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Optimal Stopping and Backward Stochastic Differential Equations

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Abstract

In this work we provide an alternative approach, based on Backward Stochastic Differential Equations, to optimal stopping theory. To this purpose, we study a particular class of Backward stochastic differential equations with jumps and a sign constraint. We prove the existence of a minimal solution by approximation via penalization. We then prove that the solution to such an equation provides an original representation for the value function of an optimal stopping problem, in the general context of non-Markovian stochastic processes. Moreover, we compare our solutions with the solutions to Reflected Backward stochastic differential equations, thus proving an explicit relationship between the former and the latter. Reflected equations are characterized by a 'reflection' constraint, which keeps the solution above a given stochastic process, and provides a classical representation for the value function of an optimal stopping problem. Therefore, our result gives an equivalence between two representations for the value function of the optimal stopping problem, thus proving a new Backward stochastic differential equations approach to optimal stopping theory, based on equations with a sign constraint on the martingale component. At the end, in a Markovian framework, we show that, as a corollary of already known results, our method provides an alternative probabilistic representation for the unique viscosity solution to an obstacle problem for parabolic partial differential equations.

Keywords: *Stochastic Differential Equations, Backward Stochastic Differential Equations, Optimal Stopping, Point Processes.*

Sommario

In questo lavoro si mostra come risolvere un problema di arresto ottimo stocastico mediante un approccio innovativo nell'ambito della teoria delle equazioni differenziali stocastiche *Retrograde*, o Backward (BSDEs). A tal fine, viene studiata una classe particolare di tali equazioni, caratterizzata dalla presenza di un termine di salto e di un vincolo di segno (che chiameremo più brevemente BSDE con vincolo di segno). Si mostra l'esistenza di una soluzione minimale mediante un metodo di approssimazione per penalizzazione. Tale soluzione è la funzione valore associata ad un problema di arresto ottimo stocastico, espressa mediante una formulazione originale comprendente cambi di misure di probabilità equivalenti. Successivamente, viene messa a confronto la soluzione proposta con la soluzione di una BSDE *Riflessa* (RBSDE, dove il termine riflessione indica la presenza di un vincolo sul processo di stato, che forza quest'ultimo a mantenersi al di sopra di un determinato processo stocastico). Come ben noto in letteratura quest'ultima è anch'essa la funzione valore di un problema di arresto ottimo, espressa mediante una formulazione classica dove, a differenza del caso precedente, la misura di probabilità è fissata. Attraverso una relazione esplicita fra le due BSDE, si prova che le due formulazioni della funzione valore sono equivalenti, e dunque l'equazione proposta si sostituisce alla RBSDE, fornendo un approccio alternativo di risoluzione del problema di arresto ottimo. In conclusione si considera il contesto Markoviano, provando come corollario di risultati già noti che il metodo proposto fornisce una rappresentazione probabilistica alternativa alla unica soluzione viscosa di un problema ad ostacoli, per equazioni differenziali paraboliche alle derivate parziali.
La tesi si presenta come elaborato prevalentemente originale. I risultati prodotti forniscono gli strumenti per un approfondimento futuro di alcuni argomenti specifici considerati.

Sia dato uno spazio di probabilità completo $(\Omega, \mathcal{F}, P)$, un processo di Wiener W in $\mathbb{R}^d$, un tempo $T > 0$ e una filtrazione $(\mathcal{F}_t)_{0 \le t \le T}$ generata da W. La BSDE si caratterizza come equazione nel senso di Ito, su un intervallo di tempo $[0, T]$, dove uno dei processi incogniti adattati alla filtrazione di riferimento assume un valore assegnato al tempo finale. Ovvero, data una coppia di processi adattati (Y, Z), che

verifica la sequente equazione

$$Y_t + \int_t^T Z_s dW_s = \xi + \int_t^T f_s ds, \quad 0 \leq t \leq T$$

dove W è il processo di Wiener in $\mathbb{R}^d$ e (ξ, f) è la coppia di oggetti chiamati rispettivamente *condizione finale* e *funzione generatrice*, (Y, Z) si dirà soluzione della BSDE se soddisfa in aggiunta alcune specifiche proprietà di misurabilità e integrabilità. Sotto opportune ipotesi sulla coppia (ξ, f), si mostra che tale soluzione esiste ed è unica.
La teoria delle BSDEs viene prevalentemente utilizzata per risolvere problemi di natura stocastica. Nel nostro lavoro, in particolare, ci occuperemo di un problema di arresto ottimo stocastico ad orizzonte temporale finito T, caratterizzato da un *funzionale guadagno* J del tipo

$$J(t, \tau) = \mathbb{E}\left[\int_t^{\tau \wedge T} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau \geq T} | \mathcal{F}_t\right], \ 0 \leq t \leq T,$$

dove τ è un tempo d'arresto rispetto alla filtrazione sottostante, mentre la tripla di processi (ξ, f, h) è presa come dato del problema. L'obiettivo è dunque quello di massimizzare, al variare dei tempi d'arresto ammissibili, il valore atteso di tale funzionale. È nota in letteratura la seguente rappresentazione

$$Y_t = ess \sup_\tau J(t, \tau), \ 0 \leq t \leq T,$$

dove Y è un processo appartenente alla tripla di processi adattati (Y, Z, K) soluzione della seguente RBSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s dW_s, \ 0 \leq t \leq T,$$

con vincolo di riflessione

$$Y_t \geq h_t, \quad 0 \leq t \leq T.$$

In particolare, la tripla di processi (Y, Z, K) viene detta soluzione della RBSDE quando soddisfa, in aggiunta all'equazione introdotta, alcune condizioni di misurabilità e integrabilità. Analogamente al caso non vincolato, sotto opportune ipotesi sui processi ξ, f e h, si mostra che tale soluzione esiste ed è unica nella classe delle soluzioni *minimali*, ovvero quelle in cui la componente Y verifica una condizione aggiuntiva di *Skohorod* del tipo

$$\int_0^T (Y_t - h_t) \, dK_t = 0.$$

Si vuole introdurre quindi una formulazione *alternativa* del problema d'arresto. A tale scopo, viene fornita una analisi preliminare sui *processi di punto* e *processi di* *punto marcato* mediante la teoria delle martingale. In particolare, il processo di punto è individuato da una sequenza $(T_n)_{n\geq 1}$ strettamente crescente di variabili aleatorie, con $T_0 = 0$, che possono essere interpretate come *tempi di salto.* A tale sequenza viene associato un *processo di conteggio* N, che ha memoria dei salti avvenuti fino all'istante di osservazione. Tale processo di conteggio è univocamente determinato da una *misura random* μ, che 'conta' i tempi di salto che occorrono ad ogni istante d'osservazione del processo. Infine, è associata a questi oggetti una *intensità stocastica* λ, che permette una analisi dei processi mediante la teoria delle martingale. Poichè, in generale, non è garantita l'esistenza di quest'ultima, si fa riferimento al seguente teorema, che svolge un ruolo chiave nella trattazione dei capitoli centrali.

Teorema 0.0.1. *Esiste un processo A tale che $A_0 = 0$, continuo a destra, prevedibile e non decrescente tale che, per ogni H processo prevedibile positivo vale*

$$\mathbb{E}\left[\int_0^\infty H_t dN_t\right] = \mathbb{E}\left[\int_0^\infty H_t dA_t\right].$$

In particolare questo implica che il processo $N - A$ è una martingala (rispetto alla sua filtrazione naturale). A è detto *compensatore* del processo N.
Dunque, si considera una variabile aleatoria η, di distribuzione esponenziale, che rappresenta un tempo aleatorio di salto. Tale variabile è un processo di punto caratterizzato da un singolo tempo di salto, e dunque un processo di conteggio N elementare, della forma

$$1(\eta \leq t),\ 0 \leq t \leq T.$$

Chiaramente la misura random μ associata è una delta di Dirac del tipo

$$\mu(t) = \delta_\eta(t),\ 0 \leq t \leq T,$$

mentre l'intensità stocastica λ viene scelta in modo tale che il processo

$$1(\eta \leq t) - t \wedge \eta,\ 0 \leq t \leq T$$

dove $t \wedge \eta$ denota il compensatore A, sia una martingala locale (rispetto alla sua filtrazione naturale).
Il funzionale guadagno da massimizzare è della forma

$$J(t,\nu) = \mathbb{E}^\nu\left[\int_t^{\eta\wedge T} f_s ds + h_\tau 1_{t\leq\eta<T} + \xi 1_{\eta\geq T} | \bar{\mathcal{F}}_t\right],\ 0 \leq t \leq T,$$

dove η *randomizza* il tempo d'arresto τ, $(\bar{\mathcal{F}}_t)_{0\leq t\leq T}$ è la filtrazione prodotto generata da W ed N, mentre ν è un processo stocastico nella classe dei processi prevedibili ed essenzialmente limitati. Ad ogni processo ν si associa una misura di probabilità $\mathbb{P}^\nu$, tale che N ha intensità stocastica

$$\tilde{\lambda}_t = \lambda_t \nu_t,\ 0 \leq t \leq T$$

sotto $\mathbb{P}^\nu$, mentre W preserva le sue proprietà di moto Browniano. Il termine $\mathbb{E}^\nu$ indica l'aspettazione condizionale rispetto alla misura di probabilità $\mathbb{P}^\nu$, e chiaramente la scelta di ν verrà fatta in modo tale da massimizzare questa aspettazione. Tale misura di probabilità esiste ed ammette la forma seguente

$$\mathbb{P}^\nu(d\omega) = \mathcal{L}^\nu_T(\omega)\mathbb{P}(d\omega),$$

dove $\mathbb{P}$ è la misura di probabilità originaria mentre $\mathcal{L}^\nu$ è della forma

$$\mathcal{L}^\nu_t = exp\left(\int_0^t ln\ \nu_s \mu(ds)\ -\ \int_0^t (\nu_s - 1)\lambda_s ds\right),\ 0 \leq t \leq T,$$

a condizione di verificare che $\mathbb{E}\left[\mathcal{L}^\nu_T\right] = 1$. Sotto le ipotesi stabilite, si mostra che tale condizione è verificata, e in particolare $\mathcal{L}^\nu$ è una martingala di quadrato integrabile, soddisfacente

$$\mathcal{L}^\nu_t = 1 + \int_0^t \mathcal{L}^\nu_{s^-}(\nu_s - 1)(\mu(ds) - \lambda_s ds),\ 0 \leq t \leq T.$$

Analogamente al caso noto, si mostra che questa formulazione originale ammette una rappresentazione del tipo

$$\bar{Y}_t = ess\sup_\nu J(t,\nu), \quad 0 \leq t \leq T,$$

dove $\bar{Y}$ è un processo appartenente alla quadrupla di processi adattati $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ soluzione di una opportuna BSDE, della forma

$$\begin{aligned} \bar{Y}_t = \xi 1_{\eta \geq T} + \int_t^{T\wedge\eta} f_s ds + h_\eta 1_{t\leq\eta<T} + \\ + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s - \bar{U}_\eta 1_{t\leq\eta<T}, \quad 0 \leq t \leq T \end{aligned}$$

e vincolo di segno del tipo

$$\bar{U}_t \leq 0,\ 0 \leq t \leq T.$$

In particolare, chiameremo questa quadrupla di processi $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ soluzione della BSDE con vincolo di segno, se soddisfa in aggiunta alcune proprietà di integrabilità e misurabilità. Sotto le stesse ipotesi sulla tripla (ξ, f, h), si mostra che tale soluzione esiste ed è unica nella classe delle soluzioni *minimali*, dove la minimalità è intesa nel senso che, per ogni altra quadrupla di processi $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ soddisfacente la BSDE con le stesse proprietà, vale che

$$\bar{Y}_t \leq \tilde{Y}_t,\ 0 \leq t \leq T.$$

Tale soluzione minimale è costruita mediante uno schema di approssimazione per penalizzazione. Come ultimo risultato originale, viene messa a confronto la soluzione proposta con la soluzione della BSDE Riflessa. In particolare, le componenti delle due BSDE sono legate dalle seguenti relazioni

$$\bar{Y}_t = Y_t 1_{[0,\eta]}(t),$$

$$\bar{U}_t = (h_t - Y_t),$$

$$\bar{Z}_t = Z_t 1_{[0,\eta]}(t),$$

$$\bar{K}_t = K_{t\wedge\eta},$$

per ogni $t \in [0,T]$. Come conseguenza di tale risultato si ottiene una equivalenza fra le due formulazioni del problema di arresto ottimo. In particolare, in $t = 0$ i valori (deterministici) dei due funzionali coincidono, e dunque la BSDE proposta può essere di fatto sostituita alla RBSDE come soluzione alternativa del problema di arresto ottimo. Come esempio d'applicazione dei nuovi risultati, si considera infine il problema in un contesto Markoviano. Il funzionale guadagno è caratterizzato da processi di diffusione di Ito, ovvero detta v la *funzione valore* associata al problema d'arresto, si ha

$$v(t,x) = sup_{\tau\in\mathcal{T}_t^T}\mathbb{E}\left[\int_t^\tau f(s,X_s^{t,x})ds + h(X_\tau^{t,x})1_{\tau<T} + g(X_\tau^{t,x})1_{\tau=T}\right],\ 0\leq t\leq T, x\in\mathbb{R}^n$$

dove f, g e h sono funzioni deterministiche a valori reali, e $X^{t,x}$ è un processo di diffusione che vale x al tempo t, soddisfacente una SDE del tipo

$$dX_s^{t,x} = x + \int_t^s b(u,X_u^{t,x})du + \int_t^s \sigma(u,X_u^{t,x})dW_u,\ t\leq s\leq T,$$

con b, σ funzioni deterministiche Boreliane su $[0,T]\times\mathbb{R}^n$, a valori rispettivamente in $\mathbb{R}^n$ e $\mathbb{R}^{n\times d}$. Si prova che, sotto opportune ipotesi per f, g, h, σ, b, la funzione valore v viene caratterizzata come la soluzione viscosa di una diseguaglianza variazionale del tipo Hamilton-Jacobi-Bellman (HJBVI)

$$min\{-\frac{\partial v(t,x)}{\partial t} - b(t,x).D_x v(t,x) + \\ + tr\left(\frac{1}{2}\sigma\sigma'(t,x)D_x^2 v(t,x)\right) - f(t,x), v(t,x) - h(x)\} = 0$$

con dato finale

$$v(T,x) = g(x).$$

Dunque, adattando la BSDE proposta al contesto Markoviano, si definisce la quadrupla di processi adattati $(\bar{Y}^{t,x}, \bar{Z}^{t,x}, \bar{K}^{t,x}, \bar{U}^{t,x})$ soluzione della BSDE

$$\begin{aligned}\bar{Y}_s^{t,x} = g(X_T^{t,x})1_{\eta \geq T} + \int_s^T f(u, X_u^{t,x})\lambda_s ds + h(X_\eta^{t,x})1_{\eta < T} + \\ + \bar{K}_T^{t,x} - \bar{K}_s^{t,x} - \int_s^T \bar{Z}_u^{t,x} dW_u - \bar{U}_\eta^{t,x} 1_{\eta < T}, \quad t \leq s \leq T,\end{aligned}$$

con vincolo di segno

$$\bar{U}_s^{t,x} \leq 0, \quad t \leq s \leq T,$$

e una scelta appropriata della variabile η. Utilizzando le stesse argomentazioni dei precedenti capitoli, sappiamo che esiste una unica soluzione nella classe delle soluzioni minimali. Quindi, sintetizzando risultati già noti sulle BSDE Riflesse e il loro legame con le HJBVI nel caso Markoviano, in aggiunta alle relazioni ottenute fra le BSDE Riflesse e le BSDE con vincolo di segno, siamo in grado di fornire l'equivalenza

$$\bar{Y}_t^{t,x} = v(t,x),$$

che porta al seguente

Teorema 0.0.2. *La funzione $v(t,x) = \bar{Y}_t^{t,x}$ è continua su $[0,T] \times \mathbb{R}^n$, ed è una soluzione viscosa della diseguaglianza variazionale di Hamilton-Jacobi-Bellman introdotta precedentemente.*

Parole chiave: Equazioni Differenziali Stocastiche, Equazioni Differenziali Stocastiche Backward, Arresto Ottimo, Processi di Punto.

Contents

Introduction

In this work we deal with optimal stopping problems for stochastic processes.

The theory of *optimal stopping* emerged as a major tool in finance around the 1970s, when the optimal stopping problem found its main application in the evaluation of stock options. The general problem discussed in this theory, is that of finding the right time τ to *stop* a given action, in order to maximize the mean of a functional, for instance

$$E\left[\int_0^{\tau\wedge T} f_s ds + h_\tau 1_{\tau<T} + \xi 1_{\tau\geq T}\right],$$

which is usually called *gain*. The functions f, h, ξ denote respectively the integral reward, the anticipated reward (available only if the action ends before T), and the terminal reward. All of these are stochastic processes. The choice of the *optimal* time τ, is made by exploiting only the information available up to time t, where t is the time of observation. The information available at time t, for $t \in [0, T]$, is modeled as a family of increasing σ-fields, i.e. a *filtration*. Thus, the time τ is a *stopping time*. Most of the general facts about optimal stopping theory can be found in the work of Peskir and Shiryaev [23]. One of the most well-developed approaches to optimal stopping problems is the one based on the theory of backward stochastic differential equations, BSDEs for short.

The notion of BSDEs in the general nonlinear case was introduced by Pardoux and Peng [20]. They studied BSDEs driven by Wiener processes, and proved the existence and uniqueness of adapted solutions, under suitable assumptions on the coefficients and on the terminal condition. Starting from that paper, BSDEs have been a very active research area due to their connections with mathematical finance, stochastic optimal control and partial differential equations (PDEs). For instance, we refer the reader to the studies in El Karoui, Mazliak [7], El Karoui, Peng, Quenez [8], for the application to control and mathematical finance, and the survey paper by Pardoux [19], for the connection between BSDEs and PDEs. The connection between BSDEs and optimal stopping theory was instead developed in the work of El Karoui, Kapoudjian, Pardoux, Peng and Quenez [6]. They studied the class of BSDEs with an additional, increasing process added to the equation, and a constraint on the solution, called an obstacle. They made the basic observation that the solution to such Reflected BSDEs (RBSDEs) provides a representation for the value function of an optimal stopping problem.

As already mentioned above, the theory of BSDEs appears to be a very powerful tool for solving PDEs, in relation to stochastic optimization problems. In particular, there is a large amount of literature concerning the optimal control of diffusive processes, dynamic programming and BSDEs. In the seminal works by Pardoux and Peng [21], for instance, it is shown that BSDEs provide a probabilistic representation for quasilinear Hamilton-Jacobi-Bellman equations. Starting from this context, a specific class of BSDEs was considered by Kharroubi and Pham [18], with the goal of providing an alternative representation for fully nonlinear Hamilton-Jacobi-Bellman PDEs. Following the idea in [17], they introduce a Poisson random measure (with finite intensity), independent of W, and they add a sign constraint on the jump component to the equation. They prove the existence of a unique minimal solution by using a penalization approach. Moreover, they provide a dual formula for the solution, which can be viewed as an alternative formulation for the value function of a stochastic control problem (inspired by [5]).

In our work, we use similar arguments to provide a particular method of solution for optimal stopping problems, again based on a BSDE approach. Following the idea developed by Elie-Kharroubi [9] for *optimal switching*, we point out an alternative formulation for the value function of our optimal stopping problem. We first replace the stopping time τ with a random variable η. Then we introduce an appropriate BSDE with (a single) jump and a sign constraint on the jump (BSDE with a sign constraint for short), similar to the one in [18], and we show that the solution represents the value function of the original optimal stopping problem, in the form that we pointed out before. Furthermore, we see that these new constrained BSDEs with jumps are explicitly related to the Reflected BSDEs in [6]. In particular, by identifying the two formulations for the value function, we provide an alternative BSDE approach for solving the optimal stopping problem.
The original results of the work are contained in chapters 3 and 4. In chapter 3, we prove the existence and uniqueness of the solution to the BSDE with a sign constraint. Then, we show that such a solution admits a representation in terms of the value function of a (non-diffusive) optimal stopping problem. In chapter 4, we compare the Reflected BSDE and the BSDE with a sign constraint, thus proving an explicit relation between the former and the latter. In particular we show that there is equivalence between the two related formulations for the value function. Therefore both BSDEs provide a solution to the optimal stopping problem. Chapters 1 and 2 contain preliminaries on BSDE theory and optimal stopping, as well as Point process theory analysed by using the martingale approach. Both chapters provide the basic

information required for the developments of the original results. In particular, we state all the well-known results without proof. In chapter 5, we reformulate the optimal stopping problem in a Markovian context, and we report the interesting fact that the BSDE with a sign constraint represents the solution to a PDE with an obstacle. Now we briefly describe the contents of every chapter.

In **Chapter 1** we present the standard optimal stopping problem for a given time horizon T. Given a probability space $(\Omega, \mathcal{F}, P)$, an $\mathbb{R}^d$-valued Brownian motion $W = (W_t)_{0\leq t\leq T}$ with its natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$, we introduce the *gain* of the problem, which consists of an $\mathcal{F}_t$-measurable integral functional of the form

$$J(t,\tau) = \mathbb{E}\left[\int_t^{\tau\wedge T} f_s ds + h_\tau 1_{\tau<T} + \xi 1_{\tau\geq T} | \mathcal{F}_t\right], \tag{0.0.1}$$

where τ denotes a stopping time with respect to the filtration $\mathbb{F}$. The functional gain is associated with the *value function* V of the problem, written as

$$V_t = ess\sup_\tau J(t,\tau), \tag{0.0.2}$$

which we denote as the *standard formulation* of the optimal stopping problem, where τ varies among the set of stopping times valued in $[t,T]$. The central result of the chapter is the characterization of the value function in terms of a solution to a particular class of BSDEs, the aforementioned class of Reflected BSDE. To this end, we give some basic notions of Backward Stochastic Differential equations (we mainly follow the discussion in Pham [25]), analysing the classical case of BSDEs driven by Brownian motion, and then introducing the class of Reflected BSDEs and their connection with optimal stopping problems.

The solution to a BSDE driven by the Brownian motion W is a pair of adapted processes (Y,Z) satisfying

$$Y_t = \xi + \int_t^T f_s ds - \int_t^T Z_s dW_s\ ,\ 0\leq t\leq T, \tag{0.0.3}$$

where f is called the *generator* and ξ is the *terminal condition.* Under a suitable integrability hypothesis on the pair (ξ, f), and the fundamental Lipschitz condition on the generator $f(t,y,z)$, with respect to (y,z), uniformly in t, we achieve existence and uniqueness for the solution in theorem 1.2.1 in an appropriate class of stochastic processes. Next, we analyse the class of BSDEs with reflection. We refer the reader to the discussion in Pham [25], and El Karoui, Kapoudjian, Pardoux, Peng and Quenez [6]. In Reflected BSDEs the solution process Y is forced to stay above a given stochastic process h, called the obstacle. An increasing process K is introduced which pushes the solution Y upwards, so that it may remain above the obstacle. In this way, we introduce the solution of a Reflected BSDE as a triple of adapted processes (Y,Z,K), thus satisfying

$$Y_t = \xi + \int_t^T f_s ds + K_T - K_t - \int_t^T Z_s dW_s, \quad 0\leq t\leq T, \tag{0.0.4}$$

with the constraint (reflection condition)

$$Y_t \geq h_t,\ 0 \leq t \leq T. \tag{0.0.5}$$

Moreover, we require the solution to satisfy the following Skohorod condition

$$\int_0^T (\,Y_t - h_t)\, dK_t = 0. \tag{0.0.6}$$

which assures the uniqueness of the solution, as stated in theorem 1.2.2. Finally, by referring to the results in El Karoui, Kapoudjian, Pardoux, Peng and Quenez [6], we identify the solution with the value function stated in (0.0.2). In particular we give the following basic identity

$$Y_t = ess\sup_{\tau} \mathbb{E}\left[\int_t^{\tau\wedge T} f_s ds + h_\tau 1_{\tau<T} + \xi 1_{\tau\geq T} | \mathcal{F}_t\right],\ 0 \leq t \leq T. \tag{0.0.7}$$

In **Chapter 2**, we first give some preliminaries on marked point processes and their analysis by using the martingale approach, following the discussions in Brémaud [2], and Jacod [16]. Point process theory turns out to be necessary in order to state an alternative formulation for the value function in (0.0.2), which involves the notions of *point process*, *stochastic intensity* and *random measure*.

The *point process* is introduced as a simple case of a more general *marked point process*, for which we refer the reader to section 2.1. The point process consists of a sequence of random variables, which can be interpreted as jump times, satisfying some specific requirements. A *random measure* is associated with this sequence, which 'counts' the number of jumps that occur at the time of observation. In this context, we also introduce the notion of *stochastic intensity*, and we provide a characterization of these objects in terms of local martingales (again, we refer the reader to section 2.1, for precise results).
Next, we move to the central aim of the chapter. We introduce a random variable η, exponentially distributed, which represents the time of a jump. In this way, it characterizes a single jump process, which is, in particular, an elementary case of point process. The associated random measure μ clearly consists of a Dirac measure, i.e.

$$\mu(t) = \delta_\eta(t), \quad 0 \leq t \leq T, \tag{0.0.8}$$

and the stochastic intensity is taken in such a way that the process

$$1(\eta \leq t) - t \wedge \eta,\ 0 \leq t \leq T \tag{0.0.9}$$

is a local martingale (w.r.t. its natural filtration). The alternative formulation for the value function in (0.0.2) takes the following form:

$$\bar{V}_t = ess\sup_{\nu} \mathbb{E}^\nu \left[\int_t^{T\wedge\eta} f_s ds + h_\eta 1_{t\leq\eta<T} + \xi 1_{\eta\geq T} | \bar{\mathcal{F}}_t\right], \tag{0.0.10}$$

where ν varies among the feasible intensities of η, $\mathbb{E}^\nu$ is the conditional expectation with respect to a specific probability measure $\mathbb{P}^\nu$, while $(\bar{\mathcal{F}}_t)_{0\le t\le T}$ is the filtration generated by the Brownian motion W and the random variable η. In particular, we observe that η *randomizes* the original stopping time τ, and we achieve the supremum by choosing the intensity of η, i.e. by controlling its law, instead of directly choosing the stopping time τ. This is possible by means of a change in probability 'à la Girsanov'. More specifically, the Girsanov theorem states that, under suitable requirements for the processes, a specific *Doléans-Dade* exponential local martingale $\mathcal{L}^\nu$ (see section 2.2 for precise results), defined as

$$\mathcal{L}_t^\nu = exp\left(\int_0^t \ln \nu_s \mu(ds) \; - \; \int_0^t (\nu_s - 1)1_{[0,\eta)}(s)ds\right), \; 0 \le t \le T, \qquad (0.0.11)$$

characterizes the Radon-Nikodim density of an equivalent change of probability measure, i.e.

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}|_{\bar{\mathcal{F}}_t} \; = \; \mathcal{L}_t^\nu, \; 0 \le t \le T. \qquad (0.0.12)$$

which induces a given change in stochastic intensity for the random measure μ, associated to η. Thus, the stochastic intensity will be chosen in such a way that the conditional expectation, with respect to the probability measure generated by the Girsanov theorem, is maximized.

Chapter 3 is the first innovative chapter of the present work. Here we focus on the BSDE driven by both W and η, of the following form

$$\begin{aligned} \bar{Y}_t = \xi 1_{\eta\ge T} + \int_t^{T\wedge\eta} f_s ds + h_\eta 1_{t\le\eta<T} + \\ + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s - \bar{U}_\eta 1_{t\le\eta<T}, \quad 0 \le t \le T, \end{aligned} \qquad (0.0.13)$$

with the sign constraint

$$\bar{U}_t \le 0, \quad 0 \le t \le T. \qquad (0.0.14)$$

The triple (ξ, f, h) is given by the problem in (0.0.1), and preserves the original assumptions with respect to the new filtration $(\bar{\mathcal{F}}_t)_{0\le t\le T}$. The quadruple of adapted processes $(\bar{Y}, \bar{Z}, \bar{K}, \bar{U})$ is such that, for any other quadruple $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{U})$ satisfying (0.0.13), (0.0.14) it holds that

$$\bar{Y}_t \le \tilde{Y}_t, \quad 0 \le t \le T, \qquad (0.0.15)$$

is called a *minimal solution* to the BSDE. Besides the usual component $(\bar{Y}, \bar{Z})$, $\bar{U}$ is a process which characterizes the jump term, and $\bar{K}$ is a nondecreasing process

that makes the constraint on $\bar{U}$ feasible.
In the sequel of this chapter, we construct a sequence of penalized BSDEs with jumps, and prove that it converges to the solution we are looking for, mainly following the discussion in Pham, Kharroubi [18]. We refer to the paper by Becherer [1], for the existence and uniqueness of the penalized solution. In order to prove convergence we first state some *a priori* estimates, then we give, in theorem 3.2.1, a proof of the existence of a unique minimal solution $(\bar{Y}, \bar{Z}, \bar{K}, \bar{U})$ to (0.0.13), (0.0.14) as the limit of the penalized sequence. In fact, the property of minimality is a corollary of a second basic result, stated in proposition 3.3.1.

The solution $\bar{Y}$ *to* (0.0.13), (0.0.14) *can be expressed, for all* $0 \leq t \leq T$, *as*

$$\bar{Y}_t = ess \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^{T \wedge \eta} f_s ds + h_\eta 1_{t \leq \eta < T} + \xi 1_{\eta \geq T} | \bar{\mathcal{F}}_t \right]. \tag{0.0.16}$$

On the other hand, by looking at the previous chapter, with the above result we provide an explicit representation for the optimal stopping problem as formulated in (0.0.10).

In **chapter 4**, we compare the Reflected BSDE and the BSDE with jumps presented in chapter 3. In particular, we prove the following relations:

$$\bar{Y}_t = Y_t 1_{[0,\eta]}(t) \tag{0.0.17}$$

$$\bar{U}_t = (h_t - Y_t) \tag{0.0.18}$$

$$\bar{Z}_t = Z_t 1_{[0,\eta]}(t) \tag{0.0.19}$$

$$\bar{K}_t = K_{t \wedge \eta}. \tag{0.0.20}$$

The equality in (0.0.17) specifies a relationship between the two associated value functions V, $\bar{V}$ in (0.0.2), (0.0.10). In particular, we show that, at the origin, they take the same deterministic value, i.e.

$$V_0 = \sup_\tau \mathbb{E} \left[\int_0^{\tau \wedge T} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau \geq T} \right], \tag{0.0.21}$$

coincides with

$$\bar{V}_0 = \sup_\nu \mathbb{E}^\nu \left[\int_0^{T \wedge \eta} f_s \lambda_s ds + h_\eta 1_{t \leq \eta < T} + \xi 1_{\eta \geq T} \right] \tag{0.0.22}$$

which trivially follows by noting that $\bar{Y}_0 = Y_0$.

The latter result, in particular, provides an alternative BSDE approach for solving the optimal stopping problem.

In **chapter 5**, we consider optimal stopping problems which admit a markovian representation, and we analyze their connection with free boundary problems. In particular, we revisit the optimal stopping problem by means of the viscosity solution approach. The main goal of the chapter is to provide a formal connection between our BSDE with a sign constraint and the HJB variational inequality associated with optimal stopping. Even though it is only a corollary of already known results, we point out this case, as it may be interesting in view of future computational tools for BSDE applications. There is a great deal of literature on the theory of viscosity solutions and we refer the reader to Crandall, Ishii and Lions [3]; here we mainly follow the discussion in Pham [25].
We first provide some basic notions on this topic, we formulate the main assumptions that are required, and we report on the central results.
The value function $v(t,x)$ of the finite horizon optimal stopping problem, for any $(t,x) \in [0,T] \times \mathbb{R}^n$, takes the form

$$v(t,x) = sup_{\tau \in \mathcal{T}_t^T} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})\,ds + h(X_\tau^{t,x})1_{\tau<T} + g(X_\tau^{t,x})1_{\tau=T}\right], \qquad (0.0.23)$$

where f g and h are deterministic real valued functions, $X^{t,x}$ is a diffusion process which takes the value x at time t, satisfying an SDE such as

$$dX_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u,\ t \leq s \leq T, \qquad (0.0.24)$$

with b, σ deterministic Borelian functions on $[0,T] \times \mathbb{R}^n$, valued respectively in $\mathbb{R}^n$ and $\mathbb{R}^{n\times d}$. It is proved that, under a suitable hypothesis for f, g, h, σ, b, the value function $v(t,x)$ is characterized as the viscosity solution to the associated HJB variational inequality

$$\begin{aligned} min\{-\frac{\partial v(t,x)}{\partial t} &- b(t,x).D_x v(t,x)+ \\ &+ tr\left(\frac{1}{2}\sigma\sigma'(t,x)D_x^2 v(t,x)\right) - f(t,x), v(t,x) - h(x)\} = 0 \end{aligned} \qquad (0.0.25)$$

with the terminal data

$$v(T,x) = g(x). \qquad (0.0.26)$$

Before providing the result of our interest, we briefly redefine the BSDE with jumps and a non-positive constraint in the markovian setting, as the quadruple of adapted processes $(\bar{Y}^{t,x}, \bar{Z}^{t,x}, \bar{K}^{t,x}, \bar{U}^{t,x})$ satisfying

$$\begin{aligned}\bar{Y}^{t,x}_s = g(X^{t,x}_T)1_{\eta\geq T} + \int_s^T f(u, X^{t,x}_u)\lambda_s ds + h(X^{t,x}_\eta)1_{\eta<T} + \\ + \bar{K}^{t,x}_T - \bar{K}^{t,x}_s - \int_s^T \bar{Z}^{t,x}_u dW_u - \bar{U}^{t,x}_\eta 1_{\eta<T}, \quad t \leq s \leq T,\end{aligned} \tag{0.0.27}$$

with the sign constraint

$$\bar{U}^{t,x}_s \leq 0, \quad t \leq s \leq T, \tag{0.0.28}$$

and an appropriate choice of the variable η. By using the same arguments as in chapter 3 we know there exists a minimal solution with the usual properties. Therefore, by means of the fundamental relation with the Reflected BSDE, introduced in chapter 4, we obtain in $s = t$ the following (deterministic) equivalence

$$\bar{Y}^{t,x}_t = v(t, x), \tag{0.0.29}$$

which finally yields the following result.

The function $v(t, x) = \bar{Y}^{t,x}_t$ *is continuous on* $[0, T] \times \mathbb{R}^n$*, and is a viscosity solution to the HJB variational inequality in* (0.0.25)*,* (0.0.26)*.*

As remarked in the conclusive chapter, we note that the connection between the HJBVI and our BSDE provides alternative tools for the solution of free boundary problems by means of the BSDE approach.

Chapter 1

Optimal stopping and the Reflected BSDE approach

The theory of optimal stopping is an important and classical field of stochastic control, with several applications in economics and finance, especially for the evaluation of American option prices.

1.1 The Stopping Problem

In the sequel, we fix $T > 0$, which is called the *terminal time* of the problem, and we introduce the following *finite time horizon* optimal stopping problem.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ satisfying the *usual conditions*. We are given three objects:

- the first is the **terminal value**, which is a random variable $\xi : \Omega \to \mathbb{R}$;
- the second is the **generator function**, which is a stochastic process $f : \Omega \times [0, T] \to \mathbb{R}$;
- the third is the **obstacle**, which is a stochastic process $h : \Omega \times [0, T] \to \mathbb{R}$;

satisfying the requirements detailed in the next sections.
We define with such objects the *functional gain* J for the optimal stopping problem, as

$$J(t, \tau) = \mathbb{E}\left[\int_t^{\tau \wedge T} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau \ge T} | \mathcal{F}_t\right], \ 0 \le t \le T. \tag{1.1.1}$$

where τ denotes a stopping time w.r.t. the filtration $\mathbb{F}$. The functional gain is associated to the *value function* V of the problem, written as

$$V_t = ess \sup_{\tau \in \mathcal{T}_t} J(t, \tau), \ 0 \le t \le T, \tag{1.1.2}$$

where $\mathcal{T}_t$ is the set of all stopping times valued in $[t;\infty)$. By introducing a functional gain which depends on the generator f, the terminal value ξ and the obstacle h, we are interested in choosing the optimal stopping time τ that maximizes its expectation value over all stopping times belonging to the set $\mathcal{T}_t$. In general, we do not have any need for the triple (ξ, f, h) but suitable integrability conditions. However, in the last section, we shall assume (ξ, f, h) as given functions of a diffusion process X, in order to give a probabilistic representation of an obstacle problem for a partial parabolic differential equation (PDE). More precisely, in the case where ξ, f, h are constructed as deterministic continuous functions composed by a diffusion process $X^{t,x}$ (starting at time t with value x), which satisfies a stochastic differential equation (SDE) like

$$\begin{cases} dX_s = b(X_s)dt + \sigma(X_s)dW_s, \\ X_t = x, \quad s \geq t, \ x \in \mathbb{R}^n \end{cases} \tag{1.1.3}$$

we can find a very interesting result in the literature. The value function $v(t,x)$, defined as

$$v(t,x) = sup_{\tau \in \mathcal{T}_t^T} \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})ds + h(X_\tau^{t,x})1_{\tau<T} + g(X_\tau^{t,x})1_{\tau=T}\right]. \tag{1.1.4}$$

can be characterized as the unique viscosity solution of a Hamilton-Jacobi-Bellman variational inequality (HJBVI), which involves an obstacle problem for a partial parabolic differential equation in a finite dimensional Euclidean space. In general, as far as optimal control and optimal stopping problems are concerned, there is a great deal of literature on the diffusive case, where, for example, several results were pointed out by means of the Backward Stochastic Differential Equation (BSDE) approach. Conversely, in the more general context of non diffusive processes, few results are available. The work of the thesis addresses such a general context: our aim is to solve a non diffusive optimal stopping problem by means of a particular BSDE approach.

1.2 Reflected BSDE

First of all, we will analyse the optimal stopping problem given in (1.1.1), (1.1.2), by means of the standard Backward Stochastic Differential Equation approach. For the following discussion, we refer the reader to the work in [6]: the paper provides a formulation for the value function (1.1.2), showing that it corresponds to a square-integrable solution of a Reflected BSDE. In the next sections, we will briefly report some general notations about BSDEs and in particular we will focus on Reflected BSDEs and their relation with the optimal stopping theory, in order to finally show the above result. We refer the reader to Appendix A for some basic notations about stochastic calculus.

In the sequel, we present some useful facts about the standard theory of BSDEs driven by Brownian motion, mainly following the discussion in [25]. Let $(W_t)_{0\leq t\leq T}$ be a d-dimensional standard Brownian motion defined in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ is the natural augmented filtration of W, i.e.

$$\mathcal{F}_t^0 = \sigma\left(W_s, 0 \leq s \leq t\right), \quad \mathcal{F}_t = \sigma\left(\mathcal{F}_t^0, \mathcal{N}\right), \tag{1.2.1}$$

where $\mathcal{N}$ denotes all negligible sets of $\mathcal{F}$. It is well known that such a construction provides a *right-continuous* filtration, hence the so-called *usual conditions* hold. We denote by

- $\mathbb{L}^2(\Omega, \mathcal{F}_T)$ the set of $\mathbb{R}$-valued, $\mathcal{F}_T$-measurable random variables x such that $\|x\|_{L^2(\Omega,\mathcal{F}_T)} := \mathbb{E}\left[|x|^2\right] < \infty$.
- $\mathbb{S}^2$ the set of $\mathbb{R}$-valued, $\mathbb{F}$-adapted and continuous processes $Y = (Y_t)_{0\leq t\leq T}$ such that $\|Y\|_{S^2} := \left(\mathbb{E}\left[sup_{0\leq t\leq T}|Y_t|^2\right]\right)^{\frac{1}{2}} < \infty$.
- $\mathbb{H}^2$ the set of $\mathbb{R}^d$-valued, $\mathbb{F}$-predictable processes $Z = (Z_t)_{0\leq t\leq T}$ such that $\|Z\|_{H^2} := \left(\mathbb{E}\left[\int_0^T |Z_t|^2\, dt\right]\right)^{\frac{1}{2}} < \infty$.

As far as measurability properties are concerned, we emphasize on the predictability property of both the processes Y and Z, as it is a necessary feature in the developments of the next chapters.
(Hypotesis H0): We are given a pair (ξ, f), called terminal condition and generator, such that

- $\xi : \Omega \to \mathbb{R}$, is an $\mathcal{F}_T$-measurable random variable $\in \mathbb{L}^2(\Omega, \mathcal{F}_T)$;
- f: $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a map satisfying
  - $\forall\ (y, z) \in \mathbb{R} \times \mathbb{R}^d$, $(\omega, t) \to f(\omega, t, y, z)$ is an $\mathbb{F}$-progressive process;
  - $(\omega, t) \to f(\omega, t, 0, 0) \in \mathbb{H}^2$;
  - for some $K > 0$ and all $y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^d$,

$$|f(t, y, z) - f(t, y', z')| \leq K(|y - y'| + |z - z'|), dP \times dt - a.s. \tag{1.2.2}$$

Let us consider the following BSDE:

$$-dY_t = f(t, Y_t, Z_t) - Z_t dW_t, \;\; Y_T = \xi. \tag{1.2.3}$$

We notice that in the Backward framework a terminal condition is given, and the solution is intended as a *pair* of stochastic processes (Y, Z).

Definition 1.2.1. A solution to the BSDE is a pair $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$ satisfying:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \ , \ 0 \leq t \leq T. \tag{1.2.4}$$

We note that, in its construction, Y_0 is $\mathcal{F}_0$-adapted, and with $\mathcal{F}_0$ being the trivial σ-fields, it follows that Y_0 is almost surely constant (with respect to P), thus deterministic.
We state an existence and uniqueness result for the above BSDE.

Theorem 1.2.1. *Given a pair (ξ, f) satisfying **Hypotesis H0**, there exists a unique solution $(Y, Z) \in \mathbb{S}^2 \times \mathbb{H}^2$ to the BSDE:*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

Let us now introduce the class of Reflected Backward Stochastic Differential Equations written in the form

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \ 0 \leq t \leq T, \tag{1.2.5}$$

with the constraint

$$Y_t \geq h_t, \ 0 \leq t \leq T, \ dP - a.s. \ on \ \Omega. \tag{1.2.6}$$

We shall impose the following assumptions:

- the pair (ξ, f), where $\xi : \Omega \to \mathbb{R}$, f: $\Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfies **(Hypothesis H0)**
- $h : \Omega \times [0, T] \to \mathbb{R}$ is an $\mathbb{F}$-adapted and continuous process satisfying $\xi \geq h_T$ and such that $\mathbb{E}\left[sup_{0 \leq t \leq T} |h_t|^2\right] < \infty$.

The Reflected BSDE is the case where the solution is forced to stay above a given stochastic process, which we have called obstacle. An increasing process is introduced which pushes the solution upwards, so that it may remain above the obstacle.

Definition 1.2.2. A minimal solution to the Reflected BSDE with a terminal condition-generator (ξ, f) and an obstacle h is a triple (Y, Z, K) of adapted processes valued respectively in $\mathbb{R}, \mathbb{R}^d, \mathbb{R}_+$ such that :

- $Z \in \mathbb{H}^2$, in particular, $\mathbb{E}\left[\int_0^T |Z_t|^2 \, dt\right] < \infty$;

- $Y \in \mathbb{S}^2$, in particular, $\mathbb{E}\left[sup_{0\leq t\leq T}|Y_t|^2\right] < \infty$;
- $K_T \in \mathbb{L}^2(\Omega, \mathcal{F}_T)$, where $K = (K_t)_{0\leq t\leq T}$ is a continuous and increasing process such that $K_0 = 0$;
- (Y, Z, K) satisfy (1.2.5), (1.2.6), with the *Skohorod* condition

$$\int_0^T (Y_t - h_t)\, dK_t = 0. \tag{1.2.7}$$

The Skohorod condition means that the push of the increasing process K is active only when the constraint is saturated, i.e. when $Y_t = h_t,\ 0 \leq t \leq T$. This condition can be viewed as a *minimality* feature (for a more detailed explanation we refer the reader to Remark 6.5.2 in [25]).

Existence and approximation via penalization

In this section we analyse the solvability properties of the RBSDE (1.2.5), (1.2.6), in particular we focus the study on the existence and uniqueness results. Existence is established in literature via two different approximation schemes: the first is a Picard-type iterative procedure, which relies on the notion of the *Snell envelope*, while the second is constructed by penalization of the constraint. We focus on the latter, as we will refer to such a scheme in the next chapters.
For each $n \in \mathbb{N}$, we consider the following sequence of penalized BSDEs, where the reflection constraint is relaxed.

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds + n\int_t^T (Y_s^n - h_s)^- ds - \int_t^T Z_s^n dW_s,\ 0 \leq t \leq T, \tag{1.2.8}$$

The function $(y - h_t)^-$ denotes the *negative part* of $y - h_t$. We note that the BSDE can be rewritten as

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s, \quad 0 \leq t \leq T, \tag{1.2.9}$$

where $f_n(t, y, z) = f(t, y, z) + n(y - h_t)^-$, for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$. From the *negative part* function properties, and the requirements on f, we have that the map f_n still satisfies the **Hypothesis H0**, then, by theorem 1.2.1, there is a unique solution (Y^n, Z^n) to the BSDE (1.2.8). We define the process

$$K_t^n = n\int_0^t (Y_s^n - h_s)^- ds, \tag{1.2.10}$$

which is, by construction, continuous, increasing, and square-integrable. It holds the following

Lemma 1.2.1. *Under* ***Hypotesis H0****, there exists a costant C, which depend only on T, such that*

$$\|Y^n\|_{S^2} + \|Z^n\|_{H^2} + \|K^n_T\|_{L^2(\Omega,\mathcal{F}_T)} \leq C, \text{ for all } n \in \mathbb{N}.$$

By exploiting the above condition, it has been proved that the sequence $(Y^n, Z^n, K^n)_n$ converges to the minimal solution of the RBSDE in (1.2.5), (1.2.6), as stated in the following

Theorem 1.2.2. *Let K^n be defined as in* (1.2.10)*, and let (Y^n, Z^n) be the solution to the penalized BSDE* (1.2.8)*, for each $n \in \mathbb{N}$. Then there exists a unique minimal solution (Y, Z, K) that satisfies the RBSDE* (1.2.5), (1.2.6)*, with the Skohorod condition* (1.2.7)*. Respectively, Y is the limit of Y^n in $\mathbb{S}^2$, Z is the limit of Z^n in $\mathbb{H}^2$, K is the limit of K^n in $\mathbb{S}^2$, that is*

$$\lim_{n\to\infty} \mathbb{E}\left[sup_{0\leq t\leq T}|Y^n_t - Y_t|^2 + \int_0^T |Z^n_t - Z_t|^2\,dt + sup_{0\leq t\leq T}|K^n_t - K_t|^2\right] = 0.$$

An explicit optimal stopping time representation

We can finally show how the notion of Reflected BSDEs is directly related to optimal stopping problems, by means of the following

Proposition 1.2.1. *Let (Y, Z, K) be a* minimal *solution to the RBSDE* (1.2.5), (1.2.6), (1.2.7)*. Then for all $t \in [0, T]$, it holds that*

$$Y_t = ess\sup_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\int_t^{\tau\wedge T} f(s, Y_s, Z_s)\,ds + h_\tau 1_{\tau<T} + \xi 1_{\tau\geq T}|\mathcal{F}_t\right] \tag{1.2.11}$$

where $\mathcal{T}$ is the set of all stopping times with respect to the filtration $\mathbb{F}$ and

$$\mathcal{T}_t = \{\tau \in \mathcal{T}; t \leq \tau < \infty\}.$$

Let us recall now the formulation in (1.1.1), (1.1.2). Let us observe that, in the case where the generator f does not depend on y, z, one can obtain both a *closed formula* for Y and an explicit solution for the optimal stopping problem. More precisely, when the triple (ξ, f, h) satisfies

- $\xi : \Omega \to \mathbb{R} \in \mathbb{L}^2(\Omega, \mathcal{F}_T)$;
- $f : \Omega \times [0, T] \to \mathbb{R} \in \mathbb{H}^2$;
- $h : \Omega \times [0, T] \to \mathbb{R}$ $\mathbb{F}$-adapted and continuous process satisfying $h_T \leq \xi$ and such that $\mathbb{E}\left[sup_{0\leq t\leq T}|h_t|^2\right] < \infty$;

we have for all $t \in [0,T]$,

$$Y_t = ess \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^{\tau \wedge T} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau \geq T} | \mathcal{F}_t\right], \tag{1.2.12}$$

where Y is the component of the triple (Y, Z, K), satisfying

$$Y_t = \xi + \int_t^T f_s ds + K_T - K_t - \int_t^T Z_s dW_s,\ 0 \leq t \leq T, \tag{1.2.13}$$

$$Y_t \geq h_t,\ 0 \leq t \leq T,\ dP - a.s., \tag{1.2.14}$$

again with the *Skohorod* condition

$$\int_0^T (Y_t - h_t)\, dK_t = 0. \tag{1.2.15}$$

Chapter 2

Optimal stopping and Point Processes

In the first part of this chapter we will provide some general features about point processes and their analysis using the martingale approach. Such information will allow us to point out in the rest of the chapter an *alternative formulation* for the optimal stopping problem presented above. By introducing a very elementary point process independent from Brownian motion, we will design a new setting for the problem, which differs from the original one in the sense that the probability space is not fixed 'a priori', as it is uniquely defined only once we have chosen among a family of equivalent probability measures.

2.1 Marked Point Process and Intensity measures

The following section is devoted to the preliminaries in marked point processes and finite variation processes. For clarity of exposition, all the results are stated in their original version, following the discussion in [2], however, we will use them in the simplified context of *one-point processes*. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration defined on the probability space $(\Omega, \mathcal{F}, P)$, we assume that $\mathbb{F}$ verifies the usual conditions. Here we first emphasize the notion of *predictable processes* (see also Appendix A).

Definition 2.1.1. (Predictable process). A process X_t (here we mean $(X_t)_{t\geq 0}$), whose mapping $(t, \omega) \to X_t(\omega)$ is measurable on $([0, \infty) \times \Omega)$ equipped with the σ-field generated by the $\mathbb{F}$-adapted and continuous processes is called $\mathbb{F}$-*predictable.*

In our application, we will only find *predictable processes* which are $\mathbb{F}$- adapted and *left-continuous*, as suggested by the following

Theorem 2.1.1. *An $\mathbb{R}^n$-valued process X_t adapted to $\mathbb{F}$ and left-continuous is $\mathbb{F}-$ predictable.*

Following the article [16], we give the pair $(E, \mathcal{E})$, where E is a Borel subset of a compact metric space and $\mathcal{E}$ consists of the Borelians of E, and we introduce the *E-marked point process* on Ω as a sequence $(T_n, X_n)_{n\geq 1}$ of random variables, where

- each T_n is a stopping time, and $T_n \leq T_{n+1}$;
- each X_n takes value in $(E, \mathcal{E})$, X_n is $\mathcal{F}_{T_n}$-measurable;
- if $T_n < \infty$, then $T_n < T_{n+1}$;

and $T_0 = 0$. We define the *random measure* $\mu(dt, dx)$ as a positive transition measure from $(\Omega, \mathcal{F})$ over $(E, \mathcal{E})$ satisfying

$$\mu(\omega, dt, dx) = \sum_{n\geq 1} 1_{T_n(\omega)<\infty} \delta_{T_n(\omega), X_n(\omega)}(dt, dx), \tag{2.1.1}$$

which we also call by abuse of notation an E-marked point process. We can associate a *counting process* with this realization, of the form $N_t(A) = \mu((0, t] \times A), with\ A \in \mathcal{E}$. When E reduces to one point, the marked point process reduces to a point process on $(0, \infty)$, which is completely described by N_t. In this context, the following definition is given.

Definition 2.1.2. ($(P, \mathcal{F}_t)$ Stochastic intensity for point process). Let N_t be a point process adapted to $\mathbb{F}$, and let λ_t be a nonnegative $\mathbb{F}$- progressive process such that for all $t \geq 0$

$$\int_0^t \lambda_s ds < \infty \quad P - a.s. \tag{2.1.2}$$

If for all nonnegative $\mathbb{F}$-predictable processes H_t, the equality

$$\mathbb{E}\left[\int_0^\infty H_s dN_s\right] = \mathbb{E}\left[\int_0^\infty H_s \lambda_s ds\right] \tag{2.1.3}$$

is verified, then we say: N_t admits the *$(P, \mathcal{F}_t)$-intensity* λ_t.

When it exists, a well-known result shows that it is always possible to define a predictable version of the stochastic intensity λ_t. Coming back to the E-marked point process defined above, we introduce further objects:

Definition 2.1.3. ($(P, \mathcal{F}_t)$ Intensity kernel for marked point process). Let $\mu(dt, dx)$ be an E-marked point process adapted to $\mathbb{F}$. Suppose for each $A \in \mathcal{E}$, $N_t(A)$ admits the $(P, \mathcal{F}_t)$- predictable intensity $\lambda_t(A)$, where $\lambda_t(\omega, dx)$ is a transition measure from $(\Omega \times [0, \infty), \mathcal{F} \times \mathcal{B}^+)$ into $(E, \mathcal{E})$. We then say that $\mu(dt, dx)$ admits the $(P, \mathcal{F}_t)$-*intensity kernel* $\lambda_t(dx)$.

Definition 2.1.4. (($P, \mathcal{F}_t$) Local characteristics of $\mu(dt, dx)$). Let $\mu(dt, dx)$ be a E-marked point process adapted to $\mathbb{F}$, with $(P, \mathcal{F}_t)$-intensity kernel $\lambda_t(dx)$ of the form

$$\lambda_t(dx) = \lambda_t \Phi_t(dx) \tag{2.1.4}$$

where λ_t is a non-negative $\mathbb{F}$-predictable process and $\Phi_t(\omega, dx)$ is a probability transition kernel from $(\Omega \times [0, \infty), \mathcal{F} \times \mathcal{B}^+)$ into $(E, \mathcal{E})$. Then $(\lambda_t, \Phi_t(dx))$ is called $(P, \mathcal{F}_t)$*-local characteristics* of $\mu(dt, dx)$.

We observe that, since $\Phi_t(dx)$ is a probability, then $\Phi_t(E) = 1$, hence we obtain the $(P, \mathcal{F}_t)$-intensity $\lambda_t = \lambda_t(E)$ for the counting process $N_t = N_t(E)$.
We now state two fundamental results for E-marked point processes.

Theorem 2.1.2. *(Projection theorem). Let $\mu(dt, dx)$ be an E-marked point process with $(P, \mathcal{F}_t)$-intensity kernel $\lambda_t(dx)$. Then for each non-negative $\mathbb{F}$-predictable process $H_t(x)$, it holds*

$$\mathbb{E}\left[\int_0^\infty \int_E H_s(x)\mu(ds, dx)\right] = \mathbb{E}\left[\int_0^\infty \int_E H_s(x)\lambda_s(dx)ds\right]. \tag{2.1.5}$$

Corollary 2.1.1. *(Integration theorem). Let $\mu(dt, dx)$ be a E-marked point process with $(P, \mathcal{F}_t)$-intensity kernel $\lambda_t(dx)$. Let $H_t(x)$, be a nonnegative $\mathbb{F}$-predictable process such that, for all $t \geq 0$,*

$$\mathbb{E}\left[\int_0^t \int_E H_s(x)\lambda_s(dx)ds\right] < \infty, \tag{2.1.6}$$

$$\left[\int_0^t \int_E H_s(x)\lambda_s(dx)ds < \infty \quad dP - a.s.\right], \tag{2.1.7}$$

defining $\tilde{\mu}(dt, dx) = \mu(dt, dx) - \lambda_t(dx)ds$,

$$\int_0^t \int_E H_s(x)\tilde{\mu}(ds, dx) \tag{2.1.8}$$

is a $(P, \mathcal{F}_t)$-martingale [local martingale].

The latter result suggests a systematic analysis of point processes with the martingale approach. Neverthless, as we can see in the literature, the existence of intensity kernel $\lambda_t(dx)$ (stochastic intensity λ_t) is not granted for marked point processes (point processes). For this reason, we need to point out a general result holding true even in such cases where intensities are not admitted.
We first look at the less general case of simple point processes: we denote A_t as the *compensator* of N_t, or equivalently, dA_t as the *dual predictable projection* of $\mu(dt)$, which satisfies the following

Theorem 2.1.3. *Let N_t be a point process adapted to the filtration $\mathbb{F}$. Then there exists a unique, right-continuous $\mathbb{F}$-predictable nondecreasing process A_t satisfying $A_0 = 0$ such that, for all H_t $\mathbb{F}$-predictable non-negative processes, it holds that*

$$\mathbb{E}\left[\int_0^\infty H_s dN_s\right] = \mathbb{E}\left[\int_0^\infty H_s dA_s\right]. \tag{2.1.9}$$

We will always assume A_t to be continuous. Moreover, In the case where N_t is non-explosive (i.e. when $T_n \to \infty$) and A_t is absolutely continuous with respect to the Lebesgue measure, that is

$$A_t = \int_0^t \lambda_s ds,\ t \geq 0$$

we have directly the characterization of the stochastic intensity λ_t. We give the following result, as a specific case of corollary 2.1.1:

Theorem 2.1.4. *If N_t admits the $(P, \mathcal{F}_t)$-intensity λ_t, then N_t is non-explosive and*

- $\tilde{\mu}((0,t]) = \mu((0,t]) - \int_0^t \lambda_s ds$ *is a $(P, \mathcal{F}_t)$-local martingale;*
- *if X_t is an $\mathbb{F}$-predictable process such that $\mathbb{E}[\int_0^t |X_s| \lambda_s ds] < \infty,\ t \geq 0,$ then $\int_0^t X_s \tilde{\mu}(ds)$ is a $(P, \mathcal{F}_t)$- martingale;*
- *if X_t is an $\mathbb{F}$-predictable process such that $\int_0^t |X_s| \lambda_s ds] < \infty,\ t \geq 0,\ P - a.s.$ then $\int_0^t X_s \tilde{\mu}(ds)$ is a $(P, \mathcal{F}_t)$- local martingale;*

We finally extend the above result to the E-marked point process $\mu(dt, dx)$. There always exists a function $\phi_t(\omega, dx), x \in E$ such that

- for every $\omega \in \Omega, t \in [0, \infty)$, the mapping $B \to \phi_t(\omega, B)$, $B \in \mathcal{E}$, is a probability measure on $(E, \mathcal{E})$;
- for every $B \subset E$, the process $(\omega, t) \to \phi_t(\omega, B)$ is $\mathbb{F}$-predictable;
- for every nonnegative $\mathbb{F}$-predictable process $H_t(x)$ we have

$$\mathbb{E}\left[\int_0^\infty \int_E H_s(x)\mu(dt, ds)\right] = \mathbb{E}\left[\int_0^\infty \int_E H_s(x)\phi_s(dx)dA_s\right]. \tag{2.1.10}$$

In the same way, we denote $\phi_t(dx)dA_t$ as the *dual predictable projection* of $\mu(dt, dx)$.

2.2 The stopping problem

In the sequel, we go back to the optimal stopping problem, with the goal of providing an original formulation by using the tools of point process theory. For this purpose, we deal with a very elementary class of point processes that we call *one-point processes.*

The randomization approach

Let us take a probability space $(\Omega', \mathcal{F}', P')$, and then introduce a random variable $\eta : \Omega' \to (0, \infty)$, exponentally distributed with mean 1, i.e. $\eta \sim exp(1)$. We define

$$N_t(\omega') := 1(\eta(\omega') \leq t), \;\; 0 \leq t \leq T, \tag{2.2.1}$$

as the process taking a null value before the occurrence of the *random time* η, and a unitary value after that time. We call $N = (N_t)_{0 \leq t \leq T}$ *one-point process* and one can easily note that it is a counting process, observing that it describes the random measure $\mu(\omega', dt)$ in (2.1.1), when the choice of stopping times $(T_n)_{n \geq 1}$ is made as $\{T_1(\omega') := \eta(\omega'),\; T_n(\omega') := \infty \; \forall n > 1\}$. We observe that, in this case, the random measure assumes the form

$$\mu(\omega', dt) = \delta_{\eta(\omega')}(dt), \quad 0 \leq t \leq T. \tag{2.2.2}$$

We then introduce $\mathbb{F}' = (\mathcal{F}'_t)_{0 \leq t \leq T}$ as the natural augmented filtration of N, i.e.

$$\mathcal{F}_t^{0'} = \sigma(N_s, 0 \leq s \leq t), \;\; \mathcal{F}_t^{'} = \sigma(F_t^{0'}, \mathcal{N}), \tag{2.2.3}$$

where $\mathcal{N}$ denotes all the negiglible sets of $\mathcal{F}'$ (it is a well-known result that such a filtration satisfies the *usual conditions*). Clearly, N is adapted to its filtration, then, by recalling theorem 2.1.3, there is a compensator $A = (A_t)_{0 \leq t \leq T}$, i.e. a non-decreasing (continuous) $\mathbb{F}'$-predictable process satisfying condition (2.1.9), which is expressed as

$$A_t = t \wedge \eta, \quad 0 \leq t \leq T. \tag{2.2.4}$$

As far as the specific form of the compensator is concerned, we merely say it is pointed out by a more general result given in [16], where an explicit form for the dual predictable projections of marked point processes is provided. We note that, by construction, A is absolutely continuous with respect to the Lebesgue measure, i.e.

$$A_t = \int_0^t dA_s = \int_0^t 1_{[0,\eta)}(s)\, ds, \;\; 0 \leq t \leq T, \tag{2.2.5}$$

hence the stochastic intensity $\lambda = (\lambda_t)_{0 \le t \le T}$ for the *one-point process* can be directly drawn as:

$$\lambda_t = 1_{[0,\eta)}(t), \quad 0 \le t \le T. \tag{2.2.6}$$

In particular, by looking at theorem 2.1.3, we get from condition (2.1.9)

$$\mathbb{E}\left[\int_0^T H_t \mu(dt)\right] = \mathbb{E}\left[\int_0^T H_t \lambda_t dt\right], \tag{2.2.7}$$

for all non-negative $\mathbb{F}'$-predictable processes $H = (H_t)_{0 \le t \le T}$ (we remind the reader that for point processes it holds $dN_t = \mu(dt)$). We finally observe that η is, by construction, a stopping time with respect to the filtration $\mathbb{F}^{'}$.
Now, we look at the previous optimal stopping problem

$$V_t = ess \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^{\tau \wedge T} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau \ge T} | \mathcal{F}_t\right], \ 0 \le t \le T. \tag{2.2.8}$$

Let us suppose that we enlarge the filtration from the one generated by W to the one generated by both W and N, which are independent. Then η can be viewed as a *randomization* of the original stopping time τ. The basic idea is the following: instead of explicitly choosing an optimal stopping time in $\mathcal{T}_t$, we randomize it by the introduction of a one-point process, and then we control it indirectly by choosing its stochastic intensity. A change in stochastic intensity will be obtained as a consequence of a change in probability measures, by means of the Girsanov theorem. In this way an alternative, weaker formulation will be given, where we keep the unique filtration fixed.
Inspired by the above discussion, we now design a qualitative formulation for the value function of the optimal stopping problem. That is, denoting with Λ the generic set of stochastic intensities for the process N, we look for a value function of the form

$$\bar{V}_t = ess \sup_{\lambda \in \Lambda} J(\lambda, t), \ 0 \le t \le T \tag{2.2.9}$$

where

$$J(t, \lambda) = \mathbb{E}^\lambda\left[\int_t^{\eta \wedge T} f_s ds + h_\eta 1_{t \le \eta < T} + \xi 1_{\eta \ge T} | \mathcal{F}_t\right], \ 0 \le t \le T. \tag{2.2.10}$$

Thus, the optimal stochastic intensity $\lambda^* \in \Lambda$, will characterize the probability measure P^{λ^*} and the associated conditional expectation $\mathbb{E}^{\lambda^*}$ through which the functional gain is maximized.
The rest of the section is devoted to the formal derivation of (2.2.9). In the sequel, we introduce the fundamental result in the change of intensities 'à la Girsanov', which is stated below in the specific context of point processes.

Theorem 2.2.1. *Let $N = (N_t)_{0\leq t\leq T}$ be a point process adapted to a filtration $\mathbb{F}$, and let $\lambda = (\lambda_t)_{0\leq t\leq T}$, be the predictable stochastic intensity of N. Let ν be the $\mathbb{F}$-predictable process, non-negative, and such that for all $0 \leq t \leq T$,*

$$\int_0^t \nu_s \lambda_s ds < \infty \quad P - a.s. \tag{2.2.11}$$

Define the process $L = (L_t)_{0\leq t\leq T}$ by

$$L_t = \left(\prod_{n\geq 1} \nu_{T_n} 1(T_n \leq t)\right) exp\left\{\int_0^t (1-\nu_s)\lambda_s ds\right\}, \quad 0 \leq t \leq T. \tag{2.2.12}$$

Then L is a P-non-negative local martingale and a P-supermartingale over $[0,T]$, satisfying

$$L_t = 1 + \int_0^t L_{s^-}(1-\nu_s)\tilde{\mu}(ds), \ 0 \leq t \leq T, \tag{2.2.13}$$

where $\tilde{\mu}(dt) = \mu(dt) - \lambda_t dt$ is the compensated measure according to the given probability P.

Corollary 2.2.1. (Direct Radon-Nikodym-derivative theorem). *The same notation as in the previous theorem. Suppose moreover that $E[L_T] = 1$. Define the probability measure $\tilde{P}$ by*

$$\frac{d\tilde{P}}{dP} = L_T. \tag{2.2.14}$$

Then N has the stochastic intensity $\tilde{\lambda}_t = \nu_t \lambda_t, 0 \leq t \leq T$, under $\tilde{P}$.

Let us conclude this section with some important notions about semimartingales. First of all, we recall from Appendix A, the definition of *finite variation process* and *quadratic variation process*. In particular, given a function x (mapping $t \in [0,T]$ into $x(t)$), its *variation* V is defined as

$$V_s^t = \sup \sum_{n=1}^N |x(t_n) - x(t_{n-1})|, \quad 0 \leq s < t \leq T,$$

where *sup* is taken over all the subdivisions $s = t_0 < t_1 < \cdots < t_N = t$. If $V_0^T < \infty$, then x is said to be a *finite variation function.*

Definition 2.2.1. (Finite variation process). We say that a process $M = (M_t)_{0\leq t\leq T}$ has finite variation if every path is càd-làg and has finite variation.

Definition 2.2.2. (Quadratic Variation for continuous local martingales). Let $M = (M_t)_{0\leq t\leq T}$ and $N = (N_t)_{0\leq t\leq T}$ be two continuous local martingales, then $< M, N >= (< M, N >_t)_{0\leq t\leq T}$ is the unique process such that $MN- < M, N >$ is a local martingale. In particular, $< M, M >$ is an increasing process, which is also denoted simply by $< M >$. Moreover, for every $0 \leq t \leq T$,

$$< M, M >_t = \lim_{\|t^n\|\to 0} \sum_{k=1}^{n} \left(M_{t_k^n} - M_{t_{k-1}^n} \right)^2 \quad \textit{in Probability.} \tag{2.2.15}$$

where t^n ranges over partitions of the interval [0, t].

Now, we see how the definition of quadratic variation can be extended to the more general case of local (discontinuous) martingale.

Definition 2.2.3. (Quadratic Variation for local martingales). Let $M = (M_t)_{0\leq t\leq T}$ and $N = (N_t)_{0\leq t\leq T}$ be two local martingales, M^c, N^c their continuous martingale parts respectively, define $[M, N]_t = M_0N_0+ < M^c, N^c >_t + \sum_{s\leq t}(\Delta M_s \Delta N_s),\ 0 \leq t \leq T$, where $\Delta M_s = M_s - M_{s^-}$. Then $[M, N] = ([M, N]_t)_{0\leq t\leq T}$ is an adapted finite variation process. In particular, $[M, M]$ is an adapted increasing process. It is also denoted simply by $[M]$.

In the case where M is itself a finite variation local martingale null in the origin, then it can be proved that

$$[M]_t = \sum_{s\leq t} (\Delta M_s)^2,\ 0 \leq t \leq T, \tag{2.2.16}$$

i.e. it is the sum of the square of its jumps.
Observe finally that in the case where M is a continuous martingale, then it holds that $[M] =< M, M >$.

Theorem 2.2.2. (Burkholder-Davis-Gundy). *Let $p \in \mathbb{R}, p \geq 1$. There exist some constants $0 < c_p < C_p < \infty$, such that, for every local martingale $M = (M_t)_{0\leq t\leq T}$ with $M_0 = 0$ and all stopping times $\tau \in \mathcal{T}_t^T$, we have*

$$c_p\mathbb{E}\left[[M]_\tau^{p/2} \right] \leq \mathbb{E}\left[sup_{0\leq t\leq \tau} |M_t| \right]^p \leq C_p\mathbb{E}\left[[M]_\tau^{p/2} \right]. \tag{2.2.17}$$

Furthermore, for continuous local martingales, this statement holds for all $0 < p < \infty$, and clearly $[M]_t =< M, M >_t,\ 0 \leq t \leq T$.

A weak formulation

Let η, N, A, λ be respectively the random time, the one-point process, the associated compensator and stochastic intensity of N, as defined in the previous section. Our goal is to obtain a formal characterization for the value function in (2.2.9). A new filtration turns out to be necessary which takes account of both the processes N and W. Recalling the two independent probability spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$, we define the *product space* $(\bar{\Omega}, \bar{\mathcal{F}})$ as

$$\bar{\Omega} = \Omega \times \Omega', \quad \bar{\mathcal{F}} = \mathcal{F} \otimes \mathcal{F}',$$

where the *product σ-field* $\mathcal{F} \otimes \mathcal{F}'$ is the one generated by rectangles of the form $A \times B$, with $A \in \mathcal{F}$, $B \in \mathcal{F}'$, i.e.

$$\mathcal{F} \otimes \mathcal{F}' = \sigma\left(\{A \times B,\ \textit{such that}\ A \in \mathcal{F}, B \in \mathcal{F}'\}\right), \tag{2.2.18}$$

endowed with the product measure $\bar{\mathbb{P}}$,

$$\bar{\mathbb{P}} = P \times P'.$$

We construct the filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \le t \le T}$ as the *usual completion* of the *product filtration* $(\bar{\mathcal{F}}^0_t)_{0 \le t \le T}$, that is

$$\bar{\mathcal{F}}_t = \sigma(\bar{\mathcal{F}}^0_{t+}, \mathcal{N}), \tag{2.2.19}$$

where

$$\bar{\mathcal{F}}^0_t = \mathcal{F}^0_t \otimes \mathcal{F}^{0'}_t = \sigma\left(\{A_t \times B_t,\ \textit{such that}\ A_t\ \in \mathcal{F}^0_t, B_t \in \mathcal{F}^{0'}_t\}\right), \tag{2.2.20}$$

while $\mathcal{N}$ denotes all the negligible sets of $\bar{\mathcal{F}}$. In this way we exploit all the results from product measure theory. Therefore, given any object belonging to the original filtered spaces $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $(\Omega', \mathcal{F}', \mathbb{F}', P')$, we know it preserves the same measurability and integrability properties w.r.t. $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$. For example, we observe that η preserves the property of being a stopping time.
We can now apply the Girsanov theorem (2.2.1) in the current setting.
Given $\mathcal{V}$, a set of ν $\mathbb{F}$-predictable processes valued in $(0, \infty)$ and essentially bounded, we define $\mathcal{L}^\nu = (\mathcal{L}^\nu_t)_{0 \le t \le T}$, for any $\nu \in \mathcal{V}$, as the *Doléans-Dade exponential local martingale* with respect to the martingale $L^\nu_\cdot = \int_0^\cdot (\nu_t - 1)\tilde{\mu}(dt)$ (we know it is a martingale from theorem 2.1.4), i.e.

$$\mathcal{L}^\nu_t = \mathcal{E}\left(L^\nu_\cdot\right)_t, \tag{2.2.21}$$

which can be written as

$$\mathcal{L}^\nu_t = exp\left(\int_0^t ln\ \nu_s \mu(ds)\ -\ \int_0^t (\nu_s - 1)\lambda_s ds\right),\ 0 \le t \le T. \tag{2.2.22}$$

Based on corollary 2.2.1, when $\mathcal{L}^\nu$ is a true martingale, i.e. when $\mathbb{E}[\mathcal{L}^\nu_T] = 1$ (looking at (2.2.13)), a probability measure $\mathbb{P}^\nu$ is defined, equivalent to $\mathbb{P}$ on $(\Omega, \bar{\mathcal{F}}_T)$ with Radon Nikodym density:

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}|_{\bar{\mathcal{F}}_t}\ =\ \mathcal{L}^\nu_t,\ 0 \le t \le T, \tag{2.2.23}$$

and we denote with $\mathbb{E}^\nu$ the expectation operator under $\mathbb{P}^\nu$. Moreover, by recalling that we have

$$dA_t = \lambda_t dt, \ 0 \le t \le T, \tag{2.2.24}$$

the effect of the Girsanov theorem is to change the compensator dA_t of dN_t under $\mathbb{P}$ to $\nu_t dA_t$ under $\mathbb{P}^\nu$ (defining a new compensated random measure as $\tilde{\mu}^\nu(dt) := \mu(dt) - \nu_s \lambda_s ds$). We note that, in this way, we choose the stochastic intensity of N by means of the equivalent change in probability measures induced by the choice of ν.
When looking at the continuous part, we have the following

Proposition 2.2.1. *Let us suppose $\mathcal{L}^\nu$ is a martingale. Then W_t remains a Brownian Motion under the equivalent probability measure $\mathbb{P}^\nu$.*

Proof. We want to use the Levý characterization theorem according to the measure of interest $\mathbb{P}^\nu$.

Theorem 2.2.3. (Levý characterization theorem). *Let $M = (M_t)_{0\le t\le T}$ be such that*

- *M is a continuous local martingale with $M_0 = 0$;*
- *M has quadratic variation $< M, M >$ such that, for all $0 \le t \le T$, it holds that $< M, M >_t = t$;*

Then M is a standard Brownian Motion.

1). We will prove the first requirement by using the Girsanov theorem in a general version for local martingales. By recalling definition 2.2.3, we can easily prove that the quadratic variation $[W, \mathcal{L}^\nu]$, with respect to the measure $\mathbb{P}$, assumes a null value. (Based on the fact that W is a continuous Lévy process, both W and $\mathcal{L}^\nu$ are null in zero, and $\mathcal{L}^\nu$ is only built by a drift and a pure jump component, i.e. $\mathcal{L}^\nu_c = 0$). Let us suppose $\mathcal{L}^\nu$ to be square-integrable (we will give some detailed proof later). Then, following the results in section 6.4 [15], $< W, \mathcal{L}^\nu >$ exists and is identified as the *dual predictable projection* of $[W, \mathcal{L}^\nu]$. Moreover, since we have just observed that

$$[W, \mathcal{L}^\nu] = 0, \tag{2.2.25}$$

it can be proved that

$$< W, \mathcal{L}^\nu >= 0. \tag{2.2.26}$$

Therefore, recalling that $\mathcal{L}^\nu$ is the density process of $\mathbb{P}^\nu$ w.r.t. $\mathbb{P}$, we get by theorem 10.14 (Girsanov for local martingales) [15], that the process W^ν, defined as

$$W^\nu_t = W_t - \int_0^t \frac{1}{\mathcal{L}^\nu_{s-}} d < W, \mathcal{L}^\nu >_s, \ \ 0 \le t \le T,$$

is a $\mathbb{P}^\nu$-local martingale, and clearly $W_0^\nu = 0$. Moreover, by recalling (2.2.26), it holds $\mathbb{P} - a.s.$ $W_t^\nu = W_t$, $0 \le t \le T$, then it is also continuous.
2). We observe that, for the quadratic variation in the case of Brownian Motion, it holds as a limit in $\mathbb{P}$-probability (see(A.3.2))

$$< W, W >_t = t. \tag{2.2.27}$$

(Based on the fact that $W_t^2 - t$ is a martingale). Hence, since $\mathbb{P}^\nu$ is equivalent to $\mathbb{P}$, we get

$$< W^\nu, W^\nu >_t = (< W, W >_t) = t, \tag{2.2.28}$$

where $< W^\nu, W^\nu >$ is the quadratic variation of W^ν under $\mathbb{P}^\nu$.

It remains to check that $\mathcal{L}^\nu$ is effectively a (square-integrable) $\mathbb{P}$-martingale.
From (2.2.13), we write $\mathcal{L}^\nu$ as $\mathcal{L}_t^\nu = 1 + \int_0^t \mathcal{L}_{s^-}^\nu(\nu_s - 1)\tilde{\mu}(ds), 0 \le t \le T$. Thus, by recalling hypothesis in theorem 2.1.4, the following condition turns out to be sufficient:

$$\mathbb{E}\left[\int_0^T \left|\mathcal{L}_{t^-}^\nu(\nu_t - 1)\right| \lambda_t dt\right] < \infty. \tag{2.2.29}$$

Proof. We know ν and λ to be (essentially) bounded. Then there exists a constant $M > 0$ such that

$$M = ess\sup_{\bar{\Omega}} \int_0^T |\nu_t - 1| \lambda_t dt. \tag{2.2.30}$$

Hence by using Holder's inequality, we get the integrability result

$$\mathbb{E}\left[\int_0^T \left|\mathcal{L}_t^\nu(\nu_t - 1)\right| \lambda_t dt\right] \le M \int_0^T \mathbb{E}\left[\left|\mathcal{L}_t^\nu\right|\right] dt \le MT. \tag{2.2.31}$$

Where we have used the fact that $\mathcal{L}_t^\nu$ is a supermartingale and $\mathcal{L}_0^\nu = 1$, and thus $\mathbb{E}\left[\left|\mathcal{L}_t^\nu\right|\right] \le 1, 0 \le t \le T$. Now, by defining S_T^ν as

$$S_T^\nu = exp\left(\int_0^T |\nu_t - 1|^2 \lambda_t dt\right), \tag{2.2.32}$$

for the same arguments, we have

$$\|S_T^\nu\|_\infty := ess\sup_{\bar{\Omega}} |S_T^\nu| < \infty. \tag{2.2.33}$$

From the explicit form (2.2.22) of $\mathcal{L}^\nu$, we have $|\mathcal{L}_t^\nu|^2 = |\mathcal{L}_t^{\nu^2}||S_T^\nu|$. Therefore, again by using Holder's inequality,

$$\mathbb{E}\left[|L_t^\nu|^2\right] \le \|S_T^\nu\|_\infty E\left[\left|L_t^{\nu^2}\right|\right] \le \|S_T^\nu\|_\infty < \infty, \tag{2.2.34}$$

from which we obtain that $\mathcal{L}^\nu$ is a $\mathbb{P}$-square-integrable martingale.
We are finally ready to provide a formal characterization for the value function of the optimal stopping problem. We introduce J^ν as the functional gain of the form

$$J^\nu(t) = \mathbb{E}^\nu\left[\int_t^{T\wedge\eta} f_s ds + h_\eta 1_{t\le\eta<T} + \xi 1_{\eta\ge T}|\bar{\mathcal{F}}_t\right],\ 0\le t\le T \tag{2.2.35}$$

or equivalently (by recalling the form of μ and λ)

$$J^\nu(t) = \mathbb{E}^\nu\left[\int_t^{T} f_s\lambda_s ds + \int_t^T h_s\mu(ds) + \xi 1_{\eta\ge T}|\bar{\mathcal{F}}_t\right],\ 0\le t\le T. \tag{2.2.36}$$

Then we write the associated *value function* $\bar{V}$ as

$$\bar{V}_t = ess\sup_{\nu\in\mathcal{V}} J^\nu(t),\ 0\le t\le T. \tag{2.2.37}$$

We remark that the new functional differs from the original one in the sense that the probability space is not fixed 'a priori', as it is uniquely defined only once we have chosen among the family of equivalent probability measures.
The reader may find strict similarities with the alternative formulation for the value function of a stochastic control problem, as presented by [18].

Chapter 3

A new BSDE approach to optimal stopping

In this section, we present a particular class of constrained BSDE with jumps. We prove the existence of a minimal solution by means of the penalization method. Moreover, we show that the solution represents the value function of the original stopping problem, in the form that we pointed out in chapter 2. In the sequel, we mainly follow the procedure in Kharroubi, Pham [18].

3.1 Notation and setting

Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, N, μ, λ be defined as in the previous chapter, with $\tilde{\mu}(dt) = \mu(dt) - \lambda_t dt$ $0 \leq t \leq T$. We give the following notation

- $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_t)$, $0 \leq t \leq T$, the sets of $\mathbb{R}$-valued $\bar{\mathcal{F}}_t$-measurable random variable x such that $\mathbb{E}\left[|x|^2\right] < \infty$.
- $\mathbb{H}^2$ the set of $\mathbb{R}^d$-valued, $\bar{\mathbb{F}}$-progressive processes $M = (M_t)_{0\leq t\leq T}$ such that $\|M\|_{H^2} := \left(\mathbb{E}\left[\int_0^T |M_t|^2 \, dt\right]\right)^{\frac{1}{2}} < \infty$.
- $\mathbb{S}^2$ the set of $\mathbb{R}$-valued càd-làg $\bar{\mathbb{F}}$-adapted processes $Y = (Y_t)_{0\leq t\leq T}$ such that $\|Y\|_{S^2} := \left(\mathbb{E}\left[sup_{0\leq t\leq T} |Y_t|^2\right]\right)^{\frac{1}{2}} < \infty$.
- $\mathbb{L}^p(W), p \geq 1$, the set of $\mathbb{R}^d$-valued, $\bar{\mathbb{F}}$-predictable processes $Z = (Z_t)_{0\leq t\leq T}$ such that $\|Z\|_{L^p(W)} := \left(\mathbb{E}\left[\int_0^T |Z_t|^p \, dt\right]\right)^{\frac{1}{p}} < \infty$.
- $\mathbb{L}^p(\tilde{\mu}), p \geq 1$, the set of $\mathbb{R}$-valued, $\bar{\mathbb{F}}$-predictable processes $U = (U_t)_{0\leq t\leq T}$ such that $\|U\|_{L^p(\tilde{\mu})} := \left(\mathbb{E}\left[\int_0^T |U_t|^p \, \lambda_t dt\right]\right)^{\frac{1}{p}} < \infty$.

- $\mathbb{K}^2$ the closed subset of $\mathbb{S}^2$ of predictable non-decreasing processes $K = (K_t)_{0\leq t\leq T}$ such that $K_0 = 0$.

As already mentioned above, we work with the product filtered space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$. Thus, we merely rewrite the triple (ξ, f, h), for all $(\omega, \omega', t) \in \bar{\Omega} \times [0, T]$, as

- $\xi(\omega, \omega') = \xi(\omega)$;
- $f_t(\omega, \omega') = f_t(\omega)$;
- $h_t(\omega, \omega') = h_t(\omega)$;

and we note it is $\bar{\mathbb{F}}$-measurable. Moreover, according to the Fubini Tonelli theorem, one can easily prove that it preserves all the integrability properties w.r.t. the corresponding spaces.
In particular, the following **(Hypothesis H1)** holds

- $\xi : \Omega \to \mathbb{R}$ $\bar{\mathcal{F}}_T$-measurable random variable $\in \mathbb{L}^2(\Omega, \bar{\mathcal{F}}_T)$;
- $f : \Omega \times [0, T] \to \mathbb{R} \in \mathbb{H}^2$, i.e.
  - the map $(\omega, t) \to f_t(\omega)$ is progressively measurable;
  - $\mathbb{E}\left[\int_0^T |f_t|^2 dt\right] < \infty$;
- $h : \Omega \times [0,\, T] \to \mathbb{R}$ $\bar{\mathbb{F}}$-adapted and continuous process satisfying $h_T \leq \xi$ and such that $\mathbb{E}\left[sup_{0\leq t\leq T} |h_t|^2\right] < \infty$;

where $\mathbb{E}$ denotes the expectation operator with respect to the probability measure $\bar{\mathbb{P}}$.

3.2 Existence and approximation via penalization

Let us now introduce our class of Backward Stochastic Differential Equations written in the form

$$\begin{aligned}\bar{Y}_t = \xi 1_{\eta\geq T} + \int_t^T f_s \lambda_s ds + \int_t^T h_s \mu(ds) + \\ + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s - \int_t^T \bar{U}_s \mu(ds), \quad 0 \leq t \leq T;\end{aligned} \tag{3.2.1}$$

with the sign constraint

$$\bar{U}_t \leq 0, \quad dt \times d\mathbb{P} \ a.s.\ on\ \bar{\Omega} \times [0, T]. \tag{3.2.2}$$

Besides the usual component $(\bar{Y}, \bar{Z})$, $\bar{U}$ is a process that characterizes the jump term, and $\bar{K}$ is a non-decreasing process that makes the constraint on $\bar{U}$ feasible.

Definition 3.2.1. Given the triple (ξ, f, h), a solution to the BSDE with sign constraint is a quadruple $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K}) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu}) \times \mathbb{K}^2$ satisfying (3.2.1), (3.2.2). Moreover, the solution $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ is *minimal* if for any other quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu}) \times \mathbb{K}^2$ satisfying (3.2.1), (3.2.2) we have

$$\bar{Y}_t \le \tilde{Y}_t, \quad 0 \le t \le T, \ \bar{\mathbb{P}} - a.s.$$

We are now interested in testing the solvability properties of such a class of equations. In particular, we will prove existence of the solution by means of a penalization approach, afterwards, we will show it is unique in the class of minimal solutions, in the sense of definition 3.2.1. On the other hand, as suggested by Remark 2.1 in Pham, Kharroubi [18], we note that when it exists, there is a unique minimal solution. More precisely, we clearly have uniqueness of the component $\bar{Y}$, while for the processes $(\bar{Z}, \bar{U})$ the property follows by identifying the Brownian parts and the finite variation parts, and finally for $\bar{K}$ by identifying the predictable parts, recalling that the jump times of the point process N are *totally inaccessible.* Let us now focus on the penalized sequence $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$, where the sign constraint is relaxed. For each $n \in \mathbb{N}$, the triple $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ satisfies a BSDE with jumps of the form

$$\begin{aligned}\bar{Y}^n_t = \xi 1_{\eta \ge T} + \int_t^T f_s \lambda_s ds + \int_t^T h_s \mu(ds) + n \int_t \left(\bar{U}^n_s\right)^+ \lambda_s ds + \\ - \int_t^T \bar{Z}^n_s dW_s - \int_t^T \bar{U}^n_s \mu(ds), \ 0 \le t \le T. \quad (3.2.3)\end{aligned}$$

The function $(u)^+ = max(u, 0)$ denotes the *positive part* of u.
Let us verify existence and uniqueness of the solution. We rewrite the equation in the following form, where we add $\pm \int_t^T (h_s - \bar{U}^n_s)\lambda_s ds$.

$$\bar{Y}^n_t = \xi 1_{\eta \ge T} + \int_t^T F^n_s(\bar{U}^n_s) ds - \int_t^T \bar{Z}^n_s dW_s - \int_t^T (\bar{U}^n_s - h_s)\tilde{\mu}(ds), \qquad (3.2.4)$$

where the generator function $F^n : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is a map defined by

$$F^n_t(u) = \left(f_t + (h_t - u) + n\,(u)^+\right)\lambda_t, \ \forall (t,u) \in [0, T] \times \mathbb{R}. \qquad (3.2.5)$$

By requirements on (f, h), and recalling that $|\lambda_t| \le 1, 0 \le t \le T$, one can easily prove that F^n satisfies the following conditions:

- $\forall u \in \mathbb{R}$, the map $(\omega, t) \to F^n_t(\omega, u)$ is progressively measurable;
- $F^n(0) = (F^n_t(0))_{0 \le t \le T} \in \mathbb{H}^2$;
- there exists $L > 0$ such that

$$|F^n_t(u) - F^n_t(u')| \le L|u - u'|, \quad d\bar{\mathbb{P}} \times dt - a.s., \quad \forall u, u' \in \mathbb{R}. \qquad (3.2.6)$$

In particular, F_n is Lipschitz in u uniformly in $t \in [0, T]$.

Therefore, by applying Proposition 3.2 in Becherer [1], we obtain that the BSDE (3.2.3) admits a unique solution $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu})$ for each $n \in \mathbb{N}$. We define the process

$$\bar{K}_t^n = n \int_0^t \left(\bar{U}_s^n\right)^+ \lambda_s ds, \ 0 \leq t \leq T, \tag{3.2.7}$$

which is by construction continuous, increasing, and square-integrable. First of all, we provide a dual representation of the penalized BSDEs in terms of the equivalent probability measure $\bar{\mathbb{P}}^\nu$. To this end, we need the following

Lemma 3.2.1. *Let $\phi \in \mathbb{L}^2(W)$ and $\psi \in \mathbb{L}^2(\tilde{\mu})$. Then for every $\nu \in \mathcal{V}$, the processes $\int_0^\cdot \phi_t dW_t$ and $\int_0^\cdot \phi_t \tilde{\mu}^\nu(ds)$ are $\bar{\mathbb{P}}^\nu$-martingales.*

Proof. 1) We look at the first object. We define

$$S_t^\phi = \int_0^t \phi_s dW_s, \ 0 \leq t \leq T.$$

Since we have shown that W remains a Brownian motion under $\bar{\mathbb{P}}^\nu$, we already know that S^ϕ is a (continuous) $\bar{\mathbb{P}}^\nu$-local martingale. Hence a sufficient condition is uniform integrability (see Appendix A) on S^ϕ w.r.t. the probability $\bar{\mathbb{P}}^\nu$. From Burkholder-Davis-Gundy theorem in 2.2.2 we obtain, for some $C > 0$,

$$\mathbb{E}^\nu \left[sup_{\ 0\leq t\leq T} \left| S_t^\phi \right| \right] \leq C \ \mathbb{E}^\nu \left[\sqrt{< S^\phi >_T} \right]. \tag{3.2.8}$$

Where $< S^\phi >$ is the quadratic variation process of the continuous local martingale S^ϕ w.r.t. $\bar{\mathbb{P}}^\nu$, which we know is indistinguishable from $< S^\phi >$ w.r.t. $\bar{\mathbb{P}}$. Now, recalling that $\mathcal{L}^\nu$ is the Radon Nikodym density (i.e. $\frac{d\bar{\mathbb{P}}^\nu}{d\bar{\mathbb{P}}} = \mathcal{L}_T^\nu$), we get the following equality

$$\mathbb{E}^\nu \left[\sqrt{< S^\phi >_T} \right] = \mathbb{E} \left[\mathcal{L}_T^\nu \sqrt{< S^\phi >_T} \right], \tag{3.2.9}$$

where

$$< S^\phi >_T = \int_0^T |\phi_s|^2 ds, \quad \bar{\mathbb{P}} - a.s..$$

Since both $\mathcal{L}_T^\nu$ and $< S^\phi >_T$ are square-integrable variables (it follows by hypothesis on ν, ϕ), from the Cauchy Schwartz inequality, we have

$$\mathbb{E} \left[\mathcal{L}_T^\nu \sqrt{< S^\phi >_T} \right] \leq C \left(\mathbb{E} \left[|\mathcal{L}_T^\nu|^2 \right] \right)^{\frac{1}{2}} \ \left(\mathbb{E} \left[< S^\phi >_T \right] \right)^{\frac{1}{2}} < \infty,$$

Which makes the condition (3.2.8) satisfied.
2). The same procedure is applied to the second object. We define

$$S_t^{\psi} = \int_0^t \psi_s \tilde{\mu^{\nu}}(ds),\ 0 \leq t \leq T,$$

where we recall that $\tilde{\mu^{\nu}}(ds)$ is the compensated random measure with respect to $\bar{\mathbb{P}}^{\nu}$, i.e. $\tilde{\mu^{\nu}}(ds) = \mu(ds) - \nu_s \lambda_s ds$. We observe that by hypothesis on ν, ψ, and since $\bar{\mathbb{P}}^{\nu}$ is equivalent to $\bar{\mathbb{P}}$, we have

$$\int_0^t |\psi_s| \, \nu_s \lambda_s ds < \infty,\ 0 \leq t \leq T,\ \bar{\mathbb{P}}^{\nu} - a.s. \tag{3.2.10}$$

Thus, by using the theorem 2.1.4, the above condition ensures that S^{ψ} is a (càd-làg) $\bar{\mathbb{P}}^{\nu}$- local martingale. As in the former case, we look for a uniform integrability of S^{ψ} in order to prove it is effectively a $\mathbb{P}^{\nu}$-martingale. By applying the same procedure, we have, for some $C > 0$,

$$\mathbb{E}^{\nu}\left[sup_{\ 0\leq t\leq T}\left|S_t^{\psi}\right|\right] \leq C\ \mathbb{E}^{\nu}\left[\sqrt{[S^{\psi}]_T}\right], \tag{3.2.11}$$

where $[S^{\psi}]$ is the quadratic variation process of the càd-làg local martingale S^{ψ} w.r.t. $\bar{\mathbb{P}}^{\nu}$, which can be proved (Theorem 12.14, [15]) to be indistinguishable from $[S^{\psi}]$ w.r.t. $\bar{\mathbb{P}}$. Therefore, as in the previous case, we have the equality

$$\mathbb{E}^{\nu}\left[\sqrt{[S^{\psi}]_T}\right] = \mathbb{E}\left[\mathcal{L}_T^{\nu}\sqrt{[S^{\psi}]_T}\right], \tag{3.2.12}$$

where (from (2.2.16), by observing that $\Delta S_t^{\psi} = \psi_t \mu(\{t\})$)

$$[S^{\psi}]_T = \int_0^T (\psi_s)^2 \mu(ds), \quad \bar{\mathbb{P}} - a.s.. \tag{3.2.13}$$

Again, since both $\mathcal{L}_T^{\nu}$ and $[S^{\psi}]_T$ are square-integrable variables (it follows by hypothesis on ν, ψ), from the Cauchy Schwartz inequality, we have

$$\mathbb{E}\left[\mathcal{L}_T^{\nu}\sqrt{[S^{\psi}]_T}\right] \leq C\left(\mathbb{E}\left[|\mathcal{L}_T^{\nu}|^2\right]\right)^{\frac{1}{2}}\ \left(\mathbb{E}\left[[S^{\psi}]_T\right]\right)^{\frac{1}{2}} < \infty.$$

Which makes the condition (3.2.11) satisfied.
We are ready to state the following

Proposition 3.2.1. *Let* $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu})$ *be a solution of* (3.2.3). *Then* $\bar{Y}^n$ *is explicitly represented as*

$$\bar{Y}_t^n = ess \sup_{\nu \in \mathcal{V}^n} J^{\nu}(t),\ 0 \leq t \leq T, \tag{3.2.14}$$

where $J^{\nu}(t)$ *denotes the functional gain as in* (2.2.36), *while* $\mathcal{V}^n$ *is the set of* ν $\bar{\mathbb{F}}$-*predictable processes valued in* $(0, n]$ *for all* $n \in \mathbb{N}$.

Proof. We fix n and apply the conditional expectation $\mathbb{E}^\nu$ to $\bar{Y}_t^n$, obtaining from lemma 3.2.1

$$\mathbb{E}^\nu\left[\bar{Y}_t^n|\bar{\mathcal{F}}_t\right] = \mathbb{E}^\nu\left[\xi 1_{\eta\geq T} + \int_t^T f_s\lambda_s ds + \int_t^T h_s\mu(ds) + n\int_t^T \left(\bar{U}_s^n\right)^+\lambda_s ds - \int_t^T \bar{U}_s^n\mu(ds)|\bar{\mathcal{F}}_t\right],$$

$$\bar{Y}_t^n = J^\nu(t) + \mathbb{E}^\nu\left[n\int_t^T \left(\bar{U}_s^n\right)^+\lambda_s ds - \int_t^T \bar{U}_s^n\mu(ds)|\bar{\mathcal{F}}_t\right], \tag{3.2.15}$$

$$\bar{Y}_t^n = J^\nu(t) + \mathbb{E}^\nu\left[\int_t^T (n\left(\bar{U}_s^n\right)^+ - \bar{U}_s^n\nu_s)\lambda_s ds|\bar{\mathcal{F}}_t\right]. \tag{3.2.16}$$

Where the last equality follows from condition (2.1.9), by recalling that N_t has stochastic compensator $\nu_t dA_t = \nu_t\lambda_t dt$ under $\bar{\mathbb{P}}^\nu$, and thus it follows that

$$\mathbb{E}^\nu\left[\int_0^T \bar{U}_t^n\mu(dt)\right] = \mathbb{E}^\nu\left[\int_0^T \bar{U}_t^n\nu_t\lambda_t dt\right]. \tag{3.2.17}$$

Now, given $u, v\in\mathbb{R}$, the function $g(u) := sup_{\ \nu\in[0,n]}\nu u$ satisfies

$$\begin{cases} g(u) = nu & \textit{if } u > 0, \\ g(u) = 0 & \textit{if } u \leq 0, \end{cases} \tag{3.2.18}$$

that is $g(u) = n(u)^+$. Thus by looking at (3.2.16), and observing that $sup_{\ \nu\in[0,n]}\nu u \geq sup_{\ \nu\in(0,n]}\nu u$, we have for any $\nu\in\mathcal{V}^n$

$$n\left(\bar{U}_t^n\right)^+ - \bar{U}_t^n\nu_t \geq 0, \quad d\bar{\mathbb{P}}^\nu\times dt - a.s., \tag{3.2.19}$$

which yields again by (3.2.16)

$$\bar{Y}_t^n \geq J^\nu(t). \tag{3.2.20}$$

Conversely, for any $\epsilon > 0$ we consider the process

$$\nu_t^\epsilon = \epsilon 1_{\bar{U}_t^n\leq 0} + n1_{\bar{U}_t^n>0}, \quad 0\leq t\leq T. \tag{3.2.21}$$

We note that ν^ϵ is ($\bar{\mathbb{P}}^\nu - a.s.$) non-negative for all $t\in[0,T]$, moreover, it is upper-bounded by n, hence it belongs to $\mathcal{V}^n$, for all $n\in\mathbb{N}$. By construction we have

$$n\left(\bar{U}_t^n\right)^+ - \bar{U}_t^n\nu_t^\epsilon \leq \epsilon\ |\bar{U}_t^n|, \quad 0\leq t\leq T,\ d\bar{\mathbb{P}}^\nu - a.s. \tag{3.2.22}$$

Therefore, by collecting the latter condition and equation (3.2.16), we have

$$\bar{Y}_t^n \leq J^{\nu^\epsilon}(t) + \epsilon\ \mathbb{E}^{\nu^\epsilon}\left[\int_t^T \left|\bar{U}_s^n\right|\lambda_s ds|\bar{\mathcal{F}}_t\right]. \tag{3.2.23}$$

Finally, by recalling that $\bar{U}^n\in\mathbb{L}^2(\tilde{\mu})$, we obtain the proof from arbitrariness of ϵ.

The rest of the section is devoted to the convergence study of the penalized sequence. Next, we establish a priori uniform estimates on $\bar{Y}^n$,$\bar{Z}^n$,$\bar{U}^n$,$\bar{K}^n$, $n\in\mathbb{N}$.

We begin with the following

Proposition 3.2.2. *The sequence* $(\bar{Y}^n)_n$ *is non-decreasing, i.e.* $\bar{Y}_t^n \leq \bar{Y}_t^{n+1}$ *for all* $t \in [0,T]$ *and all* $n \in \mathbb{N}$*. Moreover, it holds* $\bar{\mathbb{P}} - a.s.$ *that*

$$\left|\bar{Y}_t^n\right| \leq C, \quad \forall 0 \leq t \leq T. \tag{3.2.24}$$

In particular, we have

$$sup_{\ 0\leq t\leq T}|\bar{Y}_t^n|^2 \leq C, \tag{3.2.25}$$

thus, recalling that $\bar{Y}^n \in \mathbb{S}^2$,

$$\mathbb{E}\left[sup_{\ 0\leq t\leq T}|\bar{Y}_t^n|^2\right] \leq C, \quad \forall n \in \mathbb{N}. \tag{3.2.26}$$

Proof. Let us prove inequality (3.2.24), as the monotonicity trivially followed by proposition 3.2.1. According to the definition of *essential supremum*, for any $\epsilon > 0$, $n \in \mathbb{N}$, there exists a process $\nu^n \in \mathcal{V}^n$ such that

$$\bar{Y}_t^n - \epsilon \leq J^{\nu^n}(t), \quad \bar{\mathbb{P}} - a.s. \tag{3.2.27}$$

Therefore, by observing that $\mathcal{V}^n \subset \mathcal{V}, n \in \mathbb{N}$, we only need to prove $J^\nu(t) \leq C$, $\nu \in \mathcal{V}$, for some $C > 0$ (independent from ν). We define Ψ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$, $\bar{\mathcal{F}}_T$-measurable random variable such that

$$\Psi(\omega,\omega') = \int_0^T |f_s(\omega)|ds + |\xi(\omega)| + sup_{\ 0\leq t\leq T}|h_t(\omega)|, \tag{3.2.28}$$

where it is clear that Ψ is independent from ω'. We observe that, given $J^\nu(t)$, it holds

$$J^\nu(t) = \mathbb{E}^\nu\left[\int_t^{T\wedge\eta} f_s ds + \xi 1_{\eta\geq T} + h_\eta 1_{t\leq\eta<T}|\bar{\mathcal{F}}_t\right] \leq$$
$$\leq \mathbb{E}^\nu\left[\int_0^T |f_s|ds + |\xi| + sup_{\ 0\leq t\leq T}|h_t|\right] = \mathbb{E}^\nu\left[|\Psi|\right]. \tag{3.2.29}$$

We denote $\mathbb{E}_P$ as the expectation operator w.r.t. the measure P. In order to obtain the proof, it is enough to show that

$$\mathbb{E}^\nu\left[|\Psi|\right] = \mathbb{E}_P\left[|\Psi|\right], \tag{3.2.30}$$

since, by requirements on (ξ, f, h), we easily have

$$\mathbb{E}_P\left[|\Psi|^2\right] \leq C, \tag{3.2.31}$$

for some constant $C > 0$. Let us fix $j \in \mathbb{N}$, we define a deterministic function $g : \mathbb{R}^d \times \mathbb{R}^d \cdots \times \mathbb{R}^d \to \mathbb{R}^+$ such that

$$\psi_j(\omega) = g(W_{t_1}(\omega), W_{t_2}(\omega), \ldots, W_{t_j}(\omega)), \tag{3.2.32}$$

where W denotes the brownian motion, and $0 = t_1 < t_2 < \cdots < t_j = T$ is a subdivision of $[0, T]$. Applying the expectation operator, we have

$$\mathbb{E}^{\nu}\left[|\psi_j|\right] = \mathbb{E}^{\nu}\left[g(W_{t_1}, W_{t_2}, \ldots, W_{t_j})\right] = \mathbb{E}_P\left[g(W_{t_1}, W_{t_2}, \ldots, W_{t_j})\right] = \mathbb{E}_P\left[|\psi_j|\right]. \tag{3.2.33}$$

Where the last equality follows from proposition 2.2.1. Now, we obtain the proof by monotonic convergence results, observing that $\Psi(\omega)$ is a non-negative $\mathcal{F}_T$-measurable random variable, hence it can be expressed as the limit of a sequence of the aforementioned class of functions.
Now we are ready to introduce the following result of a uniform estimate.

Proposition 3.2.3. *The stated conditions hold. Then there exists a constant C depending only on T such that*

$$\mathbb{E}\left[sup_{\ 0\leq t\leq T}\left|\bar{Y}_t^n\right|^2 + \int_0^T \left|\bar{Z}_t^n\right|^2 dt + \int_0^T \left|\bar{U}_t^n\right|^2 \lambda_t dt + \left|\bar{K}_T^n\right|^2\right] < C, \quad \forall n \in \mathbb{N}. \tag{3.2.34}$$

Proof. We fix $n \in \mathbb{N}$ and we note that $\bar{Y}^n$ can be decomposed into a finite variation part A^n and a continuous martingale part M^n, i.e.

$$\bar{Y}_t^n = M_t^n + A_t^n,\ 0 \leq t \leq T, \tag{3.2.35}$$

where

$$M_t^n = -\int_t^T \bar{Z}_s^n dW_s, \tag{3.2.36}$$

while

$$A_t^n = \xi 1_{\eta\geq T} + \int_t^T f_s \lambda_s ds + \int_0^T (h_s - \bar{U}_s^n)\mu(ds) + \bar{K}_T^n - \bar{K}_t^n. \tag{3.2.37}$$

From Ito's formula for semimartingales of this form (see Appendix B), we obtain that

$$\begin{aligned} f(\bar{Y}_T^n) = f(\bar{Y}_t^n) + \int_t^T f'(\bar{Y}_{s^-}^n)dM_s^n + \frac{1}{2}\int_t^T f''(\bar{Y}_{s^-}^n)d < M^n, M^n >_s + \\ + \int_t^T f'(\bar{Y}_{s^-}^n)dA_s^n + \sum_{t\leq s\leq T}\left\{f(\bar{Y}_{s^-}^n + \Delta\bar{Y}_s^n) - f(\bar{Y}_{s^-}^n) - f'(\bar{Y}_s^n)\Delta\bar{Y}_s^n\right\}, \end{aligned} \tag{3.2.38}$$

where $f \in C^2(\mathbb{R})$, $\Delta\bar{Y}_s^n = (h_t - \bar{U}_t^n)\mu\{t\}$, and $\bar{Y}^{n-} = (\bar{Y}_{t-}^n)_{0\leq t\leq T}$ is the *left limit process* associated to $\bar{Y}^n$. Now, by applying this to $f(\bar{Y}_t^n) = |\bar{Y}_t^n|^2$, we have

$$\begin{aligned} |\xi 1_{\eta\geq T}|^2 = |\bar{Y}_t^n|^2 - 2\int_t^T \bar{Z}_s^n \bar{Y}_{s^-}^n dW_s + \int_t^T |\bar{Z}_s^n|^2 ds - 2\int_t^T f_s \bar{Y}_{s^-}^n \lambda_s ds - \\ - 2\int_t^T \bar{Y}_{s^-}^n d\bar{K}_s^n + \int_t^T \left\{|\bar{Y}_{s^-}^n + (h_s - \bar{U}_s^n)|^2 - |\bar{Y}_{s^-}^n|^2 - 2\bar{Y}_{s^-}^n(h_s - \bar{U}_s^n)\right\}\mu(ds). \end{aligned} \tag{3.2.39}$$

which can be rewritten as

$$|\xi 1_{\eta\geq T}|^2 = |\bar{Y}^n_t|^2 - 2\int_t^T \bar{Z}^n_s \bar{Y}^n_{s^-} dW_s + \int_t^T |\bar{Z}^n_s|^2 ds - 2\int_t^T f_s \bar{Y}^n_{s^-}\lambda_s ds - \\ - 2\int_t^T \bar{Y}^n_{s^-} d\bar{K}^n_s + \int_t^T |h_s - \bar{U}^n_s|^2 \mu(ds). \quad (3.2.40)$$

Since $(\bar{Y}^n, \bar{Z}^n) \in \mathbb{S}^2 \times \mathbb{L}^2(W)$, we obtain from Young's inequality that

$$\mathbb{E}\left[(\int_0^T |\bar{Z}^n_s \bar{Y}^n_s|^2 ds)^{\frac{1}{2}}\right] \leq \mathbb{E}\left[\left(sup_{\ 0\leq t\leq T}|\bar{Y}^n_t|^2\right)^{\frac{1}{2}}\left(\int_0^T |\bar{Z}^n_s|^2 ds\right)^{\frac{1}{2}}\right] \\ \leq \frac{1}{2}\mathbb{E}\left[sup_{\ 0\leq t\leq T}|\bar{Y}^n_t|^2\right] + \frac{1}{2}\mathbb{E}\left[\int_0^T |\bar{Z}^n_s|^2 ds\right] < \infty, \quad (3.2.41)$$

hence, from the Burkholder-Davis-Gundy inequality, this shows that the local martingale in (3.2.40) is a uniformly integrable martingale. By taking the expectation, we obtain

$$\mathbb{E}\left[|\bar{Y}^n_t|^2\right] + \mathbb{E}\left[\int_t^T |\bar{Z}^n_s|^2 ds\right] + \mathbb{E}\left[\int_t^T |h_s - \bar{U}^n_s|^2 \lambda_s ds\right] = \\ = \mathbb{E}\left[|\xi 1_{\eta\geq T}|^2\right] + 2\mathbb{E}\left[\int_t^T \bar{Y}^n_s d\bar{K}^n_s\right] + 2\mathbb{E}\left[\int_t^T \bar{Y}^n_s f_s \lambda_s ds\right] \quad (3.2.42)$$

In order to get (3.2.34), we need to eliminate h from the left side of the above equality. For simplicity, in the following passages we will refer to a constant C, which is not always the same but could vary from line to line.

$$\mathbb{E}\left[\int_t^T |h_s - \bar{U}^n_s|^2 \lambda_s ds\right] = \mathbb{E}\left[\int_t^T |h_s|^2 \lambda_s ds\right] + \mathbb{E}\left[\int_t^T |\bar{U}^n_s|^2 \lambda_s ds\right] + \\ - 2\mathbb{E}\left[\int_t^T h_s \bar{U}^n_s \lambda_s ds\right] \leq C + \mathbb{E}\left[\int_t^T |\bar{U}^n_s|^2 \lambda_s ds\right] + \mathbb{E}\left[\int_t^T 2|h_s \bar{U}^n_s| \lambda_s ds\right]. \quad (3.2.43)$$

Again, by applying Young's inequality to the last term

$$\mathbb{E}\left[\int_t^T |h_s - \bar{U}^n_s|^2 \lambda_s ds\right] \leq C + \mathbb{E}\left[\int_t^T |\bar{U}^n_s|^2 \lambda_s ds\right] + \\ + \frac{1}{4}\mathbb{E}\left[\int_t^T |\bar{U}^n_s|^2 \lambda_s ds\right] + \frac{1}{4}\mathbb{E}\left[\int_t^T |h_s|^2 \lambda_s ds\right] \leq \gamma\mathbb{E}\left[\int_t^T |\bar{U}^n_s|^2 \lambda_s ds\right] \quad (3.2.44)$$

Hence, on coming back to (3.2.42), with $\gamma > 1$,

$$\mathbb{E}\left[|\bar{Y}^n_t|^2\right] + \mathbb{E}\left[\int_t^T |\bar{Z}^n_s|^2 ds\right] + \gamma\mathbb{E}\left[\int_t^T |\bar{U}^n_s|^2 \lambda_s ds\right] \leq \\ \leq \mathbb{E}\left[|\xi|^2\right] + 2\mathbb{E}\left[\int_t^T \bar{Y}^n_s f_s \lambda_s ds\right] + 2\mathbb{E}\left[\int_t^T \bar{Y}^n_s d\bar{K}^n_s\right]. \quad (3.2.45)$$

First of all, we analyse the integral term

$$2\mathbb{E}\left[\int_t^T \bar{Y}_s^n f_s \lambda_s ds\right]. \tag{3.2.46}$$

From condition (3.2.24) on $\bar{Y}^n$, it holds $\bar{\mathbb{P}} - a.s.$, that

$$\int_t^T |\bar{Y}_s^n f_s| ds \leq C \int_t^T |f_s| ds, \tag{3.2.47}$$

therefore, by recalling that $f \in \mathbb{L}^2(W)$, we can apply the expectation operator, which yields

$$2\mathbb{E}\left[\int_t^T \bar{Y}_s^n f_s \lambda_s ds\right] \leq 2\mathbb{E}\left[\int_t^T |\bar{Y}_s^n f_s| ds\right] \leq 2C\mathbb{E}\left[\int_t^T |f_s| ds\right] \leq C, \quad 0 \leq t \leq T. \tag{3.2.48}$$

Let us now focus on the integral term

$$2\mathbb{E}\left[\int_t^T \bar{Y}_s^n d\bar{K}_s^n\right]. \tag{3.2.49}$$

We have $\bar{\mathbb{P}} - a.s.$, that

$$\int_t^T \bar{Y}_s^n d\bar{K}_s^n \leq sup_{\ t \leq s \leq T} |\bar{Y}_s^n||\bar{K}_T^n - \bar{K}_t^n|, \tag{3.2.50}$$

by applying the inequality $2ab \leq \frac{1}{\alpha}a^2 + \alpha b^2$ $(\alpha > 0)$, we have

$$2\int_t^T \bar{Y}_s^n d\bar{K}_s^n \leq \frac{1}{\alpha} sup_{\ t \leq s \leq T} |\bar{Y}_s^n|^2 + \alpha|\bar{K}_T^n - \bar{K}_t^n|^2. \tag{3.2.51}$$

Now, from the relation

$$\bar{K}_T^n - \bar{K}_t^n = \bar{Y}_t^n - \xi 1_{\eta \geq T} - \int_t^T f_s \lambda_s ds + \int_t^T (h_s - \bar{U}_s^n)\mu(ds) + \int_t^T \bar{Z}_s^n dW_s, \tag{3.2.52}$$

by applying the expectation to $|\bar{K}_T^n - \bar{K}_t^n|^2$, we obtain

$$\mathbb{E}\left[|K_T^n - K_t^n|^2\right] \leq C\mathbb{E}\left[|Y_t^n|^2 + |\xi|^2 + \int_t^T |f_s|^2 ds + \int_t^T |h_s - U_s^n|^2 \lambda_s ds + \int_t^T |Z_s^n|^2 ds\right]. \tag{3.2.53}$$

For the latter inequality, we have used the following results (which are recalled more precisely in the next chapter):

- $\mathbb{E}\left[\left|\int_t^T Z_s^n dW_s\right|^2\right] = \mathbb{E}\left[\int_t^T |Z_s^n|^2 \, ds\right]$

- $\mathbb{E}\left[\left|\int_t^T (h_s - U_s^n)\tilde{\mu}(ds)\right|^2\right] \leq C\mathbb{E}\left[\int_t^T |h_s - U_s^n|^2 \lambda_s ds\right]$

Hence by the requirements on (ξ, h, f), there exists a suitable constant C_1 such that

$$\mathbb{E}\left[|\bar{K}_T^n - \bar{K}_t^n|^2\right] \leq C_1 \mathbb{E}\left[|\bar{Y}_t^n|^2 + \int_t^T |\bar{U}_s^n|^2 \lambda_s ds + \int_t^T |\bar{Z}_s^n|^2 ds\right]. \tag{3.2.54}$$

Therefore we pass to the expectation in (3.2.51), obtaining from (3.2.54)

$$2\mathbb{E}\left[\int_t^T \bar{Y}_s^n d\bar{K}_s^n\right] \leq \frac{1}{\alpha}\mathbb{E}\left[sup_{\ t\leq s\leq T}|\bar{Y}_s^n|^2\right] + \alpha C_1 \mathbb{E}\left[|\bar{Y}_t^n|^2 + \int_t^T |\bar{U}_s^n|^2 \lambda_s ds + \int_t^T |\bar{Z}_s^n|^2 ds\right], \tag{3.2.55}$$

hence, by collecting the results in (3.2.48), (3.2.55), we have

$$\mathbb{E}\left[|\xi|^2\right] + 2\mathbb{E}\left[\int_t^T \bar{Y}_s^n f_s \lambda_s ds\right] + 2\mathbb{E}\left[\int_t^T \bar{Y}_s^n d\bar{K}_s^n\right] \leq C +$$
$$+\frac{1}{\alpha}\mathbb{E}\left[sup_{\ t\leq s\leq T}|\bar{Y}_s^n|^2\right] + \alpha C_1 \mathbb{E}\left[|\bar{Y}_t^n|^2 + \int_t^T |\bar{U}_s^n|^2 \lambda_s ds + \int_t^T |\bar{Z}_s^n|^2 ds\right]. \tag{3.2.56}$$

Therefore, by choosing $\alpha > 0$ such that $\alpha C_1 < 1$, we obtain after some passages that equation (3.2.45) becomes

$$\mathbb{E}\left[|\bar{Y}_t^n|^2 + \int_t^T |\bar{Z}_s^n|^2 ds + \int_t^T |\bar{U}_s^n|^2 \lambda_s ds\right] \leq C\mathbb{E}\left[sup_{\ t\leq s\leq T}|\bar{Y}_s^n|^2\right], \tag{3.2.57}$$

then, by using condition (3.2.26), for some $C > 0$

$$\mathbb{E}\left[sup_{\ 0\leq t\leq T}|\bar{Y}_t^n|^2 + \int_0^T \left|\bar{Z}_t^n\right|^2 dt + \int_0^T \left|\bar{U}_t^n\right|^2 \lambda_t dt +\right] < C, \quad \forall n \in \mathbb{N}, \tag{3.2.58}$$

which gives the required uniform estimates (3.2.34) for $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$, and also $\bar{K}^n$ by (3.2.54).
We are ready to introduce the following

Theorem 3.2.1. *Assume that the triple* (ξ, f, h) *satisfies the* ***Hypothesis H1****. Then there exists a quadruple* $(\bar{Y}, \bar{Z}, \bar{K}, \bar{U}) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu}) \times \mathbb{K}^2$ *satisfying the equation* (3.2.1)*, i.e.*

$$\bar{Y}_t = \xi 1_{\eta\geq T} + \int_t^T f_s \lambda_s ds + \int_t^T h_s \mu(ds) + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s - \int_t^T \bar{U}_s \mu(ds), \quad 0 \leq t \leq T, \tag{3.2.59}$$

with the sign constraint (3.2.2)

$$\bar{U}_t \leq 0, \quad dt \times d\mathbb{P} \ a.s. \ on \ \Omega \times [0, T]. \tag{3.2.60}$$

The quadruple $(\bar{Y}, \bar{Z}, \bar{K}, \bar{U})$ *is the limit of the sequence* $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)_n$, $n \in \mathbb{N}$, *solution of the penalized BSDE* (3.2.3), *i.e.*

$$\begin{aligned}\bar{Y}^n_t = \xi 1_{\eta \geq T} + \int_t^T f_s 1_{[0,\eta)}(s) ds + \int_0^T h_s \mu(ds) + \bar{K}^n_T - \bar{K}^n_t - \\ - \int_t^T \bar{Z}^n_s dW_s - \int_t^T \bar{U}^n_s \mu(ds), \ 0 \leq t \leq T,\end{aligned} \tag{3.2.61}$$

with $\bar{K}_n$ *defined as in* (3.2.7) *More precisely,* $\bar{Y}$ *is the pointwise limit of* $\bar{Y}^n$, $\bar{Z}$ *is the weak limit of* $\bar{Z}^n$ *in* $\mathbb{L}^2(W)$, $\bar{K}_t$ *the weak limit of* $\bar{K}^n_t$ *in* $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_t)$, $t \in [0,T]$, *while* $\bar{U}$ *is the weak limit of* $\bar{U}^n$ *in* $\mathbb{L}^2(\tilde{\mu})$.

Proof. We start by carrying out the following

Proposition 3.2.4. *The sequence* $(\bar{Y}^n)_{n \geq 0}$ *converges (increasingly) to a process* $\bar{Y} \in \mathbb{S}^2$, *i.e. such that* $\mathbb{E}\left[sup_{\ 0 \leq t \leq T} |\bar{Y}_t|^2\right] < \infty$.

Proof. It simply follows that by recalling $(\bar{Y}^n_t)_n$ to be an increasing sequence, it therefore admits a pointwise limit $\bar{Y}_t$ (a.s.), for any $t \in [0,T]$. We consider the non-negative increasing sequence $\left(\sup_{0 \leq t \leq T} |\bar{Y}^n_t|^2\right)_n$, it clearly holds $\bar{\mathbb{P}} - a.s.$ that

$$|\bar{Y}^n_t|^2 \leq \left(\sup_{0 \leq t \leq T} |\bar{Y}^n_t|^2 \right), \tag{3.2.62}$$

which yields

$$|\bar{Y}_t|^2 \leq \lim_{n \to \infty} \left(\sup_{0 \leq t \leq T} |\bar{Y}^n_t|^2 \right) \tag{3.2.63}$$

for any $t \in [0,T]$, hence we take the *supremum*

$$\left(\sup_{0 \leq t \leq T} |\bar{Y}_t|^2 \right) \leq \lim_{n \to \infty} \left(\sup_{0 \leq t \leq T} |\bar{Y}^n_t|^2 \right). \tag{3.2.64}$$

Therefore, by applying the Fatou Lemma, we obtain the required condition from (3.2.26), i.e.

$$\mathbb{E}\left[\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\bar{Y}^n_t|^2 \right] \leq \lim_{n \to \infty} \mathbb{E}\left[\sup_{0 \leq t \leq T} |\bar{Y}^n_t|^2 \right] \leq C. \tag{3.2.65}$$

Moreover, again by using proposition 3.2.2, we know the sequence $(|\bar{Y}^n_t|)_n$ to be uniformly bounded, hence dominated by an integrable variable in $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_t)$, for all $t \in [0,T]$. It then follows by dominated convergence that

$$\lim_{n \to \infty} \mathbb{E}\left[|\bar{Y}^n_t - \bar{Y}_t|^2 \right] = 0, \tag{3.2.66}$$

i.e. we obtain further convergence in $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_t)$, $0 \leq t \leq T$.
Let us go back to the main proof. From the uniform estimate in (3.2.58), we have that sequences $\bar{Z}^n$, $\bar{U}^n$ are bounded in the respective Hilbert spaces $\mathbb{L}^2(W)$, $\mathbb{L}^2(\tilde{\mu})$. Then we can extract subsequences that converge weakly in the related spaces. For all $t \in [0, T]$, we define the integral operators I_t^w, I_t^μ, such that

- I_t^w:$\mathbb{L}^2(W) \to \mathbb{L}^2(\Omega, \bar{\mathcal{F}}_t)$ such that

$$I_t^w(\bar{Z}^n) = \int_0^t \bar{Z}_s^n dW_s; \tag{3.2.67}$$

- I_t^μ:$\mathbb{L}^2(\tilde{\mu}) \to \mathbb{L}^2(\Omega, \bar{\mathcal{F}}_t)$ such that

$$I_t^\mu(\bar{U}^n) = \int_0^t \bar{U}_s^n \tilde{\mu}(ds); \tag{3.2.68}$$

and we denote $(I_t^w)_{0 \leq t \leq T}$, $(I_t^\mu)_{0 \leq t \leq T}$ as the respectively $\mathbb{R}$-valued stochastic processes (which, in particular, are both martingales).
We note that for any $t \in [0, T]$, the operators I_t^w, I_t^μ are linear (trivial), and continuous. In particular, given a stopping time $\tau \in \mathcal{T}_0^T$, we see that I_τ^w satisfies

$$\begin{aligned} \left\| I_\tau^w(\bar{Z}^n) \right\|_{\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_\tau)} = \mathbb{E}\left[|I_\tau^w(\bar{Z}^n)|^2 \right] = \mathbb{E}\left[\left| \int_0^\tau \bar{Z}_s^n dW_s \right|^2 \right] = \\ = \mathbb{E}\left[\int_0^\tau |\bar{Z}_s^n|^2 ds \right] \leq \mathbb{E}\left[\int_0^T |\bar{Z}_s^n|^2 ds \right] = \left\| \bar{Z}^n \right\|_{\mathbb{L}^2(W)}, \end{aligned} \tag{3.2.69}$$

where, by observing that the process $(I_t^w)_{0 \leq t \leq T}$ is a square integrable martingale, we have used Ito isometry (see Appendix A). While, for I_τ^μ, we obtain from the Bulkholder-Davis-Gundy inequality that for some $C > 0$

$$\begin{aligned} \left\| I_\tau^\mu(\bar{U}^n) \right\|_{\mathbb{L}^2(\Omega, \mathcal{F}_\tau)} = \mathbb{E}\left[|I_\tau^\mu(\bar{U}^n)|^2 \right] = \mathbb{E}\left[\left| \int_0^\tau \bar{U}_s^n \tilde{\mu}(ds) \right|^2 \right] \leq \sup_{0 \leq \tau \leq T} \mathbb{E}\left[\left| \int_0^\tau \bar{U}_s^n \tilde{\mu}(ds) \right|^2 \right] \leq \\ \leq C\mathbb{E}\left[[I^\mu(\bar{U}_n)]_T \right] = C\mathbb{E}\left[\int_0^T |\bar{U}_s^n|^2 \mu(ds) \right] = C \left\| \bar{U}^n \right\|_{L^2(\tilde{\mu})}, \end{aligned} \tag{3.2.70}$$

where, by observing that $(I_t^\mu)_{0 \leq t \leq T}$ is a finite variation process, we have used

$$[I^\mu(\bar{U}^n)]_T = \sum_{0 \leq t \leq T} |\Delta I_t^\mu(\bar{U}^n)^2| = \int_0^T |\bar{U}_s^n|^2 \mu(ds). \tag{3.2.71}$$

Thus, by exploiting a well-known result in functional analysis, we can state that the operators I_τ^μ, I_τ^w are weakly continuous in $L^2(\Omega, \bar{\mathcal{F}}_\tau)$. In particular, providing that $\bar{Z}^n, \bar{U}^n$ converges weakly to $\bar{Z}, \bar{U}$, we have

- $\int_0^\tau \bar{Z}^n_s dW_s \ \rightarrow \ \int_0^\tau \bar{Z}_s dW_s$,
- $\int_0^\tau \bar{U}^n_s \tilde{\mu}(ds) \ \rightarrow \ \int_0^\tau \bar{U}_s \tilde{\mu}(ds)$, $\big($*and thus* $\int_0^\tau \bar{U}^n_s \mu(ds) \ \rightarrow \ \int_0^\tau \bar{U}_s \mu(ds)\big)$.

Now, since $\bar{K}^n_\tau$ verifies

$$\bar{K}^n_\tau = \bar{Y}^n_0 - \bar{Y}^n_\tau - \int_0^\tau f_s \lambda_s ds - \int_0^\tau h_s \mu(ds) + \int_0^\tau \bar{U}^n_s \mu(ds) + \int_0^\tau \bar{Z}^n_s dW_s, \quad (3.2.72)$$

we also have the following weak convergence in $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_\tau)$

$$\bar{K}^n_\tau \rightarrow \bar{K}_\tau = \bar{Y}_0 - \bar{Y}_\tau - \int_0^\tau f_s \lambda_s ds - \int_0^\tau h_s \mu(ds) + \int_0^\tau \bar{U}_s \mu(ds) + \int_0^\tau \bar{Z}_s dW_s. \quad (3.2.73)$$

Moreover, given a non-negative square integrable random variable ϕ, we have for any σ, τ stopping times such that $\sigma \leq \tau$

$$0 \leq \mathbb{E}[(\bar{K}^n_\tau - \bar{K}^n_\sigma)\phi] = \mathbb{E}[(\bar{K}^n_\tau)\phi] - \mathbb{E}[(\bar{K}^n_\sigma)\phi] \rightarrow \mathbb{E}[(\bar{K}_\tau)\phi] - \mathbb{E}[(\bar{K}_\sigma)\phi] = \mathbb{E}[(\bar{K}_\tau - \bar{K}_\sigma)\phi],$$

thus

$$\bar{K}_\tau \geq \bar{K}_\sigma, \quad \bar{\mathbb{P}} - a.s. \text{ for all } \sigma, \ \tau. \quad (3.2.74)$$

Hence, by observing that $\bar{K}$ is *optional* (from (3.2.73), one can see it admits for all $t \in [0, T]$ a càd-làg modification), we apply the *section theorem* (see appendix A), from which we state that $\bar{K}$ is increasing. Therefore, by applying Lemma 2.2 in Peng [22], we obtain $\bar{Y}$,$\bar{K}$ as càd-làg processes, hence $(\bar{Y}_t)_{0\leq t\leq T}$ is identified (up to indinstinguishability with respect to $\mathbb{P}$) with equation in (3.2.1), that is

$$\bar{Y}_t = \xi 1_{\eta \geq T} + \int_t^T f_s \lambda_s ds + \int_t^T h_s \mu(ds) + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s - \int_t^T \bar{U}_s \mu(ds). \quad (3.2.75)$$

Moreover, $\bar{K}$ is predictable.
Proof. Since $\bar{K}$ is adapted and cádlág, it is enough to show that $\bar{K}$ satisfies the requirements of Theorem 4.33 in [15], more precisely:

- for each *totally inaccessible time* S, $\bar{K}_S = \bar{K}_{S^-}$ a.s. on $\{S < \infty\}$;
- for each *predictable time* T, $\bar{K}_T 1_{T<\infty}$ is $\mathcal{F}_{T^-}$-measurable.

Let us take the sequence $(\bar{K}^n)_n$, we know it converges weakly in $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_\tau)$, for any τ stopping time. One can easily observe that it converges weakly in $\mathbb{L}^2(\Omega, \bar{\mathcal{F}})$ as well, i.e. given $\xi \in \mathbb{L}^2(\Omega, \bar{\mathcal{F}})$,

$$\mathbb{E}\left[\bar{K}^n_\tau \xi\right] = \mathbb{E}\left[\bar{K}^n_\tau \mathbb{E}\left[\xi | \bar{\mathcal{F}}_\tau\right]\right] \rightarrow \mathbb{E}\left[\bar{K}_\tau \mathbb{E}\left[\xi | \bar{\mathcal{F}}_\tau\right]\right] = \mathbb{E}\left[\bar{K}_\tau \xi\right]. \quad (3.2.76)$$

1). Let S be a *totally inaccessible* (finite) time. We immediately obtain the first condition from the predictability of $\bar{K}^n$, i.e. it holds

$$\bar{K}^n_S = \bar{K}^n_{S^-}, \quad \bar{\mathbb{P}} - a.s., \quad (3.2.77)$$

hence, by (weak) convergence

$$\mathbb{E}\left[\bar{K}_S\xi\right] \leftarrow \mathbb{E}\left[\bar{K}^n_S\xi\right] = \mathbb{E}\left[\bar{K}^n_{S^-}\xi\right] \rightarrow \mathbb{E}\left[\bar{K}_{S^-}\xi\right] \tag{3.2.78}$$

which merely yields

$$\bar{K}_S = \bar{K}_{S^-}, \quad \bar{\mathbb{P}} - a.s. \tag{3.2.79}$$

2). Let T be a *predictable* (finite) time. Again, since $\bar{K}^n$ is predictable, $\bar{K}^n_T$ is $\bar{\mathcal{F}}_{T^-}$-measurable. Therefore

$$\mathbb{E}\left[\bar{K}^n_T\xi\right] = \mathbb{E}\left[\bar{K}^n_T\mathbb{E}\left[\xi|\bar{\mathcal{F}}_{T^-}\right]\right]. \tag{3.2.80}$$

By recalling (3.2.76), we pass to the (weak) limit on both sides of the equality, obtaining by the uniqueness of weak convergence and conditional expectation rules

$$\mathbb{E}\left[\bar{K}_T\xi\right] = \mathbb{E}\left[\bar{K}_T\mathbb{E}\left[\xi|\bar{\mathcal{F}}_{T^-}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\bar{K}_T|\mathcal{F}_{T^-}\right]\mathbb{E}\left[\xi|\bar{\mathcal{F}}_{T^-}\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\bar{K}_T|\mathcal{F}_{T^-}\right]\xi\right], \tag{3.2.81}$$

and thus by

$$\bar{K}_T = \mathbb{E}\left[\bar{K}_T|\bar{\mathcal{F}}_{T^-}\right], \quad \bar{\mathbb{P}} - a.s.. \tag{3.2.82}$$

we obtain $\bar{\mathcal{F}}_{T^-}$-measurability of $\bar{K}_T$.

It still has to be proved that the sign constraint (3.2.2) is verified. We have that $\mathbb{E}\left[|\bar{K}^n_T|^2\right] < \infty$, for all $n \in \mathbb{N}$, which clearly yields

$$\left(n\int_0^T (\bar{U}^n_s)^+\lambda_s ds\right) < \infty, \ \bar{\mathbb{P}} - a.s. \tag{3.2.83}$$

and then, necessarily

$$\lim_{n\to\infty}\int_0^T (\bar{U}^n_s)^+\lambda_s ds = 0, \quad \bar{\mathbb{P}} - a.s. \tag{3.2.84}$$

Moreover, by the uniform estimates ($\mathbb{E}\left[|\bar{K}^n_T|^2\right] \leq C$, for some $C > 0$, $n \in \mathbb{N}$) we have the sequence $\left(\int_0^T (\bar{U}^n_s)^+\lambda_s ds\right)_n$ to be uniformly bounded, hence dominated by an integrable random variable in $\mathbb{L}^2(\Omega, \bar{\mathcal{F}}_T)$. It then follows by dominated convergence that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T (\bar{U}^n_s)^+\lambda_s ds\right] = 0. \tag{3.2.85}$$

In conclusion, by calling $\Phi(\bar{U}_n) = \mathbb{E}\left[\int_0^T (\bar{U}^n_s)^+\lambda_s ds\right]$, we see it is a non-negative convex functional from $\mathbb{L}^2(\tilde{\mu})$ into $\mathbb{R}^+$ (convexity merely follows from linearity and convexity of function $(.)^+$). Thus, exploiting a well known result in functional analysis, we can state that $\Phi(\bar{U}_n)$ is weakly lower semicontinuous, i.e., provided that $\bar{U}^n$ converges weakly to $\bar{U}$, it holds that

$$\Phi(\bar{U}) \leq \liminf_{n\to\infty}\Phi(\bar{U}^n) = \lim_{n\to\infty}\mathbb{E}\left[\int_0^T (\bar{U}^n_s)^+\lambda_s ds\right] = 0. \tag{3.2.86}$$

From which we finally obtain the required condition, i.e.

$$\bar{U}_t \leq 0, \quad dt \times d\bar{\mathbb{P}} - a.s. \tag{3.2.87}$$

As far as the uniqueness property is concerned, we will prove it as a consequence of the central result stated in the following section.

3.3 An explicit optimal stopping time representation

Proposition 3.3.1. *Let $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ be the solution obtained in theorem 3.2.1, then the component $\bar{Y}$ is explicitly represented as*

$$\bar{Y}_t = ess \sup_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f_s \lambda_s ds + \int_t^T h_s \mu(ds) + \xi 1_{\eta \geq T} | \bar{\mathcal{F}}_t \right], \quad 0 \leq t \leq T. \tag{3.3.1}$$

In particular, the solution $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ is minimal, i.e. for any other quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ satisfying (3.2.1), (3.2.2)*, it holds that*

$$\bar{Y}_t \leq \tilde{Y}_t, \;\; 0 \leq t \leq T, \;\; \mathbb{P} - a.s.. \tag{3.3.2}$$

Proof. We fix $t \in [0, T]$ and denote $Y_t^* = ess \sup_{\nu \in \mathcal{V}} J^\nu(t)$. By applying the conditional expectation $\mathbb{E}^\nu$ to $\bar{Y}_t$, we obtain

$$\mathbb{E}^\nu \left[\bar{Y}_t | \bar{\mathcal{F}}_t \right] = \mathbb{E}^\nu [\xi 1_{\eta \geq T} + \int_t^T f_s \lambda_s ds + \int_t^T h_s \mu(ds) + \bar{K}_T - \bar{K}_t - \int_t^T \bar{U}_s \mu(ds) | \bar{\mathcal{F}}_t], \tag{3.3.3}$$

$$\bar{Y}_t = J^\nu(t) + \mathbb{E}^\nu \left[\bar{K}_T - \bar{K}_t | \bar{\mathcal{F}}_t \right] - \mathbb{E}^\nu \left[\int_t^T \bar{U}_s \nu_s \lambda_s ds | \bar{\mathcal{F}}_t \right], \tag{3.3.4}$$

and therefore, by recalling that $\nu > 0$, $\bar{K}$ is non-decreasing and $\bar{U}$ satisfies the sign constraint (3.2.2), we get the first inequality, i.e.

$$\bar{Y}_t \geq Y_t^*. \tag{3.3.5}$$

Conversely, since $\mathcal{V}^n \subset \mathcal{V}$, it is clear from proposition 3.2.1 that $\bar{Y}_t^n \leq Y_t^*$, for all n. Now, recalling the results in theorem 3.2.1, we know that $\bar{Y}_t$ is the pointwise limit of $\bar{Y}_t^n$, then

$$\bar{Y}_t = \lim_{n \to \infty} \bar{Y}_t^n \leq Y_t^*, \tag{3.3.6}$$

and thus

$$\bar{Y}_t = Y_t^*, \tag{3.3.7}$$

from which we have condition (3.3.1). As far as condition (3.3.2) is concerned, we observe that the inequality (3.3.5) holds for any other quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ satisfying (3.2.1), (3.2.2), i.e. $\tilde{Y}_t \geq Y_t^*$, thus we have from (3.3.7) that

$$\bar{Y}_t \leq \tilde{Y}_t \tag{3.3.8}$$

which gives the minimality of the solution. We then conclude the discussion by recalling that the property of minimality assures uniqueness of the solution $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$.

Chapter 4

From the Reflected BSDE to the BSDE with the sign constraint

From what we have largely shown in previous sections, one can see that optimal stopping is a topic closely connected to the Backward Stochastic Differential Equations theory. In this work, we have given two different characterizations of the optimal stopping problem. First we presented the standard formulation, where we emphasize the fact that the probability space on which the processes are defined is fixed; then we provided a new *alternative* formulation, in which the space varies among a suitable family of probability measures. After that, we provided an interpretation for the latter formulation of the stopping problem as the solution of a BSDE with a sign constraint, the former being greatly analysed in the literature, and already solved by the Reflected BSDE approach. Therefore, to conclude with the study, we ask ourselves whether and how a relation can be found between the two BSDEs, connected respectively with the standard and alternative formulation of the same problem. We aim to show they assume the same value in the origin, which would mean providing an alternative solution to the optimal stopping problem, in the general context of non-diffusive processes.

4.1 An explicit connection for the BSDEs

We shall compare the respective penalized BSDEs, from which the result may be obtained as the limit of the sequences, instead of directly analyzing the two BSDEs (1.2.13), (1.2.14) and (3.2.1), (3.2.2) in which, we recall, unicity involves different properties of minimality.
We observe that the two solutions are defined on different probability spaces. In particular, the first sequence $(Y^n, Z^n)_n$, which verifies

$$Y_t^n = \xi + \int_t^T f_s ds + n \int_t^T (Y_s^n - h_s)^- ds - \int_t^T Z_s^n dW_s, \quad 0 \le t \le T, \quad (4.1.1)$$

is defined on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Instead, the second sequence $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)_n$, which verifies

$$\bar{Y}_t^n = \xi 1_{\eta \geq T} + \int_t^T f_s 1_{[0,\eta)}(s) ds + \int_t^T h_s \mu(ds) + n \int_t^T \left(\bar{U}_s^n\right)^+ 1_{[0,\eta)}(s) ds + \\ - \int_t^T \bar{Z}_s^n dW_s - \int_t^T \bar{U}_s^n \mu(ds), \; 0 \leq t \leq T, \quad (4.1.2)$$

is defined on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$. More precisely, we recall that

$$\mathcal{F}_t^0 = \sigma(W_s, 0 \leq s \leq t), \quad \mathcal{F}_t = \sigma(\mathcal{F}_t^0, \mathcal{N}), \; 0 \leq t \leq T, \qquad (4.1.3)$$

where we denote $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, while

$$\bar{\mathcal{F}}_t^0 = \mathcal{F}_t^0 \otimes \mathcal{F}_t^{0'}, \quad \bar{\mathcal{F}}_t = \sigma(\bar{\mathcal{F}}_t^0, \mathcal{N}), \; 0 \leq t \leq T, \qquad (4.1.4)$$

and respectively $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{0 \leq t \leq T}$.
Therefore, in order to make the comparison possible, we will extend for each $n \in \mathbb{N}$, the pair (Y^n, Z^n) and the triple (ξ, f, h) from $(\Omega, \mathcal{F}, \mathbb{F}, P)$ to $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$. In particular, by writing

- $Y_t^n(\omega, \omega') = Y_t^n(\omega)$;
- $Z_t^n(\omega, \omega') = Z_t^n(\omega)$;
- $\xi(\omega, \omega') = \xi(\omega)$;
- $f_t(\omega, \omega') = f_t(\omega)$;
- $h_t(\omega, \omega') = h_t(\omega), \quad (\omega, \omega') \in \bar{\Omega}, \; 0 \leq t \leq T.$

we remember that (according to the Fubini Tonelli theorem and product σ-field theory) all measurability and integrability properties are preserved (with respect to the corresponding spaces). Let us define $\tilde{Y}^n$, $\tilde{Z}^n$, $\tilde{U}^n$ on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ as

$$\tilde{Y}_t^n = Y_t^n 1_{[0,\eta]}(t) \qquad (4.1.5)$$

$$\tilde{U}_t^n = (h_t - Y_t^n) \qquad (4.1.6)$$

$$\tilde{Z}_t^n = Z_t^n 1_{[0,\eta]}(t), \qquad (4.1.7)$$

for any $0 \leq t \leq T$. By recalling the properties of the solution processes Y^n and Z^n (see Chapter 1), we observe that we have a granted predictability for $\tilde{Z}^n$ and $\tilde{U}^n$.
We aim to prove that the triple $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$, as defined in (4.1.5), (4.1.6), (4.1.7) satisfies the penalized BSDE in (4.1.2).
Proof. We fix $t \in [0, T]$, and give the following partition of $\bar{\Omega}$, recalling that η takes its value in $(0, \infty)$.

- $E_1(t) = \{(\omega, \omega') \in \bar{\Omega} : t \le T \le \eta(\omega')\}$;
- $E_2(t) = \{(\omega, \omega') \in \bar{\Omega} : t \le \eta(\omega') < T\}$;
- $E_3(t) = \{(\omega, \omega') \in \bar{\Omega} : \eta(\omega') < t \le T\}$.

In the first case, it clearly holds that $\tilde{Y}^n_t = Y^n_t$, $\big(\bar{\mathbb{P}} - a.s.\ on\ E_1(t)\big)$. In particular, by substituting the penalized BSDE (4.1.1) into (4.1.5), we have

$$\tilde{Y}^n_t = Y^n_t = \xi + \int_t^T f_s ds + n \int_t^T (Y^n_s - h_s)^- ds - \int_t^T Z^n_s dW_s,$$

which can be rewritten as

$$\tilde{Y}^n_t = \xi + \int_t^T f_s ds + n \int_t^T (h_s - Y^n_s)^+ ds - \int_t^T Z^n_s dW_s,$$

thus, by (4.1.6), (4.1.7)

$$\tilde{Y}^n_t = \xi + \int_t^T f_s ds + n \int_t^T (\tilde{U}^n_s)^+ ds - \int_t^T \tilde{Z}^n_s dW_s, \ \ on\ E_1(t).$$

Thus the first proof trivially follows by looking at (4.1.2) in the case $\eta \ge T$, where

$$\int_t^T (\bar{U}^n_s)^+ 1_{[0,\eta)}(s) ds = \int_t^T (\bar{U}^n_s)^+ ds, \quad \int_t^T f_s 1_{[0,\eta)}(s) ds = \int_t^T f_s ds,$$

while

$$\int_t^T h_s \mu(ds) \ \ and \ \ \int_t^T \bar{U}^n_s \mu(ds)$$

both assume a null value.
We now look at the second case. Again, by substituting the penalized BSDE (4.1.1) into (4.1.5), we get

$$\tilde{Y}^n_t = \xi + \int_t^T f_s ds + n \int_t^T (h_s - Y^n_s)^+ ds - \int_t^T Z^n_s dW_s.$$

In fact, it still holds that $\tilde{Y}^n_t = Y^n_t$, $\big(\bar{\mathbb{P}} - a.s.\ on\ E_2(t)\big)$, but in this case we have $t \le \eta < T$. The integral terms can be split as follows

$$\tilde{Y}^n_t = \int_t^\eta f_s ds + n \int_t^\eta (h_s - Y^n_s)^+ ds - \int_t^\eta Z^n_s dW_s + \left(\xi + \int_\eta^T f_s ds + n \int_\eta^T (h_s - Y^n_s)^+ ds - \int_\eta^T Z^n_s dW_s \right),$$

thus, according to the definition of Y^n_t,

$$\tilde{Y}^n_t = \int_t^\eta f_s ds + n \int_t^\eta (h_s - Y^n_s)^+ ds - \int_t^\eta Z^n_s dW_s + Y^n_\eta.$$

Moreover, since η is a stopping time, we can write

$$\int_t^\eta Z_s^n dW_s \ \ as \ \ \int_t^T Z_s^n 1_{[0,\eta]}(s) dW_s,$$

and clearly also

$$\int_t^\eta f_s ds \ \ as \ \ \int_t^T f_s 1_{[0,\eta)}(s) ds,$$

$$\int_t^\eta (h_s - Y_s^n)^+ ds \ \ as \ \ \int_t^T (h_s - Y_s^n)^+ 1_{[0,\eta)}(s) ds.$$

Therefore, by (4.1.6), (4.1.7)

$$\tilde{Y}_t^n = \int_t^T f_s 1_{[0,\eta)}(s) ds + n \int_t^T (\tilde{U}_s^n)^+ 1_{[0,\eta)}(s) ds - \int_t^T \tilde{Z}_s^n dW_s + (h_\eta - \tilde{U}_\eta^n).$$

In conclusion, by observing that

$$(h_\eta - \tilde{U}_\eta^n) = \int_t^T (h_s - \tilde{U}_s^n) \mu(ds) \ \ on \ E_2(t),$$

we finally have

$$\tilde{Y}_t^n = \int_t^T f_s 1_{[0,\eta)}(s) ds + \int_t^T h_s \mu(ds) + n \int_t^T (\tilde{U}_s^n)^+ 1_{[0,\eta)}(s) ds - \int_t^T \tilde{Z}_s^n dW_s - \int_t^T \tilde{U}_s^n \mu(ds)$$

from which the required condition is still satisfied on $E_2(t)$.
In the last case, by construction, the process $\tilde{Y}^n$ assumes $\big(\bar{\mathbb{P}} - a.s.\big)$ a null value. Therefore, in order to prove the equivalence with the solution in (4.1.2), we need to show that for $\eta < t$ $\bar{Y}^n$ takes a null value as well.
From (4.1.2), we have

$$\bar{Y}_t^n = -\int_t^T \bar{Z}_s^n dW_s \ \ on \ E_3(t),$$

and then by applying the conditional expectation (observe that $E_i(t) \in \bar{\mathcal{F}}_t, i = 1, 2, 3$) we get

$$\bar{Y}_t^n 1_{E_3(t)} = \mathbb{E}\left[\bar{Y}_t^n 1_{E_3(t)} \middle| \bar{\mathcal{F}}_t\right] = \mathbb{E}\left[\left(-\int_t^T \bar{Z}_s^n dW_s\right) |\bar{\mathcal{F}}_t\right] 1_{E_3(t)},$$

thus obtaining

$$\bar{Y}_t^n = 0, \quad on \ E_3(t).$$

In this way we have identified $(\tilde{Y}^n, \tilde{Z}^n, \tilde{U}^n)$ with the solution $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ to the penalized BSDE (4.1.2), for each $n \in \mathbb{N}$. Thus by looking at the previous relations, we rewrite

$$\bar{Y}^n_t = Y^n_t 1_{[0,\eta]}(t), \tag{4.1.8}$$
$$\bar{U}^n_t = (h_t - Y^n_t), \tag{4.1.9}$$
$$\bar{Z}^n_t = Z^n_t 1_{[0,\eta]}(t), \tag{4.1.10}$$

and also, by recalling definitions of K^n, $\bar{K}^n$ in (1.2.10),(3.2.7)

$$\bar{K}^n_t = K^n_{t\wedge\eta}. \tag{4.1.11}$$

Hence, by passing to the (weak) limit in $\mathbb{L}^2(\bar{\Omega}, \bar{\mathcal{F}}_t)$ on both sides of the above equalities, we obtain

$$\bar{Y}_t = Y_t 1_{[0,\eta]}(t), \tag{4.1.12}$$
$$\bar{U}_t = (h_t - Y_t), \tag{4.1.13}$$
$$\bar{Z}_t = Z_t 1_{[0,\eta]}(t), \tag{4.1.14}$$
$$\bar{K}_t = K_{t\wedge\eta}. \tag{4.1.15}$$

Let us consider this result: the above conditions mean that, starting from a Reflected BSDE like (1.2.13), (1.2.14), we are able to point out, by suitable adjustements, a (minimal) solution to the BSDE with a sign constraint satisfying (3.2.1), (3.2.2). In particular, the proof provides an explicit relationship holding between the two BSDEs.

In conclusion, recalling that BSDEs (1.2.13), (1.2.14), and (3.2.1), (3.2.2) represent the value functions of the associated optimal stopping problem, we obtain that

$$V_t = Y_t = ess\sup_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\int_t^{\tau\wedge T} f_s ds + h_\tau 1_{\tau<T} + \xi 1_{\tau\geq T} | \mathcal{F}_t\right], \tag{4.1.16}$$

$$\bar{V}_t = \bar{Y}_t = ess\sup_{\nu\in\mathcal{V}} \mathbb{E}^\nu \left[\int_t^T f_s\lambda_s ds + h_\eta 1_{t\leq\eta<T} + \xi 1_{\eta\geq T} | \bar{\mathcal{F}}_t\right] \tag{4.1.17}$$

can in turn be put into relation. Again by looking at (4.1.8) (and recalling η to be valued in $(0,\infty)$), we note that it holds

$$\bar{Y}^n_0 = Y^n_0 \tag{4.1.18}$$

i.e. for all $n \in \mathbb{N}$, $\bar{Y}^n$ and Y^n assume the same (deterministic) value in the origin. Then, passing to the pointwise limit in $t = 0$ (we can do it according to theorem 3.2.1), we have

$$\bar{Y}_0 \leftarrow \bar{Y}^n_0 = Y^n_0 \rightarrow Y_0, \tag{4.1.19}$$

from which we obtain that the two value functions

$$V_0 = Y_0 = \sup_{\tau\in\mathcal{T}} E\left[\int_0^{\tau\wedge T} f_s ds + h_\tau 1_{\tau<T} + \xi 1_{\tau\geq T}\right], \tag{4.1.20}$$

$$\bar{V}_0 = \bar{Y}_0 = \sup_{\nu \in \mathcal{V}} \mathbb{E}^{\nu} \left[\int_0^T f_s \lambda_s ds + h_\eta 1_{\eta < T} + \xi 1_{\eta \geq T} \right] \tag{4.1.21}$$

assume the same value in the origin.

Chapter 5

The Markovian case

Let us now move away from a general context and focus on the Markovian context, as a large part of optimal stopping problems over finite or infinite time horizons which arise in science and finance are related to Ito's diffusion processes. In the literature, the value function of these problems is usually expressed as a generalized solution of a Hamilton-Jacobi-Bellman variational inequality (HJBVI) that involves a second order parabolic partial differential equation in a finite dimensional Euclidean space. In this chapter, dealing with finite horizon problems, we will first introduce the setting and state the fundamental conditions under which the above result holds; afterwards, we will show how, after a suitable arrangement which fits the required conditions, the stopping problem and thus the BSDE introduced into our study can in turn be expressed as a generalized solution of the HJBVI mentioned above.

5.1 Introduction

The following exposition is mainly extracted from the work in [24] (where a more complex setting is studied, concerning the optimal stopping problems of controlled jump diffusion processes), and [25].
Let us briefly recall the stochastic background for the problem.

We fix $T > 0$ and we denote $t \in [0, T]$, T as respectively the initial and terminal time of the finite horizon optimal stopping problem. Let $(W_t)_{0 \leq t \leq T}$ be a d-dimensional standard Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the natural augmented filtration of W, satisfying the *usual conditions*.
We consider the SDE valued in $\mathbb{R}^n$

$$dX_s = b(t, X_s)dt + \sigma(s, X_s)dW_s, \ \ s \geq 0, \tag{5.1.1}$$

where the process X denotes the state of the system, while b and σ are deterministic Borelian functions on $[0, T] \times \mathbb{R}^n$ valued respectively in $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$, satisfying the

assumptions detailed in the next paragraph.
The finite horizon optimal stopping problem is described by the value function

$$v(t,x) = sup\ _{\tau \in \mathcal{T}_t^T} \mathbb{E}\left[\int_t^\tau u(s, X_s^{t,x})ds + w(X_\tau^{t,x})\right], \tag{5.1.2}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$, where u and w are deterministic real valued functions satisfying the assumptions detailed in the next paragraph, $X^{t,x}$ is a diffusion process satisfying an SDE like (5.1.1) which takes value x at time t, and $\mathcal{T}_t^T$ is the set of all stopping times with respect to $\mathbb{F}$, valued in $[t,T]$.
The Hamilton-Jacobi-Bellman equation associated with this problem is a variational inequality (HJBVI) involving a second order partial parabolic differential equation such as

$$min\left\{-\frac{\partial v(t,x)}{\partial t} - \mathcal{L}v(t,x) - u(t,x), v(t,x) - w(x)\right\} = 0 \tag{5.1.3}$$

with the terminal data

$$v(T,x) = w(x). \tag{5.1.4}$$

Here $\mathcal{L}$ is the second order operator associated with the diffusion $X^{t,x}$:

$$\mathcal{L}v(t,x) = b(t,x).D_x v(t,x) + tr\left(\frac{1}{2}\sigma\sigma'(t,x)D_x^2 v(t,x)\right). \tag{5.1.5}$$

As is well known from the literature, there is generally no strong solution for the equation (5.1.3), so that we are forced to introduce the concept of *viscosity* solution as a notion of a weak solution, and to provide an existence and uniqueness result in this sense. Usually, in the theory about optimal stopping time problems of diffusion processes, one can find an additional exponential term characterizing the value function, denoted as *discount factor*, which involves a reaction term in the HJB equation. We will not take this term into consideration, as we suppose it merely assumes a unitary value.

5.2 Viscosity solutions and optimal stopping

Let us recall the concept of the strong solution for SDEs.

Definition 5.2.1. (Strong solution for SDEs). Let us take $t \in [0,T]$ and $x \in \mathbb{R}^n$. A strong solution $X^{t,x}$ for the SDE (5.1.1) is a vectorial progressively measurable process starting at time t with value x such that

$$\int_t^s |b(u, X_u^{t,x})|du + \int_t^s |\sigma(u, X_u^{t,x})|^2 du < \infty,\ \forall t \leq s \in [0,T], P-a.s., \tag{5.2.1}$$

and the following relation holds

$$X_s^{t,x} = x + \int_t^s b(u, X_u^{t,x})du + \int_t^s \sigma(u, X_u^{t,x})dW_u,\ t \le s \in [0,T],\ P - a.s. \quad (5.2.2)$$

Let us set the fundamental requirements **(Hypothesis H2)**.

We assume that b, σ, u and w are continuous deterministic functions with respect to (t, x). Furthermore, there exists $K > 0$, such that for all $t, s \in [0,T]$, $x, y \in \mathbb{R}^n$

- $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \le K|x - y|$;
- $|b(t,x)| + |\sigma(t,x)| \le K(1 + |x|)$;
- $|u(t,x) - u(s,y)| + |w(x) - w(y)| \le K\left[|t - s| + |x - y|\right]$;
- $|u(t,x)| + |w(x)| \le K(1 + |x|)$.

In particular, assumptions made on b, σ (Lipschitz and linear growth conditions) ensure existence and uniqueness for the solution to the SDE (5.1.1) in the sense of definition 5.2.1.
Let us now focus on the continuity and viscosity properties of the value function. As it is well known, by **Hypothesis H2** on coefficients b, σ, f and g, the value function v in (5.1.2) is continuous and satisfies a Lipschitz condition in x uniformly in t. In particular, it holds the following:

Proposition 5.2.1. *The stated conditions hold. Then $v \in C^0([0,T] \times \mathbb{R}^n)$. More precisely, there exists a constant $C > 0$ such that for all $t, s \in [0,T]$, $x, y \in \mathbb{R}^n$,*

$$|v(t,x) - v(s,y)| \le C\left[(1 + |x|)|t - s|^{\frac{1}{2}} + |x - y|\right]. \quad (5.2.3)$$

We now introduce the concept of a viscosity solution.

Definition 5.2.2. (Viscosity solution). We say that a continuous function $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution (subsolution) to the HJB variational inequality (5.1.3) if and only if

$$min\{-\frac{\partial \phi(t,x)}{\partial t} - b(t,x).D_x\phi(t,x) + \\ + tr\left(\frac{1}{2}\sigma\sigma'(t,x)D_x^2\phi(t,x)\right) - u(t,x), v(t,x) - w(x)\} \ge 0 (\le 0) \quad (5.2.4)$$

whenever $\phi \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and $v - \phi$ has a global minimum (maximum) at $(t,x) \in [0,T) \times \mathbb{R}^n$. v is a viscosity solution of (5.1.3) if is both super and subsolution.

Hence the following fundamental result, which is proved as a consequence of the *Dynamic programming principle* (see [24]).

Theorem 5.2.1. *Under* ***Hypothesis H2****, the value function* v *in* (5.1.2) *is a viscosity solution for the HJB variational inequality in* (5.1.3)*.*

Moreover, it can be proved that v is the *unique* viscosity solution in the class of functions satisfying a linear growth condition.
From theorem 5.2.1 a very important application then follows of optimal stopping problems: they provide a unique (weak) solution of a HJBVI, which involves a solution to a second order partial parabolic differential equation.
The rest of the section is devoted to highlighting the connection between our BSDE and the HJBVI, after giving a suitable rearrangement of the general optimal stopping problem presented in (1.1.2).
We recall the original formulation.

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be the usual filtered probability space, where $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is the augmented filtration generated by a d-dimensional Brownian Motion $(W_t)_{0 \le t \le T}$. The value function V was expressed as

$$V_t = ess \sup_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^{\tau \wedge T} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau \ge T} | \mathcal{F}_t\right]. \tag{5.2.5}$$

First of all, we put $\tau' = \tau \wedge T$, observing that τ' is still a stopping time valued in $[t, T]$. Moreover, being $\mathcal{F}_\tau$ the filtration generated by τ,[1] it holds that $\mathcal{F}_{\tau'} = \mathcal{F}_{\tau \wedge T} = \mathcal{F}_\tau$, since T is deterministic. By construction, it clearly holds that

- $\{\omega \in \Omega : \tau'(\omega) < T\} \equiv \{\omega \in \Omega : \tau(\omega) < T\}$,
- $\{\omega \in \Omega : \tau'(\omega) = T\} \equiv \{\omega \in \Omega : \tau(\omega) \ge T\}$.

Thus, we rewrite the value function in the following way,

$$V_t = ess \sup_{\tau \in \mathcal{T}_t^T} \mathbb{E}\left[\int_t^{\tau} f_s ds + h_\tau 1_{\tau < T} + \xi 1_{\tau = T} | \mathcal{F}_t\right], \tag{5.2.6}$$

noting that, for simplicity, we have called τ' again as τ.

[1] See Appendix A

It is now more evident that the results shown in the previous section can be applied to our stopping problem in the case where

- $f_s(\omega) = f(s, X_s^{t,x}(\omega))$,
- $h_s(\omega) = h(X_s^{t,x}(\omega))$,
- $\xi(\omega) = g(X_T^{t,x}(\omega))$,

with f, h deterministic continuous functions respectively on $[0,T] \times \mathbb{R}^n$, $\mathbb{R}^n$ into $\mathbb{R}$, composed with a diffusion process $X^{t,x}$, kept as in definition 5.1.1 (with coefficients b, σ satisfying **Hypothesis H2**), and g a measurable function on $\mathbb{R}^n$ satisfying a linear growth condition, with $g \geq h$. Moreover, f, h satisfy for all $(t,s) \in [0,T]$, $x, y \in \mathbb{R}$,

- $|f(t,x) - f(s,y)| + |h(x) - h(y)| \leq K\,[|t-s| + |x-y|]$,
- $|f(t,x)| + |h(x)| \leq K(1 + |x|)$,

for some $K > 0$. In this way **Hypothesis H2** is fully satisfied. In this context, the functional gain can be rewritten as

$$J(t,x,\tau) = \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})ds + h(X_\tau^{t,x})1_{\tau<T} + g(X_\tau^{t,x})1_{\tau=T}|\mathcal{F}_t\right], \tag{5.2.7}$$

Now, by observing that the diffusion process $X^{t,x}$ starts at time t with a deterministic value x, the conditional expectation w.r.t. $\mathcal{F}_t$ is itself a deterministic value, which can be rewritten as

$$J(t,x,\tau) = \mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})ds + h(X_\tau^{t,x})1_{\tau<T} + g(X_\tau^{t,x})1_{\tau=T}\right], \tag{5.2.8}$$

and finally

$$v(t,x) = sup_{\tau\in\mathcal{T}_t^T}\mathbb{E}\left[\int_t^\tau f(s, X_s^{t,x})ds + h(X_\tau^{t,x})1_{\tau<T} + g(X_\tau^{t,x})1_{\tau=T}\right]. \tag{5.2.9}$$

The reader may observe that, in this way, we pointed out a formulation similar to the one in (5.1.2), therefore theorem 5.2.1 can be applied, obtaining that the value function v in (5.2.9) is the *unique* viscosity solution to the following HJBVI

$$\begin{aligned} min\{-\frac{\partial v(t,x)}{\partial t} - b(t,x).D_x v(t,x)+ \\ + tr\left(\frac{1}{2}\sigma\sigma'(t,x)D_x^2 v(t,x)\right) - f(t,x), v(t,x) - h(x)\} = 0 \end{aligned} \tag{5.2.10}$$

with the terminal data

$$v(T, x) = g(x). \tag{5.2.11}$$

In the sequel, we sum up the latter result with the one discussed in section 1.2.2. By putting the reflected BSDEs (1.2.13), (1.2.14) in the current setting, we have a triple $(Y^{t,x}, Z^{t,x}, K^{t,x})$, satisfying

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(u, X_u^{t,x})\, du + K_T^{t,x} - K_s^{t,x} - \int_s^T Z_u^{t,x}\, dW_u, \tag{5.2.12}$$

$$Y_s^{t,x} \geq h(X_s^{t,x}),\ t \leq s \leq T,\ dP - a.s.\ on\ \Omega, \tag{5.2.13}$$

again with the *Skohorod* condition

$$\int_t^T \left(Y_s^{t,x} - h(X_s^{t,x}) \right) dK_s^{t,x} = 0. \tag{5.2.14}$$

Where f, h, g and $X^{t,x}$ are given by (5.2.9); in particular, they satisfy **Hypothesis H2**. Then, by the Markov property of $X^{t,x}$, and uniqueness of the solution to the Reflected BSDE, we have that

$$v(t, x) = Y_t^{t,x} \tag{5.2.15}$$

is a deterministic function of $(t, x) \in [0, T] \times \mathbb{R}^n$, which satisfies the following

Theorem 5.2.2. *The function* $v(t, x) = Y_t^{t,x}$ *in* (5.2.15) *is continuous on* $[0, T] \times \mathbb{R}^n$, *and is a viscosity solution to the HJB variational inequality in* (5.2.10).

An explicit solution to HJBVI

Even though it can be drawn as a direct consequence of the facts discussed above, we consider it significant to highlight the explicit connection between the BSDEs studied in this work and viscosity solutions to HJB variational inequalities.
We thus report the following results

Definition 5.2.3. Given $(t, x) \in [0, T] \times \mathbb{R}^n$, f, h, g, $X^{t,x}$ as in the previous section, we call a minimal solution to the BSDE with a sign constraint the quadruple of processes $(\bar{Y}^{t,x}, \bar{Z}^{t,x}, \bar{K}^{t,x}, \bar{U}^{t,x}) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu}) \times \mathbb{K}^2$ satisfying

$$\begin{aligned} \bar{Y}_s^{t,x} = g(X_T^{t,x}) 1_{\eta \geq T} + \int_s^T f(u, X_u^{t,x}) \lambda_s ds + \int_s^T h(X_u^{t,x}) \mu(du) + \\ + \bar{K}_T^{t,x} - \bar{K}_s^{t,x} - \int_s^T \bar{Z}_u^{t,x} dW_u - \int_s^T \bar{U}_u^{t,x} \mu(du), \quad t \leq s \leq T, \end{aligned} \tag{5.2.16}$$

with the sign constraint

$$\bar{U}_s^{t,x} \leq 0, \quad ds \times d\mathbb{P} \; a.s. \; on \; \bar{\Omega} \times [t,T]. \tag{5.2.17}$$

Moreover, for any other quadruple $(\tilde{Y}^{t,x}, \tilde{Z}^{t,x}, \tilde{K}^{t,x}, \tilde{U}^{t,x}) \in \mathbb{S}^2 \times \mathbb{L}^2(W) \times \mathbb{L}^2(\tilde{\mu}) \times \mathbb{K}^2$ satisfying (5.2.16),(5.2.17), it holds that

$$\bar{Y}_s^{t,x} \leq \tilde{Y}_s^{t,x}, \quad t \leq s \leq T, \; \bar{\mathbb{P}} - a.s..$$

Here $\eta = \eta + t : \Omega' \to (t, \infty)$ denotes a random variable exponentially distributed with mean $1 + t$, while λ, μ are respectively the stochastic intensity and the random measure related to the point process

$$N_s = 1(\eta \leq s), \quad t \leq s \leq T. \tag{5.2.18}$$

According to the same arguments of chapter 3, the following holds:

Theorem 5.2.3. *The above conditions hold. Then there exists a unique minimal solution* $(\bar{Y}^{t,x}, \bar{Z}^{t,x}, \bar{K}^{t,x}, \bar{U}^{t,x})$ *to the BSDE* (5.2.16) (5.2.17).

We now put in the current setting the relation pointed out in chapter 4 between the reflected BSDE and the BSDE with sign constraint, thus obtaining for the component $\bar{Y}^{t,x}$

$$\bar{Y}_s^{t,x} = Y_s^{t,x} 1_{[0,\eta]}(s), \quad t \leq s \leq T. \tag{5.2.19}$$

The reader may observe that, since it holds that $\{\eta(\omega') > t, \; \forall \omega' \in \Omega'\}$, then both $\bar{Y}^{t,x}$ and $Y^{t,x}$ assume the same (deterministic) value in t. Hence, by recalling theorem 5.2.2, we have

$$\bar{Y}_t^{t,x} = Y_t^{t,x} = v(t,x) \tag{5.2.20}$$

where we remember that $v(t,x)$ is the value function in (5.2.9). Thus, we state the following

Theorem 5.2.4. *The function* $v(t,x) = \bar{Y}_t^{t,x}$ *in* (5.2.20) *is continuous on* $[0,T] \times \mathbb{R}^n$*, and is a viscosity solution to the HJB variational inequality in* (5.2.10).

Which provides an interesting application of the developed alternative BSDE approach to optimal stopping problems and related PDEs of Hamilton-Jacobi-Bellman type.

Conclusions

In this work we have studied the optimal stopping problem by means of a particular BSDE approach. From a computational point of view, BSDEs appear as a very powerful tool to solve PDEs, especially of Hamilton-Jacobi-Bellman type. A large amount of literature is available in this context; for istance, we refer the reader to the works of Gobet [12], [14] and [13], concerning the numerical solution of different classes of BSDEs.
In our case, with the results introduced in these chapters, we encourage further developments in the study of BSDEs with jumps and a sign constraint and their application to stochastic control and related PDEs. In particular, as shown in chapter 5, this class of BSDEs provides, as an alternative to the Reflected BSDEs, a probabilistic representation for the viscosity solution of a HJB variational inequality, which involves a partial parabolic differential equation. Thus, the question arises how to choose between BSDEs with a sign constraint and Reflected BSDEs, and what are the advantages of the former and the latter. On the other hand, one can also find an interesting application of these results to the development of a non-Markovian context for optimal stopping problems, or to Markov processes which are not Ito-diffusion processes anymore.

Appendices

Appendix A

Stochastic Processes

In this section we are going to establish some basic notations about *filtrations*, *stopping times* and *martingales*, as they are constantly used in the rest of the work.

A.1 Measurability

Let $(\Omega, \mathcal{F})$ be a measurable space. A *filtration* is an increasing family $\mathbb{F} : (\mathcal{F}_t)_{0 \leq t \leq T}$ of σ-fields of $\mathcal{F}$, i.e. such that, for all $0 \leq s \leq t \leq T$, (in general, t could vary in a generic time interval $\mathbb{T}$: here we always refer to the interval $[0, T]$), it holds that

$$\mathcal{F}_s \subset \mathcal{F}_t. \tag{A.1.1}$$

In our work we always refer to filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ which satisfies the *usual conditions*, which means that the respective filtrations are *right continuous*, i.e.

$$\mathcal{F}_t^+ := \bigcap_{h>0} \mathcal{F}_{t+h} = \mathcal{F}_t, \tag{A.1.2}$$

and *complete*, i.e. $\mathcal{F}_0$ contains the negligible sets of $\mathcal{F}_T$. An *E-valued process* $X = (X_t)_{0 \leq t \leq T}$ is a family of random variables on $(\Omega, \mathcal{F})$ indexed by time t, which varies among $[0, T]$, and valued in a measurable space $(E, \mathcal{E})$. The canonical example of *filtration* is given by the one *generated* by the process X: we call $\mathbb{F}^X = (\mathcal{F}_t^X)_{0 \leq t \leq T}$ the following

$$\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t), \quad t \in [0, T], \tag{A.1.3}$$

that is, the smallest σ-field under which X_s is measurable for all $0 \leq s \leq t$. One can also refer to $\mathbb{F}^X$ as the *natural filtration of the process X*.
Let us now establish some measurability notations. We recall that, given a process X and an element ω of Ω, the mapping

$$X(\omega) : t \in [0, T] \to X_t(\omega) \tag{A.1.4}$$

is said to be the *path* of the process X. We refer to a *continuous* (*càd-làg*) process if and only if the mapping $X(\omega)$ is continuous (right continuous and left limited) almost surely w.r.t. the probability measure P. In the sequel, we are given a filtration $\mathbb{F} : (\mathcal{F}_t)_t$ on $(\Omega, \mathcal{F}, P)$.

Definition A.1.1. (Adapted process). A process $X = (X_t)_{0 \le t \le T}$ which is $\mathcal{F}_t$-measurable for all $t \in [0, T]$ is called $\mathbb{F}$-*adapted*.

Definition A.1.2. (Progressively measurable process). A process $X = (X_t)_{0 \le t \le T}$, whose mapping $(t, \omega) \to X_t(\omega)$ is measurable on $([0, t] \times \Omega, \mathcal{B}([0, t]) \times \mathcal{F}_t)$ for any $0 \le t \le T$, is called $\mathbb{F}$-*progressive*.

Definition A.1.3. (Optional process). A process $X = (X_t)_{0 \le t \le T}$, whose mapping $(t, \omega) \to X_t(\omega)$ is measurable on $([0, T] \times \Omega)$ equipped with the σ-field generated by the $\mathbb{F}$-adapted and càd-làg processes is called $\mathbb{F}$-*optional*.

Definition A.1.4. (Predictable process). A process $X = (X_t)_{0 \le t \le T}$, whose mapping $(t, \omega) \to X_t(\omega)$ is measurable on $([0, T] \times \Omega)$ equipped with the σ-field generated by the $\mathbb{F}$-adapted and continuous processes is called $\mathbb{F}$-*predictable*.

A.2 Stopping Times

We remind the reader that $\mathcal{F}_t$ could be interpreted as the information known at time t. When referring to an occurence of a certain event up to time t, we refer to the notion of *stopping time*: a random variable $\tau : \Omega \to \mathbb{R}^+$ such that, for all $0 \le t \le T$

$$\{\omega \in \Omega : \tau(\omega) \le t\} \in \mathcal{F}_t. \tag{A.2.1}$$

The information available at time t allows us to decide whether or not this event has occurred before t. For any $0 \le s \le t$, we call $\mathcal{T}_s^t$ the set of all stopping times (w.r.t. the filtration $\mathbb{F}$), with values in $[s, t]$. We observe that any deterministic time t is a stopping time. Moreover, given two stopping times τ, σ, then $\tau \wedge \sigma = \min(\tau, \sigma)$ is still a stopping time.
One can also define the *σ-field* $\mathcal{F}^\tau$ associated with a stopping time τ, which provides the information cumulated until τ, in the following form

$$\mathcal{F}^\tau = \{A \in \cup_{0 \le t \le T} \mathcal{F}_t : A \cap \{\tau \le t\} \in \mathcal{F}_t, \quad \forall t \in [0, T]\}, \tag{A.2.2}$$

and also, respectively

$$\mathcal{F}^{\tau^+} = \{A \in \cup_{0 \le t \le T} \mathcal{F}_t : A \cap \{\tau \le t\} \in \mathcal{F}_{t^+}, \quad \forall t \in [0, T]\}, \tag{A.2.3}$$

$$\mathcal{F}^{\tau^-} = \mathcal{F}_0 \vee \{A \in \cup_{0 \le t \le T} \mathcal{F}_t : A \cap \{\tau < t\} \in \mathcal{F}_t, \quad \forall t \in [0, T]\}. \tag{A.2.4}$$

Note that a deterministic time t is a stopping time and its associated σ-field is simply $\mathcal{F}^t$. We now introduce two specific classes of stopping times.

Definition A.2.1. (Predictable time). A random variable $T : \Omega \to \mathbb{R}^+$ is called a *predictable time*, if the following set

$$\{(\omega, t) \in \Omega \times \mathbb{R}^+ : U(\omega) < t \leq V(\omega)\}, \tag{A.2.5}$$

where U, V are two $\mathbb{R}^+$-valued functions defined on Ω such that $U \leq V$, is predictable. We note that a predictable time is a stopping time, and a constant stopping time (deterministic) is predictable.

Definition A.2.2. (Accessible time). A stopping time T is called *accessible time*, if there exists a sequence $(T_n)_n$ of predictable times such that

$$\{(\omega, t) \in \Omega \times \mathbb{R}^+ : t = T(\omega)\} \subset \bigcup_n \{(\omega, t) \in \Omega \times \mathbb{R}^+ : t = T_n(\omega)\}. \tag{A.2.6}$$

The notions of *stochastic process* and *stopping time* can be closely related: we report, for example, a useful result which allow us to compound processes with stopping times, thus preserving their own measurability properties.

Proposition A.2.1. *Let $X = (X_t)_{0 \leq t \leq T}$ be a $\mathbb{F}$-progressive stochastic process, and τ a stopping time with values in $\mathbb{R}^+$. Then the random variable $X_\tau 1_{\tau \in [0,T]}$ is $\mathcal{F}^\tau$-measurable, while the stopped process $X^\tau = (X_{t \wedge \tau})_{0 \leq t \leq T}$ is $\mathbb{F}$-progressive.*

We then state the following fundamental

Theorem A.2.1. *(Section theorem). Let X and Y be two $\mathbb{F}$-optional processes. Suppose that for any stopping time τ, we have: $X_\tau = Y_\tau$ almost surely on $\{\tau < \infty\}$, then the path of X and Y coincides almost surely, i.e. the processes X and Y are indistinguishable.*

A.3 Brownian motion and Martingales

The main example of a stochastic process is *Brownian motion*, which is an $\mathbb{R}^d$-valued continuous stochastic process $W = (W_t^1, \dots W_t^d)_{0 \leq t \leq T}$ such that:

- $W_0 = 0$;
- for all $0 \leq s \leq t \in [0, T]$, *the increment* $W_t - W_s$ is independent of $\mathcal{F}_s^W$, where $\mathbb{F}^W = (\mathcal{F}_t^W)_{0 \leq t \leq T}$ denotes the *natural filtration* of W, i.e.

$$\mathcal{F}_t^W = \sigma(W_s,\ 0 \leq s \leq t), \quad 0 \leq t \leq T;$$

- for all $0 \leq s \leq t \in [0, T]$, *the increment* $W_t - W_s$ follows a Gaussian distribution with mean 0 and covariance matrix $I_d(t - s)$.

A very helpful tool in the study of Brownian motions and other branches of stochastic theory was provided by an important class of processes: the so-called *martingales*. An $\mathbb{R}$-valued stochastic process X is called *$(P, \mathcal{F}_t)$-martingale (submartingale), (supermartingale)* over $[0, T]$ if, for all $0 \leq t \leq T$,

- X_t is adapted to $\mathcal{F}_t$;
- X_t is integrable, i.e. $\mathbb{E}\left[|X_t|\right] < \infty$;
- for all $0 \leq s \leq t$, it holds $\mathbb{E}\left[X_t|\mathcal{F}_s\right] = (\geq), (\leq) X_s$.

Where the last condition means that, given all the information up to time s, the best guess for the value of the process at time t, with $t \geq s$ is the current value X_s. The reader may also find in the production only *P-martingale*, or simply *martingale*, when we do not need to highlight the probability space. From the class of martingales one can derive the enlarged class of *local martingales*. That is, given an increasing sequence of stopping times $(\tau_n)_n$, such that $\lim_{n\to\infty} \tau_n = \infty$, an $\mathbb{R}$-valued stochastic process X is called $(P, \mathcal{F}_t)$- *local martingale* if, for all $n \geq 1$, the stopped process

$$X^{\tau_n} = (X_{t\wedge\tau_n})_{0\leq t\leq T}, \tag{A.3.1}$$

is a $(P, \mathcal{F}_t)$- martingale.

Definition A.3.1. (Quadratic variation). Let X be a continuous $(P, \mathcal{F}_t)$-local martingale. we call *quadratic variation* $< X, X >$ the unique (up to indistinguishability) process such that $X^2 - < X, X >$ is a local martingale. In particular, $< X, X >$ is an increasing process, which is also denoted simply by $< X >$. Moreover, for every $t > 0$,

$$< X, X >_t = \lim_{\|t^n\|\to 0} \sum_{k=1}^{n} \left(X_{t_k^n} - X_{t_{k-1}^n} \right)^2 \quad \textit{in Probability.} \tag{A.3.2}$$

where t^n ranges over partitions of the interval [0, t].

Moreover, in the case X is a martingale which verifies $\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^2\right] < \infty$ (in general it is not continuous), it is proved that $< X, X >$ can be identified as the unique predictable integrable increasing process such that $X^2 - < X, X >$ is a uniformly integrable martingale null in 0.

In the following, we make a brief comment on stochastic integral notations.
Let M be such that $\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|^2\right] < \infty$. We call $\mathbb{L}^2(M)$ the space of progressively measurable process α such that $\mathbb{E}\left[\int_0^T |\alpha_t|^2 d < M, M >_t\right] < \infty$: it can be proved that $\mathbb{L}^2(M)$ is a Hilbert space endowed with the scalar product

$$(\alpha, \beta)_{\mathbb{L}^2(M)} = \mathbb{E}\left[\int_0^T \alpha_t \beta_t d < M, M >_t\right]. \tag{A.3.3}$$

Moreover, the set of *simple processes* i.e. $\alpha = (\alpha_t)_{0\leq t\leq T}$, of the form

$$\alpha_t = \sum_{k=1}^{n} \alpha(k) 1_{(t_k,t_{k+1}]}(t), \quad (t_n)_n \in \mathcal{T}_t^T, \;\; \alpha(k)\mathcal{F}_{t_k} - \textit{measurable r.v.} \tag{A.3.4}$$

is dense in $\mathbb{L}^2(M)$. The *stochastic integral* of a simple process α is defined by

$$\int_0^t \alpha_s dM_s = \sum_{k=1}^{n} \alpha(k)(M_{t\wedge t_{k+1}} - M_{t\wedge t_k}), \quad 0 \leq t \leq T, \tag{A.3.5}$$

which is a r.v. $\mathcal{F}_t$-measurable and square integrable; moreover, we have the isometric relation

$$\mathbb{E}\left[\left|\int_0^T \alpha_t dM_t\right|^2\right] = \mathbb{E}\left[\int_0^T |\alpha_t|^2 d < M, M >_t\right], \tag{A.3.6}$$

which can be extended by density to all processes belonging to $\mathbb{L}^2(M)$. In particular, for $M = W$, by recalling that $\{< W, W >_t = t \;\forall\; 0 \leq t \leq T\}$, P-a.s., we get the well-known **Ito Isometry**, i.e.

$$\mathbb{E}\left[\left|\int_t^T \alpha_t dW_t\right|^2 |\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T |\alpha_t|^2 dt |\mathcal{F}_t\right], \quad 0 \leq t \leq T. \tag{A.3.7}$$

We now list some useful results.

Theorem A.3.1. *(Doob's inequality). Let X be a non-negative càd-làg $(P, \mathcal{F}_t)$-martingale. Then, for all $0 \leq t \leq T$,*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|\right]^p \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_T|^p\right], \quad \forall p \geq 1. \tag{A.3.8}$$

Proposition A.3.1. *Let M be a $(P, \mathcal{F}_t)$-local martingale. Suppose that, for all $0 \leq t \leq T$,*

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |M_s|\right] \leq \infty, \tag{A.3.9}$$

i.e. is uniformly integrable, then M is a martingale.

Proposition A.3.2. *Let M be a non-negative $(P, \mathcal{F}_t)$- local martingale, such that M_0 is integrable, i.e. $E[|M_0|] < \infty$. Then M is a $(P, \mathcal{F}_t)$-supermartingale.*

Appendix B

Ito's formula for semimartingales

In the sequel, we are given a probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$.
As we already mentioned in chapter 2, we speak about a *finite variation process* when, given a process $X = (X_t)_{0 \le t \le T}$, every path is càd-làg and with finite variation, that is, the mapping $X(\omega) : t \in [0, T] \to X_t(\omega)$ satisfies

$$\sup \sum_{n=1}^{N} |X_{t_n}(\omega) - X_{t_{n-1}}(\omega)| < \infty, \tag{B.0.1}$$

where the *supremum* is taken from among all subdivisions $(t_n)_n$ of $[0, t]$. We now give the central following

Definition B.0.2. (Semimartingale). A càd-làg $\mathbb{F}$-adapted process X is said to be a *(continuous) semimartingale* if it admits the following decomposition

$$X = X_0 + M + A$$

where M is a càd-làg (continuous) local martingale null in 0 and A is a (continuous) finite variation process.

We always work with such a class of processes. In the case X is a continuous semimartingale, its *quadratic variation* $< X, X >$ coincides with $< M, M >$ in the sense of definition (A.3.1). We now give the Ito formula for semimartingales of the form

$$X = X_0 + M + A,$$

where M is a continuous martingale and A is an adapted finite variation process. Given that f is a real valued function of class $C^{1,2}$, the process $(f(t, X_t))_{0 \le t \le T}$ is a semimartingale which satisfies

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(u,X_u)du + \int_0^t \frac{\partial f}{\partial x}(u,X_u)dM_u + \\ + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(u,X_u)d < M,M >_u + \int_0^t \frac{\partial f}{\partial x}(u,X_u)dA_u^c + \\ + \sum_{0\leq s\leq t} |f(s,X_s) - f(s,X_{s^-})|, \quad 0 \leq t \leq T. \quad \text{(B.0.2)}$$

Where $(X_{t^-})_{0\leq t\leq T}$, denotes the *left limit process* of X, i.e.

$$X_{t^-} = \lim_{s\nearrow t} X_s, \quad 0 \leq t \leq T, \quad \text{(B.0.3)}$$

while A^c denotes the continuous part of A, i.e.

$$A_t^c = A_t - \sum_{0\leq s\leq t} \Delta A_s, \quad \text{(B.0.4)}$$

where

$$\Delta A_s = A_s - A_{s^-}. \quad \text{(B.0.5)}$$

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Ringraziamenti