Strategic Behaviour Study of Flexible Generator in Hybrid DSO and P2P Markets - A Three-Phase Unbalanced Distribution Network Case

Master of Science Dissertation in
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Abstract

The high penetration of distributed energy resources (DERs) into the electric grid in medium voltage (MV) and low voltage (LV) electric networks requires further reconsideration about developing local markets at this grid level. Unlike the transmission high voltage (HV) grids, the energy transaction in the distribution network (DN) is highly affected by the active losses due to the high ratio of R/X. Therefore, distribution locational marginal prices (DLMPs) should be calculated with further accuracy by considering the three-phase unbalanced nature of DN. To this end, in this work, a linearized approximation of AC-OPF is used to obtain the DLMPs in terms of energy price at the grid supply point (GSP), total active and reactive losses, voltage support, and congestion in the feeders. In due course, the distribution system operator (DSO) ancillary services (AS) market is formulated using the linearized AC-OPF to clear the market by motivating the flexible DERs, including flexible loads (FL) and flexible generators (FG), to curtail or re-dispatch, respectively, to keep the DN operating optimally. Moreover, the grid usage price (GUP), which is the derivative of DLMP, is calculated to enable bilateral contract energy trade. A fully decentralized peer-to-peer (P2P) market framework is adopted to model the bilateral energy transaction using distributed optimization (DO) approach, the augmented direction method of multipliers (ADMM). The essential novelty of this work is to employ the Stackelberg game-theoretic (bi-level programming) approach for modelling the strategic behaviour of the DERs that can lead to gird constraint violation and consequent higher DLMPs. Accordingly, at the upper-level of the FG submits strategic offers and maximizes its revenue by manipulating the DLMP in its favour. By contrast, the DSO AS market in the lower-level problem clears the market, maximizing the social welfare by incorporating the costs for the FGs, and utility for the FLs. In addition, the optimal and safe operation of the DN is considered in the lower-level problem.

Keywords: Strategic DER; Bi-level programming; DSO ancillary services market; P2P Market; ADMM

Sommario

L'elevata penetrazione delle risorse energetiche distribuite (DERS) nella rete elettrica in reti elettriche di media tensione (MV) e bassa tensione (LV) richiede un'ulteriore riconsiderazione sullo sviluppo di mercati locali a questo livello di rete. A differenza delle griglie ad alta tensione (HV) di trasmissione, la transazione energetica nella rete di distribuzione (DN) è fortemente influenzata dalle perdite attive a causa dell'elevato rapporto di R/X. Pertanto, i prezzi marginali localizzati di distribuzione (DLMP) devono essere calcolati con ulteriore precisione considerando la natura sbilanciata trifase di DN. A tal fine, in questo lavoro, un'approssimazione linearizzata di AC-OPF viene utilizzata per ottenere i DLMP in termini di prezzo di energia nel punto di approvvigionamento della rete (GSP), perdite totali attive e reattive, supporto di tensione e congestione negli alimentatori. A tempo debito, il mercato dei servizi ausiliari (AS) del sistema di distribuzione (AS) viene formulato utilizzando l'AC-OPF linearizzato per cancellare il mercato motivando i DER flessibili, inclusi carichi flessibili (FL) e generatori flessibili (FG) o ri-dispatch, rispettivamente, per mantenere il DN funzionante in modo ottimale. Inoltre, il prezzo di utilizzo della griglia (GUP), che è il derivato di DLMP, è calcolato per consentire il commercio di energia contrattuale bilaterale. Viene adottato un framework di mercato peer-to-peer (P2P) completamente decentralizzato per modellare l'approccio di Ottimizzazione Distribuite (DO) di Ottimizzazione distribuita, il metodo della direzione aumentata dei moltiplicatori (ADMM). La novità essenziale di questo lavoro è quella di utilizzare l'approccio teorico-teorico (programmazione a due livelli) di Stackelberg per modellare il comportamento strategico dei DER che possono portare a violazioni dei vincoli GIRD e conseguenti DLMP più elevati. Di conseguenza, al livello superiore della FG presenta offerte strategiche e massimizza le sue entrate manipolando il DLMP a suo favore. Al contrario, il DSO come mercato nel problema di livello inferiore cancella il mercato, massimizzando il benessere sociale incorporando i costi per l'FGS e l'utilità per il FLS. Inoltre, il funzionamento ottimale e sicuro del DN è considerato nel problema di livello inferiore.

Parole chiave: DER strategico; Programmazione a due livelli; Mercato dei servizi ausiliari DSO; Mercato P2P; ADMM

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The work at hand is the result of hard-working days in Milano and Munich. Days that I spent far from my beloved family abroad to pursue my dreams. At this moment, I would like to express my most genuine gratitude to those who were there for me throughout my master's degree studies.

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Finally, I would like to dedicate this work to the loving memory of my grandmother, who passed away of Covid on the first day of starting this project in Munich on October 14th, 2021. May she rests in peace.

"Offen love between two people intensifies not because of beauty or some advantage, but because of sheer spiritual affinity."
by **Al-ghazali**

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Nomenclature

Notation

$\jmath$ is the imaginary unit with $\jmath^2 = -1$. Scalars are denoted by small no-bold letters, e.g. x. Vectors and matrices are in bold letters, i.e., $\mathbf{x}, \mathbf{X}$. Entries of a matrix $\mathbf{X}$ are specified by $x_{i,j}$. All vectors are column vectors. The i-th row and j-th column of a matrix $\mathbf{X}$ is denoted by $\mathbf{x}_{i,*}$ and $\mathbf{x}_{*,j}$, respectively. Entrywise matrix multiplication is denoted by $\mathbf{X} \circ \mathbf{X}$. Entries of vector $\mathbf{x}$ are specified by x_i. The obtained optimal solutions for an optimization variable $x, \mathbf{x}$ are denoted by $x^*, \mathbf{x}^*$. For complex scalars, vectors or matrices, $\Re(), \Im()$ are used to extract the real and imaginary part. The transpose of a vector or matrix is denoted by $()^\mathrm{T}$ and $\mathrm{diag}(\mathbf{x})$ constructs a diagonal matrix with entries of $\mathbf{x}$. $|\mathbf{x}|$ denotes the magnitude of a complex entity.

Acronym	**Description**
AC	Alternating Current
AC-OPF	Alternating Current Optimal Power Flow
ADMM	Alternating Direction Method of Multipliers
AS	Ancillary Services
BESS	Battery Energy Storage System
DAM	Day Ahead Market
DC	Direct Current
DC-OPF	Direct Current Optimal Power Flow
DER	Distributed Energy Resource
DERMS	Distributed Energy Resource Management System
DG	Distributed Generator
DLMP	Distribution Locational Marginal Price
DN	Distribution Network

DO	Distributed Optimization
DSO	Distribution System Operator
EV	Electric Vehicle
ESS	Energy Storage System
FTR	Financial Transmission Right
FG	Flexible Generator
FL	Flexible Load
GNE	Generalized Nash Equilibrium
GSP	Grid Supply Point
GUP	Grid Usage Price
HV	High Voltage
KKT	Karush–Kuhn–Tucker
LMP	Locational Marginal Price
LP	Linear Programming
LV	Low Voltage
MCP	Mixed Complementarity Problem
MESMO	Multi-Energy System Modeling and Optimization
MILP	Mixed-Integer Linear Programming
MPCC	Mathematical Problem with Complementarity Constraints
MPCC	Mathematical Problem with Equilibrium Constraints
MV	Medium Voltage
NE	Nash Equilibrium
OPF	Optimal Power Flow
P2P	Peer-to-Peer
PTDF	Power Transmission Distribution Factor
PV	Photovoltaic

SDP	Semi-definite Program
SG	Strategic Generator
SOCP	Second-Order-Conic Program
TSO	Transmission System Operator
Number Sets	**Description**
$\mathbb{C}$	Complex Numbers
$\mathbb{R}$	Real Numbers
$\mathcal{D}$	Set of DERs
$\mathcal{P}$	Set of Phases
$\mathcal{N}$	Set of Nodes
$\mathcal{H}$	Set of Lines
$\mathcal{T}$	Set of Time-steps
$\mathcal{S}$	Set of Seller Peers
$\mathcal{B}$	Set of Buyer Peers
Parameters	**Description**
$\mathbf{A}$	DER Model State Matrix
$\mathbf{B}$	DER Model Control Matrix
$\mathbf{C}$	DER Model Output Matrix
$\mathbf{D}$	DER Model Feed-through Matrix
$\mathbf{v}^x$	State Disturbance Vector
$\mathbf{v}^y$	Output Disturbance Vector
$\mathbf{Y}$	Network Admittance Matrix
$\mathbf{Y}^f$	Branch Admittance Matrix in "From" Direction
$\mathbf{Y}^t$	Branch Admittance Matrix in "To" Direction
$\mathbf{A}^{f/t}$	Branch Incidence Matrices in "From/To" Directions
$\mathbf{M}^{p/q,y}$	DER Active/Reactive Mapping to Output Matrix

$\mathbf{M}_p^{(\cdot)}$	Sensitivity Matrix of $(\cdot)$ Entity with Respect to DER's Active Power Injection
$\mathbf{M}_q^{(\cdot)}$	Sensitivity Matrix of $(\cdot)$ Entity with Respect to DER's Reactive Power Injection
$\hat{p}_t^{loss}$	Active Loss Constant
$\hat{q}_t^{loss}$	Reactive Loss Constant
$\lvert\hat{\mathbf{u}}_t\rvert$	Nodal Voltage Constant Magnitude Vector
$\lvert\mathbf{s}_t^{f/t}\rvert$	Branch Active Power Constant Magnitude Vector in "From"/"To" Directions
$c_t^{0,p}$	Energy Price at GSP
$\mathbf{c}_t^p$	DER Marginal Active Power Cost/Utility
Variables	**Description**
$\mathbf{p}$	DER Active Power Vector Vector
$\mathbf{q}$	DER Reactive Power Vector Vector
$\mathbf{u}$	Nodal Voltage Vector
$\mathbf{s}$	Apparent Power Injection Vector
$\mathbf{s}^f$	Branch Power in "From" Direction
$\mathbf{s}^t$	Branch Power in "To" Direction
$\mathbf{x}$	State Variable Vector
$\mathbf{u}$	Control Variable Vector
$\mathbf{y}$	Output Variable Vector
p_t^{loss}	Active Losses
q_t^{loss}	Reactive Losses
$\boldsymbol{\pi}_t$	DLMP
$c_t^{offer,p}$	SG's Strategic Offer
$\boldsymbol{\Psi}^{(\cdot)}$	$(\cdot)$ Entity Sized Binary Variables Vector
$\boldsymbol{\lambda}$	Equality Constraints Dual Variable Vector
$\boldsymbol{\mu}$	In-Equality Constraints Dual Variable Vector

Introduction

The concept of the TSO market has already matured enough to be exploited on large and commercial scales in high voltage (HV) transmission grids. Moving toward further intelligent energy networks and a zero CO2 environment, the necessity of adopting a market mechanism for incentivizing prosumers to participate in local energy market distribution has become significant [1]. In the distribution networks (DN), given the high number and variety of DERs at DN and the feeders being medium and low voltage (MV/LV) with a high ratio of resistance per reactance (R/X), the contribution of active energy losses, voltage drop, and feeder congestion are substantial [2]. Therefore, the prospective DSO local market is inherent to issues such as voltage unbalance and the possible strategic behaviour of DERs that can play a vital role in providing support for the DN in the downstream feeders [3].

Moreover, the mentioned unique topology of DN leads to the inefficiency of conventional DC optimal power flow solutions for spot pricing of energy, so-called locational marginal pricing (LMP). Therefore, adopting a proper power flow analysis is critical for tackling optimal operation problems in the distribution grids and spot pricing of energy reputed as distribution locational marginal pricing (DLMP) by considering the magnitude of the voltages at different phases as well as the reduction of active losses in the grid [4]. In particular, AC-OPF can be a proper approach to handle this matter in question. However, mathematically, this method is highly non-convex and non-linear, and it makes the solution to the OPF problem either infeasible or unreliable, with high chances of being trapped at a locally optimal solution [5]. Hence, different methods are proposed for dealing with these challenges.

In general, two promising approaches are proposed to cope with the non-convex AC-OPF; i) Relaxation approaches, in which some constraints of the OPF problem are loosened to yield a convex solution for the problem [6]. Second-order conic (SOC) programming, as one of the least tight relaxation methods, is mainly used to solve the convexified optimization problem of AC-OPF.

ii) Linearized AC-OPF where the non-linear terms of the constraints are written with reference to the active and reactive nodal injection in the radial DN. Accordingly, the Taylor series linearization method is used based on the sensitivity matrices, i.e., power transmission distribution factor (PTDF) [7]. Having linearized the power flow equations, the presentation of DLMPs will be more straightforward and more intuitive; however, the accuracy of results will depend on the flat point at which the linearization is extended. Therefore, [4, 8] an iteration-based methodology is proposed in which the flat point is updated iteratively to achieve more accurate results. Providing the configuration of DN is radial at which each node has a single unique parent, the fixed points for the linearized AC-OPF can be calibrated based on the power flow results from the nominal operation points of the DERs in the DN, which will lead to acceptable approximation [9]. As a basis for our work, the latter approach, the linearization method, will be employed in our work for OPF analysis and obtaining DLMPs.

Regarding the functionality of the DSO AS market, the market mechanism adopted in this work is based on [10]. The perception is that the DSO, as the market organizer, has already participated in the wholesale day-ahead market (DAM) organized by TSO by anticipating the power demand pattern for the following day. Accordingly, the TSO market is cleared, and the market prices are broadcast to the DSO. For the DSO AS market, the main objective is first to minimize the losses in the downstream grids, second to minimize power procurement from real-time or the reserve market, and third to ensure the safe and optimal operation of DN without voltage violation or line thermal overloading in terms of ancillary services. Therefore, the DSO AS market provides flexible DERs with incentives to re-dispatch/curtail so that the overall costs of DSO are minimized. At the same time, the operational limitations of the network are met [1, 11].

From the elastic DER's perspective, the AS market can provide them with an opportunity to sell their over-generation capacity (for RES) or to curtail their unnecessary demand (for flexible loads) to make a profit out of the market. Considering the future intelligent grids, the DERs can also include large-scale storage, flexible buildings, and flexible electric vehicle (EV) chargers that can actively play a role in the DSO AS markets [12, 13].

A) Literature Review

A[14], a bi-level programming method is proposed to motivate DERs to participate in the market proactively. The model adopts a novel undirected SOC form of AC OPF to calculate DLMPs under sufficient assumptions and conditions. To solve the bi-level programming, mixed-integer semidefinite programming is employed. Literature [15] has provided A GNB-based market mechanism to settle the energy transaction between DERs.

Also, relaxation is used to transform non-convex AC OPF to SOC programming, and bilinear terms are linearized by the linear outer approximation method. The mechanism of cooperation is mutual trust and truthful sharing of information. Similar to the work we will propose, In [16], bi-level demand management is presented for an industrial zone, including heat and power infrastructure based on DLMP derived from SOCP AC power flow. J. Conejo in [17], analyses the equilibrium point for the strategic prosumer and DSO market using SOC relaxed AC-OPF in a single-phase DN. Eventually, it concludes that the equilibrium point obtained from bilevel programming coincides with the results of the market fully controlled by DSO. Similarly, in [18], a game-theoretic bi-level programming for modelling the prosumer in the DSO market; however, a transparent methodology for the network constraints is not illustrated. Eventually, the paper [19] obtains the GNE problem of the strategic DER and DSO market by considering the SOC of AC-OPF and the GNE point is obtained based on iterative approaches.

In the majority of the works introduced above, the SOCP is adopted to solve the AC-OPF in DN to deal with the non-linearities, approximations are applied.. The theoretical drawback of this approach is the risk of obtaining a solution from the relaxed problem, which does not coincide with the original solution set [20, 21]. Moreover, mainly the relaxed AC-OPF results in a recursive formula making the interpretation of DLMPs in radial DN counterintuitive. Furthermore, the three-phase unbalanced configuration of DN is not taken into account as the most critical characteristic of DN.

B) Motivation

In the literature review section, we saw that magnificent works had been carried out to model the transactive energy management of DERs in DSO local markets. Nevertheless, a thorough methodology with a clear interpretation of DLMP's role for simultaneous incorporating to the reliable operation of the network as well as satisfying the DERs for proactively participating in the market is not provided. Moreover, in most cases, relaxations are adopted for solving the bi-level problem, reducing the methodology's reliability. In addition, the real three-phase unbalanced configuration of the DN is not adequately modelled.

1. The three-phase unbalanced network is modelled as a realistic platform to integrate DSO AS market.

2. The state-space modelling of flexible DERs is introduced to encapsulate the operational limits of flexible DERs.

3. The MCP formulation of the DSO market is obtained by deriving the joint-KKT conditions for the DSO problem to validate the Nash-equilibrium point for the DSO AS single-level market problem.
4. The strategic behaviour of large-scale FG is studied by adopting the Stackelberg game theoretic approach as a bi-level programming problem.
5. The MPEC of the bi-level problem is obtained, and its equivalent mixed-integer linear programming (MILP) is retrieved from the dual problem of the DSO market problem.
6. The feasibility of integrating the decentralized P2P market solved by ADMM into the strategic DSO AS market is studied.

In the following chapters, the methodologies and problem formulations will be outstretched.

1 | Electric Grid and DER Modelling

This chapter is devoted to introducing the electric grid model of a multi-phase distribution network (DN) and the connected distributed energy resources (DER). Accordingly, in the first step, the non-linear power flow equations for a radial DN are provided, and then they will be reformulated on a linearly approximated basis. Eventually, the state-space modelling of DERs will be developed to characterise the actual behaviour of flexible loads and generators during a specific time horizon. The principal content of this chapter is derived from [2, 4, 7, 22], and they are provided for the sake of convenient reference and access. Therefore, as indicated earlier, this work's essential contribution will be provided in chapter three.

1.1. Multi-phase Nonlinear Power Flow

Having defined a radial topology for the DN and letting the sets of nodes and branches to be defined as $\mathrm{n} \in \mathcal{N}$ and $\mathrm{b} \in \mathcal{B}$, the relationship between the number of nodes and branches is $len(\mathcal{B}) = len(\mathcal{B}) + 1$. Therefore, each node originated from a unique parent node [23]. Moreover, since we are going to study the multi-phase radial DN, the expression "branch" refers to the phases (single, double, or triple) connecting two nodes (buses) as an interface.

Figure 1.1 indicates the Π-model of an electric grid branch where $\mathbf{u}_{\mathrm{m/n}} = \left[u^p_{\mathrm{m/n}}\right]_{p\in\mathcal{P}} \in \mathbb{C}^{3\times1}$ are three phase voltage column vectors for the generic buses m and n that $p \in \mathcal{P}$ is the phase index. The net current and power injection at the nodes are shown by $\mathbf{i}_{\mathrm{m/n}} = \left[i^p_{\mathrm{m/n}}\right]_{p\in\mathcal{P}} \in \mathbb{C}^{3\times1}$ and $\mathbf{s}_{\mathrm{m/n}} = \left[s^p_{\mathrm{m/n}}\right]_{p\in\mathcal{P}} \in \mathbb{C}^{3\times1}$, respectively. Moreover, the current and power flowing in the branches in "from" and "to" directions are shown by $\mathbf{i}^{f/t}_{\mathrm{m,n}} \in \mathbb{C}^{3\times1}$ and $\mathbf{s}^{f/t}_{\mathrm{m,n}} \in \mathbb{C}^{3\times1}$. Having defined the variables, the power flowing through the branch, the nodal power balance and losses can be developed as follows in equation (1.1).

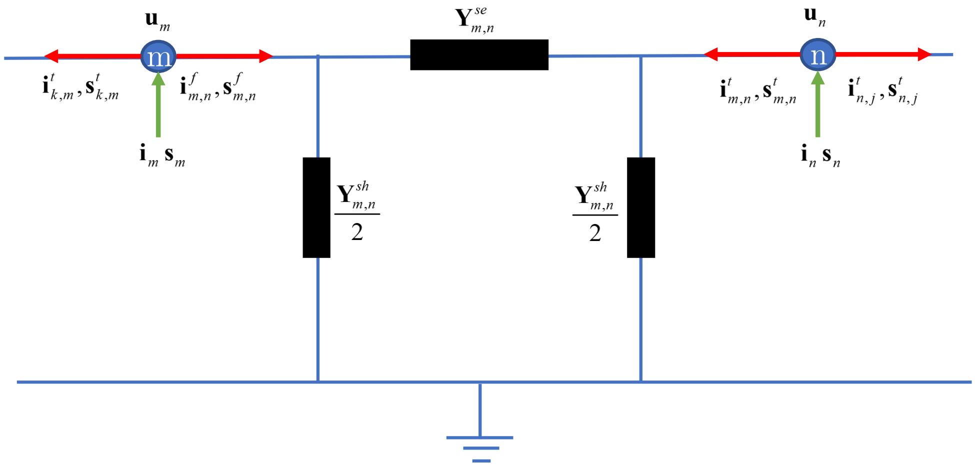

Figure 1.1: Electric Π-model of a generic branch.

$$\mathbf{s}^{f}_{\mathrm{m,n}} = \mathbf{diag}(\mathbf{u}_{\mathrm{m}})\overline{\mathbf{i}^{f}_{\mathrm{m,n}}} \qquad \mathbf{s}^{t}_{\mathrm{m,n}} = \mathbf{diag}(\mathbf{u}_{\mathrm{n}})\overline{\mathbf{i}^{t}_{\mathrm{m,n}}} \tag{1.1a}$$

$$\mathbf{i}^{f}_{\mathrm{m,n}} = \frac{\mathbf{Y}^{sh}_{\mathrm{m,n}}}{2}\mathbf{u}_{\mathrm{m}} + \mathbf{Y}^{se}_{\mathrm{m,n}}\left(\mathbf{u}_{\mathrm{m}} - \mathbf{u}_{\mathrm{n}}\right) \qquad \mathbf{i}^{t}_{\mathrm{m,n}} = \frac{\mathbf{Y}^{sh}_{\mathrm{m,n}}}{2}\mathbf{u}_{\mathrm{n}} + \mathbf{Y}^{se}_{\mathrm{m,n}}\left(\mathbf{u}_{\mathrm{n}} - \mathbf{u}_{\mathrm{m}}\right) \tag{1.1b}$$

$$\mathbf{s}_{\mathrm{m}} = \mathbf{diag}(\mathbf{u}_{\mathrm{m}})\overline{\mathbf{i}_{\mathrm{m}}} \qquad \mathbf{s}_{\mathrm{n}} = \mathbf{diag}(\mathbf{u}_{\mathrm{n}})\overline{\mathbf{i}_{\mathrm{n}}} \tag{1.1c}$$

$$\mathbf{i}_{\mathrm{m}} = \mathbf{i}^{f}_{\mathrm{m,n}} + \mathbf{i}^{t}_{\mathrm{k,m}} \qquad \mathbf{i}_{\mathrm{n}} = \mathbf{i}^{t}_{\mathrm{m,n}} + \mathbf{i}^{f}_{\mathrm{n,j}} \tag{1.1d}$$

$$\mathbf{s}^{ls}_{\mathrm{m,n}} = \mathbf{1}^{\mathrm{T}}\mathbf{s}^{f}_{\mathrm{m,n}} + \mathbf{1}^{\mathrm{T}}\mathbf{s}^{t}_{\mathrm{m,n}} = \mathbf{u}^{\mathrm{T}}_{\mathrm{m}}\overline{\mathbf{i}^{f}_{\mathrm{m,n}}} + \mathbf{u}^{\mathrm{T}}_{\mathrm{n}}\overline{\mathbf{i}^{t}_{\mathrm{m,n}}} \tag{1.1e}$$

Generalising the idea, the three-phase non-linear power flow can be extended for all nodes in a radial DN as in equation (1.2).

$$\mathbf{s}^{f} = \mathbf{diag}\left(\mathbf{A}^{f}\mathbf{u}\right)\overline{\mathbf{i}^{f}} \qquad \mathbf{s}^{f} = \mathbf{diag}\left(\mathbf{A}^{t}\mathbf{u}\right)\overline{\mathbf{i}^{t}} \tag{1.2a}$$

$$\mathbf{i}^{f} = \mathbf{Y}^{f}\mathbf{u} \qquad \mathbf{i}^{t} = \mathbf{Y}^{t}\mathbf{u} \tag{1.2b}$$

$$\mathbf{i} = \mathbf{Y}\mathbf{u} \tag{1.2c}$$

$$\mathbf{s} = \mathbf{diag}(\mathbf{u})\overline{\mathbf{i}} \tag{1.2d}$$

$$s^{ls} = \mathbf{u}^{\mathrm{T}}\overline{\mathbf{Y}\mathbf{u}} \tag{1.2e}$$

where $\mathbf{s}^{f/t}$ and $\mathbf{i}^{f/t} \in \mathbb{C}^{3\mathcal{B}\times 1}$ denote the branch power and current flows in "from" and "to" direction for the overall branches. Likewise, $\mathbf{v}$, $\mathbf{s}$, $\mathbf{i} \in \mathbb{C}^{3\mathcal{N}\times 1}$ stand for nodal voltage and the net power and current injection column vectors, respectively. The nodal and branch admittance matrices are shown by $\mathbf{Y} \in \mathbb{C}^{3\mathcal{N}\times 3\mathcal{N}}$, $\mathbf{Y}^{f/t} \in \mathbb{C}^{3\mathcal{B}\times 3\mathcal{B}}$. Eventually, the $\mathbf{A}^{f,t} \in \mathbb{C}^{3\mathcal{B}\times 3\mathcal{N}}$ are the branch incidence matrices in "from" and "to" directions. The given admittance and incidence matrices are obtained based on the following rules provided in equation (1.3).

$$\mathbf{A}^{f/t} = \left[\mathbf{A}^{f/t}_{b,n}\right]_{(b,n)\in\mathcal{B}\times\mathcal{N}} \qquad \mathbf{A}^{f/t}_{b,n} = \begin{cases} \mathbf{I} \in \mathbb{R}^{3\times 3} & \text{for } n = n_b^{f/t} \\ \mathbf{0} \in \mathbb{R}^{3\times 3} & \text{otherwise} \end{cases} \tag{1.3a}$$

$$\mathbf{Y}^{f/t} = \left[\mathbf{Y}^{f/t}_{b,n}\right]_{(b,n)\in\mathcal{B}\times\mathcal{N}} \qquad \mathbf{Y}^{f/t}_{b,n} = \begin{cases} \frac{\mathbf{Y}^{sh}_b}{2} + \mathbf{Y}^{se}_b & \text{for } n = n_b^{f/t} \\ -\mathbf{Y}^{se}_b & \text{for } n = n_b^{t/f} \\ \mathbf{0} \in \mathbb{R}^{3\times 3} & \text{otherwise} \end{cases} \tag{1.3b}$$

$$\mathbf{Y} = \left[\mathbf{Y}_{n,m}\right]_{(n,m)\in\mathcal{N}\times\mathcal{N}} \qquad \mathbf{Y}_{n,m} = \begin{cases} \sum_{b\in\mathcal{B}_n} \left(\frac{\mathbf{Y}^{sh}_b}{2} + \mathbf{Y}^{se}_b\right) & \text{for } n = m \\ \sum_{b\in\mathcal{B}_n\cap\mathbf{B}_m} -\mathbf{Y}^{se}_b & \text{otherwise} \end{cases} \tag{1.3c}$$

The indices $n_b^{f/t}$ denote the node at "from" and "to" ends of branch b, and the $\mathcal{B}_n$ stands for the sets of branches connected to node n.

An important point to note in the three-phase power flow is the Δ or Y connections of the DERs at a bus. Let figure 1.2 demonstrate a simple bus and DERs connected in Δ or Y modes. The nodal current and power injection can be extended based on Kirchhoff's laws as the following in equation (1.4).

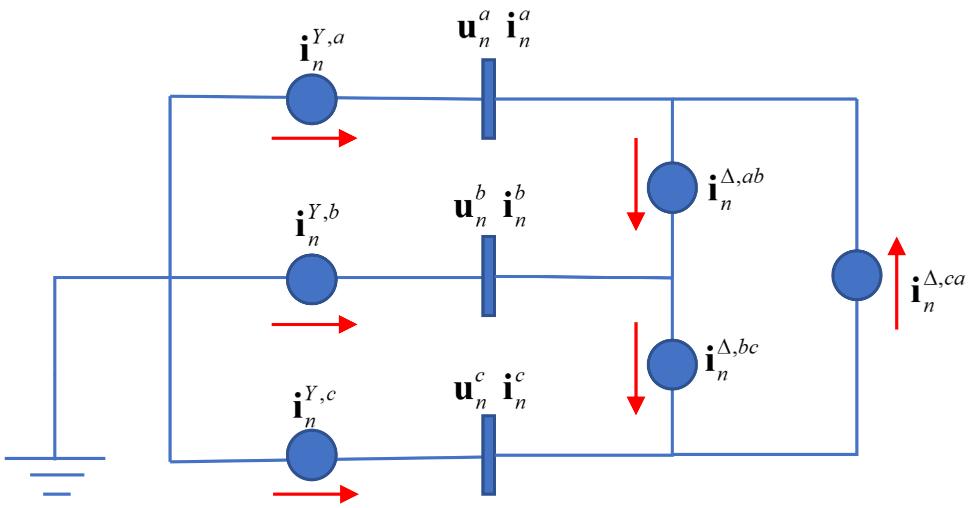

Figure 1.2: Connection configuration of DERs, Δ or Y.

$$s_n^a = u_n^a \overline{i_n^a} = u_n^a \left(\overline{i_n^{Y,a}} + \overline{i_n^{\Delta,ca}} - \overline{i_n^{\Delta,ab}}\right) \qquad s_n^{Y,a} = u_n^a \overline{i_n^{Y,a}} \qquad s_n^{\Delta,ab} = \left(u_n^a - u_n^b\right) \overline{i_n^{\Delta,ab}} \tag{1.4a}$$

$$s_n^b = u_n^b \overline{i_n^b} = u_n^b \left(\overline{i_n^{Y,b}} + \overline{i_n^{\Delta,ab}} - \overline{i_n^{\Delta,bc}}\right) \qquad s_n^{Y,b} = u_n^b \overline{i_n^{Y,b}} \qquad s_n^{\Delta,bc} = \left(u_n^b - u_n^c\right) \overline{i_n^{\Delta,bc}} \tag{1.4b}$$

$$s_n^c = u_n^c \overline{i_n^c} = u_n^c \left(\overline{i_n^{Y,c}} + \overline{i_n^{\Delta,bc}} - \overline{i_n^{\Delta,ca}}\right) \qquad s_n^{Y,c} = u_n^c \overline{i_n^{Y,c}} \qquad s_n^{\Delta,ca} = \left(u_n^c - u_n^a\right) \overline{i_n^{\Delta,ca}} \tag{1.4c}$$

Vectorising equation (1.4) and extending it to the whole three-phase network, the generalised nodal injection can be written as in equation (1.5).

$$\mathbf{s}^f = \mathbf{diag}\left(\mathbf{A}^f \mathbf{u}\right) \overline{\mathbf{i}^f} \qquad \mathbf{s}^f = \mathbf{diag}\left(\mathbf{A}^t \mathbf{u}\right) \overline{\mathbf{i}^t} \tag{1.5a}$$

$$\mathbf{i}^f = \mathbf{Y}^f \mathbf{u} \qquad \mathbf{i}^t = \mathbf{Y}^t \mathbf{u} \tag{1.5b}$$

$$\mathbf{i} = \mathbf{Y}\mathbf{u} \tag{1.5c}$$

$$\mathbf{s} = \mathbf{diag}(\mathbf{u}) \left(\overline{\mathbf{i}^Y} - (\mathbf{H})^T \overline{\mathbf{i}^\Delta}\right) \qquad \mathbf{s}^Y = \mathbf{diag}(\mathbf{u})\overline{\mathbf{i}^Y} \qquad \mathbf{s}^\Delta = \mathbf{diag}(\mathbf{H}\mathbf{u})\overline{\mathbf{i}^\Delta} \tag{1.5d}$$

$$s^{ls} = \mathbf{u}^{\mathrm{T}} \overline{\mathbf{Y}\mathbf{u}} \tag{1.5e}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{\Gamma} & & \\ & \ddots & \\ & & \mathbf{\Gamma} \end{bmatrix} \in \mathbb{R}^{3\mathcal{N} \times 3\mathcal{N}} \quad ; \quad \mathbf{\Gamma} = \begin{bmatrix} \Gamma^{aa} & \Gamma^{ab} & \Gamma^{ac} \\ \Gamma^{ba} & \Gamma^{bb} & \Gamma^{bc} \\ \Gamma^{ca} & \Gamma^{cb} & \Gamma^{cc} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \tag{1.5f}$$

where $\mathbf{H}$ stands for the aggregated delta phase mapping matrix ($\mathbf{\Gamma}$). Eventually, the given nonlinear power flow in equation (1.2) can be solved using iterative approaches such as fixed point method introduced exhaustively in [2].

1.2. Linear AC Power Flow Approximation

The global linear approximation for the nonlinear power flow equations will be represented in this section. The linearisation is performed adopting Taylor's theorem given in Theorem 1.1 where the model represents a secant plane between the no-load point and a reference point of the power flow manifold [24].

Theorem 1.1 (Taylor's theorem)**.** *If a real-valued function $f(x)$ is differentiable at the* *point $x = a$, then it has a linear approximation near this point such that:*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f"(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Employing Theorem 1.1, the nonlinear power flow equations can be extended around the reference point satisfying equation (1.5). Then each nonlinear term can be written in linear form in terms of nodal active and reactive power injections as follows in equation (1.6).

$$|\mathbf{u}_t| = |\hat{\mathbf{u}}_t| + \mathbf{M}_p^{|u|}\Delta\mathbf{p}_t + \mathbf{M}_q^{|u|}\Delta\mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (1.6a)$$

$$|\mathbf{s}_t^f| = |\hat{\mathbf{s}}_t^f| + \mathbf{M}_p^{|s^f|}\Delta\mathbf{p}_t + \mathbf{M}_q^{|s^f|}\Delta\mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (1.6b)$$

$$|\mathbf{s}_t^t| = |\hat{\mathbf{s}}_t^t| + \mathbf{M}_p^{|s^t|}\Delta\mathbf{p}_t + \mathbf{M}_q^{|s^t|}\Delta\mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (1.6c)$$

$$p_t^{loss} = \hat{p}_t^{loss} + \mathbf{M}_p^{p^{loss}}\Delta\mathbf{p}_t + \mathbf{M}_q^{p^{loss}}\Delta\mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (1.6d)$$

$$q_t^{loss} = \hat{q}_t^{loss} + \mathbf{M}_p^{q^{loss}}\Delta\mathbf{p}_t + \mathbf{M}_q^{q^{loss}}\Delta\mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (1.6e)$$

In these equation sets, $t \in \mathcal{T}$ determines the timesteps and $|\mathbf{u}_t|$ and $|\hat{\mathbf{u}}_t| \in \mathbb{R}^{\mathcal{N}\times 1}$ vectors are the nodal voltage's state variables and the reference point parameters, respectively. The vectors $|\mathbf{s}_t^{f/t}|$ and $|\hat{\mathbf{s}}_t^{f/t}| \in \mathbb{R}^{\mathcal{B}\times 1}$ respectively indicate the state variables and the reference point parameters for the branch power flow in "from", and "to" directions. Eventually, p_t^{loss}, $\hat{p}_t^{loss}$, q_t^{loss}, and $\hat{p}_t^{loss} \in \mathbb{R}$ are the total active and reactive power losses in the network and their reference value parameters. Having defined the state variable vectors and their corresponding reference points, the $\mathbf{M}_p^{(\cdot)}$ and $\mathbf{M}_q^{(\cdot)}$ are the sensitivity matrices of the corresponding state variables with respect to the active and reactive power injection of the DER sets, i.e. $\mathbf{p}_t$ and $\mathbf{q}_t \in \mathbb{R}^{\mathcal{D}}$ where Δ stands for the deviation of the active and reactive power's state variable and their respective reference values ($\Delta\mathbf{p}_t = \mathbf{p}_t - \hat{\mathbf{p}}_t$ and $\Delta\mathbf{q}_t = \mathbf{q}_t - \hat{\mathbf{q}}_t$).

To obtain the introduced sensitivity matrices, as a first step, the Jacobian metric of the complex voltage vector's sensitivity to the nodal delta and wye injections should be evaluated. Afterwards, the given matrices can be achieved by applying the partial derivative's fundamental rules. Eventually, the given matrices will be reformulated to account for the sensitivity of the state variables to the complex power injection of each DER [25]. Moving to further details, according to [2], the global sensitivity estimation of complex voltage reveals a secant plane between the no-load point $\mathbf{u}^{nl}$ and a reference operation point $\hat{\mathbf{u}}$ of the power flow manifold. This proposition is mathematically expressed in equation (1.7).

$$\mathbf{u}^{ns} = \mathbf{u}^{nl,ns} + \left(\mathbf{Y}^{ns,ns}\right)^{-1} \left(\mathbf{diag}\left(\hat{\mathbf{u}}^{ns}\right)^{-1} \overline{\mathbf{s}^{Y,ns}} + \left(\mathbf{H}^{ns,ns}\right)^{T} \mathbf{diag}\left(\hat{\mathbf{u}}^{ns}\right)^{-1} \overline{\mathbf{s}^{\Delta,ns}}\right) \tag{1.7}$$

To this end, the sensitivity of no-source nodes' complex voltage vector for the active and reactive power injection in delta and wye configurations can be developed by taking the partial derivatives of the voltage at equation (1.7) with regards to the nodal power injection in either configuration. That being said, the following sensitivity matrices can be developed in equation (1.8)

$$\left(\mathbf{M}^{u}_{p^Y}\right)^{ns,ns} = \frac{\partial \mathbf{u}^{ns}}{\partial \mathbf{p}^{Y,ns}} = (\mathbf{Y}^{ns,ns})^{-1}\mathbf{diag}(\hat{\mathbf{u}}^{ns})^{-1} \tag{1.8a}$$

$$\left(\mathbf{M}^{u}_{q^Y}\right)^{ns,ns} = \frac{\partial \mathbf{u}^{ns}}{\partial \mathbf{q}^{Y,ns}} = -j(\mathbf{Y}^{ns,ns})^{-1}\mathbf{diag}(\hat{\mathbf{u}}^{ns})^{-1} \tag{1.8b}$$

$$\left(\mathbf{M}^{u}_{p^\Delta}\right)^{ns,ns} = \frac{\partial \mathbf{u}^{ns}}{\partial \mathbf{p}^{\Delta,ns}} = (\mathbf{Y}^{ns,ns})^{-1}(\mathbf{H}^{ns,ns})^{T}\mathbf{diag}(\mathbf{H}^{\mathbf{ns,ns}}\overline{\hat{\mathbf{u}}^{ns}})^{-1} \tag{1.8c}$$

$$\left(\mathbf{M}^{u}_{q^\Delta}\right)^{ns,ns} = \frac{\partial \mathbf{u}^{ns}}{\partial \mathbf{q}^{\Delta,ns}} = -j(\mathbf{Y}^{ns,ns})^{-1}(\mathbf{H}^{ns,ns})^{T}\mathbf{diag}(\mathbf{H}^{\mathbf{ns,ns}}\overline{\hat{\mathbf{u}}^{ns}})^{-1} \tag{1.8d}$$

where the superscript ns stands for excluding the source slack nod from the grid configuration to calculate state variables. For the source node, the assumption is that the voltage is independent of the nodal injection from the no-source nodes. Therefore, the corresponding sensitivity matrices for the source node are going to be set to zero.

$$\left(\mathbf{M}^{u}_{p^Y/q^Y/p^\Delta/q^\Delta}\right)^{sr,sr} = \mathbf{0} \tag{1.9a}$$

$$\left(\mathbf{M}^{u}_{p^Y/q^Y/p^\Delta/q^\Delta}\right)^{sr,ns} = \mathbf{0} \tag{1.9b}$$

$$\left(\mathbf{M}^{u}_{p^Y/q^Y/p^\Delta/q^\Delta}\right)^{ns,sr} = \mathbf{0} \tag{1.9c}$$

The most important point to emerge from these equations is that these sensitivity matrices are written in terms of nodal power injections, $\mathbf{p}^{Y/\Delta}$, $\mathbf{p}^{Y/\Delta}$. While in the principal linearised formulations introduced in equation (1.6), the sensitivities correspond to the apparent power injection of each DER. Therefore, a modification is required to apply on equation (1.8) accounting for both delta and wye configurations of all DERs in the grid. For this purpose, the following true statements are introduced in Lemma 1.2.1 to prove statements later on.

Lemma 1.2.1 (Partial derivative rules). *Providing functions z and y are continuous and differentiable, the following rules hold true for them [2].*

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}\frac{\partial y}{\partial x} \tag{1.10a}$$

$$\frac{\partial (yz)}{\partial x} = \frac{\partial y}{\partial x}z + \frac{\partial z}{\partial x}y \tag{1.10b}$$

$$\frac{\partial |y|}{\partial x} = \frac{1}{|y|}\Re\left(\bar{y}\frac{\partial y}{\partial x}\right) \qquad \textit{where } y \in \mathbb{C}, x \in \mathbb{R} \tag{1.10c}$$

$$\frac{\partial(\boldsymbol{diag}(y)\bar{z})}{\partial x} = \boldsymbol{diag}(y)\frac{\partial \bar{z}}{\partial x} + \boldsymbol{diag}(\bar{z})\frac{\partial y}{\partial x} = \boldsymbol{diag}(y)\overline{\frac{\partial z}{\partial x}} + \boldsymbol{diag}(\bar{z})\frac{\partial y}{\partial x} \tag{1.10d}$$

$$\frac{\partial(y^T\bar{z})}{\partial x} = y^T\frac{\partial \bar{z}}{\partial x} + \bar{z}^T\frac{\partial y}{\partial x} = y^T\overline{\frac{\partial z}{\partial x}} + \bar{z}^T\frac{\partial y}{\partial x} \tag{1.10e}$$

From there, the partial derivatives chain rule equation (1.10a) can be adopted to manifest equation (1.11).

$$\mathbf{M}^u_p = \frac{\partial \mathbf{u}}{\partial \mathbf{p}} = \frac{\partial \mathbf{u}}{\partial \mathbf{p}^Y}\frac{\partial \mathbf{p}^Y}{\partial \mathbf{p}} + \frac{\partial \mathbf{u}}{\partial \mathbf{p}^\Delta}\frac{\partial \mathbf{p}^\Delta}{\partial \mathbf{p}} = \mathbf{M}^u_{p^Y}\mathbf{A}^Y + \mathbf{M}^u_{p^\Delta}\mathbf{A}^\Delta \tag{1.11a}$$

$$\mathbf{M}^u_q = \frac{\partial \mathbf{u}}{\partial \mathbf{q}} = \frac{\partial \mathbf{u}}{\partial \mathbf{q}^Y}\frac{\partial \mathbf{q}^Y}{\partial \mathbf{q}} + \frac{\partial \mathbf{u}}{\partial \mathbf{q}^\Delta}\frac{\partial \mathbf{q}^\Delta}{\partial \mathbf{q}} = \mathbf{M}^u_{q^Y}\mathbf{A}^Y + \mathbf{M}^u_{q^\Delta}\mathbf{A}^\Delta \tag{1.11b}$$

where $\mathbf{A}^{Y/\Delta}$ denote the mapping matrices of DER to the grid nodes, which are defined as follows in equation (1.12).

$$\mathbf{A}^Y = \left[a^Y_{(n,p^n),d}\right]_{((n,p^n),d)\in\mathcal{N}\times\mathcal{D}} \qquad a^Y_{(n,p^n),d} = \begin{cases} \frac{1}{N^p_d} & \text{for } n = n_d \text{ and } p^n \in \mathcal{P}_d \\ 0 & \text{otherwise} \end{cases} \tag{1.12a}$$

$$\mathbf{A}^\Delta = \left[a^\Delta_{(n,p^n),d}\right]_{((n,p^n),d)\in\mathcal{N}\times\mathcal{D}} \qquad a^\Delta_{(n,p^n),d} = \begin{cases} \frac{1}{N^p_d} & \text{for } n = n_d \text{ and } p^n \in \mathcal{P}_d \\ 0 & \text{otherwise} \end{cases} \tag{1.12b}$$

The symbol $\mathcal{P}_d$ defines the set of phases that DER d is connected to at node n. The scalar N^p_d determines the number of phases that DER d is connected to at the same node.

To this point, the desired sensitivity matrix for the nodal voltage is obtained in terms of the injections from DERs. However, It follows that in the fundamental linearised AC power flow equations shown in equation (1.6), the sensitivity is related to the magnitude of nodal voltages, which means one more consideration is necessary. Thereby, by relying on equation (1.10c) in Lemma 1.2.1, we can reach out the goal. Consequently, the sensitivities of voltage magnitude with respect to the active and reactive power injections of DERs can be expressed as follows in equation (1.13).

$$\mathbf{M}_p^{|u|} = \mathbf{diag}\left(|\hat{\mathbf{u}}|\right)^{-1} \Re\left(\mathbf{diag}(\overline{\hat{\mathbf{u}}})\mathbf{M}_p^u\right) \tag{1.13a}$$
$$\mathbf{M}_q^{|u|} = \mathbf{diag}\left(|\hat{\mathbf{u}}|\right)^{-1} \Re\left(\mathbf{diag}(\overline{\hat{\mathbf{u}}})\mathbf{M}_q^u\right) \tag{1.13b}$$

From nonlinear power flow equation (1.2) we found out $\mathbf{s}^{f/t} = \mathbf{diag}(\mathbf{A}^{f/t}\mathbf{u})\overline{\mathbf{Y}^{f/t}\mathbf{u}}$. We are looking for the sensitivities of absolute value of branch power flow to the DER injections ($\mathbf{M}_{p/q}^{|s^{f/t}|}$) in "from" and "to" directions. By utilizing equation (1.10a) in Lemma 1.2.1, the following expressions can be implied.

$$\mathbf{M}_p^{|s^{f/t}|} = \frac{\partial|\mathbf{s}^{f/t}|}{\partial\mathbf{p}} = \frac{\partial|\mathbf{s}^{f/t}|}{\partial\mathbf{u}}\frac{\partial\mathbf{u}}{\partial\mathbf{p}} = \mathbf{M}_u^{|s^{f/t}|}\mathbf{M}_p^u \tag{1.14a}$$
$$\mathbf{M}_q^{|s^{f/t}|} = \frac{\partial|\mathbf{s}^{f/t}|}{\partial\mathbf{q}} = \frac{\partial|\mathbf{s}^{f/t}|}{\partial\mathbf{u}}\frac{\partial\mathbf{u}}{\partial\mathbf{q}} = \mathbf{M}_u^{|s^{f/t}|}\mathbf{M}_q^u \tag{1.14b}$$

Applying Lemma 1.2.1 on the nonlinear power flow equation (1.5) we can leverage the following results.

$$\mathbf{M}_u^{s^{f/t}} = \mathbf{diag}\left(\mathbf{A}^{f/t}\hat{\mathbf{u}}\right)\mathbf{Y}^{f/t} + \mathbf{diag}\left(\overline{\mathbf{Y}^{f/t}\hat{\mathbf{u}}}\right)\mathbf{A}^{f/t} \tag{1.15a}$$
$$\mathbf{M}_u^{|s^{f/t}|} = \frac{\partial|\mathbf{s}^{f/t}|}{\partial\mathbf{u}} = \mathbf{diag}\left(|\hat{\mathbf{s}}^{f/t}|\right)^{-1}\Re\left(\mathbf{diag}(\overline{\hat{\mathbf{s}}^{f/t}})\mathbf{M}_u^{s^{f/t}}\right) \tag{1.15b}$$

Equation (1.15a) is emerged based on the rule provided by the handy equation (1.10d) whereas equation (1.15b) is expressed with the help of equation (1.10c).

Finally, the sensitivity of active and reactive power losses to the DERs' power vector can be derived in a similar way as follows.

$$\mathbf{M}_p^{p^{loss}} = \frac{\partial p^{loss}}{\partial \mathbf{p}} = \Re\left(\frac{\partial s^{loss}}{\partial \mathbf{p}}\right) = \Re\left(\frac{\partial s^{loss}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{p}}\right) = \Re\left(\mathbf{M}_u^{s^{loss}}\mathbf{M}_p^u\right) \tag{1.16a}$$

$$\mathbf{M}_q^{p^{loss}} = \frac{\partial p^{loss}}{\partial \mathbf{q}} = \Re\left(\frac{\partial s^{loss}}{\partial \mathbf{q}}\right) = \Re\left(\frac{\partial s^{loss}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{q}}\right) = \Re\left(\mathbf{M}_u^{s^{loss}}\mathbf{M}_q^u\right) \tag{1.16b}$$

$$\mathbf{M}_p^{q^{loss}} = \frac{\partial q^{loss}}{\partial \mathbf{p}} = \Im\left(\frac{\partial s^{loss}}{\partial \mathbf{p}}\right) = \Im\left(\frac{\partial s^{loss}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{p}}\right) = \Im\left(\mathbf{M}_u^{s^{loss}}\mathbf{M}_p^u\right) \tag{1.16c}$$

$$\mathbf{M}_q^{q^{loss}} = \frac{\partial q^{loss}}{\partial \mathbf{q}} = \Im\left(\frac{\partial s^{loss}}{\partial \mathbf{q}}\right) = \Im\left(\frac{\partial s^{loss}}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{q}}\right) = \Im\left(\mathbf{M}_u^{s^{loss}}\mathbf{M}_q^u\right) \tag{1.16d}$$

The sensitivity of apparent power losses with regards to the voltage vector can be attained via taking the partial derivative of the general power loss relationship indicated in equation (1.5) by relying on lemma 1.2.1 (equation (1.10e)).

1.3. State-Space Modelling of DER

The section is developed based on the work proposed in [22] where it aims to model the flexibility or non-flexibility of the loads or generators in a standard encapsulated state-space model. The state-space formulation of DER is going to ensure the desecrated time dependency of flexible DERs and for non-flexible DERs the scheduling time series are going to be applied on their nominal operation values to dispatch them. Accordingly, the model variables can be expressed as: the active and reactive power injections of DERs ($\mathbf{p}/\mathbf{q}$), state vector ($\mathbf{x}$), control vector ($\mathbf{u}$) and output vector ($\mathbf{y}$). Hence, the standard discrete state-space modelling of flexible DERs can be expressed as it follows in equation (1.17).

$$\mathbf{x}_{d,t^0} = \mathbf{x}_d^0 \qquad \forall d \in \mathcal{D} \tag{1.17a}$$

$$\mathbf{x}_{d,t+1} = \mathbf{A}_d\mathbf{x}_{d,t} + \mathbf{B}_d\mathbf{u}_{d,t} + \mathbf{v}_d^x \qquad \forall d \in \mathcal{D}, \forall t \in \mathcal{T} - \{t^n\} \tag{1.17b}$$

$$\mathbf{y}_{d,t} = \mathbf{C}_d\mathbf{x}_{d,t} + \mathbf{D}_d\mathbf{u}_{d,t} + \mathbf{v}_d^y \qquad \forall d \in \mathcal{D}, \forall t \in \mathcal{T} \tag{1.17c}$$

$$\underline{\mathbf{y}}_{d,t} \leq \mathbf{y}_{d,t} \leq \overline{\mathbf{y}}_{d,t} \qquad \forall d \in \mathcal{D}, \forall t \in \mathcal{T} \tag{1.17d}$$

$$p_d = \mathbf{M}_d^{p,y}\mathbf{y}_{d,t} \qquad q_d = \mathbf{M}_d^{q,y}\mathbf{y}_{d,t} \qquad \forall d \in \mathcal{D}, \forall t \in \mathcal{T} \tag{1.17e}$$

where $d \in \mathcal{D}$ is the sets of flexible DER in this section and $t \in \mathcal{T}$ stands for the set of time-steps so that t^0 and t^n are the initial and final time-steps, respectively. Matrix $\mathbf{A}_d$ and $\mathbf{B}_d$ are the state and control matrices. Whereas $\mathbf{c}_d$ and $\mathbf{D}_d$ are represent the output and feed-through matrices, sequentially. Moreover, $\mathbf{v}_d^{x/y}$ indicate the disturbance vectors for state and output equations that is considered as parameter in our model. Last but not least, $\mathbf{M}_d^{p,y}$ and $\mathbf{M}_d^{q,y}$ are the mapping matrices, with binary ($\{0,1\}$) values for its arrays, corresponding the flexible DERs to the output vector; where it will take 1 if the flexible DER is connected to the DN and 0, otherwise. the components of the state, control and output vectors and their corresponding matrices are going to be obtained for flexible loads and generators in the following subsections.

1.3.1. State-Space Components of Flexible Loads

The flexible load model stands for a general load with controllable active and reactive power demand. So the main characteristics that its model should include are a) the maximum load curtailing time b) the maximum curtailing power. Having said that, we start with the following differential equation for the consumed energy by the flexible load d shown by e_d as:

$$\dot{e}_d = p_d - p_d^{bas} \tag{1.18}$$

where p_d and p_d^{bas} represent the instant and baseline power demands, respectively. The baseline power demands refers to the scheduled nominal power time series of the load that is defined as:

$$p_{d,t} = p_d^{nm} p_{d,t}^{dispatch} \qquad q_{d,t} = q_d^{nm} \frac{q_d^{nm}}{p_d^{nm}} p_{d,t}^{dispatch} \qquad \forall t \in \mathcal{T} \tag{1.19}$$

where $p_{d,t}^{dispatch} \in [0,1]$ is a parameter determining the dispatch of flexible load for the given time horizon before curtailing and p_d^{nm} is the nominal capacity of load. Should we want to write equation (1.18) in discrete time domain, the following result can be obtained in equation (1.20):

$$e_{d,t+1} = e_{d,t} + \Delta t \left(p_{d,t} - p_{d,t}^{base} \right) \qquad \forall t \in \mathcal{T} - \{t^n\} \tag{1.20}$$

The scalar Δt denotes the time-step interval hours. therefrom, the operational constraint for the flexible load can be expressed as:

$$-e_d^{pu}\left|p_d^{nm}\right|\Delta t \leq e_{d,t} \leq e_d^{pu}\left|p_d^{nm}\right|\Delta t \qquad \forall t \in \mathcal{T} \tag{1.21a}$$

$$-p_d^{-,pu} p_{d,t}^{bas} \leq -p_{d,t} \leq -p_d^{+,pu} p_{d,t}^{bas} \qquad \forall t \in \mathcal{T} \tag{1.21b}$$

The symbol e_d^{pu} is the maximum curtailing time period of the flexible load d and its unit is hours. The scalars $p_d^{-,pu}$ and $p_d^{+,pu}$ stand for the per-unit minimum and maximum power curtailing factor with respect to the baseline power $p_{d,t}^{bas}$. Since the demand value is negative, equation (1.21b) is multiplied by minus to model the curtailing limit properly.

Eventually, the encapsulated state-space equations can be attained by arranging the operational limits of flexible loads based on the general equation shown in equation (1.17).

$$x_{d,t} = \left[e_{d,t}\right] \qquad u_{d,t} = \left[p_{d,t}\right] \qquad y_{d,t} = \begin{bmatrix} e_{d,t} \\ p_{d,t} \end{bmatrix} \qquad \forall t \in \mathcal{T} \tag{1.22}$$

The entries of the state-space matrices can be derived based on equations (1.20) and (1.21) as:

$$A_d = [1] \qquad B_d = [\Delta t] \qquad v_{d,t}^{x} = \left[-p_{d,t}^{bas}\Delta t\right] \quad \forall t \in \mathcal{T} \tag{1.23a}$$

$$C_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad D_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad v_{d,t}^{y} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \forall t \in \mathcal{T} \tag{1.23b}$$

$$x_d^0 = \left[e_{d,t^N}\right] \qquad \overline{y}_{d,t} = \begin{bmatrix} -e_d^{pu}\left|p_d^{nm}\right| \\ p_d^{-,pu} p_{d,t}^{bas} \end{bmatrix} \qquad \underline{y}_{d,t} = \begin{bmatrix} e_d^{pu}\left|p_d^{nm}\right| \\ p_d^{+,pu} p_{d,t}^{bas} \end{bmatrix} \quad \forall t \in \mathcal{T} \tag{1.23c}$$

$$M_d^{p,y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad M_d^{q,y} = \begin{bmatrix} 0 & \frac{q_d^{nm}}{p_d^{nm}} \end{bmatrix} \tag{1.23d}$$

To this end, all the necessary vectors and matrices of the state-space equation for the flexible load are defined and the next section is going to be allocated to defining the matrix entries for a flexible generator's state-space equations.

1.3.2. State-Space Components of Flexible Generators

Similar to the state-space model of flexible loads, the model for the flexible generator should be able to implicitly formulate the main characteristics of the generator and preserve its time dependency, if exists. As the name regards, the flexible generator should be capable of increasing or decreasing its output as per grid request. However, there are practical limitations for the generator to change its power outputs with respect to its nominal or scheduled operation point. The fundamental operation limit of a flexible generator which are its minimum and maximum operating limits can be mathematically expressed as follows in equation (1.24).

$$p_d^{-,pu} p_{d,t}^{bas} \leq p_{d,t} \leq p_d^{+,pu} p_{d,t}^{bas} \qquad \forall t \in \mathcal{T} \tag{1.24}$$

where the scalar $p_{d,t}^{bas}$ is the baseline power injection of flexible generator which is similar to the concept given for the flexible load in Section 1.3.1. The symbols $p_d^{+/-,pu}$ indicate the upper and lower limits on the deviation from the baseline scheduled power injection of flexible generator. considering the operation limit given in equation (1.24), we can enclose the given constraint in the state-space model format.

$$x_{d,t} = [] \qquad u_{d,t} = [p_{d,t}] \qquad y_{d,t} = [p_{d,t}] \qquad \forall t \in \mathcal{T} \tag{1.25}$$

The matrix entries of the state-space equations can be derived as it follows in equation (1.26)

$$A_d = [] \qquad B_d = [] \qquad v_{d,t}^{x} = [] \quad \forall t \in \mathcal{T} \tag{1.26a}$$

$$C_d = [] \qquad D_d = [1] \qquad v_{d,t}^{y} = [] \quad \forall t \in \mathcal{T} \tag{1.26b}$$

$$x_d^0 = [] \qquad \overline{y}_{d,t} = \left[p_d^{+,pu} p_{d,t}^{bas}\right] \qquad \underline{y}_{d,t} = \left[p_d^{-,pu} p_{d,t}^{bas}\right] \quad \forall t \in \mathcal{T} \tag{1.26c}$$

$$M_d^{p,y} = [1] \qquad M_d^{q,y} = \begin{bmatrix} \frac{q_d^{nm}}{p_d^{nm}} \end{bmatrix} \tag{1.26d}$$

Up to this point, all the necessary requirements for modelling the three-phase electric grid and DERs are introduced and integrated. The next chapter is going to be allocated to modelling the optimal operation of the grid and DERs.

2 | Modelling Coordinated Hybrid DSO and Peer-to-Peer Market

This chapter introduces a centralised local market for DSO where the chief objective is to provide the DN with ancillary services (AS) and motivate DERs toward network-friendly behaviour. Regarding the AS, the idea is to reduce losses in the system while managing the line congestion and nodal voltage support. Persuading the flexible DERs to contribute to the system efficiency, the distribution locational marginal prices (DLMP) as a scheme for the spot pricing of energy are going to be obtained based on the DN and DER model proposed in Chapter 1.

There are essential assumptions that should be taken into account in this model base on Kai Zhang's work [26]. As a matter of fact, the loads at the distribution level are mainly supplied by the TSO market following the day-ahead market (DAM) clearing. So, the spot pricing of the energy is taken place by the TSO market in the first place at the grid supply point (GSP). Moreover, DSO clears the market prior to the wholesale market so it can provide the TSO with AS and/or over-the-top energy [1, 11].

With respect to the DERs' properties, their cost or utility are considered a linear function, where the DSO has the authority to set operation points for the DER's re-dispatch program in the AS market. The same is true for the flexible loads, except that they should be curtailed as per DSO's request.

2.1. DSO Market Primal Problem

In this section, the market's primal optimisation problem will be developed based on the grid and DER models we obtained in the previous chapter. It should be noted that the DERs (loads and generators) are categorised by either "fixed" or "flexible" outputs. Regarding the fixed DERs, they operate inelastically based on their scheduled setpoints regardless of the decisions made by the optimisation problems. On the other hand, the elastic DERs can be re-dispatched up to their operation limitations, followed by the optimisation problem results. In the following subsections, the mathematical model of each component (DER and DN) is going to be introduced alongside their objective functions. Generally, the standard convex optimisation problem can be formulated with an objective function to be optimised subject to equality and non-equality constraints as follows:

$$\min_{\mathbf{x}} \quad F(\mathbf{x}) \tag{2.1a}$$

$$\text{s.t.} \quad \mathbf{Ax} - \mathbf{b} \leq \mathbf{0} \qquad : \boldsymbol{\mu} \tag{2.1b}$$

$$\mathbf{Cx} - \mathbf{d} = \mathbf{0} \qquad : \boldsymbol{\lambda} \tag{2.1c}$$

Developing a similar standard convex optimization, the optimal operation of gird and DERs can be obtained with the objective of maximizing social welfare. The constraints of the problem are associated with **i)** the DER operational limits demonstrated by state-space modelling, and **ii)** the distribution network operational limits.

2.1.1. DER Operational Constraints

As indicated in chapter 1, the operational limits of the DERs (flexible loads and generators) can be expressed in an encapsulated state-space equation format. Arranging the equations in the standard format, the following state-space modelling constraints can be provided for the DER.

$$\mathbf{x}_{t^0}^{storage} = \mathbf{x}_{t^n}^{storage} \qquad : \boldsymbol{\lambda}^{x,storage} \tag{2.2a}$$

$$\mathbf{x}_{t^0}^{non-storage} = \hat{\mathbf{x}} \qquad : \boldsymbol{\lambda}^{x,non-storage} \tag{2.2b}$$

$$\mathbf{x}_{t+1} = \mathbf{Ax}_t + \mathbf{Bu}_t + \mathbf{v}_t^x \qquad : \boldsymbol{\lambda}_t^x \qquad \forall t \in \mathcal{T} - \{t^n\} \tag{2.2c}$$

$$\mathbf{y}_t = \mathbf{Cx}_t + \mathbf{Du}_t + \mathbf{v}_t^y \qquad : \boldsymbol{\lambda}_t^y \qquad \forall t \in \mathcal{T} \tag{2.2d}$$

$$\mathbf{p}_t = \hat{\mathbf{p}}_t^{fixed} + \mathbf{M}^{p,y}\mathbf{y}_t \qquad : \boldsymbol{\lambda}_t^{p,y} \qquad \forall t \in \mathcal{T} \tag{2.2e}$$

$$\mathbf{q}_t = \hat{\mathbf{q}}_t^{fixed} + \mathbf{M}^{q,y}\mathbf{y}_t \qquad : \boldsymbol{\lambda}_t^{q,y} \qquad \forall t \in \mathcal{T} \tag{2.2f}$$

$$\underline{\mathbf{y}}_t \leq \mathbf{y}_t \leq \overline{\mathbf{y}}_t \qquad : \underline{\boldsymbol{\mu}}_t^y, \overline{\boldsymbol{\mu}}_t^y \qquad \forall t \in \mathcal{T} \tag{2.2g}$$

The state-space equations given in equation (2.2) represent the qualities of all flexible DERs in the system, and their components are explained exhaustively in Section 1.3.

However, there are some extra comments that should be left for further clarification. Accordingly, the vectors given by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the dual variables vectors of the corresponding constraint. Also, in the case of modelling energy storage systems, the constraint given in equation (2.2a) ensures that the energy level in the storage at the end of the day is equal to its initial level in the simulation. Equation (2.2b) initialize the flexible DER in the simulation whereas, equations (2.2c) and (2.2d) are the state and output equations.

An important point about mapping the output vector to the active and reactive power of DERs via equations (2.2e) and (2.2f) is that $\mathbf{p}/\mathbf{q} \in \mathbb{R}^{\mathcal{D}\times 1}$ are the active and reactive power vectors of all DER sets including flexible/non-flexible loads and generators ($\mathcal{D}$). In general, these vectors are variables in our problem; however, the arrays associated with the fixed DERs inside these vectors must be parameterized. To apply the idea, the scheduled active/reactive power vectors for the sets of fixed DERs are adopted with the same dimension of all DERs, $\hat{\mathbf{p}}/\hat{\mathbf{q}}^{fixed} \in \mathbb{R}^{\mathcal{D}\times 1}$. Whereas the arrays respective to the flexible DER inside this column vector are set to zero. Merging this vector with the matrix product of $\mathbf{M}^{p/q,y} \in \mathbb{R}^{\mathcal{D}\times \mathcal{O}}$ and $\mathbf{y}_t \in \mathbb{R}^{\mathcal{O}\times 1}$ reveals the active/reactive power general vector ($\mathbf{p}/\mathbf{q}_t$) which some of its arrays will be inherently variables derived from the output vector and some are parameters related to the fixed DERs.

2.1.2. Electric Grid Constraints

This section will introduce the linearized approximation model for AC-OPF attained in Section 1.2 as the distribution network operational constraints of the DSO market optimization problem. Similar to the state-space constraints, the co-related dual variables are expressed by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ in equation (2.3).

$$p_t^{loss} = \hat{p}_t^{loss} + \mathbf{M}_p^{p^{loss}} \mathbf{p}_t + \mathbf{M}_q^{p^{loss}} \mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (2.3a)$$

$$p_t^0 + \mathbf{1}^{\mathrm{T}} \mathbf{p}_t = p_t^{loss} \qquad : \lambda_t^p \qquad \forall t \in \mathcal{T} \qquad (2.3b)$$

$$q_t^{loss} = \hat{q}_t^{loss} + \mathbf{M}_p^{q^{loss}} \mathbf{p}_t + \mathbf{M}_q^{q^{loss}} \mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (2.3c)$$

$$q_t^0 + \mathbf{1}^{\mathrm{T}} \mathbf{q}_t = q_t^{loss} \qquad : \lambda_t^q \qquad \forall t \in \mathcal{T} \qquad (2.3d)$$

$$|\mathbf{u}_t| = |\hat{\mathbf{u}}_t| + \mathbf{M}_p^{|u|} \mathbf{p}_t + \mathbf{M}_q^{|u|} \mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (2.3e)$$

$$|\underline{\mathbf{u}}_t| \leq |\mathbf{u}_t| \leq |\overline{\mathbf{u}}_t| \qquad : \underline{\boldsymbol{\mu}}_t^{|u|}, \overline{\boldsymbol{\mu}}_t^{|u|} \qquad \forall t \in \mathcal{T} \qquad (2.3f)$$

$$|\mathbf{s}_t^f| = |\hat{\mathbf{s}}_t^f| + \mathbf{M}_p^{|s^f|} \mathbf{p}_t + \mathbf{M}_q^{|s^f|} \mathbf{q}_t \qquad \forall t \in \mathcal{T} \qquad (2.3g)$$

$$|\underline{\mathbf{s}_t^f}| \leq |\mathbf{s}_t^f| \leq |\overline{\mathbf{s}_t^f}| \qquad : \underline{\boldsymbol{\mu}}_t^{|s^f|}, \overline{\boldsymbol{\mu}}_t^{|s^f|} \qquad \forall t \in \mathcal{T} \qquad (2.3h)$$

$$|\mathbf{s}_t^t| = |\hat{\mathbf{s}}_t^t| + \mathbf{M}_p^{|s^t|}\mathbf{p}_t + \mathbf{M}_q^{|s^t|}\mathbf{q}_t \qquad \forall t \in \mathcal{T} \tag{2.3i}$$

$$|\underline{\mathbf{s}_t^t}| \leq |\mathbf{s}_t^t| \leq |\overline{\mathbf{s}_t^t}| \qquad : \underline{\boldsymbol{\mu}}_t^{|s^t|}, \overline{\boldsymbol{\mu}}_t^{|s^t|} \qquad \forall t \in \mathcal{T} \tag{2.3j}$$

The active and reactive power transaction at the PSP is expressed by p_t^0 and q_t^0 as variables. The parameters with over-line indicate the maximum level of the corresponding variable vector and the ones with under-line are the minimum levels of them. Worth bearing in mind that there is a slight modification in the linearized AC-OPF given in equation (2.3) in comparison with the one introduced in equation (1.6). As the original idea of the linearized approximation method based on the Taylor series suggests, the sensitivity matrices are multiplied by the deviation of the active/reactive power, $\mathbf{M}_{p/q}^{(\cdot)}\Delta\mathbf{p}/\mathbf{q}$. Consider the following simplification for a generic linearized function:

$$f(x) = \hat{f}(x) + \frac{\partial f(x)}{\partial x}\underbrace{\Delta x}_{(x-\hat{x})} = \underbrace{\hat{f}^{new}(x)}_{\hat{f}(x)-\frac{\partial f(x)}{\partial x}\hat{x}} + \frac{\partial f(x)}{\partial x}x$$

In order to avoid using Δ, the same simplification method is applied to the linearized AC-OPF constraints. Therefore, basically, the reference values for the state variables for the problem are already updated.

2.1.3. DSO Objective Function

The objective function of the DSO market is to maximize social welfare by minimizing the operational cost of flexible generators and maximizing the utility benefit of the flexible loads. Moreover, the problem should also take into account minimizing the cost of purchasing energy from the PSP as well as power losses in the distribution network. Therefrom, the following objective function can be developed for the DSO market problem subject to the DER and grid operational constraints.

$$\underset{p_t^0, q_t^0, \mathbf{p}_t, \mathbf{q}_t}{Minimize} \sum_{t \in \mathcal{T}} \left\{ c_t^{p,0} p_t^0 + c_t^{q,0} q_t^0 + \left(\mathbf{c}_t^{p,DER}\right)^{\mathrm{T}} \mathbf{p}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}} \mathbf{q}_t \right\} \tag{2.4a}$$

s.t. equations (2.2) and (2.3) (2.4b)

It is possible to eliminate the variables related to the active and reactive power of PSP (p_t^0/q_t^0) by adopting the following simplification derived from equations (2.3b) and (2.3d).

$$p_t^0 = -\mathbf{1}^{\mathrm{T}}\mathbf{p}_t + p_t^{loss} \quad \overset{\times c_t^{p,0}}{\Rightarrow} \quad c_t^{p,0}p_t^0 = -c_t^{p,0}\mathbf{1}^{\mathrm{T}}\mathbf{p}_t + c_t^{p,0}p_t^{loss} \tag{2.5a}$$

$$q_t^0 = -\mathbf{1}^{\mathrm{T}}\mathbf{q}_t + q_t^{loss} \quad \overset{\times c_t^{q,0}}{\Rightarrow} \quad c_t^{q,0}p_t^0 = -c_t^{q,0}\mathbf{1}^{\mathrm{T}}\mathbf{q}_t + c_t^{q,0}q_t^{loss} \tag{2.5b}$$

In this way the primal optimization problem of DSO market can be expressed as it follows:

$$\underset{p_t^{loss},q_t^{loss},\mathbf{p}_t,\mathbf{q}_t}{Minimize} \quad \sum_{t\in\mathcal{T}} \left\{ \begin{array}{l} c_t^{p,0}p_t^{loss} - c_t^{p,0}\mathbf{1}^{\mathrm{T}}\mathbf{p}_t + \left(\mathbf{c}_t^{p,DER}\right)^{\mathrm{T}}\mathbf{p}_t \\ +c_t^{q,0}q_t^{loss} - c_t^{q,0}\mathbf{1}^{\mathrm{T}}\mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}}\mathbf{q}_t \end{array} \right\} \tag{2.6a}$$

$$\textbf{s.t.} \quad p_t^{loss} = \hat{p}_t^{loss} + \mathbf{M}_p^{p^{loss}}\mathbf{p}_t + \mathbf{M}_q^{p^{loss}}\mathbf{q}_t \qquad : \lambda_t^{p^{loss}} \qquad \forall t\in\mathcal{T} \tag{2.6b}$$

$$q_t^{loss} = \hat{q}_t^{loss} + \mathbf{M}_p^{q^{loss}}\mathbf{p}_t + \mathbf{M}_q^{q^{loss}}\mathbf{q}_t \qquad : \lambda_t^{q^{loss}} \qquad \forall t\in\mathcal{T} \tag{2.6c}$$

$$\text{equations (2.2) and (2.3e) to (2.3j)} \tag{2.6d}$$

As equation (2.6) indicates the objective function is modified to model the active and reactive power exchange with the upstream grid implicitly by replacing them with active and reactive losses (p_t^{loss} and q_t^{loss}). Therefore, the equality constraints related to the active and reactive power balance, equations (2.3b) and (2.3d), are replaced by equations (2.6b) and (2.6c), and their corresponding dual variables ($\lambda_t^{p^{loss}}$, and $\lambda_t^{p^{loss}}$).

2.2. DSO Market KKT Optimality Conditions and Dual Objective Function

Every convex optimization problem can be written in form of the sets of equations to be solved which are called "Karush–Kuhn–Tucker (KKT)" conditions [27].

Lagrangian function:

$$L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = F(\mathbf{x}) + \boldsymbol{\lambda}^{\mathrm{T}}(\mathcal{C}\mathbf{x}-\mathbf{d}) + \boldsymbol{\mu}^{\mathrm{T}}(\mathbf{A}\mathbf{x}-\mathbf{b}) \tag{2.7a}$$

KKT conditions:

$$\nabla_{\mathbf{x}}L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}) = \mathbf{0} \tag{2.7b}$$

$$(\mathcal{C}\mathbf{x}-\mathbf{d}) = \mathbf{0} \tag{2.7c}$$

Complementary slackness:

$$\mathbf{0} \le \mathrm{diag}\,(\boldsymbol{\mu}) \perp (-\mathbf{A}\mathbf{x}+\mathbf{b}) \ge \mathbf{0} \tag{2.7d}$$

Accordingly, the KKT conditions are necessary and sufficient for the optimality of a convex optimization problem. Therefore, a standard convex optimization problem is given in equation (2.1), the KKT conditions can be obtained by testing the partial derivative of the Lagrangian function to the primal and dual variables, equation (2.7).

Equation (2.7d) indicates the complementary sub-condition of the KKT condition and it makes the sets of equations nonlinear. To this end, Fortuny-Amat so-called Big-M method can be employed to linearize the KKT's complementarity equations by adopting an auxiliary binary variable as [28]:

$$\mathbf{0} \leq (-\mathbf{A}\mathbf{x} + \mathbf{b}) \leq \mathrm{diag}(\mathbf{M})\boldsymbol{\Psi} \tag{2.8a}$$

$$\mathbf{0} \leq \boldsymbol{\mu} \leq \mathrm{diag}(\mathbf{M})\left(\boldsymbol{\Psi} - \mathbf{1}\right) \tag{2.8b}$$

where the vector $\mathbf{M}$ is a suitably big number and $\boldsymbol{\Psi}$ is the auxiliary binary variable. Bear in mind that this linearization transforms the linear programming (LP) problem into mixed-integer linear programming (MILP).

Eventually, for every primal problem for a standard convex optimization, there is a dual problem and objective function. Based on the strong duality theorem, for such a problem, the results for the primal problem are identical to the ones obtained from the dual problem at the optimal point. In other words, the dual objective function is equal to the primal one at the optimum point [27]. This fact is expressed in equation (2.9) for a standard minimization convex problem.

Dual Problem:

$$\underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{max} \quad \left\{ \underset{\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu}}{min} \quad L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = F(\mathbf{x}) + \boldsymbol{\lambda}^{\mathrm{T}}(\mathcal{C}\mathbf{x} - \mathbf{d}) + \boldsymbol{\mu}^{\mathrm{T}}(\mathbf{A}\mathbf{x} - \mathbf{b}) \right\} \tag{2.9a}$$

$\Rightarrow$ **Dual OF:**

$$\underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{max} \quad D(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -(\mathbf{d})^{\mathrm{T}}\boldsymbol{\lambda} - (\mathbf{b})^{\mathrm{T}}\boldsymbol{\mu} \tag{2.9b}$$

At optimum point of $\mathbf{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*$*:*

$$F(\mathbf{x}^*) = D(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*) \tag{2.9c}$$

2.2.1. DSO Market Problem KKT Optimality Condition

Let us first, write the Lagrangian function for the DSO market as follows:

$$
\begin{aligned}
L(p_t^{loss}, q_t^{loss}, \mathbf{p}_t, \mathbf{q}_t, \boldsymbol{\lambda}, \boldsymbol{\mu}) = &+ \sum_{t\in\mathcal{T}} \Big\{ c_t^{p,0} p_t^{loss} - c_t^{p,0} \mathbf{1}^{\mathrm{T}} \mathbf{p}_t + \left(\mathbf{c}_t^{p,DER}\right)^{\mathrm{T}} \mathbf{p}_t \\
&\quad + c_t^{q,0} q_t^{loss} - c_t^{q,0} \mathbf{1}^{\mathrm{T}} \mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}} \mathbf{q}_t \Big\} \\
&+ \left(\mathbf{x}_{t^0}^{storage} - \mathbf{x}_{t^n}^{storage}\right)^{\mathrm{T}} \boldsymbol{\lambda}^{x,storage} \\
&+ \left(\mathbf{x}_{t^0}^{non-storage} - \hat{\mathbf{x}}\right)^{\mathrm{T}} \boldsymbol{\lambda}^{x,non-storage} \\
&+ \sum_{t\in\mathcal{T}-\{t^n\}} \left(\mathbf{x}_{t+1} - \mathbf{A}\mathbf{x}_t - \mathbf{B}\mathbf{u}_t - \mathbf{v}_t^x\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^x \\
&+ \sum_{t\in\mathcal{T}} \left(\mathbf{y}_t - \mathbf{C}\mathbf{x}_t - \mathbf{D}\mathbf{u}_t - \mathbf{v}_t^y\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^y \\
&+ \sum_{t\in\mathcal{T}} \left(\mathbf{p}_t - \hat{\mathbf{p}}_t^{fixed} - \mathbf{M}^{p,y}\mathbf{y}_t\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{p,y} \\
&+ \sum_{t\in\mathcal{T}} \left(\mathbf{q}_t - \hat{\mathbf{q}}_t^{fixed} - \mathbf{M}^{q,y}\mathbf{y}_t\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{q,y} \\
&+ \sum_{t\in\mathcal{T}} \left(-\mathbf{y}_t + \underline{\mathbf{y}}_t\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^y + \sum_{t\in\mathcal{T}} \left(\mathbf{y}_t - \overline{\mathbf{y}}_t\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^y \\
&+ \sum_{t\in\mathcal{T}} \left(p_t^{loss} - \hat{p}_t^{loss} - \mathbf{M}_p^{p^{loss}}\mathbf{p}_t - \mathbf{M}_q^{p^{loss}}\mathbf{q}_t\right)^{\mathrm{T}} \lambda_t^{p^{loss}} \\
&+ \sum_{t\in\mathcal{T}} \left(q_t^{loss} - \hat{q}_t^{loss} - \mathbf{M}_p^{q^{loss}}\mathbf{p}_t - \mathbf{M}_q^{q^{loss}}\mathbf{q}_t\right)^{\mathrm{T}} \lambda_t^{q^{loss}} \\
&+ \sum_{t\in\mathcal{T}} \left(|\underline{\mathbf{u}}_t| - |\hat{\mathbf{u}}_t| - \mathbf{M}_p^{|u|}\mathbf{p}_t - \mathbf{M}_q^{|u|}\mathbf{q}_t\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|u|} \\
&+ \sum_{t\in\mathcal{T}} \left(-|\overline{\mathbf{u}}_t| + |\hat{\mathbf{u}}_t| + \mathbf{M}_p^{|u|}\mathbf{p}_t + \mathbf{M}_q^{|u|}\mathbf{q}_t\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|u|} \\
&+ \sum_{t\in\mathcal{T}} \left(|\underline{\mathbf{s}_t^f}| - |\hat{\mathbf{s}}_t^f| - \mathbf{M}_p^{|s^f|}\mathbf{p}_t - \mathbf{M}_q^{|s^f|}\mathbf{q}_t\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|s^f|} \\
&+ \sum_{t\in\mathcal{T}} \left(-|\overline{\mathbf{s}_t^f}| + |\hat{\mathbf{s}}_t^f| + \mathbf{M}_p^{|s^f|}\mathbf{p}_t + \mathbf{M}_q^{|s^f|}\mathbf{q}_t\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|s^f|} \\
&+ \sum_{t\in\mathcal{T}} \left(|\underline{\mathbf{s}_t^t}| - |\hat{\mathbf{s}}_t^t| - \mathbf{M}_p^{|s^t|}\mathbf{p}_t - \mathbf{M}_q^{|s^t|}\mathbf{q}_t\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|s^t|} \\
&+ \sum_{t\in\mathcal{T}} \left(-|\overline{\mathbf{s}_t^t}| + |\hat{\mathbf{s}}_t^t| + \mathbf{M}_p^{|s^t|}\mathbf{p}_t + \mathbf{M}_q^{|s^t|}\mathbf{q}_t\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|s^t|} \qquad (2.10a)
\end{aligned}
$$

Following the standard procedure expressed in equation (2.7), by taking the partial derivative of the Lagrangian function with respect to the primal and dual variables the following KKT optimality conditions will be revealed.

$$\frac{\partial L(\cdot)}{\partial \mathbf{x}_{t^0}} = \boldsymbol{\lambda}^{x,storage} + \boldsymbol{\lambda}^{x,non-storage} - (\mathbf{A})^{\mathrm{T}} \boldsymbol{\lambda}^{x}_{t^0} - (\mathbf{C})^{\mathrm{T}} \boldsymbol{\lambda}^{y}_{t^0} = \mathbf{0} \tag{2.11a}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{x}_{t}} = \boldsymbol{\lambda}^{x}_{t-1} - (\mathbf{A})^{\mathrm{T}} \boldsymbol{\lambda}^{x}_{t} - (\mathbf{C})^{\mathrm{T}} \boldsymbol{\lambda}^{y}_{t} = 0 \qquad \forall t \in \mathcal{T} - \{t^0, t^n\} \tag{2.11b}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{x}_{t^n}} = \boldsymbol{\lambda}^{x}_{t^n-1} - \boldsymbol{\lambda}^{x,storage} - (\mathbf{C})^{\mathrm{T}} \boldsymbol{\lambda}^{y}_{t^n} = \mathbf{0} \tag{2.11c}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{u}_{t}} = - (\mathbf{B})^{\mathrm{T}} \boldsymbol{\lambda}^{x}_{t} - (\mathbf{D})^{\mathrm{T}} \boldsymbol{\lambda}^{y}_{t} = \mathbf{0} \qquad \forall t \in \mathcal{T} - \{t^n\} \tag{2.11d}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{u}_{t^n}} = - (\mathbf{D})^{\mathrm{T}} \boldsymbol{\lambda}^{y}_{t^n} \tag{2.11e}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{y}_{t}} = \boldsymbol{\lambda}^{y}_{t} + \overline{\boldsymbol{\mu}}^{y}_{t} - \underline{\boldsymbol{\mu}}^{y}_{t} - (\mathbf{M}^{p,y})^{\mathrm{T}} \boldsymbol{\lambda}^{p,y}_{t} - (\mathbf{M}^{q,y})^{\mathrm{T}} \boldsymbol{\lambda}^{q,y}_{t} \qquad \forall t \in \mathcal{T} \tag{2.11f}$$

$$\frac{\partial L(\cdot)}{\partial p^{loss}_{t}} = c^{p,0}_{t} + \lambda^{p^{loss}}_{t} \qquad \forall t \in \mathcal{T} \tag{2.11g}$$

$$\frac{\partial L(\cdot)}{\partial q^{loss}_{t}} = c^{q,0}_{t} + \lambda^{q^{loss}}_{t} \qquad \forall t \in \mathcal{T} \tag{2.11h}$$

$$\begin{aligned}\frac{\partial L(\cdot)}{\partial \mathbf{p}_{t}} &= \mathbf{c}^{p,DER}_{t} + \boldsymbol{\lambda}^{p,y}_{t} \\ &- c^{p,0}_{t}\mathbf{1} \\ &- \left(\mathbf{M}^{p^{loss}}_{p}\right)^{\mathrm{T}} \lambda^{p^{loss}}_{t} - \left(\mathbf{M}^{q^{loss}}_{p}\right)^{\mathrm{T}} \lambda^{q^{loss}}_{t} \\ &+ \left(\mathbf{M}^{|u|}_{p}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}^{|u|}_{t} - \underline{\boldsymbol{\mu}}^{|u|}_{t}\right) \\ &+ \left(\mathbf{M}^{|s^f|}_{p}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}^{|s^f|}_{t} - \underline{\boldsymbol{\mu}}^{|s^f|}_{t}\right) + \left(\mathbf{M}^{|s^t|}_{p}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}^{|s^t|}_{t} - \underline{\boldsymbol{\mu}}^{|s^t|}_{t}\right) \qquad \forall t \in \mathcal{T}\end{aligned} \tag{2.11i}$$

$$\begin{aligned}\frac{\partial L(\cdot)}{\partial \mathbf{q}_{t}} &= \mathbf{c}^{q,DER}_{t} + \boldsymbol{\lambda}^{q,y}_{t} \\ &- c^{q,0}_{t}\mathbf{1} \\ &- \left(\mathbf{M}^{p^{loss}}_{q}\right)^{\mathrm{T}} \lambda^{p^{loss}}_{t} - \left(\mathbf{M}^{q^{loss}}_{q}\right)^{\mathrm{T}} \lambda^{q^{loss}}_{t} \\ &+ \left(\mathbf{M}^{|u|}_{q}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}^{|u|}_{t} - \underline{\boldsymbol{\mu}}^{|u|}_{t}\right) \\ &+ \left(\mathbf{M}^{|s^f|}_{q}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}^{|s^f|}_{t} - \underline{\boldsymbol{\mu}}^{|s^f|}_{t}\right) + \left(\mathbf{M}^{|s^t|}_{q}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}^{|s^t|}_{t} - \underline{\boldsymbol{\mu}}^{|s^t|}_{t}\right) \qquad \forall t \in \mathcal{T}\end{aligned} \tag{2.11j}$$

The corresponding complementary slackness of the inequality constraints for the given problem can be linearized using the proposed Big-M method (equation (2.8)) and written as:

$$\mathbf{0} \leq (-\mathbf{y}_t + \overline{\mathbf{y}}_t) \leq \operatorname{diag}(\boldsymbol{\mathcal{M}}^y)\left(\overline{\boldsymbol{\Psi}}_t^y\right) \qquad \forall t \in \mathcal{T} \tag{2.12a}$$

$$\mathbf{0} \leq (\overline{\boldsymbol{\mu}}_t^y) \leq \operatorname{diag}(\boldsymbol{\mathcal{M}}^y)\left(\mathbf{1}^y - \overline{\boldsymbol{\Psi}}_t^y\right) \qquad \forall t \in \mathcal{T} \tag{2.12b}$$

$$\mathbf{0} \leq \left(\mathbf{y}_t - \underline{\mathbf{y}}_t\right) \leq \operatorname{diag}(\boldsymbol{\mathcal{M}}^y)\left(\underline{\boldsymbol{\Psi}}_t^y\right) \qquad \forall t \in \mathcal{T} \tag{2.12c}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^y\right) \leq \operatorname{diag}(\boldsymbol{\mathcal{M}}^y)\left(\mathbf{1}^y - \underline{\boldsymbol{\Psi}}_t^y\right) \qquad \forall t \in \mathcal{T} \tag{2.12d}$$

$$\mathbf{0} \leq (-|\mathbf{u}_t| + |\overline{\mathbf{u}}_t|) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\overline{\boldsymbol{\Psi}}_t^{|u|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12e}$$

$$\mathbf{0} \leq \left(\overline{\boldsymbol{\mu}}_t^{|u|}\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\mathbf{1}^{|u|} - \overline{\boldsymbol{\Psi}}_t^{|u|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12f}$$

$$\mathbf{0} \leq (|\mathbf{u}_t| - |\underline{\mathbf{u}}_t|) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\underline{\boldsymbol{\Psi}}_t^{|u|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12g}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^{|u|}\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\mathbf{1}^{|u|} - \underline{\boldsymbol{\Psi}}_t^{|u|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12h}$$

$$\mathbf{0} \leq \left(-|\mathbf{s}_t^f| + |\overline{\mathbf{s}_t^f}|\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\overline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12i}$$

$$\mathbf{0} \leq \left(\overline{\boldsymbol{\mu}}_t^{|s^f|}\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\mathbf{1}^{|s^f|} - \overline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12j}$$

$$\mathbf{0} \leq \left(|\mathbf{s}_t^f| - |\underline{\mathbf{s}_t^f}|\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\underline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12k}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^{|s^f|}\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\mathbf{1}^{|s^f|} - \underline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12l}$$

$$\mathbf{0} \leq \left(-|\mathbf{s}_t^t| + |\overline{\mathbf{s}_t^t}|\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\overline{\boldsymbol{\Psi}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12m}$$

$$\mathbf{0} \leq \left(\overline{\boldsymbol{\mu}}_t^{|s^t|}\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\mathbf{1}^{|s^t|} - \overline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12n}$$

$$\mathbf{0} \leq \left(|\mathbf{s}_t^t| - |\underline{\mathbf{s}_t^t}|\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\underline{\boldsymbol{\Psi}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12o}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \leq \operatorname{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\mathbf{1}^{|s^t|} - \underline{\boldsymbol{\Psi}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \tag{2.12p}$$

So far, the MILP-based complete KKT condition and the respective linearized complementary slackness are presented in equations (2.11) and (2.12). A particular point to bear in mind is that the variables are column vectors and their size is relative to their corresponding variable type; therefore, their size is also projected to the associated binary variables $\mathbf{Psi}_t^{(\cdot)}$. A case in point, equation (2.12e) represents the linearized complementary condition for the voltage upper limit, where $|\mathbf{u}|_t \in \mathbb{R}^{\mathcal{N}\times 1}$; therefore, $\overline{\boldsymbol{\Psi}}_t^{|u|} \in \mathbb{R}^{\mathcal{N}\times 1}$, $\boldsymbol{\mathcal{M}}^{|u|} \in \mathbb{R}^{\mathcal{N}\times 1}$, and $\mathbf{1}^{|u|} \in \mathbb{R}^{\mathcal{N}\times 1}$.

2.2.2. DSO Dual Objective Function and Strong Duality Theorem Test

Earlier, we put an argument based on the strong duality theorem that every convex optimization problem has a dual problem that its objective function level is equal to the primal problem a the optimum point [27]. Therefore, we employ the standard procedure provided in equation (2.9) to attain the dual objective function for the DSO market optimization problem as follows.

$$
\begin{aligned}
\underset{\boldsymbol{\lambda}_t^*,\boldsymbol{\mu}_t^*}{Maximize} \quad & + \left(-\hat{\mathbf{x}}\right)^{\mathrm{T}} \boldsymbol{\lambda}^{x,non-storage} + \sum_{t\in\mathcal{T}-\{t^n\}} \left(-\mathbf{v}_t^x\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^x + \sum_{t\in\mathcal{T}} \left(-\mathbf{v}_t^y\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^y \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(-\hat{\mathbf{p}}_t^{fixed}\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{p,y} + \left(-\hat{\mathbf{q}}_t^{fixed}\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{q,y} \right\} \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(\underline{\mathbf{y}}_t\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^y + \left(-\overline{\mathbf{y}}_t\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^y \right\} \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(-\hat{p}_t^{loss}\right)^{\mathrm{T}} \lambda_t^{p^{loss}} + \left(-\hat{q}_t^{loss}\right)^{\mathrm{T}} \lambda_t^{q^{loss}} \right\} \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(|\underline{\mathbf{u}}_t| - |\hat{\mathbf{u}}_t|\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|u|} + \left(-|\overline{\mathbf{u}}_t| + |\hat{\mathbf{u}}_t|\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|u|} \right\} \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(|\underline{\mathbf{s}}_t^f| - |\hat{\mathbf{s}}_t^f|\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|s^f|} + \left(-|\overline{\mathbf{s}}_t^f| + |\hat{\mathbf{s}}_t^f|\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|s^f|} \right\} \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(|\underline{\mathbf{s}}_t^t| - |\hat{\mathbf{s}}_t^t|\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|s^t|} + \left(-|\overline{\mathbf{s}}_t^t| + |\hat{\mathbf{s}}_t^t|\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|s^t|} \right\} \\
= \underset{p_t^{loss*},q_t^{loss*},\mathbf{p}_t^*,\mathbf{q}_t^*}{Minimize} \quad & + \sum_{t\in\mathcal{T}} \left\{ c_t^{p,0} p_t^{loss} - c_t^{p,0} \mathbf{1}^{\mathrm{T}} \mathbf{p}_t + \left(\mathbf{c}_t^{p,DER}\right)^{\mathrm{T}} \mathbf{p}_t \right. \\
& \left. + c_t^{q,0} q_t^{loss} - c_t^{q,0} \mathbf{1}^{\mathrm{T}} \mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}} \mathbf{q}_t \right\}
\end{aligned} \tag{2.13a}
$$

The important statement expressed in equation (2.13) is always true as far as the DSO market optimization problem is convex.

2.3. Distribution Locational Marginal Prices (DLMP) and Grid Using Prices (GUP)

As discussed earlier, the main goal of developing the DSO market is to minimize the distribution network's overall power losses by solving AC-OPF. Moreover, it is supposed to incentivize the flexible DER for providing AS for the system through spot pricing of energy in the network (DLMP).

By convention, obtaining DLMPs is subject to the sensitivity of objective function to the nodal active power injection. Considering the KKT condition of the DSO market problem in equation (2.11), the given sensitivity of the Lagrangian function with respect to the DER active power injection,equation (2.11i), indicates the concept that we are seeking. Hence, we will derive the DLMPs out of it in the following.

$$\frac{\partial L(\cdot)}{\partial \mathbf{p}_t} = \mathbf{c}_t^{p,DER} + \boldsymbol{\lambda}_t^{p,y} \tag{2.14a}$$

$$- c_t^{p,0}\mathbf{1} - \left(\mathbf{M}_p^{p^{loss}}\right)^{\mathrm{T}} \lambda_t^{p^{loss}} - \left(\mathbf{M}_p^{q^{loss}}\right)^{\mathrm{T}} \lambda_t^{q^{loss}} + \left(\mathbf{M}_p^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) + \left(\mathbf{M}_p^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) + \left(\mathbf{M}_p^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \quad \forall t \in \mathcal{T} \tag{2.14b}$$

Having considered equation (2.14), genuinely, from the flexible DER prospective, the DLMP can be expressed as:

$$\boldsymbol{\pi}_t^{p,DER} = \mathbf{c}_t^{p,DER} + \overline{\boldsymbol{\mu}}_t^{p,DER} - \underline{\boldsymbol{\mu}}_t^{p,DER} \qquad \forall t \in \mathcal{T} \tag{2.15a}$$

where $\mathbf{c}_t^{p,DER}$ is the marginal offer or bid of flexible DER $\overline{\boldsymbol{\mu}}_t^{p,DER}$ and $\underline{\boldsymbol{\mu}}_t^{p,DER}$ are the dual variables regarding the binding of DER to its maximum and minimum operational limit at time-step $t \in \mathcal{T}$. In Section 1.3, we comprehensively discussed how the operational limits of the flexible DERs are implicitly modelled by state-space matrix equations. Therefore, the DLMP the flexible DER viewpoint can be re-written as:

$$\boldsymbol{\pi}_t^{p,DER} = \mathbf{c}_t^{p,DER} + \underbrace{\overline{\boldsymbol{\mu}}_t^{p,DER} - \underline{\boldsymbol{\mu}}_t^{p,DER}} = \mathbf{c}_t^{p,DER} + \boldsymbol{\lambda}_t^{p,y} \qquad \forall t \in \mathcal{T} \tag{2.16a}$$

Coinciding equation (2.16) with the sensitivity of the Lagrangian function to the DER active power in equation (2.14), the following results will satisfy the DLMP formulation from the network perspective.

$$\begin{aligned}
\boldsymbol{\pi}_t^{p,DER} &= \mathbf{c}_t^{p,DER} + \boldsymbol{\lambda}_t^{p,y} = \\
&+ c_t^{p,0}\mathbf{1} \\
&+ \left(\mathbf{M}_p^{p^{loss}}\right)^{\mathrm{T}} \lambda_t^{p^{loss}} + \left(\mathbf{M}_p^{q^{loss}}\right)^{\mathrm{T}} \lambda_t^{q^{loss}} \\
&- \left(\mathbf{M}_p^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&- \left(\mathbf{M}_p^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_p^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (2.17\text{a})\\
\Rightarrow \quad \boldsymbol{\pi}_t^{p,grid} &= c_t^{p,0}\mathbf{1} \\
&+ \left(\mathbf{M}_p^{p^{loss}}\right)^{\mathrm{T}} \lambda_t^{p^{loss}} + \left(\mathbf{M}_p^{q^{loss}}\right)^{\mathrm{T}} \lambda_t^{q^{loss}} \\
&- \left(\mathbf{M}_p^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&- \left(\mathbf{M}_p^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_p^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (2.17\text{b})
\end{aligned}$$

Eventually, taking into account the KKT condition given in equations (2.11g) and (2.11h), it yields that $\lambda_t^{p^{loss}} = -c_t^{p,0}$ and $\lambda_t^{q^{loss}} = -c_t^{q,0}$ $\forall t \in \mathcal{T}$. Hence, the final expression of DLMP from the grid prospective can be indicated as in equation (2.18).

$$\begin{aligned}
\boldsymbol{\pi}_t^{p} &= c_t^{p,0}\mathbf{1} \\
&- \left(\mathbf{M}_p^{p^{loss}}\right)^{\mathrm{T}} c_t^{p,0} - \left(\mathbf{M}_p^{q^{loss}}\right)^{\mathrm{T}} c_t^{q,0} \\
&- \left(\mathbf{M}_p^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&- \left(\mathbf{M}_p^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_p^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (2.18\text{a})
\end{aligned}$$

Respectively, the DLMP is the superposition of the energy price at the PSP ($\boldsymbol{\pi}_t^0$), the contribution of losses in the DN ($\boldsymbol{\pi}_t^{loss}$), the voltage violation at nodes ($\boldsymbol{\pi}_t^{|u|}$), and the line thermal capacity violation $\boldsymbol{\pi}_t^{|s|}$.

$$\boldsymbol{\pi}_t^{p} = \boldsymbol{\pi}_t^{0} + \boldsymbol{\pi}_t^{loss} + \boldsymbol{\pi}_t^{|u|} + \boldsymbol{\pi}_t^{|s|} \qquad \forall t \in \mathcal{T} \quad (2.19a)$$

$$\boldsymbol{\pi}_t^{0} = +c_t^{p,0}\mathbf{1} \qquad \forall t \in \mathcal{T} \quad (2.19b)$$

$$\boldsymbol{\pi}_t^{loss} = -\left(\mathbf{M}_p^{p^{loss}}\right)^{\mathrm{T}} c_t^{p,0} - \left(\mathbf{M}_p^{q^{loss}}\right)^{\mathrm{T}} c_t^{q,0} \qquad \forall t \in \mathcal{T} \quad (2.19c)$$

$$\boldsymbol{\pi}_t^{|u|} = -\left(\mathbf{M}_p^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \qquad \forall t \in \mathcal{T} \quad (2.19d)$$

$$\boldsymbol{\pi}_t^{|s|} = -\left(\mathbf{M}_p^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_p^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (2.19e)$$

To enable the bilateral P2P decentralized market concept, it is necessary to evaluate the fee that the peers are to endure due to using the infrastructures provided by the DSO for their independent energy transactions. Therefore, the grid usage price (GUP) will be derived from DLMP and deployed to apply the fees on the decentralized market which we will discuss in Section 2.4.

To this end, based on [26], the GUP is the deviation of energy price at the buyer node and the seller node. Therefore, the mathematical expression of GUP can be developed as follows:

$$\begin{aligned}\Pi_{s,b,t}^{p} = \pi_{b,t}^{p} - \pi_{s,t}^{p} = \\ &+ \left(-M_b^{p^{loss}} + M_s^{p^{loss}}\right) c_t^{p,0} + \left(-M_b^{q^{loss}} + M_b^{q^{loss}}\right) c_t^{q,0} \\ &+ \left(-\mathbf{M}_b^{|u|} + \mathbf{M}_s^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\ &+ \left(-\mathbf{M}_b^{|s^f|} + \mathbf{M}_s^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) \\ &+ \left(-\mathbf{M}_b^{|s^t|} + \mathbf{M}_s^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (2.20a)\end{aligned}$$

Equation (2.20) represents the GUP for two generic buyer and seller nodes in the DN. An important point to consider is that the sensitivity matrices accommodate for a single DER, either s or b. Therefore, they experience redundant in size so that for the buyer DER $M_b^{p/q^{loss}} \in \mathbb{R}$, $\mathbf{M}_b^{|u|} \in \mathbb{R}^{1\times\mathcal{N}}$ and $\mathbf{M}_b^{|s^{f/t}|} \in \mathbb{R}^{1\times\mathcal{H}}$. Where $\mathcal{N}$ denotes the sets of nodes and $\mathcal{H}$ stands for the sets of branches. By generalizing the idea for all seller and buyer peers in the DN, the GUP can be written as:

$$\mathbf{\Pi}_t^p = \mathbf{1}_{s,t}(\boldsymbol{\pi}_{b,t}^p)^{\mathrm{T}} - \boldsymbol{\pi}_{s,t}^p(\mathbf{1}_{b,t})^{\mathrm{T}} \qquad \forall t \in \mathcal{T} \tag{2.21a}$$

where $\mathbf{\Pi}_{s,b,t}^p \in \mathbb{R}^{\mathcal{S}\times\mathcal{B}}$, and $\mathcal{S}$ indicates the sets of seller peers and $\mathcal{B}$ represents the buyer ones. So far, the GUP matrix given in equation (2.21) can be employed to apply the fees for independent energy transactions in the DN in a decentralized. This concept will be discussed exhaustively in Section 2.4.

2.4. Peer-to-Peer (P2P) Market Integration

As discussed earlier, the main purpose of developing the DSO AS market is to provide the system with higher reliability and support in terms of keeping the operation of the network in nominal frequency and accommodating loss minimization, voltage support as well as congestion management. Furthermore, the European Union Strategy Energy Technology (SET) Plan envisions prosumer-centric smart energy networks that enable independent energy transactions for small-scall DER [29]. In this way, the passive consumers will be motivated to play an active role in supporting the network constraints; therefore, the power quality will be increased. However, since the majority of the flexible DER are renewable-based, their probabilistic behaviour can potentially endanger the system inertia. Moreover, providing the DER fair feed-in tariff is essential for motivating the flexible DER for curtailment response. Therefore, the supervision of grid operators, in this case, DSO, is highly recommended for enabling flexibility in the system.

On the other hand, prosumers prefer to share their excess energy in their community due to the advancements in IoT and blockchain sector. This way, they will have the opportunity to liberally trade energy in a bilateral P2P manner while preserving their local information. Therefore, they will pay more premium on investments in green energy [30, 31]. In this sense, the deployed P2P energy transaction in the Brooklyn microgrid has demonstrated the efficiency of this scheme. In fact, the results enabled the prosumers to attain higher profit and incentives for investment in RES and energy storage systems (ESS) [32].

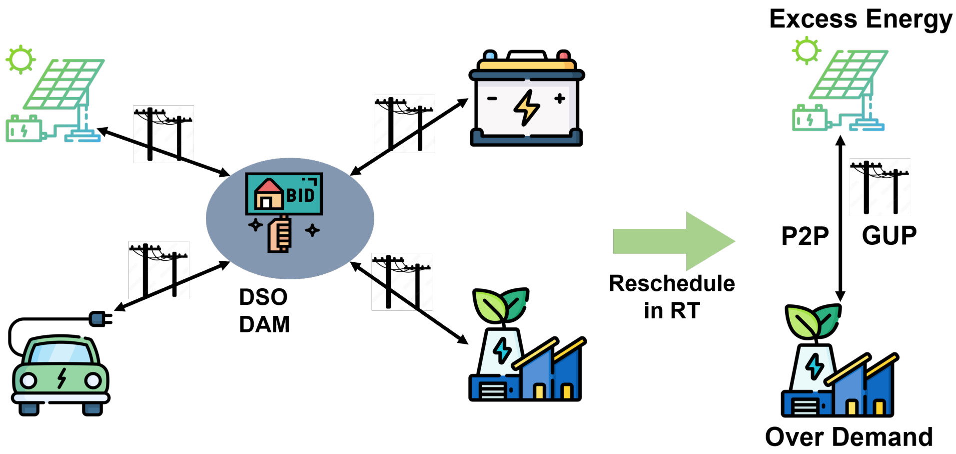

Figure 2.1: DSO day ahead market and P2P energy transaction.

The work conducted by Zhang demonstrates an efficient market mechanism that enables the prosumer-centric P2P energy transaction [26]. The idea is that the DSO market implements a day-ahead market; based on that, the DER is assumed to bind to its scheduled values. However, in real-time, some DERs may be unable to keep up with their scheduled values due to the unexpected over-demand or over-supply. Furthermore, the procurement of energy from the upstream grid will lead to a potential cost for a consumer. On the other hand, a producer may have excess energy that is insufficient to be sold to the upstream DSO grid. Therefrom, the P2P market mechanism can be deployed to connect them on a bilateral basis. From the DSO perspective, we introduced a GUP in Section 2.3 that plays the role of charging the DERs for their bilateral energy transaction, which DSO imposes due to providing network infrastructures. Figure 2.1 illustrate the concept figuratively.

The P2P energy market can be developed as an optimization problem with the objective of maximizing the social welfare of the peers as well as the profit of the DSO as the provider of the necessary infrastructure. Yet, it is important to bear in mind that in order to exploit the P2P prosumer-centric bilateral energy transaction, the information privacy of peers is essential. Therefore, the conventional-centric optimization problem cannot meet the requirements [4]. To this end, distributed optimization methods can play a vital role in solving the problem, especially since the idea of P2P energy transactions is decentralized. A potentially efficient approach for dealing with the problem will be presented in the following.

2.4.1. Distributed Optimization (ADMM)

An optimization problem can be decomposed into multiple optimization problems providing the necessary conditions are met [33]. There are different methods for decomposing an optimization problem according to its structure. Two of the main approaches are based on i) Benders decomposition and ii) Lagrangian relaxation. The former relies on the complicating variables, which if replaced with iterative parameters, the problem will break down into multiple problems. About the latter, the problem includes complicated constraints that, if relaxed the problem can be developed into multiple sub-problems [33]. This work scopes the second method, Lagrangian relaxation-based decomposition. To be specific, one of the derivatives of this method, i.e. alternating direction method of multipliers (ADMM), will be exploited and discussed in further detail.

As an efficient decomposition method, ADMM can provide the sub-problems with information privacy on the grounds that it only needs the data exchange of the proxy variables. Therefore, the sensitive local information of the sub-problems, such as the utility functions and constraints, will not be revealed. This property makes the ADMM a robust method to adopt in our work to meet the expectations of the DERs ad the decentralized peers of the market. Moving to the mathematical representation, consider the following convex optimization problem.

$$\min_{\mathbf{x},\mathbf{y}}. \quad f(\mathbf{x}) + g(\mathbf{y}) \tag{2.22a}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{c} \qquad : \boldsymbol{\lambda} \in \mathbb{R}^n \tag{2.22b}$$

where $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^l$ are the vector primal variables of the problem and $\boldsymbol{\lambda} \in \mathbb{R}^n$ is the respective dual variable vector. Accordingly the $\mathbf{A} \in \mathbb{R}^{n \times k}$, $\mathbf{B} \in \mathbb{R}^{n \times l}$, and $\mathbf{c} \in \mathbb{R}^n$ are the parameters of the problem. The augmented Lagrangian function for the primal problem can be developed as:

$$L_\rho(\mathbf{x}, \mathbf{y}, \boldsymbol{\lambda}) = f(\mathbf{x}) + g(\mathbf{y}) + \boldsymbol{\lambda}^{\mathrm{T}} (\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{c}) + \frac{1}{2}\rho \|\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} - \mathbf{c}\| \tag{2.23}$$

Then, the problem can be decomposed into multiple sub-problems and solved according to the following algorithm given in Algorithm 2.1.

Algorithm 2.1 ADMM solution algorithm for a generic convex problem.

while *ADMMconverged* **do**

$$\mathbf{x}(k+1) := \underset{\mathbf{x}}{\text{argmin}}\ L_\rho(\mathbf{x}, \mathbf{y}(k), \boldsymbol{\lambda}(k)) \tag{2.24a}$$

$$\mathbf{y}(k+1) := \underset{\mathbf{y}}{\text{argmin}}\ L_\rho(\mathbf{x}(k+1), \mathbf{y}, \boldsymbol{\lambda}(k)) \tag{2.24b}$$

$$\boldsymbol{\lambda}(k+1) := \lambda(k) + \rho\left(\mathbf{A}\mathbf{x}(k+1) + \mathbf{B}\mathbf{y}(k+1) - \mathbf{c}\right) \tag{2.24c}$$

end while

As it regards the ADMM algorithm works based on the alternating solution update of the sub problems revealing $\mathbf{x}, \mathbf{y}$, then calculating $\boldsymbol{\lambda}$ based on them. If the problem sub-problems are convex, ADMM will converge to the optimal solution. Moreover, it is probable that the non-convex problem converges using ADMM [34].

2.4.2. ADMM-based P2P Market Formulation

As the name regards, the P2P energy transaction stands for connection and interaction of all sets of seller DERs and the buyer ones in a directed graph manner as shown in figure 2.2.

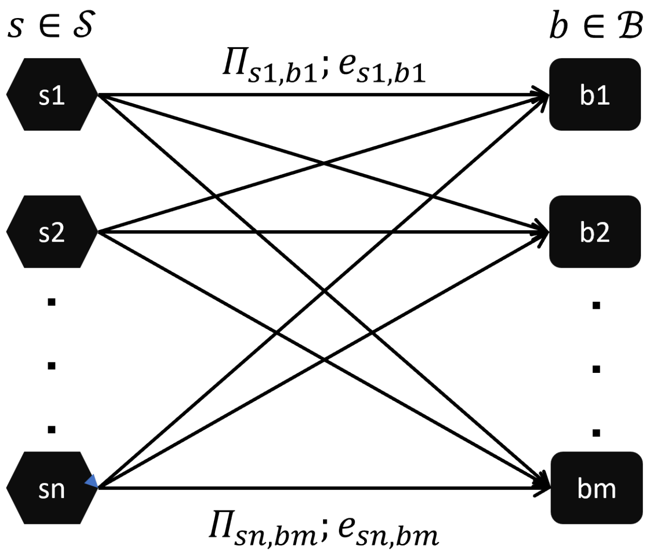

Figure 2.2: DSO day ahead market and P2P energy transaction.

As the directed graph represents, the DERs can either be sellers or buyers at each time interval; simultaneous buying and selling are not authorised. The quantity of energy transacted between seller and buyer is indicated by $e_{s,b,t}$ for time interval t and by popularizing it to all P2P energy transactions in the grid it can be expressed by energy transaction matrix $\mathbf{E}_t \in \mathbb{R}^{S\times B} \forall t \in \mathcal{T}$. The grid using price for a specific seller and buyer is shown by $\Pi_{s,b,t}$ and similarly the GUP matrix for the whole graph can be written as $\mathbf{\Pi}_t \in \mathcal{R}^{S\times B} \forall t \in \mathcal{T}$.

It is worth explaining again that the GUP is the difference between the spot price of energy (DLMP) at the buyer and seller nodes represented by the following formulation.

$$
\begin{aligned}
\Pi^p_{s,b,t} &= \pi^p_{b,t} - \pi^p_{s,t} = \\
&+ \left(-M_b^{p^{loss}} + M_s^{p^{loss}}\right) c_t^{p,0} + \left(-M_b^{q^{loss}} + M_b^{q^{loss}}\right) c_t^{q,0} \\
&+ \left(-\mathbf{M}_b^{|u|} + \mathbf{M}_s^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&+ \left(-\mathbf{M}_b^{|s^f|} + \mathbf{M}_s^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) \\
&+ \left(-\mathbf{M}_b^{|s^t|} + \mathbf{M}_s^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (2.25\text{a})
\end{aligned}
$$

For the whole grid:

$$
\mathbf{\Pi}_t^p = \mathbf{1}_{s,t}(\boldsymbol{\pi}^p_{b,t})^{\mathrm{T}} - \boldsymbol{\pi}^p_{s,t}(\mathbf{1}_{b,t})^{\mathrm{T}} \qquad \forall t \in \mathcal{T} \qquad (2.25\text{b})
$$

Noting equation (2.25a) for a generic GUP, it is evident that the GUP inherits some attributes of the DLMP. Correspondingly, GUP incorporates the active and reactive losses, the voltage of the nodes and the congestion in the distribution feeders. Therefore, while transacting energy among the peers, the objective function of each peer will contribute to the system's safety and optimal operation.

The main assumption made here for developing the P2P market based on the distributed optimisation methods, in our case ADMM method, is that the prosumers (peers) are supposed not to behave strategically and collaborate as a whole to maximise the overall social welfare of the group. This statement holds valid as long as the number of participants in the market is sufficient so that the strategic behaviour of s single peer would not lead to its surplus welfare [10]. To this end, the centric problem for maximising the social welfare of the whole participants can be developed as follows.

$$\underset{\mathbf{p}_t^s,\mathbf{p}_t^b,\mathbf{E}_t}{Maximize} - \sum_{t\in\mathcal{T}} \left\{ (\mathbf{c}_t^s)^{\mathrm{T}}\mathbf{p}_t^s - (\mathbf{c}_t^b)^{\mathrm{T}}\mathbf{p}_t^b + \mathbf{1}_s^{\mathrm{T}} \cdot (\mathbf{\Pi}_t \circ \mathbf{E}_t) \cdot \mathbf{1}_b \right\} \tag{2.26a}$$

$$\text{s.t.} \quad \mathbf{E}_t\mathbf{1}_b = \mathbf{p}_t^s \qquad : \boldsymbol{\lambda}_t^s \in \mathbb{R}^{\mathcal{S}} \quad \forall t \in \mathcal{T} \tag{2.26b}$$

$$(\mathbf{E}_t)^{\mathrm{T}}\mathbf{1}_s = \mathbf{p}_t^b \qquad : \boldsymbol{\lambda}_t^b \in \mathbb{R}^{\mathcal{B}} \quad \forall t \in \mathcal{T} \tag{2.26c}$$

$$\underline{\mathbf{p}_t^s} \le \mathbf{p}_t^s \le \overline{\mathbf{p}_t^s} \qquad : \underline{\boldsymbol{\mu}_t^s}, \overline{\boldsymbol{\mu}_t^s} \in \mathbb{R}^{\mathcal{S}} \quad \forall t \in \mathcal{T} \tag{2.26d}$$

$$\underline{\mathbf{p}_t^b} \le \mathbf{p}_t^b \le \overline{\mathbf{p}_t^b} \qquad : \underline{\boldsymbol{\mu}_t^b}, \overline{\boldsymbol{\mu}_t^b} \in \mathbb{R}^{\mathcal{B}} \quad \forall t \in \mathcal{T} \tag{2.26e}$$

$$\mathbf{E}_t \ge \mathbf{0} \qquad : \mathbf{\Omega}_t \in \mathbb{R}^{\mathcal{S}\times\mathcal{B}} \quad \forall t \in \mathcal{T} \tag{2.26f}$$

As equation (2.26) indicates, the centric problem aims at maximising the revenue for the seller peer and minimising the cost for the buyer node. Furthermore, it contributes to optimising the energy transacting via the distribution feeders by the GUP factor. It should be noted that $(\mathbf{\Pi}_t \circ \mathbf{E}_t)$ corresponds to the Hadamard product of GUP matrices for the whole seller and buyer peers and the energy transacted between them. Moreover, equation (2.26b) represents the power balance for the seller node, indicating that the sum of energy transacted from the seller node to the buyer one should be equal to its energy injection. Similar reasoning for the buyer nodes is provided in equation (2.26c). The inequality constraints in equations (2.26d) and (2.26e) represent the maximum and minimum power that each seller/buyer peer can inject/withdraw. Eventually, equation (2.26f) ensures that the power can only flow in one direction, as discussed and demonstrated in figure 2.2. The respective dual variables of the problem are provided on the right side of each constraint.

As aforementioned, the notion behind the P2P market is to preserve the information privacy of peers by solving the problem in a fully distributed manner. To this end, the ADMM as a robust distributed optimisation approach shall be employed to decompose the centric problem into multiple sub-problems to be solved. Considering a generic seller node $i \in \mathcal{S}$ the augmented Lagrangian function for this particular peer can be formulated as follows.

$$\begin{aligned} L_{i,t}^{s,admm}\left(p_{i,t}^s, \mathbf{e}_{i,*,t}\right) = \sum_{t\in\mathcal{T}} \Bigg\{ & c_{i,t}^s p_{i,t}^s + \frac{1}{2}\mathbf{e}_{i,*,t}\mathbf{\Pi}_{i,*,t}^{\mathrm{T}} + \mathbf{\Lambda}_{i,*,t}\left(\mathbf{e}_{i,*,t} - \mathbf{e}_{i,*,t}^{+}\right)^{\mathrm{T}} \\ & + \frac{1}{2}\rho\left(\mathbf{e}_{i,*,t} - \mathbf{e}_{i,*,t}^{+}\right)\left(\mathbf{e}_{i,*,t} - \mathbf{e}_{i,*,t}^{+}\right)^{\mathrm{T}} \Bigg\} \end{aligned} \tag{2.27}$$

According to equation (2.27), $\mathbf{e}_{i,*}, \mathbf{\Pi}_{i,*} \in \mathbb{R}^{\mathcal{S}}$ stand for i-th row of the energy transaction and GUP matrices ($\mathbf{E}$). The factor $\frac{1}{2}$ in $\frac{1}{2}\mathbf{e}_{i,*,t}\mathbf{\Pi}_{i,*,t}^{\mathrm{T}}$ term is due to dividing the GUP into two parts for seller and buyer peers. The equality dual variables of the whole problem are encapsulated in $\mathbf{\Lambda}_t \in \mathbb{R}^{\mathcal{S}\times\mathcal{B}}$ which characterises the information regarding to the energy trade price and it should be identical for all seller and buyers upon the convergence. In order to solve the ADMM sub-problems iteratively a container parameter for storing the energy transaction results at each iteration is designated to be $\mathbf{E}_t^{+}$, the energy transaction local copy, which upon the consensus it is going to be equal to $\mathbf{E}_t$. Sequentially, $\mathbf{e}_{i,*,t}$ represents the i-th row in the energy transaction matrix, $\mathbf{E}_t$. Last but not least, ρ is the penalty factor constant adopted for ADMM approach.

For an individual seller i, the power balance and inequality constraints can be written as follows.

$$\mathbf{e}_{i,*,t}\mathbf{1}_b = p_{i,t}^{s} \qquad : \lambda_{i,t}^{s} \quad \forall t \in \mathcal{T} \tag{2.28a}$$

$$\underline{p}_{i,t}^{s} \leq p_{i,t}^{s} \leq \overline{p}_{i,t}^{s} \qquad : \underline{\mu}_{i,t}^{s}, \overline{\mu}_{i,t}^{s} \quad \forall t \in \mathcal{T} \tag{2.28b}$$

$$\mathbf{e}_{i,*,t}^{\mathrm{T}} \geq \mathbf{0} \qquad : \mathbf{\Omega}_{i,*,t} \quad \forall t \in \mathcal{T} \tag{2.28c}$$

Adopting a similar procedure, the augmented Lagrangian function for an individual buyer peer $j \in \mathcal{S}$ can be developed as in equation (2.29).

$$\begin{aligned} L_{j,t}^{b,admm}\left(p_{j,t}^{b}, \mathbf{e}_{*,j,t}\right) = \sum_{t\in\mathcal{T}} \Big\{ & -c_{j,t}^{b}p_{j,t}^{b} + \frac{1}{2}\mathbf{e}_{*,j,t}^{\mathrm{T}}\mathbf{\Pi}_{*,j,t} + \mathbf{\Phi}_{*,j,t}^{\mathrm{T}}\left(\mathbf{e}_{*,j,t} - \mathbf{e}_{*,j,t}^{+}\right) \\ & + \frac{1}{2}\rho\left(\mathbf{e}_{*,j,t} - \mathbf{e}_{*,j,t}^{+}\right)^{\mathrm{T}}\left(\mathbf{e}_{*,j,t} - \mathbf{e}_{*,j,t}^{+}\right) \Big\} \end{aligned} \tag{2.29}$$

where $\mathbf{\Phi}_t \in \mathbb{R}^{\mathcal{S}\times\mathbf{B}}$ denotes the corresponding Lagrangian dual variable from the buyer prospective which at the end of the day it must be equal to the $\mathbf{\Lambda}_t$ upon the consensus. Hereby, the constraints for the respective buyer peer can be formulated as:

$$\mathbf{e}_{*,j,t}^{\mathrm{T}}\mathbf{1}_s = p_{j,t}^{b} \qquad : \lambda_{j,t}^{b} \quad \forall t \in \mathcal{T} \tag{2.30a}$$

$$\underline{p}_{j,t}^{b} \leq p_{j,t}^{b} \leq \overline{p}_{j,t}^{b} \qquad : \underline{\mu}_{j,t}^{b}, \overline{\mu}_{j,t}^{b} \quad \forall t \in \mathcal{T} \tag{2.30b}$$

$$\mathbf{e}_{*,j,t} \geq \mathbf{0} \qquad : \mathbf{\Omega}_{*,j,t} \quad \forall t \in \mathcal{T} \tag{2.30c}$$

Up to this point, the sub-problems for the seller and buyer peers are developed based on the ADMM distributed optimisation mechanism. Accordingly, the sub-problems are to be solved in parallel to each other, and their optimal local results should be announced so that in the consecutive iteration, an optimum global point can be reached for the whole P2P market problem. Therefrom, Algorithm 2.2 should be effective in carrying out the promised consensus for the P2P market.

Algorithm 2.2 ADMM solution algorithm for P2P market problem.

1: DSO Announces an initial GUP to the peers.
2: **while** *ADMMconverged* **do**
3: Maximise the social welfare sub-problems of the seller and buyer peers in parallel: for seller i:

$$\hat{p}_{i,t}^{s(k+1)}, \hat{\mathbf{e}}_{i,*,t}^{(k+1)} := \text{argmin } L_{i,t}^{s,admm}\left(p_{i,t}^{s}, \mathbf{e}_{i,*,t}\right)$$

$$\begin{aligned}
\text{s.t.} \quad & \mathbf{e}_{i,*,t}\mathbf{1}_b = p_{i,t}^s && : \lambda_{i,t}^s \quad \forall t \in \mathcal{T} \\
& \underline{p}_{i,t}^s \leq p_{i,t}^s \leq \overline{p}_{i,t}^s && : \underline{\mu}_{i,t}^s, \overline{\mu}_{i,t}^s \quad \forall t \in \mathcal{T} \\
& \mathbf{e}_{i,*,t}^{\mathrm{T}} \geq \mathbf{0} && : \mathbf{\Omega}_{i,*,t} \quad \forall t \in \mathcal{T}
\end{aligned}$$

for buyer j:

$$\tilde{p}_{j,t}^{b(k+1)}, \tilde{\mathbf{e}}_{*,j,t}^{(k+1)} := \text{argmin } L_{j,t}^{b,admm}\left(p_{j,t}^{b}, \mathbf{e}_{*,j,t}\right)$$

$$\begin{aligned}
\text{s.t.} \quad & \mathbf{e}_{*,j,t}^{\mathrm{T}}\mathbf{1}_s = p_{j,t}^b && : \lambda_{j,t}^b \quad \forall t \in \mathcal{T} \\
& \underline{p}_{j,t}^b \leq p_{j,t}^b \leq \overline{p}_{j,t}^b && : \underline{\mu}_{j,t}^b, \overline{\mu}_{j,t}^b \quad \forall t \in \mathcal{T} \\
& \mathbf{e}_{*,j,t} \geq \mathbf{0} && : \mathbf{\Omega}_{*,j,t} \quad \forall t \in \mathcal{T}
\end{aligned}$$

Regarding attaining the consensus, initially, the DSO announces the GUP to the peers. The peers for each seller or buyer node start solving their own local sub-problems based on the GUP from the grid. Then, they obtain results for their local variables, including active power injection and power transactions to/from other peers. The power transaction results i.e. $\hat{\mathbf{e}}_{i,*,t}^{(k+1)}$ for the seller nodes and $\tilde{\mathbf{e}}_{*,j,t}^{(k+1)}$ for the buyers, are broadcast among the peers by themselves so they can update the Lagrangian multipliers for the next iteration.

4:
- Seller i broadcasts $\hat{\mathbf{e}}_{i,*,t}^{(k+1)}$ to all buyers and $\hat{p}_{i,t}^{s(k+1)}$ to the DSO;
- Buyer j broadcasts $\tilde{\mathbf{e}}_{*,j,t}^{(k+1)}$ to all sellers and $\tilde{p}_{j,t}^{b(k+1)}$ to the DSO;

5: DSO solves AC-OPF by acquiring $\hat{p}_{i,t}^{s(k+1)}$ and $\tilde{p}_{j,t}^{b(k+1)}$ and updates GUP.

6: Peers update the local copies for their energy transaction:

$$\mathbf{e}_{i,*,t}^{+(k+1)} = \frac{1}{2}\left(\hat{\mathbf{e}}_{i,*,t}^{(k+1)} + \tilde{\mathbf{e}}_{i,*,t}^{(k+1)}\right) \qquad \forall t \in \mathcal{T}$$

$$\mathbf{e}_{*,j,t}^{+(k+1)} = \frac{1}{2}\left(\hat{\mathbf{e}}_{*,j,t}^{(k+1)} + \tilde{\mathbf{e}}_{*,j,t}^{(k+1)}\right) \qquad \forall t \in \mathcal{T}$$

7: Update Lagrangian multipliers:

$$\mathbf{\Lambda}_{i,*,t}^{(k+1)} = \mathbf{\Lambda}_{i,*,t}^{(k)} + \rho\left(\hat{\mathbf{e}}_{i,*,t}^{(k+1)} - \mathbf{e}_{i,*,t}^{+(k+1)}\right) \qquad \forall t \in \mathcal{T}$$

$$\mathbf{\Phi}_{*,j,t}^{(k+1)} = \mathbf{\Phi}_{*,j,t}^{(k+1)} + \rho\left(\tilde{\mathbf{e}}_{*,j,t}^{(k+1)} - \mathbf{e}_{*,j,t}^{+(k+1)}\right) \qquad \forall t \in \mathcal{T}$$

8: **end while**

At the same time, the DSO acquires the new power injection/withdraw i.e. $\hat{\mathbf{p}}_{i,t}^{s(k+1)}$ for the seller nodes and $\tilde{\mathbf{p}}_{j,t}^{b(k+1)}$ to re-calculate the DLMP and derive GUP. Afterwards, the convergence of the problem is checked. If the ADMM is not converged, the new GUP and updated Lagrangian multipliers are re-utilised to follow the same procedure until a Nash equilibrium is attained.

2.4.3. Coordinated P2P and DSO Markets

As suggested in Section 2.4.2 and algorithm 2.2, the market-clearing process for DSO and P2P market takes place at each iteration. Subsequently, the DLMP and its derivative, GUP, are updated at each iteration until a Nash equilibrium point satisfies both markets. Therefore, there is a coordination between these markets which also we can consider it a two-stage optimisation problem.

In the first stage, the DSO clears the AS market by rescheduling the flexible DERs to ensure the minimisation of losses in the distribution network, support the nodal voltage, and manage the congestion in the feeders. Following that, the DLMPs are calculated, which is necessary to incentivise the flexible DERs to curtail and contribute to the power quality and safety of the DN. The GUP, which is the derivative of the DLMP and corresponds to the financial transmission right of DSO, is calculated in this stage and broadcast to the P2P market participants. GUP inherits the attributes of DLMP in the sense of loss minimisation, voltage and line overload management.

In the second stage, the P2P market is cleared using a distributed optimisation approach (ADMM). The results of each iteration of the ADMM, including the power transaction between the peers and their power injection/withdrawal, are published among the peers and the DSO market coordinator, respectively. The coordination of the DSO and P2P market to accommodate the power injection in the first stage continues until the DSO and P2P market participants are satisfied with the equilibrium.

The following flowchart in figure 2.3 well demonstrates the workflow of the P2P and DSO AS market coordination.

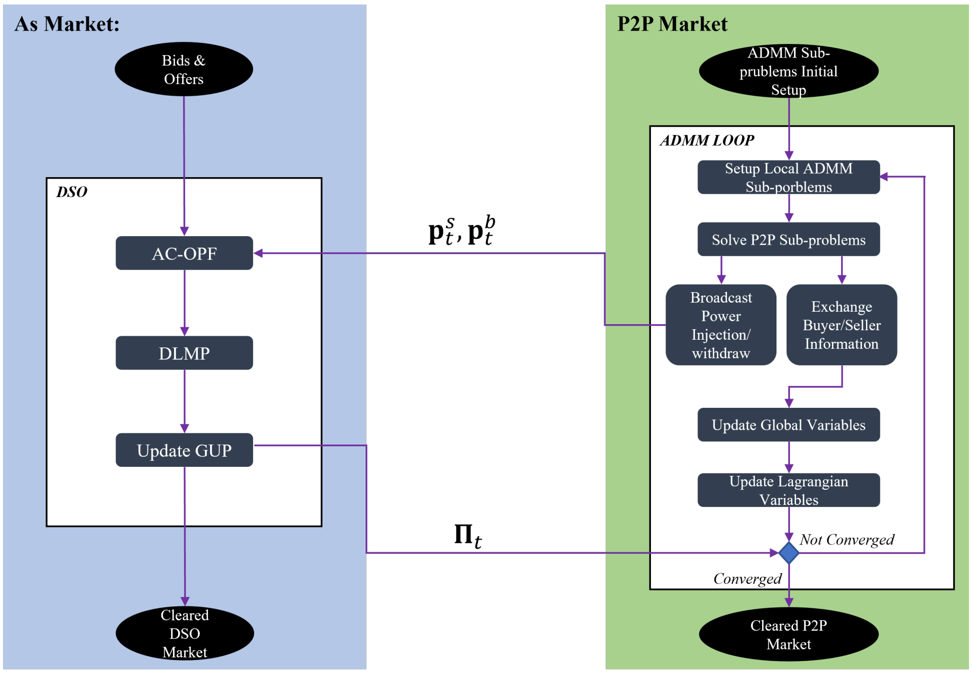

Figure 2.3: DSO AS and P2P market coordination flowchart with ADMM.

An important point to bear in mind here is that the centric P2P market introduced in equation (2.26) is a convex problem which guarantees the convergence of the ADMM subproblems. Nevertheless, the resulting multi-stage problem from DSO AS and P2P market coordination leads to a non-convex problem on the grounds that the GUP is derived from the KKT conditions of the first stage problem. Knowing that the KKT optimality conditions are by nature non-convex problems, we can deduce that the whole coordination problem is non-convex. Hence, the ADMM algorithm solution may not guarantee the global optimal solution [27, 35, 36].

3 | Strategic Behaviour Modelling of Flexible Generator

The most significant contribution of this work will be accommodated in this chapter, where the strategic behaviour of flexible generators in the proposed hybrid DSO AS and P2P market will be studied. Therefore, we will discuss how the distribution network configuration can intensify or abate the strategic behaviour of a flexible generator. Accordingly, the effect of the strategic generator on the DLMPs and consequently on the GUP will be taken into the examination.

To model the strategic behaviour of a flexible generator, we consider it to be an independent decision-maker with a personal objective function alongside the DSO market coordinator that pursues its own unique objective as another autonomous decision-maker. Hence, we will have a problem with multiple decision-makers and multiple objective functions that may have disputes. Therefore, the conventional optimisation approaches cannot effectively model the problem's properties. To this end, the game-theoretical approaches can be functional in solving the problem where there are different players (decision-makers) with distinguished goals [37, 38].

The remainder of this chapter will be allocated to i) Investigation of Stackelberg game functionality and ii) Mathematical integration of strategic generator.

3.1. Stackelberg Game Functionality

Game theory was introduced as a unique standard mathematical field by John Neumann in 1928. Following his work, in 1950, John Nash conducted an extensive study on players' strategies to categorise the games in cooperative and noncooperative approaches taken by the players and he studied the existence of equilibrium points. Therefore, the Nash Equilibrium in the game theory field is named after him for his exhaustive studies [38, 39].

It is essential to bear in mind that if a single decision-maker (player) seeks an individual pay-off maximisation, the principle is called an optimisation problem. In a condition that the same player has multiple pay-off functions to maximise, the concept will be a multi-objective optimisation problem. On the other hand, providing that multiple decision-makers are pursuing an exclusive objective function, the problem will be classified as team theory. Last but not least, the game theory is introduced when multiple players are maximising their unique pay-off function. The different problem concepts are provided in Table 3.1.

Table 3.1: Different decision making disciplines.

	1 payoff	n payoffs
1 player	Optimization	Multi-objective optimization
n players	Team theory	Game theory

In general, the games can be categorised from different perspectives. For instance, we can classify them as cooperative and noncooperative games. In a game, if players are binding to a cooperation commitment to maximising their pay-off, it is called a cooperative game. Having said that, if the players do not form an alliance and seek their selfish pay-off maximisation, the game will be called a noncooperative game [40].

Furthermore, the games can be classified as simultaneous or dynamic games. In the former, the rational and intelligent players usually by not have information regarding the other player's decisions come with a decision at the same time to maximise their own pay-off. On the contrary, in dynamic or sequential games, the later rational player takes action by having sufficient information about the decisions of earlier players. In other words, a player takes action, and subsequently, the later players react to the earlier decisions [39].

Also, it is possible to have a combination of different categories in a game. In this work, the concept of the Stackelberg Game will be covered that specifically can be attributed to the noncooperative games played sequentially. In other words, the Stackelberg game, also known as the leader-follower game, is a dynamic noncooperative game. As the name regards, the mathematical programming of this game consists of at least two levels. In the first level, the leader makes a decision, and the rational follower players react accordingly. Hence, the Stackelberg game is also called bi-level programming [40].

Before disclosing further details about the Stackelberg game, it is necessary to recap mixed complementarity programming (MCP) and introduce the Nash equilibrium (NE) problem and generalised Nash equilibrium (GNE) concepts, which are the basics of our discussion in the following sections.

3.1.1. Mixed Complementarity Problem (MCP)

Having multiple functions $F: \quad \mathbb{R}^n \rightarrow \mathbb{R}^n$, the MCP written by definition to find $x \in \mathbb{R}^{\mathcal{N}_1}$ and $y \in \mathbb{R}^{\mathcal{N}_2}$ such that for all i we have:

$$0 \leq F_i(x,y) \perp x_i \geq 0 \qquad \forall i \in \mathcal{N}_1 \tag{3.1a}$$

$$F_j(x,y) = 0; \quad y_j \in \mathbb{R} \qquad \forall j \in \mathcal{N}_2 \tag{3.1b}$$

Regarding the general formulation of a standard MCP in equation (3.1), it consists of non-equality functions in which its product with the corresponding variables must be equal to zero while the variable and the respective inequality function should be non-zero. Moreover, there are equality constraints that the domain of its corresponding input variable should be free. Comparing MCP general formulation with the KKT optimality conditions obtained for a general convex problem with equality and non-equality constraints (Section 2.1, equation (2.1)), it turns out that the resulted KKTs are genuinely MCP. For the sake of convenience, the standard convex problem and its KKTs are re-written in the following.

$$\min_{\mathbf{x}}. \quad F(\mathbf{x}) \tag{3.2a}$$

$$\text{s.t.} \quad \mathbf{Ax} - \mathbf{b} \leq \mathbf{0} \qquad : \boldsymbol{\mu} \tag{3.2b}$$

$$\mathbf{Cx} - \mathbf{d} = \mathbf{0} \qquad : \boldsymbol{\lambda} \tag{3.2c}$$

Lagrangian function: $L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = F(\mathbf{x}) + \boldsymbol{\lambda}^{\mathrm{T}}(\mathcal{C}\mathbf{x} - \mathbf{d}) + \boldsymbol{\mu}^{\mathrm{T}}(\mathbf{Ax} - \mathbf{b})$ (3.2d)

KKT conditions:

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \mathbf{0} \tag{3.2e}$$

$$(\mathcal{C}\mathbf{x} - \mathbf{d}) = \mathbf{0} \tag{3.2f}$$

Complementarity slackness: $\mathbf{0} \leq \operatorname{diag}(\boldsymbol{\mu}) \perp (-\mathbf{Ax} + \mathbf{b}) \geq \mathbf{0}$ (3.2g)

As regards, the KKT conditions given in equation (3.2) have the properties of MCP. So, in general, the KKT condition of a convex problem is usually referred to as the MCP formulation of the corresponding convex problem [37].

3.1.2. Nash Equilibrium Problems

By definition given in [37], a mathematical problem is called equilibrium when the associated constraints emerged from the KKT condition belonging to the multiple optimisation problems are interrelated. The mathematical expression of an equilibrium problem is provided as follows.

Consider $i \in \mathcal{N}$ and optimization problems below:

$$\underset{\mathbf{x}^i}{\text{Minimize}}\; f^i(\mathbf{x}^1, \ldots, \mathbf{x}^n) \tag{3.3a}$$

$$s.t. \quad h^i(\mathbf{x}^1, \ldots, \mathbf{x}^n) = 0 \qquad : \lambda^i \forall i \in \mathcal{N} \tag{3.3b}$$

$$g^i(\mathbf{x}^1, \ldots, \mathbf{x}^n) \leq 0 \qquad : \mu^i \forall i \in \mathcal{N} \tag{3.3c}$$

where the vector $\mathbf{x}^i \in \mathbb{R}^{\mathcal{N}^i}$ are the optimization variables of i-th problem. Accordingly, by setting $\mathcal{N}^T = \sum_{i \in \mathcal{N}} n^i$ the pay-off function and constraints of i-th problem is allowed to be as $f^i : \mathbb{R}^{\mathcal{N}^T} \rightarrow \mathbb{R}$, $h^i : \mathbb{R}^{\mathcal{N}^T} \rightarrow \mathbb{R}^{\mathcal{M}^i_E}$, and $g^i : \mathbb{R}^{\mathcal{N}^T} \rightarrow \mathbb{R}^{\mathcal{M}^i_I}$. Where for a NE problem $\mathbf{x}^i_E \in \mathbb{R}^{\mathcal{M}^i_E}$, $\mathbf{x}^i_I \in \mathbb{R}^{\mathcal{M}^i_I}$, and $\{\mathcal{M}^i_E, \mathcal{M}^i_I\} \in \mathcal{N}^i$.

For an equilibrium problem, the mathematical expression given above means that the decision variables of i-th problem belong to vector $\mathbf{x}^i$. Nevertheless, the corresponding pay-off function (f^i), equality (h^i), and non-equality constraints (h^i) pertain to the decision sets of all other problems $\mathcal{N}$. In particular, the decision variables specific in problem $i \in \mathcal{N}$ is going to be considered as a parameter in problem $j \in \mathcal{N}$ and both problems are considering the decisions of other problem for solving their own problem mutually. Therefrom, all the problems are interrelated, resulting in an equilibrium problem [37]. A more generalised version of equilibrium problems is known as a generalised equilibrium problem. In a GNE, alongside the conditions given above, it turns out that the constraints of an individual player take the decision variables of other problems as an unknown argument. Therefore, other players' actions are not considered parameters in the consecutive player's problem. More details about the GNE problems can be found in [41].

A general equilibrium problem and its properties are well-explained up to this point. Now, it is essential to analyse the solution to such problems. To this end, the solution to an equilibrium problem is called Nash equilibria. Such a solution needs to satisfy the following condition.

$$f^i(\mathbf{x}^{i,NE};\mathbf{x}^{-i,NE}) \geq f^i(\mathbf{x}^{i,*};\mathbf{x}^{-i,NE}) \quad \forall \mathbf{x}^i \in \mathbb{R}^{\mathcal{N}^i}\, and\, i \in \mathcal{N} \tag{3.4}$$

Equation (3.4) denotes that if the equilibrium problem emerges to a Nash equilibria solution, the deviation of any player i from that particular solution results in a lower pay-off for that player.

Let us derive the joint KKT condition for the problem given in equation (3.3) to investigate the solution methodology and NE point of the problem.

$$\begin{aligned}
\nabla_{\mathbf{x}^i} f^i(\mathbf{x}^1,\ldots,\mathbf{x}^n) - (\lambda^i)^{\mathrm{T}} \nabla_{\mathbf{x}^i} h^i(\mathbf{x}^1,\ldots,\mathbf{x}^n) & \\
-(\mu^i)^{\mathrm{T}} \nabla_{\mathbf{x}^i} g^i(\mathbf{x}^1,\ldots,\mathbf{x}^n) = 0 & \qquad \forall i \in \mathcal{N} && (3.5a)\\
h^i(\mathbf{x}^1,\ldots,\mathbf{x}^n) = 0 & \qquad \forall i \in \mathcal{N} && (3.5b)\\
0 \leq \mu^i \perp -g^i(\mathbf{x}^1,\ldots,\mathbf{x}^n) \geq 0 & \qquad \forall i \in \mathcal{N} && (3.5c)
\end{aligned}$$

where $\mu^i \in \mathbb{R}^{\mathcal{M}_I^i}$ and $\lambda^i \in \mathbb{R}^{\mathcal{M}_E^i}$ are the dual Lagrangian variables related to the inequality and equality constraints of each problem, respectively.

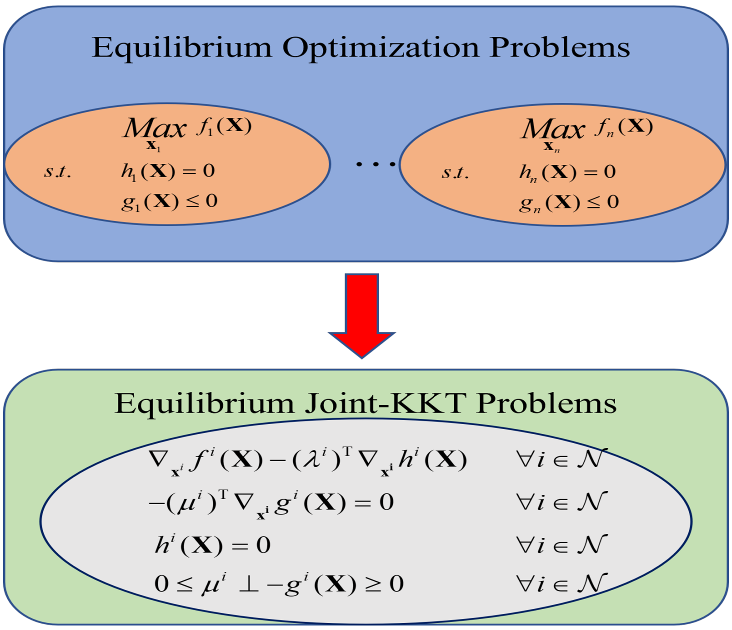

Figure 3.1: Transition from equilibrium problem to the equivalent KKT problem.

An important point to take into account is that the existence and uniqueness of the Nash equilibrium point to be obtained from solving the equivalent KKT conditions of an

Equilibrium problem is subject to the following conditions given in Theorem 3.1.

Theorem 3.1. *Let $\bar{\mathbf{X}} \in \mathbb{R}^{\mathcal{N}T}$ be the feasible solution of equilibrium problem (3.3). Assuming that $(\bar{\mathbf{X}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{\mu}}) \in (\mathbb{R}^{\mathcal{N}T}, \mathbb{R}^{\mathcal{M}_E}, \mathbb{R}^{\mathcal{M}_I})$ solves the equivalent KKT problem (3.5). Providing that f^i, g^i are convex and h^i is an affine function, then the solution to the equivalent KKT condition will be identical to the Nash equilibrium solution of the original problem as a global optimum.*

So far, we have discussed the equilibrium problems and investigated the necessary conditions for the existence and uniqueness of its solution as Nash equilibria. Also, we saw that when we derive the joint-KKT conditions for the equilibrium, we come up with the MCP problem. Now, the question is, how can we ensure that the solution of an optimization problem with a single objective function and the decision variables vectors attributed is the Nash equilibrium? To answer the question, we must first decompose the optimization problem into multiple sub-problems to reformat it into an equilibrium problem. Afterwards, the MCP of the equilibrium problem is derived and compared to the KKT optimality conditions of the original optimization problem. If the MCP and KKT conditions are identical, we can say that the results attained for the optimization problem is a Nash equilibrium, and every player is satisfied with them, and if they deviated to other decisions, their pay-off would be lower [33].

3.1.3. Stackelberg Game and Bi-level Programming

Earlier, a brief history of the Stackelberg game was discussed. We argued that it is classified as sequential (dynamic) non-cooperative games in which the first player takes action by having information about the limitations of the second player. Subsequently, the second rational player reacts and makes decisions by having sufficient information about the former player's actions. In this game, the first player, as a leader, starts the game, and she is well aware of the second player's capabilities as the follower. Therefore, the leader's action influences the follower's optimal reaction, and for the leader to maximize her payoff, it is essential to take into account the qualifications of the follower and anticipate the follower's reaction [40].

It is worth mentioning that such a leader-follower game can have multiple followers as independent problems with constraints interrelated (equilibrium problem). The standard generic bi-level programming of such a leader-follower game is mathematically expressed in equation (3.6) with multiple followers $i \in \mathcal{N}$.

$$\underset{\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}}{\text{Minimize}} \quad F(\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) \tag{3.6a}$$

$$s.t. \quad H(\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) = 0 \tag{3.6b}$$

$$G(\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) \leq 0 \tag{3.6c}$$

$$\forall i \in \mathcal{N} \quad \begin{cases} \underset{\mathbf{y}^i}{\text{Minimize}} \quad f^i(\mathbf{x},\mathbf{Y}) \\ s.t. \\ \quad h^i(\mathbf{x},\mathbf{Y}) = 0 \qquad : \lambda^i \\ \quad g^i(\mathbf{x},\mathbf{Y}) \leq 0 \qquad : \mu^i \end{cases} \tag{3.6d}$$

In problem (3.6), upper-level is expressed by equations (3.6a) to (3.6c) where the corresponding leader objective function is denoted by equation (3.6a) while equations (3.6b) and (3.6c) are the standard equality and inequality constraints of the upper-level problem. Furthermore, the sets of $\mathcal{N}$ optimization problems given in equation (3.6d) are the lower-level follower problems restricting the upper-level leader problem.

Regarding decision variable structure of the problem, the vector variable $\mathbf{x} \in \mathbb{R}^{\mathcal{N}_0}$ is the decision variable specific to the upper-level which is seen as the action of the leader broadcast to the followers as parameter. The vector variables $(\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) \in (\mathbb{R}^{\mathcal{M}_y^T},\mathbb{R}^{\mathcal{M}_E^T},\mathbb{R}^{\mathcal{M}_I^T})$ are the concatenated decision variables of all followers' problem and their corresponding concatenated Lagrangian variables for the equality and inequality constraints, respectively. Where the total number of decision variables for the followers is given by $\mathcal{M}_y^T = \sum_{i\in\mathcal{N}} m_y^i$ for m_y^i denoting the size of decision variable vector of i-th problem. Similarly the size of all followers' equality and inequality constraints is calculated by $\mathcal{M}_E^T = \sum_{i\in\mathcal{N}} m_E^i$ and $\mathcal{M}_I^T = \sum_{i\in\mathcal{N}} m_I^i$, respectively.

Similar to the equilibrium problem (3.3) that we talked about in Section 3.1.2, each follower problem in equation (3.6d) may anticipate the decisions made by other followers as $\mathbf{Y}$ and consider the action of the leader as a parameter argument, $\mathbf{x}$, and come up with the reaction $\mathbf{y}^i$.

It is obvious that the objective function of the leader problem should be in contradiction with the ones associated with the followers. Otherwise, the bi-level programming can be reduced to a single-level problem by merging the constraints of lower and upper-level problems and adding up the objective functions.

Providing that the lower-level problems are convex, their joint-KKT optimality conditions can be derived and replaced with their original problem by an optimal solution guaranteed. Consequently, the constraints of the upper-level problem will not consist of optimization problems but MCP-based constraints. Such single-level problem formulation is called mathematical programming with complementarity constraints (MPCC), and in case the follower problems are equilibrium problems, they will be denoted as mathematical programming with equilibrium constraints (MPEC) which are the topic of Section 3.1.4.

3.1.4. MPCC/MPEC Transformation of Bi-level Programming

Providing the follower optimization problems (3.6d) constraining the leader problem (3.6a) are convex problems, their KKT conditions are necessary and sufficient for guaranteeing the optimum solution [42]. Therefore, the original lower-level problems can be replaced by their KKT conditions as follows.

$$\underset{\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}}{\text{Minimize}} \quad F(\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) \tag{3.7a}$$

$$s.t. \quad H(\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) = 0 \tag{3.7b}$$

$$G(\mathbf{x},\mathbf{Y},\mathbf{\Lambda},\mathbf{\Phi}) \leq 0 \tag{3.7c}$$

$$\nabla_{\mathbf{x}^i} f^i(\mathbf{x},\mathbf{X}) + (\lambda^i)^{\mathrm{T}} \nabla_{\mathbf{x}^i} h^i(\mathbf{x},\mathbf{Y}) + (\mu^i)^{\mathrm{T}} \nabla_{\mathbf{x}^i} g^i(\mathbf{x},\mathbf{Y}) = 0 \qquad \forall i \in \mathcal{N} \tag{3.7d}$$

$$h^i(\mathbf{x},\mathbf{Y}) = 0 \qquad \forall i \in \mathcal{N} \tag{3.7e}$$

$$0 \leq \mu^i \perp - g^i(\mathbf{x},\mathbf{Y}) \geq 0 \qquad \forall i \in \mathcal{N} \tag{3.7f}$$

As it regards, the constraints given in equations (3.7d) to (3.7f) are mixed complementarity problems MCP for the upper-level problem. Therefore, such a problem is called mathematical programming with complementarity constraints (MPCC). In cases the lower-level follower problems form an equilibrium problem, the MPCC will be called mathematical programming with equilibrium constraints (MPEC). A more general concept of MPEC is the case in which multiple leaders are forming an equilibrium problem in the upper level as well. Such a multi-leader and multi-follower problem is known as equilibrium programming with equilibrium constraints (EPEC). It is important to remember that the feasible region of such an MPCC problem might not be a convex set. Furthermore, the complementarity constraints given in equation (3.7f) might not genuinely meet the constraint qualifications. Hence, solving such a problem can potentially be challenging [38].

It is worth reminding that the MCP substantially constitutes non-linear programs. However, as we discussed earlier in chapter 2 they can be linearized by adopting Fortuny-Amat (also known as Big-M) method, which is explained in equation (2.8).

3.2. Strategic Flexible Generator Modelling

In chapter 2, the structure of the DSO market and its mathematical formulation were explained. There, the perception was that every DER participating in DSO AS market is a price-taker, and they do not have the capability to contend with the "market power", or at least they do not tend[43]. Thereby, the flexible DERs submit their bids or offer truthfully according to their actual marginal costs or utilities. As a result, the market-clearing outcome is viable for every market participant. This situation, i.e. the trustful behaviour of market players, can hold, providing the number of market participants is higher and their capacity (production or consumption) is relatively symmetrical, which usually is the case in real-world scenarios [44]. Furthermore, on a condition that the power transmission capacity of the network is sufficient enough, we can expect that the outcome of the market is a Nash equilibrium point, and any player would lead to a lower pay-off if they played otherwise [45].

In recent years, the expansion of energy market liberalization at the distribution network level has proliferated. Especially, governmental organizations such as European Union Strategy Energy Technology (SET) Plan motivate private investors and the residents to exploit the renewable energy-based DERs toward building carbon-zero societies [29]. To this end, the DSO markets are becoming widespread; nevertheless, as mentioned in chapter 1, the structure of the DN requires further considerations. This is mainly due to the vast contribution of active losses, a higher number of consumers and the consequent unbalanced electricity network [2]. Furthermore, the radial structure of the distribution network with comparatively long feeders leads to voltage drop along with the feeders and probable line congestion [11, 46].

The structure of a three-phase unbalanced distribution network is studied in chapter 1. Where we provided an approximation-based linear AC-OPF model by analyzing the sensitivity of system losses, nodal voltage, and line loading with respect to active and reactive power injections throughout the DN in Section 1.2. There we observed that, unlike transmission grids, the higher R/X ration in DN results in higher sensitivity of the aforementioned system variables to the active power injection. Having considered that, if a relatively sizeable flexible generator, e.g. commercial DG, PV, BESS, etc., is located downstream of a DN feeder, the active power injected by this particular DER can have an essential impact on the reduction of the losses, stabilizing the voltage, and eliminating the overloading of the lines [47].

The regulations provided by DSO assign limitations for hosting capacity (HC) such that the range of rapid voltage change (RVC) at the hosting node should not exceed the following range $|V_{DG,n} - V_n| \leq 4 \div 5\%$. This means the HC of a node in the DN should be assigned so that when the DER is out of order, the voltage deviation at the node does not exceed the RVC range [48]. Even though such regulations can help the DSO operate in safety margins by reducing its dependency on the DER, the presence of a liberalized DSO market binding and operating at the edge of limitations still can lead to a remarkable pay-off surplus for a flexible generator. In other words, a potential flexible generator with a relatively high impact on the system safety of the distribution network can behave strategically in the proposed DSO market without violating any network restrictions.

When a producer tends to act strategically, it is not a "price-taker" anymore. Yet, he can exercise market power and be a "price-maker". A strategic producer expects to change the market outcome to his own advantage by adopting a proper strategy when offering. This means he submits untruthful offers that are not necessarily equal to the marginal cost. Therefore, the strategic generator seeks its own objective and benefits independently. In the following sections, the mathematical expression of such behaviour will be developed.

3.2.1. Bi-level Programming of Strategic Producer

The structure of the pool energy market requires sequential decision-making. In the first place, the market participants submit their bids or offers, and then the system operator clears the market as the price setter. As we discussed earlier, a potential producer can behave strategically and offer untruthfully so that by comparatively higher offers, they can change the result of the market-clearing for their own benefit. Considering the DSO AS market as a case, for such a strategic market player, it is necessary to anticipate the market-clearing price and the network structure and maximize its revenue by submitting offers that are not necessarily equal to their marginal cost. Accordingly, the optimization problem for a single strategic producer can be developed initially as:

$$\underset{p_t^{sg}, c_t^{offer,p^{sg}}, \pi_t^{sg}}{Maximize} \quad \sum_{t \in \mathcal{T}} \left\{ p_t^{sg} \pi_t^{sg} - p_t^{sg} c_t^{mrg,p^{sg}} \right\} \tag{3.8a}$$

$$s.t. \quad c_t^{offer,p^{sg}} \geq 0 \qquad : \mu_t^{c^{offer}} \qquad \forall t \in \mathcal{T} \tag{3.8b}$$

Explicitly, the objective function of problem (3.8) maximizes the revenue of the strategic generator (SG) by maximizing the difference between SG's income from power sold in market clearing price (DLMP) and marginal cost for the same amount of power. Therefore, it tries to increase the DLMP at its location in the grid by offering strategically. Respectively, p^{sg} is the decision variable denoting the power dispatch of SG. The parameter $c_t^{mrg,p^{sg}}$ stands for the marginal cost of the SG, and it is an input of the problem. The strategic offering price is given by $c_t^{offer,p^{sg}}$, and the market-clearing price (DLMP) for SG is represented by π_t^{sg}, which is a variable that the SG should anticipate.

On the other hand, the DSO market takes the offered price from all market participants, including the strategic producer, and clears the market. For DSO as another player, the objective is to optimize social welfare by dispatching the flexible generators and curtailing the flexible loads such that the flexible generator's cost is minimized and the utility of flexible loads is maximized. Besides, the losses in the network are to be minimized while the problem is binding to the voltage and congestion constraints. Correspondingly, the optimization problem for the DSO market, which we discussed comprehensively in chapter 2 is re-expressed by implicating the SG's offer.

$$\underset{p_t^{loss},q_t^{loss},\mathbf{p}_t,\mathbf{q}_t}{Minimize} = + \sum_{t\in\mathcal{T}} \Big\{ c_t^{p,0} p_t^{loss} - c_t^{p,0}\mathbf{1}^{\mathrm{T}}\mathbf{p}_t + \left(\mathbf{c}_t^{p,DER*}\right)^{\mathrm{T}}\mathbf{p}_t + c_t^{q,0} q_t^{loss} - c_t^{q,0}\mathbf{1}^{\mathrm{T}}\mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}}\mathbf{q}_t \Big\} \tag{3.9a}$$

$$s.t. \quad \mathbf{x}_{t^0}^{storage} = \mathbf{x}_{t^n}^{storage} \quad : \boldsymbol{\lambda}^{x,storage} \tag{3.9b}$$

$$\mathbf{x}_{t^0}^{non-storage} = \hat{\mathbf{x}} \quad : \boldsymbol{\lambda}^{x,non-storage} \tag{3.9c}$$

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{v}_t^x \quad : \boldsymbol{\lambda}_t^x \quad \forall t \in \mathcal{T} - \{t^n\} \tag{3.9d}$$

$$\mathbf{y}_t = \mathbf{C}\mathbf{x}_t + \mathbf{D}\mathbf{u}_t + \mathbf{v}_t^y \quad : \boldsymbol{\lambda}_t^y \quad \forall t \in \mathcal{T} \tag{3.9e}$$

$$\mathbf{p}_t = \hat{\mathbf{p}}_t^{fixed} + \mathbf{M}^{p,y}\mathbf{y}_t \quad : \boldsymbol{\lambda}_t^{p,y} \quad \forall t \in \mathcal{T} \tag{3.9f}$$

$$\mathbf{q}_t = \hat{\mathbf{q}}_t^{fixed} + \mathbf{M}^{q,y}\mathbf{y}_t \quad : \boldsymbol{\lambda}_t^{q,y} \quad \forall t \in \mathcal{T} \tag{3.9g}$$

$$\underline{\mathbf{y}}_t \leq \mathbf{y}_t \leq \overline{\mathbf{y}}_t \quad : \underline{\boldsymbol{\mu}}_t^y, \overline{\boldsymbol{\mu}}_t^y \quad \forall t \in \mathcal{T} \tag{3.9h}$$

$$p_t^{loss} = \hat{p}_t^{loss} + \mathbf{M}_p^{p^{loss}}\mathbf{p}_t + \mathbf{M}_q^{p^{loss}}\mathbf{q}_t \quad : \lambda_t^{p^{loss}} \quad \forall t \in \mathcal{T} \tag{3.9i}$$

$$q_t^{loss} = \hat{q}_t^{loss} + \mathbf{M}_p^{q^{loss}}\mathbf{p}_t + \mathbf{M}_q^{q^{loss}}\mathbf{q}_t \quad : \lambda_t^{q^{loss}} \quad \forall t \in \mathcal{T} \tag{3.9j}$$

$$|\mathbf{u}_t| = |\hat{\mathbf{u}}_t| + \mathbf{M}_p^{|u|}\mathbf{p}_t + \mathbf{M}_q^{|u|}\mathbf{q}_t \quad \forall t \in \mathcal{T} \tag{3.9k}$$

$$|\underline{\mathbf{u}}_t| \leq |\mathbf{u}_t| \leq |\overline{\mathbf{u}}_t| \quad : \underline{\boldsymbol{\mu}}_t^{|u|}, \overline{\boldsymbol{\mu}}_t^{|u|} \quad \forall t \in \mathcal{T} \tag{3.9l}$$

$$|\mathbf{s}_t^f| = |\hat{\mathbf{s}}_t^f| + \mathbf{M}_p^{|s^f|}\mathbf{p}_t + \mathbf{M}_q^{|s^f|}\mathbf{q}_t \qquad \forall t \in \mathcal{T} \tag{3.9m}$$

$$|\underline{\mathbf{s}_t^f}| \leq |\mathbf{s}_t^f| \leq |\overline{\mathbf{s}_t^f}| \qquad : \underline{\boldsymbol{\mu}}_t^{|s^f|}, \overline{\boldsymbol{\mu}}_t^{|s^f|} \qquad \forall t \in \mathcal{T} \tag{3.9n}$$

$$|\mathbf{s}_t^t| = |\hat{\mathbf{s}}_t^t| + \mathbf{M}_p^{|s^t|}\mathbf{p}_t + \mathbf{M}_q^{|s^t|}\mathbf{q}_t \qquad \forall t \in \mathcal{T} \tag{3.9o}$$

$$|\underline{\mathbf{s}_t^t}| \leq |\mathbf{s}_t^t| \leq |\overline{\mathbf{s}_t^t}| \qquad : \underline{\boldsymbol{\mu}}_t^{|s^t|}, \overline{\boldsymbol{\mu}}_t^{|s^t|} \qquad \forall t \in \mathcal{T} \tag{3.9p}$$

As it regards, the optimization problem (3.9) models the state-space model of DERs encapsulating the operational limits of DERs as well as the linearized AC-OPF constraints associated with the DN limitations. Obviously, the problem is the same as the one given in Section 2.1 except for a single term in the objective function. The offer/bids vector of DER presented in the objective function ($\mathbf{c}_t^{p,DER*}$) involves the strategic offer of SG, i.e. $c_t^{offer,p^{sg}}$. Nevertheless, it is important to note that all offers/bids are seen as parameters from the DSO AS market perspective to clear the market and calculate DLMP.

It is evident that the problem for strategic DER (3.8) and DSO market problem (3.9) persist in contradictory objective functions, and both of them are independent decision-makers. A game-theoretical problem-solving approach can be adapted to model the overall problem and satisfy both players by finding a Nash equilibrium if it exists. Moreover, the decision-making is sequential for the problems on the grounds that the SG submits offers as the first player and the market DSO clears the market in the following. So, the proposed game must be classified as dynamic. To this end, Stackelberg game-theoretical approach proposed in Section 3.1 can be a potential tool. Accordingly, we can propose the following bi-level programming to seek the Nash equilibrium point for the SG and DSO.

$$\underset{\substack{\mathbf{x}_t, \mathbf{u}_t, \mathbf{y}_t \\ p_t^{loss}, q_t^{loss}, \mathbf{p}_t, \mathbf{q}_t \\ c_t^{offer,p^{sg}}, \pi_t^{sg}, \boldsymbol{\Lambda}_t, \boldsymbol{\Phi}_t}}{Maximize} \quad \sum_{t \in \mathcal{T}} \left\{ p_t^{sg}\pi_t^{sg} - p_t^{sg} c_t^{mrg,p^{sg}} \right\} \tag{3.10a}$$

$$s.t. \quad c_t^{offer,p^{sg}} \geq 0 \qquad : \mu_t^{c^{offer}} \qquad \forall t \in \mathcal{T} \tag{3.10b}$$

$$\begin{aligned} \underset{p_t^{loss}, q_t^{loss}, \mathbf{p}_t, \mathbf{q}_t}{Minimize} = \sum_{t \in \mathcal{T}} \Big\{ & c_t^{p,0} p_t^{loss} - c_t^{p,0}\mathbf{1}^{\mathrm{T}}\mathbf{p}_t + \left(\mathbf{c}_t^{p,DER*}\right)^{\mathrm{T}} \mathbf{p}_t \\ & + c_t^{q,0} q_t^{loss} - c_t^{q,0}\mathbf{1}^{\mathrm{T}}\mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}} \mathbf{q}_t \Big\} \end{aligned} \tag{3.10c}$$

$$s.t. \quad Equations \text{ (3.9b) } to \text{ (3.9p)} \tag{3.10d}$$

Problem (3.10) describes the bi-level problem for a strategic DER in the DSO market where in the upper level, the SG tries to maximize its pay-off by manipulating the market-clearing prices, i.e. DLMP, via submitting a strategic offer. In the lower-level problem, DSO clears the market based on the offers as parameters, regardless that the offer submitted by the SG is seen variable for the upper level. So fundamentally, the DLMP from the lower-level problem (π_t^{sg}) and the strategic offer from the upper-level problem ($c_t^{offer,p^{sg}}$) are the complicating variables linking the two problems. This fact is illustrated in figure 3.2

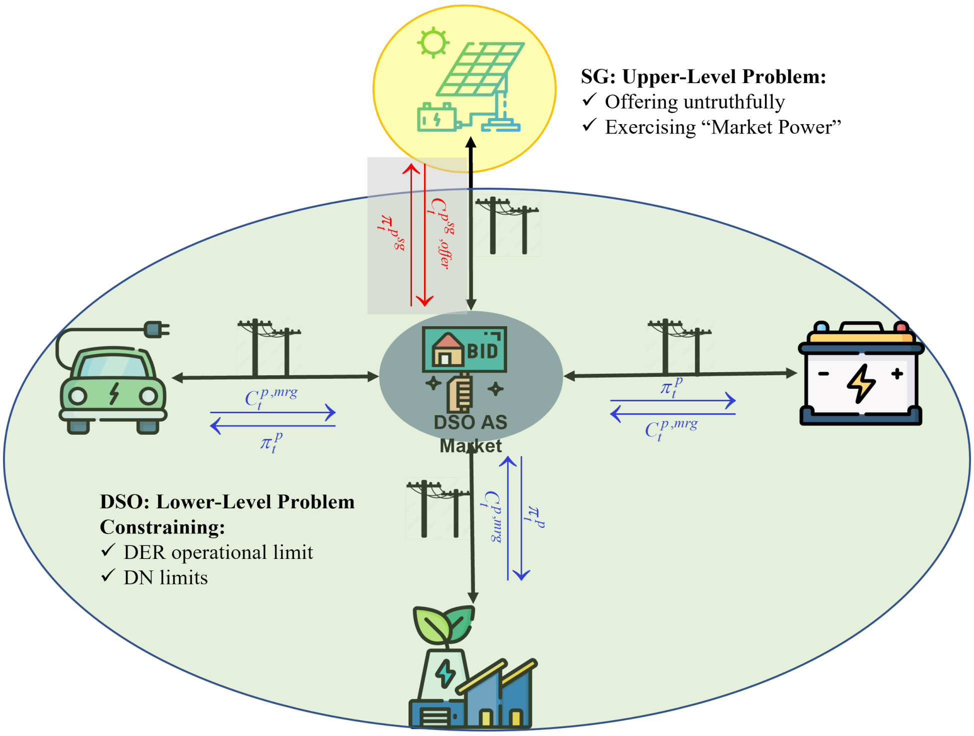

Figure 3.2: Structure of bi-level programming of SG in the DSO market.

Furthermore, the interrelation of the variables is evidently such that the upper level, in addition to its genuine variables, embraces the primal and dual variables of the lower-level problem. That being the case, $\mathbf{\Lambda}_t$ and $\mathbf{\Phi}_t$ denote all the Lagrangian variables corresponding to the equality and inequality constraints in the lower-level problem, respectively.

This particular bi-level problem incorporates significant properties to be noted. Firstly, the lower-level problem is a convex optimization problem which is solved for multiple time steps, and the constraints are affiliated with each other during the time horizon. Therefore, it can be considered an equilibrium problem. Secondly, in the objective function of leader problem $p_t^{sg}\pi_t^{sg}$ term is the product of two variable term constituting bi-linear term.

3.2.2. MPEC Formulation of Strategic DER and DSO AS Market Problem

Since the follower DSO problem is a convex equilibrium problem, its KKT condition is necessary and sufficient for optimality. In Section 3.1.4 we discussed that the convex follower problem of the bi-level program could be replaced by its equivalent KKT conditions, and the result will be a single-level MPCC/MPEC problem. Integrating the idea into the bi-level programming of strategic DER and DSO market by knowing that the DSO problem is an equilibrium problem, the MPEC formulation of the problem can be derived as follows in equation (3.11).

$$\underset{\substack{\mathbf{x}_t, \mathbf{u}_t, \mathbf{y}_t \\ p_t^{loss}, q_t^{loss}, \mathbf{p}_t, \mathbf{q}_t \\ c_t^{offer,p^{sg}}, \pi_t^{sg}, \boldsymbol{\Lambda}_t, \boldsymbol{\Phi}_t, \boldsymbol{\Psi}_t}}{Maximize} \quad \sum_{t \in \mathcal{T}} \left\{ p_t^{sg} \pi_t^{sg} - p_t^{sg} c_t^{mrg,p^{sg}} \right\} \tag{3.11a}$$

$$s.t. \quad c_t^{offer,p^{sg}} \geq 0 \qquad : \mu_t^{c^{offer}} \qquad \forall t \in \mathcal{T} \tag{3.11b}$$

KKT condition of lower-lower level problem:

$$\frac{\partial L(\cdot)}{\partial \mathbf{x}_{t^0}} = \boldsymbol{\lambda}^{x,storage} + \boldsymbol{\lambda}^{x,non-storage} - (\mathbf{A})^{\mathrm{T}} \boldsymbol{\lambda}_{t^0}^{x} - (\mathbf{C})^{\mathrm{T}} \boldsymbol{\lambda}_{t^0}^{y} = \mathbf{0} \tag{3.11c}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{x}_{t}} = \boldsymbol{\lambda}_{t-1}^{x} - (\mathbf{A})^{\mathrm{T}} \boldsymbol{\lambda}_{t}^{x} - (\mathbf{C})^{\mathrm{T}} \boldsymbol{\lambda}_{t}^{y} = \mathbf{0} \qquad \forall t \in \mathcal{T}/\{t^0, t^n\} \tag{3.11d}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{x}_{t^n}} = \boldsymbol{\lambda}_{t^n-1}^{x} - \boldsymbol{\lambda}^{x,storage} - (\mathbf{C})^{\mathrm{T}} \boldsymbol{\lambda}_{t^n}^{y} = \mathbf{0} \tag{3.11e}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{u}_{t}} = -(\mathbf{B})^{\mathrm{T}} \boldsymbol{\lambda}_{t}^{x} - (\mathbf{D})^{\mathrm{T}} \boldsymbol{\lambda}_{t}^{y} = \mathbf{0} \qquad \forall t \in \mathcal{T}/\{t^n\} \tag{3.11f}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{u}_{t^n}} = -(\mathbf{D})^{\mathrm{T}} \boldsymbol{\lambda}_{t^n}^{y} = \mathbf{0} \tag{3.11g}$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{y}_{t}} = \boldsymbol{\lambda}_{t}^{y} + \overline{\boldsymbol{\mu}}_{t}^{y} - \underline{\boldsymbol{\mu}}_{t}^{y} - (\mathbf{M}^{p,y})^{\mathrm{T}} \boldsymbol{\lambda}_{t}^{p,y} - (\mathbf{M}^{q,y})^{\mathrm{T}} \boldsymbol{\lambda}_{t}^{q,y} = 0 \qquad \forall t \in \mathcal{T} \tag{3.11h}$$

$$\frac{\partial L(\cdot)}{\partial p_t^{loss}} = c_t^{p,0} + \lambda_t^{p^{loss}} = \mathbf{0} \qquad \forall t \in \mathcal{T} \tag{3.11i}$$

$$\frac{\partial L(\cdot)}{\partial q_t^{loss}} = c_t^{q,0} + \lambda_t^{q^{loss}} = \mathbf{0} \qquad \forall t \in \mathcal{T} \tag{3.11j}$$

$$\frac{\partial L(\cdot)}{\partial p_t^{sg}} = c_t^{offer,p^{sg}} + \lambda_t^{p^{sg},y} - c_t^{p,0} - \left(\mathbf{M}_{p^{sg}}^{p^{loss}}\right)\lambda_t^{p^{loss}} - \left(\mathbf{M}_{p^{sg}}^{q^{loss}}\right)\lambda_t^{q^{loss}} + \left(\mathbf{M}_{p^{sg}}^{|u|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) + \left(\mathbf{M}_{p^{sg}}^{|s^f|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) + \left(\mathbf{M}_{p^{sg}}^{|s^t|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) = \mathbf{0} \quad \forall t \in \mathcal{T} \quad (3.11k)$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{p}_t^{nsg}} = \mathbf{c}_t^{p^{nsg},DER} + \boldsymbol{\lambda}_t^{p^{nsg},y} - c_t^{p,0}\mathbf{1}^{p^{nsg}} - \left(\mathbf{M}_{p^{nsg}}^{p^{loss}}\right)^{\mathrm{T}}\lambda_t^{p^{loss}} - \left(\mathbf{M}_{p^{nsg}}^{q^{loss}}\right)^{\mathrm{T}}\lambda_t^{q^{loss}} + \left(\mathbf{M}_{p^{nsg}}^{|u|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) + \left(\mathbf{M}_{p^{nsg}}^{|s^f|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) + \left(\mathbf{M}_{p^{nsg}}^{|s^t|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) = \mathbf{0} \quad \forall t \in \mathcal{T} \quad (3.11l)$$

$$\frac{\partial L(\cdot)}{\partial \mathbf{q}_t} = \mathbf{c}_t^{q,DER} + \boldsymbol{\lambda}_t^{q,y} - c_t^{q,0}\mathbf{1} - \left(\mathbf{M}_{q}^{p^{loss}}\right)^{\mathrm{T}}\lambda_t^{p^{loss}} - \left(\mathbf{M}_{q}^{q^{loss}}\right)^{\mathrm{T}}\lambda_t^{q^{loss}} + \left(\mathbf{M}_{q}^{|u|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) + \left(\mathbf{M}_{q}^{|s^f|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) + \left(\mathbf{M}_{q}^{|s^t|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) = \mathbf{0} \quad \forall t \in \mathcal{T} \quad (3.11m)$$

Equality constraints in lower-level problem (3.9) (3.11n)

Eventually, the KKT condition's linearized complementarity slackness associated with the inequality constraints:

$$\mathbf{0} \leq (-\mathbf{y}_t + \overline{\mathbf{y}}_t) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^y\right)\left(\overline{\boldsymbol{\Psi}}_t^y\right) \quad \forall t \in \mathcal{T} \quad (3.12a)$$

$$\mathbf{0} \leq (\overline{\boldsymbol{\mu}}_t^y) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^y\right)\left(\mathbf{1}^y - \overline{\boldsymbol{\Psi}}_t^y\right) \quad \forall t \in \mathcal{T} \quad (3.12b)$$

$$\mathbf{0} \leq \left(\mathbf{y}_t - \underline{\mathbf{y}}_t\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^y\right)\left(\underline{\boldsymbol{\Psi}}_t^y\right) \quad \forall t \in \mathcal{T} \quad (3.12c)$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^y\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^y\right)\left(\mathbf{1}^y - \underline{\boldsymbol{\Psi}}_t^y\right) \quad \forall t \in \mathcal{T} \quad (3.12d)$$

$$\mathbf{0} \leq (-|\mathbf{u}_t| + |\overline{\mathbf{u}}_t|) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\overline{\boldsymbol{\Psi}}_t^{|u|}\right) \quad \forall t \in \mathcal{T} \quad (3.12e)$$

$$\mathbf{0} \leq \left(\overline{\boldsymbol{\mu}}_t^{|u|}\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\mathbf{1}^{|u|} - \overline{\boldsymbol{\Psi}}_t^{|u|}\right) \quad \forall t \in \mathcal{T} \tag{3.12f}$$

$$\mathbf{0} \leq \left(|\mathbf{u}_t| - |\underline{\mathbf{u}}_t|\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\underline{\boldsymbol{\Psi}}_t^{|u|}\right) \quad \forall t \in \mathcal{T} \tag{3.12g}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^{|u|}\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|u|}\right)\left(\mathbf{1}^{|u|} - \underline{\boldsymbol{\Psi}}_t^{|u|}\right) \quad \forall t \in \mathcal{T} \tag{3.12h}$$

$$\mathbf{0} \leq \left(-|\mathbf{s}_t^f| + |\overline{\mathbf{s}_t^f}|\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\overline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \quad \forall t \in \mathcal{T} \tag{3.12i}$$

$$\mathbf{0} \leq \left(\overline{\boldsymbol{\mu}}_t^{|s^f|}\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\mathbf{1}^{|s^f|} - \overline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \quad \forall t \in \mathcal{T} \tag{3.12j}$$

$$\mathbf{0} \leq \left(|\mathbf{s}_t^f| - |\underline{\mathbf{s}_t^f}|\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\underline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \quad \forall t \in \mathcal{T} \tag{3.12k}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^{|s^f|}\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^f|}\right)\left(\mathbf{1}^{|s^f|} - \underline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \quad \forall t \in \mathcal{T} \tag{3.12l}$$

$$\mathbf{0} \leq \left(-|\mathbf{s}_t^t| + |\overline{\mathbf{s}_t^t}|\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\overline{\boldsymbol{\Psi}}_t^{|s^t|}\right) \quad \forall t \in \mathcal{T} \tag{3.12m}$$

$$\mathbf{0} \leq \left(\overline{\boldsymbol{\mu}}_t^{|s^t|}\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\mathbf{1}^{|s^t|} - \overline{\boldsymbol{\Psi}}_t^{|s^f|}\right) \quad \forall t \in \mathcal{T} \tag{3.12n}$$

$$\mathbf{0} \leq \left(|\mathbf{s}_t^t| - |\underline{\mathbf{s}_t^t}|\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\underline{\boldsymbol{\Psi}}_t^{|s^t|}\right) \quad \forall t \in \mathcal{T} \tag{3.12o}$$

$$\mathbf{0} \leq \left(\underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \leq \mathrm{diag}\left(\boldsymbol{\mathcal{M}}^{|s^t|}\right)\left(\mathbf{1}^{|s^t|} - \underline{\boldsymbol{\Psi}}_t^{|s^t|}\right) \quad \forall t \in \mathcal{T} \tag{3.12p}$$

As regards the MPEC problem resulting from replacing the lower-level DSO problem with the respective linearized KKT conditions is a mixed-integer non-linear program (MINLP). The non-linearity is originated from the bi-linear term in the objective function related to the upper-level strategic DER problem.

The KKT conditions related to the MPEC problem, equations (3.11) and (3.12), is similar to the one provided in Section 2.2.1 and equations (2.11) and (2.12) in all terms with the exception of the derivative of the Lagrangian function with respect to the strategic DERs' active power variable in equation (3.11k). Following that, the term consists of the strategic offer of SG ($c_t^{offer,p^{sg}}$), which in the MPEC formulation, it is the variable of the problem and should be determined upon the Nash equilibrium point.

The formulation of DLMP and its properties are explained comprehensively in Section 2.3. To recap, we explained that the DLMP is obtained from the partial derivative of the Lagrangian function of the DSO problem with respect to active power injection/withdrawal of DER. The corresponding KKT is expressed in equation (3.11k) for the SG in the strategic market. The concept of DLMP indicates that the DLMP from the DER perspective should be equal to the DLMP obtained from the grid limitations. Therefore, we can write the following relationship for the SG by considering the equation (3.11k).

$$\begin{aligned}
\pi_t^{p^{sg},DER} &= c_t^{offer,p^{sg}} + \overbrace{\overline{\mu}_t^{p^{sg},DER} - \underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}} = \\
&+ c_t^{p,0} \\
&+ \left(M_{p^{sg}}^{p^{loss}}\right)\lambda_t^{p^{loss}} + \left(M_{p^{sg}}^{q^{loss}}\right)\lambda_t^{q^{loss}} \\
&- \left(\mathbf{M}_{p^{sg}}^{|u|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&- \left(\mathbf{M}_{p^{sg}}^{|s^f|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_{p^{sg}}^{|s^t|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (3.13a) \\
\Rightarrow \quad \pi_t^{p^{sg},grid} &= c_t^{p,0} \\
&+ \left(M_{p^{sg}}^{p^{loss}}\right)\lambda_t^{p^{loss}} + \left(M_{p^{sg}}^{q^{loss}}\right)\lambda_t^{q^{loss}} \\
&- \left(\mathbf{M}_{p^{sg}}^{|u|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&- \left(\mathbf{M}_{p^{sg}}^{|s^f|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_{p^{sg}}^{|s^t|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (3.13b)
\end{aligned}$$

By considering the KKTs conditions associated with the active power losses variable, equation (3.11i), the final formulation of DLMP is yield as:

$$\begin{aligned}
\pi_t^{p^{sg}} &= c_t^{p,0} \\
&- \left(M_{p^{sg}}^{p^{loss}}\right)c_t^{p,0} - \left(M_{p^{sg}}^{q^{loss}}\right)c_t^{p,0} \\
&- \left(\mathbf{M}_{p^{sg}}^{|u|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
&- \left(\mathbf{M}_{p^{sg}}^{|s^f|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_{p^{sg}}^{|s^t|}\right)^{\mathrm{T}}\left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \qquad (3.14a)
\end{aligned}$$

In the DSO problem, the operation limits of the DER are constrained by the state-space modelling. Therefore, dual variables related to the operational limits of SG i.e. $\overline{\mu}_t^{p^{sg},DER}$ and $\underline{\mu}_t^{p^{sg},DER}$, are implicitly denoted by $\lambda_t^{p^{sg},y}$ which corresponding to the DER's state-space output equations. Last but not least, equation (3.13) reveals how the decision variable $\pi_t^{p^{sg}}$ in the upper-level problem, which genuinely belongs to the lower-level, is linked with the exclusive strategic offer variable of the upper-level problem by:

$$\pi_t^{p^{sg},DER} = c_t^{offer,p^{sg}} + \overbrace{\overline{\mu}_t^{p^{sg},DER} - \underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}} \qquad \forall t \in \mathcal{T} \qquad (3.15)$$

Up to this point, the MPEC of the bi-level problem is obtained, and its complementarity constraints are transformed to MILP. However, the bi-linear term of the MPEC's objective function yields MINLP. For such a problem, finding a globally optimal solution by the current commercial solvers is not guaranteed.

3.2.3. Linearizing the Bi-leaner MPEC for Strategic DER

The source of non-linearity in the MPEC formulation of SG problem is attributed to $\pi_t^{p^{sg},DER} p_t^{sg}$ term in the objective function. Based on the KKT conditions, we obtained the relationship between the DLMP and strategic offer given in equation (3.15). By multiplying the sides of this equation with p_t^{sg} and summing up the terms over the time horizon, the bi-linear term can be retrieved and written as:

$$\sum_{t\in\mathcal{T}} \pi_t^{p^{sg},DER} p_t^{sg} = \sum_{t\in\mathcal{T}} \left\{ c_t^{offer,p^{sg}} p_t^{sg} + \left(\overbrace{\overline{\mu}_t^{p^{sg},DER} - \underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}} \right) p_t^{sg} \right\} \tag{3.16}$$

At the first glance at equation (3.16), it is evident that the first term in the right side of equality, i.e. $\sum_{t\in\mathcal{T}} c_t^{offer,p^{sg}} p_t^{sg}$, belongs to the objective function of lower-level DSO problem (3.10c). Accordingly, if we could find a linear equivalent term for the right side of equation (3.16), the problem is solved.

According to the strong duality theorem expressed in equation (2.9), at the global optimal point, the solution of the dual problem is identical to the primal problem, providing that a problem is convex [27]. Relying on this theorem, we can state the equivalent dual objective function for the lower-level DSO problem at the optimal solution point in equation (3.17).

$$\begin{aligned}
\underset{p_t^{loss*},q_t^{loss*},\mathbf{p}_t^*,\mathbf{q}_t^*}{Minimize} \quad & + \sum_{t\in\mathcal{T}} \Bigg\{ c_t^{p,0} p_t^{loss} - c_t^{p,0} \mathbf{1}^{\mathrm{T}} \mathbf{p}_t + \left(\mathbf{c}_t^{p,DER*}\right)^{\mathrm{T}} \mathbf{p}_t \\
& \qquad + c_t^{q,0} q_t^{loss} - c_t^{q,0} \mathbf{1}^{\mathrm{T}} \mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}} \mathbf{q}_t \Bigg\} \\
= \underset{\boldsymbol{\lambda}_t^*,\boldsymbol{\mu}_t^*}{Maximize} \quad & + \left(-\hat{\mathbf{x}}\right)^{\mathrm{T}} \boldsymbol{\lambda}^{x,non-storage} + \sum_{t\in\mathcal{T}-\{t^n\}} \left(-\mathbf{v}_t^x\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^x + \sum_{t\in\mathcal{T}} \left(-\mathbf{v}_t^y\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^y \\
& + \sum_{t\in\mathcal{T}} \left\{ \left(-\hat{\mathbf{p}}_t^{fixed}\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{p,y} + \left(-\hat{\mathbf{q}}_t^{fixed}\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{q,y} \right\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{t\in\mathcal{T}}\left\{\left(\underline{\mathbf{y}}_t\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{y}+(-\overline{\mathbf{y}}_t)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{y}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(-\hat{p}_t^{loss}\right)^{\mathrm{T}}\lambda_t^{p^{loss}}+\left(-\hat{q}_t^{loss}\right)^{\mathrm{T}}\lambda_t^{q^{loss}}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{(|\underline{\mathbf{u}}_t|-|\hat{\mathbf{u}}_t|)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{|u|}+(-|\overline{\mathbf{u}}_t|+|\hat{\mathbf{u}}_t|)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{|u|}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(|\underline{\mathbf{s}_t^f}|-|\hat{\mathbf{s}}_t^f|\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{|s^f|}+\left(-|\overline{\mathbf{s}_t^f}|+|\hat{\mathbf{s}}_t^f|\right)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{|s^f|}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(|\underline{\mathbf{s}_t^t}|-|\hat{\mathbf{s}}_t^t|\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{|s^t|}+\left(-|\overline{\mathbf{s}_t^t}|+|\hat{\mathbf{s}}_t^t|\right)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{|s^t|}\right\}
\end{aligned}
\tag{3.17a}
$$

The utility/cost functions of the DERs are given by $\left(\mathbf{c}_t^{p,DER*}\right)^{\mathrm{T}}\mathbf{p}_t$ as matrix multiplication of two transposed and column and straight column vectors. Besides, we already know this matrix product implicitly include $c_t^{offer,p^{sg}}p_t^{sg}$ term. So, let us, point out the following mathematical trick for this term.

$$
\begin{aligned}
c_t^{offer,p^{sg}}p_t^{sg}=c_t^{offer,p^{sg}}p_t^{sg}&+(\overbrace{\overline{\mu}_t^{p^{sg},DER}-\underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}})p_t^{sg}\\
&-(\overbrace{\overline{\mu}_t^{p^{sg},DER}-\underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}})p_t^{sg}
\end{aligned}
\tag{3.18}
$$

Now, by decoupling the utility/cost function of the DER's active power in equation (3.17) and replacing $c_t^{offer,p^{sg}}p_t^{sg}$ with its equivalent terms given in equation (3.18) we can manipulate the relationship obtained based on the strong duality theorem as:

$$
\sum_{t\in\mathcal{T}}\left\{
\begin{aligned}
&c_t^{p,0}p_t^{loss}-c_t^{p,0}\mathbf{1}^{\mathrm{T}}\mathbf{p}_t+\overbrace{\left(\mathbf{c}_t^{p^{nsg},DER}\right)^{\mathrm{T}}\mathbf{p}_t^{nsg}+c_t^{mrg,p^{sg}}p_t^{sg}}^{\left(\mathbf{c}_t^{p,DER}\right)^{\mathrm{T}}\mathbf{p}_t}\quad\textbf{(I)}\\
&\textbf{(II)}-c_t^{mrg,p^{sg}}p_t^{sg}+\underbrace{c_t^{offer,p^{sg}}p_t^{sg}+(\overbrace{\overline{\mu}_t^{p^{sg},DER}-\underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}})p_t^{sg}}_{\pi_t^{p^{sg},DER}p_t^{sg}}
\end{aligned}
\right.
$$

$$
\left.\mathbf{(III)} - (\overbrace{\overline{\mu}_t^{p^{sg},DER} - \underline{\mu}_t^{p^{sg},DER}}^{\lambda_t^{p^{sg},y}})p_t^{sg} + c_t^{q,0} q_t^{loss} - c_t^{q,0} \mathbf{1}^{\mathrm{T}} \mathbf{q}_t + \left(\mathbf{c}_t^{q,DER}\right)^{\mathrm{T}} \mathbf{q}_t \right\}
$$

$$
\begin{aligned}
&= (-\hat{\mathbf{x}})^{\mathrm{T}} \boldsymbol{\lambda}^{x,non-storage} + \sum_{t\in\mathcal{T}-\{t^n\}} (-\mathbf{v}_t^x)^{\mathrm{T}} \boldsymbol{\lambda}_t^x + \sum_{t\in\mathcal{T}} (-\mathbf{v}_t^y)^{\mathrm{T}} \boldsymbol{\lambda}_t^y \\
&+ \sum_{t\in\mathcal{T}} \left\{ \left(-\hat{\mathbf{p}}_t^{fixed}\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{p,y} + \left(-\hat{\mathbf{q}}_t^{fixed}\right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{q,y} \right\} \\
&+ \sum_{t\in\mathcal{T}} \left\{ \left(\underline{\mathbf{y}}_t\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^y + (-\overline{\mathbf{y}}_t)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^y \right\} \\
&+ \sum_{t\in\mathcal{T}} \left\{ \left(-\hat{p}_t^{loss}\right)^{\mathrm{T}} \lambda_t^{p^{loss}} + \left(-\hat{q}_t^{loss}\right)^{\mathrm{T}} \lambda_t^{q^{loss}} \right\} \\
&+ \sum_{t\in\mathcal{T}} \left\{ (|\underline{\mathbf{u}}_t| - |\hat{\mathbf{u}}_t|)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|u|} + (-|\overline{\mathbf{u}}_t| + |\hat{\mathbf{u}}_t|)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|u|} \right\} \\
&+ \sum_{t\in\mathcal{T}} \left\{ \left(|\underline{\mathbf{s}_t^f}| - |\hat{\mathbf{s}}_t^f|\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|s^f|} + \left(-|\overline{\mathbf{s}_t^f}| + |\hat{\mathbf{s}}_t^f|\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|s^f|} \right\} \\
&+ \sum_{t\in\mathcal{T}} \left\{ \left(|\underline{\mathbf{s}_t^t}| - |\hat{\mathbf{s}}_t^t|\right)^{\mathrm{T}} \underline{\boldsymbol{\mu}}_t^{|s^t|} + \left(-|\overline{\mathbf{s}_t^t}| + |\hat{\mathbf{s}}_t^t|\right)^{\mathrm{T}} \overline{\boldsymbol{\mu}}_t^{|s^t|} \right\}
\end{aligned} \tag{3.19a}
$$

As term **(I)** indicates in equation (3.19), by adding and substituting the cost associated with the marginal price of SG, the original non-strategic objective function of the DSO market is retrieved. Besides, term **(II)** is yielding the genuine objective function of the upper-level SG problem (3.11). Eventually, **(III)** demonstrates additional bi-linear terms, which are the product of the SG's operational limit Lagrangian variables and its active power. Now, by simplifying equation (3.19) keeping only the upper-level objective function on one side of the equality, the equivalent objective function for the upper-level problem can be attained, which still is not linear due to (**III**) term. To this end, let us unwrap the state-space modelling of SG, which constitute the maximum and minimum operational limits of the SG and write its corresponding complementarity slackness as one of the terms related to the KKT condition of the DSO market problem.

The explicit operational limit of SG:

$$
\underline{p_t^{sg}} \le p_t^{sg} \le \overline{p_t^{sg}} \qquad : \underline{\mu}_t^{p^{sg}}, \overline{\mu}_t^{p^{sg}} \qquad \forall t \in \mathcal{T} \tag{3.20a}
$$

The MCP associated with KKT conditions:

$$\underline{\mu}_t^{p^{sg}}, \overline{\mu}_t^{p^{sg}} \geq 0 \qquad \forall t \in \mathcal{T} \qquad (3.20b)$$

$$\left(\underline{p_t^{sg}} - p_t^{sg}\right) \underline{\mu}_t^{p^{sg}} = 0 \qquad \forall t \in \mathcal{T} \qquad (3.20c)$$

$$\left(-\overline{p_t^{sg}} + p_t^{sg}\right) \overline{\mu}_t^{p^{sg}} = 0 \qquad \forall t \in \mathcal{T} \qquad (3.20d)$$

By noting equations (3.20c) and (3.20d) we can conclude the following impression:

$$p_t^{sg} \underline{\mu}_t^{p^{sg}} = \underline{p_t^{sg}} \underline{\mu}_t^{p^{sg}} \qquad \forall t \in \mathcal{T} \qquad (3.21a)$$

$$p_t^{sg} \overline{\mu}_t^{p^{sg}} = \overline{p_t^{sg}} \overline{\mu}_t^{p^{sg}} \qquad \forall t \in \mathcal{T} \qquad (3.21b)$$

Therefore, the bi-linear terms given in (**III**) can be expressed as the following linear term:

$$(\overline{\mu}_t^{p^{sg},DER} - \underline{\mu}_t^{p^{sg},DER}) p_t^{sg} = (\overline{p_t^{sg}} \overline{\mu}_t^{p^{sg}} - \underline{p_t^{sg}} \underline{\mu}_t^{p^{sg}}) \qquad \forall t \in \mathcal{T} \qquad (3.21c)$$

Finally, by putting all the findings together, the mixed-integer linear MPEC formulation of strategic DER in the DSO market can be expressed in the problem (3.22).

$$\underset{\substack{\mathbf{x}_t, \mathbf{u}_t, \mathbf{y}_t \\ p_t^{loss}, q_t^{loss}, \mathbf{p}_t, \mathbf{q}_t \\ c_t^{offer,p^{sg}}, \pi_t^{sg}, \mathbf{\Lambda}_t, \mathbf{\Phi}_t, \mathbf{\Psi}_t}}{Maximize} \quad \sum_{t \in \mathcal{T}} \left\{ p_t^{sg} \pi_t^{sg} - p_t^{sg} c_t^{mrg,p^{sg}} \right\} =$$

$$- \sum_{t \in \mathcal{T}} \left\{ c_t^{p,0} p_t^{loss} - c_t^{p,0} \mathbf{1}^{\mathrm{T}} \mathbf{p}_t + \left(\mathbf{c}_t^{p,DER*} \right)^{\mathrm{T}} \mathbf{p}_t + c_t^{q,0} q_t^{loss} - c_t^{q,0} \mathbf{1}^{\mathrm{T}} \mathbf{q}_t + \left(\mathbf{c}_t^{q,DER} \right)^{\mathrm{T}} \mathbf{q}_t \right\}$$

$$+ \sum_{t \in \mathcal{T}} \left\{ \overline{p_t^{sg}} \overline{\mu}_t^{p^{sg}} - \underline{p_t^{sg}} \underline{\mu}_t^{p^{sg}} \right\}$$

$$+ (-\hat{\mathbf{x}})^{\mathrm{T}} \boldsymbol{\lambda}^{x,non-storage}$$

$$+ \sum_{t \in \mathcal{T} - \{t^n\}} (-\mathbf{v}_t^x)^{\mathrm{T}} \boldsymbol{\lambda}_t^x + \sum_{t \in \mathcal{T}} (-\mathbf{v}_t^y)^{\mathrm{T}} \boldsymbol{\lambda}_t^y$$

$$+ \sum_{t \in \mathcal{T}} \left\{ \left(-\hat{\mathbf{p}}_t^{fixed} \right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{p,y} + \left(-\hat{\mathbf{q}}_t^{fixed} \right)^{\mathrm{T}} \boldsymbol{\lambda}_t^{q,y} \right\}$$

$$
\begin{aligned}
&+\sum_{t\in\mathcal{T}}\left\{\left(\underline{\mathbf{y}}_t\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{y}+\left(-\overline{\mathbf{y}}_t\right)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{y}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(-\hat{p}_t^{loss}\right)^{\mathrm{T}}\lambda_t^{p^{loss}}+\left(-\hat{q}_t^{loss}\right)^{\mathrm{T}}\lambda_t^{q^{loss}}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(|\underline{\mathbf{u}}_t|-|\hat{\mathbf{u}}_t|\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{|u|}+\left(-|\overline{\mathbf{u}}_t|+|\hat{\mathbf{u}}_t|\right)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{|u|}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(|\underline{\mathbf{s}}_t^{f}|-|\hat{\mathbf{s}}_t^{f}|\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{|s^f|}+\left(-|\overline{\mathbf{s}}_t^{f}|+|\hat{\mathbf{s}}_t^{f}|\right)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{|s^f|}\right\}\\
&+\sum_{t\in\mathcal{T}}\left\{\left(|\underline{\mathbf{s}}_t^{t}|-|\hat{\mathbf{s}}_t^{t}|\right)^{\mathrm{T}}\underline{\boldsymbol{\mu}}_t^{|s^t|}+\left(-|\overline{\mathbf{s}}_t^{t}|+|\hat{\mathbf{s}}_t^{t}|\right)^{\mathrm{T}}\overline{\boldsymbol{\mu}}_t^{|s^t|}\right\} \qquad (3.22a)\\
&s.t. \quad \textit{constraints of MPEC given in equations } (3.11) \textit{ and } (3.12) \qquad (3.22b)
\end{aligned}
$$

The final formulation of the MPEC problem in equation (3.22) is a mixed-integer linear program (MILP), and it can be solved by off-the-shelf powerful linear programming solvers such as CPLEX and GUROBI [49, 50].

The final contribution of this chapter is integrating the P2P market with the distributed optimization (ADMM) formulation into the current strategic bi-level problem (3.22). Fundamentally, the algorithm for solving such a problem is similar to the one we adopted in Section 2.4.2, algorithm 2.2. However, the fundamental difference is that instead of DSO non-strategic market, the bi-level problem will clear the market and derive the DLMP and GUP.

4 | Case Study and Results

The contribution of this chapter is to conduct a comprehensive numerical simulation to investigate the validity of the theoretical findings in chapters 2 and 3. The IEEE 34-node three-phase unbalanced DN test feeder is taken under examination. The network is distinguished by long feeders, and it is a part of Arizona DN. The nominal voltage and frequency of the feeder are determined by 24.9 kV and 60 Hz, respectively, which are also determined as the base values for per unit indications [51]. The base apparent power of the system is also considered to be 1 MW. It should be noted that there are slight modifications with respect to the original model. The distributed loads along the feeder have considered two spot loads shared equally between the parent and the offspring nodes. Furthermore, multiple distributed generators (PV generators) are included in the network to implement the idea of the DSO AS market in the presence of producer, and consumer DERs [52]. The feeder configuration is illustrated in figure 4.1; however, further details about the geometry of the lines, transformer data, and the connection type of the loads can be found in reference [51].

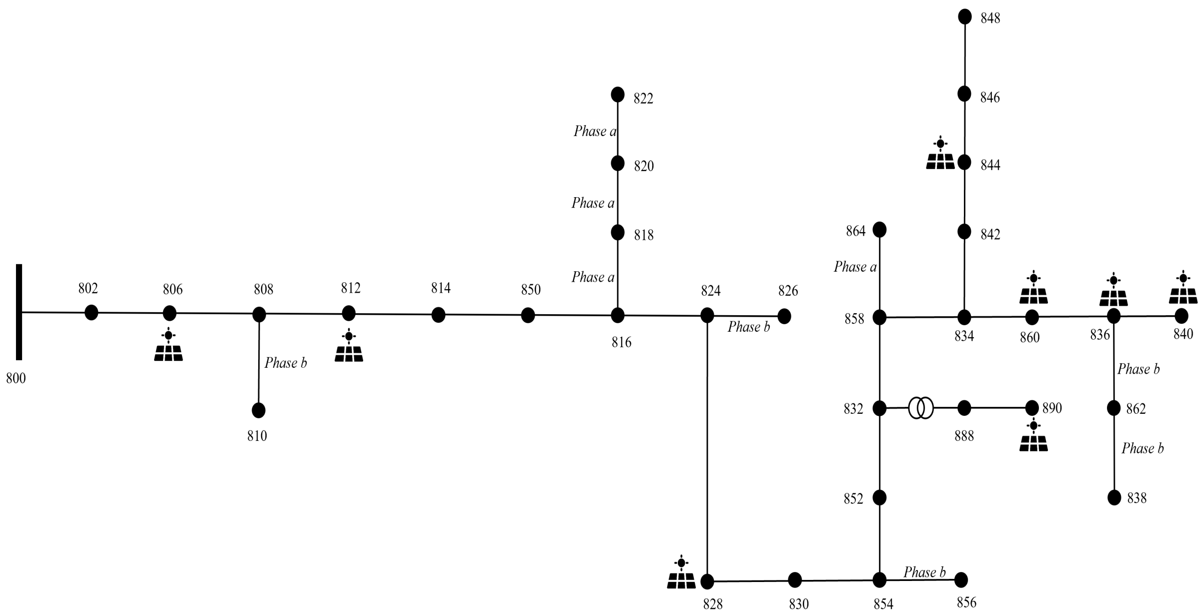

Figure 4.1: IEEE 34-node three-phase DN test feeder with PV generations.

The optimization problems introduced in previous chapters are carried out in a Python environment [53]s. The "Multi-Energy System Modeling and Optimization (MESMO)" tool is employed to model the respective LP and MIP optimization problems and solve them via GUROBI, or CPLEX solvers [49, 50]. MESMO is an open-source Python package developed by TUMCREATE, Institute for High-Performance Computing A*STAR in Singapore and the Chair of ENS at the Technical University of Munich (TUM). This software is capable of modelling the nominal or optimal operation of multi-scaled electric, and thermal distribution networks with high penetration of DERs, including EVs, DGs, ESS, and flexible buildings [25].

Regarding the parameters adopted for constraining the optimization problem, the normally the voltage at each node is to be limited between $v_{n,t} = [0.9, 1.1]$ p.u. The feeder's loading cannot exceed the line's 100 per cent capacity. However, these limitations may change deliberately to evaluate how the grid limitations can resonate with the strategic behaviour of SG in DN. Such modifications will be expressed if any results have emerged from them. Moreover, the type of DERs is considered to be either non-storage flexible load (FL) or PV-based flexible generator (FG). In addition, inelastic loads and generators in the DSO market cannot be curtailed or dispatched. The loads' profile is designated to be mixed-commercial and residential with the following behaviour during 24 hours figure 4.2. Eventually, the generic PV generation profile for the same day is illustrated in the same graph.

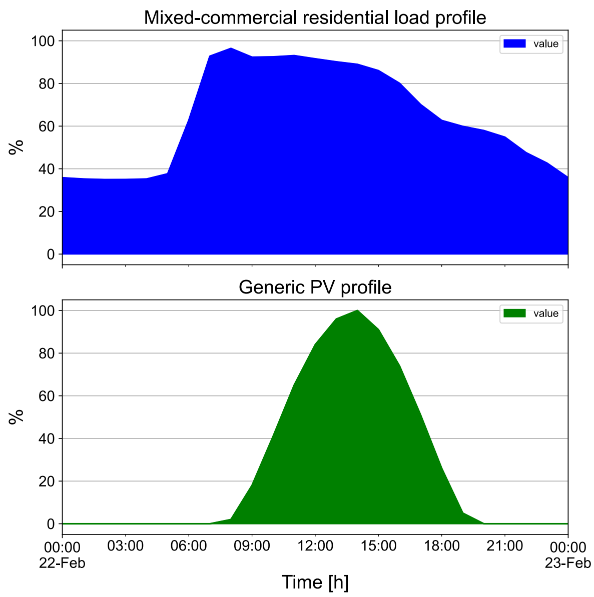

Figure 4.2: Mixed-commercial residential load and generic PV generation profiles.

The nominal capacity of active and reactive power of DERs are given in [52]. The PV generator located at bus 860 has a comparatively sizeable nominal capacity, and it is located in the downstream feeder where the voltage drop is higher; therefore, providing the potential condition for the DER to offer strategically.

The marginal cost of all PV generators is supposed to be 0.1 [$/kWh], and the price time-series of the energy at GSP for a day which are derived from the DSO participating in the pool TSO market are portrayed in figure 4.3.

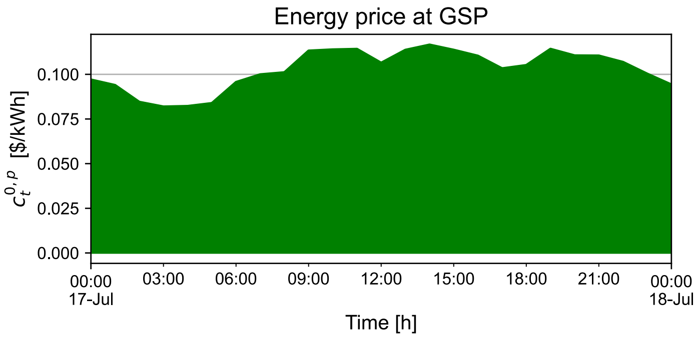

Figure 4.3: Energy price time-series at the grid supply point.

Regarding the optimization parameter sets, setting the value of big-M ($\mathcal{M}$), which is used for linearizing the MCP slackness, requires essential and careful considerations. According to the multiple tests we conducted, by setting very high values for $\mathcal{M}$, the MIP problem leads to a locally optimal solution which is incorrect; meanwhile, when small values are set for it, either the problem does not converge, or it becomes infeasible. However, in general, for our test case, $\mathcal{M}$ =1E+5 demonstrated acceptable accuracy for the MIP problem on the grounds that the scale of the problem is enormous and the number of variables and constraints is significant. The rest of this chapter is organized into two sections. In Section 4.1 we will discuss the non-strategic DSO market and its corresponding KKT problem to validate the identical solution to both problems. Section 4.2 will evaluate the strategic behaviour of FG in the DSO AS market.

4.1. Numerical Results for Non-Strategic DSO AS Market and Corresponding KKT Conditions

In chapter 2 we studied the structure of the DSO AS market as a convex problem. Also, in Section 2.2.1, we proved that since the problem is convex, the KKT conditions are necessary and sufficient to guarantee the optimality of the problem. Therefore, solving the corresponding MCP of the KKT conditions should reveal identical results to the primal problem of the DSO AS market. Therefore, we expect that the dispatching and curtailment of FG and FL should be the same in both cases. Subsequently, the active and reactive losses, nodal voltages, and line loading in the "To" and "From" directions, the consequent DLMPs should emerge as a unique solution to both problems. The simulation will be analyzed for the report time of 2:00 PM.

4.1.1. DSO Primal vs KKT Problems at Peak Demand and PV Generation

As the load and generation profile suggests, at 2 PM, the PV generators operate at maximum capacity, and the demand is almost the highest for the mixed-commercial residential load type. We will evaluate the optimization results for both DSO primal and KKT problems.

Accordingly, figure 4.4 portrays the active power dispatch of flexible DERs (both FG and FL) in per-unit.

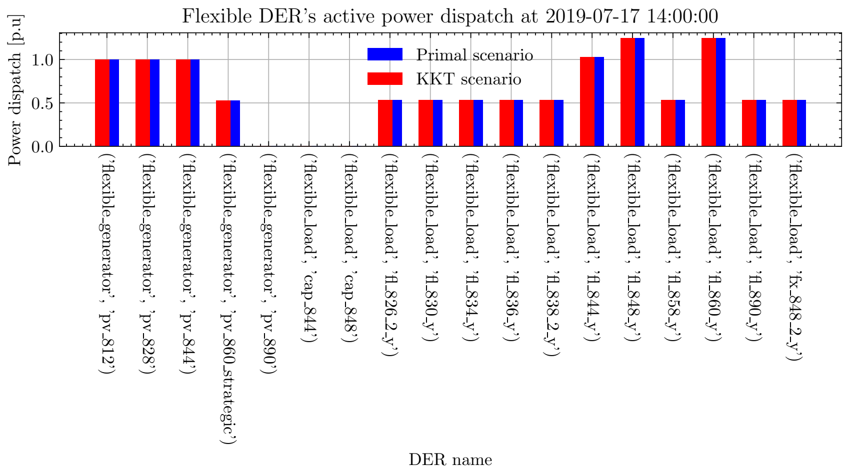

Figure 4.4: Flexible DERs active power dispatch in primal problem vs KKT problem.

Considering figure 4.3, the energy price at GSP has its highest value at the report time. However, PV generators offer energy at a lower cost. Therefore, most of the FGs contribute to the grid's power compensation. On the other hand, the curtailing price of the FLs is also high. Hence, most of them are not curtailed at all. In addition, as the voltage profile suggests in figure 4.5, the high penetration of fixed PV generators has led to an over-voltage along the feeder. To this end, the FLs must be dispatched at their maximum level to regulate the voltage in the allowed range.

Since the voltage tends to bind to its maximum level, considering the DLMP relationship given in equation (4.1) it is evident such voltage level will attenuate the DLMP in downstream buses.

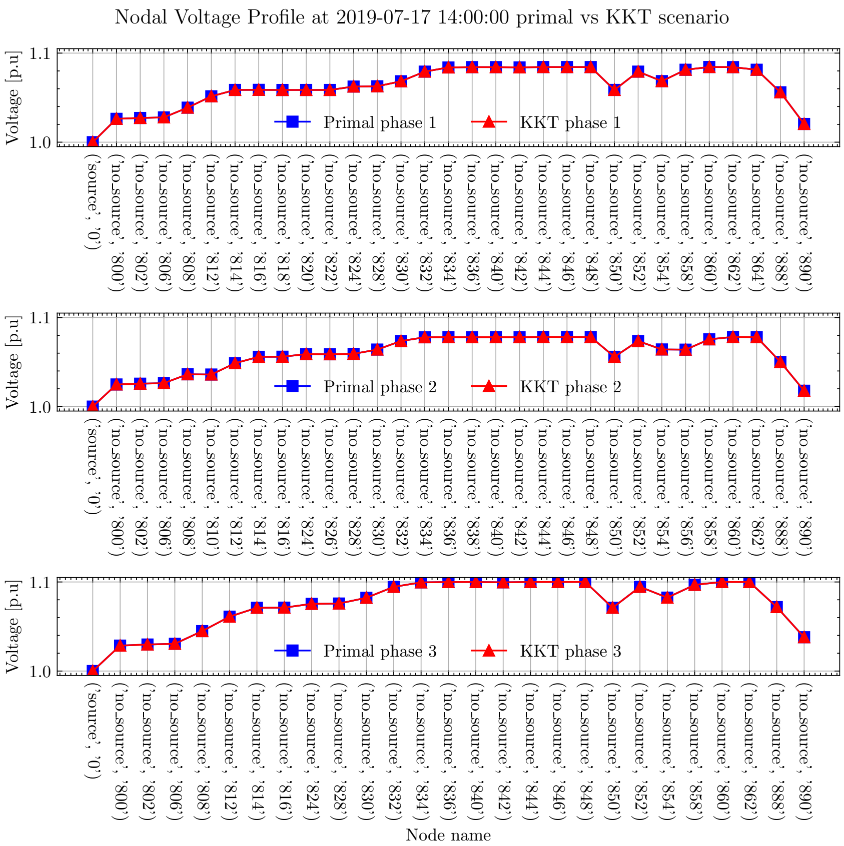

Figure 4.5: Voltage profile at 2:00 PM for three-phase nodes in primal problem vs KKT.

$$
\begin{aligned}
\boldsymbol{\pi}_t^p = {} & c_t^{p,0}\mathbf{1} \\
& - \left(\mathbf{M}_p^{p^{loss}}\right)^{\mathrm{T}} c_t^{p,0} - \left(\mathbf{M}_p^{q^{loss}}\right)^{\mathrm{T}} c_t^{q,0} \\
& - \left(\mathbf{M}_p^{|u|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|u|} - \underline{\boldsymbol{\mu}}_t^{|u|}\right) \\
& - \left(\mathbf{M}_p^{|s^f|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^f|} - \underline{\boldsymbol{\mu}}_t^{|s^f|}\right) - \left(\mathbf{M}_p^{|s^t|}\right)^{\mathrm{T}} \left(\overline{\boldsymbol{\mu}}_t^{|s^t|} - \underline{\boldsymbol{\mu}}_t^{|s^t|}\right) \qquad \forall t \in \mathcal{T} \quad (4.1a)
\end{aligned}
$$

Having said that, figure 4.6 demonstrates that there is no line congestion in either direction. So, it is anticipated that the contribution of the line congestion to the DLMPs is zero. Furthermore, the active losses peak at the given report time as shown in figure 4.8 which is another factor for reducing DLMPs. It should also be considered that since the ration of resistance per reactance, R/X, the higher consumption or production of active power will lead to higher levels of active losses. In the following, the contribution of energy price at GSP, active losses, voltage level, and the line congestion to the overall DLMPs at the nodes are demonstrated for each phase, figures 4.9 to 4.11.

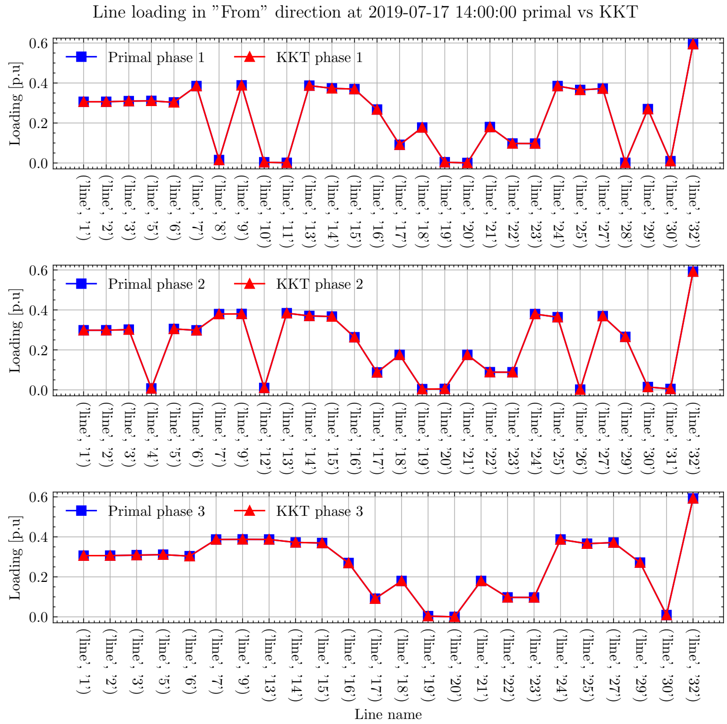

Figure 4.6: Feeder loading for three-phase lines at 2 PM in "From" direction, primal problem vs KKT.

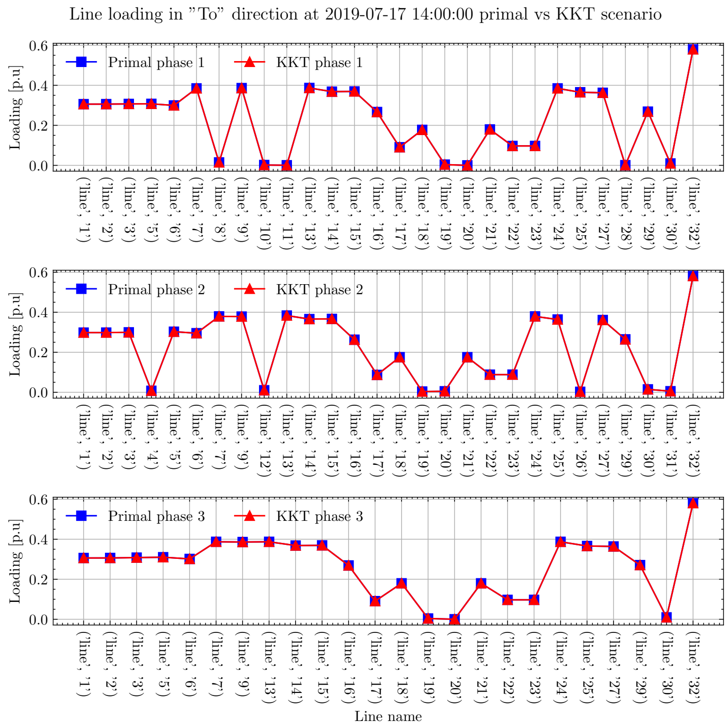

Figure 4.7: Feeder loading for three-phase lines at 2 PM in "To" direction, primal problem vs KKT.

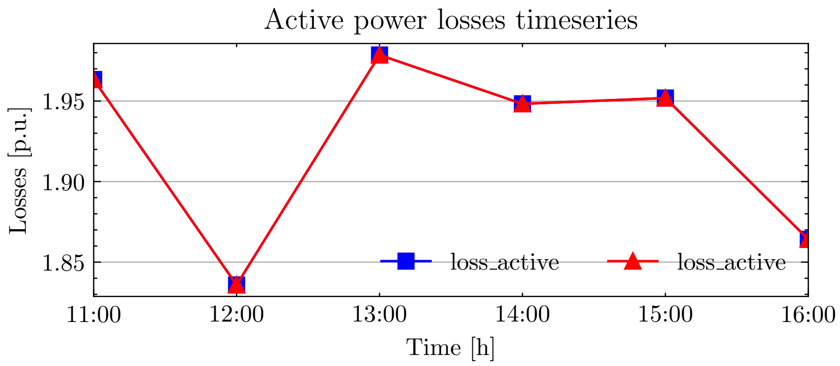

Figure 4.8: Active power losses in primal problem vs KKT.

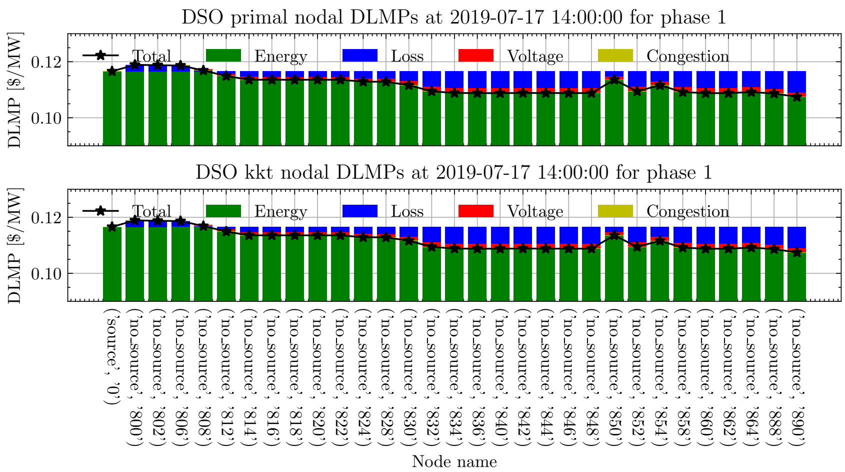

Figure 4.9: DLMP components and total values for phase 1, DSO primal vs KKT problem.

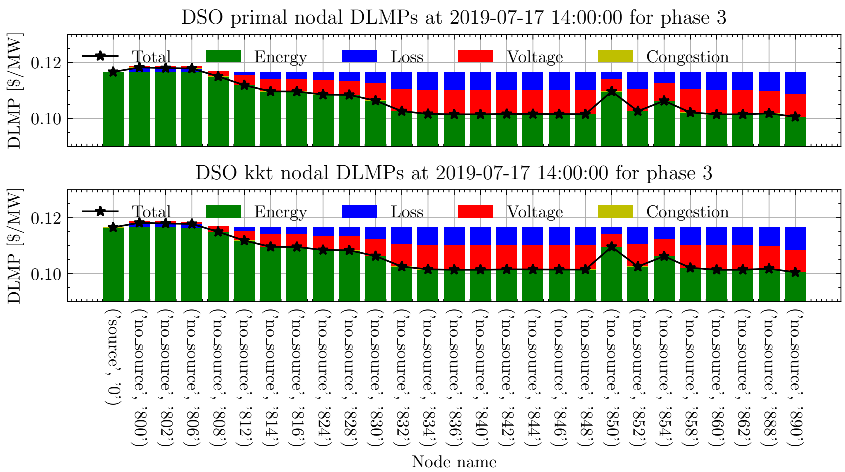

Figure 4.11: DLMP components and total values for phase 3, DSO primal vs KKT problem.

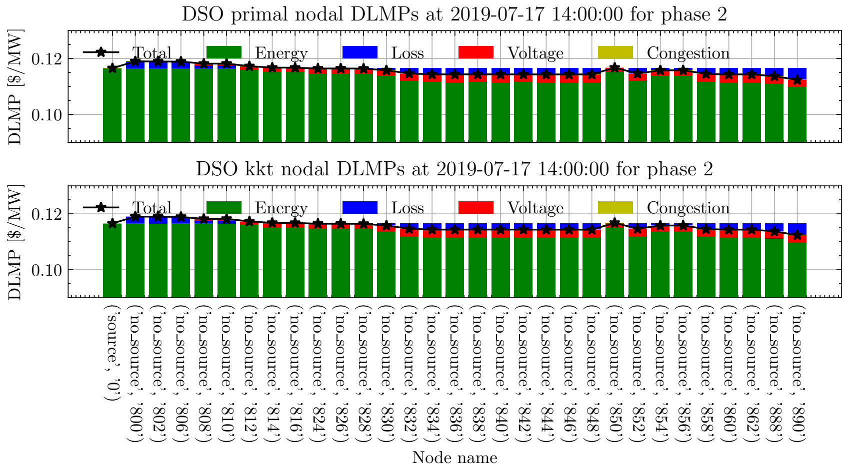

Figure 4.10: DLMP components and total values for phase 2, DSO primal vs KKT problem.

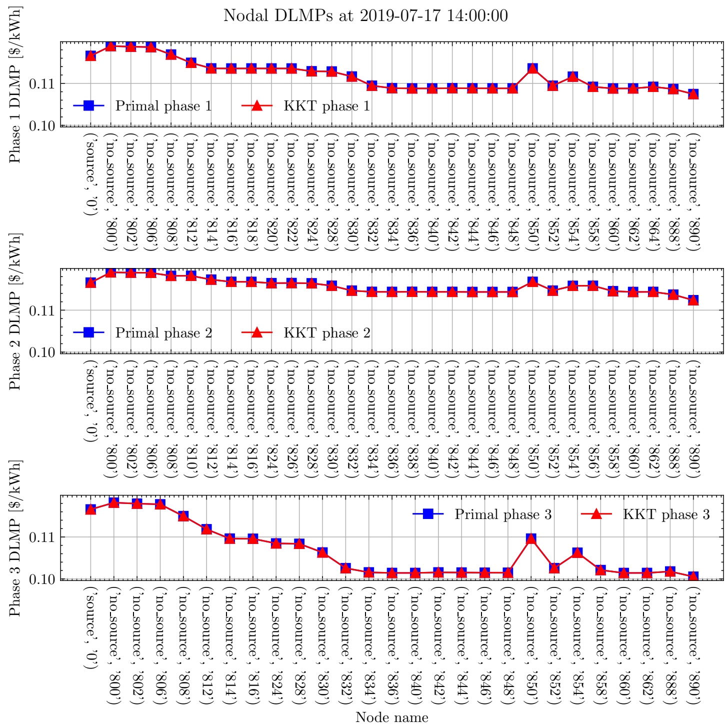

Figure 4.12: Nodal DLMP for the three-phase DN at 2:00 PM, DSO primal vs KKT problem.

As it regards, the total DLMP is reduced noticeably due to the negative components of voltage and active losses. Besides, the results confirm that there is no congestion contribution to the DLMP. Last but not least, since the network is unbalanced, the DLMP for different phases takes distinct values; this fact can emerge from figure 4.12.

4.2. Numerical Results for Strategic VS Non-Strategic DSO AS Market

As explained in chapter 3, the structure of MPEC formulation obtained for modelling the strategic behaviour of DER in the DSO AS market is similar to the KKT equivalent problem, and its results were discussed in Section 4.1. Nevertheless, in the MPEC of the strategic model, firstly, the marginal prices of the SG are variable; secondly, an equivalent linear objective function is assigned for the problem (3.22).

Therefrom, by noting problem (3.22) we will investigate the strategic behaviour of SG and its impact on the DN variables and characteristics. The same as Section 4.1, the report time is set to 2 PM, and due to the high computational burden, the simulation is only carried out from 11 AM to 4 PM. Moving to the optimization results as a comparison between strategic and non-strategic scenarios, the results for active power dispatching of flexible DERs are depicted in figure 4.13

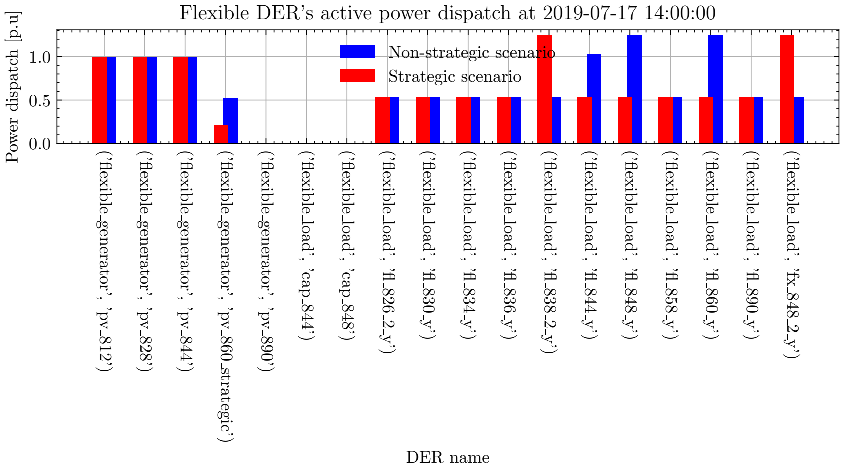

Figure 4.13: Flexible DERs active power dispatch in strategic vs non-strategic problems.

Evidently, since the SG offers higher prices, the DSO decides to dispatch it lower in comparison with the non-strategic scenario. Besides, overall, the FLs are curtailed more to compensate for the lower dispatch of the SG.

Figure 4.14 illustrates the strategic offer of SG in the DSO AS market compared to the level of its real marginal cost.

As expected, since the contribution of SG to the network reliability is crucial, the SG tends to submit offers higher than its real marginal cost. It is interesting that at 12 PM, the submitted offer is even lower than its marginal cost, which can lead to potential

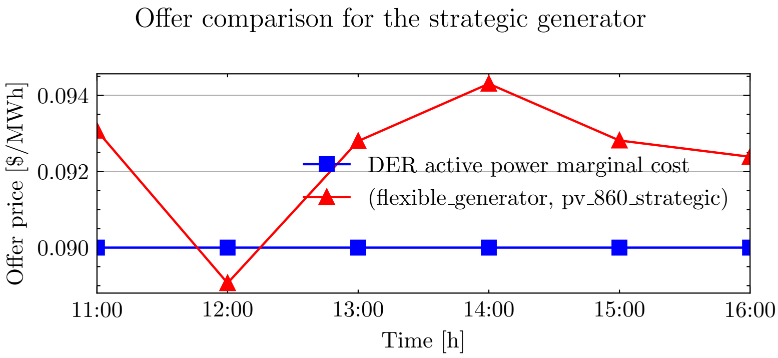

Figure 4.14: Strategic offers of SG vs real marginal cost.

profit loss for the SG. However, it should be noted that given the active power dispatch comparison presented in figure 4.15, the SG is not contributing active power to the DN at all.

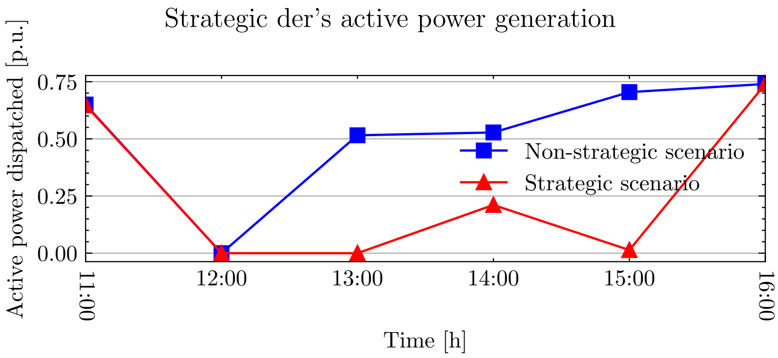

Figure 4.15: Active power dispatch time-series of SG in Strategic vs non-strategic problems.

It is evident that when the SG offers higher prices, it leads to higher costs for the DSO. Therefore, if DSO can replace its contribution with other resources, the result will be decreased in dispatching of SG as shown in figure 4.16.

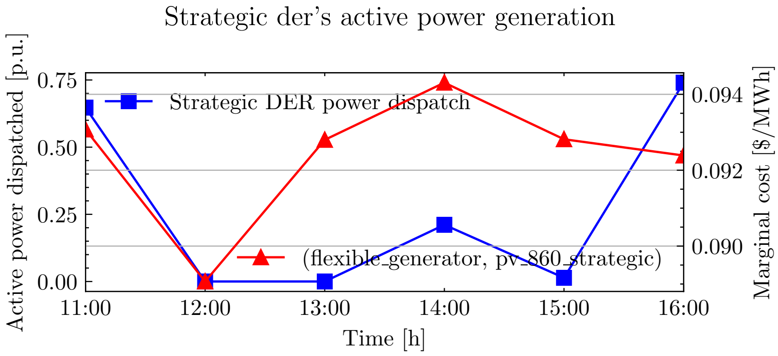

Figure 4.16: SG's active power dispatch and offer in the strategic scenarios.

In the following, the impact of SG's strategic behaviour on the network parameters will be investigated. Accordingly, the losses are reduced during the market-clearing period in the strategic scenario due to the strategic behaviour of SG leading to different dispatching patterns of flexible DERs, figure 4.17.

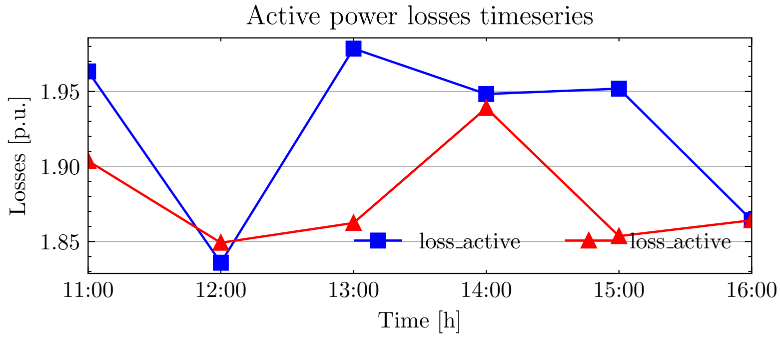

Figure 4.17: Active power losses in strategic vs non-strategic problems.

Following that, the voltage profile of the DN is portrayed in figure 4.18.

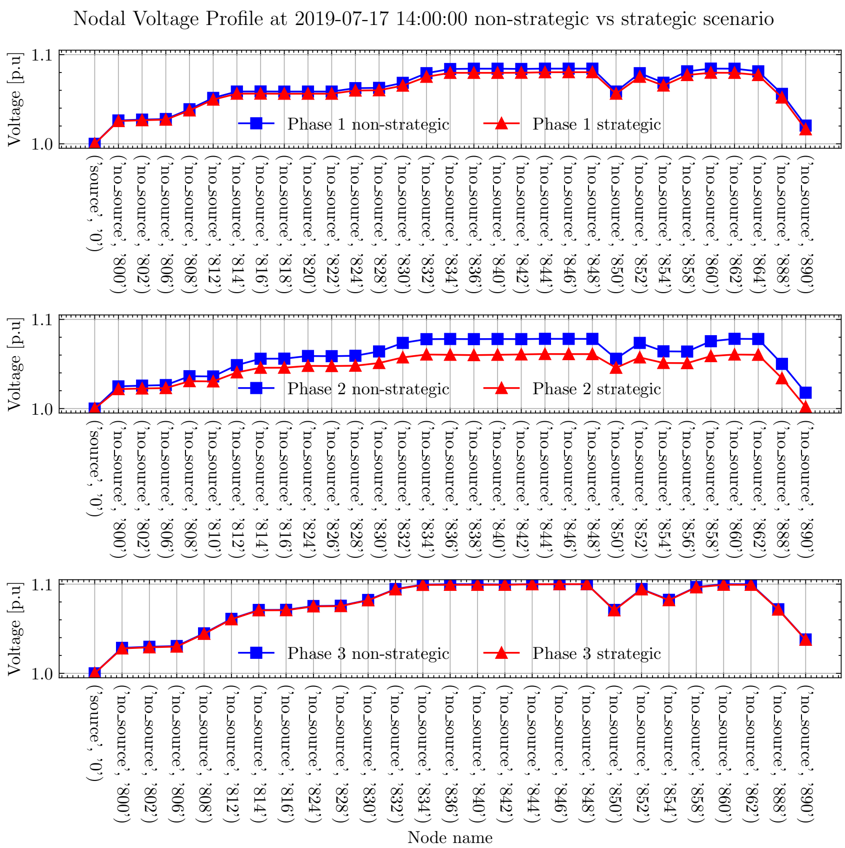

Figure 4.18: Voltage profile at 2:00 PM for three-phase nodes in strategic vs non-strategic problems.

As regards the strategic behaviour of SG results in a lower voltage profile throughout the feeder. Noting equation (4.1) for the DLMP, it is natural for the voltage to have lower values on the grounds that it will lead to higher DLMPs in favour of SG. It is important to note that at 2 PM, the voltage drop for the second phase is more remarkable compared to other phases, and this is due to the unbalanced nature of DN.

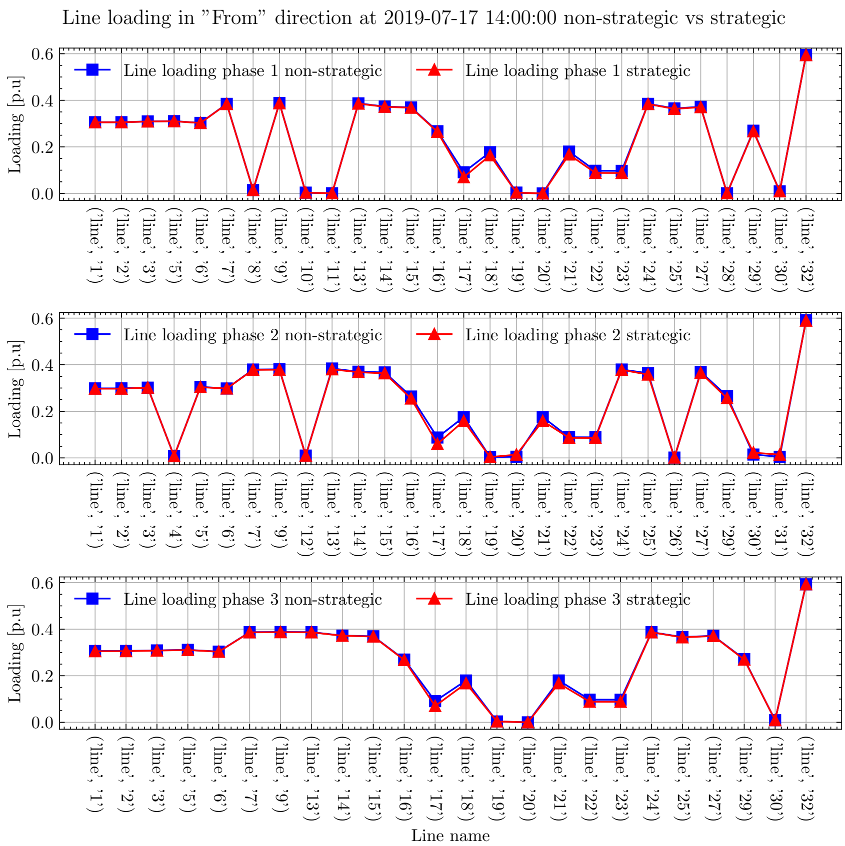

Figure 4.19: Line loading in "From" direction at 2:00 PM for three-phase feeder in strategic vs non-strategic problems.

With regards to the leading of the lines in "From" and "To" directions, figures 4.19 and 4.20 yield that the loading of the lines does not experience dramatic changes and it is limited in the acceptable domain without violating the maximum or minimum limits. Therefore, the anticipation is that the nodal DLMPs will not be affected by the congestion components.

The bar graphs express the components of the nodal DLMPs at 2 PM for different phases in figures 4.21 to 4.23. The most significant facts to emerge from them are that the main contribution to the nodal DLMPs is energy price at the root node, energy losses, and the voltage components in both scenarios. In the meantime, the results confirm that the feeder congestion contribution to the DLMPs is zero for both scenarios.

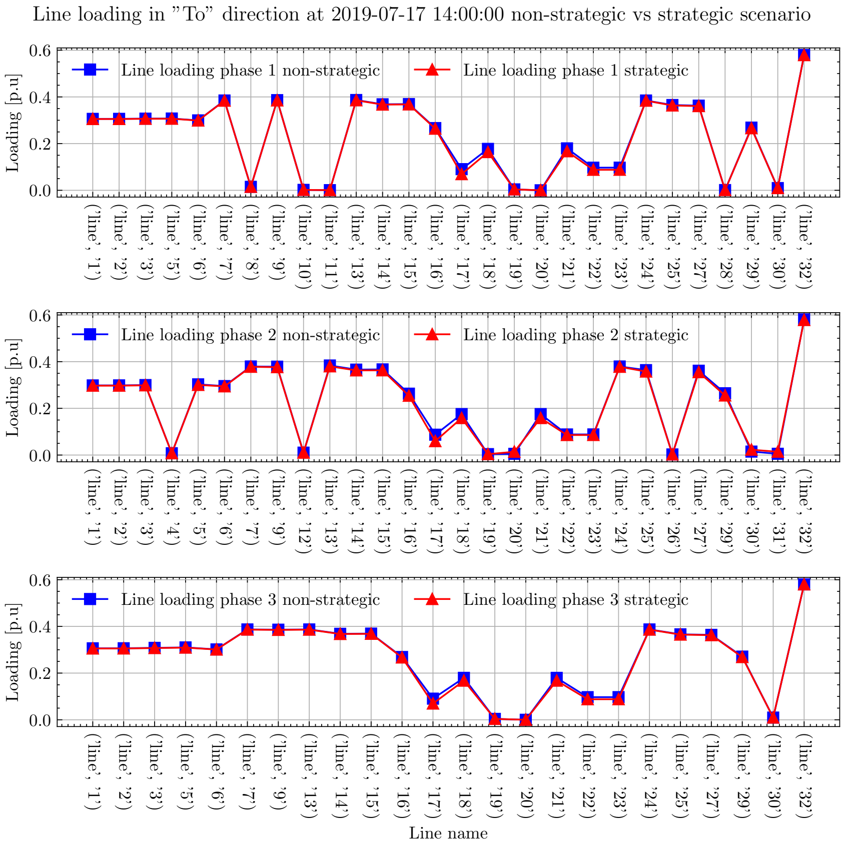

Figure 4.20: Line loading in "To" direction at 2:00 PM for three-phase feeder in strategic vs non-strategic problems.

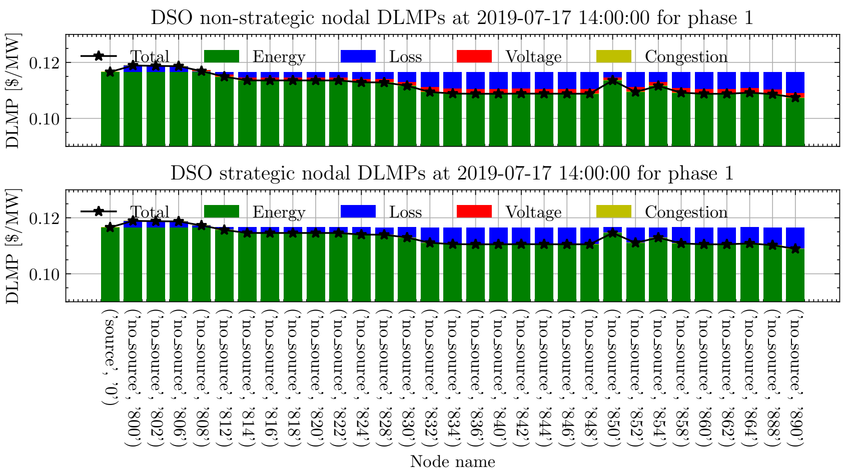

Figure 4.21: DLMP components and total values for phase 1, strategic vs non-strategic problems.

However, the voltage level is not binding to its maximum ceiling due to the reduced voltage profile along the feeder in the strategic scenario. Therefore, the negative volt-

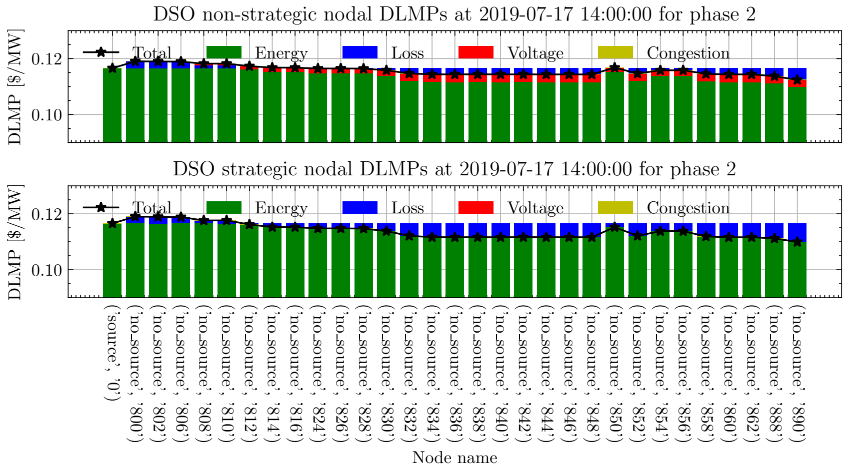

Figure 4.22: DLMP components and total values for phase 2, strategic vs non-strategic problems.

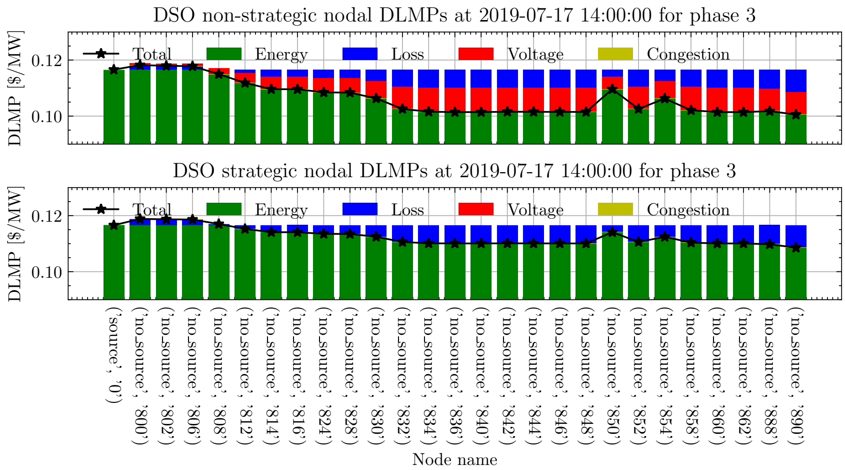

Figure 4.23: DLMP components and total values for phase 3, strategic vs non-strategic problems.

age components in the nodal DLMPs are cancelled out. Consequently, the total nodal DLMPs are increased for all phases for the third phase; this statement is more evident in figures 4.23 and 4.24.

With respect to the DLMPs assigned for the SG's node, the values are depicted in figure 4.25. As it regards, overall, the DLMP time-series for the strategic bus 860 is increased. Therefore, we can conclude that the SG successfully manipulates the DLMPs to his benefit.

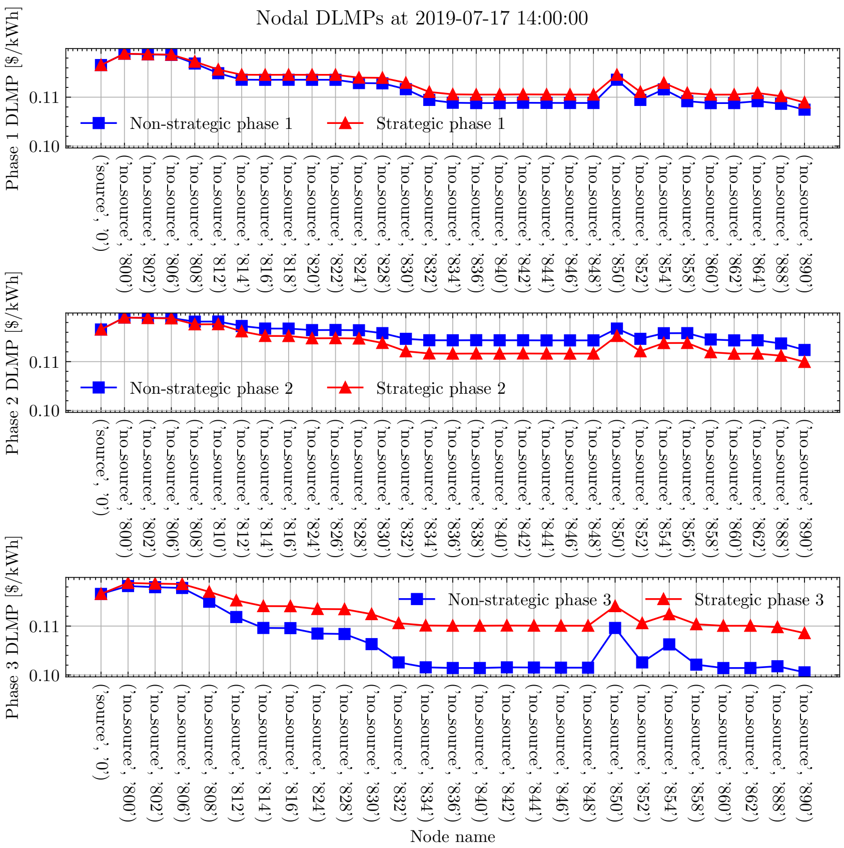

Figure 4.24: Total nodal DLMPs at 2 PM, strategic vs non-strategic problems.

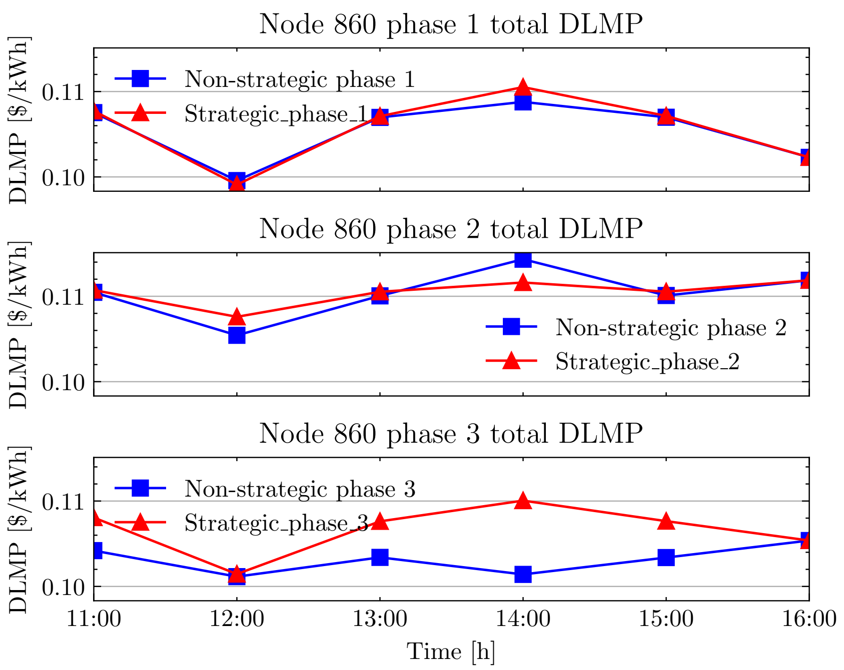

Figure 4.25: DLMP time-series at strategic node 860, strategic vs non-strategic problems.

The final word is that the total DLMPs increase due to submitting strategic offers by SG that are higher than the real marginal cost. Nevertheless, it does not necessarily mean that the strategic DER will make a higher profit because the DSO has the authority not to dispatch the SG or dispatch it very low. Therefore, even though, as the first decision-maker in such a sequential game, the SG is the potential to make more profit, the DSO as the market regulator, may also take actions that lead to even lower expected profit for SG.

4.3. Results for Hybrid P2P and Strategic DSO Market

In this section, the P2P market is integrated into the strategic DSO market to evaluate the decentralized optimization approach using ADMM for clearing the coordinated P2P and DSO AS market. In addition, the impact of the strategic behaviour of SG on the overall P2P energy transaction will be investigated.

As explained in Section 2.4.3, the idea for clearing the hybrid DSO and P2P market is that the DSO clears the market and broadcasts the GUP to the seller and buyer peers. Based on that, the P2P market is settled fully in a distributed manner using ADMM, and the power injection/withdrawal into/from the nodes by the peers is announced to the DSO to reevaluate the GUP. The process continues until a consensus is attained.

Moving to the simulation results, the optimization time-step is limited to the peak hours between 1 PM and 3 PM to avoid the enormous computational burden. Accordingly, some of the DERs that were considered fixed in the previous sections, are designated to be the seller and buyer peers of the P2P market.

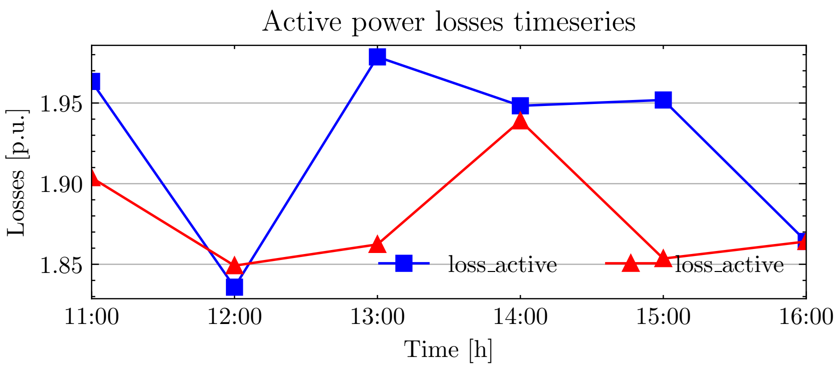

Figure 4.26: Active power losses in hybrid strategic vs non-strategic DSO and P2P markets at 2 PM.

By running the simulation and letting the report time be 2 PM, the results shown in figure 4.27 were obtained for the flexible DERs' active power dispatch in the DSO strategic market. The overall system losses are also depicted in figure 4.26.

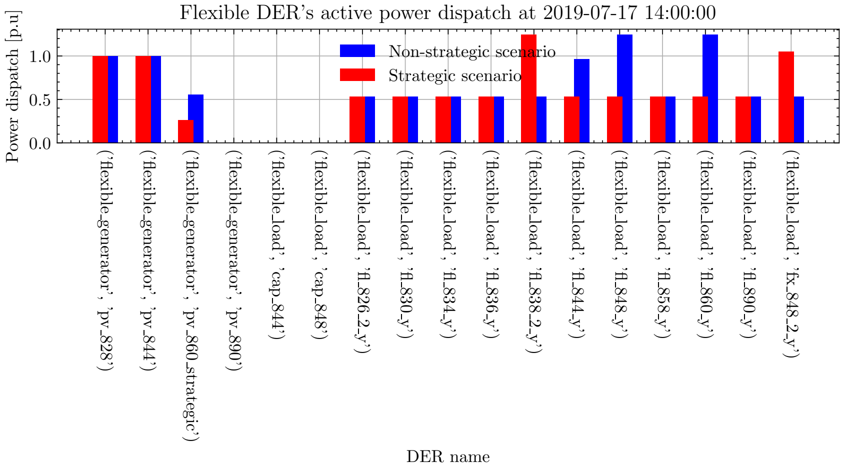

Figure 4.27: Flexible DERs' active power dispatch in hybrid strategic vs non-strategic DSO and P2P markets at 2 PM.

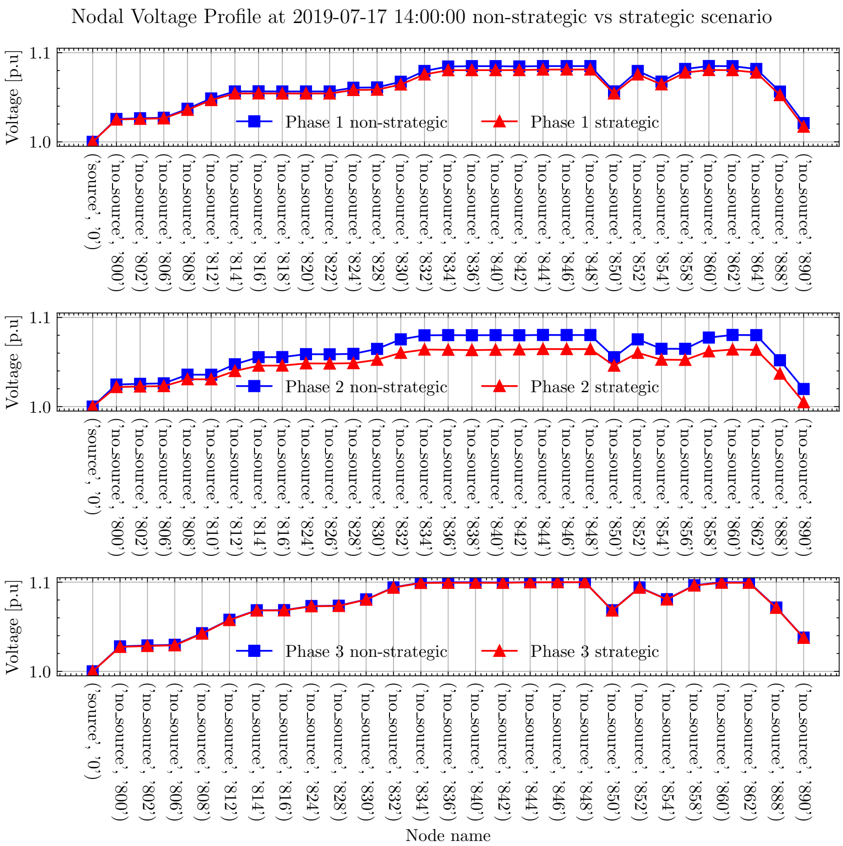

Figure 4.28: Voltage profile in hybrid strategic vs non-strategic DSO and P2P markets at 2 PM.

As discussed in the previous section, the strategic behaviour of SG is affecting the overall system behaviour, and this is evident in the figures. Noting figure 4.28, the voltage profile of the nodes is decreased due to the strategic behaviour of SG so it can increase the DLMPs.

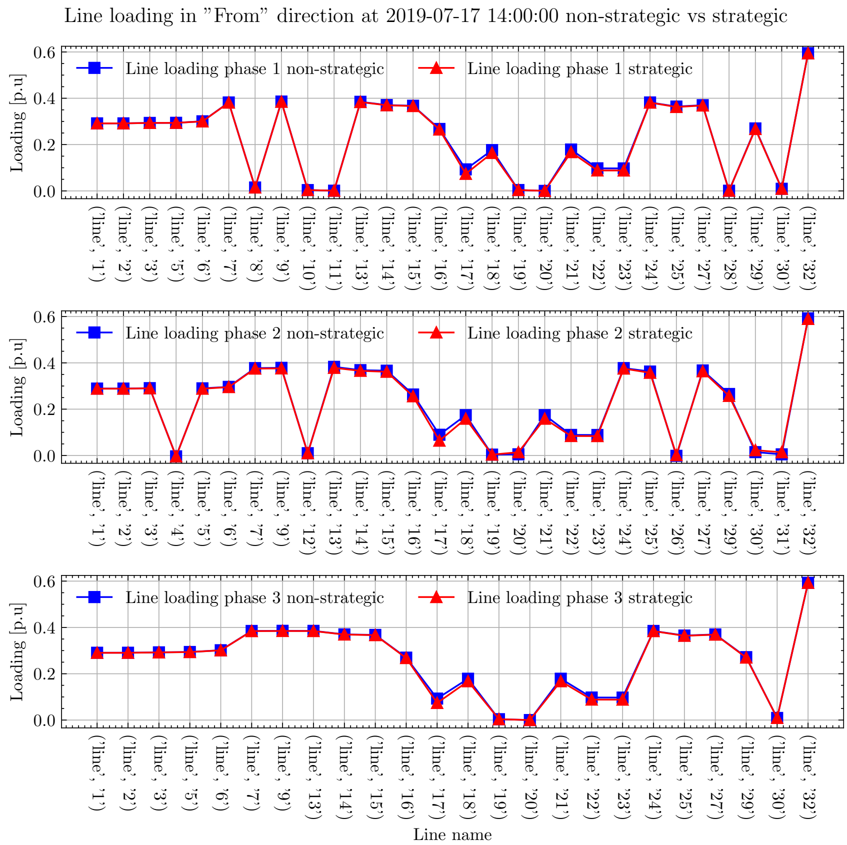

Figure 4.29: Line loading "From" in hybrid strategic vs non-strategic DSO and P2P markets at 2 PM.

The loading of the lines in the "From" direction is illustrated in figure 4.29. Accordingly, there is no congestion in the feeders at the report time, so it is expected that the contribution of the congestion into the DLMPs should be zero. The same reasoning holds for the thermal loading of the lines in the "To" direction.

Figures 4.30 to 4.32 demonstrate the contribution of different components to the DLMPs for three-phase nodes. As it regards, the SG can successfully increase the DLMPs not only in his node but also throughout the DN. Another important fact to emerge is that the most significant portion of the DLMPs is the energy price at GSP, and the negative voltage contribution is evident in the non-strategic scenario. However, the strategic behaviour leads to voltage component cancellation for all nodes with different phases. Therefore, the respective DLMPs enjoy an increase.

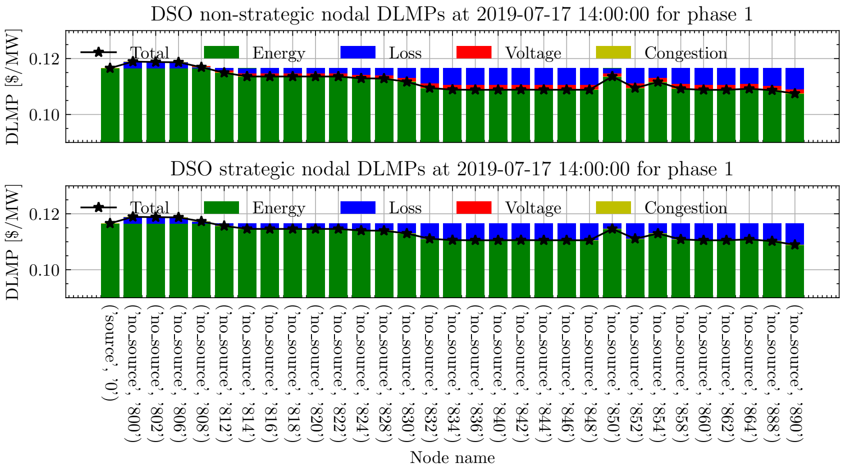

Figure 4.30: DLMP components for phase 1 in hybrid DSO and P2P markets at 2 PM.

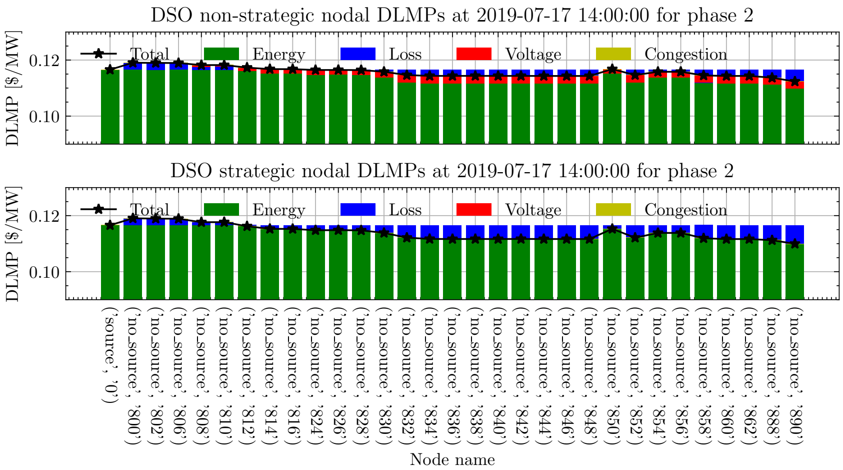

Figure 4.31: DLMP components for phase 2 in hybrid DSO and P2P markets at 2 PM.

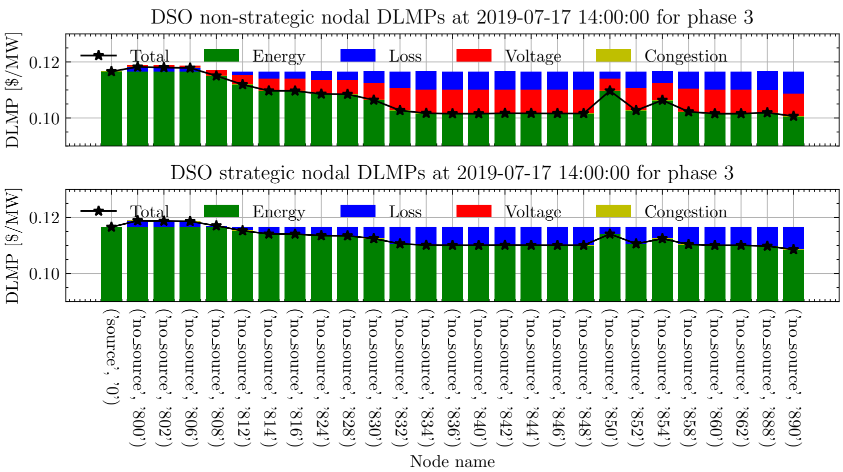

Figure 4.32: DLMP components for phase 3 in hybrid DSO and P2P markets at 2 PM.

The DLMP time-series from the SG's perspective is depicted in figure 4.33 demonstrating how the SG has benefited from the strategic behaviour.

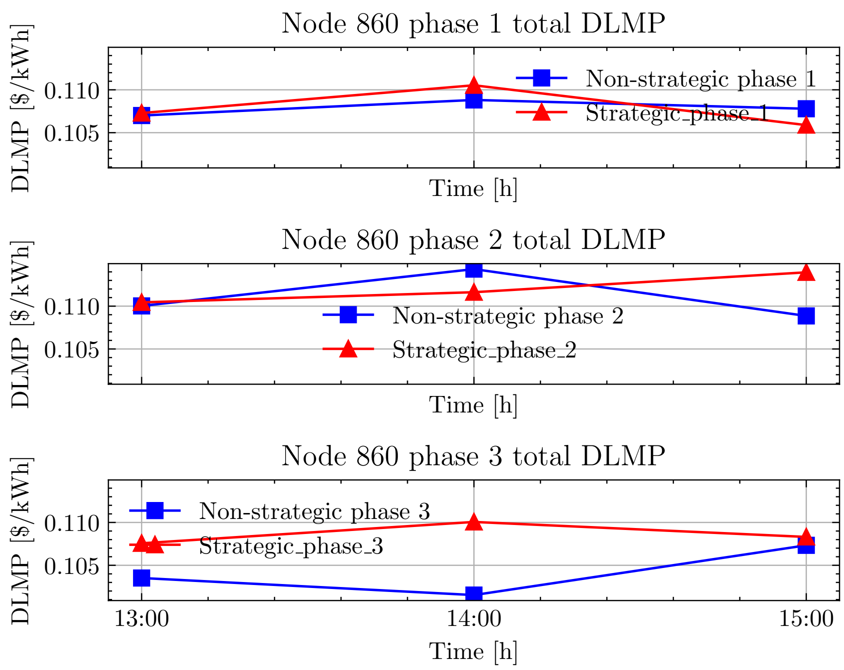

Figure 4.33: Three-phase DLMP time series at strategic node in hybrid DSO and P2P markets.

Overall, such deviation of results and system parameters with respect to the strategic scenario is due to the following behaviour of SG in the DN. According to figure 4.34, when SG behaves greedily by offering higher prices, the DSO as the market coordinator reduces the power dispatch at 2 PM and keeps the system operating optimally. However, at 3 PM, even though the submitted offer was higher, the DSO increased the dispatching.

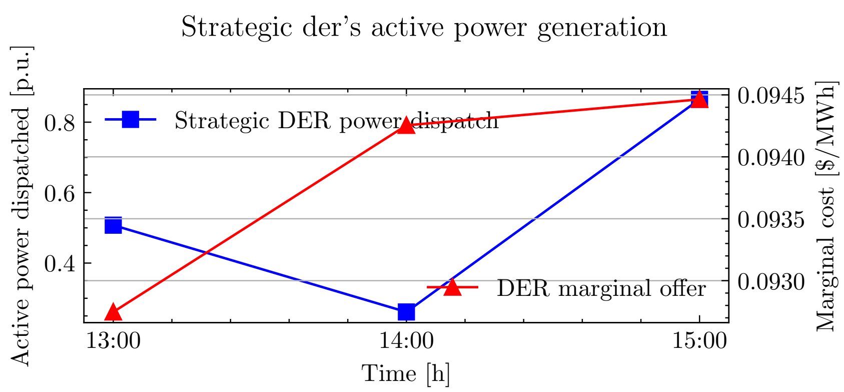

Figure 4.34: Strategic generation and offer of SG in hybrid DSO and P2P markets.

In the following, the results for the P2P market will be discussed. The most important result to investigate first is the GUP obtained from the coordination of DSO and P2P markets. The GUP is a derivative of DLMPs; therefore, any manipulation in the DLMPs will be projected in the GUPs. According to that, the GUP for non-strategic and strategic scenarios for the hybrid P2P and DSO AS market is illustrated in the figure 4.35.

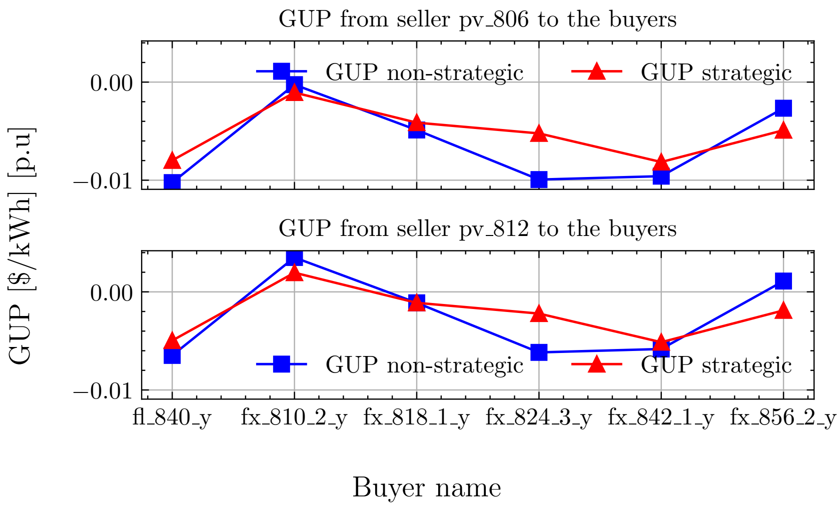

Figure 4.35: GUP for P2P market at 2 PM, non-strategic vs strategic scenario.

It should be noted that the x labels and tiles of figure 4.35 indicate the name and location of the seller and buyer nodes in the IEEE test feeder given in figure 4.1. Besides, the GUPs, which is the deviation of DLMP at the buyer node and the seller one, are given based on the DERs perspective. As anticipated, the GUP is also impacted by the DLMP increase in the strategic scenario.

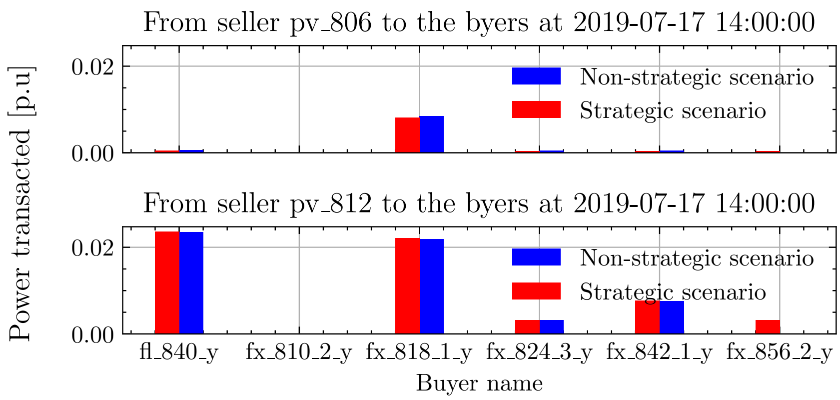

Figure 4.36: Energy transaction in P2P market at 2 PM, strategic vs non-strategic scenario.

Figure 4.36 illustrates how much energy is transacted from the sellers to the buyer peers. The most significant fact to conclude from the GUP and the energy transaction figures are that since the DLMP is lower downstream of the feeder, the respective GUP from the buyer's perspective will be lower. Therefore, the buyer nodes tend to procure energy from "pv_812" which has lower GUP from the buyer perspective. Moreover, the strategic behaviour of SG has led to a noticeable difference in energy transactions among the peers, especially for "fx_865_2_y" DER, which is located well in the tail of the feeder.

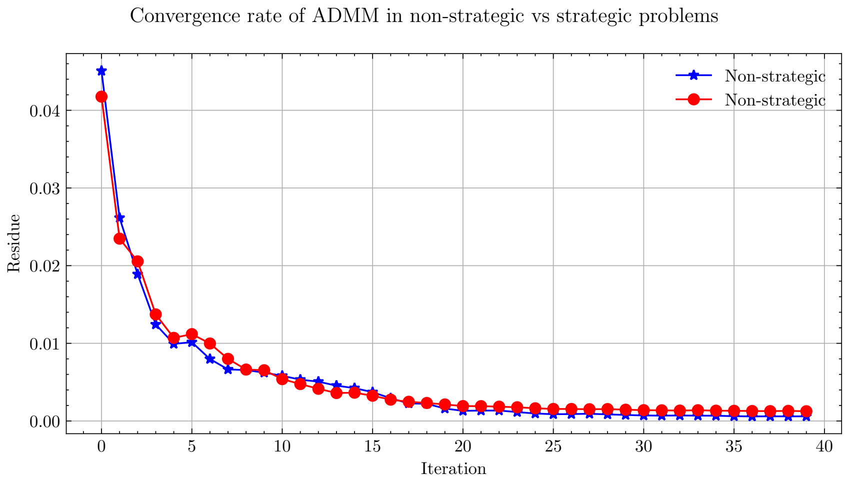

Figure 4.37: Convergence of ADMM in strategic vs non-strategic scenarios.

Last but not least, figure 4.37 shows the convergence rate of ADMM for its functionality in solving the coordinated P2P and DSO AS market in strategic and non-strategic scenarios. Accordingly, since the problem of the strategic market is a mixed-integer linear MPEC, the ADMM convergence ratio is slightly lower compared to the DSO non-strategic linear problem. Overall, the problem in the strategic scenario takes a longer time to be solved due to its complexity and the significant number of real and integer variables.

Conclusion

In this work, we investigated the interaction between the DERs and DSO market by adopting a comprehensive model of three-phase unbalanced DN. A linearized approximation method for AC-OPF was proposed to settle down the market and calculate the DLMPs. The radial structure of DN led to acceptable accuracy of the linearized AC-OPF. Accordingly, we saw that the DLMPs effectively regulate the losses, voltages and congestion throughout the network. In the next step, the GUP was obtained and assigned as the FTR to coordinate the bilateral energy transactions in a P2P decentralized manner using ADMM. The last stage of the project included the strategic behaviour study of FG in the DSO market. The bi-level programming was adopted to model the problem where SG tried to manipulate the market outcome to maximize revenue by submitting strategic offers in the upper-level problem. On the other hand, the DSO market, as the lower-level follower player, cleared the market by minimizing the costs. The bi-level problem was reformulated as single-level MPEC, and the equivalent linearized components substituted the nonlinear terms by adopting the strong duality theorem and big-M method. The optimization results revealed that the strategic behaviour of SG leads to a lower voltage profile throughout the feeder. Consequently, the voltage component of the DLMP is affected, leading to the higher DLMP in favour of the SG. The same results were valid in the hybrid DSO AS and P2P market, so the GUP and the consequent energy transaction among the peers experience a noticeable change when there is a strategic DER in the DN. In this work, the coordination of DSO and TSO is not taken into account. From there, as future work, it can be taken under the scope to investigate the interaction of DSO local and TSO wholesale markets.

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