabla D^k u(0)[Y]^k \mathrm{dx} \right| \\ &\leq \sum_{l=1}^{k} \binom{k}{l} \|Y\|^l_{L^\infty} \left\| \nabla D^{k-l} u(0)[Y]^{k-l} \right\|_{L^2} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2} \end{aligned}$$

By defining $S_k := \dfrac{1}{k!} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2}$, we have:

$$S_k \leq \sum_{l=1}^{k} \frac{\|Y\|^l_{L^\infty}}{l!} S_{k-l}. \tag{3.22}$$

We prove by induction that

$$S_k \le C_k \|Y\|_{L^\infty}^k S_0, \tag{3.23}$$

where $\{C_k\}_{k\ge 1}$ are defined by the recursion as

$$\begin{cases} C_0 = 1 \\ C_k = \sum_{l=1}^k \frac{1}{l!} C_{k-l}. \end{cases} \tag{3.24}$$

If $k = 1$, (3.23) easily follows from (3.22). Now, let us suppose that (3.23) is verified for every S_j with $j = 1, \dots, k-1$. Then, using (3.22), the inductive hypothesis and the definition of the coefficients C_k in (3.24),

$$\begin{aligned} S_k &\le \sum_{l=1}^k \frac{\|Y\|_{L^\infty}^l}{l!} S_{k-l} = \sum_{l=1}^{k-1} \frac{\|Y\|_{L^\infty}^l}{l!} S_{k-l} + \frac{\|Y\|_{L^\infty}^k}{k!} S_0 \\ &\le \sum_{l=1}^{k-1} \frac{\|Y\|_{L^\infty}^l}{l!} C_{k-l} \|Y\|_{L^\infty}^{k-l} S_0 + \frac{\|Y\|_{L^\infty}^k}{k!} S_0 \\ &= \|Y\|_{L^\infty}^k \left(\sum_{l=1}^{k-1} \frac{C_{k-l}}{l!} + \frac{1}{k!} \right) S_0 = \|Y\|_{L^\infty}^k C_k S_0, \end{aligned}$$

so that (3.23) is verified. In [15], the authors show by induction that $C_k \le \left(\frac{1}{\log 2}\right)^k$ $\forall\, k \ge 0$. Hence,

$$S_k \le \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k S_0.$$

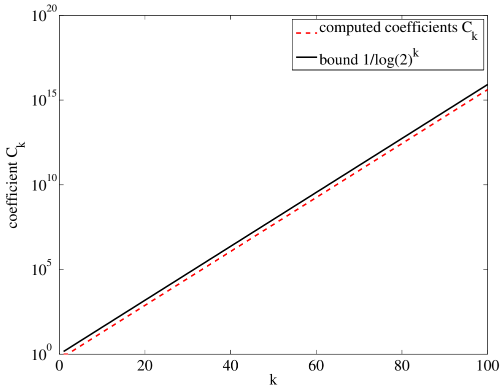
In conclusion,

$$\begin{aligned} \left\| D^k u(0)[Y]^k \right\|_{H^1} &\le \sqrt{C_P^2 + 1} \left\| \nabla D^k u(0)[Y]^k \right\|_{L^2} \le \sqrt{C_P^2+1}\, S_0 \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k k! \\ &\le \left(\sqrt{C_P^2+1}\, C_P \left\| u^0 \right\|_{H^1(D)} \right) \left(\frac{\|Y\|_{L^\infty}}{\log 2} \right)^k k!, \end{aligned}$$

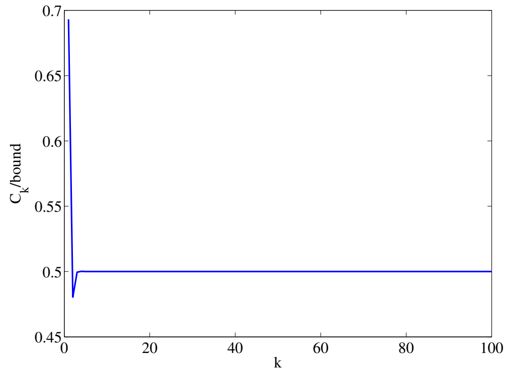
so that (3.21) is proved with $C = \sqrt{C_P^2+1}\, C_P \left\| u^0 \right\|_{H^1(D)}$. Moreover, since $\|Y\|_{L^\infty(D)} \in L^q(\Omega, \mathbb{P})$ for any $q > 0$, we conclude that $D^k u(0)[Y]^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ for any $p > 0$. $\square$

In Figure 3.2 we plot in semilogarithmic scale the coefficients C_k defined in (3.24) and we point out their exponential growth by comparing them with the sequence $\left\{(\log 2)^{-k}\right\}_{k\ge 0}$. Figure 3.2(b) shows that $\left\{C_k \,(\log 2)^k\right\}_k$ stabilizes on the constant 0.5.

Combining (3.21) and (3.15) we give an estimate for the $L^1(\Omega; H^1(D))$ and $L^2(\Omega; H^1(D))$ norm of the Taylor polynomial $T^K u$.



(a)



(b)

Figure 3.2: *3.2(a): Comparison between the computed coefficients* $\{C_k\}_k$ (3.24) *and the upper bound* $\{(\log 2)^{-k}\}_k$ *plotted in semilogarithmic scale. 3.2(b): Plot of* $\left\{C_k (\log 2)^k\right\}_k$ *as a function of* k*.*

Theorem 3.4.7. *Suppose that Assumption A2 is fulfilled. Then*

$$\|T^K u\|_{L^1(\Omega;H^1(D))} \leq C_1 \sum_{k=0}^{K} \frac{1}{\sigma^2} \left(\frac{\sigma}{\log 2}\right)^k k\,(k-1)!!, \tag{3.25}$$

$$\|T^K u\|_{L^2(\Omega;H^1(D))} \leq C_2 \sum_{k=0}^{K} \frac{1}{\sigma} \left(\frac{\sigma}{\log 2}\right)^k \sqrt{2k\,(2k-1)!!}, \tag{3.26}$$

where $C_1 = C_1(D, \|u^0\|_{H^1(D)})$, $C_2 = C_2(D, \|u^0\|_{H^1(D)})$.

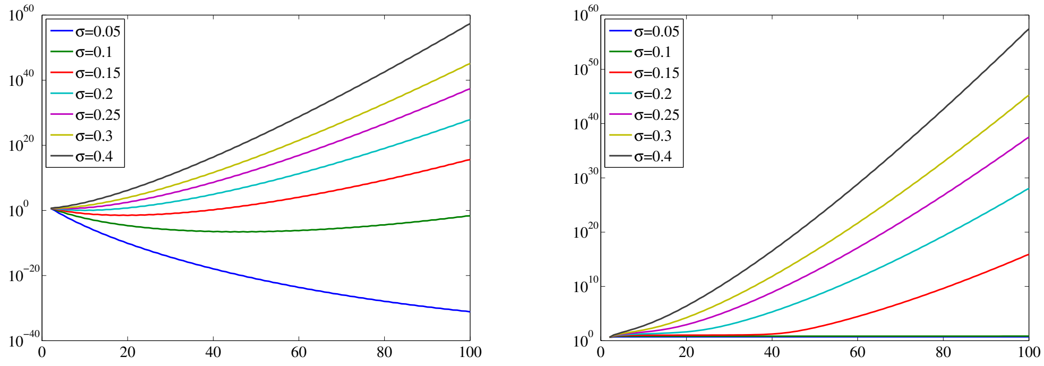
Proof. Applying the L^1-norm in probability to both sides of (3.21) and using (3.15),

$$\begin{aligned}
\|T^K u\|_{L^1(\Omega;H^1(D))} &\leq \sum_{k=0}^{K} \frac{1}{k!} \left\|D^k u(0)[Y]^k\right\|_{L^1(\Omega;H^1(D))} \\
&\leq C \sum_{k=0}^{K} \frac{1}{k!} \left(\frac{1}{\log 2}\right)^k k!\,\mathbb{E}\,\|Y\|_{L^\infty}^k \\
&\leq C_1 \sum_{k=0}^{K} \left(\frac{1}{\log 2}\right)^k \sigma^{k-2}\, k\,(k-1)!!,
\end{aligned}$$

so that (3.25) is verified. We prove (3.26) analogously. □

The behavior of both the upper bounds (3.25) and (3.26) is given by the product of an exponential term and a bifactorial term. Figure 3.3(a) shows that there exists k_σ depending on σ such that, for $k \leq k_\sigma$ the behavior of the sequence $\{s_k\} = \left\{(\log 2^{-1}\sigma)^k\;(k-1)!!\right\}$ is dominated by the exponential term and the sequence is decreasing, whereas, for $k > k_\sigma$, the bifactorial term prevails and the sequence starts increasing. This fact reflects on the behavior of the upper bound (3.25) which starts diverging for K_σ such that $s_{K_\sigma} \geq 1$ (see Figure 3.3(b)). Figures 3.4(a) and 3.4(b) confirm that the asymptotic behavior depends on the bifactorial. Moreover, we note that

$K_\sigma^1 \simeq 2K_\sigma^2$, where K_σ^1 and K_σ^2 refer to (3.25) and (3.26) respectively. Recall that we already observed this behavior in Section 3.2.2.



(a) (b)

Figure 3.3: *3.3(a): Sequence* $\left\{\sigma^{-2}\left(\log 2^{-1}\sigma\right)^k\, k\,(k-1)!!\right\}$ *as a function of* k *in semilogarithmic scale for different values of the standard deviation of the random field* Y*. 3.3(a): Upper bound* (3.25) *as a function of the order of the Taylor polynomial* K *for different value of* σ*.*

It is easy to deduce a result also for the $L^p\left(\Omega; H^1(D)\right)$-norm of the Taylor polynomial. In view of Chapter 4, where we approximate the mean and the two-points correlation of u using its Taylor polynomial, we focus on the $L^1\left(\Omega; H^1(D)\right)$ and $L^2\left(\Omega; H^1(D)\right)$ norms.

Under Assumption A1 and exploiting the upper bound (3.14) instead of (3.15), we obtain similar results as in Theorem 3.4.7, where a behavior σ^k is predicted, but the constant R_γ depending on the covariance function of Y is involved:

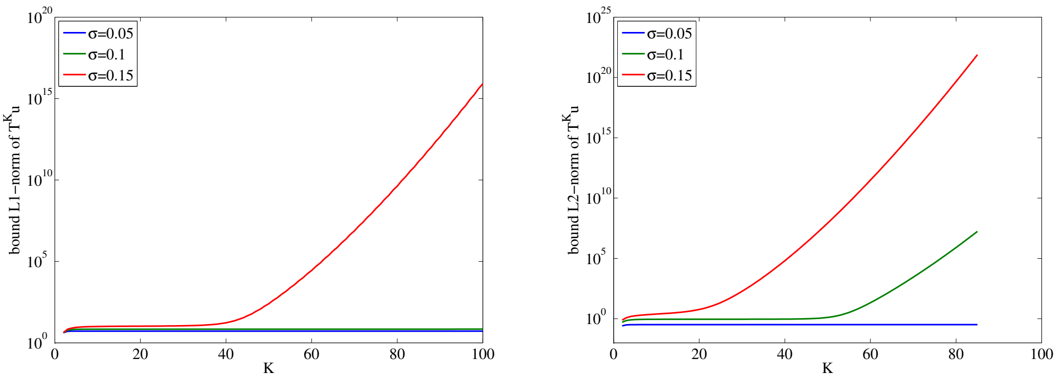
$$\left\|T^K u\right\|_{L^1(\Omega;H^1(D))} \le C_1 \sum_{k=0}^{K} \left(\sqrt{R_\gamma}\right)^k \left(\frac{\sigma}{\log 2}\right)^k (k-1)!!, \tag{3.27}$$

$$\left\|T^K u\right\|_{L^2(\Omega;H^1(D))} \le C_2 \sum_{k=0}^{K} \left(\sqrt{R_\gamma}\right)^k \left(\frac{\sigma}{\log 2}\right)^k \sqrt{(2k-1)!!}. \tag{3.28}$$

3.4.3 Upper bound on the norm of the Taylor residual

The problem solved by $D^K u(tY)[Y]^K$, $t \in (0,1)$, is: given $f \in L^2(D)$ and all lower order derivatives $D^j u(tY)[Y]^j \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$, $j < K$, find $D^K u(tY)[Y]^K \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$ s.t.

$$\int_D \mathrm{e}^{tY} \nabla_x D^K u(tY)[Y]^K \cdot \nabla_x v\, dx = \tag{3.29}$$
$$-\sum_{l=1}^{K} \binom{K}{l} \int_D Y^l \mathrm{e}^{tY} \nabla_x D^{K-l} u(tY)[Y]^{K-l} \cdot \nabla_x v\, dx$$



(a) (b)

Figure 3.4: *3.4(a): Upper bound* (3.25) *for thee values of σ. 3.4(b): Upper bound* (3.26) *for thee values of σ.*

$\forall\, v \in H^1_{\Gamma_D}(D)$, a.s. in Ω. Following an analogous reasoning as in Theorem 3.4.7, we find that problem (3.29) is well-posed and

$$\|D^K u(tY)[Y]^K\|_{H^1(D)} \leq C\, \mathrm{e}^{t\|Y\|_{L^\infty}} \left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^K K! < +\infty, \quad \forall K \geq 1 \tag{3.30}$$

where $C = C(C_P, \|u^0\|_{H^1(D)})$.

Theorem 3.4.8. *Under Assumption A2 it holds*

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C\,(K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!} \tag{3.31}$$

where $C = C\left(D, \|u^0\|_{H^1(D)}\right)$.

Proof. Using (3.30), we find

$$\begin{aligned}\left\|R^K u\right\|_{H^1(D)} &\leq \frac{1}{K!}\int_0^1 (1-t)^K \left\|D^{K+1}u(tY)[Y]^{K+1}\right\|_{H^1(D)} dt\\ &\leq C\,(K+1)\left(\frac{\|Y\|_{L^\infty}}{\log 2}\right)^{K+1}\int_0^1 (1-t)^K \mathrm{e}^{t\|Y\|_{L^\infty}}\, dt.\end{aligned}$$

Let

$$I_K := \int_0^1 (1-t)^K \mathrm{e}^{t\|Y\|_{L^\infty}}\, dt. \tag{3.32}$$

By induction, we show that

$$I_K = \frac{K!}{\|Y\|^{K+1}_{L^\infty}} \sum_{j=K+1}^{+\infty} \frac{\|Y\|^j_{L^\infty}}{j!}. \tag{3.33}$$

Indeed, for $K = 0$, using the integration by parts formula we find:

$$I_0 = \int_0^1 \mathrm{e}^{t\|Y\|_{L^\infty}}\, dt = \frac{\left(\mathrm{e}^{\|Y\|_{L^\infty}} - 1\right)}{\|Y\|_{L^\infty}} = \frac{1}{\|Y\|_{L^\infty}} \sum_{j=1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}.$$

Suppose now that relation (3.33) holds for $K-1$. Then, integrating by parts,

$$\begin{aligned} I_K &= \left[(1-t)^K \frac{\mathrm{e}^{t\|Y\|_{L^\infty}}}{\|Y\|_{L^\infty}}\right]_0^1 + \frac{K}{\|Y\|_{L^\infty}} \int_0^1 (1-t)^{K-1} \mathrm{e}^{t\|Y\|_{L^\infty}}\, dt \\ &= -\frac{1}{\|Y\|_{L^\infty}} + \frac{K}{\|Y\|_{L^\infty}} I_{K-1} \\ &= -\frac{1}{\|Y\|_{L^\infty}} + \frac{K}{\|Y\|_{L^\infty}} \frac{(K-1)!}{\|Y\|_{L^\infty}^K} \sum_{j=K}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!} \\ &= \frac{K!}{\|Y\|_{L^\infty}^{K+1}} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}. \end{aligned}$$

Hence,

$$\left\|R^K u(Y,x)\right\|_{H^1(D)} \leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!},.$$

Observe that, since $\sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!} \leq \mathrm{e}^{\|Y\|_{L^\infty}}$ and $\mathbb{E}\left[\mathrm{e}^{\|Y\|_{L^\infty}}\right] < +\infty$, the dominated convergence theorem states that

$$\mathbb{E}\left[\sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}\right] = \sum_{j=K+1}^{+\infty} \frac{\mathbb{E}\left[\|Y\|_{L^\infty}^j\right]}{j!}.$$

Using (3.15), we conclude

$$\begin{aligned} \left\|R^K u(Y,x)\right\|_{L^1(\Omega;H^1(D))} &\leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\mathbb{E}\left[\|Y\|_{L^\infty}^j\right]}{j!} \\ &\leq C\ (K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\sigma^{j-2}}{(j-2)!!}. \end{aligned}$$

□

The direct computation of I_K allows the simplification of the term $\|Y\|_{L^\infty}^{K+1}$. On the other hand, it obliges us to handle with the term $\sum_{j=K+1}^{+\infty} \frac{\|Y\|_{L^\infty}^j}{j!}$, and it is not immediate to show the boundness of its statistical moments except for the expected value. A less sharp estimate than (3.31) can be obtained noticing that $I_K \leq \frac{\mathrm{e}^{\|Y\|_{L^\infty}}}{K+1}$. In this case the term $\|Y\|_{L^\infty}^{K+1}$ does not simplify and we have to bound the statistical moments of $\|Y\|_{L^\infty}^{K+1} \mathrm{e}^{\|Y\|_{L^\infty}}$. The advantage is that, using the Cauchy-Schwarz inequality and the Fernique's theorem, we can prove upper bounds for the $L^p(\Omega; H^1(D))$-norm of the residual, for every $p > 0$. We state here only the result for the first two statistical moments of $R^K u$:

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C \left(\frac{1}{\log 2}\right)^{K+1} \sigma^K \sqrt{(2K+2)(2K+1)!!}, \tag{3.34}$$

$$\left\|R^K u\right\|_{L^2(\Omega;H^1(D))} \leq C \left(\frac{1}{\log 2}\right)^{K+1} \sigma^{K+1/2} \left(4(K+1)(4K+3)!!\right)^{1/4}. \tag{3.35}$$

Under Assumption A1, using (3.14) instead of (3.15), both with the exact computation of I_K and with its estimate, we predict that the $L^1(\Omega; H^1(D))$-norm of $R^K u$ behaves as σ^{K+1} and, more generally, the $L^p(\Omega; H^1(D))$-norm of $R^K u$ behaves as σ^{K+1} as a function of σ. The following bound is obtained with the exact computation of I_K

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C\,(K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{\left(\sqrt{R_\gamma}\,\sigma\right)^j}{j!!}, \tag{3.36}$$

whereas, with the estimate for the integral I_K, we obtain:

$$\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} \leq C \left(\sqrt{R_\gamma}\right)^{K+1} \left(\frac{\sigma}{\log 2}\right)^{K+1} \sqrt{(2K+1)!!}, \tag{3.37}$$

$$\left\|R^K u\right\|_{L^2(\Omega;H^1(D))} \leq C \left(\sqrt{R_\gamma}\right)^{K+1} \left(\frac{\sigma}{\log 2}\right)^{K+1} \sqrt{(4K+3)!!}. \tag{3.38}$$

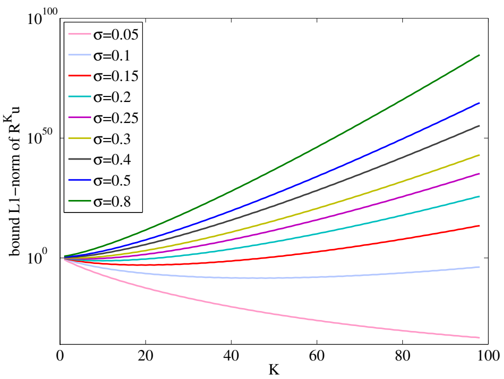


Figure 3.5: *Estimate* (3.31) *as a function of K, for different values of the standard deviation σ.*

In Figure 3.5 we plot in semilogarithmic scale the estimate (3.31) as a function of the order of the residual K. We observe the same behavior as in the Figure 3.3(a), where a sequence given by the product of an exponential term and a bifactorial term is plotted. We highlight that we simply predicted and did not actually prove the divergence of the Taylor series. To do that, it is necessary to show the divergence of a lower bound for the norm of the residual $R^K u$. Nevertheless, in Section 3.6.1 we focus on the simple case of a single random variable and we perform numerical tests which confirm the divergence of the Taylor series.

3.4.4 Optimal K and minimal error

In the previous section we established theoretical estimates on the norm of the Taylor residual, which are divergent for every $\sigma > 0$. Since our results are upper bounds, we did not actually prove the divergence of the Taylor series, but we simply predicted it. Nevertheless, the numerical experiments in Section 3.6 confirm the divergence of the Taylor series. We observed that, for K small enough (3.31) is decreasing as a function of K, whereas for big K (3.31) starts increasing and eventually goes to infinity (see Figure 3.5). Hence, there exists an optimal value of K depending on σ, K^{σ}_{opt}, which can be estimated as the argmin of the right-hand side in (3.31), and beyond which adding new terms to the Taylor expansion will deteriorate the accuracy instead of improving it. The estimate (3.31) states that, for every $\sigma > 0$ fixed, the minimal error err^{σ}_{min} we can commit using a perturbation approach is bounded by the right-hand side of (3.31) evaluated in K^{σ}_{opt}. Here, we provide an approximation for both K^{σ}_{opt} and err^{σ}_{min}.

Proposition 3.4.9. *Let* $0 < \sigma \leq \dfrac{\log^2 2}{5}$*. Then, the optimal order of the Taylor expansion can be estimated as*

$$\bar{K}^{\sigma} := \left\lfloor \frac{1}{\alpha^2} \right\rfloor - 4, \tag{3.39}$$

where $\alpha := \dfrac{\sigma}{\log 2}$.

We derive an estimate on the minimal error err^{σ}_{min} as

$$err^{\sigma}_{min} \leq b(\bar{K}^{\sigma}) \tag{3.40}$$

where $b(K)$ is the right-hand side in (3.31).

To prove Proposition 3.4.9 we need the following lemma.

Lemma 3.4.10. *Let* $0 < \gamma < 1$ *real. Then, for every integer* $N \geq 1$*,*

$$\sum_{n \geq N} \frac{\gamma^n}{n!!} \leq \frac{1}{1-\gamma} \frac{\gamma^N}{N!!}. \tag{3.41}$$

Proof. Using that $\left\{ \dfrac{\gamma^n}{n!!} \right\}$ is a decreasing sequence and $0 < \gamma < 1$, we easily conclude:

$$\sum_{n \geq N} \frac{\gamma^n}{n!!} \leq \frac{1}{N!!} \sum_{n \geq N} \gamma^n = \frac{1}{N!!} \frac{\gamma^N}{1-\gamma}.$$

□

Proof of Proposition 3.4.9. The first step of the proof consists in showing that

$$\left\| R^K u \right\|_{L^1(\Omega; H^1(D))} \leq C \frac{1}{(\log 2)^2 (1-\sigma)} v(K), \tag{3.42}$$

where $v(K) = \alpha^{K-1}(K+2)!!$, $\alpha = \dfrac{\sigma}{\log 2}$ and C independent of K. Starting from (3.31) and using Lemma 3.4.10 we find:

$$\begin{aligned}\left\|R^K u\right\|_{L^1(\Omega;H^1(D))} &\leq C\frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1}(K+1)!\,\frac{\sigma^{K-1}}{(K-1)!!}\\&= C\frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1}\sigma^{K-1}(K+1)K\frac{(K-1)!}{(K-1)!!}\\&= C\frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1}\sigma^{K-1}(K+1)K(K-2)!!\\&\leq C\frac{1}{1-\sigma}\left(\frac{1}{\log 2}\right)^{K+1}\sigma^{K-1}(K+2)!!,\end{aligned}$$

so that (3.42) is proved. To find the argmin of $v(K)$, we consider $\log(v(K))$:

$$\log(v(K)) = \begin{cases}(2n-3)\log\alpha + \log(2n)!!, & \text{if } K = 2n-2,\\(2n-4)\log\alpha + \log(2n-1)!!, & \text{if } K = 2n-3.\end{cases}$$

We analyze the two cases K odd or even separately. Firstly, suppose $K = 2n-2$. Using that $(2n)!! = 2^n n!$ and $\mathrm{e}\left(\frac{n}{\mathrm{e}}\right)^n \leq n! \leq \mathrm{e}\, n\left(\frac{n}{\mathrm{e}}\right)^n$, we find

$$\begin{aligned}\log(v(n)) &= (2n-3)\log\alpha + n\log 2 + \log(n!)\\&\leq (2n-3)\log\alpha + n\log 2 + \log\left(\mathrm{e}n\left(\frac{n}{\mathrm{e}}\right)^n\right)\\&= (2n-3)\log\alpha + n\log 2 + \log n + n\log n - n + 1.\end{aligned}$$

On the other hand, if $K = 2n-3$, we use that $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$:

$$\begin{aligned}\log(v(n)) &= (2n-4)\log\alpha + \log(2n!) - n\log 2 - \log(n!)\\&\leq (2n-4)\log\alpha + \log(2\mathrm{e}n\left(\frac{2n}{\mathrm{e}}\right)^{2n}) - n\log 2 - \log(\mathrm{e}\left(\frac{n}{\mathrm{e}}\right)^n)\\&= (2n-4)\log\alpha + n\log 2 + \log n + n\log n - n + \log 2.\end{aligned}$$

We observe that, for both K odd and even,

$$\log(v(n)) \leq w(n) + \bar{C}$$

where

$$w(n) := 2n\log\alpha + n\log 2 + (n+1)\log(n+1) - n \tag{3.43}$$

and $\bar{C}$ is the positive constant

$$\bar{C} = \begin{cases}-3\log\alpha + 1, & \text{if } K = 2n-2,\\-4\log\alpha + \log 2, & \text{if } K = 2n-3.\end{cases} \tag{3.44}$$

Note that we have bounded $(n+1)\log n$ with $(n+1)\log(n+1)$ in view of having a simpler derivative $\dfrac{d}{dn}w(n)$. We look for the argmin$(w(n))$ by imposing $\dfrac{d}{dn}w(n) = 0$, that is

$$2\log\alpha + \log 2 + \log(n+1) + 1 - 1 = 0,$$

Table 3.1: *This Table contains the optimal K^{σ}_{opt} as argmin of the right-hand side of* (3.31)*, its estimate* $\bar{K}^{\sigma}$ (3.39) *and the estimate of err^{σ}_{min}.*

σ	K^{σ}_{opt}	$\bar{K}^{\sigma}$	$b(\bar{K}^{\sigma})$
0.10	45	44	2.7131e-09
0.15	19	17	9.9610e-04
0.18	11	11	1.8118e-02
0.20	9	8	5.1292e-02

which implies $n = \left\lfloor \dfrac{1}{2\alpha^2} \right\rfloor - 1$, so that we can choose $\bar{K}^{\sigma} = \left\lfloor \dfrac{1}{\alpha^2} \right\rfloor - 4$. $\square$

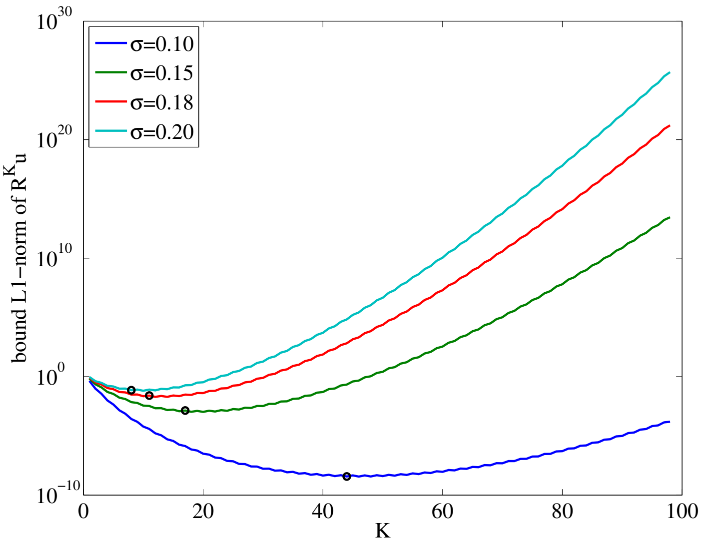


Figure 3.6: *Semilogarithmic plot of the right-hand side of* (3.31) $b(K)$ *(continuous line) and of the points* $\left(\bar{K}^{\sigma}, b(\bar{K}^{\sigma})\right)$ *(black dot) for different values of* σ*.*

In Table 3.1 we report the optimal K^{σ}_{opt} as argmin of the right-hand side in (3.31), its estimate $\bar{K}^{\sigma}$ (3.39), and the estimate of minimal error in $L^1(\Omega; H^1)$-norm (right-hand side of (3.40)) for different values of σ. Figure 3.6 represents $b(K)$ (continuous line) and the points $\left(\bar{K}^{\sigma}, b(\bar{K}^{\sigma})\right)$ (black dot) for different values of σ. We take the values $b(\bar{K}^{\sigma})$ as an estimate of the minimal error we can commit by performing a perturbation approach as in the previous section.
As Table 3.1 and Figure 3.6 suggest, the estimate of the optimal K (3.39) is quite sharp. Moreover, the smaller is σ, the bigger is K^{σ}_{opt} and the smaller is the minimal error we commit.

3.5 Finite number of independent random variables

In Section 3.4 we analyzed the Darcy problem where the permeability is described as an infinite-dimensional random field with lognormal distribution: $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, $Y(\omega, x)$ centered Gaussian random field. We performed a perturbation analysis around the zero mean of the Gaussian field Y and we studied the approximation properties of the Taylor polynomial. In this setting, the Taylor polynomial is not computable, since it involves the Gateaux derivatives of the stochastic solution u with respect to Y. Here we consider a simpler framework, in which the permeability field is modeled using a finite number of independent random variables. This situation can be achieved for example by approximating the Gaussian field $Y(\omega, x)$ by a N-terms Karhunen-Loève expansion (see Proposition 3.3.4). In this case the Taylor polynomial is explicitly computable.

3.5.1 Gaussian random vector

In this section we consider a permeability tensor $a(\omega, x)$ satisfying the following assumption:

Assumption A3. *The permeability tensor is modeled as* $a(\omega, x) = e^{Y(\omega,x)}$*, with* $Y(\omega, x)$ *centered, given by*

$$Y(\omega, x) = \sigma \sum_{n=1}^{N} \sqrt{\tilde{\lambda}_n} Y_n(\omega)\phi_n(x) \tag{3.45}$$

where $0 < \sigma < 1$*,* $\phi(x) = (\phi_1(x), \dots, \phi_N(x)) \in (L^\infty(D))^N$ *is a vector of* L^2*-orthonormal functions, and* $\mathbf{Y} = (Y_1, \dots, Y_N)$ *is a vector of* N *independent standard Gaussian random variables.*

In view of the Doob-Dynkin lemma [84], the solution u can be written as $u(\omega, x) = u(\mathbf{Y}(\omega), x)$, where $\mathbf{Y}(\omega) = (Y_1(\omega), \dots, Y_N(\omega))$. We define the space $L^p_\rho\left(\mathbb{R}^N; H^1(D)\right)$ as the space of functions $v : \mathbb{R}^N \times D \to \mathbb{R}$ such that

$$\|v\|_{L^p_\rho(\mathbb{R}^N;H^1(D))} := \left(\int_{\mathbb{R}^N} \|v(\mathbf{Y}, \cdot)\|^p_{H^1} d\rho\right)^{1/p} < \infty,$$

where ρ_n is the standard Gaussian probability density $\forall\, n$, and $\rho = \prod_{n=1}^N \rho_n$ is the joint probability density of the vector $\mathbf{Y}$. $L^p_\rho\left(\mathbb{R}^N; H^1(D)\right)$ is a Banach space, and for $p = 2$ it is a Hilbert space.

Then, problem (3.4) becomes:

Darcy problem - independent random variables

Given $f \in L^2(D)$*, find* $u \in L^p_\rho\left(\mathbb{R}^N; H^1(D)\right)$ *s.t.* $u|_{\Gamma_D} = g$ and

$$\int_D e^{\sigma \sum_{n=1}^N \sqrt{\tilde{\lambda}_n} Y_n \phi_n(x)} \nabla u(\mathbf{Y}, x) \cdot \nabla v(x)\, dx = \int_D f(x)v(x)\, dx \tag{3.46}$$

$$\forall v \in H^1_{\Gamma_D}(D), \quad \rho - \text{a.s. in } \mathbb{R}^N.$$

Let us denote with $\partial^{\mathbf{k}}_{\mathbf{Y}} u(\mathbf{Y}, x)$ the $\mathbf{k}$-th order partial derivative of the infinitely many times differentiable solution $u : \mathbb{R}^N \to H^1(D)$ evaluated in $(\mathbf{Y}, x)$,

$$\partial^{\mathbf{k}}_{\mathbf{Y}} u(\mathbf{Y}, x) = \frac{\partial^{k_1+\ldots+k_N} u(Y_1, \dots, Y_N, x)}{(\partial Y_1)^{k_1} \ldots (\partial Y_N)^{k_N}}. \tag{3.47}$$

The multivariate Taylor polynomial of u of degree K in a neighborhood of $\mathbf{0}$ is given by:

$$T^K u(\mathbf{Y}, x) = \sum_{|\mathbf{k}| \leq K} \frac{\partial_{\mathbf{Y}}^{\mathbf{k}} u(\mathbf{0}, x)}{\mathbf{k}!} \mathbf{Y}^{\mathbf{k}}, \tag{3.48}$$

where $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{N}^N$ is a multi index of length N, $\mathbf{k}! := \prod_{n=1}^N k_n!$, $|\mathbf{k}| := \sum_{n=1}^N k_n$ and $\mathbf{Y}^{\mathbf{k}} := \prod_{n=1}^N Y_n^{k_n}$. The integral expression of the K-th order residual $R^K u(\mathbf{Y}, x) = u(\mathbf{Y}, x) - T^K u(\mathbf{Y}, x)$ is

$$R^K u(\mathbf{Y}, x) := \sum_{|\mathbf{k}| = K+1} \frac{K+1}{\mathbf{k}!} \mathbf{Y}^{\mathbf{k}} \int_0^1 (1-t)^K \partial_{\mathbf{Y}}^{\mathbf{k}} u(t\mathbf{Y}, x) dt. \tag{3.49}$$

In what follows we apply the results of Sections 3.4.2 and 3.4.3 to obtain upper bounds for the norms of the Taylor polynomial and Taylor residual. In this finite dimensional setting, the Gateaux derivative $D^K u(0)[Y]^K$ simplifies:

$$D^K u(0)[Y]^K = \sum_{|\mathbf{k}| = K} \partial_{\mathbf{Y}}^{\mathbf{k}} u(\mathbf{0}, x) \mathbf{Y}^{\mathbf{k}}$$

Provided that ϕ_n is Hölder continuous with exponent $0 < \gamma \leq 1$ for every n, the theoretical estimate (3.14) applies, leading to the following upper bound

$$\mathbb{E}\left[\|Y\|_{L^\infty}^k\right] \leq C\, \sigma^k\, (R_{\gamma,N})^{k/2}\, (k-1)!!. \tag{3.50}$$

where $R_{\gamma,N} := \sum_{n=1}^N \tilde{\lambda}_n \, \|\phi_n\|^2_{C^{0,\gamma}}$, and C is a positive constant independent of k and σ. Moreover, the theoretical estimates on the norm of the Taylor polynomial (3.27), (3.28) and Taylor residual (3.36), (3.37) and (3.38) still hold with $R_{\gamma,N}$ instead of R_γ.

Note that, letting $N \to +\infty$, we recover the estimates given in the infinite dimensional setting ($Y(\omega, x)$ random field), provided that Assumption A1 is satisfied.

3.5.2 Bounded random vector

Whether the permeability tensor is modeled as a lognormal random field or is described using a finite number of independent Gaussian random variables, in both cases we predicted the divergence of the Taylor series of the stochastic solution u. The common characteristic of the lognormal infinite and finite dimensional setting is that the random field/vector is unbounded, that is, it can assume every real positive value. On the contrary, in the case of independent *bounded* random variables, the convergence of the Taylor series to u has been proved in [9]. (See also [33]). In particular, the authors study the Darcy problem (3.4) where $a(\omega, x)$ is given by

$$a(\omega, x) = \mathbb{E}[a](x) + \sum_{n=1}^N \phi_n(x) Y_n(\omega), \tag{3.51}$$

with $Y_n(\Omega) \subset [-\gamma_n, \gamma_n]$, $0 < \gamma_n < +\infty$ $\forall n$, and $\phi_n \in L^\infty(D)$ $\forall n$. If there exist two constants α_1, α_2 such that $0 < \alpha_1 \leq a(\omega, x) \leq \alpha_2 < \infty$ a.s. in Ω, then the stochastic

Darcy problem is well-posed. Moreover, if

$$\xi := \left\| \frac{\sum_{n=1}^{N} |\phi_n(x)|\gamma_n}{\mathbb{E}[a](x)} \right\|_{L^\infty} < 1, \tag{3.52}$$

then, for every $p > 0$,

$$\left\|T^K u\right\|_{L^p(\Omega;H^1)} \leq C_1 \sum_{k=0}^{K} \xi^k = C_1 \frac{1-\xi^{K+1}}{1-\xi}, \tag{3.53}$$

$$\left\|u - T^K u\right\|_{L^p(\Omega;H^1)} = \left\|R^K u\right\|_{L^p(\Omega;H^1)} \leq C_2 \sum_{k=K+1}^{\infty} \xi^k = C_2 \frac{1}{1-\xi} \xi^{K+1}, \tag{3.54}$$

where $C_1 = C_1(\alpha_1, C_P, f)$, $C_2 = C_2(\alpha_1, C_P, f)$.

The parameter ξ describes the variability of the bounded permeability field (3.51). In the bounded setting, we conclude that, if the variability of $a(\omega, x)$ is small enough, that is if ξ is small enough, the Taylor series is convergent.

3.6 Single random variable - Numerical results

In the previous sections, we predicted the divergence of the Taylor series of the stochastic solution u in the case where the permeability field $a(\omega, x)$ is described as a lognormal random field or vector. Recall that the Taylor polynomial is directly computable only in the finite-dimensional setting. Here we consider a simple case, where $a(\omega, x) = \mathrm{e}^{\phi(x)Y(\omega)}$, with $Y \simeq \mathcal{N}(0, \sigma^2)$. We compute the Taylor polynomial of u and perform some numerical tests, which confirm the divergence of the Taylor series $\forall\, \sigma > 0$. We also consider the case of a bounded random variable $Y \in [\gamma, \gamma]$ and show that the Taylor series is indeed convergent in this case, as recalled in Section 3.5.2.

3.6.1 Gaussian setting

Let us suppose $N = 1$, $Y \sim \mathcal{N}(0, \sigma^2)$, with $0 < \sigma < 1$ and $\phi \in L^\infty(D)$. Theorem 3.4.6 states that the boundary value problem solved by the k-th derivative of u, $\partial_Y^k u(0, x)$ is well-posed, and

$$\left\|\partial_Y^k u(0, x)\right\|_{H^1(D)} \leq C \left(\frac{\|\phi\|_{L^\infty}}{\log 2}\right)^k k!, \tag{3.55}$$

where $C = C(C_P, f)$. In the same way, (3.30) implies

$$\left\|\partial_Y^k u(tY, x)\right\|_{H^1(D)} \leq C\, \mathrm{e}^{t|Y|\|\phi\|_{L^\infty}} \left(\frac{\|\phi\|_{L^\infty}}{\log 2}\right)^k k!. \tag{3.56}$$

Using the upper bound (3.55) and the value of the statistical moments of $|Y|$

$$\mathbb{E}\left[|Y|^p\right] = C\, \sigma^p (p-1)!!, \quad C = \begin{cases} 1 & \text{if } p \text{ is even} \\ \sqrt{\frac{2}{\pi}} & \text{if } p \text{ is odd} \end{cases} \tag{3.57}$$

we deduce

$$\left\| T^K u \right\|_{L^p_\rho(\mathbb{R};H^1(D))} \leq C \sum_{k=0}^{K} \left(\frac{\|\phi\|_{L^\infty}\, \sigma}{\log 2} \right)^k ((pk-1)!!)^{1/p} \tag{3.58}$$

where $T^K u(Y,x) := \displaystyle\sum_{k=0}^{K} \frac{\partial_Y^k u(0,x)}{k!} Y^k$ is the K-th order Taylor polynomial and $C = C\left(C_P, \|f\|_{L^2(D)}\right)$. Similarly, using (3.56), we derive the following estimate, analogous to (3.31):

$$\left\| R^K u \right\|_{L^1_\rho(\mathbb{R};H^1(D))} \leq C\, (K+1)! \left(\frac{1}{\log 2} \right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{(\|\phi\|_{L^\infty}\, \sigma)^j}{j!!}, \tag{3.59}$$

where $R^K u(Y,x) := \dfrac{1}{K!} \displaystyle\int_0^1 (1-t)^K \partial_Y^{K+1} u(tY,x) Y^{K+1} dt$ is the K-th order integral residual and $C = C(D, \|f\|_{L^2(D)})$. Note that the theoretical estimate (3.59) is obtained by direct computation of the integral $I_K = \displaystyle\int_0^1 (1-t)^K \mathrm{e}^{t|Y|\|\phi\|_{L^\infty}}$:

$$I_K = \frac{K!}{(|Y|\, \|\phi\|_{L^\infty})^{K+1}} \sum_{j=K+1}^{+\infty} \frac{(|Y|\, \|\phi\|_{L^\infty})^j}{j!}$$

(see (3.32) and (3.33)). As observed in the infinite dimensional setting, the integral I_K can be bounded by $I_K \leq \dfrac{\mathrm{e}^{|Y|\|\phi\|_{L^\infty}}}{K+1}$. In this way, using the Cauchy-Schwarz inequality for $\mathbb{E}\left[|Y|^{p(K+1)}\, \mathrm{e}^{p|Y|\|\phi\|_{L^\infty}} \right]$, an upper bound for the $L^p_\rho(\mathbb{R}; H^1(D))$-norm of the residual can be deduced:

$$\left\| R^K u \right\|_{L^p_\rho(\mathbb{R};H^1(D))} \leq C \left(\frac{\|\phi\|_{L^\infty}\, \sigma}{\log 2} \right)^{K+1} ((2p(K+1)-1)!!)^{1/2p}, \tag{3.60}$$

where $C = C(D, \|f\|_{L^2(D)})$.

We develop some numerical computations in a 1D case, with $D = [0,1]$, homogeneous Dirichlet boundary conditions imposed on $\Gamma_D = \{0,1\}$, $f(x) = x$ and $\phi(x) = \cos(\pi x)$. The problems solved by $u^0(x)$ and $\partial_Y^k u(0,x)$ respectively are:

$$\int_0^1 (u^0(x))' v'(x) dx = \int_0^1 f(x) v(x) dx, \tag{3.61}$$

$\forall v \in H^1_0([0,1])$, and

$$\int_0^1 (\partial_Y^k u(0,x))' v'(x) dx = -\sum_{l=1}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \int_0^1 \phi(x)^l (\partial_Y^{k-l} u(0,x))' v'(x) dx, \tag{3.62}$$

$\forall v \in H^1_0([0,1])$, $\forall k \geq 1$. Let $\{\varphi_i\}_{i=1}^N$ be the piecewise linear finite element basis. Applying the linear finite element method (FEM) to problem (3.61), we end up with the following system:

$$AU^0 = F^0, \tag{3.63}$$

where the stiffness matrix is tridiagonal, symmetric and its generic element is given by $A_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)dx$, the right-hand side is a vector whose j-th element is $F_j^0 = \int_0^1 f(x)\varphi_j(x)dx$, and U^0 is the unknown vector. Applying the linear FEM to the k-th problem (3.62), we end up with the following system:

$$AU^k = -\sum_{l=1}^{k} \binom{k}{l} F^l U^{k-l}, \tag{3.64}$$

where the stiffness matrix is the same as in (3.63) and the right-hand side contains the solutions $U^0, \ldots, U^{k-1}$ of the l-th problem for $l = 0, \ldots, k-1$, and the matrices F^l, $l = 1, \ldots, k$, where $F_{ij}^l = \int_0^1 (\phi(x))^l \varphi_j'(x)\varphi_i'(x)dx$.

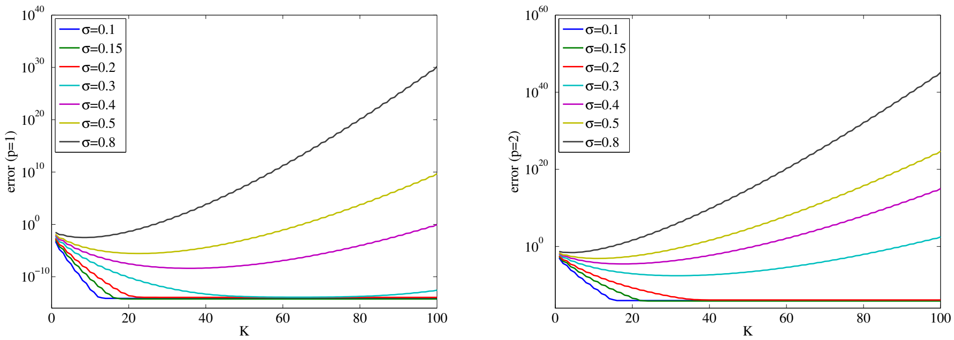


Figure 3.7: *Error $\|u_h - T^K u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}$ computed by linear FEM in space and high order Hermite quadrature formula, for $p = 1$ (left) and $p = 2$ (right).*

Let $Y = \bar{Y}$ be fixed and let us denote with $u_h(\bar{Y}, x)$ the linear FEM solution of the Darcy problem (3.46) collocated in $Y = \bar{Y}$, so that

$$T^K u_h(\bar{Y}, x) = \sum_{k=0}^{K} \sum_{i=1}^{N} \frac{U_i^k}{k!} \varphi_i(x) \bar{Y}^k.$$

In Figure 3.7 we plot in semilogarithmic scale the errors $\|u_h - T^K u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}$ ($p = 1, 2$) computed by linear FEM in space and high order Hermite quadrature formula, for different values of the standard deviation $0 < \sigma < 1$. Note that we have chosen the same spatial discretization both for u_h and $T^K u_h$, so that we observe only the truncation error of the Taylor series. This figure numerically confirm both the divergence of the Taylor series $\forall\, \sigma$, and the existence of an optimal order of the Taylor polynomial K^σ_{opt} depending on σ (see Section 3.4.4). Moreover, the higher is p, the worse is the behavior of the norm of the residual, since it starts diverging for a smaller K.

Figure 3.8 compares the computed error $\|u_h - T^K u_h\|_{L^1_\rho(\mathbb{R};L^2(D))}$ with the theoretical estimate for the $L^1_\rho(\mathbb{R}; H^1(D))$ norm of the residual (3.59). Figure 3.9 compares the theoretical upper bound for the $L^1_\rho(\mathbb{R}; H^1(D))$ and $L^2_\rho(\mathbb{R}; H^1(D))$ norms of the Taylor polynomial (see (3.58)) with the same quantities computed by linear FEM in space and

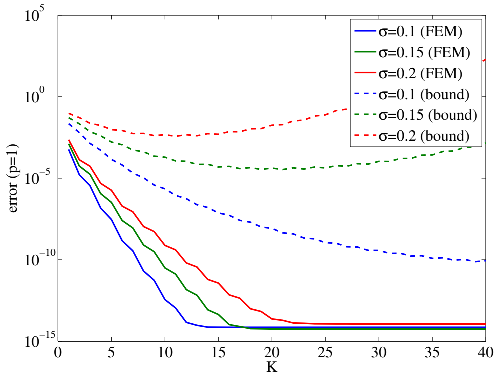


Figure 3.8: *Comparison between the computed error* $\|u_h - T^K u_h\|_{L^1_\rho(\mathbb{R};L^2(D))}$ *and the theoretical estimate* (3.59).

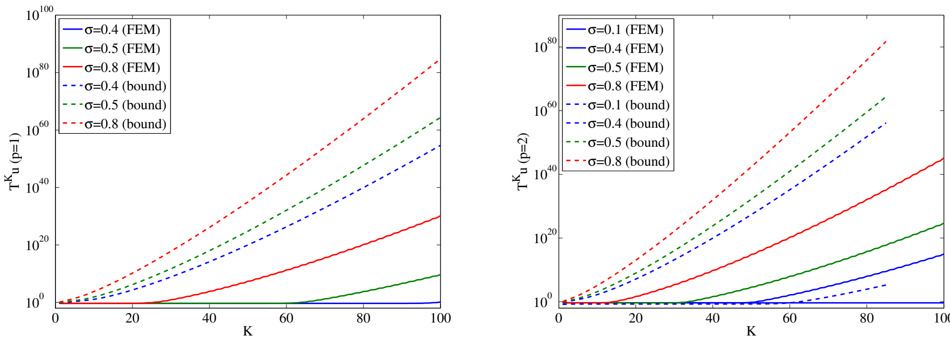


Figure 3.9: *Comparison between the computed norm* $\|T^K u_h\|_{L^p_\rho(\mathbb{R};L^2(D))}$ *and its theoretical estimate* (3.58)*, for* $p = 1$ *(left) and* $p = 2$ *(right).*

a high order Hermite quadrature formula. Both the estimates for the Taylor polynomial (3.58) and the Taylor residual (3.59) are quite pessimistic. This is a consequence of the estimate on $\|\partial^k_Y u(0,x)\|_{H^1(D)}$, which is itself very pessimistic.

With the aim of improving the theoretical bounds on the norm of the Taylor polynomial and residual, we assume that the growth of the derivatives follows the ansatz:

$$\|\partial^k_Y u(0,x)\|_{L^2(D)} \sim \left(\frac{\gamma \|\phi\|_{L^\infty}}{\log 2}\right)^k k! \tag{3.65}$$

for a suitable value of γ. Then we try to fit the value of γ starting from the numerical results obtained. In this specific example, the fitting procedure gives $\gamma = \dfrac{1}{3.5}$. Nevertheless, we highlight that the choice of γ strongly depends on $\phi(x)$, whereas it seems not to depend on the loading term $f(x)$. In Figure 3.10 we plot in semilogarithmic scale the quantity $\|\partial^k_Y u(0,x)\|_{L^2(D)}$ computed by linear FEM, compared with the theoretical estimate (3.55) and the fitted one (3.65) with $\gamma = \frac{1}{3.5}$. The agreement of the computed norm $\|\partial^k_Y u(0,x)\|_{L^2(D)}$ with the fitted estimate (3.65) is remarkable, which strongly indicates that the ansatz (3.65) is appropriate.

We then use the fitted value $\gamma = \dfrac{1}{3.5}$ in the estimate (3.58) of the norm of the Taylor

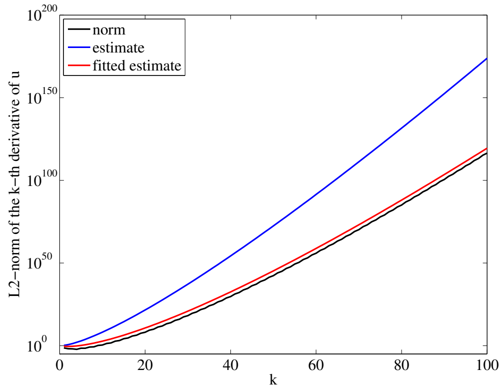


Figure 3.10: *Comparison between the quantity $\left\|\partial_Y^k u(0,x)\right\|_{L^2(D)}$ computed by linear FEM, its theoretical estimate* (3.55) *and the fitted one* (3.65) *with $\gamma = \dfrac{1}{3.5}$.*

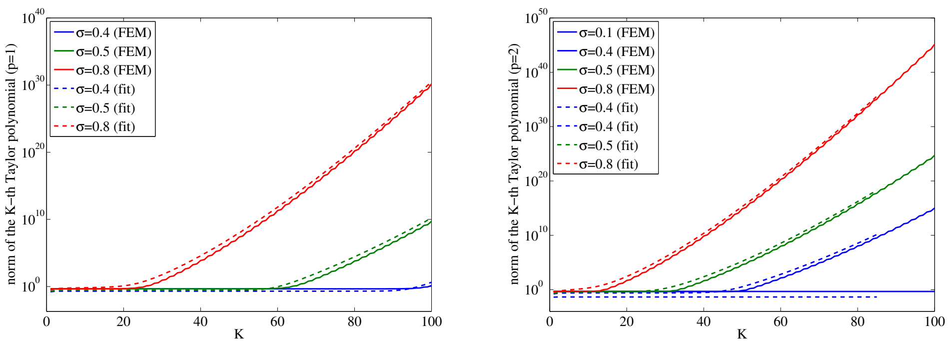


Figure 3.11: *Comparison between the computed quantity $\left\|T^K u_h\right\|_{L^p_\rho(\mathbb{R};L^2(D))}$ and its theoretical estimate* (3.66) *with the fitted value $\gamma = \dfrac{1}{3.5}$, for $p = 1$ (left) and $p = 2$ (right).*

polynomial

$$\left\|T^K u\right\|_{L^p_\rho(\mathbb{R};H^1(D))} \leq C \sum_{k=0}^{K} \left(\frac{\gamma\,\sigma\,\;\|\phi\|_{L^\infty}}{\log 2} \right)^k ((pk-1)!!)^{1/p} \tag{3.66}$$

as well as on the norm of the residual (3.59)

$$\left\|R^K u\right\|_{L^1_\rho(\mathbb{R};H^1(D))} \leq C\;(K+1)! \left(\frac{1}{\log 2}\right)^{K+1} \sum_{j=K+1}^{+\infty} \frac{(\|\phi\|_{L^\infty}\,\gamma\,\sigma)^j}{j!!}. \tag{3.67}$$

Figures 3.11 and 3.12 compare the computed quantities $\left(\left\|T^K u_h\right\|_{L^p_\rho(\mathbb{R};H^1(D))}\right.$ for $p = 1, 2$ and $\left.\left\|R^K u_h\right\|_{L^1_\rho(\mathbb{R};H^1(D))}\right.$ respectively$\left.\right)$ with the fitted bounds (3.66) and (3.67) respectively. We underline that, with the ansatz (3.65) on the growth of the derivatives we are able to sharply predict the optimal order of the Taylor polynomial K^σ_{opt}.

Finally, we analyze the behavior of the error $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ as a function of σ. Figure 3.13 shows this error in semilogarithmic scale. Observe that the exponential behavior σ^{K+1} predicted in (3.37), is confirmed.

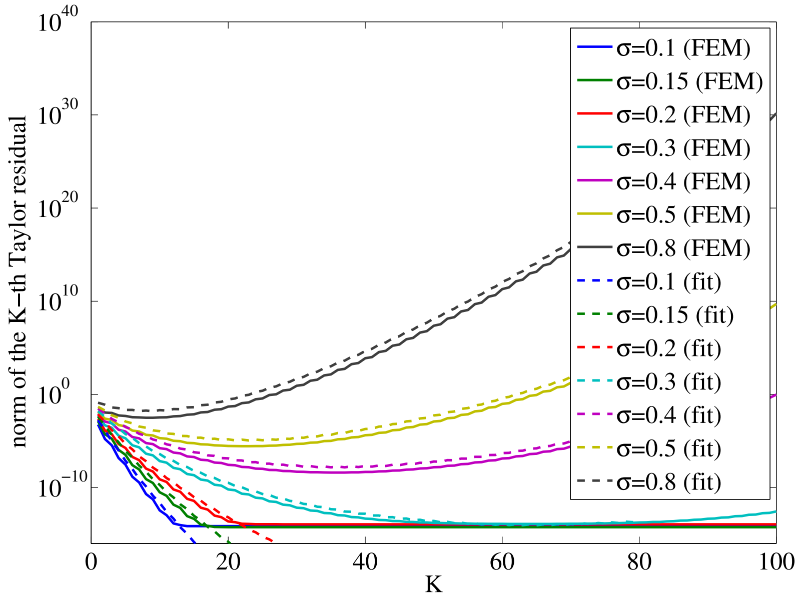


Figure 3.12: *Comparison between the computed quantity* $\left\|R^K u_h\right\|_{L^1(\Omega;L^2(D))}$ *and its theoretical estimate* (3.67) *with the fitted value* $\gamma = \dfrac{1}{3.5}$.

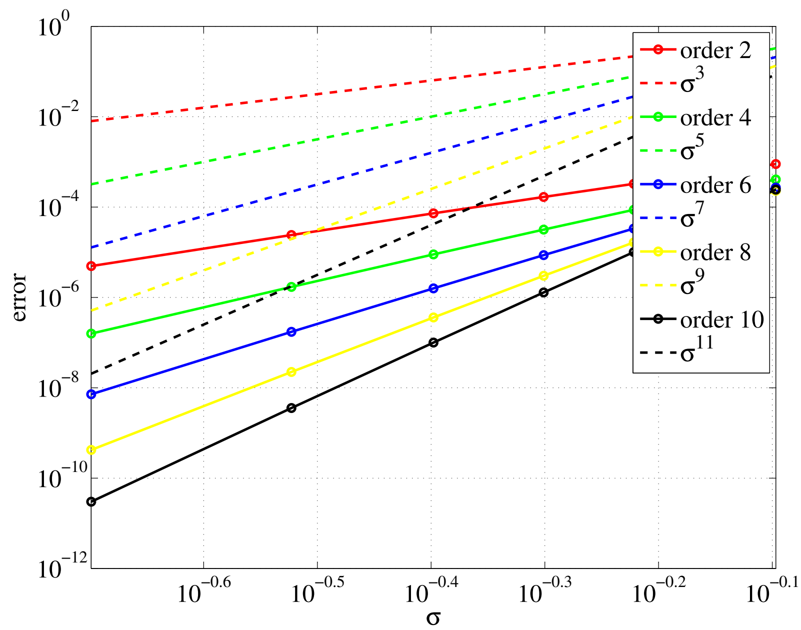


Figure 3.13: *Error* $\left\|\mathbb{E}\left[u_h\right] - \mathbb{E}\left[T^K u_h\right]\right\|_{L^2(D)}$ *as a function of* σ.

3.6.2 Uniform setting

Let Y be a uniform random variable $Y \sim \mathcal{U}([-\gamma,\gamma])$, ρ the uniform density function and $\phi \in L^\infty(D)$. Let us suppose that the permeability coefficient $a(\omega,x) = \mathbb{E}[a](x) + \phi(x)Y(\omega)$ satisfies $0 < \alpha_1 \leq a(\omega,x) \leq \alpha_2 < \infty$ a.s. in Ω, so that the stochastic Darcy problem is well-posed.

The upper bounds on $\left\|T^K u\right\|_{L^p_\rho(\mathbb{R};H^1(D))}$ (3.53) and $\left\|u - T^K u\right\|_{L^p_\rho(\mathbb{R};H^1(D))}$ (3.54) can be achieved starting from the following estimate on the norm of the k-th derivative of u:

$$\left\|\partial^k_Y u(0,x)\right\|_{H^1(D)} \leq C \left\|\frac{\phi(x)}{\mathbb{E}[a](x)}\right\|^k_{L^\infty(D)} k!, \tag{3.68}$$

where $C = C(C_P, f, \alpha_1)$. This theoretical bound (3.68) is obtained using a recursive argument similar to that in the proof of Theorem 3.4.6.

Here we present some numerical examples which confirm the convergence of the Taylor series, provided that the variability of the field is small enough (see condition (3.52)). Let $D = [0,1]$, $\phi(x) = (x^2+1)$, $\mathbb{E}[a](x) = 10(x^2+1)$, and suppose that homogeneous Dirichlet boundary conditions are imposed on $\Gamma_D = \{0,1\}$. Instead of using the upper bound (3.68), we compute $\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)}$ by linear FEM. The problem solved by $\partial^k_Y u(0,x)$ is:

$$\int_0^1 (\mathbb{E}[a](x)\partial^k_Y u(0,x))' v'(x)dx = -k\int_0^1 \phi(x)(\partial^{k-1}_Y u(0,x))'v'(x)dx, \tag{3.69}$$

$\forall\, v \in H^1_0([0,1]),\ k \geq 1$, and its discretization is

$$AU^k = -k\left(FU^{k-1}\right), \tag{3.70}$$

where both the matrices A and F are tridiagonal and symmetric, and their generic elements respectively are

$$A_{ij} = \int_0^1 \mathbb{E}[a](x)\varphi'_i(x)\varphi'_j(x)dx$$

and

$$F_{jl} = \int_0^1 \phi(x)\varphi'_l(x)\varphi'_j(x)dx,$$

where $\{\varphi_i\}_{i=1}^N$ is the linear FEM basis. Note that the right-hand side of (3.70) contains only U^{k-1}, contrary to the case of a lognormal variable, which contains $U^0,\ldots,U^{k-1}$ instead.

Let $Y = \bar{Y}$ be fixed, $u_h(\bar{Y},x)$ be the linear FEM solution of the Darcy problem (3.46) collocated in $Y = \bar{Y}$, and

$$T^K u_h(\bar{Y},x) = \sum_{k=0}^K \sum_{i=1}^N \frac{U^k_i}{k!}\varphi_i(x)\bar{Y}^k.$$

In Figure 3.14 we show in semilogarithmic scale the norm $\left\|T^K u_h\right\|_{L^1_\rho(\mathbb{R};L^2(D))}$ computed by linear FEM in space and a high order Legendre quadrature formula compared with its theoretical estimate (3.53). Figure 3.15 compares the error

$$\left\|u_h - T^K u_h\right\|_{L^p_\rho(\mathbb{R};L^2(D))}, \quad p = 1,2$$

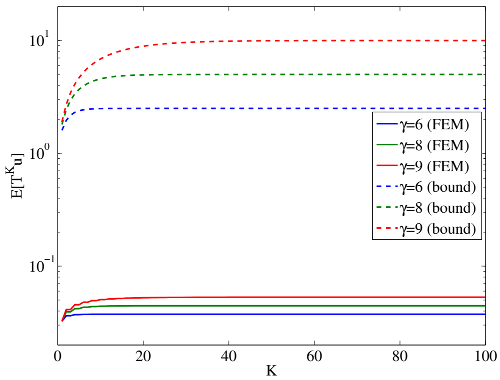


Figure 3.14: *Comparison between the norm* $\left\|T^K u_h\right\|_{L^1_\rho(\mathbb{R};L^2(D))}$ *computed by linear FEM in space and a high order Legendre quadrature formula, and its theoretical estimate* (3.53).

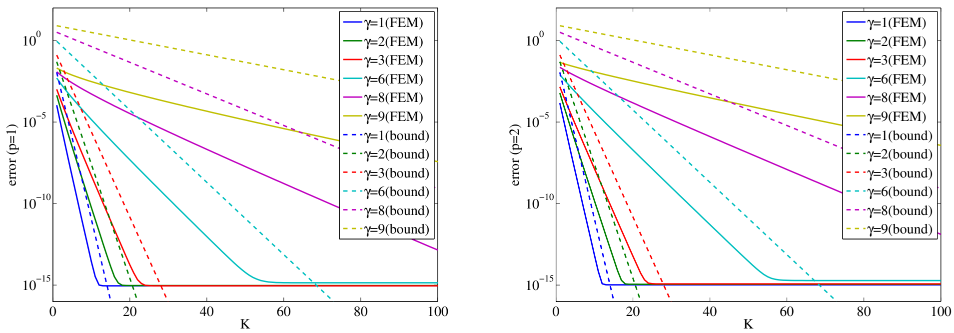


Figure 3.15: *Comparison between the error* $\left\|u_h - T^K u_h\right\|_{L^p_\rho(\mathbb{R};L^2(D))}$ *computed by linear FEM and a high order Legendre quadrature formula, and its theoretical estimate* (3.54)*, for* $p = 1$ *(left) and* $p = 2$ *(right).*

computed by linear FEM in space and a high order Legendre quadrature formula with its theoretical estimate (3.54). Note that we have chosen the same spatial discretization both for u_h and $T^K u_h$, so that we observe only the truncation error of the Taylor series. We conclude that the theoretical estimates for both the norm of the Taylor polynomial (3.53) and the norm of the Taylor residual (3.54) are quite pessimistic. As in the Gaussian setting, we identify the cause in the pessimistic estimate of the $\left\|\partial^k_Y u(0,x)\right\|_{H^1(D)}$ (3.68). Figure 3.16(a) compares the norm $\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)}$ computed by linear FEM with its theoretical upper bound (3.68), and in Figure 3.16(b) we plot in semilogarithmic scale the ratio between the theoretical upper bound and the computed norm. We believe that, by a fitting procedure on the estimate of $\left\|\partial^k_Y u(0,x)\right\|_{L^2(D)}$, better a priori/a posteriori upper bounds for the norm of the Taylor polynomial and the Taylor residual can be provided, as done in the Gaussian setting.

3.7 Conclusions

The present work addresses the Darcy problem describing the single-phase flow in a bounded randomly heterogeneous porous medium occupying the domain $D \subset \mathbb{R}^d$,

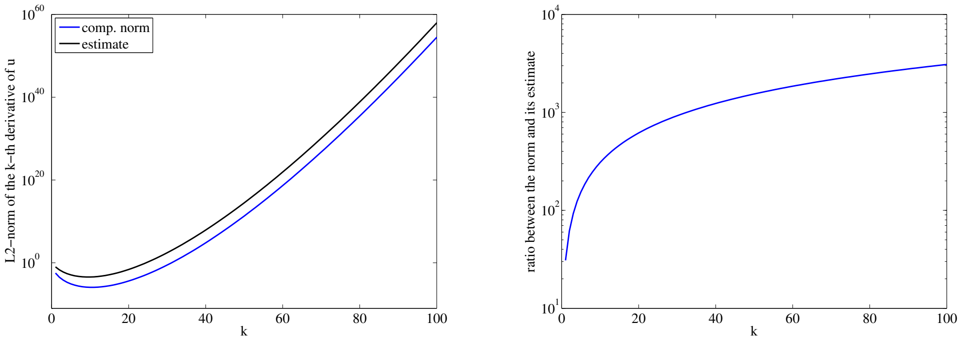


Figure 3.16: *3.16(a): Comparison between the norm* $\left\|\partial_Y^k u(0,x)\right\|_{L^2(D)}$ *computed by linear FEM and its theoretical upper bound* (3.68). *3.16(b): Ratio between the upper bound of* $\left\|\partial_Y^k u(0,x)\right\|_{L^2(D)}$ (3.68) *and the computed norm* $\left\|\partial_Y^k u(0,x)\right\|_{L^2(D)}$.

$d = 2,3$, where the permeability tensor is modeled as a lognormal random field: $a(\omega,x) = \mathrm{e}^{Y(\omega,x)}$. Under the assumption of small variability of the field Y, we perform a perturbation analysis and we study the approximation properties of the Taylor polynomial of order K. We predict the divergence of the Taylor series, and we confirm it by numerical examples with just one random variable. We state the existence of an optimal order of the Taylor polynomial and provide a way to compute it. Finally, we compare our results to the results obtained in [9], where the authors model the permeability tensor as a linear combination of bounded random variables.

3.8 Appendix

Proof of Lemma 3.3.5. For simplicity, let us suppose the field Y centered and isotropic, so that $\Lambda = \lambda_2 I$, where I is the identity matrix and λ_2 the variance of $\dfrac{\partial Y}{\partial x_i}$ independent of i. The proof can be extended to the case of non-isotropic fields. With the aim of using (3.16), we explicit the expected value of the Euler characteristic φ:

$$\mathbb{E}\left[\varphi(A_u)\right] = \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \sum_{k=1}^{d} \frac{\begin{pmatrix} d \\ k \end{pmatrix} T^k \lambda_2^{k/2}}{(2\pi)^{(k+1)/2}\sigma^k} H_{k-1}\left(\frac{u}{\sigma}\right) + \psi\left(\frac{u}{\sigma}\right), \tag{3.71}$$

where λ_2 is the variance of $\dfrac{\partial Y}{\partial x_i}$ (independent on i thanks to the isotropy assumption), $\psi(x)$ is the tail probability function of a standard Gaussian random variable, given by

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_x^{+\infty} e^{-\frac{t^2}{2}}\, dt, \tag{3.72}$$

and H_k is the k-th Hermite polynomial, defined as

$$H_k(x) = k! \sum_{j=0}^{[k/2]} \frac{(-1)^j x^{k-2j}}{j!(k-2j)!2^j}, \quad k \geq 0. \tag{3.73}$$

In particular, if $d = 2$,

$$\mathbb{E}\left[\varphi(A_u)\right] = \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \left(\frac{2T\sqrt{\lambda_2}}{2\pi\sigma} + \frac{T^2\lambda_2}{(2\pi)^{3/2}\sigma^2}\frac{u}{\sigma}\right) + \psi\left(\frac{u}{\sigma}\right), \tag{3.74}$$

and, if $d = 3$,

$$\mathbb{E}\left[\varphi(A_u)\right] = \mathrm{e}^{-\frac{u^2}{2\sigma^2}} \left(\frac{3T\sqrt{\lambda_2}}{2\pi\sigma} + \frac{3T^2\lambda_2}{(2\pi)^{3/2}\sigma^2}\frac{u}{\sigma} + \frac{T^3\lambda_2^{3/2}}{(2\pi)^2\sigma^3}\left(\frac{u^2}{\sigma^2} - 1\right)\right) + \psi\left(\frac{u}{\sigma}\right). \tag{3.75}$$

We want to compute the statistical moments of $\|Y\|_{L^\infty(D)}$:

$$\mathbb{E}\left[\|Y\|^k_{L^\infty(D)}\right] = \mathbb{E}\left[|\sup_x Y(x)|^k\right] = k\int_0^{+\infty} u^{k-1}\mathbb{P}\left\{|\sup_x Y(x)| > u\right\} du. \tag{3.76}$$

We observe that

$$\begin{aligned}
\mathbb{P}\left\{|\sup_x Y(x)| > u\right\} &= \mathbb{P}\left\{\sup_x Y(x) > u \cup \sup_x Y(x) < -u\right\} \\
&= \mathbb{P}\left\{\sup_x Y(x) > u\right\} + \mathbb{P}\left\{\sup_x Y(x) < -u\right\} \\
&\leq \mathbb{P}\left\{\sup_x Y(x) \geq u\right\} + 1 - \mathbb{P}\left\{\sup_x Y(x) \geq -u\right\} \\
&\leq \mathbb{E}\left[\varphi(A_u)\right] + 1 - \mathbb{E}\left[\varphi(A_{-u})\right] + o\left(\mathrm{e}^{-\frac{\alpha u^2}{2\sigma^2}}\right),
\end{aligned}$$

where α is a constant bigger than 1 (see [1]). In particular, if $d = 2$ and if we neglect the term $o\left(\mathrm{e}^{-\frac{\alpha u^2}{2\sigma^2}}\right)$,

$$\begin{aligned}
\mathbb{P}\left\{|\sup_x Y(x)| > u\right\} &\leq 2\frac{T^2\lambda_2}{(2\pi)^{3/2}}\mathrm{e}^{-\frac{u^2}{2\sigma^2}}\frac{u}{\sigma^3} + \psi\left(\frac{u}{\sigma}\right) - \psi\left(-\frac{u}{\sigma}\right) + 1 \\
&= 2\frac{T^2\lambda_2}{(2\pi)^{3/2}}\mathrm{e}^{-\frac{u^2}{2\sigma^2}}\frac{u}{\sigma^3} + 2\psi\left(\frac{u}{\sigma}\right)
\end{aligned}$$

where we used that $\psi\left(-\frac{u}{\sigma}\right) = 1 - \psi\left(\frac{u}{\sigma}\right)$. In the same way, if $d = 3$ and if we neglect the term $o\left(\mathrm{e}^{-\frac{\alpha u^2}{2\sigma^2}}\right)$,

$$\mathbb{P}\left\{|\sup_x Y(x)| > u\right\} \leq 2\frac{3T^2\lambda_2}{(2\pi)^{3/2}}\mathrm{e}^{-\frac{u^2}{2\sigma^2}}\frac{u}{\sigma^3} + 2\psi\left(\frac{u}{\sigma}\right).$$

With the aim of using (3.76), both in the 2D and in the 3D case, we need to compute two integrals:

$$\begin{aligned}
I_1 &:= \int_0^{+\infty} u^k\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du, \\
I_2 &= \int_0^{+\infty} u^{k-1}\psi\left(\frac{u}{\sigma}\right)\, du.
\end{aligned}$$

Thanks to the integration by parts formula,

$$
\begin{aligned}
I_1 &= \left[-\sigma^2\, u^{k-1}\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\right]_0^{+\infty} + \sigma^2(k-1)\int_0^{+\infty} u^{k-2}\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du \\
&= \sigma^2(k-1)\int_0^{+\infty} u^{k-2}\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du \\
&= \sigma^4(k-1)(k-3)\int_0^{+\infty} u^{k-4}\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du \\
&= \begin{cases} \sigma^k(k-1)!!\displaystyle\int_0^{+\infty} \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du = \frac{\sqrt{2\pi}}{2}\sigma^{k+1}(k-1)!! & \text{if } k \text{ is even} \\ \sigma^{k-1}(k-1)!!\displaystyle\int_0^{+\infty} u\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du = \sigma^{k+1}(k-1)!! & \text{if } k \text{ is odd} \end{cases}
\end{aligned} \tag{3.77}
$$

On the other hand, to compute I_2 we observe that $\psi(x) < \frac{1}{x}\phi(x)$ if $x > 0$, where $\phi(x) = \dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$.

$$
\begin{aligned}
I_2 &< \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} u^{k-1}\frac{\sigma}{u}\phi\left(\frac{u}{\sigma}\right) = \frac{\sigma}{2\pi}\int_0^{+\infty} u^{k-2}\, \mathrm{e}^{-\frac{u^2}{2\sigma^2}}\, du \\
&= \begin{cases} \frac{1}{2\sqrt{2\pi}}\sigma^k(k-3)!! & \text{if } k \text{ is even} \\ \frac{\sqrt{2\pi}}{2}\sigma^k(k-3)!! & \text{if } k \text{ is odd} \end{cases}
\end{aligned} \tag{3.78}
$$

Taking into account (3.77) and (3.78), we conclude both in the 2D and in the 3D case that

$$
\begin{aligned}
\mathbb{E}\left[\|Y\|^k_{L^\infty(D)}\right] &\le C\left(\frac{k}{\sigma^3}\sigma^{k+1}(k-1)!! + k\sigma^k(k-3)!!\right) \\
&= C\left(\sigma^{k-2}k\,(k-1)!! + \sigma^k k\,(k-3)!!\right) \\
&\le C\,\sigma^{k-2}k\,(k-1)!!,
\end{aligned}
$$

where $C = C(T, \lambda_2)$. Hence, (3.15) is proved. □

CHAPTER 4

Derivation and analysis of the moment equations

4.1 Introduction

This chapter focuses on the stochastic Darcy problem introduced and analyzed in Chapter 3

$$-\operatorname{div}(e^{Y(\omega,x)}\nabla u(\omega,x)) = f(x), \tag{4.1}$$

$Y(\omega,x)$ being a centered Gaussian random field with small standard deviation.

In the approach adopted here, the entire field, and not a finite dimensional approximation of it, is considered. As a consequence, the Taylor series is not computable, since it involves $u^k := D^k u(0)[Y]^k$, the k-th Gateaux derivative of u with respect to Y, for $k \geq 0$. Here we propose to approximate the statistics of u with the statistics of the K-th order Taylor polynomial $T^K u = \displaystyle\sum_{k=0}^{K} \frac{u^k}{k!}$.

We focus our attention mainly on the expected value $\mathbb{E}[u]$. In Chapter 3, Section 3.4.4, we have predicted that a good choice for the order K is $K = K^{\sigma}_{opt}$. We derive the problem solved by $\mathbb{E}\left[u^k\right]$. Its solution requires to solve a recursive problem for the $(l+1)$-points correlations $\mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right]$, with $l = k, k-1, \ldots, 1$. Note that $\mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right]$ is defined on the tensor product domain $D^{\times(l+1)}$, so that we have to handle a high dimensional problem.

The outline of the chapter is the following. Section 4.2 focuses on the derivation of the first moment equations. In particular, we highlight the recursive structure of the problem. In Section 4.3 we state some Hölder type regularity results for the correlations $\mathbb{E}\left[Y^{\otimes k}\right]$ and $\mathbb{E}\left[v\otimes Y^{\otimes k}\right]$, with $v \in V$, V Banach space. In Section 4.4 we prove both the well-posedness and some regularity results for the first moment problem. Finally, Section 4.5 gives some hints on the study of the m-th moment equations, for $m \geq 2$.

4.2 Derivation of the recursive equations for the first moment

Let $Y(\omega, x)$ be a centered Gaussian random field. As done in Chapter 3, Section 3.3.4, we define $\sigma^2 = \dfrac{1}{|D|}\displaystyle\int_D \mathbb{V}ar\left[Y(\omega,x)\right]dx$. Note that if $Y(\omega,x)$ is stationary, then its variance is independent of $x \in D$ and coincides with σ^2. By a little abuse of notation, we refer to σ as the standard deviation of Y also in the case where Y is non-stationary.

Under the assumption of small standard deviation $0 < \sigma < 1$ of the Gaussian random field $Y(\omega, x)$, the idea of the perturbation technique is to approximate the statistical moments of the stochastic solution using its Taylor polynomial. More concretely, we approximate the mean of u using the expected value of the K-th order Taylor polynomial

$$\mathbb{E}\left[u(Y,x)\right] \approx \mathbb{E}\left[T^K u(Y,x)\right] = \sum_{k=0}^{K} \frac{\mathbb{E}\left[u^k\right]}{k!}. \tag{4.2}$$

We refer to $\mathbb{E}\left[u^k\right]$ as the k-th order correction of the mean of u and to $\mathbb{E}\left[T^K u(Y,x)\right]$ as the K-th order approximation of the mean of u.

The aim of the present section is to describe the structure of the problem solved by the k-th order correction of the mean, assuming that every quantity is well-defined and every problem is well-posed. We will detail these theoretical aspects in the next sections.

The approximation of order 0, u^0, is deterministic and is the unique solution of problem (3.19). The k-th ($k \geq 1$) order correction $\mathbb{E}\left[u^k\right]$ is well-defined since $u^k \in L^p\left(\Omega; H^1_{\Gamma_D}(D)\right)$, $p > 0$ (Theorem 3.4.6), belongs to $H^1_{\Gamma_D}(D)$, and is the unique solution of the following problem, obtained applying the expected value to both sides of problem (3.20):

k-th order correction problem - weak formulation

$$\boxed{\begin{aligned}&\int_D \nabla\mathbb{E}\left[u^k\right](x)\cdot\nabla v(x)\,dx\\ &\qquad = -\sum_{l=1}^{k}\binom{k}{l}\int_D \mathbb{E}\left[\nabla u^{k-l}Y^l\right](x)\cdot\nabla v(x)\,dx \quad \forall v \in H^1_{\Gamma_D}(D).\end{aligned}} \tag{4.3}$$

It is not possible to directly solve (4.3) since it involves the unknown quantities

$$\mathbb{E}\left[\nabla u^{k-l}Y^l\right](x), \quad l = 1, \dots, k.$$

Each term $\mathbb{E}\left[\nabla u^{k-l}Y^l\right]$ is defined as the evaluation on $\mathrm{diag}(D^{\times(l+1)})$, that is the diagonal of the tensorized domain $D^{\times(l+1)} := \underbrace{D \times \ldots \times D}_{l+1 \text{ times}}$, of

$$\left(\nabla \otimes \mathrm{Id}^{\otimes l}\right)\ \mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right],$$

where the $(l+1)$-points correlation function $\mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right]$ is defined as

$$\mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right](x_1, x_2 \ldots, x_{l+1}) := \int_\Omega u^{k-l}(\omega,x_1)\otimes Y(\omega,x_2)\otimes\ldots\otimes Y(\omega,x_{l+1})d\mathbb{P}(\omega)$$

and the tensor operator $\nabla \otimes \mathrm{Id}^{\otimes l} = \nabla \otimes \underbrace{\mathrm{Id} \otimes \ldots \otimes \mathrm{Id}}_{l \text{ times}}$ is the gradient operator on the first variable x_1 and the identity operator on all the other variables. Note that, if $l = 0$, then $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] = \mathbb{E}\left[u^k\right]$.

Given a function $v(x_1, \ldots, x_n, \ldots, x_{n+s}, \ldots, x_l)$, $n + s \leq l$ positive integers, we introduce the following notation:

$$\begin{aligned} &\left(\mathrm{Tr}_{|n:s}\right) v(x_1, \ldots, x_{n-1}, x_n, x_{n+s}, \ldots, x_l) \\ &\quad := v(x_1, \ldots, x_{n-1}, \underbrace{x_n, \ldots, x_n}_{s \text{ times}}, x_{n+s}, \ldots, x_l). \end{aligned} \tag{4.4}$$

In particular, $\mathbb{E}\left[\nabla u^{k-l} Y^l\right] = \mathrm{Tr}_{|1:l+1} \mathbb{E}\left[\nabla u^{k-l} \otimes Y^{\otimes l}\right]$. In Lemma 4.3.3 we will show that $\mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l} \otimes Y^{\otimes l}\right]$ is square integrable over $D^{\times(l-s+1)}$ for each $1 \leq s \leq l$.

We have observed that, to obtain the k-th order correction $\mathbb{E}\left[u^k\right]$, we need to derive the boundary value problem solved by the $(l+1)$-points correlation function $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$ for $l = 1, \ldots, k$. To derive the equation for $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$ we start from the equation solved by u^{k-l} (see equation (3.20)) and multiply both sides by $Y(\omega, x_2) \cdots Y(\omega, x_{l+1})$:

$$\begin{aligned} &-\left(\mathrm{div} \otimes \mathrm{Id}^{\otimes l}\right) \nabla u^{k-l}(\omega, x_1) \otimes Y(\omega, x_2) \otimes \cdots \otimes Y(\omega, x_{l+1}) \\ &= \sum_{s=1}^{k-l} \begin{pmatrix} k-l \\ s \end{pmatrix} \left(\mathrm{div} \otimes \mathrm{Id}^{\otimes l}\right) \left(\nabla u^{k-l-s} Y^s\right)(\omega, x_1) \otimes Y(\omega, x_2) \otimes \cdots \otimes Y(\omega, x_{l+1}) \end{aligned}$$

for a.e. $(x_1, \ldots, x_{l+1}) \in D^{\times(l+1)}$, a.s. in Ω. Taking the expectation on both sides and using the integration by parts formula on the first variable we obtain the problem:

$(\mathbf{l+1})$-points correlation problem - weak formulation

Given all the lower order terms

$\mathbb{E}\left[u^{k-l-s} \otimes Y^{\otimes(s+l)}\right] \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes(s+l)}$ for $s = 1, \ldots, k-l$,

find $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$ s.t.

$$\begin{aligned} &\int_{D^{\times(l+1)}} \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) \mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \cdot \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) v \, dx_1 \ldots dx_{l+1} = \\ &-\sum_{s=1}^{k-l} \begin{pmatrix} k-l \\ s \end{pmatrix} \int_{D^{\times(l+1)}} \mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l-s} \otimes Y^{\otimes(s+l)}\right] \cdot \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) v \, dx_1 \ldots dx_{l+1} \end{aligned}$$

$$\forall\, v \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$$

(4.5)

Note that problem (4.3) is a particular case of problem (4.5) with $l = 0$.

To summarize, the problem we have in hand has a recursive structure. Indeed, to obtain the K-th order approximation of $\mathbb{E}[u]$ (4.2), we need to accomplish the following steps:

Table 4.1: *K-th order approximation of the mean. The first column contains the input terms and the first row contains the k-th order corrections, for $k = 0, \ldots, K$. To compute $\mathbb{E}\left[T^K u(Y,x)\right]$, we need all the elements in the upper triangular part of the table, that is all the elements in the k-th diagonals with $k = 0, \ldots, K$.*

$\mathbb{E}\left[u^0\right] = u^0$	$\mathbb{E}\left[u^1\right] = 0$	$\mathbb{E}\left[u^2\right]$	$\mathbb{E}\left[u^3\right] = 0$	$\iddots$
$\mathbb{E}\left[u^0 \otimes Y\right] = u^0 \otimes \mathbb{E}\left[Y\right] = 0$	$\mathbb{E}\left[u^1 \otimes Y\right]$	$\mathbb{E}\left[u^2 \otimes Y\right] = 0$	$\iddots$	$\iddots$
$\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right] = u^0 \otimes \mathbb{E}\left[Y^{\otimes 2}\right]$	$\mathbb{E}\left[u^1 \otimes Y^{\otimes 2}\right] = 0$	$\iddots$	$\iddots$	$\iddots$
$\mathbb{E}\left[u^0 \otimes Y^{\otimes 3}\right] = u^0 \otimes \mathbb{E}\left[Y^{\otimes 3}\right] = 0$	$\iddots$	$\iddots$	$\iddots$	$\iddots$

for $k = 0, \ldots, K$

Compute $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right]$

for $l = k-1, k-2, \ldots, 0$

Solve the boundary value problem (4.5) to obtain the $(l+1)$-points correlation function $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$

end

The result for $l = 0$ is the $k - th$ order correction $\mathbb{E}\left[u^k\right]$ to the mean $\mathbb{E}\left[u\right]$

end

This procedure is depicted in Table 4.1. The first column contains the input terms of our recursion, namely the deterministic function u^0 and all the statistical k-points correlations of the Gaussian random field Y multiplied by u^0. The first row contains the increasing order correction to the mean, that is the output terms of our recursive problem. The k-th diagonal is composed of $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$ with $l \in \{0, \ldots, k\}$. So, for example, the first diagonal contains $\mathbb{E}\left[u^0 \otimes Y\right]$ and $\mathbb{E}\left[u^1\right]$, and the second diagonal contains $\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right]$, $\mathbb{E}\left[u^1 \otimes Y\right]$ and $\mathbb{E}\left[u^2\right]$. Each non-zero term $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$, can be obtained only if we have previously computed all the terms before in the k-th diagonal, that is $\mathbb{E}\left[u^0 \otimes Y^{\otimes k}\right]$, $\mathbb{E}\left[u^1 \otimes Y^{\otimes (k-1)}\right]$, …, $\mathbb{E}\left[u^{k-l-1} \otimes Y^{\otimes (l+1)}\right]$, since these terms enter in the problem solved by $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$ as right-hand side in (4.5). Since we assumed $\mathbb{E}\left[Y\right](x) = 0$ w.l.o.g., all the $(2k+1)$-points correlations of Y vanish. As a consequence, all the terms in the odd diagonals vanish. Finally, we observe that this table has a triangular structure, in the sense that, to compute the K-th order approximation of the mean, we need all the elements in the upper triangular part of the table:

$$\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right], \quad k \in \{0, \ldots, K\}, \ l \in \{0, \ldots, k\}.$$

4.3 Regularity results for the correlations $\mathbb{E}\left[Y^{\otimes k}\right]$, $\mathbb{E}\left[v \otimes Y^{\otimes k}\right]$

In this section we provide some results on the regularity of the correlations $\mathbb{E}\left[Y^{\otimes k}\right]$ and $\mathbb{E}\left[v \otimes Y^{\otimes k}\right]$ with $v \in V$, V being a Banach space.

4.3.1 Hölder spaces with mixed regularity

Let $D \subset \mathbb{R}^d$ be an open bounded domain. Given $x \in D$, we denote with $|x|$ the euclidean norm of x, as well as, given $(x_1, \dots, x_k) \in D^{\times k}$ we denote with $|(x_1, \dots, x_k)|$ the euclidean norm of $(x_1, \dots, x_k)$.

The γ-Hölder space $\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$ $(0 < \gamma \leq 1)$ consists of all continuous functions $v : \bar{D}^{\times k} \to \mathbb{R}$ with finite seminorm

$$|v|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})} := \sup_{\substack{\mathbf{x}, \mathbf{x}+\mathbf{h} \in \bar{D}^{\times k} \\ \mathbf{h} > 0}} |D^{\gamma}_{\mathbf{h}} v(x_1, \dots, x_k)| < +\infty,$$

where $\mathbf{h} = (h_1, \dots, h_k)$, $\mathbf{x} = (x_1, \dots, x_k)$, and $D^{\gamma}_{\mathbf{h}}$ is the linear operator defined as

$$D^{\gamma}_{\mathbf{h}} v(x_1, \dots, x_k) := \frac{v(x_1 + h_1, \dots, x_k + h_k) - v(x_1, \dots, x_k)}{|\mathbf{h}|^{\gamma}}. \tag{4.6}$$

If $|v|_{\mathcal{C}^{0,1}(\bar{D}^{\times k})} < +\infty$, the function v is said Lipschitz regular. The space $\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$ is a Banach space with the natural norm

$$\|v\|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})} := \|v\|_{\mathcal{C}^{0}(\bar{D}^{\times k})} + |v|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})}.$$

Generalizing the spaces $\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$, and $\mathcal{C}^{r}(\bar{D}^{\times k})$ $(r \in \mathbb{N})$ with mixed regularity (see e.g. [80]), we define the Hölder space with mixed regularity $\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})$ as follows. It is the space of all continuous functions $v : \bar{D}^{\times k} \to \mathbb{R}$ with finite seminorm

$$|v|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} := \sup_{\substack{\mathbf{x}, \mathbf{x}+\mathbf{h} \in \bar{D}^{\times k} \\ \mathbf{h} > 0}} \left|D^{\gamma,mix}_{\mathbf{h}} v(x_1, \dots, x_k)\right| < +\infty,$$

where $D^{\gamma,mix}_{\mathbf{h}}$ is the mixed counterpart of (4.6):

$$D^{\gamma,mix}_{\mathbf{h}} v(x_1, \dots, x_k) := D^{\gamma}_{1,h_1} \cdots D^{\gamma}_{k,h_k} v(x_1, \dots, x_k), \tag{4.7}$$

with

$$D^{\gamma}_{i,h_i} v(x_1, \dots, x_k) := \frac{v(x_1, \dots, x_i + h_i, \dots, x_k) - v(x_1, \dots, x_k)}{|h_i|^{\gamma}}.$$

$\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})$ is a Banach space with the norm

$$\|v\|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})} := \|v\|_{\mathcal{C}^{0}(\bar{D}^{\times k})} + |v|_{\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})}.$$

In other words, $\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k})$ is the space of functions $v : \bar{D}^{\times k} \to \mathbb{R}$ γ-Hölder regular in every direction separately. In the same way, given V a functional (Hilbert or Banach) space, $\mathcal{C}^{0,\gamma,mix}\left(\bar{D}^{\times k}; V\right)$ denotes the space of functions $v : \bar{D}^{\times k} \to V$ with γ-Hölder regularity in every direction separately. Clearly, $\mathcal{C}^{0,\gamma,mix}(\bar{D}^{\times k}) \subset \mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$. The following lemma states in which case the inverse inclusion holds.

Lemma 4.3.1. *Let* $0 < \gamma \leq 1$. *Then*

$$\mathcal{C}^{0,\gamma}(\bar{D}^{\times k}) \subset \mathcal{C}^{0,\gamma/k,mix}(\bar{D}^{\times k}). \tag{4.8}$$

Proof. We prove (4.8) for $k=2$. Let $v \in \mathcal{C}^{0,\gamma}(\bar{D}^{\times 2})$.

$$
\begin{aligned}
&|v|_{\mathcal{C}^{0,\gamma/2,mix}(\bar{D}^{\times 2})} \\
&= \sup_{\mathbf{x},\mathbf{h}} \left| D^{\gamma/2}_{1,h_1} D^{\gamma/2}_{2,h_2} v(x_1,x_2) \right| \qquad (4.9)\\
&= \sup_{\mathbf{x},\mathbf{h}} \frac{|v(x_1+h_1,x_2+h_2) - v(x_1+h_1,x_2) - v(x_1,x_2+h_2) + v(x_1,x_2)|}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}} \qquad (4.10)
\end{aligned}
$$

Let us define $w(x_1,x_2;h_1,h_2) := v(x_1+h_1,x_2+h_2) - v(x_1+h_1,x_2) - v(x_1,x_2+h_2) + v(x_1,x_2)$ so that

$$
\begin{aligned}
(4.10) &= \sup_{\mathbf{x},\mathbf{h}} \frac{|w(x_1,x_2;h_1,h_2)|}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}} \\
&\le \max\left\{ \sup_{\mathbf{x},|h_1|<|h_2|} \frac{|w(x_1,x_2;h_1,h_2)|}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}}, \sup_{\mathbf{x},|h_1|\ge|h_2|} \frac{|w(x_1,x_2;h_1,h_2)|}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}} \right\}.
\end{aligned}
$$

We start considering

$$
\begin{aligned}
&\sup_{\mathbf{x},|h_1|<|h_2|} \frac{|w(x_1,x_2;h_1,h_2)|}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}} \\
&\le \sup_{\mathbf{x},|h_1|<|h_2|} \frac{1}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}} \Bigg(|h_1|^{\gamma} \frac{|v(x_1+h_1,x_2+h_2) - v(x_1,x_2+h_2)|}{|h_1|^{\gamma}} \\
&\qquad + |h_1|^{\gamma} \frac{|v(x_1+h_1,x_2) - v(x_1,x_2)|}{|h_1|^{\gamma}} \Bigg) \\
&\le \sup_{\mathbf{x},|h_1|<|h_2|} \frac{|h_1|^{\gamma/2}}{|h_2|^{\gamma/2}} \left(\left|D^{\gamma}_{1,h_1} v(x_1,x_2+h_2)\right| + \left|D^{\gamma}_{1,h_1} v(x_1,x_2)\right| \right) \\
&\le 2\,|v|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times 2})}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\sup_{\mathbf{x},|h_1|\ge|h_2|} \frac{|w(x_1,x_2;h_1,h_2)|}{|h_1|^{\gamma/2}\,|h_2|^{\gamma/2}} \\
&\le \sup_{\mathbf{x},|h_1|\ge|h_2|} \frac{|h_2|^{\gamma/2}}{|h_1|^{\gamma/2}} \left(\left|D^{\gamma}_{2,h_2} v(x_1+h_1,x_2)\right| + \left|D^{\gamma}_{2,h_2} v(x_1+h_1,x_2)\right| \right) \\
&\le 2\,|v|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times 2})}.
\end{aligned}
$$

Hence, we conclude (4.8) for $k=2$. In the general case, given $\mathbf{h} = (h_1,\dots,h_k)$, and i^* such that $|h_{i^*}| \le |h_i|\ \forall\, i \ne i^*$, it holds

$$
\begin{aligned}
&\sup_{\substack{\mathbf{x}\\ |h_{i^*}|\le|h_i|, i\ne i^*}} \frac{|w(x_1,\dots,x_k;h_1,\dots,h_k)|}{\prod_{i=1}^k |h_i|^{\gamma/k}} \\
&\le \sup_{\substack{\mathbf{x}\\ |h_{i^*}|\le|h_i|, i\ne i^*}} \frac{|h_{i^*}|^{\gamma}}{\prod_{i=1}^k |h_i|^{\gamma/k}} \frac{|w(x_1,\dots,x_k;h_1,\dots,h_k)|}{|h_{i^*}|^{\gamma}} \\
&\le 2^{k-1}\,|v|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})}.
\end{aligned}
$$

□

4.3.2 Hölder mixed regularity of $\mathbb{E}\left[Y^{\otimes k}\right]$ and $\mathbb{E}\left[v \otimes Y^{\otimes k}\right]$

The following proposition states a Hölder mixed regularity result for the k-points correlation of the Gaussian random field $Y(\omega, x)$.

Proposition 4.3.2. *Let Y be a centered Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. Suppose $v \in V$, V Banach space. Then, for every positive integer k, the following properties hold:*

P1 $\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^{0,t/2,mix}(\bar{D}^{\times k})$,

P2 $\mathbb{E}\left[v \otimes Y^{\otimes k}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times k}; V\right)$.

Proof. Let us start proving property **P1**. Using the definition of the k-points correlation function of Y we have

$$\begin{aligned}
&D^{t/2}_{1,h_1}\mathbb{E}\left[Y^{\otimes k}\right](x_1,\dots,x_k)\\
&= \frac{\mathbb{E}\left[Y^{\otimes k}\right](x_1+h_1,\dots,x_k) - \mathbb{E}\left[Y^{\otimes k}\right](x_1,\dots,x_k)}{|h_1|^{t/2}}\\
&= \frac{\mathbb{E}\left[Y(\omega,x_1+h_1)\otimes Y(\omega,x_2)\otimes\cdots\otimes Y(\omega,x_k)\right]}{|h_1|^{t/2}}\\
&\quad + \frac{\mathbb{E}\left[Y(\omega,x_1)\otimes Y(\omega,x_2)\otimes\cdots\otimes Y(\omega,x_k)\right]}{|h_1|^{t/2}}\\
&= \mathbb{E}\left[\frac{Y(\omega,x_1+h_1)-Y(\omega,x_1)}{|h_1|^{t/2}}\otimes Y(\omega,x_2)\otimes\cdots\otimes Y(\omega,x_k)\right].
\end{aligned}$$

Repeating iteratively the same steps,

$$D^{t/2}_{k,h_k}\cdots D^{t/2}_{1,h_1}\mathbb{E}\left[Y^{\otimes k}\right](x_1,\dots,x_k) = \mathbb{E}\left[\bigotimes_{j=1}^{k}\frac{Y(\omega,x_j+h_j)-Y(\omega,x_j)}{|h_j|^{t/2}}\right].$$

Now,

$$\begin{aligned}
\left|\mathbb{E}\left[Y^{\otimes k}\right]\right|_{\mathcal{C}^{0,t/2,mix}(\bar{D}^{\times k})} &= \sup_{\substack{x_1\dots,x_k\\ h_1,\dots,h_k}}\left|D^{t/2}_{k,h_k}\cdots D^{t/2}_{1,h_1}\mathbb{E}\left[Y^{\otimes k}\right]\right|\\
&= \sup_{\substack{x_1\dots,x_k\\ h_1,\dots,h_k}}\left|\mathbb{E}\left[\bigotimes_{j=1}^{k}\frac{Y(\omega,x_j+h_j)-Y(\omega,x_j)}{|h_j|^{t/2}}\right]\right|\\
&\leq \sup_{\substack{x_1\dots,x_k\\ h_1,\dots,h_k}}\mathbb{E}\left[\bigotimes_{j=1}^{k}\frac{|Y(\omega,x_j+h_j)-Y(\omega,x_j)|}{|h_j|^{t/2}}\right].
\end{aligned}$$

Using the Hölder inequality, we have

$$\mathbb{E}\left[\bigotimes_{j=1}^{k}\frac{|Y(\omega,x_j+h_j)-Y(\omega,x_j)|}{|h_j|^{t/2}}\right] \leq \prod_{j=1}^{k}\mathbb{E}\left[\frac{|Y(\omega,x_j+h_j)-Y(\omega,x_j)|^k}{|h_j|^{kt/2}}\right]^{1/k},$$

so that

$$\sup_{\substack{x_1\dots,x_k\\h_1,\dots,h_k}} \mathbb{E}\left[\bigotimes_{j=1}^{k}\frac{|Y(\omega,x_j+h_j)-Y(\omega,x_j)|}{|h_j|^{t/2}}\right]$$
$$\leq \prod_{j=1}^{k}\sup_{x_j,h_j}\mathbb{E}\left[\frac{|Y(\omega,x_j+h_j)-Y(\omega,x_j)|^k}{|h_j|^{kt/2}}\right]^{1/k}.$$

We conclude using the result (3.7) in Proposition 3.3.2 for $p = k/2$. To prove **P2** we observe that, for every $(x_2,\dots,x_{k+1}) \in D^{\times k}$ fixed,

$$x_1 \mapsto v(x_1)\mathbb{E}\left[Y(\omega,x_2)\otimes\cdots\otimes Y(\omega,x_{k+1})\right]$$

is a function in V parametric in $(x_2,\dots,x_{k+1})$. Moreover, the function

$$(x_2,\dots,x_{k+1}) \mapsto v(x_1)\mathbb{E}\left[Y(\omega,x_2)\otimes\cdots\otimes Y(\omega,x_{k+1})\right] := \varphi(x_1,x_2,\dots,x_k)$$

is such that

$$|\varphi|_{\mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times k};V\right)} := \sup_{\substack{x_2,\dots,x_{k+1}\\h_2,\dots,h_{k+1}}}\left\|D^{t/2,mix}_{(h_2,\dots,h_{k+1})}\varphi(\cdot,x_2,\dots,x_{k+1})\right\|_V < \infty.$$

□

Note that the regularity result in **P1** for $k = 2$ is in agreement with our assumption $Cov_Y \in \mathcal{C}^{0,t}(\overline{D\times D})$, $0 < t \leq 1$ (see Lemma 4.3.1).

4.3.3 Trace regularity results

The following two propositions provide regularity results for the traces of functions with Hölder regularity.

Lemma 4.3.3. *Let Y be a centered Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D\times D})$, $0 < t \leq 1$, and $v \in L^2(D)$, so that*

$$\mathbb{E}\left[v\otimes Y^{\otimes k}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times k};L^2(D)\right) \subset \mathcal{C}^{0,t/2}\left(\bar{D}^{\times k};L^2(D)\right).$$

Then, it holds

$$\left\|\mathrm{Tr}_{|1:s}\mathbb{E}\left[v\otimes Y^{\otimes k}\right]\right\|_{L^2(D^{\times(k-s+2)})} \leq C\left\|\mathbb{E}\left[v\otimes Y^{\otimes k}\right]\right\|_{\mathcal{C}^{0,t/2}\left(\bar{D}^{\times k};L^2(D)\right)}, \tag{4.11}$$

with $C = C(D,s)$, for any integers $1 \leq s \leq k+1$, where $\mathrm{Tr}_{|1:s}$ is the trace operator introduced in (4.4).

Proof. To lighten the notations, let us define $\varphi := \mathbb{E}\left[v\otimes Y^{\otimes k}\right]$. We have

$$\begin{aligned}
&\left\|\mathrm{Tr}_{|1:s}\varphi(x_1,\ldots,x_s,x_{s+1},\ldots,x_{k+1})\right\|_{L^2_{x_1}(D)}\\
&= \|\varphi(\underbrace{x_1,\ldots,x_1}_{s \text{ times}},x_{s+1},\ldots,x_{k+1})\|_{L^2_{x_1}(D)}\\
&\leq \|\varphi(x_1,\ldots,x_1,x_{s+1},\ldots,x_{k+1})-\varphi(x_1,x_2,\ldots,x_s,x_{s+1},\ldots,x_{k+1})\|_{L^2_{x_1}(D)}\\
&\qquad + \|\varphi(x_1,x_2,\ldots,x_s,x_{s+1},\ldots,x_{k+1})\|_{L^2_{x_1}(D)}\\
&= \sqrt{\int_D \frac{\left|\varphi(x_1,\ldots,x_1,x_{s+1},\ldots,x_{k+1})-\varphi(x_1,\ldots,x_{k+1})\right|^2}{\left|(x_2-x_1,\ldots,x_s-x_1)\right|^t}\left|(x_2-x_1,\ldots,x_s-x_1)\right|^t dx_1}\\
&\qquad + \|\varphi(x_1,\ldots,x_{k+1})\|_{L^2_{x_1}(D)}\\
&\leq (s-1)^{t/4}diam(D)^{t/2}\left|\varphi\right|_{\mathcal{C}^{0,t/2}\left(\bar{D}^{\times k};L^2(D)\right)} + \|\varphi(x_1,\ldots,x_{k+1})\|_{L^2_{x_1}(D)}
\end{aligned}$$

where in the last inequality we have used that

$$\left|(x_2-x_1,\ldots,x_s-x_1)\right|^2 = \sum_{j=2}^{s}\left|x_j-x_1\right|^2 \leq (s-1)diam(D)^2,$$

$diam(D)$ being the diameter of the domain D. Using that

$$\mathbb{E}\left[v\otimes Y^{\otimes k}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times k};L^2(D)\right) \subset L^2(D^{\times k+1}),$$

we obtain

$$\begin{aligned}
&\int_{D^{\times(k+1)}}\left|\mathrm{Tr}_{|1:s}\varphi\right|^2 dx_1\cdots dx_{k+1}\\
&\leq 2(s-1)^{t/2}diam(D)^t\left|D\right|^k\left|\varphi\right|^2_{\mathcal{C}^{0,t/2}\left(\bar{D}^{\times k};L^2(D)\right)} + 2\left\|\varphi\right\|^2_{L^2(D^{\times(k+1)})}\\
&\leq 2\left((s-1)^{t/2}diam(D)^t\left|D\right|^k + \left|D\right|^k\right)\left\|\varphi\right\|^2_{\mathcal{C}^{0,t/2}\left(\bar{D}^{\times k};L^2(D)\right)}
\end{aligned}$$

so that (4.11) is verified with $C = \sqrt{2\left|D\right|^k\left((s-1)^{t/2}diam(D)^t+1\right)}$. ☐

Lemma 4.3.4. *Let $\varphi \in \mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$, $0<\gamma\leq 1$, and k positive integer. Then $\mathrm{Tr}_{|1:k}\varphi \in \mathcal{C}^{0,\gamma}(\bar{D})$ and*

$$\left|\mathrm{Tr}_{|1:k}\varphi\right|_{\mathcal{C}^{0,\gamma}(\bar{D})} \leq k^{\gamma/2}\left|\varphi\right|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})}. \tag{4.12}$$

Proof. Let $\varphi \in \mathcal{C}^{0,\gamma}(\bar{D}^{\times k})$. Using that $\left|(x,\ldots,x)-(y,\ldots,y)\right|^2 = k\left|x-y\right|^2$ for every $(x,\ldots,x),(y,\ldots,y) \in D^{\times k}$, we have

$$\begin{aligned}
\frac{\left|\mathrm{Tr}_{|1:k}\varphi(x)-\mathrm{Tr}_{|1:k}\varphi(y)\right|}{\left|x-y\right|^{\gamma}} &= \frac{\left|\varphi(x,\ldots,x)-\varphi(y,\ldots,y)\right|}{\left|x-y\right|^{\gamma}}\\
&= k^{\gamma/2}\frac{\left|\varphi(x,\ldots,x)-\varphi(y,\ldots,y)\right|}{\left|(x,\ldots,x)-(y,\ldots,y)\right|^{\gamma}}\\
&\leq k^{\gamma/2}\left|\varphi\right|_{\mathcal{C}^{0,\gamma}(\bar{D}^{\times k})}.
\end{aligned}$$

☐

4.4 Well-posedness and regularity results for the equations for the first moment

Here we prove the well-posedness result for problem (4.5), which also implies the well-posedness of the k-th correction problem (4.3).

Theorem 4.4.5. *Let Y be a centered Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. Then, problem* (4.5) *is well-posed for every $k \geq 0$ and $l = 0, \dots, k-1$ integers.*

Proof. Problem (4.5) is of the form: find $w \in \mathcal{V} := H^1_{\Gamma_D}(D) \otimes (L^2(D))^{\otimes l}$ such that

$$\mathcal{A}(w, v) = \mathcal{L}(v) \quad \forall\, v \in \mathcal{V},$$

where $\mathcal{A} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is the bilinear form

$$\mathcal{A}(w, v) := \int_{D^{\times(l+1)}} \nabla \otimes \mathrm{Id}^{\otimes l} w(x_1, \dots, x_{l+1}) \cdot \nabla \otimes \mathrm{Id}^{\otimes l} w(x_1, \dots, x_{l+1})\, dx_1 \dots dx_{l+1},$$

and $\mathcal{L} : \mathcal{V} \to \mathbb{R}$ is the linear form

$$\mathcal{L}(v) := -\sum_{s=1}^{k-l} \begin{pmatrix} k-l \\ s \end{pmatrix} \int_{D^{\times(l+1)}} \mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l-s} \otimes Y^{\otimes(s+l)}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes l} v\, dx_1 \dots dx_{l+1}.$$

It is easy to verify that $\mathcal{A}$ is continuous and coercive. Moreover, using Lemma 4.3.3 with $v = \nabla u^{k-l-s}$ we obtain immediately the continuity of $\mathcal{L}$, so that we conclude the existence of a unique solution thanks to the Lax Milgram lemma. □

Moreover, it holds the following regularity result.

Theorem 4.4.6. *Let Y be a centered Gaussian random field with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, $0 < t \leq 1$. Moreover, suppose that the domain is convex $\mathcal{C}^{1,t/2}$, and $u^0 \in \mathcal{C}^{1,t/2}(\bar{D})$. Then, for every positive integers k and s,*

$$\mathbb{E}\left[u^k \otimes Y^{\otimes s}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{1,t/2}(\bar{D})\right).$$

Proof. We prove the theorem by induction on k. If $k = 0$, we know that $\mathbb{E}\left[u^0 \otimes Y^{\otimes s}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{1,t/2}(\bar{D})\right)$ $\forall s$, thanks to property **P2** in Proposition 4.3.2 with $V = \mathcal{C}^{1,t/2}(\bar{D})$. Now, suppose by induction that $\mathbb{E}\left[u^l \otimes Y^{\otimes s}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{1,t/2}(\bar{D})\right)$ for every s and $l = 1, \dots, k-1$. The problem solved by $\mathbb{E}\left[u^k \otimes Y^{\otimes s}\right]$ is

$$\begin{aligned}
&\int_{D^{\times(s+1)}} \nabla \otimes \mathrm{Id}^{\otimes s} \mathbb{E}\left[u^k \otimes Y^{\otimes s}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes s} v\, dx_1 \dots dx_{s+1} \\
&\quad = -\sum_{j=1}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \int_{D^{\times(s+1)}} \mathrm{Tr}_{|1:j+1} \mathbb{E}\left[\nabla u^{k-j} \otimes Y^{\otimes(s+j)}\right] \cdot \nabla \otimes \mathrm{Id}^{\otimes s} v\, dx_1 \dots dx_{s+1}
\end{aligned}$$

$\forall\, v \in H^1_{\Gamma_D}(D) \otimes (L^2(D))^{\otimes s}$ (see (4.5)). By induction, we know that

$$\mathbb{E}\left[\nabla u^{k-j} \otimes Y^{\otimes(j+s)}\right] \in \mathcal{C}^{0,t/2,mix}(\bar{D}^{\times(j+s+1)}).$$

We parametrize $\mathbb{E}\left[\nabla u^{k-j} \otimes Y^{\otimes(j+s)}\right]$ with respect to the last s variables, and for every fixed $(\bar{x}_{j+2}, \ldots, \bar{x}_{j+s+1}) \in D^{\times s}$ we define

$$\begin{aligned} v(x_1, \ldots, x_{j+1}) &:= \mathbb{E}\left[\nabla u^{k-j} \otimes Y^{\otimes^{(j+s)}}\right](x_1, \ldots, x_{j+1}, \bar{x}_{j+2}, \ldots, \bar{x}_{j+s+1}) \\ &\in \mathcal{C}^{0,t/2,mix}(\bar{D}^{\times(j+1)}). \end{aligned}$$

Lemma 4.3.4 states that $\mathrm{Tr}_{|1:j+1}(v) \in \mathcal{C}^{0,t/2}(\bar{D})$, so that

$$\mathrm{Tr}_{|1:j+1}\mathbb{E}\left[\nabla u^{k-j} \otimes Y^{\otimes(s+j)}\right] \in \mathcal{C}^{0,t/2,mix}(\bar{D}^{\times(s+1)}).$$

Using a shift theorem for the Hölder regularity (see [44]), we conclude that

$$\mathbb{E}\left[u^k \otimes Y^{\otimes s}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{1,t/2}(\bar{D})\right).$$

□

Remark 4.4.7. *In general, if $u^0 \in \mathcal{C}^{r_1,t/2}(\bar{D})$ and $Y \in L^p\left(\Omega; \mathcal{C}^{r_2,t/2}(\bar{D})\right)$, $r_1, r_2 \geq 0$, $\forall\, p$, then Propositions 4.3.2 and 4.3.4 easily generalize. Define $r := \min\{r_1 - 1, r_2\}$. If the domain D is $\mathcal{C}^{r+1,t/2}$, Theorem 4.4.6 generalizes so that*

$$\mathbb{E}\left[u^k \otimes Y^{\otimes s}\right] \in \mathcal{C}^{r_2,t/2,mix}\left(\bar{D}^{\times s}; \mathcal{C}^{r+1,t/2}(\bar{D})\right).$$

4.5 Moment equations for the two-points correlation of u

In Sections 4.2 and 4.4 we have analyzed the recursive problem for the K-th order approximation of the expected value of u. Here we give some details on how these results generalize if the two-points correlation of u is taken into account.

Under the assumption of small standard deviation $0 < \sigma < 1$, the K-th order ($K \geq 0$ integer) approximation of the two points correlation of u is:

$$\mathbb{E}\left[u \otimes u\right] \approx \sum_{k_1+k_2 \leq K} \frac{\mathbb{E}\left[u^{k_1} \otimes u^{k_2}\right]}{k_1!\; k_2!}. \tag{4.13}$$

We refer to $\mathbb{E}\left[u^{k_1} \otimes u^{k_2}\right]$ as the correction of order (k_1, k_2) of $\mathbb{E}\left[u \otimes u\right]$.

The correction of order $(0,0)$ is the tensor product $u^0 \otimes u^0$, where u^0 is deterministic and is the unique solution of problem (3.19). $\mathbb{E}\left[u^{k_1} \otimes u^{k_2}\right] \in H^1_{\Gamma_D}(D) \otimes H^1_{\Gamma_D}(D)$ is the unique solution of the following problem:

$(\mathbf{k_1}, \mathbf{k_2})$-th order correction problem - weak formulation

$$\begin{aligned} &\int_{D^{\times 2}} (\nabla \otimes \nabla)\, \mathbb{E}\left[u^{k_1} \otimes u^{k_2}\right] \cdot (\nabla \otimes \nabla)\, v \, dx_1 \, dx_2 \\ &= \sum_{l_1=1}^{k_1} \sum_{l_2=1}^{k_2} \begin{pmatrix} k_1 \\ l_1 \end{pmatrix} \begin{pmatrix} k_2 \\ l_2 \end{pmatrix} \\ &\int_{D^{\times 2}} \mathrm{Tr}_{|1:l_1+1} \mathrm{Tr}_{|l_1+2:l_1+l_2+2} \mathbb{E}\left[\nabla u^{k_1-l_1} \otimes Y^{\otimes l_1} \otimes \nabla u^{k_2-l_2} \otimes Y^{\otimes l_2}\right] \cdot (\nabla \otimes \nabla)\, v \, dx_1 \, dx_2 \\ &\qquad \forall v \in H^1_{\Gamma_D}(D) \otimes H^1_{\Gamma_D}(D). \end{aligned} \tag{4.14}$$

Problem (4.14) is obtained tensorizing the problems solved by u^{k_1} and u^{k_2} (3.20) and then taking the expected value. As in the case of problem (4.3), before solving (4.14) we need to compute all the terms

$$\mathrm{Tr}_{|_{1:l_1+1}}\mathrm{Tr}_{|_{l_1+2:l_1+l_2+2}}\mathbb{E}\left[\nabla u^{k_1-l_1}\otimes Y^{\otimes l_1}\otimes \nabla u^{k_2-l_2}\otimes Y^{\otimes l_2}\right],$$

so that a recursion on the (l_1+l_2+2)-points correlations

$$\mathbb{E}\left[u^{k_1-l_1}\otimes Y^{\otimes l_1}\otimes u^{k_2-l_2}\otimes Y^{\otimes l_2}\right] \tag{4.15}$$

is needed. Note that, if $l_1 = l_2 = 0$, the problem solved by the correlation (4.15) coincides with problem (4.14). Lemma 4.3.3 generalizes so that the well-posedness of problem solved by the correlation (4.15) can be deduced.

Note that the recursive procedure which ends up with the approximation of the two-points correlation function can be repeated with suitable modifications, in order to achieve an approximation of the m-th moment problem, with $m \geq 3$ integer. In the next chapter we will describe an algorithm able to solve the first moment problem. By suitable modifications of the code, it will also be able to solve the m-th moment problem.

4.6 Conclusions

This chapter addresses the deterministic equations solved by the statistical moments of the unique solution of the stochastic Darcy problem with deterministic loading term and lognormal permeability coefficient. We mainly focus our attention on the first statistical moment, which we approximate using $\mathbb{E}\left[T^K u\right] = \sum_{k=0}^{K} \frac{\mathbb{E}\left[u^k\right]}{k!}$. We study the structure of the problem we have to solve to compute $\mathbb{E}\left[T^K u\right]$. In particular, for each $k = 0, \dots, K$ fixed, a recursion on the $(l+1)$-points correlation $\mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right]$, $l = 1, \dots, k$, is needed. We first state a well-posedness result for this recursive problem. Then, using a shift theorem for elliptic problems, we provide a Hölder-type regularity result: under the assumptions $u^0 \in \mathcal{C}^{0,t/2}(D)$ and D $\mathcal{C}^{1,t/2}$, we prove that $\mathbb{E}\left[u^{k-l}\otimes Y^{\otimes l}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times l}; \mathcal{C}^{1,t/2}(\bar{D})\right)$. We end the chapter giving some details on the structure of the higher order moment equations.

CHAPTER 5

Low-rank approximation of the moment equations

5.1 Introduction

In Chapter 4 we have studied the structure and well posedness of the first moment problem. Given a centered Gaussian random field Y with covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, we have shown that, for any $k \geq 0$ and $0 \leq l \leq k$ integers, there exists a unique $(l+1)$-points correlation function $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$ solution of problem (4.5). Moreover, under additional regularity assumptions on the domain D and u^0, we have proved that $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \in \mathcal{C}^{0,t/2,mix}\left(\bar{D}^{\times l}; \mathcal{C}^{1,t/2}(\bar{D})\right)$.

Here, we discretize problem (4.5) with *full tensor product finite elements*, using piecewise linear and piecewise constant finite elements to approximate $H^1_{\Gamma_D}(D)$ and $L^2(D)$ respectively. In this way, each $(l+1)$-points correlation $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$ is represented as a tensor of order $l+1$, and the discrete formulation of problem (4.5) is a linear system involving high order tensors. Since the number of entries of a tensor is exponential in its order, it is possible to explicitly store only tensors with small order. For this reason, we exploit a data-sparse or low-rank format (*TT-format*) to represent and make computations between high order tensors.

Alternatively, a *sparse tensor product finite element* discretization may be pursued. See e.g. [21] for an exhaustive presentation, as well as [28,56,91,97] for an application to the moment equations. We will not investigate this approach here, however.

Instead of taking into account the entire field Y, it can be approximated using a finite number of independent random variables, by performing a truncated Fourier or KL expansion. In this case, the Taylor polynomial can be explicitly computed. In Section 5.7.3 we compare the numerical complexity of this method with that of the TT-algorithm we propose.

The outline of the chapter is the following. Section 5.2 fixes the notations for the tensor calculus. In Section 5.3 we derive the finite element formulation of problem (4.5). Section 5.4 gives a brief overview on the most used low-rank techniques to represent or approximate a high order tensor. Of particular interest is the TT-format, the data-sparse technique used for our computations. Sections 5.5 and 5.6 are devoted to the description of the algorithms used to compute the k-points correlation function $\mathbb{E}\left[Y^{\otimes k}\right]$ and the solution of problem (4.5) respectively. We end the chapter with some numerical tests and comments on the complexity of our algorithm.

5.2 Notations for tensor calculus

Let $d \geq 1$ integer. A tensor of order d is a d-dimensional array, that is an element of the tensor product of d vector spaces. We focus on real tensors $\mathcal{X} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ with entries $\mathcal{X}(i_1, \ldots, i_d) \in \mathbb{R}$. In particular, a tensor of order one is a vector and a tensor of order two is a matrix.

Definition 5.2.1. *Let $\mathcal{X} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ be a tensor of order d and $U \in \mathbb{R}^{m \times n_\mu}$ be a matrix ($\mu \in \{1, \ldots, d\}$). The μ-th mode product of the tensor $\mathcal{X}$ with the matrix U is denoted by $\mathcal{X} \times_\mu U$, is an element of $\mathbb{R}^{n_1 \times \ldots \times n_{\mu-1} \times m \times n_{\mu+1} \times \ldots \times n_d}$, and is given by*

$$\left(\mathcal{X} \times_\mu U\right)(i_1, \ldots, i_{\mu-1}, j, i_{\mu+1}, \ldots, i_d) = \sum_{i_\mu=1}^{n_\mu} \mathcal{X}(i_1, \ldots, i_\mu, \ldots, , i_d) U(j, i_\mu). \quad (5.1)$$

The notation introduced in Definition 5.2.1 is standard: see e.g. [65] and the references therein. On the other hand, to our knowledge, there is no standard notation for the generalization of formula (5.1) acting on two tensors.

Definition 5.2.2. *Let*

$$\mathcal{Y} \in \mathbb{R}^{m_1 \times \ldots \times m_{s-1} \times n_s \times \ldots \times n_{s+r-1} \times m_{s+r} \times \ldots \times m_d}$$
$$\mathcal{X} \in \mathbb{R}^{n_s \times \ldots \times n_{s+r-1} \times h}$$

be tensors of order d and $r+1$ respectively, with entries

$$\mathcal{Y}(k_1, \ldots, k_{s-1}, i_s, \ldots, i_{s+r-1}, k_{s+r}, \ldots, k_d), \quad \mathcal{X}(i_s, \ldots, i_{s+r-1}, j),$$

with $s, r \geq 1$ and $d \geq s + r$. We define the tensor of order $d - r + 1$

$$\mathcal{Z} := \mathcal{X} \times_{s,r} \mathcal{Y} \in \mathbb{R}^{m_1 \times \ldots \times m_{s-1} \times h \times m_{s+r} \times \ldots \times m_d}$$

as the saturation of the first r indices of $\mathcal{X}$ with the r indices of $\mathcal{Y}$ from position s to $s + r - 1$:

$$\begin{aligned}&\mathcal{Z}(k_1, \ldots, k_{s-1}, j, k_{s+r}, \ldots, k_d) \qquad (5.2)\\ &= \sum_{i_s=1}^{n_s} \cdots \sum_{i_{s+r-1}=1}^{n_{s+r-1}} \mathcal{X}(i_s, \ldots, i_{s+r-1}, j) \mathcal{Y}(k_1, \ldots, k_{s-1}, i_s, \ldots, i_{s+r-1}, k_{s+r}, \ldots, k_d).\end{aligned}$$

The operation in Definition 5.2.2 satisfies the following property:

$$\mathcal{X} \times_{s,r} \left(\mathcal{Y} \times_{p,q} \mathcal{Z}\right) = \begin{cases} \mathcal{Y} \times_{p-r+1,q} \left(\mathcal{X} \times_{s,r} \mathcal{Z}\right) & \text{if } p \geq r+s, \\ \mathcal{Y} \times_{p,q} \left(\mathcal{X} \times_{s+q-1,r} \mathcal{Z}\right) & \text{if } s > p. \end{cases}$$

Notice, in particular, that

$$\mathcal{X} \times_{s,1} (\mathcal{Y} \times_{p,1} \mathcal{Z}) = \mathcal{Y} \times_{p,1} (\mathcal{X} \times_{s,1} \mathcal{Z}) \quad \forall\, s \neq p.$$

Note that, if $\mathcal{X}(j, i_\mu)$ is a matrix and $\mathcal{Y}(i_1, \ldots, i_\mu \ldots, i_d)$ is a tensor of order d, then

$$\mathcal{Y} \times_\mu \mathcal{X} = \mathcal{X}^T \times_{\mu,1} \mathcal{Y}$$

for any $\mu = 1, \ldots, d$, so that Definition 5.2.2 generalizes the μ-th mode product.

Moreover, in the particular case where $\mathcal{X}(i, j)$ and $\mathcal{Y}(i, k)$ are both matrices, the saturation (5.2) is nothing else than the classical matrix-matrix multiplication:

$$\begin{aligned} \mathcal{X} \times_{1,1} \mathcal{Y} &= \mathcal{X}^T \mathcal{Y}, \\ \mathcal{X} \times_{2,1} \mathcal{Y} &= \mathcal{Y}\mathcal{X}. \end{aligned}$$

We end the section introducing a standard operation between matrices, the *Kronecker product* (see e.g. [53, 65]).

Definition 5.2.3. *Let $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2}$ and $\mathcal{Y} \in \mathbb{R}^{m_1 \times m_2}$ be matrices. The Kronecker product $\mathcal{X} \otimes \mathcal{Y}$ is a tensor of order four, size (n_1, m_1, n_2, m_2) and entries:*

$$(\mathcal{X} \otimes \mathcal{Y})(i_1, i_2, j_1, j_2) = \mathcal{X}(i_1, j_1)\mathcal{Y}(i_2, j_2). \tag{5.3}$$

Definition 5.2.3 easily generalizes to tensors $\mathcal{X}$, $\mathcal{Y}$ of the same order.

5.3 FE discretization of the first moment equation

To lighten the presentation, from now on we suppose $g = 0$, that is homogeneous Dirichlet boundary conditions are imposed on Γ_D in the stochastic Darcy problem. Let $\mathcal{T}_h$ be a regular partition of the domain $D \subseteq \mathbb{R}^2$ ($D \subseteq \mathbb{R}^3$ respectively) into triangles (tetrahedrons respectively) such that the union of all the elements of the partition is the closure of the domain and the intersection of any two of them is empty or is a common edge or vertex (face or edge or vertex respectively). Let us denote with h the discretization parameter, i.e. the maximum diameter of the triangles/tetrahedrons in $\mathcal{T}_h$. To discretize the Hilbert spaces $H^1_{\Gamma_D}(D)$ and $L^2(D)$ we use piecewise linear and piecewise constant finite elements respectively:

$$V_h = span \{\phi_n\}_{n=1}^{N_v} \subset H^1_{\Gamma_D}(D) \tag{5.4}$$

$$W_h = span \{\psi_i\}_{i=1}^{N_e} \subset L^2(D) \tag{5.5}$$

where $\{\phi_n\}_n$ is the Lagrangian $\mathbb{P}_1$ basis, $\{\psi_i\}_i$ is the piecewise constant basis, N_v is the number of vertices excluding those on Γ_D, and N_e is the number of elements of the triangulation. Having the two bases $\{\phi_n\}_n$ and $\{\psi_i\}_i$, for every integer $l \geq 1$, we can construct a basis for the tensor product space $H^1_{\Gamma_D}(D) \otimes (L^2(D))^{\otimes l}$ (see problem (4.5)):

$$V_h \otimes (W_h)^{\otimes l} = span \{\phi_n \otimes \psi_{i_1} \otimes \ldots \otimes \psi_{i_l},\ n = 1, \ldots, N_v,\ i_1, \ldots, i_l = 1, \ldots, N_e\}.$$

5.3.1 0-th order problem: FEM formulation

The 0-th order problem is the deterministic Laplacian problem: given $f \in L^2(D)$, find $u^0(x) \in H^1_{\Gamma_D}(D)$ such that

$$\int_D \nabla u^0(x) \cdot \nabla v(x)\, dx = \int_D f(x) v(x)\, dx \quad \forall v \in H^1_{\Gamma_D}(D).$$

See equation (3.19).

Its finite element formulation is: find $u^0(x) = \sum_{n=1}^{N_v} u^0(n)\phi_n(x)$ such that

$$\sum_{n=1}^{N_v} u^0(n) \underbrace{\int_D \nabla\phi_n(x) \cdot \nabla\phi_m(x)\, dx}_{A(m,n)} = \underbrace{\int_D f(x)\phi_m(x)\, dx}_{F(m)} \quad \forall\, m = 1, \dots, N_v. \tag{5.6}$$

where A is the stiffness matrix for the $\mathbb{P}_1$ basis. The linear system (5.6) can be compactly written as

$$A\, U^0 = F, \tag{5.7}$$

where $U^0 = \{u^0(n)\}_{n=1}^{N_v}$.

5.3.2 2-nd order problem: FEM formulation

Given the 3-points correlation function $\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right]$, to derive the second order correction $\mathbb{E}\left[u^2\right]$ we need to solve the problem for the 2-points correlation $\mathbb{E}\left[u^1 \otimes Y\right]$ first (see Table 4.1).

Problem for $\mathbb{E}\left[u^1 \otimes Y\right]$: FEM formulation

Recall the equation for $\mathbb{E}\left[u^1 \otimes Y\right]$ (see problem (4.5) with $l = 1$ and $k = 2$): given $\mathbb{E}\left[u^0 \otimes Y^{\otimes 2}\right] \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes 2}$, find $\mathbb{E}\left[u^1 \otimes Y\right] \in H^1_{\Gamma_D}(D) \otimes L^2(D)$ such that

$$\boxed{\begin{aligned} &\int_D \int_D (\nabla \otimes \mathrm{Id})\, \mathbb{E}\left[u^1 \otimes Y\right](x_1, x_2) \cdot (\nabla \otimes \mathrm{Id})\, v(x_1, x_2)\, dx_1\, dx_2 \\ &= -\int_D \int_D \mathrm{Tr}_{|1:2} \mathbb{E}\left[\nabla u^0 \otimes Y^{\otimes 2}\right](x_1, x_2) \cdot (\nabla \otimes \mathrm{Id})\, v(x_1, x_2)\, dx_1\, dx_2 \end{aligned}}$$

for every $v \in H^1_{\Gamma_D}(D) \otimes L^2(D)$, where the trace operator $\mathrm{Tr}_{|1:s}$ has been introduced in Chapter 4, equation (4.4).

Its finite element formulation is: given

$$\begin{aligned} &\mathrm{Tr}_{|1:2} \mathbb{E}\left[\nabla u^0 \otimes Y^2\right](x_1, x_2) \\ &\quad = \sum_{n=1}^{N_v} \sum_{i_1=1}^{N_e} \sum_{i_2=1}^{N_e} \mathcal{C}_{u^0 \otimes Y^2}(n, i_1, i_2) \nabla\phi_n(x_1) \otimes \psi_{i_1}(x_1) \otimes \psi_{i_2}(x_2), \end{aligned}$$

find $\mathbb{E}\left[u^1 \otimes Y\right](x_1, x_2) = \sum_{n=1}^{N_v} \sum_{i=1}^{N_e} \mathcal{C}_{u^1 \otimes Y}(n,i)\, \phi_n(x_1) \otimes \psi_i(x_2)$ such that

$$
\begin{aligned}
&\sum_{n,i} \mathcal{C}_{u^1 \otimes Y}(n,i) \int_D \int_D (\nabla \phi_n(x_1) \otimes \psi_i(x_2)) \cdot (\nabla \phi_m(x_1) \otimes \psi_j(x_2))\, dx_1\, dx_2 \\
&\quad = - \sum_{n,i_1,i_2} \mathcal{C}_{u^0 \otimes Y^{\otimes 2}}(n, i_1, i_2) \\
&\qquad \int_D \int_D (\nabla \phi_n(x_1) \otimes \psi_{i_1}(x_1) \otimes \psi_{i_2}(x_2)) \cdot (\nabla \phi_m(x_1) \otimes \psi_j(x_2))\ dx_1\, dx_2
\end{aligned} \tag{5.8}
$$

$\forall\, \phi_m \otimes \psi_j \in V_h \otimes W_h$. We split the integral in the left-hand side of (5.8) as

$$
\begin{aligned}
&\int_D \int_D (\nabla \phi_n(x_1) \otimes \psi_i(x_2)) \cdot (\nabla \phi_m(x_1) \otimes \psi_j(x_2))\, dx_1\, dx_2 \\
&= \underbrace{\left(\int_D \nabla \phi_n(x_1) \cdot \nabla \phi_m(x_1)\, dx_1 \right)}_{A(n,m)} \underbrace{\left(\int_D \psi_i(x_2)\, \psi_j(x_2)\, dx_2 \right)}_{M(i,j)}
\end{aligned}
$$

where A is the stiffness matrix for the $\mathbb{P}_1$ basis, and M is the mass matrix for the $\mathbb{P}_0$ basis. In the same way, we split the integral in the right-hand side of (5.8) as

$$
\begin{aligned}
&\int_D \int_D (\nabla \phi_n(x_1) \otimes \psi_{i_1}(x_1) \otimes \psi_{i_2}(x_2)) \cdot (\nabla \phi_m(x_1) \otimes \psi_j(x_2))\, dx_1\, dx_2 \\
&= \underbrace{\left(\int_D \psi_{i_1}(x_1)\, \nabla \phi_n(x_1) \cdot \nabla \phi_m(x_1)\, dx_1 \right)}_{\mathcal{B}^1(n,i_1,m)} \underbrace{\left(\int_D \psi_{i_2}(x_2) \psi_j(x_2)\, dx_2 \right)}_{M(i_2,j)}.
\end{aligned}
$$

Note that the tensor of order three $\mathcal{B}^1 \in \mathbb{R}^{N_v \times N_e \times N_v}$ is a weighted stiffness matrix and represents the discrete analogous of the trace operator $\mathrm{Tr}_{|1:2}$.

The linear system solved by $\mathcal{C}_{u^1 \otimes Y}$ can be written in a compact form using the notation introduced in Section 5.2:

$$
M \times_{2,1} (A \times_{1,1} \mathcal{C}_{u^1 \otimes Y}) = -M \times_{2,1} \left(\mathcal{B}^1 \times_{1,2} \mathcal{C}_{u^0 \otimes Y^{\otimes 2}} \right),
$$

which is equivalent to

$$
\boxed{A \times_{1,1} \mathcal{C}_{u^1 \otimes Y} = -\mathcal{B}^1 \times_{1,2} \mathcal{C}_{u^0 \otimes Y^{\otimes 2}}} \tag{5.9}
$$

Problem for $\mathbb{E}[u^2]$: FEM formulation

Recall the equation for $\mathbb{E}[u^2]$: given $\mathbb{E}[u^0 \otimes Y^{\otimes 2}] \in H^1_{\Gamma_D}(D) \otimes (L^2(D))^{\otimes 2}$ and $\mathbb{E}[u^1 \otimes Y] \in H^1_{\Gamma_D}(D) \otimes L^2(D)$, find $\mathbb{E}[u^2] \in H^1_{\Gamma_D}(D)$ such that

$$
\boxed{
\begin{aligned}
&\int_D \nabla \mathbb{E}\left[u^2\right](x) \cdot \nabla v(x)\, dx = \\
&-2 \int_D \mathrm{Tr}_{|1:3} \mathbb{E}\left[\nabla u^0 \otimes Y^{\otimes 2}\right](x) \cdot \nabla v(x)\, dx - \int_D \mathrm{Tr}_{|1:2} \mathbb{E}\left[\nabla u^1 \otimes Y\right](x) \cdot \nabla v(x)\, dx
\end{aligned}
}
$$

for every $v \in H^1_{\Gamma_D}(D)$. See problem (4.5) with $l = 0$ and $k = 2$.

Its finite element formulation is: given

$$\mathrm{Tr}_{|1:2}\mathbb{E}\left[\nabla u^1 \otimes Y\right](x) = \sum_{n=1}^{N_v}\sum_{i=1}^{N_e} \mathcal{C}_{u^1\otimes Y}(n,i)\nabla\phi_n(x)\psi_i(x),$$
$$\mathrm{Tr}_{|1:3}\mathbb{E}\left[\nabla u^0 \otimes Y^{\otimes 2}\right](x) = \sum_{n=1}^{N_v}\sum_{i_1=1}^{N_e}\sum_{i_2=1}^{N_e} \mathcal{C}_{u^0\otimes Y^{\otimes 2}}(n,i_1,i_2)\nabla\phi_n(x)\psi_{i_1}(x)\psi_{i_2}(x),$$

find $\mathbb{E}\left[u^2\right](x) = \sum_{n=1}^{N_v} \mathcal{C}_{u^2}(n)\ \phi_n(x)$ such that

$$\begin{aligned}
&\sum_n \mathcal{C}_{u^2}(n)\int_D \nabla\phi_n(x)\cdot\nabla\phi_m(x)\,dx \\
&\qquad = -2\sum_{n,i}\mathcal{C}_{u^1\otimes Y}(n,i)\int_D \psi_i(x)\nabla\phi_n(x)\cdot\nabla\phi_m(x)\,dx \\
&\qquad\quad - \sum_{n,i_1,i_2}\mathcal{C}_{u^0\otimes Y^{\otimes 2}}(n,i_1,i_2)\int_D \psi_{i_1}(x)\psi_{i_2}(x)\nabla\phi_n(x)\cdot\nabla\phi_m(x)\,dx
\end{aligned} \tag{5.10}$$

$\forall\ \phi_m \in V_h$. We define the tensor of order four $B^2 \in \mathbb{R}^{N_v\times N_e\times N_e\times N_v}$ as

$$\mathcal{B}^2(n,i_1,i_2,m) := \int_D \psi_{i_1}(x)\ \psi_{i_2}(x)\ \nabla\phi_n(x)\cdot\nabla\phi_m(x)\,dx. \tag{5.11}$$

Since $\{\psi_j\}$ is the piecewise constant finite element basis for $L^2(D)$, we observe that

$$\mathcal{B}^2(n,i_1,i_2,m) = \delta_{i_1,i_2}\mathcal{B}^1(n,i_1,m) = \begin{cases} 0, & \text{if } i_1 \neq i_2 \\ \mathcal{B}^1(n,i_1,m), & \text{if } i_1 = i_2 \end{cases}$$

where δ_{i_1,i_2} is the Kronecker delta, so that $\mathcal{B}^2$ is the discrete counterpart of the continuous operator $\mathrm{Tr}_{|1:3}$. Splitting the integrals in (5.10) as done for the equation for $\mathcal{C}_{u^1\otimes Y}$, we derive the compact expression for the finite element discretization of the equation for $\mathcal{C}_{u^2}$:

$$\boxed{A\,\mathcal{C}_{u^2} = -2\mathcal{B}^1 \times_{1,2} \mathcal{C}_{u^1\otimes Y} - \mathcal{B}^2 \times_{1,3} \mathcal{C}_{u^0\otimes Y^2}} \tag{5.12}$$

5.3.3 k-th order problem: FEM formulation

Here we derive the finite element formulation for the k-th order problem generalizing the results obtained with $k = 2$.

Recall the recursive k-th order problem: given all the lower order terms

$$\mathbb{E}\left[u^{k-l-s}\otimes Y^{\otimes(s+l)}\right] \in H^1_{\Gamma_D}(D)\otimes\left(L^2(D)\right)^{\otimes(s+l)}$$

for $s = 1, \ldots, k-l$, find $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$ such that

$$\boxed{\begin{aligned}&\int_{D^{\times(l+1)}} \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) \mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right] \cdot \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) v \, dx_1 \ldots dx_{l+1} = \\ &-\sum_{s=1}^{k-l} \binom{k-l}{s} \int_{D^{\times(l+1)}} \mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l-s} \otimes Y^{\otimes(s+l)}\right] \cdot \left(\nabla \otimes \mathrm{Id}^{\otimes l}\right) v \, dx_1 \ldots dx_{l+1}\end{aligned}}$$

$\forall\, v \in H^1_{\Gamma_D}(D) \otimes \left(L^2(D)\right)^{\otimes l}$.

The finite element discretizations of the known term $\mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l-s} \otimes Y^{\otimes(s+l)}\right]$ and the unknown $\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right]$ are respectively:

$$\begin{aligned}&\mathrm{Tr}_{|1:s+1} \mathbb{E}\left[\nabla u^{k-l-s} \otimes Y^{\otimes(s+l)}\right](x_1, \ldots, x_{l+1}) \\ &\quad = \sum_{n, i_1, \ldots, i_{s+l}} \mathcal{C}_{u^{k-l-s} \otimes Y^{\otimes(s+l)}}(n, i_1, \ldots, i_{s+l}) \\ &\qquad\qquad \nabla\phi_n(x_1)\psi_{i_1}(x_1) \cdots \psi_{i_s}(x_1) \otimes \psi_{i_{s+1}}(x_2) \otimes \cdots \otimes \psi_{i_{s+l}}(x_{l+1}), \\ &\mathbb{E}\left[u^{k-l} \otimes Y^{\otimes l}\right](x_1, \ldots, x_{l+1}) \\ &\quad = \sum_{n, i_1, \ldots, i_l} \mathcal{C}_{u^{k-l} \otimes Y^{\otimes l}}(n, i_1, \ldots, i_l) \phi_n(x_1) \otimes \psi_{i_1}(x_2) \otimes \cdots \otimes \psi_{i_l}(x_{l+1}).\end{aligned}$$

Generalizing the definitions of $\mathcal{B}^1 \in \mathbb{R}^{N_v \times N_e \times N_v}$ and $\mathcal{B}^2 \in \mathbb{R}^{N_v \times N_e \times N_e \times N_v}$, we introduce the tensor of order $s+2$, $\mathcal{B}^s \in \mathbb{R}^{N_v \times N_e \times \ldots \times N_e \times N_v}$, as

$$\mathcal{B}^s(n, i_1, \ldots, i_s, m) := \int_D \psi_{i_1}(x) \ldots \psi_{i_s}(x) \, \nabla\phi_n(x) \cdot \nabla\phi_m(x) \, dx. \tag{5.13}$$

Since we discretize the space $L^2(D)$ with piecewise constants, we have

$$\mathcal{B}^s(n, i_1, \ldots, i_s, m) = \delta_{i_{s-1}, i_s} \mathcal{B}^{s-1}(n, i_1, \ldots, i_{s-1}, m) = \ldots = \delta_{i_1, \ldots, i_s} \mathcal{B}^1(n, i_1, m),$$

so that $\mathcal{B}^s$ is a highly sparse tensor.

Repeating the same steps done in the case $k = 2$, we obtain the compact form of the recursive k-th order problem:

$$\boxed{A \times_{1,1} \mathcal{C}_{u^{k-l} \otimes Y^{\otimes l}} = -\sum_{s=1}^{k-l} \binom{k-l}{s} \mathcal{B}^s \times_{1,s+1} \mathcal{C}_{u^{k-l-s} \otimes Y^{\otimes(s+l)}}} \tag{5.14}$$

Observe that the linear system (5.14) coincides with (5.9) for $k = 2$ and $l = 1$, and with (5.12) for $k = 2$ and $l = 0$.

Remark 5.3.4. *In general, the structure of the tensor $\mathcal{B}^s$ strongly depends on the choice of the basis functions. With the aim of developing a black-box solver for the k-th order problem, we would need as input for the algorithm the stiffness matrix A and the tensors $\mathcal{B}^s$.*

5.4 Low-rank formats

The finite element discretization of the k-th order problem (5.14) involves high order tensors. The number of entries of a tensor grows exponentially in its order, so that it is possible to explicitly store only tensors of small order d. This phenomenon is known as *curse of dimensionality*. For large d it is necessary to use data-sparse or low-rank formats. We refer to [65] and the references therein for an introduction on the argument. In what follows, we give a brief overview on the most used techniques to represent or approximate a high order tensor ($d \geq 3$).

5.4.1 Classical formats

Let $\mathcal{X} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ be a tensor of order d. The *Canonical Polyadic (CP) format* of the tensor $\mathcal{X}$ is given by a linear combination of rank one tensors:

$$\mathcal{X}(i_1, \ldots, i_d) = \sum_{\alpha=1}^{r} U_1(i_1, \alpha) \cdots U_d(i_d, \alpha), \quad \forall i_j = 1, \ldots, n_j,\ j = 1, \ldots, d,$$

where r is a positive integer called canonical rank and $U_j = U_j(i_j, \alpha)$, $j = 1, \ldots, d$, are $n_j \times r$ matrices known as canonical factors. The storage of a tensor in CP format requires little memory if r is small enough. The major difficulties with respect to the CP format come from the numerical point of view. Indeed, on the one hand, the computation of the canonical rank is an NP (non polynomial) hard problem. On the other hand, there are no robust algorithms to compute the canonical representation of a given tensor with a fixed accuracy.

Remark 5.4.5. *Suppose $D^{\times d} \subset \mathbb{R}^d$ and $u \in V^{\otimes d}$ defined on $D^{\times d}$, V Hilbert space. Given a multivariate Lagrangian basis $\phi_{\mathbf{i}}$, $\mathbf{i} = (i_1 \ldots, i_d)$, associated to the grid $\{(x_{i_1}, \ldots, x_{i_d})\}$, $i_j = 1, \ldots, n_j \;\forall\, j$, the function u can be approximated as*

$$u(x_1, \ldots, x_d) = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} \mathcal{X}_u(i_1, \ldots, i_d) \phi_{i_1}(x_1) \cdots \phi_{i_d}(x_d), \tag{5.15}$$

$\mathcal{X}_u$ being a tensor of order d and size $n_1 \times \ldots \times n_d$.

Remark 5.4.6. *In the same setting as remark 5.4.5, let $V_0 \subset V_1 \subset \ldots \subset V_L \subset \ldots \subset V$ be a sequence of dense and nested finite dimensional subspaces of V, and $W_j := V_{j-1}^{\perp}$ in V_j, i.e. the orthogonal complement of V_{j-1} in V_j. Define the sparse tensor product approximation of $V^{\otimes d}$ as*

$$V_L^{(d)} := \sum_{|\mathbf{l}| \leq L} W_{l_1} \otimes \ldots \otimes W_{l_d}.$$

If we denote with $\{\psi_{l,i}\}_{i=1,\ldots,n_l}$ a basis of W_l, and with $\Lambda(L) = \{\mathbf{l} :\ |\mathbf{l}| \leq L\}$ the set of multi indices with absolute value less or equal to L, then

$$V_L^{(d)} = span\left\{\psi_{l_1,i_1} \otimes \ldots \otimes \psi_{l_d,i_d},\ \mathbf{l} \in \Lambda(L),\ i_j = 1, \ldots, n_{l_j}\right\}.$$

Each element $u \in V_L^{(d)}$ can be represented as

$$u(x_1, \ldots, x_d) = \sum_{\mathbf{l} \in \Lambda(L)} \sum_{i_1 \cdots i_d} \alpha_{\mathbf{l}}\, \psi_{l_1,i_1}(x_1) \otimes \ldots \otimes \psi_{l_d,i_d}(x_d). \tag{5.16}$$

Formula (5.16) *gives a CP representation of $\mathcal{X}_u$ with rank $r = \#\Lambda(L)$, where $\mathcal{X}_u$ is introduced in* (5.15). *Hence, a sparse grid representation can be seen as a particular* *case of low-rank representation where the basis is fixed. Whereas, when a low-rank representation of a tensor is looked for, the basis is not a priori fixed.*

The *Tucker format* of $\mathcal{X}$ is defined as the multiplication of a tensor $\mathcal{C}$ and matrices $U_1, \ldots, U_d$

$$\mathcal{X} = \mathcal{C} \times_1 U_1 \times_2 U_2 \ldots \times_d U_d, \tag{5.17}$$

where $\mathcal{C} \in \mathbb{R}^{r_1 \times \ldots \times r_d}$ is called core tensor, $(r_1, \ldots, r_d)$ is a tuple of positive integers called Tucker rank and $U_j = U_j(i_j, \alpha_j)$, $j = 1, \ldots, d$, are $n_j \times r_j$ matrices known as mode frames for the Tucker tensor representation. Elementwise, (5.17) becomes:

$$\mathcal{X}(i_1, \ldots, i_d) = \sum_{\alpha_1=1}^{r_1} \ldots \sum_{\alpha_d=1}^{r_d} \mathcal{C}(\alpha_1, \ldots, \alpha_d) U_1(i_1, \alpha_1) U_2(i_2, \alpha_2) \ldots U_d(i_d, \alpha_d),$$

for every $i_j = 1, \ldots, n_j$, $j = 1, \ldots, d$. The CP format can be seen as a particular case of the Tucker format, with $r_1 = \ldots = r_d = r$ and $\mathcal{C}(\alpha_1, \ldots, \alpha_d) = \delta_{\alpha_1, \ldots, \alpha_d}$, where $\delta_{\alpha_1, \ldots, \alpha_d}$ is the usual Kronecker delta.

We refer to the Matlab Tensor Toolbox [12] for a Matlab implementation of tensors in CP and Tucker format. We refer to [23, 57] and [96] for a deeper introduction on CP and Tucker formats respectively.

The major problem of the Tucker format relies in the storage of the core tensor, which still suffers from the curse of dimensionality. To overcome this problem, two different approaches have been proposed in recent years: the hierarchical Tucker (HT) decomposition and the Tensor Train (TT) decomposition. They are both based on the singular value decomposition (SVD) and require the storage of some auxiliary three dimensional arrays instead of a d dimensional tensor.

5.4.2 Hierarchical Tucker format

Before defining the Hierarchical Tucker (HT) format we need to introduce some preliminary concepts.

Definition 5.4.7 (Dimension tree)**.** *Let d be a positive integer. A dimension tree for d is a binary tree such that:*

- *the root is $\{1, \ldots, d\}$;*
- *each leaf node is a singleton $t = \{\mu\}$, $\mu \in \{1, \ldots, d\}$;*
- *each parent node $t = \{\mu_1, \ldots, \mu_q\}$ is the disjoint union of two successors:*

$$t = s_1 \cup s_2, \quad s_1 = \{\mu_1, \ldots, \mu_r\}, s_2 = \{\mu_{r+1}, \ldots, \mu_q\}.$$

Given a dimension tree $\mathcal{T}$, we denote with $\mathcal{L}(\mathcal{T})$ the set of leaves and with $\mathcal{I}(\mathcal{T}) = \mathcal{T} \setminus \mathcal{L}(\mathcal{T})$ the set of non-leaf nodes. A dimension tree is called *canonical dimension tree* if the separation index of every two successors is $r = [q/2]$, the integer part of $q/2$. See Figure 5.1 for an example of canonical dimension tree for $d = 5$.

For every node t in a dimension tree it is possible to define its complementary $t' := \{1, \ldots, d\} \setminus t$.

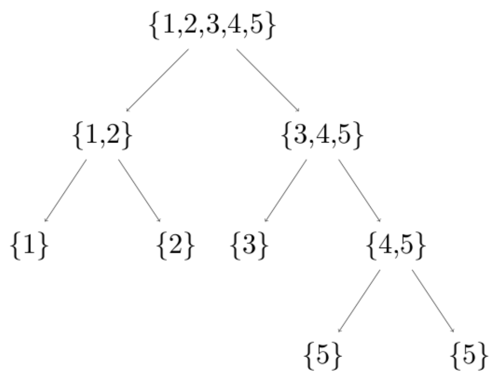


Figure 5.1: *Canonical dimensional tree for* $d = 5$ *with root* $\{1, 2, 3, 4, 5\}$ *and leaves* $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$ *and* $\{5\}$.

Definition 5.4.8. *Let* $t = \{\mu_1, \dots, \mu_q\} \in \mathcal{T}$ *be a node in a dimension tree for* d *and* $\mathcal{X} \in \mathbb{R}^{n_1 \times \dots \times n_d}$. *We define the* t*-matricization of* $\mathcal{X}$ *by merging all the indices in* t *into one row index, and all the indices in* t' *into one column index:*

$$\mathcal{X}^{(t)} \in \mathbb{R}^{N_t \times N_{t'}},$$
$$\mathcal{X}^{(t)}\left((i_\mu)_{\mu \in t}, (i_\nu)_{\nu \in t'}\right) = \mathcal{X}(i_1, \dots, i_d),$$

where $N_t := \prod_{\mu \in t} n_\mu$ *and* $N_{t'} := \prod_{\nu \in t'} n_\nu$.

The HT decomposition of a tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ is realized by performing a SVD decomposition of the matricization $\mathcal{X}^{(t)}$ for each interior node $t \in \mathcal{I}(\mathcal{T})$, $\mathcal{T}$ being a dimensional tree for d. The reason why this works is explained in the following proposition.

Proposition 5.4.9 (Nestedness of matricization)**.** *Let* $\mathcal{X} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ *and* $t = s_1 \cup s_2$ *be a non-leaf node in* $\mathcal{T}$*, a dimension tree for* d*. Then*

$$span\left\{\mathcal{X}^{(t)}\right\} \subset span\left\{\mathcal{X}^{(s_1)} \otimes \mathcal{X}^{(s_2)}\right\},$$

where $\mathcal{X}^{(j)}$ *denotes the vector subspace image of* $\mathcal{X}^{(t)}$ *for* $j = t, s_1, s_2$.

Let $U_l = \{(U_l)_i\}_{i=1}^{r_l} \in \mathbb{R}^{N_l \times r_l}$ be a basis of the image of $\mathcal{X}^{(l)}$ with $r_l = rank(\mathcal{X}^{(l)})$, for $l = t, s_1, s_2$. Proposition 5.4.9 states the existence of a so called *transfer array* $B_t \in \mathbb{R}^{r_{s_1} \times r_{s_2} \times r_t}$ such that

$$(U_t)_i = \sum_{k_1=1}^{r_{s_1}} \sum_{k_2=1}^{r_{s_2}} B_t(k_1, k_2, i)(U_{s_1})_{k_1} \otimes (U_{s_2})_{k_2}.$$

for every $i = 1, \dots, r_t$.

The recursive application of Proposition 5.4.9 guarantees that to represent a tensor into the HT-format, it is sufficient to store the matrix U_t for each leaf node $t \in \mathcal{L}(\mathcal{T})$, and the transfer array B_t for each non-leaf node $t \in \mathcal{I}(\mathcal{T})$:

$$\left((U_t)_{t \in \mathcal{L}(\mathcal{T})}, (B_t)_{t \in \mathcal{I}(\mathcal{T})}\right).$$

The set of non-negative integers $(r_t)_{t\in\mathcal{T}}$ is called *hierarchical rank.*

Since the cardinality of $\mathcal{I}(\mathcal{T})$ is $d-1$, the storage complexity of $\mathcal{X}$ in HT-format is bounded by

$$(d-1)r^3 + r\sum_{\mu=1}^{d} n_\mu, \tag{5.18}$$

where $r := \max\{r_t \;:\; t \in \mathcal{T}\}$. Note that the storage complexity is linear in d (provided that r does not explode in d), so that the curse of dimensionality is overpassed. We refer to the Matlab Toolbox [66] for the implementation of the HT-format and to [49,54] and the references therein for a more exhaustive dissertation.

5.4.3 Tensor Train format

Let $\mathcal{X} \in \mathbb{R}^{n_1\times\dots\times n_d}$ be a d dimensional tensor. Its *tensor train (TT)* representation is given by

$$\mathcal{X}(i_1,\dots,i_d) = \sum_{\alpha_1=1}^{r_1}\dots\sum_{\alpha_{d-1}=1}^{r_{d-1}} G_1(i_1,\alpha_1)G_2(\alpha_1,i_2,\alpha_2)\dots G_d(\alpha_{d-1},i_d), \tag{5.19}$$

where $G_j \in \mathbb{R}^{r_{j-1}\times n_j\times r_j}$, $j = 1,\dots,d$, are tree dimensional arrays called *cores* of the TT-decomposition ($r_0 = r_d = 1$), and the set $(r_1,\dots,r_{d-1})$ is known as *TT-rank.* The storage complexity is $O\left((d-2)nr^2 + 2rn\right)$ where $n = \max\{n_1,\dots,n_d\}$, $r = \max\{r_1,\dots,r_d\}$.

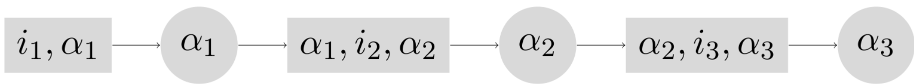


Figure 5.2: *Representation of the TT-format of a tensor (see formula* (5.19)*).*

Figure 5.2 depicts the TT-format of the tensor $\mathcal{X}$ in (5.19) and highlights its linear structure. There are two kinds of indices: the auxiliary indices $\alpha_1,\dots,\alpha_{d-1}$, contained in the circles, and the physical indices $i_1,\dots,i_d$, contained in the rectangles. The arrows represent a linear relation. As a curiosity we specify that the name tensor train decomposition comes from the fact that Figure 5.2 actually looks like a train.

Thanks to its linear structure, the TT-format can be seen as a generalization of the CP format. The main advantage with respect to the CP format is that in [81] the author provides an algorithm to compute in an efficient way the TT-decomposition of a given tensor and, consequently, the TT-ranks. This algorithm, called TT-SVD algorithm, is based on a recursive application of the SVD.

Note that the TT-format is a particular case of the HT-format, with dimension tree $\mathcal{T}$ of the form represented in Figure 5.3, that is for each non-leaf node $t \in \mathcal{I}(\mathcal{T})$, $t = s_1 \cup s_2$ where either s_1 or s_2 is a leaf. The HT-format implies the use of recursive algorithms which may cause more difficulties in the implementation. On the other hand, the TT-format, thanks to its linear structure, is much more easy to handle with. For this reason, we chose to develop a code which employs only TT-format representations of tensors (see Sections 5.5 and 5.6).

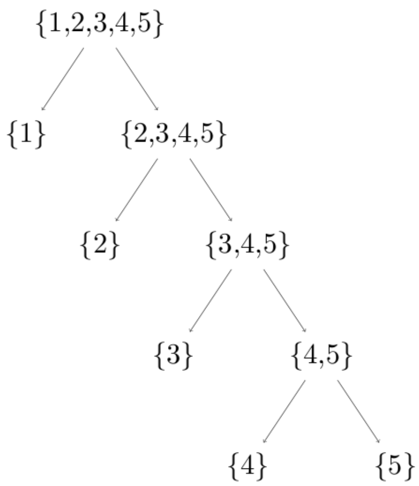


Figure 5.3: *Dimension tree $\mathcal{T}$ for $d = 5$. Given a tensor of order d, its HT representation with respect to $\mathcal{T}$ is equal to its TT representation.*

Many linear algebra operations (like addition, scalar product, μ-th mode product, etc), together with the rounding operation, that consists in reducing the TT-ranks while maintaining the accuracy, are implemented in the Matlab TT-Toolbox available at `http://spring.inm.ras.ru/osel/?page_id=24`.

5.5 Computation of the correlations of Y in TT-format

In Section 5.3 we have discretized the first moment equation. In the next section we describe a recursive algorithm developed in TT-format, able to solve the first moment equation. The input terms are the k-points correlations

$$\mathbb{E}\left[Y^{\otimes k}\right](x_1, \ldots, x_k) := \int_\Omega Y(\omega, x_1) \otimes \cdots \otimes Y(\omega, x_k)\, d\mathbb{P}. \tag{5.20}$$

Here we provide a way to compute these correlations in TT-format, recalling the main results obtained in [67].

Let us start with the following proposition.

Proposition 5.5.10. *Let $Y(\omega, x)$ be a centered second order Gaussian random field and*

$$Y_N(\omega, x) = \sigma \sum_{j=1}^{N} \sqrt{\tilde{\lambda}_j}\, \phi_j(x)\, \xi_j(\omega) \tag{5.21}$$

its truncated Karhunen-Loève expansion (see Proposition 3.3.4) with a prescribed tolerance $tol > 0$, where $\{\xi_j\}$ are independent mean-free Gaussian random variables, $\{\phi_j\}$ is a orthonormal basis for $L^2(D)$ and $\sigma^2 = \dfrac{1}{|D|}\displaystyle\int_D \mathbb{V}ar\left[Y_N(\omega, x)\right] dx$. Then, for *every positive integer k, the k-points correlation function of Y_N is such that*

$$\mathbb{E}\left[Y_N^{\otimes k}\right] = \begin{cases} 0, & \text{if } k \text{ odd} \\ \sigma^k \displaystyle\sum_{\substack{j_1,\dots,j_{k/2}=1 \\ j_1 \le \dots \le j_{k/2}}}^{N} \prod_{l=1}^{N} \tilde{\lambda}_l^{m_{\mathbf{j}}(l)} \left(2m_{\mathbf{j}}(l)-1\right)!! \left(\sum_{\mathbf{i}\in P_{\mathbf{j}}} \bigotimes_{\mu=1}^{k} \phi_{i_\mu}\right), & \text{if } k \text{ even} \end{cases} \tag{5.22}$$

where $m_{\mathbf{i}}(l)$ is the multiplicity of l in the multi-index $\mathbf{i} = (i_1, \dots, i_k)$

$$m_{\mathbf{i}}(l) := \#\{n \,:\, i_n = l\} \tag{5.23}$$

and $P_{\mathbf{j}}$ is the set of all unique permutations of the multi-index $(j_1, j_1, j_2, j_2, \dots, j_{k/2}, j_{k/2})$.

Proof. For completeness, we report here the proof, which can also be found in [67]. For convenience in the presentation, let $\sigma = 1$. Using the truncated KL expansion of Y (5.21) in the definition of the k-points correlation (5.20), and exploiting the independence of the random variables $\{\xi_j\}$,

$$\begin{aligned} \mathbb{E}\left[Y_N^{\otimes k}\right](x_1,\dots,x_k) &= \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \mathbb{E}\left[\prod_{\mu=1}^{k} \sqrt{\tilde{\lambda}_{i_\mu}} \xi_{i_\mu}(\omega)\right] \bigotimes_{\mu=1}^{k} \phi_{i_\mu} \\ &= \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \underbrace{\prod_{l=1}^{N} \tilde{\lambda}_l^{m_{\mathbf{i}}(l)/2} \mathbb{E}\left[\xi_l(\omega)^{m_{\mathbf{i}}(l)}\right]}_{\mathcal{C}_k(i_1,\dots,i_k)} \bigotimes_{\mu=1}^{k} \phi_{i_\mu}, \end{aligned} \tag{5.24}$$

where $m_{\mathbf{i}}(l)$ is the multiplicity of l in the multi-index $\mathbf{i} = (i_1, \dots, i_k)$, defined in (5.23). Observe that, for every multi-index $\mathbf{i} \in \{1, \dots, N\}^k$ and every integer $l \in \{1, \dots, N\}$, $\mathbb{E}\left[\xi_l(\omega)^{m_{\mathbf{i}}(l)}\right] \neq 0$ only if $m_{\mathbf{i}}(l)$ is even, so that $\mathbb{E}\left[Y^{\otimes k}\right](x_1, \dots, x_k) \neq 0$ only if k is even. Let us now suppose k and all the multiplicities $m_i(l)$ even, so that the multi-index $(i_1, \dots, i_k)$ is a permutation of the multi-index $\mathbf{j} = (j_1, j_1, j_2, j_2, \dots, j_{k/2}, j_{k/2})$, for $j_1, \dots, j_{k/2} \in \{1, \dots, N\}$. Then, the k-points correlation function $\mathbb{E}\left[Y_N^{\otimes k}\right]$ is given by

$$\begin{aligned} \mathbb{E}\left[Y_N^{\otimes k}\right] &= \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \prod_{l=1}^{N} \tilde{\lambda}_l^{m_{\mathbf{i}}(l)/2} \left(m_{\mathbf{i}}(l)-1\right)!! \bigotimes_{\mu=1}^{k} \phi_{i_\mu} \\ &= \sum_{\substack{j_1,\dots,j_{k/2}=1 \\ j_1 \le \dots \le j_{k/2}}}^{N} \prod_{l=1}^{N} \tilde{\lambda}_l^{m_{\mathbf{j}}(l)} \left(2m_{\mathbf{j}}(l)-1\right)!! \left(\sum_{\mathbf{i}\in P_{\mathbf{j}}} \bigotimes_{\mu=1}^{k} \phi_{i_\mu}\right), \end{aligned} \tag{5.25}$$

where $P_{\mathbf{j}}$ is the set of all unique permutations of $(j_1, j_1, j_2, j_2, \dots, j_{k/2}, j_{k/2})$. □

Let k be even. The technique used to compute the TT-format of the k-points correlation $\mathbb{E}\left[Y_N^{\otimes k}\right]$ is composed of three steps.

Step 1 According to Proposition 5.5.10, first of all we compute the KL-expansion of the field $Y(\omega, x)$, that is the set of decreasing non-negative eigenvalues $\{\tilde{\lambda}_j\}_{j=1}^N$ and the eigenvector matrix $\phi = (\phi_1, \dots, \phi_N)$. At this step, we compute a sufficiently high number N of modes.

Step 2 Given the order k and the vector of decreasing non-negative eigenvalues $\{\tilde{\lambda}_j\}_{j=1}^N$, we aim at computing the TT-format of the the core tensor $\mathcal{C}_k$ defined in (5.24). Since the storage of the exact TT representation of tensor $\mathcal{C}_k$ becomes expensive for k moderately large, an approximation $\tilde{\mathcal{C}}_k$ of $\mathcal{C}_k$ satisfying

$$\left\|\mathcal{C}_k - \tilde{\mathcal{C}}_k\right\|_F \leq tol,$$

is computed via the Matlab function **constr_tt**, tol being a prescribed tolerance and $\|\cdot\|_F$ denoting the Frobenius norm. We refer to [67] for a description of this Matlab function. Note that, since the basis $\{\phi_j\}_j$ is orthonormal in $L^2(D)$, then $\left\|\mathbb{E}\left[Y_N^{\otimes k}\right]\right\|_{(L^2(D))^{\otimes k}} = \sigma^k \|\mathcal{C}_k\|_F$.

Step 3 After having further approximated the TT-tensor $\tilde{\mathcal{C}}_k$ using the sub-routine **tt_round** from the TT-toolbox, we multiply $\tilde{\mathcal{C}}_k$ for the eigenvector matrix $\phi = (\phi_1, \ldots, \phi_N)$ using the sub- routine **ttm** from the TT-toolbox. Multiplying for σ^k, we finally obtain the TT-format of $\mathbb{E}\left[Y_N^{\otimes k}\right]$, denoted as $\mathcal{C}^{TT}_{Y^{\otimes k}}$.

Algorithm 1 summarizes steps 2 and 3, and can be applied to a d-dimensional domain D. On the other hand, we have implemented the KL-expansion only in the case $D \subset \mathbb{R}$. Once the KL-expansion is available for $D \subset \mathbb{R}^d$, the entire computation of $\mathbb{E}\left[Y_N^{\otimes k}\right]$ may be performed in precisely the same way as with $D \subset \mathbb{R}$. Observe that the cost of computing $\mathcal{C}^{TT}_{Y^{\otimes k}}$ and its storage is independent of the number of degrees of freedom N_h and depends only on the decay of the eigenvalues $\{\tilde{\lambda}_j\}$ or equivalently, the truncation level N needed to achieve a prescribed tolerance.

Algorithm 1 Function **compute_moment_Y**, which computes the even k-points correlations of a centered Gaussian random field.

Require: order of the correlation k, eigenvector matrix $\phi = (\phi_1, \ldots, \phi_N)$, eigenvalues $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N)$ and tolerances tol_1, tol_2
Ensure: TT-tensor $\mathcal{C}^{TT}_{Y^{\otimes k}}$
  $\tilde{\mathcal{C}}_k$=**constr_tt**($k, \tilde{\lambda}, tol_1$)
  $\mathcal{C}^{TT}_{Y^{\otimes k}}$=**tt_round**($\tilde{\mathcal{C}}_k, tol_2$)
  for $j = 1 : k$ **do**
    $\mathcal{C}^{TT}_{Y^{\otimes k}}$=**ttm**($\mathcal{C}^{TT}_{Y^{\otimes k}}, j, \phi^T$)
  end for
  $\mathcal{C}^{TT}_{Y^{\otimes k}} = \sigma^k \mathcal{C}^{TT}_{Y^{\otimes k}}$

Remark 5.5.11. *Note that the tensor $\mathbb{E}\left[Y_N^{\otimes k}\right]$ is supersymmetric, i.e. it is invariant under any permutation of its entries. The TT-tensor $\mathcal{C}^{TT}_{Y^{\otimes k}}$ is constructed in such a way that the following symmetries are preserved:*

$$\mathcal{C}^{TT}_{Y^{\otimes k}}(i_1, \ldots, i_{k/2}, i_{k/2+1}, \ldots, i_k) = \mathcal{C}^{TT}_{Y^{\otimes k}}(i_k, \ldots, i_{k/2+1}, i_{k/2}, \ldots, i_1),$$
$$\mathcal{C}^{TT}_{Y^{\otimes k}}(\ldots, i_n, \ldots, i_m, \ldots) = \mathcal{C}^{TT}_{Y^{\otimes k}}(\ldots, i_m, \ldots, i_n, \ldots),$$

$\forall\, n, m \leq k/2,$ *or* $\forall\, n, m > k/2.$

5.6 Computation of the first moment approximation in TT-format

In the previous section we have explained how to compute the correlations of $Y(\omega, x)$ in TT-format. These are the input data of the recursive first moment problem described in Table 4.1. Here we illustrate how to solve in TT-format the linear system (5.14) for $k = 2, \dots, K$ and $l = k-1, k-2, \dots, 0$, where K is the prescribed order of approximation for the computation of the mean. Recall that, for any k, if $l = 0$, the solution of problem (5.14) is the finite element discretization of k-th order correction $\mathbb{E}\left[u^k\right]$.

Let k, l be fixed. The main ingredients needed to numerically solve problem (5.14) are the construction in TT-format of the tensor $\mathcal{B}^s$ for $s = 1, \dots, k-l$, denoted with $\mathcal{B}^{TT,s}$, and the implementation of the saturation $\times_{r,s}$ between two TT-tensors $\forall\, r, s$ (see definition 5.2.2).

Tensor $\mathcal{B}^s$ in TT-format: $\mathcal{B}^{TT,s}$

First of all, we explicitly construct $\mathcal{B}^1$ and convert it in TT-format using the sub-routine **tt_tensor** from the TT-toolbox, obtaining $\mathcal{B}^{TT,1}$. Recall that the tensor $\mathcal{B}^s$ satisfies the following relation

$$\mathcal{B}^s(n, i_1, \dots, i_s, m) = \delta_{i_1,\dots,i_s} \mathcal{B}^1(n, i_1, m).$$

We have implemented the function **add_modes**, which receives as inputs a positive integer s and a TT-tensor $\mathcal{X}^1 \in \mathbb{R}^{n_1 \times \dots \times n_d}$, and returns a TT-tensor $\mathcal{X}^{s+1} \in \mathbb{R}^{n_1 \times \dots \times n_1 \times n_2 \dots \times n_d}$ defined adding s modes to $\mathcal{X}^1$ as follows:

$$\mathcal{X}^{s+1}(i_1, \dots, i_s, i_{s+1}, \dots, i_{s+d}) = \delta_{i_1,\dots,i_{s+1}} \mathcal{X}^1(i_{s+1}, \dots, i_{s+d}). \tag{5.26}$$

Note that, after reordering the entries of $\mathcal{B}^{TT,1}$ as $\mathcal{B}^{TT,1}(i_1, n, m)$, then

$$\mathcal{B}^{TT,s} = \mathtt{add\_modes}(\mathcal{B}^{TT,1}, \mathtt{s} - 1).$$

To realize the operation (5.26), we apply the function **extract_cell** which, given a TT-tensor $\mathcal{X}^1$, returns a cell array $\mathcal{X}^1_{cell}$ containing the reordered cores $G_1, \dots, G_d$ of $\mathcal{X}^1$:

$$\mathcal{X}^1_{cell}\{j\}(l, i_1, i_2) = G_j(i_1, l, i_2), \quad \mathcal{X}^1_{cell}\{j\} \in \mathbb{R}^{n_j \times r_{j-1} \times r_j} \quad \forall\, j,$$

with $r_0 = r_d = 1$.

We construct $\mathcal{X}^{s+1}_{cell}$, the cell array containing the cores of $\mathcal{X}^{s+1}$ as follows.

- $\mathcal{X}^{s+1}_{cell}\{1\}$ is the identity matrix of size $(n_1 \times n_1)$.
- $\mathcal{X}^{s+1}_{cell}\{j\}$ is the identity tensor I of size $(n_1 \times n_1 \times n_1)$ for $j = 2, \dots, s$.
- $\mathcal{X}^{s+1}_{cell}\{s+1\}$ is the saturation of the first index of I with the first index of the matrix $\mathcal{X}^1_{cell}\{1\}$. To compute this saturation, we use the sub-routine **ttt** of HT-Toolbox.
- $\mathcal{X}^{s+1}_{cell}\{j\} = \mathcal{X}^1_{cell}\{j-s\}$ for $j = s+2, \dots, s+d$.

Finally, we convert $\mathcal{X}^{s+1}_{cell}$ in TT-format using the sub-routine **tt_tensor**. See Algorithm 2.

Algorithm 2 Function **addmodes**, which adds modes to the left of a given TT-tensor.

Require: TT-tensor $\mathcal{X}^1(i_1, \ldots, i_d)$ and a positive integer s.
Ensure: TT-tensor $\mathcal{X}^{s+1}(i_1, \ldots, i_s, i_{s+1}, \ldots, i_{s+d})$ as in (5.26).

$\mathcal{X}^1_{cell}$ = **extract_cell**($\mathcal{X}^1$)
Set $n_1 = size(\mathcal{X}^1_{cell}\{1\}, 1)$, d=order of the TT-tensor $\mathcal{X}^1$
Initialize $\mathcal{X}^{s+1}_{cell}$ as a cell array of dimension $s + d$
Set $\mathcal{X}^{s+1}_{cell}\{1\} = \texttt{eye}(n_1, n_1)$
for $j = 2 : s$ **do**
  Set $\mathcal{X}^{s+1}_{cell}\{j\} = \texttt{eye}(n_1, n_1, n_1)$
end for
Set $\mathcal{X}^{s+1}_{cell}\{s + 1\}$ = **ttt**($\mathcal{X}^1_{cell}\{1\}, \texttt{eye}(n_1, n_1, n_1), 1, 1$)
for $j = s + 2 : s + d$ **do**
  Set $\mathcal{X}^{s+1}_{cell}\{j\} = \mathcal{X}^1_{cell}\{j - s\}$
end for
Set $\mathcal{X}^{s+1}$=**tt_tensor**($\mathcal{X}^{s+1}_{cell}$)

Saturation $\times_{s,r}$ between two TT-tensors

Given two TT-tensors $\mathcal{X} \in \mathbb{R}^{n_s \times \ldots \times n_{s+r-1} \times h}$, $\mathcal{Y} \in \mathbb{R}^{m_1 \times \ldots \times m_{s-1} \times n_s \times \ldots \times n_{s+r-1} \times m_{s+r} \times \ldots \times m_d}$, the function **apply** performs the multiplication between $\mathcal{X}$ and $\mathcal{Y}$ along prescribed nodes. In particular, the call $\mathcal{Z} = \texttt{apply}(\mathcal{Y}, \mathcal{X}, \texttt{s}, \texttt{r})$ realizes $\mathcal{Z} = \mathcal{X} \times_{s,r} \mathcal{Y}$, that is

$$\begin{aligned}&\mathcal{Z}(k_1, \ldots, k_{s-1}, j, k_{s+r}, \ldots, k_d)\\ &= \sum_{i_s=1}^{n_s} \cdots \sum_{i_{s+r-1}=1}^{n_{s+r-1}} \mathcal{X}(i_s, \ldots, i_{s+r-1}, j)\mathcal{Y}(k_1, \ldots, k_{s-1}, i_s, \ldots, i_{s+r-1}, k_{s+r}, \ldots, k_d),\end{aligned}$$

where $\mathcal{Z} \in \mathbb{R}^{m_1 \times \ldots \times m_{s-1} \times h \times m_{s+r} \times \ldots \times m_d}$ is a $d - r + 1$ order tensor. See Definition 5.2.2.

The implementation of this operation makes use of the function **extract_cell**. Let $\mathcal{X}_{cell}$ and $\mathcal{Y}_{cell}$ be cell arrays containing the cores of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then $\mathcal{Z}_{cell}$ is constructed as follows.

- Compute the tensor of order 3 $temp1$ as the saturation of the first index of $\mathcal{Y}_{cell}\{s\}$ with the first index of the matrix $\mathcal{X}_{cell}\{1\}$. This saturation is realized with the sub-routine **ttt** of HT-Toolbox.
- Matricize $temp1$ using the sub-routine **matricize** of the HT-Toolbox.
- Perform a loop on $k = 2, \ldots, r$, and at each step of the loop:

  Compute the tensor of order 4 $temp2$ as the saturation of the first index of $\mathcal{Y}_{cell}\{s + k - 1\}$ with the first index of $\mathcal{X}_{cell}\{k\}$

  Reshape and matricize $temp2$, where **reshape** is a Matlab sub-routine

  Compute the matrix - matrix product $temp1 = temp1 * temp2$
- Set $\mathcal{Z}_{cell} = [\mathcal{Y}_{cell}(1 : s - 1); temp1; \mathcal{Y}_{cell}(s + r : d)]$.

See Algorithm 3.

Algorithm 3 Function **apply**, which performs the saturation of two TT-tensors along prescribed nodes.

Require: TT-tensors $\mathcal{X} \in \mathbb{R}^{n_s\times\ldots\times n_{s+r-1}\times h}$, $\mathcal{Y} \in \mathbb{R}^{m_1\times\ldots\times m_{s-1}\times n_s\times\ldots\times n_{s+r-1}\times m_{s+r}\times\ldots\times m_d}$ and positive integers $s,\ r$.
Ensure: TT-tensor $\mathcal{Z} \in \mathbb{R}^{m_1\times\ldots\times m_{s-1}\times h\times m_{s+r}\times\ldots\times m_d}$ obtained as $\mathcal{Z} = \mathcal{X} \times_{s,r} \mathcal{Y}$

$\mathcal{X}_{cell}$ = **extract_cell**($\mathcal{X}$), $\mathcal{Y}_{cell}$ = **extract_cell**($\mathcal{Y}$)
Set $temp1$ = **ttt**($\mathcal{Y}_{cell}\{s\}, \mathcal{X}^T_{cell}\{1\}, 1, 1$)
Set $temp1$ = **matricize**($temp1, 2, [3\ 1]$)
for $k = 2 : r$ **do**
    Set $sz\_y = size(\mathcal{Y}_{cell}\{s+k-1\})$, $sz\_x = size(\mathcal{X}_{cell}\{k\})$
    Set $temp2$ = **ttt**($\mathcal{Y}_{cell}\{s+k-1\}, \mathcal{X}_{cell}\{k\}, 1, 1$)
    Set $temp2$ = **reshape**($temp2, [sz\_y(2:3), sz\_x(2:3)]$)
    Set $temp2$ = **matricize**($temp2, [1\ 3], [2\ 4]$)
    Set $temp1 = temp1 * temp2$
end for
Set $sz\_temp1 = [size(\mathcal{X}_{cell}\{r\}, 3), size(\mathcal{Y}_{cell}\{s\}, 2), size(\mathcal{Y}_{cell}\{s+r-1\}, 3)]$
Set $temp1$ = **dematricize**($temp1, sz\_temp1, 2, [31]$)
Set $temp1$ = **ttm**($temp1, \mathcal{X}_cell\{r+1\}, 1$)
Set $\mathcal{Z}_{cell} = [\mathcal{Y}_{cell}(1:s-1); temp1; \mathcal{Y}_{cell}(s+r:d)]$
Set $\mathcal{Z}$=**tt_tensor**($\mathcal{Z}_{cell}$)

Recursive first moment problem in TT-format

We are now ready to describe the code developed to solve the recursive K-th order approximation problem. This code computes all the terms in Table 4.1. The inputs are the order K of the approximation $\mathbb{E}\left[T^K u\right]$ of $\mathbb{E}\left[u\right]$ we want to achieve and the number of elements N_h of the partition of D.

After solving the 0-th order approximation problem, we compute the third order tensor $\mathcal{B}^1$ as well as the tensors $\mathcal{B}^{TT,k}\ \forall\ k = 2,\ldots,K$ using the function **addmodes** (see Algorithm 2), and store them into the structure SB. Moreover, we compute the input terms in the recursion, that is $\mathcal{C}^{TT}_{Y^{\otimes k}}\ \forall\ k = 2,\ldots,K$ even, with the function **compute_moment_Y** (see Algorithm 1), and $\mathcal{C}^{TT}_{u^0\otimes Y^{\otimes k}}$ with the sub-routine of the TT-toolbox **kron**, which implements the Kronecker products between TT-tensors. We store $\mathcal{C}^{TT}_{u^0\otimes Y^{\otimes k}}$ into the structure SY.

Then, we initialize a structure $STab$ in which we are going to store all the correlations of Table 4.1. We start storing in $STab$ all the correlations $\mathcal{C}^{TT}_{u^0\otimes Y^{\otimes k}}$ for $k = 0,\ldots,K$. We make two nested loops on the columns $c = 2,\ldots,K+1$ and rows $r = 1,\ldots,K+2-c$ of Table 4.1, so that we identify the term we are going to compute, that is $\mathcal{C}^{TT}_{u^{c-1}\otimes Y^{\otimes r-1}} = \mathcal{C}^{TT}_{u^{k-l}\otimes Y^{\otimes l}}$ with $l = r-1$ and $k = c+r-2$. We extract from $STab$ the elements in the k-th diagonal of the table, that is $\mathcal{C}^{TT}_{u^{k-l-s}\otimes Y^{\otimes(s+l)}}$ for $s = 1,\ldots,k-l$. To solve the linear system (5.14) we proceed in two steps: we compute the loading term $-\sum_{s=1}^{k-l} \mathcal{B}^{TT,s} \times_{1,s+1} \mathcal{C}^{TT}_{u^{k-l-s}\otimes Y^{\otimes(s+l)}}$ using the function **apply**, and then we multiply the loading term for the inverse of the stiffness matrix A^{-1} along the first direction, using the sub-routine **ttm**. For a large problem, it is not feasible to compute A^{-1}. In general, one should modify the code implementing a function that, given a matrix B and a TT-tensor v solve $B^{-1}v$, without explicitly constructing B^{-1}.

Finally, the K-th order correction is obtained summing the elements of the first row of Table 4.1 for $k = 0:2:K$, divided by the factorial term $k!$. See Algorithm 4.

Algorithm 4 Implementation of the recursion in Table 4.1.

Require: Order of the approximation K, number of elements of the mesh N_h, standard deviation σ, covariance function Cov_Y of $\frac{1}{\sigma}Y$, tolerances tol_1, tol_2, tol_3, load function f
Ensure: K-th order approximation of $\mathbb{E}[u]$ in TT-format, that is $\sum_{k=0}^{K} \frac{1}{k!}\mathcal{C}_{u^k}$
  Solve the deterministic problem for $\mathcal{C}_{u^0}$, set $\mathcal{C}_{u^0}^{TT}$ =**tt_tensor**($\mathcal{C}_{u^0}$) and save it in a structure $SY(1) = \mathcal{C}_{u^0}^{TT}$
  Construct $\mathcal{B}^1$, set $\mathcal{B}^{TT,1}$ =**tt_tensor**($\mathcal{B}^1$) and save it in a structure $SB(1) = \mathcal{B}^{TT,1}$
  Compute the KL-expansion of Y with a prescribed tolerance tol_1, and derive N, $\phi = \{\phi_j\}_{j=1}^N$, $\tilde{\lambda} = \{\tilde{\lambda}_j\}_{j=1}^N$
  for k=2:K **do**
    Construct $\mathcal{B}^{TT,k}$=**addmodes**($\mathcal{B}^{TT,1}, k-1$) and set $SB(k) = \mathcal{B}^{TT,k}$
    Construct $\mathcal{C}_{Y^{\otimes k}}^{TT}$ = **compute_moment_Y**($k, \phi, \tilde{\lambda}, tol_2, tol_3$)
    Compute $\mathcal{C}_{u^0 \otimes Y^{\otimes k}}^{TT}$ =**kron**($\mathcal{C}_{u^0}^{TT}, \mathcal{C}_{Y^{\otimes k}}^{TT}$) and set $SY(k) = \mathcal{C}_{u^0 \otimes Y^{\otimes k}}^{TT}$
  end for
  Initialize the structure $STab$ and store $\mathcal{C}_{u^0 \otimes Y^{\otimes k}}^{TT}$ for every k
  for c=2:K+1 **do**
    for r=1:K+2-c **do**
      extract from $STab$ the correlations $\mathcal{C}_{u^{k-l-s} \otimes Y^{\otimes (s+l)}}^{TT}$ for $s = 1, \ldots, k-l$, where $k = r + c - 2$ and $l = r - 1$
      Compute $temp = -\sum_{s=1}^{k-l} \mathcal{B}^{TT,s} \times_{1,s+1} \mathcal{C}_{u^{k-l-s} \otimes Y^{\otimes (s+l)}}^{TT}$
      Compute $\mathcal{C}_{u^{k-l} \otimes Y^{\otimes l}}^{TT}$ = **ttm**($temp, 1, A^{-1}$) and store it in $STab$
    end for
  end for
  Compute $\sum_{k=0}^{K} \frac{\mathcal{C}_{u^k}}{k!}$

Remark 5.6.12. *The computation of the stiffness matrix A and of the tensor $\mathcal{B}^1$ is performed in the one dimensional case $D = [0,1]$, with piecewise linear and constant finite elements to discretize $H^1_{\Gamma_D}(D)$ and $L^2(D)$ respectively. In general, we can see our algorithm as a black-box method which needs as inputs the stiffness matrix A and the tensor $\mathcal{B}^1$, and returns the K-th order approximation of the expected value of the stochastic solution.*

5.7 Storage requirements of the TT-algorithm

The storage complexity of a tensor of order k in TT-format highly depends on the TT-rank $(r_1, \ldots, r_{k-1})$. See Section 5.4.3. In the first part of this section we numerically study the storage complexity of the input data of our algorithm, that is the k-points correlations of Y in TT-format $\mathcal{C}_{Y^{\otimes k}}^{TT}$, for $k = 0, \ldots, K$ even. In the second part we aim at understanding how this complexity spreads throughout the recursive problem described in Table 4.1. All our computations are performed in the one dimensional case $D = [0,1]$.

The storage requirement is a limiting aspect of our implementation, and prevents us to grow significantly in K. We identify the problem in the lack of the implementation of sparse tensors in Matlab, and hence of sparse tensors in TT-format.

The TT-ranks strongly affect also the computational cost of the recursive algorithm. Indeed, they correspond to the number of linear systems to be solved. In Section 5.7.3 we investigate the computational cost of the TT-algorithm, and compare it to the computational cost of the direct computation of the truncated Taylor polynomial.

5.7.1 Storage requirements of the correlations of Y

As described in Section 5.5, to compute the k-points correlation of Y in TT-format $\mathcal{C}^{TT}_{Y^{\otimes k}}$, for $k \leq K$ even, we firstly have to perform the truncated KL-expansion of Y with a prescribed accuracy tol (see formula (5.21)). In all this section we take $tol = 10^{-16}$, so that the *complete KL-expansion* corresponding to the discretized field Y_h piecewise constant on the mesh $\mathcal{T}_h$ is considered.

Let N be the number of random variables which parametrize the field Y_h. Then $N \leq N_h$, where N_h is the dimension of the finite element subspace V_h used to discretize the deterministic problem. In [67] the authors show that, if $\mathcal{C}^{TT}_{Y^{\otimes k}}$ is computed exactly, then its TT-rank satisfies:

$$r_p = \begin{pmatrix} N+p-1 \\ p \end{pmatrix} \tag{5.27}$$

for $p = 1, \ldots, k/2$. For the symmetry in the construction of $\mathcal{C}^{TT}_{Y^{\otimes k}}$ (see Remark 5.5.11), $r_p = r_{k-p}$ for $p = 1, \ldots, k/2$. The storage of the exactly computed TT-tensor $\mathcal{C}^{TT}_{Y^{\otimes k}}$ becomes costly for k moderately large. Using Algorithm 1, we construct an approximation of this TT-tensor, which we still denote with $\mathcal{C}^{TT}_{Y^{\otimes k}}$. The output of the call $\texttt{compute\_moment\_Y(k,}\phi\texttt{,}\tilde{\lambda}\texttt{,}tol_1\texttt{,}tol_2\texttt{)}$ is a TT-tensor whose TT-ranks are bounded by (5.27).

As an example, let us consider a one dimensional domain $D = [0,1]$ discretized with $N_h = 200$ subintervals of length $h = 1/N_h$, and the Gaussian covariance function

$$Cov_Y(x_1, x_2) = \mathrm{e}^{-\frac{\|x_1 - x_2\|^2}{L^2}}, \quad (x_1, x_2) \in D \times D \tag{5.28}$$

with correlation length $L = 0.2$. The field is parametrized by $N = 27$ random variables. In Figure 5.4 we compare the upper bound in (5.27) (black line) with the TT-ranks of the approximated $\mathcal{C}^{TT}_{Y^{\otimes 4}} = \texttt{compute\_moment\_Y(k,}\phi\texttt{,}\tilde{\lambda}\texttt{,}tol\texttt{,}tol\texttt{)}$ computed for different tolerances tol. The smaller is the tolerance, the higher are the TT-ranks. In Figure 5.5 (left) the same type of plot is done for $k = 2, 4, 6, 8, 10$.

Figure 5.5 (right) is obtained with $N_h = 200$ and the exponential covariance function

$$Cov_Y(x_1, x_2) = \mathrm{e}^{-\frac{\|x_1 - x_2\|}{L}}, \quad (x_1, x_2) \in D \times D \tag{5.29}$$

with correlation length $L = 0.2$. $N = 200$ random variables are considered. In the exponential case, the TT-ranks grow faster than in the Gaussian setting.

For each k, the TT-rank of $\mathcal{C}^{TT}_{Y^{\otimes k}}$ is a vector of length $k-1$ with maximum $r^{(k)}_{max} = \max\{r_p |\ p = 1, \ldots, k-1\}$ in position $k/2$. Here we want to study the growth of $r^{(k)}_{max}$ as a function of the tolerance tol imposed in the Matlab function **compute_moment_Y** ($tol_1 = tol_2 = tol$) both for the Gaussian and the exponential covariance function.

Note that, given a TT-tensor $\mathcal{X}$, the sub-routine **tt_round** of the TT-Toolbox is an SVD-based algorithm which approximates $\mathcal{X}$ with a prescribed accuracy tol according to the following criterion

$$\sqrt{\sum_{j \geq J} \left(\nu^{(p)}_j\right)^2} \leq \sqrt{\sum_{j \geq 1} \left(\nu^{(p)}_j\right)^2} \frac{tol}{\sqrt{k-1}}, \tag{5.30}$$

where $\nu^{(p)}_j$ are the singular values of the p-th matricization of the TT-tensor, for $p = 1, \ldots, k-1$.

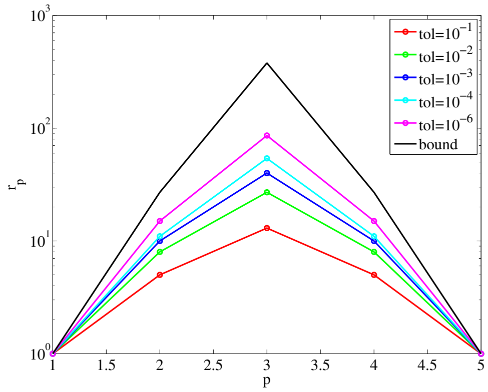


Figure 5.4: *Semilogarithmic plot of the upper bound for the TT-ranks in* (5.27) *(black line) compared with the TT-ranks of the approximated* $\mathcal{C}^{TT}_{Y^{\otimes 4}}$ *computed for different tolerances* tol.

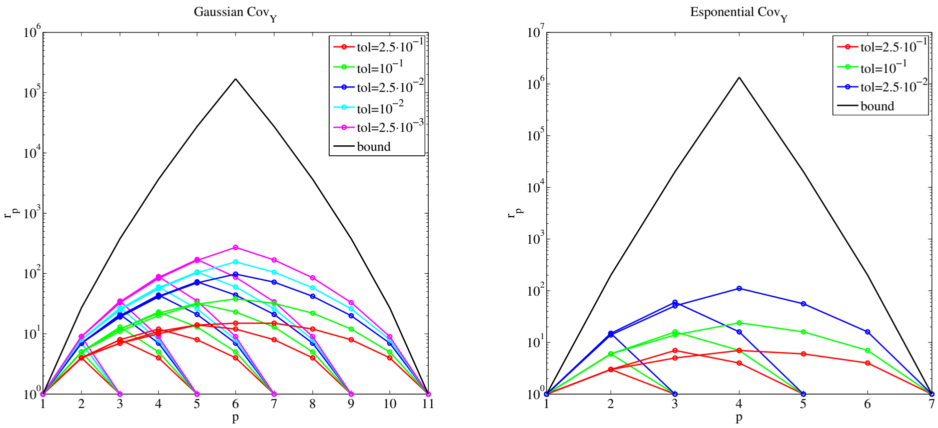


Figure 5.5: *Semilogarithmic plot of the upper bound for the TT-ranks in* (5.27) *(black line) compared with the TT-ranks of the approximated* $\mathcal{C}^{TT}_{Y^{\otimes k}}$ *computed for different tolerances. On the left,* $k = 2, 4, 6, 8, 10$ *and* Cov_Y *Gaussian, on the right,* $k = 2, 4, 6$ *and* Cov_Y *exponential.*

Let us start with the Gaussian covariance function (5.28) with correlation length $L = 0.2$. Let $N_h = 1000$ and $N = 27$. The eigenvalues of a Gaussian covariance function are such that $\tilde{\lambda}_j \sim \mathrm{e}^{-\alpha j^2}$, with α positive constant depending on L. See Figure 5.6. If $k = 2$, then $\nu_j^{(1)} = \tilde{\lambda}_j \sim \mathrm{e}^{-\alpha j^2}$, so that (5.30) becomes:

$$\sum_{j \geq J} \mathrm{e}^{-2\alpha j^2} \leq \frac{tol^2}{k-1} s,$$

where $s = \sum_{j \geq 1} \mathrm{e}^{-2\alpha j^2} < \infty$. Since $\mathrm{e}^{-2\alpha J^2} \leq \sum_{j \geq J} \mathrm{e}^{-2\alpha j^2}$, it follows

$$J \geq \sqrt{\frac{1}{\alpha} \log \frac{1}{tol} - \frac{1}{2\alpha} \log \frac{s}{k-1}},$$

so that

$$r^{(2)}_{max} \sim \sqrt{\frac{1}{\alpha} \log \frac{1}{tol} - \frac{1}{2\alpha} \log \frac{s}{k-1}}.$$

If the singular values of all matricizations of the TT-tensor $\mathcal{X}$ behave in the same way, $\nu_j^{(p)} \sim \mathrm{e}^{-\alpha j^2}$, we would infer

$$r_{max}^{(k)} \sim \sqrt{\frac{1}{\alpha}\log\frac{1}{tol} - \frac{1}{2\alpha}\log\frac{s}{k-1}}. \tag{5.31}$$

In Figure 5.7 we compare the behavior of $r_{max}^{(k)}$ as a function of tol ($k = 2, 4, 6, 8, 10$) with the ansatz (5.31). The behavior is correctly predicted for $k = 2$, but for $k \geq 4$ a dimensionality effect occurs, so that the bigger is k, the faster $r_{max}^{(k)}$ grows.

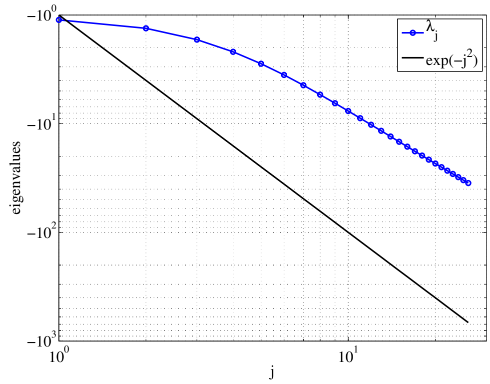


Figure 5.6: *Behavior of the eigenvalues of the Gaussian covariance function* (5.28)*, with* $L = 0.2$*,* $N_h = 1000$ *and* $N = 27$*.*

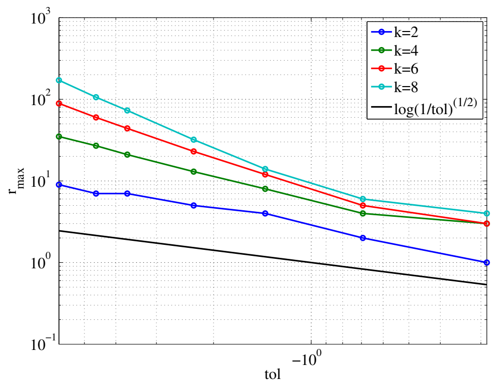


Figure 5.7: *Plot of* $r_{max}^{(k)}$ *as a function of tol, for a Gaussian covariance function.*

Now, let us take the exponential covariance function (5.29) with correlation length $L = 0.2$, $N_h = N = 1000$. The eigenvalues are such that $\tilde{\lambda}_j \sim \dfrac{1}{j^2}$ (see Figure 5.8), so that inequality (5.30) for $k = 2$ becomes:

$$\sum_{j \geq J} \frac{1}{j^4} \leq \frac{tol^2}{k-1} s,$$

where $s = \sum_{j\geq 1} \frac{1}{j^4} < \infty$. Using that $\frac{1}{3(J+1)^3} \leq \sum_{j\geq J} \frac{1}{j^4}$, we obtain

$$J \geq \frac{1}{tol^{2/3}} \left(\frac{k-1}{3s}\right)^{1/3} - 1. \tag{5.32}$$

As in the Gaussian case, $r_{max}^{(2)}$ behaves as a function of tol as the right-hand side in (5.32), but for $k \geq 4$ a dimensionality effect occurs. See Figure 5.9.

Remark 5.7.13. *In Chapter 4 we have studied the Hölder mixed regularity of the k-points correlation function $\mathbb{E}\left[Y^{\otimes k}\right]$. If the covariance function $Cov_Y \in \mathcal{C}^{0,t}(\overline{D \times D})$, then $\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^{0,t/2,mix}(\bar{D}^{\times k})$ (Proposition 4.3.2). The exponential covariance function* (5.29) *is Hölder continuous with exponent $t = 1$, i.e. it is Lipschitz, so that $\mathbb{E}\left[Y^{\otimes k}\right] \in \mathcal{C}^{0,1/2,mix}(\bar{D}^{\times k})$. This is in agreement with the bound $r_{max}^{(k)} \leq tol^{-2}$ observed in Figure 5.9.*

Remark 5.7.14. *Given a d-variate, 2π-periodic function f with mixed Sobolev regularity s, in [89] the authors show that the storage complexity for achieving accuracy tol by representing f in HT-format satisfies worse rates than those obtained with sparse grids. Actually, sparse grids are expressly constructed to approximate multivariate functions with mixed Sobolev regularity.*

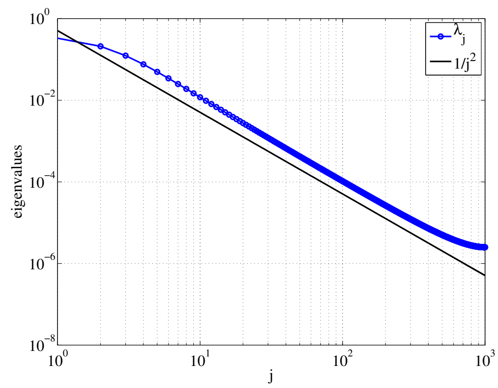


Figure 5.8: *Behavior of the eigenvalues of the exponential covariance function* (5.29), *with* $L = 0.2$, $N_h = 1000$ *and* $N = 1000$.

5.7.2 Storage requirements of the recursion

We perform some numerical tests to study the storage complexity of the correlations involved in K-th order problem, that is the elements of Table 4.1. In all the plots presented here, we have reordered the indices in the TT-correlation $\mathcal{C}^{TT}_{u^{k-l}\otimes Y^{\otimes l}}$ so that the index relative to u^{k-l} is the last one.

Let us take $N_h = 100$ and the Gaussian covariance function of Y of the form (5.28) with $L = 0.2$. As in the previous section, a complete KL-expansion is performed, which can be truncated at $N = 26$ with machine precision.

In Figure 5.10 we plot the TT-ranks of the correlations needed to solve the K-th order problem, for $K = 2, 4, 6$. For example, in the picture with $K = 2$ we plot the

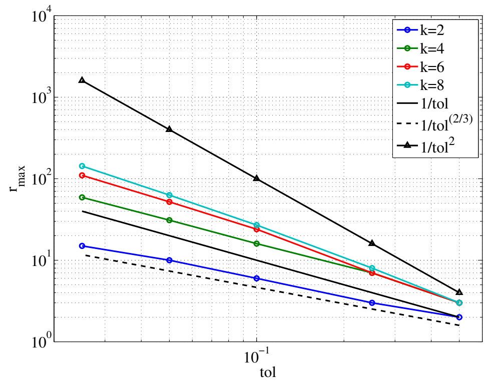


Figure 5.9: *Plot of $r_{max}^{(k)}$ as a function of tol, for an exponential covariance function, with $N_h = N =$ 1000.*

TT-ranks of $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 2}}$ (blue line), $\mathcal{C}^{TT}_{u^1 \otimes Y}$ (red line) and $\mathcal{C}^{TT}_{u^2}$ (green line). The black line is the upper bound in (5.27). Note that the blue and the red lines coincide, since to obtain $\mathcal{C}^{TT}_{u^1 \otimes Y}$ we simply saturate two indices of $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 2}}$ with the TT-tensor $\mathcal{B}^{TT,1}$ and then multiply for the inverse of the stiffness matrix A. On the other hand, when the right-hand side of problem (5.14) is given by the summation of more than one term, the **tt_round** (with $tol = 10^{-14}$) is applied to this summation in order to avoid the growth of the TT-ranks. From Figure 5.10 we deduce that the storage requirement is decreasing along each diagonal of Table 4.1, and the largest memory is needed to store the input term $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes K}}$. To obtain Figure 5.10 we have used tolerances $tol_1 = tol_2 = tol = 10^{-10}$ in the function **compute_moment_Y**. Figure 5.11 represents the TT-ranks of the correlations needed to solve the 6-th order problem, where different tolerances $tol_1 = tol_2 = tol$ are used in **compute_moment_Y**. As expected, the smaller tol is, the higher the TT-ranks are.

5.7.3 Comparison with the computation of the truncated Taylor series

Let N be the number of random variables we take into account in the KL-expansion, so that the random field Y is parametrized by the Gaussian random vector $\mathbf{Y} = (Y_1 \ldots, Y_N)$. In the approach we propose, the first moment problem is derived and solved in TT-format. The most natural alternative is to directly compute the Taylor polynomial, and use it to approximate $\mathbb{E}[u]$. Here we compare these two methods.

Computation of $T^K u$

The K-th order Taylor polynomial is

$$T^K u(\mathbf{Y}, x) = \sum_{n=0}^{K} \sum_{|\mathbf{k}|=n} \frac{\partial^{\mathbf{k}}_{\mathbf{Y}} u(\mathbf{0}, x)}{\mathbf{k}!} \mathbf{Y}^{\mathbf{k}}.$$

The number of partial derivatives of order n computable for a function of N variables is $\begin{pmatrix} N+n-1 \\ n \end{pmatrix}$. As a consequence, to compute $T^K u(\mathbf{Y}, x)$, we have to solve M_1

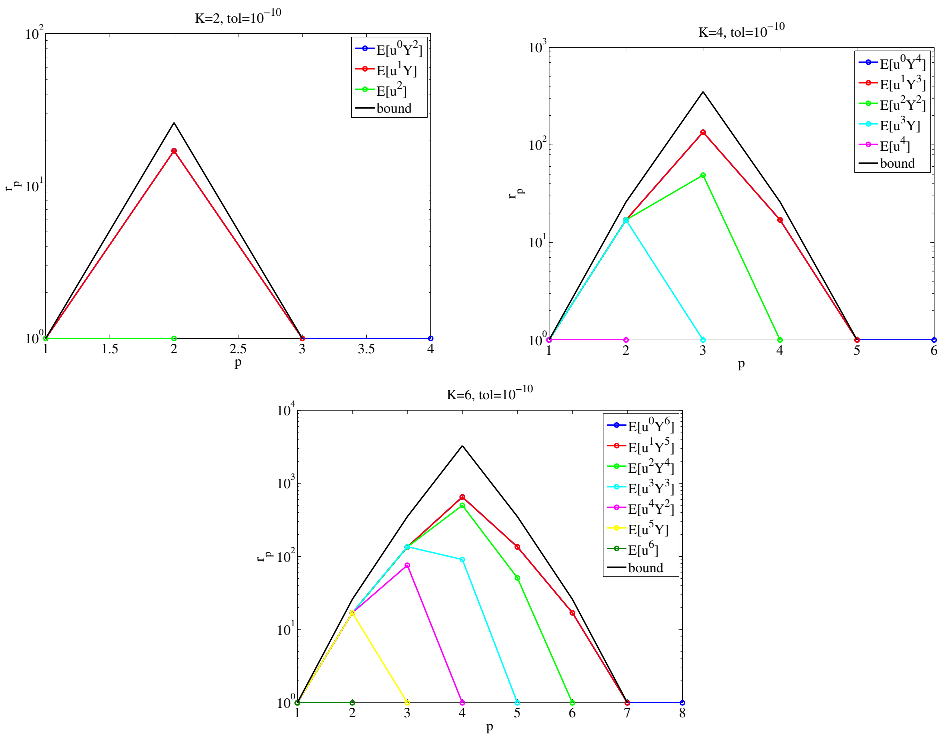


Figure 5.10: *Semilogarithmic plot of the TT-ranks of the correlations needed to solve the K-th order problem, for $K = 2, 4, 6$.*

linear problems, where

$$M_1 = \sum_{n=0}^{K} \begin{pmatrix} N+n-1 \\ n \end{pmatrix} = \begin{pmatrix} N+K \\ K \end{pmatrix} \tag{5.33}$$

where the last equality follows by induction.

Moment equations

The K-th order approximation of $\mathbb{E}\left[u\right]$ is

$$\mathbb{E}\left[T^K u(\mathbf{Y}, x)\right] = \sum_{n=0:2:K} \frac{\mathbb{E}\left[u^n\right]}{n!} = \sum_{n=0:2:K} \frac{1}{n!} \mathbb{E}\left[\sum_{|\mathbf{k}|=n} \partial^{\mathbf{k}}_{\mathbf{Y}} u(\mathbf{0}, x) \mathbf{Y}^{\mathbf{k}}\right].$$

Suppose that each correlation $\mathbb{E}\left[Y^{\otimes k}\right]$ is constructed without exploiting any symmetry. Let us take for example $n = 4$. Starting from $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 4}}$, to derive $\mathcal{C}^{TT}_{u^4}$ we have to perform the following saturations:

$$\mathcal{B}^{TT,1} \times_{1,2} \mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 4}}, \quad \mathcal{B}^{TT,2} \times_{1,3} \mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 4}}, \quad \mathcal{B}^{TT,3} \times_{1,4} \mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 4}}, \quad \mathcal{B}^{TT,4} \times_{1,5} \mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 4}},$$

and then multiply each of them for the inverse of the stiffness matrix A (i.e. solving a linear system) in the suitable direction. Let $(1, 1, r_1, r_2, r_3, 1)$ be the TT-rank of $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes 4}}$. Recalling the equality (5.27), we deduce that the maximum number of linear

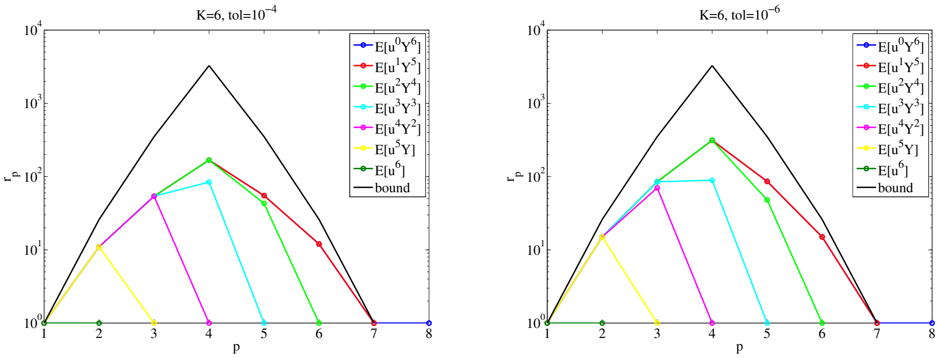


Figure 5.11: *Semilogarithmic plot of the TT-ranks of the correlations needed to solve the* 6*-th order problem, where different tolerances* $tol_1 = tol_2 = tol$ *are used in* ***compute_moment_Y***

systems we have to solve, if no compression by **tt_round** is performed, is

$$1 + r_1 + r_2 + r_3 = 1 + \begin{pmatrix} N \\ 1 \end{pmatrix} + \begin{pmatrix} N+1 \\ 2 \end{pmatrix} + \begin{pmatrix} N+2 \\ 3 \end{pmatrix}$$
$$= \sum_{p=0}^{3} \begin{pmatrix} N+p-1 \\ p \end{pmatrix} = \begin{pmatrix} N+3 \\ 3 \end{pmatrix}.$$

In general, to compute the K-th order approximation $\mathbb{E}\left[u^K\right]$, we have to solve M_2 linear systems, where

$$\begin{aligned} M_2 &= \sum_{n=2:2:K} \sum_{p=0}^{n-1} \begin{pmatrix} N+p-1 \\ p \end{pmatrix} + 1 = \sum_{n=2:2:K} \begin{pmatrix} N+n-1 \\ n-1 \end{pmatrix} + 1 \\ &= \sum_{n=2:2:K} \begin{pmatrix} (N+1)+(n-1)-1 \\ n-1 \end{pmatrix} + 1 \\ &= \sum_{m=1:2:K-1} \begin{pmatrix} (N+1)+m-1 \\ m \end{pmatrix} + 1 \\ &\leq \sum_{m=0}^{K-1} \begin{pmatrix} (N+1)+m-1 \\ m \end{pmatrix} = \begin{pmatrix} N+K \\ K-1 \end{pmatrix} \end{aligned} \tag{5.34}$$

We conclude that the cost of the moment equations is smaller than the cost of the direct computation of $T^K u$.

We can improve both M_1 and M_2. In the direct computation of the Taylor polynomial, we can exploit that $\mathbb{E}\left[\mathbf{Y}^{\mathbf{k}}\right] \neq 0$ iff k_i is even for each $i = 1, \ldots, N$. It is not difficult to see that, to compute $T^K u(\mathbf{Y}, x)$ we need all the partial derivatives $\partial_{\mathbf{Y}}^{\mathbf{k}} u(\mathbf{0}, x)$ for $|\mathbf{k}| \leq K/2$, and only some of the partial derivatives $\partial_{\mathbf{Y}}^{\mathbf{k}} u(\mathbf{0}, x)$ for $|\mathbf{k}| > K/2$. We consider the reduced complexity

$$M_1' = \sum_{n=0}^{K/2} \begin{pmatrix} N+n-1 \\ n \end{pmatrix} = \begin{pmatrix} N+K/2 \\ K/2 \end{pmatrix} \tag{5.35}$$

On the other hand, the complexity M_2 can be improved using the symmetry of the TT-ranks, that is $r_p = r_{n-p}$ for $p = 1, \dots, n/2-1$. (See Remark 5.5.11 and Figure 5.4).

$$\begin{aligned}
M_{2,sym} &= \sum_{n=2:2:K} \left(2 \sum_{p=0}^{n/2-1} \binom{N+p-1}{p} + \binom{N+n/2-1}{n/2} - 1 \right) + 1 \\
&= \sum_{n=2:2:K} \left(2 \binom{N+n/2-1}{n/2-1} + \binom{N+n/2-1}{n/2} - 1 \right) + 1 \\
&= \sum_{n=2:2:K} \left(\binom{N+n/2-1}{n/2-1} \frac{N+n}{n/2} - 1 \right) + 1 \\
&\leq \sum_{m=0}^{K/2} \binom{N+m}{m} (N+2) = (N+2) \binom{N+K/2}{K/2-1} \\
&\ll \binom{N+K}{K-1}.
\end{aligned}$$

Moreover, the TT-format offers the possibility to dramatically reduce the computational cost thanks to the sub-routine **tt_round**. In Figure 5.12 we compare the two reduced complexities M_1' in (5.35) and

$$M_2' = \sum_{n=2:2:K} \sum_{p=0}^{n-1} r_p + 1 \tag{5.36}$$

where the TT-ranks of $\mathcal{C}^{TT}_{Y^{\otimes k}}$ are symmetric and computed using different tolerances in the function **compute_moment_Y**. A Gaussian covariance function with $L = 0.2$, and a number of intervals $N_h = 200$ are considered. From this comparison, we deduce that the moment equation approach is convenient from the point of view of the computational cost. However, note that this is not a completely fair comparison, since an anisotropic Taylor polynomial can be computed, evaluating only the most important derivatives, up to a prescribed tolerance. How an optimally truncated Taylor expansion compares with our moment equations approach with compressed TT-format is still an open question.

5.8 Numerical tests

In this section we perform some numerical tests and solve the stochastic Darcy problem with deterministic loading term $f(x) = x$ in the one dimensional domain $D = [0, 1]$, both for a Gaussian and exponential covariance function. Homogeneous Dirichlet boundary conditions are imposed on $\Gamma_D = \{0, 1\}$.

Truncated KL - Gaussian covariance function

Let $Y(\omega, x)$ be a stationary centered Gaussian random field with Gaussian covariance function

$$Cov_Y(x_1, x_2) = \sigma^2 \, \mathrm{e}^{-\frac{\|x_1 - x_2\|^2}{L^2}}, \quad (x_1, x_2) \in D \times D$$

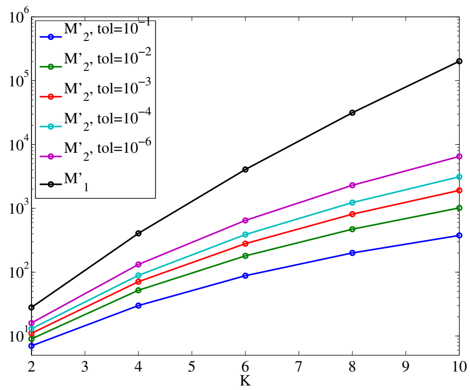


Figure 5.12: *Semilogarithmic plot of* M_1' (5.35) *and* M_2' (5.36)*, where* $\{r_p\}_p$ *is the TT-rank of the* $\mathcal{C}^{TT}_{u^0 \otimes Y^{\otimes n}}$ *computed by* ***compute_moment_Y*** *with different tolerances imposed. A Gaussian covariance function with* $L = 0.2$*, and a number of intervals* $N_h = 200$ *are considered.*

where $0 < \sigma < 1$ and $L = 0.2$ are the standard deviation and the correlation length, respectively, of $Y(\omega, x)$. Let us take a uniform discretization of the spatial domain $D = [0, 1]$ in $N_h = 100$ intervals ($h = 1/N_h$). As first step, we perform the truncated KL-expansion of $Y(\omega, x)$ with a tolerance $tol = 10^{-4}$, so that $N = 11$ random variables are considered and the 99% of variance of the field is captured.

Using the function **compute_moment_Y** and Algorithm 4 with tolerance in the TT-computations given by 10^{-16}, we compute the 6-th order approximation of $\mathbb{E}[u]$, that is, we solve the recursive problem (5.14) for $k = 0, \dots, 6$. As reference solution we consider the mean of u computed via the collocation method on a Smolyak sparse grid with 12453 collocation points, on the same spatial discretization $(N_h = 100)$. See Figure 5.13.

Note that the error comes from different contributions: the truncation of the KL-expansion, the TT approximation, the truncation of the Taylor series and the FEM approximation.

Here, we start from the same truncated KL-expansion both to compute the collocation solution and the TT solution, use the same FE grid, and the TT computations are done at machine precision, that is with tolerance 10^{-16}. Hence, we observe only the error due to the truncation of the Taylor series.

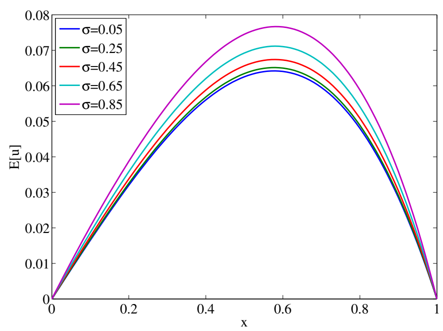


Figure 5.13: $\mathbb{E}[u]$ *computed via the collocation method, for different values of* σ.

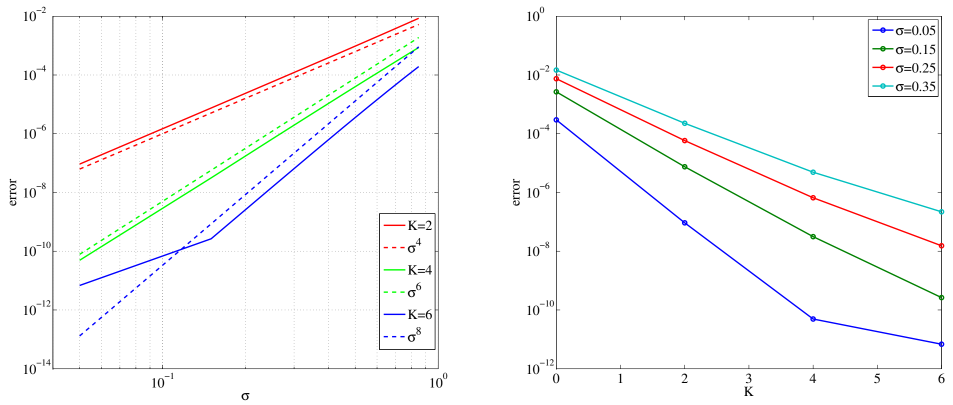
Table 5.1: *Numerically observed order of* $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ *as a function of* σ.

	$K=0$	$K=1$	$K=2$	$K=3$	$K=4$	$K=5$	$K=6$
$\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$	2	2	4	4	6	6	8

In Chapter 3 we have shown that $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)} = O(\sigma^{K+1})$. Moreover, since $Y(\omega, x)$ is centered, then $\mathcal{C}_{Y^{\otimes(2k+1)}} = 0$ (see proposition 5.5.10), so that

$$\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)} = \begin{cases} O(\sigma^{K+2}) & \text{if } K \text{ is even,} \\ O(\sigma^{K+1}) & \text{if } K \text{ is odd.} \end{cases} \tag{5.37}$$

In Table 5.1 we summarize the numerically observed orders of the computed error $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$, and in Figure 5.14(a) we plot in logarithmic scale the error as a function of σ. The behavior predicted in (5.37) is confirmed. Figure 5.14(b) represents the computed error $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ as a function of K (at least up to $K = 6$) for different values of σ. It turns out that for $\sigma < 1$ it is always useful to take into account higher order corrections. As we can see in Figure 5.14 (blue lines), the maximum precision that the method can reach is 10^{-12}. We believe that this can be due to the precision in the computation of the reference solution.



(a) (b)

Figure 5.14: *5.14(a) Logarithmic plot of the computed error* $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ *as a function of* σ. *5.14(b) Semilogarithmic plot of the computed error* $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ *as a function of* K *for different* σ.

Truncated KL - Gaussian covariance function - Conditioned expected value

Suppose that $Y(\omega, x)$ is a conditioned field to N_{oss} available point wise observations. As observed in Chapter 3, Section 3.3.3, the covariance function Cov_Y is non-stationary, but still Hölder continuous, so that we can conclude the well-posedness of the stochastic Darcy problem. All the results in Chapter 3 apply to $Y'(\omega, x) := Y(\omega, x) - \mathbb{E}\left[Y\right](x)$, where $\mathbb{E}\left[Y\right](x)$ is the conditioned expected value of Y. Following the same steps done in Chapter 4 we can deduce the recursive first moment problem, which now involves the $(l+1)$-points correlations $\mathbb{E}\left[u^{k-l} \otimes (Y')^{\otimes l}\right]$, where $u^{k-l} := D^{k-l}u(\mathbb{E}[Y])[Y']^{k-l}$ is the Gateaux derivative of u in $\mathbb{E}[Y]$ evaluated along the vector $\underbrace{(Y', \dots, Y')}_{k-l \text{ times}}$.

Considering the case of $Y(\omega, x)$ conditioned to available observations is very relevant in applications. Indeed, suppose the domain D contains an heterogeneous porous medium. Although it is not possible to know its permeability everywhere, from the practical point of view it is possible to measure it in a certain number of fixed points. Hence, the natural model considered in the geophysical literature describes the permeability as a conditioned lognormal random field. See e.g. [51, 52, 86]. The more observations are available, the smaller the total variance of the field will be. This, actually, favors the use of perturbation methods.

As in the previous numerical test, let $N_h = 100$ be the number of subintervals of D, and $tol = 10^{-4}$ the tolerance imposed in the truncation of the KL-expansion. The N_{oss} observations available are evenly distributed in $D = [0, 1]$. To capture the 99% of variability of the field, $N = 9$ and $N = 8$ random variables are needed in the cases $N_{oss} = 3$ and $N_{oss} = 5$ respectively. Note that, the highest is the number of observations, the smaller is the number of random variables needed to reach the same level of accuracy in the KL-expasion. The reference solution is computed via the collocation method (Smolyak grid with 5965 and 3905 collocation points for $N_{oss} = 3$ and $N_{oss} = 5$ respectively). See Figure 5.15.

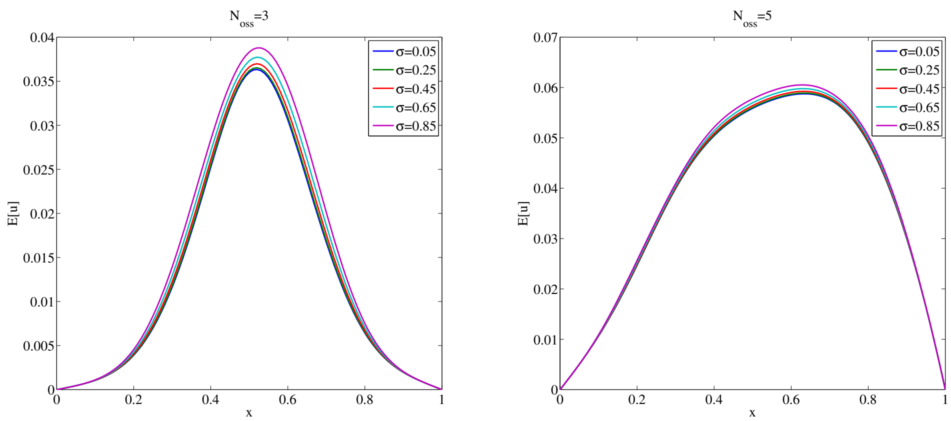


Figure 5.15: $\mathbb{E}[u]$ *computed via the collocation method, with* $N_{oss} = 3$ *(left) and* $N_{oss} = 5$ *(right).*

Figure 5.16 represents the behavior of the error $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ as a function of σ, with $N_{oss} = 3$ (left) and $N_{oss} = 5$ (right). The same rate as for $N_{oss} = 0$ (see (5.37)) is observed. In Figure 5.17 we plot the error as a function of K. The error is about 1 order of magnitude smaller for $N_{oss} = 3$ (compared to $N_{oss} = 0$) and 2 orders of magnitude smaller for $N_{oss} = 5$.

Complete KL - Exponential covariance function

Let $Y(\omega, x)$ be a stationary centered Gaussian random field with exponential covariance function

$$Cov_Y(x_1, x_2) = \sigma^2 \, \mathrm{e}^{-\frac{\|x_1 - x_2\|}{L}}, \quad (x_1, x_2) \in D \times D$$

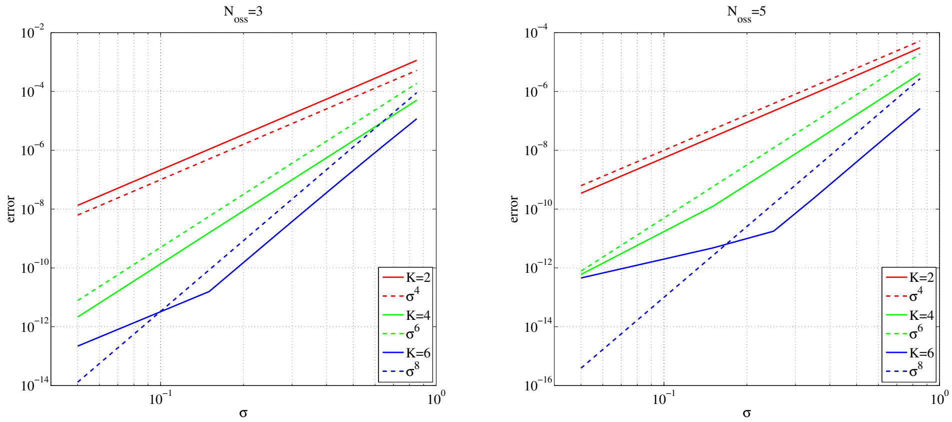


Figure 5.16: *Logarithmic plot of the computed error* $\left\|\mathbb{E}\left[u-T^K u\right]\right\|_{L^2(D)}$ *as a function of* σ*, with* $N_{oss}=3$ *(left) and* $N_{oss}=5$ *(right).*

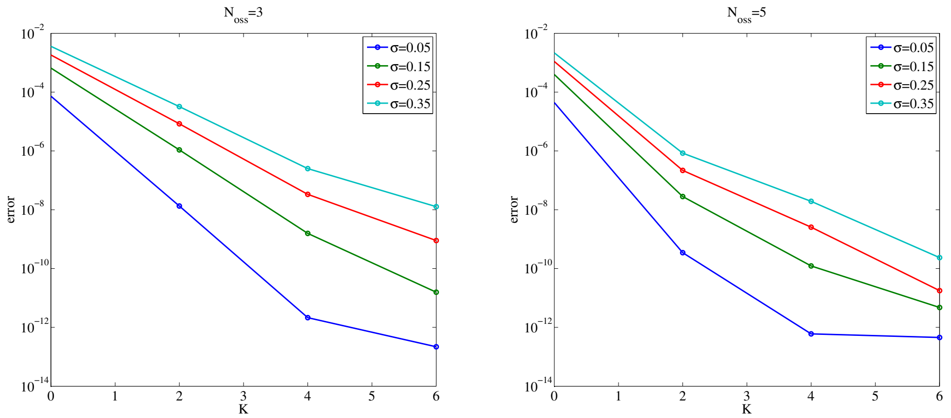


Figure 5.17: *Semilogarithmic plot of the computed error* $\left\|\mathbb{E}\left[u-T^K u\right]\right\|_{L^2(D)}$ *as a function of* K*, with* $N_{oss}=3$ *(left) and* $N_{oss}=5$ *(right).*

with $0<\sigma<1$ and $L=0.2$. Let $N_h=100$. We compute the KL-expansion with tolerance $tol=10^{-4}$, so that $N=100$ random variables are considered and the 100% of variance of the field is captured. Then, we compute the second order correction of $\mathbb{E}[u]$ with our TT code. Since $N=100$, a collocation method becomes unfeasible. By a qualitative comparison with $\mathbb{E}[u]$ computed via the Monte Carlo method with $M=10000$ samples, we show that our algorithm is effective and provides a valid solution also in this framework. See Figure 5.18, where the second order correction is compared with the Monte Carlo method for $\sigma=0.05$ (left) and $\sigma=0.65$ (right). In plotting the Monte Carlo solution, we have also added error bars representing $\pm\sigma_{MC}$, where σ_{MC} is the estimated standard deviation of the Monte Carlo estimator. We observe that the TT-solution is always contained in the confidence interval of the Monte Carlo solution.

Complete KL - Gaussian covariance function

Let us consider a stationary Gaussian random field $Y(\omega,x)$ with Gaussian covariance function of correlation length $L=0.2$. Let $N_h=100$. Instead of truncating the

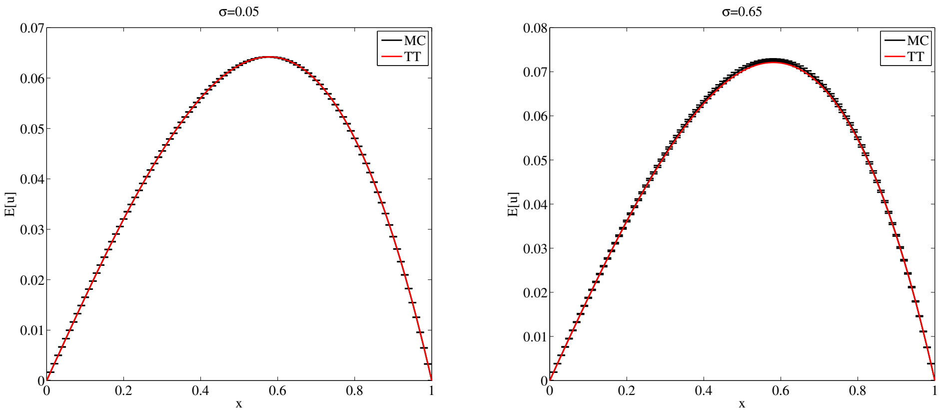


Figure 5.18: *Comparison between the second order correction computed via our TT-code, and* $\mathbb{E}[u]$ *computed via the Monte Carlo method (*$M = 10000$ *samples) for* $\sigma = 0.05$ *(left) and* $\sigma = 0.65$ *(right).*

KL-expansion as done before, we consider the complete KL, that is we compute the expansion with the accuracy 10^{-16}. $N = 26$ random variables have to be considered to capture the 100% of variance of $Y(\omega, x)$ (up to machine precision). We run our TT-code imposing different tolerances in the function **compute_moment_Y**. In this way, we can observe how the error $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ depends on the TT-approximation.

Figure 5.19 represents the computed error $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ as a function of the standard deviation σ, for different tolerances, with $K = 2, 4, 6$. The tolerance 10^{-1} (blue line) is such that the predicted behavior (5.37) is not observed, even for $K = 2$. Whereas, the tolerance 10^{-8} (magenta line) guarantees the predicted behavior both for $K = 2$, $K = 4$ and $K = 6$.

In Figure 5.20 we plot the error as a function of K, for different tolerances, with $\sigma = 0.05$ (left) and $\sigma = 0.55$ (right). The total error is the sum of two contributions: the truncation of the Taylor expansion and the tolerance used in TT-computations, which should ideally be balanced. In Figure 5.20 we see that, the larger σ is, the smaller the tolerance in the TT-computations has to be to equilibrate the truncation error.

We finally attempt to investigate how the total error depends on the complexity of the algorithm. In particular, we numerically study the dependance of the error on the complexity under the assumption that the complexity of the recursive algorithm is dominated by the number of linear systems we have to solve in the recursion, that is M_2' in (5.36). Figure 5.21 represents the logarithmic plot of the error $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ as a function of M_2' for different tolerances in the function **compute_moment_Y**, with $\sigma = 0.05, 0.25, 0.85$. We compare it with the quantity $\dfrac{\sigma_{MC}}{\sqrt{M_2'}}$ (black line), which gives an idea of the behavior of a the Monte Carlo estimate. Note that, for small σ (e.g. $\sigma = 0.05$), the smaller the tolerance imposed is, the higher the accuracy reached. This is not the case if we let σ grow. Indeed, the TT-error is no more the most influencing component of the error, which is dominated, instead, by the truncation error.

For a fixed truncation level, there is therefore an optimal choice of the tolerance tol_{opt}. Figure 5.21 shows that, if the optimal tolerance is chosen, the performance of

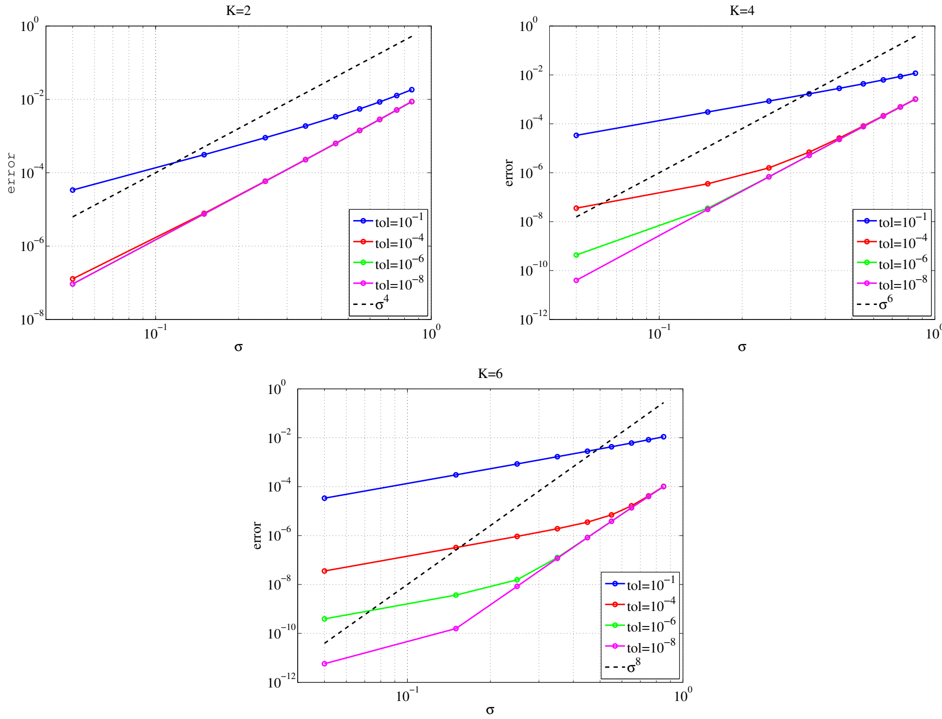


Figure 5.19: *Logarithmic plot of the computed error* $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ *as a function of the standard deviation* σ*, for different tolerances.*

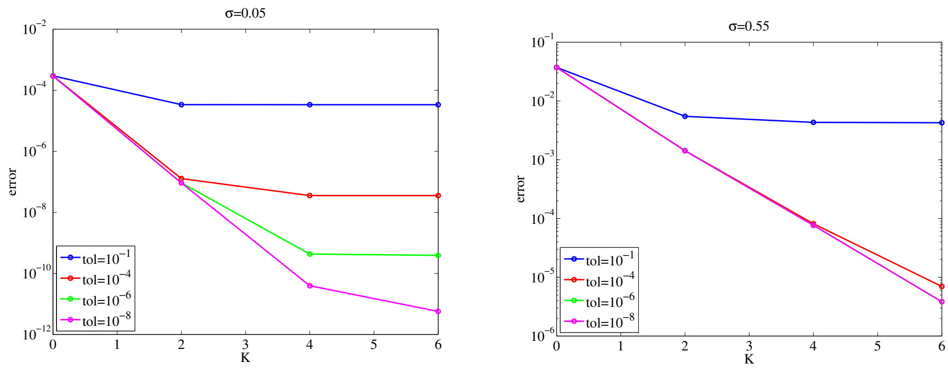


Figure 5.20: *Semilogarithmic plot of the computed error* $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ *as a function of* K*, for different tolerances*

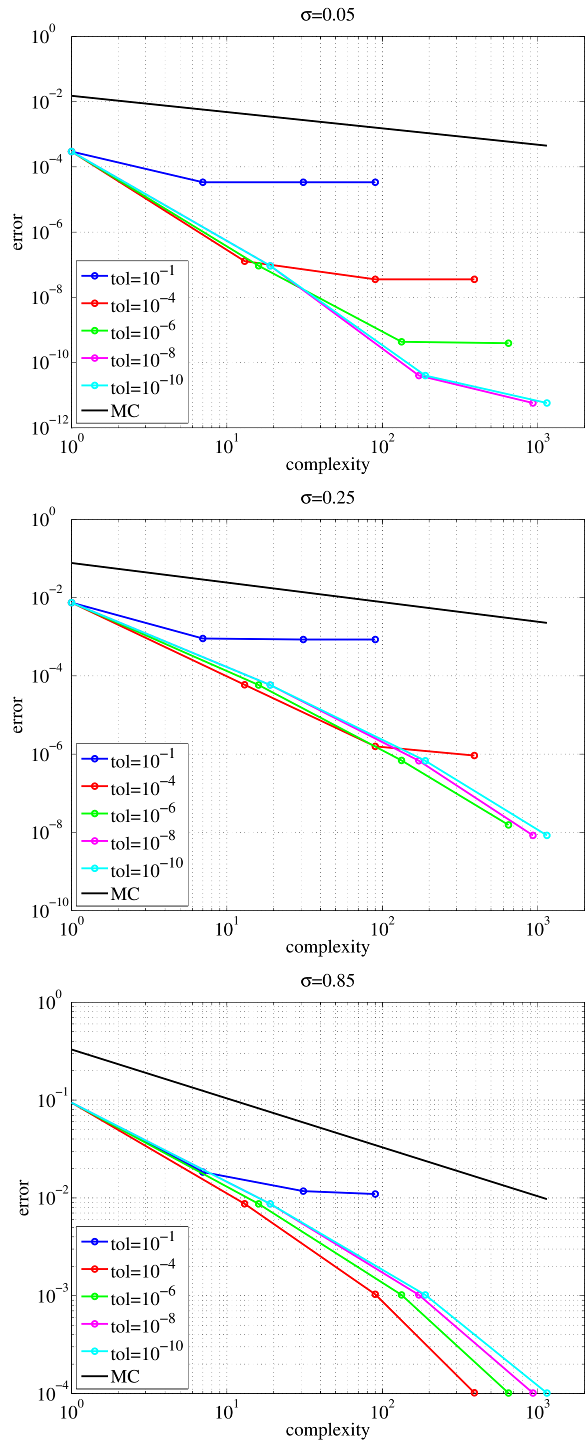


Figure 5.21: *Logarithmic plot of the computed error* $\left\|\mathbb{E}\left[u - T^K u\right]\right\|_{L^2(D)}$ *as a function of* M_2' *for different tolerances in the TT-computations. The black line gives an idea of the behavior a the Monte Carlo estimate.*

the moment equations is far superior to a standard Monte Carlo method. The question of how to determine a priori the optimal tolerance as a function of K and σ is still open and under investigation.

5.9 Conclusions

We have derived the full tensor product finite element formulation of the recursive problem solved by the $(l+1)$-points correlation functions $\mathcal{C}_{u^{k-l}\otimes Y^{\otimes l}}$, for $k = 0, \dots, K$ and $l = 0, \dots, k$. Since the number of entries of a tensor is exponential in its order, we have introduced a data-sparse format (the TT-format) to store the tensors and make computations. We have developed an algorithm in TT-format, able to compute the K-th order approximation $\mathbb{E}\left[T^K u\right]$.

We have studied the storage requirements of the algorithm we propose. The parameter we have taken into account is the TT-rank. We have performed some numerical tests to understand how the TT-rank of the input correlations $\mathcal{C}^{TT}_{Y^{\otimes k}}$ $(k = 0, \dots, K)$ depends on k. Moreover, we have shown the evolution of the TT-rank along each diagonal of Table 4.1.

We have run our code both in the case of an a-priori truncated KL-expansion and a "complete" (untruncated) KL-expansion. If a truncated KL-expansion is performed, we have compared the solution provided by our recursive TT-code with a reference solution obtained by stochastic collocation. We have numerically observed the behavior of the error $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ as a function of σ predicted in Chapter 3, that is $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)} = O(\sigma^{K+1})$. We have also studied the error $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ as function of K.

More relevant results are obtained when the complete KL-expansion is considered. We have numerically investigated the dependence of the error $\left\|\mathbb{E}\left[u\right] - \mathbb{E}\left[T^K u\right]\right\|_{L^2(D)}$ on the tolerance imposed in the function **compute_moment_Y**. From the numerical tests, we deduce that the convenient tolerance to consider depends both on K and σ.

The method we propose here is able to reach the same accuracy as a collocation method. Moreover, in the case where the collocation method in unfeasible, it still provides a valid solution, and turns out to be much more performing than the Monte Carlo method. Thanks to the TT-approximations, we can treat also the case where $Y(\omega, x)$ is not approximated, but its complete KL-expansion is considered (up to machine precision).

The limiting aspect of our method are the storage requirements, which prevent us to grows significantly in the order of the Taylor polynomial K. We believe that a great improvement will follow from the implementation of sparse tensors toolboxes, which are still missing in Matlab.

Conclusions and future work

In this thesis we have considered linear PDEs with randomness arising either in the forcing term or in the coefficient. We have modeled uncertain input terms as random variables or random fields with known probability laws. Given complete knowledge on the input terms, we have studied the moment equations, i.e. the deterministic equations solved by the statistical moments of the stochastic solution of the SPDE, and solved them to make inference on the stochastic solution of the PDE.

In the case of randomness arising in the loading term, the stochastic counterpart of the Hodge Laplace problem in mixed form has been analyzed. The moment equations are derived exactly, and their well-posedness is proved via a tensorial inf-sup condition. Both full and sparse tensor product finite element discretizations are analyzed. We have proved the stability of both the discretizations, and have shown that a sparse approximation provides almost the same rate of accuracy as a full approximation, with a drastic reduction in the number of degrees of freedom.

We have studied the Darcy boundary value problem modeling the fluid flow in a heterogeneous porous medium, where the permeability is described as a lognormal random field: $a(\omega, x) = \mathrm{e}^{Y(\omega,x)}$, $Y(\omega, x)$ being a Gaussian random field. Under the assumption of small variability of Y, we have performed a perturbation analysis. We have explored the approximation properties of the K-th Taylor polynomial, predicted the divergence of the Taylor series, and provided an estimate of the optimal order of the Taylor polynomial.

The expected value of the solution has been approximated by the expected value of its Taylor polynomial, which in turn solves a recursive deterministic problem. We have stated the well-posedness of this problem, and obtained Hölder-type regularity results.

We have proposed an algorithm in TT-format able to solve the first moment problem at a prescribed order K (the degree of the Taylor polynomial). The numerical tests performed highlight that the same level of accuracy as a stochastic collocation method can be obtained with our algorithm in cases with only few random variables. However, our approach does not need an a priori truncation of the random field and still provides valid solutions in cases where the stochastic collocation method is unfeasible.

We have studied how the complexity of the algorithm depends on the precision tol achieved in the TT-computations. We have numerically predicted the existence of an optimal tol depending both on the order of approximation K and the standard deviation of the field Y. If the optimal tolerance is chosen, the performance of the moment equations is far superior to a standard Monte Carlo method.

Summarizing, we have applied the perturbation technique to the Darcy problem with lognormal permeability tensor, derived the moment equations and solved them with a recursive algorithm in TT-format. We conclude the superiority of this technique with respect to both a gPC expansion and Monte Carlo methods. Indeed, our TT-code is able to handle high dimensional problems in the probability space and even infinite dimensional random fields. On the other hand, if the optimal tolerance in TT-computations is chosen, our TT-code is much more performing than a standard Monte Carlo method.

However, we underline that there are still open questions to be investigated. The numerical tests we have performed highlight the importance of the correct choice of the tolerance in the TT-computations. Yet, we do not provide any a priori strategy to determine the optimal tolerance. This topic is still open and under current investigation.

On the other hand, our algorithm does not allow a significative growth in the order K of the correction (degree of the Taylor polynomial). This is due to the storage requirements of the TT-tensors involved in the recursion, which strongly depend on the TT-ranks. The dependence of the TT-ranks on the dimension of the tensor is an interesting topic we are working on. We believe that a great improvement will follow from the implementation of sparse tensors toolboxes, which are still missing in Matlab or other programming languages.

Bibliography

[1] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007.

[2] H. Amann and J. Escher. *Analysis. II*. Birkhäuser Verlag, Basel, 2008. Translated from the 1999 German original by Silvio Levy and Matthew Cargo.

[3] A. Ambrosetti and G. Prodi. *A primer of nonlinear analysis*, volume 34 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1993 original.

[4] T. W. Anderson. *An introduction to multivariate statistical analysis*. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2003.

[5] D. N. Arnold. Differential complexes and numerical stability. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 137–157, Beijing, 2002. Higher Ed. Press.

[6] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006.

[7] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc. (N.S.)*, 47(2):281–354, 2010.

[8] I. Babuška and A. K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The mathematical foundations of the finite element method with applications to partial differential equations (Proc. Sympos., Univ. Maryland, Baltimore, Md., 1972)*, pages 1–359. Academic Press, New York, 1972.

[9] Ivo Babuška and Panagiotis Chatzipantelidis. On solving elliptic stochastic partial differential equations. *Comput. Methods Appl. Mech. Engrg.*, 191(37-38):4093–4122, 2002.

[10] I. Babuška, F. Nobile, and R. Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 45(3):1005–1034, 2007.

[11] I. Babuška, R. Tempone, and G. E. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM Journal on Numerical Analysis*, 42(2):800 – 825, 2005.

[12] B. W. Bader, T. G. Kolda, et al. Matlab tensor toolbox version 2.5. Available online, January 2012.

[13] A. Barth, C. Schwab, and N. Zollinger. Multi-level Monte Carlo finite element method for elliptic PDEs with stochastic coefficients. *Numerische Mathematik*, 119:123–161, 2011.

[14] J. Bear and A. H.-D. Cheng. *Modeling Groundwater Flow and Contaminant Transport*, volume 23. Springer Netherlands, 2010.

[15] J. Beck, R. Tempone, F. Nobile, and L. Tamellini. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Math. Models Methods Appl. Sci.*, 22(9):1250023, 33, 2012.

[16] V. I. Bogachev. *Gaussian measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[17] F. Bonizzoni, A. Buffa, and F. Nobile. Moment equations for the mixed formulation of the hodge laplacian with stochastic data. Technical Report 31/2012, MOX - Modeling and Scientific Computing, Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, 2012.

[18] A. Bossavit. *Computational electromagnetism. Variational formulations, complementarity, edge elements*. Electromagnetism. Academic Press Inc., San Diego, CA, 1998.

[19] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[20] A. Buffa. Remarks on the discretization of some noncoercive operator with applications to heterogeneous Maxwell equations. *SIAM J. Numer. Anal.*, 43(1):1–18 (electronic), 2005.

[21] H-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147–269, 2004.

[22] R. E. Caflisch. Monte carlo and quasi-monte carlo methods. *Acta Numerica*, 7:1–49, 1998.

[23] J. D. Carroll and J. J. Chang. Analysis of individual differences in multidimensional scaling via an n-way generalization of "eckart-youn" decomposition. *Psychometrika*, 35:283–319, 1970.

[24] J. Charrier. Strong and weak error estimates for elliptic partial differential equations with random coefficients. *SIAM Journal on Numerical Analysis*, 50(1):216–246, 2012.

[25] J. Charrier and A. Debussche. Weak truncation error estimates for elliptic pdes with lognormal coefficients. *Stochastic Partial Differential Equations: Analysis and Computations*, 2013.

[26] J. Charrier, R. Scheichl, and A.L. Teckentrup. Finite element error analysis of elliptic pdes with random coefficients and its application to multilevel monte carlo methods. BICS Preprint 2/11, University of Bath, 2011.

[27] Dan Cheng and Yimin Xiao. The mean euler characteristic and excursion probability of gaussian random fields with stationary increments. *arXiv preprint arXiv:1211.6693*, 2012.

[28] A. Chernov. Sparse polynomial approximation in positive order Sobolev spaces with bounded mixed derivatives and applications to elliptic problems with random loading. *Appl. Numer. Math.*, 62(4):360–377, 2012.

[29] A. Chkifa, A. Chohen, and R. DeVore. Sparse adaptive taylor approximation algorithms for parametric and stochastic elliptic pdes. *ESAIM: Mathematical Modelling and Numerical Analysis*, 47:253–280, 2013.

[30] S. H. Christiansen, H. Z. Munthe-Kaas, and B. Owren. Topics in structure-preserving discretization. *Acta Numer.*, 20:1–119, 2011.

[31] S. H. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus. *Math. Comp.*, 77(262):813–829, 2008.

[32] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup. Multilevel Monte Carlo methods and applications to elliptic PDEs with random coefficients. *Comput. Vis. Sci.*, 14(1):3–15, 2011.

[33] A. Cohen, R. DeVore, and C. Schwab. Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. *Found. Comput. Math.*, 10(6):615–646, 2010.

[34] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[35] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[36] G. Dagan. *Flow and Transport in Porous Formations*. Springer-Verlag Heidelberg Berlin New York, 1989.

[37] R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao. *A minicourse on stochastic partial differential equations*, volume 1962 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[38] P. Fernandes and G. Gilardi. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Methods Appl. Sci.*, 7(7):957–991, 1997.

[39] H. J. H. Franssen, A. Alcolea, M. Riva, M. Bakr, M. van der Wiel, F. Stauffer, and A. Guadagnini. A comparison of seven methods for the inverse modelling of groundwater flow. Application to the characterisation of well catchments. *Advances in Water Resources*, 32(6):851 – 872, 2009.

[40] Philipp Frauenfelder, Christoph Schwab, and Radu Alexandru Todor. Finite elements for elliptic problems with stochastic coefficients. *Comput. Methods Appl. Mech. Engrg.*, 194(2-5):205–228, 2005.

[41] J. Galvis and M. Sarkis. Approximating infinity-dimensional stochastic Darcy's equations without uniform ellipticity. *SIAM J. Numer. Anal.*, 47(5):3624–3651, 2009.

[42] B. Ganapathysubramanian and N. Zabaras. Sparse grid collocation schemes for stochastic natural convection problems. *Journal of Computational Physics*, 225(1):652 – 685, 2007.

[43] R. G. Ghanem and P. D. Spanos. *Stochastic finite elements: a spectral approach*. Springer-Verlag, New York, 1991.

[44] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag (Berlin and New York), 1983.

[45] M. Giles. Improved multilevel Monte Carlo convergence using the Milstein scheme. In *Monte Carlo and quasi-Monte Carlo methods 2006*, pages 343–358. Springer, Berlin, 2008.

[46] M. B. Giles. Multilevel Monte Carlo path simulation. *Oper. Res.*, 56(3):607–617, 2008.

[47] C. J. Gittelson. Stochastic Galerkin discretization of the log-normal isotropic diffusion problem. *Math. Models Methods Appl. Sci.*, 20(2):237–263, 2010.

[48] I. G. Graham, F. Y. Kuo, D. Nuyens, R. Scheichl, and I. H. Sloan. Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications. *J. Comput. Phys.*, 230(10):3668–3694, 2011.

[49] L. Grasedyck. Hierarchical singular value decomposition of tensors. *SIAM J. Matrix Anal. Appl.*, 31(4):2029–2054, 2009/10.

[50] Mircea Grigoriu. *Stochastic calculus*. Birkhäuser Boston Inc., Boston, MA, 2002. Applications in science and engineering.

[51] A. Guadagnini and S. P. Neuman. Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains: 1. Theory and computational approach. *Water Resour. Res.*, 35(10):2999–3018, 1999.

[52] A. Guadagnini and S. P. Neuman. Nonlocal and localized analyses of conditional mean steady state flow in bounded, randomly nonuniform domains: 2. Computational examples. *Water Resour. Res.*, 35(10):3019–3039, 1999.

[53] W. Hackbusch. *Tensor spaces and numerical tensor calculus*, volume 42. Springer, 2012.

[54] W. Hackbusch and S. Kühn. A new scheme for the tensor representation. *Journal of Fourier Analysis and Applications*, 15:706–722, 2009.

[55] H. Harbrecht, R. Schneider, and C. Schwab. Multilevel frames for sparse tensor product spaces. *Numer. Math.*, 110(2):199–220, 2008.

[56] H. Harbrecht, R. Schneider, and C. Schwab. Sparse second moment analysis for elliptic problems in stochastic domains. *Numerische Mathematik*, 109:385–414, 2008.

[57] R. A. Harshman. Foundations of the parafac procedure: Models and conditions for an "explanatory" multi-modal factor analysis. *UCLA Working Papers in Phonetics*, 16:1 – 84, 1970.

[58] S. Heinrich. Multilevel monte carlo methods. In Svetozar Margenov, Jerzy Wasniewski, and Plamen Yalamov, editors, *Large-Scale Scientific Computing*, volume 2179 of *Lecture Notes in Computer Science*, pages 58–67. Springer Berlin / Heidelberg, 2001.

[59] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002.

[60] R. Hiptmair, C. Schwab, and C. Jerez-Hanckes. Sparse tensor edge elements. Technical report, Swiss Federal Institute of Technology Zurich, 2012.

[61] H. Hoel, E. Schwerin, A. Szepessy, and R. Tempone. Adaptive multilevel monte carlo simulation. In *Numerical Analysis of Multiscale Computations*, volume 82 of *Lecture Notes in Computational Science and Engineering*, pages 217–234. Springer Berlin Heidelberg, 2012.

[62] H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic partial differential equations: A modeling, white noise functional approach*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1996.

[63] J. Kampé de Fériet. Random solutions of partial differential equations. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III*, pages 199–208, Berkeley and Los Angeles, 1956. University of California Press.

[64] S. Kesavan. *Topics in functional analysis and applications*. John Wiley & Sons Inc., New York, 1989.

[65] T. Kolda and B. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, 2009.

[66] D. Kressner and C. Tobler. htucker - a matlab toolbox for tensors in hierarchical tucker format. Preprint submitted to ACM Trans. Math. Software, 2012.

[67] R. Kumar, D. Kressner, F. Nobile, and C. Tobler. Low-rank tensor approximation for high order correlation functions of gaussian random fields. In preparation.

[68] F. Y. Kuo, C. Schwab, and I. H. Sloan. Quasi-monte carlo finite element methods for a class of elliptic partial differential equations with random coefficients. *SIAM Journal on Numerical Analysis*, 50(6):3351–3374, 2012.

[69] M. Ledoux and M. Talagrand. *Probability in Banach spaces*. Classics in Mathematics. Springer-Verlag, Berlin, 2011.

[70] P. Lévy. *Processus stochastiques et mouvement brownien*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1992. Followed by a note by M. Loève, Reprint of the second (1965) edition.

[71] W. A. Light and E. W. Cheney. *Approximation theory in tensor product spaces*, volume 1169 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[72] M. Loève. *Probability theory. I*. Springer-Verlag, New York, fourth edition, 1977. Graduate Texts in Mathematics, Vol. 45.

[73] M. Loève. *Probability theory. II*. Springer-Verlag, New York, fourth edition, 1978. Graduate Texts in Mathematics, Vol. 46.

[74] W. S. Massey. *A basic course in algebraic topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[75] H. G. Matthies and A. Keese. Galerkin methods for linear and nonlinear elliptic stochastic partial differential equations. *Comput. Methods Appl. Mech. Engrg.*, 194(12-16):1295–1331, 2005.

[76] P. Monk. *Finite element methods for Maxwell's equations*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2003.

[77] H. Niederreiter. *Random number generation and quasi-Monte Carlo methods*, volume 63 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[78] F. Nobile, R. Tempone, and C. G. Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2411–2442, 2008.

[79] F. Nobile, R. Tempone, and C. G. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2309–2345, 2008.

[80] E. Novak and K. Ritter. High-dimensional integration of smooth functions over cubes. *Numer. Math.*, 75(1):79–97, 1996.

[81] I. V. Oseledets. Tensor-train decomposition. *SIAM J. Sci. Comput.*, 33(5):2295–2317, 2011.

[82] I. V. Oseledets and E. E. Tyrtyshnikov. Recursive decomposition of multidimensional tensors. *Dokl. Akad. Nauk*, 427(1):14–16, 2009.

[83] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[84] M. M. Rao and R. J. Swift. *Probability theory with applications*, volume 582 of *Mathematics and Its Applications (Springer)*. Springer, New York, second edition, 2006.

[85] M. Reed and B. Simon. *Methods of modern mathematical physics: Functional analysis*, volume I. Academic Press Inc., New York, second edition, 1980.

[86] M. Riva, A. Guadagnini, and M. De Simoni. Assessment of uncertainty associated with the estimation of well catchments by moment equations. *Advances in Water Resources*, 29(5):676 – 691, 2006.

[87] C. P. Robert and G. Casella. *Monte Carlo statistical methods*. Springer Texts in Statistics. Springer-Verlag, New York, second edition, 2004.

[88] R. A. Ryan. *Introduction to tensor products of Banach spaces*. Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2002.

[89] R. Schneider and A. Uschmajew. Approximation rates for the hierarchical tensor format in periodic sobolev spaces. Technical Report 06.2013, MATHICSE - Ecole Polytechnique Fédérale de Lausanne, 2013.

[90] J. Schöberl. A posteriori error estimates for Maxwell equations. *Math. Comp.*, 77(262):633–649, 2008.

[91] C. Schwab and C. J. Gittelson. Sparse tensor discretizations of high-dimensional parametric and stochastic pdes. *Acta Numerica*, 20:291–467, 2011.

[92] C. Schwab and R. A. Todor. Sparse finite elements for elliptic problems with stochastic loading. *Numer. Math.*, 95(4):707–734, 2003.

[93] D. M. Tartakovsky and S. P. Neuman. Transient flow in bounded randomly heterogeneous domains: 1. Exact conditional moment equations and recursive approximations. *Water Resour. Res.*, 34(1):1–12, 1998.

[94] A.L. Teckentrup, R. Scheichl, M.B. Giles, and E. Ullmann. Further analysis of multilevel monte carlo methods for elliptic pdes with random coefficients. *Numerische Mathematik*, pages 1–32, 2013.

[95] R. A. Todor and C. Schwab. Convergence rates for sparse chaos approximations of elliptic problems with stochastic coefficients. *IMA Journal of Numerical Analysis*, 27(2):232–261, April 2007.

[96] L. R. Tucker. Some mathematical notes on three-mode factor analysis. *Psychometrika*, 31:279–311, 1966.

[97] T. von Petersdorff and C. Schwab. Sparse finite element methods for operator equations with stochastic data. *Appl. Math.*, 51(2):145–180, 2006.

[98] D. Xiu and J. S. Hesthaven. High-order collocation methods for differential equations with random inputs. *SIAM Journal on Scientific Computing*, 27(3):1118–1139, 2005.

[99] M. Ye, S. P. Neuman, A. Guadagnini, and D. M. Tartakovsky. Nonlocal and localized analyses of conditional mean transient flow in bounded, randomly heterogeneous porous media. *Water Resour. Res.*, 40(5), May 2004.

[100] D. Zhang. *Stochastic Methods for Flow in Porous Media Coping with Uncertainties*. Academic Press, 2002.