Politecnico di Milano
Department of Mathematics
Doctoral Programme In Mathematical Models and Methods in Engineering

HNN-extensions of Finite Inverse Semigroups

Doctoral Dissertation of:
Mohammed Abu Ayyash

Supervisor:
Prof. Alessandra Cherubini

Co-advisor:
Dr. Emanuele Rodaro

Tutor:
Prof. Alessandra Cherubini

The Chair of the Doctoral Program:
Prof. Roberto Lucchetti

2014 – XXVII

Dedication

الَى روح امي و روح ابي و اروَاح اخوتي زريق و احمد

ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor Prof. Alessandra Cherubini for the continuous support and tutoring of my Ph.D study and research, for her patience, motivation, enthusiasm, and immense knowledge. She helped and guided me through these years of research and in writing this thesis. I would also like to thank my co-advisor Dr. Emanuele Rodaro who was always available for helping, tutoring, explaining ideas and giving his smart suggestions face to face or through skype for many hours. I could not have imagined having better advisors and mentors for my Ph.D study.

My sincere thanks also to all the Faculty and staff of Politecnico di Milano, especially the Department of Mathematics and the PhD school for their cooperation and support. I thank my colleagues in Politecnico di Milano for their help and in Bethlehem University for encouraging me to gain the opportunity to have my PhD from Italy.

A great thanks and blessing to Br. Robert Smith the former vice president of Bethlehem University for his friendship and cordiality.

I would like to thank the Italian Ministry of Foreign Affairs and Bethlehem University, that through the Italian project E-PLUS (Enhancement of the Palestinian University System) they actually allowed me to embrace this opportunity, and also the PRIN project 2011 "Automi e Linguaggi Formali: Aspetti Matematici e Applicativi" in whose activities I was involved.

Last but not the least, I would like to express my deepest gratitude and acknowledgement to my family: my wife Mariam for taking care of our children and our house alone in Palestine during my studies in Italy. Without these sacrifices this thesis would have not been achieved. Beside my wife, I really owe gratitude to my children for being away from them during my studies and I hope God will help me to compensate them. A word of acknowledgement to my sisters, my nephews and my nieces for supporting me spiritually throughout my studies.

Abstract

The concept of HNN-extensions of groups was introduced by Higman, Neumann and Neumann in 1949. HNN-extensions and amalgamated free products have played a crucial role in combinatorial group theory, especially for algorithmic problems.

In inverse semigroup theory there are many ways of constructing HNN-extension in order to ensure the embeddability of the original inverse semigroup in the new one. For instance, Howie used unitary subsemigroups , N.D. Gilbert used ordered ideals and Yamamura put some conditions on idempotents. In this thesis we adopt Yamamura's definition. Let $S^* = [S; A, B]$ be an HNN-extension of inverse semigroups.

We show that the word problem of HNN-extensions of inverse semigroups can be undecidable even under some nice conditions on S, A and B. Then we consider HNN-extension S^* with S finite, because under such hypothesis the word problem is decidable and we prove that the Schützenberger graph of the elements of S^* is a context-free graph, showing that the language recognized by the Schützenberger automaton is a deterministic context-free language. Moreover, we construct the grammar generating this language.

We characterize the HNN-extensions of finite inverse semigroups which are completely semisimple inverse semigroups, using a characterization of HNN-extensions of finite inverse semigroups which have a copy of the bicyclic monoid as subsemigroup.

Furthermore, we give some properties of the Schützenberger graph of the elements of HNN-extensions of finite inverse semigroups mainly focusing on properties of the hosts, i.e. minimal finite subgraphs that contain all essential information about the automaton. We use the description of such Schützenberger automata and the Bass-Serre theory to study the maximal subgroups of the HNN-extensions of finite inverse semigroups.

Contents

Introduction

THIS thesis is devoted to the study of HNN-extensions of inverse semigroups in the sense of Yamamura. Inverse semigroups were introduced independently by Viktor Vladimirovich Wagner in the Soviet Union in 1952, and by Gordon Preston in Great Britain in 1954 and now days their theory is a major part of semigroup theory. Even if one could look at inverse semigroups as a generalization of the notion of groups, their are not a merely generalization because they naturally arise whenever one deals with partial one to one transformation on sets. For this reason inverse semigroup theory has deep connections with important mathematical disciplines, like geometry, functional analysis, number theory. Moreover the seminal works of Munn and Scheiblich which independently described the structure of free inverse monoids give raise to quite big amount of work on algorithmic problems in inverse semigroups. Namely from their descriptions the decidability of word problem for free inverse semigroups was proved. Their results were widely generalized by Stephen who provided a tool for dealing with word problem for presentation of inverse semigroups via graphical methods that make use of graph and automata theory and are closely tied to aspects of geometric and combinatorial group theory. More recently inverse semigroup theory attracted the attention of computer scientiests not only for algorithmic questions but become also relevant because notions and tools of inverse semigroup theory are a "proper" language for concretely modelling time-sensitive interactive systems and for dealing with questions in solid state physics like quasi-crystals.

In combinatorial group theory HNN-extension and amalgams are important tools for building new groups. As Cohn remarked these constructions are in a certain sense two faces of the same medals. The notion of amalgams can be quite naturally introduced also for inverse semigroups. The same does not happen for HNN-extensions, becase the embeddability of HNN-extensions of inverse semigroups does not hold in general, and suitable conditions are requested to achieve embedability, for instance Howie in [23] put the unitary condition on the subsemigroups, Gilbert in [14] put the assumption that the subsemigroups are ordered ideals, and each stable letter corresponds to one of the idempotents in an associated subsemigroup. Yamamura in [50] put the assumption that associated subsemigroups are monoids, and only one stable letter is required. In this thesis we consider HNN-extensions of inverse semigroups according to Yamamura's definition. Our results are quite often obtained along the lines of similar results for amalgams, but differ from them for several technical differences. Our main tool is the construction of Schützenberger graphs for the words of S^*, HNN-extension of finite inverse semigroup S illustrated by Rodaro and Cherubini in [40].
The first chapter of the thesis is devoted to introduce all the needed background, preliminary topics and basic notions relevant to the subject and Chapter 2 focuses on algorithmic issues. In particular, we prove that the word problem for HNN-extensions of inverse semigroup is undecidable even if we put nice conditions on the HNN-extension. Since as pointed out by Margolis, Meakin and Sapir, the insolvability of the word problem means that the family of inverse semigroups cannot be studied as a "whole", while the solvability of the word problem indicates that it is possible to study structural properties of the elements of the family, we consider for the rest of the thesis HNN-extensions of finite inverse semigroups, where the word problem is decidable. For such HNN-extension we prove that the languages recognized by the Schützenberger automata of words in HNN-extensions of finite inverse semigroup are context-free languages, using the notion of context-free graph, then we construct the deterministic context-free grammars which generate the same language, so giving an alternative proof of the decidability of word problem for HNN-extensions of finite inverse semigroups. This approach could give more information on the complexity of the word problem with respect to the proof by Rodaro and Cherubini.
In the last chapters we consider structural properties of HNN-extensions of finite inverse semigroups. First of all we characterize HNN-extension of finite inverse semigroups having a copy of the bicyclic subsemigroup

as subsemigroup. Since it is well known that an inverse semigroup is completely semisimple if and only if it does not contain a copy of the bicyclic subsemigroup, we derive a characterization of HNN-extensions of finite inverse semigroups which are completely semisimple. Then we use Basse-Serre theory of groups acting on graphs to describe the maximal subgroups as fundamental groups of certain graph of groups.

CHAPTER 1

Elementary Definitions and Basic Notions

1.1 Definition and basic properties

1.1.1 Inverse semigroups

A semigroup is an algebraic structure that consists of a non empty set S with an associative binary operation $(\cdot)$, a semigroup S has an identity 1 if $1 \in S$ and, for all x in S, $x \cdot 1 = x = 1 \cdot x$ and is called commutative if it satisfies the commutative property. A semigroup S that has 1 is called monoid. In the sequel we denote by S^1 the semigroup S itself if S is a monoid, otherwise we add to S a new element 1 and we put, for all $a \in S \cup \{1\}$, $a \cdot 1 = 1 \cdot a = a$, and S^1 will be referred to as semigroup with adjoint identity 1. In an analogous way, we may define the semigroup with adjoint zero S^0 with the condition that $0.a = a.0 = 0$ for all $a \in S$. We sometimes omit $(\cdot)$ and we write ab instead of $a \cdot b$. An element e of a semigroup S such that $e = e^2$ is called an idempotent of S. A semigroup all whose elements are idempotents is called band, a commutative band is called semilattice.
An element a in a semigroup S is called regular if there exists an element b in S such that $aba = a$. Then it is also $a(bab)a = a$ and $bab = (bab)a(bab)$,

so there is a $b' \in S$ such that $ab'a = a,\ b'ab' = b'$, such a b' is called an inverse of a. Hence a regular element $a \in S$ has always an inverse, but this inverse is in general not unique. Clearly each idempotent of S is a regular element. A semigroup S whose elements are all regular is called regular semigroup. A semigroup S is called an inverse semigroup if each element $a \in S$ has a unique inverse, denoted by a^{-1}. Equivalently, an inverse semigroup S is a regular semigroup S whose idempotents commute. Hence the set of idempotents of an inverse semigroup forms a semilattice, called the semilattice of idempotents of S and denoted $E(S)$. An inverse semigroup with an identity element is called an inverse monoid.

In an inverse semigroup S, it is obvious that the inverse of a^{-1} is the element a, thus the operation $a \mapsto a^{-1}$ is an involution on S, and the inverse of a product $a_1a_2...a_k,\ a_i \in S$, is equal to the product $a_k^{-1}...a_2^{-1}a_1^{-1}$. In each inverse semigroup S the relation $\leq$ defined by putting $a \leq_S b$ ($a \leq b$, for short, if no confusion arises) for $a, b \in S$ if and only if there exists an idempotent $e \in S$ such that $a = e \cdot b$ is a partial order relation, called the natural partial order on the inverse semigroup S. Let S be an inverse semigroup and let $a, b \in S$ such that $a \leq b$, if $b \in E(S)$ then $be = a$ for some $e \in E(S)$ thus $a \in E(S)$. However, in general $a \in E(S)$ does not imply that $b \in E(S)$. If this latter implication also holds for all comparable pairs of elements of S, the semigroup S is called and E-unitary semigroup. An idempotent is called primitive if it is non-zero and is minimal in the set of non-zero idempotents with respect to the order defined on S.

A non-empty subset of a semigroup S is called a subsemigroup if it is closed with respect to the binary operation on S and is called inverse subsemigroup if it is also closed under inverses. A non-empty subset I of a semigroup S is called a left ideal if $IS = \{as | a \in I, s \in S\}$ is a subset of I, a right ideal if $SI \subseteq I$, and it is called (two-sided) ideal if it is both left and right ideal. Moreover, $S^1a = \{sa \mid s \in S\} \cup \{a\}$, aS^1 and S^1aS^1 are similarly defined; $S^1a,\ aS^1,\ S^1aS^1$ are called respectively right, left and two-sided principal ideals of S generated by a. Evidently every (left, right, two-sided) ideal is a subsemigroup but the converse is not in general true. A semigroup without zero is called simple if it has no proper ideals. A semigroup S with zero is called 0-simple if $\{0\}$ and S are the only ideals and $S^2 \neq \{0\}$.

Another important concept in semigroup and inverse semigroup theory is the notion of the congruence relation which is an equivalence relation ρ on the inverse semigroup S that satisfies the condition, if for any $a, b, c \in S$ with $a\rho b$ then $ac\rho bc$ and $ca\rho cb$. Then it also follows that $a\rho b$ implies $a^{-1}\rho b^{-1}$. The set S/ρ of all the ρ-equivalence classes of S with the multiplication $a\rho b\rho = (ab)\rho$ forms a (inverse) semigroup called the quotient semigroup of S by ρ. Finally, we conclude these basic notions by the definition of semigroup homomorphism which is a mapping that preserves multiplication, i.e., the mapping $\phi : S \to S'$ is a homomorphism if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in S$. Evidently $\phi(1) = 1$, $\phi(0) = 0$ and $\phi(a^{-1}) = (\phi(a))^{-1}$, in particular the homomorphism $\rho^{\natural} : S \to S/\rho$ is called the natural homomorphism induced by ρ. Note also that a homomorphic image of an inverse semigroup is again an inverse semigroup.

1.1.2 Green's Relations

The notion of ideals mentioned above leads to the consideration of certain relations on a semigroup introduced by J. A. Green (1951) and known as Green's relations. These relations play a fundamental role in the development of semigroup theory. They are defined as follows:

$$\begin{aligned} &a\mathcal{L}b \text{ if } S^1a = S^1b\\ &a\mathcal{R}b \text{ if } aS^1 = bS^1\\ &a\mathcal{J}b \text{ if } S^1aS^1 = S^1bS^1\\ &a\mathcal{H}b \text{ if } a\mathcal{L}b \text{ and } a\mathcal{R}b \end{aligned}$$

The relation $\mathcal{L}$ and $\mathcal{R}$ commute, and this allows us to define the fifth relation $\mathcal{D}$ on S by $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. All the above defined relations are equivalence relations on S. We denote the $\mathcal{L}, (\mathcal{R}, \mathcal{H}, \mathcal{J}, \mathcal{D})$-class containing the element a by L_a (R_a, H_a, J_a, D_a), respectively.
Notice that the first three relations are defined in terms of principal ideals, thus the inclusion order among principal ideals induces a corresponding order among the equivalence classes as follows:

$$\begin{aligned} L_a \leq L_b &\quad \text{if } S^1a \subseteq S^1b,\\ R_a \leq R_b &\quad \text{if } aS^1 \subseteq bS^1,\\ J_a \leq J_b &\quad \text{if } S^1aS^1 \subseteq S^1bS^1. \end{aligned}$$

Furthermore, it is well known that for any inverse semigroup S and for all $e, f \in E(S), J_e \leq J_f$ if and only if $g \leq e$ for some $g\mathcal{D}f$. Since $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$ and $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$ then it immediately follows that $\mathcal{D} \subseteq \mathcal{J}$. Our interest will be in finite inverse semigroups, then it is important to remind the reader of some relevant results.

Proposition 1.1.1. [25] *If S is a periodic semigroup then $\mathcal{J} = \mathcal{D}$.* ■

Thus in the sequel when we will refer to finite or periodic inverse semigroups we will use freely the relation $\mathcal{D}$ instead of $\mathcal{J}$ and vice versa. Since we will widely use Green's relation we shortly recall some properties which can be easily found in any book on semigroup theory see for instance [25, 36, 28].

Proposition 1.1.2. *Let S be a semigroup and let $a, b \in S$;*

1. *(Green's lemma)*
   - *let $a\mathcal{R}b$ and let s, s' in S^1 be elements such that $as = b, bs' = a$. Then the mappings $\rho_s : L_a \to L_b$ defined by $\rho_s(x) = xs$ for $x \in L_a$ and $\rho_{s'} : L_b \to L_a$ $\rho_{s'}(y) = ys'$ for $y \in L_b$ are mutually inverses $\mathcal{R}-$class preserving bijections.*
   - *Let $a\mathcal{L}b$ and let s, s' in S^1 be elements such that $sa = b, s'b = a$. Then the mappings $\lambda_s : R_a \to R_b$ defined by $\lambda_s(x) = sx$ for $x \in R_a$ and $\lambda_{s'} : R_b \to R_a$ defined by $\lambda_{s'}(y) = s'y$ for $y \in R_b$) are mutually inverses $\mathcal{L}-$ class preserving bijections.*
2. *If a and b are $\mathcal{D}-$ equivalent elements in a semigroup S, then*
$$\mid H_a \mid = \mid H_b \mid .$$
3. *(Green's Theorem) If H is an $\mathcal{H}-$class in semigroup S then either $H^2 \cap H = \emptyset$ or H is a subgroup of S.*
4. *If e is an idempotent in a semigroup S, then H_e is a maximal subgroup of S. An $\mathcal{H}-$class in S can not contain more than one idempotent.*
5. *If a $\mathcal{D}-$class D contains a regular element, then every element in D is regular.*
6. *If e is an idempotent in a semigroup S and a, b are elements such that $a\mathcal{L}e$ and $e\mathcal{R}b$, then $a \; \mathcal{R} \; ab \; \mathcal{L} \; b$.*
7. *Let S be an inverse semigroup. Then,*
   - *i) $a\mathcal{R}b$ if and only if $aa^{-1} = bb^{-1}$*
   - *ii) $a\mathcal{L}b$ if and only if $a^{-1}a = b^{-1}b$*
   - *iii) If $e, f \in E(S)$, then $e\mathcal{D}f$ if and only if there exists an element $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$*
   - *iv) Each $\mathcal{L}-$class and each $\mathcal{R}-$class in S contains a unique idempotent,*

8. *If H_1 and H_2 are two group $\mathcal{H}-$classes in the same $\mathcal{D}-$class D of an inverse semigroup S, then there is an element a in D such that the mapping $\phi : H_1 \to H_2$ defined by $\phi(x) = a^{-1}xa$ for $x \in H_1$ and the mapping $\lambda : H_2 \to H_1$ defined by $\lambda(y) = aya^{-1}$ for $y \in H_2$ are mutually inverse isomorphisms.* ■

Thus, the maximal subgroups of a semigroup S are exactly the $\mathcal{H}-$classes that contain idempotents, and any two maximal subgroups in the same $\mathcal{D}-$class are isomorphic.

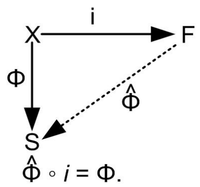
Representation of Inverse Semigroups
Vagner [46] and Preston [37] proved the following remarkable result.

Theorem 1.1.1 (Vagner-Preston representation theorem)**.** *Any inverse semigroup can be faithfully represented as an inverse semigroup of partial one-to-one mappings on some set.* ■

Let X be a non-empty set, and let S_X denote the set of all partial injective maps on X (including the empty map). S_X under the operation defined as follows : for $\alpha, \beta \in S_X$, we define $\beta \circ \alpha$ on the set $Dom(\beta \circ \alpha) = \alpha^{-1}(Ran(\alpha) \cap Dom(\beta))$ with range $Ran(\beta \circ \alpha) = \beta(Ran(\alpha) \cap Dom(\beta))$ and $(\beta \circ \alpha)x = \beta(\alpha x)$ for each $x \in Dom(\beta \circ \alpha)$, is an inverse semigroup called the symmetric inverse semigroup. Under Vagner-Preston representation any inverse semigroup S is embedded into a symmetric inverse semigroup S_X with $X = S$ by associating to each element $a \in S$ the map $\tau_a : Saa^{-1} \to Sa^{-1}a$ defined by $\tau_a(x) = x \cdot a$, for all $x \in Saa^{-1}$. Similarly, any inverse semigroup S can be homomorphically mapped into $S_{E(S)}$, the symmetric inverse semigroup of all partial injective mappings on the set of idempotents $E(S)$, by sending an element $a \in S$ to the mapping $\rho_a : E(S)aa^{-1} \to E(S)a^{-1}a$ defined by $\rho_a(x) = a^{-1} \cdot x \cdot a$.

1.1.3 Presentation of Inverse Semigroups

Let S be an inverse semigroup and X a subset of S. Let $X^{-1} = \{x^{-1} \mid x \in X\}$. If every element of S can be written as a product of finitely many elements of $X \cup X^{-1}$, we say that S is generated by X and X is called a set of generators of S. Given a set of abstract generators X and a mapping $i : X \to F$ of X into an inverse semigroup F, we say that F is the free inverse semigroup generated by the set X if it satisfies the usual universal property, i.e., for every inverse semigroup S and for every mapping $\Phi : X \to S$ there is a unique homomorphism $\hat{\Phi} : F \to S$ such that $\hat{\Phi} \circ i = \Phi$, that is, the following diagram commute.



We denote the free inverse semigroup on X by $FIS(X)$. Free inverse semigroups are the free objects in the category of inverse semigroups, and they are unique up to isomorphism. The existence of free inverse semigroup on any set X was firstly proved by Vagner [46]. A more concrete construction was given by Scheiblich [42]. A graph theoretical method to study free inverse monoids (and semigroups) was introduced by Munn [33] who showed that elements of a free inverse monoid (semigroup) correspond to birooted word trees called Munn trees (see Section 1.2.4.). A brief summary of Vagner's construction is as follows: Let X be a non-empty set, let X^{-1} be a disjoint set with the bijection $x \mapsto x^{-1}$ for all $x \in X$. Let $FS(X) = (X \cup X^{-1})^+$ denote the free semigroup on $X \cup X^{-1}$, $(X \cup X^{-1})^+$ may be represented as the semigroup of all non-empty words over $X \cup X^{-1}$ with respect to concatenation of words. For every $x \in X$ let $(x^{-1})^{-1} = x$ then for $w = x_1 x_2 ... x_n \in FS(X)$ where $x_i \in X \cup X^{-1}$ for each $i = 1, 2, ...n$, we denote $x_n^{-1} x_{n-1}^{-1} ... x_1^{-1}$ by w^{-1}.
Let ν be the congruence relation on $FS(X)$ generated by the relation

$$\{(uu^{-1}u, u) \mid u \in FS(X)\} \cup \{(uu^{-1}vv^{-1}, vv^{-1}uu^{-1}) \mid u, v \in FS(X)\}$$

Then we have :

Theorem 1.1.2. *The semigroup $F = FS(X)/\nu$ is the free inverse semigroup $FIS(X)$ on the set X.* ■

The congruence ν is called the Vagner congruence on $(X \cup X^{-1})^+$. A *presentation of an inverse semigroup* S is a pair $\langle X \mid R \rangle$ where X is a non-empty set (of generators) and R is a binary relation on the set $(X \cup X^{-1})^+$ such that $S \cong FS(X)/\tau$, where τ is the smallest congruence relation on $FS(X)$ containing $R \cup \nu$. If an inverse semigroup S is presented by a set of generators X and relations R, we write $S = Inv\langle X|R \rangle$, and we call $\langle S|R \rangle$ a presentation of S. If the set X is finite, then S is called finitely generated; if R is finite, S is called finitely related; and in the case when both X and R are finite, we say that S is finitely presented.
Now, let S be an arbitrary inverse semigroup generated by a set X and let $\rho : FIS(X) \to S$ be a surjective homomorphism from the free inverse semigroup $FIS(X)$ to S. It is well known that $S \cong FIS(X)/Ker(\rho)$

where $Ker(\rho)$ is the congruence relation consisting of $\{(v, w) \mid v, w \in F, \rho(v) = \rho(w)\}$. Thus we have the following well known theorem.

Theorem 1.1.3. *Every inverse semigroup S is a homomorphic image of the free inverse semigroup $FIS(S)$.* ■

The aim of combinatorial inverse semigroup theory is to extract information on an inverse semigroup starting from its presentation. There are several fundamental problems in combinatorial inverse semigroup theory, for instance:

- The word problem: Given a presentation $\langle X|R\rangle$ for an inverse semigroup $S = Inv\langle X|R\rangle$, is there an algorithm to decide whether any two given words u and v over $X \cup X^{-1}$ are equals or not in S? If such an algorithm exists we say that S has decidable (or solvable) word problem.
- Membership problem: Given an inverse subsemigroup H of an inverse semigroup $Inv\langle X|R\rangle$ is there an algorithm to decide for any given word u over $X \cup X^{-1}$ whether or not u is in H? If such an algorithm exists we say that H has solvable membership problem.
- The isomorphism problem: Given two finitely presented inverse semigroups is there an algorithm to decide whether or not they are isomorphic?

Moreover when a problem is proved to be decidable, the question about the complexity of decidability algorithms naturally arises. For more information about decidability, algorithms and their complexity we refer the reader to any standard book on computational theory and complexity (see Davis [12] or Hopcroft and Ullman [22]). The concept of an effectively calculable function may also be found in these texts.
There are finitely presented inverse semigroups with unsolvable word problem, namely the existence of finitely presented groups with unsolvable word problem was proved independently by Novikov [34] and Boone [5]. Munn [33] and Scheiblich [42] proved that the word problem for the free inverse monoids is solvable. Stephen [44] introduced a graph-theoretical approach to word problems for inverse monoids (see also Meakin [32]). For a supplementary knowledge about algorithmic problems on groups, semigroups and inverse semigroups we refer the reader to Margolis, Meakin and Sapir [30].
The concepts of a free inverse monoid and a presentation for an inverse monoid S can be defined in a similar manner.

1.1.4 HNN-extension

There are many ways of obtaining an inverse semigroup and their presentations from given inverse semigroups such as free product, semidirect product, amalgam and HNN-extension of inverse semigroups. Our interest is in the latest one. The concept of HNN-extension was originally introduced for groups by Higman, Neumann and Neumann [20], who showed that if A and B are isomorphic subgroups of a group G, then it is possible to find a group H where G embeds such that A and B are conjugate to each other.
Although the general idea of HNN-extension of inverse semigroups is similar to the idea HNN-extension of groups, there are many differences. In Higman, Neumann and Neumann original work, the embeddability of a group into its HNN-extension is one of the most important features of the construction, and is a consequence of Britton's lemma which provides normal forms for elements of an HNN-extension. However, when one deals with (inverse) semigroups some extra conditions on the original (inverse) semigroup S and on the associated (inverse) subsemigroups are required to the embeddability property. Consequently, many different definitions of HNN-extension arose in the years, and most of them have some rigid conditions to obtain the embedding of the original semigroup in its HNN-extension. For instance Howie [23] introduced a concept of HNN-extension of a semigroup and proved an embeddability property in such extensions in the case that the associated subsemigroups are unitary. Embeddability of one type of HNN-extension of inverse semigroup is shown in T.E.Hall [18] and Ash [2]. There are at least three types of HNN-extensions of inverse semigroups that are currently used. A comprehensive overview of all three definitions as well as an extensive discussion of embeddability and the relationship with amalgamated free products can be found in Yamamura's work [47]. Recently, Gilbert [14] using the isomorphism between the categories of inverse semigroups and inductive groupoids has shown a way to construct HNN-extensions of inverse semigroups which extends the definition given for groups. It is possible to extend an inductive groupoid G to an inverse semigroup $S(G)$; using this features and the definition of HNN-extension for grupoid Gilbert defines naturally the HNN-extensions of inverse semigroups whose associated inverse semigroups are order ideals. In this thesis we use the definition of HNN-extension for inverse semigroups introduced by Yamamura in [47]. We choose this definition because it is close to the definition of HNN-extension of groups, guarantees the embeddability of the original inverse semigroup S in its HNN-extension, and has strongly relations with the definition of amalgamated free product

of inverse semigroups, analogously to the group case.

Definition 1.1.1. [A.Yamamura]. Let $S = Inv\langle X|R\rangle$ be an inverse semigroup. Let $\varphi : A \to B$ be an isomorphism of an inverse subsemigroup A onto an inverse subsemigroup B where $e \in A \subseteq eSe$ and $f \in B \subseteq fSf$ (or $e \notin A \subseteq eSe$ and $f \notin B \subseteq fSf$, for some $e, f \in E(S)$). Then the inverse semigroup

$$S^* = Inv\langle S, t|t^{-1}at = \varphi(a), t^{-1}t = f, tt^{-1} = e, \forall a \in A\rangle$$

is called the HNN-extension of S associated with $\varphi : A \to B$ and it is denoted by $[S; A, B; \varphi]$ or shortly $[S; A, B]$. We denote the set of all t-rules $\{t^{-1}at = \varphi(a), \forall a \in A, t^{-1}t = f, tt^{-1} = e,\}$ by R_{HNN}. Then $S^* = Inv\langle X, t|R \cup R_{HNN}\rangle$.

HNN-extensions have a natural universal mapping property which is described in the next proposition. The universal mapping property can also be used to define the notion of HNN-extension.

Proposition 1.1.3. *Let $S^* = [S; A, B]$ be an HNN-extension of an inverse semigroup S. Then S^* has the following universal mapping property:*

i) There is a unique homomorphism $\sigma : S \to S^*$ such that $t^{-1}\sigma(a)t = \sigma(\varphi(a))\forall a \in A$ and $t^{-1}t = \sigma(f), tt^{-1} = \sigma(e)$.

ii) For each inverse semigroup T and homomorphism $\rho : S \to T$ such that there exists $p \in T$ with $p^{-1}\rho(a)p = \rho(\varphi(a))\forall a \in A, pp^{-1} = \rho(e)$ and $p^{-1}p = \rho(f)$, there is a unique homomorphism $\rho' : S^* \to T$ such that $\rho'(t) = p$ and $\rho' \circ \sigma = \rho$. ■

Yamamura in [47] showed that S embeds into S^* provided that it satisfies the light conditions on e and f given in definition 1.1.1.

Proposition 1.1.4. *A.Yamamura. S embeds into S^* provided that $e \in A \subseteq eSe$ and $f \in B \subseteq fSf$ (or $e \notin A \subseteq eSe$ and $f \notin B \subseteq fSf$ for some $e, f \in E(S)$).* ■

The interest in the structure of HNN-extension comes from the role that HNN-extensions play in studying algorithmic problems in inverse semigroups (see [47]). Moreover, Yamamura in [48] introduced a more general definition of HNN-extension and showed that free inverse semigroups and the bicyclic semigroup are HNN-extensions of semilattices

1.2 Combinatorial Tools

Geometrical methods play an important role in the combinatorial inverse semigroup theory. The seminal word of Munn [33] has shown how biroted trees allows to solve the word problem for free inverse semigroups, working like a normal form of elements of the semigroups. A generalization of Munn trees due to Stephen [44] has become a central notion in dealing with word problem of inverse semigroups (especially for amalgamated free products and HNN-extension of some families of inverse semigroups). Moreover, Schützenberger graphs connect decidability problems with formal language theory useful for algorithmic purposes (see Chapter 2), and they can be used for extracting structural properties of the some types of inverse semigroups (see Chapters 3 and 5). Schützenberger graphs are one of the main tools to prove our results; therefore, the remaining sections of this chapter are devoted to introduce Schützenberger graphs: we start with general notions on graphs, automata, inverse word graphs, then we define Schützenberger graphs and automata and we summarize their important properties, and lastly we describe a sequence of steps for the construction of Schützenberger graphs for HNN-extensions.

1.2.1 Basic Notions of Graph Theory

A graph Γ is the pair $(V(\Gamma), Ed(\Gamma))$ of disjoint set, where $V(\Gamma)$ is a non-empty set called the set of vertices and $Ed(\Gamma)$ is called the set of edges. We say that two vertices p and q are adjacent if there exists an edge $y \in Ed(\Gamma)$ such that p is its initial vertex which we will denote it by $0(y)$ and q is its terminal vertex denoted by $1(y)$. An element $y \in Ed(\Gamma)$ is called an oriented (a directed) edge, An edge y^{-1} is called the inverse (reverse) of y if $0(y^{-1}) = 1(y)$ and $1(y^{-1}) = 0(y)$. In general y^{-1} (sometimes is denoted by $\bar{y}$) is not an edge of the graph. Moreover, a graph satisfies that, for each edge y: $(y^{-1})^{-1} = y$ and $y^{-1} = y$ if and only if $0(y) = 1(y)$. An orientation of a graph Γ is a subset $\mathcal{O}_+$ (positive orientation) of $Ed(\Gamma)$ such that if $\mathcal{O}_+$ contains both y^{-1} or y then $y^{-1} = y$, $\mathcal{O}_-$ consists of the inverse edges of $\mathcal{O}_+$. A non-oriented (undirected) edge y is denoted by $y = \{p, q\}$. An oriented (directed) graph (diagraph) is a graph Γ together with its orientation $\mathcal{O}$. The size of a graph is measured by the cardinality of its set of vertices. A graph Γ' such that $V(\Gamma') \subseteq V(\Gamma)$ and $Ed(\Gamma') \subseteq Ed(\Gamma)$ is called a subgraph of Γ.

Given a direct graph Γ, a path in Γ of length $n \geq 1$ is a finite sequence $y_1, y_2, ..., y_n$ of edges from $Ed(\Gamma)$ satisfying the property $1(y_i) = 0(y_{i+1})$, for all $1 \leq i \leq n-1$, sometimes we describe a path by mentioning the enrolled vertices, for instance the path $v_1, v_2, ..., v_n$, where $v_i \in V(\Gamma)$ for all i, $1 \leq i \leq n$, such a path is also called a $(v_1 - v_n)$-path. A loop is a path such that $v_1 = v_n$. If Γ is a not oriented graph, a path is a path of Γ for

some orientation of Γ. A path is called reduced if it does not contain a loop, otherwise the path is called reducible. A graph Γ is said to be connected if for any two vertices $v_1, v_2 \in V(\Gamma)$ there is a path from v_1 to v_2. A maximal connected subgraph Γ' of Γ is called a connected component of Γ.
A tree is a connected non-empty non-oriented graph without loops, with some abuse of notations we call a directed graph a tree if its underlying undirected graph is a tree. A subgraph that is a tree is called a subtree. A unique reduced path connecting two different vertices in a tree is called a geodesic. A spanning tree of a graph Γ is a subgraph Υ that is a tree and that contains every vertex of the graph Γ.

1.2.2 Inverse Word Graphs

We will focus on a particular labelled diagraph called an *inverse word graph* that satisfies the following definition.

Definition 1.2.1. Let X be a non-empty set (an alphabet). An inverse word graph over the set X is a connected diagraph Γ such that

- $Ed(\Gamma) \subseteq V(\Gamma) \times (X \cup X^{-1}) \times V(\Gamma)$
- $(v_1, x, v_2) \in Ed(\Gamma)$ if and only if $(v_2, x^{-1}, v_1) \in Ed(\Gamma)$

Thus an inverse word graph is an inverse graph whose edges are labelled in $X \cup X^{-1}$ such that, for each edge labelled by $x \in X$, x^{-1} is the label of its unique reverse edge, we call such a graph an X-labelled inverse word graph. A path π of length $n \geq 1$ in an inverse word graph is a sequence of the form

$$\pi = v_0, x_1, v_2, x_2, v_3, ..., v_{n-1}x_{n-1}, v_n,$$

i.e a sequence of consecutive connected edges labelled by $x_1, x_2, ..., x_{n-1} \in X \cup X^{-1}$. We say the path π from the vertex v_0 to the vertex v_n is labelled by the word (string) $x_1 x_2 ... x_{n-1} \in (X \cup X^{-1})^+$. There is always a trivial path from a vertex v to itself labelled by the empty word $\epsilon \in (X \cup X^{-1})^*$.

An inverse word graph Γ is said to be deterministic if all distinct edges exiting from the same vertex are labelled by different letters, i.e., if $(v, x, v'), (v, x, v'') \in Ed(\Gamma)$ then $v' = v''$. A graph Γ is said to be injective if $(v', x, v), (v'', x, v) \in E(\Gamma)$ then $v' = v''$. One can easily show that an inverse word graph is deterministic if and only if it is injective. As a convention we will denote the path from the vertex v_1 to the vertex v_2 labelled by the word $w \in (X \cup X^{-1})^*$ by the triple (v_1, w, v_2).

Let Γ and Γ' be two X-labelled inverse word graphs, a homomorphism $\phi : \Gamma \to \Gamma'$ of Γ in Γ', is a pair of functions (ϕ_V, ϕ_E), where $\phi_V : V(\Gamma) \to V(\Gamma')$ and $\phi_E : Ed(\Gamma) \to Ed(\Gamma')$ such that $\phi((v_1, x, v_2)) = (\phi_V(v_1), x, \phi_V(v_2))$, that is, a homomorphism ϕ is a map on vertices which preserves incidence, orientation and labeling. The homomorphism ϕ is said to be a monomorphism (an embedding) if both ϕ_V and ϕ_E are injective and ϕ is called epimorphism if both its components are surjective. Finally, ϕ is called isomorphism if both its components are bijective. We will use ϕ for ϕ_V or ϕ_E when there is no danger of confusion. If ϕ is a homomorphism from Γ to itself then ϕ is called an endomorphism and we write $\phi \in End(\Gamma)$ and if $\phi : \Gamma \to \Gamma$ is an isomorphism then it is called an automorphism and we write that $\phi \in Aut(\Gamma)$.
The quotient of an inverse word graph Γ by an equivalence relation $\eta \subseteq V(\Gamma) \times V(\Gamma)$ is the graph Γ/η where $V(\Gamma/\eta) = V(\Gamma)/\eta$ and $Ed(\Gamma/\eta) = \{(v_1\eta, x, v_2\eta) | (v_1, x, v_2) \in Ed(\Gamma)\}$. Thus the vertices of the inverse word graph Γ/η are the equivalence classes of the vertices of Γ with respect to η. The equivalence relation η induces the natural homomorphism $\pi : \Gamma \to \Gamma/\eta$.

1.2.3 Inverse Automaton

An inverse graph over X with two chosen distinguished vertices called roots, or more precisely initial vertex (state) and final vertex (state), is called an inverse automaton (with input alphabet X). Inverse automata are one of our most important tools and this section is devoted to them.

Definition 1.2.2. Let X be a non-empty set. Then an inverse automaton $\mathcal{A}$ over X is an ordered triple (α, Γ, β), where Γ is an inverse word graph over X and α and β are vertices of Γ.

The vertex α is called the initial root of $\mathcal{A}$ while β is called the terminal root, therefore $\mathcal{A} = (\alpha, \Gamma, \beta)$ is a birooted inverse word graph over X and it is possible that $\alpha = \beta$. The language $L[\mathcal{A}]$ recognized by $\mathcal{A}$ is the set of all words in $(X \cup X^{-1})^*$ which label a path from α to β.
Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ and $\mathcal{A}' = (\alpha', \Gamma', \beta')$ be two inverse automata. A homomorphism $\phi : \mathcal{A} \to \mathcal{A}'$ of inverse word automata is a homomorphism of their underlying word graphs Γ and Γ' such that $\phi(\alpha) = \alpha'$ and $\phi(\beta) = \beta'$. Homomorphisms of inverse word automata and language inclusions are connected by the following result (Theorems 2.4 and 2.5 of [44]).

Proposition 1.2.1. *Let $\phi : \Gamma \to \Gamma'$ be a homomorphism of inverse word graphs Γ and Γ' on X and let $\alpha, \beta \in V(\Gamma)$. If $w \in (X \cup X^{-1})^+$ labels a*

path from α to β in Γ then w labels a path from $\phi(\alpha)$ to $\phi(\beta)$ in Γ'. So if $\mathcal{A} = (\alpha, \Gamma, \beta)$ and $\mathcal{A}' = (\phi(\alpha), \Gamma', \phi(\beta))$ then $L[\mathcal{A}] \subseteq L[\mathcal{A}']$.
Conversely, let $\mathcal{A} = (\alpha, \Gamma, \beta)$ and $\mathcal{A}' = (\alpha', \Gamma', \beta')$ be deterministic inverse automata over X. If $L[\mathcal{A}] \subseteq L[\mathcal{A}']$, then there is a homomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi(\alpha) = \alpha'$ and $\phi(\beta) = \beta'$. ■

The quotient of an inverse automaton $\mathcal{A} = (\alpha, \Gamma, \beta)$ by an equivalence relation $\eta \subseteq V(\Gamma) \times V(\Gamma)$ is the inverse automaton $\mathcal{A}/\eta = (\alpha\eta, \Gamma/\eta, \beta\eta)$. A determinized form of an inverse automaton $\mathcal{A}$ is defined to be the quotient $\mathcal{A}/\rho$ where ρ is the minimal equivalence relation on $V(\Gamma)$ such that $\mathcal{A}/\rho$ is deterministic. The determinized form of the automaton $\mathcal{A}/\eta$ is called a DV-quotient of $\mathcal{A}$.

1.3 Schützenberger Graphs

The combinatorial group theory is closely related to the geometry of certain digraphs with edges labeled from the generating set. An examples of this connection is provided by the Cayley graph of a group. Stephen [44] introduced a family of digraphs which generalize both the notions of Cayley graphs for groups and Munn trees for free inverse semigroups and allow a deeper vision of several topics of combinatorial inverse semigroups theory.

Definition 1.3.1 (Schützenberger graphs). Let $S = Inv\langle X|R\rangle \cong (X \cup X^{-1})^+/\tau$ and let w be a word in $(X \cup X^{-1})^+$. The Schützenberger graph of w relative to the presentation $\langle X|R\rangle$ is the graph $S\Gamma(X, R; w)$ whose vertices are the elements of the $\mathcal{R}$-class $R_{w\tau}$ of $w\tau$, and whose edges are of the form

$$\{(v_1, x, v_2) \mid v_1, v_2 \in R_{w\tau} \text{ and } v_1(x\tau) = v_2\}.$$

It is clear that for any two words w and w' representing the same element s in S, i.e., $w\tau = s = w'\tau$ (in other words, $w = w'(mod\tau)$), the Schützenberger graphs $S\Gamma(X, R; w)$ and $S\Gamma(X, R; w')$ are isomorphic. Therefore the Schützenberger graph depends only on the element s represented by the word w so with a light abuse of notation we will use also the notation $S\Gamma(X, R; w\tau)$ (for short $S\Gamma(w)$) to denote $S\Gamma(X, R; w)$ and we freely speak of Schützenberger graphs of the element $w\tau$ of S. It should be also noted that any two elements from same $\mathcal{R}-$class determine the same Schützenberger graph.
The Schützenberger graphs introduced above have many useful properties.

Theorem 1.3.1. *(Theorem 3.1 of [44]) Let w,w' be words in $(X \cup X^{-1})^+$, and let $s, s' \in \mathcal{R}_{w\tau}$. Then:*

1. $S\Gamma(X, R; w\tau)$ *is a deterministic inverse word graph over* X*;*
2. $w\tau \leq w'\tau$ *if and only if* w' *labels a path in* $S\Gamma(X, R; w\tau)$ *from* $ww^{-1}\tau$ *to* $w\tau$.
3. $s'(w'\tau) = s$ *if and only if* w' *labels a path in* $S\Gamma(X, R; w\tau)$ *from* s' *to* s. ■

The Schützenberger automaton of w relative $Inv\langle X \mid R\rangle$ can now be defined as the inverse automaton

$$\mathcal{A}(X, R; w\tau) = (ww^{-1}\tau, S\Gamma(X, R; w\tau), w\tau).$$

Stephen proved that the language of $\mathcal{A}(X, R; w\tau)$ is the set $\{z \in (X \cup X^{-1})^+ \mid w\tau \leq z\tau\}$ which we denote by $[w]_\tau \uparrow$ or by $L[\mathcal{A}(X, R; w\tau)]$.
The fundamental relation between the word problem for an inverse semigroup S given by a presentation $\langle X \mid R\rangle$ and the Schützenberger automaton of a word w relative to $\langle X \mid R\rangle$ is well exhibited in the following theorem by Stephen [44].

Theorem 1.3.2. *Let* w_1 *and* w_2 *be two words in* $(X \cup X^{-1})^+$*. The following statements are equivalent;*

1. $w_1\tau = w_2\tau$*;*
2. $w_1 \in L[\mathcal{A}(X, R; w_2\tau)]$ *and* $w_2 \in L[\mathcal{A}(X, R; w_2\tau)]$*;*
3. $L[\mathcal{A}(X, R; w_1\tau)] = L[\mathcal{A}(X, R; w_2\tau)]$*;*
4. $\mathcal{A}(X, R; w_1\tau) \cong \mathcal{A}(X, R; w_2\tau)]$. ■

Green's relations, natural partial order, Schützenberger group of an inverse semigroup S can be described via morphisms of Schützenberger graphs (see[44]):

Theorem 1.3.3. *Let* $S = Inv\langle X \mid R\rangle = (X \cup X^{-1})^+/\tau$*, and let* w_1*,*$w_2 \in (X \cup X^{-1})^+$*, then*

1. $w_1\tau\mathcal{D}w_2\tau$ *if and only if* $S\Gamma(X, R; w_1\tau)$ *and* $S\Gamma(X, R; w_2\tau)$ *are isomorphic;*
2. $w_1\tau\mathcal{R}w_2\tau$ *if and only if there exists an isomorphism* $\phi : S\Gamma(X, R; w_1\tau) \to S\Gamma(X, R; w_2\tau)$ *such that* $(w_1w_1^{-1}\tau)\phi = w_2w_2^{-1}\tau$*;*
3. $w_1\tau\mathcal{L}w_2\tau$ *if and only if there exists an isomorphism* $\phi : S\Gamma(X, R; w_1\tau) \to S\Gamma(X, R; w_2\tau)$ *such that* $(w_1\tau)\phi = w_2\tau$*;*

4. *$w_1\tau = w_2\tau$ if and only if and only if there exists an isomorphism ϕ : $S\Gamma(X, R : w_1\tau) \to S\Gamma(X, R, w_2\tau)$ such that $(w_1 w_2^{-1})\phi = w_2 w_2^{-1}\tau$ and $(w_1\tau)\phi = w_2\tau$;*

5. *$w_1\tau \geq w_2\tau$ if and only if there exists a morphism ϕ : $S\Gamma(X, R, w_1\tau) \to S\Gamma(X, R; w_2\tau)$ such that $(w_1 w_1^{-1}\tau)\phi = w_2 w_2^{-1}\tau$ and $(w_1\tau)\phi = w_2\tau$;*

6. *The Schützenberger group of $\mathcal{R}_{w_1\tau}$, that is the group $H_{w_1 w_1^{-1}\tau}$, is isomorphic to the group of automorphisms of $S\Gamma(X, R; w_1\tau)$.* ■

Stephen proved his results widely using the following notion of approximate automaton.

Definition 1.3.2 (Approximate automaton)**.** An inverse automaton $\mathcal{A}$ over X approximates the automaton (is an approximate automaton for) $\mathcal{A}(X, R; w\tau)$ if

- $L[\mathcal{A}] \subseteq L[\mathcal{A}(X, R; w\tau)]$, and
- there exists a word $w' \in L[\mathcal{A}]$ such that $w'\tau = w\tau$.

We write $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to denote that $\mathcal{A}$ approximates $\mathcal{A}'$.
Stephen [44] proved that any inverse automaton $\mathcal{A}$ over X can approximate at most one Schützenberger automaton. It is obvious that a Schützenberger automaton approximates itself.
The following lemma from [44] shows which Schützenberger automaton is approximated by an approximate automaton when we change its roots.

Lemma 1.3.1. *Let w be a word in $(X \cup X^{-1})^+$ and let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be any inverse automaton approximating the Schützenberger automaton $\mathcal{A}(X, R; w\tau)$. Also, let w_1 be the label word of an $\alpha' - \alpha$ path in Γ, and let w_2 be the label word of a $\beta - \beta'$ path in Γ, for some $\alpha', \beta' \in V(\Gamma)$. Then the inverse automaton $(\alpha', \Gamma, \beta')$ approximates the Schützenberger automaton $\mathcal{A}'(X, R; w_1 w w_2\tau)$.* ■

Therefore, if we do not consider the roots of the automaton, we can talk about an approximate graph relative to the presentation $\langle X|R\rangle$ meaning an inverse graph over X such that for each choice of a pair of vertices determines an inverse automaton approximating some Schützenberger automaton relative to $\langle X|R\rangle$.
A product of two disjoint automata $\mathcal{A}'_1 = (\alpha_1, \Gamma_1, \beta_1)$ and $\mathcal{A}'_2 = (\alpha_2, \Gamma_2, \beta_2)$ is the automaton $\mathcal{A}'_1 \times \mathcal{A}'_2 = (\alpha_1\eta, (\Gamma_1 \cup \Gamma_1)/\eta, \beta_2\eta)$, where η is the equivalence on $V(\Gamma_1 \cup \Gamma_2)$ generated by $\{(\beta_1, \alpha_2)\}$.

Lemma 1.3.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be inverse automaton over X that approximate $\mathcal{A}(X, R; w_1\tau)$ and $\mathcal{A}(X, R; w_2\tau)$ respectively. Then $\mathcal{A}_1 \times \mathcal{A}_2$ is an inverse automaton over X that approximate $\mathcal{A}(X, R; w_1 w_2 \tau)$.* ■

1.3.1 Elementary Constructions

Let $S = Inv\langle X \mid R\rangle$. Let $w = x_1 x_2 ... x_n$ be a word in $(X \cup X^{-1})^+$. In this section we briefly describe Stephen's iterative procedure for building the Schützenberger automaton $\mathcal{A}(X, R; w)$. See [44] for more details. The procedure begins by building the linear automaton of w, $lin(w) = (\alpha_w, \Gamma_w, \beta_w)$, where Γ_m is a path of length n labelled consecutively by the letters of w with two distinguished vertices, α_w the initial vertex of the path and β_w the terminal vertex of the path.

$$V(\Gamma_w) = \{\alpha_w = v_0, v_1, v_2, ..., v_{n-1}, v_n = \beta_w\} \text{ and}$$
$$Ed(\Gamma_w) = \{(v_i, x_{i+1}, v_{i+1}), (v_{i+1}, x_{i+1}^{-1}, v_i) \mid 0 \leq i \leq n-1\}$$

The importance of linear automata is established in the next result.

Proposition 1.3.1. *[44] Let $w \in (X \cup X^{-1})^+$. The linear automaton $lin(w) = (\alpha_w, \Gamma_w, \beta_w)$, approximate $\mathcal{A}(X, R; w)$.* ■

Iteratively applying the two construction defined below, elementary expansion and elementary determination, starting from $lin(w) = (\alpha_w, \Gamma_w, \beta_w)$ we obtain a sequence of inverse automata over X.

- **(Elementary Expansion)** Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be an inverse automaton over X, and suppose that for some relation (r, s) in R there is a path from a vertex v_1 to a vertex v_2 of Γ labelled by r but no path from v_1 to v_2 labelled by s. Then, an elementary expansion of $\mathcal{A}$ relative to $Inv\langle X \mid R\rangle$ is obtained from $\mathcal{A}$ by sewing a disjoint copy of the underlying graph Γ_s of the linear automaton $lin(s) = (\alpha_s, \Gamma_s, \beta_s)$ to Γ by identifying α_s with v_1 and β_s with v_2.
  Thus, an elementary expansion of the automaton $\mathcal{A}$ yields in a new automaton $\mathcal{A}'$ which is obtained by attaching a linear path labelled by the missing part of a relation from R to a side of the relation that is already there. Clearly, there is an embedding $\mathcal{A} \hookrightarrow \mathcal{A}'$.

- **(Elementary Determination or edge folding)** Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be an inverse automaton over X, and suppose the existence of a pair of edges emanating from the same initial vertex and having the same label. Then, an elementary determination of $\mathcal{A}$ relative to $Inv\langle X \mid R\rangle$

is obtained from $\mathcal{A}$ by taking the quotient by the least equivalence relation on Γ identifying the pair of edges.

Again, there is a natural homomorphism from $\mathcal{A}$ to the automaton $\mathcal{A}'$ obtained by an elementary determination of $\mathcal{A}$.
Both elementary expansion and elementary determination preserve the property of approximating $\mathcal{A}(X, R; w)$, thus applying these two operations to $lin(w) = (\alpha_w, \Gamma_w, \beta_w)$ gives rise to a sequence of automata approximating $\mathcal{A}(X, R, w)$.

Proposition 1.3.2. *Let* $w \in (X \cup X^{-1})^+$. *Let*

$$\mathcal{A}_1 \to \mathcal{A}_2 \to \mathcal{A}_3 \to ... \to \mathcal{A}_n \to ...$$

be a sequence of inverse automata that approximate $\mathcal{A}(X, R; w)$*, there is a homomorphism* $\Phi_i : \mathcal{A}_i \to \mathcal{A}_{i+1}$ *for all* $i \geq 1$*. The direct limit of the sequence is again an approximate automaton for* $\mathcal{A}(X, R; w)$*.* ■

Furthermore each automaton in the sequence is a better approximate automaton than the previous ones in the sense that the language recognized by the n-th automaton contains the language recognized by the $(n-1)$-th one for each n. We say that a deterministic inverse automaton $\mathcal{A}$ over X is closed relative to the presentation $\langle X \mid R\rangle$ if $\mathcal{A}$ does not allow any elementary expansion. Any deterministic inverse automaton $\mathcal{A}$ over X that approximates $\mathcal{A}(X, R; w\tau)$ and is closed relative to $Inv\langle X \mid R\rangle$ is isomorphic to $\mathcal{A}(X, R; w\tau)$ [44]. Also, any Schützenberger automaton is deterministic and closed relative to $\langle X \mid R\rangle$ and approximates itself. Therefore, the choice of any pair of vertices for the roots of a Schützenberger graph always determines a Schützenberger automaton (up to isomorphism). The result which summarizes the above discussion is the following

Theorem 1.3.4. *[44] Let* $w \in (X \cup X^{-1})^+$*, and let* $\{\mathcal{A}_i\}$ *be the directed system obtained from* $lin(w)$ *by repeated application of elementary expansions and elementary determinations. Then the direct limit of this system is* $\mathcal{A}(X, R, w)$*.* ■

According to Stephen [45] the direct limit (colimit) in the above theorem is called the closure of $lin(w)$ with respect to the presentation $Inv\langle X \mid R\rangle$ and it is denoted by $cl_R(lin(w))$. For any inverse word graph Γ (automaton $\mathcal{A}$) the direct system consisting of all objects that can be obtained by an arbitrary sequence of elementary expansions and elementary determinations, starting from Γ ($\mathcal{A}$) has a colimit, this colimit is the closure of Γ ($\mathcal{A}$) with respect to $Inv\langle X \mid R\rangle$ denoted by $cl_R(\Gamma)$ ($cl_R(\mathcal{A})$). Note that, in general, Stephen's procedure is not effective. It follows from Theorem 1.3.2

that if for all $w \in (X \cup X^{-1})^+$ we obtain a closed deterministic inverse automaton after finitely many applications of elementary expansions and derterminations, then the inverse semigroup $S = \langle X \mid R \rangle$ has decidable word problem.

1.3.2 Munn trees and free inverse semigroups

Recall that $FIS(X)$, the free inverse semigroup on X, can be represented as a quotient of $(X \cup X^{-1})^+$, i.e., $FIS(X) = (X \cup X^{-1})^+/\nu$, where ν is the Vagner congruence defined in Section 1.1.3. If the set X is finite, then the free inverse semigroup is finitely presented $FIS(X) = Inv\langle X|\emptyset\rangle$, so birooted Munn trees are exactly the Schützenberger automata relative to the presentation $\langle X|\emptyset\rangle$. Since we have only to do determinations to produce Munn trees, it easily follows that the word problem for $FIS(X)$ is decidable: i.e., one can decide for two given words $w_1, w_2 \in (X \cup X^{-1})^+$, whether $w_1\nu = w_2\nu$ or $w_1\nu \neq w_2\nu$. We will describe the solution given by Munn [33] and we illustate it by an example.
If $w \in (X \cup X^{-1})^+$, then for the linear automaton $lin(w) = (\alpha, L_w, \beta)$ of w (possibly nondeterministic) with two distinguished roots (a small arrow will be drown into the vertex α to indicate that it is the initial vertex of L_w, and for the terminal vertex an arrow will be drown out the vertex β). The determinized form of L_w is a tree Υ_w called the *Munn tree* of w. The Munn tree of w is obtained by a finite sequence of determinations on L_w.

Example 1.3.1. *Let* $w = a^{-1}abbb^{-1}a^{-1}$.

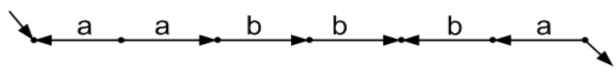


Figure 1.1: *linear graph* (α, L_w, β) *of* $w = a^{-1}abbb^{-1}a^{-1} \in (X \cup X^{-1})^+$

Applying each possible edge folding we obtain the following Munn tree of w

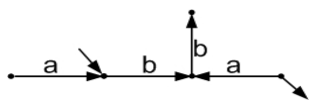


Figure 1.2: *Munn tree* $(\alpha_w, \Upsilon_w, \beta_w)$ *of* $w = a^{-1}abbb^{-1}a^{-1} \in FIS(X)$

The solution to the word problem for $FIS(X)$ is given by

Theorem 1.3.5. *Munn [33] Let $w, w' \in (X \cup X^{-1})^+$. Then $w\nu = w'\nu$ if and only if $(\alpha_w, \Upsilon_w, \beta_w) \cong (\alpha_{w'}, \Upsilon_{w'}, \beta_{w'})$.* ■

Of course Munn trees also provide information about the Green's relations and the natural partial order of $FIS(X)$.

Proposition 1.3.3. *Let $w, w' \in (X \cup X^{-1})^+$.*

1. *$w\nu$ is an idempotent of $FIS(X)$ if and only if $\alpha_w = \beta_w$ in $(\alpha_w, \Upsilon_w, \beta_w)$.*
2. *$w\nu \geq w'\nu$ if and only if there exists a birooted graph morphism $f : (\alpha_w, \Upsilon_w, \beta_w) \to (\alpha_{w'}, \Upsilon_{w'}, \beta_{w'})$*
3. *$w\nu \mathcal{R} w'\nu$ if and only if there exists a rooted graph isomorphism $f : (\alpha_w, \Upsilon_w) \to (\alpha_{w'}, \Upsilon_{w'})$.* ■

In statement (2), note that every graph morphism between trees is injective. Thus, for example, $w\nu \geq w'\nu$ in $FIS(X)$ if and only if $(\alpha_w, \Upsilon_w, \beta_w)$ is a (birooted) subgraph of $(\alpha_{w'}, \Upsilon_{w'}, \beta_{w'})$.
We can see $FIS(X)$ as *a* semigroup of birooted trees with multiplication of two birooted automata.

Proposition 1.3.4. *$FIS(X)$ is isomorphic to the semigroup of all finite birooted trees labelled over X.* ■

We refer the reader to [36] and [28] for more information about Munn trees.

1.4 Schützenberger Graphs of HNN-extensions of Finite Inverse Semigroups

In case of amalgams and HNN-extensions,it is sometimes useful to reorder the sequence of determinations and expansions to be iteratively applied to the linear graph of a word w in order to obtain the Schützenberger automaton of w, so that they can be ordered in 5 constructions. This reordering has also the advantage of providing more information about the shape of Schützenberger graphs of the words of amalgams and of the HNN-extensions of inverse semigroups. For more detailed descriptions of these constructions we refer the reader to [26, 38, 40]. Here we briefly recall the constructions applied for getting the Schützenberger graphs of words in HNN-extensions of finite inverse semigroups. The first four constructions

which we summarize in Section 2.2.3 produce a finite subgraph which approximates the Schützenberger graph under construction and contains all the information needed to proceed to the last step which consists only of the last construction 5. Such construction has to be applied possibly infinitely many times and it has an important role in our work. For this reson it will be described in more details in Section 1.4.2.
Before starting with the construction we give a "pre-elaboration" of the word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ to put it in a suitable form w' so that w and w' represent the same element of S^*. Let $w = s_1 t^{k_1} s_2 t^{k_2} ... t^n s_{n+1}$ where $s_i \in (X \cup X^{-1})^*$ for all $1 \leq i \leq n+1$, $k_i \in \mathbb{Z} \setminus \{0\}$ for all $1 \leq i \leq n$, then we expand w by replacing each t by etf where $t\tau = (etf)\tau$ to obtain

$$w' = s_1(etf)^{k_1} s_2(etf)^{k_2} ... (etf)^n s_{n+1}$$

Note that $w\tau = w'\tau$. For instance if $w = s_1 t^2 s_2 t^{-2} s_3 t s_4$ then the expanded form w' of w will be

$$\begin{aligned} w' &= s_1(etf)^2 s_2(etf)^{-2} s_3(etf) s_4 \\ &= (s_1 e)t(fe)t(fs_2 f)t^{-1}(ef)t^{-1}(es_3 e)t(fs_4). \end{aligned}$$

In this way in w' each occurrence of t has exponents in $\{-1, 1\}$ and any pair of occurrences of t/t^{-1} are separated by a factor containing some element of S.

Before proceeding to describe the constructions let us fix some notations and terminology. Let Γ be an inverse word automaton on $\overline{X}$. A maximal connected inverse subgraph on X of Γ, with at least one edge, is called *S-lobe* (or shortly *lobe*) and an S-lobe that consists only of one vertex is called a trivial S-lobe. With a slight abuse of notation we say that in some S-lobe Δ there is a path (v_1, s, v_2), for some $s \in S$, if there is some $u \in (X \cup X^{-1})^+$ with $u\tau = s$ such that (v_1, u, v_2) is a path in Δ. A *t-edge* is an edge of Γ labeled by t. Two vertices v_1, v_2 are called *t-adjacent* if they are connected by a t-edge, i.e. if either (v_1, t, v_2) or (v_2, t, v_1) is an edge of Γ. A t-edge is called *extremal* if it has a vertex belonging neither to S-lobe nor to another t-edge, i.e., an extremal t-edge is an edge connecting an S-lobe to a trivial S-lobe. Two S-lobes Δ_1, Δ_2 are called *adjacent* if there are two t-adjacent vertices $v_1 \in V(\Delta_1), v_2 \in V(\Delta_2)$, each of these vertices is called an *intersection vertex* of Γ. If there is an edge (v_i, t, v_{i+1}), then the ordered pair (v_i, v_{i+1}) is called an *intersection pair*. The *lobe graph* of the inverse word graph Γ is the digraph $\Upsilon(\Gamma)$ such that its vertices are the S-lobes of Γ and the edges represent the adjacency, i.e. there is an edge $y \in \Upsilon(\Gamma)$ with $0(y) = \Delta_1$ and $1(y) = \Delta_2$ if and only if Δ_1 and

Δ_2 are adjacent lobes in Γ and (v_1, v_2) with $v_1 \in V(\Delta_1), v_2 \in V(\Delta_2)$ is an intersection pair. We say that an inverse word graph Γ on $\overline{X}$ has the *t-linking property* if for each non extremal t-edge (v_1, t, v_2) there are loops (v_1, e, v_1) and (v_2, f, v_2). For more information we refer the reader to [26, 38, 40].

In the following figure the lobe graph of the expanded form of $w = s_1t^2s_2t^{-2}s_3ts_4$ is represented.

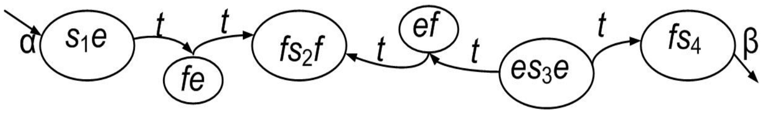


Figure 1.3: *Lobes of expanded form of w'*

1.4.1 t-Cactoid, t-Opuntoid and t-Core

Definition 1.4.1. We say that an inverse word graph Γ is a *t-cactoid* graph if it satisfies the following properties:

- its lobe graph $\Upsilon(\Gamma)$ is a finite tree,
- it has the t-linking property and
- there is only one t-edge between two adjacent lobes (vertices of the lobe tree)

We remark that the linear graph of the expanded form of each word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ is a cactoid graph, so in the sequel we assume that the first construction is applied to a cactoid graph. A t-cactoid automaton is an automaton whose underlying graph is t-cactoid. We remark that , since we assumed that S is finite then it has finite $\mathcal{R}-$classes.

1. **Construction 1: (Closure relative to $\langle \overline{X} | R \rangle$)**
   Let (α, Γ, β) be a t-cactoid automaton. This construction is based on two steps:
   - Close each S-lobe of Γ with respect to the presentation $\langle X | R \rangle$;
   - Apply a determination (edge folding) to the t-edges.
     After applying these steps iteratively for all lobes of the graph we obtain a t-cactoid graph whose lobes are DV-quotients of Schützenberger graphs relative to the presentation $\langle X | R \rangle$. The new graph has at most as many lobes as Γ. If we take as initial and terminal vertices of the new graph the natural images of α and

β we get a t-cactoid inverse automaton, moreover if the original automaton was an approximate automaton of the Schützenberger automaton $\mathcal{A}(X, R \cup R_{HNN}; w)$ for some word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ then the same does the new one.

2. **Construction 2: (The Loop equality property)**
   Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be a t-cactoid inverse automaton over $\overline{X}$, whose lobes are DV-quotients of Schützenberger graphs relative to the presentation $\langle X|R\rangle$. This construction is divided into two phases

   i) Let (v_1, v_2) be an intersection pair between two S-lobes Δ_1 and Δ_2 of Γ and define

   $$\mathcal{L}_A(v_1, \Delta_1) = \{a \in A | a \text{ labels a loop based at } v_1 \text{ in } \Delta_1\} \text{ and}$$

   $$\mathcal{L}_B(v_2, \Delta_2) = \{b \in B | b \text{ labels a loop based at } v_2 \text{ in } \Delta_2\}$$

   Since $\mathcal{A}$ has the t-linking property then $\mathcal{L}_A(v_1, \Delta_1)$ and $\mathcal{L}_B(v_2, \Delta_2)$ are non-empty. Moreover $\mathcal{L}_A(v_1, \Delta_1) \neq \emptyset$ and $\mathcal{L}_B(v_2, \Delta_2) \neq \emptyset$ are finite inverse subsemigroups of A and B. Thus each of them has a minimal idempotent denoted by $f_A(v_1, \Delta_1)$ and $f_B(v_2, \Delta_2)$. We say that $\mathcal{A}$ has *L property* if
   $$\varphi(f_A(v_1, \Delta_1)) = f_B(v_2, \Delta_2)$$
   for each intersection pair (v_1, v_2).
   Suppose that the $\mathcal{A}$ does not satisfy the L property. Then there is an intersection pair (v_1, v_2) between two S-lobes Δ_1 and Δ_2 such that

   $$\varphi(f_A(v_1, \Delta_1)) \notin \mathcal{L}_B(v_2, \Delta_2) \text{ or } \varphi^{-1}(f_B(v_2, \Delta_2)) \notin \mathcal{L}_A(v_1, \Delta_1)$$

   Then perform the following steps: For the first case attach to the S-lobe Δ_2 a new S-lobe consisting of the Schützenberger automaton $(v_2, S\Gamma(X, R; \varphi(f_A(v_1, \Delta_1))), v_2)$ and for the second case attach the lobe $(v_1, S\Gamma(X, R; \varphi^{-1}(f_B(v_2, \Delta_2))), v_1)$. Formally we obtain new automata by forming the products

   $$\Gamma' = (v_2, \Gamma, v_2) \times (v_2, S\Gamma(X, R; \varphi(f_A(v_1, \Delta_1))), v_2)$$

   $$\Gamma'' = (v_1, \Gamma, v_1) \times (v_1, S\Gamma(X, R; \varphi^{-1}(f_B(v_2, \Delta_2))), v_1)$$

   for the two cases respectively. Then apply construction 1 and repeat this step for all non-extremal t-edges where $\varphi(f_A(v_1, \Delta_1)) \neq f_B(v_2, \Delta_2)$ $(\varphi^{-1}(f_B(v_2, \Delta_2)) \neq f_A(v_1, \Delta_1))$.

ii) The L property implies that for each intersection pair (v_1, v_2) with $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ and for each path (v_1, a, v) with $a \in A$ in Δ_1 there is also a path $(v_2, \varphi(a), v'')$ in Δ_2 and vice versa. However, this one-to-one correspondence between paths in Δ_1 and paths in Δ_2 labelled respectively by isomorphic elements from A and B does not imply in general that for each loop based at v_1 labelled a for some $a \in A$ ($\varphi^{-1}(b)$ for some $b \in B$) in Δ_1, $\varphi(a)$ (b) labels a loop based at v_2 in Δ_2, i.e., in general the last equality of the following equation does not hold:
$\varphi(\mathcal{L}_A(v_1, \Delta_1)) = \{\varphi(a) | a \in \mathcal{L}_A(v_1, \Delta_1)\} = \mathcal{L}_B(v_2, \Delta_2)$
When the above equation is satisfied for all intersection pairs of $\mathcal{A}$ then $\mathcal{A}$ is said to have the *loop equality property*
If the automaton $\mathcal{A}$ satisfies the L property but it does not satisfy the loop equality property, there is an intersection pair (v_1, v_2) with $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ and an element $a \in A$ such that $a \in \mathcal{L}_A(v_1, \Delta_1)$ but $\varphi(a) \notin \mathcal{L}_B(v_2, \Delta_2)$ or $b \in B$ such that $b \in \mathcal{L}_B(v_2, \Delta_2)$ but $\varphi^{-1}(b) \notin \mathcal{L}_A(v_1, \Delta_1)$. If $a \in \mathcal{L}_A(v_1, \Delta_1)$ ($b \in \mathcal{L}_B(v_2, \Delta_2)$) and (v_2, a, v^*) ((v_1, b, v^*)) is a path in Δ_2 (Δ_1) then perform the following step.
Form the V-quotient of the S-lobe Δ_2 (Δ_1) by identifying v_2 (v_1) with v^*, then apply Construction 1 and Construction 2 phase (i) and then repeat Construction 2 phase (ii) for every such intersection pairs in $\mathcal{A}$.

The construction produces a t-cactoid graph satisfying the loop equality property whose lobes are closed DV-quotients of Schützenberger graphs relative to the presentation $\langle X|R\rangle$. Moreover, the number of lobes does not increase. Take as initial and final vertices the natural images of α and β, getting a new automaton. This new automaton approximates $\mathcal{A}(X, R \cup R_{HNN}; w)$ for some $w \in (\overline{X} \cup \overline{X}^{-1})^+$ if the automaton $\mathcal{A} = (\alpha, \Gamma, \beta)$ dose the same.

3. **Construction 3: (Related pair separation property)** A t-cactoid automaton $\mathcal{A}$ is said to have the *related pair separation property* if it has the loop equality property and for any S-lobe Δ there is no path (v, a, v') ((v, b, v')) with $a \in A$ ($b \in B$) where v and v' are intersection vertices in $V(\Delta)$ such that (v, v_1), (v', v_2) ((v_1, v), (v_2, v')) are intersection pairs between Δ and two different S-lobes Δ_1 and Δ_2.
Let $\mathcal{A}$ be a t-cactoid automaton whose lobes are closed DV-quotients of Schützenberger graphs relative to $\langle X|R\rangle$ which satisfies the loop equality property but does not satisfy the related pair separation prop-

erty. Then there is an S-lobe Δ and two vertices $v, v' \in V(\Delta)$ connected by a path labelled by $a \in A$ ($b \in B$) such that there are two edges (v, v_1), (v', v_2) ((v_1, v), (v_2, v')) with $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ for some different S-lobes Δ_1 and Δ_2 which are adjacent to the S-lobe Δ. Assume the first case occurs, then since $\mathcal{A}$ has the loop equality property, there are a path from v_1 to v_1' in Δ_1 labelled by $\varphi(a)$ and another one is from v_2 to v_2' in Δ_2 also labelled by $\varphi(a)$ (the other case is analogous). Then perform the following step:
Separate the automaton $\mathcal{A}$ at the vertex v_2 and replace v_2 by $v_2(0)$ in Δ and by $v_2(2)$ in Δ_2 to obtain two components T_1 and T_2 which contain respectively $v_2(0)$ and $v_2(2)$. Build the automaton

$$(v_1', \Sigma, v_1') = (v_1', T_1, v_1') \times (v_2(2), T_2, v_2(2))$$

Denote by $\mathcal{B}$ the automaton $(\alpha', \Sigma, \beta')$, where α' and β' are the natural images of α and β. Then apply constructions 1 and 2 to $\mathcal{B}$ to obtain a new automaton $\mathcal{A}'$. Repeat construction 3 for pair of intersection vertices in a same lobe connected by a path labelled by elements in A and B. The obtained automaton is a t-cactoid automaton with the related pair separation property whose lobes are closed DV-quotient of Schützenberger graphs relative to $\langle X|R\rangle$ and it approximates $\mathcal{A}(X, R \cup R_{HNN}; w)$ for some $w \in (\overline{X} \cup \overline{X}^{-1})^+$ if the automaton $\mathcal{A} = (\alpha, \Gamma, \beta)$ does the same.

4. **Construction 4: (t-saturation and t- assimilation properties)**
Let $\mathcal{A}$ be an inverse automaton whose S-lobes are closed finite inverse graphs relative to $\langle X|R\rangle$. Let v be a vertex in an S-lobe Δ such that (v, a, u) ((v, b, u)) is a path with $a \in A$ ($b \in B$). Since $A \subseteq eSe$ ($B \subseteq fSf$) then e (f) labels a loop based at v and f (e) labels a loop at u. Assume at this phase that the vertex v is an intersection vertex while u is not. Thus we have to make an expansion according to the relation $t^{-1}at = \varphi(a), a \in A$ ($t\varphi^{-1}(a)t^{-1} = a, a \in B$). This will be done by two steps

   - *t-saturation property*
   Let v be an intersection vertex in the the S-lobe Δ then $\mathcal{L}_A(v, \Delta) \neq \emptyset$ ($\mathcal{L}_B(v, \Delta) \neq \emptyset$). Let (v, a, u) be a path as above, then e (f) labels a loop based at v and f (e) labels a loop at u because $e = tt^{-1}$ and $f = tt^{-1}$. Since v is already a vertex of a t-edge t then u must be a vertex of a t-edge thus we have to implant an extremal t-edge (u, t, u') ((u', t, u)) at u according whether the loop based at u is labelled by e or f, the vertex u is called related

vertex to the intersection vertex v. Applying this procedure to all such paths will lead to an automaton that has a *t-saturation property*

- *t- assimilation property*
  Let $\mathcal{A}$ be an automaton with the related pair separation property and t-saturation property. Let Δ_1, Δ_2 be two adjacent lobes connected by the t-edge (v_1, t, v_2) (the case (v_2, t, v_1) is analogous), and let x_1, x_2 be two related vertices at the intersection pair (v_1, v_2). Thus there are t-edges (x_1, t, x_1') and (x_2, t, x_2') for which there is no t-edge connecting x_1 with x_2. Hence, there are paths $(v_1, t^{-1}at, x_1)$ and $(v_2, \varphi(a), x_2)$, where $a \in A$. Because the relation $t^{-1}at = \varphi(a), a \in A$ is in R_{HNN} we have to identify x_1' with x_2 and x_2' with x_1, i.e., we then have a t-edge connecting x_1 with x_2. After applying this step to all related vertices we obtain an automaton that has a *t- assimilation property*.

The iterative applications of these four constructions will end after a finite number of steps into a finite inverse word automaton called *t-Core* whose underlying inverse word graph has a particular shape known as t-opuntoid graph which is characterized by the following properties

- each S-lobe is a closed DV-quotient of some Schützenberger graph relative to $\langle X|R\rangle$;
- it has the loop equality property;
- it has the related pair separation property;
- it has the t-saturation property;
- it has the t-assimilation property.

An inverse word subgraph Γ' of Γ is called t-subopuntoid if Γ' is a t-opuntoid graph and its lobes are also lobes of Γ.

1.4.2 Construction 5: Complete t-opuntoid

In general t-$Core(w)$ is not closed with respect to $\langle \overline{X}|R \cup R_{HNN}\rangle$. Indeed, in any vertex v of t-$Core(w)$ for which there is a loop labelled by some element of A or B an expansion must be applied (because of the relations of the form $t^{-1}at = \varphi(a)$). This vertex, by the saturation property, is the vertex of at most two t-edges, and the expansion must be done in case one of these edges is an extremal

t-edge. In this case we distinguish between the two ending vertices of that t-edge and we call the vertex in the S-lobe a *bud*, while the other vertex is called a t-*bud*. Note that a bud can be a vertex of a t-edge which is not extremal, while a t-bud always belongs to an extremal t-edge. An *opuntia* is a t-opuntoid graph having only one S-lobe, for an S-lobe Δ we denote by $O_p(\Delta)$ the smallest t-opuntoid containing an isomorphic copy of Δ, i.e. $O_p(\Delta)$ is obtained taking an isomorphic copy of Δ and adding all the t-edges to respect the t-saturation property. In a t-opuntoid Γ, if v is a bud of Γ then either the set $\mathcal{L}_A(v, \Delta)$ or $\mathcal{L}_B(v, \Delta)$ is non-empty. In [40] it is proved that a specific finite series of expansions and determinations, known as Construction 5, applied to a t-bud v of a t-opuntoid Γ generates (after finitely many steps) a new t-opuntoid Γ' obtained from Γ by adding a new opuntia connected to v by a t-edge whose direction depends on which one between $\mathcal{L}_A(v, \Delta)$ and $\mathcal{L}_B(v, \Delta)$ is non-empty. Because this construction plays an important role in the sequel we will explain its implementation under the assumption that S is finite:

5. (**Construction 5: Complete t-opuntoid**) Let (α, Γ, β) be an t-opuntoid automaton and let (v, y, v') be an extremal edge with $y \in \{t, t^{-1}\}$, such that v is a bud and v' is its corresponding t-bud of Γ. Assume that $v \in V(\Delta)$ for some lobe Δ of Γ, and assume that $\mathcal{L}_A(v, \Delta) \neq \emptyset$, (the other case $\mathcal{L}_B(v, \Delta) \neq \emptyset$ is analogous). Let (x, Σ, x) be the Schützenberger automaton of $\varphi(f_A(v, \Delta))$ relative to $\langle X|R\rangle$, and consider the set which is called the *net*:

$$N(x, \Sigma) = \{y \in V(\Sigma) | (x, \varphi(u), y) \text{ is a path in } \Sigma \text{ for some } u \in \mathcal{L}_A(v, \Delta)\}.$$

Let ρ be the least equivalence relation on $V(\Sigma)$ which identifies all the elements of $N(x, \Sigma)$ with x and such that Σ/ρ is deterministic and put $\Delta' = \Sigma/\rho$. It can be proved that Δ' is a finite inverse word graph closed with respect to $\langle X|R\rangle$ such that $\varphi(\mathcal{L}_A(v, \Delta)) = \mathcal{L}_B(x\rho, \Delta')$ where $x\rho$ denotes the ρ-class of x.
In [40] it is proved that there is a one-to-one correspondence between the set

$$B^{\Delta}_{\Delta'} = \{p \in V(\Delta) | (v, u, p) \text{ is a path in } \Delta \text{ for some } u \in A\}$$

and

$$R_{\Delta'} = \{p' \in V(\Delta') | (x\rho, \varphi(u), p') \text{ is a path in } \Delta' \text{ for some } u \in A\}$$

given by the map $\psi : B^{\Delta}_{\Delta'} \to R_{\Delta'}$ which associates to a vertex p in $B^{\Delta}_{\Delta'}$, such that (v, a, p) is a path in Δ for some $a \in A$, the vertex p' for which $(x\rho, \varphi(a), p')$ is a path in Δ', the set $R_{\Delta'}$ is called the root of the lobe Δ' while $B^{\Delta}_{\Delta'}$ is the set of buds in Δ corresponds to the set $R_{\Delta'}$. Then let $\Gamma' = (v, \Gamma, v) \uplus (x\rho, \Delta', x\rho)$ be the inverse word graph defined by the disjoint union of (v, Γ, v), $(x\rho, \Delta', x\rho)$ identifying $x\rho$ with the t-bud v, plus all the t-edges $(p, t, \psi(p))$ for all $p \in B^{\Delta}_{\Delta'}$. Moreover, to maintain the structure of t-opuntoid we need to satisfy the t-saturation property, thus for each vertex v such that $\mathcal{L}_A(v, \Delta') \neq \emptyset$ ($\mathcal{L}_B(v, \Delta') \neq \emptyset$) we add the extremal t-edge $(v, t, v')((v, t^{-1}, v))$ then we apply the construction of the assimilation property. Consequently, we may say that $B^{\Delta}_{\Delta'} = V(O_P(\Delta')) \cap V(\Delta)$ and $R_{\Delta'} = V(O_P(\Delta)) \cap V(\Delta')$.

This construction can be applied iteratively to obtain a directed system of t-opuntoid automata. In [40] it has proved that the direct limit of such system is $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$. From the applications of constructions 1-4 it is obtained a finite deterministic t-opuntoid automaton t-$Core(w)$ whose lobes are closed DV-quotients of Schützenberger graphs relative to the presentation $\langle X|R\rangle$ and such that t-$Core(w)$ is an approximate automaton of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$. Each application of Construction 5 reduces the numbers of buds relative to t-$Core(w)$ but in the mean time introduces new lobes and eventually new buds. Let $\mathcal{A}_0 = t - Core(w)$ and put $\mathcal{A}_{i+1}$ the automaton obtained from $\mathcal{A}_i$ by an application of Construction 5 to some t-bud of $\mathcal{A}$. In this way we obtain a sequence:

$$t - Core(w) = \mathcal{A}_0 \hookrightarrow \mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \hookrightarrow ...$$

of automata with $\mathcal{A}_i \rightsquigarrow \mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$. The injective homomorphisms $\phi_{ij} : \mathcal{A}_i \to \mathcal{A}_j$ with the sequence $\{\mathcal{A}_i\}_{i \geq 0}$ form a direct limit. In Theorem 1 of [40] it is proved that

$$\mathcal{A}(\overline{X}, R_{HNN} \cup R; w) = \lim_{\to} \mathcal{A}_i$$

and it is a complete t-opuntoid automaton whose S-lobes are closed DV-quotients of Schützenberger graphs relative to $\langle X|R\rangle$.

The set of lobes belonging to $cl_R(\Gamma)$ but not to Γ is called the set of *external lobes* of Γ and denoted by $\mathcal{E}(\Gamma)$. Let Θ, Θ' be two adjacent lobes of an t-opuntoid graph Γ, following [38, 40], we say that Θ' *directly feeds off* Θ, in symbols $\Theta \mapsto \Theta'$, if Θ' can be obtained from Θ by applying Construction 5 at some intersection pair $(v, v')((v', v))$ between Θ and Θ'. Moreover for each pair of lobes Θ, Θ' of Γ we say that Θ' feeds off Θ, $\Theta \to \Theta'$, if Θ and Θ' are related in the transitive closure of $\mapsto$. Clearly for each lobe $\Theta' \in \mathcal{E}(\Gamma)$ there is a lobe Θ of Γ such that $\Theta \to \Theta'$.

Let Γ be a t-opuntoid graph whose lobes are closed DV-quotients of

Schützenberger graphs relative to the presentation $\langle X|R\rangle$, and let Δ be a lobe of Γ. Since S is finite then at each vertex v of Δ there exists the minimum idempotent from S labelling a loop at v in Δ denoted by $e(v, \Delta)$.

We conclude this chapter by investigating some properties of the relation ρ introduced in Construction 5 and we consistently refer to the notation used there. We remind the following

Lemma 1.4.1. *Fix the notation as in the statement of Construction 5. Let $a \in A$ be such that $(x, \varphi(a), y)$ is a path of Σ for some $y \in N(x, \Sigma)$ and let n be the smallest integer such that $(\varphi(a))^n$ is an idempotent of B. Then $(\varphi(a))^n$ labels a loop based at x in Σ. Denote by x_i the vertex in Σ at the end of the path starting at x and labeled by $(\varphi(a))^i$, for $i = 1, \dots, n-1$. Let Σ' be the DV-quotient of Σ obtained by identifying all of the vertices $x_1, x_2, \dots, x_{n-1}$ with x and then determinizing. Then two vertices q_1 and q_2 of Σ are identified in Σ' if and only if there is some word z that labels a path in Σ from x_i to q_1 and a path in Σ from x_j to q_2 for some i, j. Furthermore, a word $s \in (X \cup X^{-1})^+$ labels a path in Σ starting at q_1 if and only if s labels a path in Σ starting at q_2.* ■

Recall the following proposition from [40].

Proposition 1.4.1. *If $(q\rho, w, p\rho)$ is a path in $\Delta' = \Sigma/\rho$ for some $w \in (X \cup X^{-1})^+$, then for all $q' \in q\rho$ there is a path (q', w, p') in Σ with $p' \in p\rho$. Moreover, $x\rho = N(x, \Sigma)$.* ■

This proposition leads to a description of ρ in terms of the subgroup of the automorphism group of Σ that fixes the net $N(x, \Sigma)$.

Proposition 1.4.2. *With the notation of Construction 5, let $z, r \in V(\Sigma)$. Then $z\rho r$ if and only if there is an automorphism $\phi \in Aut(\Sigma)$ with $\phi(N(x, \Sigma)) \subseteq N(x, \Sigma)$ such that $\phi(z) = r$. Furthermore we have the following properties:*

1. *$x\rho = N(x, \Sigma)$,*
2. *Let $\pi_\rho : \Sigma \to \Delta'$ be the natural homomorphism induced by ρ. Then the set $\pi_\rho^{-1}(R'_\Delta)$ is the set of vertices of Σ connected to x by some $\varphi(u), u \in A$ (we recall that $R'_\Delta = V(O_P(\Delta)) \cap V(\Delta')$).*

Proof Let $z\rho r$ and denote by $e(z, \Sigma), e(r, \Sigma)$ the minimum idempotents of S labeling loops based respectively at z and r in the Schützenberger graph Σ. By Proposition 1.4.1 it easily follows that $e(z, \Sigma) = e(r, \Sigma)(= e(z\rho, \Sigma/\rho))$. Thus by Theorem 1.3.3 $(r, \Sigma, r) \cong (z, \Sigma, z)$, i.e there exists $\phi \in Aut(\Sigma)$ with $\phi(z) = r$. Moreover, since $z\rho r$, then by Lemma 1.4.1,

there are two vertices $q, q' \in N(x, \Sigma)$ and a word $h \in (X \cup X^{-1})^*$ such that $(q, h, z), (q', h, r)$ are two paths in Σ. Since $\phi(z) = r$ then $\phi(q) = q'$. Now let $q'' \in N(x, \Sigma)$. By definition of the net there is $a \in \mathcal{L}_A(v, \Delta)$ such that $(q, \varphi(a), q'')$ is a path in Σ. Hence the image of $(q, \varphi(a), q'')$ by ϕ is the path $(q', \varphi(a), \phi(q''))$ and so, $q' \in N(x, \Sigma)$ and $a \in \mathcal{L}_A(v, \Delta)$ imply $\phi(q'') \in N(x, \Sigma)$. So $\phi(N(x, \Sigma)) \subseteq N(x, \Sigma)$.
Conversely let $\phi \in Aut(\Sigma)$ such that $\phi(N(x, \Sigma)) \subseteq N(x, \Sigma)$ and $\varphi(z) = r$. Since Σ is connected for any $q \in N(x, \Sigma)$ there is a path (q, s, z) in Σ. The image of this path through ϕ is the path $(\varphi(q), s, r)$ where $\phi(q) \in N(x, \Sigma)$ and so $z\rho t$ by Lemma 1.4.1.
We prove claim (1). The inclusion $N(x, \Sigma) \subseteq x\rho$ is trivial. Let $x' \in x\rho$, we have just proved that there is an automorphism $\phi \in Aut(\Sigma)$ with $\phi(x) = x'$ and $\phi(N(x, \Sigma)) \subseteq N(x, \Sigma)$, whence $x \in N(x, \Sigma)$ implies $x' \in N(x, \Sigma)$.
Now we prove second statement. If $(x, \varphi(a), y)$ is a path in Σ with $a \in A$ then $(\pi_\rho(x), \varphi(a), \pi_\rho(y)) = (v, \varphi(a), y\rho)$ is a path in $\Delta' \cong \Sigma/\rho$. Conversely let $(x\rho, \varphi(a), z\rho)$ be a path in Δ' with $a \in A$. By Proposition 1.4.1 for all z' with $z'\rho z$ there is a path (x', a, z') in Σ with $x' \in x\rho = v$. By claim (1) $x' \in N(x, \Sigma)$, thus there is a path $(x, \varphi(\overline{a}), x')$ in Σ with $\overline{a} \in \mathcal{L}_A(v, \Delta)$, whence $(x, \varphi(\overline{a})\varphi(a), z') = (x, \varphi(\overline{a}a), z')$ is a path of Σ with $\varphi(\overline{a}a) \in U$. ■

Let Σ be an inverse subgraph of the closed inverse graph Γ over X, and let T be a set of relations over X^+. The graph $\Gamma \bigoplus cl_T(\Sigma)$ is a quotient of the disjoint union $\Gamma \cup cl_T(\Sigma')$, where Σ' is an isomorphic copy of Σ, via the isomorphism $\eta' : \Sigma \to \Sigma'$; $\eta'' : \Sigma \to cl_T(\Sigma')$ is the composition of η' and the natural map from $\Sigma' \to cl_T(\Sigma')$; and $\eta = \{(v, v\eta'') : v \in V(\Sigma)\}$ for more information of the operator $\bigoplus$ see [45]. Using this notation to prove the following useful Lemma.

Lemma 1.4.2. *Let Γ be a closed inverse word graph with respect to $\langle Y|T\rangle$. Let $v_1, v_2 \in V(\Gamma)$ and $w \in L[(v_1, \Gamma, v_2)]$. Then there exits a homomorphism*

$$\phi : \mathcal{A}(Y, T; w) \to (v_1, \Gamma, v_2)$$

Proof It is known that $\Gamma \oplus cl_T(lin(w))$ is an approximate graph of $cl_T(\Gamma) = \Gamma$ and $cl_T(lin(w)) \cong S\Gamma(Y, T; w)$, thus $L[\mathcal{A}(Y, T; w)] \subseteq L[(v_1, \Gamma, v_2)]$ since $\mathcal{A}(Y, T; w)$ is an approximate automaton of (v_1, Γ, v_2). Therefore, the statement follows from Theorem 1.3.3. ■ We also have the following proposition.

Proposition 1.4.3. *[39] Let $w, w' \in (X \cup X^{-1})^+$ and let*

$$\phi : S\Gamma(X, R; w) \to S\Gamma(X, R; w')$$

be a homomorphism (isomorphism) of Schützenberger graphs relative to $\langle X|R\rangle$*. Then* ϕ *can be extended to a homomorphism (isomorphism)* Φ *between their closures relative to* $\langle \overline{X}|R_{HNN} \cup R\rangle$*:*

$$\Phi : S\Gamma(w) \to S\Gamma(w')$$

Proof Let α be the initial vertex of the Schützenberger automaton $\mathcal{A}(X, R; w)$, let $\alpha' = \phi(\alpha)$. Thus it is clear that ϕ is also a homomorphism from $\mathcal{A}(X, R; e(\alpha'\Delta'))$ to $\mathcal{A}(X, R; e(\alpha, \Delta))$ where Δ', Δ are lobes in $S\Gamma(w'), S\Gamma(w)$ containing α', α respectively. In particular it is an isomorphism if and only if $e(\alpha, \Delta) = e(\alpha', \Delta')$. Since ϕ preserves the labeling, there is a loop labeled by $e(\alpha, \Delta)$ based at α' in $S\Gamma(X, R, e(\alpha', \Delta'))$ whence $e(\alpha, \Delta) \geq e(\alpha', \Delta')$ in S. Since S embeds into S^*, we get $e(\alpha, \Delta) \geq e(\alpha', \Delta')$ in S^*. Hence, by Theorem 1.3.3, there is a homomorphism $\Phi : S\Gamma(e(\alpha, \Delta)) \to S\Gamma(e(\alpha', \Delta'))$ with $\Phi(\alpha) = \alpha'$ which is an isomorphism if and only if $e(\alpha, \Delta) = e(\alpha', \Delta')$, i.e. if and only if ϕ is an isomorphism. Since $\Phi|_{S\Gamma(X,R;e(\alpha,\Delta))}(\alpha) = \phi(\alpha)$, then, by the determinism of $S\Gamma(e(\alpha, \Delta)), S\Gamma(e(\alpha', \Delta'))$, and the fact that two homomorphisms that coincides on a vertex will be equal, we have Φ is an extension of ϕ. ■

CHAPTER 2

Algorithmic issues on HNN-extension of inverse semigroups

THIS chapter contains two main results. The first one proves that the word problems in Yamamura's HNN-extension of inverse semigroups can be undecidable even under some nice conditions on the original semigroup S. The second one concerns with HNN-extension of finite inverse semigroups, where the word problem was proved to be decidable in [40], and shows that the language recognized by the Schützenberger automaton of any word w with respect to the standard presentation of an HNN-extension $[S, A, B]$. with S finite is a deterministic context-free language. Our approach follows similar ideas developed in [9]. The chapter is organized as follows. In the first section we prove that the word problem of an HNN-extension $[S, A, B]$ can be undecidable even if we assume the following conditions: S has finite $\mathcal{R}$-classes, the membership to A and B has the solvable membership problem and the isomorphisms φ, φ^{-1} are computable can be undecidable. This is achieved by reducing this problem to a recent result of undecidability for amamlgams of inverse semigroups [41]. This fact is in the contrast with the group case where under the same conditions for HNN-extension of groups the word problem is decidable (see Corollary 2.2 of [29]). In Section 2.2 the first subsection is devoted to some basic notion on formal language theory, mainly focusing on context free languages, their recognizing automata and their transition graphs. Subsection 2.2.2 gives a brief summary on the main features of the so called context-free graphs in the sense of [31]. We show that Schützenberger graphs of HNN-extensions of finite inverse semigroups are context-free graphs, this proves that the languages of these automata are context-free. In Section 2.2.3 we explicitly exhibit a pushdown automaton recognizing these languages. This approach has some advantages from the computational complexity point of view.

2.1 Undecidability of HNN-extension under some nice conditions

The word problem in Yamamura's HNN-extension $[S; A, B; \varphi]$ has been proved to be decidable in some cases, for instance when S is finite [40], when S is a free inverse semigroup and A and B are finitely generated [26], and in the lower bounded case [26] under the conditions that S has solvable word problem, A and B have solvable membership problem, φ is computable and other three technical conditions guaranteing the existence of algorithms for deciding whether or not the sets $U_A(e) = \{u \in A | u \geq e\}$, $U_B(e) = \{u \in B | u \geq e\}$ are empty for any idempotent e of S, of an algorithm for computing the minimum idempotents of $U_A(e)$, $U_B(e)$, and lastly of an algorithm to decide whether a path connecting two given vertices of the Schützenberger graph relative to $[S; A, B; \varphi]$ is labelled by an element from A or B. The result present here has an opposite trend. Indeed, we prove that even assuming nice conditions on the tuple S, A, B, φ the word problem for Yamamura's HNN-extension of inverse semigroups may be undecidable.

It is noteworthy that, although inverse semigroups seem to be very closed to groups the result here obtained is in contrast with the group case. Indeed, the word problem for HNN-extension for groups has been proved to be solvable when the original group S has a solvable word problem, the subgroups A and B have solvable membership problem, and φ, φ^{-1} are computable (see Corollary 2.2 of [29]). Our main result, Theorem 2.1.1, relies on the undecidability under similar conditions of the word problem for amalgams of inverse semigroups proved in [41], and a result in [10] which shows that amalgams are isomorphic to certain quotients of some particular Yamamura's HNN-extensions.

2.1.1 Amalgam vs HNN-extension

Let $S_1 = Inv\langle X_1|R_1\rangle$, $S_2 = Inv\langle X_2|R_2\rangle$ with $X_1 \cap X_2 = \emptyset$. We recall the following definitions

Definition 2.1.1. Let $S_1,\ S_2$ be disjoint semigroups. The free product of S_1 and S_2 is an inverse semigroup $S = S_1 * S_2$ such that:

(i) there is a homomorphism $\sigma_i : S_i \to S$ for every $i \in \{1, 2\}$,

(ii) for every inverse semigroup T and for every pair $\{\phi_i : S_i \to T, i = 1, 2\}$ of homomorphisms, there is a unique homomorphism $\phi : S \to T$ such that $\sigma_i \circ \phi = \phi_i$ for each $i \in \{1, 2\}$.

Definition 2.1.2. Let $S_1,\ S_2$ be two inverse disjoint semigroups and let $\omega_i : U \hookrightarrow S_i$, $i \in \{1, 2\}$ be monomorphisms. The amalgam of S_1 and S_2 with core U is denoted by $[S_1, S_2; U, \omega_1, \omega_2]$ or for short $[S_1, S_2; U]$.
The free product of S_1 and S_2 amalgamating U (or amalgamated free product of S_1 and S_2 with core U) is an inverse semigroup $S = S_1 *_U S_2$ such that:

(i) there is a homomorphism $\sigma_i : S_i \to S$ for every $i \in \{1, 2\}$ and $\sigma_1 \circ \omega_1 = \sigma_2 \circ \omega_2$,

(ii) for every inverse semigroup T and for every pair $\{\phi_i : S_i \to T, i = 1, 2\}$ of homomorphisms such that $\omega_1 \circ \phi_1 = \omega_2 \circ \phi_2$, there is a unique homomorphism $\phi : S \to T$ such that $\sigma_i \circ \phi = \phi_i$ for each $i \in \{1, 2\}$.

Notice that the free product and the amalgamated free product of inverse semigroups are the coproduct and the pushout in the category of inverse semigroups, respectively. Thus their existence and uniqueness come from universal algebra arguments.

It is known that the amalgam $[S_1, S_2; U]$ of inverse semigroups is strongly embedded in $S_1 *_U S_2$. In the sequel we assume S_i to be finitely presented by $\langle X_i|R_i\rangle$. Then $S_1 * S_2$ is presented by $\langle X|R\rangle$ with $X = X_1 \cup X_2$, $R = R_1 \cup R_2$, and $S_1 *_U S_2$ is presented by $\langle X|R \cup R_W\rangle$ with X, R as above and $R_W = \{(\omega_1(u), \omega_1(u))|u \in U\}$ (where with a slight abuse of notation $\omega_i(u)$ denotes a word in $(X_i \cup X_i^{-1})^+$) these presentations are called the standard presentations of $S_1 * S_2$ and $S_1 *_U S_2$.
Let Γ be an inverse word graph (automaton) on X, a *lobe* of Γ is a maximum connected inverse subgraph whose edges are labelled only by $(X_1 \cup X_1^{-1})$ for short a lobe of color 1 or by $(X_2 \cup X_2^{-1})$ for short a lobe of color 2. Two lobes are said to be adjacent if they have at least one common vertex. The lobe graph is an undirected graph whose vertices are lobes of Γ and

whose edges correspond to the adjacency of lobes. In [8] it is shown that the Schützenberger graphs of words with respect to the standard presentation of $S_1 *_U S_2$ of finite inverse semigroups are opuntoid graphs. Similarly, in Section 1.4 t-opuntoid graphs have been introduced to describe the shape of $\mathcal{A}(\overline{Y}, R_{HNN}; w)$ when S is finite. If the finiteness hypothesis is missed, then according to Theorem 3.6 of [10] and Theorem 2.2.1 of [26] each Schützenberger graph relative to the standard presentation of $S_1 *_U S_2$ and of the HNN-extension S^* have a shape satisfying the following weaker conditions.

Definition 2.1.3. A deterministic inverse word graph on X whose lobe graph is a tree is called *weak opuntoid* graph (automaton). An inverse word graph (automaton) on $\overline{X}$ is called *weak t-opuntoid* if it is deterministic and its lobe graph is a tree.

To each amalgam $[S_1; S_2, U; \omega_1, \omega_2]$ we may associate a Yamamura's HNN-extension. We briefly recall this construction, and we refer the reader to [10] for further details. Let $e_i \notin X_i$ and consider $S_i^{e_i}$, i.e., the semigroup S_i with adjoint identity e_i, then $S_i^{e_i}$ is presented by $\langle X_i \cup e_i | R_i^1 \rangle$, where $R_i^1 = R_i \cup \{e_i^2 = e_i, e_i x_i = x_i e_i = x_i | x_i \in X_i\}$. Let U^1 be the inverse semigroup obtained adjoining the identity 1 to U, the embeddings $\omega_i = U \hookrightarrow S_i$ can be extended to an embedding $\omega_i^1 = U^1 \hookrightarrow S_i^{e_i}$ by putting $\omega_i^1(1) = e_i$ and $\omega_i^1(u) = \omega_i(u)$ for all $u \in U$. In this way we have built the amalgam $[S_1^{e_1}, S_2^{e_2}; U^1, \omega_1^1, \omega_2^1]$. Note that $U_1^{e_1}, U_2^{e_2}$ embed into $S_1^{e_1} * S_2^{e_2}$, and so with a slight abuse of notation we identify these semigroups with their embedding in $S_1^{e_1} * S_2^{e_2}$. Note that they are isomorphic subsemigroups by the map $(\omega_1^1)^{-1} \circ \omega_2^1$. Therefore, the Yamamura's HNN-extension associated to $[S_1^{e_1}; S_2^{e_1}, U^1; \omega_1^1, \omega_2^1]$ is given by the HNN-extension

$$[S_1^{e_1} * S_2^{e_2}; U_1^{e_1}, U_2^{e_2}; (\omega_1^1)^{-1} \circ \omega_2^1]$$

which is presented by $\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$, where $\overline{X} = \{X_1 \cup \{e_1\}\} \cup \{X_2 \cup \{e_2\}\} \cup \{t\}$, $\overline{R} = R_1^1 \cup R_2^1$, and

$$R_{HNN} = \{(\omega_1^1)^{-1} \circ \omega_2^1(u_1) = t^{-1} u_1 t, u_1 \in U\} \cup \{tt^{-1} = e_1, t^{-1} t = e_2\}$$

Let $S^* = Inv\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$, in [10, Theorem 1] it is proved that

$$S^*/\rho \cong (S_1^{e_1} *_{U^1} S_2^{e_2}) \cong (S_1 *_U S_2)^1 \tag{2.1}$$

where $(S_1 *_U S_2)^1$ denotes $S_1 *_U S_2$ with adjoint identity 1 and ρ is the congruence on S^* generated by the relation $t = e_1$, $t = e_2$. Moreover, in [10] it is proved that the Schützenberger automaton of a word $w \in (X \cup$

$X^{-1})^*$ relative to the standard presentation of $S_1 *_U S_2$ can be obtained from the Schützenberger automaton relative to $\langle \overline{X} | \overline{R} \cup R_{HNN} \rangle$ of a special word $w' \in (\overline{X} \cup \overline{X}^{-1})^*$ associate to w. Precisely, consider its factorization

$$w = w_1 w_2 ... w_{2n-1} w_{2n}$$

where $w_1 \in (X_1 \cup X_1^{-1})^*$, $w_{2i} \in (X_2 \cup X_2^{-1})^+$, $w_{2i+1} \in (X_1 \cup X_1^{-1})^+$, $1 \leq i \leq n-1$ and $w_{2n} \in (X_2 \cup X_2^{-1})^*$. The following associated word

$$w' = w_1 e_1 t e_2 w_2 e_2 t^{-1} e_1 ... e_2 t^{-1} e_1 w_{2n-1} e_1 t e_2 w_{2n}$$

is called the *separated normal form* of w. It is significant to recall the following results from [10].

Lemma 2.1.1 (Lemma 3.4 [10]). *Let $w \in (X \cup X^{-1})^+$, the Schützenberger automaton $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R} \cup \{t = e_1, t = e_2\}, w)$ can be obtained from the Schützenberger automaton $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w')$ of the separated normal form w' of w by identifying the initial and terminal vertices of each t-edge.* ■

Proposition 2.1.1 (Corollary 3.5 [10]). *The Schützenberger automaton of a word $w \in (X \cup X^{-1})^+$ relative to the presentation $\langle X | R \cup R_W \rangle$ of $S_1 *_U S_2$ can be obtained by deleting all the loops labelled by e_1, e_2 and t in $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R} \cup \{t = e_1, t = e_2\}, w)$.* ■

2.1.2 The undecidability result

In this section we prove that the word problem of HNN-extensions is undecidable when one restricts to the following constrains: S has finite $\mathcal{R}$-classes, the membership problem for A, B in S is decidable, $A \cong B$ is a free inverse semigroup with zero and of finite rank and φ and its inverse are computable functions. The following proposition is a consequence of Lemma 2.1.1 and Proposition 2.1.1.

Proposition 2.1.2. *The Schützenberger automaton of a word $w \in (X \cup X^{-1})^+$ relative to the standard presentation of $S_1^{e_1} *_{U^1} S_2^{e_2}$ of the amalgam $[S_1^{e_1}, S_2^{e_2}; U^1, \omega_1^1, \omega_2^1]$ can be obtained from the Schützenberger automaton $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w')$ of the separated normal form w' of w by identifying the initial and terminal vertices of each t-edge and then deleting all the obtained loops labelled by t.* ■

By Lemma 3.3 of [10], the automaton $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w')$ of Lemma 2.1.1 has a particular shape as a weak t-opuntoid on $\overline{X}$ related to the associated HNN-extension. This shape is described in the following definition.

Definition 2.1.4. A weak t-opuntoid automaton (graph) on $\overline{X}$ is called separated if the following two conditions hold:

i) each S-lobe has edges labeled either by elements in $X_1 \cup X_1^{-1} \cup \{e_1, e_1^{-1}\}$, or $X_2 \cup X_2^{-1} \cup \{e_2, e_2^{-1}\}$; in the first case we say that the S-lobe is colored 1 and 2 in the other case;

ii) each t-edge is pointing from an S-lobe of color 1 to a t-adjacent one colored 2.

We denote by $\mathcal{C}_t$ the set of all separated weak t-opuntoid graphs on $\overline{X}$ related to the associated HNN-extension, and $\mathcal{C}$ the set of all weak opuntoid graphs on $X \cup \{e_1\} \cup \{e_2\}$. As suggested by Proposition 2.1.2 there is a bijection between these two sets.

Lemma 2.1.2. *The map $\psi : \mathcal{C}_t \to \mathcal{C}$ defined by identifying the initial vertex with the terminal vertex of each t-edge and then erasing the formed loops, is a bijection.*

Proof Let Γ, Γ' be two isomorphic separated weak t-opuntoid graphs in $\mathcal{C}_t$ and let $\phi : \Gamma \to \Gamma'$ be the graph isomorphism between Γ and Γ'. Since ϕ preserves labelling and incidence then there is one-to-one correspondence between the S-lobes of Γ and Γ' such that the restriction of ϕ to any S-lobe Δ in Γ will be an isomorphism between the S-lobes Δ and $\phi(\Delta)$. By the definition of ψ, we have that ψ preserves adjacency and it does not affect the S-lobes thus $\psi(\Delta)$ is isomorphic to $\psi(\phi(\Delta))$ hence $\psi(\Gamma) \cong \psi(\Gamma')$. Note that, in this last passage we have implicitly used the fact (contained in the definition of separated weak t-opuntoid automaton) that each vertex of a t-edge belongs to exactly one t-edge (by the determinism of the weak t-opuntoid and condition ii) of Definition 2.1.4). Thus, this prevents that in the acts of making these identification two different S-lobes of the same color are identified in the lobe graph see Figure 2.1.
On the other hand, one can define the inverse of ψ by separating adjacent lobes of the weak opuntoid automaton in $\mathcal{C}$ and implanting t-edges from the lobe of color 1 to its adjacent lobe of color 2 between each intersection vertices. Similarly one can show that this map is well defined and that it is actually the inverse of ψ. ∎

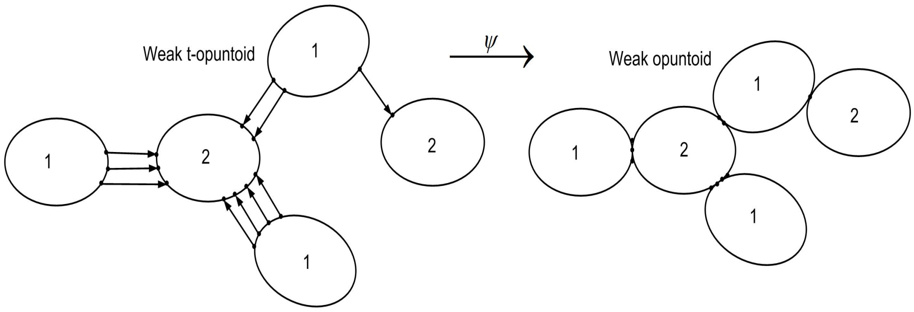


Figure 2.1: *The map ψ from a weak t-opuntoid graph into a weak opuntoid graph*

When ψ is seen as a morphism of automata, then ψ is not an isomorphism anymore. Indeed, if α, α' (β, β') are connected by a t-edge in Γ, then in the process of making quotient of this edge, these vetices are identified, Hence, two different automata (α, Γ, β) and $(\alpha', \Gamma', \beta')$ are eventually identified. However, this ambiguity is controlled by property ii) of Definition 2.1.4. In fact, since a weak t-opuntoid graph is deterministic and by ii) of Definition 2.1.4, each vertex v can have at most one associate t-adjacent vertex $\overline{v}$ see Figure 2.2.

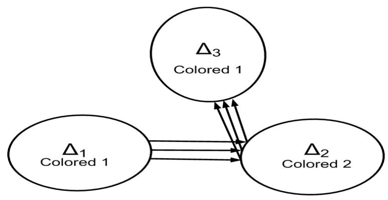


Figure 2.2: *A t-opuntoid that is not separated*

It is also evident that these two vertices are identified in an intersection vertex by the map ψ. Thus, we can prove the following proposition.

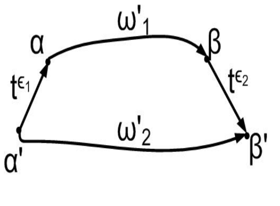
Lemma 2.1.3. *Let $w_1, w_2 \in (X \cup X^{-1})^+$ with w_1' and w_2' their corresponding separated normal forms, respectively. Let $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_1') = (\alpha, \Gamma_1, \beta)$, $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_2') = (\alpha', \Gamma_2, \beta')$ be the corresponding Schützenberger automata which are separated weak t-opuntoid automata with the property that:*

$$\psi\left((\alpha, \Gamma_1, \beta)\right) = \psi\left((\alpha', \Gamma_2, \beta')\right) = (\overline{\alpha}, \overline{\Gamma}, \overline{\beta})$$

then there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ such that

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*$$

Proof By Lemma 2.1.2 we have that the underlying graphs of $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_1')$ and $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_2')$ are isomorphic. Hence, without loss of generality, we can assume $\Gamma_1 = \Gamma_2 = \Gamma$. Furthermore, by the previous observation $(\alpha', t^{\epsilon_1}, \alpha)$ and $(\beta, t^{\epsilon_2}, \beta')$ are t-edges in Γ for some $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$, where $\epsilon = 0$ means that the two vertices are equal. Now, using the fact that (α, Γ, β) and $(\alpha', \Gamma, \beta')$ are the Schützenberger automata of w_1', w_2', respectively, and using the language properties of Schützenberger automata see Section 1.3, we obtain $t^{-\epsilon_1} w_2' t^{-\epsilon_2} \geq w_1'$, hence $t^{\epsilon_1} t^{-\epsilon_1} w_2' t^{-\epsilon_2} t^{\epsilon_2} \geq t^{\epsilon_1} w_1' t^{\epsilon_2}$. By the definition of order we have $w_2' \geq t^{\epsilon_1} t^{-\epsilon_1} w_2' t^{-\epsilon_2} t^{\epsilon_2}$ and by the definition of $L[\mathcal{A}]$ we get $t^{\epsilon_1} w_1' t^{\epsilon_2} \geq w_2'$ from which it follows



$$w_2' \geq t^{\epsilon_1} t^{-\epsilon_1} w_2' t^{-\epsilon_2} t^{\epsilon_2} \geq t^{\epsilon_1} w_1' t^{\epsilon_2} \geq w_2',$$

Paths that may be in $\Gamma = \Gamma_1 = \Gamma_2$

i.e., $t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2'$ in S^*. ■

Before proving the main result we need three more lemmas. The following one is an easy observation.

Lemma 2.1.4. *Let* $w_1, w_2 \in (X \cup X^{-1})^*$*, then* $w_1 = w_2$ *in* $S_1^{e_1} *_{U^1} S_2^{e_2}$ *if and only if* $w_1 = w_2$ *in* $S_1 *_U S_2$. ■

Lemma 2.1.5. *Let* $w_1, w_2 \in (X \cup X^{-1})^+$ *with* w_1' *and* w_2' *their corresponding separated normal forms, respectively. Then* $w_1 = w_2$ *in* $S_1 *_U S_2$ *if and only if there are* $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ *such that*

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*$$

Proof The “if” part is a consequence of (2.1) and Lemma 2.1.4. To prove the "only if" part let $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_1') = (\alpha, \Gamma_1, \beta)$, $\mathcal{A}(\overline{X}, R_{HNN} \cup \overline{R}, w_2') = (\alpha', \Gamma_2, \beta')$ be the corresponding Schützenberger automata. Since $w_1 = w_2$ in $S_1 *_U S_2$, then by Lemma 2.1.4 we have $w_1 = w_2$ in $S_1^{e_1} *_{U^1} S_2^{e_2}$. Thus, by Proposition 2.1.2, and the definition of the morphism ψ we get

$$\psi\left((\alpha, \Gamma_1, \beta)\right) = \psi\left((\alpha', \Gamma_2, \beta')\right)$$

Hence, by Lemma 2.1.3 the statement follows. ■

The last lemma regards the finiteness of the $\mathcal{R}$-classes in the free product of inverse semigroups, for completeness we report the result here.

Lemma 2.1.6 (Proposition 5.1 [27])**.** *If* S_1, S_2 *are semigroups with finite* $\mathcal{R}$*-classes then the free product* $S_1 * S_2$ *has finite* $\mathcal{R}$*-classes.* ■

We are now in position to prove the main theorem of this section.

Theorem 2.1.1. *The word problem for Yamamura’s HNN-extensions of inverse semigroups* $[S; A, B; \varphi]$ *is undecidable even if we assume the following conditions:*

- *S has finite $\mathcal{R}$-classes (therefore solvable word problem);*
- *the membership problem for A, B in S is decidable, and $A \cong B$ is a free inverse semigroup with zero and finite rank;*
- *φ and its inverse are computable functions.*

Proof Assume, contrary to our statement, that the word problem in HNN-extensions with the conditions stated in the theorem is decidable. Consider any amalgam of inverse semigroups $[S_1, S_2; U, \omega_1, \omega_2]$ such that:

- S_1, S_2 have finite $\mathcal{R}$-classes;
- U is a free inverse semigroup with zero of finite rank;
- the membership problem of $\omega_i(U)$ is decidable in S_i for $i = 1, 2$;
- ω_1, ω_2 and their inverses are computable functions;

Rodaro and Silva proved (Theorem 1, [41]) that such amalgams can have undecidable word problem. We consider the associated Yamamura's HNN-extension

$$S^* = [S_1^{e_1} * S_2^{e_2}; U_1^{e_1}, U_2^{e_2}; (\omega_1^1)^{-1} \circ \omega_2^1] \tag{2.2}$$

By Lemma 2.1.6 we have that $S_1^{e_1} * S_2^{e_2}$ has finite $\mathcal{R}$-classes. Furthermore, $U_1^{e_1} \cong U_2^{e_2}$ is a free inverse semigroup with zero of finite rank, and both $(\omega_1^1)^{-1} \circ \omega_2^1$ and $(\omega_2^1)^{-1} \circ \omega_1^1$ are computable functions. Since the membership problem of $\omega_i(U)$ is decidable in S_i for $i = 1, 2$, then the same holds for $U_1^{e_1}, U_2^{e_2}$ in $S_1^{e_1} * S_2^{e_2}$. Therefore, (2.2) is a Yamamura's HNN-extensions satisfying the conditions of the statement, and by our assumptions it has solvable word problem. However, by Lemma 2.1.5 we can decide whether or not two words $w_1, w_2 \in (X \cup X^{-1})^*$ are equal in $S_1 *_U S_2$ by simply building (effectively) the associated separated normal forms w_1', w_2', and then using the decidability of the word problem for the HNN-extension (2.2) to effectively test whether or not there are $\epsilon_1, \epsilon_2 \in \{0, 1, -1\}$ such that

$$t^{\epsilon_1} w_1' t^{\epsilon_2} = w_2' \text{ in } S^*$$

Since there are finitely many cases to consider, this is a decidable task, hence the word problem for $S_1 *_U S_2$ is decidable. However, this contradicts Theorem 1 of [41]. ■

2.2 The language of Schützenberger automaton of HNN-extension of finite inverse semigroup

The previous result shows that there is no hope to have a general decidability result for HNN-extension under "mild conditions" of the initial semigroups. Thus we now pass to analyze a case where this problem is decidable, namely we will focus on HNN-extensions of finite inverse semigroups and henceforth when we refer to an HNN-extension we always consider an HNN-extension of finite inverse semigroups. First of all we prove that in these HNN-extensions the word problem is decidable by using an alternative approach with respect to the approach given in [40]. Namely, first we prove that the Schützenberger graph $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ of each word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ relative to the standard presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$ of an HNN-extension of a finite inverse semigroup is a *context-free graph* in the sense of [31]. Then we give a construction of a grammar of a push-down automaton recognizing the same language as $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$. This fact, besides giving an alternative proof of the decidability of the word problem for S^* frames this problem in the class of algorithmic problems that can be solved in polynomial time.

2.2.1 Basic notions of language theory

In this subsection we give a glimpse of the theory of formal languages mainly focusing on context-free languages, generating grammars and push-down automata. We also give some basic notions on the transition graphs of context-free languages. For an excellent introduction to the subject see [22, 24]. Let Σ be a finite set. A *language* on the alphabet Σ is a subset of the free monoid generated by Σ.
The most common devices to generate languages on an alphabet Σ are grammars, these are formally defined as follows:

Definition 2.2.1. A grammar G is a 4-tuple $G = (V, \Sigma, P, S)$ where

1. V and Σ are finite disjoint sets, called respectively the non terminal and terminal alphabet,
2. $P \subset (V \cup \Sigma)^* V (V \cup \Sigma)^* \times (V \cup \Sigma)^*$ is a finite set of productions,
3. $S \in V$ is the starting symbol or axiom.

Each production is written in the form (α, β) or $\alpha \to \beta$ with $\alpha \in (V \cup \Sigma)^* V (V \cup \Sigma)^*$, $\beta \in (V \cup \Sigma)^*$. The strings α and β are called the left-hand side and the right-hand side of the production $\alpha \to \beta$, respectively.

Productions are also called rewriting rules because, roughly speaking, they replace factors of words α with β. According on the types of productions Chomsky introduced a hierarchy of four main different types of grammars.

Definition 2.2.2. A grammar G is

- of type 3 (or regular), if all its productions are of the following form $A \to a$, $A \to aB$ ($A \to Ba$) with $A, B \in V$ and $a \in \Sigma \cup \{\varepsilon\}$;
- of type 2 (or context-free), if all its productions are of the form $A \to \gamma$ with $A \in V$ and $\gamma \in (V \cup \Sigma)^*$;
- of type 1 (or context-sensitive), if in all productions $\alpha \to \beta$, $|\alpha| < |\beta|$ (except for the production $S \to \varepsilon$);
- of type 0 (or unrestricted) if no restriction is posed on the form of productions.

Let G be a grammar, $\Rightarrow$ is a relation between words on the alphabet $V \cup \Sigma$ defined as follows $v \Rightarrow w$ if $v = v_1 \alpha v_2$, $w = v_1 \beta v_2$ and $\alpha \to \beta$ is a production in P, in such case we say that w is derived from v with one step of computation; $\Rightarrow^n$ and $\Rightarrow^*$ mean the n-th power and the reflexive and transitive closure of $\Rightarrow$, respectively.

Definition 2.2.3. Let G be a grammar. The language generated by G is $L(G) = \{v \in \Sigma^+ | S \Rightarrow^+ v\}$. A language $L(G)$ is called regular, context-free, context-sensitive, recursively enumerable according to its generating grammar type 3,2,1,0, respectively.

Of course each regular language is context-free, each context-free language is context-sensitive and each context-sensitive language is recursively enumerable, moreover each inclusion is proper.

Example 2.2.1. *Consider $G = (V, \Sigma, P, S)$ where $V = \{S\}, \Sigma = \{a, b\}$ and P consists of the following rules:*

$$S \to aSb$$
$$S \to ab$$

One can easily use induction to show that $L(G) = \{a^n b^n : n \geq 1\}$ Equivalently, we may get the same language by defining productions P as

$$S \to aSb$$
$$S \to \varepsilon$$

It is clear by the form of the productions of G that $L(G)$ is a context-free language but it is not regular.

Basically a context-free language is not regular if it needs some sort of memory or ability to count, for instance, in the above example you need to count how many a's are in the sting to know how many b's should be introduced. Another example of context-free but not regular language is the language of matching parenthesis which is not regular because one needs to remember the last opened but not closed parenthesis in order to close it. Each family of languages can be defined by means of a suitable machine counterpart. Finite automata recognize regular languages, push-down automata recognize context-free languages, linear bounded automata recognize context-sensitive languages and Turing machine recognize recursively enumerable languages.

In the sequel only push down automata (PDA) are presented in some detail. A PDA is essentially a finite state automaton augmented with an auxiliary tape (stack) with one head which can read, write and erase symbols. This auxiliary tape works as a pushdown store with a last in, first out (LIFO) policy, like a stack of plates in some cafeterias. The transition of a PDA from state to state depends not only on what it is seen in the input tape and on the state where the machine is, but also on what it is seen on the top of the stack. There are two versions of PDA's, a deterministic PDA (DPDA) and non deterministic PDA (NPDA), hence a context-free language can be deterministic (DCFL) or non deterministic (NCFL). On the contrary of what happens for regular languages where the classes of deterministic and non deterministic languages coincide, for the context-free case, NCFL contains strictly DCFL. Our interest in this thesis is in DPDA's. We also use a definition of PDA in which no move can be done with empty stack. There are two natural ways of acceptance for PDA: the acceptance by empty stack and the acceptance by final states. In the first case the accepted languages is the set of inputs for which there is a sequence of moves that empties the stack, in the latter the accepted language is the set of inputs that allow a computation entering in a final state. These two definitions of acceptance are equivalent, see [24, 22]. We have the following well known theorem.

Theorem 2.2.1. *A language is accepted by a pushdown automaton if and only if it is a context-free language.*

In the sequel, we use the PDA accepting by final states, so we now give the formal definition of a PDA accepting by final states and such that each move reads the current character of input and the top character of the stack content. In other words no ε-move is allowed on the input and on the stack. The latter is not a restriction because all moves that neither read nor write some character on the stack can be simulated by a move reading and rewriting the top character. Moreover, as usual convention, we do not allow our machine to continue running when it empties its stack. The machine starts with an initial stack symbol $\perp$ on the stack.

Definition 2.2.4. [24, 22] A pushdown automaton $\mathcal{P}$ is a 7-tuple $\mathcal{P} = (Q, Y, \Sigma, \delta, q_0, \perp, F)$, where

1. Q is a finite set of states;

2. Y is a finite alphabet, called input alphabet;

3. Σ is a finite alphabet, called the stack alphabet;

4. $q_0 \in Q$ is the initial state;

5. $\perp$ is the initial stack symbol;

6. $F \subseteq Q$ is the set of final states;

7. $\delta \subseteq (Q \times Y \times \Sigma) \times (Q \times \Sigma^*)$ is the transition relation.

The meaning of $((q, a, A), (q', u)) \in \delta$ with $q, q' \in Q$, $a \in Y$, $A \in \Sigma$, and $u \in \Sigma^*$ is that when the PDA is in the state q, it reads the input symbol a and the character A from the top of the stack, it can go to the state q', replace A by u, and move the reading head of the input tape one square to the right. The symbols in u are placed on the stack from right to left so that the leftmost symbol of u becomes the top of the stack.
A configuration of $\mathcal{P}$ is a triple (q, γ, z) with $q \in Q$, $\gamma \in \Sigma^+$ and $z \in Y^*$. One step of computation of $\mathcal{P}$, denoted by $\models$, is defined as follows. Let (q, γ, z) a configuration of $\mathcal{P}$ where $\gamma = A\gamma'$, $z = az'$ and $A \in \Sigma$, $a \in Y$. Then $(q, \gamma, z) \models (q', u\gamma', z')$, if $((q, a, A), (q', u)) \in \delta$. As usual $\models^n$ and $\models^*$ denote respectively a computation of n steps, and the reflexive and transitive closure of $\models$. A word $w \in Y^+$ is accepted by $\mathcal{P}$ if $(q_0, \perp, w) \models^* (q_f, \gamma, \varepsilon)$ for some $q_f \in F$, $\gamma \in \Sigma^+$ and ε empty word.

The transition graph of $\mathcal{P}$ is the labelled digraph on Y whose vertices are the possible configurations of $\mathcal{P}$ such that there is an edge labelled by $a \in Y$ from the vertex v_1 to the vertex v_2 if and only if, reading the letter a, the pushdown automaton moves from the configuration v_1 to the configuration v_2 . The following example is well known.

Example 2.2.2. *The PDA $\mathcal{P}$ recognizes the language $\{a^n b^n : n \geq 0\}$ is defined by* $\mathcal{P} = (\{\alpha, q, \beta\}, \{a, b\}, \{X, \bot\}, \delta, \alpha, \bot, \beta)$

$$\delta(\alpha, a, \bot) = (\alpha, X\bot)$$
$$\delta(\alpha, a, X) = (\alpha, XX)$$
$$\delta(\alpha, b, X) = (q, \varepsilon)$$
$$\delta(q, b, X) = (q, \varepsilon)$$
$$\delta(q, \varepsilon, \bot) = (\beta, \bot)$$

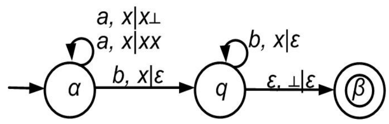


Figure 2.3: *The graphical description of the PDA that recognizes the language $\{a^n b^n : n \geq 0\}$ (using final state acceptance)*

A total state of the PDA $\mathcal{P}$ is a pair (q, γ) where $q \in Q$ is the current state of $\mathcal{P}$ and $\gamma \in \Sigma^*$ is the content of the stack. Now we associate to any pushdown automaton $\mathcal{P}$ a connected labeled graph $\Gamma(\mathcal{P})$, which is the complete transition graph of the machine $\mathcal{P}$ in the sense that $\Gamma(\mathcal{P})$ is a picture of all the possible total states of $\mathcal{P}$ with all the ways that allow to reach these total states. For constructing a PDA transition graph we will modify the above notion of computation to be consistent with graph notations and we write $(q, \gamma) \models^a_{\mathcal{P}} (q', \gamma')$ if $\gamma = A\gamma''$, $\gamma' = u\gamma''$ and $((q, a, A), (q', u)) \in \delta$. We write $\mathcal{P} \models^* (q, \gamma)$ if there exists some word $z \in Y^*$ such that when $\mathcal{P}$ can reach the total state (q, γ) starting from the initial total state $(q_0, \bot)$ after reading z.

Example 2.2.3. *Consider the simple PDA $\mathcal{P}$ with $Q = \{q_0\}$ and $X = \Sigma = \{0, 1\}$. $\mathcal{P}$ starts with initial total state $(q_0, \bot)$ and on reading an input symbol simply adds that symbol to the stack. It is clear that the graph $\Gamma(\mathcal{P})$ is the infinite binary tree as depicted in Figure 2.4.*

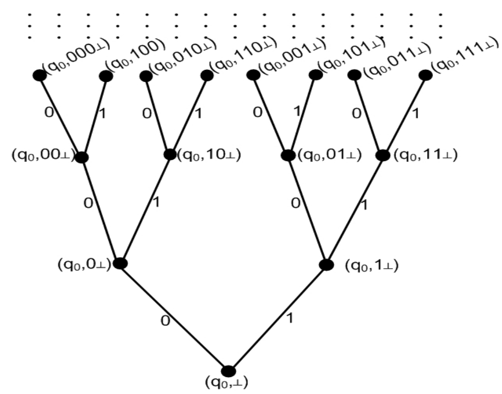


Figure 2.4: $\Gamma(\mathcal{P})$ *the complete transition graph of an infinite binary tree*

In Figure 2.4 the vertices are labelled by total states that contain the corresponding stack contents.

Example 2.2.4. *Consider the PDA $\mathcal{P}$ with $X = \{a, b, c\}, \Sigma = \{\mathsf{z}\}$ and $Q = \{q_0, q_1\}$ where the action of $\mathcal{P}$ is the following. $\mathcal{P}$ starts in the total state $(q_0, \perp)$. In either state q_0 or q_1, on reading the input a, $\mathcal{P}$ does not change state and adds z to the stack. When $\mathcal{P}$ is in state q_0, on reading input b, $\mathcal{P}$ changes states to q_1 and leaves the stack unchanged. Finally, when $\mathcal{P}$ is in state q_0, reading c, $\mathcal{P}$ changes to state q_1 and removes the top symbol from the stack. Formally,*
$(q_i, \gamma) \models^a (q_i, \mathsf{z}\gamma)$, $i = 1, 2$, $(q_0, \gamma) \models^b (q_1, \gamma)$ and $(q_0, \mathsf{z}\gamma) \models^c (q_1, \gamma)$.
The graph $\Gamma(\mathcal{P})$ is the one-way infinite braced ladder illustrated in Figure 2.7.

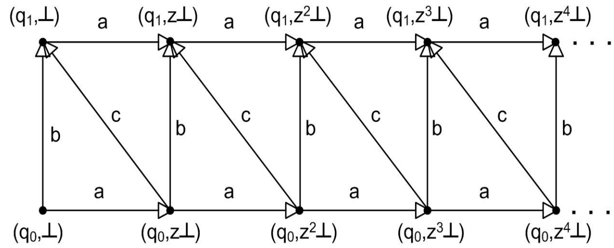


Figure 2.5: $\Gamma(\mathcal{P})$ *the complete transition graph (infinite braced ladder)*

2.2.2 Context-free graphs

We recall from [31] the following definition.

Definition 2.2.5. A *finitely generated graph* is a labelled graph Γ having the following properties:

1. Γ is a connected graph with a distinguished vertex ν_0, called the *origin* of Γ;

2. Γ has a fixed upper bound on the degree of vertices;

3. The alphabet of labels of Γ is finite.

It is clear that for each pushdown automaton $\mathcal{P}$, $\Gamma(\mathcal{P})$ satisfies all conditions of the above definition provided that the initial total state $(q_0, \perp)$ is selected as origin.
Moreover when we consider a direct graph whose label alphabet is of the form $X \cup X^{-1}$ for each edge e labelled $a \in X$ from (q, γ) to (q', γ') we can see the inverse edge e^{-1} with label a^{-1} from (q', γ') to (q, γ) as a *reverse transition* of $\mathcal{P}$, then the Cayley graph of a group and the Schützenberger graph of a word w relative to an inverse semigroup presentation are finitely generated graphs with origin 1 and ww^{-1} respectively.

Note that the graph $\Gamma(\mathcal{P})$ does not depend on the set of final states of $\mathcal{P}$ and thus it is not closely correlated with the language which $\mathcal{P}$ accepts, moreover it is an infinite graph. We are interested in a subfamily of finitely generated graphs which, roughly speaking, "are finitely behaved at infinity"[31].
The "ways of going to infinity" for an infinite graph is studied in the theory of ends. So we start this subsection giving the definition of ends and some relevant results with some examples.

Definition 2.2.6. Let Γ be a graph, a *line* in Γ is a sequence $\{v_i, i \in \mathbb{Z}\}$ of distinct vertices such that v_i is adjacent to v_{i+1} for all $i \in \mathbb{Z}$, a *ray* (also called a *half-line*) in Γ is a sequence $\{v_i \in \mathbb{N}\}$ of distinct vertices such that v_i is adjacent to v_{i+1} for all $i \in \mathbb{N}$.

The ends can be seen as equivalence classes of rays.

Definition 2.2.7. Two rays R_1 and R_2 are said to be in the same end if there is a ray R_3 in Γ which contains infinitely many vertices from both R_1 and R_2.

This definition becomes very simple in the special case where Γ is a tree, then two rays are in the same end if and only if their intersection is a ray. We may rephrase the definition by saying that R_1 and R_2 are in the same end if and only if for every finite set $F \subseteq V(\Gamma)$ there is a path in $\Gamma \setminus F$ connecting a vertex in R_1 to a vertex in R_2. Equivalently, R_1 and R_2 are in the same end if and only if there are infinitely many disjoint paths connecting vertices in R_1 to vertices in R_2. Now it is easy to check that being in the same end is an equivalence relation. Its equivalence classes are called the *ends* of Γ and the set of ends is denoted by $\Omega(\Gamma)$
Example 2.2.5.

1. *The infinite grid $\mathbb{Z} \times \mathbb{Z}$ does not "branch" at all and has only one end.* (It is easy to find a ray in $\mathbb{Z} \times \mathbb{Z}$ that contains all the vertices. Every other ray must be in the same end as that ray.)
2. *The infinite line Z has two ends.*
3. *The $3-valent$ regular tree T_3 clearly has a lot of "branching".*
   It is one of the special properties of trees that given a vertex v and an end ω there is precisely one ray in ω that starts at v. Hence T_3 has $2^{\aleph_o}$ ends.

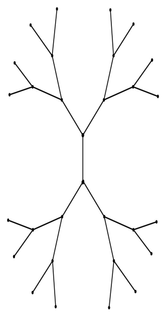


Figure 2.6: *3-valent-tree*

Let Γ be a finitely generated graph with origin ν_0. Let $n \geq 0$, Γ_n denotes the subgraph of Γ consisting of all the vertices whose distance from ν_0 is less than n (Thus Γ_0 is empty and Γ_1 consists of ν_0 and its incident edges). With $\Lambda_n(\nu)$ we denote the connected component of $\Gamma \setminus \Gamma_n$ which contains ν. A vertex p of $\Lambda_n(\nu)$ is called *frontier point* of $\Lambda_n(\nu)$ if $d(\nu_0, p) = n$, the set of frontier points of $\Lambda_n(\nu)$ will be denoted by $\Phi_n(\nu)$. Since the dergree of the vertices of Γ has an upper bound, each set $\Phi_n(\nu)$ is finite. By the connectedness property we have $\Lambda_n(\nu) = \Lambda_n(\mu)$ if and only if $\Phi_n(\nu) = \Phi_n(\mu)$. The number of ends of a connected locally finite graph Γ is the limit for n going to infinity of the number of infinite components of $\Gamma \setminus \Gamma^{(n)}$.
Let ν_1 and ν_2 be vertices of Γ. An *end-isomorphism* between the two subgraphs $\Lambda_n(\nu_1)$ and $\Lambda_n(\nu_2)$ is a label preserving graph isomorphism $\psi : \Lambda_n(\nu_1) \to \Lambda_n(\nu_2)$ such that $\psi(\Phi_n(\nu_1)) = \Phi_n(\nu_2)$. Of course an end-isomorphism is a V-isomorphism according to [44]. The following definition is central in this subsection.

Definition 2.2.8. [31] A graph Γ is context-free if Γ is a finitely generated graph such that $\{\Lambda_n(\nu) \mid \nu \in V(\Gamma)\}$ has only finitely many isomorphism classes under end-isomorphisms.

To illustrate the above definition, consider the labeled infinite binary tree $T = \Gamma$, as depicted in Figure 2.4. There are 2^n components of $T \setminus \Gamma^{(n)}$ but there is only one isomorphism class and only one frontier point. Regardless of the value of n, a component of $T \setminus \Gamma^{(n)}$ is end-isomorphic to the whole tree T.
Then, let Γ be the graph $\Gamma(\mathcal{P})$ of Figure 2.5, it is easy to see that there is only one component of $\Gamma \setminus \Gamma^{(n)}$ for any n. There are two isomorphism classes, that of the whole graph and the isomorphism type of $\Gamma \setminus \Gamma^{(n)}$ for $n \geq 1$. A component $\Gamma \setminus \Gamma^{(n)}$, $n \geq 1$, has two frontier points.

The following well known result links transition graphs of pushdown automata with context-free graphs.

Theorem 2.2.2. *[31] A finitely generated graph Γ is context-free if and only if Γ is the complete transition graph of some pushdown automaton.* ■

So we can prove that the language recognized by the Schützenberger automaton $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ of each word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ relative to the standard presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$ of an HNN-extension of a finite inverse semigroup is a context-free language by showing that its underlying graph is a *context-free graph* in the sense of Definition 2.2.8. First we recall some definitions and give some specific notations for this section.
Note (see Subsection 1.4.2) that when Construction 5 is applied to a bud

ν of an opuntoid graph Γ, Γ is unaffected, and the Construction depends locally on the opuntia $O_p(\Delta)$ containing the bud ν and not on the whole graph Γ. So, for two generic opuntia $O_P(\Delta)$, $O_p(\Delta')$ we write $O_P(\Delta) \xrightarrow{p,t,q} O_P(\Delta')$ ($O_P(\Delta) \xrightarrow{p,t^{-1},q} O_p(\Delta')$), for short $\Delta \xrightarrow{p,t,q} \Delta'$ ($\Delta \xrightarrow{p,t^{-1},q} \Delta'$), whenever applying Construction 5 to the $t - bud\ q$ such that $(p,t,q)((p,t^{-1},q))$ is an extremal t-edge with a bud $p \in V(\Delta)$ we obtain a t-opuntoid consisting of two adjacent S-lobes Δ, Δ'. In Subsection 1.4.2 we called *external lobe* any S-lobe which results from the application of Construction 5. In any external lobe there is an important set of vertices defined in Subsection 1.4.2 called the root of the lobe. We call *external lobe type* of the standard presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$ of an HNN-extension of a finite inverse semigroup, any pair (Ω, Ξ) where Ω is a deterministic inverse word graph which is a quotient of a Schützenberger automaton of some idempotent of A or B with respect to $\langle X | R \rangle$, and $\Xi \subseteq V(\Omega)$ is a maximal subset of vertices connected to each other by paths labelled by elements of A (or all connected by paths labelled by elements in B). Since S is finite there are finitely many (up to isomorphism) quotients of Schützenberger automata of some idempotents of A or B relative to $\langle \overline{X} | R_{HNN} \cup R \rangle$, and for each quotient there are finitely many maximal sets of vertices which can be selected as distinguished vertices, so there are finitely many lobe types. Therefore, we can order them as pairs: $(\Omega_1, \Xi_1), (\Omega_2, \Xi_2), \ldots, (\Omega_K, \Xi_K)$. There is an injective map σ from the set of the external lobes of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ into the set $\{1, 2, ..., K\}$ such that for any external lobe Δ of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ there is an isomorphism η_Δ between Δ and $\Omega_{\sigma(\Delta)}$ with $\eta_\Delta(R_\Delta) = \Xi_{\sigma(\Delta)}$. We say that the integer $\sigma(\Delta)$ is the *type of the external lobe* Δ. Roughly speaking lobes types represent all the graphs that can occur in some Schützenberger graph as external lobes with a distinguished set of vertices which represent the possible roots of that lobe. We also add to the previous set of external lobe types the element (Ω_0, Ξ_0) where Ω_0 is the underlying graph of $t - Core(w)$, $\Xi_0 = \{\emptyset\}$. By convention η_{Ω_0} is the identity. We denote by $\mathcal{B}_j$, with $0 \leq j \leq K$ the set of buds of Ω_j. We refer to the pair (Ω_j, Ξ_j) as the lobe of type j. If Construction 5 is applied to an opuntia $O_p(\Delta)$ at an extremal t-edge (ν, y, ν') with $\nu \in V(\Delta)$ and $y \in \{t, t^{-1}\}$ yielding an external opuntia $O_p(\Delta')$ with $\nu' \in V(\Delta')$, then there are two lobe types Ω_h, Ω_k with $0 < h \leq K$ and $0 \leq k \leq K$, and two graph isomorphisms η_Δ and $\eta_{\Delta'}$ such that $\eta_\Delta(\Delta) = \Omega_k$, $\eta_\Delta(\nu) = p$, and $\eta_{\Delta'}(\Delta) = \Omega_h$, $\eta_{\Delta'}(\nu') = q$, where $p \in \mathcal{B}_k$ and $q \in \Xi_h$. Accordingly, we say that lobe type Ω_h feeds off the lobe type Ω_k, and we write it as $k \xrightarrow{p,t,q} h$ ($k \xrightarrow{p,t^{-1},q} h$). Intuitively,

$\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ is built starting from Ω_0 by iteratively patching lobes from $\{\Omega_1, \ldots, \Omega_K\}$. More precisely, if Γ is the t-opuntoid obtained by this patching procedure at a certain step, then if p is a bud of some S-lobe of Γ isomorphic to some Ω_k, at the next step we add the S-lobe Ω_h.
Let Δ be an external lobe of $S\Gamma(X, R; w)$, let $\nu \in R_\Delta$, and let Δ' be the adjacent lobe of Δ in ν; according to [39] we call *feed off branch* (for short *branch*) of Δ the t-subopuntoid subgraph $Br(\Delta)$ of $S\Gamma(X, R; w)$ whose underlying lobe tree is the connected component of $\Upsilon(S\Gamma(\overline{X}, R_{HNN} \cup R; w)) \setminus \{\Delta'\}$ containing Δ. Since the S-lobes in $Br(\Delta)$ are built applying iteratively Construction 5 which is defined locally, it is not difficult to check that the following lemma, analogous to Lemma 1 of Section 3 of [9], holds.

Lemma 2.2.1. *Two external lobes of the same type have isomorphic branches. Hence in $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ there are only finitely many branches up to isomorphism.* ■

Since the lobe graph $\Upsilon(S\Gamma(w))$ of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ is a tree, for each pair of lobes Δ, Δ' of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ there is a unique reduced lobe path $\Delta = \Lambda_0, \ldots, \Lambda_n = \Delta'$ connecting Δ to Δ'. Since we consider the underlying graph Ω_0 of $Core(w)$ as a unique lobe (although formally it is not alobe), we call *geodesic* from a lobe Δ to the "lobe" Ω_0 the shortest reduced lobe path of $\Upsilon(S\Gamma(w))$ from Δ to a lobe of Ω_0. If Δ is not an external lobe we assume that the geodesic is formed by the unique lobe Δ itself. Let $\Lambda_s, \Lambda_{s-1}, \ldots, \Lambda_0$ be such a geodesic, where Λ_0 is a lobe of the t-subopuntoid graph Ω_0 and $\Lambda_s = \Delta$. If $s > 0$, each Λ_{i+1} is obtained from Λ_i by applying Construction 5 at some bud $\nu_i \in V(\Lambda_i)$ forming the intersection pair (ν_i, ν_{i+1}) $((\nu_{i+1}, \nu_i))$, where $\nu_{i+1} \in V(\Lambda_{i+1})$.

Before proceeding to define the PDA associated to considered Schützenberger graph we show with using geometric arguments, that the Schützenberger automaton of a word on $\overline{X} \cup \overline{X}^{-1}$ relative to the standard presentation of S^* is a context-free graph, whence its language is a DCFL. We will start by the following definition.

Definition 2.2.9. For a word $w \in (\overline{X} \cup \overline{X}^{-1})^*$, $\|w\|$ is the number of t and t^{-1} occurring in w, obviously $\|w\| \leq |w|$. Let (α, Γ, β) be a t-opuntoid automaton, and let $\nu \in V(\Gamma)$, the *norm* $\|\nu\|$ *of* ν is $\|u\|$ where u is the word labelling the shortest path connecting α to ν.
Note that, since each S-lobe Δ of the t-opuntoid automaton (α, Γ, β) is a connected graph and the lobe graph is a tree, then all the vertices of an S-lobe Δ have the same norm. Hence, we can define *norm* $\|\Delta\|$ *of the S-lobe*

Δ as the norm of any vertex in $V(\Delta)$. The *distance between two vertices* $\nu_1, \nu_2 \in V(\Gamma)$ is the length of the shortest word labelling a path connecting ν_1 to ν_2 and it is denoted by $d(\nu_1, \nu_2)$, such paths are called *geodesics*.

Note that when the graph Γ is connected then $d(.,.)$ is a metric on $V(\Gamma)$ and in particular $\|\nu\| \leq d(\alpha, \nu)$.

Definition 2.2.10. (Base lobe) Let $\Gamma = S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ be the Schützenberger graph of $w \in (\overline{X} \cup \overline{X}^{-1})^{+}$ and let $\nu \in V(\Gamma)$. For every $\mu \in \Lambda_n(\nu)$ let $\Delta(\mu)$ denote the S-lobe of Γ which contains μ. An S-lobe $\Delta(\mu)$ with minimum norm is called the *base* of the subgraph $\Lambda_n(\nu)$ and it is denoted by $B(\nu)$.

We have the following proposition.

Proposition 2.2.1. *The base $B(\nu)$ is unique, $V(\Lambda_n(\nu)) \cap V(B(\nu)) \neq \emptyset$, and $\Lambda_n(\nu)$ is contained in $Br(B(\nu))$. Furthermore, if $p \in V(B(\nu))$, then for any $q \in \Phi_n(\nu)$ we have:*

$$d(p, q) \leq 2|S|$$

Proof Since $\Lambda_n(\nu)$ is connected, each path in $\Lambda_n(\nu)$ connecting any two vertices in $\Lambda_n(\nu)$ induces a lobe path in $\Upsilon(\Gamma)$. Therefore, the subgraph Ω of $\Upsilon(\Gamma)$ formed by all the lobes $\Delta(\mu)$ with $\mu \in \Lambda_n(\nu)$ is a subtree of $\Upsilon(\Gamma)$ and there is a unique lobe in Ω with minimum norm, corresponding to $B(\nu)$. If Δ' denotes the lobe adjacent to $B(\nu)$ with norm $\|B(\nu)\| - 1$, it is evident that Ω is contained in the lobe graph whose underlying lobe tree is the connected component of $\Gamma \setminus \{\Delta'\}$ containing $B(\nu)$. This corresponds to the fact that $\Lambda_n(\nu)$ is contained in $Br(B(\nu))$. As a direct consequence of the definition of base we get $V(\Lambda_n(\nu)) \cap V(B(\nu)) \neq \emptyset$. Let us prove the last statement of the proposition. Since each lobe Δ is a quotient of a Schützenberger automaton of some element in S, and the vertices of a Schützenberger automaton of a word w are the elements of the $\mathcal{R}$-class of the element of S represented by w, we have

$$\max\{d(v, v') : v, v' \in V(\Delta)\} \leq |S|.$$

Suppose that (α, u, q) is the geodesic connecting α to q. Since $\Upsilon(\Gamma)$ is a tree and the base $B(\nu)$ is the lobe with smallest norm containing an element of $\Lambda_n(\nu)$, there is at least a vertex $q' \in V(B(\nu))$ belonging to the path (α, u, q). Therefore, if $n = d(\alpha, \nu)$ and $p' \in V(\Lambda_n(\nu)) \cap V(B(\nu))$, we get

$$n \leq d(\alpha, p') \leq d(\alpha, q') + d(q', p') \leq d(\alpha, q') + |S|$$

where in the last inequality we have used $d(q', p') \leq |S|$. Hence $d(\alpha, q') \geq n - |S|$, and since q' belongs to the geodesic (α, u, q) we also have $n = d(\alpha, q) = d(\alpha, q') + d(q', q)$, from which we get $d(q', q) \leq |S|$. Therefore we have the claim $d(p, q) \leq d(p, q') + d(q', q) \leq 2|S|$. ■

To prove the main theorem we need to consider a subset of vertices of $V(\Lambda_n(\nu))$ larger than the set of frontier points $\Phi_n(\nu)$. We call the S-*neighbor* of ν the set $\mathcal{N}(\nu) = \{q \in V(\Lambda_n(\nu)) : \exists p \in \Phi_n(\nu), d(p, q) \leq 4|S|\}$. This set essentially determines $\Lambda_n(\nu)$ up to isomorphism as the following lemma shows.

Lemma 2.2.2. *Let $\psi : Br(B(\nu)) \to Br(B(\nu'))$ be an isomorphism such that $\psi(\mathcal{N}(\nu)) = \mathcal{N}(\nu')$ and $\psi(\Phi_n(\nu)) = \Phi_n(\nu')$, then ψ is also an isomorphism $\psi : \Lambda_n(\nu) \to \Lambda_n(\nu')$.*

Proof Since $\psi(\mathcal{N}(\nu)) = \mathcal{N}(\nu')$ it is enough to prove that for any vertex $p \in \Lambda_n(\nu)$ with $d(q, p) \geq 4|S|$ for any $q \in \Phi_n(\nu)$, we have $\psi(p) \in V(\Lambda_n(\nu'))$. In fact, to show that $\psi : \Lambda_n(\nu) \to \Lambda_n(\nu')$ is an isomorphism it is enough to consider ψ^{-1} and repeat the argument. To prove that $\psi(p) \in V(\Lambda_n(\nu'))$ it is enough to show that $d(\alpha, \psi(p)) \geq m$ with $m = d(\alpha, \nu')$. Indeed, assume, contrary to the claim, that there is a path (p', u, p) with $p' \in \Phi_n(\nu)$ such that $\psi(p) \notin V(\Lambda_n(\nu'))$ and $d(\alpha, \psi(p)) \geq m$. Without loss of generality we can take the path of minimal length with this property. Thus if w is a word which is a prefix of u, then the vertex p'' such that (p', w, p'') is a path in Γ has the property that $\psi(p'') \in V(\Lambda_n(\nu'))$. Therefore, if we prove that $d(\alpha, \psi(p)) \geq m$, then we arrive to the contradiction $\psi(p) \in \Lambda_n(\nu')$. We devote the rest of the proof to prove the claim $d(\alpha, \psi(p)) \geq m$. Let (α, u_1, p'), (p', u_2, p) be geodesics. Since $\psi(\Phi_n(\nu)) = \Phi_n(\nu')$, $\psi(p') \in \Phi_n(\nu')$, hence there is a geodesic $(\alpha, w_1, \psi(p'))$, and it is easy to see that $(\psi(p'), u_2, \psi(p))$ is also a geodesic. Let $(\alpha, h, \psi(p))$ be a geodesic, since $\Upsilon(\Gamma)$ is a tree, by Proposition 2.2.1 there is a vertex $s \in V(B(\nu'))$ belonging to the geodesic $(\alpha, h, \psi(p))$. By the triangular inequality we have:

$$d(p', p) = d(\psi(p'), \psi(p)) \leq d(s, \psi(p)) + d(s, \psi(p')) \leq d(s, \psi(p)) + 2|S|$$

where the last inequality follows by $d(s, \psi(p')) \leq 2|S|$, since by Proposition 2.2.1 $s \in V(B(\nu'))$ and $\psi(p') \in \Phi_n(\nu')$. Thus, since $(\alpha, h, \psi(p))$ is a geodesic, we get:

$$d(\alpha, s) = d(\alpha, \psi(p)) - d(s, \psi(p)) \leq d(\alpha, \psi(p)) - d(p, p') + 2|S|. \quad (2.2)$$

Using again the triangular inequality we get:

$$m = d(\alpha, \psi(p')) \leq d(\alpha, s) + d(s, \psi(p')) \leq d(\alpha, s) + 2|S| \quad (2.3)$$

where we have also used Proposition 2.2.1 in the last inequality. By inequalities (2.2), (2.3) we get:

$$d(\alpha, \psi(p)) \geq d(\alpha, s) + d(p, p') - 2|S| \geq m + d(p, p') - 4|S|.$$

Hence, since $p' \in \Phi_n(\nu)$, we get $d(p, p') \geq 4|S|$, and so $d(\alpha, \psi(p)) \geq m$. ■

Theorem 2.2.3. *Let $[S; A, B]$ be an HNN-extension of a finite inverse semigroups $S = \langle X|R\rangle$ and let $w \in (\overline{X} \cup \overline{X}^{-1})^+$. The Schützenberger graph $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ is a context-free graph.*

Proof We have already noted that, $\Gamma = S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ is a finitely generated graph. It remains to prove that the set $\mathcal{C} = \{\Lambda_n(\nu) | \nu \in V(\Gamma)\}$ has only finitely many isomorphism classes under end-isomorphisms. Since $t - Core(w)$ is finite, there are finitely many subgraphs $\Lambda_n(\nu)$ of the collection $\mathcal{C}$ for which $V(\Lambda_n(\nu)) \cap V(t - Core(w)) \neq \emptyset$. Therefore, it is enough to prove that the subset $\mathcal{C}'$ of $\mathcal{C}$ formed by the subgraphs $\Lambda_n(\nu)$ for which $B(\nu)$ is an external lobe, has only finitely many isomorphism classes under end-isomorphisms. By Lemma 2.2.1 Γ has finitely many branches of external lobes up to isomorphisms. Therefore, by Lemma 2.2.2 the possible elements of $\mathcal{C}'$ up to end-isomorphism are determined by the possible S-neighbors. By Proposition 2.2.1 the frontier points are located at distance at most $2|S|$ with respect to any vertex in a base lobe. Hence, by the definition, the vertices of an S-neighbor are located at distance at most $6|S|$ from any vertex of a base lobe, whence the possible configurations of the S-neighbors are finitely many up to isomorphism. ■

We therefore obtain the following result.

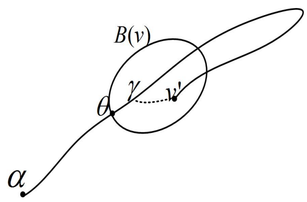
Corollary 2.2.1. *Let $[S; A, B]$ be an HNN-extension of finite inverse semigroup $S = \langle X|R\rangle$ and let $w \in (\overline{X} \cup \overline{X}^{-1})^+$, then the language recognized by the Schützenberger automaton $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ is a context-free language.*

We may reach the result of Theorem 2.2.3 by considering a narrower "neighborhood" than the one used in Lemma 2.2.2 by following similar techniques as in [9], but first let us prove that the base $B(\nu)$ contains a frontier point.

Proposition 2.2.2. *Let $\Gamma = S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ be the Schützenberger graph of $w \in (\overline{X} \cup \overline{X}^{-1})^+$, let $\nu \in V(\Gamma)$ and let $d(\alpha, \nu) = n$. The base $B(\nu)$ contains at least a vertex $\mu \in \Phi_n(\nu)$ and $\Phi_n(\nu) = \Phi_n(\mu)$.*

Proof let $\nu' \in B(\nu) \cap \Lambda_n(\nu)$, if $d(\alpha, \nu') = n$ then we are done, so suppose $d(\alpha, \nu') > n$, let π be the geodesic from α to ν' and let $\theta \in R_{B(\nu)} \cap \pi$. There are two cases to be considered. Firstly, the geodesic π has no parts except the geodesic from α to θ that do not belong to $B(\nu)$, i.e., the geodesic from θ to ν' is in $B(\nu)$, in this case there is a vertex μ in the geodesic π such that $d(\alpha, \mu) = n$. If $\mu \notin V(B(\nu))$, then $\mu \in V(\Delta)$ for some Δ. Thus $\Delta \neq B(\nu)$ and $\|\Delta\| \leq \|B(\nu)\|$, but this violates the definition of $B(\nu)$. Hence $\mu \in B(\nu) \cap \Phi_n(\nu)$.

Secondly, the geodesic from α to ν' has parts other than the geodesic from α to θ, which do not belong to $B(\nu)$, this case occurs when $min\{d(\gamma, \nu') : \gamma \in V(B(\nu)) \cap \pi,$ the path from γ to ν' is in $B(\nu)\}$ is greater than the length of the part of the geodesic π from γ to ν', see the depicted figure. Thus there is a vertex μ in a geodesic from γ to ν' contained in $B(\nu)$ such that $d(\alpha, \mu) = n$.



The last statement is obvious from the definition of $\Phi_n(\nu)$ and the connectedness of $\Lambda_n(\nu)$. ■

Lemma 2.2.1 shows in general that the distance between a frontier point and a vertex in the base lobe is at most $2|S|$. However the Lemma 2.2.3 shows that the distance[1] between a lobe containing a frontier point and the base lobe is at most $|S|$; consequently, the distance between a frontier point and a vertex in the base lobe is at most $3|S|$ according to Lemma 2.2.3.

Lemma 2.2.3. *Let $S^* = [S; A, B]$ be an HNN-extension of finite inverse semigroup $S = \langle X|R \rangle$. Let $w \in (\overline{X} \cup \overline{X}^{-1})^+$, then for each lobe Δ of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ the following properties hold:*

1. *Δ is a subgraph of $\Lambda_n(\nu)$ if and only if $R_\Delta \subseteq V(\Lambda_n(\nu))$;*
2. *if $V(\Delta) \cap \Phi_n(\nu) \neq \emptyset$, the length of the reduced lobe path from $B(\nu)$ to Δ is at most $|S|$.*

Proof It is obvious that $R_\Delta \subseteq V(\Delta) \subseteq V(\Lambda_n(\nu))$ and this proves the only part of the first property. To prove the other direction assume that $R_\Delta \subseteq V(\Lambda_n(\nu))$. To show that Δ is a subgraph of $\Lambda_n(\nu)$ it is sufficient to show that $V(\Delta) \subseteq V(\Lambda_n(\nu))$, so let $\nu' \in V(\Delta)$ and let $n = d(\alpha, \nu)$, since the lobe graph of Γ is a tree then each path connecting α to $\nu' \in V(\Delta)$ must contain at least one vertex $\mu \in R_\Delta$, hence $d(\alpha, \nu') \geq d(\alpha, \mu) \geq n$, thus $\nu' \in V(\Gamma \setminus \Gamma_n)$, so $\Delta \subset \Gamma \setminus \Gamma_n$. Moreover, since $\Lambda_n(\nu)$ is connected graph and $\mu \in R_\Delta \subseteq V(\Lambda_n(\nu))$, there is a path connecting ν to μ in

[1] the distance of two vertices in the tree

$\Lambda_n(\nu) \subseteq \Gamma \setminus \Gamma_n$, since μ and ν' are vertices of Δ there is a path from μ to ν' in Δ and because μ and ν are also vertices in $\Lambda_n(\nu)$ hence there is also a path from ν to ν' in $\Gamma \setminus \Gamma_n$ containing μ. Since $\Lambda_n(\nu)$ is the connected graph containing ν therefore $\nu' \in V(\Lambda_n(\nu))$. For proving the second property let Δ be a lobe of Γ such that $V(\Delta) \cap \Phi_n(\nu) \neq \emptyset$. If $\Delta = B(\nu)$ then the reduced lobe path from $B(\nu)$ to Δ contains no t or t^{-1} thus the statement holds. Assume $\Delta \neq B(\nu)$ let $\mu \in V(\Delta) \cap \Phi_n(\nu)$. Then $d(\alpha, \mu) = d(\alpha, \nu) = m$ for some non negative integer m. Let γ be the shortest path from α to μ. Since $B(\nu)$ has the shortest lobe path to α, the lobe graph of Γ is a tree and $\Lambda_n(\nu)$ is a connected subgraph of Γ, then γ must use $B(\nu)$ therefore it contains two intersection vertices ν_1 and ν_2 of $B(\nu)$"exactly two different vertices because γ is the shortest path". Assume that $h = d(\nu_2, \alpha) > d(\nu_1, \alpha)$, of course $h \leq m$. Let $\nu' \in \Phi_n(\nu) \cap V(B(\nu))$ and since ν_2 and ν' are vertices of $B(\nu)$ then there is path from ν' to ν_2, let $k = d(\nu', \nu_2)$. Then the length of the path from α to ν' containing ν_2 will be longer than the shortest path from α to ν'. It follows that $m \leq h + k$, and so $m - h \leq k$. Since S is finite and $B(\nu)$ is a lobe then the length of any shortest path connecting any two of its vertices will not exceed $|S|$ then $m - h \leq k = d(\nu_2, \nu') \leq |S|$. It follows that $d(\nu_2, \mu) = m - h \leq |S|$. Let z be the word of S^* that labels the part of the path γ which is the path from ν_2 to μ, so it is also the shortest path connecting ν_2 and μ. Now as $\|z\|$ denotes the number of t, t^{-1} occurred in a reduced lobe path labeled by z then $\|z\| \leq d(\nu_2, \mu) = m - h \leq |S|$. ∎

Then we can now give an alternative proof of Theorem 2.2.3 as follows:

Proof (*Alternative Proof of Theorem* 2.2.3)
It is known that $\Gamma = S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ is a finitely generated graph. It remains to prove that the set $\{\Lambda_n(\nu) | \nu \in V(\Gamma)\}$ has only finitely many isomorphism classes under end-isomorphisms. Since the lobe $B(\nu)$ is unique we proceed in considering two cases: either $B(\nu)$ is an external lobe, or it is not. The first case implies that $B(\nu)$ is a lobe of $t - Core(w)$, then by Proposition 2.2.2 $B(\nu)$ contains a vertex ν' of $\Phi_n(\nu)$ which implies that $\Phi_n(\nu) = \Phi_n(\nu')$ hence $\Lambda_n(\nu) = \Lambda_n(\nu')$. Since $t - Core(w)$ is a finite automaton, there are finitely many $\Lambda_n(\nu)$ whose base is not an external lobe. For the second case, Lemma 2.2.1 implies that Γ has finitely many branches of external lobes up to isomorphisms. So let $\nu_1, \nu_2 \in V(\Gamma)$, such that the bases $B(\nu_1), B(\nu_2)$ of $\Lambda_n(\nu_1)$ and $\Lambda_n(\nu_2)$ respectively, are external lobes of the same lobe type thus their corresponding branches

are isomorphic. Suppose that $\Lambda_n(\nu_1) \cong \Lambda_n(\nu_2)$ and let η be the isomorphism of $\Lambda_n(\nu_1)$ onto $\Lambda_n(\nu_2)$, it is an end-isomorphism if and only if $\eta(\Phi_n(\nu_1)) = \Phi_n(\nu_2)$. So for an external lobe $B(\nu)$, in the branch of its lobe type the number of different (up to end-isomorphism) $\Lambda_n(\nu)$ subgraphs is determined by the number of subsets of $V(\Lambda_n(\nu))$ that can be $\Phi_n(\nu)$. By lemma 2.2.3 the length of a reduced lobe path from $B(\nu)$ and any a lobe containing a vertex from $\Phi_n(\nu)$ is at most $|S|$ and since the lobe graph of $Br(B(\nu))$ is a tree, thus we have at most $|S|$ such lobes. The vertex of $\Phi_n(\nu)$ in such lobe can be chosen of at most $|S|$ vertices, this implies that $|\Phi_n(\nu)| \leq |S|^{|S|}$ whence there are finitely many sets of vertices in $\Lambda_n(\nu)$ that satisfy the above conditions. So Γ has finitely many classes up to end-isomorphisms. ■

2.2.3 A construction of a PDA that simulates the Schützenberger automaton

In this section we build a pushdown automaton that recognizes the language $L = L[\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)]$ to have information on the characteristics of the language L and on the size of the grammar generating it. Such information can be relevant in computational questions and it is not provided by the previous approach.

Definition 2.2.11. For each geodesic $\Lambda_s, \Lambda_{s-1}, \ldots, \Lambda_0$ of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ we call *geodesic type* a sequence of tuples:

$$(j_{s-1}, j_s, p_{s-1}, y_s, q_s)(j_{s-2}, j_{s-1}, p_{s-2}, y_{s-1}, q_{s-1}) \ldots (j_1, j_2, p_1, y_2, q_2)(0, j_1, p_0, y_1, q_1)(0, 0, \alpha, -, \alpha)$$

such that:

- for each i with $0 \leq i \leq s$, j_i is the type of the lobe Λ_i of the geodesic and so according to our convention $j_0 = 0$;
- for each i with $1 \leq i \leq s$, $y_i \in \{t, t^{-1}\}$;
- for each i with $1 \leq i \leq s$, $j_{i-1} \xrightarrow{p_{i-1}, y_i, q_i} j_i$;
- for each i with $1 \leq i \leq s$ there is an isomorphism η_{Λ_i} from Λ_i onto Ω_{j_i} such that for $i \geq 2$ $(\eta^{-1}_{\Lambda_{i-1}}(p_{i-1}), y_i, \eta^{-1}_{\Lambda_i}(q_i))$ is a t-edge connecting Λ_{i-1} to Λ_i, and $(p_0, y_1, \eta^{-1}_{\Lambda_1}(q_1))$ is a t-edge connecting a lobe of Ω_0 to Λ_1.

If (α, z, ν) is a path of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$, we call *geodesic type associated with the path* (α, z, ν) a *geodesic type* from the lobe Δ containing ν to Ω_0. Note that, since two adjacent lobes have in general more than one intersection pairs, there are also many geodesic types associated to a given path, all of them are formed by sequences of 5-tuples having the same length and each i-th tuples of the sequences have the same first two components. In the following definition we describe the pushdown automaton which simulates the behavior of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$.

Definition 2.2.12. The *pushdown automaton associated with* $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ is the automaton $\mathcal{P}_w = (Q, \overline{X} \cup \overline{X}^{-1}, \Sigma_w, \delta, q_0, \perp, F)$ where:

- $Q = \bigcup_{h \in [0,K]} V(\Omega_h)$ is the (finite) *set of states*;
- $\overline{X} \cup \overline{X}^{-1}$ is the *input alphabet*;
- $\Sigma_w = \{(i, j, p, y, q) | i, j \in [0, K], p \in \mathcal{B}_i, q \in \Xi_j, i \xrightarrow{p,y,q} j, y \in \{t, t^{-1}\}\}$ is the (finite) *stack alphabet*, whose elements in the sequel will be often denoted by capital letters;
- $q_0 = \alpha \in V(\Omega_0)$ is the *initial state*;
- $\perp = (0, 0, \alpha, -, \alpha)$ is the *initial stack symbol*;
- $F = \{\beta\}$ with $\beta \in V(\Omega_0)$ is the *final state* of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$;
- $\delta : Q \times (\overline{X} \cup \overline{X}^{-1}) \times \Sigma_w \to Q \times \Sigma_w^*$ is the transition function defined as following. Let $M = (i, j, p, y, q) \in \Sigma_w$, $x \in \overline{X} \cup \overline{X}^{-1}$ and $q_1, q_2 \in Q$ then:

  (i) *Inside a lobe*. If (q_1, x, q_2) is an edge of the lobe Ω_j, then $\delta(q_1, x, M) = (q_2, M)$.

  (ii) *Pass into a new lobe*. If (q_1, x, q_2) is not an edge of the lobe Ω_j (i.e. $x \in \{t, t^{-1}\}$), and either $x \neq y^{-1}$ or $q_1 \notin \Xi_j$, then $\delta(q_1, x, M) = (q_2, NM)$ with $N = (j, h, q_1, x, q_2)$ if and only if $j \xrightarrow{q_1,x,q_2} h$.

  (iii) *Go back into an already visited lobe*. If (q_1, x, q_2) is not an edge of the lobe Ω_j with $x = y^{-1}$ and $q_1 \in \Xi_j$, if $y = t^{-1}(y = t)$ let (q, u, q_1) be a path of Ω_j for some $u \in A$ $(u \in B)$, then $\delta(q_1, x, M) = (q_2, \epsilon)$ where q_2 is the vertex of Ω_i such that $(p, \varphi(u), q_2)$(or $(p, \varphi^{-1}(u), q_2)$) is a path in Ω_i.

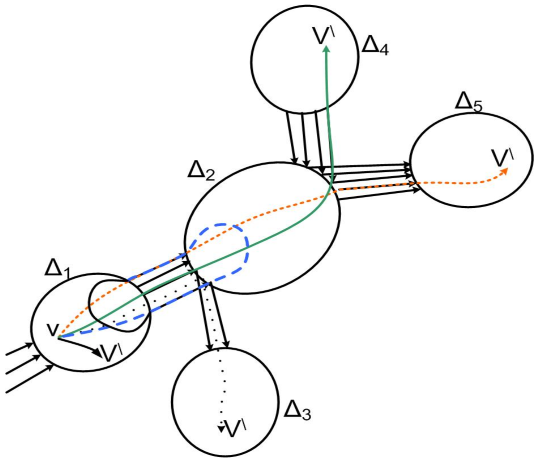


Figure 2.7: *A part of a Schützenberger graph representing all possible transitions of the rules of δ in $\mathcal{P}_w$*

We remark that since the construction of $\mathcal{P}_w$ depends on the edges in $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ which is deterministic, then $\mathcal{P}_w$ is a deterministic pushdown automaton.

Theorem 2.2.4. *With the above definition*

$$L[\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)] = L[\mathcal{P}_w].$$

Proof We show that a word $z \in (\overline{X} \cup \overline{X}^{-1})^+$ labels a path (α, z, ν) in $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ if and only if there is a computation

$$(q_0, \perp, zz') \models^* (q, \prod_{j=k}^{1} M_j \perp, z')$$

of $\mathcal{P}_w$ where $z' \in (\overline{X} \cup \overline{X}^{-1})^*$ and $\prod_{j=k}^{1} M_j$ is the geodesic type associated to the path (α, z, ν).

Let us prove the "only if part". Assume that $(q_0, \perp, zz') \models^n (q, \gamma, z')$ is a computation of $\mathcal{P}_w$ with $z' \in (\overline{X} \cup \overline{X}^{-1})^*$, $\gamma = \prod_{j=k}^{1}(h_{j-1}, h_j, p_{j-1}, y_j, q_j) \perp$. We prove by induction on n that there is a lobe Δ of type h_k and a vertex $\nu \in V(\Delta)$ such that $q = \eta_\Delta(\nu)$ and (α, z, ν) is the path of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ with γ as associated geodesic type. If $n = 1$, then the computation consists of a unique transition that can be either of type (i) or of type (ii). In the first case the transition is performed in $Core(w)$ with $q \in V(\Omega_0)$, $\gamma = \perp$, $z \in \overline{X} \cup \overline{X}^{-1}$, $(q_0, z, q) \in E(\Omega_0)$ and

the geodesic type associated to the path (q_0, z, q) is $(0, 0, \alpha, -, \alpha)$. If the computation is of type (ii), then $\gamma = (0, h_1, \alpha, y_1, q) \perp$ with $0 \xrightarrow{\alpha, y_1, q} h_1$, $q \in \Xi_{h_1}$ and $z = y_1$. Therefore, there exist a vertex $q_0 \in V(\Omega_0)$ and an external lobe Δ of type h_1, such that $\eta_\Delta(\Delta) = \Omega_{h_1}, \eta_\Delta(R_\Delta) = \Xi_{h_1}$ for some graph isomorphism η_Δ such that $(\eta_\Delta^{-1}(q_0), z, \eta_\Delta^{-1}(q))$ is a t-edge connecting a lobe of Ω_0 with Δ. Hence, the statement is satisfied because $(0, h_1, \alpha, y_1, q)(0, 0, \alpha, -, \alpha)$ is the geodesic type associated to the path $(\alpha, z, \eta_\Delta^{-1}(q))$.
Suppose that the statement holds for each derivation of length $n-1$ and consider a derivation of length n, let $z''az' \in (\overline{X} \cup \overline{X}^{-1})^+$ where $z'' \in (\overline{X} \cup \overline{X}^{-1})^+$, $|z''| = n-1$ and $a \in \overline{X} \cup \overline{X}^{-1}$. Since at each step of the computation an input character is read, we have:

$$(q_0, \perp, zz') = (q_0, \perp, z''az') \models^{n-1} (q', \gamma', az') \models (q, \gamma, z'),$$

where $z = z''a$ and $\gamma' = \left(\prod_{j=s}^{1}(h_{j-1}, h_j, p_{j-1}, y_j, q_j)\right) \perp$ for some $s \leq n-1$.
By induction hypothesis there are a lobe Δ' of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ and a vertex $\nu' \in V(\Delta')$ such that $\eta_{\Delta'}(\nu') = q'$ and (α, z'', ν') is a path of the Schützenberger automaton whose associated geodesic type is the sequence γ'. We now consider the n-th step which could be any one of the transitions defined in $\mathcal{P}_w$. We consider the following cases.

i) Suppose that the transition uses rule (i), then $q', q \in V(\Omega_{h_s})$, $(q', a, q) \in E(\Omega_{h_s})$. So there is a vertex $\nu \in V(\Delta')$ such that $\eta_{\Delta'}(\nu) = q$ and $(\nu', a, \nu) \in E(\Delta')$. Hence, $\gamma' = \gamma$ is the geodesic type associated to both $(\alpha, z''a, \nu)$ and (α, z'', ν'). Thus (α, z, ν) is the path in $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ corresponding to the computation $(q_0, \perp, zz') \models^* (q, \gamma, z')$.

ii) Assume that the n-th step of the computation uses rule (ii). Let $M = (h_s, h_{s+1}, q', y_{s+1}, q)$ be the item that is pushed into the stack according to the computation $(q', \gamma', az') \models (q, M\gamma', z')$, then we have two possibility: $q' \notin \Xi_{h_s}$ hence $q' \in \mathcal{B}_{h_s} \setminus \Xi_{h_s}$, with $h_s \xrightarrow{q', a, q} h_{s+1}$ and $y_{s+1} = a$, or $a \neq y_{s+1}^{-1}$ whence $q' \in \mathcal{B}_{h_s} \cap \Xi_{h_s}$, with $h_s \xrightarrow{q', a, q} h_{s+1}$ and $a = y_{s+1}$. Hence $a \in \{t, t^{-1}\}$, $h_s \xrightarrow{q', a, q} h_{s+1}$ with $y_{s+1} = a$, and there is a map $\eta_{\Delta'}$ from Δ' to Ω_{h_s} and a vertex $\nu' \in V(\Delta')$ such that $\eta_{\Delta'}(\nu') = q'$. Thus, there is a lobe Δ adjacent to Δ', a vertex $\nu \in V(\Delta)$ and an isomorphism η_Δ from Δ to $\Omega_{h_{s+1}}$ such that $\eta_\Delta(\nu) = q$, $\eta_\Delta(R_\Delta) = \Xi_{h_{s+1}}$ and (ν', a, ν) is a t-edge connecting Δ'

to Δ. Hence, $(\alpha, z''a, \nu)$ is a path of the Schützenberger automaton, and $\gamma = M\gamma'$ is its associated geodesic type.

iii) Assume the transition function acts according to rule (iii). Let $\gamma' = N\gamma$ with $N = (h_{s-1}, h_s, p, y_s, p')$, where the n-th step of the derivation is $(q', N\gamma, az') \models (q, \gamma, z')$. In this case $q' \in \Xi_{h_s}$ and there is a path (p', u, q') of Ω_{h_s} for some $u \in A(B)$ and $a = y_s^{-1}$. Accordingly $(p, \varphi(u), q)((p, \varphi^{-1}(u), q))$ is a path in $\Omega_{h_{s-1}}$. Since $h_{s-1} \xrightarrow{p,a^{-1},p'} h_s$, there are two lobes Δ and Δ' of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ and there exist two isomorphisms η_Δ from Δ to $\Omega_{h_{s-1}}$ and $\eta_{\Delta'}$ from Δ' to Ω_{h_s} such that $\eta_\Delta^{-1}(p) = r$, $\eta_{\Delta'}(R_{\Delta'}) = \Xi_{h_s}$, $\eta_{\Delta'}^{-1}(p') = r'$ and $\Delta \xrightarrow{r,a^{-1},r'} \Delta'$. Let $\eta_{\Delta'}^{-1}(q') = \nu'$ and $\eta_\Delta^{-1}(q) = \nu$, hence (ν, a^{-1}, ν') is a t-edge connecting Δ to Δ', and so (α, z, ν) is a path of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ whose associated geodesic type is γ.

Let us now prove the "if part". Let $\pi = (\alpha, z, \nu)$ be a path in $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ with $\nu \in V(\Delta)$ for some lobe Δ of the Schützenberger automaton. We prove by induction on $|z|$ that in $\mathcal{P}_w$ there is the computation $(q_0, \perp, zz') \models^{|z|} (\eta_\Delta(\nu), \sigma, z')$ where

$$\sigma = (h_{s-1}, h_s, p_{s-1}, y_s, q_s) \dots (0, h_1, p_0, y_1, q_1)(0, 0, \alpha, -, \alpha)$$

with $s \leq |z|$ is a geodesic type associated to the path π.
If $|z| = 0$ the statement clearly holds. Thus, assume that the statement holds for each word of length less than $|z|$. Let $z'' \in (\overline{X} \cup \overline{X}^{-1})^*$, $a \in \overline{X} \cup \overline{X}^{-1}$ such that $z = z''a$ and let $\pi' = (\alpha, z'', \nu')$ be a path in the Schützenberger automaton with $\nu' \in V(\Delta')$, where Δ' is either an external lobe of the Schützenberger automaton or Ω_0. Let $\sigma' = (h_{j-1}, h_j, p_{j-1}, y_j, q_j) \dots (0, h_1, p_0, y_1, q_0)(0, 0, \alpha, -, \alpha)$ be a geodesic type associated to π'. By induction hypothesis, in the automaton $\mathcal{P}_w$ there is a computation $(q_0, \perp, z''az') \models^{|z''|} (\eta_{\Delta'}(\nu'), \sigma', az')$. Since the Schützenberger automaton is deterministic and $\pi = (\alpha, z, \nu)$ is a path of the Schützenberger automaton, we get that (ν', a, ν) is an edge of the Schützenberger automaton. We proceed considering the different possible positions of the vertex ν'.

- Suppose that $\nu' \in R_{\Delta'}$. If $a \notin \{t, t^{-1}\}$, the edge $(\nu', a, \nu) \in E(\Delta')$, so taking $\Delta = \Delta'$ and applying the first rule (i) on the configuration $(\eta_{\Delta'}(\nu'), \sigma', az')$ we get $(\eta_{\Delta'}(\nu'), \sigma', az') \models (\eta_{\Delta'}(\nu), \sigma', z')$. Hence $(q_0, \perp, zz') \models^{|z|} (\eta_\Delta(\nu), \sigma', z')$ where σ' is a geodesic type associated to π. If $a = y_j^{-1}$ then we have to turn back to an already visited lobe. Thus, let Δ be a lobe of type h_{j-1}, using rule (iii) we get $(\eta_{\Delta'}(\nu'), \sigma', az') \models (\eta_{\Delta}(\nu), \sigma'', z')$, where $\sigma'' = (h_{j-1}, h_{j-2}, q_{j-1}, y_{j-2}, p_{j-2}) \dots (0, 0, \alpha, -, \alpha)$ is a geodesic type associated to π. Lastly, if $a = y_j$ then $q' = \eta_{\Delta'}(\nu') \in \mathcal{B}_{h_j}$ and $\eta_{\Delta'}(\Delta') = \Omega_{h_j}$. Thus, applying rule (ii) we obtain the computation $(\eta_{\Delta'}(\nu'), \sigma', az') \models (\eta_{\Delta}(\nu), M\sigma, z')$ where $M\sigma$ with $M = (h_j, h_{j+1}, q', y_{j+1}, q)$ is the geodesic type associated to π.

- Assume $\nu' \notin R_{\Delta'}$. If $\eta_{\Delta'}(\nu') \in \mathcal{B}_{h_j} \setminus \Xi_{h_j}$ and $a = y_j$ then we are entering a new lobe Δ of type say h_{j+1}. Applying rule (ii) we obtain the computation $(\eta_{\Delta'}(\nu'), \sigma', az') \models (\eta_{\Delta}(\nu), M\sigma', z')$ with $M = (h_j, h_{j+1}, \eta_{\Delta'}(\nu'), y_{j+1}, q_{h_{j+1}})$. Hence, $(q_0, \perp, zz') \models^{|z|} (\eta_{\Delta}(\nu), M\sigma', z')$ and $M\sigma'$ is a geodesic type associated to π. Finally if $\eta_{\Delta'}(\nu') \notin \mathcal{B}_{h_j}$ with $a \notin \{t, t^{-1}\}$ then Δ coincides with Δ', whence using rule (i) we get $(\eta_{\Delta'}(\nu'), \sigma', az') \models (\eta_{\Delta'}(\nu), \sigma', z')$. Thus, $(q_0, \perp, zz') \models^{|z|} (\eta_{\Delta'}(\nu), \sigma', z')$ where σ' is a geodesic type associated to π.

■

Theorem 2.2.4 can be used to obtain an alternative proof of Theorem 2 of [8].

Theorem 2.2.5. *([8], Theorem 2) The word problem in HNN-extension of finite inverse semigroup is decidable.*

Proof Let $w, w' \in (\overline{X} \cup \overline{X}^{-1})^+$, and denote by $\mathcal{P}_w$ and $\mathcal{P}_{w'}$ the pushdown automata associated to the Schützenberger automata of w and w', respectively. We know that $w\tau = w'\tau$ if and only if $w \in L[\mathcal{P}_{w'}]$ and $w' \in L[\mathcal{P}_w]$. Hence, the word problem for HNN-extension of finite inverse semigroups reduces to a membership problem for deterministic context-free languages and the membership of a word to a deterministic context-free language can be solved in linear time with respect to the length of the word [19]. ■

Since the membership problem for DCFL is linear in the length of a word [19], the algorithmic cost for the word problem is given by the construction of the grammar. Assuming that the presentation for S does not belong to the input, the construction of all the lobe types can be assumed as a constant cost. Thus the effective cost is given by the construction of the $t - Core(w)$. By counting the number of steps performed in Constructions 1-4 described in Section it is straightforward to prove that the construction of $t - Core(w)$ requires a number of steps polynomial in $|w|$, from which we derive the following.

Corollary 2.2.2. *There is a polynomial time algorithm for the word problem for HNN-extension of finite inverse semigroup.* ■

CHAPTER 3

Bicyclic Subsemigroups

THE first description of the *bicyclic monoid* was given by Evgenii Lyapin in 1953. Bicyclic monoid defined by the presentation $\langle p, q | pq = 1 \rangle$, plays an important role in semigroup theory because of its many remarkable properties and generalizations (see [1, 6, 16, 21, 35]). For instance completely 0-simple (simple) semigroups, i.e. 0-simple (simple) semigroups with a primitive idempotent, form a well known class of semigroups of the Rees-Sushkevich variety (see Theorems 3.2.3 and 3.3.1 of [25]). This class is characterized in terms of bicyclic semigroups, namely a semigroup S is completely 0-simple (simple) if and only if S contains a nonzero idempotent and it does not contain any bicyclic subsemigroup (Proposition 4.7 of [15]). Moreover, *completely semisimple* inverse semigroups, i.e inverse semigroups whose principal factors are completely simple or completely 0-simple (see [25]) are also characterized in terms of bicyclic monoids such that an inverse semigroup is completely semisimple if and only if it does not contain a copy of the bicyclic semigroup. The chapter is arranged as follows. The first section is devoted to recalling some basic notions and some relevant results about bicyclic monoids and completely semisimple inverse semigroups. In the second section we give a characterization for the relation ρ which was introduced in Section 1.4.2 and we enhance this section with more analogous results as in [39] about feeding off branches and the lifting property. In the third section we give a characterization for the HNN-extension of finite inverse semigroups to have a bicyclic inverse subsemigroup. Finally, in the last section we a characterize the HNN-extension of finite inverse semigroups that are completely semisimple inverse semigroup.

3.1 Basic Notions

There are many ways of constructing the bicyclic semigroup, and various notations for referring to it. Lyapin in 1953 denoted it with the symbol P; Clifford and Preston used the notion $\mathcal{C}$; while most recent papers have tended to use B. The bicyclic semigroup $B(a;b)$ is defined as the monoid generated by two elements a and b subject only to the condition that $ba = 1$. It follows that the elements can all be written in the standard form a^ib^j where $i; j \geq 0$. We can write out the elements of B in the following array

1	b	b^2	b^3	b^4	$\cdots$
a	ab	ab^2	ab^3	ab^4	$\cdots$
a^2	a^2b	a^2b^2	a^2b^3	a^2b^4	$\cdots$
a^3	a^3b	a^3b^2	a^3b^3	a^3b^4	$\cdots$
a^4	a^4b	a^4b^2	a^4b^3	a^4b^4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The multiplication on B is defined as follows:

$$a^kb^la^mb^n = \begin{cases} a^{k+m-l}b^n, & \text{if } l \leq m \\ a^kb^{l-m+n}, & \text{if } l > m \end{cases}$$

The above two cases can be put together as follows:

$a^kb^la^mb^n = a^{k-l+r}b^{n-m+r}$, where $r = max\{l, m\}$.

Since the above multiplication has only two a, b symbols we can focus on the powers and therefore the monoid B is isomorphic to the monoid $\mathbb{N}^0 \times \mathbb{N}^0$, where $\mathbb{N}^0 = \mathbb{N} \cup \{0\}$. It is easy to see that B is an inverse semigroup: the element a^ib^j has inverse a^jb^i. The idempotents of B are of the form a^nb^n, Green's relations $\mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are given by

$$a^i b^j \mathcal{L} a^k b^l \text{ if and only if } j = l$$
$$a^i b^j \mathcal{R} a^k b^l \text{ if and only if } i = k \text{ and}$$
$$a^i b^j \mathcal{H} a^k b^l \text{ if and only if } j = l \text{ and } i = k$$

In the array, the rows are the $\mathcal{R}$-classes of B, the columns are the $\mathcal{L}$-classes and the $\mathcal{H}$-classes are singletons. There is only one $\mathcal{D}$-class; that is, B is a bisimple monoid (hence simple).
A simple (respectively 0-simple) semigroup having a primitive idempotent is completely simple (respectively completely 0-simple) this motivates the following definition.

Definition 3.1.1. A semigroup whose all principal factors are completely simple or completely 0-simple is called completely semisimple.

An inverse semigroup is said to be completely semisimple when the natural partial order is the equality when restricted to any $\mathcal{D}$-class, i.e., if D is a $\mathcal{D}$-class of an inverse semigroup S then the restriction of the natural order to D is either equality, or for every $b \in D$, there exists $a \in D$ such that $a < b$. In particular an inverse semigroup S is either completely semisimple, or there exists a pair of distinct $\mathcal{D}$-related elements $a, b \in S$ such that $a < b$. This can be summarized in the following theorem.

Proposition 3.1.1. *For an inverse semigroup S the following conditions are equivalent:*

1. *S does not contain a copy of the bicyclic monoid;*
2. *S is completely semisimple;*
3. *for all $f, g \in E(S)$ with $g\mathcal{D}f$, $f \leq g$ implies $f = g$.* ■

We now give another characterization of an inverse semigroup to be completely semisimple inverse semigroups in terms of Schützenberger automata in the following proposition whose proof can be found in [39]. First let us remind the reader of the notation $^\omega$ that denotes the standard *infinite iteration*, i.e. for a given word $w \in (Y \cup Y^{-1})^+$, w^ω is the infinite word having w^n as a prefix for all the integers $n \geq 1$ (see [35]).

Proposition 3.1.2. *[39] Let $S = Inv\langle Y|T\rangle$ be an inverse semigroup. The following are equivalent:*

1. *S is completely semisimple;*

2. $End(S\Gamma(Y,T;w)) = Aut(S\Gamma(Y,T;w))$ *for all* $w \in (Y \cup Y^{-1})^+$*;*

3. *for all words* $w \in (Y \cup Y^{-1})^+$ *if* w^ω *labels a path in* $\mathcal{A}(Y,T;w)$ *starting from the initial state* α *of* $\mathcal{A}(Y,T;w)$*, then also* w^{-1} *labels a path starting from* α*.* ■

We have immediately the following corollary

Corollary 3.1.1. *An inverse semigroup* $S = Inv\langle Y|T\rangle = (Y \cup Y^{-1})^+/\eta$ *has a bicyclic subsemigroup if and only if there is a word* $w \in (Y \cup Y^{-1})^+$ *such that* w^ω *labels a path in* $\mathcal{A}(Y,T;w)$ *starting from the initial state* α *of* $\mathcal{A}(Y,T;w)$*, but* w^{-1} *does not label any path starting from* α*.* ■

3.2 Feeding off branches and their lifting properties

In this section we will frequently use the concept of feeding off that comes from the use of Construction 5. Without loss of generality consider the case $\Sigma = S\Gamma(X,R;\varphi(f))$ while the case $\Sigma = S\Gamma(X,R;\varphi^{-1}(f))$ is analogous and whenever there are differences we will consider them separately.

3.2.1 Feeding off branch

A consequence of Construction 5 and the feeding off concept we have the following important proposition.

Proposition 3.2.1. *Let* Γ *be a t-opuntoid graph and let* Δ, Δ' *be two lobes of* Γ *such that* $\Delta \mapsto \Delta'$ *let* $(\nu,\nu')((\nu,\nu'))$ *be an intersection pair of* Δ *and* Δ' *with* $\nu \in V(\Delta)$ *and* $\nu' \in V(\Delta')$*. Then*

- *if* $f = f_A(\nu,\Delta)$ *then*
$$\varphi(f) = e(\nu',\Delta') = f_B(\nu',\Delta') \in B$$
- *if* $\Delta \cong S\Gamma(X,R,f)$ *where* $f \in A$*, then*
$$\Delta' \cong S\Gamma(X,R,\varphi(f)) \text{ and } \Delta' \mapsto \Delta$$

Proof Let $e(\nu',\Delta')$ be the minimal idempotent labelling a loop at ν' in Δ', then by Proposition 1.4.1 $(x, e(\nu',\Delta'), x)$ is also a loop in (x,Σ,x) which is a Schützenberger automaton of $\varphi(f)$, thus $e(\nu',\Delta') \geq \varphi(f)$. Moreover, $\varphi(f)$ labels a loop at ν' in Δ' hence by the minimality of $e(\nu',\Delta')$ we also have $e(\nu',\Delta') \leq \varphi(f)$, whence $e(\nu',\Delta') = \varphi(f)$. Since $\varphi(f) \in B$ we also have $e(\nu',\Delta') = f_B(\nu',\Delta')$.
To prove the second assertion, assume that $(\nu,\Delta,\nu) = \mathcal{A}(X,R;f)$, then

for all $u \in \mathcal{L}_A(\nu, \Delta)$ we have $\varphi(u) \geq \varphi(f)$. Since $\Sigma = S\Gamma(X, R; \varphi(f))$ thus $\varphi(u)$ labels a loop at x in Σ. Hence $N(x, \Sigma) = \{x\}$, therefore ρ is the identity relation, whence $\Delta' \cong \Sigma = S\Gamma(X, R; \varphi(f))$. Thus both Δ, Δ' are Schützenberger graphs relative to the presentation $\langle X|R\rangle$. Applying Construction 5 to the t-opuntoid graph Δ' in ν' we then obtain Δ. Whence $\Delta' \mapsto \Delta$. ■

Definition 3.2.1 (Feeding off branch)**.** Let Δ, Δ' be two lobes of $S\Gamma(w)$ such that $\Delta' \in \mathcal{E}(t\text{-}Core(w))$, $\Delta \mapsto \Delta'$ and each reduced lobe path in $\Upsilon(S\Gamma(w))$ from a lobe of $t\text{-}Core(w)$ to Δ' contains Δ. The *feeding off branch* of Δ' with respect to Δ, denoted by $B_\Delta(\Delta')$, is the t-subopuntoid subgraph of $S\Gamma(\overline{X}, R_{HNN}; w)$ whose lobe graph is the connected component of $\Upsilon(S\Gamma(w)) \setminus \{\Delta\}$ containing Δ' and we shortly say f-*branch*.

Take the opuntia $O_P(\Delta')$ of Δ' which is in $\mathcal{E}(t\text{-}Core(w))$ and apply iteratively construction 5 to all t-buds of $O_P(\Delta')$ except those were previously buds in Δ then we will obtain the lobes of $B_\Delta(\Delta')$. Therefore, we can equivalently redefine $B_\Delta(\Delta')$ in terms of the closure operator $cl_{R_{HNN}\cup R}(\circ)$ as follows: consider $cl_{R_{HNN}\cup R}(\Delta')$, since $\Delta \mapsto \Delta'$ in $S\Gamma(w)$ then for each $\nu' \in R_{\Delta'}, \mathcal{L}_A(\nu', \Delta') \neq \emptyset$ ($\mathcal{L}_B(\nu', \Delta') \neq \emptyset$), so by the assimilation property of $cl_{R_{HNN}\cup R}(\Delta')$, there is a lobe Ω in $cl_{R_{HNN}\cup R}(\Delta')$ adjacent to Δ' such that the set of intersection pairs between Δ' and Ω is equal to the set of intersection pairs between Δ and Δ', i.e.,

$$R_\Omega = V(O_P(\Delta')) \cap V(\Omega) = V(O_P(\Delta')) \cap V(\Delta) = B^{\Delta}_{\Delta'}. \tag{3.1}$$

Therefore we may say that $B_\Delta(\Delta')$ is the t-subopuntoid subgraph of $cl_{R_{HNN}\cup R}(\Delta')$ whose lobe graph is the connected component of $\Upsilon(cl_{R_{HNN}\cup R}(\Delta')) \setminus \{\Omega\}$ containing Δ'.
Since $t\text{-}Core(w)$ is finite, the following lemma easily follows.

Lemma 3.2.1. *Let $\Delta_1, \Delta_2, ..., \Delta_k, ...$ be an infinite reduced lobe path in $\Upsilon(S\Gamma(w))$. Then there is an integer $N_0 > 1$ such that, for all $N \geq N_0 > 1$, Δ_N is an external lobe of $t\text{-}Core(w)$ and*

$$\Delta_{N-1} \mapsto \Delta_N \mapsto ... \mapsto \Delta_j \mapsto \Delta_{j+1} \mapsto ...$$

Moreover the infinite reduced lobe path $\Delta_{N-1}, \Delta_N, ...$ belongs to $B_{\Delta_{N-1}}(\Delta_N)$. ■

Let $B_\Delta(\Delta')$ be an f-branch of $S\Gamma(w)$. Recall that $\Delta' = \Sigma/\rho$ where $\Sigma = S\Gamma(X, R; \varphi(f))$ and $f = f_A(\nu, \Delta)$. It is obvious that $cl_{R_{HNN}\cup R}(\Sigma) =$

$S\Gamma(\varphi(f))$. It follows by the assimilation property and by the second statement of Proposition 1.4.2 that there is a unique lobe Λ in $S\Gamma(\varphi(f))$ such that $\Sigma \mapsto \Lambda$ and

$$\pi_\rho^{-1}(V(O_P(\Delta')) \cap V(O_P(\Delta))) = V(O_P(\Lambda)) \cap V(O_P(\Sigma)) \qquad (3.2)$$

Note that by Proposition 3.2.1 each lobe Σ and Λ feeds off each other then we have $\Lambda \cong S\Gamma(X, R; f)$ and $\Sigma = S\Gamma(X, R; \varphi(f))$ thus $cl_{R_{HNN} \cup R}(\Lambda) = S\Gamma(f) = S\Gamma(\varphi(f))$.

Definition 3.2.2 (lifted f-branch)**.** Let $B_\Delta(\Delta')$ be an f-branch of $S\Gamma(w)$ where $\Delta' = \Sigma/\rho$. Let Λ be adjacent lobe of Σ in $S\Gamma(f)$ satisfying equation (3.2). Then the f-branch $B_\Lambda(\Sigma)$ is called the *lifted* f-*branch* of $B_\Delta(\Delta')$.

3.2.2 The Ξ relation

In this section we give a generalization for the relation ρ by introducing the relation Ξ on the vertices of $S\Gamma(f)$. Furthermore we show that $B_\Delta(\Delta') \cong B_\Lambda(\Sigma)/\Xi$. We now introduce the relation Ξ on the vertices of $S\Gamma(f)$.

Definition 3.2.3 (Ξ)**.** Let $z, r \in V(S\Gamma(f))$. Then $z\Xi r$ if and only if there are two paths $(q, h, z), (q', h, r)$ in $S\Gamma(\varphi(f))$ with $q, q' \in N(x, \Sigma)$ where $h \in (\overline{X} \cup \overline{X}^{-1})^*$.

Proposition 3.2.2. *Let $z, r \in V(S\Gamma(f))$, then $z\Xi r$ if and only if there is an automorphism $\Psi \in Aut(S\Gamma(f))$ such that $\Psi(z) = r$ and $\Psi(N(x, \Sigma)) \subseteq N(x, \Sigma)$. Furthermore, we have the following properties:*

- $\Xi \cap (V(\Sigma) \times V(\Sigma)) = \rho$
- $x\Xi = N(x, \Sigma)$

Proof let $z\Xi r$, then by the definition of Ξ there are two vertices $q, q' \in N(x, \Sigma)$ and a word $h \in (\overline{X} \cup \overline{X}^{-1})^*$ such that $(q, h, z), (q', h, r)$ are paths in $S\Gamma(f)$. Since $q, q' \in N(x, \Sigma)$ then $q\rho q'$ and by Proposition 1.4.2 there is an automorphism $\psi \in Aut(\Sigma)$ such that $\psi(q) = q'$ and $\psi(N(x, \Sigma)) \subseteq N(x, \Sigma)$. By Proposition 1.4.3 we can extend ψ to an automorphism $\Psi \in Aut(S\Gamma(\varphi(f))) = Aut(S\Gamma(f))$ because $cl_{R_{HNN} \cup R} = (\Sigma)S\Gamma(\varphi(f))$. Moreover, $\Psi(N(x, \Sigma)) \subseteq N(x, \Sigma)$ and $\Psi(z) = r$ because the automorphism Ψ extends ψ and $S\Gamma(f)$ is deterministic.
Conversely, let $\Psi(z) = r$ for some $\Psi \in Aut(S\Gamma(f))$ with $\Psi(N(x, \Sigma)) \subseteq N(x, \Sigma)$. Let $q \in N(x, \Sigma)$ and $s \in (\overline{X} \cup \overline{X}^{-1})^*$ be a word such that (q, s, z) is a path in $S\Gamma(f)$, thus $(\Psi(q), s, \Psi(z)) = (\Psi(q), s, r)$ is a path in $S\Gamma(f)$ with $\Psi(q) \in N(x, \Sigma)$, whence $z\Xi r$.

Now we prove property (1). By Lemma 1.4.1 $\rho \subseteq \Xi \cap (V(\Sigma) \times V(\Sigma))$. To prove the opposite inclusion, let $c, d \in V(\Sigma)$ with $c\Xi d$. Then, by the first statement of this proposition there is an automorphism $\Psi \in Aut(S\Gamma(\varphi(f)))$ such that $\Psi(c) = d$ and $\Psi(N(x, \Sigma)) \subseteq N(x, \Sigma)$. Thus consider the restriction $\Psi_{|_\Sigma}$. We claim that $\Psi_{|_\Sigma} \in End(\Sigma)$. In fact let $y \in V(\Sigma)$, since Σ is connected there is a path (c, h, y) in Σ with $h \in (X \cup X^{-1})^*$. Thus applying Ψ to the path we get $(d, h, \Psi(y))$. Since $d \in V(\Sigma)$ and $h \in (X \cup X^{-1})^*$ thus $\Psi(y)$ must be in Σ. Furthermore, since Σ is finite and $\Psi_{|_\Sigma}$ is injective we get $\Psi_{|_\Sigma} \in Aut(\Sigma)$. Since $\Psi_{|_\Sigma}(c) = d$, $N(x, \Sigma) \subseteq V(\Sigma)$ and $\Psi_{|_\Sigma}(N(x, \Sigma)) \subseteq N(x, \Sigma)$, then by Proposition 1.4.2 we obtain that $c\rho d$. We now prove property (2), It is clear that $N(x, \Sigma) \subseteq x\Xi$. Inorder to prove the other inclusion let $x' \in x\Xi$, There is an automorphism $\Psi \in Aut(S\Gamma(f))$ with $\Psi(x) = x'$ and $\Psi(N(x, \Sigma)) \subseteq N(x, \Sigma)$. Since $x \in N(x, \Sigma)$ then $x' = \Psi(x) \in N(x, \Sigma)$. ■

Lemma 3.2.2. *Ξ is an equivalence relation on $V(S\Gamma(\varphi(f)))$.*

Proof It is obvious that Ξ is a reflexive and a symmetric relation. To prove that it is transitive, let $x_1\Xi x_2$ and $x_2\Xi x_3$ then by definition of Ξ there are four paths (q_1, r, x_1), (q_2, r, x_2), (p_1, s, x_2) and (p_2, s, x_3) in $S\Gamma(\varphi(f))$ with $r, s \in (\overline{X} \cup \overline{X}^{-1})^*$ and $q_1, q_2, p_1, p_2 \in N(x, \Sigma)$. Since $q_2, p_1 \in N(x, \Sigma)$ then there exists $u \in \mathcal{L}_A(\nu, \Delta)$ ($\mathcal{L}_B(\nu, \Delta)$) such that $(q_2, \varphi(u), p_1)$ is a path in Σ. Hence $(p_1, sr^{-1}\varphi(u), p_1)$ is a loop at p_1 in $S\Gamma(\varphi(f))$. Since $p_1\Xi p_2$ by Proposition 3.2.2 there is an automorphism $\Psi \in Aut(S\Gamma(\varphi(f)))$ such that $\Psi(p_1) = p_2$. Thus $(\Psi(p_1), sr^{-1}\varphi(u), \Psi(p_1)) = (p_2, sr^{-1}\varphi(u), p_2)$ is also a loop based at p_2. Since $S\Gamma(\varphi(f)))$ is deterministic and (p_2, s, x_3) is a path in $S\Gamma(\varphi(f)))$ then $(p_2, (\varphi(u))^{-1}r, x_3)$ is a path in $S\Gamma(\varphi(f))$. Let $p_3 \in V(\Sigma)$ such that $(p_2, (\varphi(u))^{-1}, p_3)$ $((p_2, (\varphi^{-1}(u))^{-1}, p_3))$ is a path in Σ. Since $u \in \mathcal{L}_A(\nu, \Delta)$ then $p_3 \in N(x, \Sigma)$. Therefore, (q_1, r, x_1) and (p_3, r, x_3) are paths in $S\Gamma(\varphi(f))$ with $q_1, p_3 \in N(x, \Sigma)$. Whence $x_1\Xi x_3$. ■

Proposition 3.2.3 (Lifting Property)**.** *Let $(p\Xi, w, q\Xi)$ be a path in $S\Gamma(\varphi(f))/\Xi$. Then for each $p' \in p\Xi$ there is a path (p', w, q') in $S\Gamma(\varphi(f))$ with $q' \in q\Xi$. Moreover, $S\Gamma(\varphi(f))/\Xi$ is a closed inverse word graph with respect to the standard presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$.*

Proof Let $(p\Xi, w, q\Xi)$ be a path in $S\Gamma(\varphi(f))/\Xi$, We can decompose w as $w = w_1 w_2 ... w_m$ such that (p_i, w_i, q_i) for all $i = 1, ..., m$ are paths in $S\Gamma(\varphi(f))$ with $p_1 \in p\Xi, q_m \in q\Xi$ and $q_i\Xi p_{i+1}, q_i \neq p_{i+1}$ for $i = 1, ..., m-1$. Let m be the minimum number of factors among the decompositions fulfilling the above properties, and let us prove that $m = 1$.

Suppose that $m > 1$, then $q_1 \Xi p_2$ and $q_1 \neq p_2$. By Proposition 3.2.2 there is an automorphism $\Psi \in Aut(S\Gamma(\varphi(f)))$ with $\Psi(p_2) = q_1$. Now apply Ψ to the decomposition to obtain

$$(\Psi(p_1), w_1, \Psi(q_1)), (q_1, w_2, \Psi(q_2)), (\Psi(p_3), w_3, \Psi(q_3)), ..., (\Psi(p_m), w_m, \Psi(q_m)).$$

Using $\Psi(p_2) = q_1$ and putting (p_1, w_1, q_1) instead of $(\Psi(p_1), w_1, \Psi(q_1))$ in the second decomposition to form the decomposition:

$$(p_1, w_1, q_1), (q_1, w_2, \Psi(q_2)), (\Psi(p_3), w_3, \Psi(q_3)), ..., (\Psi(p_m), w_m, \Psi(q_m)).$$

Which satisfies the previous properties but with $m-1$ components, a contradiction. Hence, $m = 1$. Therefore, there is a path (p_1, w, q_1) in $S\Gamma(\varphi(f))$ with $p_1 \in p\Xi$ and $q_1 \in q\Xi$. Using Proposition 3.2.2, for all $p' \in p\Xi$ there is an automorphism $\xi \in Aut(S\Gamma(\varphi(f)))$ with $\xi(p_1) = p'$ and by applying ξ to the path (p_1, w, q_1) we get the path $(p', w, \xi(q_1))$ in $S\Gamma(\varphi(f))$ with[1] $\xi(q_1)\Xi q_1$.
To prove the last statement let (w_1, w_2) be a relation in $\langle \overline{X} | R_{HNN} \cup R \rangle$. Suppose that $(q\Xi, w_1\tau, p\Xi)$ is a path in $S\Gamma(\varphi(f))/\Xi$, then by the above lifting property $(q, w_1\tau, p)$ is a path in $S\Gamma(\varphi(f))$, which is a closed inverse word graph hence $(q, w_2\tau, p)$ is also a path in $S\Gamma(\varphi(f))$ whence $(q\Xi, w_2\tau, p\Xi)$ is also a path in $S\Gamma(\varphi(f))/\Xi$. ■

Corollary 3.2.1. *Denote by* $\pi_\Xi : S\Gamma(\varphi(f)) \to S\Gamma(\varphi(f))/\Xi$ *the natural homomorphism induced by* Ξ *and let*

$$R'_\Delta = \{q\Xi \in V(S\Gamma(\varphi(f))/\Xi) | \exists u, u \in A, (x\Xi, \varphi(u), q\Xi) \textit{is a path in } S\Gamma(\varphi(f))/\Xi\}.$$

Then $\pi_\Xi^{-1}(R'_\Delta) = R_\Sigma$

Proof Let $q\Xi \in R'_\Delta$ and let $u \in A$ such that $(x\Xi, \varphi(u), q\Xi)$ is a path in $S\Gamma(\varphi(f))/\Xi$. Since $q\Xi q$ then by the lifting property we have $(x', \varphi(u), q)$ is a path in $S\Gamma(\varphi(f))$ with $x' \in x\Xi$. By Proposition 3.2.2, since $x\Xi = N(x, \Sigma)$ we get $x' \in N(x, \Sigma)$, thus by the definition of $N(x, \Sigma)$ there is $\overline{u} \in \mathcal{L}_A(\nu, \Delta)$ such that $(x, \varphi(\overline{u}), x')$ is a path in Σ. Hence $(x, \varphi(\overline{u}u), q)$ is a path in $S\Gamma(\varphi(f))$ such that $\varphi(\overline{u}u) \in B$, thus $\pi_\Xi^{-1}(q\Xi) = q$ is connected to x in Σ by a path labelled by $\varphi(\overline{u}u) \in B$ hence $q \in R_\Sigma$. Whence $\pi_\Xi^{-1}(R'_\Delta) \subseteq R_\Sigma$.
To show the other inclusion, let $q \in R_\Sigma$ and let $u \in A$ such that $(x, \varphi(u), q)$ is a path in $S\Gamma(\varphi(f))$. Hence $(x\Xi, \varphi(u), q\Xi)$ is a path in $S\Gamma(\varphi(f))/\Xi$ whence $q\Xi \in R'_\Delta$, i.e., $q \in \pi_\Xi^{-1}(R'_\Delta)$. ■

[1] $q_1 \Xi q_1$ thus there is $x_1 \in N(x, \Sigma)$ and $s \in (\overline{X} \cup \overline{X}^{-1})^*$ such that (q_1, s, x_1) is a path in $S\Gamma(\varphi(f))$ hence $(\xi(q_1), s, \xi(x_1))$ is also a path in $S\Gamma(\varphi(f))$ and since $\xi(N(x, \Sigma)) \subseteq N(x, \Sigma)$ whence by the definition of Ξ we get $\xi(q_1)\Xi q_1$

We also conclude from this corollary that

$$\pi_{\Xi}^{-1}(V(O_P(\Delta')) \cap V(O_P(\Omega)) = V(O_P(\Sigma)) \cap V(O_P(\Lambda)).$$

In particular we have

$$\pi_{\Xi}^{-1}(V(O_P(\Delta')) \cap V(\Omega) = V(O_P(\Sigma)) \cap V(\Lambda)$$

We now prove that $cl_{R_{HNN}\cup R}(\Delta')$ and $S\Gamma(\varphi(f))/\Xi$, are isomorphic by showing that the automata $(x\rho, cl_{R_{HNN}\cup R}(\Delta'), x\rho)$ and $(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)$ accept the same language.

Lemma 3.2.3.

$$L[(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)] = \{w \in (\overline{X} \cup \overline{X}^{-1})^+ : \exists u_1, u_2 \in \mathcal{L}_A(\nu, \Delta), \varphi(u_1)(w\tau)\varphi(u_2) \geq \varphi(f)\}$$

Proof Let $r \in L[(x\Xi, S\Gamma(\varphi(f)), x\Xi)]$ then $(x\Xi, r, x\Xi)$ is a loop in $S\Gamma(\varphi(f))/\Xi$. By Proposition 3.2.3 there is a path (q, r, q') in $S\Gamma(\varphi(f))$ with $q, q' \in x\Xi$, hence by property 2 of Proposition 3.2.2 $q, q' \in N(x, \Sigma)$. By the definition of $N(x, \Sigma)$ there are $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$ such that $(x, \varphi(u_1), q)$ and $(x, \varphi(u_2), q')$ are paths in Σ. Therefore, $\varphi(u_1)(r)\tau\varphi(u_2)^{-1}$ labels a loop based at x in $S\Gamma(\varphi(f))$. Whence $\varphi(u_1)(r)\tau\varphi(u_2)^{-1} \geq \varphi(f)$. To prove the other inclusion, let $r \in (\overline{X} \cup \overline{X}^{-1})^+$ such that $\varphi(u_1)(r)\tau\varphi(u_2)^{-1} \geq \varphi(f)$ for some $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$, let $(s_1)\tau = \varphi(u_1)$ and $(s_2)\tau = \varphi(u_2)$ then $(x, s_1 r s_2, x)$ is a loop based at x in $S\Gamma(\varphi(f))$. If we denote by x_1, x_2 the vertices of Σ such that $(x, s_1, x_1), (x_2, s_2, x)$ are paths in Σ. Since $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$ we have $x_1, x_2 \in N(x, \Sigma)$. By Proposition 3.2.2 $x_1, x_2 \in x\Xi$ thus s_1 labels the loop $(x\Xi, s_1, x_1\Xi)$ and s_2 labels the loop $(x_2\Xi, s_2, x\Xi)$ in $S\Gamma(\varphi(f))/\Xi$ hence r labels the loop $(x\Xi, r, x\Xi)$ in $S\Gamma(\varphi(f))/\Xi$, i.e., $r \in L[(x\Xi, S\Gamma(\varphi(f)), x\Xi)]$. ■

Lemma 3.2.4.

$$L[(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)] \subseteq L[(\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu')]$$

Proof Let $r \in L[x\Xi, S\Gamma(\varphi(f)), x\Xi]$ then by the previous Lemma there are $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$ such that $\varphi(u_1)(r\tau)\varphi(u_2) \geq \varphi(f)$. Since $\varphi(f)$ labels a loop based at ν' in Δ' then by Lemma 1.4.2 there is a homomorphism $\phi : \mathcal{A}(\overline{X}, R_{HNN} \cup R; \varphi(f)) \to (\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu')$, so $\varphi(u_1)(r\tau)\varphi(u_2)$ labels a loop at ν' in $cl_{R_{HNN}\cup R}(\Delta')$, and since $\varphi(u_1), \varphi(u_2)$ are labelling loops based at ν' in $cl_{R_{HNN}\cup R}(\Delta')$, thus r labels a loop based at ν' in $cl_{R_{HNN}\cup R}(\Delta')$. Whence $r \in L[(\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu')]$. ■

Lemma 3.2.5.

$L[(\nu', \Delta', \nu')] = \{r \in (X \cup X^{-1})^+ : \exists u_1, u_2 \in \mathcal{L}_A(\nu, \Delta), \varphi(u_1)(r\tau)\varphi(u_2) \geq \varphi(f)\}$

Proof Let $h \in L[(\nu', (\Delta'), \nu')]$, then by Proposition 1.4.1 we have $(\nu', h\tau, \nu') = (x\rho, h\tau, x\rho)$ is a path in Σ/ρ which implies that there is a path (q, h, q') in Σ with $q, q' \in x\rho$. Since $x\rho = N(x, \Sigma)$, then by property 1 of Proposition 1.4.2 we have $q, q' \in N(x, \Sigma)$. Thus there are $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$ such that $(x, \varphi(u_1), q)$ and $(x, \varphi(u_2)^{-1}, q')$ are paths in Σ. Therefore, $(x, \varphi(u_1)(h\tau)(\varphi(u_2)^{-1})^{-1}, x) = (x, \varphi(u_1)(h\tau)\varphi(u_2), x)$ is a loop in Σ based at x, hence $\varphi(u_1)(h\tau)\varphi(u_2) \geq \varphi(f)$. Hence $L[(\nu', \delta', \nu')] \subseteq \{r \in ((X \cup X^{-1}))^+ : \exists u_1, u_2 \in \mathcal{L}_A(\nu, \Delta), \varphi(u_1)(r\tau)\varphi(u_2) \geq \varphi(f)\}$. Conversely, let $r \in \{r \in (X \cup X^{-1})^+ : \exists u_1, u_2 \in \mathcal{L}_A(\nu, \Delta), \varphi(u_1)(r\tau)\varphi(u_2) \geq \varphi(f)\}$ thus $\varphi(u_1)(r\tau)\varphi(u_2) \geq \varphi(f)$ for some $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$. Since $\varphi(u_1), \varphi(u_2)$ are labelling loops based at ν' in Δ', hence $(r\tau)$ labels a loop based at ν' in Δ', whence $r \in L[(\nu', \Delta', \nu')]$. ■

We recall the following lemma from (Stephen [45] Lemma 3.4) for proving the reverse inclusion of the Lemma 3.2.4

Lemma 3.2.6. *Let $S = Inv\langle X|T\rangle = (X \cup X^{-1})^+/\tau$. If $\mathcal{A}(\alpha, \Gamma, \beta)$ is a connected inverse X automaton, then*

$$L[cl_T(\mathcal{A})] = \{w \in (X \cup X^{-1})^+ : w\tau \geq h\tau, \textit{for some } h \in L[\mathcal{A}]\}.$$

Lemma 3.2.7.

$$L[(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)] \supseteq L[(\nu', cl_{R_{HNN} \cup R}(\Delta'), \nu')].$$

Proof By Lemma 3.2.3 and by the previous lemma it is sufficient to prove that:

$$\{w \in (\overline{X} \cup \overline{X}^{-1})^+ : w\tau \geq h\tau, \text{for some } h \in L[(\nu', \Delta', \nu')]\} \subseteq$$
$$\{w \in (\overline{X} \cup \overline{X}^{-1})^+ : \exists u_1, u_2 \in \mathcal{L}_A(\nu, \Delta), \varphi(u_1)(w\tau)\varphi(u_2) \geq \varphi(f)\}$$

Let $r \in \{w \in (\overline{X} \cup \overline{X}^{-1})^+ : w\tau \geq h\tau, \text{for some } h \in L[(\nu', \Delta', \nu')]\}$. Thus $r\tau \geq h\tau$ for some $h \in L[(\nu', (\Delta'), \nu')]$, then by Lemma 3.2.5 there are $u_1, u_2 \in \mathcal{L}_A(\nu, \Delta)$ such that $\varphi(u_1)(h\tau_S)\varphi(u_2) \geq \varphi(f)$. Since S embeds into S^* then $\varphi(u_1)(r\tau)\varphi(u_2) \geq \varphi(u_1)(h\tau)\varphi(u_2) \geq \varphi(f)$ hence $r \in \{w \in (\overline{X} \cup \overline{X}^{-1})^+ : \exists u_1, u_2 \in \mathcal{L}_A(\nu, \Delta), \varphi(u_1)(w\tau)\varphi(u_2) \geq \varphi(f)\}$, from which it follows the claim. ■ Therefore from Lemmas 3.2.4 and 3.2.7 we have

$$L[(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)] = L[(\nu', cl_{R_{HNN} \cup R}(\Delta'), \nu')].$$

Theorem 3.2.1. *Let Δ, Δ' be two lobes of $S\Gamma(w)$ such that $\Delta \mapsto \Delta'$ and let $f \in E(A)$ such that $(\nu', \Delta', \nu') \cong (x\rho, \Sigma/\rho, x\rho)$ where $\Sigma = S\Gamma(X, R; \varphi(f))$ and ρ is the least equivalence relation on Σ which identifies the net $N(x, \Sigma)$ and makes Σ/ρ deterministic. Then*

$$(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi) \cong (\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu').$$

Proof From Proposition 3.2.3 $(x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)$ is a deterministic inverse word automaton. Since $(\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu')$ is closed with respect to the standard presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$, it is also a deterministic inverse word automaton thus by Proposition 1.2.1, there are homomorphisms $\phi : (x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi) \to (\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu')$ with $\phi(x\Xi) = \nu'$ and $\phi' : (\nu', cl_{R_{HNN}\cup R}(\Delta'), \nu') \to (x\Xi, S\Gamma(\varphi(f))/\Xi, x\Xi)$ with $\phi'(\nu') = x\Xi$ hence the result follows. ■

From the above theorem we identify $S\Gamma(\varphi(f))/\Xi$ with $cl_{R_{HNN}\cup R}(\Delta')$. Let Λ be the lobe of $cl_{R_{HNN}\cup R}(\Delta')$ that fulfills condition (3.1) then $R'_\Delta = V(O_P(\Delta)) \cap V(\Delta') = R_{\Delta'}$. Hence

$$\pi_\Xi^{-1}(R_{\Delta'}) = R_\Sigma \tag{3.3}$$

Theorem 3.2.2. *Let $B_\Lambda(\Sigma)$ be the lifted f-branch of $B_\Delta(\Delta')$, then*

$$B_\Delta(\Delta') \cong B_\Lambda(\Sigma)/\Xi.$$

Moreover, if $(z\Xi, w, s\Xi)$ is a path in $B_\Delta(\Delta')$, then for each $z' \in z\Xi$ there is a path (z', w, s') in $B_\Lambda(\Sigma)$ with $s' \in s\Xi$.

Proof Since $B_\Delta(\Delta')$ is in $cl_{R_{HNN}\cup R}(\Delta') = S\Gamma(\varphi(f))/\Xi$, then consider the map $\xi : B_\Lambda(\Sigma) \to B_\Delta(\Delta')$ defined by $\xi_V(y) = y\Xi$ for all $y \in V(B_\Lambda(\Sigma))$ and $\xi_{Ed}((z, a, q)) = (z\Xi, a, q\Xi)$ for all $(z, a, q) \in Ed(B_\Lambda(\Sigma))$. We will show that ξ is an isomorphism. First we prove that it is well defined on Σ. From Proposition 3.2.2 $\Xi \cap (V(\Sigma) \times V(\Sigma)) = \rho$ thus $\Delta' = \Sigma/\rho = \Sigma/\Xi$ then $\xi_{|_\Sigma} : \Sigma \to \Delta'$ is the natural mapping π_Ξ hence it is well defined, we show that ξ is well defined on $B_\Lambda(\Sigma) \setminus \{\Sigma\}$. Suppose $y\Xi \notin V(B_\Delta(\Delta'))$ for some $y \in V(B_\Lambda(\Sigma))$, i.e., assume that $y\Xi \in cl_{R_{HNN}\cup R}(\Delta') \setminus V(B_\Delta(\Delta'))$ for some $y \in V(B_\Lambda(\Sigma))$. Since $y\Xi \in cl_{R_{HNN}\cup R}(\Delta') \setminus V(B_\Delta(\Delta'))$ then by the definition of f-branch $B_\Delta(\Delta')$ we have that $y\Xi$ belongs to the t-subopuntoid subgraph of $cl_{R_{HNN}\cup R}(\Delta')$ whose lobe graph is a connected component of $\Upsilon(cl_{R_{HNN}\cup R}(\Delta') \setminus \{\Delta'\})$ containing Ω. For any $z\Xi \in V(O_P(\Delta')) \cap V(\Omega)$ there is a path $(z\Xi, s, y\Xi)$ such that s can be decomposed as $t^{k_0}s_1t^{k_1}s_2t^{k_2}s_3t^{k_3}...s_mt^{k_m}$ for some integers $m > 0, k_0 \leq 0, k_i \in \mathbb{Z} - \{0\}, i = 1, ..., m$. Note that $k_0 \leq 0$ because we are considering $\Sigma = S\Gamma(X, R; \varphi(f))$. Moreover, we can choose

$z\Xi \in V(O_P(\Delta')) \cap V(\Omega)$ so that all the prefixes $g \neq \epsilon$ of s and the final vertex $q\Xi$ of the path $(z\Xi, g, q\Xi)$ does not belong again to $V(O_P(\Delta')) \cap V(\Omega)$. By Proposition 3.2.3 and by Equation (3.3) there exists a path (z', s, y) in $S\Gamma(\varphi(f))$ such that $z' \in \pi_{\Xi}^{-1}V(O_P(\Delta')) \cap V(\Omega) = V(O_P(\Sigma)) \cap V(\Lambda)$. We have cases to be considered according to the word s. Case $i)$ $k_0 = 0$, for the prefix s_1 of s the final vertex r in the path (z', s_1, r) belongs to Λ, because $s_1 \in (X \cup X^{-1})^+$. Thus the path $(r, t^{k_1}s_2t^{k_2}s_3t^{k_3}...s_mt^{k_m}, y)$ connects a vertex $r \in V(\Lambda)$ to the vertex $y \in V(B_\Lambda(\Sigma))$. Since $\Upsilon(S\Gamma(\varphi(f)))$ is a tree, thus there is a prefix s' of $t^{k_1}s_2t^{k_2}s_3t^{k_3}...s_mt^{k_m}$ such that (r, s', h) is a path with $h \in V(O_P(\Sigma)) \cap V(\Lambda)$. Therefore, s_1s' is a non-empty prefix of s such that the final vertex $h\Xi$ of the path $(z\Xi, s_1s', h\Xi)$ belongs to $V(O_P(\Delta')) \cap V(\Omega)$ which is a contradiction. In case $ii)$ $k_0 \leq -1$, there is a lobe Λ' adjacent to Λ other than Σ with $z' \in V(O_P(\Lambda')) \cap V(\Lambda)\ R_\Lambda$, we may have a vertex r with non-empty prefix of s of this case such that r is a vertex of the t-subopuntoid subgraph of $cl_{R_{HNN} \cup R}(\Sigma) \setminus \{\Sigma\}$ containing Λ and is connecting $y \in V(B_\Lambda(\Sigma))$. Following an argument similar to case (i), we get a contradiction. Whence $\xi_V(y) = y\Xi \in V(B_\Delta(\Delta')$, for all $y \in V(B_\Lambda(\Sigma))$. We now prove that ξ_{Ed} is well defined, i.e., we prove that $\xi_{Ed}((z, a, q)) = (z\Xi, a, q\Xi) \in B_\Delta(\Delta')$ for all $(z, a, q) \in Ed(B_\Lambda(\Sigma))$. Since $\xi_V(z), \xi_V(q) \in V(B_\Delta(\Delta'))$ then either $\xi_{Ed}((z, a, q)) = (z\Xi, a, q\Xi) \in B_\Delta(\Delta')$ or $z\Xi, q\Xi \in V(O_P(\Delta')) \cap V(\Omega)$ and $a \in (X \cup X^{-1})^+$ but the latest will imply that $(z, a, q) \in Ed(\Lambda)$ which is a contradiction.

Now we show that ξ is an epimorphism. First we prove that ξ_V is epimorphism, showing that for any $z\Xi \in V(B_\Delta(\Delta'))$, $z\Xi \subseteq V(B_\Lambda(\Sigma))$. Let $y\Xi \in V(O_P(\Delta')) \cap V(O_P(\Delta)))$, then $y\Xi \subseteq V(O_P(\Lambda)) \cap V(O_P(\Sigma)) \subseteq V(B_\Lambda(\Sigma))$. Consider the case $z\Xi \in V(B_\Delta(\Delta')) \setminus V(O_P(\Delta')) \cap V(O_P(\Delta))$. Assume, contrary to our claim, that there exists $z\Xi \in V(B_\Delta(\Delta')) \setminus V(O_P(\Delta')) \cap V(O_P(\Delta)))$ such that $z \notin V(B_\Lambda(\Sigma))$. We can choose the path $(y\Xi, s, z\Xi)$ with $y\Xi \in V(O_P(\Delta')) \cap V(O_P(\Delta)))$, we may take $y\Xi \in V(O_P(\Delta)) \cap V(\Delta')$ and s with $s = s_1t^{k_1}s_2t^{k_2}s_3t^{k_3}...s_mt^{k_m}$ such that for any prefix s' of s the final vertex $r\Xi$ of the path $(z\Xi, s', r\Xi)$ is not in $V(O_P(\Delta')) \cap V(O_P(\Delta)))$. From Proposition 3.2.3 and by Equation 3.3 there exists a path (y', s, z) in $S\Gamma(\varphi(f))$ such that $y' \in y\Xi \subseteq V(O_P(\Lambda)) \cap V(O_P(\Sigma)) \subseteq V(B_\Lambda(\Sigma))$, in particular $y' \in V(O_P(\Lambda)) \cap V(\Sigma)$. Then (y', s_1, r) is a path in Σ and $(r, t^{k_1}s_2t^{k_2}s_3t^{k_3}...s_mt^{k_m}, z)$ is a path connecting $r \in V(\Sigma)$ with $z \notin V(B_\Lambda(\Sigma))$. Since $\Upsilon(S\Gamma(\varphi(f)))$ is a tree, there exists a prefix l of $t^{k_1}s_2t^{k_2}s_3t^{k_3}...s_mt^{k_m}$ such that (r, l, z') is a path with $z' \in V(O_P(\Lambda)) \cap V(O_P(\Sigma))$. Hence $(y\Xi, s_1s', z'\Xi)$ is a path in $S\Gamma(\varphi(f))/\Xi$ where $s_1s' \neq \epsilon$ is a prefix of s and $z'\Xi \in$

$V(O_P(\Delta')) \cap V(O_P(\Delta))$ which is a contradiction. We now show that ξ_{Ed} is epimorphism. Suppose that $(z\Xi, a, q\Xi)$ is an edge of $B_\Delta(\Delta')$, then by Proposition 3.2.3 and ξ_V is an epimorphism, we have that (z, a, q') is an edge in $S\Gamma(\varphi(f))$ with $q' \in q\Xi$ and $z, q' \in V(B_\Lambda(\Sigma))$. Suppose that $(z, a, q') \in Ed(S\Gamma(\varphi(f))) \setminus Ed(B_\Lambda(\Sigma))$. Since $S\Gamma(\varphi(f))$ is a t-opuntoid graph, then z, q' must be in $V(O_P(\Lambda)) \cap V(\Sigma)$, hence $(z, a, q') \in Ed(\Lambda)$, i.e., $(z\Xi, a, q\Xi) \in Ed(\Omega)$. Since $B_\Delta(\Delta') = B_\Omega(\Delta')$ we get the contradiction $(z\Xi, a, q\Xi) \notin Ed(B_\Delta(\Delta'))$, whence ξ_{Ed} is also an epimorphism. ∎

3.3 Bicyclic subsemigroups in HNN-extension of finite inverse semigroup

In this section we give a characterizations of HNN-extension of finite inverse semigroup to have a bicyclic subsemigroup, depending on the existence of two idempotents e and f in $A \cup B$ that are $\mathcal{D}$-related in S and $e < f$. In addition to this algebraic characterization, we also give a geometric one depending on the existence of an infinite path labelled by w^ω having the word w^n as a prefix for all integers $n \geq 1$. Let us begin with the following useful lemma.

Lemma 3.3.1. *Let $w \in (\overline{X} \cup \overline{X}^{-1})^+$ be a word in $(\alpha, S\Gamma(w), \beta)$ such that there is a path starting from α labelled by w^ω but w^{-1} does not label a path at α. Then*

- *if $(\alpha, w^i, \beta_i), (\alpha, w^j, \beta_j)$ are two paths in $S\Gamma(w)$ with $i \neq j$, then $\beta_i \neq \beta_j$.*
- *if α and β are not intersection vertices, then we can find a word h such that h^ω labels a path in $\mathcal{A}(\overline{X}, R_{HNN} \cup R; h) = (\delta, S\Gamma(h), \mu)$ starting at the intersection vertex δ but h^{-1} does not label a path starting at δ.*
- *if g is an idempotent labelling a loop at a vertex β_i then g is also labelling a loop at the vertex β_j, $\forall j \geq i$.*
- *the minimum idempotent that labels a loop at β_i is $w^{-i}w^i$.*

Proof To prove the first statement, assume for some $i > j$ we have $\beta_i = \beta_j$ then by the fact that $S\Gamma(w)$ is a deterministic inverse word graph, we have that w^{-1} will label a path starting at α but this contradicts the hypothesis. Hence the path π starting at α labelled by w^ω is an infinite path.

To show the second statement, assume that α and β are not intersection

vertices thus the factorization of w is of the form $s_1t^{k_1}s_2t^{k_2}...t^{k_n}s_{n+1}$ such that $s_i \in (X \cup X^{-1})^+, k_i \in \mathbb{Z} - \{0\}$ for all i. Let $h = (t^{k_1}s_2t^{k_2}...t^{k_{n-1}}s_ns_1)^2$ and put $\mathcal{A}(\overline{X}, R_{HNN} \cup R; h) = (\delta, S\Gamma(h), \mu)$. Let $\alpha' = \alpha s_1$ and $\beta' = \alpha' h$ then by Proposition 1.2.1, there is a homomorphism $(\delta, S\Gamma(h), \mu) \to (\alpha', S\Gamma(w), \beta')$ because there is a path in $S\Gamma(w)$ labelled by h starting at α'. Since w^{-1} does not label a path in $S\Gamma(w)$ starting at α and $S\Gamma(w)$ is deterministic then we conclude that h^{-1} does not label a path in $S\Gamma(h)$ starting at δ. Let $\delta' = \delta t^{k_1}s_2t^{k_2}...t^{k_{n-1}}s_n$, then it is easy to find a path in $S\Gamma(h)$ starting at δ' and labelled by w, thus there is also a homomorphism $(\alpha, S\Gamma(w), \beta) \to (\delta', S\Gamma(w), \mu')$ with $\mu' = \delta' w$. Hence, there is also a path labelled by w^ω starting at δ' in $S\Gamma(h)$. Therefore, by the definition of h there is also a path starting at δ labelled by h^ω.
In the third statement consider the graphs $S\Gamma(w^i)$ and $S\Gamma(w^j)$ where $j \geq i$. Since (α, w^i, β_i) is a path in $(\alpha, S\Gamma(w^j), \beta_j)$ thus there is a homomorphism $\gamma : (\alpha, S\Gamma(w^i), \beta_i) \to (\alpha, S\Gamma(w^j), \beta_j)$ hence if g labels a loop at β_i then by γ, g labels a loop at β_j.
To prove the last claim, let $e = ww^{-1}$ and f be the minimum idempotent labeling a loop at β_i then $w^{-i}w^i = w^{-i+1}w^{-1}w^i = w^{-i+1}(w^{-1}ww^{-1})w^i = w^{-i}ww^{-1}w^i = u^{-1}eu$ where $u = w^i$. Since $u^{-1}eu$ labels a loop at β_i thus $u^{-1}eu \geq f$ hence $fu^{-1}eu = f$ and since ww^{-1} is the minimum idempotent labeling a loop at α in $(\alpha, S\Gamma(w), \beta)$ then $ufu^{-1} \geq e$ thus $eufu^{-1} = e$. Hence $w^{-i}w^i = u^{-1}eu = u^{-1}eufu^{-1}u = u^{-1}euu^{-1}uf = u^{-1}euf = f$. ■

Theorem 3.3.1. *Let $w \in (\overline{X} \cup \overline{X}^{-1})^+$ be a word in $(\alpha, S\Gamma(w), \beta)$ such that there is a path starting at α labelled by w^ω but w^{-1} does not label a path at α. Then in $S^* = [S, A; B]$ there are two idempotents $f \in E(A) \cup E(B)$ and $e \in E(S^*)$ such that $e\mathcal{D}f$ and $e < f$*

Proof From statement 1 of Lemma 3.3.1 we may assume that all $\beta_i's$ are intersection vertices. Since each lobe of $S\Gamma(w)$ is finite, then the infinite path π spans an infinite subtree T of $\Upsilon(S\Gamma(w))$. From T we can extract an infinite reduced lobe path $\Delta_1, \Delta_2, ..., \Delta_j, ...$ containing a subsequence $\{\beta_{i_j}, j \geq 0\}$. By Lemma 3.2.1 there is a sufficiently large integer N such that $\Delta_N \mapsto \Delta_{N+1} \mapsto ... \mapsto \Delta_j \mapsto ....$ Thus we can choose two integers $m \geq 0, M \geq N$ such that $\Delta_M \mapsto \Delta_{M+1} \mapsto ... \mapsto \Delta_j \mapsto ...$ with $(\beta_{i_m}, v_{i_{m+1}})$ is an intersection pair of Δ_M and Δ_{M+1} where β_{i_m} is an intersection vertex of Δ_M and $v_{i_{m+1}}$ is an intersection vertex of Δ_{M+1} and all but finitely many vertices of the infinite sequence $\{\beta_{i_j}, j \geq m\}$ are contained in the f-branch $B_{\Delta_M}(\Delta_{M+1})$. Since $\{\beta_{i_j}, j \geq m\}$ is an infinite set and each lobe is finite, then there is a sufficiently large integer s such that $(\beta_{i_s}, w, \beta_{i_s+1})$ is a path of $B_{\Delta_M}(\Delta_{M+1})$ and $i_s \geq i_m$. Let f be the minimum idempotent of A (B) labelling a loop based at β_{i_m} and let $F \in (\overline{X} \cup \overline{X}^{-1})^+$ such that $F\tau = f$. By statement 2 of Lemma 3.3.1 F labels a loop based at $\beta_{i_s}, \beta_{i_s+1}$ and so since $B_{\Delta_M}(\Delta_{M+1})$ is connected there is a path $(\beta_{i_m}, FyFwF, \beta_{i_s+1})$, for some word $y \in (\overline{X} \cup \overline{X}^{-1})^+$, contained in $B_{\Delta_M}(\Delta_{M+1})$. Put $\lambda = FyFwF$, by Theorem 3.2.2 in the lifted f-branch $B_\Lambda(\Sigma)$ associated to the branch $B_{\Delta_M}(\Delta_{M+1})$ there is also a path (x, λ, z) with $x\Xi = \beta_{i_m}$ and $z\Xi = \beta_{i_s+1}$. Since $B_\Lambda(\Sigma)$ is a t-subopuntoid subgraph of $S\Gamma(f)$ we get that $(\lambda\lambda^{-1})\tau \geq f$, and since $F\tau = f$ then

$$
\begin{aligned}
\lambda\tau &= f(y\tau)f(w\tau)f \\
\Rightarrow \quad (\lambda\lambda^{-1})\tau &= f(y\tau)f(w\tau)f(f(w\tau)^{-1}f(y\tau)^{-1}f) \\
\Rightarrow \quad f(\lambda\lambda^{-1})\tau &= ff(y\tau)f(w\tau)f(f(w\tau)^{-1}f(y\tau)^{-1}f) \\
&= f(y\tau)f(w\tau)f(f(w\tau)^{-1}f(y\tau)^{-1}f) \\
\Rightarrow \quad f(\lambda\lambda^{-1})\tau &= (\lambda\lambda^{-1})\tau
\end{aligned}
$$

Hence we have $f \geq (\lambda\lambda^{-1})\tau$. Whence $(\lambda\lambda^{-1})\tau = f$. By a similar computation, we obtain that $(\lambda^{-1}\lambda)\tau \leq f$. We prove that $(\lambda^{-1}\lambda)\tau < f$. Suppose, contrary to the claim that $(\lambda^{-1}\lambda)\tau = f$. Since $(x, S\Gamma(f), x)$ is the Schützenberger automaton of f then λ^{-1} labels the path $(x, F^{-1}w^{-1}F^{-1}y^{-1}F^{-1}, q)$. Since F^{-1} labels a loop at x then the path $(x, w^{-1}F^{-1}, y_1)$ is also a path in $S\Gamma(f)$. Moreover, since $F\tau = f$ then F^{-1} labels a loop at y_1 hence (x, w^{-1}, y_1) is a path in $S\Gamma(f)$. As $F\tau = f = (\lambda\lambda^{-1})\tau$ labels a loop at y_1, by Lemma 1.4.2 we can iterate this procedure to obtain a sequence of vertices $\{y_i, i \geq 0\}$ where $y_0 = x$ and (y_i, w^{-1}, y_{i+1}) are paths in $S\Gamma(f)$ for all i. Therefore $(w^{-1})^\omega$ labels a path at $y_0 = x$ in $S\Gamma(f)$. Since f labels a loop at β_{i_m} in $S\Gamma(f)$ then $L[(x, S\Gamma(f), x)] \subseteq L[(\beta_{i_m}, S\Gamma(f), \beta_{i_m})]$. Hence $(w^{-1})^\omega$ labels a path at β_{i_m} in $S\Gamma(f)$. By the definition of β_{i_m} and since $S\Gamma(f)$ is deterministic we obtain that w^{-1} labels a path at α which contradicts the statement of the theorem. Thus we have $(\lambda\lambda^{-1})\tau = f \in E(A)$ $(E(B)), (\lambda^{-1}\lambda)\tau = e < f$ with $f\mathcal{D}e$ in S^* (because $(\lambda\lambda^{-1})\tau = f$ and $(\lambda^{-1}\lambda)\tau = e$). ■

Corollary 3.3.1. *Let $S^* = [S, A; B]$ be an HNN-extension of finite inverse semigroup S. Then the following statements are equivalent*

1. *S^* has a bicyclic subsemigroup*
2. *There is a word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ such that w^ω, but not w^{-1} labels a path starting at the initial state of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$*
3. *There is a word $w \in (\overline{X} \cup \overline{X}^{-1})^+$ such that w^ω, but not $(w^{-1})^\omega$ labels a path starting at the initial state of $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$*

4. *There are two idempotents $f \in E(A)$ $(E(B))$ and $e \in E(S^*)$ such that $f\mathcal{D}e$ and $e < f$*

Proof 1) $\Rightarrow$ 2) Corollary 3.1.1.
2) $\Rightarrow$ 3) is obvious.
3) $\Rightarrow$ 2) Assume that w^{-1} labels a path at α of $(\alpha, S\Gamma(w), \beta) = \mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$, thus $(w^{-1}w)\tau$ labels a loop at α hence $(w^{-1}w)\tau \geq (ww^{-1})\tau$. Since w^{ω} labels a path at α and $(w^{-1}w)\tau$ is the minimum idempotent labelling a loop at β then $(ww^{-1})\tau \geq (w^{-1}w)\tau$. Therefore $(ww^{-1})\tau = (w^{-1}w)\tau$. Similarly, for all $i \geq 1$ we can show that $(w^{-i}w^{i})\tau \geq (w^{-1}w)\tau = (ww^{-1})\tau$, whence $(w^{-1})^{\omega}$ labels also a path at α in $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w)$ which contradicts the assumption.
2) $\Rightarrow$ 4) Theorem 3.3.1
4) $\Rightarrow$ 1) Proposition 3.1.1. ∎

3.4 Complete semisimplicity problem for HNN-extension of finite inverse semigroup

The following lemma gives a stronger condition than condition (2) of Proposition 3.1.2 for $S^* = [S, A; B]$ to be completely semisimple when we restrict to the finite case.

Lemma 3.4.1. *Let $S^* = [S, A; B]$ be an HNN-extension of a finite inverse semigroup. Then S^* is completely semisimple if and only if for all $g \in E(A) \cup E(B)$:*

$$Aut(S\Gamma(g)) = End(S\Gamma(g)).$$

Proof The "only if" part follows directly from Proposition 3.1.2. For the "if" part, let $Aut(S\Gamma(g)) = End(S\Gamma(g))$ for all $g \in E(A) \cup E(B)$. Let $f \in E(A) \cup E(B)$ and $e \in E(S^*)$ such that $e\mathcal{D}f$ and $e \leq f$. Following Corollary 3.3.1, for showing that S^* is completely semisimple we prove that $e = f$. Let $(x, S\Gamma(f), x) = \mathcal{A}(\overline{X}, R_{HNN} \cup R; f), (y, S\Gamma(e), y) = \mathcal{A}(\overline{X}, R_{HNN} \cup R; e)$. Since $e \leq f$ then by Theorem 1.3.3 there is a homomorphism $\zeta : (x, S\Gamma(f), x) \to (y, S\Gamma(e), y)$ with $\zeta(x) = y$ and again by Theorem 1.3.3, since $e\mathcal{D}f$ then there an isomorphism $\mu : S\Gamma(e) \to S\Gamma(f)$, thus $\mu \circ \zeta \in End(S\Gamma(f))$. Hence $\mu \circ \zeta \in Aut(S\Gamma(f))$ with $(\mu \circ \zeta)(x) = y$. Now the fact that μ is an isomorphism implies that ζ is also an isomorphism and since x and y are arbitrary we have $\mathcal{A}(\overline{X}, R_{HNN} \cup R; e) \cong \mathcal{A}(\overline{X}, R_{HNN} \cup R; f)$ hence $e = f$. ∎

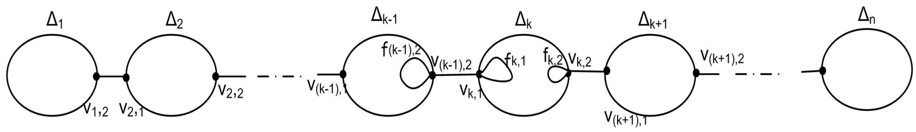
We denote by $J_e \geq J_f$ for the usual order between the $\mathcal{J}-$classes. It is well known that for any inverse semigroup S and for all $e, f \in E(S)$, $J_e \geq$

J_f if and only if $e \geq g$ for some $g\mathcal{D}f$. We denote $\mathcal{D}^S, \mathcal{D}^A$ and $\mathcal{D}^B$ the $\mathcal{D}$-relation on S, A and B, respectively; we also use the same convention for the other Green's relations and we use the same notation for their equivalence classes, for instance the $\mathcal{H}-$classes of an element g in S, A and B are denoted by H_g^S, H_g^A and H_g^B, respectively. In the next result we give a characterization for S^* to be completely semisimple. Before we proceeding we need the following definition.

Definition 3.4.1 (Preorder)**.** For every $f, e \in S$ we denote $e \prec_S f$ if and only if $J_e^S \geq J_f^S$ in S and we define $\prec_{AB}$ on $E(A) \cup E(B)$ to be the set $\{(a, \varphi(a)), (\varphi(a), a) | a \in E(A)\} \cup \{(b, \varphi^{-1}(b)), (\varphi^{-1}(b)), b) : b \in E(B)\}$ and we denote by $\prec$ the transitive closure of $\prec_S \cup \prec_{AB}$.

Theorem 3.4.1. *Let $S^* = [S, A; B]$ be an HNN-extension of a finite inverse semigroup. Then S^* is completely semisimple if and only if $\prec \cap \succ_S \subseteq \prec_S$.*

Proof We start with the "if part", so assume $\prec \cap \succ_S \subseteq \prec_S$, let $f \in E(A) \cup E(B)$ and let $\Sigma_1 \cong S\Gamma(X, R; f)$ which is the underlying graph of t-$Core(f)$. Let $\Psi \in End(S\Gamma(f))$ then by Lemma 3.4.1 we show that $\Psi \in Aut(S\Gamma(f))$. So let Σ be the lobe of $S\Gamma(f)$ containing the image of Σ_1 under Ψ. If $\Sigma = \Sigma_1$ then $\Psi_{|\Sigma_1}$ is an endomorphism on Σ_1 which is a Schützenberger automaton relative to the presentation $\langle X|R\rangle$ of S. Since S is completely semisimple then by Proposition 3.1.2 $\Psi_{|\Sigma_1}$ is an automorphism hence by Proposition 1.4.3 $\Psi_{|\Sigma_1}$ may be extended to an automorphism of $S\Gamma(f)$ which coincides with Ψ because they agree on the lobe Σ_1. Therefore, we assume $\Sigma \neq \Sigma_1$, which implies that Σ is an external lobe of $t - Core(f)$. Thus $\Sigma_1 \to \Sigma$ and let $\Sigma_1 = \Delta_1, \Delta_2, ..., \Delta_n = \Sigma$ be the reduced lobe path from Σ_1 to Σ. Since Σ_1 is the underlying graph of t-$Core(f)$ then for all $1 \leq k \leq n$, $\Delta_k \mapsto \Delta_{k+1}$. We put $v_{k,1}, v_{k,2} \in V(\Delta_k)$ for the intersection vertices of the lobe Δ_k such that $(v_{k,2}, v_{(k+1),1})$ $((v_{(k+1),1}, v_{k,2}))$ is an intersection pair between the lobes Δ_k, Δ_{k+1} for all $1 \leq k \leq n-1$, let $f_{k,i} = f_A(v_{k,i}, \Delta_k)(f_B(v_{k,i}, \Delta_k)), i = 1, 2$ (see the following figure)



Thus, by Proposition 3.2.1 we have $f_{(k+1),1} = \varphi(f_{k,2})(\varphi^{-1}(f_{k,2})) = e(v_{(k+1),1}, \Delta_{k+1})$ that is $(f_{k,2}, \varphi(f_{k,2}))((f_{k,2}, \varphi^{-1}(f_{k,2})) \in \prec_{AB}$ hence $f_{k,2} \prec_{AB} f_{(k+1),1}$ for all $1 \leq k \leq n$. Since $f_{k,1}\mathcal{D}^S e(v_{k,2}, \Delta_k)$ and

$e(v_{k,2}, \Delta_k) \leq f_{k,2}$ for all $1 \leq k \leq n$ then $f_{k,1} \prec_S f_{k,2}$ consequently we have constructed the sequence

$$f_{1,2} \prec_{AB} f_{2,1} \prec_S f_{2,2} \prec_{AB} \ldots \prec_S f_{(n-1),2} \prec_{AB} f_{n,1} \tag{3.4}$$

Note that this sequence can have terms like $\ldots f_{(k-1),2} \prec_{AB} f_{k,1} \prec_{AB} f_{(k+1),2} \ldots$. Let $v_n = \Psi(v_{1,2})$ where $v_{1,2} \in V(\Delta_1)$ and $v_n \in V(\Delta_n)$, since $\Psi_{|\Sigma} : \Delta_1 \to \Delta_n$ is a homomorphism of inverse word graphs, then we have $e(v_n, \Delta_n) \leq e(v_{1,2}, \Delta_1) = f_{1,2}$ the last equality holds by Proposition 3.2.1 because the Schützenberger graph Δ_1 is also feeding off Δ_2. Now $f_{n,1}\mathcal{D}^S e(v_n, \Delta_n)$ and $e(v_n, \Delta_n) \leq f_{1,2}$ imply $f_{n,1} \prec_S f_{1,2}$, whence that $f_{k,2} \prec f_{k,1} \prec_S f_{k,2}$ for all $f_{k,i}, i = 1, 2$ in Equation (3.4). Hence, applying the condition $\prec \cap \succ_S \subseteq \prec_S$ we obtain[1] $f_{k,2} \prec_S f_{k,1}$ hence $f_{k,2}\mathcal{J}^S f_{k,1}$. Since $f_{k,1}\mathcal{D}^S e(v_{k,2}, \Delta_k)$, i.e., $f_{k,1}\mathcal{J}^S e(v_{k,2}, \Delta_k)$ then $f_{k,2}\mathcal{J}^S f_{k,1}\mathcal{J}^S e(v_{k,2}, \Delta_k)$, and since $e(v_{k,2}, \Delta_k) \leq f_{k,2}$ we have $e(v_{k,2}, \Delta_k) = f_{k,2}$. Next, from $f_{1,2} \prec f_{n,1} \prec_S f_{1,2}$, $\prec \cap \succ_S \subseteq \prec_S$ and using similar argument we get $f_{1,2}\mathcal{J}^S f_{n,1}\mathcal{J}^S e(v_n, \Delta_n)$. Since $e(v_n, \Delta_n) \leq f_{1,2}$, we have $e(v_n, \Delta_n) = f_{1,2}$, and from $e(v_n, \Delta_n) \leq e(v_{1,2}, \Delta_1) \leq f_{1,2}$ we get $e(v_{1,2}, \Delta_1) = f_{1,2}$, hence $e(v_n, \Delta_n) = e(v_{1,2}, \Delta_1) = f_{1,2}$. Furthermore, since $\Sigma_1 = \Delta_1$ is the Schützenberger graph $S\Gamma(X, R; f), \Delta_1 \mapsto \Delta_2$ and $e(v_{2,1}, \Delta_2) = f_{2,2} = \varphi(f_{1,2})$ $(\varphi(f_{1,2}))$ $((\varphi^{-1}(f_{1,2})))$ then by Proposition 3.2.1 we have that Δ_2 is isomorphic to $S\Gamma(X, R; \varphi^{-1}(f_{1,2}))$. Repeating this argument for all the lobes Δ_k, we get that $\Sigma = \Delta_n$ is also isomorphic to $S\Gamma(X, R; \varphi f_{n,1})$. Moreover, since $e(v_n, \Delta_n) = e(v_{1,2}, \Delta_1) = f_{1,2}$, we get $(v_{1,2}, \Sigma_1, v_{1,2}) \cong (v_n, \Sigma, v_n)$ under the isomorphism $\psi : \Sigma_1 \to \Sigma$ with $\psi(v_{1,2}) = v_n$. Hence, by Proposition 1.4.3 ψ can be extended to an automorphism $\Psi' \in Aut(S\Gamma(f))$, since $\Psi(v_1) = \Psi'(v_1)$ then $\Psi = \Psi'$, hence $\Psi \in Aut(S\Gamma(f))$. Therefore, we have proved that $Aut(S\Gamma(f)) = End(S\Gamma(f))$ for all $f \in E(A) \cup E(B)$ and it follows by Lemma 3.4.1 that $S^* = [S, A; B]$ is completely semisimple.

Conversely, let f, g be idempotents in $E(A) \cup E(B)$ with $f \prec g$ and $g \prec_S f$, we show that $f \prec_S g$. Since $f \prec g$ then there are idempotents $f_1, f_2, \ldots, f_n \in E(A) \cup E(B)$, where $f_1 = f$, $f_n = g$, and:

$$f_1 \prec_{AB} f_2 \prec_S f_3 \prec_{AB} f_4 \prec_S, \ldots, \prec_S f_{n-1} \prec_{AB} f_n.$$

Therefore, for all f_k, f_{k+1} such that $f_k \prec_S f_{k+1}$ there is a homomorphism ϕ_k from $S\Gamma(X, R; f_{k+1})$ to $S\Gamma(X, R; f_k)$. By Proposition 1.4.3

[1] since $f_{k,2} \prec_S f_{k,1}$ then $J^S_{f_{k,2}} \leq J^S_{f_{k,1}}$ and since $f_{k,2} \prec_S f_{k,1}$ thus $J^S_{f_{k,1}} \leq J^S_{f_{k,2}}$ hence $J^S_{f_{k,1}} = J^S_{f_{k,2}}$, i.e., $f_{k,2}\mathcal{J}^S f_{k,1}$

ϕ_k extends to a homomorphism from $S\Gamma(f_{k+1})$ to $S\Gamma(f_k)$ and is also denoted by ϕ_k. In case $f_k \prec_{AB} f_{k+1}$, by definition of $\prec_{AB}$ we have either $f_{k+1} = \varphi(f_k)$, $f_k = \varphi^{-1}(f_{k+1})$, $\varphi(f_{k+1}) = f_k$ or $\varphi^{-1}(f_k) = f_{k+1}$ We consider the case $f_{k+1} = \varphi(f_k)$ (the others are similar). The Schützenberger graph $S\Gamma(X, R; f_k)$ is an underlying graph of a t-$Core(f_k)$ of $S\Gamma(f_k)$, hence $cl_{R_{HNN} \cup R}(S\Gamma(X, R; f_k)) = S\Gamma(f_k)$ and also $S\Gamma(X, R; f_{k+1}) = S\Gamma(X, R; \varphi(f_k))$ is also an underlying graph of a t-$Core(f_k)$ thus $S\Gamma(f_{k+1}) = cl_{R_{HNN} \cup R}(S\Gamma(X, R; f_{k+1})) = S\Gamma(f_k)$ hence $S\Gamma(f_{k+1}) = S\Gamma(f_k)$. Moreover, since $f_n = g \prec_S f = f_1$ then there is a homomorphism, say ϕ_n, from $S\Gamma(X, R; f_1)$ to $S\Gamma(X, R; f_n)$ which can also be extended to a homomorphism from $S\Gamma(f_1)$ to $S\Gamma(f_n)$. The composition $\phi = \phi_1 \circ ... \circ \phi_n$ defines a homomorphism from $S\Gamma(f_n)$ to $S\Gamma(f_1)$. Thus $\phi \circ \phi_n \in End(S\Gamma(f_1))$ and $\phi_n \circ \phi \in End(S\Gamma(f_n))$. Since S^* is completely semisimple and $f_1, f_n \in E(A) \cup E(B)$ then by Proposition (8) $\phi_n \circ \phi \in Aut(S\Gamma(f_1))$ and $\phi_n \circ \phi \in Aut(S\Gamma(f_n))$. Therefore, since $\phi_n \circ (\phi \circ (\phi_n \circ \phi)^{-1})$ and $((\phi_n \circ \phi)^{-1}) \circ \phi) \circ \phi_n$ are the identities on $S\Gamma(f_1)$ and $S\Gamma(f_n)$, respectively, then ϕ_n is an isomorphism from $S\Gamma(f_1)$ to $S\Gamma(f_n)$. Thus by Theorem 1.3.3 $f_1 \mathcal{D}^S f_n$, thus $J^S_{f_1} = J^S_{f_n}$, whence $f_1 = f \prec_S f_n = g$. ■

CHAPTER 4

Hosts

In Chapter 1 we explored the construction of the Schützenberger graph for a word w in the HNN-extension S^* of a finite inverse semigroup S, and we have shown that the underlying graph Ω_0 of the automaton t-$Core(w)$ is a t-opuntoid finite graph containing all the needed information to build $S\Gamma(w)$ by iteratively applying Construction 5. However, other finite subgraphs of $S\Gamma(w)$ share with Ω_0 such properties, the minimal ones of these finite t-subopuntoid subgraphs of $S\Gamma(w)$ are called hosts. More formally each host is a t-subopuntoid subgraph that has a finite lobe tree with the property that no pair of lobes feeds off each other, and each lobe of $S\Gamma(w)$ which is not a lobe of the host feeds off some lobe in the host. These hosts store a lot information about $S\Gamma(w)$ and a Schützenberger graph $S\Gamma(w)$ is a complete t-opuntoid graph that possess a host. The set $Host(\Gamma)$ of all lobes of all hosts in a t-opuntoid graph Γ will play a significant role in extracting valuable properties of Γ. The results of this chapter are similar to some results on amalgam of finite inverse semigroups in [7, 38] and on lower bounded HNN-extensions [26] and are fundamentals in order to obtain the main results of Chapter 5. The chapter is organized as follows. In the first section we present the definition of host and we collect some relevant results. In the second section first we exhibit the role that any host plays in forming morphisms between t-opuntoid graphs that posses hosts and then we introduce the notion of the host set $Host(\Gamma)$ which we use to reveal some properties about the automorphism group of Γ. The last two sections are devoted to study the number of hosts in $Host(\Gamma)$, mainly focusing on the cases where $Host(\Gamma)$ is composed by a unique host or it is infinite.

4.1 Hosts of complete t-opuntoids

Let Γ be a t-opuntoid graph, a *parasite* of Γ is an S-lobe Δ' of Γ which is t-adjacent precisely to one other S-lobe Δ of Γ and such that Δ' feeds off Δ.
In the sequel we often use the word lobe to denote an S-lobe.

We now proceed by the following definition of a host.

Definition 4.1.1. A t-subopuntoid subgraph Γ' of a t-opuntoid graph Γ is called a *host* of Γ if:

- its lobe tree is finite,
- it is parasite-free,
- every S-lobe of Γ not belonging to Γ' feeds off some S-lobe of Γ'.

A host of a t-opuntoid automaton is a host of its underlying graph.
It is obvious that a host Θ of a t-opuntoid graph Γ is a minimal t-subopuntoid subgraph of Γ such that $cl_{R_{HNN}\cup R}(\Theta) \supseteq \Gamma$.

Lemma 4.1.1. *Let Γ be a t-opuntoid graph. Then a host of Γ is a maximal t-subopuntoid subgraph Σ of Γ with the property that none of the lobes of Σ feeds off other lobes of Σ.*

Proof Let Σ be a host of Γ. By definition none of the lobes of Σ feeds off other lobes of Σ. Assume that Γ' is a t-subopuntoid subgraph of Γ with finitely many lobes that properly contains Σ. Hence $\Upsilon(\Sigma)$ is a proper subtree of $\Upsilon(\Gamma')$, and therefore there exists at least one lobe Λ' of Γ' that is not a lobe of Σ. By the definition of host, the lobe Λ' must feed off some lobe Λ of Σ. Thus there exists a reduced lobe path p in Γ, $p : \Lambda = \Delta_1, \Delta_2, ..., \Delta_m = \Lambda'$, from Λ to Λ', where $\Delta_i \to \Delta_{i+1}$ for $1 \leq i \leq m-1$. Since Γ' is connected and both Λ and Λ' are in Γ', the whole path p is in Γ'. That is Γ' has a lobe which feeds off another one in Γ' in other words Γ' could not be a host. Hence Σ is a maximal t-subopuntoid subgraph of Γ such that no lobe of Σ feeds off another lobe of Σ. ■

In the sequel we say that an S- lobe of a t-opuntoid graph is *extremal* if it has a unique t-adjacent S-lobe.

Lemma 4.1.2. *Every t-opuntoid graph with finitely many lobes has a host.*

Proof let Γ be a t-opuntoid graph with finitely many lobes. If none of the lobes of Γ feeds off some other lobes of Γ, then Γ is a host of itself. Otherwise, there exists an extremal lobe Δ' of Γ that feeds off its unique adjacent lobe Δ. The subgraph $\Gamma' = \Gamma \setminus \{\Delta'\}$ is a t-subopuntoid subgraph of Γ. If none of the lobes of Γ' feeds off some other lobes of Γ', then Γ' is a host of Γ; otherwise continuing in this manner we obtain a successive sequence $\Gamma_1, \Gamma_2, ..., \Gamma_n$ of t-subopuntoids subgraphs of Γ such that Γ_i is obtained from Γ_{i-1} by removing an extremal lobe of Γ_i that feeds off a lobe of Γ_i. Since Γ has finitely many lobes then the sequence will terminate in a t-subopuntoid subgraph Γ_n which does not contain a lobe that feeds off some other lobe of Γ_n. It is possible that Γ_n consists only of one lobe. We now show that Γ_n is a host of Γ. It is sufficient to show that every lobe of Γ not in Γ_n feeds off a lobe of Γ_n. Then let Λ' be a lobe in $\Gamma \setminus \Gamma_n$. For any lobe Λ of Γ_n there exists a unique reduced lobe path $\Lambda = \Delta_1, \Delta_2, ..., \Delta_m = \Lambda'$ from Λ to Λ', where $m \geq 2$. We show by induction on m that Λ' feeds off Λ. If $m = 2$ then Λ' was removed from Γ_j for some $j < n$ thus Λ' was an extremal lobe that feeds off its adjacent lobe Λ, hence $\Lambda \mapsto \Lambda'$. If $m > 2$, Λ' is an extremal lobe that feeds off its adjacent lobe in Γ_k for some $k < n$, because otherwise Λ' would not have been removed, thus we have $\Delta_{m-1} \mapsto \Delta_m = \Lambda'$ and by induction hypotheses, Δ_{m-1} feeds off Λ hence $\Lambda \to \Delta_{m-1} \mapsto \Lambda'$. ■

Corollary 4.1.1. *Let $w \in (\overline{X} \cup \overline{X}^{-1})^+$, then t-$Core(w)$ relative to the presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$ has a host.* ■

Corollary 4.1.2. *Let $S^* = [S; A, B]$ be an HNN-extension of a finite inverse semigroup S. Then the Schützenberger automata relative to the standard presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$ are complete t-opuntoid automata which posses a host and whose S-lobes are closed DV-quotients of Schützenberger automata relative to the presentation $\langle X | R \rangle$.* ■

Lemma 4.1.3. *Let $S\Gamma(w)$ be the Schützenberger graph of $w \in (\overline{X} \cup \overline{X}^{-1})^+$ with respect to the presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$. If the host of $S\Gamma(w)$ is not unique then every host is an S-lobe and the unique lobe path between two hosts consists entirely of S-lobes that are hosts.*

Proof Let Θ, Θ' are two distinct hosts of $S\Gamma(w)$. Suppose that they share an S-lobe Δ. Let Δ' be an extremal S-lobe of Θ, if it is also a lobe of Θ' then the unique lobe path connecting Δ to Δ' is a path of Θ' because $\Upsilon(\Theta')$ and $\Upsilon(\Theta)$ are subtrees of the tree $\Upsilon(S\Gamma(w))$, thus if all the extremal lobes of Θ are in Θ' this implies that $\Theta \subseteq \Theta'$ which contradicts the fact that Θ' is host. Hence, there is an extremal lobe Δ'' of Θ not contained in Θ'. The unique reduced lobe path p connecting Δ to Δ'' is a path of Θ because $\Upsilon(\Theta)$ is a subtree of $\Upsilon(S\Gamma(w))$. Since Δ'' is not a lobe of Θ' then $\overline{\Delta} \to \Delta''$ for some lobe $\overline{\Delta}$ of Θ'. Since p is the unique reduced lobe path connecting Δ to Δ'' so $\overline{\Delta}$ is a lobe of Θ with $\overline{\Delta} \to \Delta''$ which contradicts the minimality of Θ. Hence, the two hosts Θ, Θ' have no common lobe. Suppose that Θ' has more than one lobe, then there are two extremal lobes Δ' and Δ'' and there is a reduced lobe path $q : \Delta' = \Delta_1, ...\Delta_n = \Delta''$ in Θ'. Since Θ is a host and Δ', Δ'' are not in Θ then there are two lobes $\overline{\Delta}, \overline{\overline{\Delta}}$ of Θ and two reduced lobe paths p', p'' in $S\Gamma(w)$ such that $\overline{\Delta} \to \Delta', \overline{\overline{\Delta}} \to \Delta''$ and $p' : \Delta'_1, ...\Delta'_m$, $p'' : \Delta''_1, ...\Delta''_k$ with $\Delta'_m = \Delta', \overline{\Delta} = \Delta'_1, \Delta'_i \to \Delta'_{i+1}$ for $1 \leq i \leq m-1$ and $\Delta''_k = \Delta'', \overline{\overline{\Delta}} = \Delta''_1, \Delta''_i \to \Delta''_{i+1}$ for $1 \leq i \leq k-1$. Let Δ'_i with $i < m$ be a lobe in p' belonging to Θ', since Θ' is a t-opuntoid graph and Δ' is an extremal lobe of Θ', then the lobe subpath $\Delta'_i, ...\Delta'_m$ of p' belongs entirely to Θ'. Hence $\Delta'_i \to \Delta'$ where Δ' is an extremal lobe of Θ' and Δ'_i is a lobe of Θ', but this contradicts the minimality of Θ'. Therefore, all the lobes of p', except $\Delta'_m = \Delta'$, are not in Θ', analogously for p''. Hence p', p'' are paths whose lobes except $\Delta'_m = \Delta'$ and $\Delta''_k = \Delta''$, do not belong to Θ'. Since $\overline{\Delta}, \overline{\overline{\Delta}}$ are lobes of Θ, which has no lobe in common with Θ', then there is a reduced lobe path r in Θ connecting $\overline{\Delta}$ to $\overline{\overline{\Delta}}$ whose all lobes do not belong to Θ'. Therefore the path $(p')^{-1}rp''$ is a lobe path connecting Δ' to Δ'' which has no lobe other than Δ' and Δ'' in Θ'. Hence $(p')^{-1}rp''$ and q are two distinct lobe paths connecting the same lobes Δ', Δ'', against the fact that $\Upsilon(S\Gamma(w))$ is a tree. Hence Θ' has a unique lobe and similarly the other host of $S\Gamma(w)$ has a unique lobe. Let Δ be a lobe of the path $p = \Delta_1, ...\Delta_n$ connecting the two hosts $\Theta = \Delta_1$ and $\Theta' = \Delta_n$. Since Θ is a host then $\Delta_i \mapsto \Delta_{i+1}$ for $1 \leq i < n$, whence $\Delta \to \Theta'$. Let $\hat{\Delta}$ be any lobe in $S\Gamma(w)$, then $\Theta' \to \hat{\Delta}$ because Θ' is a host, and since $\to$ is transitive, $\Delta \to \Theta'$ and $\Delta \to \hat{\Delta}$, it follows that Δ is a host. ■

One of the consequences of Lemma 4.1.3 is that a host of a complete t-opuntoid graph Γ with more than one lobe is the unique host of Γ.

4.2 Homomorphisms of hosts and Schützenberger graphs

We remark that several of the results presented in this section are very similar to the ones obtained for opuntoid graphs in the case of amalgams, the main difference being that the lobe tree of a t-opuntoid graph has a natural orientation given by the t-edges, while in the case of opuntoid graphs the orientation of the lobe trees is conventionally fixed as the one going from vertices representing lobes colored by 1 to vertices representing lobes colored by 2 (see [8]).

4.2.1 Extension of homomorphisms of hosts

The relation between homomorphism of t-opuntoid graphs and the homomorphism of their underlying lobe graphs is described in the following two lemmas. The main idea in proving the following two lemmas is that homomorphisms of t-opuntoid graphs preserve labelling, adjacency and they map lobes into lobes. Using these facts the following lemma is trivial.

Lemma 4.2.1. *Let Γ and Γ' be t-opuntoid graphs. Then every homomorphism $\gamma : \Gamma \to \Gamma'$ induces a homomorphism of their lobe graphs $\gamma' : \Upsilon(\Gamma) \to \Upsilon(\Gamma')$.*

Proof Labelling of the edges is preserved under any homomorphism of inverse word graphs. Thus, every homomorphism $\gamma : \Gamma \to \Gamma'$ maps t-edges of Γ to t-edges of Γ' and lobes of Γ to lobes of Γ'. Also adjacent lobes are mapped into adjacent lobes. It follows that a homomorphism γ induces a homomorphism of the lobe graphs $\gamma' : \Upsilon(\Gamma) \to \Upsilon(\Gamma')$. ■

This homomorphism of the underlying lobe graphs commutes with the operation of contracting lobes of a t-opuntoid graph Γ into vertices of $\Upsilon(\Gamma)$, i.e. if Δ' is the vertex of the lobe graph $\Upsilon(\Gamma)$ corresponding to a lobe Δ of Γ, then $\gamma'(\Delta') = (\gamma(\Delta))'$. We remark that the lobe graphs of Schützenberger graphs are oriented graphs and every graph homomorphism on oriented graph preserves orientation, so γ' preserve the orientation. We will usually denote the homomorphism γ' by γ and the vertex Δ' by Δ.
In particular, for isomorphism of t-opuntoid graphs we have the following lemma.

Lemma 4.2.2. *Let Γ, Γ' be t-opuntoid graphs and let $\psi : \Gamma \to \Gamma'$ be a homomorphism. Then ψ is an isomorphism if and only if it induces an isomorphism of their lobe trees and maps lobes isomorphically onto lobes.* ■

Corollary 4.2.1. *Let Γ be t-opuntoid graph, then a homomorphism on Γ is an automorphism if and only if it induces an automorphism of the lobe tree and maps lobes isomorphically onto lobes.* ■

Let Γ be a t-opuntoid automaton with finitely many lobes. It is straightforward to check that the automorphism group of Γ embeds in the automorphism group of some lobe of Γ because an automorphism of a finite tree fixes a vertex or an edge (see [3, Section 27.1.3]) and in case it fixes an edge (since we are working on oriented graphs) it fixes the vertices of this edge.

Lemma 4.2.3. *Let Γ be a t-opuntoid graph with finitely many lobes. Then $Aut(\Gamma)$ is embedded into the automorphism group of some lobe of Γ.*

Proof By Lemma 4.1.3 every automorphism of Γ induces an automorphism of the lobe tree $\Upsilon(\Gamma)$. Since Γ has finitely many lobes, $\Upsilon(\Gamma)$ is finite oriented tree. Every automorphism of finite oriented tree must stabilize some vertex, and therefore some vertex Δ of $\Upsilon(\Gamma)$ is mapped to itself. That is an automorphism of Γ induces an automorphism of the lobe Δ. Since automorphisms that agree on a vertex are equal, thus two automorphism of Γ induce the same automorphism of Δ (i.e. they agree on Δ), then they must be equal. Thus $Aut(\Gamma)$ embeds into $Aut(\Delta)$ and the embedding is defined by $\gamma \mapsto \gamma_{|\Delta}$. ■

Note that any isomorphism of t-opuntoid graph maps a host onto a host. The following two results show the importance of hosts on extending a homomorphism from a host to whole t-subopuntoid graphs.
Let Γ be a t-opuntoid graph and let Θ be a t-subopuntoid subgraph of Γ. For any lobe Δ not belonging to Θ the notation $\Theta \cup \Delta$ will denote the least t-subopuntoid subgraph of Γ which contains Θ and Δ. We have the following lemma.

Lemma 4.2.4. *Let Γ, Γ' be two complete t-opuntoid graphs and let $\Theta \subseteq \Gamma$ and $\Theta' \subseteq \Gamma'$ be two t-subopuntoid subgraphs both containing a host of Γ and Γ', respectively. Let $\phi : \Theta \to \Theta'$ be an isomorphism between Θ and Θ'. Let Δ, Λ be adjacent lobes of Γ where Δ belongs to Θ and Λ does not belong to Θ. Then there is a lobe Λ' in Γ' adjacent to $\Delta' = \phi(\Delta)$ such that ϕ can be extended to an isomorphism Φ from $\Theta \cup \Lambda$ onto $\Theta' \cup \Lambda'$.*

Proof Let (v, y) $((y, v))$ be an intersection pair between Δ and Λ. Since Θ contains a host and Λ is adjacent to Δ then $\Delta \mapsto \Lambda$, so $\Lambda = \Sigma/\rho$, where $(x, \Sigma, x) = \mathcal{A}(x, R; \varphi(f_A(v, \Delta)))$ or $(x, \Sigma, x) = \mathcal{A}(X, R; \varphi^{-1}(f_B(v, \Delta)))$, and ρ is the least equivalence relation on $V(\Sigma)$ such that $\rho \supseteq \{(x, \bar{v}) \mid \bar{v} \in N(x, \Sigma)\}$, the corresponding $V-$quotient Σ/ρ is deterministic, $N(x, \Sigma) =$

$\{x' \in V(\Sigma) | (x, \varphi(u), x')$ is a path in Σ with $u \in \mathcal{L}_A(v, \Delta)\}$ and $y = x\rho$. Since $f_A(v, \Delta) \in A$ and ϕ preserves labels, then in Δ' there is a loop based on $\phi(v)$ labelled by $f_A(v, \Delta)$. Since Γ' is complete there exists a lobe Λ' adjacent to Δ' with $(\phi(v), y')$ intersection pair between Δ' and Λ' (otherwise $(\phi(v), t, y')$ would be an extremal t-edge which will contradict the completeness of Γ'). assume that Δ', Λ' are adjacent lobes in Θ', then $\Delta = \phi^{-1}(\Delta')$ and $\phi^{-1}(\Lambda')$ are adjacent lobes in Θ at the same intersection vertex of Δ thus by the properties of t-opuntoid graph $\Theta^{-1}(\Lambda')$ and Λ are identified hence $\phi^{-1}(\Lambda') = \Lambda$ whence Λ is a lobe in Θ, a contradiction. Therefore, Λ' is not a lobe of Θ'. Again, since Θ' contains a host and Δ', Λ' are adjacent, then $\Delta' \mapsto \Lambda'$. So let $\Lambda' = \overline{\Sigma}/\sigma$ where $(\lambda, \overline{\Sigma}, \lambda) = \mathcal{A}(X, R, \varphi(f_A(\phi(v), \Delta')))$ and σ is the least equivalence relation on the vertices of $\overline{\Sigma}$ such that $\sigma \supseteq \{(\lambda, \bar{\bar{v}}) | \bar{\bar{v}} \in N(\lambda, \overline{\Sigma})\}$, $\overline{\Sigma}/\sigma$ is deterministic and $y' = \overline{\lambda}\sigma$. Now ϕ is an isomorphism and it preserves labels, hence $\mathcal{L}_A(v, \Delta) = \mathcal{L}_A(\phi(v), \Delta')$ and $f_A(\phi(v), \Delta') = f_A(v, \Delta)$, so $\overline{\Sigma} = S\Gamma(X, R, f_A(\phi(v), \Delta')) \cong S\Gamma(X, R, f_A(v, \Delta)) = \Sigma$. Therefore there is one-to-one correspondence between $N(x, \Sigma)$ and $N(\lambda, \overline{\Sigma})$ which implies that $(y, \Sigma/\rho, y) \cong (y', \overline{\Sigma}/\sigma, y')$. Let ψ be the isomorphism from $(y, \Sigma/v\rho, y)$ onto $(y', \overline{\Sigma}/\sigma, y')$ and define $\Phi : \Theta \cup \Upsilon \to \Theta' \cup \Upsilon'$ by

$$\Phi(\eta) = \begin{cases} \phi(\eta), & \text{if } \eta \in \Theta \\ \psi(\eta), & \text{if } \eta \in \Lambda \\ (\phi(v), t, \psi(y)), & \text{if } \eta = (v, t, y) \end{cases}$$

For the case $(x, \Sigma, x) = \mathcal{A}(x, R, \varphi^{(-1)}(f_B(v, \Delta)))$ follow similar argument as above to obtain Φ as follows

$$\Phi(\eta) = \begin{cases} \phi(\eta), & \text{if } \eta \in \Theta \\ \psi(\eta), & \text{if } \eta \in \Lambda \\ (\phi(v), t^{-1}, \psi(y)), & \text{if } \eta = (v, t^{-1}, y) \end{cases}$$

It obvious that Φ is an isomorphism from $\Theta \cup \Lambda$ onto $\Theta' \cup \Lambda'$ with $\Phi_{|\Theta} = \phi$. ■

From the above lemma we deduce the following property.

Proposition 4.2.1. *Let Γ, Γ' be two complete t-opuntoid graphs and let Θ and Θ' be t-subopuntoid subgraphs both containing a host of Γ and Γ' respectively. Let $\phi : \Theta \to \Theta'$ be an isomorphism. Then there is an isomorphism $\phi^* : \Gamma \to \Gamma'$ which extends ϕ.*

Proof Let $\mathcal{P}$ be the set of the pairs $(\psi, \bar{\Gamma})$ where $\bar{\Gamma}$ is a t-subopuntoid subgraph of Γ which contains Θ and let ψ be a graph homomorphisms that maps $\bar{\Gamma}$ isomorphically onto a t-subopuntoid subgraphs $\bar{\Gamma}'$ of Γ' which contains Θ' and such that $\psi_{|_\Theta} = \phi$. Then $\mathcal{P}$ is clearly nonempty since it contains at least the pair (ϕ, Θ). Moreover there is a natural partial order on $\mathcal{P}$ defined by $(\psi_1, \Gamma_1) \leq (\psi_2, \Gamma_2)$ if and only if Γ_1 is a t-subopuntoid subgraph of Γ_2 and $\psi_2|_{\Gamma_1} = \psi_1$. By the Hausdorff's maximality theorem there exists a maximal totally ordered chain $\Omega = \{(\psi_\alpha, \Gamma_\alpha)\}_{\alpha \in \{I\}}$; let (ϕ^*, Γ^*) be the pair defined by $\Gamma^* = \cup_\alpha \Gamma_\alpha$ and if $v \in V(\Gamma_\alpha)$ then $\phi^*(v) = \phi_\alpha(v)$. It is easy to show that the element (ϕ^*, Γ^*) belongs to $\mathcal{P}$ and in particular it is a maximal element of the chain Ω. If $\Gamma^* \neq \Gamma$ then there are two adjacent lobes Δ, Δ' with Δ in Γ^* and Δ' not in Γ^* whence by proposition 4.2.4 we can extend ϕ^* from Γ^* to $\Gamma^* \cup \Delta'$ but this contradicts the maximality of (ϕ^*, Γ^*), whence $\Gamma^* = \Gamma$ and ϕ^* is an isomorphism between the t-opuntoid graph Γ onto the t-subopuntoid subgraph $\phi^*(\Gamma)$. Let ${\phi^*}^{-1}$ be the isomorphism from $\phi^*(\Gamma)$ onto Γ. Suppose that $\phi^*(\Gamma) \neq \Gamma'$ since both $\Gamma, \phi^*(\Gamma)$ contain a host then we can repeat the above argument to extend ${\phi^*}^{-1}$ to an isomorphism ${\phi^*}^{-1}$ from Γ' onto Γ but this would contradicts the maximality of the pair (ϕ^*, Γ^*) whence $\phi^*(\Gamma) = \Gamma'$. ∎

Corollary 4.2.2. *Let Γ and Γ' be complete t-opuntoid graphs that have hosts. Let Σ be any host of Γ. Then every isomorphism of Σ onto some host of Γ' extends uniquely to an isomorphism of Γ onto Γ'.*

Proof Let γ be an isomorphism of Σ onto some host Σ' of Γ'. Then, by Proposition 4.2.1, γ extends uniquely to a homomorphism $\delta : \Gamma \to \Gamma'$. The inverse $\gamma^{-1} : \Sigma' \to \Sigma$ extends uniquely to a homomorphism $\delta' : \Gamma' \to \Gamma$. Since $(\delta \circ \delta')|_\Sigma = \gamma \circ \gamma^{-1} = id_\Sigma$, the composition $\delta \circ \delta'$ and the identity isomorphism id_Γ agree on Σ, and hence they must be equal. Thus the homomorphism δ is also an isomorphism, with inverse δ'. ∎

The following corollary follows directly from Corollary 4.2.2

Corollary 4.2.3. *Let Γ be a complete t-opuntoid graph that has hosts. Let Σ be any host of Γ. Then every isomorphism of Σ onto some host of Γ extends uniquely to an automorphism of Γ.* ∎

4.2.2 The Host Set

The main role in studying the automorphism group of a Schützenberger graph is played by the set of all its hosts, so we introduce the following definition.

Definition 4.2.1. The least t-subopuntoid subgraph of a t-opuntoid graph Γ which contains all the hosts of Γ is called a *host set* and is denoted by *Host*(Γ).

Lemma 4.2.5. *Let Γ be a complete t-opuntoid graph that has a host. Let Γ' denote the subgraph that consists of all the lobes of every host of Γ together with all connecting t-edges. Then Γ' is a t-subopuntoid subgraph of Γ .*

Proof If Γ has only one host then Γ' is this host, and is therefore a t-subopuntoid subgraph of Γ. If Γ has more than one host then, by Lemma 4.1.3 every host is a lobe, and the unique reduced lobe path connecting any two hosts consists entirely of hosts. Thus Γ' is a t-subopuntoid subgraph of Γ. ■

Note that the t-subopuntoid subgraph Γ' of Γ of the above lemma is the least t-subopuntoid subgraph that contains all the hosts of Γ, hence lemma 4.2.5 gives an equivalent definition of Host(Γ). Now, for a given complete t-opuntoid graph Γ, we give some results linking *Host*(Γ) to the automorphism group of Γ.

Theorem 4.2.1. *Let Γ be a complete t-opuntoid graph which posses a host. Then $Aut(\Gamma) \cong Aut(Host(\Gamma))$*

Proof We show that the map $\lambda : Aut(\Gamma) \to Aut(Host(\Gamma))$ defined by $\lambda(\psi) := \psi_{|_{Host(\Gamma)}}$ is a group isomorphism from $Aut(\Gamma)$ onto $Aut(Host(\Gamma))$. It is clear that for any $\psi \in Aut(\Gamma)$, if $\Delta \to \Delta'$ then $\psi(\Delta) \to \psi(\Delta')$ since ψ preserves edge labels and adjacency between lobes. Hence ψ sends hosts into hosts thus it is a homomorphism $\psi : Host(\Gamma) \to Host(\Gamma)$. Since $\psi^{-1} \in Aut(\Gamma)$ from the same argument $\psi^{-1} : Host(\Gamma) \to Host(\Gamma)$ whence $\psi_{|_{Host(\Gamma)}}$ belongs to $Aut(Host(\Gamma))$. It is clear that λ is a group homomorphism and since two automorphisms that coincide on a vertex are equal we deduce that λ is a monomorphism. Now we show that λ is surjective. Indeed each $\Phi \in Aut(Host(\Gamma))$ can be seen as an isomorphism between two t-subopuntoid subgraphs containing a host and so, since Γ is complete, from Proposition 4.2.1 Φ can be extended to an automorphism $\psi \in Aut(\Gamma)$ whence $\Phi = \psi_{|_{Host(\Gamma)}} = \lambda(\psi)$. ■

Corollary 4.2.4. *Let Γ be a complete t-opuntoid graph that has a host. If Γ has finite number of hosts then $Aut(\Gamma)$ is embedded into the automorphism group of some lobe.*

Proof If Γ has a finite number of hosts, the previous lemma implies that Γ' =Host(Γ) is a t-subopuntoid subgraph of Γ with finitely many lobes.

Then by Lemma 4.2.3, $Aut(\Gamma')$ is embedded into of some lobe Δ of Γ'. Hence, by Theorem 4.2.1, $Aut(\Gamma)$ is embedded into $Aut(\Delta)$. ∎

Corollary 4.2.5. *Let Θ_1,Θ_2 be two distinct hosts of the t-opuntoid graph Γ. Then for each isomorphism $\phi : \Theta_1 \to \Theta_2$ there is an automorphism $\Phi \in Aut(Host(\Gamma)) \cong Aut(\Gamma)$ which extends ϕ.*

Proof Since Γ has more than one host, from Lemma 4.2.1 we know that Θ_1,Θ_2 are two lobes. if $\phi : \Theta_1 \to \Theta_2$ is an isomorphism, hence from Proposition 4.2.1 there is an automorphism $\Phi \in Aut(\Gamma)$ which extends ϕ. Therefore from the proof of Theorem 4.2.3 we have that $\Phi \mid_{Host(\Gamma)}$ is an automorphism which extends ϕ. ∎

Lemma 4.2.6. *Let Δ_1, Δ_2 be two adjacent lobes that are hosts of $S\Gamma(w)$ and suppose that Δ_1 is a Schützenberger graph of some idempotent of S relative to the presentation $\langle X|R \rangle$ then Δ_2 is a Schützenberger graph of some idempotent of $A \cup B$.*

Proof Let (ν_1, ν_2) be an intersection pair of Δ_1 and Δ_2. Since Δ_1 is a host thus $\Delta_1 \mapsto \Delta_2$ with $\Delta_2 = \Sigma/\rho$ where Σ is the underlying graph of Schützenberger graph of $f = \varphi(f_A(\nu_1, \Delta_1))$ $(\varphi^{-1}(f_B(\nu_1, \Delta_1)))$ and ρ is the least equivalence relation defined on $V(\Sigma)$ such that $\{(\nu_2, \nu')|\nu' \in N(\nu_2, \Delta_2)\} \subseteq \rho$ and Σ/ρ is deterministic, we show that $N(\nu_2, \Delta_2) = \{\nu_2\}$ then $\Delta_2 = \Sigma$ is the Schützenberger graph of $f \in A$ (B). Suppose that $u \in \mathcal{L}_A(\nu_1, \Delta_1)$ $(\mathcal{L}_B(\nu_1, \Delta_1))$. Since Δ_1 is a Schützenberger graph we have $u \geq e(\nu_1, \Delta_1)$(minimum idempotent of S that labels a loop at ν_1 in Δ_1) thus by Proposition 3.2.1 $u \geq e(\nu_1, \Delta_1) = f_A(\nu_1, \Delta_1)$ $(f_B(\nu_1, \Delta_1))$ hence $\varphi(u) \geq \varphi(f_A(\nu_1, \Delta_1)) = f_B(\nu_2, \Delta_2)$ $(\varphi^{-1}(u) \geq \varphi^{-1}(f_B(\nu_1, \Delta_1)) = f_A(\nu_2, \Delta_2))$, it follows that $\varphi(u)(\varphi^{-1}(u))$ labels a loop in Σ at ν_2 whence $N(\nu_2, \Delta_2) = \{\nu_2\}$, therefore $\Delta_2 = \Sigma$ is a Schützenberger of f. ∎

Corollary 4.2.6. *If $S\Gamma(w)$ contains more than one host and $Host(S\Gamma(w))$ contains a lobe which is a Schützenberger graph relative to the presentation $\langle X|R \rangle$ then each lobe of $Host(S\Gamma(w))$ is Schützenberger graph of some idempotent of A (B).*

Proof Let Δ' and Δ be two lobes of $Host(S\Gamma(w))$ and assume that Δ is a Schützenberger graph relative to the presentation $\langle X|R \rangle$. Since $Host(S\Gamma(w))$ is connected there is a reduced lobe path in $Host(S\Gamma(w))$ connecting Δ' to Δ formed by

$$\Delta = \Delta_1, ..., \Delta_n = \Delta',$$

then by lemma 4.2.6 all the lobes $\Delta_{i's}$ are Schützenberger graphs of idempotents in A (B) relative to the presentation $\langle X|R \rangle$ and also Δ' is a

Schützenberger graph of an idempotent of A or (B) relative to the presentation $\langle X|R\rangle$. ∎

4.3 Number of hosts in Schützenberger graphs

In this section we characterize Schützenberger graphs of S^* with more than one host showing that each host is a Schützenberger graph of idempotents of $A \cup B$ and we characterize infinite host set.

Theorem 4.3.1. *Let $S^* = [S; A, B]$ be an HNN- extension of a finite inverse semigroup, let $w \in (X \cup X^{-1})^+$. The following are equivalent:*

1. *$S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ has more than one host.*
2. *Each host of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ is the Schützenberger graph of some idempotent of $A \cup B$ relative to the presentation $\langle X|R\rangle$ of S .*
3. *$ww^{-1}\mathcal{D}^{S^*} f$ for some idempotent $f \in E(A) \cup E(B)$.*

Proof 1) $\Rightarrow$ 3). Assume that $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ has more than one host. Then, by Lemma 4.1.3 there are (at least) two adjacent lobes of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ which are hosts. Let (ν_1, ν_2) be an intersection pair between these adjacent hosts Δ_1 and Δ_2, and assume without loss of generality that $f' = f_A(\nu_1, \Delta_1)$ and $f = \varphi(f') = f_B(\nu_2, \Delta_2)$. By definition of host $cl_{R_{HNN} \cup R}(\Delta_2) = S\Gamma(\overline{X}, R_{HNN} \cup R; w)$. Moreover, $\Delta_1 \mapsto \Delta_2$, so by Theorem 3.2.1 $(\nu_2, S\Gamma(\overline{X}, R_{HNN} \cup R; w), \nu_2) \cong (x\Xi, S\Gamma(\overline{X}, R_{HNN} \cup R; f)/\Xi, x\Xi)$, where $(\nu_2, \Delta_2, \nu_2) \cong \Sigma/\rho, \Sigma = S\Gamma(X, R; f)$ and $\nu_2 = x\rho$. Let e be an idempotent in S^* labelling a loop based at ν_2 in $cl_{R_{HNN} \cup R}(\Delta_2) = S\Gamma(\overline{X}, R_{HNN} \cup R; w)$. By Proposition 3.2.3 e labels also a loop based at x in $S\Gamma(\overline{X}, R_{HNN} \cup R; f)$, whence $e \geq f$. So f is the minimum idempotent labelling a loop based at ν_2 in $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$ whence $S\Gamma(\overline{X}, R_{HNN} \cup R; w) = S\Gamma(\overline{X}, R_{HNN} \cup R; ww^{-1}) \cong S\Gamma(\overline{X}, R_{HNN} \cup R; f)$, then by Theorem 1.3.3 $ww^{-1}\mathcal{D}^{S^*} f$.
3) $\Rightarrow$ 2). Let $f \in E(A) \cup E(B)$. Without loss of generality we can assume $f \in E(A)$ (the case $f \in E(B)$ is similar). Put $\Delta' = S\Gamma(X, R; f)$. By Theorem 1.3.3 $S\Gamma(\overline{X}, R_{HNN} \cup R; ww^{-1}) \cong S\Gamma(\overline{X}, R_{HNN} \cup R; f)$. Obviously $S\Gamma(\overline{X}, R_{HNN} \cup R; f)$ is obtained by iterated applications of Construction 5 to Δ', thus Δ' is a host of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$. Now let Δ be any host of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$. Let $P : \Delta' = \Delta_1, \Delta_2, \ldots, \Delta_n = \Delta$ be a reduced lobe path of length n connecting Δ' with Δ. We prove that Δ is a Schützenberger graph of some idempotent of $A \cup B$ by induction on n. If $n = 1$ the statement is trivially true. Assume that for

all $1 \leq k < n$, Δ_k is a Schützenberger graph of some idempotent of $A \cup B$. Since Δ' and $\Delta_n = \Delta$ are hosts, by Proposition 4.1.3 Δ_{n-1} is a host and by induction hypothesis it is a Schützenberger graph of some idempotent u of $A \cup B$, i.e $\Delta_{n-1} = S\Gamma(X, R; u)$. Let (ν_{n-1}, ν_n) be an intersection pair of Δ_{n-1} and Δ_n. Since $\Delta_{n-1} \mapsto \Delta_n$, then by Proposition 3.2.1 $e(\nu_n, \Delta_n) = f_B(\nu_n, \Delta_n) = \varphi(f_A(\nu_{n-1}, \Delta_{n-1}))$. Since Δ_{n-1} is a Schützenberger graph, $(\nu_{n-1}, \Delta_{n-1}, \nu_{n-1}) \cong \mathcal{A}(X, R; f_A(\nu_{n-1}, \Delta_{n-1}))$ and so by Corollary 4.2.6 $\Delta_n \cong S\Gamma(X, R; \varphi(f_A(\nu_{n-1}, \Delta_{n-1})))$.
2) $\Rightarrow$ 1). Let Δ be a host of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$. Then $\Delta \cong S\Gamma(X, R; f)$ for some $f \in E(A) \cup E(B)$. Assume as before $f \in E(A)$. Then $f = f_A(\nu, \Delta) = e(\nu, \Delta)$ for some $\nu \in V(\Delta)$. Applying Construction 5 at ν one gets a new lobe $\Delta' \cong S\Gamma(X, R; \varphi(f))$, therefore, by Proposition 3.2.1 we also have $\Delta' \mapsto \Delta$. Let Λ be any lobe of $S\Gamma(\overline{X}, R_{HNN} \cup R; w)$. Since Δ is a host, then Λ feeds off Δ that in turns directly feeds off Δ'. So Λ feeds off Δ' and hence Δ' is a host. ■

Proposition 4.3.1. $S\Gamma(w)$ *has more than one host and each lobe is a Schützenberger graph of some idempotent in* $A \cup B$ *relative to the presentation* $\langle X|R\rangle$ *if and only if* w *is* $\mathcal{J}^{S^*}$*-related to some* $w' \in A \cup B$.

Proof ($\Leftarrow$) If $w\mathcal{J}^{S^*}w'$ with $w' \in A \cup B$, then by Theorem 1.3.3 $S\Gamma(w) \cong S\Gamma(w')$ and so $Host(S\Gamma(w)) \cong Host(S\Gamma(w'))$. Since $w' \in A \cup B$, then we can build $\mathcal{A}(\overline{X}, R_{HNN} \cup R; w')$ either by starting from the linear automaton of the word w' or starting from the linear automaton of the word $\varphi(w')$ ($\varphi^{-1}(w')$). In the first case $Core(w') = \mathcal{A}(X, R; w')$, in the second case $Core(w') = \mathcal{A}(X, R; \varphi(w'))$ ($\mathcal{A}(X, R; \varphi^{-1}(w'))$). Since the core contains a host, then both $\mathcal{A}(X, R; w')$ and $\mathcal{A}(X, R; \varphi(w'))(\mathcal{A}(X, R; \varphi^{-1}(w')))$ are hosts of $S\Gamma(w')$, hence $S\Gamma(w)$ has more than one host. Since $\mathcal{A}(X, R; w')$ is a Schützenberger graph and a host, then by Corollary 4.2.6 $Host(S\Gamma(w'))$ has lobes which are Schützenberger graphs of idempotents of $A \cup B$ relative to the presentation $\langle X|R\rangle$ and so is $Host(S\Gamma(w))$.
Conversely, suppose that $S\Gamma(w)$ has more than one host and $Host(S\Gamma(w))$ has lobes that are Schützenberger graphs relative to the presentation $\langle X|R\rangle$, hence there is a lobe Δ of $Host(S\Gamma(w))$ which is both a host and a Schützenberger graph relative to the presentation $\langle X|R\rangle$. Moreover, by Lemma 4.2.6 Δ is a Schützenberger graph of some element $w' \in A \cup B$ relative to the presentation $\langle X|R\rangle$. Apply Construction 5 iteratively to the automaton (α, Δ, β) to obtain the complete automaton $S\Gamma(w')$ relative to the presentation $\langle \overline{X}|R_{HNN} \cup R\rangle$. Since Δ is a host of $S\Gamma(w)$, then $S\Gamma(w') \cong S\Gamma(w)$ whence by Theorem 1.3.3 $w\mathcal{J}^{S^*}w'$. ■

Proposition 4.3.2. *If $w \in S$ then $Host(S\Gamma(w))$ has lobes that are Schützenberger graphs relative to the presentation $\langle X|R\rangle$.*

Proof Since $w \in S$ then the underlying graph of $\mathcal{A}(X, R; w)$ is a host of $S\Gamma(w)$. So if $S\Gamma(w)$ has a unique host then it is a Schützenberger graph relative to the presentation $\langle X|R\rangle$. If $S\Gamma(w)$ has more than one host then each host is a lobe. Since $\mathcal{A}(X, R; w)$ is a host and a lobe, then all the hosts are Schützenberger graphs relative to the presentation $\langle X|R\rangle$. ■

We can conclude from Proposition 4.3.2 and Lemma 4.1.3 the following:

Corollary 4.3.1. *Let $w \in S$ then*

- *$S\Gamma(X, R; w)$ relative to the presentation $\langle X|R\rangle$ embeds onto a host of $S\Gamma(w)$ and each host has a unique lobe.*
- *If $w' \in S^*$ with $w\mathcal{J}^{S^*}w'$ then $Host(S\Gamma(w))$ has lobes that are Schützenberger graphs relative to the presentation $\langle X|R\rangle$.*

Proof The first statement follows from the proof of Proposition 4.3.2, Lemma 4.1.3 and by the fact that if one host of $S\Gamma(w)$ is a lobe then each host is also a lobe.
For the second statement, let $w\mathcal{J}^{S^*}w'$ then $S\Gamma(w) \cong S\Gamma(w')$ so $Host(S\Gamma(w)) \cong Host(S\Gamma(w'))$ thus the result is a consequence of Proposition 4.3.2. ■

Corollary 4.3.2. *If $w \in A \cup B$ then $Host(S\Gamma(w))$ has more than one lobe and each host is a lobe which is a Schützenberger graph of elements in $A \cup B$.*

Proof By the first part of the proof of Proposition 4.3.2 $S\Gamma(w)$ has more than one host and at least one of them is a Schützenberger graph of an element in $A \cup B$ hence by Lemma 4.2.6 and by Lemma 4.1.3 each host is a lobe and a Schützenberger graph of an element in $A \cup B$. ■

4.4 Infinite Host Set

The rest of this chapter is devoted to characterize infinite Schützenberger graphs.

Definition 4.4.1. Let Δ, Δ' be two lobes of a t-opuntoid graph such that $\Delta' = \phi(\Delta)$ for some isomorphism ϕ. Let $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ be the reduced lobe path connecting Δ to Δ' and let (ν_0, ν_1) be an intersection pair between the lobes Δ_0 and Δ_1 with $\nu_0 \in V(\Delta_0)$ and $\nu_1 \in V(\Delta_1)$. The

isomorphism ϕ is called a *shift-isomorphism* (and Δ, Δ' are called shift-isomorphic by ϕ) if $\phi(\nu_0) \notin V(O_P(\Delta_{n-1})) \cap V(\Delta_n)$.
The lobes Δ, Δ' are called *successive isomorphic* lobes if no Δ_i $(0 < i < n)$ is isomorphic to Δ_0.

We have the following lemma.

Lemma 4.4.1. *Let $S^* = [S; A, B]$ with S finite inverse semigroup and let $e \in E(S^*)$ such that $e\mathcal{D}^{S^*} f$ for some idempotent $f \in E(A)$ $(E(B))$. Let Δ, Δ' be two distinct lobes of $Host(S\Gamma(e))$ such that $\Delta' = \phi(\Delta)$ for some $\phi \in Aut(Host(S\Gamma(e)))$. Let $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ be the reduced lobe path connecting Δ to Δ'. Then either for some j with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$ $\phi|_{\Delta_j}$ is a shift-isomorphism or, for all j with $0 \leq j \leq \lfloor \frac{n}{2} \rfloor$, $\phi(\Delta_j) = \Delta_{n-j}$.*

Proof By Lemma 4.1.3 and Theorem 4.3.1, each Δ_h, $0 \leq h \leq n$, is a host which is the Schützenberger graph of some idempotent of $E(A) \cup E(B)$ relative to the presentation $\langle X|R \rangle$. We prove the statement by induction on $n \geq 2$. The base of induction is trivial. If $\phi|_{\Delta_0}$ is a shift-isomorphism of Δ_0 onto Δ_n the statement is trivially true. So assume that $\phi|_{\Delta_0}$ is not a shift-isomorphism. Let (ν_0, ν_1) be an intersection pair between the lobes Δ_0 and Δ_1 with $\nu_0 \in V(\Delta_0)$ and $\nu_1 \in V(\Delta_1)$, then $\phi(\nu_0) \in V(O_P(\Delta_{n-1})) \cap V(\Delta_n)$. Moreover by Proposition 3.2.1 we get $f_0 = f_A(\nu_0, \Delta_0) = e(\nu_0, \Delta_0) \in E(A)$ (the case $f_0 \in E(B)$ is similar) thus $f_n = f_A(\phi(\nu_0), \Delta_n) = e(\phi(\nu_0), \Delta_n) \in E(A)$, and if $(\phi(\nu_0), \phi(\nu_1))$ denotes the intersection pair between the lobes Δ_{n-1} and Δ_n with $\phi(\nu_0) \in V(\Delta_n)$ and $\phi(\nu_1) \in V(\Delta_{n-1})$, we have $(\phi(\nu_1), \Delta_{n-1}, \phi(\nu_1)) \cong \mathcal{A}(X, R; \varphi(f_0)) \cong (\nu_1, \Delta_1, \nu_1)$, hence $\phi(\Delta_1) = \Delta_{n-1}$. Since the reduced lobe path from Δ_1 to Δ_{n-1} has length $n-1$ the statement holds by induction hypothesis. ■

Proposition 4.4.1. *Let $S^* = [S; A, B]$ with S finite inverse semigroup and let $e \in E(S^*)$ such that $e\mathcal{D}^{S^*} f$ for some idempotent $f \in E(A) \cup E(B)$. Then the following are equivalent*

1. *$Host(S\Gamma(e))$ is infinite;*
2. *$Host(S\Gamma(e))$ has infinitely many lobes;*
3. *there are two isomorphic hosts of $S\Gamma(e)$ which are not successive isomorphic lobes;*
4. *there is a shift-isomorphism between two hosts of $S\Gamma(e)$.*

Proof The equivalence between 1) and 2) is trivial.
1) $\Rightarrow$ 3). By Theorem 4.3.1 each lobe of $Host(S\Gamma(e))$ is a Schützenberger

graph of some idempotent of $E(A) \cup E(B)$ relative to the presentation $\langle X|R\rangle$. Since S is finite, there are finitely many Schützenberger graphs of idempotents of $E(A) \cup E(B)$ relative to the presentations $\langle X|R\rangle$. Since all the lobes are finite, each lobe has finitely many adjacent lobes and so the degree of each vertex of the lobe tree $\Upsilon(Host(S\Gamma(e)))$ is finite. Therefore there is an infinite reduced lobe path in $\Upsilon(Host(S\Gamma(e)))$ in which there are at least three isomorphic lobes, whence there are two isomorphic hosts which are not successive.

$3) \Rightarrow 4)$. Suppose that $S\Gamma(e)$ has two isomorphic hosts Δ and Δ' which are not successive isomorphic lobes. Thus, in the reduced lobe path $\Delta = \Delta_0, \Delta_1, \dots, \Delta_n = \Delta'$ connecting them, there is a lobe Δ_h with $1 \leq h \leq n-1$ isomorphic to both Δ and Δ'. We can assume without loss of generality that no Δ_j, with $1 \leq j \leq n-1, j \neq h$, is isomorphic to Δ. Since adjacent lobes are not isomorphic then $n \geq 4$ is even and let $k = n/2$. Let ϕ be the isomorphism sending Δ onto Δ'. By Propositions 4.2.1 and 4.2.1 ϕ can be extended to an automorphism $\overline{\phi} \in Aut(Host(S\Gamma(e)))$. Assume that for each lobe Λ of $Host(S\Gamma(e))$ $\overline{\phi}|_{\Lambda}$ is not a shift-isomorphism of Λ onto some host Λ'. Then by Lemma 4.4.1 for all j with $0 \leq j < k$, $\overline{\phi}(\Delta_j) = \Delta_{2k-j}$. Then $k = h$, otherwise both Δ_h and Δ_{n-h} would be isomorphic to Δ, hence $\overline{\phi}|_{\Delta_k} \in Aut(\Delta_k)$. Let (ν_{k-1}, ν_k) be an intersection pair between the lobes Δ_{k-1} and Δ_k with $\nu_{k-1} \in V(\Delta_{k-1})$ and $\nu_k \in V(\Delta_k)$, then $(\overline{\phi}(\nu_{k-1}), \overline{\phi}(\nu_k))$ is an intersection pair between the lobes Δ_k and some lobe Δ'_{k+1} adjacent to Δ_k with $\Delta'_{k+1} \neq \Delta_{k-1}$. Now let $\psi : \Delta \to \Delta_k$ be an isomorphism. If ψ is a shift-isomorphism then we are done, otherwise if (ν_0, ν_1) is an intersection pair between the lobes Δ_0 and Δ_1, then $(\psi(\nu_0), \psi(\nu_1))$ is an intersection pair between the lobes Δ_k and Δ_{k-1}. Using the fact that ψ preserves the labelling, it is easy to see that ψ is actually a bijection between the set of all intersection pairs of $V(\Delta_0)$, $V(\Delta_1)$ and the set of all intersection pairs of $V(\Delta_k)$, $V(\Delta_{k-1})$. Thus let (ν'_0, ν'_1) be an intersection pair between $V(\Delta_0)$ and $V(\Delta_1)$ such that $(\psi(\nu'_0), \psi(\nu'_1))$ is an intersection pair between Δ_k and Δ_{k-1}. Hence $(\overline{\phi}(\psi(\nu'_0)), \overline{\phi}(\psi(\nu'_k)))$ is an intersection pair between Δ_k and Δ'_{k+1} , thus $(\overline{\phi}(\psi(\nu'_0)), \overline{\phi}(\psi(\nu'_k)))$ is not an intersection pair of Δ_{k-1} and Δ_k. Therefore, the map $\overline{\phi} \circ \psi : \Delta \to \Delta_t$ defined by $(\overline{\phi} \circ \psi)(\nu) = \overline{\phi}(\psi(\nu))$ is a shift-isomorphism from Δ to Δ_k.

$4) \Rightarrow 2)$. Assume by contradiction that $Host(S\Gamma(e))$ has two shift-isomorphic lobes, and finitely many lobes. Let Δ, Δ' be two shift-isomorphic lobes by ϕ in $Host(S\Gamma(e))$ such that the reduced lobe path $\Delta = \Delta_0, \Delta_1, \dots, \Delta_n = \Delta'$ from Δ to Δ' is of maximal length. By Lemma 4.1.3 and Theorem 4.3.1 each Δ_j, $0 \leq j \leq n$ is a host and the Schützenberger graph of some idempotent of A (B). Moreover by Proposition 3.2.1, $(\nu_{j-1}, \Delta_{j-1}, \nu_{j-1}) \cong \mathcal{A}(X, R; f)$, $(\nu_j, \Delta_j, \nu_j) \cong \mathcal{A}(X, R, \varphi(f))$ where (ν_{j-1}, ν_j) is an intersection pair between Δ_{j-1}, Δ_j, and $f = f_A(\nu_{j-1}, \Delta_{j-1}) = e(\nu_{j-1}, \Delta_{j-1}) \in A$, and $\varphi(f) = f_B(\nu_j, \Delta_j) = e(\nu_j, \Delta_j) \in B$. By Propositions 4.2.1 and Theorem 4.2.1 the isomorphism $\phi : \Delta \to \Delta'$ can be extended to an automorphism $\overline{\phi} \in Aut(Host(S\Gamma(e)))$. We prove by induction on h that, for all h with $0 \leq h \leq n$, $\overline{\phi}$ maps the t-subopuntoid subgraph $\Theta_h = \bigcup_{0 \leq j \leq h} \Delta_j$ of $Host(S\Gamma(e))$ onto a t-subopuntoid subgraph of $Host(S\Gamma(e))$ whose lobes, except eventually Δ_n, are all different from the lobes of Θ_h. Moreover $\overline{\phi}$ is a shift-isomorphism between Δ_h and $\overline{\phi}(\Delta_h)$. The base of induction is trivial. So let $\Theta_{h-1} = \bigcup_{0 \leq j \leq h-1} \Delta_j$ and put $\Delta_{n+j} = \overline{\phi}(\Delta_j)$ for all $0 \leq j \leq h-1$. By Lemma 4.2.2 $\overline{\phi}(\Theta_{h-1}) = \bigcup_{0 \leq j \leq h-1} \Delta_{n+j}$ and for all j with $0 \leq j \leq h-2$, Δ_{n+j} is adjacent to Δ_{n+j+1}. Moreover by induction hypothesis Θ_{h-1} and $\overline{\phi}(\Theta_{h-1})$ have disjoint sets of lobes and $\overline{\phi}$ is a shift-isomorphism of Δ_{h-1} onto Δ_{n+h-1}. Let (ν_{h-1}, ν_h) be an intersection pair between Δ_{h-1} and Δ_h with $\nu_h \in V(\Delta_h)$ and $\nu_{h-1} \in V(\Delta_{h-1})$ and let $f = f_A(\nu_{h-1}, \Delta_{h-1})$ thus $\varphi(f) = f_B(\nu_h, \Delta_h)$. By Lemma 4.2.4 $\overline{\phi}$ maps $\Theta_h = \Theta_{h-1} \cup \Delta_h$, onto $\overline{\phi}(\Theta_{h-1}) \cup S\Gamma(X, R; \varphi(f_A(\overline{\phi}(\nu_{h-1}), \overline{\phi}(\Delta_{h-1}))))$. So $S\Gamma(X, R; \varphi(f_A(\overline{\phi}(\nu_{h-1}), \overline{\phi}(\Delta_{h-1})))) = \Delta_{n+h}$ does not coincide with any lobe of Θ_h. Moreover, if (ν_h, ν_{h+1}) is an intersection pair between Δ_h and Δ_{h+1} then $(\overline{\phi}(\nu_h), \overline{\phi}(\nu_{h+1}))$ is not an intersection pair between Δ_{n+h-1} and Δ_{n+h}, so $\overline{\phi}$ is a shift isomorphism between Δ_h and Δ_{n+h}. In particular for $h = n$, $\overline{\phi}$ is a shift-isomorphism of Δ_n onto Δ_{2n} and $\overline{\phi}^2$ is a shift-isomorphism of Δ_0 onto Δ_{2n}, against the assumption that the reduced lobe path connecting $\Delta = \Delta_0$ to $\Delta' = \Delta_n$ is a path of maximal length among the reduced lobe paths connecting two hosts which are isomorphic under a shift-isomorphism. Then $Host(S\Gamma(e))$ has infinitely many lobes. ■

From the above proposition we derive the following corollary.

Corollary 4.4.1. *Let $S^* = [S; A, B]$ with S finite inverse semigroup and let $e \in E(S^*)$ such that $e\mathcal{D}^{S^*} f$ for some idempotent $f \in E(A) \cup E(B)$. Then $Host(S\Gamma(e))$ is infinite if and only if in $Host(S\Gamma(e))$ there is a reduced lobe path $P : \Delta_0, \ldots, \Delta_{t-1}, \Delta_t, \Delta_{t+1}, \ldots \Delta_{2t}$ with $\Delta_0 \cong \Delta_t \cong \Delta_{2t}$.*

Proof By Proposition 4.4.1 if $Host(S\Gamma(e))$ is infinite then there is a shift-isomorphism between two lobes of $Host(S\Gamma(e))$. Let Δ and Δ' be such lobes and let $\phi : \Delta \to \Delta'$ be a shift-isomorphism. If P

is the reduced lobe path from Δ to Δ', by the same argument of the proof of implication $4) \Rightarrow 2)$ of Proposition 4.4.1, $P \cup \phi(P)$ is a reduced lobe path in $Host(S\Gamma(e))$ satisfying the statement. Conversely, let $P : \Delta_0, \ldots, \Delta_{t-1}, \Delta_t, \Delta_{t+1}, \ldots \Delta_{2t}$ with $\Delta_0 \cong \Delta_t \cong \Delta_{2t}$ be a reduced lobe path of $Host(S\Gamma(e))$, then in P there are two non successive isomorphic lobes, and so $Host(S\Gamma(e))$ is infinite by Proposition 4.4.1. ■

CHAPTER 5

Maximal Subgroups

WE use the description of the Schützenberger automata for HNN-extensions of finite inverse semigroups given by Rodaro, Cherubini in [40] which we summarized in Section 1.4 to obtain algebraic results for such HNN-extension. Schützenberger automata, in the case of an HNN-extension of a finite inverse semigroup, are automata with special structure possessing finite subgraphs, that contain all essential information about the automaton. Using this crucial fact, and the Bass-Serre theory as in Jacayova [26], we show that a maximal subgroup of an HNN-extension of a finite inverse semigroup is isomorphic to the fundamental group of a graph of groups, moreover when the idempotent of the maximal subgroup is in S then we give a presentation of that maximal subgroup as fundamental group based on the $\mathcal{D}$-structure of S, otherwise when idempotent of the maximal subgroup does not belong to S then we show that the maximal subgroup is isomorphic to a certain subgroup of the original semigroup S. Moreover in the first case of idempotents in S we can distinguish between idempotents in $A \cup B$ or in $S \setminus (A \cup B)$ to get more information along the line of Cherubini, Jacayova and Rodaro [7]. We start the chapter with a brief review of relevant aspects of the Bass-Serre theory, in particular the group action on graphs and the definition of graph of groups. In Section 5.3 we give the construction of the fundamental group of a graph of groups based first on t-opuntoid graphs of words of S^* and the other based on the $\mathcal{D}$-structure of the original inverse semigroup S. The last section is devoted to proof the main results concerning the two sorts of the maximal subgroups of S^*.

5.1 Group action

In this section we give the definition and a brief description of group action on sets, on graphs and on a special type of graphs called quotient graphs.

5.1.1 On Sets

Definition 5.1.1. Let G be a group and S be a non-empty set. The *left action* of G on S is a map $\rho : G \times S \to S$ denoted by $\rho(g, s) = g \cdot s$ and satisfying the two conditions:

1- $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s, \quad \forall g_1, g_2 \in G, \forall s \in S$, where $g_1 g_2$ denotes the operation in G,

2- $1 \cdot s = s \;, \quad \forall s \in S$, where 1 denotes the identity of G.

The right action is defined similarly.
One can easily verify that each fixed $g \in G$ acts as a bijection map ρ_g on S hence the (left) action of g is a permutation on S, it follows that there is a group homomorphism from G into the full symmetric group $Sym(S)$. If this homomorphism is injective then the action is called *faithfull*. The following notion are useful

- Orbit: Let $s \in S$, the *orbit* of s is the set $G \cdot s = \{g \cdot s | g \in G\}$. Define the equivalence relation $\sim$ on S by $s_1 \sim s_2$ if and only if there exists $g \in G$ such that $s_1 = g \cdot s_2$, thus each orbit is an equivalence class of S with respect to $\sim$.

- Stabilizer: Let $s \in S$, the *stabilizer* of s in G, denoted by $Stab_G(s)$ or s^G, is the set $Stab_G(s) = \{g \in G | g \cdot s = s\}$ which is a subgroup of G.

It is easy to check that if two elements $x, y \in S$ are in the same orbit, i.e. $x = g \cdot y$ for some $g \in G$, then the stabilizers $Stab_G(x)$ and $Stab_G(y)$ are isomorphic, according to the relation $Stab_G(y) = gStab_G(x)g^{-1}$, i.e. if $x \in G \cdot y$ then $Stab_G(x) = Stab_G(y)$.

5.1.2 On Graphs

The action of a group on a graph X is a simultaneous action of the group on the set of the vertices and the set of edges of a graph that respects the graph structure. More precisely let $X = (V(X), Ed(X))$ be a graph, an action of the group G acts on X is an action on the set $V(X) \cup Ed(X)$ which is compatible with the structure of the graph X, in other words the action preserves incidence, i.e., for each $y \in Ed(X)$, $0(g \cdot y) = g \cdot 0(y)$, $1(g \cdot y) = g \cdot 1(y)$ and $g \cdot \bar{y} = \overline{g \cdot y}$, where as usual $\bar{y}$ is the inverse edge of y. In addition the action will be called *without inversions* if $\bar{y} \neq g \cdot y$ for all $y \in Ed(X)$ and all $g \in G$. In other words G acts on X by an automorphism. In our work we will consider actions without inversions and groups acting on trees.

5.1.3 On Quotient Graphs [11, 43]

Let G be a group acting on the graph X without inversions. The quotient graph of the action of G on X is the graph denoted by $(G \setminus X)$ whose vertex set is the set of the orbits of $V(X)$ denoted by $V(G \setminus X) = \{G \cdot v | v \in V(X)\}$, and whose edge set is the set of orbits of $Ed(X)$, it is denoted by $Ed(G \setminus X) = \{G \cdot y | y \in Ed(X)\}$ such that $0(G \cdot y) = G \cdot 0(y)$, $1(G \cdot y) = G \cdot 1(y)$ and $\overline{G \cdot y} = G \cdot \bar{y}$ for all $y \in Ed(X)$. These graphs can possibly have multiple edges and loops. Besides that, there are two maps associated to the quotient graph $(G \setminus X)$: a map from $V(X)$ onto $V(G \setminus X)$ defined by $v \longmapsto G \cdot v$ and a map from $Ed(X)$ onto $Ed(G \setminus X)$ defined by $y \longmapsto G \cdot y$. We will focus on actions when the graph X is a tree, we conclude this section by recalling the following theorem from [43]

Theorem 5.1.1. *Let G be a group acting on a connected tree X without inversions. Then every subtree Υ of $(G \backslash X)$ can be lifted to a subtree of X.* ■

5.2 Fundamental Group of Graph of Groups

The corner stone in building fundamental group is the concept of graph of groups which is defined as follows:

Definition 5.2.1 (Graph of Groups)**.** Let X be a connected non-empty graph with vertex set $V(X)$ and edge set $Ed(X)$. A *Graph of Groups* $(\mathcal{G}(-), X)$ consists of the graph X and a map $\mathcal{G}$ that assigns a vertex group G_v to each vertex $v \in V(X)$, and an edge group G_y to each edge $y \in Ed(X)$ with a family of monomorphisms $\sigma_y : G_y \to G_{0(y)}, y \in Ed(X)$ subjected to the condition $G_y = G_{\bar{y}}$.

We remark that a family of monomorphisms $\tau_y : G_y \to G_{1(y)}, y \in Ed(X)$ can be used instead of the family $\sigma_y : G_y \to G_{0(y)}, y \in Ed(X)$ under the condition $\sigma_y = \tau_{\bar{y}}$.

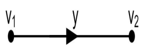
Serre in [43] defines the fundamental group based on a graph of groups in two equivalent ways, the first depends on one vertex of the graph and the other uses a spanning tree. For our work we will follow the second definition and therefore the fundamental group will be defined as following.

Definition 5.2.2 (Fundamental Group)**.** Let $(\mathcal{G}(-), X)$ be a graph of groups and let Υ be a maximal (spanning) subtree of the underlying graph of X. Consider the quotient group $\mathcal{F}(\mathcal{G}(-), X) = \langle\{G_v | v \in V(X)\} \cup \{y | y \in Ed(X)\}\rangle / N$, where $\langle\{G_v | v \in V(X)\} \cup \{y | y \in Ed(X)\}\rangle$ is the group generated by the disjoint union of the vertex groups and the edges of X and N is the normal closure of the set $\{\bar{y}y | y \in Ed(X)\} \cup \{y^{-1}\sigma_y(a)y\tau_y^{-1}(a) | y \in Ed(X), a \in G_y\}$. The fundamental group $\pi(\mathcal{G}(-), X, \Upsilon)$ of $(\mathcal{G}(-), X)$ is the quotient group $\mathcal{F}(\mathcal{G}(-), X)/M$, where M is the normal closure of the set $\{y | y \in \Upsilon\}$; that is, $\pi(\mathcal{G}(-), X, \Upsilon)$ is obtained from $\mathcal{F}(\mathcal{G}(-), X)$ by adding the relation $\{y = 1 | y \in \Upsilon\}$.

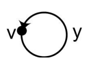
Note that, according to proposition 18 of [11], $\pi(\mathcal{G}(-), X, \Upsilon)$ is independent (up to isomorphism) of the choice of the maximal subtree Υ; therefore we may denote the fundamental group by $\pi(\mathcal{G}(-), X)$.

Example 5.2.1. *Let X be any graph, and let $G_v = G_y =< 1 >$ for every vertex and every edge of X, let $\mathcal{O}$ be the orientation of the edges of the graph X, i.e., $\mathcal{O} \subseteq Ed(X)$ such that for each pair $y, \bar{y} \in Ed(X)$ one and only one of them belongs to $\mathcal{O}$. Then the fundamental group $\pi(\mathcal{G}(-), X) =< \{y | y \in A\} >$ with no relation, where $A = \mathcal{O} - (Ed(X) \cap Ed(\Upsilon))$. That is, $\pi(\mathcal{G}(-), X)$ is the free group whose generators are the positive edges of X not in the maximal tree.*

Example 5.2.2. *Let X be a tree and let $G_y =< 1 >, \forall y \in Ed(X)$, take Υ to be X then $\pi(\mathcal{G}(-), X) = *_{v \in V(X)} G_v$ is the free product of the groups $G_v, v \in V(X)$.*

Example 5.2.3. *Let X be the segment* 

then $\pi(\mathcal{G}(-), X) = [G_{v_1}, G_{v_2}; G_y]$ is the free product of the groups G_{v_1} and G_{v_2} amalgamating G_y via the monomorphisms σ_y and τ_y.

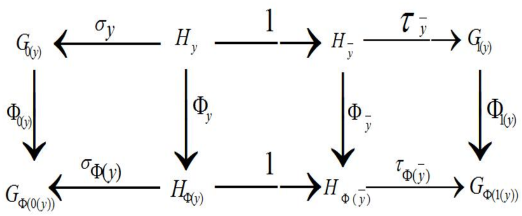
Example 5.2.4. *Let X be the loop* 

then $\pi(\mathcal{G}(-), X) = \langle G_v, y | y\sigma_y(g)y^{-1} = \tau_y(g)\rangle$, $g \in G_y$ is the HNN-extension of the group G_v via the monomorphisms $\sigma_y : G_y \to G_{0(y)} = G_v$ and $\tau_y : G_y \to G_{1(y)} = G_v$.

These examples give an indication of the connection between graph of groups and fundamental group on one side and on the other side with free groups, free product, amalgamated free product of groups, HNN-extension of groups.
The fundamental group contains the vertex groups and the edge groups as subgroups. This important property is consequence of Bass and Serre theorem and is stated in the following theorem.

Theorem 5.2.1. *Let $(\mathcal{G}(-), X)$ be a graph of groups. Then each vertex group G_v and each edge group G_y is embedded into the fundamental group $\pi(\mathcal{G}(-), X)$.* ■

Let $(\mathcal{G}(-), X)$ and $(\mathcal{H}(-), Y)$ be two graphs of groups. An *isomorphism* between them is a graph isomorphism $\Phi : X \to Y$ and a family of group isomorphisms $\{\Phi_v : G_v \to H_{\Phi(v)} | v \in V(X)\}$ and $\{\Phi_y : G_y \to H_{\Phi(y)} | y \in Ed(X)\}$, where $G_{v's}$, $G_{y's}$ are respectively the vertex and edge groups of $(\mathcal{G}(-), X)$ and $H_{v's}$, $H_{y's}$ are respectively the vertex and edge groups of $(\mathcal{H}(-), Y)$ provided that $\overline{\Phi_y} = \Phi_{\bar{y}}$ and the following diagram commutes



$$\Phi_{0(y)} \circ \sigma_y = \sigma_{\Phi(y)} \circ \Phi_y \text{ and } \Phi_{1(y)} \circ \tau_{\bar{y}} = \tau_{\Phi(\bar{y})} \circ \Phi_{\bar{y}}$$

We write $(\mathcal{G}(-), X) \cong (\mathcal{H}(-), Y)$ to indicate that the two graphs of groups are isomorphic. Isomorphism of graphs of groups will lead to an isomorphism of their corresponding fundamental groups.
A graph of groups $(\mathcal{H}(-), Y)$ is conjugate to $(\mathcal{G}(-), X)$ if $X = Y$, $G_y = H_y$, $G_v = H_v$, i.e, they have the same underlying graph, the same edge groups, the same vertex groups and the monomorphisms are the same but $\tau'_y : H_y \to H_{1(y)}$ which is defined by $\tau'_y(a) = g^{-1}\tau_y(a)g$ for all $a \in H_y$, for a fixed $g \in G_{1(y)}$. A graph of groups $(\mathcal{H}(-), Y)$ is said to be *conjugate isomorphic* to graph of groups $(\mathcal{G}(-), X)$ if it is isomorphic to a conjugate of $(\mathcal{G}(-), X)$. We conclude by the following theorem of [11].

Theorem 5.2.2. *Let $(\mathcal{G}(-), X)$ and $(\mathcal{H}(-), Y)$ be conjugate isomorphic graphs of groups. Then $\pi(\mathcal{G}(-), X)$ is isomorphic to $\pi(\mathcal{H}(-), Y)$.*

5.3 Bass-Serre Theory

Our goal is to have a presentation for the maximal subgroups of HNN-extensions of finite inverse semigroups. Our technique is based on the use of Bass-Serre theory via graph of groups and actions of groups as illustrated above. In this section we give a a short overview of this theory, in later section we will detail the main result of the theory when X is a complete t-opuntoid graph.

Let G be a group acting on the non-empty graph X, and let $Y = (G \setminus X)$ be the quotient graph of X under the action of G. We start describing how to build the graph of groups $(\mathcal{G}(-), Y)$. Let T be a maximal subtree of $Y = (G \setminus X)$ and let $\mathcal{O}$ be an orientation of the edges of Y, i.e., let $\mathcal{O}$ be subset of $Ed(Y)$ such that for each pair of opposite edges y and $\bar{y}$ in $Ed(Y)$, exactly one of the them belongs to $\mathcal{O}$. By Theorem 5.1.1 there exists a subtree T' of X that maps onto the tree T by the lifting homomorphism j. Given a vertex v of T and an edge y of T, let $j(v)$ and $j(y)$ denote the lift of v and y, respectively, into the tree T'. The stabilizer groups $Stab_G(j(v))$ and $Stab_G(j(y))$ are assigned to each vertex v of T and to each edge y of T, respectively. Since any element of G stabilizing an edge y of X stabilizes the initial and terminal vertices of y as well, it is clear that $G_y = Stab_G(j(y)) \leq Stab_G(0(j(y)) = Stab_G(j(0(y)) = G_{0(y)}$, and $G_y \leq G_{1(y)}$. Since y is an edge of the tree T, then one can take the above inclusions as the monomorphisms σ_y and τ_y. Furthermore, since T is a maximal subtree of Y, it contains all the vertices of Y, and thus it remains to assign groups to the edges of Y not included in the tree T. Let us first consider the positive edges of $E(Y) - E(T)$, i.e., the edges of $E(Y) - E(T)$ included in $\mathcal{O}$. For each edge y in $\mathcal{O}$ there exists an edge x in X whose initial vertex belongs to the set of vertices of T', $0(x) = j(0(y))$, and which maps onto y. Denote this edge by $j(y)$. The group assigned to the positive edge y will now be the stabilizer group $Stab_G(j(y))$, with the monomorphism σ_y being again the inclusion of $Stab_G(j(y))$ into $Stab_G(0(x))$. The group $Stab_G(j(y))$ is not necessarily a subgroup of the group $Stab_G(j(1(y)))$. However the vertices $1(x)$ and its lift $j(1(y))$ belong to the same vertex orbit of G, and thus there exists an element g_y of G mapping $1(x)$ to $j(1(y))$, and the two groups $Stab_G(1(x))$ and $Stab_G(j(1(y))$ are isomorphic inside of G under conjugation by g_y. This allows us to define the monomorphism $\tau_y : Stab_G(j(y)) \to Stab_G(1(y))$ by setting $\tau_y = g_y \cdot \iota \cdot (g_y)^{-1}$, where ι is the inclusion $Stab_G(j(y)) \leq Stab_G(1(x))$. Finally, to extend our group assignments to the negative edges of y, we simply notice that we have already assigned groups and monomorphisms to all of their positive counterparts,

and thus, for any negatively oriented edge y we put $G_y = G_{\bar{y}} = Stab_G(\bar{y})$, $\sigma_y = \tau_{\bar{y}}$, and $\tau_y = \sigma_{\bar{y}}$. This completes the construction of a graph of groups $(\mathcal{G}(-), Y)$ from the action of group G on a graph X.
Now, let $\Phi : \pi(\mathcal{G}(-), (G \setminus X)) \to G$ be the group homomorphism defined by the inclusions $G_v \to G$ and the mapping $\Phi(y) = g_y$, sending the element y to $g_y \in G$ and mapping $1(x)$ onto $j(1(y))$. In the particular case of groups acting on trees we have a deeper result.

Theorem 5.3.1 (Structure theorem of Bass-Serre theory)**.** *Let G be a group acting without inversions on a connected graph X. Then X is a tree if and only if $\Phi : \pi(\mathcal{G}(-), (G \setminus X)) \to G$ is an isomorphism.* ■

Thus, the above theorem allows one to get information on the structure of the group G provided that an action of G on a tree is known.

Our interest is to describe maximal subgroups of HNN-extensions of finite inverse semigroups and to do that in the next subsection we apply Basse-Serre theory using the particular shape of the Schützenberger graphs of such HNN-extensions, then in subsection 5.3.2 we associate to an HNN-extension a graph of groups whose vertices are $\mathcal{D}$-classes of the inverse semigroup S. Then we show that these approaches lead to isomorphic fundamental groups because they are based on conjugate isomorphic graphs of groups.

5.3.1 Graph of groups from a complete t-opuntoid graph

In Chapter 4, we have shown that each Schützenberger graph Γ of a word in $(\overline{X} \cup \overline{X}^{-1})^+$ relative to the standard presentation of an HNN-extension of a finite inverse semigroup is a complete t-opuntoid graph that has a host. Let $\Gamma' = Host(\Gamma)$ be the set of all hosts of Γ and denote by $\Upsilon(\Gamma')$ the lobe graph of Γ' which is an oriented tree. By Lemma 4.2.1, every automorphism of Γ' induces an automorphism of $\Upsilon(\Gamma')$, thus we can define an action of the automorphism group $Aut(\Gamma')$ of Γ' on the tree $\Upsilon(\Gamma')$. Denote the group $Aut(\Gamma')$ by G. Now we can define the action of G on $\Upsilon(\Gamma')$ in the following obvious way, for every automorphism $\gamma \in G$ and for every vertex Δ of $\Upsilon(\Gamma')$ let $\gamma \cdot \Delta = \gamma(\Delta)$ the image of Δ under γ and for every edge (Δ_1, Δ_2) of $\Upsilon(\Gamma')$ let $\gamma \cdot (\Delta_1, \Delta_2) = (\gamma(\Delta_1), \gamma(\Delta_2))$. Since $\Upsilon(\Gamma')$ is an oriented tree and every automorphism preserves its orientation, no edge can be mapped in its inverse and thus G acts on $\Upsilon(\Gamma')$ without inversions. The quotient graph $Y = (G \setminus \Upsilon(\Gamma'))$ is the graph of orbits of $\Upsilon(\Gamma')$ under the action of G. Since all the edges of $\Upsilon(\Gamma')$ belonging to the same orbit have the same orientation, the quotient graph $Y = (G \setminus \Upsilon(\Gamma'))$ inherits the orientation from the tree $\Upsilon(\Gamma')$. Now we have all the ingredients to construct a graph of groups (see Section 5.3). We have the connected graph $\Upsilon(\Gamma')$ on which the group $G = Aut(\Gamma')$ acts without inversions and we have assigned the orientation to the edges of the quotient graph Y. The next step is to choose some spanning tree T of Y and define a lifting function $j : Y \to \Upsilon(\Gamma')$. The lifting function is used to define vertex groups and edge groups. The vertex group $G(\Delta)$, for each vertex Δ of Y is the stabilizer group $Stab_G(j(\Delta))$ under the action of G on the lifting of Δ and the edge group $G(y)$ for each edge $y = (\Delta_1, \Delta_2)$ of Y is the stabilizer group $Stab_G(j(y))$ under the action of G on its lifting. The monomorphism $\sigma_y : G(y) \to G(0(y))$ is the inclusion and the monomorphism $\tau_y : G(y) \to G(1(y))$ is defined by $\gamma \mapsto \gamma_y \gamma (\gamma_y)^{-1}$, where γ_y is an automorphism from G mapping $1(j(y))$ to $j(1(y))$. Note that, if an edge y belongs to the maximal tree T, we take γ_y to be the identity. In that case τ_y is also just an inclusion. Since $\Upsilon(\Gamma')$ is a tree, by the Structure Theorem 5.3.1 of Bass-Serre Theory G is isomorphic to the fundamental group $\pi(\mathcal{G}(-), Y)$ of the graph of groups $(\mathcal{G}, Y)$.

Theorem 5.3.2. *Let $S^* = [S; A, B]$ be the HNN-extension of the finite inverse semigroup S, H_e be the maximal subgroup in S^* containing the idempotent $e \in S^*$ and let $(\mathcal{G}(-), Y)$ be the above defined graph of groups. Then $H_e \cong \pi(\mathcal{G}(-), Y)$.* ■

5.3.2 Graph of groups from the $\mathcal{D}$-structure

We will describe another construction of graphs of groups associated with S^* using the $\mathcal{D}$-structure of the inverse semigroup S. This construction is a generalization of one used in Yamamura's paper [49]. Similar constructions for certain classes of amalgams of inverse semigroups were first introduced by Haataja, Margolis and Meakin in [17] and generalized by Bennett in [4] and also is used by Jajcayova in [26] to give the characterization of the maximal subgroups of the lower bounded HNN-extension of inverse semigroups. Our graph is similar to [26]. The construction depends mainly on the $\mathcal{D}$-classes of S. This graph is a graph whose vertices are the $\mathcal{D}$-classes of S and whose edges are the $\mathcal{D}$-classes of A and of B. Then $\mathcal{H}$-class groups of S are assigned to the vertices and edges of the graph to construct the graph of groups. The detailed construction is as follows:

Let S be an inverse semigroup. We will denote a $\mathcal{D}$-class of S by D^S and $\mathcal{D}$-class of A, (respectively B), by D^A, (respectively D^B). Note that if φ is the HNN isomorphism between subsemigroups A and B, the image $\varphi(D^A)$ of a $\mathcal{D}$-class D^A of A under φ, is a $\mathcal{D}$-class of B, and conversely, $\varphi^{-1}(D^B)$ is a $\mathcal{D}$-class of A for any $\mathcal{D}$-class D^B of B. For every D^A (and every D^B) there is unique D^S such that $D^A \subset D^S$ (and $D^B \subset D^S$).

The graph Z is defined as follows : The set $V(Z)$ of all vertices of Z consists of all $\mathcal{D}$-classes of S. The set of all positively oriented edges $Ed_+(Z)$ contains all the triples (D_1^S, D^A, D_2^S) where D^A is a $\mathcal{D}$-class of A, the initial vertex D_1^S is the unique $\mathcal{D}$-class of S containing D^A and the terminal vertex D_2^S is the $\mathcal{D}$-class of S containing $\varphi(D^A)$. The set of all negatively oriented edge $Ed_-(Z)$ contains all the triples (D_1^S, D^B, D_2^S) where D^B is a $\mathcal{D}$-class of B, the initial vertex D_1^S is the $\mathcal{D}$-class of S containing D^B and the terminal vertex D_2^S is the $\mathcal{D}$-class of S containing $\varphi^{-1}(D^B)$. The set of all edges of Z is then $Ed(Z) = Ed_+(Z) \cup Ed_-(Z)$. The inverse $(D^A)^{-1}$ is $\varphi(D^A)$ and $(D^B)^{-1}$ is $\varphi^{-1}(D^B)$.

Now we need to assign a group to each vertex and edge. For each $D^S \in V(Z)$, specify an $\mathcal{H}$-class group of S in D^S and denote it by $H(D^S)$. Similarly, for each $D^A \in Ed_+(Z)$, specify an $\mathcal{H}$-class group A in D^A and denote it by $H(D^A)$. For $D^B \in Ed_-(Z)$, set $H(D^B) = H(\varphi^{-1}(D^B)) = H((D^B)^{-1})$.

Finally, we have to define monomorphisms from the edge group to the vertex groups. Let $y = (D_1^S, D^A, D_2^S)$ be some positive edge from Z. Then $D^A \subset D_1^S$. Let K be the $\mathcal{H}$-class group of S containing $H(D^A)$. Then $K \cong H(D_1^S)$ since they are both $\mathcal{H}$-class groups with the same $\mathcal{D}$-class of S. Then, by Green's Lemma, there exists an element $d \in D_1^S$ such that $dKd^{-1} = H(D_1^S)$. The map $\sigma_{D^A} : H(D^A) \to H(D_1^S)$ is defined by $h \mapsto dhd^{-1}$, for $h \in H(D^A)$. It can be easily verified that σ_{D^A} is an injective group homomorphism.

For an edge y, we also have $\varphi(D^A) \subset D_2^S$. Let K denote the $\mathcal{H}$-class group of D_2^S containing $H(\varphi(D^A))$. Then $K \cong H(D_2^S)$, and there is an element $d \in D_2^S$ such that $dKd^{-1} = H(D_2^S)$, the map $\tau_{D^A} : H(D^A) \to H(D_2^S)$ is defined by $h \to d(\varphi(h))d^{-1}$, for $h \in H(D^A)$. Again, it is not hard to show that τ_{D^A} is an injective group homomorphism.

For a negatively oriented edge $y = (D_1^S, D^B, D_2^S)$, the maps σ_{D^B} and τ_{D^B} are defined by $\sigma_{D^B} = \tau_{(D^B)^{-1}}$ and $\tau_{D^B} = \sigma_{(D^B)^{-1}}$. The construction of the graphs of groups $(\mathcal{H}, Z)$ is now complete.

For each idempotent $e \in E(S^*)$, let Z_e be the component of Z containing (as a vertex) the $\mathcal{D}$-class of e in S. We denote the restriction of $(\mathcal{H}(-), Z)$ to Z_e by $(\mathcal{H}(-), Z_e)$. Thus $(\mathcal{H}(-), Z_e)$ is a connected graph of groups. Therefore, we obtain the fundamental group $\pi(\mathcal{H}(-), Z_e)$ which is determined by an idempotent e of S. So far we had prepared the environment for introducing the main section of this chapter.

5.4 Maximal subgroups

Although we have seen in Theorem 5.3.2 that for a general idempotent, e of S^* the maximal subgroup containing e is isomorphic to the fundamental group of graph of groups which based on the t-opuntoid graph of e we explore more in details such groups depending on the properties of hosts of the Schützenberger graphs of the idempotents. The discussion will be divided into two parts according to the types of the idempotents of the corresponding maximal subgroups. Namely the idempotents are either elements of the original inverse semigroup S or are new idempotents, i.e., elements of $S^* \setminus S$, and the maximal subgroups are accordingly divided into two types. In subsection 5.4.1 we consider the case of original idempotents. In the last section we discuss the simpler case where e is not $\mathcal{D}$-related to any element of S.

5.4.1 Maximal subgroups containing original idempotents

Theorem 5.4.1. *Let S^* be an HNN-extension of a finite inverse semigroup S, and let e be an idempotent of S. Then the maximal subgroup of S^* containing e is isomorphic to the fundamental group of the graph of groups $(\mathcal{H}(-), Z_e)$.*

Proof Let Γ denote the Schützenberger graph of e relative to presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$. Then, by Corollary 4.1.2, Γ is a complete t-opuntoid graph that has a host. We can construct a graph of groups $(\mathcal{G}(-), Y)$ as illustrated in Section 5.3.1 thus $\pi(\mathcal{G}(-), Y) \cong Aut(\Gamma)$. Hence, $\pi(\mathcal{G}(-), Y)$ is isomorphic to the maximal subgroup H_e of S^* containing e. On the other hand, we construct another fundamental group based on graph of groups $(\mathcal{H}(-), Z_e)$ by using the $\mathcal{D}$-structure of the semigroup S, as described in Section 5.3.2.

The proof is established by showing that the graphs of groups $(\mathcal{G}(-), Y)$ and $(\mathcal{H}(-), Z_e)$ are conjugate isomorphic, which we show in Lemma 5.4.1. Then by Theorem 5.2.2 the fundamental group $\pi(\mathcal{G}(-), Y)$ is isomorphic to the fundamental group $\pi(\mathcal{H}(-), Z_e)$. ∎

Lemma 5.4.1. *Let S^* be an HNN-extension of a finite inverse semigroup S, and let e be an idempotent of S. Then the graphs of groups $(\mathcal{G}(-), Y)$ and $(\mathcal{H}(-), Z_e)$ are conjugate isomorphic.*

Proof To show that $(\mathcal{G}(-), Y)$ is conjugate isomorphic to $(\mathcal{H}(-), Z_e)$, we need a graph isomorphism Φ between Y and Z_e and isomorphisms between the corresponding vertex and edge groups such that these group isomorphisms commute with the corresponding edge monomorphisms up to conjugation by an element of $H(\Phi(0(y)))$.

As above, let Γ be the Schützenberger graph of e relative to presentation $\langle \overline{X} | R_{HNN} \cup R \rangle$, and let $\Gamma' = Host(S\Gamma(e))$. By the first statement of Corollary 4.3.1 the Schützenberger graph $S\Gamma(X, R; e)$ of e relative to $\langle X | R \rangle$ is (isomorphic to) a host of Γ and each host is a lobe. Moreover, by the second statement of the same corollary each host of $S\Gamma(e)$ is a lobe. Since $S\Gamma(X, R; e)$ is a host and a lobe then by Corollary 4.2.6 each lobe in $\Gamma' = Host(\Gamma)$ is a Schützenberger graph relative to the presentation $\langle X | R \rangle$.

We define a graph map $\Phi : Y \to Z$ as follows: Let Δ be a vertex of Y. Recall that its lift $j(\Delta)$ is a vertex of $\Upsilon(\Gamma')$, and hence there is a lobe in Γ' corresponding to $j(\Delta)$. We will also refer to this lobe of Γ' as $j(\Delta)$. Put $\Phi(\Delta)$ equal to the $\mathcal{D}$-class of the idempotent $e(v, j(\Delta))$ in S, where v is any vertex of $j(\Delta)$ in Γ'. This map is well-defined, because if we take any other vertex v' of $j(\Delta)$ then the graphs $S\Gamma(X, R, e(v, j(\Delta)))$ and $S\Gamma(X, R, e(v', j(\Delta)))$ are both isomorphic to $j(\Delta)$ and thus isomorphic to each other, hence, $e(v', j(\Delta))\mathcal{D}^S e(v, j(\Delta))$.

Now we define Φ on edges, let $y = (\Delta_1, \Delta_2)$ be a positive edge of Y. Then $j(\Delta_1)$ and $j(\Delta_2)$ are adjacent lobes of the t-subopuntoid subgraph $\Gamma' = Host(\Gamma)$. Thus there exists a t-edge z with $0(z) \in V(j(\Delta_1))$ and $1(z) \in V(j(\Delta_2))$. Since $n(\Gamma) = 1$, both $j(\Delta_1)$ and $j(\Delta_2)$ must be hosts of Γ and must therefore feed off each other. Since $j(\Delta_1) \mapsto j(\Delta_2)$, any t-edge $z = (0(z), t, 1(z))$ connecting $j(\Delta_1)$ and $j(\Delta_2)$ satisfies the following equality: $\varphi(f_A(e(0(z), j(\Delta_1)))) = e(1(z), j(\Delta_2))$, where φ is the isomorphism between the inverse subsemigroups A and B. Hence, $e(1(z), j(\Delta_2))$ is an idempotent from B. Similarly since $j(\Delta_2) \mapsto j(\Delta_1)$, $\varphi^{-1}((f_B(e(1(z), j(\Delta_2))) = e(0(z), j(\Delta_1))$ for any t-edge z connecting $j(\Delta_1)$ and $j(\Delta_2)$. Since $e(1(z), j(\Delta_2)) \in E(B)$, $f_B(e(1(z), j(\Delta_2))) = e(1(z), j(\Delta_2))$, and therefore $e(0(z), j(\Delta_1)) = \varphi^{-1}(e(1(z), j(\Delta_2)))$. Thus $e(0(z), j(\Delta_1))$ is an idempotent from A. The case $z = (0(z), t^{-1}, 1(z))$ is analogous.

Next, we put $\Phi(\Delta_1, \Delta_2) = (D_1^S, D^A, D_2^S)$, such that D^A is the $\mathcal{D}$-class of A containing $e(0(z), j(\Delta_1))$. The map Φ is well-defined for positive edges, because if we take any other t-edge z' connecting the lobes $j(\Delta_1)$ and $j(\Delta_2)$ in Γ', by the related pair separation property of Γ', the vertices $0(z)$ and $0(z')$ of $j(\Delta_1)$ are connected by a path labeled by an element u from A. Then, applying standard results in [44, Lemma 5.1], $(0(z), j(\Delta_1), 0(z')) \cong \mathcal{A}(X, R, e(0(z), j(\Delta_1)))u)$, and $e(0(z), j(\Delta_1))\mathcal{R}e(0(z), j(\Delta_1))u\mathcal{L}e(0(z'), j(\Delta_1))$. Hence, $e(0(z), j(\Delta_1))\mathcal{D}e(0(z'), j(\Delta_1))$.

By the definition of a positive edge in Z, we have $0(\Phi(y)) = D_1^S \supseteq D^A$ and $1(\Phi(y)) = D_2^S \supseteq \varphi(D^A)$, where D_1^S and D_2^S are the unique $\mathcal{D}$-classes of S containing D^A and $\varphi(D^A)$, respectively. Now, $\Phi(0(y)) = \Phi(\Delta_1) = D^S_{e(v,j(\Delta_1))} = D^S_{e(0(z),j(\Delta_1))} \supseteq D^A_{e(0(z),j(\Delta_1))} = D^A$. Thus $\Phi(0(y)) = D_1^S$, as required. Also, $\Phi(1(y)) = \Phi(\Delta_2) = D^S_{e(v,j(\Delta_2))} = D^S_{e(1(z),j(\Delta_2))} \supseteq D^B_{e(1(z),j(\Delta_2))} = \varphi(D^A)$, and thus $\Phi(1(y)) = D_2^S$.

If $y = (\Delta_1, \Delta_2)$ is a negative edge of Y, we put $\Phi(\Delta_1, \Delta_2) = (D_1^S, D^B, D_2^S)$, where D^B is the $\mathcal{D}$-class of B containing $e(1(z), j(\Delta_2))$. Similarly as above, we can also show that for the negative edges, Φ is well-defined map, with $\Phi(0(y)) = D_1^S$ and $\Phi(1(y)) = D_2^S$. Thus we have established that the map $\Phi : Y \to Z$ defines a graph morphism.

Now we will show that Φ is an injective graph morphism. Suppose that Δ and Δ' are vertices of Y for which $\Phi(\Delta) = \Phi(\Delta')$. Let v and v' be vertices of $j(\Delta)$ and $j(\Delta')$ respectively. Then the idempotents $e(v, j(\Delta))$ and $e(v', j(\Delta'))$ are $\mathcal{D}$-related in S. Hence, $S\Gamma(X, R; e(v, j(\Delta))) \cong S\Gamma(X, R; e(v', j(\Delta')))$. Thus, $j(\Delta)$ and $j(\Delta')$ are isomorphic to each other. Both $j(\Delta)$ and $j(\Delta')$ are lobes of Γ', and since $n(\Gamma) = 1$, they are hosts of Γ. By Corollary 4.2.3, any isomorphism between them extends to an automorphism of Γ. It follows that $j(\Delta)$ and $j(\Delta')$ lie in the same orbit. Hence $\Delta = \Delta'$ and Φ is injective on vertices.

Now suppose that y_1 and y_2 are two positive edges of Y such that $\Phi(y_1) = \Phi(y_2) = (D_1^S, D^A, S_2^S)$. Then their lifts $j(y_1) = (0(j(y_1)), 1(j(y_1)))$ and $j(y_2) = (0(j(y_2)), 1(j(y_2)))$ are edges of $\Upsilon(\Gamma')$. This means that the lobes $0(j(y_1)), 1(j(y_1)), 0(j(y_2))$ and $1(j(y_2))$ are lobes of the t-opuntoid graph Γ'. Since $\Phi(y_1) = \Phi(y_2)$, by the definition of the graph morphism Φ, the idempotents $e(0(z_1), 0(j(y_1)))$ and $e(0(z_2), 0(j(y_2)))$ associated with the edges y_1 and y_2, are $\mathcal{D}$-related in the subsemigroup A. Here z_1 is any t-edge of Γ connecting the lobes $0(j(y_1))$ and $1(j(y_1))$ and z_2 is any t-edge of Γ connecting the lobes $0(j(y_2))$ and $1(j(y_2))$. Since the two idempotents are $\mathcal{D}$-related in A, there exists an element u from A such that $e(0(z_1), 0(j(y_1)))\mathcal{R}u\mathcal{L}e(0(z_2), 0(j(y_2)))$. Since $D^A \subseteq D_1^S$, all the three elements $e(0(z_1), 1(j(y_1)))$, u and $e(0(z_2), 0(j(y_2)))$ are in the same $\mathcal{D}$-class of S. Using Stephen's results, we can find a vertex v_1 in the lobe $0(j(y_1))$ such that the automata $(0(z_1), 0(j(y_1)), v_1)$, and $\mathcal{A}(X, R, u)$ are isomorphic. Then the word u labels the path between $0(z_1)$ and v_1 in $0(j(y_1))$. By the related pair separation property of the graph Γ, the vertex v_1 must also be an intersection vertex of $0(j(y_1))$, and there exists a t-edges z_1' connecting $0(j(y_1))$ and $1(j(y_1))$ such that $0(z_1') = v_1$. Thus the edges z_1, z_1' connect the same pair of lobes, $0(j(y_1)), 1(j(y_1))$, in Γ, and therefore are both (together with all other edges connecting $0(j(y_1))$ and $1(j(y_1))$) is sent into the same edge $j(y_1)$ in $\Upsilon(\Gamma')$. Similarly, we can find a vertex v_2 in the lobe $0(j(y_2))$ such that $(v_2, 0(j(y_2)), 0(z_2)) \cong \mathcal{A}(X, R, u)$. The word u labels the path from v_2 to $0(z_2)$ in $0(j(y_2))$, which implies that there exists a t-edge z_2' connecting $0(j(y_2))$ and $1(j(y_2))$ with $0(z_2') = v_2$. Again, the image of both z_2 and z_2' in $\Upsilon(\Gamma')$ is the edge $j(y_2)$. Now, $(0(z_1), 0(j(y_1)), v_1) \cong (v_2, 0(j(y_2)), 0(z_2))$, because their are both isomorphic to the automaton $\mathcal{A}(X, R, u)$. Thus there exists an isomorphism γ of the lobes $0(j(y_1))$ and $0(j(y_2))$ sending the vertex $0(z_1)$ to the vertex $0(z_2)$, hence it sends the edge z_1 to the edge z_2. Recall that the lobes $0(j(y_1))$ and $0(j(y_2))$ are from Γ', therefore they are both hosts of Γ (since $n(\Gamma) = 1$). By Corollary 4.2.3 every isomorphism between $0(j(y_1))$ and $0(j(y_2))$ extends uniquely to an automorphism of Γ and thus $\gamma \in Aut(\Gamma)$, then the induced automorphism γ of $\Upsilon(\Gamma')$ maps $j(y_1)$ to $j(y_2)$, and hence $j(y_1)$ and $j(y_2)$ lie in the same G-orbit. Therefore, $y_1 = y_2$, and Φ is also one-to-one on edges.

The graph Y is connected and thus $\Phi(Y)$ is a connected subgraph of Z. We want to show that $\Phi(Y) = Z_e$, where e is our idempotent from S. Let Δ be some host of Γ isomorphic to $S\Gamma(X, R; e)$. Then $\Phi(\Delta)$ is the $\mathcal{D}$-class of e in S. Thus $\Phi(Y)$ is a connected subgraph of Z_e. Suppose that (D_1^S, D^A, D_2^S) is a positive edge of Z_e with $D_1^S = \Phi(\Delta_1)$ for some vertex Δ_1 of Y. Let v be a vertex of the lobe $j(\Delta_1)$ of Γ'. Let f be an idempotent of A that belongs to D^A. Since $D^A \subset D_1^S$, then both $e(v, j(\Delta_1))$ and f belongs to D_1^S. That implies $S\Gamma(X, R; e(v, j(\Delta_1))) \cong S\Gamma(X, R; f)$. Since $S\Gamma(X, R; e(v, j(\Delta_1)))$ is isomorphic to Δ_1, we may choose v such that $e(v, j(\Delta_1)) = f$. Now, since f is an idempotent of A and Γ is a complete t-opuntoid graph, the vertex v must be an intersection vertex, i.e. there is a t-edge z such that $0(z) = v$. Denote by Λ the lobe of Γ adjacent to $j(\Delta_1)$ containing $1(z)$. Since $j(\Delta_1)$ is a lobe of Γ' and it is a host of Γ, then $e(1(z), \Lambda) = \varphi(f)$. Thus we have $e(0(z), j(\Delta_1)) = f$ and $e(1(z), \Lambda) = \varphi(f)$, which implies that $j(\Delta_1) \mapsto \Lambda$ and $\Lambda \mapsto j(\Delta_1)$. From this it follows that Λ is also a host of Γ. It is thus a lobe of Γ' and there is a vertex of $\Upsilon(\Gamma')$ corresponding to it. Hence the edge $(j(\Delta_1), \Lambda)$ is an edge of $\Upsilon(\Gamma')$. Therefore, there is an edge $y = (\Delta_1, \Delta_2)$ of Y such that $j(y) = (j(\Delta_1), j(\Delta_2))$ and $(j(\Delta_1), \Lambda)$ are in the same orbit. That means that there is an automorphism of Γ' that maps Λ to $j(\Delta_2)$ and $j(\Delta_1)$ to itself. This automorphism maps the t-edge z connecting $j(\Delta_1)$ and Λ to a t-edge z' connecting $j(\Delta_1)$ and $j(\Delta_2)$. The intersection vertex $0(z)$ of $j(\Delta_1)$ is mapped to another intersection vertex $0(z')$ of $j(\Delta_1)$, and the intersection vertex $1(z)$ of Λ is mapped to an intersection vertex $1(z')$ of $j(\Delta_2)$. Then $e(0(z'), j(\Delta_1)) = e(0(z), j(\Delta_1)) = f$ and $e(1(z'), j(\Delta_2)) = e(1(z), \Lambda) = \varphi(f)$. This implies, using the definition of the map Φ and the fact that f is an idempotent of the $\mathcal{D}$-class D^A, that the map Φ sends the edge $y = (\Delta_1, \Delta_2)$ of Y to the edge (D_1^S, D^A, D_2^S) of Z_e. Using similar arguments we can show that for a negative edge (D_1^S, D^B, D_2^S) of Z_e there is an edge y of Y such that $\Phi(y) = (D_1^S, D^B, D_2^S)$. We may now conclude that $\Phi(Y)$ is maximal connected subgraph of Z_e, and since Z_e is connected, $\Phi(Y) = Z_e$. Thus the map $\Phi : Y \to Z$ defines a graph monomorphism onto Z_e.

Next, we define the vertex group isomorphism. Let Δ be a vertex of Y. Recall that the image of Δ under the graph isomorphism Φ is the $\mathcal{D}$-class of S containing the idempotent $e(v, (j(\Delta))$, where v is any vertex of $j(\Delta)$. The vertex group of Δ is $G(\Delta) = Stab(j(\Delta))$ under the action of $G = Aut(\Gamma)$ on $\Upsilon(\Gamma')$. Let the group $H(\Phi(\Delta))$ assigned to vertex $\Phi(\Delta)$ of Z_e, be the $\mathcal{H}$-class group of S in $\Phi(\Delta)$ with identity g, denoted by H_g. Both the idempotents g and $e(v, j(\Delta))$ belong to the $\mathcal{D}$-class $\Phi(\Delta)$. Hence the graphs $S\Gamma(X, R; e(v, j(\Delta)))$ and $S\Gamma(X, R; g)$ are isomorphic, and we can choose the vertex v such that $e(v, j(\Delta)) = g$. Since $S\Gamma(X, R; e(v, j(\Delta))) \cong j(\Delta)$, we get $j(\Delta) \cong S\Gamma(X, R; g)$. Denote this isomorphism by ψ. Note that $v = \psi^{-1}(g)$ for v such that $e(v, j(\Delta)) = g$. From above we see that $Aut(j(\Delta)) \cong Aut(S\Gamma(X, R; g))$ under the map defined by $\gamma \mapsto \psi\gamma\psi^{-1}$. Every automorphism of Γ that stabilizes Δ (i.e. an element from $G(\Delta)$ induces an automorphism of $j(\Delta)$ and conversely, since the lobe $j(\Delta)$ is a host of Γ, by Corollary 4.2.3, every automorphism of $j(\Delta)$ extends uniquely to an automorphism of Γ. Thus $G(\Delta) \cong Aut(j(\Delta))$ under the map: $\gamma \mapsto \gamma_{|j(\Delta)}$. From Theorem 1.3.3, $Aut(S\Gamma(X, R; g)) \cong H_g$, defined by $\gamma \mapsto \gamma(g)$. The composition of these isomorphisms defines an isomorphisms $\Phi : G(\Delta) \to H_g$, where $\gamma \mapsto (\psi\gamma_{|j(\Delta)}\psi^{-1})$. Note that for $v \in V(j(\Delta))$ such that $e(v, j(\Delta)) = g$, $\psi^{-1}(g) = v$, and hence the element $(\psi\gamma_{|j(\Delta)}\psi^{-1})(g) = (\psi\gamma_{|j(\Delta)})(\psi^{-1}(g)) = (\psi\gamma_{|j(\Delta)})(v)$. The Schützenberger automation $\mathcal{A}(X, R; \psi\gamma(v))$ is isomorphic to $(v, j(\Delta), \gamma(v))$, under the isomorphism ψ^{-1} and thus Φ is independent from ψ and may be defined by $\gamma \mapsto s$, where $\mathcal{A}(X, R; s) \cong (v, j(\Delta), \gamma(v))$, and v is any vertex of $j(\Delta)$ such that $e(v, j(\Delta)) = g$.

We now define the edge group isomorphisms. Let $y = (\Delta_1, \Delta_2)$ be a positive edge of Y. Let $\Phi(y)$ be equal to (D_1^S, D^A, D_2^S). Recall from the definition of the graph isomorphism Φ, that D^A is the $\mathcal{D}$-class of A containing the idempotent $e(0(z), j(\Delta_1)) \in E(A)$, where z is any t-edge connecting $j(\Delta_1)$ and $j(\Delta_2)$ in Γ'. The group $G(y)$ assigned to the edge y has been defined as $Stab(j(y))$ under the action of $G = Aut(\Gamma)$ on $\Upsilon(\Gamma')$. The group $H(\Phi(y))$ assigned to $\Phi(y)$ has been defined as a specified $\mathcal{H}$-class group of A in D^A. Let f denote the identity of the group $H(\Phi(y))$. If z is any t-edge of Γ' connecting the lobes $j(\Delta_1)$ and $j(\Delta_2)$, then $e(0(z), j(\Delta_1))$ is an idempotent of A contained in D^A. Thus $e(0(z), j(\Delta_1))\mathcal{D}f$ in A, and there exists an element $u \in A$ such that $e(0(z), j(\Delta_1))\mathcal{R}u\mathcal{L}f$. Since $D^A \subseteq D_1^S$, all three elements, $e(0(z), j(\Delta_1))$, f and u, belong to the same $\mathcal{D}$-class of S, and hence we can find a vertex v of $j(\Delta_1)$ such that $e(v, j(\Delta_1)) = f$. Moreover, $(0(z), j(\Delta_1), v) \cong \mathcal{A}(X, R; u)$ and the word u labels the path between the vertices $0(z)$ and v in $j(\Delta_1)$. Because of the separation property of the complete t-opuntoid graph Γ, the vertex v is also an intersection vertex of $j(\Delta_1)$, and there exists a t-edge z' in Γ' connecting the lobes $j(\Delta_1)$ and $j(\Delta_2)$ with $0(z') = v$. Thus by choosing $z = z'$, we have $e(0(z), j(\Delta_1)) = f$. In the previous part of this proof, we established the isomorphism between the vertex groups $G(\Delta_1)$ and $H(D_1^S)$ as $\gamma \mapsto s$, where $\mathcal{A}(X, R; s) \cong (v, j(\Delta_1), \gamma(v))$ and v is any vertex of $j(\Delta_1)$ such that $e(v, j(\Delta_1)) = g$. By Green's Lemma $H(D_1^S)$ is isomorphic to the $\mathcal{H}$-class group of f in S, H_f^S. Thus there is an element $d \in D_1^S$ such that $H(D_1^S) = dH_f^S d^{-1}$. This defines isomorphism from the vertex group $G(\Delta_1)$ onto H_f^S, where $\gamma \mapsto dsd^{-1}$.

Now, if $\gamma \in G(y) \subseteq Aut(\Gamma)$, then $\gamma(z)$ must also be a t-edge connecting the lobes $j(\Delta_1)$ and $j(\Delta_2)$ in Γ'. Again, by the related pair separation property of Γ, the path from $0(z)$ to $0(\gamma(z)) = \gamma(0(z))$ in $j(\Delta_1)$ is labeled by an element u from A. Then we have that $(0(z), j(\Delta_1), \gamma(0(z))) \cong \mathcal{A}(X, R; fu)$. By Theorem 1.3.3, the element fu is $\mathcal{H}$-related to the element f and thus $fu \in H(\Phi(y))$. Therefore, the map $\Phi : \gamma \mapsto dsd^{-1}$ sends elements of $G(y)$ into $H(\Phi(y))$ and we have showon that $\Phi(G(y)) \subseteq H(\Phi(y))$.

Next, we will establish the opposite inclusion. Let s be an element of $H(\Phi(y))$, and let $\gamma \in G(\Delta_1)$ such that $\mathcal{A}(X, R; s) \cong (0(z), j(\Delta_1), \gamma(0(z)))$. Then s labels the path from $0(z)$ to $\gamma(0(z))$ in $j(\Delta_1)$. By the related pair separation property of Γ, the vertex $\gamma(0(z))$ is an intersection vertex of $j(\Delta_1)$ thus there is a t-edge z' in Γ' connecting the lobes $j(\Delta_1)$ and $j(\Delta_2)$. This implies that $\gamma \in G(y)$ and hence $H(\Phi(y)) \subseteq \Phi(G(y))$. Thus, for each positive edge y, we have an isomorphism between edge groups $\Phi : G(y) \to H(\Phi(y))$ defined by $\gamma \mapsto dsd^{-1}$, where $\mathcal{A}(X, R, s) \cong (0(z), j(\Delta_1), 0(z)\gamma)$ for any t-edge z connecting $j(\Delta_1)$ and $j(\Delta_2)$ in Γ' such that $e(0(z), j(\Delta_1)) = f$. The element d is any element of the $\mathcal{D}$-class $D_1^S = 0(\Phi(y))$ such that dfd^{-1} is equal to the identity of the vertex group $H(D_1^S)$. Note that isomorphisms between edge groups for negative edges are also determined, because the group assigned to a negative edge is equal to the group assigned to the corresponding positive edge.

Finally, we show that the isomorphisms between the vertex groups and edge groups of $(\mathcal{G}, Y)$ and $(\mathcal{H}, Z_e)$ commute with the appropriate edge monomorphism. Let $y = (0(y), 1(y))$ be a positive edge of Y and let $\Phi(y) = (D_1^S, D^A, D_2^S)$. Let f and g denote the identities of the groups $H(\Phi(y))$ and $H(\Phi(1(y)))$, respectively. Recall that the group monomorphism $\tau_y : G(y) \to G(1(y))$ is defined by $\gamma \mapsto \gamma_y \gamma \gamma_y^{-1}$, where γ_y is an automorphism from G mapping $1(j(y))$ to $j(1(y))$. The vertex group morphism $\Phi : G(1(y)) \to H(\Phi(1(y)))$ is defined by $\gamma \mapsto s_1$, where $\mathcal{A}(X, R; s_1) \cong (v, j(1(y), \gamma(v)))$ and v is any vertex of $j(1(y))$ such that $e(v, j(1(y))) = g$. Thus the composition $\Phi \circ \tau_y : G(y) \to H(\Phi(1(y))) = H(1(\Phi(y)))$ is defined by $\gamma \mapsto s_1$, where $\mathcal{A}(X, R; s_1) \cong (v, 1(j(y)), (\gamma_y \gamma \gamma_y^{-1})(v))$ and v is any vertex of $1(j(y))$ such that $e(v, 1(j(y))) = g$. Since γ_y maps $1(j(y))$ isomorphically onto $j(1(y))$, we can redefine this map as : $\gamma \to s_1$, where $\mathcal{A}(X, R; s_1) \cong (v, j(1(y)), \gamma(v))$ and v is any vertex of $j(1(y))$ such that $e(v, j(1(y))) = g$.

Next recall that the edge group isomorphism $\Phi : G(y) \to H(\Phi(y))$ has been defined by $\gamma \mapsto d_1 s_2 d_1^{-1}$, where $\mathcal{A}(R, X; s_2) \cong (0(z), j(0(y)), \gamma(0(z)))$ for any t-edge z connecting the lobes $j(0(y))$ and $j(1(y))$ in Γ' such that $e(0(z), j(0(y))) = f$. The element d_1 is any element of the $\mathcal{D}$-class $D_1^S = 0(\Phi(y))$ such that $d_1 f d_1^{-1}$ is equal to the identity of the vertex group $H(D_1^S)$. The group monomorphism $\tau_{\Phi(y)} : H(\Phi(y)) \to H(1(\Phi(y)))$ has been defined as $h \mapsto d_2(\varphi(h))d_2^{-1}$ where d_2 is any element from D_2^S such that $d_2 \varphi(f) d_2^{-1} = g$. Thus the composition $\tau_{\Phi(y)} \circ \Phi : G(y) \to H(1(\Phi(y))) = H(\Phi(1(y)))$ is defined by : $\gamma \mapsto d_2 \varphi(d_1 \varphi(s_2) d_1^{-1}) d_2^{-1} = d s_2 d^{-1}$, where $\mathcal{A}(X, R; s_2) \cong (0(z), j(0(y)), \gamma(0(z)))$ for any t-edge z connecting the lobes $j(0(y))$ and $j(1(y))$ such that $e(0(z), j(0(y))) = f$. The element $d = d_2 \varphi(d_1) \in D_2^S$.

To show that $\Phi \circ \tau_y = \tau_{\Phi(y)} \circ \Phi$, we need to show that $s_1 = d\varphi(s_2)d^{-1}$. Since Γ has the loop equality property, the facts that $e(0(z), j(0(y))) = f$ and f is an idempotent of A, imply that $e(1(z), j(1(y))) = \varphi(f)$. The elements $\varphi(f)$ and d are $\mathcal{D}$-related in S, hence, there exists a vertex v' in the lobe $j(1(y))$ such that $(v', j(1(y)), 1(z)) \cong \mathcal{A}(X, R; d)$. This implies that $e(v', j(1(y))) = dd^{-1} = g$. Thus, we can choose the vertex v, from the definition of the morphism $\Phi \circ \tau$, to be v'. Hence the automaton $(v, j(1(y)), 1(z))$ is isomorphic to $\mathcal{A}(X, R; d)$. Next, since the element $s_2 \in A$ labels the path from $0(z)$ to $\gamma(0(z))$ in the lobe $j(0(y))$, by the loop equality property, we have that $\varphi(s_2) \in B$ labels the path from $1(z)$ to $\gamma(1(z))$ in the lobe $j((1(y))$ and $(1(z), j(1(y)), \gamma(1(z))) \cong \mathcal{A}(X, R; \varphi(s_2))$. The automaton $(\gamma(v), j(1(y)), \gamma(1(z)))$ is isomorphic to $(v, j(1(y)), 1(z))$, and it is therefore also isomorphic to $\mathcal{A}(X, R; d)$. Thus, interchanging the initial and terminal vertices, $(\gamma(1(z)), j1(y)), \gamma(v)) \cong \mathcal{A}(X, R; d^{-1})$. Now the product of the three automata : $(v, j(1(y)), 1(z)) \times (1(z), j(1(y)), \gamma(1(z))) \times (\gamma(1(z)), j(1(y)), \gamma(v)) = (v, j(1(y)), \gamma(v))$, is, by Stephen's results, isomorphic to $\mathcal{A}(X, R; d\varphi(s_2)d_2^{-1})$. Hence $d\varphi(s_2)d_2^{-1} = s_1$ and $\Phi \circ \tau_y = \tau_{\Phi(y)} \circ \Phi$, and this concludes the proof of the lemma. ■

5.4.2 Maximal subgroups containing idempotent from $A \cup B$

In this section we follow ideas from [7] in order to get better description of the maximal subgroups of S^*. While in Section 5.4.1 we have the condition that the original idempotent e is in S, in this section we further subdivide into two cases: $e \in A \cup B$ or $e \in S \setminus (A \cup B)$ and then we have two different case. We have the following theorem.

Theorem 5.4.2. *Let e be an idempotent in S. With the above notations, let $Y = H_e^{S^*} \setminus \Upsilon_e$. Then*

1. *each $\Delta \in V(Y)$ is a Schützenberger graph $S\Gamma(X, R; e(\nu, \Delta))$ for some $\nu \in V(\Delta)$;*
2. *Y is finite;*
3. *$(\Delta_1, t, \Delta_2) \in Ed(Y)$ (the other case $(\Delta_1, t^{-1}, \Delta_2) \in E(Y)$ is similar) if and only if the following conditions hold*
   - *e is $\mathcal{D}^{S^*}$-related to some idempotent of A;*
   - *$\Delta_1 \cong S\Gamma(X, R; f)$ and $\Delta_2 \cong S\Gamma(X, R; \varphi(f))$ for some $f \in E(A)$;*
   - *there is a lobe Δ_2' of $Host(S\Gamma(e))$ adjacent to Δ_1 in Υ_e and an automorphism $\Psi \in Aut(Host(S\Gamma(e)))$ such that $\Psi(\Delta_2') = \Delta_2$ and $e(\nu', \Delta_2') = \varphi(f)$ where (ν, ν'), with $\nu \in V(\Delta_1)$ and $\nu' \in V(\Delta_2')$, is an intersection pair between Δ_1 and Δ_2'.*

Proof Assume that e is an idempotent of S. Let Δ_0 be the underlying graph of $Core(e)$, Then Δ_0 is a host of $\Gamma = S\Gamma(\overline{X}, R_{HNN} \cup R; e)$. If e is not $\mathcal{D}^{S^*}$-related to any idempotent of A (B) then it is the unique host, otherwise Γ has more than one host, each host is a lobe and a Schützenberger graph of some idempotent of A (B) relative to the presentation $\langle X|R\rangle$ by

Theorem 4.3.1, so statement 1 is proved.
If e is not $\mathcal{D}^{S^*}$-related to any idempotent of $A \cup B$ then Y is trivially finite. So assume that e is $\mathcal{D}^{S^*}$-related to some idempotent of $A \cup B$. Since by Theorem 4.3.1 all hosts of $S\Gamma(e)$ are Schützenberger graphs of idempotents of $A \cup B$, there are at most $|A \cup B|$ different hosts up to isomorphisms. From Proposition 4.2.1 and Theorem 4.2.1 it follows that two isomorphic hosts lie in the same $H_e^{S^*}$-orbit, hence $|V(Y)| \leq |A \cup B|$ and so statement 2 holds.
Let us prove the last statement. The "if" part is trivial. We prove the "only if" part. Let $(\Delta_1, t, \Delta_2) \in E(Y)$, since $Core(e)$ is a host and it consists only of one lobe then every host is a lobe, thus every lobe in $Host(\Gamma)$ is a host, hence Γ has more than one host. Therefore, by Theorem 4.3.1, e is $\mathcal{D}^{S^*}$-related to some idempotent of A (B). Moreover, there is $(\Delta_1', t, \Delta_2') \in Ed(\Upsilon_e)$ and is in the same $H_e^{S^*}$-orbit of (Δ_1, t, Δ_2). The lobes Δ_1', Δ_2' are adjacent, feed off each other. Let (q', p') $((p', q'))$ be an intersection pair with $q' \in V(\Delta_1')$ and $p' \in V(\Delta_2')$. By Theorem 4.3.1 $\Delta_1' \cong S\Gamma(X, R; e(q', \Delta_1'))$ with $e(q', \Delta_1') \in E(A)$ $(E(B))$ and $\Delta_2' \cong S\Gamma(X, R; e(p', \Delta_2'))$ with $e(p', \Delta_2') \in E(B)$ $(E(A))$. Moreover, there is an automorphism ϕ of $Host(S\Gamma(e))$ such that $\phi(\Delta_1') = \Delta_1$, then Δ_1 and $\phi(\Delta_2')$ are adjacent lobes such that $\phi(q') = q \in \Delta_1$ and $\phi(p') = p \in \phi(\Delta_2')$. Hence $e(q, \Delta_1) = e(q', \Delta_1') \in E(A)$ $(E(B))$. $\phi(\Delta_2') \cong S\Gamma(X, R; e(p', \phi(\Delta_2')))$ with $e(p', \phi(\Delta_2')) \in E(B)$ $(E(A))$ is a host and $(\Delta_1, t, \phi(\Delta_2')) \in Ed(\Upsilon_e)$ lies in the same $H_e^{S^*}$-orbit of (Δ_1, t, Δ_2). Then there is an automorphism ψ of $Host(S\Gamma(e))$ such that $\psi(\phi(\Delta_2')) = \Delta_2$, $\psi(\Delta_1) = \Delta_1$, $\psi(q') = \nu \in \Delta_1$ and $\psi(p') = \nu' \in \Delta_2$, thus $e(\nu, \Delta_1) \in E(A)$ $(E(B))$ and $e(\nu', \Delta_2) \in E(B)$ $(E(A))$ hence $\Delta_1 \cong S\Gamma(X, R; e(\nu, \Delta_1))$ and $\Delta_2 \cong S\Gamma(X, R; e(\nu', \Delta_2))$. ■

We have the following corollary.

Corollary 5.4.1. *Let e be $\mathcal{D}$-related in S^* to some idempotent g of $S \setminus (A \cup B)$, then the maximal subgroup $H_e^{S^*} \cong H_g^S$.*

Proof From the previous Theorem we conclude that the graph of groups, in the case when e is an original idempotent but it is not $\mathcal{D}$-related in the HNN-extension to any idempotent in $A \cup B$, consists of just a single vertex. This vertex corresponds to the Schützenberger graph of some g in S. Thus $Aut(Host(S\Gamma(g))) = Aut(S\Gamma(X, R; g))$ and by Theorems 1.3.3, 4.2.1 we have $H_g^{S^*} \cong Aut(S\Gamma(g)) = Aut(Host(S\Gamma(g))) = Aut(S\Gamma(X, R; g)) \cong H_g^S$. Hence, $H_g^{S^*} \cong H_g^S$. ■

We now consider e $\mathcal{D}$-related in S^* to some idempotent of $A \cup B$. We use Theorem 5.4.2 and the graph of groups $(\mathcal{G}(-), Y)$ to obtain a better description of the associated groups which are stabilizers of vertices and edges in Υ_e .

Corollary 5.4.2. *Let e be $\mathcal{D}$-related in S^* to some idempotent of $A \cup B$ then the stabilizers of the vertices of the graph of groups $(\mathcal{G}(-), Y)$ are maximal subgroups of idempotents of $A \cup B$ in the original semigroup S.*

Proof Let $e\mathcal{D}^{S^*}f$ with $f \in A \cup B$ thus $S\Gamma(e)$ has more than one host and each host is the Schützenberger graph of an element in $E(A) \cup E(B)$. By Theorem 5.4.2 vertices of Υ_e are Schützenberger graphs of idempotents belonging to $A \cup B$, thus by Theorem 1.3.3 we have for each vertex Δ in Υ_e, $Aut(\Delta) = Aut(S\Gamma(X, R; g)) \cong H_g^S$, for some $g \in E(A) \cup E(B)$. By Proposition 4.2.1 and the definition of $Stab_G(\Delta)$ we conclude that $Aut(\Delta) = Stab_G(\Delta)$. It follows that $Stab_G(\Delta) \cong H_g^S$. ■

The next proposition gives us a description of the stabilizer of an edge in Υ_e, but first we need the following lemma.

Lemma 5.4.2. *Let $(\nu, \Delta, \nu) = \mathcal{A}(X, R; f)$ for some $f \in A$. Let $I(\nu, \Delta) = \{y \in V(\Delta) : (\nu, u, y)$ is a path in Δ for some $u \in A\}$. Then*

$$H_f^A \cong \{\eta \in Aut(\Delta) : \eta(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$$

The case $f \in B$ is analogous

Proof Theorem 1.3.3 shows that $H_f^S \cong Aut(\Delta)$ by the isomorphism $m \mapsto \phi_m$ defined by $\phi_m(v) = m^{-1}v$. Since H_f^A is a subgroup of H_f^S, then H_f^A is also a subgroup of $Aut(\Delta)$. We claim that the map $u \mapsto \psi_u$ defined by $\psi_u(v) = u^{-1}v$ with $v \in V(\Delta)$, $u \in H_f^A$ is an isomorphism from H_f^A onto $\{\eta \in Aut(\Delta) : \eta(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$. To show that $\psi_u \in \{\eta \in Aut(\Delta) : \eta(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$ it is enough to prove that $\psi_u(\nu) \in I(\nu, \Delta)$ since each element of $I(\nu, \Delta)$ is connected to ν by some element of A and $\psi_u \in Aut(\Delta)$. Since f is the unity of H_f^A we get $u = fuf$, moreover since $\nu = f$ we get:

$$\psi_u(\nu) = (fu^{-1}f)f = f(fu^{-1}f) = \nu(fu^{-1}f)$$

so by [44] $fu^{-1}f \in A$ labels a path connecting ν to $\psi_u(\nu)$, whence $\psi_u(\nu) \in I(\nu, \Delta)$. It remains to show that $u \mapsto \psi_u$ is surjective. Let $\psi \in \{\eta \in Aut(\Delta) : \eta(I(\nu, \Delta)) \subseteq I(\nu, \Delta)\}$ then there is some $u \in A$ which labels a path in Δ connecting ν to $\psi(\nu)$. Since ψ is an automorphism also fuf labels a path connecting ν to $\psi(\nu)$. Note that the element $fu^{-1}f \in H_f^A$. Thus, consider the automorphism $\psi_{fu^{-1}f}$, then:

$$\psi_{fu^{-1}f}(\nu) = \psi_{fu^{-1}f}(f) = f(fuf) = \nu(fuf) = \psi(\nu)$$

and so $\psi_{fu^{-1}f} = \psi$ since they coincide on a vertex. ■

Proposition 5.4.1. *Let* Δ_1, Δ_2 *be two adjacent lobes of* $Host(S\Gamma(e))$ *and let* $f = f_A(\nu_1, \Delta_1) = \varphi^{-1}(f_B(\nu_2, \Delta_2))$ *for some intersection pair* (ν_1, ν_2) *between* Δ_1 *and* Δ_2*. Let* $z = (\nu_1, t, \nu_2)$*, then* $Stab_G(z) \cong H_f^A$ *(*$G = H_e^{S^*}$*).*

Proof Using Proposition 4.2.1 and Theorem 4.2.1, it is straightforward to check that $Stab_G(z)$ is isomorphic to $\{\eta \in Aut(\Delta_1) : \eta(V(\Delta_1) \cap V(O_p(\Delta_2))) \subseteq V(\Delta_1) \cap V(O_p(\Delta_2))\}$. By the assimilation property we have

$$V(\Delta_1) \cap V(O_p(\Delta_2)) = \{y \in V(\Delta_1) : (\nu_1, u, y) \text{ is a path in } \Delta_1 \text{ for some } u \in A\}.$$

and so $Stab_G(z) \cong H_f^A$ by Lemma 5.4.2. ■

For the clarity of the presentation we sumarize the previous facts in the following theorem.

Theorem 5.4.3. *With the above notations, if* e *is* $\mathcal{D}^{S^*}$*-related to some idempotent of* A *(*B*), then*

$$H_e^{S^*} \cong \pi(\mathcal{G}, Y)$$

where Y *is finite. Moreover, the group* G_v*,* $v \in V(Y)$*, is a maximal subgroup in* S *of some idempotent of* A *(*B*), while* G_y*,* $y \in E(Y)$*, is a maximal subgroup in* A *(*B*).* ■

Since Y is finite, from [13, page 14] follows that, $H_e^{S^*}$ is built by iteratively performing an amalgamated free product for each edge belonging to the maximal subtree Υ of Y, followed by HNN-extensions for each edge not in Υ. Therefore the next natural steps is to characterize whether Y is a tree or not. This clearly gives us a characterization which reveals whether the construction of $H_e^{S^*}$ involves just iterated group amalgams, or it also involves HNN-extensions. First we characterize the case where $H_e^{S^*}$ is finite in the following proposition.

Proposition 5.4.2. *Let* $e \in E(S^*)$ *with* $e\mathcal{D}^{S^*}f$ *for some* $f \in E(A) \cup E(B)$*. Then* $H_e^{S^*}$ *is finite if and only if* $H_e^{S^*} \cong H_g^S$*, for some* $g \in E(A) \cup E(B)$*.*

Proof By Propositions 4.2.1 , 4.4.1 and Theorem 4.2.1, $H_e^{S^*}$ is finite if and only if $Host(S\Gamma(f))$ is finite. If $Host(S\Gamma(e))$ is finite, since the automorphism group of a finite tree fixes a vertex or an edge (see [3, Subsection 27.1.3]), then it is straightforward to check that in this case each automorphism ϕ of $Host(S\Gamma(e))$ has to fix a lobe $\Delta = S\Gamma(X, R; g)$, $g \in E(A) \cup E(B)$. Since each such lobe is a host, thus $Aut(Host(S\Gamma(e))) \cong Aut(\Delta)$, whence $H_e^{S^*} \cong H_g^S$. The converse is trivial. ■

For infinite maximal subgroups we have the following characterization.

Theorem 5.4.4. *Let $e \in E(S^*)$ with $e\mathcal{D}^{S^*}f$ for some $f \in E(A) \cup E(B)$. Then the following are equivalent:*

1. $H_e^{S^*}$ *is infinite;*
2. *there is a sequence $f = f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2} \ldots, f_{2k-2,1}$ of idempotents of $A \cup B$ for some $k > 1$ such that:*
   - $f\mathcal{D}^{S^*}f_{1,2}$,
   - *for each $1 < i \leq 2k-1$, $f_{i,1}$ and $f_{i,2}$ are $\mathcal{D}^S$-related but are not $\mathcal{D}$-related in A (B),*
   - $f_{1,2}\mathcal{D}^S f_{k,1}\mathcal{D}^S f_{2k-2,1}$.
3. $Y = H_e^{S^*} \backslash \Upsilon_e$ *is not a tree.*

Proof Again, by Propositions 4.4.1, 4.2.1 and Theorem 4.2.1, $H_e^{S^*}$ is infinite if and only if $Host(S\Gamma(f))$ is infinite. Moreover by Corollary 4.4.1 $Host(S\Gamma(f))$ is infinite if and only if there is a reduced lobe path

$$P : \Delta_1, \Delta_2 \ldots, \Delta_k, \ldots, \Delta_{2k-1}$$

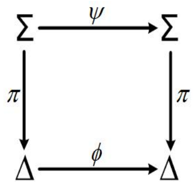
with $\Delta_1 \cong \Delta_t \cong \Delta_{2k-1}$. We prove the equivalence $1 \Leftrightarrow 2$ by showing that this geometric characterization is equivalent to the algebraic conditions described in the statement 2.
1) $\Rightarrow$ 2) Take any intersection pair $(\nu_{i,2}, \nu_{(i+1),1})$ between Δ_i and Δ_{i+1} with $\nu_{i,2} \in V(\Delta_i)$ and $\nu_{i+1,1} \in V(\Delta_{i+1})$ for $1 \leq i \leq 2k-1$ of P. By Proposition 3.2.1 and without loss of generality we have the following sequence $e(\nu_{1,2}, \Delta_1) = f_{C_{1,2}}(\nu_{1,2}, \Delta_1), e(\nu_{2,1}, \Delta_2) = f_{C_{2,1}}(\nu_{2,1}, \Delta_2), \ldots, e(\nu_{(2k-1),1}, \Delta_{2k-1}) = f_{C_{(2k-1),1}}(\nu_{(2k-1),1}, \Delta_{2k-1})$
of idempotents of $A \cup B$ with $C_{i,j}$ $1 \leq i \leq 2k-1, j = 1,2$ is either A or B and $e(\nu_{1,2}, \Delta_1)\mathcal{D}^{S^*}f$. Put $f_{i,j} = e(\nu_{i,j}, \Delta_i)$ $1 \leq i \leq 2k-1, j = 1,2$. Since $\Delta_1 \cong \Delta_k \cong \Delta_{2k-1}$, then $f_{1,2}\mathcal{D}^S f_{k,1}\mathcal{D}^S f_{(2k-1),1}$. Moreover it is straightforward to check that this sequence satisfies also the other conditions of statement 2.
2) $\Rightarrow$ 1) $\Delta_1 = S\Gamma(X, R; f_{1,2})$ is a host of $\Gamma = S\Gamma(\overline{X}, R_{HNN} \cup R; f) \cong S\Gamma(\overline{X}, R_{HNN} \cup R; e)$. Let $\nu_{1,2} \in V(\Delta_1)$ such that $e(\nu_{1,2}, \Delta_1) = f_{1,2}$. Since Γ is complete $\nu_{1,2}$ is an intersection vertex of Δ_1, so let Δ_2 be the lobe of Γ adjacent to the lobe Δ_1 at an intersection pair $(\nu_{1,2}, \nu_{2,1})$ with $\nu_{1,2} \in V(\Delta_1)$ and $\nu_{2,1} \in V(\Delta_2)$. Then $\Delta_2 = S\Gamma(X_2, R_2; f_{2,1})$, where $f_{2,1} = \varphi(f_{1,2})$ $(\varphi^{-1}(f_{1,2}))$, is a host by Proposition 3.2.1. Since $f_{2,1}\mathcal{D}^S f_{2,2}$ and $f_{2,1}$ is not $\mathcal{D}$-related in A (B) to $f_{2,2}$, then there is a vertex $\nu_{2,2} \in V(\Delta_2)$ which is not connected to $\nu_{2,1}$ by any path labeled by an element in A (B) and $e(\nu_{2,2}, \Delta_2) = f_{2,2}$. Note that $\nu_{2,2}$ can be the same vertex $\nu_{2,1}$ with

$f_{1,2} \in E(A)$ $(E(B))$ but $f_{2,1} \in E(B)$ $(E(A))$. Thus $\nu_{2,2}$ is an intersection vertex of Δ_2, and so there is a lobe Δ_3, different from Δ_1, such that the $(\nu_{2,2}, \nu_{3,1})$ is an intersection pair between Δ_2 and Δ_3 with $\nu_{2,2} \in V(\Delta_2)$, $\nu_{3,1} \in V(\Delta_3)$, and $\Delta_3 \cong S\Gamma(X, R; f_{3,1})$ is a host by Proposition 3.2.1. Using $f_{3,1}\mathcal{D}^S f_{3,2}$ and the fact that $f_{3,1}$, $f_{3,2}$ are not $\mathcal{D}$-related in A (B) we get that there is a vertex $\nu_{3,2}$ with $e(\nu_{3,2}, \Delta_3) = f_{3,2}$ which is an intersection vertex of Δ_3. Thus there is a lobe Δ_4 for which there is an intersection pair $(\nu_{3,2}, \nu_{4,1})$ between Δ_3 and Δ_4. Continuing in this way we build a reduced lobe path $P : \Delta_1, \ldots, \Delta_t, \ldots, \Delta_{2k-1}$ such that $(\nu_{i,2}, \nu_{(i+1),1})$ is an intersection pair between Δ_i and Δ_{i+1} with $\nu_{i,2} \in V(\Delta_i)$ and $\nu_{(i+1),1} \in V(\Delta_{i+1})$ for $1 \leq i \leq 2k-2$, with $e(\nu_{1,2}, \Delta_1) = f_{C_{1,2}}(\nu_{1,2}, \Delta_1), e(\nu_{2,1}, \Delta_2) = f_{C_{2,1}}(\nu_{2,1}, \Delta_2), \ldots, e(\nu_{2k-1}, \Delta_{2k-1}) = f_{C_{2k-1,1}}(\nu_{2k-1,1}, \Delta_{2k-1})$. Since $f_{1,2}\mathcal{D}^S f_{k,1}\mathcal{D}^S f_{(2k-1),1}$, we get $\Delta_1 \cong \Delta_k \cong \Delta_{2k-1}$.
1) $\Rightarrow$ 3) By Proposition 4.4.1 there is shift-isomorphism ϕ. Hence, there is a reduce lobe path $P : \Delta_1, \ldots, \Delta_{2k-1}$ such that Δ_1 is sent onto Δ_{2k-1} by ϕ and the automorphism on the lobe graph induced by ϕ does not map the edge (ν_1, t, ν_2) $((\nu_1, t^{-1}, \nu_2))$ connecting the lobes Δ_1 and Δ_2 into the edge $(\nu_{2k-1}, t, \nu_{2k-2})$ $((\nu_{2k-1}, k^{-1}, \nu_{2k-2}))$ connecting the lobes Δ_{2k-1} and Δ_{2k-2}. Therefore, these two edges do not belong to the same $H_e^{S^*}$-orbit, but $\phi(\Delta_k) = \Delta_{2k-1}$ thus Δ_k and Δ_{2k-1} are in the same vertex $H_e^{S^*}$-orbit, so the path P' induced by the path P in Y is a non-trivial loop because it contains a non-empty edge. Hence Y is not a tree.
3) $\Rightarrow$ 1) A reduced loop P' in Y lifts to a reduced lobe path $P : \Delta_1, \ldots, \Delta_{2k-1}$ in $Host(S\Gamma(f))$ for some $k > 1$ and with Δ_1, Δ_{2k-1} belonging to the same $H_e^{S^*}$-orbit. Hence there is an automorphism $\phi \in Aut(Host(S\Gamma(f)))$ which sends Δ_1 onto Δ_{2k-1}. Furthermore, any automorphism does not send the t-edge connecting Δ_1 to Δ_2 into the t-edge connecting Δ_{2k-1} to Δ_{2k-2}, otherwise P' would not be reduced. Hence, $\phi|_{\Delta_1} : \Delta_1 \to \Delta_{2k-1}$ is a shift-isomorphism. Therefore, by Proposition 4.4.1 $Host(S\Gamma(f))$ is infinite. ■

From the above theorem we obtain that Y is a tree if and only if $H_e^{S^*}$ is finite.

5.4.3 Maximal subgroups containing new idempotents

Let e be not $\mathcal{D}^{S^*}$-related to any idempotent of S. Then $S\Gamma(\overline{X}, R_{HNN} \cup R; e)$ has a unique host that is a t-subopuntoid subgraph of the underlying graph of $Core(e)$. Thus $H_e^{S^*}$ stabilizes some lobes Δ of the host. Since this lobe is finite for any $\nu \in V(\Delta)$ there is a minimum idempotent, namely $f = e(\nu, \Delta)$, labelling a loop based at ν. Thus, by [8, Lemma 2] (ν, Δ, ν) is a DV-quotient of the Schützenberger automaton $\mathcal{A}(X, R; f) = (\alpha, \Sigma, \alpha)$ called in [8] the maximum determinizing Schützenberger automaton of (ν, Δ, ν). Denoting by $\pi : (\alpha, \Sigma, \alpha) \to (\nu, \Delta, \nu)$ the natural homomorphism induced by this quotient we show that we can lift an automorphism ϕ of Δ to an automorphism ψ of Σ for which the following diagram commutes:



Theorem 5.4.5. *Let (ν, Δ, ν) be a closed inverse automaton relative to the presentation $\langle X|R\rangle$. With the above notation let (α, Σ, α) be the maximum determinizing Schützenberger automaton of (ν, Δ, ν) where $\pi(\alpha) = \nu$. Then every automorphism $\phi \in Aut(\Delta)$ can be lifted to an automorphism $\psi \in Aut(\Sigma)$ such that $\phi \circ \pi = \pi \circ \psi$. Moreover there is a group epimorphism from the subgroup $H := \{\psi \in Aut(\Sigma) : \exists \phi \in Aut(\Delta), \phi \circ \pi = \pi \circ \psi\}$ of $Aut(\Sigma)$ onto $Aut(\Delta)$ with kernel $N = H \cap S$ where $S = \{\psi \in Aut(\Sigma) : \psi(\pi^{-1}(\nu)) \subseteq \pi^{-1}(\nu)\}$.*

Proof Let $\phi \in Aut(\Delta)$, $\nu' = \phi(\nu)$ and let $w \in (X \cup X^{-1})^*$ be a word that labelling the path (ν, w, ν') in Δ. Since ϕ is a label preserving automorphism, then $e(\nu', \Delta) = e(\nu, \Delta) = f$ hence $ww^{-1} = w^{-1}w = f$. Since $(\alpha, \Sigma, \alpha) = \mathcal{A}(X, R; f)$ there is also a path (α, w, α') for some $\alpha' \in V(\Sigma)$ and by the minimality of $e(\nu', \Delta)$ we get $e(\alpha', \Sigma) = f$. Therefore $(\alpha', \Sigma, \alpha')$ and (α, Σ, α) are Schützenberger automata that accept the same language, hence by Theorem 1.3.3 there is an automorphism $\psi \in Aut(\Sigma)$ such that $\psi(\alpha) = \alpha'$. We prove that ψ is the automorphism satisfying the lifting property $\phi \circ \pi = \pi \circ \psi$. For this purpose let μ be a vertex of Σ and let $r \in (X \cup X^{-1})^+$ be a word labelling a path (α, r, μ) in Σ, so applying the automorphism ψ this path goes to (α', r, μ') with $\mu' = \psi(\mu)$. Consider $\pi(\mu)$ then clearly $(\nu, r, \pi(\mu))$ is a path in Δ thus the image of this path by ϕ is $(\nu', r, \phi(\pi(\mu)))$, hence $(\nu, wr, \phi(\pi(\mu)))$ is also a path in Δ. Consider now $\pi(\mu')$, since (α, wr, μ') is a path in Σ then $(\nu, wr, \pi(\mu'))$ is also a path in Δ, whence by the determinism of Δ we get $\phi(\pi(\mu)) = \pi(\mu') = \pi(\psi(\mu))$.
Let $H := \{\psi \in Aut(\Sigma) : \exists \phi \in Aut(\Delta), \phi \circ \pi = \pi \circ \psi\}$. It is straightforward checking that H is a subgroup of $Aut(\Delta)$.

For any $\psi \in H$, the relation $\pi \circ \psi \circ \pi^{-1} \subseteq V(\Delta) \times V(\Delta)$ is a function, since by definition of H there is a ϕ such that $\phi \circ \pi = \pi \circ \psi$ and so, taking into account that π is surjective, then for any right inverse π^{-1} we have

$$(\pi \circ \psi) \circ \pi^{-1} = (\phi \circ \pi) \circ \pi^{-1} = (\pi \circ \pi^{-1}) \circ \phi = 1_\Delta \circ \phi = \phi$$

So there is a map $\lambda : H \to Aut(\Delta)$ defined by $\lambda(\psi) = \pi^{-1} \circ \psi \circ \pi$. Moreover λ is surjective since by the first statement of the theorem for any $\phi \in Aut(\Delta)$ there is a $\psi \in Aut(\Sigma)$ such that $\phi \circ \pi = \pi \circ \psi$ and so $\psi \in H$ and $\lambda(\psi) = \phi$. It remains to show that λ is a homomorphism. Let $\psi_i \in H$ and let $\phi_i \in Aut(\Delta)$ such that $\psi_i \circ \pi = \pi \circ \phi_i$ for $i = 1, 2$, by the definitions we get

$$\begin{aligned} \lambda(\psi_1 \circ \psi_2) &= \pi \circ (\psi_1 \circ \psi_2) \circ \pi^{-1} = \\ &= (\pi \circ \psi_1) \circ (\pi^{-1} \circ \phi_2) = (\pi \circ \psi_1 \circ \pi^{-1}) \circ \phi_2 = \\ &= \lambda(\psi_1) \circ \phi_2 = \lambda(\psi_1) \circ \lambda(\psi_2) \end{aligned}$$

The last statement is a routine calculus which involves only the definitions of H and λ. ■

Note that without the finiteness condition of the inverse semigroup S, in general it is not possible to define the maximum determinizing Schützenberger automaton of a closed inverse word automaton relative to the presentation $\langle X|R\rangle$ of the inverse semigroup S. It is also quite easy to produce an example where it is not possible to lift an automorphism of a closed DV-quotient Δ of a Schützenberger automaton Σ to an automorphism of Σ (see [38]). Moreover the subgroup H in the previous theorem is in general a proper subgroup of $Aut(\Sigma)$. To prove this fact it is enough to consider the dihedral group $D_4 = Gp\langle r, s|r^2, s^2, (rs)^4\rangle$ which is clearly a finite inverse semigroup with only one Schützenberger graph which is the Cayley graph of D_4. If in the Cayley graph of D_4 we identify the identity e with s and then we determinize, we obtain an inverse word graph Δ with $Aut(\Delta) \cong Gp\langle\sigma|\sigma^2\rangle$. It is easy to show that σ can be lifted up to the automorphism $(sr)^2$, however the automorphism (sr) is not in H.

The next theorem covers the case.

Theorem 5.4.6. *Let $e \in E(S^*)$ with S finite inverse semigroups and suppose that e is not $\mathcal{D}^{S^*}$-related to any idempotent of S. Therefore $H_e^{S^*}$ is a homomorphic image of some subgroup of the maximal subgroup H_g^S of S for some $g \in E(S)$.*

Proof In this case, $Host(S\Gamma(e))$ consists of a unique host with finitely many lobes. Thus using Theorems 1.3.3, 4.2.1 and Lemma 4.2.3 implies that $H_e^{S^*}$ is isomorphic to a subgroup of the automorphism group of some lobe Δ of $Host(S\Gamma(e))$. By Theorem 5.4.5 $Aut(\Delta)$ is an homomorphic image of $Aut(\Sigma)$ where $\Sigma = S\Gamma(X, R; g)$ for some $g \in E(S)$. Therefore $H_e^{S^*}$ is a homomorphic image of $Aut(\Sigma) \cong H_g^S$. In particular $H_e^{S^*} \cong H_g^S/N$ where N is the normal subgroup described in Theorem 5.4.5. ■

Conclusions

In this thesis we dealt with algorithmic and structural issues of HNN-extensions of inverse semigroups focusing on the case that the original inverse semigroups are finite. We considered the definition of Yamamura.

In the first part we proved that the word problem for HNN-extensions of inverse semigroups is undecidable even under some nice conditions. Then we restricted ourself to the case of HNN-extensions of finite inverse semigroup where the word problem was proved to be decidable, and we shown that the languages recognized by the Schützenberger graphs of the words of HNN-extensions of finite inverse semigroup are context-free languages then we built the context-free grammars for the deterministic pushdown automata which recognize such languages. This result allowed us to give an alternative proof of the above mentioned decidability result.

In the second part we were interested in the algebraic structural properties of the HNN-extension of finite inverse semigroups. We gave a characterization for HNN-extension of finite inverse semigroup containing a copy of a bicyclic subsemigroup, we also gave a characterization for HNN-extensions of finite inverse semigroups which are completely semisimple and we almost gave a comprehensive study of the maximal subgroups of HNN-extensions of finite inverse semigroups. We used Basse-Serre theory and the structural properties of the Schützenberger graphs to prove that the maximal subgroups are isomorphic to the fundamental groups of certain graph of groups. Furthermore, we divided these maximal subgroups into types according the idempotent of subgroup. In case the idempotent is from the original inverse semigroup we proved that the maximal subgroups are isomorphic to fundamental group of graph of groups based on the $\mathcal{D}$-structure of the original inverse semigroup. In case the idempotent is in $A \cup B$ then the stabilizers of the vertices of the graph of groups, are maximal subgroups of idempotents of $A \cup B$ in the original semigroup S, while the stabilizers of the edges are the maximal subgroups in A or in B. In case the idempotent is a new one then the maximal subgroup is isomorphic to a subgroup of the original inverse semigroup. Lastly, the thesis contains some results about the graph theoretical structure of the Schützenberger graphs of the words of HNN-extension of finite inverse semigroups that are useful for our studying, for instance we introduced an equivalence relation on the vertices of the Schützenberger graph that generalizes the relation ρ, we gave a characterization for the hosts, the finite t-subopuntoid subgraphs of the Schützenberger graphs that contain all the information for building the Schützenberger graphs, to contain more than one host, and we gave a characterization for the Schützenberger graphs to be infinite graphs. We also gave alternative proofs for some results.

Further work is twofold. In the framework of Yamamura's definition of HNN-extension (which seems to be the one closer to the corresponding definition in group theory), one could try to find other conditions on the original inverse semigroup S in order the word problem to be decidable. Moreover, other decidability problems could be considered in HNN-extensions of finite inverse semigroups, for instance membership problem or else the solvability of some classes of equations on HNN-extensions of finite inverse semigroups. Then, since there is no unique definition of HNN-extension of inverse semigroups, one could consider the other definitions of HNN-extensions given in the literature (e.g. the Gilbert's, see [14] and [50]) and look for the problems considered in this thesis for these definitions. Such a work could also clarify the relationship among these variants of HNN-extensions of inverse semigroups.

Bibliography

[1] C. L. Adair. A generalization of the bicyclic semigroup. *Semigroup forum*, 21:13–25, 1980.

[2] C. Ash. Embedding theorem using amalgamation bases. *Semigroups (T. E. Hall, P. R. Jones, G. B. Preston, eds.) (Academic Press)*, pages 167–176, 1980.

[3] L. Babai. *Automorphism groups, isomorphism, reconstruction*. R.L. Graham, M. Grotschel, L. Lovász (Eds.), Handbook of Combinatorics, North- Holland, 1995.

[4] P. Bennett. On the structure of inverse semigroup amalgams. *International Journal of Algebra and Computation*, 7 (5):577–604, 1997.

[5] W. W. Boone. Certain simple unsolvable problems of group theory. *Indag. Math.,(16)231-237, 492-497(1954);(17)252-256,571-577(1955); (19) 22-27, 227-232.*, 1957.

[6] K. Byleen, J. C Meakin, and F. Pastijn. The fundamental four-spiral semigroup. *Journal of Algebra*, 54:6–26, 1978.

[7] A. Cherubini, T. Jajcayova, and E. Rodaro. Maximal subgroups of amalgams of finite inverse semigroups. *To appear in the Simigroup Forum DOI 10.1007/s00233-014-9601-4*.

[8] A. Cherubini, J. C. Meakin, and B. Piochi. Amalgams of finite inverse semigroups. *Journal of Algebra*, 285:706–725, 2005.

[9] A. Cherubini, C. Nuccio, and E. Rodaro. Amalgams of finite inverse semigroups and deterministic context-free languages. *Semigroup Forum*, 85(1):129–146, 2012.

[10] A. Cherubini and E. Rodaro. Amalgams vs Yamamura's HNN-extensions of inverse semigroups. *Algebra Colloquium*, 18(04):647–657, 2011.

[11] J. Daniel and E. Cohen. *Combinatorial group theory:A topological approach*. Cambridge University Press, 1989.

[12] M. Davis. *Computability and unsolvability*. Dover Publications, INC. New York, 1958.

[13] W. Dicks and M. J. Dunwoody. *Groups acting on graphs*. Cambridge studies in advanced mathematics 17, 1989.

[14] N. D. Gilbert. HNN-extensions of inverse semigroups and groupoids. *Journal of Algebra*, 272:27–45, 2004.

[15] P. A. Grillet. *Semigroups: an introduction to structure theory*. CRC Press, 1995.

[16] P. A. Grillet. On the fundamental double four-spiral semigroup. *Bull. Belg. Math. Soc. Simon Stevin*, 3(2):201–208, 1996.

[17] S. Haataja, S. Margolis, and J.C. Meakin. On the structure of fully regular semigroup amalgams. *Journal of Algebra*, 183:38–54, 1996.

[18] T. E. Hall. Inverse and regular semigroups and amalgamations: A brief survey. *Proc. Sympos. Semigroups, Nothern Illinois Univ., De Kalb*, I11:49–79, 1979.

[19] M.A. Harrison. *Introduction to formal languages theory*. Addison-Wesley, 1978.

[20] G. Higman, B. H. Neumann, and H. Neumann. Embedding theorems for groups. *J. London Math. Soc.*, 26:247–254, 1949.

[21] J. W. Hogan. The α- bicyclic semigroup as a topological semigroup. *Semigroup Forum*, 28:265–271, 1984.

[22] J.E. Hopcroft, R. Motwani, and J.D. Ullman. *Introduction to automation theory, languages and computation*. Addison-Wesley, 2001.

[23] J. M. Howie. Embedding theorems for semigroups. *Quart. J. Math (2)*, 14:254–258, 1963.

[24] J. M. Howie. *Automata and languages*. Clarendon Press. Oxford, 1991.

[25] J. M. Howie. *Fundamentals of semigroup theory*. Oxford Science Pubblications, 1995.

[26] T. Jajcayova. HNN-extensions of inverse semigroups. *PhD Thesis at University of Nebraska-Lincoln Department of Mathematics and Statistics*, 1997.

[27] P. R. Jones, S. W. Margolis, J. Meakin, and J. B. Stephen. Free products of inverse semigroups II. *Glasgow Math. J.*, 33(3):373–387, 1991.

[28] M. Lawson. *Inverse semigroups*. World Scientific Publishing Co. Pte. Ltd, 1998.

[29] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory "Classics in Mathematics" series, reprint of the 1977 edition*. World Scientific Publishing Co. Pte. Ltd, 2001.

[30] S. Margolis, J. C. Meakin, and M. Sapir. Algorithmic problems in groups, semigroups and inverse semigroups. *Semigroups, Formal Languages and Groups (J. Fountain ed. 1995)* NATO *adv. sci. inst. ser. c Math. Phys. Sci.*, 466:147–214, 1995.

[31] D.E. Müller and P.E. Schupp. Context-free languages, groups, the theory of ends, second order logic. *TCS*, 37:51–75, 1985.

[32] J. C. Meakin. An invitation to inverse semigroups. *Proc. of the conference on Ordered structures and algebra of computer languages (K.P.Shum and P.C.Yuen eds.)*, pages 91–115, 1993.

[33] W. D. Munn. Free inverse semigroups. *Proc. London Math. Soc.*, 29 (3):385–404, 1974.

[34] P. S. Novikov. On the algorithmic unsolvability of the word problem in group theory. *Trudy. Math. Inst. Steklov*, (3) 29:44–143, 1955.

[35] D. Perrin and J. E. Pin. *Infinit words automata, semigroups, logic and games*. Elsevier Accademic Press, 2004.

[36] M. Petrich. *Inverse semigroups*. A Wiley-Interscience Publication John Wiley & Sons, 1984.

[37] G. B. Preston. Representations of inverse semi-groups. *J. London Math. Soc.*, 29:411–419, 1954.

[38] E. Rodaro. HNN-extensions and amalgams of finite inverse semigroups. *PhD Thesis at Università degli Studi di Milano, Department of Matematics Federigo Enriques*, 2006.

[39] E. Rodaro. Bicyclic subsemigroups in amalgams of finite inverse semigroups. *International Journal of Algebra and Computation*, 20(01):89–113, 2010.

[40] E. Rodaro and A. Cherubini. Decidability of the word problem in Yamamura's HNN-extensions of finite inverse semigroups. *Semigroup Forum*, 77(2):163–186, 2008.

[41] E. Rodaro and P.V. Silva. Amalgam of inverse semigroups and reversible two-counter machines. *Journal of Pure and Applide Algebra*, 217(4):585–597, 2013.

[42] H. E. Scheiblich. Free inverse semigroups. *Proc. Amer. Soc.*, 38:1–7, 1973.

[43] Jean-Pierre Serre. *Trees*. Springer-Verlag Berlin Heidelberg, 1980.

[44] J. B. Stephen. Presentation of inverse monoids. *Journal of Pure and Applied Algebra*, 198:81–112, 1990.

[45] J. B. Stephen. Amalgamated free products of inverse semigroups. *Journal of Algebra*, 208:339–424, 1998.

[46] V. V. Vagner. Generalized heaps and generalized groups with a transitive compatibility relation. *International Journal of Algebra and Compution*, 70:25–39, 1961.

[47] A. Yamamura. HNN-extensions of inverse semigroups and applications. *International Journal of Algebra and Computation*, 7 (5):605–624, 1997.

[48] A. Yamamura. HNN-extensions of semilattices. *International Journal of Algebra and Computation*, 09:555–596, 1999.

[49] A. Yamamura. Locally full HNN-extensions of inverse semigroups. *J. Aust. Math. Soc.*, 72 (2):605–624, 2001.

[50] A. Yamamura. Embedding theorems for HNN-extensions of inverse semigroups. *Journal of Pure and Applied Algebra*, 210:521–535, 2007.