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Spaces of entire functions, sampling theorems and applications

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Abstract

Sampling theory is the study of spaces of continuous functions that can be exactly rebuilt from samples taken on a discrete set of points, and has a great number of practical applications in communication engineering and signal processing. The most important function spaces connected to the classical sampling theory are the Paley-Wiener spaces, which are spaces of bandlimited functions that have many properties very useful for sampling. The bandlimited functions can generally be rebuilt from samples taken on a sequence of equidistant points.

In real applications, a signal effective bandwidth can vary in time. Adjusting the sampling rate accordingly should improve the sampling efficiency and information storage. While this old idea has been pursued in numerous publications, some fundamental problems are not fully solved yet. The most important regards how to take samples on non-uniform intervals or at a time-varying rate preserving the possibility to perfectly and stably reconstruct the signal.

In this work we introduce new properties and new sampling formulas for some spaces of entire functions, namely the de Branges spaces and the Paley-Wiener spaces, based on non-uniform sampling sets strongly different from these of classical results, and we study their applications to signal processing.

Then we study new spaces of entire functions that generalize the classical Paley-Wiener spaces, in particular the time-varying bandlimit spaces, recently introduced by Kempf and Martin. We analyze the classes of operators connected to these spaces and we investigate the connections between these spaces and the de Branges spaces.

Moreover we introduce a new class of time-varying bandlimit spaces, which are unitarily isomorphic to the Kempf-Martin spaces, but with some different important properties, that make them more controllable and inter-

pretable.

Finally we study the relation between the de Branges spaces and the solution of the inverse problem of a canonical systems, which is strongly connected to the properties of these spaces.

Summary

After the first two introductory chapters, in this work we present many results that can mainly be divided in four different parts:

- in the first part we develop new non-uniform sampling formulas for the de Branges spaces and the Paley-Wiener spaces, and then we express these formulas in terms of the sampling sequence points;
- in the the second part we develop a generalization of the Fourier Transform for the de Branges spaces, and then we investigate the isomorphism between these spaces and the Kempf-Martin spaces; thanks to this we derive new characterizations for the Kempf-Martin spaces and we give a simpler and more general proof of their most important properties;
- in the third part we investigate the consequences of the results of the first two parts on the applications of the concept of time-varying bandlimit, and we propose a new family of time-varying bandlimit spaces and a new generalized sampling theory;
- in the fourth part we describe an improvement to the Romanov algorithm for the solution of the canonical inverse problem.

In *Chapter* 1 we introduce the state of art about spaces of entire functions, sampling theorems, and applications. Moreover we explain the motivations for the research presented in this work.

In *Chapter* 2 we introduce all the already known results that we use in the next chapters, mainly regarding the reproducing kernel Hilbert spaces, the Hardy spaces, the model spaces, the de Branges spaces and the Paley-Wiener spaces.

In the first part of the work (*Chapters* 3 and 4) we introduce new properties and sampling formulas for the de Branges spaces. Moreover we connect

these results with the Paley-Wiener spaces, and we present new non-uniform sampling formulas for these spaces. The main aspect of these sampling formulas is that they are based on a set of non-uniform sampling sequences that is strongly different from the one of classical Paley-Wiener-Levinson result. Moreover we give a characterization of the sequences for which this sampling formula is valid. In particular:

- *Theorem* 3.5 shows a sampling formula for a de Branges space $\mathcal{B}(E)$ based on the sequence of zeros of $\Theta(z)-1$, where $\Theta(z)$ is any meromorphic inner function that is divided by the meromorphic inner function $\Phi(z) = \frac{E^{\#}(z)}{E(z)}$;

- *Theorem* 4.2 is a particular case of *Theorem* 3.5 and shows a sampling formula for the Paley-Wiener space $\mathcal{PW}_a$ based on the sequence of zeros of $\Theta(z) - 1$, where $\Theta(z)$ is any meromorphic inner function that is divided by the meromorphic inner function $\Phi(z) = e^{2\pi iz}$;

- *Theorem* 4.14 shows under which conditions a given sequence verifies the requirements of *Theorem* 4.2;

- *Theorem* 4.17 shows a sampling formula for the Paley-Wiener space $\mathcal{PW}_a$ based on a sequence $\{t_n\}_n$ such that $t_n \neq \frac{\pi}{a}n$ only for a finite number of n's, but without constraints on the corresponding t_n's;

- *Theorem* 4.18 shows a sampling formula for the Paley-Wiener space $\mathcal{PW}_a$ based on a sequence $\{t_n\}_n$ such that $\left|\frac{\pi}{a}n - t_n\right| \leq \delta \ \forall n \in \mathbb{Z}$ for some $\delta < \frac{\pi}{2a}$, under a condition that we show to be verified for an infinite number of sequences;

- *Theorem* 4.19 shows a sampling formula for the Paley-Wiener space $\mathcal{PW}_a$ based on a sequence $\{t_n\}_n$ such that $t_0 = 0$, $\left|\frac{\pi}{a}n - t_n\right| \leq \delta$ if $|n| < M$ for some $\delta < \frac{\pi}{2a}$ and some integer $M > 0$, and $\left|\frac{\pi}{a}n - t_n\right| \leq \frac{\delta_1}{\frac{\pi}{a}|n|}$ if $|n| \geq M$, for some δ_1 such that $0 < \delta_1 \leq \frac{\pi}{a}M\delta$.

Finally, in *Section* 4.5 we show that the constraints of the sampling sequences in *Theorems* 4.2, 4.17 and 4.19 are more useful for real applications with respect to those of the classical Paley-Wiener-Levinson, since they are more flexible on a finite subsequence of the sampling sequence, and there always exists a finite subsequence of the sampling sequence such that the reconstruction performed on it has any desired precision.

In the second part of the work (*Chapters* 5 and 6), first of all we present a generalization of the Fourier Transform for the de Branges spaces (see *Theorem* 5.1). This transform induces an isometric isomorphism between every de Branges space and a corresponding subspace of the space $\mathcal{L}^2(\mathbb{R})$, similar to how the original Fourier Transform induces an isometric isomorphism between the Paley-Wiener space $\mathcal{PW}_a$ and $\mathcal{L}^2[-a, a]$. Then, we introduce the Kempf-Martin spaces using the theory of symmetric operators, according to the arguments presented in [40], and we show that there exists an isometric multiplier between the Kempf-Martin spaces and the de Branges spaces (see *Theorem* 6.2). This isomorphism allows us to find a necessary and sufficient conditions for a function to belong to a Kempf-Martin space. Finally, we give an alternative and equivalent definition of the Kempf-Martin spaces based on this isomorphism, and we derive and improve all the main results presented in [40] from the properties of the de Branges spaces, without using the theory of simple symmetric operators.

In the third part (*Chapter* 7) we consider all the results of both the previous two parts, and we investigate their consequences on the concept of time-varying bandlimit and on the sampling theory. In *Section* 7.1 we explain the concept of time-varying bandlimit for the de Branges spaces and the Kempf-Martin spaces, and we give its formal definition. In *Sections* 7.2 and 7.3 we explain how the time-varying bandlimit functions can be interpreted as the result of the application of a distortion in the time domain to the functions of a space of a subaspace of $\mathcal{L}^2(\mathbb{R})$. Moreover, thanks to this observation, we define a new family of spaces of time-varying bandlimit functions $\mathcal{V}(\Theta)$, each of which is associated to a meromorphic inner function $\Theta(z)$. The main properties of these spaces are the following.

- There exists an isometric multiplier between every space $\mathcal{V}(\Theta)$ and a Kempf-Martin space, and thanks to this the spaces $\mathcal{V}(\Theta)$ maintain many of the properties of the Kempf-Martin spaces.

- We can associate to every time-varying bandlimit function $F(z) \in \mathcal{V}(\Theta)$ a normalized frequency representation.

- The normalized frequency representation of $F(z) \in \mathcal{V}(\Theta)$ is obtained by applying a weighted Fourier Transform to $F(z)$. This transform induces an unitary isomorphism between $\mathcal{V}(\Theta)$ and a subspace of $\mathcal{L}^2(\mathbb{R})$.

- The spaces $\mathcal{V}(\Theta)$ have many properties that are analogous to the ones of Paley-Wiener spaces for bandlimited functions.

- The spaces $\mathcal{V}(\Theta)$ result to be more interpretable and controllable than other time-varying bandlimit spaces since their functions can be represented by a summation of simpler functions.

In *Section* 7.4 we recall the Shannon sampling method, and then we introduce a generalized sampling method for time-varying bandlimit functions based on the spaces $\mathcal{V}(\Theta)$. Since an arbitrary signal effective bandwidth can change in time, the goal of this method is to improve the sampling efficiency by adjusting the sampling rate according to the signal effective time-varying bandwidth, taking samples of a signal only as frequently as necessary. The generalized sampling method is mainly composed by the following 4 steps.

1. Analyze the frequency of the raw signals of interest in order to choose a suitable time-varying bandlimit space.

2. Filter the raw signal to obtain a signal with the desired time-varying bandlimit.

3. Store the samples on the chosen sampling sequence.

4. Rebuild the filtered function from the discrete samples using the reconstruction formula of time-varying bandlimit spaces.

Finally, in the fourth part (*Chapter* 8) we introduce the canonical systems, the canonical inverse problem, and the solution given by Romanov (see [46], *Section* 7 (p. 37)), which is constructive, iterative and not explicit. Then we present an improvement of the results of Romanov. In particular, in *Theorem* 8.3 we gives an explicit solution to *Theorem* 6 in [46] (p. 21), which is the main result on which Romanov's arguments are based. Then, in *Theorem* 8.6 we apply this result to the algorithm for solving the inverse problem proposed by Romanov. Also our solution is iterative and not explicit, but unlike that of Romanov, the result of each iteration is explicit in terms of the results of the previous iteration.

Contents

CHAPTER 1

Introduction and motivation

Sampling theory is the study of spaces of continuous functions that can be exactly rebuilt from samples taken on a discrete set of points, and has a great number of practical applications in communication engineering and signal processing. The classical sampling theory is strongly connected with the space of bandlimited functions. Given $a > 0$, a function $F \in \mathcal{L}^1(\mathbb{R})$ is said to be a-bandlimited if its Fourier transform vanishes outside the closed interval $[-a, a]$. The frequency upper bound a is known as the bandlimit and $2a$ is referred to as the bandwidth. The Fourier transform of a function $F \in \mathcal{L}^1(\mathbb{R})$, denoted by $\mathcal{F}(F)$, is defined as

$$\hat{F}(z) = \mathcal{F}(F)(z) = \int_{-\infty}^{+\infty} F(x)e^{-ixz}dx.$$

The classical Whittaker-Kotelnikov-Shannon sampling theorem (see [47]) states that an a-bandlimited function $F(t)$ can be completely rebuilt for all $t \in \mathbb{R}$ from its values $\{F(t_n)\}_n$ on a sequence of equidistant sampling points $\{t_n\}_n$, with $t_{n+1} - t_n = \frac{\pi}{a}$, by the following sampling formula

$$F(t) = \sum_n \frac{\sin(a(t-t_n))}{a(t-t_n)} F(t_n). \tag{1.1}$$

The function $G(t, t_n) = \frac{\sin(a(t-t_n))}{a(t-t_n)}$ is referred as the *sampling kernel.*

The most important functions spaces connected to the classical sampling theory are the Paley-Wiener spaces. Given $a > 0$, the Paley-Wiener space of parameter a (referred as $\mathcal{PW}_a$) is a reproducing kernel Hilbert space and is defined as the set of all entire functions square integrable on $\mathbb{R}$ and such that $|F(z)| \leq Ce^{a|z|}$ $\forall z \in \mathbb{C}$, for some positive constant C. The reason why these spaces are strongly connected to the classical sampling theory is that the Fourier transform induces an isomorphism between $\mathcal{PW}_a$ and $\mathcal{L}^2[-a,a]$ (in particular $\mathcal{L}^2[-a,a]$ is the image of $\mathcal{PW}_a$ via Fourier transform). Then all the functions of the space $\mathcal{PW}_a$ are a-bandlimited functions, and they can be rebuilt using formula (1.1) with $t_n = \frac{n\pi}{a}$ and sampling kernel given by

$$G(t,t_n) = \frac{\sin\left(a\left(t-t_n\right)\right)}{a\left(t-t_n\right)}. \tag{1.2}$$

The Paley-Wiener spaces have many properties that are very useful for sampling, the main ones being the fact that the set of functions $\{G(t,t_n)\}_n$ is an orthonormal basis for $\mathcal{PW}_a$, and that $\frac{a}{\pi}G(w,z)$ is the reproducing kernel of $\mathcal{PW}_a$.

The classical sampling theory has been generalized in several directions including, for example, non-uniform sampling or derivative sampling (see [52]). Non-uniform sampling consists in exactly rebuild a function starting from samples taken at irregular intervals. Non-uniform sampling is very important since it comes natural in many applications, for example in automotive industry, data communication, medicine or astronomy. Given the large number of applications, the research is now focusing on new methods to rebuild signals through non-uniform sampling based on sampling sequences with weaker and more flexible constraints than those already known. One of the most important results in non-uniform sampling is the Paley-Wiener-Levinson theorem (see [31]). It asserts that, given a sequence of reals $\{t_n\}_n$ such that

$$\delta := \sup_{n\in Z}\left|t_n - \frac{n\pi}{a}\right| < \frac{\pi}{4a},$$

then for any $F \in \mathcal{PW}_a$ we have

$$F(t) = \sum_n \frac{S(t)}{S'\left(t_n\right)\left(t-t_n\right)} F\left(t_n\right) \quad (t \in \mathbb{R}), \tag{1.3}$$

where

$$S(t) = (t-t_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right)\left(1-\frac{t}{t_{-n}}\right).$$

A generalization of the Paley-Wiener spaces is given by the well-known de Branges spaces, that are Hilbert spaces of entire functions. Given a Hermite Biehler function $E(z)$ (i.e. an entire function such that $|E(z)| > |E(\overline{z})|$ for $z \in \mathbb{C}^+$), the corresponding de Branges space is given by $\mathcal{B}(E) = E\mathcal{K}(\Theta)$, where $\Theta(z) = \frac{\overline{E(\overline{z})}}{E(z)}$ is a meromorphic inner function and $\mathcal{K}(\Theta)$ is the model space $\mathcal{H}^2 \ominus \Theta\mathcal{H}^2$. Here, $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{C}^+)$ denotes the classical Hardy space of the upper half plane. Let $\{t_n\}_n$ be the sequence of real points such that $\Theta(t_n) = 1$ (that are generally not equidistant), then every function $F \in \mathcal{B}(E)$ can be exactly and uniquely rebuilt starting form its values on the points $\{t_n\}_n$ with the following sampling formula:

$$F(z) := \sum_n \frac{K_{\mathcal{B}(E)}\left(t_n, z\right)}{K_{\mathcal{B}(E)}\left(t_n, t_n\right)} F\left(t_n\right),$$

where

$$K_{\mathcal{B}(E)}(w, z) = \frac{E(z)\overline{E(w)} - \overline{E(\overline{z})}E(\overline{w})}{2\pi i(\overline{w} - z)}$$

is the reproducing kernel of $\mathcal{B}(E)$.

In practical applications, the bandlimit a is necessarily the largest frequency that occurs in the set of signals considered. The larger is the value of a, the smaller is the spacing $\frac{\pi}{a}$ needed between every two consecutive samples. Even if a given signal appears to have low frequency for most of its duration, and to have high frequencies only for a short time interval, the samples need to be taken at a high rate for all time in order to apply the Shannon sampling formula. This is obviously inefficient and motivates the extension of signal processing methods such as filtering, sampling and reconstruction to the setting of time-varying bandwidth.

The first and principal problem is to define what exactly is a time-varying bandlimit. The traditional notion of bandlimit is determined by the Fourier transform of the entire signal, hence it is non-local and depends on the signal global behaviour. This makes it difficult to give a precise definition of the concept of a time-varying bandlimit. In the literature there are several approaches to the definition of variable bandlimit, see for example [1], [2], [10], [18], [26], [50].

Among all these definitions, the most interesting is probably the one recently introduced by Kempf and Martin in [40]. The Kempf-Martin spaces are based on a non-Fourier generalized sampling theory and use as mathematical engine the functional analytical theory of selfadjoint extensions of symmetric operators with deficiency indices (1,1) in Hilbert spaces. These spaces have a sampling formula that is analogous to the one of the Paley-Wiener spaces. Furthermore, in the paper by Kempf and Martin these spaces shy away from a formal definition since are defined through their reproducing kernel, and a more in-depth characterization seems to be desirable.

The Kempf-Martin definition of time-varying bandlimit is based on the observation that, in conventional Shannon sampling theory, the constant bandlimit a is inversely proportional to the constant spacing $\frac{\pi}{a}$ of the standard Nyquist sampling sequences. Kempf and Martin then identify the sample points in each of these sampling sequences $\left(t_n(\alpha) = (n+\alpha)\frac{\pi}{a}\right)$ for $\alpha \in [0, 1)$, appearing in the Shannon sampling formula, with the simple eigenvalues of a self-adjoint operator Z_α. They further observe that the family $\{Z_\alpha \mid \alpha \in [0,1)\}$ is the one-parameter family of self-adjoint extensions of a single symmetric linear transformation Z, which is simple, regular, with deficiency indices $(1,1)$, and acts as multiplication by the independent variable on a dense domain in $\mathcal{PW}_a$. One can combine the spectra of these self-adjoint extensions to define a smooth, strictly increasing bijection on the real line, $t(n+\alpha) := t_n(\alpha)$. If γ denotes the compositional inverse of t, we observe that

$$\pi\gamma'(t) = a \tag{1.4}$$

is the bandlimit. The derivative $\gamma'(t)$ is then a measure of the local density of the sampling sequences $\{t_n(\alpha)\}_n$ near the point t, and it is proportional to the constant bandlimit in the case of Shannon sampling.

The crucial observation is that the spectra of the self-adjoint extensions of such a symmetric operator T do not need to be equidistant. Hence it is possible to generalize Shannon sampling theory using the representation theory of regular simple symmetric linear transformations with deficiency indices $(1,1)$. Kempf and Martin show that any such symmetric T is unitarily equivalent to multiplication by the independent variable in a local bandlimit space $\mathcal{KM}(T)$, a Hilbert space of functions on $\mathbb{R}$ with the same special reconstruction properties as the Paley-Wiener spaces $\mathcal{PW}_a$ of a-bandlimited functions.

Kempf and Martin prove that any $F \in \mathcal{KM}(T)$ can be rebuilt from its samples taken on any sampling sequence $\{t_n(\alpha)\}_n$, $\alpha \in [0,1)$, where the $t_n(\alpha)$ are the simple eigenvalues of a self-adjoint extension, T_α, of T. The local density of the sampling sequences $\{t_n(\alpha)\}_n$ then provides a natural notion of time-varying bandlimit that recovers the classical definition in the case where $\mathcal{KM}(T) = \mathcal{PW}_a$.

Since the Kempf-Martin spaces $\mathcal{KM}(T)$ represent a very promising solution and at the same time they still have an insufficient characterization, in this work we decided to study this version of time-varying bandlimit functions.

Another important open problem that is strictly connected with the de Branges spaces and the so-called canonical inverse problem. A canonical system is a differential equation of the form

$$J\frac{dY}{dx} = zH(x)Y, \tag{1.5}$$

where

- $H(x)$ is a function $(0,L) \to \mathrm{Mat}_2(\mathbb{R}), 0 < L \leq \infty$, such that $H(x)$ is positive semidefinite a.e. for $x \in (0,L)$, and that $H \in \mathcal{L}^1(0,L')$ for all $L' < L$.
- $Y = \begin{bmatrix} Y_+ \\ Y_- \end{bmatrix} \in \mathbb{C}^2$;

- $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$;
- $z \in \mathbb{C}$ is a parameter.

In [11] de Branges shows that if $Y(x,z)$ is the solution of (1.5), then $E_x(z) = Y_+(x,z) + iY_-(x,z)$ is a Hermite Biehler function of z for each $x \in (0,L)$. Given any $E(z)$, the problem of building $H(x)$ such that $E_L(z) = E(z)$ is known as the canonical inverse problem, and an algorithm to solve it was proposed by Romanov in [46]. His work can be considered a far-reaching generalization of the Stiltjes algorithm in the inverse spectral theory of Jacobi matrices. Unlike many other one-dimensional inverse spectral theories, it is not perturbative, which means that there is no underlying problem with well-understood eigenfunctions to be compared with. Romanov's algorithm is mainly based on the result of *Theorem* 6 in [46] (p. 21), which proves that for any polynomial Hermite Biehler function $E(z)$ with no real zeros and such that $E(0) = 1$, we have

$$\frac{1}{2}\begin{pmatrix} E(z) + E^{\#}(z) \\ \frac{1}{i}\Big(E(z) - E^{\#}(z)\Big) \end{pmatrix} = M_1(z)\dots M_n(z)\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $n = \deg(E)$ and the M_j's are 2×2 square matrices such that $M_j(z) = I + zR_j, \det R_j = \operatorname{tr} R_j = 0, R_{12} \geq 0, R_{21} \leq 0$.

The problem with this theorem is that it gives an algorithm to build the matrices $\{M_j\}_{j=1,\dots,n}$ without giving their explicit expression in terms of $E(z)$. Since Romanov's algorithm for the solution of the canonical inverse problem is based on an iteration on the degree of $E(z)$ in which this theorem is applied to each step, its downside is that it does not give an explicit solution, and that neither the result of each iteration is explicit. Hence for this reason we think that this problem is still considered not fully solved.

CHAPTER 2

Preliminary definitions and results

In this section we introduce the functions, the spaces and well-known results that will be used in all the next chapters. For more details see [11], [25], [30], [34] and [42].

Before starting with definitions and theorems, we clarify here the meaning of some notations that we will use throughout this work.

- Let $\mathcal{U}_1$,$\mathcal{U}_2$ be two vector subspaces of some vector space $\mathcal{V}$.
  - The sum $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$ is defined to be the set of all possible sums $u_1 + u_2$ with $u_1 \in \mathcal{U}_2$, $u_2 \in \mathcal{U}_2$.
  - The direct sum $\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$ is equal to the sum $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$ in the case $\mathcal{U}_1 \perp \mathcal{U}_2$.
  - $\mathcal{V} \ominus \mathcal{U}_1$ denotes the orthogonal complement of $\mathcal{U}_1$ in $\mathcal{V}$.
- Let $\{\mathcal{U}_n\}_{n\geq 0}$ be an infinite set of vector subspaces of the same vector space $\mathcal{V}$, with $\mathcal{U}_n \perp \mathcal{U}_m$ $\forall m, n \geq 0$, $m \neq n$. Then $\bigoplus_{n>0} \mathcal{U}_n$ denotes the closure of the subspace formed by all the possible sums $\sum_{n\geq 0} u_n$, with $u_n \in \mathcal{U}_n$ and $u_n \neq 0$ only for a finite number of n.
- $\sum_n$ stands for $\sum_{n\in\mathbb{Z}}$ (and obviously $\sum_{n\neq m}$ stands for $\sum_{n\in\mathbb{Z}, n\neq m}$).
- $\{\cdot\}_n$ stands for $\{\cdot\}_{n\in\mathbb{Z}}$.
- $\mathcal{L}^2(\Omega)$ is the space of functions square integrable on Ω.
- $\{x\}$ denotes the fractional part of the real value x: $\{x\} = x - \lfloor x \rfloor$.

2.1 Reproducing kernel Hilbert spaces

Definition 2.1. A *reproducing kernel Hilbert space* (RKHS) $\mathcal{H}$ on a subset $\Omega \subset \mathbb{C}$ is a Hilbert space of functions on Ω with the property that point evaluation at any $z \in \Omega$ defines a bounded linear functional δ_z on $\mathcal{H}$.

By the Riesz representation lemma, for any $z \in \Omega$ there is a unique point evaluation vector $K_z \in \mathcal{H}$ so that for any $F \in \mathcal{H}$ we have

$$F(z) = \delta_z(F) = \langle F, K_z \rangle_{\mathcal{H}} .$$

The reproducing kernel of $\mathcal{H}$ is the symmetric function $K(z,w) : \Omega \times \Omega \to \mathbb{C}$ is defined by

$$K_{\mathcal{H}}(w,z) := \langle K_w, K_z \rangle_{\mathcal{H}} , \tag{2.1}$$

and is a positive definite funcion, which means that for any $n \in \mathbb{N}, z_1, \ldots, z_n \in \Omega$, and $c_1, \ldots, c_n \in \mathbb{R}$ we have

$$\sum_{i,j=1}^{n} c_i c_j K(z_i, z_j) = \left\langle \sum_{i=1}^{n} c_i K_{z_i}, \sum_{j=1}^{n} c_j K_{z_j} \right\rangle_{\mathcal{H}} \geq 0.$$

The classical theory of RKHS of Aronszajn and Moore (see [44]) shows that there is a bijective correspondence between the positive kernel functions $K(z,w)$ on $\Omega \times \Omega$ and the reproducing kernel Hilbert spaces $\mathcal{H}$ on Ω. Indeed, given any positive kernel function $K(z,w)$ there always exsists a RKHS $\mathcal{H}(K)$ which has $K(z,w)$ as its reproducing kernel, while for every RKHS $\mathcal{H}(K)$ the reproducing kernel$K_{\mathcal{H}}(z,w)$ is unique.

2.2 Hardy spaces and meromorphic inner functions

Definition 2.2. For $p > 0$, the *Hardy space* $\mathcal{H}^p(\mathbb{C}^+)$ on the upper half-plane $\mathbb{C}^+$ is defined as the space of holomorphic functions $F(z)$ on $\mathbb{C}^+$ such that

$$\|F\|_{\mathcal{H}^p} := \sup_{y>0} \left(\int_{-\infty}^{+\infty} |F(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty.$$

The *Hardy space* $\mathcal{H}^\infty(\mathbb{C}^+)$ is defined as the space of holomorphic functions $F(z)$ on $\mathbb{C}^+$ such that

$$\|F\|_{\mathcal{H}^\infty} := \sup_{z \in \mathbb{C}^+} |F(z)| < \infty.$$

Definition 2.3. A *Blaschke product* $B \in \mathcal{H}^\infty(\mathbb{C}^+)$ is a product of the form

$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{z}_k}{z_k} \frac{z - z_k}{z - \overline{z}_k}, \tag{2.2}$$

where the zeros $\{z_k\}_{k \geq 1}$ obey the Blaschke condition

$$\sum_{k \geq 1} \frac{\Im(z_k)}{|z_k|^2} < \infty.$$

Definition 2.4. A *meromorphic inner function* on the upper half plane is a meromorphic function $\Theta : \mathbb{C} \to \mathbb{C}$ which is holomorphic in the upper half plane and such that $|\Theta(z)| < 1$ for $z \in \mathbb{C}^+$, $|\Theta(x)| = 1$ for $x \in \mathbb{R}$.

Every meromorphic inner function $\Theta(z)$ obviously belongs to $\mathcal{H}^\infty(\mathbb{C}^+)$, and can be factored uniquely as

$$\Theta(z) = \gamma e^{ibz} \prod_{k=1}^{\infty} \frac{\overline{z_k}}{z_k} \frac{z - z_k}{z - \overline{z}_k} = \gamma e^{ibz} B(z), \tag{2.3}$$

where $b \in \mathbb{R}$, $b \geq 0$, $\gamma \in \mathbb{C}$, $|\gamma| = 1$, and $B(z) = \prod_{k=1}^{\infty} \frac{\overline{z_k}}{z_k} \frac{z-z_k}{z-\overline{z}_k}$ is a Blaschke product with no accumulation point. The value of b in (2.3) is referred as the *logarithmic residue* of $\Theta(z)$.

Definition 2.5. The *phase function* $\tau : \mathbb{R} \to \mathbb{R}$ of a meromorphic inner function $\Theta(z)$ is the unique differentiable function such that $\Theta(t) = e^{2\pi i \tau(t)}$ for $t \in \mathbb{R}$, with $\tau'(t) > 0$ $\forall t \in \mathbb{R}$.

For our purpose the case of an inner function with logarithmic residue $b = 0$ and a finite number of zeros in its Blaschke product is a degenerate and not interesting case. Hence in the next chapters, when we will define a meromorphic inner function, we will always tacitly assume that either $b > 0$ or the number of zeros of the Blaskhe product is infinite. With this assumption, it is easy to see that the image of the phase function $\tau(t)$, as t varies in $\mathbb{R}$, is the whole real line. In the next chapters we will also consider many times the sequence $\{t_n\}_n$ of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$, which obviously is the sequence of real points where $\tau(t)$ assumes integer values. Therefore with our assumptions this sequence is infinite, and we will always consider its indexes so that $\tau(t_n) = n$.

Definition 2.6. The *spectral function* $t : \mathbb{R} \to \mathbb{R}$, of a non-constant meromorphic inner function $\Theta(z)$ is defined as the inverse of the phase function: $t(x) = \tau^{-1}(x)$ $\forall x \in \mathbb{R}$.

2.3 Herglotz functions

Definition 2.7. A function $F : \mathbb{C}^+ \to \mathbb{C}$ is called a *Herglotz function* (or Nevanlinna-Herglotz function) if $F(z)$ is analytic on $\mathbb{C}^+$ and such that $\Im(F(z)) \geq 0$ for all $z \in \mathbb{C}^+$.

For every Herglotz function $F(z)$ the following representation holds:

$$F(z) = c + dz + \int_{\mathbb{R}} \left(\frac{1}{w - z} - \frac{w}{1 + w^2} \right) d\mu(w), \quad z \in \mathbb{C}^+, \tag{2.4}$$

where $c \in \mathbb{R}$, $d \in \mathbb{R}^+$ and $\mu(w)$ is a positive regular Borel measure obeying the Herglotz condition

$$\int_{\mathbb{R}} \frac{d\mu(w)}{1 + w^2} < \infty.$$

In particular we have

$$
\begin{aligned}
c &= \mathrm{Re}(F(i)),\\
d &= \lim_{y\to\infty} \frac{F(iy)}{iy} \geq 0.
\end{aligned} \tag{2.5}
$$

This representation is referred as the Nevanlinna representation of a Herglotz function.

There exists a bijective correspondence between the Herglotz functions and the set of all contractive, analytic functions in $\mathbb{C}^+$. Given a Herglotz function $F(z)$, the corresponding contractive analytic function is given by

$$
m(z) = \frac{F(z)-i}{F(z)+i} = \frac{-iF(z)-1}{-iF(z)+1}, \tag{2.6}
$$

while given a contractive analytic function $m(z)$ then the correspondent Herglotz function is given by

$$
F(z) = i\left(\frac{1+m(z)}{1-m(z)}\right). \tag{2.7}
$$

The function $m(z)$ is inner if and only if the positive Borel measure μ of $F(z)$ is singular with respect to the Lebesgue measure. Moreover, $m(z)$ is a meromorphic inner function if and only if μ is a purely discrete measure. A purely discrete measure is given by

$$
\mu := \sum_n w_n \delta_{t_n}, \tag{2.8}
$$

where

- $\{w_n\}_n$ is a sequence of strictly positive weights;
- $\{t_n\}_n$ is a purely discrete, strictly increasing sequence with no finite accumulation point;
- $\{\delta_{t_n}\}_n$ is a sequence of Dirac delta masses:

$$
\delta_{t_n}(x) = \begin{cases} 1 & x = t_n; \\ 0 & x \neq t_n. \end{cases}
$$

If μ is a purely discrete measure, the representation (2.4) becomes

$$
F(z) = c + dz + \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) w_n, \quad z \in \mathbb{C}^+, \tag{2.9}
$$

and the Herglotz condition is given by

$$
\sum_n \frac{w_n}{1+t_n^2} < \infty.
$$

2.4 Functions of bounded type

Definition 2.8. A function $F : \Omega \to \mathbb{C}$, which is analytic in a region $\Omega \subseteq \mathbb{C}$, is said to be of *bounded type* in Ω if $F(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are analytic and bounded in Ω and $Q(z)$ is not identically zero.

The following theorem is an important known result about functions of bounded type (see *Theorem* 9 in [11], p. 22).

Theorem 2.9 (de Branges)**.** *Let $F(z)$ be a function which is analytic in the upper half plane and which does not have the origin as a limit point of zeros. A necessary and sufficient condition for $F(z)$ to be of bounded type in the half-plane is that*

$$F(z) = B(z)e^{-ihz}e^{G(z)} \tag{2.10}$$

where $B(z)$ is a Blaschke product, h is a real number, and $G(z)$ is a function analytic in the upper half-plane such that

$$\Re(G(x+iy)) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

for some real valued function $\mu(x)$ such that

$$\int_{-\infty}^{+\infty} \frac{|d\mu(t)|}{1+t^2} < \infty.$$

The real number h in the representation of F given in equation (2.10) is referred as the *mean type* of F. Thanks to *Theorem* 10 in [11], we have that the mean type h of a a bounded type function F is given by

$$h = \limsup_{y\to+\infty} y^{-1}\log|F(iy)|. \tag{2.11}$$

We observe that any analytic function $G(z)$ bounded in a region $\Omega \subseteq \mathbb{C}$ is of bounded type there since it can be written as $G(z) = \frac{P(z)}{Q(z)}$ with $P(z) = G(z)$ and $Q(z) = 1$. Considering that in the upper half plane any meromorphic inner function $\Theta(z)$ is bounded, we obtain that it is also a function of bounded type, and hence it must have the form given in (2.10). Since a meromorphic inner function must also have the form given in (2.3), we easily get that the mean type h in (2.10) is equal to $-b$ in (2.3), and then by (2.11) we obtain that the logarithmic residue of $\Theta(z)$ is given by

$$b = -\limsup_{y\to+\infty} y^{-1}\log|\Theta(iy)|. \tag{2.12}$$

It is easy to see that the reciprocal of a function of bounded type on a region Ω is of bounded type on Ω, and that the product of two functions of bounded type on a region Ω is of bounded type on Ω. We give now some well-known examples of functions of bounded type.

- Every polynomial is of bounded type in any bounded region of $\mathbb{C}$. Every polynomial is also of bounded type on $\mathbb{C}^+$, since any polynomial $F(z)$ of degree n can be expressed as $F(z) = \frac{P(z)}{Q(z)}$ with

$$P(z) = \frac{F(z)}{(z+i)^n},$$
$$Q(z) = \frac{1}{(z+i)^n},$$

  and both $P(z)$ and $Q(z)$ are bounded on $\mathbb{C}^+$. Also the reciprocal of every polynomial is of bounded type in every bounded region of $\mathbb{C}$, and on $\mathbb{C}^+$.

- The functions $\sin(z)$ and $\cos(z)$ are of bounded type on $\mathbb{C}^+$. Indeed, for example, we have $\sin(z) = \frac{P(z)}{Q(z)}$ with

$$P(z) = \sin(z)e^{iz},$$
$$Q(z) = e^{iz},$$

  and both $P(z)$ and $Q(z)$ are bounded on $\mathbb{C}^+$.

- Every Herglotz function $F(z)$ is of bounded type on $\mathbb{C}^+$ since can be written as $F(z) = \frac{P(z)}{Q(z)}$ with

$$P(z) = \frac{F(z)}{F(z)+i},$$
$$Q(z) = \frac{1}{F(z)+i},$$

  and both $P(z)$ and $Q(z)$ are bounded on $\mathbb{C}^+$.

2.5 Hermite Biehler functions

Definition 2.10. A *Hermite Biehler function* $E : \mathbb{C} \to \mathbb{C}$ is an entire function such that $|E(z)| > |E^{\#}(z)|$ for every $z \in \mathbb{C}^+$, where $E^{\#}(z) = \overline{E(\overline{z})}$.

An important and well-known result about these functions is the following theorem (see [25]).

Theorem 2.11. *Any meromorphic inner function*

$$\Theta(z) = \gamma e^{ibz} \prod_{k=1}^{\infty} \frac{\overline{z_k}}{z_k} \frac{z - z_k}{z - \overline{z}_k} = \gamma e^{ibz} B(z)$$

can be represented as

$$\Theta(z) = \frac{E^{\#}(z)}{E(z)}, \quad z \in \mathbb{C}$$

where $E(z)$ is a Hermite Biehler function, given by

$$E(z) = e^{-i\frac{b}{2}z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\overline{z}_k}\right) \exp\left\{\sum_{n=1}^{k} \frac{1}{n} \Re\left(\frac{1}{\overline{z}_k^n}\right) z^n\right\}.$$

Definition 2.12. Given any meromorphic inner function $\Theta(z)$, we define as a *de Branges function* of $\Theta(z)$ every Hermite Biehler funtion $E(z)$ such that $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$.

2.6 Model spaces

If $F_1 \in \mathcal{H}^\infty$ and $F_2 \in \mathcal{H}^2$, we observe that $G := F_1 F_2 \in \mathcal{H}^2$. Indeed we have

$$\begin{aligned}
\|G\|_{\mathcal{H}^2} &= \sup_{y>0} \left(\int_{-\infty}^{+\infty} |F_1(x+iy)|^2 |F_2(x+iy)|^2 dx\right)^{\frac{1}{2}} \\
&\leq \sup_{y>0} \left(\sup_{z \in \mathbb{C}^+} |F_1(z)|\right) \left(\int_{-\infty}^{+\infty} |F_2(x+iy)|^2 dx\right)^{\frac{1}{2}} \\
&= \|F_1\|_{\mathcal{H}^\infty} \|F_2\|_{\mathcal{H}^2} \\
&< \infty.
\end{aligned}$$

Then, given a meromorphic inner function $\Theta(z)$, it is easy to see that $\Theta\mathcal{H}^2$ is a subspace of $\mathcal{H}^2$.

Definition 2.13. Given a meromorphic inner function $\Theta(z)$, the *model space* $\mathcal{K}(\Theta)$ on the upper-half plane is defined as

$$\mathcal{K}(\Theta) := \mathcal{H}^2 \ominus \Theta\mathcal{H}^2.$$

Any model space $\mathcal{K}(\Theta)$ is a reproducing kernel Hilbert space of analytic functions on $\mathbb{C}^+$ with reproducing kernel

$$K_{\mathcal{K}(\Theta)}(z, w) := \frac{i}{2\pi} \frac{1 - \Theta(z)\overline{\Theta(w)}}{z - \bar{w}}, \quad z, w \in \mathbb{C}^+.$$

Let $\Theta_1(z)$ and $\Theta_2(z)$ be two meromorphic inner functions, and set $\Theta(z) = \Theta_1(z)\Theta_2(z)$. Then, for the model space $\mathcal{K}(\Theta) = \mathcal{H}^2 \ominus \Theta\mathcal{H}^2$ the following direct sum decomposition is true:

$$\mathcal{K}(\Theta) = \left(\mathcal{H}^2 \ominus \Theta_0\mathcal{H}^2\right) \oplus \Theta_0 \left(\mathcal{H}^2 \ominus \Theta_1\mathcal{H}^2\right) = \mathcal{K}(\Theta_0) \oplus \Theta_0 \mathcal{K}(\Theta_1). \quad (2.13)$$

Definition 2.14. Let $\Theta_1(z)$ and $\Theta_2(z)$ be two meromorphic inner functions. We say that $\Theta_1(z)$ *divides* $\Theta_2(z)$ if $\frac{\Theta_2(z)}{\Theta_1(z)}$ is again a meromorphic inner function. We define as *least common multiple* of $\Theta_1(z)$ and $\Theta_2(z)$ a meromorphic inner function $\Phi(z)$ that is divided by $\Theta_1(z)$ and $\Theta_2(z)$, and that divide any other meromorphic inner function divided by both of them. In this case we write $\Phi = \mathrm{LCM}(\Theta_1, \Theta_2)$.

The following facts are true:

$$\begin{aligned} &\Theta_1(z) \text{ divides } \Theta_2(z) \text{ if and only if } \Theta_1\mathcal{H}^2 \supseteq \Theta_2\mathcal{H}^2; \\ &\Theta_1\mathcal{H}^2 \cap \Theta_2\mathcal{H}^2 = \Phi\mathcal{H}^2, \text{ where } \Phi = \mathrm{LCM}(\Theta_1, \Theta_2). \end{aligned} \tag{2.14}$$

2.7 De Branges spaces

Definition 2.15. Given a Hermite Biehler function $E(z)$, the *de Branges space* $\mathcal{B}(E)$ is defined as the set of all entire functions $F(z)$ such that

$$\|F\|^2_{\mathcal{B}(E)} = \int_{-\infty}^{+\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt < \infty$$

and such that both ratios $\frac{F(z)}{E(z)}$ and $\frac{F^{\#}(z)}{E(z)}$ are of bounded type and of non-positive mean type in the upper half-plane (see [11], p. 50).

The space $\mathcal{B}(E)$ is a vector space over the complex numbers, with scalar product defined by

$$\langle F, G\rangle_{\mathcal{B}(E)} = \int_{-\infty}^{+\infty} \frac{F(t)\overline{G(t)}}{|E(t)|^2} dt.$$

Thanks to *Theorem* 19 in [11] (p. 50), we know that the space $\mathcal{B}(E)$ is a reproducing kernel Hilbert space with reproducing kernel $K_{\mathcal{B}(E)}(w, z)$ given by

$$K_{\mathcal{B}(E)}(w, z) = \frac{E(z)E^{\#}(\overline{w}) - E^{\#}(z)E(\overline{w})}{2\pi i(\overline{w} - z)}. \tag{2.15}$$

The reproducing kernel can be written also as

$$\begin{aligned} K_{\mathcal{B}(E)}(w, z) &= \frac{B(z)\overline{A(w)} - A(z)\overline{B(w)}}{\pi(z - \overline{w})} \\ &= \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z - \overline{w})}, \end{aligned}$$

where we set

$$A(z) = \frac{1}{2}(E(z) + E^{\#}(z)), \quad B(z) = \frac{i}{2}(E(z) - E^{\#}(z)).$$

Notice that

$$\overline{A(z)} = A(\overline{z}) \quad \text{and} \quad \overline{B(z)} = B(\overline{z}).$$

If we let $w \to \overline{z}$, we obtain

$$\begin{aligned} K_{\mathcal{B}(E)}(\overline{z}, z) &= \lim_{w \to \overline{z}} \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z - \overline{w})} \\ &= \lim_{w \to \overline{z}} \frac{B(z)A(\overline{w}) - B(z)A(z) + B(z)A(z) - A(z)B(\overline{w})}{\pi(z - \overline{w})} \\ &= \frac{-B(z)A'(z) + A(z)B'(z)}{\pi}. \end{aligned}$$

It is easy to check that

$$-B(z)A'(z)+A(z)B'(z)=-\frac{i}{2}\left(E^{\#'}(z)E(z)-E^{\#}(z)E'(z)\right),$$

so that

$$K_{\mathcal{B}(E)}(\overline{z},z)=\frac{E^{\#'}(z)E(z)-E^{\#}(z)E'(z)}{2\pi i}. \tag{2.16}$$

Two important results about the de Branges spaces are *Theorem* 20 and *Theorem* 22 in [11] (p. 53, 55). We report here the statements of these theorems, for sake of completeness and in order to express them clearly according to the notations used in this work.

Theorem 2.16 (de Branges)**.** *A necessary and sufficient condition for an entire function $F(z)$ to belong to a de Branges space $\mathcal{B}(E)$ is that*

$$\|F(t)\|^2=\int_{-\infty}^{+\infty}\left|\frac{F(t)}{E(t)}\right|^2 dt<\infty$$

and that $|F(z)|^2\leq\|F(t)\|^2K(z,z)$ for all $z\in\mathbb{C}$.

Theorem 2.17 (de Branges)**.** *Let $\mathcal{B}(E)$ be a de Branges space and let $\tau(t)$ be the phase function of $\Theta(z)=\frac{E^{\#}(z)}{E(z)}$. For $\theta\in[0,1)$, let $\{t_n(\theta)\}_n$ be the sequence of solutions of $\tau(t)=\theta \mod 1$. Then the sequence $\left\{\frac{K_{\mathcal{B}(E)}(t_n(\theta),z)}{E(t_n(\theta))}\right\}_n$ is an orthogonal set in $\mathcal{B}(E)$.*

The next theorem introduce the well-known sampling formula of the de Branges space.

Theorem 2.18 (de Branges)**.** *Let $\mathcal{B}(E)$ be a de Branges space, and let $\Theta(z)=\frac{E^{\#}(z)}{E(z)}$ be its corresponding meromorphic inner function. Consider the sequence $\{t_n\}_n$ of solutions of $\Theta(t)=1$ for $t\in\mathbb{R}$. Then for every $F\in\mathcal{B}(E)$ the following sampling formula is verified:*

$$F(z)=\sum_n\frac{K_{\mathcal{B}(E)}\left(t_n,z\right)}{K_{\mathcal{B}(E)}\left(t_n,t_n\right)}F\left(t_n\right). \tag{2.17}$$

The series converges in norm of $\mathcal{B}(E)$, and converges uniformly on the compact subsets of $\mathbb{C}$.

The reproducing kernel $K_{\mathcal{B}(E)}$ is given by (2.15), and hence we get

$$K_{\mathcal{B}(E)}(t_n,z)=\frac{E(z)E^{\#}(t_n)-E^{\#}(z)E(t_n)}{2\pi i(t_n-z)}.$$

Since $\Theta(t_n)=\frac{E^{\#}(t_n)}{E(t_n)}=\frac{\overline{E(t_n)}}{E(t_n)}=1$, we obtain $E(t_n)\in\mathbb{R}$ and $E(t_n)=E^{\#}(t_n)$. Therefore

$$K_{\mathcal{B}(E)}(t_n,z)=\frac{E(t_n)\left(E(z)-E^{\#}(z)\right)}{2\pi i(t_n-z)}. \tag{2.18}$$

From (2.16) we have

$$K_{\mathcal{B}(E)}(\overline{u},u) = \frac{E^{\#'}(u)E(u) - E^{\#}(u)E'(u)}{2\pi i}, \tag{2.19}$$

and, using $E(t_n) \in \mathbb{R}$, we get

$$K_{\mathcal{B}(E)}(t_n,t_n) = E(t_n)\frac{E^{\#'}(t_n) - E'(t_n)}{2\pi i}. \tag{2.20}$$

We observe that

$$\Theta'(z) = \frac{E^{\#'}(z)E(z) - E^{\#}(z)E'(z)}{E(z)^2}. \tag{2.21}$$

Since $E(t_n) \in \mathbb{R}$ we have $E(t_n) = E^{\#}(t_n)$, and then

$$\Theta'(t_n) = \frac{E^{\#'}(t_n) - E'(t_n)}{E(t_n)}.$$

Hence (2.20) becomes

$$K_{\mathcal{B}(E)}(t_n,t_n) = \frac{E(t_n)^2\Theta'(t_n)}{2\pi i}. \tag{2.22}$$

Now, using (2.18) and (2.22), we obtain that (2.17) can be rewritten as

$$\begin{aligned} F(z) &= \sum_n \frac{K_{\mathcal{B}(E)}\left(t_n,z\right)}{K_{\mathcal{B}(E)}\left(t_n,t_n\right)} F\left(t_n\right) \\ &= \sum_n \frac{E(z)(1-\Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} F(t_n). \end{aligned} \tag{2.23}$$

Another important well-known result about de Branges spaces is the following.

Theorem 2.19. *Let $E(z)$ be a Hermite Biehler function and let $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$ be its corresponding meromorphic inner function. Then we have that*

$$\mathcal{B}(E) = E\mathcal{K}(\Theta)$$

where $\mathcal{K}(\Theta) = \mathcal{H}^2 \ominus \Theta\mathcal{H}^2$ is the model space correspondent to $\Theta(z)$.

2.8 Paley-Wiener spaces

Definition 2.20. Given $a > 0$, the *Paley-Wiener space* with parameter a, referred as $\mathcal{PW}_a$, is the set of all the entire functions $F(z)$ square integrable on $\mathbb{R}$ and such that $|F(z)| \leq Ce^{a|z|}$ for some constant C.

The space $\mathcal{PW}_a$ is a Hilbert space with scalar product given by

$$\langle F, G\rangle_{\mathcal{PW}_a} = \int_{-\infty}^{\infty} F(x)\overline{G(x)}dx,$$

and then with norm

$$\|F\|_{\mathcal{PW}_a} = \|F\|_{\mathcal{L}^2(\mathbb{R})}.$$

The Paley-Wiener theorem shows that $\mathcal{PW}_a$ is a separable Hilbert space, and the Fourier transform induces a unitary (up to a rescaling factor $\frac{1}{2\pi}$) isomorphism from $\mathcal{PW}_a$ onto $\mathcal{L}^2[-a, a]$. Indeed, for $F_1, F_2 \in \mathcal{PW}_a$ and $G_1 = \mathcal{F}(F_1), G_2 = \mathcal{F}(F_2) \in \mathcal{L}^2[-a, a]$ we have

$$\langle F_1, F_2\rangle_{\mathcal{PW}_a} = \frac{1}{2\pi}\langle G_1, G_2\rangle_{\mathcal{L}^2[-a,a]}. \tag{2.24}$$

Thanks to this isomorphism we can represent $\mathcal{PW}_a$ also in the following way:

$$\mathcal{PW}_a = \left\{F(z) :\ F(z) = \mathcal{F}^{-1}(G) = \frac{1}{2\pi}\int_{-a}^{a} G(y)e^{izy}\, dy\,, \qquad G \in \mathcal{L}^2[-a, a]\right\}$$

where $\mathcal{F}$ is the Fourier transform. The space $\mathcal{PW}_a$ is a reproducing kernel Hilbert space, with reproducing kernel given by

$$K_{\mathcal{PW}_a}(w, z) = \frac{a}{\pi}\,\mathrm{sinc}(a(w - \overline{z})) = \frac{\sin(a(w - \overline{z}))}{\pi(w - \overline{z})},$$

such that

$$F(z) = \langle F(w), K_z(w)\rangle_{\mathcal{PW}_a} \quad \forall z, w \in \mathbb{C}.$$

For every $F \in \mathcal{PW}_a$ the following sampling formula is verified:

$$F(z) = \sum_n F\left(n\frac{\pi}{a}\right)\mathrm{sinc}\left(a\left(z - n\frac{\pi}{a}\right)\right), \tag{2.25}$$

and the set

$$\left\{\mathrm{sinc}\left(a\left(z - n\frac{\pi}{a}\right)\right)\right\}_n$$

is an orthonormal basis of the space $\mathcal{PW}_a$. Moreover, for every $F \in \mathcal{PW}_a$ we have

$$\|F\|^2_{\mathcal{PW}_a} = \sum_n |F(n)|^2.$$

A very important and well-known aspect of the Paley-Wiener space $\mathcal{PW}_a$ is that it coincides with the de Branges space $\mathcal{B}(E)$ associated to the function $E(z) = e^{-iaz}$.

CHAPTER 3

Sampling formulas for the de Branges spaces

3.1 Sampling formulas

In this section we introduce some new sampling formulas for the de Branges spaces, different from the classical one given in (2.17). These formulas are mainly derived from some inclusion properties for the de Branges spaces, and are the basis on which we will build all the non-uniform sampling formulas for the Paley-Wiener spaces in the next chapter.

Theorem 3.1. *Let $E_0(z), E_1(z)$ be Hermite Biehler functions, and let $E_2(z) = E_0(z)E_1(z)$. Recalling that $E_2(z)$ is also a Hermite Biehler function, we have that $\mathcal{B}(E_0)E_1$ is a closed subspace of $\mathcal{B}(E_2)$.*

Proof. We give two different proofs of this theorem.

Proof 1. To show that $\mathcal{B}(E_0)E_1$ is a close subspace of $\mathcal{B}(E_2)$ it is sufficient to prove that for every $G \in \mathcal{B}(E_0)$ the entire function $F(z) = G(z)E_1(z)$ is such that $F \in \mathcal{B}(E_2)$ with equality of norms. We have

$$\begin{aligned}\int_{-\infty}^{+\infty} \left|\frac{F(t)}{E_2(t)}\right|^2 dt &= \int_{-\infty}^{+\infty} \left|\frac{F(t)}{E_0(t)E_1(t)}\right|^2 dt \\ &= \int_{-\infty}^{+\infty} \left|\frac{G(t)}{E_0(t)}\right|^2 dt \qquad (3.1) \\ &= \|G\|^2_{\mathcal{B}(E_0)} \\ &< \infty.\end{aligned}$$

Let $K_{\mathcal{B}(E_0)}(z,w)$ be the reproducing kernel of $\mathcal{B}(E_0)$. Thanks to (2.15), for $z = x+iy$, with $y \neq 0$, we have that

$$K_{\mathcal{B}(E_0)}(z,z) = \frac{|E_0(z)|^2 - |E_0(\overline{z})|^2}{4\pi y}. \tag{3.2}$$

Using *Theorem* 2.16, for every $G \in \mathcal{B}(E_0)$ we have also

$$\|G\|^2_{\mathcal{B}(E_0)} \geq \frac{|G(z)|^2}{K_{\mathcal{B}(E_0)}(z,z)}.$$

Then,

$$\begin{aligned}
K_{\mathcal{B}(E_2)}(z,z)\|F\|^2_{\mathcal{B}(E_2)} &= K_{\mathcal{B}(E_2)}(z,z)\|G\|^2_{\mathcal{B}(E_0)} \\
&\geq \frac{K_{\mathcal{B}(E_2)}(z,z)}{K_{\mathcal{B}(E_0)}(z,z)}|G(z)|^2 \\
&= \frac{|E_2(z)|^2 - |E_2(\overline{z})|^2}{|E_0(z)|^2 - |E_0(\overline{z})|^2}\frac{|F(z)|^2}{|E_1(z)|^2} \\
&= \frac{|E_0(z)|^2 - \frac{|E_1(\overline{z})|^2}{|E_1(z)|^2}|E_0(\overline{z})|^2}{|E_0(z)|^2 - |E_0(\overline{z})|^2}|F(z)|^2.
\end{aligned}$$

Then, for $z = x+iy$ and $y > 0$, we observe that $\frac{|E_1(\overline{z})|}{|E_1(z)|} < 1$ and $|E_0(z)|^2 - |E_0(\overline{z})|^2 > 0$, and hence we get

$$\begin{aligned}
K_{\mathcal{B}(E_2)}(z,z)\|F\|^2_{\mathcal{B}(E_2)} &\geq \frac{|E_0(z)|^2 - \frac{|E_1(\overline{z})|^2}{|E_1(z)|^2}|E_0(\overline{z})|^2}{|E_0(z)|^2 - |E_0(\overline{z})|^2}|F(z)|^2 \\
&\geq \frac{|E_0(z)|^2 - |E_0(\overline{z})|^2}{|E_0(z)|^2 - |E_0(\overline{z})|^2}|F(z)|^2 \\
&= |F(z)|^2.
\end{aligned}$$

For $y < 0$ we observe that $\frac{|E_1(\overline{z})|}{|E_1(z)|} > 1$ and $|E_0(z)|^2 - |E_0(\overline{z})|^2 < 0$, and then we finally obtain

$$\begin{aligned}
K_{\mathcal{B}(E_2)}(z,z)\|F\|^2_{\mathcal{B}(E_2)} &\geq \frac{|E_0(z)|^2 - \frac{|E_1(\overline{z})|^2}{|E_1(z)|^2}|E_0(\overline{z})|^2}{|E_0(z)|^2 - |E_0(\overline{z})|^2}|F(z)|^2 \\
&\geq \frac{|E_0(z)|^2 - |E_0(\overline{z})|^2}{|E_0(z)|^2 - |E_0(\overline{z})|^2}|F(z)|^2 \\
&= |F(z)|^2.
\end{aligned}$$

Thanks to continuity of $K_{\mathcal{B}(E_2)}(z,z)$ and $F(z)$ we have that $K_{\mathcal{B}(E_2)}(z,z)\|F\|^2_{\mathcal{B}(E_2)} \geq |F(z)|^2$ is true also for $y = 0$, and then

$$K_{\mathcal{B}(E_2)}(z,z)\|F\|^2_{\mathcal{B}(E_2)} \geq |F(z)|^2 \quad \forall z \in \mathbb{C}. \tag{3.3}$$

Thanks to *Theorem* 2.16, (3.1) and (3.3), we obtain $F \in \mathcal{B}(E_2)$ and $\|F\|_{\mathcal{B}(E_2)} = \|G\|_{\mathcal{B}(E_0)}$, and then $\mathcal{B}(E)E_1$ is a closed subspace of $\mathcal{B}(E_2)$.

Proof 2. For $i = 0, 1, 2$ we set $\Theta_i(z) = \frac{E_i^{\#}(z)}{E_i(z)}$, and we observe that $\Theta_2(z) = \Theta_0(z)\Theta_1(z)$. Thanks to *Theorem* 2.19 we have that the de Branges spaces $\mathcal{B}(E_0), \mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are given by

$$\begin{aligned}\mathcal{B}(E_0) &= E_0\mathcal{K}(\Theta),\\ \mathcal{B}(E_1) &= E_1\mathcal{K}(\Theta_1),\\ \mathcal{B}(E_2) &= E_2\mathcal{K}(\Theta_2),\end{aligned}$$

where $\mathcal{K}(\Theta_i)$ is the model space $\mathcal{H}^2 \ominus \Theta_i\mathcal{H}^2$, $i = 0, 1, 2$. Thanks to (2.14) we have $\Theta_0\Theta_1\mathcal{H}^2 \subseteq \Theta_0\mathcal{H}^2$, and then we get

$$\mathcal{K}(\Theta_0) = \mathcal{H}^2 \ominus \Theta_0\mathcal{H}^2 \subseteq \mathcal{H}^2 \ominus \Theta_0\Theta_1\mathcal{H}^2 = \mathcal{K}(\Theta_0\Theta_1) = \mathcal{K}(\Theta_2).$$

Hence we finally obtain

$$\mathcal{B}(E_0)E_1 = E_0E_1\mathcal{K}(\Theta_0) = E_2\mathcal{K}(\Theta_0) \subseteq E_2\mathcal{K}(\Theta_2) = \mathcal{B}(E_2).$$

The equality of norm can be derived as in *Proof* 1 by (3.1), and then we can conclude that $\mathcal{B}(E)E_1$ is a closed subspace of $\mathcal{B}(E_2)$. □

Theorem 3.2. *Let $\Theta_0(z)$ and $\Theta_1(z)$ be two meromorphic inner functions such that $\Theta_2(z) := LCM(\Theta_0, \Theta_1) = \Theta_0(z)\Theta_1(z)$. Let $E_0(z)$, $E_1(z)$ be respectively de Branges functions of $\Theta_0(z)$, $\Theta_1(z)$, and let $E_2(z) = E_0(z)E_1(z)$. Then*

$$\mathcal{B}(E_0)E_1 + \mathcal{B}(E_1)E_0 = \mathcal{B}(E_2).$$

Proof. It is easy to see that $E_2(z)$ is a de Branges function of $\Theta(z)$, infact we have

$$\frac{E_2^{\#}(z)}{E_2(z)} = \frac{E_0^{\#}(z)}{E_0(z)}\frac{E_1^{\#}(z)}{E_1(z)} = \Theta_0(z)\Theta_1(z) = \Theta_2(z).$$

Thanks to *Theorem* 2.19 we have that the de Branges spaces $\mathcal{B}(E_0), \mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ are given by

$$\begin{aligned}\mathcal{B}(E_0) &= E_0\mathcal{K}(\Theta),\\ \mathcal{B}(E_1) &= E_1\mathcal{K}(\Theta_1),\\ \mathcal{B}(E_2) &= E_2\mathcal{K}(\Theta_2),\end{aligned}$$

where $\mathcal{K}(\Theta_i)$ is the model space $\mathcal{H}^2 \ominus \Theta_i\mathcal{H}^2$, $i = 0, 1, 2$. Thanks to (2.14) we have $\Theta_0\mathcal{H}^2 \cap \Theta_1\mathcal{H}^2 = \Theta_2\mathcal{H}^2$, and then we get

$$\begin{aligned}\mathcal{K}(\Theta_2) &= \mathcal{H}^2 \ominus \Theta_2\mathcal{H}^2\\ &= \mathcal{H}^2 \ominus (\Theta_0\mathcal{H}^2 \cap \Theta_1\mathcal{H}^2).\end{aligned} \tag{3.4}$$

Given two subspaces $\mathcal{U}_0$ and $\mathcal{U}_1$ of the same vector space $\mathcal{U}$, it is well-known that

$$\mathcal{U} \ominus (\mathcal{U}_0 + \mathcal{U}_1) = (\mathcal{U} \ominus \mathcal{U}_0) \cap (\mathcal{U} \ominus \mathcal{U}_1). \tag{3.5}$$

Then we have

$$\mathcal{H}^2 \ominus (\mathcal{K}(\Theta_0) + \mathcal{K}(\Theta_1)) = \Theta_0\mathcal{H}^2 \cap \Theta_1\mathcal{H}^2,$$

and hence

$$\mathcal{K}(\Theta_0) + \mathcal{K}(\Theta_1) = \mathcal{H}^2 \ominus (\Theta_0\mathcal{H}^2 \cap \Theta_1\mathcal{H}^2). \tag{3.6}$$

Therefore, by (3.4) and (3.6) we get

$$\mathcal{K}(\Theta_2) = \mathcal{K}(\Theta_0) + \mathcal{K}(\Theta_1).$$

Finally we obtain

$$\begin{aligned}
\mathcal{B}(E_0)E_1 + \mathcal{B}(E_1)E_0 &= E_0E_1\mathcal{K}(\Theta_0) + E_0E_1\mathcal{K}(\Theta_1) \\
&= E_2\mathcal{K}(\Theta_0) + E_2\mathcal{K}(\Theta_1) \\
&= E_2\mathcal{K}(\Theta_2) \\
&= \mathcal{B}(E_2).
\end{aligned}$$

□

Theorem 3.3. *Let $E(z), E_1(z)$ be two Hermite Biehler functions such that $\left|\frac{E(x)}{E_1(x)}\right| \le M \ \forall x \in \mathbb{R}$ for some $M > 0$, that $\frac{E(z)}{E_1(z)}$ is of bounded type on $\mathbb{C}^+$, and that*

$$\limsup_{y\to+\infty} y^{-1} \log\left|\frac{E(iy)}{E_1(iy)}\right| \le 0.$$

Then we have

$$\mathcal{B}(E) \subseteq \mathcal{B}(E_1), \tag{3.7}$$

and therefore for every $G \in \mathcal{B}(E)$ we obtain

$$G(z) = \sum_n \frac{E_1(z)(1-\Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n - z)} G(t_n),$$

where

$$\Theta_1(z) = \frac{E_1^{\#}(z)}{E_1(z)}.$$

The convergence of the series is uniform on the compact subsets of $\mathbb{C}$.

Proof. Consider $F \in \mathcal{B}(E)$. By definition of de Branges spaces (see *Section* 2.7), we have that $F \in \mathcal{B}(E_1)$ if and only if F is such that

$$\int_{-\infty}^{+\infty} \left|\frac{F(t)}{E_1(t)}\right|^2 dt < \infty,$$

and that both the ratios $\frac{F(z)}{E_1(z)}, \frac{F^{\#}(z)}{E_1(z)}$ are of bounded type and of non-positive mean type in the upper half-plane. We have

$$\int_{-\infty}^{+\infty} \left|\frac{F(t)}{E_1(t)}\right|^2 dt \leq M \int_{-\infty}^{+\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt = M||F||_{\mathcal{B}(E)} < \infty.$$

Moreover we observe that

$$\frac{F(z)}{E_1(z)} = \frac{F(z)}{E(z)}\frac{E(z)}{E_1(z)}, \quad \frac{F^{\#}(z)}{E_1(z)} = \frac{F^{\#}(z)}{E(z)}\frac{E(z)}{E_1(z)}.$$

Since $F \in \mathcal{B}(E)$ we have that $\frac{F(z)}{E(z)}$ and $\frac{F^{\#}(z)}{E(z)}$ are of bounded type, while $\frac{E(z)}{E_1(z)}$ is of bounded type on $\mathbb{C}^+$ by hypothesys. Hence we obtain that both $\frac{F(z)}{E_1(z)}$ and $\frac{F^{\#}(z)}{E_1(z)}$ are of bounded type. Thanks to (2.11) we have that the mean type h of $\frac{F}{E_1}$ verifies

$$\begin{aligned}
h &= \limsup_{y\to+\infty} y^{-1} \log\left|\frac{F(iy)}{E_1(iy)}\right| \\
&\leq \limsup_{y\to+\infty} y^{-1} \log\left|\frac{F(iy)}{E(iy)}\right| + \limsup_{y\to+\infty} y^{-1} \log\left|\frac{E(iy)}{E_1(iy)}\right| \\
&\leq \limsup_{y\to+\infty} y^{-1} \log\left|\frac{F(iy)}{E(iy)}\right| \\
&\leq 0,
\end{aligned}$$

where in the last step we used the fact that $\frac{F(z)}{E(z)}$ is of non-positive mean type. Hence $\frac{F(z)}{E_1(z)}$ is of non-positive mean type, and similarly we get that also $\frac{F^{\#}(z)}{E_1(z)}$ is of non-positive mean type.

Therefore $F \in \mathcal{B}(E_1)$, and (3.7) is proved. Thanks to *Theorems* 2.18 and (2.23), for every $G \in \mathcal{B}(E)$ we obtain

$$G(z) = \sum_n \frac{E_1(z)(1-\Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n - z)} G(t_n),$$

and that the convergence is uniform on the compact subsets of $\mathbb{C}$. $\square$

Theorem 3.4. *Let $E(z)$ be a Hermite Biehler function, and let $\Theta(z)$ be any meromorphic inner function of the form*

$$\Theta(z) = \Phi(z)\Phi_1(z), \tag{3.8}$$

where $\Phi(z) = \frac{E^{\#}(z)}{E(z)}$ and $\Phi_1(z)$ is any meromorphic inner function. Let $\{t_n\}_n$ be the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$ (with $t_n < t_{n+1}$ $\forall n \in \mathbb{Z}$). Then for every $G \in \mathcal{B}(E)$ we have

$$G(z) = \sum_n \frac{E(z)(1-\Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} G(t_n). \tag{3.9}$$

The series converges in norm of $\mathcal{B}(E)$.

Proof. If $\Phi_1(z) = 1$, then set $E_1(z) = 1$, otherwise let $E_1(z)$ be the Hermite Biehler function defined in *Theorem* 2.11 such that

$$\Phi_1(z) = \frac{E_1^{\#}(z)}{E_1(z)}.$$

Let $E_2(z) = E(z)E_1(z)$. Given any $G \in \mathcal{B}(E)$, we set $F(z) = G(z)E_1(z)$. We have $F \in \mathcal{B}(E_2)$: it is a consequence of *Theorem* 3.1 in the case $E_1(z) \neq 1$, while it is obvious in the case $E_1(z) = 1$ (since $E(z) = E_2(z)$). Thanks to *Theorem* 2.18, we know that a generic Hermite Biehler function $F \in \mathcal{B}(E_2)$, for $t \in \mathbb{R}$, obeys

$$F(z) = \sum_n \frac{K_{\mathcal{B}(E_2)}(t_n, z)}{K_{\mathcal{B}(E_2)}(t_n, t_n)} F(t_n). \tag{3.10}$$

Then we get

$$G(z)E_1(z) = \sum_n \frac{K_{\mathcal{B}(E_2)}(t_n, z)}{K_{\mathcal{B}(E_2)}(t_n, t_n)} G(t_n)E_1(t_n),$$

and therefore

$$G(z) = \sum_n \frac{K_{\mathcal{B}(E_2)}(t_n, z)E_1(t_n)}{K_{\mathcal{B}(E_2)}(t_n, t_n)E_1(z)} G(t_n). \tag{3.11}$$

Recalling (2.23) we obtain

$$\begin{aligned} G(z) &= \sum_n \frac{E_1(t_n)E_2(z)\left(1 - \Theta(z)\right)}{E_1(z)E_2(t_n)\Theta_2'(t_n)(t_n - z)} G(t_n) \\ &= \sum_n \frac{E(z)(1 - \Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} G(t_n). \end{aligned}$$

For any $N \in \mathbb{Z}$ with $N > 0$ we observe that

$$F_N := \left(F(z) - \sum_{n=-N}^{N} \frac{K_{\mathcal{B}(E_2)}(t_n, z)}{K_{\mathcal{B}(E_2)}(t_n, t_n)} F(t_n) \right) \in \mathcal{B}(E_2),$$

and that

$$G_N := \left(G(z) - \sum_{n=-N}^{N} \frac{K_{\mathcal{B}(E_2)}(t_n, z)E_1(t_n)}{K_{\mathcal{B}(E_2)}(t_n, t_n)E_1(z)} G(t_n) \right) = \frac{F_N}{E_1(z)} \in \mathcal{B}(E),$$

with

$$\|G_N\|_{\mathcal{B}(E)} = \|F_N\|_{\mathcal{B}(E_2)}.$$

Since (3.10) converges in norm of $\mathcal{B}(E_2)$ by *Theorem* 2.18, we can conclude that (3.9) convergences in norm of $\mathcal{B}(E)$. $\square$

From *Theorem* 2.18 we already know that every function of a de Branges space $\mathcal{B}(E)$ can be rebuilt exactly and uniquely given its values on the sequence of real points $\{t_n\}_n$ for which $\Phi(t_n) = \frac{E^\#(t_n)}{E(_nt)} = 1$. The important aspect of the sampling formula in *Theorem* 3.4 is that it shows that a function of a de Branges space $\mathcal{B}(E)$ can be rebuilt exactly and uniquely also given its values on the sequence of real points $\{t_n\}_n$ for which $\Theta(t_n) = 1$ for any meromorphic inner function $\Theta(z)$ divided by $\Phi(z)$. Hence for every function of a de Branges space there are infinite different sequences from which the function itself can be rebuilt exactly.

Theorem 3.4 can be proved also as a consequence of (2.13) with a more direct proof. However in this thesis we preferred to prove this theorem using the proposed proof as it is more consistent with the general approach of this thesis and with the methodology used to prove the other sampling formulas in the following paragraphs.

3.2 Orthogonal bases

At this point, a natural question is it the set of sampling kernels

$$\left\{ \frac{E(z)(1-\Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} \right\}_n,$$

appearing in the recontruction formula of *Theorem* 3.4, is an orthogonal basis of $\mathcal{B}(E)$. The next theorem shows that the answer is positive if the function $\Phi_1(z)$ in (3.8) is a Blaschke product multiplied by a constant γ with $|\gamma| = 1$, i.e. $\Phi_1(z)$ is a meromorphic inner function with logarthmic residue $b = 0$.

Theorem 3.5. *Let $E(z)$ be a Hermite Biehler function, and let $\Theta(z)$ be any meromorphic inner function of the form*

$$\Theta(z) = \gamma\Phi(z)B(z),$$

where $\Phi(z) = \frac{E^\#(z)}{E(z)}$, $B(z)$ is a Blaschke product and $|\gamma| = 1$. Let $\{t_n\}_n$ be the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$. Then the set

$$\{F_n(z)\}_n = \left\{ \frac{E(z)(1-\Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} \right\}_n \tag{3.12}$$

is an orthogonal basis of $\mathcal{B}(E)$.

Proof. Let $E_1(z)$ be a de Branges function of the meromorphic inner function $\gamma B(z)$, so that $\gamma B(z) = \frac{E_1^\#(z)}{E_1(z)}$. Setting $E_2(z) = E(z)E_1(z)$ we obtain $\Theta(z) = \frac{E_2^\#(z)}{E_2(z)}$. Thanks to *Theorem* 3.4 we have that the set (3.12) is a subset of

$\mathcal{B}(E)$. First of all we show that it is an orthogonal set in $\mathcal{B}(E)$. Now, consider the functions

$$F_{2,n}(z) = \frac{E_2(z)(1-\Theta(z))}{E_2(t_n)\Theta'(t_n)(t_n - z)} = \frac{K_{\mathcal{B}(E_2)}(t_n, z)}{K_{\mathcal{B}(E_2)}(t_n, t_n)} \in \mathcal{B}(E_2) \quad \forall n \in \mathbb{Z}.$$

We observe that $F_n(z) = \frac{E_1(t_n)}{E_1(z)} F_{2,n}(z)$ $\forall n \in \mathbb{Z}$. Therefore for all $n_a, n_b \in \mathbb{Z}$ we get

$$\begin{aligned}
\langle F_{n_a}, F_{n_b}\rangle_{\mathcal{B}(E)} &= \int_{-\infty}^{+\infty} \frac{F_{n_a}(t)\overline{F_{n_b}(t)}}{|E(t)|^2} dt \\
&= \int_{-\infty}^{+\infty} \frac{F_{n_a}(t)}{E(t)} \overline{\frac{F_{n_b}(t)}{E(t)}} dt \\
&= \int_{-\infty}^{+\infty} \frac{E_1(t_{n_a})F_{2,n_a}(t)}{E(t)E_1(t)} \overline{\frac{E_1(t_{n_b})F_{2,n_b}(t)}{E(t)E_1(t)}} dt \\
&= E_1(t_{n_a})\overline{E_1(t_{n_b})} \int_{-\infty}^{+\infty} \frac{F_{2,n_a}(t)}{E_2(t)} \overline{\frac{F_{2,n_b}(t)}{E_2(t)}} dt \\
&= E_1(t_{n_a})\overline{E_1(t_{n_b})} \int_{-\infty}^{+\infty} \frac{F_{2,n_a}(t)\overline{F_{2,n_b}(t)}}{|E_2(t)|^2} dt \\
&= E_1(t_{n_a})\overline{E_1(t_{n_b})} \langle F_{2,n_a}, F_{2,n_b}\rangle_{\mathcal{B}(E_2)}.
\end{aligned} \tag{3.13}$$

Since the set

$$\{F_{2,n}(z)\}_n = \left\{ \frac{K_{\mathcal{B}(E_2)}(t_n, z)}{K_{\mathcal{B}(E_2)}(t_n, t_n)} \right\}_n$$

is an orthogonal set in $\mathcal{B}(E_2)$ thanks to *Theorem* (2.17), we get that $\{F_n(z)\}_n$ is an orthogonal set in $\mathcal{B}(E)$. Finally, thanks to *Theorem* 3.4 we obtain that the only function that can be perpendicular to all the elements of the sequence $\{F_n(z)\}_n$ is the null vector, hence $\{F_n(z)\}_n$ is a complete orthogonal set in $\mathcal{B}(E)$, and therefore an orthogonal basis. □

The following theorem introduce another orthogonal set for the de Branges space $\mathcal{B}(E)$.

Theorem 3.6. *Let E(z) be a Hermite Biehler function. Consider any two Hermite Biehler functions $E_1(z), E_2(z)$ such that $E(z) = E_1(z)E_2(z)$, and let $\{t_n\}_n$ be the sequence of solutions of the meromorphic inner function $\Theta_1(z) = \frac{E_1^{\#}(z)}{E_1(z)} = 1$ on the real line. Then the functions $\{F_n(z)\}_n = \left\{\frac{K_{\mathcal{B}(E_1)}(t_n,z)E_2(z)}{E_1(t_n)}\right\}_n$ are an orthogonal set in $\mathcal{B}(E)$.*

Proof. Thanks to *Theorem* (2.17) we have that the functions $\{G_n(z)\}_n = \left\{\frac{K_{\mathcal{B}(E_1)}(t_n,z)}{E_1(t_n)}\right\}_n$ are an orthogonal set in $\mathcal{B}(E_1)$. Thanks to *Theorem* 3.1, we have that $\mathcal{B}(E_1)E_2 \subseteq \mathcal{B}(E)$, and then we get $F_n(z) = \frac{K_{\mathcal{B}(E_1)}(t_n,z)E_2(z)}{E_1(t_n)} \in B(E)$ $\forall n \in \mathbb{Z}$. Moreover we have

$$\begin{aligned}
\langle F_n, F_m \rangle_{\mathcal{B}(E)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_n(x)\overline{F_m(x)} \frac{1}{|E(x)|^2} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F_n(x)}{E(x)} \overline{\frac{F_m(x)}{E(x)}} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K_{\mathcal{B}(E_1)}(t_n, x)}{E_1(t_n)E_1(x)} \overline{\frac{K_{\mathcal{B}(E_1)}(t_m, x)}{E_1(t_n)E_1(x)}} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K_{\mathcal{B}(E_1)}(t_n, x)}{E_1(t_n)} \frac{\overline{K_{\mathcal{B}(E_1)}(t_m, x)}}{\overline{E_1(t_n)}} \frac{1}{|E_1(x)|^2} dx \\
&= \langle G_n, G_m \rangle_{\mathcal{B}(E_1)} = 0.
\end{aligned}$$

□

CHAPTER 4

Sampling formulas for the Paley-Wiener spaces

The well-known sampling formula (2.25) for the Paley Wiener spaces has the limit of being valid only for sequences of sampling points where the distance between a point and the next one is always the same. These sequences are referred as *uniform sequences*. However, there is another well-known sampling formula for the Paley-Wiener spaces which works also on non-uniform sequences. It is a generalization of the sampling formula (2.25), in the sense that if we apply it to a uniform sampling sequence we obtain exactly the sampling formula (2.25). This new sampling formula is described in the following Paley-Wiener-Levinson theorem (see [31]).

Theorem 4.1 (Paley-Wiener-Levinson)**.** *Let* $\{t_n\}_n$ *be a sequence of reals such that*

$$D := \sup_{n\in Z}\left|t_n - \frac{n\pi}{a}\right| < \frac{\pi}{4a}$$

and let $S(t)$ *be the entire function defined by*

$$S(t) = (t - t_0)\prod_{n=1}^{\infty}\left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right)$$

Then, for any $F \in \mathcal{PW}_a$

$$F(t) = \sum_n \frac{S(t)}{S'(t_n)(t - t_n)} F(t_n) \quad (t \in \mathbb{R}), \tag{4.1}$$

and the series on the right hand side converges uniformly on compact subsets of $\mathbb{R}$*.*

In this chapter, we introduce some different sampling formulas for the Paley-Wiener spaces, which are based on sets of non-uniform sequences with different characteristics from those of the Paley-Wiener-Levinson theorem.

4.1 Sampling formulas for non-uniform sampling

We introduce here the first new sampling formula for non-uniform sampling, which is mainly base on the result of *Theorem* 3.4.

Theorem 4.2. *Fix any $a > 0$ and consider the Paley-Wiener space $\mathcal{PW}_a$. Let $\Theta(z) = \gamma e^{ibz} B(z)$ be a meromorphic inner function according to the representation given in* (2.3)*, with logarithmic residue $b \geq 2a$. Let $\{t_n\}_n$ be the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$. Then for every $G \in \mathcal{PW}_a$ we have*

$$G(z) = \sum_n \frac{(1 - \Theta(z))e^{ia(t_n - z)}}{(t_n - z)\Theta'(t_n)} G(t_n). \tag{4.2}$$

The series converges in norm of $\mathcal{PW}_a$.

Proof. Consider the Hermite Biehler function

$$E(z) = e^{-iaz},$$

so that $\mathcal{B}(E) = \mathcal{PW}_a$, and let $\Phi(z) = \frac{E^\#(z)}{E(z)} = e^{2iaz}$. Let $\Theta(z) = \gamma e^{ibz} B(z)$ be a meromorphic inner function with logarithmic residue $b \geq 2a$. Then also $\Phi_1(z) = e^{-i2az}\Theta(z) = e^{i2(b-2a)z} B(z)\Theta(z)$ is a meromorphic inner function since $b - 2a > 0$, and we have

$$\Theta(z) = e^{2iaz}\Phi_1(z) = \Phi(z)\Phi_1(z).$$

Hence $E(z)$ and $\Theta(z)$ satisfy the conditions required in *Theorem* 3.4, and then for every $G \in \mathcal{PW}_a$ we obtain

$$\begin{aligned} G(z) &= \sum_n \frac{E(z)(1 - \Theta(z))}{E(t_n)(t_n - z)\Theta'(t_n)} G(t_n). \\ &= \sum_n \frac{e^{-iaz}(1 - \Theta(z))}{e^{-iat_n}(t_n - z)\Theta'(t_n)} G(t_n). \\ &= \sum_n \frac{(1 - \Theta(z))e^{ia(t_n - z)}}{(t_n - z)\Theta'(t_n)} G(t_n). \end{aligned} \tag{4.3}$$

By the same theorem we obtain also that the series converges in norm of $\mathcal{PW}_a$. $\square$

As already seen for the de Branges spaces in *Theorem* 3.5, we are also able to establish when the set of sampling kernels $\left\{\frac{(1-\Theta(z))e^{ia(t_n-z)}}{(t_n-z)\Theta'(t_n)}\right\}_n$ is an orthonormal basis of $\mathcal{PW}_a$, finding out an infinite number of orthonormal bases. This is done by the next theorem.

Theorem 4.3. *Fix any $a > 0$ and consider the Paley-Wiener space $\mathcal{PW}_a$. Let $\Theta(z)$ be a meromorphic inner function of the form $\Theta(z) = \gamma e^{2iaz}B(z)$, according to the representation given in (2.3). Let $\{t_n\}_n$ be the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$. Then the set*

$$\{G_n\}_n = \left\{ \frac{(1-\Theta(z))e^{ia(t_n-z)}}{(t_n - z)\Theta'(t_n)} \right\}_n$$

is an orthogonal basis of $\mathcal{PW}_a$.

Proof. Consider the Hermite Biehler function

$$E(z) = e^{-iaz},$$

and let $\Phi(z) = \frac{E^{\#}(z)}{E(z)} = e^{2iaz}$. As we already pointed it, we have $\mathcal{B}(E) = \mathcal{PW}_a$. Then we have

$$\Theta(z) = \gamma e^{2iaz}B(z) = \gamma\Phi(z)B(z).$$

Hence $E(z)$ and $\Theta(z)$ satisfy the conditions required in *Theorem* 3.5, and we get that

$$\left\{ \frac{E(z)(1-\Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} \right\}_n \tag{4.4}$$

is an orthogonal basis of $\mathcal{PW}_a$. Finally, proceeding similarly to (4.3) we easily get

$$\{G_n\}_n = \left\{ \frac{(1-\Theta(z))e^{ia(t_n-z)}}{(t_n - z)\Theta'(t_n)} \right\}_n = \left\{ \frac{E(z)(1-\Theta(z))}{E(t_n)\Theta'(t_n)(t_n - z)} \right\}_n . \tag{4.5}$$

□

Example 4.4.

Consider $\Theta(z) = e^{2\pi iz}$, so that in Theorem 4.2 we have $\{t_n\}_n = \{n\}$. We set $a = \pi$, and then for any $G \in \mathcal{PW}_\pi$ we obtain

$$\begin{aligned} G(z) &= \sum_n \frac{(1-e^{2\pi iz})e^{i\pi(n-z)}}{2\pi i(n-z)} G(n) \\ &= \sum_n \frac{(1-e^{2\pi i(z-n)})e^{i\pi(n-z)}}{2\pi i(n-z)} G(n) \\ &= \sum_n \frac{\sin(\pi(n-z))}{\pi(n-z)} G(n), \end{aligned}$$

that is the classical sampling formula for the Paley-Wiener space $\mathcal{PW}_\pi$.

Now we want to compare the results of *Theorem* 4.2 with those of the Paley-Wiener-Levinson theorem (Theorem 4.1). Fix $a \in \mathbb{R}$, $a > 0$. Given any $F \in \mathcal{PW}_a$, thanks to these two theorems we have two different sampling formulas for non-uniform sampling, that are valid on two different families of non-uniform sequences. Kadec showed that the set of sampling kernels in (4.1) is a Riesz basis for $\mathcal{PW}_a$, while thanks to *Theorem* 4.3 we have seen that the set of sampling kernels in (4.2) is an orthogonal basis if $\Theta(z) = \gamma e^{2iaz} B(z)$, where $B(z)$ is a Blaschke product. Hence it makes sense to compare the properties of the sampling sequences of the Paley-Wiener-Levinson theorem with the properties of the sequences $\{t_n\}_n$ of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$, for $\Theta(z) = \gamma e^{2iaz} B(z)$, since in both cases the corresponding sampling kernels are bases of $\mathcal{PW}_a$.

It is easy to see that for every sequence $\{t_n\}_n$ in *Theorem* 4.1 we have

$$|t_n - t_m| > \left(|m-n| - \frac{1}{2}\right)\frac{\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m,$$

while in the sequences in *Theorem* 4.2 there is no for lower bounds for $|t_n - t_m|$. Indeed given any $\epsilon > 0$ small as desired and any integer $M < \infty$ big as desired, it is possible to find a suitable sequence $\{t_n\}_n$ such that M different elements are contained in a real interval of length ϵ. To see this, we need to show that it is possible to find a meromorphic inner function $\Theta(z)$ such that M different elements of the sequence $\{t_n\}_n$ of solutions of $\Theta(t) = 1$ are contained in a real interval of length ϵ. We recall that the phase function $\tau(t)$ of a meromorphic inner function $\Theta(z)$ is the unique differentiable function such that $\Theta(t) = e^{2\pi i \tau(t)}$ for $t \in \mathbb{R}$, with $\tau'(t) > 0$ $\forall t \in \mathbb{R}$ and $\tau(t_n) = n$ $\forall n \in \mathbb{Z}$ (so that $\Theta(t_n) = 1$). Then, given any Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{z_k}}{z_k} \frac{z - z_k}{z - \overline{z}_k}, \qquad \sum_n \frac{\Im z_n}{|z_n|^2} < \infty,$$

for $\Theta(z) = \gamma e^{2iaz} B(z)$ we obtain

$$\tau'(t) = \frac{1}{2\pi i} \frac{\Theta'(z)}{\Theta(z)} = \frac{a}{\pi} + \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\Im(z_k)}{|z_k - t|^2}.$$

Fix any $c \in \mathbb{R}$ and any $\epsilon > 0$, and consider the interval $[c, c+\epsilon]$. Without loss of generality we set $\gamma = 1$, and let $B(z)$ be the Blaschke product of $N = \left\lceil \frac{M\pi}{\epsilon} \right\rceil$ distinct zeros $\{z_k\}_{k=1,\dots,N}$ such that $c \leq \Re(z_k) \leq c + \epsilon$, $\Im(z_k) = \delta < 1$ for all

$k = 1, ..., N$. Hence for $t \in [c, c+\epsilon]$ we get

$$\begin{aligned}\tau'(t) &= \frac{1}{2\pi i}\frac{\Theta'(z)}{\Theta(z)} \\ &= \frac{a}{\pi} + \frac{1}{\pi}\sum_{k=1}^{N}\frac{\Im(z_k)}{|z_k - t|^2} \\ &\geq \frac{1}{\pi}\sum_{k=1}^{N}\frac{\delta}{(\Re(z_k) - t)^2 + \delta^2} \\ &> \frac{1}{\pi}\sum_{k=1}^{N}\frac{\delta}{\epsilon^2 + \delta^2} \\ &> \frac{N}{\pi} \\ &\geq \frac{M}{\epsilon},\end{aligned}$$

where we observed that, since $\delta < 1$, $\frac{\delta}{\epsilon^2+\delta^2} > 1$ for enough small ϵ. Then we have

$$\tau(c+\epsilon) - \tau(c) = \int_c^{c+\epsilon}\tau'(s)dt > \int_c^{c+\epsilon}\frac{M}{\epsilon}dt = M,$$

and we obtain that $\tau(t) \in \mathbb{N}$ at least M times in $[c, c+\epsilon]$ since it is a continuous function. Then $\Theta(t) = e^{2\pi i\tau(t)} = 1$ at least M times in $[c, c+\epsilon]$, and then there are M elements of the sequence in an interval of length ϵ, as desired.

Now we compare the upper bounds of $|t_n - t_m|$. For every sequence $\{t_n\}_n$ in *Theorem* 4.1 we easily get

$$|t_n - t_m| < \left(|m-n| + \frac{1}{2}\right)\frac{\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m,$$

while for *Theorem* 4.2 we have

$$|t_n - t_m| < \frac{|m-n|\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m.$$

To see this, it is sufficient to show that

$$t_{n+1} - t_n \leq \frac{\pi}{a} \quad \forall n \in \mathbb{Z}.$$

We have

$$\tau'(t) = \frac{1}{2\pi}\frac{\Theta'(z)}{\Theta(z)} = \frac{a}{\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\Im(z_k)}{|z_k - t|^2} \geq \frac{a}{\pi} \tag{4.6}$$

Now consider the spectral function $t(s) = \tau^{-1}(s)$ ($t(s)$ exists and is well defined since $\tau'(t) > 0 \ \forall t \in \mathbb{R}$), then $t(n) = t_n$ and $t(n+1) = t_{n+1}$, and hence we obtain

$$t_{n+1} - t_n = \int_n^{n+1} t'(s)ds = \int_n^{n+1}\frac{1}{\tau'(t(s))}ds \leq \int_n^{n+1}\frac{\pi}{a}ds = \frac{\pi}{a}.$$

Hence we can conclude that the difference in the upper bound of $|t_n - t_m|$ gives a little more a flexibility to the sampling sequences of the Paley-Wiener-Levinson theorem, while difference in the lower bound gives much more flexibility to the sampling sequences of *Theorem* (4.2).

4.2 Representation of a meromorphic inner function

The sampling kernels in *Theorem* 4.2 and *Theorem* 4.3 are expressed in terms of the meromorphic inner function $\Theta(z)$. Hence now it is interesting to establish when for a given a sequence $\{t_n\}_n$ there exists a sampling formula of the form given in *Theorem* 4.2, and express it in terms of the sequence itself. This means to find necessary and sufficient conditions for a given sequence $\{t_n\}_n$ to be the sequence of solutions of $\Theta(t) = 1$ on the real line for some meromorphic inner function $\Theta(z)$ with logarithmic residue $b > 0$.

For this purpose the first step is to give a representation of any meromorphic inner function $\Theta(z)$ in terms of the sequence $\{t_n\}_n$ of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$. We obtain this fundamental result in *Theorem* 4.6. Then we use this result to prove *Theorem* 4.8, where the representation of a de Branges function $E(z)$ of $\Theta(z)$ is given in terms of the same sequence $\{t_n\}_n$. This representation will be very useful in the next sections.

Before introducing these theorems, we need the following deifinition.

Definition 4.5. Let $\{t_n\}_n \subset \mathbb{R}$ and $\{t'_n\}_n \subset (0, \infty)$ be two sequences with the following properties:

1. $\{t_n\}_n$ is a strictly increasing sequence with no finite accumulation point;
2. $t'_n > 0 \ \forall n \in \mathbb{Z}$;
3. $\sum_n \frac{t'_n}{1+t_n^2} < \infty$;
4. $\sum_n t'_n = +\infty$.

A couple of sequences $(\{t_n\}_n, \{t'_n\}_n)$ that verifies this properties is referred as a *bandlimit pair*. Moreover we define a *normalized bandlimit pair* as a bandlimit pair such that $\sum_n \frac{t'_n}{1+t_n^2} = \pi$.

Obviously, given an bandlimit pair, we can obtain a normalized bandlimit pair multiplying all the elements of the sequence $\{t'_n\}_n$ by $\frac{\pi}{\sum_n \frac{t'_n}{1+t_n^2}}$.

Theorem 4.6. *A function $\Theta : \mathbb{C} \to \mathbb{C}$ is a meromorphic inner function if and only if there exist a bandlimit pair $(\{t_n\}, \{t'_n\})$ and a complex number α, $\Im(\alpha) > 0$, so that $\Theta(z)$ is given by the formula*

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}. \tag{4.7}$$

In particular any meromorphic inner function $\Theta(z)$ has infinite different representations of the type given in (4.7)*, one for every different value of* $\sum_n \frac{t'_n}{t_n^2+1}$ *(that can be any positive real number since* $(\{t_n\}, \{t'_n\})$ *is a bandlimit pair). Setting* $\beta = \sum_n \frac{t'_n}{t_n^2+1}$*, the elements of the sequence* $\{t_n\}_n$ *are all and only the solutions of* $\Theta(t) = 1$ *for* $t \in \mathbb{R}$*, the elements of the sequence* $\{t'_n\}_n$ *are given by* $t'_n = \frac{2i\Im(\alpha)}{\Theta'(t_n)}$*, and* α *is given by*

$$\alpha = -\frac{2\beta\Im(\Theta(i))}{1-|\Theta(i)|^2} + i\beta\left(\frac{|\Theta(i)|^2 - 2\Re(\Theta(i)) + 1}{1-|\Theta(i)|^2}\right). \tag{4.8}$$

Proof. Thanks to *Corollary* 4.6 in [40] (p. 1628) we have that a function $\Theta(z)$ on $\mathbb{C}$ is a meromorphic inner function obeying $\Theta(i) = 0$ if and only if there is a bandlimit pair $(\{t_n\}, \{t'_n\})$ so that $\Theta(z)$ is given by

$$\Theta(z) = \frac{z-i}{z+i}\frac{\sum_n \frac{1}{t_n-z}\frac{1}{t_n-i}t'_n}{\sum_n \frac{1}{t_n-z}\frac{1}{t_n+i}t'_n} = \frac{\sum_n t'_n\left(\frac{1}{t_n-z} - \frac{1}{t_n-i}\right)}{\sum_n t'_n\left(\frac{1}{t_n-z} - \frac{1}{t_n+i}\right)},$$

where $\{t_n\}_n$ consists in the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$, and the sequence $\{t'_n\}_n$ is such that $t'_n = \frac{2i\Im(\alpha)}{\Theta'(t_n)}$.

First we show that any meromorphic inner function $\Theta(z)$ can be expressed with the representation given in (4.7). Let $w = \Theta(i)$ (clearly w must safisty $|w| < 1$ since $\Theta(z)$ is inner). We recall here that for any fixed $h \in \mathbb{D}$, the Mobius transformation, given by

$$F_h(z) := \frac{z-h}{1-z\bar{h}}$$

is an analytic automorphism of the unit disk with compositional inverse F_{-h}. In particular, given any meromorphic inner function $\Psi(z)$, then the composition $F_h \circ \Psi$ is again a meromorphic inner function (called the *Frostman shift* of $\Psi(z)$). If we set $h = w$ in the Mobius transformation, then the meromorphic inner function $\Phi(z)$, given by

$$\Phi(z) = \frac{\overline{w}-1}{w-1}\left(F_w(z) \circ \Theta(z)\right) = \frac{\overline{w}-1}{w-1}\left(\frac{\Theta(z)-w}{1-\Theta(z)\bar{w}}\right),$$

satisfies $\Phi(i) = 0$. Moreover it is easy to check that for $t \in \mathbb{R}$ we have $\Psi(t) = 1$ if and only if $\Theta(t) = 1$. Hence we can apply the above mentioned *Corollary* 4.6 in [40], and we can write

$$\Phi(z) = \frac{z-i}{z+i}\frac{\sum_n \frac{1}{t_n-z}\frac{1}{t_n-i}t'_n}{\sum_n \frac{1}{t_n-z}\frac{1}{t_n+i}t'_n} = \frac{\sum_n t'_n\left(\frac{1}{t_n-z} - \frac{1}{t_n-i}\right)}{\sum_n t'_n\left(\frac{1}{t_n-z} - \frac{1}{t_n+i}\right)},$$

for a bandlimit pair $(\{t_n\},\{t'_n\})$. Now, since the inverse of $F_w(z)$ is given by $F_{-w}(z)$, we get

$$
\begin{aligned}
\Theta(z) = F_{-w}(z)\circ\left(\frac{w-1}{\overline{w}-1}\Phi(z)\right) &= \frac{\frac{w-1}{\overline{w}-1}\frac{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n-i}\right)}{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n+i}\right)}+w}{1+\overline{w}\frac{w-1}{\overline{w}-1}\frac{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n-i}\right)}{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n+i}\right)}} \\
&= \frac{(w-1)\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n-i}\right)+(|w|^2-w)\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n+i}\right)}{(\overline{w}-1)\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n+i}\right)+(|w|^2-\overline{w})\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{1}{t_n-i}\right)} \\
&= \frac{(|w|^2-1)\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)+i(-2w+|w|^2+1)\sum_n\frac{t'_n}{t_n^2+1}}{(|w|^2-1)\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)+i(2\overline{w}-|w|^2-1)\sum_n\frac{t'_n}{t_n^2+1}} \qquad (4.9)\\
&= \frac{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)-i\beta\left(\frac{|w|^2-2w+1}{1-|w|^2}\right)}{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)-i\beta\left(\frac{-|w|^2+2\overline{w}-1}{1-|w|^2}\right)} \\
&= \frac{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)-\frac{2\beta\Im(w)}{1-|w|^2}-i\beta\left(\frac{|w|^2-2\Re(w)+1}{1-|w|^2}\right)}{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)-\frac{2\beta\Im(w)}{1-|w|^2}+i\beta\left(\frac{|w|^2-2\Re(w)+1}{1-|w|^2}\right)}.
\end{aligned}
$$

Now we set $\alpha=-\frac{2\beta\Im(w)}{1-|w|^2}+i\beta\left(\frac{|w|^2-2\Re(w)+1}{1-|w|^2}\right)$, and we observe that $\Im(\alpha)=\beta\left(\frac{|w|^2-2\Re(w)+1}{1-|w|^2}\right)=\beta\left(\frac{\Im(w)^2+(\Re(w)-1)^2}{1-|w|^2}\right)>0\ \forall w$ such that $|w|<1$. Hence we obtain

$$
\Theta(z)=\frac{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)+\overline{\alpha}}{\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)+\alpha},
$$

as we wanted to show. By construction the sequence $\{t_n\}_n$ consists in the sequence of solutions of $\Theta(t)=1$ for $t\in\mathbb{R}$, and this can easily verified also in the new representation of $\Theta(z)$. We can also verify that $w=\Theta(i)$, indeed

$$
\begin{aligned}
\Theta(i) &= \frac{\sum_n t'_n\left(\frac{1}{t_n-i}-\frac{t_n}{t_n^2+1}\right)+\overline{\alpha}}{\sum_n t'_n\left(\frac{1}{t_n-i}-\frac{t_n}{t_n^2+1}\right)+\alpha} \\
&= \frac{i\sum_n\left(\frac{t'_n}{t_n^2+1}\right)-i\beta\left(1+\frac{2w(\overline{w}-1)}{1-|w|^2}\right)}{i\sum_n\left(\frac{t'_n}{t_n^2+1}\right)-i\beta\left(1+\frac{2(\overline{w}-1)}{1-|w|^2}\right)} \\
&= \frac{\left(-\frac{2w(\overline{w}-1)}{1-|w|^2}\right)}{\left(-\frac{2(\overline{w}-1)}{1-|w|^2}\right)} \\
&= w.
\end{aligned}
$$

Moreover we have

$$
\Theta'(z)=\frac{2i\Im(\alpha)\sum_n\frac{t'_n}{(t_n-z)^2}}{\left(\sum_n t'_n\left(\frac{1}{t_n-z}-\frac{t_n}{t_n^2+1}\right)+\alpha\right)^2},
$$

then we observe

$$\Theta'(t_n) = \lim_{t \to t_n} \Theta'(t) = \frac{2i\Im(\alpha)}{t_n'},$$

and hence we obtain $t_n' = \frac{2i\Im(\alpha)}{\Theta'(t_n)}$.

Now we show the inverse implication: any $\Theta(z)$ with the form given in (4.7) is a meromorphic inner function. We observe that

$$\begin{aligned}\Theta(i) &= \frac{i\beta + \overline{\alpha}}{i\beta + \alpha} \\ &= \left(\frac{\Re(\alpha)^2 - \Im(\alpha)^2 + \beta^2}{\Re(\alpha)^2 + (\Im(\alpha) + \beta)^2}\right) - i\left(\frac{2\Re(\alpha)\Im(\alpha)}{\Re(\alpha)^2 + (\Im(\alpha) + \beta)^2}\right),\end{aligned} \tag{4.10}$$

and hence we get

$$|\Theta(i)|^2 = 1 - \frac{4\beta\Im(\alpha)}{\Re(\alpha)^2 + (\Im(\alpha) + \beta)^2} < 1,$$

since $\beta, \Im(\alpha) > 0$. Setting $w = \Theta(i)$, by (4.10) we get

$$\begin{aligned}-\frac{2\beta\Im(w)}{1 - |w|^2} &= \frac{4\beta\Re(\alpha)\Im(\alpha)}{4\beta\Im(\alpha)} \\ &= \Re(\alpha)\end{aligned}$$

and

$$\begin{aligned}&\beta\left(\frac{|w|^2 - 2\Re(w) + 1}{1 - |w|^2}\right) \\ &= \beta\frac{2(\Re(\alpha)^2 + (\Im(\alpha) + \beta)^2) - 4\beta\Im(\alpha) - 2(\Re(\alpha)^2 - \Im(\alpha)^2 + \beta^2)}{4\beta\Im(\alpha)} \\ &= \frac{4\Im(\alpha)^2 + 4\beta\Im(\alpha) - 4\beta\Im(\alpha)}{4\Im(\alpha)} \\ &= \Im(\alpha).\end{aligned}$$

Therefore we have

$$\alpha = -\frac{2\beta\Im(w)}{1 - |w|^2} + i\beta\left(\frac{|w|^2 - 2\Re(w) + 1}{1 - |w|^2}\right),$$

and hence (4.8) is verified. Then, thanks to (4.9), we obtain

$$\Theta(z) = F_{-w}(z) \circ \left(\frac{w - 1}{\overline{w} - 1}\Phi(z)\right),$$

where

$$\Phi(z) = \frac{\sum_n t_n'\left(\frac{1}{t_n - z} - \frac{1}{t_n - i}\right)}{\sum_n t_n'\left(\frac{1}{t_n - z} - \frac{1}{t_n + i}\right)}.$$

Thanks again to *Corollary* 4.6 in [40], $\Phi(z)$ is a meromorphic inner function which satisfy $\Phi(i) = 0$, and the same is obviously true for $\frac{w-1}{\overline{w}-1}\Phi(z)$. Therefore, thanks to the above described properties of the Mobius tranfsorm, we obtain that also

$$\Theta(z) = F_{-w}(z) \circ \left(\frac{w-1}{\overline{w}-1}\Phi(z)\right)$$

is a meromorphic inner function, as we wanted to show. Finally it is easy to see that the elements of the sequence $\{t_n\}_n$ are all and only the solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$, while the expression for $\{t'_n\}_n$ can be verified in the same way of the other implication.

□

Example 4.7.

Consider $\Theta(z) = e^{2\pi i z}$. Then in *Theorem* 4.6 we set $\beta = \pi$ and we obtain

$$\begin{aligned}
\Theta(i) &= e^{-2\pi}, \\
\alpha &= i\pi\left(\frac{e^{-4\pi} - 2e^{-2\pi} + 1}{1 - e^{-4\pi}}\right) = i\pi\tanh(\pi) \\
\{t_n\}_n &= \{n\}, \\
\{t'_n\}_n &= \left\{\frac{e^{-4\pi} - 2e^{-2\pi} + 1}{1 - e^{-4\pi}}\right\} = \{\tanh(\pi)\}.
\end{aligned}$$

Then, recalling that $\sum_n \left(\frac{1}{n-z} - \frac{n}{n^2+1}\right) = -\pi\cot(\pi z)$, by (4.7) we get

$$\begin{aligned}
\Theta(z) &= \frac{\tanh(\pi)\sum_n \left(\frac{1}{n-z} - \frac{n}{n^2+1}\right) - i\pi\tanh(\pi)}{\tanh(\pi)\sum_n \left(\frac{1}{n-z} - \frac{n}{n^2+1}\right) + i\pi\tanh(\pi)}, \\
&= \frac{-\pi\cot(\pi z) - i\pi}{-\pi\cot(\pi z) + i\pi} \\
&= e^{2\pi i z},
\end{aligned}$$

as expected.

Theorem 4.8. *Let the couple* $(\{t_n\}_n, \{t'_n\}_n)$ *be a bandlimit pair such that*

$$\sum_{n\neq 0} \frac{1}{|t_n|^{q+1}} < \infty,$$

for some $q \in \mathbb{Z}$, $q \geq 0$. *Let*

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}.$$

be the meromorphic inner function associated to $(\{t_n\}_n, \{t'_n\}_n)$ *according to Theorem 4.6. Let* $E(z)$ *be given by*

$$E(z) = z^c \left(\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \alpha \right) \prod_{n \in \mathbb{Z}, t_n \neq 0} \left(1 - \frac{z}{t_n} \right) e^{u_p(z)}, \tag{4.11}$$

$$c = \begin{cases} 1, & \textit{if } \exists n \mid t_n = 0 \\ 0, & \textit{otherwise} \end{cases}$$

$$u_q(z) = \sum_{m=1}^{q} \left(\frac{z}{t_n} \right)^m,$$

and p *is the smallest nonnegative integer for which the series*

$$\sum_{n \neq 0} \frac{1}{|t_n|^{p+1}}$$

is convergent. Then the function $E(z)$ *is a de Branges function of* $\Theta(z)$*. Moreover, if* $t'_n = d$ *for some constant* d *and all* $n \in \mathbb{Z}$*, then the product in* (6.13) *converges for* $u_q(z) = \frac{z}{t_n}$*, i.e. for* $q = 1$*.*

Proof. By (2.4) we have that

$$S(z) = \left(\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \alpha \right)$$

is a Herglotz function, i.e. an analytical function with non-negative real part on $\mathbb{C}^+$. Thanks to [51] (p. 56) we have that the product

$$P(z) = z^c \prod_{n \in \mathbb{Z}, t_n \neq 0} \left(1 - \frac{z}{t_n} \right) e^{u(z)}$$

converges uniformly to an entire function whose zeros are all and only the elements of the sequence $\{t_n\}_n$. Hence

$$E(z) = S(z)P(z)$$

is an entire function. We observe that

$$P(z) = P^{\#}(z)$$

and that

$$S^{\#}(z) = \left(\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \overline{\alpha} \right).$$

We set $z = x + iy$ and we consider the case $y > 0$. Recalling that $\Im(\alpha) > 0$, we have

$$\begin{aligned} \Re(S(z)) &= \Re(S^{\#}(z)), \\ |\Im(S(z))| &= \left| \frac{y}{(t_n - x)^2 + y^2} + \Im(\alpha) \right| \\ &> \left| \frac{y}{(t_n - x)^2 + y^2} - \Im(\alpha) \right| \\ &= |\Im(S^{\#}(z))|, \end{aligned} \tag{4.12}$$

and then

$$|S(z)| > |S^{\#}(z)|.$$

Hence we obtain

$$|E(z)| = |S(z)||P(z)| > |S^{\#}(z)||P(z)| = |S^{\#}(z)||P^{\#}(z)| = |E^{\#}(z)|.$$

and therefore $E(z)$ is a Hermite Biehler function. Finally, we have

$$\frac{E^{\#}(z)}{E(z)} = \frac{S^{\#}(z)P^{\#}(z)}{S(z)P(z)} = \frac{S^{\#}(z)}{S(z)} = \Theta(z),$$

and then $E(z)$ is a de Branges function of $\Theta(z)$.

Finally we observe that, if $t'_n = d$ for some constant d and all $n \in \mathbb{Z}$, then

$$\sum_n \frac{1}{t_n^2 + 1} = \frac{1}{d} \sum_n \frac{d}{t_n^2 + 1} < \infty,$$

since $(\{t_n\}_n, \{t'_n = d\}_n)$ is a bandlimit pair. Hence we get

$$\sum_{t_n \neq 0} \frac{1}{t_n^2} \leq \sum_{t_n \neq 0, |t_n| < 1} \frac{1}{t_n^2} + \sum_{|t_n| \geq 1} \frac{2}{t_n^2 + 1} \leq \sum_{t_n \neq 0, |t_n| < 1} \frac{1}{t_n^2} + 2 \sum_n \frac{1}{t_n^2 + 1} < \infty,$$

where in the last step we considered the fact that the number of t_n for which $|t_n| < 1$ is finite since the sequence $\{t_n\}_n$ has no accumulation points. Then $p \leq 1$. Since the product

$$\prod_{n \in \mathbb{Z}, t_n \neq 0} \left(1 - \frac{z}{t_n}\right) e^{u_q(z)} \tag{4.13}$$

converges for all $q \geq p$ (see [51] (p. 55)), we can conclude that in our case it converges for $q = 1$. □

4.3 Properties of meromorphic inner functions

In this section, given a meromorphic funtcion $\Theta(z)$, we study the important relations between the sequence $\{t_n\}_n$ of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$, and other properties of $\Theta(z)$ (in particular the logarithmic residue and the properties of the phase and the spectral function). These results will be very useful for the next sections.

Lemma 4.9. *Let the couple $(\{t_n\}_n, \{t'_n\}_n)$ be a bandlimit pair, and let*

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}.$$

be a meromorphic inner function, according to Theorem 4.6. Let $\tau(x)$ be its phase function, and let $t(x)$ be its spectral function. Then we have

$$\begin{aligned} \tau'(x) &= \frac{\Im(\alpha) \sum_n \frac{t'_n}{(t_n - x)^2}}{\pi \left(\sum_n t'_n \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)^2 + \Im(\alpha)^2}, \\ t'(n) &= \frac{\pi}{\Im(\alpha)} t'_n \quad \forall n \in \mathbb{Z}. \end{aligned}$$

Proof. By the definition the phase function $\tau(x)$ is the unique function that verifies

$$\Theta(x) = e^{2\pi i \tau(x)} \qquad \forall x \in \mathbb{R}.$$

Then we get

$$\begin{aligned} \tau'(x) &= \frac{\Theta'(x)}{2\pi i \Theta(x)} \\ &= \frac{2i\Im(\alpha) \sum_n \frac{t'_n}{(t_n - x)^2}}{2\pi i \left(\sum_n t'_n \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}\right)\left(\sum_n t'_n \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right) + \alpha\right)} \qquad (4.14) \\ &= \frac{\Im(\alpha) \sum_n \frac{t'_n}{(t_n - x)^2}}{\pi \left(\sum_n t'_n \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)^2 + \Im(\alpha)^2}. \end{aligned}$$

Moreover

$$\begin{aligned} t'(n) &= \frac{1}{\tau'(t(n))} \\ &= \frac{1}{\tau'(t_n)} \\ &= \lim_{x \to t_n} \frac{\pi \left(\sum_n t'_n \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)^2 + \Im(\alpha)^2}{\Im(\alpha) \sum_n \frac{t'_n}{(t_n - x)^2}} \qquad (4.15) \\ &= \frac{\pi}{\Im(\alpha)} t'_n. \end{aligned}$$

□

Proposition 4.10. *Let the couple $(\{t_n\}_n, \{t'_n\}_n)$ be a bandlimit pair such that $t'_n = t'$ $\forall n \in \mathbb{Z}$ for some constant $t' > 0$, and let*

$$\Theta(z) = \frac{t' \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{t' \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}.$$

be any of the associated meromorphic inner function according to Theorem 4.6, with phase function $\tau(x)$. If $|t_n - \frac{\pi}{a}n| \leq \delta$ $\forall n \in \mathbb{Z}$, for some $\delta < \frac{\pi}{2a}$ and $a > 0$, then there exist two real constants A and B such that $A \geq \frac{a}{\pi}$ and $0 < B \leq \frac{a}{\pi}$ and that

$$B \leq \tau'(x) \leq A \quad \forall x \in \mathbb{R}.$$

Proof. By *Lemma* 4.9 we have

$$\tau'(x) = \frac{\Im(\alpha) \sum_n \frac{t'_n}{(t_n - x)^2}}{\pi \left(\sum_n t'_n \left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2 + 1} \right) + \Re(\alpha) \right)^2 + \Im(\alpha)^2} \tag{4.16}$$

First we show that there exists $A > 0$ such that

$$\tau'(x) \leq A \quad \forall x \in \mathbb{R}.$$

Consider a generic $n_0 \in \mathbb{N}$ and set

$$s_{n_0}(x) = \sum_{n \geq 1} t' \left(\frac{1}{t_{n_0+n} - x} + \frac{1}{t_{n_0-n} - x} \right).$$

For $|x - \frac{\pi}{a}n_0| \leq \frac{\pi}{2a}$ we get

$$\begin{aligned} |s_{n_0}(x)| &= \left| \sum_{n \geq 1} t' \left(\frac{t_{n_0+n} + t_{n_0-n} - 2x}{(t_{n_0+n} - x)(t_{n_0-n} - x)} \right) \right| \\ &\leq 2 \left(\delta + \frac{\pi}{2a} \right) t' \sum_{n \geq 1} \frac{1}{\left(\frac{\pi}{a}n - \delta - \frac{\pi}{2a} \right)^2}, \end{aligned} \tag{4.17}$$

where we observed that $|t_{n_0+n} - x| \geq |t_{n_0+n} - \frac{\pi}{a}n_0| - |\frac{\pi}{a}n_0 - x| \geq (\frac{\pi}{a}n - \delta) - \frac{\pi}{2a}$ (with obviously $\frac{\pi}{a}n - \delta - \frac{\pi}{2a} > 0$ for any $n \geq 1$). Moerover we have

$$\begin{aligned} |t_{n_0+n} + t_{n_0-n} - 2x| &\leq |t_{n_0+n} + t_{n_0-n} - 2n_0| + |2n_0 - 2x| \\ &\leq 2\delta + \frac{\pi}{a}. \end{aligned} \tag{4.18}$$

Setting

$$C = 2 \left(\delta + \frac{\pi}{2a} \right) t' \sum_{n \geq 1} \frac{1}{\left(\frac{\pi}{a}n - \delta - \frac{\pi}{2a} \right)^2} + \sum_n \frac{t_n}{t_n^2 + 1} + |\Re(\alpha)|,$$

and $\epsilon = \min \left(\frac{t'}{C}, \frac{\pi}{2a} - \delta \right)$, for x s.t. $|x - t_{n_0}| < \epsilon$ we observe that

$$\frac{t'}{|x - t_{n_0}|} > \frac{t'}{\epsilon} \geq C,$$

and that

$$\left| x - \frac{\pi}{a}n_0 \right| \leq |x - t_{n_0}| + \left| t_{n_0} - \frac{\pi}{a}n_0 \right| \leq \left(\frac{\pi}{2a} - \delta \right) + \delta = \frac{\pi}{2a}.$$

Then we obtain

$$
\begin{aligned}
\tau'(x) =& \frac{\Im(\alpha)}{\pi}\left|\frac{\frac{t'}{(t_{n_0}-x)^2}+\sum_{n\geq 1}\frac{t'}{(t_{n+n_0}-x)^2}}{\left(\frac{t'}{(t_{n_0}-x)}+s_{n_0}(x)-\sum_n \frac{t_n}{t_n^2+1}+\Re(\alpha)\right)^2+\Im(\alpha)^2}\right| \\
\leq& \frac{\Im(\alpha)}{\pi}\frac{t'\left(\frac{1}{|t_{n_0}-x|^2}+\sum_{n\geq 1}\frac{1}{\left(\frac{\pi}{a}n-\delta-\frac{\pi}{2a}\right)^2}\right)}{\left(\frac{t'}{|t_{n_0}-x|}-C\right)^2+\Im(\alpha)^2}.
\end{aligned}
\tag{4.19}
$$

The function

$$
g(s)=\frac{\Im(\alpha)}{\pi}\frac{t'\left(\frac{1}{|s|^2}+\sum_{n\geq 1}\frac{1}{\left(\frac{\pi}{a}n-\delta-\frac{\pi}{2a}\right)^2}\right)}{\left(\frac{t'}{|s|}-C\right)^2+\Im(\alpha)^2}
$$

is such that

$$
\lim_{s\to 0} g(s)=\frac{\Im(\alpha)}{\pi t'},
$$

and moreover in the interval $-\epsilon \leq s \leq \epsilon$ we have $\frac{t'}{|s|}-C>0$ by the choice of ϵ. Then $g(s)$ is bounded in the set $-\epsilon\leq s\leq \epsilon$ and there exists $A_1>0$ such that

$$
g(s)\leq A_1.
$$

Since $g(s)$ doesn't depend on n_0, also A_1 doesn't depend on n_0. Then for $|t_{n_0}-x|<\epsilon$ we obtain

$$
\tau'(x)\leq A_1.
$$

On the other side, for x such that $|x-\frac{\pi}{a}n_0|\leq \frac{\pi}{2a}$ and $|x-t_{n_0}|\geq \epsilon$ we easily get

$$
\tau'(x)\leq \frac{t'\left(\frac{1}{\epsilon^2}+\sum_{n\geq 1}\frac{1}{\left(\frac{\pi}{a}n-\delta-\frac{\pi}{2a}\right)^2}\right)}{\pi\Im(\alpha)}.
$$

and also $A_2=\frac{t'\left(\frac{1}{\epsilon^2}+\sum_{n\geq 1}\frac{1}{\left(\frac{\pi}{a}n-\delta-\frac{\pi}{2a}\right)^2}\right)}{\pi\Im(\alpha)}$ doesn't depend on the choice of n_0. Then we have obtained

$$
\tau'(x)\leq A=\max(A_1,A_2).
$$

Now show that there exist $B>0$ such that

$$
\tau'(x)\geq B \quad \forall x\in\mathbb{R}.
$$

Consider a generic $n_0 \in \mathbb{N}$. For $|x - \frac{\pi}{a}n_0| \leq \frac{\pi}{2a}$ we get

$$\begin{aligned}
\tau'(x) =& \frac{\Im(\alpha)}{\pi} \left| \frac{\frac{t'}{(t_{n_0}-x)^2} + \sum_{n\geq 1} \frac{t'}{(t_{n+n_0}-x)^2}}{\left(\frac{t'}{(t_{n_0}-x)} + s_{n_0}(x) - \sum_n \frac{t_n}{t_n^2+1} + \Re(\alpha)\right)^2 + \Im(\alpha)^2} \right| \\
\geq& \frac{\Im(\alpha)}{\pi} \frac{t'\left(\frac{1}{|t_{n_0}-x|^2} + \sum_{n\geq 1} \frac{1}{\left(\frac{\pi}{a}n+\delta+\frac{\pi}{2a}\right)^2}\right)}{\left(\frac{t'}{|t_{n_0}-x|} + C\right)^2 + \Im(\alpha)^2}.
\end{aligned}$$

Consider the function $h(s)$ given by

$$h(s) = \frac{\Im(\alpha)}{\pi} \frac{t'\left(\frac{1}{|s|^2} + \sum_{n\geq 1} \frac{1}{\left(\frac{\pi}{a}n+\delta+\frac{\pi}{2a}\right)^2}\right)}{\left(\frac{t'}{|s|} + C\right)^2 + \Im(\alpha)^2}.$$

It is easy to check that

$$\lim_{s\to 0} h(s) = \frac{\Im(\alpha)}{\pi t'} > 0,$$

and then $h(s) > 0$ in the closed interval $-\frac{\pi}{2a} \leq s \leq \frac{\pi}{2a}$. Since $h(s)$ is continuous, there exists $B > 0$ such that in the same interval we have

$$h(s) \geq B.$$

Whereas $h(s)$ doesn't depend on n_0, neither does B, and then we conclude that

$$\tau'(x) \geq B \quad \forall x \in \mathbb{R}.$$

Finally, to see that $A \geq \frac{a}{\pi}$ and that $B \leq \frac{a}{\pi}$ it is sufficient to observe that since

$$n\frac{\pi}{a} - \delta \leq t_n \leq n\frac{\pi}{a} + \delta \quad \forall n \in \mathbb{Z},$$

we get

$$\tau\left(n\frac{\pi}{a} - \delta\right) \leq n \leq \tau\left(n\frac{\pi}{a} + \delta\right) \quad \forall n \in \mathbb{Z},$$

and hence

$$\lim_{n\to+\infty} \frac{n}{\tau(n\frac{\pi}{a})} = 1. \tag{4.20}$$

Now, for $\tau'(x) \geq B > \frac{a}{\pi}$ $\forall x \in \mathbb{R}$ we would have

$$\begin{aligned}
\lim_{n\to+\infty} \frac{n}{\tau(n\frac{\pi}{a})} &= \lim_{n\to+\infty} \frac{n}{\tau(0) + \int_0^{n\frac{\pi}{a}} \tau'(s)ds} \\
&\leq \lim_{n\to+\infty} \frac{n}{\tau(0) + Bn\frac{\pi}{a}} \\
&= \frac{a}{B\pi} \\
&< 1,
\end{aligned} \tag{4.21}$$

while for $\tau'(x) \leq A < \frac{a}{\pi}$ $\forall x \in \mathbb{R}$ we would have

$$\begin{aligned}\lim_{n\to+\infty} \frac{n}{\tau(n\frac{\pi}{a})} &= \lim_{n\to+\infty} \frac{n}{\tau(0) + \int_0^{n\frac{\pi}{a}} \tau'(s)ds} \\ &\geq \lim_{n\to+\infty} \frac{n}{\tau(0) + An\frac{\pi}{a}} \\ &= \frac{a}{A\pi} \\ &> 1.\end{aligned} \tag{4.22}$$

Hence we have shown that $A \geq \frac{a}{\pi}$ and that $B \leq \frac{a}{\pi}$. $\square$

Proposition 4.11. *Let the couple $(\{t_n\}_n, \{t'_n\}_n)$ be a bandlimit pair such that $t'_n = t'$ $\forall n \in \mathbb{Z}$ for some constant $t' > 0$, and let*

$$\Theta(z) = \frac{t' \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{t' \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}.$$

be any of the associated meromorphic inner function according to Theorem 4.6, with phase function $\tau(x)$. If $|t_n - \frac{\pi}{a}n| \leq \delta$ $\forall n \in \mathbb{Z}$ for some $\delta < \frac{\pi}{2a}$ and $a > 0$, then there exists $D > 0$ such that

$$|\tau''(x)| \leq D, \quad \forall x \in \mathbb{R}.$$

Proof. Deriving the formula for $\tau'(x)$ given in *Lemma* 4.9 we get

$$\begin{aligned}\tau''(x) =& \frac{\Im(\alpha)}{\pi} \frac{\sum_n \frac{2t'}{(t_n-x)^3} \left(\left(\sum_n t'\left(\frac{1}{t_n-x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)^2 + \Im(\alpha)^2\right)}{\left(\left(\sum_n t'\left(\frac{1}{t_n-x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)^2 + \Im(\alpha)^2\right)^2} \\ &- \frac{\Im(\alpha)}{\pi} \frac{\left(\sum_n \frac{t'}{(t_n-x)^2}\right)^2 \left(2\left(\sum_n t'\left(\frac{1}{t_n-x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)\right)}{\left(\left(\sum_n t'\left(\frac{1}{t_n-x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha)\right)^2 + \Im(\alpha)^2\right)^2} \\ =& \tau'(x)^2 \frac{\pi}{\Im(\alpha)} \left(\frac{\sum_n \frac{2t'}{(t_n-x)^3}}{\left(\sum_n \frac{2t'}{(t_n-x)^2}\right)^2}(g(x)^2 + \Im(\alpha)^2) - 2g(x)\right),\end{aligned} \tag{4.23}$$

where

$$g(x) = \sum_n t'\left(\frac{1}{t_n - x} - \frac{t_n}{t_n^2+1}\right) + \Re(\alpha).$$

Consider a generic $n_0 \in \mathbb{N}$, and $|x - \frac{\pi}{a} n_0| \leq \frac{\pi}{2a}$. Set:

$$\begin{aligned} g_1(x) &= \sum_{n\geq 1} t' \left(\frac{1}{t_{n_0+n} - x} + \frac{1}{t_{n_0-n} - x} \right), \\ g_2(x) &= \sum_{n\neq 0} t' \left(\frac{1}{(t_{n_0+n} - x)^2} \right), \\ g_3(x) &= \sum_{n\neq 0} t' \left(\frac{1}{(t_{n_0+n} - x)^3} \right). \end{aligned}$$

Recalling (4.18), we observe that

$$\begin{aligned} |g_1(x)| &\leq 2\left(\delta + \frac{\pi}{2a}\right) t' \sum_{n\geq 1} \frac{1}{\left(\frac{\pi}{a} n - \delta - \frac{\pi}{2a}\right)^2} =: G_1, \\ |g_2(x)| &\leq t' \sum_{n\neq 0} \frac{1}{\left(\frac{\pi}{a} n - \delta - \frac{\pi}{2a}\right)^2} =: G_2, \\ |g_3(x)| &\leq t' \sum_{n\neq 0} \frac{1}{\left(\frac{\pi}{a} n - \delta - \frac{\pi}{2a}\right)^3} =: G_3, \end{aligned}$$

and we underline that the constants on the right side don't depend on n_0. Now we set

$$f(x) := \frac{\left(\sum_n \frac{2t'}{(t_n - x)^3}\right)(g(x)^2 + \Im(\alpha)^2)}{\left(\sum_n \frac{2t'}{(t_n - x)^2}\right)^2} - 2g(x),$$

so that by (4.23) we have

$$\tau''(x) = \tau'(x)^2 \frac{\pi}{\Im(\alpha)} f(x).$$

We observe that

$$\begin{aligned} f(x) =& \frac{\left(2\frac{t'}{\left(t_{n_0} - x\right)^3} + 2g_3(x)\right)\left(\left(\frac{t'}{t_{n_0} - x} + g_1(x) + \Re(\alpha)\right)^2 + \Im(\alpha)^2\right)}{\left(\frac{t'}{\left(t_{n_0} - x\right)^2} + g_2(x)\right)^2} \\ &- 2\left(\frac{t'}{t_{n_0} - x} + g_1(x) + \Re(\alpha)\right). \end{aligned}$$

Multiplying the numerator and denominator of the fraction by $(t_{n_0} - x)^4$ we get

$$f(x) = \frac{\left(2\frac{t'}{\left(t_{n_0} - x\right)} + 2p_3(x)\right)\left((t' + p_2(x))^2 + p_4(x)\right)}{(t' + p_2(x))^2} - 2\left(\frac{t'}{t_{n_0} - x} + p_1(x)\right),$$

where

$$\begin{aligned}
p_1(x) &= (t_{n_0}-x)^2(g_1(x)+\Re(\alpha)),\\
p_2(x) &= g_2(x)\,(t_{n_0}-x)^2,\\
p_3(x) &= 2g_3(x)\,(t_{n_0}-x)^2,\\
p_4(x) &= (t_{n_0}-x)^2\,\Im(\alpha)^2.
\end{aligned}$$

Now we can write

$$f(x) = \frac{t'}{t_{n_0}-x}s_1(x)+s_2(x),$$

where

$$\begin{aligned}
s_1(x) =& \frac{2\left((t'+p_2(x))^2+p_4(x)\right)}{(t'+p_2(x))^2}-2,\\
s_2(x) =& \frac{2p_3(x)\left((t'+p_2(x))^2+p_4(x)\right)}{(t'+p_2(x))^2}-2p_1(x).
\end{aligned}$$

With a straightforward calculation we get

$$s_1(x) = 2\left(\frac{-2t'p_2(x)+p_2(x)^2+2t'p_1(x)-p_1(x)^2+p_4(x)}{(t'+p_2(x))^2}\right),$$

and then

$$\begin{aligned}
\left|\frac{t'}{t_{n_0}-x}s_1(x)\right| \leq& 2\left(\frac{2t'G_2\frac{\pi}{2a}+G_2^2\left(\frac{\pi}{2a}\right)^3+2t'\frac{\pi}{2a}(G_1+|\Re(\alpha)|)}{t'^2}\right)\\
&+2\left(\frac{\left(\frac{\pi}{2a}\right)^3(G_1+|\Re(\alpha)|)^2+\left(\frac{\pi}{2a}\right)^2\Im(\alpha)^2}{t'^2}\right)\\
=:& C_1,
\end{aligned}$$

where we used the fact that $g_2(x)>0\ \forall x\in\mathbb{R}$. It is fundamental to observe that C_1 doesn't depend on n_0. Moreover we have

$$\begin{aligned}
|s_2(x)| \leq& \frac{2G_3\frac{\pi}{2a}}{t'^2}\left(\left(t'+\left(\frac{\pi}{2a}\right)^2(G_1+|\Re(\alpha)|)\right)^2+\left(\frac{\pi}{2a}\right)^2\Im(\alpha)^2\right)\\
&+2\left(G_1+|\Re(\alpha)|\right),\\
=:& C_2
\end{aligned}$$

where we used again the fact that $g_2(x)>0\ \forall x\in\mathbb{R}$. We get that also C_2 doesn't depend on n_0. Then for $|x-\frac{\pi}{a}n_0|\leq\frac{\pi}{2a}$ we have obtained

$$|\tau''(x)| \leq \frac{\pi A^2}{\Im(\alpha)}(C_1+C_2) =: D.$$

Since D doesn't depend on n_0, and considering that the union of all the intervals $|x - \frac{\pi}{a} n_0| \leq \frac{\pi}{2a}$ as n_0 varies in $\mathbb{Z}$ is the whole real line, we finally get

$$|\tau''(x)| \leq D, \quad \forall x \in \mathbb{R}.$$

□

Lemma 4.12. *Consider a meromorphic inner function $\Theta(z)$ given by*

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}, \tag{4.24}$$

where $(\{t_n\}, \{t'_n\})$ is a bandlimit pair and $\Im(\alpha) > 0$. Let b be the logarithmic residue of $\Theta(z)$. Then

$$b = -\limsup_{y \to +\infty} y^{-1} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2 + 1} \right) + \overline{\alpha} \right|. \tag{4.25}$$

Proof. For $y > 0$ we observe that

$$\Im \left(\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \alpha \right) = \sum_n \left(\frac{y}{t_n^2 + y^2} \right) + \Im(\alpha) > \Im(\alpha).$$

Since $\Im(\alpha) > 0$, we get

$$\lim_{y \to +\infty} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \alpha \right| > 0.$$

and hence

$$\limsup_{y \to +\infty} y^{-1} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \alpha \right| = 0.$$

Thanks to (2.12) we obtain

$$\begin{aligned}
b &= -\limsup_{y \to +\infty} y^{-1} \log |\Theta(iy)| \\
&= -\limsup_{y \to +\infty} y^{-1} \log \left| \frac{\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1}\right) + \alpha} \right| \\
&= -\limsup_{y \to +\infty} y^{-1} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha} \right| \\
&\quad - \limsup_{y \to +\infty} y^{-1} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \alpha \right| \\
&= -\limsup_{y \to +\infty} y^{-1} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha} \right|,
\end{aligned}$$

as we wanted to show. □

Proposition 4.13. *Consider a meromorphic inner function $\Theta(z)$ given by*

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) + \alpha}, \tag{4.26}$$

where $(\{t_n\}, \{t'_n\})$ is a bandlimit pair and $\Im(\alpha) > 0$. Let b be the logarithmic residue of $\Theta(z)$. If

$$\lim_{y \to +\infty} \Im \left(\frac{\sum_n \frac{t'_n}{(t_n - iy)^3}}{\sum_n \frac{t'_n}{(t_n - iy)^2}} \right) = c > 0, \tag{4.27}$$

then $b \geq 2c > 0$.

Proof. Consider the representation of $\Theta(z)$ given in (4.7). A simple calculation gives

$$-i\frac{\Theta'(iy)}{\Theta(iy)} = -i\frac{2i\Im(\alpha)\sum_n \frac{t'_n}{(t_n - iy)^2}}{\left(\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \alpha\right)\left(\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha}\right)}.$$

Recalling that

$$\lim_{y \to +\infty} \left(\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha} \right) = 0,$$
$$\lim_{y \to +\infty} \left(\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \alpha \right) = 2i\Im(\alpha),$$

we obtain

$$\begin{aligned}
\lim_{y \to +\infty} \left(-i\frac{\Theta'(iy)}{\Theta(iy)} \right) &= \lim_{y \to +\infty} \left(-i\frac{\sum_n \frac{t'_n}{(t_n - iy)^2}}{\sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha}} \right) \\
&= \lim_{y \to +\infty} \left(-2i\frac{\sum_n \frac{t'_n}{(t_n - iy)^3}}{\sum_n \frac{t'_n}{(t_n - iy)^2}} \right),
\end{aligned}$$

where in the last step we used L'Hopital's rule. Hence we get

$$\begin{aligned}
\lim_{y \to +\infty} \Re \left(-i\frac{\Theta'(iy)}{\Theta(iy)} \right) &= \lim_{y \to +\infty} \Re \left(-2i\frac{\sum_n \frac{t'_n}{(t_n - iy)^3}}{\sum_n \frac{t'_n}{(t_n - iy)^2}} \right) \\
&= 2 \lim_{y \to +\infty} \Im \left(\frac{\sum_n \frac{t'_n}{(t_n - iy)^3}}{\sum_n \frac{t'_n}{(t_n - iy)^2}} \right) \\
&\geq 2c \\
&> 0
\end{aligned}$$

Then for every $\epsilon > 0$ we set $b_\epsilon = 2c - \epsilon$, and there exists $M_\epsilon > 0$ such that $-\Re\left(i\frac{\Theta'(iy)}{\Theta(iy)}\right) > b_\epsilon \ \forall y > M_\epsilon$. Thanks to this we obtain

$$\begin{aligned}
b_\epsilon(y - M_\epsilon) = \int_{M_\epsilon}^{y} b_\epsilon \, ds \leq \int_{M_\epsilon}^{y} -\Re\left(i\frac{\Theta'(is)}{\Theta(is)}\right) ds = -\Re\left(\int_{M_\epsilon}^{y} i\frac{\Theta'(is)}{\Theta(is)} ds\right) \\
= -\Re\left(\log(\Theta(iy)) - \log(\Theta(iM_\epsilon))\right) = -\Re\left(\log\left(\frac{\Theta(iy)}{\Theta(iM_\epsilon)}\right)\right) \\
= -\log\left|\frac{\Theta(iy)}{\Theta(iM_\epsilon)}\right| = -\log|\Theta(iy)| + \log|\Theta(iM_\epsilon)| \\
\leq -\log|\Theta(iy)|,
\end{aligned}$$

where in the last step we observed that $\log|\Theta(iM_\epsilon)| < 0$ since $|\Theta(iM_\epsilon)| < 1$. Then we get

$$-y^{-1}\log|\Theta(iy)| + y^{-1}M_\epsilon b_\epsilon \geq b_\epsilon.$$

Recalling (2.12), we have that b is given by

$$\begin{aligned}
b &= \limsup_{y\to+\infty}\left(-y^{-1}\log|\Theta(iy)|\right) \\
&= \limsup_{y\to+\infty}\left(-y^{-1}\log|\Theta(iy)| + y^{-1}M_\epsilon b_\epsilon\right) \\
&\geq b_\epsilon \\
&> 0.
\end{aligned}$$

Hence we get $b \geq 2c - \epsilon$ for every $\epsilon > 0$, and we can conclude that

$$b \geq 2c > 0.$$

□

4.4 Sampling formulas in terms of the sampling points

In this section we introduce many new sampling formulas for non uniform sampling. In *Theorems* 4.14 we derive the expression of the sampling formula of *Theorem* 4.2 in terms of the sampling points, as we had set out in *Section* 4.2. Moreover we introduce also new sampling formulas directly expressed in terms of the sampling points (*Theorems* 4.17, 4.18 and 4.19). These sampling formulas result to have less strong constraints than Paley-Wiener-Levinson theorem for a finite, but big as desired, subsequence of the sampling sequence.

Theorem 4.14. *Let $\{t_n\}_n \subset \mathbb{R}$ and $\{t'_n\}_n \subset (0, \infty)$ be two sequences with the following properties:*

1. *$\{t_n\}_n$ is a strictly increasing sequence with no finite accumulation point;*

2. $t'_n > 0 \ \forall n \in \mathbb{Z}$;
3. $\sum_n \frac{t'_n}{1+t_n^2} < \infty$;
4. $\sum_n t'_n = +\infty$;
5. $b := -\limsup_{y\to+\infty} y^{-1} \log \left| \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha} \right| > 0$,
   where
   $\alpha = -\lim_{y\to+\infty} \sum_n t'_n \left(\frac{1}{t_n+iy} - \frac{t_n}{t_n^2+1} \right)$.

Then for every $a \in \mathbb{R}$ s.t. $0 < a \leq \frac{b}{2}$ and for every $G \in \mathcal{PW}_a$ we have

$$G(z) = \sum_n \frac{t'_n e^{ia(t_n - z)}}{(t_n - z)\left(\sum_m t'_m \left(\frac{1}{t_m - z} - \frac{t_m}{t_m^2+1} \right) + \alpha \right)} G(t_n).$$

The series converges in norm of $\mathcal{PW}_a$.

Proof. First of all we observe that a necessary condition for property (5) is that

$$\lim_{y\to+\infty} \sum_n t'_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha} = 0,$$

and that α actually verifies this condition.

Thanks to properties (1)-(4) we have that $(\{t_n\}, \{t'_n\})$ is a bandlimit pair, and thanks to *Theorem* 4.6 we get that the function

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) + \overline{\alpha}}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) + \alpha} \tag{4.28}$$

is a meromorphic inner function such that $\{t_n\}_n$ is the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$. By *Lemma* 4.12 we have that the logarithmic residue of $\Theta(z)$ is equal to b. Again thanks to *Theorem* 4.6 we have

$$\Theta'(t_n) = \frac{2i\Im(\alpha)}{t'_n},$$

and by a simple calculation we get

$$1 - \Theta(z) = \frac{2i\Im(\alpha)}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) + \alpha}.$$

Fix any $a \in \mathbb{R}$ s.t. $0 < a \leq \frac{b}{2}$ (we can do this since $b > 0$). Hence by *Theorem* 4.2 for every $G \in \mathcal{PW}_a$ we finally obtain

$$\begin{aligned} G(z) &= \sum_n \frac{(1 - \Theta(z)) e^{ia(t_n - z)}}{(t_n - z)\Theta'(t_n)} G(t_n) \\ &= \sum_n \frac{t'_n e^{ia(t_n - z)}}{(t_n - z)\left(\sum_m t'_m \left(\frac{1}{t_m - z} - \frac{t_m}{t_m^2+1} \right) + \alpha \right)} G(t_n). \end{aligned}$$

By the same theorem we obtain also that the series converges in norm of $\mathcal{PW}_a$. $\square$

Thanks to *Theorem* 4.14, given a Paley-Wiener function and a suitable bandlimit pair $(\{t_n\}, \{t'_n\})$ (where the points of the sequence $\{t_n\}_n$ are generally not equidistant), it is possible to rebuild exactly and uniquely the function from its values on the points on the sequence $\{t_n\}_n$, and the sampling formula is expressed only in terms of these values and the bandlimit pair.

Example 4.15.

We consider the bandlimit pair

$$\begin{aligned}\{t_n\}_n &= \{n\}\\ \{t'_n\}_n &= \{\tanh(\pi)\}.\end{aligned}$$

Properties (1), (2), (4) of *Theorem* 4.14 are easily verified. For property (3) we have

$$\sum_n \frac{t'_n}{1+t_n^2} = \tanh(\pi)\sum_n \frac{1}{1+n^2} = \pi\tanh(\pi)\coth(\pi) = \pi.$$

For property (5) we have

$$\begin{aligned}\overline{\alpha} &= -\tanh(\pi)\lim_{y\to+\infty}\sum_n\left(\frac{1}{n-iy}-\frac{n}{n^2+1}\right)\\ &= -i\pi\tanh(\pi)\lim_{y\to+\infty}\coth(\pi y)\\ &= -i\pi\tanh(\pi),\end{aligned}$$

and then $\Im(\alpha) > 0$. Moreover for $y > 0$ we obtain

$$\begin{aligned}\left|\sum_n t'_n\left(\frac{1}{t_n-iy}-\frac{t_n}{t_n^2+1}\right)+\overline{\alpha}\right| &= \left|\tanh(\pi)\left(\sum_n\left(\frac{1}{n-iy}-\frac{n}{n^2+1}\right)-i\pi\right)\right|\\ &= \tanh(\pi)\left|i\pi\coth(\pi y)-i\pi\right|\\ &= \frac{2\pi\tanh(\pi)}{e^{2\pi y}-1}\end{aligned}$$

Consequently there exists $A > 0$ such that for y big enough we get

$$\left|\sum_n t'_n\left(\frac{1}{t_n-iy}-\frac{t_n}{t_n^2+1}\right)+\overline{\alpha}\right| < Ae^{-2\pi y}.$$

Hence also property (5) of *Theorem* 4.14 is verified, with $c = 2\pi$, and then the theorem can be applied. We set $a = \frac{c}{2} = \pi$, and hence for any $G \in \mathcal{PW}_\pi$

we obtain

$$
\begin{aligned}
G(z) &= \sum_n \frac{t_n' e^{ia(t_n - z)}}{(t_n - z)\left(\sum_n t_n' \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha\right)} G(t_n) \\
&= \sum_n \frac{\tanh(\pi) e^{i\pi(n-z)}}{(n - z)\left(\tanh(\pi)\sum_n \left(\frac{1}{n - z} - \frac{n}{n^2+1}\right) + i\pi\tanh(\pi)\right)} G(n) \\
&= \sum_n \frac{e^{i\pi(n-z)}}{(n-z)\left(-\pi\cot(\pi z) + i\pi\right)} G(n) \\
&= \sum_n \frac{(1 - e^{2\pi i z}) e^{i\pi(n-z)}}{2\pi i (n-z)} G(n) \\
&= \sum_n \frac{(1 - e^{-2\pi i (n-z)}) e^{i\pi(n-z)}}{2\pi i (n-z)} G(n) \\
&= \sum_n \frac{\sin(\pi(n-z))}{\pi(n-z)} G(n),
\end{aligned}
$$

that is the classical sampling formula for the Paley-Wiener space $\mathcal{PW}_\pi$.

The result of *Theorem* 4.14 is mainly based on *Theorem* 3.1. But in *Chapter* 3 we proved also other inclusion properties. *Theorems* 4.17, 4.18 and 4.19 introduce different sampling formulas derived mainly from the inclusion property in *Theorem* 3.3. Before introducing these 3 theorems we need to recall the well-known Phragmen-Lindelof theorem (see [51], p. 80).

Theorem 4.16 (Phragmen-Lindelof)**.** *Let $F(z)$ be continuous on a closed sector of opening π/μ and analytic in the open sector. Suppose that on the bounding rays of the sector,*

$$|F(z)| \leq M,$$

and that for some $\nu < \mu$,

$$|F(z)| \leq e^{r^\nu}$$

whenever z lies inside the sector and $|z| = r$ is sufficiently large. Then $|F(z)| \leq M$ throughout the sector.

Theorem 4.17. *Let $a > 0$ and let $\mathbb{K} = \{n_k\}_{k=0,\dots,K}$ be a finite set of consecutive integers of any size. Let $\{t_n\}_n$ be a strictly increasing sequence such that $t_n = \frac{\pi}{a} n$ $\forall n \in \mathbb{Z} \setminus \mathbb{K}$, and that*

$$\frac{\pi}{a}(n_0 - 1) < t_{n_0} < \dots < t_{n_K} < \frac{\pi}{a}(n_K + 1).$$

Then $\forall G \in \mathcal{PW}_a$ the following sampling formula holds:

$$G(z) = \sum_n \left(\prod_{m \neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z - t_n}{\frac{\pi}{a} m}} \right) G(t_n),$$

and the convergence of the series is uniform on the compact subsets of $\mathbb{C}$.

Proof. We prove the theorem supposing that $t_0 = 0$. The proof for the case $t_0 \neq 0$ can easily derived from this one.

It is immediate to see that the couple $(\{t_n\}, \{t'_n = 1\})$ is a bandlimit pair, since it verifies all the conditions required in *Definition* 4.5. Let $E(z) = e^{-iaz}$ (so that $\mathcal{PW}_a = \mathcal{B}(E)$) and

$$E_1(z) = \left(\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \alpha \right) z \prod_{n \neq 0} \left(\frac{t_n - z}{\frac{\pi}{a} n} \right) e^{\frac{z}{\frac{\pi}{a} n}}, \tag{4.29}$$

where $\alpha = \sum_{n \in \mathbb{K}} \left(\frac{t_n}{t_n^2+1} - \frac{\frac{\pi}{a} n}{\left(\frac{\pi}{a} n\right)^2 + 1} \right) + ia$. The product in (4.29) converges, since

$$z \prod_{n \neq 0} \left(\frac{\frac{\pi}{a} n - z}{\frac{\pi}{a} n} \right) e^{\frac{z}{\frac{\pi}{a} n}} = \frac{\sin(az)}{a}, \tag{4.30}$$

and $t_n \neq \frac{\pi}{a} n$ only for a finite number of $n \in \mathbb{Z}$. Proceeding as in the proof of *Theorem* 4.8, we easily obtain that $E(z)$ is a Hermite Biehler function. We easily observet that

$$E_1(z) = \left(\sum_n \left(\frac{1}{t_n - z} - \frac{\frac{\pi}{a} n}{\left(\frac{\pi}{a} n\right)^2 + 1} \right) + ia \right) z \prod_{n \neq 0} \left(\frac{t_n - z}{\frac{\pi}{a} n} \right) e^{\frac{z}{\frac{\pi}{a} n}}$$

We want to show that $E(z)$ and $E_1(z)$ satisfy the conditions of Theorem 3.3, which means that $\left|\frac{E(x)}{E_1(x)}\right| < M$ for all $x \in \mathbb{R}$ and for some $M > 0$, that $\frac{E(z)}{E_1(z)}$ is of bounded type on $\mathbb{C}^+$ and that

$$\limsup_{y \to +\infty} y^{-1} \log \left| \frac{E(iy)}{E_1(iy)} \right| \leq 0.$$

We start by proving that $\left|\frac{E(x)}{E_1(x)}\right| < M$ for all $x \in \mathbb{R}$ and for some $M > 0$. We can write

$$\begin{aligned} E_1(z) = &\sum_{n \neq 0} \left(\frac{z}{\frac{\pi}{a} n} \prod_{m \neq 0, n} \left(\frac{t_m - z}{\frac{\pi}{a} m} \right) e^{\frac{z}{\frac{\pi}{a} m}} - \frac{\frac{\pi}{a} n}{\left(\frac{\pi}{a} n\right)^2 + 1} z \prod_{m \neq 0} \left(\frac{t_m - z}{\frac{\pi}{a} m} \right) e^{\frac{z}{\frac{\pi}{a} m}} \right) \\ &- \prod_{m \neq 0} \left(\frac{t_m - z}{\frac{\pi}{a} m} \right) e^{\frac{z}{\frac{\pi}{a} m}} + iaz \prod_{m \neq 0} \left(\frac{t_m - z}{\frac{\pi}{a} m} \right) e^{\frac{z}{\frac{\pi}{a} m}}. \end{aligned}$$

First suppose that $t_{n_0} \neq \frac{\pi}{a} n_0$ for some $n_0 \neq 0$, and that $t_n = \frac{\pi}{a} n$ for $n \neq n_0$. Then

$$E_1(z) = \frac{z}{\frac{\pi}{a} n_0} \prod_{m \neq 0, n_0} \left(\frac{\frac{\pi}{a} m - z}{\frac{\pi}{a} m} \right) e^{\frac{z}{\frac{\pi}{a} m}} + \frac{t_{n_0} - z}{\frac{\pi}{a} n_0 - z} S(z),$$

where

$$\begin{aligned}
S(z) = &- \frac{\frac{\pi}{a}n_0}{\left(\frac{\pi}{a}n_0\right)^2+1} z \prod_{m\neq 0}\left(\frac{\frac{\pi}{a}m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}} \\
&+ \sum_{n\neq n_0}\left(\frac{z}{\frac{\pi}{a}n}\prod_{m\neq 0,n}\left(\frac{\frac{\pi}{a}m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2+1} z \prod_{m\neq 0}\left(\frac{\frac{\pi}{a}m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}}\right) \\
&- \prod_{m\neq 0}\left(\frac{\frac{\pi}{a}m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}} + iaz \prod_{m\neq 0}\left(\frac{\frac{\pi}{a}m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}}.
\end{aligned}$$

Then, for $x \in \mathbb{R}$, since trivially

$$\lim_{x\to\pm\infty} \frac{t_{n_0} - x}{\frac{\pi}{a}n_0 - x} = 1,$$

we get

$$\begin{aligned}
&\lim_{x\to\pm\infty} |E_1(x)| \\
&= \lim_{x\to\pm\infty} \left|\left(\sum_n \left(\frac{1}{\frac{\pi}{a}n - x} - \frac{\frac{\pi}{a}n}{(\frac{\pi}{a}n)^2+1}\right) + ia\right) x \prod_{m\neq 0}\left(\frac{\frac{\pi}{a}m - x}{\frac{\pi}{a}m}\right) e^{\frac{x}{\frac{\pi}{a}m}}\right| \\
&= \lim_{x\to\pm\infty} \left|-a(-i + \cot(ax))\frac{1}{a}\sin(ax)\right| \\
&= \lim_{x\to\pm\infty} \left|-e^{-iax}\right| \\
&= 1.
\end{aligned}$$

Considering that $E_1(x)$ is a continuous function with no zeros on the real line, we can conclude that it has a lower bound $M > 0$ such that $E_1(x) > M$ $\forall x \in \mathbb{R}$.

Now, suppose $n_k \neq 0$ $\forall k = 0, ..., K$. We iterate on $k = 0, ..., K$, beginning from $k = 0$. We start considering the sequence $\{t_n\}_n = \{n\}$, and for every iteration we replace n_k with t_{n_k}. Let $\{t_{k,n}\}_n$ be the sequence obtained after having replaced n_k with t_{n_k}, and define

$$E_{1,k}(z) = \left(\sum_n \left(\frac{1}{t_{k,n} - z} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2+1}\right) + ia\right) z \prod_{n\neq 0}\left(\frac{t_{k,n} - z}{n}\right) e^{\frac{z}{\frac{\pi}{a}n}},$$

so that $E_{1,K}(z) = E_1(z)$. The first iteration is clearly given by the case described above, with $t_{0,n_0} = n_0$ and $t_{0,n} = n$ for $n \neq n_0$. For $k \geq 1$ we get

$$E_{1,k}(z) = \frac{z}{\frac{\pi}{a}n_k}\prod_{m\neq 0,n_k}\left(\frac{t_{k-1,m} - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}} + \frac{t_{k,n_k} - z}{\frac{\pi}{a}n_k - z} S_k(z),$$

where

$$\begin{aligned} S_k(z) = &- \frac{\frac{\pi}{a}n_k}{\left(\frac{\pi}{a}n_k\right)^2+1} z \prod_{m\neq 0}\left(\frac{t_{k-1,m}-z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}} \\ &+ \sum_{n\neq n_k}\left(\frac{z}{n}\prod_{m\neq 0,n}\left(\frac{t_{k-1,m}-z}{\frac{\pi}{a}m}\right)e^{\frac{z}{\frac{\pi}{a}m}} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2+1} z \prod_{m\neq 0}\left(\frac{t_{k-1,m}-z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}}\right) \\ &- \prod_{m\neq 0}\left(\frac{t_{k-1,m}-z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}} + iaz\prod_{m\neq 0}\left(\frac{t_{k-1,m}-z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}}. \end{aligned}$$

Therefore, proceeding as above, for $x \in \mathbb{R}$ we trivially get

$$\lim_{x\to\pm\infty} \frac{t_{k,n_k}-x}{\frac{\pi}{a}n_k - x} = 1,$$

and hence, observing that $t_{k-1,n_k} = \frac{\pi}{a}n_k$, we obtain

$$\begin{aligned} &\lim_{x\to\pm\infty} |E_{1,k}(x)| \\ &= \lim_{x\to\pm\infty}\left|\left(\sum_n\left(\frac{1}{t_{k-1,n}-x} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2+1}\right)+ia\right) x\prod_{m\neq 0}\left(\frac{t_{k-1,m}-x}{\frac{\pi}{a}m}\right)e^{\frac{z}{\frac{\pi}{a}m}}\right| \\ &= \lim_{x\to\pm\infty}|E_{1,k-1}(z)| \\ &= 1. \end{aligned}$$

Then in particular we get $\lim_{x\to\pm\infty}|E_1(x)| = \lim_{x\to\pm\infty}|E_{1,K}(x)| = 1$. Considering that $E_1(x)$ is a continuous function with no zeros on the real line, similarly to above we can conclude that $E_1(x)$ has a lower bound $M > 0$ such that

$$|E_1(x)| > M \quad \forall x \in \mathbb{R}.$$

Therefore we finally obtain that

$$\left|\frac{E(x)}{E_1(x)}\right| < M \quad \forall x \in \mathbb{R}. \tag{4.31}$$

Now we want to show that $\frac{E(z)}{E_1(z)}$ is of bounded type on the upper half plane. We recall (4.30) and we write

$$\begin{aligned} \frac{E(z)}{E_1(z)} &= \left(\left(\sum_n\left(\frac{1}{t_n-z} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2+1}\right)+ia\right) z\left(\prod_{n\neq 0}\left(\frac{t_n-z}{\frac{\pi}{a}n}\right)e^{\frac{z}{\frac{\pi}{a}n}}\right)e^{iaz}\right)^{-1} \\ &= \left(N(z)\frac{\sin(az)}{a}e^{iaz}\prod_{n\in K}\left(\frac{t_n-z}{\frac{\pi}{a}n-z}\right)\right)^{-1}, \end{aligned}$$

where

$$N(z) = \left(\sum_n \left(\frac{1}{t_n - z} - \frac{\frac{\pi}{a} n}{\left(\frac{\pi}{a} n\right)^2 + 1} \right) + ia \right).$$

We observe that $\sin(az)e^{iaz}$ is obsiously bounded on $\mathbb{C}^+$. We have

$$N(x+iy) = \left(\sum_n \left(\frac{t_n - x + iy}{(t_n - x)^2 + y^2} - \frac{\frac{\pi}{a} n}{\left(\frac{\pi}{a} n\right)^2 + 1} \right) + ia \right),$$

and hence we observe that $\Im(N(x+iy)) \geq a > 0$ for all $y \geq 0$. Then we get that $|N(z)| \geq a$ for all $z \in \mathbb{C}^+$. Now we write

$$\frac{E(z)}{E_1(z)} = \frac{a}{\sin(az)e^{iaz}N(z)} \prod_{n \in K} \left(\frac{\frac{\pi}{a} n - z}{t_n - z} \right).$$

Since $\frac{1}{N(z)}$ and $\sin(az)e^{iaz}$ are bounded on $\mathbb{C}^+$, we get that $\frac{a\frac{1}{N(z)}}{\sin(az)e^{iaz}}$ is of bounded type on $\mathbb{C}^+$. The product $\prod_{n \in K} \left(\frac{\frac{\pi}{a}n - z}{t_n - z} \right)$ is of bounded type since all the polynomials and their reciprocals are of bounded type on $\mathbb{C}^+$, as we pointed out in section 2.4. Then we conclude that $\frac{E_1(z)}{E(z)}$ is of bounded type on $\mathbb{C}^+$ since the product of two functions of bounded type is obviously of bounded type.

It remains to show that $\frac{E_1(z)}{E(z)}$ is of non-positive mean type. We have that

$$\begin{aligned} &\limsup_{y \to +\infty} y^{-1} \log |E_1(iy)| \\ &= \limsup_{y \to +\infty} y^{-1} \left(\log |s(y) + ia| + \log |\sin(iay)| + f(y) \right) \end{aligned} \tag{4.32}$$

where

$$\begin{aligned} s(y) &= \sum_n \left(\frac{1}{t_n - iy} - \frac{\frac{\pi}{a} n}{\left(\frac{\pi}{a} n\right)^2 + 1} \right), \\ f(y) &= \sum_{n \in \mathbb{K}} \log \left| \frac{t_n - iy}{\frac{\pi}{a} n} \right| - \log \left| \frac{\frac{\pi}{a} - iy}{\frac{\pi}{a} n} \right|. \end{aligned}$$

It is easy to check that

$$\begin{aligned} \limsup_{y \to +\infty} y^{-1} \log |\sin(iay)| &= a, \\ \limsup_{y \to +\infty} y^{-1} f(y) &= 0. \end{aligned} \tag{4.33}$$

Define δ as the maximum distance between two successive elements of $\{t_n\}_n$:

$$\delta = \max_{n \in \mathbb{Z}} (t_{n+1} - t_n).$$

We observe that δ is for sure finite since by definition $t_{n+1} - t_n \neq \frac{\pi}{a}$ only for a finite number of $n \in \mathbb{Z}$. Then for $y > 0$ we have

$$\begin{aligned}
\log|s(y) + ia| &= \log\left|\sum_n \left(\frac{t_n + iy}{t_n^2 + y^2} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2 + 1}\right) + ia\right| \\
&\geq \log\left|\Im\left(\sum_n \left(\frac{t_n + iy}{t_n^2 + y^2} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2 + 1}\right) + ia\right)\right| \\
&= \log\left(a + y\sum_n \frac{1}{t_n^2 + y^2}\right) \\
&\geq \log\left(a + y\sum_n \frac{1}{\left(\frac{\pi}{a}n + \delta\right)^2 + y^2}\right) \\
&= \log\left(a - \frac{ia}{2}\left(\cot(a(\delta - iy)) - \cot(a(\delta + iy))\right)\right).
\end{aligned}$$

Since

$$\lim_{y\to\infty} a - \frac{ia}{2}\left(\cot(a(\delta - iy)) - \cot(a(\delta + iy))\right) = 2a,$$

we get

$$\begin{aligned}
&\limsup_{y\to+\infty} y^{-1}\log\left|\sum_n \left(\frac{1}{t_n - iy} - \frac{\frac{\pi}{a}n}{\left(\frac{\pi}{a}n\right)^2 + 1}\right) + ia\right| \\
&\geq \limsup_{y\to+\infty} y^{-1}\log\left(a - \frac{ia}{2}\left(\cot(a(\delta - iy)) - \cot(a(\delta + iy))\right)\right) \\
&= 0.
\end{aligned} \tag{4.34}$$

From (4.32), (4.33), (4.34) we obtain

$$\limsup_{y\to+\infty} y^{-1}\log|E_1(iy)| \geq a.$$

Then we can conclude that

$$\begin{aligned}
\limsup_{y\to+\infty} y^{-1}\log\left|\frac{E(iy)}{E_1(iy)}\right| &= \limsup_{y\to+\infty} y^{-1}\log\left|\frac{e^{ay}}{E_1(iy)}\right| \\
&= \limsup_{y\to+\infty} y^{-1}\left(ay - \log|E_1(iy)|\right) \\
&= a - \limsup_{y\to+\infty}\left(y^{-1}\log|E_1(iy)|\right) \\
&\leq 0.
\end{aligned} \tag{4.35}$$

We have shown that the conditions of *Theorem* 3.3 are satisfied for $E(z)$ and $E_1(z)$. Then for every $G \in \mathcal{PW}_a$ we get

$$G(z) = \sum_n \frac{E_1(z)(1 - \Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n - z)}G(t_n),$$

where

$$\Theta_1(z) = \frac{E_1^{\#}(z)}{E_1(z)} = \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha}, \tag{4.36}$$

and the convergence of the series is uniform on the compact subsets of $\mathbb{C}$. Thanks to *Theorem* 4.6 we have

$$\Theta_1'(t_n) = 2i\Im(\alpha) = 2ia,$$

and by (4.29), for $n \neq 0$ we have

$$E_1(t_n) = t_n \frac{e^{\frac{z}{\frac{\pi}{a}n}}}{\left(\frac{\pi}{a}n\right)} \prod_{m\neq 0,n} \left(\frac{t_m - t_n}{\frac{\pi}{a}m}\right) e^{\frac{t_n}{\frac{\pi}{a}m}},$$

and for $n = 0$ we have

$$E_1(0) = \prod_{m\neq 0} \left(\frac{t_m}{\left(\frac{\pi}{a}m\right)}\right).$$

Moreover, thanks to (4.36) and (4.29), by a simple calculation we get

$$E_1(z)(1-\Theta_1(z)) = 2iaz \prod_{m\neq 0} \left(\frac{t_m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}}.$$

Hence we finally obtain

$$\begin{aligned}
G(z) &= \sum_n \frac{E_1(z)(1-\Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n - z)} G(t_n) \\
&= \left(\prod_{m\neq 0} \frac{t_m - z}{t_m} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(0) + \sum_{n\neq 0} \frac{z}{t_n} \left(\prod_{m\neq 0,n} \frac{t_m - z}{t_m - t_n} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(t_n). \\
&= \sum_n \left(\prod_{m\neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(t_n).
\end{aligned}$$

□

Theorem 4.18. *Fix $a > 0$, and let $\{t_n\}_n$ be a sequence such that $\left|\frac{\pi}{a}n - t_n\right| \leq \delta$ $\forall n \in \mathbb{Z}$ for some $\delta < \frac{\pi}{2a}$. Let $A > 0$ be the constant defined in Proposition 4.10 with respect to the bandlimit pair $(\{t_n\}_n, \{t_n'\}_n)$ with $t_n' = 1$ $\forall n \in \mathbb{Z}$. Suppose that there exists a constants $K > 0$ such that*

$$\left|\chi(x) \prod_{n\neq 0} \left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \geq K, \qquad \textit{for} \quad t_{n_0} + \frac{1}{2A} \leq x \leq t_{n_0+1} - \frac{1}{2A}, \quad \forall n_0 \in \mathbb{Z}, \tag{4.37}$$

where $\chi(x) = x$ if $t_0 = 0$ and $\chi(x) = \left(1 - \frac{x}{t_0}\right)$ if $t_0 \neq 0$. Then for every $G \in \mathcal{PW}_a$ we get

$$G(z) = \sum_n \left(\prod_{m \neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z - t_n}{\frac{\pi}{a} m}} \right) G(t_n),$$

and the convergence of the series is uniform on the compact subsets of $\mathbb{C}$.

Proof. We prove the theorem supposing that $t_0 = 0$. The proof for the case $t_0 \neq 0$ can easily derived from this one.

Let $E(z) = e^{-iaz}$. Consider the couple $(\{t_n\}_n, \{t'_n\}_n)$ with $t'_n = 1\ \forall n \in \mathbb{Z}$: it is easy to see that it is a bandlimit pair since it verifies all the conditions required in *Section* 4.2. Let $\Theta(z)$ be a meromorphic inner function associated to this bandlimit pair accoding to *Theorem* 4.6, given by:

$$\Theta_1(z) = \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \overline{\alpha}}{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \alpha}, \tag{4.38}$$

where $\alpha = \sum_{n \neq 0} \left(\frac{t_n}{t_n^2 + 1} - \frac{1}{t_n} \right) + ia$. Thanks to *Theorem* 4.8, we know that the function

$$\tilde{E}_1(z) = \left(\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \alpha \right) \tag{4.39}$$

is a de Branges function of $\Theta(z)$. We define

$$\begin{aligned}
E_1(z) =& \tilde{E}_1(z) \prod_{n \neq 0} e^{\frac{z}{\frac{\pi}{a} n} - \frac{z}{t_n}} \\
=& \left(\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \sum_{n \neq 0} \left(\frac{t_n}{t_n^2 + 1} - \frac{1}{t_n} \right) + ia \right) \\
& z \prod_{n \neq 0} \left(\frac{t_n - z}{t_n} \right) e^{\frac{z}{t_n}} \prod_{n \neq 0} e^{\frac{z}{\frac{\pi}{a} n} - \frac{z}{t_n}} \\
=& \left(-\frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{t_n - z} - \frac{1}{t_n} \right) + ia \right) z \prod_{n \neq 0} \left(\frac{t_n - z}{t_n} \right) e^{\frac{z}{\frac{\pi}{a} n}}.
\end{aligned}$$

The product $\prod_{n\neq 0} e^{\frac{z}{\frac{\pi}{a}n}-\frac{z}{t_n}}$ converges since

$$\begin{aligned}\left|\prod_{n\neq 0} e^{\frac{z}{\frac{\pi}{a}n}-\frac{z}{t_n}}\right| &= \prod_{n\neq 0} e^{\frac{z\left|\frac{\pi}{a}n-t_n\right|}{\frac{\pi}{a}|nt_n|}} \\ &\leq \prod_{n\neq 0} e^{\frac{z}{2|nt_n|}} \\ &\leq \prod_{n\geq 1} e^{\frac{z}{\frac{\pi}{a}n\left(n-\frac{1}{2}\right)}} \\ &= e^{\frac{az}{\pi}\sum_{n\geq 1}\frac{1}{n\left(n-\frac{1}{2}\right)}} \\ &= e^{\frac{az\log(16)}{\pi}},\end{aligned}$$

where we used the fact that $\sum_{n\geq 1}\frac{1}{n\left(n-\frac{1}{2}\right)} = \log(16)$. It is easy to see that also $E_1(z)$ is a de Branges function of $\Theta(z)$, since

$$\frac{E_1^{\#}(z)}{E_1(z)} = \frac{\tilde{E}_1^{\#}(z)\prod_{n\neq 0} e^{\frac{z}{\frac{\pi}{a}n}-\frac{z}{t_n}}}{\tilde{E}_1(z)\prod_{n\neq 0} e^{\frac{z}{\frac{\pi}{a}n}-\frac{z}{t_n}}} = \frac{\tilde{E}_1^{\#}(z)}{\tilde{E}_1(z)} = \Theta(z).$$

We want to show that $E(z)$ and $E_1(z)$ satisfy the conditions of Theorem 3.3, which means that $\left|\frac{E(x)}{E_1(x)}\right| < M$ for all $x\in\mathbb{R}$ and for some $M>0$, that $\frac{E(z)}{E_1(z)}$ is of bounded type on $\mathbb{C}^+$ and that

$$\limsup_{y\to+\infty} y^{-1}\log\left|\frac{E(iy)}{E_1(iy)}\right| \leq 0.$$

We start by proving that $\frac{E(x)}{E_1(x)} < M$ for all $x\in\mathbb{R}$ and for some $M>0$. We recall that the phase function of $\Theta_1(z)$ is the unique function $\tau(x)$ such that $\forall x\in\mathbb{R}$ we get

$$\Theta_1(x) = e^{2\pi i\tau(x)}.$$

Setting $f(x) = -\frac{1}{x} + \sum_{n\neq 0}\left(\frac{1}{t_n - x} - \frac{1}{t_n}\right)$, we get

$$e^{2\pi i\tau(x)} = \frac{f(x) - ia}{f(x) + ia},$$

and hence

$$2\pi i\tau(x) = \log\left(\frac{f(x)-ia}{f(x)+ia}\right) + 2\pi i n(x),$$

where $n(x)$ is an integer that depends on x. Using the well-known identity

$$\operatorname{arccot}(x) = \frac{i}{2}\log\left(\frac{x-i}{x+i}\right)$$

we obtain

$$\begin{aligned}\tau(x) &= \frac{1}{2\pi i}\log\left(\frac{f(x)-ia}{f(x)+ia}\right)+n(x)\\ &= \frac{1}{2\pi i}\log\left(\frac{\frac{f(x)}{a}-i}{\frac{f(x)}{a}+i}\right)+n(x)\\ &= -\frac{1}{\pi}\operatorname{arccot}\left(\frac{f(x)}{a}\right)+n(x).\end{aligned}$$

Then, recalling that $\cot(-x) = -\cot(x)$ and that $\cot(x) = \cot(x+n\pi)$ $\forall n \in \mathbb{Z}$, we get

$$\cot(\pi\tau(x)) = -\frac{f(x)}{a},$$

which means

$$f(x) = -a\cot(\pi\tau(x)).$$

Now for $n \neq 0$ we set

$$g_n(x) = \left(\frac{t_n - x}{t_n}\right)e^{\frac{x}{t_n}},$$

and we observe that, for $x \neq t_n$ $\forall n \in \mathbb{Z}$, $g_n(x)$ is differentiable and we have

$$\frac{g_n'(x)}{g_n(x)} = \frac{1}{t_n} - \frac{1}{t_n - x}.$$

Then, defining $g(x) = x\prod_{n\neq 0} g_n(x)$, we get

$$\frac{g'(x)}{g(x)} = \frac{1}{x} + \sum_{n\neq 0}\frac{1}{t_n} - \frac{1}{t_n - x},$$

and hence

$$\frac{g'(x)}{g(x)} = -f(x) = a\cot(\pi\tau(x)). \tag{4.40}$$

Now, fix $n_0 > 0$, and consider x such that $t_{n_0} < x < t_{n_0+1}$. We recall that the spectral function $t(r)$ of $\Theta(z)$ is given by the inverse of the phase function ($t(r) = \tau^{-1}(r)$), and we observe that $t(n) = t_n$ $\forall n \in \mathbb{Z}$. We analyze the case $t\left(n_0 + \frac{1}{2}\right) \leq x < t_{n_0+1}$. Solving (4.40) we get

$$g(x) = c_{n_0} e^{\int_{t\left(n_0+\frac{1}{2}\right)}^{x} a\cot(\pi\tau(s))ds},$$

where the real constant c_{n_0} is given by

$$c_{n_0} = g\left(t\left(n_0 + \frac{1}{2}\right)\right).$$

We recall that by *Proposition* 4.10 we have $B \leq \tau'(x) \leq A$ $\forall x \in \mathbb{R}$ for some $A > 0, B > 0$, and that by *Proposition* 4.11 we have $|\tau''(x)| \leq D$,

$\forall x \in \mathbb{R}$, for some $D > 0$. We recall also the well-known formula of the second derivative of the inverse function:

$$g(x) = f^{-1}(x) \Longrightarrow g''(x) = -\frac{f''(g(x))}{f'(g(x))^3}.$$

Then for the function $t(r) = \tau^{-1}(r)$ we get

$$|t''(r)| = \left|\frac{\tau''(t(x))}{\tau'(t(x))^3}\right| \leq \frac{D}{B^3} =: M_0.$$

Now, by *Lemma* 4.9 we have $t'(n) = \frac{\pi}{a}t'_n = \frac{\pi}{a}\ \forall n \in \mathbb{Z}$, then we observe that $t'(r) > 0\ \forall r \in \mathbb{Z}$ since $t(r)$ is strictly increasing, and hence for $t(n_0 + \frac{1}{2}) \leq r < t_{n_0+1}$ we get

$$t'(r) = t'(n_0+1) - \int_r^{n_0+1} t''(s)ds \leq \frac{\pi}{a} + M_0(n_0+1-r).$$

Recalling that $\cot(\pi r) \leq 0$ for $n_0 + \frac{1}{2} \leq r < n_0 + 1$, we obtain

$$\begin{aligned}
|g(x)| &= |c_{n_0}|e^{\int_{t\left(n_0+\frac{1}{2}\right)}^{x} a\cot(\pi\tau(s))ds} \\
&= |c_{n_0}|e^{\int_{n_0+\frac{1}{2}}^{\tau(x)} a\cot(\pi r)t'(r)dr} \qquad (4.41)\\
&\geq |c_{n_0}|e^{\int_{n_0+\frac{1}{2}}^{\tau(x)} a\cot(\pi r)\left(\frac{\pi}{a}+M_0(n_0+1-r)\right)dr}.
\end{aligned}$$

Now we set

$$\begin{aligned}
h(r) = &\left(1 + \frac{a}{\pi}M_0(n_0+1-r)\right)\log(\sin(\pi(r-n_0))) \\
&+ \frac{aM_0}{\pi}\int_{n_0+\frac{1}{2}}^{r} \log(\sin(\pi(s-n_0)))ds,
\end{aligned}$$

and we observe that

$$\begin{aligned}
\frac{d}{dr}h(r) =&a\cot(\pi(r-n_0))\left(\frac{\pi}{a} + M_0(n_0+1-r)\right) \\
&- \frac{aM_0}{\pi}\log(\sin(\pi(r-n_0))) \\
&+ \frac{aM_0}{\pi}\log(\sin(\pi(r-n_0))) \\
=&a\cot(\pi r)\left(\frac{\pi}{a} + M_0(n_0+1-r)\right),
\end{aligned}$$

where in the last step we used the fact that $\cot(x+n\pi) = \cot(x)\ \forall n \in \mathbb{Z}$.

Hence we can write

$$\begin{aligned}
&\int_{n_0+\frac{1}{2}}^{\tau(x)} a\cot(\pi r)\left(\frac{\pi}{a} + M_0(n_0+1-r)\right) dr \\
&= \Big[\left(1+\frac{a}{\pi}M_0(n_0+1-r)\right)\log(\sin(\pi(r-n_0))) \\
&\quad +\frac{aM_0}{\pi}\int_{n_0+\frac{1}{2}}^{r}\log(\sin(\pi(s-n_0)))ds\Big]_{n_0+\frac{1}{2}}^{\tau(x)} \\
&= \left(1+\frac{a}{\pi}M_0(n_0+1-\tau(x))\right)\log(\sin(\pi(\tau(x)-n_0))) \\
&\quad + \frac{aM_0}{\pi}\int_{n_0+\frac{1}{2}}^{\tau(x)}\log(\sin(\pi(s-n_0)))ds.
\end{aligned}$$

Now we set

$$\begin{aligned}
h_1(r) = &\left(\frac{a}{\pi}M_0(n_0+1-r)\right)\log(\sin(\pi(r-n_0))) \\
&+ \frac{aM_0}{\pi}\int_{n_0+\frac{1}{2}}^{r}\log(\sin(\pi(s-n_0)))ds,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{n_0+\frac{1}{2}}^{\tau(x)} a\cot(\pi r)\left(\frac{\pi}{a} + M_0(n_0+1-r)\right) dr \\
&= \log(\sin(\pi(\tau(x)-n_0))) + h_1(\tau(x)),
\end{aligned}$$

and then

$$\begin{aligned}
|g(x)| &\geq |c_{n_0}|e^{\int_{n_0+\frac{1}{2}}^{\tau(x)} a\cot(\pi r)\left(\frac{\pi}{a}+M_0(n_0+1-r)\right)dr} \\
&= |c_{n_0}|\sin(\pi(\tau(x)-n_0))e^{h_1(\tau(x))}.
\end{aligned}$$

Now for $t\left(n_0+\frac{1}{2}\right) \leq x < t_{n_0+1}$ we have

$$\begin{aligned}
|E_1(x)| =& |(f(x)+i\pi)g(x)|\prod_{n\neq 0} e^{\frac{z}{\frac{\pi}{a}n}-\frac{z}{t_n}} \\
\geq& |c_{n_0}|\,|-a\cot(\pi(\tau(x))+ia|\sin(\pi(\tau(x)-n_0)) \\
&e^{h_1(\tau(x))}\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}} \\
=& |c_{n_0}|\,|-a\cot(\pi(\tau(x)-n_0))+ia|\sin(\pi(\tau(x)-n_0)) \\
&e^{h_1(\tau(x))}\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}} \\
=& |c_{n_0}|ae^{h_1(\tau(x))}\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}}.
\end{aligned}$$

Obviously we have $h_1(n_0+\frac{1}{2}) = 0$, and moreover

$$\begin{aligned}h_1(n_0+1) =& \frac{aM_0}{\pi}\int_{n_0+\frac{1}{2}}^{n_0+1}\log(\sin(\pi(s-n_0)))ds\\ =& \frac{aM_0}{\pi}\int_{\frac{1}{2}}^{1}\log(\sin(\pi s))ds\\ =& -\frac{aM_0\log(2)}{2\pi}.\end{aligned}$$

Since

$$\frac{dh_1(r)}{dr} = a\cot(\pi r)\left(M_0(n_0+1-r)\right) < 0, \quad n_0+\frac{1}{2} \le r < n_0+1,$$

we have that

$$-\frac{aM_0\log(2)}{2\pi} \le h_1(r) \le 0, \quad n_0+\frac{1}{2} \le r < n_0+1,$$

which means

$$-\frac{aM_0\log(2)}{2\pi} \le h_1(\tau(x)) \le 0, \quad t\left(n_0+\frac{1}{2}\right) \le x < t_{n_0+1}.$$

Hence for $t\left(n_0+\frac{1}{2}\right) \le x < t_{n_0+1}$ we get

$$\begin{aligned}|E_1(x)| &\ge |c_{n_0}|ae^{h_1(\tau(x))}\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}}\\ &\ge |c_{n_0}|a2^{-\frac{aM_0}{2\pi}}\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}}.\end{aligned}$$

Now, by definition of c_{n_0} we have

$$\begin{aligned}c_{n_0} &= g\left(t\left(n_0+\frac{1}{2}\right)\right)\\ &= t\left(n_0+\frac{1}{2}\right)\prod_{n\neq 0}\left(\frac{t_n - t\left(n_0+\frac{1}{2}\right)}{t_n}\right)e^{\frac{t\left(n_0+\frac{1}{2}\right)}{t_n}}\\ &= t\left(n_0+\frac{1}{2}\right)\prod_{n\neq 0}\left(\frac{t_n - t\left(n_0+\frac{1}{2}\right)}{t_n}\right)e^{\frac{t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}}\prod_{n\neq 0} e^{\frac{t\left(n_0+\frac{1}{2}\right)}{t_n}-\frac{t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}}.\end{aligned}$$

Using *Proposition* 4.10 and recalling that $t'(r) = \frac{1}{\tau'(t(r))}$ we get

$$\begin{aligned}t\left(n_0+\frac{1}{2}\right) &\ge t_{n_0} + \int_0^{\frac{1}{2}} t'(r)dr\\ &\ge t_{n_0} + \frac{1}{2A},\end{aligned}$$

and

$$\begin{aligned} t\left(n_0+\frac{1}{2}\right) &\leq t_{n_0+1}-\int_0^{\frac{1}{2}} t'(r)dr \\ &\leq t_{n_0+1}-\frac{1}{2A}. \end{aligned}$$

Then we have

$$t_{n_0}+\frac{1}{2A}\leq t\left(n_0+\frac{1}{2}\right)\leq t_{n_0+1}-\frac{1}{2A},$$

and by (4.37) there exists K such that

$$\begin{aligned} |c_{n_0}| &= \left| t\left(n_0+\frac{1}{2}\right)\prod_{n\neq 0}\left(\frac{t_n-t\left(n_0+\frac{1}{2}\right)}{t_n}\right)e^{\frac{t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}}\right| \prod_{n\neq 0} e^{\frac{t\left(n_0+\frac{1}{2}\right)}{t_n}-\frac{t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}} \\ &\geq K\prod_{n\neq 0} e^{\frac{t\left(n_0+\frac{1}{2}\right)}{t_n}-\frac{t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}}. \end{aligned}$$

From this we obtain

$$\begin{aligned} |c_{n_0}|\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}} &\geq K\prod_{n\neq 0} e^{\frac{t\left(n_0+\frac{1}{2}\right)}{t_n}-\frac{t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}}\prod_{n\neq 0} e^{\frac{x}{\frac{\pi}{a}n}-\frac{x}{t_n}} \\ &= K\prod_{n\neq 0} e^{\frac{t\left(n_0+\frac{1}{2}\right)-x}{t_n}+\frac{x-t\left(n_0+\frac{1}{2}\right)}{\frac{\pi}{a}n}} \\ &= K\prod_{n\neq 0} e^{\frac{(\frac{\pi}{a}n-t_n)\left(t\left(n_0+\frac{1}{2}\right)-x\right)}{t_n\frac{\pi}{a}n}} \\ &\geq K\prod_{n\neq 0} e^{\frac{-\delta}{2B(\frac{\pi}{a}|n|-\delta)\frac{\pi}{a}|n|}} \\ &= K\prod_{n\geq 1} e^{\frac{-\delta}{B(\frac{\pi}{a}n-\delta)\frac{\pi}{a}n}} \\ &= Ke^{\frac{1}{B}\sum_{n\geq 1}\frac{1}{\frac{\pi}{a}n(\frac{\pi}{a}n-\delta)}} \\ &=: M_1>0, \end{aligned} \tag{4.42}$$

where we used the fact that

$$\left|t\left(n_0+\frac{1}{2}\right)-x\right| = \left|\int_{n_0+\frac{1}{2}}^x t'(s)ds\right| \leq \int_{n_0+\frac{1}{2}}^{n_0+1}|t'(s)|ds\leq\frac{1}{2B},$$

since $t'(s)=\frac{1}{\tau'(t(s))}<\frac{1}{B}$ thanks to *Proposition* 4.10, and that $\sum_{n\geq 1}\frac{1}{\frac{\pi}{a}n(\frac{\pi}{a}n-\delta)}$ obviously converges. It is important to underline that M_1 is independent on n_0.

Now consider the case $t_{n_0}<x\leq t\left(n_0+\frac{1}{2}\right)$. Similarly to above we have

$$g(x)=c_{n_0}e^{\int_{t\left(n_0+\frac{1}{2}\right)}^x \pi\cot(\pi\tau(s))ds}.$$

Recalling that $\cot(\pi r) > 0$ for $n_0 < r \leq n_0 + \frac{1}{2}$, we obtain

$$\begin{aligned} |g(x)| &= |c_{n_0}| e^{\int_{t\left(n_0+\frac{1}{2}\right)}^{x} a\cot(\pi\tau(s))ds} \\ &= |c_{n_0}| e^{\int_{n_0+\frac{1}{2}}^{\tau(x)} a\cot(\pi r)t'(r)dr}. \\ &= |c_{n_0}| e^{-\int_{\tau(x)}^{n_0+\frac{1}{2}} a\cot(\pi r)t'(r)dr}. \end{aligned}$$

We observe that

$$t'(r) = t'(n_0) + \int_{n_0}^{r} t''(s)ds \leq \frac{\pi}{a} + M_0(r - n_0),$$

and hence we get

$$|g(x)| \geq |c_{n_0}| e^{-\int_{\tau(x)}^{n_0+\frac{1}{2}} a\cot(\pi r)\left(\frac{\pi}{a}+M_0(r-n_0)\right)dr}.$$

From this inequality, proceeding similarly to the case $t\left(n_0 + \frac{1}{2}\right) \leq x < t_{n_0+1}$ after equation (4.41), we obtain that there exists M_3, independent on n_0, such that

$$|E_1(x)| \geq M_2 > 0.$$

Then, for $t_{n_0} < x < t_{n_0+1}$, we finally have

$$|E_1(x)| \geq \min(M_1, M_2) =: M_3 > 0, \tag{4.43}$$

where M_3 doesn't depend on n_0.

For the case $n_0 < 0$ we consider the function the function $g(x)$ on the interval $t_{n_0-1} < x < t_{n_0}$ and, proceeding in a completely analogous way to the case $n_0 > 0$, we get the same result of (4.43). Then we set $M_4 = \inf\{|E_1(x)|, t_{-1} < x < t_1\}$, and we observe that $M_4 > 0$ since $E_1(z)$ doesn't have zeros on the real line. Finally we set $M = \min\{M_3, M_4\} > 0$, and recalling that $E_1(z)$ is continuous everywhere, and hence in particular on $z = t_n$ $\forall n \in \mathbb{Z}$, we can conclude that

$$|E_1(x)| \geq M \quad \forall x \in \mathbb{R}. \tag{4.44}$$

Therefore we finally obtain that

$$\left|\frac{E(x)}{E_1(x)}\right| < M \quad \forall x \in \mathbb{R}. \tag{4.45}$$

Now we want to show that $\frac{E(z)}{E_1(z)}$ is of bounded type on the upper half plane. We write

$$\begin{aligned} \frac{E(z)}{E_1(z)} &= \left(\left(\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + ia\right) z \left(\prod_{n\neq 0} \left(\frac{t_n - z}{t_n}\right) e^{\frac{z}{\frac{\pi}{a}n}}\right) e^{iaz}\right)^{-1} \\ &= \left(N(z)P(z)e^{-iaz}\right)^{-1}, \end{aligned}$$

where

$$P(z) = z\left(\prod_{n\neq 0}\left(\frac{t_n - z}{t_n}\right)e^{\frac{z}{\frac{\pi}{a}n}}\right)e^{2iaz},$$

$$N(z) := \left(\sum_n\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + ia\right).$$

We can write $N(z)$ as

$$N(x+iy) = \left(\sum_n\left(\frac{t_n - x + iy}{(t_n - x)^2 + y^2} - \frac{t_n}{t_n^2+1}\right) + ia\right),$$

and hence we observe that $\Im(N(x+iy)) \geq a > 0$ for all $y \geq 0$. Then we get that $|N(z)| \geq a$ for all $z \in \mathbb{C}^+$.

Now we want to apply the Phragmen-Lindelof theorem (i.e., *Theorem* (4.16)) to the function $P(z)$ on the closed sector $\Omega_1 = \{z = x+iy : x \geq 0, y \geq 0\}$. For $x \in \mathbb{R}$ we have

$$|P(x)| = \left|\frac{E_1(x)}{E(x)}\right|\frac{1}{|N(x)|} \leq \frac{1}{aM} =: P_1.$$

Moreover, for $y \geq 0$ we get

$$\begin{aligned}
|P(iy)| &= e^{-2ay}y\prod_{n\geq 1}\left|1 - \frac{iy}{t_n}\right|\left|1 - \frac{iy}{t_{-n}}\right| \\
&= e^{-2ay}y\prod_{n\geq 1}\left(1 + \frac{y^2}{t_n^2}\right)^{\frac{1}{2}}\left(1 + \frac{y^2}{t_{-n}^2}\right)^{\frac{1}{2}} \\
&\leq e^{-2ay}y\prod_{n\geq 1}\left(1 + \frac{y^2}{\frac{\pi^2}{a^2}(n-\frac{1}{2})^2}\right) \\
&= e^{-2ay}y\cosh(ay).
\end{aligned}$$

The function $e^{-2iay}y\cosh(ay)$ is obviously bounded for $y > 0$, and then there exists $P_2 > 0$ so that $|P(z)| < P_2$ for $y \geq 0$. Given a sequence of complex numbers $\{z_n\}_n$ all different from zero, the greatest lower bound of positive numbers γ for which the series

$$\sum_n \frac{1}{|z_n|^\gamma}$$

is convergent is called the *exponent of convergence* of the sequence $\{z_n\}_n$ (see definition in [51] p. 66). For all $\gamma > 1$ the zeros of $P(z)$ satisfy

$$\sum_{n\neq 0}\frac{1}{|t_n|^\gamma} \leq \sum_{n\neq 0}\frac{1}{\left(\left|\frac{\pi}{a}n\right| - \frac{\pi}{2a}\right)^\gamma} < \infty,$$

while for $\gamma = 1$ we have

$$\sum_{n\neq 0} \frac{1}{\left|\frac{\pi}{a}n\right| + \frac{\pi}{2a}} \leq \sum_{n\neq 0} \frac{1}{|t_n|},$$

and the sum on the right side diverges since the sum on the left side obviously diverges. Then the exponent of convergence of the zeros of $P(z)$ is $\lambda = 1$, and thanks to *Theorem* 6 in [51] (p. 69) we get that the canonical product

$$\prod_{n\neq 0} \left(\frac{t_n - z}{t_n}\right) e^{\frac{z}{t_n}}$$

has order $\rho = 1$. From this we easily obtain that also $P(z)$ has order $\rho = 1$. Therefore $|P(r)| \leq e^{r^{1+\epsilon}}$ for every $\epsilon > 0$. In particular we can take $\epsilon = \frac{1}{2}$, so that $|P(|z|)| \leq e^{|z|^{\frac{3}{2}}}$.

Finally we are in the conditions to apply *Theorem* (4.16) to the function $P(z)$ on the sector Ω_1. Indeed we have shown that $|P(x)| \leq P_1$ for $x \in \mathbb{R}$ and $|P(iy)| \leq P_2$ for $y \geq 0$. Since the bounding rays of the sector Ω_1 are the semiaxis $\{y = 0, x \geq 0\}$ and $\{x = 0, y \geq 0\}$, setting $P_0 = \max(P_1, P_2)$, we get $|P(z)| \leq P_0$ on the bounding rays of Ω_1. Moreover the opening of Ω_1 is $\frac{\pi}{2}$ and $|P(z)| \leq e^{|z|^{\frac{3}{2}}}$ for all $z \in \mathbb{C}$, hence $\nu = \frac{3}{2}$ and $\mu = 2$ according to the definition of ν and μ in the statement of *Theorem* (4.16), and we have $\nu < \mu$ as required. Then all the conditions of the theorem are satisfied for the sector Ω_1, and we get $|P(z)| \leq P_0$ for $z \in \Omega_1$. Similarly we get $|P(z)| \leq P_0$ for $z \in \Omega_2 := \{z = x + iy : x \leq 0, y \geq 0\}$, and therefore $|P(z)| \leq P_0$ for all $z \in \mathbb{C}^+$. We can conclude that $\frac{E_1(z)}{E(z)}$ is of bounded type on $\mathbb{C}^+$, since

$$\frac{E(z)}{E_1(z)} = \frac{e^{iaz}\frac{1}{N(z)}}{P(z)},$$

where e^{iaz}, $\frac{1}{N(z)}$ and $P(z)$ are all bounded on $\mathbb{C}^+$.

It remains to show that $\frac{E_1(z)}{E(z)}$ is of non-positive mean type. We have that

$$\begin{aligned}
&\limsup_{y\to+\infty} y^{-1} \log|E_1(iy)| \\
&= \limsup_{y\to+\infty} y^{-1} \left(\log|s_1(y) + \alpha| + \log|y| + s_2(y) + \sum_{n\neq 0} \frac{z}{\frac{\pi}{a}n}\right),
\end{aligned} \tag{4.46}$$

where

$$s_1(y) = \sum_n \left(\frac{1}{t_n - iy} - \frac{t_n}{t_n^2 + 1}\right),$$

$$s_2(y) = \sum_{n\neq 0} \log\left|\frac{t_n - iy}{t_n}\right|,$$

$$\alpha = \sum_{n\neq 0} \left(\frac{1}{t_n} - \frac{t_n}{t_n^2 + 1}\right) + ia.$$

We observe that

$$\begin{aligned}
s_2(y) &= \sum_{n\neq 0} \log\left|\frac{t_n - iy}{t_n}\right| \\
&= \sum_{n\neq 0} \log\left(1+\frac{y^2}{t_n^2}\right)^{\frac{1}{2}} \\
&\geq \frac{1}{2}\sum_{n\neq 0} \log\left(1+\frac{y^2}{(\frac{\pi}{a}(|n|+\frac{1}{2}))^2}\right) \\
&= \sum_{n\geq 1} \log\left(1+\frac{y^2}{(\frac{\pi}{a}(n+\frac{1}{2}))^2}\right) \\
&= \log\prod_{n\geq 1} \left(1+\frac{y^2}{(\frac{\pi}{a}(n+\frac{1}{2}))^2}\right) \\
&= \log\frac{\pi^2\cosh(ay)}{\pi^2+4a^2y^2}.
\end{aligned}$$

Since

$$\limsup_{y\to+\infty} y^{-1}\frac{\pi^2\cosh(ay)}{\pi^2+4a^2y^2} = a,$$

we get

$$\limsup_{y\to+\infty} y^{-1}s_2(y) \geq a. \tag{4.47}$$

Moreover for $y>0$ we have

$$\begin{aligned}
\log|s_1(y)+\alpha| &= \log\left|\sum_n \left(\frac{t_n+iy}{t_n^2+y^2} - \frac{t_n}{t_n^2+1}\right)+\alpha\right| \\
&\geq \log\left|\Im\left(\sum_n \left(\frac{t_n+iy}{t_n^2+y^2} - \frac{t_n}{t_n^2+1}\right)+\alpha\right)\right| \\
&= \log\left(a + y\sum_n \frac{1}{t_n^2+y^2}\right) \\
&\geq \log\left(a + y\sum_n \frac{1}{\left(\frac{\pi}{a}n+\delta\right)^2+y^2}\right) \\
&= \log\left(a - \frac{ia}{2}\left(\cot(a(\delta-iy)) - \cot(a(\delta+iy))\right)\right).
\end{aligned}$$

Since

$$\lim_{y\to\infty} a - \frac{ia}{2}\left(\cot(a(\delta-iy)) - \cot(a(\delta+iy))\right) = 2a,$$

we get

$$\begin{aligned}
&\limsup_{y\to+\infty} y^{-1}\log|s_1(y)+\alpha| \\
&\geq \limsup_{y\to+\infty} y^{-1}\log\Big(a-\frac{ia}{2}\left(\cot(a(\delta-iy))-\cot(a(\delta+iy))\right)\Big) \\
&= 0.
\end{aligned}\tag{4.48}$$

Obviously we have

$$\begin{aligned}
\limsup_{y\to+\infty} y^{-1}\log|y| &= 0, \\
\sum_{n\neq 0}\frac{z}{\frac{\pi}{a}n} &= 0.
\end{aligned}\tag{4.49}$$

From (4.46), (4.47), (4.48), (4.49) we obtain

$$\limsup_{y\to+\infty} y^{-1}\log|E_1(iy)| \geq a.$$

Then we can conclude that

$$\begin{aligned}
\limsup_{y\to+\infty} y^{-1}\log\left|\frac{E(iy)}{E_1(iy)}\right| &= \limsup_{y\to+\infty} y^{-1}\log\left|\frac{e^{ay}}{E_1(iy)}\right| \\
&= \limsup_{y\to+\infty} y^{-1}\left(ay-\log|E_1(iy)|\right) \\
&= a-\limsup_{y\to+\infty}\left(y^{-1}\log|E_1(iy)|\right) \\
&\leq 0.
\end{aligned}\tag{4.50}$$

We have shown that the conditions of *Theorem* 3.3 are satisfied for $E(z)$ and $E_1(z)$. Then for every $G\in\mathcal{PW}_a$ we get

$$G(z)=\sum_n\frac{E_1(z)(1-\Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n-z)}G(t_n),$$

and the convergence of the series is uniform on the compact subsets of $\mathbb{C}$. Thanks to (4.38) and *Theorem* 4.6 we have

$$\Theta'(t_n)=2ia,$$

and by (4.39), for $n\neq 0$ we have

$$E_1(t_n)=t_n\frac{e^{\frac{z}{\frac{\pi}{a}n}}}{\left(\frac{\pi}{a}n\right)}\prod_{m\neq 0,n}\left(\frac{t_m-t_n}{\frac{\pi}{a}m}\right)e^{\frac{t_n}{\frac{\pi}{a}m}},$$

and for $n=0$ we have

$$E_1(0)=\prod_{m\neq 0}\left(\frac{t_m}{\left(\frac{\pi}{a}m\right)}\right).$$

Moreover, thanks to (4.38) and (4.39), by a simple calculation we get

$$E_1(z)(1-\Theta_1(z)) = 2iaz \prod_{m\neq 0}\left(\frac{t_m - z}{\frac{\pi}{a}m}\right) e^{\frac{z}{\frac{\pi}{a}m}}.$$

Hence we finally obtain

$$\begin{aligned}
G(z) &= \sum_n \frac{E_1(z)(1-\Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n - z)} G(t_n) \\
&= \left(\prod_{m\neq 0} \frac{t_m - z}{t_m} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(0) + \sum_{n\neq 0} \frac{z}{t_n}\left(\prod_{m\neq 0,n} \frac{t_m - z}{t_m - t_n} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(t_n). \\
&= \sum_n \left(\prod_{m\neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(t_n).
\end{aligned}$$

□

Theorem 4.19. *Fix $a>0$, and let $\{t_n\}_n$ be a sequence such that $\left|\frac{\pi}{a}n - t_n\right| \leq \delta$ if $|n| < M$ for some $\delta < \frac{\pi}{2a}$ and some integer $M > 0$, and $\left|\frac{\pi}{a}n - t_n\right| \leq \frac{\delta_1}{\frac{\pi}{a}|n|}$ if $|n| \geq M$, for some δ_1 such that $0 < \delta_1 \leq \frac{\pi}{a}M\delta$. Then for every $G \in \mathcal{PW}_a$ we get*

$$G(z) = \sum_n \left(\prod_{m\neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z-t_n}{\frac{\pi}{a}m}}\right) G(t_n),$$

and the convergence of the series is uniform on the compact subsets of $\mathbb{C}$.

Proof. We prove the theorem supposing that $t_0 = 0$, and the proof for the case $t_0 \neq 0$ can easily obtained from this one.

First of all we observe that for $|n| \geq M$ we have $\left|\frac{\pi}{a}n - t_n\right| \leq \frac{\delta_1}{\frac{\pi}{a}|n|} \leq \frac{\frac{\pi}{a}M\delta}{\frac{\pi}{a}M} = \delta$. Then, considering the result of *Theorem* 4.18, it is sufficient to show that for all $\epsilon > 0$ there exists a constants $K_\epsilon > 0$ such that

$$\left| x \prod_{n\neq 0}\left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}} \right| \geq K_\epsilon, \qquad \text{for} \quad t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad \forall n_0 \in \mathbb{Z}.$$

We define

$$f(x) = x \prod_{n\neq 0}\left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}}.$$

First we consider $n_0 \geq M$ and x such that $0 < t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon$, for some fixed ϵ. We observe that

$$\frac{d}{ds}\left(1 - \frac{x}{s}\right) = \frac{x}{s^2} > 0 \qquad \text{for } x > 0.$$

For $n > n_0 + 1$ we have $\frac{x}{t_n} < \frac{t_{n_0+1}}{t_n} < 1$ and then $1 - \frac{x}{t_n} > 0$, while for $n < 0$ we easily get $1 - \frac{x}{t_n} > 0$. Then we obtain

$$\left|\left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \geq \left|\left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}|n|}}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \qquad \text{for } n \leq -M, n > n_0 + 1,$$
$$\left|\left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \geq \left|\left(1 - \frac{x}{\frac{\pi}{a}n - \delta}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \qquad \text{for } -M < n < 0.$$

On the other hand for $0 < n < n_0$ we get $\frac{x}{t_n} > \frac{t_{n_0}}{t_n} > 1$, hence $1 - \frac{x}{t_n} < 0$, and we obtain

$$\left|\left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \geq \left|\left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \qquad \text{for } M \leq n < n_0,$$
$$\left|\left(1 - \frac{x}{t_n}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \geq \left|\left(1 - \frac{x}{\frac{\pi}{a}n + \delta}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \qquad \text{for } 0 < n < M.$$

Now we define

$$\begin{aligned}
f_1(x) =& \prod_{n \leq -M} \left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}|n|}}\right) e^{\frac{x}{\frac{\pi}{a}n}}, \\
f_2(x) =& \prod_{-M < n < 0} \left(1 - \frac{x}{\frac{\pi}{a}n - \delta}\right) e^{\frac{x}{\frac{\pi}{a}n}}, \\
f_3(x) =& \prod_{0 < n < M} \left(1 - \frac{x}{\frac{\pi}{a}n + \delta}\right) e^{\frac{x}{\frac{\pi}{a}n}}, \\
f_{4,n_0}(x) =& \left(1 - \frac{x}{t_{n_0}}\right)\left(1 - \frac{x}{t_{n_0+1}}\right) \prod_{M \leq n < n_0} \left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}} \\
& \prod_{n > n_0+1} \left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}, \\
f_{n_0}(x) =& x f_1(x) f_2(x) f_3(x) f_{4,n_0}(x).
\end{aligned}$$

and then we have

$$|f(x)| \geq |f_{n_0}(x)|, \qquad \text{for } t_{n_0} + \epsilon < x < t_{n_0+1} - \epsilon, \quad \forall n_0 > M. \tag{4.51}$$

Moreover we define

$$\begin{aligned}
g_1(x) &= \prod_{n\le -M}\left(1-\frac{x}{\frac{\pi}{a}n-\frac{\delta_1}{\frac{\pi}{a}|n|}}\right)e^{\frac{x}{\frac{\pi}{a}n}},\\
g_2(x) &= \prod_{-M<n<0}\left(1-\frac{x}{\frac{\pi}{a}n-\frac{\delta_1}{\frac{\pi}{a}|n|}}\right)e^{\frac{x}{\frac{\pi}{a}n}},\\
g_3(x) &= \prod_{0<n<M}\left(1-\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right)e^{\frac{x}{\frac{\pi}{a}n}},\\
g_4(x) &= \prod_{n\ge M}\left(1-\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right)e^{\frac{x}{\frac{\pi}{a}n}},\\
g(x) &= xg_1(x)g_2(x)g_3(x)g_4(x)\\
&= x\prod_{n<0}\left(1-\frac{x}{\frac{\pi}{a}n-\frac{\delta_1}{\frac{\pi}{a}|n|}}\right)e^{\frac{x}{\frac{\pi}{a}n}}\prod_{n>0}\left(1-\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right)e^{\frac{x}{\frac{\pi}{a}n}}.
\end{aligned}$$

Now, for $t_{n_0}+\epsilon<x<t_{n_0+1}-\epsilon$ we have

$$\frac{f_{n_0}(x)}{g(x)}=\frac{f_1(x)f_2(x)f_3(x)f_{4,n_0}(x)}{g_1(x)g_2(x)g_3(x)g_4(x)},$$

and we observe that

$$\begin{aligned}
\frac{f_1(x)}{g_1(x)} &= 1\\
\frac{f_2(x)}{g_2(x)} &= \frac{\prod_{-M<n<0}\left(1-\frac{x}{\frac{\pi}{a}n-\delta}\right)e^{\frac{x}{\frac{\pi}{a}n}}}{\prod_{-M<n<0}\left(1-\frac{x}{\frac{\pi}{a}n-\frac{\delta_1}{\frac{\pi}{a}|n|}}\right)e^{\frac{x}{\frac{\pi}{a}n}}},\\
\frac{f_3(x)}{g_3(x)} &= \frac{\prod_{0<n<M}\left(1-\frac{x}{\frac{\pi}{a}n+\delta}\right)e^{\frac{x}{\frac{\pi}{a}n}}}{\prod_{0<n<M}\left(1-\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}|n|}}\right)e^{\frac{x}{\frac{\pi}{a}n}}},\\
\frac{f_{4,n_0}(x)}{g_4(x)} &= \frac{\left(1-\frac{x}{t_{n_0}}\right)\left(1-\frac{x}{t_{n_0+1}}\right)\prod_{n>n_0+1}\left(1-\frac{x}{\frac{\pi}{a}n-\frac{\delta_1}{\frac{\pi}{a}n}}\right)e^{\frac{x}{\frac{\pi}{a}n}}}{\left(1-\frac{x}{\frac{\pi}{a}n_0}\right)\left(1-\frac{x}{\frac{\pi}{a}(n_0+1)}\right)\prod_{n>n_0+1}\left(1-\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right)e^{\frac{x}{\frac{\pi}{a}n}}}.
\end{aligned}$$

Since the products in $\frac{f_2(x)}{g_2(x)}$ and $\frac{f_3(x)}{g_3(x)}$ are finite and all the factors in these products are continuous and different from 0 in the compact sets $t_{n_0}+\epsilon\le x\le t_{n_0+1}-\epsilon$ for $n_0>M$, we get that there exist two constants $G_{2,\epsilon},G_{3,\epsilon}>0$

such that

$$\left|\frac{f_2(x)}{g_2(x)}\right| \geq G_{2,\epsilon}, \quad \left|\frac{f_3(x)}{g_3(x)}\right| \quad \geq G_{3,\epsilon}, \quad \text{for } t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad \forall n_0 > M.$$

For $t_{n_0} + \epsilon < x < t_{n_0+1} - \epsilon$ and $n > n_0 + 1$ we observe that

$$\left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}} > 0, \quad \left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}} \quad > 0,$$

and that

$$\frac{d}{dx}\left(\left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right) = -\frac{e^{\frac{x}{\frac{\pi}{a}n}}\left(\frac{\delta_1}{\frac{\pi}{a}n} + x\right)}{\frac{\pi}{a}n\left(\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}n}\right)} < 0,$$

$$\frac{d}{dx}\left(\left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right) = -\frac{e^{\frac{x}{\frac{\pi}{a}n}}\left(-\frac{\delta_1}{\frac{\pi}{a}n} + x\right)}{\frac{\pi}{a}n\left(\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}\right)} < 0.$$

Hence, recalling that

$$t_{n_0} \geq \frac{\pi}{a}n_0 - \frac{\delta_1}{\frac{\pi}{a}n_0}, \qquad t_{n_0+1} \leq \frac{\pi}{a}(n_0+1) + \frac{\delta_1}{\frac{\pi}{a}(n_0+1)},$$

we get

$$\begin{aligned}
&\left|\left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \\
&\geq \prod_{n > n_0+1}\left(1 - \frac{\frac{\pi}{a}(n_0+1) + \frac{\delta_1}{\frac{\pi}{a}(n_0+1)} - \epsilon}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{\frac{\pi}{a}(n_0+1) + \frac{\delta_1}{\frac{\pi}{a}(n_0+1)} - \epsilon}{\frac{\pi}{a}n}},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}\right| \\
&\leq \prod_{n > n_0+1}\left(1 - \frac{\frac{\pi}{a}n_0 - \frac{\delta_1}{\frac{\pi}{a}n_0} + \epsilon}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{\frac{\pi}{a}n_0 - \frac{\delta_1}{\frac{\pi}{a}n_0} + \epsilon}{\frac{\pi}{a}n}}.
\end{aligned}$$

Observing also that

$$\left|x - \frac{\pi}{a}n_0\right| \leq |x - t_{n_0}| + \left|t_{n_0} - \frac{\pi}{a}n_0\right| \leq \left(\frac{\pi}{a} - \epsilon\right) + \frac{\delta_1}{\frac{\pi}{a}n_0},$$

we obtain

$$\left|\frac{f_{4,n_0}(x)}{g_4(x)}\right| \geq \frac{\left(\frac{\epsilon}{\frac{\pi}{a}n_0+\frac{\delta_1}{\frac{\pi}{a}n_0}}\right)\left(\frac{\epsilon}{\frac{\pi}{a}(n_0+1)+\frac{\delta_1}{\frac{\pi}{a}(n_0+1)}}\right)}{\left(\frac{\frac{\pi}{a}-\epsilon+\frac{\delta_1}{\frac{\pi}{a}n_0}}{\frac{\pi}{a}n_0}\right)\left(\frac{\frac{\pi}{a}-\epsilon+\frac{\delta_1}{\frac{\pi}{a}(n_0+1)}}{\frac{\pi}{a}(n_0+1)}\right)} P(n_0) := \quad H(n_0),$$

where

$$P(n_0) = \prod_{n>n_0+1} \frac{\left(1-\frac{\frac{\pi}{a}(n_0+1)+\frac{\delta_1}{\frac{\pi}{a}(n_0+1)}-\epsilon}{\frac{\pi}{a}n-\frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{\frac{\pi}{a}(n_0+1)+\frac{\delta_1}{\frac{\pi}{a}(n_0+1)}-\epsilon}{\frac{\pi}{a}n}}}{\left(1-\frac{\frac{\pi}{a}n_0-\frac{\delta_1}{\frac{\pi}{a}n_0}+\epsilon}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{\frac{\pi}{a}n_0-\frac{\delta_1}{\frac{\pi}{a}n_0}+\epsilon}{\frac{\pi}{a}n}}}.$$

As n_0 goes to $+\infty$, we easily get

$$\lim_{n_0\to+\infty} H(n_0) = \frac{\epsilon^2}{\left(\frac{\pi}{a}-\epsilon\right)^2} \lim_{n_0\to+\infty} P(n_0) \quad = \frac{\epsilon^2}{\left(\frac{\pi}{a}-\epsilon\right)^2}.$$

Then, given any $\epsilon_0 < \frac{\epsilon^2}{\left(\frac{\pi}{a}-\epsilon\right)^2}$, there exists N_ϵ such that

$$\left|\frac{f_{4,n_0}(x)}{g_4(x)}\right| \geq \frac{\epsilon^2}{(\frac{\pi}{a}-\epsilon)^2} - \epsilon_0, \quad \text{for} \quad t_{n_0}+\epsilon \leq x \leq t_{n_0+1}-\epsilon, \quad \forall n_0 > N_\epsilon.$$

Define

$$\begin{aligned} H_{1,\epsilon} &= \min\{H(n_0),\ M < n_0 \leq N_\epsilon\}, \\ H_{2,\epsilon} &= \frac{\epsilon^2}{(\frac{\pi}{a}-\epsilon)^2} - \epsilon_0, \\ G_{4,\epsilon} &= \min\{H_{1,\epsilon}, H_{2,\epsilon}\}. \end{aligned}$$

Hence we have

$$\left|\frac{f_{4,n_0}(x)}{g_4(x)}\right| \geq G_{4,\epsilon}, \qquad \text{for} \quad t_{n_0}+\epsilon \leq x \leq t_{n_0+1}-\epsilon, \quad \forall n_0 > M,$$

and finally we get

$$\begin{gathered} \left|\frac{f_{n_0}(x)}{g(x)}\right| \geq G_{2,\epsilon}G_{3,\epsilon}G_{4,\epsilon} =: G_\epsilon, \\ \text{for} \quad t_{n_0}+\epsilon \leq x \leq t_{n_0+1}-\epsilon, \quad \forall n_0 > M. \end{gathered} \tag{4.52}$$

Now we observe that

$$
\begin{aligned}
g(x) =& x \prod_{n<0} \left(1 - \frac{x}{\frac{\pi}{a}n - \frac{\delta_1}{\frac{\pi}{a}|n|}}\right) e^{\frac{x}{\frac{\pi}{a}n}} \prod_{n>0} \left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) e^{\frac{x}{\frac{\pi}{a}n}}. \\
=& x \prod_{n\geq 1} \left(1 + \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) \left(1 - \frac{x}{\frac{\pi}{a}n + \frac{\delta_1}{\frac{\pi}{a}n}}\right) \qquad (4.53)\\
=& - x \operatorname{csch}\left(a\sqrt{\delta_1}\right)^2 \sin\left(\frac{1}{2}\left(-ax + a\sqrt{-4\delta_1 + x^2}\right)\right) \\
& \sin\left(\frac{1}{2}\left(ax + a\sqrt{-4\delta_1 + x^2}\right)\right),
\end{aligned}
$$

where the last step is obtained as follows. First of all we recall the following well-known products

$$
\begin{aligned}
\sin(x) &= x \prod_{n=1}^{+\infty} \left(1 - \frac{x}{\pi n}\right)\left(1 + \frac{x}{\pi n}\right), \\
\operatorname{csch}(x) &= \frac{1}{x} \prod_{n=1}^{+\infty} \frac{n^2}{n^2 + \frac{x^2}{\pi^2}}.
\end{aligned}
$$

Then we have

$$
\operatorname{csch}\left(a\sqrt{\delta_1}\right) = \frac{1}{a\sqrt{\delta_1}} \prod_{n=1}^{+\infty} \frac{n^2}{n^2 + \frac{a^2\delta_1}{\pi^2}}. \qquad (4.54)
$$

Moreover, setting

$$
h(x) = \sin\left(\frac{1}{2}\left(-ax + a\sqrt{-4\delta_1 + x^2}\right)\right) \sin\left(\frac{1}{2}\left(ax + a\sqrt{-4\delta_1 + x^2}\right)\right),
$$

we obtain

$$
\begin{aligned}
h(x) =& \left(\frac{1}{2}\left(-ax + a\sqrt{-4\delta_1 + x^2}\right)\right)\left(\frac{1}{2}\left(ax + a\sqrt{-4\delta_1 + x^2}\right)\right) \\
& \prod_{n=1}^{+\infty} \left(1 - \frac{-ax + a\sqrt{-4\delta_1 + x^2}}{2\pi n}\right)\left(1 + \frac{-ax + a\sqrt{-4\delta_1 + x^2}}{2\pi n}\right) \\
& \left(1 - \frac{ax + a\sqrt{-4\delta_1 + x^2}}{2\pi n}\right)\left(1 + \frac{ax + a\sqrt{-4\delta_1 + x^2}}{2\pi n}\right) \\
=& - a^2\delta_1 \prod_{n=1}^{+\infty} \left(\left(1 + \frac{ax}{2\pi n}\right)^2 - \frac{a^2x^2 - 4a^2\delta_1}{(2\pi n)^2}\right) \\
& \left(\left(1 - \frac{ax}{2\pi n}\right)^2 - \frac{a^2x^2 - 4a^2\delta_1}{(2\pi n)^2}\right) \\
=& - a^2\delta_1 \prod_{n=1}^{+\infty} \left(1 + \frac{ax}{\pi n} + \frac{a^2\delta_1}{(\pi n)^2}\right)\left(1 - \frac{ax}{\pi n} + \frac{a^2\delta_1}{(\pi n)^2}\right) \\
=& - a^2\delta_1 \prod_{n=1}^{+\infty} \left(\frac{n^2 + \frac{a^2\delta_1}{\pi^2}}{n^2} + \frac{ax}{\pi n}\right)\left(\frac{n^2 + \frac{a^2\delta_1}{\pi^2}}{n^2} - \frac{ax}{\pi n}\right).
\end{aligned}
$$

Hence, using (4.54), we get

$$\begin{aligned}
&- x \operatorname{csch}\left(a\sqrt{\delta_1}\right)^2 h(x) \\
&= x \prod_{n=1}^{+\infty}\left(\frac{n^2}{n^2+\frac{a^2\delta_1}{\pi^2}}\right)^2 \left(\frac{n^2+\frac{a^2\delta_1}{\pi^2}}{n^2}+\frac{ax}{\pi n}\right)\left(\frac{n^2+\frac{a^2\delta_1}{\pi^2}}{n^2}-\frac{ax}{\pi n}\right) \\
&= x \prod_{n\geq 1}\left(1+\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right)\left(1-\frac{x}{\frac{\pi}{a}n+\frac{\delta_1}{\frac{\pi}{a}n}}\right),
\end{aligned}$$

and then the last step of (4.53) is proved. Now we observe that

$$\lim_{x\to+\infty}\left(-ax+a\sqrt{-4\delta_1+x^2}\right)=0, \tag{4.55}$$

and that

$$\begin{aligned}
&\lim_{x\to+\infty} -x\sin\left(\frac{1}{2}\left(-ax+a\sqrt{-4\delta_1+x^2}\right)\right) \\
&= \lim_{x\to+\infty} -\frac{x}{2}\left(-ax+a\sqrt{-4\delta_1+x^2}\right) \\
&= \lim_{x\to+\infty} -\frac{x}{2}\left(\frac{-a^2x^2+a^2(-4\delta_1+x^2)}{ax+a\sqrt{-4\delta_1+x^2}}\right) \\
&= \lim_{x\to+\infty} \frac{2xa^2\delta_1}{ax+a\sqrt{-4\delta_1+x^2}} \\
&= a\delta_1.
\end{aligned} \tag{4.56}$$

Moreover for every $\epsilon_1>0$ it is easy to see that there exist two constants $A_{1,\epsilon_1}, B_{1,\epsilon_1}>0$ such that such that for $n_0>A_{1,\epsilon_1}$ we have

$$\begin{aligned}
&\left|\sin\left(\frac{1}{2}\left(ax+a\sqrt{-4\delta_1+x^2}\right)\right)\right| > B_{1,\epsilon_1}, \\
&\text{for}\quad \frac{\pi}{a}n_0+\epsilon_1<x<\frac{\pi}{a}(n_0+1)-\epsilon_1.
\end{aligned} \tag{4.57}$$

Then, thanks to (4.56) and (4.57) we have that there exist two constants $A_{2,\epsilon_1}, B_{2,\epsilon_1}>0$ such that such that for $n_0>A_{2,\epsilon_1}$ we have

$$|g(x)|>B_{2,\epsilon_1}, \qquad \text{for}\quad \frac{\pi}{a}n_0+\epsilon_1<x<\frac{\pi}{a}(n_0+1)-\epsilon_1.$$

Fix some $\epsilon_1<\frac{\epsilon}{2}$, and take $A_{3,\epsilon_1}>0$ such that $\left|t_n-\frac{\pi}{a}n\right|<\epsilon_1$ for $n>A_{3,\epsilon_1}$. Then for $n_0>A_\epsilon:=\max\{A_{3,\epsilon_1},A_{2,\epsilon_1}\}$ we get

$$\begin{aligned}
t_{n_0}+\epsilon &\geq \frac{\pi}{a}n_0-\epsilon_1+\epsilon \geq \frac{\pi}{a}n_0+\epsilon_1, \\
t_{n_0+1}-\epsilon &\leq \frac{\pi}{a}(n_0+1)+\epsilon_1-\epsilon \leq \frac{\pi}{a}(n_0+1)-\epsilon_1,
\end{aligned}$$

and hence

$$|g(x)| \geq B_\epsilon := B_{2,\epsilon_1}, \qquad \text{for} \quad t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad \forall n_0 > A_\epsilon. \quad (4.58)$$

Thanks to (4.51), (4.52) and (4.58) we get

$$\begin{aligned} &|f(x)| \geq |f_{n_0}(x)| \geq G_\epsilon B_\epsilon =: D_{1,\epsilon}, \\ &\text{for} \quad t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad \forall n_0 > A_\epsilon. \end{aligned}$$

For $0 < n_0 \leq A_\epsilon$ we observe that the function $f(x)$ is different from 0 and continuous in the finite union of compact sets

$$t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad 0 < n_0 \leq A_\epsilon.$$

and hence there exists $D_{2,\epsilon}$ such that

$$|f(x)| \geq D_{2,\epsilon}, \qquad \text{for} \quad t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad 0 < n_0 \leq A_\epsilon.$$

Then we obtain

$$|f(x)| \geq \max\{D_{1,\epsilon}, D_{2,\epsilon}\} := K_{1,\epsilon}, \qquad \text{for} \quad t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad \forall n_0 > 0.$$

For the case $n_0 < 0$, with arguments completely analogous to the case $n_0 > 0$ we obtain that there exists $K_{2,\epsilon}$ such that

$$|f(x)| \geq K_{2,\epsilon}, \qquad \text{for} \quad t_{n_0-1} + \epsilon \leq x \leq t_{n_0} - \epsilon, \quad \forall n_0 > 0.$$

Moreover we set

$$K_{3,\epsilon} = \inf\{|f(x)|, (t_{-1} + \epsilon < x < -\epsilon) \cup (\epsilon < x < t_1 - \epsilon)\},$$

and hence we finally get

$$\begin{aligned} &|f(x)| \geq \max\{K_{1,\epsilon}, K_{2,\epsilon}, K_{3,\epsilon}\} := K_\epsilon, \\ &\text{for} \quad t_{n_0} + \epsilon \leq x \leq t_{n_0+1} - \epsilon, \quad \forall n_0 \in \mathbb{Z}. \end{aligned}$$

□

As we did for the result of *Theorem* 4.2, it is now interesting to compare the results of *Theorem* 4.17 and *Theorem* 4.19 with those of the Paley-Wiener-Levinson theorem (Theorem 4.1), and with those of *Theorem* 4.2 too (we don't mention *Theorem* 4.14 since the sampling sequences of this theorem are exactly the same of those of *Theorem* 4.2).

We have already observed that for the sequences for which the Paley-Wiener-Levinson theorem is valid obeys the following constraints

$$\begin{aligned} |t_n - t_m| &> \left(|m-n| - \frac{1}{2}\right)\frac{\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m, \\ |t_n - t_m| &< \left(|m-n| + \frac{1}{2}\right)\frac{\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m, \end{aligned}$$

while for the sequences of *Theorem* 4.2 we have

$$|t_n - t_m| < \frac{|m-n|\pi}{a} \quad \forall n, m \in \mathbb{Z},$$

and that given any $\epsilon > 0$ small as desired and any integer $M < \infty$ big as desired, it is possible to find a suitable sequence $\{t_n\}_n$ such that M elements are contained in an interval of $\mathbb{R}$ of length ϵ.

In *Theorem* 4.17, the set $\mathbb{K}$ is a finite set of consecutive integers of any size and without constraints. Hence we easily see that we have no constraints on the maximum or minimum distance between two successive elements. The downside is of course that all the sampling points t_n for $n \notin \mathbb{K}$ are fixed to equidistant values.

In *Theorem* 4.19 we have that $\left|\frac{\pi}{a}n - t_n\right| \leq \delta \ \forall n \in \mathbb{Z}$, for some $\delta < \frac{\pi}{2a}$, than it is easy to see that

$$\begin{aligned} |t_n - t_m| &> (|m-n| - 1)\,\frac{\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m, \\ |t_n - t_m| &< (|m-n| + 1)\,\frac{\pi}{a} \quad \forall n, m \in \mathbb{Z}, n \neq m. \end{aligned}$$

In particular also in this case we have that there is no lower bound for $|t_n - t_m|$, but unlike *Theorem* 4.17 and *Theorem* 4.19 we can have at most 2 elements that stay in the same interval of length less than $\frac{\pi}{\alpha}$, since $|t_{n+1} - t_{n-1}| > \frac{\pi}{\alpha}$.

We obtained that *Theorem* 4.17 and *Theorem* 4.19 have less strong constraints than Paley-Wiener-Levinson theorem for a finite, but big as desired, subsequence of the sampling sequence. The downside is that they have stronger constraints on all the others sampling points. However, in the next section we show that the constraints of the sampling sequences in *Theorem* 4.17 and *Theorem* 4.19 are more useful for real applications.

4.5 Approximation of the sampling formulas

In real applications very often it is required to reconstruct a signal only on a predetermined compact subset of $\mathbb{C}$ and with a predetermined precision, since the sum cannot be performed on all the infinte set of sampling points. In this section we show that for this purpose the sampling formulas of *Theorems* 4.17 and 4.19 are better then those of the Paley-Wiener-Levinson theorem. The main reason for this derives from the fact that the sampling sequence of all these theorems (including those of the Paley-Wiener-Levinson theorem) converge uniformly on the compact subsets of $\mathbb{C}$, along with the fact that *Theorems* 4.17 and 4.19 allow more felxibility for a finite, but big as desired, subsequence of the sampling sequence. We already know that the sampling formulas of these theorems converge uniformly on the compact subsets of $\mathbb{C}$, as a consequence of *Theorem* 2.18. However we give here an explicit proof of the uniform convergence for the case of *Theorem* 4.17, which is the most interesting for the purpose of this section, in order to obtain also a numerical estimate of the error obtained performing the recostruction only on a finite subsequence of the sampling sequence. We start with the following well-known Lemma and we include a proof also for it, for sake of a precise reference and for sake of completeness.

Lemma 4.20. *Fix any $a > 0$ and let $\{t_n\}_n$ be an increasing sequence of reals for which there exists $\epsilon > 0$ such that $|t_{n+1} - t_n| \geq \epsilon$ $\forall n \in \mathbb{Z}$. Then for every $F \in \mathcal{PW}_a$ we have*

$$\sum_n |F(t_n)|^2 < \frac{4}{\pi\epsilon}\|F\|_{\mathcal{PW}_a}.$$

Proof. We recall that all the functions in $\mathcal{PW}_a$ are entire. Given $x_0 \in \mathbb{R}$, thanks to the mean value property we have that

$$|F(x_0)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} \left|F\left(x_0 + re^{i\theta}\right)\right|^2 d\theta$$

holds for all $r > 0$. Therefore, setting $z = x + iy$ we get

$$|F(x_0)|^2 \leq \frac{1}{\pi\delta^2}\iint_{|z-z_{0,m}|\leq\delta} |F(z)|^2 dxdy$$

for every $x_0 \in \mathbb{R}$ and every $\delta > 0$ (multiply both sides by r and integrate between 0 and δ). Then we obtain

$$\begin{aligned}
\sum_n |F(t_n)|^2 &\leq \frac{1}{\pi\delta^2}\sum_n \iint_{|z|\leq\delta} |F(t_n + z)|^2 dxdy \\
&\leq \frac{1}{\pi\delta^2}\sum_n \int_{-\delta}^{\delta}\int_{-\delta}^{\delta} |F(t_n + x + iy)|^2 dxdy \\
&= \frac{1}{\pi\delta^2}\sum_n \int_{-\delta}^{\delta}\int_{t_n-\delta}^{t_n+\delta} |F(x + iy)|^2 dxdy.
\end{aligned}$$

Now, take $\delta < \frac{\epsilon}{2}$. Then the intervals $(t_n - \delta, t_n + \delta)$ are pairwise disjoint, and hence

$$\begin{aligned}
\sum_n |F(t_n)|^2 &\leq \frac{1}{\pi\delta^2}\sum_n \int_{-\delta}^{\delta}\int_{t_n-\delta}^{t_n+\delta} |F(x + iy)|^2 dxdy \\
&\leq \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{-\infty}^{+\infty} |F(x + iy)|^2 dxdy \\
&\leq \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta} \|F\|_{\mathcal{PW}_a} dy \\
&= \frac{2}{\pi\delta}\|F\|_{\mathcal{PW}_a}.
\end{aligned} \tag{4.59}$$

Since (4.59) is valid for every $\delta < \frac{\epsilon}{2}$, we finally get

$$\sum_n |F(t_n)|^2 \leq \frac{4}{\pi\epsilon} \|F\|_{\mathcal{PW}_a}.$$

$\square$

Theorem 4.21. *Let $a > 0$ and let $\mathbb{K} = \{n_k\}_{k=0,\ldots,K}$ be a finite set of consecutive integers of any size. Let $\{t_n\}_n$ be a strictly increasing sequence such that $t_n = \frac{\pi}{a} n \ \forall n \in \mathbb{Z} \setminus \mathbb{K}$, and that*

$$\frac{\pi}{a}(n_0 - 1) < t_{n_0} < \ldots < t_{n_K} \quad < \frac{\pi}{a}(n_K + 1).$$

Given $G \in \mathcal{PW}_a$ and $N > 0$, set

$$G_N(z) = \sum_{n=-N}^{N} \left(\prod_{m \neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z - t_n}{\frac{\pi}{a} m}} \right) G(t_n).$$

Then, for any compact subset $\Omega \subset \mathbb{C}$, $G_N(z)$ converges uniformly to $G(z)$ in Ω as $N \to +\infty$, and in particular there exists $C > 0$ (dependent only on $\{t_n\}_n$) such that for every $G \in \mathcal{PW}_a$ we have

$$\sup_{z \in \Omega} |G_N(z) - G(z)| \leq \frac{C}{(N - N_0)^{\frac{1}{2}}} \|G\|_{\mathcal{PW}_a}, \quad N > N_0,$$

where N_0 is the smallest positive integer for which $|t_{N_0}| > x_0$ and $|t_{-N_0}| > x_0$, with $x_0 = \max_{x+iy \in \Omega} |x|$.

Proof. We are exactly on the same conditions of *Theorem* (4.17). From the proof of this theorem we know that

$$\begin{aligned} G(z) &= \sum_n \frac{E_1(z)(1 - \Theta_1(z))}{E_1(t_n)\Theta_1'(t_n)(t_n - z)} G(t_n) \\ &= \sum_n \left(\prod_{m \neq n} \frac{t_m - z}{t_m - t_n} e^{\frac{z - t_n}{\frac{\pi}{a} m}} \right) G(t_n). \end{aligned}$$

where

$$E_1(z) = \left(\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + ia \right) z \prod_{n \neq 0} \left(\frac{t_n - z}{\frac{\pi}{a} n} \right) e^{\frac{z}{\frac{\pi}{a} n}},$$

and

$$\Theta_1(z) = \frac{E_1^{\#}(z)}{E_1(z)} = \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) - ia}{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1} \right) + ia} \tag{4.60}$$

We set

$$M_1 := \sup_{z\in\Omega} |E(z)(1-\Theta(z))| = \sup_{z\in\Omega} |E(z) - E^{\#}(z)| < 2\sup_{z\in\Omega} |E(z)|$$

By *Theorem* (4.17) we know also that $\lim_{x\to\pm\infty} |E_1(x)| = 1$, and thanks to the continuity of $E(x)$ on $\mathbb{R}$ we get

$$M_2 := \sup_{x\in\mathbb{R}} \left|\frac{1}{E(x)}\right| < \infty.$$

Obviously we have

$$\begin{aligned} G_N(z) - G(z) = &\sum_{n=N+1}^{+\infty} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n - z)} G(t_n) \\ &+ \sum_{n=-\infty}^{-N-1} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n - z)} G(t_n). \end{aligned}$$

Let N_0 be the smallest positive integer for which $|t_{N_0}| > x_0$ and $|t_{-N_0}| > x_0$, where $x_0 = \max_{x+iy\in\Omega} |x|$. By *Theorem* 4.6 and (4.60) we have that $\Theta_1'(t_n) = 2ia$. Hence for $z \in \Omega$ and $N \geq N_0$ we obtain

$$\left|\sum_{n=N+1}^{+\infty} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n - z)} G(t_n)\right| \leq \frac{M_1 M_2}{a} \sum_{n=N+1}^{+\infty} \frac{|G(t_n)|}{|t_n - x_0|}.$$

Using the Holder inequality we get

$$\sum_{n=N+1}^{+\infty} \frac{|G(t_n)|}{|t_n - x_0|} \leq \left(\sum_{n=N+1}^{+\infty} |G(t_n)|^2\right)^{\frac{1}{2}} \left(\sum_{n=N+1}^{+\infty} \frac{1}{|t_n - x_0|^2}\right)^{\frac{1}{2}}.$$

We set $\epsilon = \min\{\frac{\pi}{a}n, \min\{|t_{n+1} - t_n|\}_{n\in\mathbb{K}}\}$, and we easily observe that $|t_{n+1} - t_n| \geq \epsilon\ \forall n \in \mathbb{Z}$. Then by *Lemma* 4.20 we get

$$\sum_{n=N+1}^{+\infty} |G(t_n)|^2 < \frac{4}{\pi\epsilon} \|G\|_{\mathcal{PW}_a}.$$

Moreover we have

$$\begin{aligned} \sum_{n=N+1}^{+\infty} \frac{1}{|t_n - x_0|^2} &\leq \sum_{n=N+1}^{+\infty} \frac{1}{|t_n - t_{N_0} + t_{N_0} - x_0|^2} \\ &\leq \sum_{n=N+1}^{+\infty} \frac{1}{\epsilon|n - N_0|^2} \\ &= \frac{1}{\epsilon^2} \sum_{n=1}^{+\infty} \frac{1}{(n + N - N_0)^2} \\ &\leq \frac{1}{\epsilon^2(N - N_0)}. \end{aligned}$$

Therefore we obtain

$$\left|\sum_{n=N+1}^{+\infty} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n-z)} G(t_n)\right| \leq \frac{C}{(N-N_0)^{\frac{1}{2}}} \|G\|_{\mathcal{PW}_a},$$

where $C = \frac{4M_1M_2}{\pi a \epsilon}$, and then

$$\lim_{N\to+\infty} \sup_{z\in\Omega} \left|\sum_{n=N+1}^{+\infty} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n-z)} G(t_n)\right| = 0.$$

Similarly we get

$$\left|\sum_{n=-\infty}^{-N-1} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n-z)} G(t_n)\right| \leq \frac{C}{(N-N_0)^{\frac{1}{2}}} \|G\|_{\mathcal{PW}_a},$$

and

$$\lim_{N\to+\infty} \sup_{z\in\Omega} \left|\sum_{n=-\infty}^{-N-1} \frac{E_1(z)(1-\Theta(z))}{E_1(t_n)\Theta'(t_n)(t_n-z)} G(t_n)\right| = 0.$$

Then we finally obtain

$$\sup_{z\in\Omega} |G_N(z) - G(z)| \leq \frac{C}{(N-N_0)^{\frac{1}{2}}} \|G\|_{\mathcal{PW}_a},$$

and

$$\lim_{N\to+\infty} \sup_{z\in\Omega} |G_N(z) - G(z)| = 0.$$

$\square$

CHAPTER 5

Generalization of the Fourier transform

For $F \in L^1(\mathbb{R})$ the Fourier transform $\mathcal{F}(F)$ of $F(x)$ is defined as

$$\hat{F}(x) := \mathcal{F}(F)(x) = \int_{-\infty}^{+\infty} F(t)e^{-itx}dt, \quad x \in \mathbb{R}.$$

Let $F \in L^1(\mathbb{R})$ be such that $\hat{F} \in L^1(\mathbb{R})$. Then the Fourier inversion theorem states that

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{itx}\hat{F}(t)dt. \tag{5.1}$$

As pointed out in *Section* 2.8, the Fourier transform induces a unitary isomorphism between the Paley-Wiener space $\mathcal{PW}_a$ and the space $\mathcal{L}^2[-a,a]$, which has far-reaching consequences. In this section we present a generalization of the Fourier transform for the de Branges spaces, that define an isomorphism between these spaces and a class of subspaces of $\mathcal{L}^2(\mathbb{R})$.

Given an infinite set of subspaces $\mathcal{U}_n$ of the same vector space $\mathcal{U}$, with a small abuse of notation in this section we will use the symbol $\bigoplus_n \mathcal{U}_n$ to denote the closure of the subspace formed by all the possible sums $\sum_{n\geq 0} u_n$, with $u_n \in \mathcal{U}_n$ and $u_n \neq 0$ only for a finite number of n, before showing that the subspaces $\{\mathcal{U}_n\}_n$ are pairwise perpendicular, and only later we will show that they are actually pairwise perpendicular.

5.1 Generalized Fourier transform

Let $\Theta(z) = \gamma e^{2ibz}B(z)$ be a meromorphic inner function according to the representation given in (2.3), with logarithmic residue $2b > 0$. To see this, we proceed as follows. We set

$$\Theta_0(z) = e^{2ibz},$$
$$\Theta_1(z) = \gamma B(z),$$

so that $\Theta(z) = \Theta_0(z)\Theta_1(z)$. For $n > 0$, we set

$$\begin{aligned}\tilde{\mathcal{K}}_n(\Theta) &:= \Theta_0^n \mathcal{K}(\Theta_0) \cap \left(\bigoplus_{m=1}^{n} \Theta_0^m \Theta_1 \mathcal{K}(\Theta_0) \right)^{\perp} \\ &= e^{2inbz}\mathcal{K}\left(e^{2inbz}\right) \cap \left(\bigoplus_{m=1}^{n} e^{2imbz}\gamma B(z)\mathcal{K}\left(e^{2nibz}\right) \right)^{\perp}.\end{aligned} \tag{5.2}$$

We will show later (see(5.12)) that, excluding the degenerate cases $\Theta_0(z) = 1$ or $\Theta_1(z) = 1$, we have

$$\tilde{\mathcal{K}}_n(\Theta) \neq \{0\}$$

for at least one value of $n > 0$. For $n > 0$ we set

$$\mathcal{L}^2_\Theta[b(2n-1), b(2n+1)] \ := \mathcal{F}\left(e^{-ibz}\tilde{\mathcal{K}}_n(\Theta)\right). \tag{5.3}$$

We recall that the Fourier transform induces an isomorphism from $\mathcal{PW}_b$ onto $\mathcal{L}^2[-b, b]$, and that the following property of the Fourier Tranfsorm is true:

$$\mathcal{F}(F(z)e^{is_0 z})(s) = \mathcal{F}(F(z))\,(s - s_0)\,. \tag{5.4}$$

Then it is easy to see that $\mathcal{L}^2[b(2n-1), b(2n+1)] = \mathcal{F}\left(e^{2inbz}\mathcal{PW}_b\right)$. Observing that

$$e^{-ibz}\tilde{\mathcal{K}}_n(\Theta) \subseteq e^{2inbz}\left(e^{-ibz}\mathcal{K}(e^{2inbz})\right) = e^{2inbz}\mathcal{PW}_b$$

we obtain

$$\mathcal{L}^2_\Theta[b(2n-1), b(2n+1)] \subseteq \mathcal{L}^2[b(2n-1), b(2n+1)]. \tag{5.5}$$

According to (5.5), for $F, G \in \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)]$ we define the scalar product

$$\begin{aligned}\langle F, G\rangle_{\mathcal{L}^2_\Theta[b(2N-1),b(2N+1)]} &:= \langle F, G\rangle_{\mathcal{L}^2[b(2N-1),b(2N+1)]} \\ &= \int_{b(2N-1)}^{b(2N+1)} F(x)\overline{G(x)}dx.\end{aligned} \tag{5.6}$$

Moreover we define

$$\mathcal{L}^2_\Theta := \mathcal{L}^2[-b, b] \oplus \left(\bigoplus_{n>0} \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)] \right), \tag{5.7}$$

We recall that $\bigoplus_{n>0} \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)]$ denotes the closure of the subspace formed by all the possible sums $\sum_{n>0} F_n$, with $F_n \in \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)]$ and $F_n \neq 0$ only for a finite number of $n \in \mathbb{Z}$. Then we easily get that $\mathcal{L}^2_\Theta \subseteq \mathcal{L}^2[-b, +\infty)$, and hence we can endow $\mathcal{L}^2_\Theta$ with the scalar product and the norm of $\mathcal{L}^2[-b, +\infty)$.

Theorem 5.1. *Let $\Theta(z) = \gamma e^{2ibz} B(z)$ be a meromorphic inner function according to the representation given in (2.3), with logarithmic residue $2b > 0$. Let $E(z)$ be a de Branges function of $\Theta(z)$. For $F \in \mathcal{B}(E)$ consider the transform $\tilde{\mathcal{F}}_E$ given by*

$$\tilde{\mathcal{F}}_E(F)(z) := \mathcal{F}\left(\frac{F(t)}{E(t)e^{ibt}}\right)(z) = \int_{-\infty}^{+\infty} F(t)\frac{e^{-i(z+b)t}}{E(t)}dt. \tag{5.8}$$

Then

$$\tilde{\mathcal{F}}_E : \quad \mathcal{B}(E) \to \mathcal{L}^2_\Theta$$

is a unitary (up to a rescaling factor $\frac{1}{2\pi}$) isomorphism, and

$$\langle F_1, F_2 \rangle_{\mathcal{B}(E)} = \frac{1}{2\pi}\langle \tilde{\mathcal{F}}_E(F_1), \tilde{\mathcal{F}}_E(F_2)\rangle_{\mathcal{L}^2_\Theta} \tag{5.9}$$

for all $F_1, F_2 \in \mathcal{B}(E)$.

Proof. We set

$$\begin{aligned}\Theta_0(z) &= e^{2ibz},\\ \Theta_1(z) &= \gamma B(z),\end{aligned}$$

so that $\Theta(z) = \Theta_0(z)\Theta_1(z)$, and we observe that $\Theta(z) = LCM(\Theta_0(z), \Theta_1(z))$. Obviously $E_0(z) = e^{-ibz}$ is a de Branges function of $\Theta_0(z)$. Set $E_1(z) = \frac{E(z)}{E_0(z)} = E(z)e^{ibz}$. Since $E(z)$ is Hermite Biehler and then entire, also $E_1(z)$ is entire. Moreover $\frac{E_1^\#(z)}{E_1(z)} = \frac{E^\#(z)E_0(z)}{E_0^\#(z)E(z)} = \frac{\Theta(z)}{\Theta_0(z)} = \Theta_1(z)$, and then $\frac{|E_1(z)|}{|E_1^\#(z)|} = \frac{1}{|\Theta_1(z)|} > 1$ on the upper half plane.

Hence $E_1(z)$ is a Hermite Biehler function, and is a de Branges function of $\Theta_1(z)$. In this way we have obtained that $E_0(z)$ and $E_1(z)$ are respectively de Branges functions of $\Theta_0(z)$ and $\Theta_1(z)$ such that $E(z) = E_1(z)E_0(z)$. Then we can apply Theorem 3.2, and we obtain

$$\mathcal{B}(E) = \mathcal{B}(E_0)E_1 + \mathcal{B}(E_1)E_0.$$

Recalling Theorem 2.19 we get

$$\begin{aligned}\mathcal{B}(E) &= E\mathcal{K}(\Theta)\\ &= E\mathcal{K}(\Theta_0\Theta_1)\\ &= E(\mathcal{K}(\Theta_0) + \mathcal{K}(\Theta_1)).\end{aligned} \tag{5.10}$$

We observe that
$$\mathcal{H}^2 = \bigoplus_{n=0}^{\infty} \Theta_0^n \mathcal{K}(\Theta_0),$$
and then we get
$$\begin{aligned}
\mathcal{K}(\Theta_0\Theta_1) &= \mathcal{H}^2 \ominus \Theta_0\Theta_1\mathcal{H}^2 \\
&= \left(\bigoplus_{n=0}^{\infty} \Theta_0^n \mathcal{K}(\Theta_0)\right) \ominus \left(\bigoplus_{n=1}^{\infty} \Theta_0^n \Theta_1 \mathcal{K}(\Theta_0)\right). \\
&= \left(\bigoplus_{n=0}^{\infty} \Theta_0^n \mathcal{K}(\Theta_0)\right) \cap \left(\bigoplus_{n=1}^{\infty} \Theta_0^n \Theta_1 \mathcal{K}(\Theta_0)\right)^{\perp}. \\
&= \lim_{N\to\infty} \left(\left(\bigoplus_{n=0}^{N} \Theta_0^n \mathcal{K}\left(\Theta_0\right)\right) \cap \left(\bigoplus_{n=1}^{\infty} \Theta_0^n \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right).
\end{aligned}$$
We recall that, given three subspaces of finite-dimension $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \subset \mathcal{H}^2$, we have
$$(\mathcal{U}_1 \bigoplus \mathcal{U}_2) \bigcap \mathcal{U}_3 = (\mathcal{U}_1 \cap \mathcal{U}_3) \bigoplus (\mathcal{U}_2 \cap \mathcal{U}_3).$$
Hence
$$\begin{aligned}
\mathcal{K}(\Theta_0\Theta_1) &= \lim_{N\to\infty} \left(\left(\bigoplus_{n=0}^{N} \Theta_0^n \mathcal{K}\left(\Theta_0\right)\right) \cap \left(\bigoplus_{n=1}^{\infty} \Theta_0^n \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right) \\
&= \lim_{N\to\infty} \bigoplus_{n=0}^{N} \left(\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=1}^{\infty} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right).
\end{aligned}$$
We observe that
$$\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=n+1}^{\infty} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right) = \{0\},$$
so that
$$\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=0}^{\infty} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp} = \Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=0}^{n} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}.$$
Therefore we obtain
$$\begin{aligned}
\mathcal{K}(\Theta_0\Theta_1) &= \lim_{N\to\infty} \bigoplus_{n=0}^{N} \left(\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=1}^{\infty} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right) \\
&= \lim_{N\to\infty} \bigoplus_{n=0}^{N} \left(\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=1}^{n} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right) \\
&= \bigoplus_{n=0}^{\infty} \left(\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=1}^{n} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right). \\
&= \mathcal{K}\left(\Theta_0\right) \oplus \left(\bigoplus_{n=1}^{\infty} \left(\Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=1}^{n} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}\right)\right).
\end{aligned}$$

Recalling (5.2) we get

$$\mathcal{K}(\Theta_0\Theta_1) = \mathcal{K}\left(\Theta_0\right) \oplus \left(\bigoplus_{n>0} \tilde{\mathcal{K}}_n(\Theta)\right),$$

and by (5.10) we finally obtain

$$\begin{aligned}\mathcal{B}(E) &= E\mathcal{K}(\Theta_0\Theta_1) \\ &= E\mathcal{K}(\Theta_0) \oplus \left(\bigoplus_{n>0} E\tilde{\mathcal{K}}_n(\Theta)\right).\end{aligned} \tag{5.11}$$

Thanks to this we can also observe that, excluding the degenerate cases $\Theta_0(z) = 1$ or $\Theta_1(z) = 1$, we have

$$\tilde{\mathcal{K}}_n(\Theta) \neq \{0\} \tag{5.12}$$

for at least one value of $n > 0$, because otherwise we would have

$$E\mathcal{K}(\Theta) = \mathcal{B}(E) = E\mathcal{K}(\Theta_0),$$

which is impossible for $\Theta_1(z) \neq 1$.

As already pointed out, the Fourier transform induces an isomorphism from $\mathcal{PW}_b$ onto $\mathcal{L}^2[-b, b]$, and then

$$\tilde{\mathcal{F}}_E(E\mathcal{K}(\Theta_0)) = \mathcal{F}(E_0\mathcal{K}(\Theta_0)) = \mathcal{F}(\mathcal{PW}_b) = \mathcal{L}^2[-b, b]. \tag{5.13}$$

Moreover, for $n > 0$ we get

$$\begin{aligned}\tilde{\mathcal{F}}_E(E\tilde{\mathcal{K}}_n(\Theta)) &= \mathcal{F}(E_0\tilde{\mathcal{K}}_n(\Theta)) \\ &= \mathcal{F}\left(e^{2inbz}(\mathcal{PW}_b \cap e^{-ibz}\mathcal{K}(\gamma B(z)))\right) \\ &= \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)].\end{aligned}$$

Then we finally obtain

$$\begin{aligned}\tilde{\mathcal{F}}_E(\mathcal{B}(E)) &= \tilde{\mathcal{F}}_E(E\mathcal{K}(\Theta_0)) \oplus \left(\bigoplus_{n>0} \tilde{\mathcal{F}}_E(E\tilde{\mathcal{K}}_n(\Theta))\right) \\ &= \mathcal{L}^2[-b, b] \oplus \left(\bigoplus_{n>0} \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)]\right) \\ &= \mathcal{L}^2_\Theta.\end{aligned}$$

Now it remains only to prove that $\tilde{\mathcal{F}}_E$ is a unitary isomorphism. We set $F_n \in E\tilde{\mathcal{K}}_n(\Theta)$ and $F_m \in E\tilde{\mathcal{K}}_m(\Theta)$ for some $n, m \geq 1$, $n \neq m$. Then we can write

$$\begin{aligned}F_n(z) &= e^{2inbz}E_1(z)H_n(z), \\ F_m(z) &= e^{2imbz}E_1(z)H_m(z),\end{aligned}$$

for some $H_n(z), H_m(z) \in E_0\mathcal{K}(\Theta) = \mathcal{PW}_b$. Then

$$\begin{aligned}\langle F_n, F_m\rangle_{\mathcal{B}(E)} &= \int_{-\infty}^{+\infty} \frac{F_n(t)\overline{F_m(t)}}{|E(t)|^2}dt \\ &= \int_{-\infty}^{+\infty} e^{2ib(n-m)t}\frac{H_n(t)\overline{H_m(t)}|E_1(t)^2|}{|E(t)|^2}dt \\ &= \int_{-\infty}^{+\infty} e^{2ib(n-m)t}H_n(t)\overline{H_m(t)}dt.\end{aligned}$$

Now it is easy to see that $e^{2inbt}H_n(t), e^{2imbt}H_m(t) \in \mathcal{PW}_{(2q+1)b}$, where $q = \max(m,n)$, and then we have

$$\begin{aligned}\langle F_n, F_m\rangle_{\mathcal{B}(E)} &= \int_{-\infty}^{+\infty} e^{2ib(n-m)t}H_n(t)\overline{H_m(t)}dt \\ &= \langle e^{2inbt}H_n(t), e^{2imbt}H_m(t)\rangle_{\mathcal{PW}_{(2q+1)b}} \\ &= \langle \mathcal{F}(e^{2inbt}H_n(t)), \mathcal{F}(e^{2imbt}H_m(t))\rangle_{\mathcal{PW}_{(2q+1)b}} \\ &= 0,\end{aligned} \tag{5.14}$$

where in the last step we observed that the support of $\mathcal{F}(e^{2inbt}H_n(t))$ is $[(2n-1)b,(2n+1)b]$, while the support of $\mathcal{F}(e^{2imbt}H_m(t))$ is $[(2m-1)b,(2m+1)b]$, and since n and m are two integers with $n \neq m$, the intersection of the two supports is null. Hence we have $E\tilde{\mathcal{K}}_n(\Theta) \perp E\tilde{\mathcal{K}}_m(\Theta)$ for $n \neq m$. In an analogous way we obtain $E\mathcal{K}(\Theta_0) \perp E\tilde{\mathcal{K}}_n(\Theta)$ $\forall n > 0$. Now, thanks to (5.11), every $F \in \mathcal{B}(E)$ can be expressed as

$$F(z) = \sum_{n=0}^{+\infty} F_n(z),$$

for some $N \geq 0$, $F_0 \in E\mathcal{K}(\Theta_0)$, $F_n \in E\tilde{\mathcal{K}}_n(\Theta)$ for all n such that $1 \leq n \leq N$ and $F_N \neq 0$. Setting

$$\begin{aligned}F_1(z) &= \sum_{n=0}^{+\infty} F_{n,1}(z), \\ F_2(z) &= \sum_{n=0}^{+\infty} F_{n,2}(z),\end{aligned}$$

thanks to (5.14) we obtain

$$\langle F_1, F_2\rangle_{\mathcal{B}(E)} = \sum_{n=0}^{+\infty}\langle F_{n,1}, F_{n,2}\rangle_{\mathcal{B}(E)}. \tag{5.15}$$

Similarly, given $G_n \in \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)], G_m \in \mathcal{L}^2_\Theta[b(2m-1), b(2m+1)]$, we easily get

$$\langle G_n, G_m\rangle_{\mathcal{L}^2_\Theta} = 0, \tag{5.16}$$

since the intersection of the support of $G_n(z)$ with the support of $G_m(z)$ is null. Given $G_1, G_2 \in \mathcal{L}^2_\Theta$ we can write

$$G_1(z) = \sum_{n=0}^{+\infty} G_{n,1}(z),$$
$$G_2(z) = \sum_{n=0}^{+\infty} G_{n,2}(z),$$

where $G_{0,1}, G_{0,2} \in L^2[-b,b]$ and $G_{n,1}, G_{n,2} \in L^2_\Theta[b(2n-1), b(2n+1)]$ for all $n \geq 1$. Then by (5.16) we obtain

$$\langle G_1, G_2 \rangle_{\mathcal{L}^2_\Theta} = \sum_{n=0}^{+\infty} \langle G_{n,1}, G_{n,2} \rangle_{\mathcal{L}^2_\Theta}. \tag{5.17}$$

Thanks to (5.15) and (5.17), to show that $\tilde{\mathcal{F}}_E$ is a unitary isomorphism between $\mathcal{B}(E)$ and $\mathcal{L}^2_\Theta$ it is sufficient to prove that it is a unitary isomorphism between every orthogonal subspace $EK(\Theta_0), \{E\tilde{\mathcal{K}}_n(\Theta)\}_{n>0}$ and the corresponding image $\mathcal{L}^2[-b,b], \{\mathcal{L}^2_\Theta[b(2n-1), b(2n+1)]\}_{n>0}$. Thanks to (5.13) we have

$$\tilde{\mathcal{F}}_E(EK(\Theta_0)) = \mathcal{F}(PW_b) = \mathcal{L}^2[-b,b],$$

and this is obviously a unitary isomorphism since $\mathcal{F}$ is a unitary isomorphism between $\mathcal{PW}_b$ and $\mathcal{L}^2[-b,b]$. For all $n > 0$ we have

$$\tilde{\mathcal{F}}_E\left(E\tilde{\mathcal{K}}_n(\Theta)\right) = \mathcal{F}\left(E_0\tilde{\mathcal{K}}_n(\Theta)\right) = \mathcal{L}^2_\Theta[b(2n-1), b(2n+1)].$$

For all $n > 0$ we observe that $e^{-2inbz}E_0\tilde{\mathcal{K}}_n(\Theta) \subseteq \mathcal{PW}_b$, and that $\mathcal{F}$ is a unitary isomorphism between $e^{-2inbz}E_0\tilde{\mathcal{K}}_n(\Theta)$ and $\mathcal{F}\left(e^{-2inbz}E_0\tilde{\mathcal{K}}_n(\Theta)\right) \subseteq \mathcal{L}^2[-b,b]$. We recall that by definition we have $\tilde{\mathcal{F}}_E\left(e^{-2inbz}E\tilde{\mathcal{K}}_n(\Theta)\right) = \mathcal{F}\left(e^{-2inbz}E_0\tilde{\mathcal{K}}_n(\Theta)\right)$. Then, given $F_1, F_2 \in e^{-2inbz}E\tilde{\mathcal{K}}_n(\Theta)$ and $G_1, G_2 \in \tilde{\mathcal{F}}_E\left(e^{-2inbz}E\tilde{\mathcal{K}}_n(\Theta)\right)$ such that

$$G_1(z) = \tilde{\mathcal{F}}_E(F_1)(z) = \mathcal{F}\left(\frac{F_1(t)}{E(t)e^{ibt}}\right)(z),$$
$$G_2(z) = \tilde{\mathcal{F}}_E(F_2)(z) = \mathcal{F}\left(\frac{F_2(t)}{E(t)e^{ibt}}\right)(z),$$

by (2.24) we have

$$\langle F_1, F_2 \rangle_{\mathcal{B}(E)} = \left\langle \frac{F_1(t)}{E(t)e^{ibt}}, \frac{F_2(t)}{E(t)e^{ibt}} \right\rangle_{\mathcal{PW}_b} = \frac{1}{2\pi}\langle G_1, G_2 \rangle_{\mathcal{L}^2[-b,b]}. \tag{5.18}$$

Thanks to (5.4) we have that multiplying a function by e^{2inbz} before applying the Fourier transform induces a translation by $-2nb$ in the image function.

Therefore, given

$$U_1(z) = e^{2inbz} F_1(z) \in E\tilde{\mathcal{K}}_n(\Theta),$$
$$U_2(z) = e^{2inbz} F_2(z) \in E\tilde{\mathcal{K}}_n(\Theta),$$

we get

$$\begin{aligned}
\tilde{\mathcal{F}}_E(U_1)(z) = \tilde{\mathcal{F}}_E(e^{2inbt}F_1(t))(z) \quad &= \mathcal{F}\left(\frac{F_1(t)e^{2inbz}}{E(t)e^{ibt}}\right)(z) = G_1(z-2nb),\\
\tilde{\mathcal{F}}_E(U_2)(z) = \tilde{\mathcal{F}}_E(e^{2inbt}F_1(t))(z) \quad &= \mathcal{F}\left(\frac{F_2(t)e^{2inbz}}{E(t)e^{ibt}}\right)(z) = G_2(z-2nb).
\end{aligned}$$

By (5.18) we obtain

$$\begin{aligned}
\langle U_1, U_2\rangle_{\mathcal{B}(E)} &= \langle e^{2inbt}F_1(t), e^{2inbt}F_2(t)\rangle_{\mathcal{B}(E)}\\
&= \left\langle \frac{F_1(t)e^{2inbt}}{E(t)e^{ibt}}, \frac{F_2(t)e^{2inbt}}{E(t)e^{ibt}}\right\rangle_{\mathcal{L}^2(\mathbb{R})}\\
&= \left\langle \frac{F_1(t)}{E(t)e^{ibt}}, \frac{F_2(t)}{E(t)e^{ibt}}\right\rangle_{\mathcal{L}^2(\mathbb{R})}\\
&= \left\langle \frac{F_1(t)}{E(t)e^{ibt}}, \frac{F_2(t)}{E(t)e^{ibt}}\right\rangle_{\mathcal{PW}_b} \qquad (5.19)\\
&= \frac{1}{2\pi}\langle G_1, G_2\rangle_{\mathcal{L}^2[-b,b]}\\
&= \frac{1}{2\pi}\langle G_1(z-2nb), G_2(z-2nb)\rangle_{\mathcal{L}^2[(2n-1)b,(2n+1)b]}.\\
&= \frac{1}{2\pi}\langle \tilde{\mathcal{F}}_E(U_1), \tilde{\mathcal{F}}_E(U_2)\rangle_{\mathcal{L}^2_\Theta[(2n-1)b,(2n+1)b]}.
\end{aligned}$$

Then the transformation between $E\tilde{\mathcal{K}}_n(\Theta)$ and $\mathcal{L}^2_\Theta[b(2n-1), b(2n+1)]$ is a unitary (up to a rescaling factor $\frac{1}{2\pi}$) isomorphism for all $n > 0$. Hence we can conclude that $\tilde{\mathcal{F}}_E$ is a unitary isomorphism between $\mathcal{B}(E)$ and $\mathcal{L}^2_\Theta$, with

$$\langle F_1, F_2\rangle_{\mathcal{B}(E)} = \frac{1}{2\pi}\langle \tilde{\mathcal{F}}_E(F_1), \tilde{\mathcal{F}}_E(F_2)\rangle_{\mathcal{L}^2_\Theta}$$

for all $F_1, F_2 \in \mathcal{B}(E)$. □

In the conditions of *Theorem* 5.1, we can consider the transform

$$\tilde{\mathcal{F}}_E(F)(z) = \int_{-\infty}^{+\infty} F(t)\frac{e^{-i(z+b)t}}{E(t)}dt.$$

as a generalization of the Fourier transform for the de Branges spaces. Indeed, when $E(z) = e^{-ibz}$ we have $\mathcal{B}(E) = \mathcal{PW}_b$, and the transform $\tilde{\mathcal{F}}_E(F(t))(z)$ becames equal to the Fourier transform since $\frac{e^{-ibt}}{E(t)} = 1$.

The transform $\tilde{\mathcal{F}}_E$ can be easily inverted using the Fourier inversion theorem.

Theorem 5.2. *Let* $\Theta(z) = \gamma e^{2ibz}B(z)$ *be a meromorphic inner function according to the representation given in* (2.3)*, with logarithmic residue* $2b > 0$*. Let* $E(z)$ *be a de Branges function of* $\Theta(z)$*, and let* $F(z) \in \mathcal{B}(E)$ *be such that*

$$\Phi = \tilde{\mathcal{F}}_E(F) \in L^1(\mathbb{R}).$$

Then we have

$$F(z) = \tilde{\mathcal{F}_E}^{-1}(\Phi)(z) = \frac{E(z)e^{ibz}}{2\pi}\int_{-b}^{+\infty} e^{itz}\Phi(t)dt. \tag{5.20}$$

Proof. It is a straightforward consequence of the Fourier inversion theorem (5.1) applied to (5.8). The integral is between $-b$ and $+\infty$ since $\Phi \in \mathcal{L}^2_\Theta$ and then $\Phi(t) = 0$ for $t < -b$ (see (5.7)).

□

A simple but important consequence of this inversion theorem is that $F \in \mathcal{B}(E)$ if and only if there exists $\Phi \in \mathcal{L}^2_\Theta$ such that

$$F(z) = \frac{E(z)e^{ibz}}{2\pi}\int_{-b}^{+\infty} e^{itz}\Phi(t)dt. \tag{5.21}$$

5.2 Orthogonal subspaces of the de Branges spaces

The result of *Theorem* (5.1) has many important consequences. In this section we show a suddivision of the de Branges spaces in othogonal subspaces and some other immediate properties, then in the next chapters we will see other imortant applications.

For $N > 0$ we set

$$\mathcal{B}_N(E) = E\tilde{\mathcal{K}}_N(\Theta),$$

while for $N = 0$ we set

$$\mathcal{B}_0(E) = E\mathcal{K}(\Theta_0).$$

We endow $\mathcal{B}_N(E)$ with the same scalar product of $\mathcal{B}(E)$, and we recall that the scalar product of $\mathcal{L}^2_\Theta[b(2N-1), b(2N+1)]$ is given by (5.6). By (5.19) we have that $\tilde{\mathcal{F}}_E$ is a unitary isomorphism between $\mathcal{B}_N(E)$ and $\mathcal{L}^2_\Theta[b(2N-1), b(2N+1)]$. Hence $F \in \mathcal{B}_N(E)$ if and only if there exists $\Phi \in \mathcal{L}^2_\Theta[b(2N-1), b(2N+1)]$ such that

$$F(z) = \frac{E(z)e^{ibz}}{2\pi}\int_{(2N-1)b}^{(2N+1)b} e^{itz}\Phi(t)dt.$$

Theorem 5.3. *Let* $\Theta(z) = \gamma e^{2ibz}B(z)$ *be a meromorphic inner function according to the representation given in* (2.3)*, with logarithmic residue* $2b > 0$*. Let* $E(z)$ *be a de Branges function of* $\Theta(z)$*, and let* $F \in \mathcal{B}(E)$ *such that*

$$\Phi = \tilde{\mathcal{F}}_E(F) \in L^1(\mathbb{R}).$$

Then

$$|F(x+iy)| \leq (2y)^{-\frac{1}{2}}|E(x+iy)|e^{yb}\|F\|_{\mathcal{B}(E)} \quad \forall x \in \mathbb{R}, y > 0.$$

Proof. Thanks to (5.20) we have that

$$F(z) = \frac{E(z)e^{ibz}}{2\pi}\int_{-b}^{+\infty} e^{itz}\Phi(t)dt.$$

Using Holder inequality and recalling (5.9) we get

$$\begin{aligned}
|F(x+iy)| &\leq \frac{|E(x+iy)|}{2\pi}\int_{-b}^{+\infty}\left|e^{it(x+iy)}\right||\Phi(t)|dt \\
&\leq \frac{|E(x+iy)|}{2\pi}\left(\int_{-b}^{+\infty}\left|e^{it(x+iy)}\right|^2 dt\right)^{\frac{1}{2}}\left(\int_{-b}^{+\infty}|\Phi(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \frac{|E(x+iy)|}{2\pi}\left(\int_{-b}^{+\infty} e^{-2yt}dt\right)^{\frac{1}{2}}\|\Phi\|_{\mathcal{L}^2_\Theta} \\
&\leq |E(x+iy)|\left(\left[\frac{-1}{2y}e^{-2yt}\right]_{t=-b}^{+\infty}\right)^{\frac{1}{2}}\|F\|_{\mathcal{B}(E)} \\
&= (2y)^{-\frac{1}{2}}|E(x+iy)|e^{yb}\|F\|_{\mathcal{B}(E)},
\end{aligned}$$

where we used the fact that $\|F\|_{\mathcal{B}(E)} = \|\Phi\|_{\mathcal{L}^2_\Theta}$ thanks to the unitary isomorphism between $\mathcal{B}(E)$ and $\mathcal{L}^2_\Theta$ described in *Theorem* 5.1. □

Theorem 5.4. *Let $\Theta(z) = \gamma e^{2ibz}B(z)$ be a meromorphic inner function according to the representation given in* (2.3)*, with logarithmic residue $2b > 0$. Let $E(z)$ be a de Branges function of $\Theta(z)$, and let $F \in \mathcal{B}_N(E)$. Then*

$$|F(x+iy)| \leq (2b)^{\frac{1}{2}}|E(x+iy)|e^{b(N+1)|y|}\|F\|_{\mathcal{B}(E)}.$$

Proof. The proof is very similar to that of the previous theorem. We set

$$\Phi = \tilde{\mathcal{F}}_E(F),$$

and we observe that $\Phi \in \mathcal{L}^2_\Theta[b(2N-1), b(2N+1)]$, hence the support of $\Phi(z)$ is the inverval $[b(2N-1), b(2N+1)]$ and therefore $\Phi \in \mathcal{L}^1(\mathbb{R})$. Then, thanks to (5.20) we have

$$F(z) = \frac{E(z)e^{ibz}}{2\pi}\int_{b(2N-1)}^{b(2N+1)} e^{itz}\Phi(t)dt.$$

Using Holder inequality and recalling (5.9) we get

$$\begin{aligned}
|F(x+iy)| &\leq \frac{|E(x+iy)|}{2\pi}\int_{b(2N-1)}^{b(2N+1)}\left|e^{it(x+iy)}\right||\Phi(t)|dt \\
&\leq \frac{|E(x+iy)|}{2\pi}\left(\int_{b(2N-1)}^{b(2N+1)}\left|e^{it(x+iy)}\right|^2 dt\right)^{\frac{1}{2}}\left(\int_{b(2N-1)}^{b(2N+1)}|\Phi(t)|^2 dt\right)^{\frac{1}{2}} \\
&= \frac{|E(x+iy)|}{2\pi}\left(\int_{b(2N-1)}^{b(2N+1)} e^{-2yt}dt\right)^{\frac{1}{2}}\|\Phi\|_{\mathcal{L}^2_\Theta} \\
&\leq |E(x+iy)|\left(\int_{b(2N-1)}^{b(2N+1)} e^{2b(2N+1)|y|}dt\right)^{\frac{1}{2}}\|F\|_{\mathcal{B}(E)} \\
&= (2b)^{\frac{1}{2}}|E(x+iy)|e^{b(N+1)|y|}\|F\|_{\mathcal{B}(E)}.
\end{aligned}$$

□

CHAPTER 6

Kempf-Martin Spaces

The whole Kempf-Martin theory presented in [40] is based on the theory of simple symmetric operators, which is the pillar upon which all the article is built. Indeed, the proof of all the crucial results about the Kempf-Martin spaces, such as the definition of the reproducing kernel, the sampling formula, the Livsic characteristic function properties and its explicit formulation, make extensive use of this theory. However, for our purpose it is not necessary to go here into the details of this theory, but it is sufficient to summerize the definitions and the fundamental results that are important to understand the Kempf-Martin spaces. We will give this overview in *Sections* 6.1, 6.2 and 6.3. In *Section* 6.4 we will investigate the isomorphism between the Kempf-Martin spaces and the de Branges spaces, and thanks to this isomorphism we will derive a necessary and sufficient condition for a function to belong to a Kempf-Martin space. In *Section* 6.5 we will give an alternative and equivalent definition of the Kempf-Martin spaces based on the same isomorphism, and we will derive and improve all the main results presented in [40] from the properties of the de Branges spaces, without using the theory of simple symmetric operators.

6.1 Symmetric linear transformations

Let $\mathcal{H}$ be a separable Hilbert space. Let T be a linear transformation T defined on a domain $\mathrm{Dom}(T) \subset \mathcal{H}$.

The *adjoint* operator of T is defined as follows. Suppose that $\mathrm{Dom}(T)$ is dense in $\mathcal{H}$. Let $\mathrm{Dom}^*(T)$ be the set of all $\psi \in \mathcal{H}$ such that there is a pair (ψ, ψ^*) with

$$\langle T\phi, \psi\rangle = \langle \phi, \psi^*\rangle \quad \forall \phi \in \mathrm{Dom}(T).$$

Then the adjoint operator of T, denoted by T^*, is defined to be

$$T^*\psi = \psi^* \quad \text{on } \mathrm{Dom}^*(T).$$

The linear transformation T is called:

1. *symmetric* if
$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \forall x, y \in \mathrm{Dom}(T);$$
2. *self-adjoint* if
   - $\mathrm{Dom}(T)$ is dense in $\mathcal{H}$,
   - $T = T^*$;
3. *densely defined* if $\mathrm{Dom}(T)$ is dense in $\mathcal{H}$;
4. *simple* if there is no non-trivial proper subspace $S \subset \mathcal{H}$ so that the restriction of T to Dom $(T) \cap S$ is self-adjoint;
5. *regular* if $T - tI$ is bounded below on $\mathrm{Dom}(T)$ for all $t \in \mathbb{R}$;
6. *closed if* the graph of T is closed in $\mathcal{H} \oplus \mathcal{H}$;

The deficiency indices, (n_+, n_-) of a linear transformation T are defined as

$$n_\pm := \dim\left(\mathrm{Ker}\left(T^* \mp i\right)\right).$$

We will use the notation $\mathcal{S}$ to denote the family of all closed simple symmetric linear transformations with equal indices $(1,1)$ defined on a domain in some separable Hilbert space. $\mathcal{S}^R$ will denote the subfamily of all closed regular simple symmetric transformations with indices $(1,1)$. Note that any symmetric T always has a minimal closed extension, so there is no loss of generality in assuming that T is closed.

Consider the map

$$b(z) := \frac{z-i}{z+i}$$

with compositional inverse

$$b^{-1}(z) = i\frac{1-z}{1+z}$$

The map b is an analytic bijection of the open upper half-plane $\mathbb{C}^+$onto the open unit disk $\mathbb{D}$. Moreover b is a bijection of the real line $\mathbb{R}$ onto $\mathbb{T}\backslash\{1\}$, where $\mathbb{T}$ is the unit circle.

Let $\mathcal{V}$ denote the family of all completely non-unitary (c.n.u.) partial isometries with deficiency indices $(1,1)$ acting on a separable Hilbert space. Here the defect or deficiency indices of a partial isometry V are defined by $n_+ := \dim(\mathrm{Ker}(V))$ and $n_- := \dim\left(\mathrm{Ran}(V)^\perp\right)$. As shown in many standard texts (see for example [5], [45]), the map $T \mapsto b(T)$ defines a bijection of $\mathcal{S}_n$ (closed simple symmetric linear transformations with indices (n,n)) onto $\mathcal{V}_n$. Namely, given any $T \in \mathcal{S}_n$ we can define $b(T)$ as an isometric linear transformation from $\mathrm{Ran}(T+i)$ onto $\mathrm{Ran}(T-i)$. We can then view $V = b(T)$ as a partial isometry on $\mathcal{H}$ with initial space $\mathrm{Ker}(V)^\perp = \mathrm{Ran}(T+i)$. Conversely, given any $V \in \mathcal{V}_n$, we can define $b^{-1}(V) = T$ on the domain $\mathrm{Ran}\left((V-I)V^*V\right)$, and then $T \in \mathcal{S}_n$ and $T = b^{-1}(b(T))$.

6.2 Self-adjoint extensions

Given $T \in \mathcal{S}$ let $V = b(T) \in \mathcal{V}$. We can build a one parameter family of unitary extensions of V as follows. Fix two vectors $\phi_\pm$ of equal norm such that

$$\phi_+ \in \mathrm{Ker}(V) = \mathrm{Ker}\left(T^* - i\right) = \mathrm{Ran}(T+i)^\perp$$

and

$$\phi_- \in \mathrm{Ran}(V)^\perp = \mathrm{Ker}\left(T^* + i\right) = \mathrm{Ran}(T-i)^\perp.$$

Define

$$U(\alpha) := V + \frac{\alpha}{\|\phi_+\|^2}\left\langle \cdot, \phi_+\right\rangle \phi_-; \alpha \in \mathbb{T} \quad \text{and} \quad U_\theta := U\left(e^{i2\pi\theta}\right); \theta \in [0,1),$$

where $\mathbb{T}$ is the unit circle in the complex plane. The set of all $U(\alpha)$ (or U_θ) is the one-parameter family of all unitary extensions of V on $\mathcal{H}$. The $U(\alpha)$ extend V in the sense that $U(\alpha)V^*V = V$ for all $\alpha \in \mathbb{T}$, they agree with V on its initial space. We write $V \subseteq U(\alpha)$ to denote that $U(\alpha)$ extends V in this way. Similarly, the subset notation $T \subset S$ for closed linear transformations T, S denotes that $\mathrm{Dom}(T) \subset \mathrm{Dom}(S)$ and $S|_{\mathrm{Dom}(T)} = T$, i.e. S is an extension of T. We then define

$$T(\alpha) := b^{-1}(U(\alpha)), \quad T_\theta = T\left(e^{i2\pi\theta}\right)$$

so that $T \subset T(\alpha) \subset T^*$ for all $\alpha \in \mathbb{T}$. The functional calculus implies that each $T(\alpha)$ is a densely defined self-adjoint operator if and only if 1 is not an eigenvalue of $U(\alpha)$, and the set of all $T(\alpha)$ (for which this expression is defined) is the set of all self-adjoint extensions of T. Note the assumption that V be c.n.u. implies that 1 is an eigenvalue to at most one $U(\alpha)$.

Given a transformation $T \in \mathcal{S}^R$, every different choice of deficiency vectors $\phi_\pm$ defines a different parameterization $\{T_\theta\}_{\theta\in[0,1)}$ of the self-adjoint extensions of T. Indeed the same self-adjoint extensions of T can be associated to two different values of $\theta \in [0,1)$ in the parametrizations derived from two different pairs of deficiency vectors $\phi_{\pm}$. Moreover, for every two different pairs of deficiency vectors $\phi_{\pm}$ there is always a not null subset of self-adjoint extensions of T that are associated to two different values of $\theta \in [0,1)$ in the corresponding parametrizations.

Lemma 2.2 in [40] shows that, given a transformation $T \in \mathcal{S}^R$, for each $\theta \in [0,1)$ the spectrum $\sigma(T_\theta)$ of the self-adjoint extension T_θ (i.e. the set of all the eigenvalues of T_θ) is given by

$$\sigma\left(T_\theta\right) = \{t_n(\theta)\},$$

where $\{t_n(\theta)\}$ is a strictly increasing sequence of eigenvalues of multiplicity one with no finite accumulation point. The spectra of all the self-adjoint extensions of T have the following properties:

- $t_n(\theta) = \sigma\left(T_\theta\right) \cap [t_n(0), t_{n+1}(0))$;
- $\sigma\left(T_\theta\right) \cap \sigma\left(T_\beta\right) = \emptyset$ for $\theta \neq \beta$;
- $\bigcup_{\theta \in [0,1)} \sigma\left(T_\theta\right) = \mathbb{R}$.

Hence the spectra of all the self-adjoint of T extensions cover the real line exactly once.

Theorem 2.6 in [40] shows that the function $t(x)$, $x \in \mathbb{R}$, defined by

$$t(n+\theta) = t_n(\theta), \qquad n \in \mathbb{Z}, \theta \in [0,1), \tag{6.1}$$

turns out to be a smooth, strictly increasing function on $\mathbb{R}$, referred as the spectral function of T. Moreover, defining $t'_n(\theta) = t'(n+\theta)$, for any $\theta \in [0,1)$ the couple $(\{t_n(\theta)\}_n, \{t'_n(\theta)\}_n)$ is a bandlimit pair.

Finally, *Theorem* 2.8 in [40] shows that there exsists a bijective correspondence between the transformations $T \in \mathcal{S}^R$ and the bandlimit pairs $\left(\{t_n\}_n, \{t'_n\}_n\right)$. Indeed, given a transformation $T \in \mathcal{S}^R$ and a couple of deficiency vectors $\phi_{\pm}$, then $\left(\{t_n(0)\}_n, \{t'_n(0)\}_n\right)$ is a bandlimit pair. Conversely, given any bandlimit pair $\left(\{t_n\}_n, \{t'_n\}_n\right)$ it is possible to build a couple of vectors $\phi_{\pm}$ for which there exists a unique transformation $T \in \mathcal{S}^R$ so that $\phi_{\pm}$ are equal norm deficiency vectors for T and that $\left(\{t_n(0)\}_n, \{t'_n(0)\}_n\right) = \left(\{t_n\}_n, \{t'_n\}_n\right)$.

6.3 Definition of the Kempf-Martin spaces

In the previous sections we summerized the definitions and the results that are necessary to define the Kempf-Martin spaces. We recall that, as we pointed out in *Section* 2.1, given any positive kernel function $K(z,w)$, there always exsist a RKHS $\mathcal{H}(K)$ which has $K(z,w)$ as its reproducing kernel. For every bandlimit pair $(\{t_n\}_n, \{t'_n\}_n)$, *Proposition* 2.18 in [40] defines a unique corresponding positive kernel funtion expressed in terms of the bandlimit pair itself. Since every linear transformation $T \in \mathcal{S}^R$ is associated to a unique corresponding bandlimit pair, it can be associated also to a unique positive kernel function $K_T(t,s)$. The Kempf-Martin space $\mathcal{KM}(T)$ associated to T is then defined as the unique RKHS that has $K_{\mathcal{KM}(T)}(t,s) := K_T(t,s)$ as reproducing kernel. Sometimes we will write $\mathcal{KM}(\{t_n\}_n, \{t'_n\}_n)$ in place of $\mathcal{KM}(T)$, obviously meaning the Kempf-Martin space whose kernel is the one defined by the bandlimit pair $(\{t_n\}_n, \{t'_n\}_n)$.

Since *Proposition* 2.18 in [40] is the fundamental result for the definition of the Kempf-Martin spaces, we report its precise statement as presented by Kempf and Martin.

Theorem 6.1 (Kempf-Martin). *For any $T \in \mathcal{S}^R$ and a fixed equal-norm deficiency vectors $\phi_\pm \in \operatorname{Ker}(T^* \mp i)$, there exists a choice of orthonormal eigenbases $\{\phi_n(\theta) \mid \theta \in [0,1)\}_n$ of eigenvectors for T_θ so that if $\phi_t := \phi_{\lfloor \tau(t) \rfloor}([\tau(t)])$, then for $s, t \in \mathbb{R}$*

$$\begin{aligned} K_{\mathcal{KM}(T)}(t,s) &:= \langle \phi_t, \phi_s \rangle \\ &= f(t)(-1)^{\lfloor \tau(t) \rfloor} \left(\sum_k \frac{t'(k)}{(t-t_k)(s-t_k)} \right) (-1)^{\lfloor \tau(s) \rfloor} f(s) \end{aligned} \tag{6.2}$$

is a smooth, real-valued, positive kernel function on $\mathbb{R} \times \mathbb{R}$, where

$$f(t) := \left(\sum_n \frac{t'(n)}{(t-t_n)^2} \right)^{-\frac{1}{2}}.$$

Another crucial result for the Kempf-Martin theory is *Theorem* 2.24 in [40]. It shows that any $F(t) \in \mathcal{KM}(T)$ obeys the sampling formula

$$F(t) = \sum_n K_{\mathcal{KM}(T)}\left(t_n(\theta), t\right) F\left(t_n(\theta)\right), \tag{6.3}$$

for all $\theta \in [0,1)$. Therefore the Kempf-Martin spaces have the same special reconstruction properties as the Paley-Wiener spaces of bandlimited functions: any $F \in \mathcal{KM}(T)$ can be reconstructed perfectly from its samples taken on $\{t_n(\theta)\}$. It turns out that the classical Paley-Wiener spaces are a special case of the Kempf-Martin spaces.

For every $T \in \mathcal{S}^R$, *Section* 3 in [40] defines the Livsic characteristic function $\Theta(z)$ associated to T, which is a meromorphic inner function such that $\Theta(i) = 0$, with the following special property. For every $\theta \in [0,1)$, the sequence $\{t_n(\theta)\}$ associated to T according to (6.1) is the set of solutions of

$$\Theta(t) = e^{i2\pi\theta}, \quad t \in \mathbb{R}, \tag{6.4}$$

as shown in *Corollary* 3.18.

The first part of *Section* 4 in [40] shows that $\Theta(z)$ can be expressed as

$$\Theta(z) = \frac{z-i}{z+i} \frac{\sum_n \frac{1}{t_n - z} \frac{1}{t_n - i} t'_n}{\sum_n \frac{1}{t_n - z} \frac{1}{t_n + i} t'_n} = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{1}{t_n - i} \right)}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{1}{t_n + i} \right)},$$

where $(\{t_n\}_n, \{t'_n\}_n)$ is the bandlimit pair associated to T. Moreover the spectral function $t(x)$ of $\Theta(z)$ coincides with the spectral function of T defined in (6.1), and $t'(n) = t'_n$.

Finally, *Theorem* 4.8 in [40] connects the theory of Kempf-Martin spaces to the theory of meromorphic model spaces of Hardy spaces by showing that any Kempf-Martin space $\mathcal{KM}(T)$, with Livsic characteristic function $\Theta(z)$, is the image of the model space $\mathcal{K}(\Theta) := \mathcal{H}^2 \ominus \Theta\mathcal{H}^2$ under the multiplication by a fixed function $M(t)$, and that this multiplication defines an onto isometry.

6.4 Characterization of the Kempf-Martin Spaces

In [40], the definition of the Kempf-Martin spaces is given through their reproducing kernel, without a really in-depth characterizations of the functions that belong to these spaces. In this section we investigate the isomorphism between the Kempf-Martin space and the de Branges spaces, which has many far-reaching consequences that we will see also in the next chapters. Thanks to this isomorphism, we also give a necessary and sufficient condition for a function belong to a Kempf-Martin space.

Theorem 6.2. *Given any regular simple symmetric linear transformation T with deficiency indices $(1,1)$, let $E(z)$ be a de Branges function of the Livsic characteristic function $\Theta(z)$ of $\mathcal{KM}(T)$. Then there exists an isometric multiplier $N(t)$ from the Kempf-Martin space $\mathcal{KM}(T)$ onto the restriction on $\mathbb{R}$ of the de Branges space $\mathcal{B}(E)$:*

$$\mathcal{KM}(T)N(t) = \mathcal{B}(E)|_{\mathbb{R}}, \quad t \in \mathbb{R}. \tag{6.5}$$

The multiplier is given by

$$N(t) = i\sqrt{K_{\mathcal{B}(E)}(t,t)} = i|E(t)|\sqrt{\tau'(t)},$$

where $K_{\mathcal{B}(E)}(w,z)$ is the reproducing kernel of $\mathcal{B}(E)$ and $\tau(t)$ is the phase function of $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$. Moreover it is isometric since, for $F_1(t), F_2(t) \in \mathcal{KM}(T)$ and $G_1(t) = N(t)F_1(t), G_2(t) = N(t)F_2(t) \in \mathcal{B}(E)$, we have

$$\langle G_1, G_2 \rangle_{\mathcal{B}(E)} = \langle F_1, F_2 \rangle_{\mathcal{KM}(T)}. \tag{6.6}$$

Proof. From *Theorem* 4.8 in [40] (p. 1628) we have that on the real line there exists an isometric multiplier $M(t)$ from $\mathcal{K}(\Theta)$ onto $\mathcal{KM}(T)$, where $\Theta(z)$ is the Livsic characteristic function of T (see *Section* 3.12 in [40], p. 1620), and $\mathcal{K}(\Theta)$ is the model space $\mathcal{H}^2 \ominus \Theta\mathcal{H}^2$. We have

$$M(t) = 2\pi\,(1 - \Theta(t))^{-1}\,(-1)^{\lfloor \tau(t) \rfloor} f(t),$$

where

$$f(t) = \left(\sum_n \frac{t'(n)}{(t - t(n))^2} \right)^{-\frac{1}{2}},$$

then $\forall F(t) \in \mathcal{KM}(T)$ there exists $H \in \mathcal{K}(\Theta)$ such that

$$F(t) = M(t)H(t), \quad t \in \mathbb{R}, \tag{6.7}$$

and $\forall H \in \mathcal{K}(\Theta)$ there exists $F(t) \in \mathcal{KM}(T)$ such that (6.7) is verified. Let $E(z)$ be a de Branges function of $\Theta(z)$. We already know that the de Branges space $\mathcal{B}(E)$ is given by

$$\mathcal{B}(E) = E\mathcal{K}(\Theta).$$

Hence, given any $G \in \mathcal{B}(E)$ we can write

$$G(t) = E(t)H(t) = \frac{E(t)}{M(t)}F(t), \quad t \in \mathbb{R} \tag{6.8}$$

for some $H \in \mathcal{K}(\Theta)$ and $F(t) \in \mathcal{KM}(T)$, and given any $F(t) \in \mathcal{KM}(T)$ there exist $H \in \mathcal{K}(\Theta)$ and $G \in \mathcal{B}(E)$ so that (6.8) is verified. Hence we obtained that there exists a multiplier $N(t)$ from $\mathcal{KM}(T)$ onto $\mathcal{B}(E)$, given by $N(t) = \frac{E(t)}{M(t)}$. We have:

$$N(t) = \frac{E(t)}{M(t)} = \frac{E(t) - E^{\#}(t)}{2\pi(-1)^{\lfloor \tau(t) \rfloor} f(t)}.$$

Now, from equation (11) in [40], p. 1612 (see aldo (3.45) and (3.38) in [19]), we have

$$f(t) = (-1)^{\lfloor \tau(t) \rfloor} \frac{\sin(\pi\tau(t))}{\pi} \sqrt{t'(\tau(t))}.$$

Moreover, for $t \in \mathbb{R}$ we have:

$$\begin{aligned}
(E(t) - E^{\#}(t))^2 &= E(t)E^{\#}(t) \left(\frac{E(t)}{E^{\#}(t)} + \frac{E^{\#}(t)}{E(t)} - 2 \right) \\
&= |E(t)|^2 \left(\frac{1}{\Theta(t)} + \Theta(t) - 2 \right) \\
&= |E(t)|^2 \left(e^{-2\pi i\tau(t)} + e^{2\pi i\tau(t)} - 2 \right) \\
&= |E(t)|^2 \left(2\cos(2\pi\tau(t)) - 2 \right) \\
&= -4|E(t)|^2 \left(\frac{1 - \cos(2\pi\tau(t))}{2} \right) \\
&= -4|E(t)|^2 \sin(\pi\tau(t))^2.
\end{aligned}$$

Hence

$$\begin{aligned} N(t) &= \frac{2i|E(t)|\sin(\pi\tau(t))}{2\sin(\pi\tau(t))\sqrt{t'(\tau(t))}} \\ &= \frac{i|E(t)|}{\sqrt{t'(\tau(t))}} \\ &= i|E(t)|\sqrt{\tau'(t)} \\ &= i|E(t)|\sqrt{\frac{\Theta'(t)}{2\pi i\Theta(t)}} \\ &= i\sqrt{\frac{E^{\#\prime}(t)E(t) - E^{\#}(t)E'(t)}{2\pi i}} \end{aligned} \tag{6.9}$$

Using (2.16) we finally obtain

$$N(t) = i\sqrt{K_{\mathcal{B}(E)}(t,t)}.$$

Moreover, from (6.9) we have that $N(t) = i|E(t)|\sqrt{\tau'(t)}$, then for $F_1(t), F_2(t) \in \mathcal{KM}(T)$ we set

$$\begin{aligned} G_1(t) &= i|E(t)|\sqrt{\tau'(t)}F_1(t) \in \mathcal{B}(E), \\ G_2(t) &= i|E(t)|\sqrt{\tau'(t)}F_2(t) \in \mathcal{B}(E), \end{aligned}$$

and we obtain:

$$\begin{aligned} \langle G_1, G_2\rangle_{\mathcal{B}(E)} &= \int_{\mathbb{R}} G_1(t)\overline{G_2(t)}\frac{1}{|E(t)|^2}dt \\ &= \int_{\mathbb{R}} F_1(t)\overline{F_2(t)}\tau'(t)dt \\ &= \langle F_1, F_2\rangle_{\mathcal{KM}(T)}. \end{aligned} \tag{6.10}$$

and as a consequence

$$\|G_1\|_{\mathcal{B}(E)} = \|F_1\|_{\mathcal{KM}(T)}.$$

This shows that with a rescaling we have an isometry between $\mathcal{B}(E)$ and $\mathcal{KM}(T)$. $\square$

Corollary 6.3. *For every regular simple symmetric linear transformation T with deficiency indices $(1,1)$, let $\Theta(z)$ be the Livsic characteristic function $\Theta(z)$ of $\mathcal{KM}(T)$. Then on the real line there exists an isometric multiplier $\tilde{N}(t)$ from the Kempf-Martin space $\mathcal{KM}(T)$ onto the model space space $\mathcal{K}(\Theta)$, given by*

$$\tilde{N}(t) = \sqrt{\frac{i\Theta'(t)}{2\pi}}. \tag{6.11}$$

Proof. From *Theorem* 4.8 in [40] (p. 1628) we already know that on the real line there exists an isometric multiplier between $\mathcal{KM}(T)$ and $\mathcal{K}(\Theta)$. Hence we just need to show that the multiplier has the expression given in (6.11). Let $E(z)$ be a de Branges function of $\Theta(z)$. Then, since $B(E) = E\mathcal{K}(\Theta)$, thanks to *Theorem* 6.2, on the real line there exists an isometric multiplier from the Kempf-Martin space $\mathcal{KM}(T)$ onto the model space space $\mathcal{K}(\Theta)$, given by $\tilde{N}(t) = \frac{N(t)}{E(t)}$. Now, recalling (6.9), we obtain

$$\begin{aligned}
\tilde{N}(t) &= \frac{N(t)}{E(t)} = i\frac{|E(t)|}{E(t)}\sqrt{\tau'(t)} = i\frac{\sqrt{E(t)E^{\#}(t)}}{E(t)}\sqrt{\tau'(t)} = i\sqrt{\frac{E^{\#}(t)}{E(t)}\tau'(t)} \\
&= i\sqrt{\Theta(t)\tau'(t)} = i\sqrt{e^{2\pi i\tau(t)}\tau'(t)} = i\sqrt{\frac{1}{2\pi i}\frac{d}{dt}\left(e^{2\pi i\tau(t)}\right)} \\
&= \sqrt{\frac{i\Theta'(t)}{2\pi}}
\end{aligned}$$

□

The next theorems give a necessary and sufficient condition for a function $F(x)$ to belong to $\mathcal{KM}(T)$.

Theorem 6.4. *Let $(\{t_n\}_n, \{t'_n\}_n)$ be a bandlimit pair such that*

$$-\limsup_{y\to\infty} y^{-1}\log\left|\sum_n t'_n\left(\frac{1}{t_n - iy} - \frac{1}{t_n - i}\right)\right| > 0, \tag{6.12}$$

and that

$$\sum_{n\neq 0}\frac{1}{|t_n|^{q+1}} < \infty$$

for some $q \in \mathbb{Z}$, $q \geq 0$. Let Let $\mathcal{KM}(\{t_n\}_n, \{t'_n\}_n)$ be the corresponding Kempf-Martin space. Let $E(z)$ be given by

$$E(z) = z^c\left(\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha\right)\prod_{n\in\mathbb{Z}, t_n\neq 0}\left(1 - \frac{z}{t_n}\right)e^{u(z)}, \tag{6.13}$$

$$c = \begin{cases} 1, & \text{if } \exists n \mid t_n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$u(z) = \sum_{m=1}^{p}\left(\frac{z}{t_n}\right)^m,$$

and p is the smallest nonnegative integer for which the series

$$\sum_{n\neq 0}\frac{1}{|t_n|^{p+1}}$$

is convergent. Then a function $F(t)$, $t \in \mathbb{R}$, belongs to $\mathcal{KM}(\{t_n\}_n, \{t'_n\}_n)$ if and only if there exists a function $G \in \mathcal{L}^2_\Theta$ such that

$$F(t) = \frac{1}{i\sqrt{K_{\mathcal{B}(E)}(t,t)}}\tilde{\mathcal{F}}_E(G(t)),$$

Proof. Thanks to *Theorem* 6.2 we have that

$$H(t) = N(t)F(t) = i\sqrt{K_{\mathcal{B}(E)}(t,t)}F(t) \in \mathcal{B}(E).$$

Let T be the regular simple symmetric linear transformation with deficiency indices $(1,1)$ corresponding to the bandlimit pair $(\{t_n\}_n, \{t'_n\}_n)$, and let $\Theta(z)$ be the Livsic characteristic function of T. By *Theorem* 4.4 and *Proposition* 4.5 in [40] (p. 1627) we have that $\Theta(z)$ has the form

$$\begin{aligned}\Theta(z) &= \frac{z - i\sum_n \frac{1}{t_n - z}\frac{1}{t_n - i}t'_n}{z + i\sum_n \frac{1}{t_n - z}\frac{1}{t_n + i}t'_n}\\ &= \frac{\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{1}{t_n - i}\right)}{\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{1}{t_n + i}\right)}\\ &= \frac{\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1}\right) + \overline{\alpha}}{\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1}\right) + \alpha},\end{aligned}$$

where $\alpha = i\sum_n \frac{t'_n}{t_n^2+1}$. Then by *Theorem* 4.8 we get that $E(z)$ given in (6.13) verifies $\Theta(z) = \frac{E^\#(z)}{E(z)}$. Thanks to *Lemma* 4.12 and to (6.12) we obtain

$$\begin{aligned}b &= -\limsup_{y\to+\infty} y^{-1}\log\left|\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1}\right) + \overline{\alpha}\right|\\ &= -\limsup_{y\to+\infty} y^{-1}\log\left|\sum_n t'_n\left(\frac{1}{t_n - z} - \frac{1}{t_n - i}\right)\right|\\ &> 0.\end{aligned}$$

Hence we can apply *Theorem* 5.1 to $H(t)$, and we obtain that $H(t) \in \mathcal{B}(E)$ if and only if there exists a function $G \in \mathcal{L}^2_\Theta$ such that

$$H(t) = \tilde{\mathcal{F}}_E(G(t)).$$

Since $F(t) = \frac{1}{i\sqrt{K_{\mathcal{B}(E)}(t,t)}}H(t)$, the proof is complete. □

6.5 Alternative definition of the Kempf-Martin spaces

According to the unitary isomorphism between the Kempf-Martin spaces and the de Branges spaces proved in *Theorem* 6.2, we introduce the following equivalent definition of the Kempf-Martin spaces. For every Hermite Biehler function $E(z)$, the corresponding Kempf-Martin space $\mathcal{KM}(E)$ is given by

$$\mathcal{KM}(E) = \frac{\mathcal{B}(E)|_{\mathbb{R}}}{i\sqrt{K_{\mathcal{B}(E)}(t,t)}} = \frac{\mathcal{B}(E)|_{\mathbb{R}}}{i|E(t)|\sqrt{\tau'(t)}}, \quad t \in \mathbb{R}. \tag{6.14}$$

Since the original Kempf-Martin spaces are defined only on the real line, here the functions of the de Branges space $\mathcal{B}(E)$ are considered to be restricted on the real line. Obviously, the Kempf-Martin space $\mathcal{KM}(E)$ coincides with the Kempf-Martin space $\mathcal{KM}(T)$ with Livsic characteristic function given by $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$. The Livsic characteristic function of the original Kempf-Martin spaces is such that $\Theta(i) = 0$, but nothing prevents us from extending the definition also to functions for which $\Theta(i) \neq 0$.

The goal of this section is to show that, starting from this equivalent definition of the Kempf-Martin spaces, we can derive and improve all the most important results for the Kempf-Martin spaces presented in [40] without using the theory of simple symmetric operators.

We will prove these results in a different order than that given in [40]. In particular first we will derive the result of *Theorem* 2.24 and the main results of *Section* 3. Only after this we will give an equivalent proof of *Propostion* 2.18. Finally we will derive the main results of *Section* 4.

Theorem 2.24 in [40] derives the sampling formula (6.3), while the main goal of *Section* 3 in [40] is to define the Livsic characteristic function $\Theta(z)$ of a Kempf-Martin space, and to prove *Corollary* 3.18, which states that, for every $\theta \in [0, 1)$, the sequence $\{t_n(\theta)\}_n$ defined in (6.1) is the set of solutions of

$$\Theta\left(t\right) = e^{i2\pi\theta}, \quad t \in \mathbb{R}. \tag{6.15}$$

For the alternative definition of the Kempf-Martin spaces, all these results are obtained in the next theorem, and we improve them showing that the sampling formula (6.3) converges in norm, and uniformly on the intervals of $\mathbb{R}$.

Theorem 6.5. *Let $E(z)$ be a Hermite Biehler function, and let $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$. For every $\theta \in [0, 1)$, let the sequence $\{t_n(\theta)\}$ be the set of solutions of*

$$\Theta\left(t\right) = e^{i2\pi\theta}, \quad t \in \mathbb{R}, \tag{6.16}$$

Then for every $\theta \in [0, 1)$ and $F(t) \in \mathcal{KM}(E)$ the following sampling formula holds:

$$F(t) = \sum_n K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right) F\left(t_n(\theta)\right).$$

The series converges in norm, and uniformly on the intervals of $\mathbb{R}$.

Proof. Given a Hermite Biehler function $E(z)$, the classical sampling formula (2.17) for the de Branges space $\mathcal{B}(E)$ is based on the sampling sequence $\{t_n\}_n$

of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$, where $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$. Now consider the inner function $\Theta_\theta(z) = e^{-2\pi i\theta}\Theta(z)$, for some $\theta \in [0, 1)$. Then $E_\theta(z) = e^{i\pi\theta}E(z)$ is a de Branges function of $\Theta_\theta(z)$. We observe that the corresponding de Branges space $\mathcal{B}(E_\theta) = E_\theta\mathcal{K}(\Theta_\theta)$ is obviously equal to $\mathcal{B}(E) = E\mathcal{K}(\Theta)$, since both $E(z)$ and $\Theta(z)$ are just multiplied by a constant. Indeed, by (2.15) we can easily see that also the reproducing kernel remains the same:

$$\begin{aligned}
K_{\mathcal{B}(E_\theta)}(w, z) &= \frac{E_\theta(z)E_\theta^{\#}(\overline{w}) - E_\theta^{\#}(z)E_\theta(\overline{w})}{2\pi i(\overline{w} - z)} \\
&= \frac{e^{i\pi\theta}E(z)e^{-i\pi\theta}E^{\#}(\overline{w}) - e^{-i\pi\theta}E^{\#}(z)e^{i\pi\theta}E(\overline{w})}{2\pi i(\overline{w} - z)} \\
&= \frac{E(z)E^{\#}(\overline{w}) - E^{\#}(z)E(\overline{w})}{2\pi i(\overline{w} - z)} \\
&= K_{\mathcal{B}(E)}(w, z).
\end{aligned}$$

Therefore, for all $\theta \in [0, 1)$, every function $G \in \mathcal{B}(E)$ can be rebuilt exactly also with samples taken on the sampling sequence $\{t_n(\theta)\}$ of soultions of $\Theta(t) = e^{2\pi i\theta}$ for $t \in \mathbb{R}$, and by (2.17) we get

$$G(z) = \sum_n \frac{K_{\mathcal{B}(E)}\left(t_n(\theta), z\right)}{K_{\mathcal{B}(E)}\left(t_n(\theta), t_n(\theta)\right)} G\left(t_n(\theta)\right). \tag{6.17}$$

Now, given $F \in \mathcal{KM}(E)$, according to (6.14) set $G(t) = N(t)F(t) \in \mathcal{B}(E)$, where

$$N(t) = i\sqrt{K_{\mathcal{B}(E)}(t, t)}. \tag{6.18}$$

By (6.17) we have

$$F(t)N(t) = \sum_n \frac{K_{\mathcal{B}(E)}\left(t_n(\theta), t\right)}{K_{\mathcal{B}(E)}\left(t_n(\theta), t_n(\theta)\right)} F\left(t_n(\theta)\right) N(t_n(\theta)).$$

and then

$$F(t) = \sum_n \frac{N(t_n(\theta))K_{\mathcal{B}(E)}\left(t_n(\theta), t\right)}{N(t)K_{\mathcal{B}(E)}\left(t_n(\theta), t_n(\theta)\right)} F\left(t_n(\theta)\right). \tag{6.19}$$

For $G \in \mathcal{B}(E)$, with $G(t) = F(t)N(t)$ for $F(t) \in \mathcal{KM}(E)$ and $t \in \mathbb{R}$, we get

$$\begin{aligned}
F(t)N(t) &= G(t) \\
&= \left\langle G(s), K_{\mathcal{B}(E)}(t, s)\right\rangle_{\mathcal{B}(E)}, \\
&= \left\langle F(s)N(s), K_{\mathcal{B}(E)}(t, s)\right\rangle_{\mathcal{B}(E)},
\end{aligned}$$

and then

$$\begin{aligned}
F(t) &= \left\langle F(s)N(s), \frac{K_{\mathcal{B}(E)}(t, s)}{\overline{N(t)}}\right\rangle_{\mathcal{B}(E)} \\
&= \left\langle F(s), \frac{K_{\mathcal{B}(E)}(t, s)}{\overline{N(t)}N(s)}\right\rangle_{\mathcal{KM}(E)}.
\end{aligned}$$

Hence, recalling that the repoducing kernel of a RKHS is unique, we obtain that the reproducing kernel of $\mathcal{KM}(E)$ is given by

$$K_{\mathcal{KM}(E)}(t,s) \;\; = \frac{K_{\mathcal{B}(E)}(t,s)}{\overline{N(t)}N(s)}. \tag{6.20}$$

Therefore, by (6.19) and (6.20) we get

$$\begin{aligned} F(t) &= \sum_n \frac{N(t_n(\theta))^2 K_{\mathcal{B}(E)}\left(t_n(\theta),t\right)}{N(t)N(t_n(\theta))K_{\mathcal{B}(E)}\left(t_n(\theta),t_n(\theta)\right)} F\left(t_n(\theta)\right). \\ &= \sum_n \frac{K_{\mathcal{KM}(E)}\left(t_n(\theta),t\right)}{K_{\mathcal{KM}(E)}\left(t_n(\theta),t_n(\theta)\right)} F\left(t_n(\theta)\right). \end{aligned} \tag{6.21}$$

Moreover, by (6.18) and (6.20) we get

$$\begin{aligned} K_{\mathcal{KM}(E)}(t,t) &= \frac{K_{\mathcal{B}(E)}(t,t)}{|N(t)|^2} \\ &= \frac{K_{\mathcal{B}(E)}(t,t)}{K_{\mathcal{B}(E)}(t,t)} \\ &= 1, \end{aligned} \tag{6.22}$$

and hence we finally get

$$F(t) = \sum_n K_{\mathcal{KM}(E)}\left(t_n(\theta),t\right) F\left(t_n(\theta)\right).$$

Now, for $M \in \mathbb{Z}$, $M > 0$ we set

$$\begin{aligned} G_M(t) &= \left(G(t) - \sum_{n=-M}^{M} \frac{K_{\mathcal{B}(E)}(t_n(\theta),t)}{K_{\mathcal{B}(E)}(t_n(\theta),t_n(\theta))} G(t_n(\theta)) \right) \in \mathcal{B}(E), \\ F_M(t) &= \left(F(t) - \sum_{n=-M}^{M} \frac{N(t_n(\theta))K_{\mathcal{B}(E)}(t_n(\theta),t)}{N(t)K_{\mathcal{B}(E)}(t_n(\theta),t_n(\theta))} F(t_n(\theta)) \right) \\ &= \frac{G_M(t)}{N(t)} \in \mathcal{KM}(E), \end{aligned}$$

with

$$\|G_M\|_{\mathcal{B}(E)} = \|F_M\|_{\mathcal{KM}(E)}.$$

Given any interval $[\alpha,\beta] \subset \mathbb{R}$, by *Theorem* 2.18 we have

$$\begin{aligned} \lim_{M\to+\infty} \sup_{t\in[\alpha,\beta]} |G_M(t)| &= 0, \\ \lim_{M\to+\infty} \|G_M\|_{\mathcal{B}(E)} &= 0. \end{aligned}$$

Therefore, since $G_M(t) = F_M(t)N(t)$ and $N(t)$ has neither poles nor zeros on $\mathbb{R}$, we obtain

$$\begin{aligned}\lim_{M\to+\infty}\sup_{t\in[\alpha,\beta]}|F_M(t)| &= \lim_{M\to+\infty}\sup_{t\in[\alpha,\beta]}\left|\frac{F_M(t)N(t)}{N(t)}\right| \\ &\le \frac{1}{\inf_{t\in[\alpha,\beta]}|N(t)|}\lim_{M\to+\infty}\sup_{t\in[\alpha,\beta]}|G_M(t)| \\ &= 0,\end{aligned}$$

and

$$\lim_{M\to+\infty}\|F_M\|_{\mathcal{KM}(E)} = \lim_{M\to+\infty}\|G_M\|_{\mathcal{B}(E)} = 0.$$

Hence the series converges in norm, and uniformly on the intervals of $\mathbb{R}$. $\square$

The main goal of *Section* 2 in [40] is to use the theory of simple symmetric operators to prove the crucial result of *Proposition* 2.18 (*Theorem* 6.1 in this work). For the equivalent definition of the Kempf-Martin spaces, we derive the same result in the next theorem. Before proving the theorem, we need the following lemma, which is an interlocutory but fundamental result.

Lemma 6.6. *Let $E(z)$ be a Hermite Biehler function, and let t be the spectral function of $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$. Let $K_{\mathcal{KM}(E)}$ be the reproducing kernel of the Kempf-Martin space $\mathcal{KM}(E)$. Then, for every $\theta \in [0,1)$, the sequence $\{K_{\mathcal{KM}(E)}(t(n+\theta), z)\}_n$ is an orthonormal basis of $\mathcal{KM}(E)$. Moreover for every $n, m \in \mathbb{Z}$ and $\theta, \beta \in [0,1)$ we have*

$$\begin{aligned}&\Big\langle K_{\mathcal{KM}(E)}(t(n+\theta),\cdot), K_{\mathcal{KM}(E)}(t(m+\beta),\cdot)\Big\rangle_{\mathcal{KM}(E)} \\ &= \frac{(-1)^{n+m}\sin(\pi(\beta-\theta))}{t_m(\beta)-t_n(\theta)}\frac{\sqrt{t'(n+\theta)t'(m+\beta)}}{\pi}.\end{aligned} \tag{6.21}$$

Proof. By (6.20) we have

$$\begin{aligned}&K_{\mathcal{KM}(E)}(r,s) \\ &= \frac{K_{\mathcal{B}(E)}(r,s)}{\overline{N(r)}N(s)} \\ &= \frac{1}{|E(r)|\sqrt{\tau'(r)}}K_{\mathcal{B}(E)}(r,s)\frac{1}{|E(s)|\sqrt{\tau'(s)}} \\ &= \frac{1}{|E(r)|\sqrt{\tau'(r)}}\frac{E(s)E^{\#}(r)-E^{\#}(s)E(r)}{2\pi i(s-r)}\frac{1}{|E(s)|\sqrt{\tau'(s)}}.\end{aligned}$$

We observe that

$$\begin{aligned}
&(E(s)E^{\#}(r) - E^{\#}(s)E(r))^2 \\
=&E(s)E(r)E^{\#}(s)E^{\#}(r)\left(\frac{E(s)E^{\#}(r)}{E^{\#}(s)E(r)} + \frac{E^{\#}(s)E(r)}{E(s)E^{\#}(r)} - 2\right) \\
=&|E(s)|^2|E(r)|^2\left(\frac{\Theta(r)}{\Theta(s)} + \frac{\Theta(s)}{\Theta(r)} - 2\right) \\
=&|E(s)|^2|E(r)|^2\left(e^{-2\pi i(\tau(s)-\tau(r))} + e^{2\pi i(\tau(s)-\tau(r))} - 2\right) \\
=&|E(s)|^2|E(r)|^2\left(2\cos(2\pi(\tau(s)-\tau(r))) - 2\right) \\
=&-4|E(s)|^2|E(r)|^2\left(\frac{1-\cos(2\pi(\tau(s)-\tau(r)))}{2}\right) \\
=&-4|E(s)|^2|E(r)|^2\sin(\pi(\tau(s)-\tau(r)))^2.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
K_{\mathcal{KM}(E)}(r,s) &= \frac{1}{|E(r)|\sqrt{\tau'(r)}}\frac{E(s)E^{\#}(r) - E^{\#}(s)E(r)}{2\pi i(s-r)}\frac{1}{|E(s)|\sqrt{\tau'(s)}} \\
&= (-1)^{\lfloor \tau(s)-\tau(r)\rfloor}\sqrt{t'(\tau(r))}\frac{\sin(\pi(\tau(s)-\tau(r)))}{\pi(s-r)}\sqrt{t'(\tau(s))}.
\end{aligned}$$

Now, set $n = \lfloor \tau(r)\rfloor, \theta = \{\tau(r)\}, m = \lfloor \tau(s)\rfloor, \beta = \{\tau(s)\}$ so that $r = t(n+\theta)$, $s = t(m+\beta)$. Then we obtain

$$\begin{aligned}
&K_{\mathcal{KM}(E)}(t(n+\theta), t(m+\beta)) \\
=&\frac{(-1)^{m-n}\sin(\pi(m+\beta-n-\theta))}{t(m+\beta)-t(n+\theta)}\frac{\sqrt{t'(n+\theta)t'(m+\beta)}}{\pi} \\
=&\frac{(-1)^{n+m}\sin(\pi(\beta-\theta))}{t_m(\beta)-t_n(\theta)}\frac{\sqrt{t'(n+\theta)t'(m+\beta)}}{\pi}.
\end{aligned}$$

Now fix any $\theta \in [0,1)$ and consider the sequence $\{K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\}_n$. Recalling (2.1) we get

$$\begin{aligned}
&\left\langle K_{\mathcal{KM}(E)}(t(n+\theta),\cdot), K_{\mathcal{KM}(E)}(t(m+\beta),\cdot)\right\rangle_{\mathcal{KM}(E)} \\
&= K_{\mathcal{KM}(E)}(t(n+\theta), t(m+\beta)) \\
&= \frac{(-1)^{n+m}\sin(\pi(\beta-\theta))}{t_m(\beta)-t_n(\theta)}\frac{\sqrt{t'(n+\theta)t'(m+\beta)}}{\pi},
\end{aligned} \tag{6.22}$$

and then (6.21) is proved. It remains to show that $\{K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\}_n$ is an othonormal basis. We observet that

$$\begin{aligned}
&\left\langle K_{\mathcal{KM}(E)}(t(n+\theta),\cdot), K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\right\rangle_{\mathcal{KM}(E)} \\
&= \lim_{\beta\to\theta} \frac{\sin(\pi(\beta-\theta))}{t_n(\beta)-t_n(\theta)} \frac{t'(n+\theta)}{\pi} \\
&= \lim_{\beta\to\theta} \frac{\sin(\pi(\beta-\theta))}{t(n+\beta)-t(n+\theta)} \frac{t'(n+\theta)}{\pi} \\
&= \lim_{\beta\to\theta} \frac{\pi}{t'(n+\theta)} \frac{t'(n+\theta)}{\pi} \qquad \text{(by l'Hopital)} \\
&= 1.
\end{aligned} \tag{6.23}$$

The result of (6.23) could be derived as an immediate consequence of (6.22), but for sake of completeness we prefer to prove it explicitly also in this case. Moreover, for $n, m \in \mathbb{Z}$, $n \neq m$, we get

$$\begin{aligned}
&\left\langle K_{\mathcal{KM}(E)}(t(n+\theta),\cdot), K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\right\rangle_{\mathcal{KM}(E)} \\
&= \lim_{\beta\to\theta} \frac{\sin(\pi(\beta-\theta))}{t_n(\beta)-t_m(\theta)} \frac{t'(n+\theta)}{\pi} \\
&= 0.
\end{aligned} \tag{6.24}$$

Hence, by (6.23) and (6.24) we have obtained that the elements of the sequence $\{K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\}_n$ are pairwise orthonormal. Then $\{K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\}_n$ is an orthonormal sequence in $\mathcal{KM}(E)$. Thanks to *Theorem* 6.5, for every $F \in \mathcal{KM}(T)$ we have

$$\begin{aligned}
F(t) &= \sum_n K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right) F\left(t_n(\theta)\right) \\
&= \sum_n K_{\mathcal{KM}(E)}(t(n+\theta), t) F\left(t_n(\theta)\right).
\end{aligned}$$

Then, for every $\theta \in [0,1)$ the only function that is perpendicular to all the elements of the sequence $\{K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\}_n$ is the null vector, hence $\{K_{\mathcal{KM}(E)}(t(n+\theta),\cdot)\}_n$ is a complete orthonormal sequence in $\mathcal{KM}(E)$, and therefore an orthonormal basis. □

Theorem 6.7. *Let $E(z)$ be a Hermite Biehler function and let $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$. For $s, t \in \mathbb{R}$, the reproducing kernel of the space $\mathcal{KM}(E)$ can be expressed as*

$$K_{\mathcal{KM}(E)}(t,s) = f(t)(-1)^{\lfloor \tau(t) \rfloor} \left(\sum_n \frac{t'(n)}{(t-t_n)(s-t_n)} \right) (-1)^{\lfloor \tau(s) \rfloor} f(s),$$

where

$$f(t) := \left(\sum_n \frac{t'(n)}{(t-t_n)^2} \right)^{-\frac{1}{2}},$$

and $\{t_n\}_n$ is the set of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$.

Proof. For semplicity we set

$$\phi_n(\theta) := K_{\mathcal{KM}(E)}(t(n+\theta), \cdot).$$

Fix any $\alpha \in [0,1)$ so that $\theta \neq \alpha$. Then, expanding $\phi_n(\theta)$ in the orthonormal basis $\{\phi_n(\alpha)\}_n$ we get

$$\begin{aligned} 1 &= \langle \phi_n(\theta), \phi_n(\theta) \rangle_{\mathcal{KM}(E)} \\ &= \sum_k \frac{\sin^2(\pi(\alpha-\theta))}{(t_k(\alpha) - t_n(\theta))^2} \frac{t'(n+\theta) t'(k+\alpha)}{\pi^2}. \end{aligned}$$

Solving for $t'(n+\theta)$ we obtain

$$t'(n+\theta) = \frac{\pi^2}{\sin^2(\pi(\alpha-\theta))} f_\alpha\left(t_n(\theta)\right)^2, \tag{6.25}$$

where

$$f_\alpha(t) := \left(\sum_k \frac{t'(k+\alpha)}{(t - t_k(\alpha))^2} \right)^{-\frac{1}{2}}.$$

Expanding $\langle \phi_n(\theta), \phi_m(\beta) \rangle_{\mathcal{KM}(E)}$ in the orthonormal basis $\{\phi_k(\alpha)\}_{k\in\mathbb{Z}}$ and using (6.21) and (6.25) we have

$$\begin{aligned} \langle \phi_n(\theta), \phi_m(\beta) \rangle_{\mathcal{KM}(E)} &= \sum_k \langle \phi_n(\theta), \phi_k(\alpha) \rangle_{\mathcal{KM}(E)} \langle \phi_k(\alpha), \phi_m(\beta) \rangle_{\mathcal{KM}(E)} \\ &= \sum_k \frac{(-1)^{n+m}}{\pi^2} \frac{t'(k+\alpha)}{(t_n(\theta) - t_k(\alpha))(t_m(\beta) - t_k(\alpha))} \\ &\quad \sin(\pi(\alpha-\theta)) \sin(\pi(\alpha-\beta)) \sqrt{t'(n+\theta) t'(m+\beta)} \\ &= \sum_k (-1)^{n+m} \frac{t'(k+\alpha)}{(t(n+\theta) - t(k+\alpha))(t(m+\beta) - t(k+\alpha))} \\ &\quad f_\alpha(t(n+\theta)) f_\alpha(t(m+\beta)). \end{aligned}$$

Finally we get

$$\begin{aligned} K_{\mathcal{KM}(E)}(t,s) &= \langle K_{\mathcal{KM}(E)}(t,\cdot), K_{\mathcal{KM}(E)}(s,\cdot) \rangle_{\mathcal{KM}(E)} \\ &= \left\langle \phi_{\lfloor \tau(t) \rfloor}(\tau(t) - \lfloor \tau(t) \rfloor), \phi_{\lfloor \tau(s) \rfloor}(\tau(s) - \lfloor \tau(s) \rfloor) \right\rangle_{\mathcal{KM}(E)} \\ &= (-1)^{\lfloor \tau(t) \rfloor} f_\alpha(t) \left(\sum_k \frac{t'(k+\alpha)}{(t - t_k(\alpha))(s - t_k(\alpha))} \right) f_\alpha(s) (-1)^{\lfloor \tau(s) \rfloor} \end{aligned}$$

for any $\alpha \in [0,1)$, and setting for $\alpha = 0$ the proof is over. $\square$

The goal of the first part of *Section* 4 in [40] (up to *Section* 4.6, pp. 1624-1628) is to derive a representation for the Livsic characteristic function in terms of the sequence $\{t_n\}_n$, obtained in equation (25) (p. 1628):

$$\Theta(z) = \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n - i} \right) t'_n}{\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n + i} \right) t'_n}. \tag{6.26}$$

In the proof of this result, an important role is played by the equation

$$\sum_n \frac{t'_n}{t_n^2+1} = \pi, \tag{6.27}$$

which is the key property of the normalized time-varying bandlimit pairs $(\{t_n\}_n, \{t'_n\}_n)$ and then a fundamental pillar of all the Kempf-Martin theory. For the alternative definition of the Kempf-Martin spaces we derive a much more general result in the following theorem, for which (6.26) and (6.27) are the particular case given by $z_0 = i$.

Theorem 6.8. *Let $\Theta(z)$ be a meromorphic inner function, and let $t(x)$ be its spectral function. For all $n \in \mathbb{Z}$ set $t_n = t(n)$ and $t'_n = t'(n)$. Then, for any $n \in \mathbb{Z}$, $\Theta(z)$ can be expressed as*

$$\Theta(z) = \frac{\frac{t'_n}{t_n - z} + \sum_{m\neq n}\left(\frac{t'_m}{t_m - z} - \frac{t'_m}{t_m - t_n}\right) - i\pi}{\frac{t'_n}{t_n - z} + \sum_{m\neq n}\left(\frac{t'_m}{t_m - z} - \frac{t'_m}{t_m - t_n}\right) + i\pi}. \tag{6.28}$$

Moreover, for every zero z_0 of $\Theta(z)$, $\Theta(z)$ can be written as

$$\Theta(z) = \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n - z_0}\right) t'_n}{\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n - \overline{z_0}}\right) t'_n}, \tag{6.29}$$

and the following equality holds:

$$\sum_n \frac{\Im(z_0) t'_n}{(t_n - \Re(z_0))^2 + \Im(z_0)^2} = \pi. \tag{6.30}$$

Proof. Let $E(z)$ be a de Branges function of $\Theta(z)$, and let $\mathcal{KM}(E)$ be its associated Kempf-Martin space. Then, by (6.25), for every $\theta \in [0,1)$ and $\alpha \in [0,1)$ such that $\alpha \neq \theta$, we have

$$t'(n+\theta) = \frac{\pi^2}{\sin^2(\pi(\alpha - \theta))}\left(\sum_m \frac{t'(m+\alpha)}{(t_n(\theta) - t_m(\alpha))^2}\right)^{-1}.$$

Setting $\alpha = 0$ we get

$$\left(\sum_m \frac{t'_m}{(t(n+\theta) - t_m)^2}\right) t'(n+\theta) = \frac{\pi^2}{\sin^2(\pi\theta)}.$$

Given $\beta \in (0,1)$ and ϵ such that $0 < \epsilon < \beta$, we have

$$\int_\epsilon^\beta \left(\sum_m \frac{t'_m}{(t(n+\theta) - t_m)^2}\right) t'(n+\theta)$$
$$= -\frac{t'_n}{t_n(\beta) - t_n} + \frac{t'_m}{t_n(\epsilon) - t_n} + \sum_{m\neq n}\left(-\frac{t'_m}{t_n(\beta) - t_m} + \frac{t'_m}{t_n(\epsilon) - t_m}\right),$$

and

$$\int_{\epsilon}^{\beta} \frac{\pi^2}{\sin^2(\pi\theta)} = -\,\pi\cot(\pi\beta) + \pi\cot(\pi\epsilon).$$

Therefore we get

$$\begin{aligned} &-\frac{t'_n}{t_n(\beta)-t_n} + \frac{t'_m}{t_n(\epsilon)-t_n} + \sum_{m\neq n}\left(-\frac{t'_m}{t_n(\beta)-t_m} + \frac{t'_m}{t_n(\epsilon)-t_m}\right) \\ &= -\pi\cot(\pi\beta) + \pi\cot(\pi\epsilon). \end{aligned} \tag{6.31}$$

As ϵ goes to 0^+, $t_n(\epsilon) - t_n \to \epsilon t'_n$ asymptotically. Hence the term $\frac{t'_n}{t_n(\epsilon)-t_n}$ on the left hand side asymptotically goes to $\frac{1}{\epsilon}$ as $\epsilon \to 0^+$. On the right hand side, $\pi\cot(\pi\epsilon) = \pi\frac{\cos(\pi\epsilon)}{\sin(\pi\epsilon)}$ asymptotically goes to $\frac{1}{\epsilon}$ as $\epsilon \to 0^+$. Then, as $\epsilon \to 0^+$ we can cancel the simple poles $\frac{1}{\epsilon}$ on both sides, and we obtain

$$\frac{t'_n}{t_n(\beta)-t_n} + \lim_{\epsilon\to 0^+}\sum_{m\neq n}\left(\frac{t'_m}{t_n(\beta)-t_m} - \frac{t'_m}{t_n(\epsilon)-t_m}\right) = \pi\cot(\pi\beta).$$

We observe that

$$\frac{t'_m}{t_n(\beta)-t_m} - \frac{t'_m}{t_n(\epsilon)-t_m} = \frac{t_n(\beta)-t_n(\epsilon)}{(t_n(\beta)-t_m)(t_n(\epsilon)-t_m)},$$

and that

$$(t_n(\beta)-t_m)(t_n(\epsilon)-t_m) > 0 \qquad \forall m \neq n,$$

since $t_n < t_n(\beta) < t_{n+1}$. Hence

$$\frac{t'_m}{t_n(\beta)-t_m} - \frac{t'_m}{t_n(\epsilon)-t_m} = \frac{t_n(\beta)-t_n(\epsilon)}{(t_n(\beta)-t_m)(t_n(\epsilon)-t_m)} > 0 \qquad \forall m \neq n,$$

and by the monotonic convergence theorem we get

$$\lim_{\epsilon\to 0^+}\sum_{m\neq n}\left(\frac{t'_m}{t_n(\beta)-t_m} - \frac{t'_m}{t_n(\epsilon)-t_m}\right) = \sum_{m\neq n}\left(\frac{t'_m}{t_n(\beta)-t_m} - \frac{t'_m}{t_n-t_m}\right).$$

Therefore we obtain

$$\frac{t'_n}{t_n(\beta)-t_n} + \sum_{m\neq n}\left(\frac{t'_m}{t_n(\beta)-t_m} - \frac{t'_m}{t_n-t_m}\right) = \pi\cot(\pi\beta). \tag{6.32}$$

Setting

$$f_{m,n} = \begin{cases} 0 & \text{if } m = n, \\ \frac{t'_m}{t_n-t_m} & \text{if } m \neq n, \end{cases}$$

we get

$$\sum_{m}\left(\frac{t'_m}{t_n(\beta)-t_m} - f_{m,n}\right) = \pi\cot(\pi\beta). \tag{6.33}$$

Moreover, since $t_n(\beta)$ covers the real line exactly once as n varies in $\mathbb{Z}$ and β varies $[0,1)$, we can set $x = t_n(\beta)$ and for all $x \in \mathbb{R}$ we can write

$$\sum_m \left(\frac{t'_m}{x - t_m} - f_{m,n} \right) = \pi \cot(\pi\tau(x)), \tag{6.34}$$

where we used the fact that $\cot(\pi\tau(x)) = \cot(\pi(n+\beta)) = \cot(\pi\beta)$. Now we define

$$f(x) = \sum_m \left(\frac{t'_m}{x - t_m} - f_{m,n} \right),$$

so that

$$f(x) = \pi \cot(\pi\tau(x)).$$

Hence, recalling that $-\cot(x) = \cot(-x)$, we have

$$\operatorname{arccot}\left(-\frac{f(x)}{\pi} \right) = -\pi\tau(x),$$

and using the well-known identity

$$\operatorname{arccot}(x) = \frac{i}{2} \log\left(\frac{x-i}{x+i} \right)$$

we obtain

$$2\pi i\tau(x) = \log\left(\frac{-f(x) - i\pi}{-f(x) + i\pi} \right),$$

and therefore

$$\begin{aligned} \Theta(x) &= e^{2\pi i\tau(x)} \\ &= \frac{-f(x) - i\pi}{-f(x) + i\pi} \\ &= \frac{\sum_m \left(\frac{t'_m}{t_m - x} + f_{m,n} \right) - i\pi}{\sum_m \left(\frac{t'_m}{t_m - x} + f_{m,n} \right) + i\pi}. \end{aligned} \tag{6.35}$$

It is easy to see that by extending the last expression to all $\mathbb{C}$ we obtain a meromorphic function, and hence (6.28) is proved.

Now, let z_0 be any zero of $\Theta(z)$. In order to obtain $\Theta(z_0) = 0$, by (6.35) we need

$$\sum_m \left(\frac{t'_m}{t_m - z_0} + f_{m,n} \right) - i\pi = 0. \tag{6.36}$$

Since $\Theta(\overline{z}) = \frac{\overline{E(z)}}{E(\overline{z})} = \frac{\overline{E(z)}}{E^{\#}(z)} = \frac{1}{\overline{\Theta(z)}}$, we get the $\Theta(z)$ has a pole for $z = \overline{z_0}$, and therefore by (6.35) we need also

$$\sum_m \left(\frac{t'_m}{t_m - \overline{z_0}} + f_{m,n} \right) + i\pi = 0. \tag{6.37}$$

Hence we finally get

$$\begin{aligned}\Theta(x) &= \frac{\sum_m \left(\frac{t'_m}{t_m - x} + f_{m,n}\right) - i\pi - \left(\sum_m \left(\frac{t'_m}{t_m - z_0} + f_{m,n}\right) - i\pi\right)}{\sum_m \left(\frac{t'_m}{t_m - x} + f_{m,n}\right) + i\pi - \left(\sum_m \left(\frac{t'_m}{t_m - \overline{z_0}} + f_{m,n}\right) + i\pi\right)} \\ &= \frac{\sum_m \left(\frac{1}{t_m - x} - \frac{1}{t_m - z_0}\right) t'_m}{\sum_m \left(\frac{1}{t_m - x} - \frac{1}{t_m - \overline{z_0}}\right) t'_m}.\end{aligned}$$

Moreover, by (6.36) and (6.37) we get

$$\begin{aligned}2i\pi &= \sum_m \left(\frac{t'_m}{t_m - z_0} + f_{m,n}\right) - \sum_m \left(\frac{t'_m}{t_m - \overline{z_0}} + f_{m,n}\right) \\ &= \sum_m \left(\frac{t'_m}{t_m - z_0} - \frac{t'_m}{t_m - \overline{z_0}}\right) \\ &= 2i\Im(z_0) \sum_m \frac{t'_m}{(t_m + \Re(z_0))^2 + \Im(z_0)^2},\end{aligned}$$

and then we obtain

$$\sum_m \frac{\Im(z_0) t'_m}{(t_m - \Re(z_0))^2 + \Im(z_0)^2} = \pi.$$

□

From this theoren we can also derive the following result.

Theorem 6.9. *Let $F(z)$ be any Herglotz function such that*

$$\Theta(z) = \frac{F(z) - i}{F(z) + i} \tag{6.38}$$

is a meromorphic inner function. Then $F(z)$ can be uniquely represented as

$$F(z) = \sum_n \left(\frac{1}{t_n - z} - \frac{t_n - \Re(z_0)}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}\right) w_n, \quad z \in \mathbb{C}^+, \tag{6.39}$$

where

- $\{t_n\}_n$ *is the sequence of poles of $F(z)$ on the real line;*
- $w_n = \lim_{z \to t_n} \frac{(F(z)+i)^2}{F'(z)}$*;*
- z_0 *is any point for which $F(z_0) = i$.*

Moreover the following equality holds:

$$\sum_n \frac{\Im(z_0) w_n}{(t_n - \Re(z_0))^2 + \Im(z_0)^2} = 1. \tag{6.40}$$

Proof. We already know that there exists an infinite number of Herglotz functions that verify (6.38). Indeed, according to *Section* 2.3, there exists a bijection between the Herglotz functions corresponding to a purely discrete measure and the meromorphic inner functions. In particular, given Herglotz function $F(z)$ corresponding to a purely discrete measure, the meromorphic inner function $\Theta(z)$ associated to $F(z)$ is given by (6.38):

$$\Theta(z) = \frac{F(z) - i}{F(z) + i}.$$

If z_0 is any zero of $\Theta(z)$, by *Theorem* 6.8 we have

$$\begin{aligned}
\Theta(z) &= \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n - z_0}\right) t_n'}{\sum_n \left(\frac{1}{t_n - z} - \frac{1}{t_n - \overline{z_0}}\right) t_n'} \\
&= \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n - \Re(z_0)}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}\right) t_n' - \sum_n \frac{i\Im(z_0) t_n'}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}}{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n - \Re(z_0)}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}\right) t_n' + \sum_n \frac{i\Im(z_0) t_n'}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}} \\
&= \frac{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n - \Re(z_0)}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}\right) t_n' - i\pi}{\sum_n \left(\frac{1}{t_n - z} - \frac{t_n - \Re(z_0)}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}\right) t_n' + i\pi},
\end{aligned}$$

where in the last step we used (6.30). Hence by (2.7) we get

$$\begin{aligned}
F(z) &= i\left(\frac{1 + \Theta(z)}{1 - \Theta(z)}\right) \\
&= \sum_n \left(\frac{1}{t_n - z} - \frac{t_n - \Re(z_0)}{(t_n - \Re(z_0))^2 + \Im(z_0)^2}\right) \frac{t_n'}{\pi}.
\end{aligned}$$

Since $\{t_n\}_n$ is the sequence of solutions of $\Theta(t) = 1$ for $t \in \mathbb{R}$ we get it is also the sequence of poles of $F(z)$ on the real line. We observe that

$$\Theta'(z) = \frac{2iF'(z)}{(F(z) + i)^2},$$

and hence, recalling *Theorem* 4.6, we have

$$t_n' = \frac{2i\pi}{\Theta'(t_n)}.$$

Therefore we obtain

$$w_n := \frac{t_n'}{\pi} = \frac{2i}{\Theta'(t_n)} = \lim_{z \to t_n} \frac{(F(z) + i)^2}{F'(z)}.$$

Since z_0 is any point for which $\Theta(z_0) = 0$, by (6.38) we easily get that it is any point for which $F(z_0) = i$. Realling (6.30), we finally obtain

$$\sum_n \frac{\Im(z_0) w_n}{(t_n - \Re(z_0))^2 + \Im(z_0)^2} = \sum_n \frac{\Im(z_0) \frac{t_n'}{\pi}}{(t_n - \Re(z_0))^2 + \Im(z_0)^2} = 1.$$

$\square$

It's interesting to notice that, given a Herglotz function $F(z)$ corresponding to a purely discrete measure and such that $F(i) = i$ and $\lim_{y\to\infty} \frac{F(iy)}{iy} = 0$, then (2.9) is a particular case of *Theorem* 6.9. Indeed, by (2.9) and (2.5) we get

$$F(z) = \sum_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) w_n, \quad z \in \mathbb{C}^+,$$

which is exactly the same equation representation of $F(z)$ that we obtain setting $z_0 = i$ in (6.39). Obviously we have

$$\begin{aligned} i &= F(i) \\ &= \sum_n \left(\frac{1}{t_n - i} - \frac{t_n}{t_n^2 + 1} \right) w_n \\ &= i \sum_n \left(\frac{1}{t_n^2 + 1} \right) w_n, \end{aligned}$$

and then also (6.40) is verified. Therefore we obtain that in this case the positive weights $\{w_n\}_n$ of the purely discrete measure (2.8) are given by $w_n = \lim_{z\to t_n} \frac{(F(z)+i)^2}{F'(z)}$.

Finally it is important to observe that also the generalized sampling theory described in *Chapters* 5,6 in [19] can be totally derived without the use of the theory of simple symmetric operators. We don't go more in details since it essentially uses all the results already proved for the equivalent definition of the Kempf-Martin space. Indeed, in particular it is based on the sequences t_n that, given any $\alpha \in [0, 1)$, verify the equation

$$\frac{t_n'}{t - t_n} - \sum_{m \neq n} \frac{t_m' \, (t - t_n)}{(t - t_m)\,(t_n - t_m)} = \pi \cot(\pi\alpha),$$

which we already obtained in (6.32).

CHAPTER 7

Time-varying bandlimit functions and applications

7.1 Definition of time-varying bandlimit

The bandlimit of a bandlimited function is strictly connected with the density of the sampling points, indeed the well-known Nyquist theorem states that an a-bandilimted function can be rebuilt extactly from samples taken with a uniform sampling such that $t_{n+1} - t_n \leq \frac{\pi}{2a}$. The classical notion of bandlimit for any Paley-Wiener space $\mathcal{PW}_a$ can be interpreted as a measure of the density of any of the Nyquist sampling sequence. Since $\mathcal{PW}_a = e^{-iaz}\mathcal{K}\left(e^{2iaz}\right)$, the phase function $\tau(t)$ of $\Theta(t) = e^{2iat}$ is simply $\tau(t) = \frac{a}{\pi}t$. It follows that the bandlimit a is given by

$$a = \pi\tau'.$$

Working in analogy with the classical Paley-Wiener spaces $\mathcal{PW}_a$, we can construct a precise and meaningful definition of time-varying bandlimit for any de Branges space and Kempf-Martin space. Given a Hermite Biehler function $E(z)$, by *Theorem* 6.5 we know that, for every $\theta \in [0, 1)$, all the functions of the de Branges space $\mathcal{B}(E)$ can be rebuilt exactly with samples taken on the sampling sequence $\{t_n(\theta)\}_n$ of solutions of $\Theta(t) = e^{i2\pi\theta}$, $t \in \mathbb{R}$. Similarly, thanks to (6.3), we know that all the functions of Kempf-Martin space $\mathcal{KM}(T)$ can be rebuilt exactly with samples taken on the sampling sequence $\{t_n(\theta)\}_n$ of solutions of $\Theta(t) = e^{i2\pi\theta}$, $t \in \mathbb{R}$, where $\Theta(z)$ is the

Livsic characteristic function of T. In both cases, for every $\theta \in [0,1)$, the sampling sequence $\{t_n(\theta)\}$ obviously satisfies

$$\begin{aligned} \tau(t_n(\theta)) &= n + \theta, \\ t_n(\theta) &= t(n+\theta), \end{aligned} \tag{7.1}$$

where $t(x)$ is the spectral function of $\Theta(z)$. Since $n \in \mathbb{Z}$ and $\theta \in [0,1)$, we have that, as n and θ vary in their domains, $n+\theta$ takes once and only once every real value. In particular, given $x \in \mathbb{R}$, we have $x = n+\theta$ with $n = \lfloor x \rfloor$ and $\theta = \{x\}$, and hence

$$t(x) = t_{\lfloor x \rfloor}(\{x\}).$$

Moreover, since $t(x)$ is a strictly increasing function, it is easy to see that every real value $t \in \mathbb{R}$ belongs to one and only one sampling sequence $\{t_n(\theta)\}_n$, and precisely it is the element with index $n = \lfloor \tau(t) \rfloor$ of the sequence corresponding to $\theta = \{\tau(t)\}$. Thanks to all these observations, the spectral function $t(x)$ can be interpreted as the function that describes how the value of $t_n(\theta)$ varies as $\theta \in [0,1)$ and $n \in \mathbb{Z}$ vary in their domain.

Given a meromorphic inner function $\Theta(z)$, the value of $\tau'(t) > 0$ determines how quickly the phase of $\Theta(t)$ is rotating on the real line, and hence measures the local density of points $\{t_n(\theta)\}_n$, as n and θ vary in their domain. Therefore, it is natural to extend the notion of bandlimit to the time-varying setting by defining the time-varying bandlimit $a(t)$ of de Branges spaces and Kempf-Martin spaces as

$$a(t) = \pi\tau'(t),$$

so that this definition is totally coherent with the classical notion of bandlimit for the Paley-Wiener spaces. This formal definition of the time-varying bandlimit is the same given in [40].

However, this formal definition doesn't give an easy interpretation of the concept of time-varying bandlimit. Indeed, it difficult to give a precise interpretation of the concept of time-varying bandlimit, since the traditional notion of bandlimit is determined by the Fourier transform of the entire signal and hence it is time-independent and non-local.

In [19] the following interpretation of time-varying bandlimit is given. The Nyquist sampling rate of a bandlimited signal is the critical sampling rate below which there is insufficient information to recover the signal and above which redundance exists. It is defined as the inverse of twice the bandwidth of the signal. In principle, if the information density vary in time, also the Nyquist rate can vary in time. Hence the time-varying bandlimit of a signal can be interpreted as half of the inverse of the time-varying Nyquist rate of the signal.

In the next section we introduce a new family of time-varying-bandlimit spaces which are consistent with this interpretation, but which also allow a direct interpretation of the concept of time-varying bandlimit in the time domain.

7.2 Time-varying bandlimit spaces $\mathcal{V}(\Theta)$

In this section we define the spaces spaces $\mathcal{V}(\Theta)$, which are a new family of reproducing kernel Hilbert spaces of time-varying bandlimit functions, and we derive an useful expression for their reproducing kernel.

Let $\Theta(z) = \gamma e^{2ibz} B(z)$ be a meromorphic inner function according to the representation given in (2.3), with logarithmic residue $2b > 0$ and phase function $\tau(t)$. Consider the space $\mathcal{V}(\Theta)$ given by

$$\mathcal{V}(\Theta) := \left\{ F : F(t) = \frac{e^{-ibt}}{\sqrt{\tau'(t)}} G(t), \quad \text{for } G \in \mathcal{K}(\Theta),\ t \in \mathbb{R} \right\}.$$

The scalar product and the norm of $\mathcal{V}(\Theta)$ are given by

$$\begin{aligned} \langle F, G \rangle_{\mathcal{V}(\Theta)} &= \int_{-\infty}^{+\infty} F(t)\overline{G}(t)\tau'(t)dt, \\ \|F\|^2_{\mathcal{V}(\Theta)} &= \int_{-\infty}^{+\infty} |F(t)|^2 \tau'(t)dt. \end{aligned}$$

Recalling that the functions of $\mathcal{K}(\Theta)$ are holomorphic and without poles on the real line, and that $\tau(t)$ is analytic on the real line and such that $\tau'(t) > 0$ $\forall t \in \mathbb{R}$, we can conclude that all the functions of the space $\mathcal{V}(\Theta)$ are analyitc on $\mathbb{R}$.

Let $E(z)$ be a de Branges function of $\Theta(z)$. Recalling *Theorem* 2.19, it is easy to see that

$$\mathcal{V}(\Theta) = \frac{e^{-ibt}}{E(t)\sqrt{\tau'(t)}} \mathcal{B}(E)|_{\mathbb{R}}, \quad t \in \mathbb{R}, \tag{7.2}$$

where the functions of $\mathcal{B}(E)$ are considered to be restricted on the real line. Given $F_1(t), F_2(t) \in \mathcal{B}(E)$, and $G_1(t) = \frac{e^{-ibt}}{E(t)\sqrt{\tau'(t)}} F_1(t)$, $G_2(t) = \frac{e^{-ibt}}{E(t)\sqrt{\tau'(t)}} F_2(t) \in \mathcal{V}(\Theta)$ we observe that

$$\begin{aligned} \langle G_1, G_2 \rangle_{\mathcal{V}(\Theta)} &= \int_{-\infty}^{+\infty} G_1(t)\overline{G_2(t)}\tau'(t)dt, \\ &= \int_{-\infty}^{+\infty} F_1(t)\overline{F_2(t)} \frac{1}{|E(t)|^2} \frac{1}{\tau'(t)} \tau'(t)dt, \\ &= \int_{-\infty}^{+\infty} F_1(t)\overline{F_2(t)} \frac{1}{|E(t)|^2} dt, \\ &= \langle F_1, F_2 \rangle_{\mathcal{B}(E)}. \end{aligned} \tag{7.3}$$

Hence the multiplication by $\frac{1}{E(t)\sqrt{\tau'(t)}}$ induces a unitary isomorphism between $\mathcal{B}(E)|_{\mathbb{R}}$ and $\mathcal{V}(\Theta)$.

Thanks to (6.14) and (7.2) we also have that

$$\begin{aligned}\mathcal{V}(\Theta) &= ie^{-ibt}\frac{|E(t)|}{E(t)}\mathcal{KM}(E) \\ &= ie^{-ibt}\sqrt{\Theta(t)}\mathcal{KM}(E),\end{aligned} \tag{7.4}$$

where $E(z)$ is a de Branges function of $\Theta(z)$ and $\mathcal{KM}(E)$ is the correspondent Kempf-Martin space. Given $F_1, F_2 \in \mathcal{KM}(E)$, and $G_1(t) = i\sqrt{\gamma B(t)}F_1(t), G_2 = i\sqrt{\gamma B(t)}F_2(t) \in \mathcal{V}(\Theta)$, by (6.10) and (7.3) we obviously have

$$\langle G_1, G_2\rangle_{\mathcal{V}(\Theta)} = \langle F_1, F_2\rangle_{\mathcal{KM}(E)}.$$

Theorem 7.1. *Let $\Theta(z)$ be a meromorphic inner function given by*

$$\Theta(z) = \frac{\sum_n t_n'\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \overline{\alpha}}{\sum_n t_n'\left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + \alpha},$$

according to the representation given in (4.7)*, with phase function $\tau(t)$. Then $\mathcal{V}(\Theta)$ is a RKHS with kernel given by*

$$K_{\mathcal{V}(\Theta)}(t,s) = \overline{f(t)}\frac{g(t)-g(s)}{t-s}f(s), \quad t, s \in \mathbb{R},$$

where

$$\begin{aligned}f(t) &= (-1)^{\lfloor \tau(t)\rfloor}e^{-ibt}\left(\frac{g(t)+\overline{\alpha}}{g'(t)(g(t)+\alpha)}\right)^{\frac{1}{2}}, \\ g(t) &= \sum_n t_n'\left(\frac{1}{t_n - t} - \frac{t_n}{t_n^2+1}\right).\end{aligned}$$

In particular we have

$$K_{\mathcal{V}(\Theta)}(t,t) = 1 \quad \forall t \in \mathbb{R}.$$

Proof. Let $E(z)$ be a de Branges function of $\Theta(z)$, and let $\mathcal{KM}(E)$ be the correspondent Kempf-Martin space. We observe that

$$\begin{aligned}\Theta(t) &= \frac{g(t)+\overline{\alpha}}{g(t)+\alpha}, \\ g'(t) &= \sum_n \frac{t_n'}{(t_n - t)^2}, \\ g(t) - g(s) &= \sum_n \frac{t_n'(t-s)}{(t-t_n)(s-t_n)},\end{aligned} \tag{7.5}$$

Then by *Theorem* 6.7 and (7.4) we get

$$\begin{aligned}K_{\mathcal{V}(\Theta)}(t,s) &= e^{ibt}\overline{\Theta(t)}^{\frac{1}{2}}K_{\mathcal{KM}(E)}(t,s)e^{-ibs}\Theta(s)^{\frac{1}{2}} \\ &= \overline{f(t)}\frac{g(t)-g(s)}{t-s}f(s), \quad t, s \in \mathbb{R},\end{aligned} \tag{7.6}$$

where

$$\begin{aligned} f(t) &= (-1)^{\lfloor \tau(t) \rfloor} e^{-ibt} \Theta(t)^{\frac{1}{2}} \left(\sum_n \frac{t_n'}{(t_n - t)^2} \right)^{-\frac{1}{2}} \\ &= (-1)^{\lfloor \tau(t) \rfloor} e^{-ibt} \left(\frac{g(t) + \overline{\alpha}}{g'(t)(g(t)+\alpha)} \right)^{\frac{1}{2}}. \end{aligned}$$

Moreover we have

$$\begin{aligned} f(t)\overline{f(t)} &= |\Theta(t)|^{\frac{1}{2}} \left(\sum_n \frac{t_n'}{(t_n - t)^2} \right)^{-1} \\ &= \left(\sum_n \frac{t_n'}{(t_n - t)^2} \right)^{-1} \end{aligned} \tag{7.7}$$

and then

$$\begin{aligned} K_{\mathcal{V}(\Theta)}(t,t) &= \left(\frac{t_n'}{(t_n - t)^2} \right)^{-1} \lim_{t \to s} \frac{g(t) - g(s)}{t - s} \\ &= \frac{1}{g'(t)} \lim_{s \to t} \frac{g(t) - g(s)}{t - s} \\ &= 1. \end{aligned} \tag{7.8}$$

□

7.3 Characterization and motivation of the spaces $\mathcal{V}(\Theta)$

In this section we give some important characterizations of the spaces $\mathcal{V}(\Theta)$: we derive a family of sampling formulas, we introduce a family of orthonormal basis, and we show that a weighted version of the Fourier transform induces a unitary isomorphism between these spaces and a class of subspaces of $\mathcal{L}^2(\mathbb{R})$. Moreover we explain why these spaces are suitable for sampling and recostrunction of time-varying bandlimit functions.

Theorem 7.2. *Let $\Theta(z)$ be a meromorphic inner function given by*

$$\Theta(z) = \frac{\sum_n t_n' \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \overline{\alpha}}{\sum_n t_n' \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2 + 1} \right) + \alpha},$$

according to the representation given in (4.7), *with phase function $\tau(t)$ and spectral function $t(x)$. For $\theta \in [0,1)$ and $n \in \mathbb{Z}$ set $t_n(\theta) = t(n+\theta)$. Then for every $\theta \in [0,1)$ and every function $G \in \mathcal{V}(\Theta)$ the following sampling formula holds:*

$$\begin{aligned} G(t) &= \sum_n K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right) G(t_n(\theta)) \\ &= \sum_n \overline{f(t_n(\theta))} \frac{g(t_n(\theta)) - g(t)}{t_n(\theta) - t} f(t) G(t_n(\theta)), \end{aligned} \tag{7.9}$$

where

$$f(t) = (-1)^{\lfloor \tau(t) \rfloor} e^{-ibt} \left(\frac{g(t) + \overline{\alpha}}{g'(t)(g(t) + \alpha)} \right)^{\frac{1}{2}},$$

$$g(t) = \sum_n t'_n \left(\frac{1}{t_n - t} - \frac{t_n}{t_n^2 + 1} \right).$$

The series converges in norm, and uniformly on the intervals of $\mathbb{R}$*. For the case* $\theta = 0$ *the sampling formula becomes*

$$G(t) = \sum_n (-1)^n e^{-ibt_n} \frac{\sqrt{t'_n}}{t_n - t} f(t) G(t_n).$$

Proof. Let $E(z)$ be a de Branges function of $\Theta(z)$, and let $\mathcal{KM}(E)$ be the correspondent Kempf-Martin space. Given $G \in \mathcal{V}(\Theta)$ such that $G(t) = ie^{-ibt}\sqrt{\Theta(t)}F(t)$ with $F \in \mathcal{KM}(E)$, thanks to *Theorem* 6.5 we have

$$\begin{aligned} -ie^{ibt}\Theta(t)^{-\frac{1}{2}}G(t) &= F(t) \\ &= \sum_n K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right) F\left(t_n(\theta)\right) \\ &= \sum_n K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right) (-i) e^{ibt_n(\theta)} \Theta(t_n(\theta))^{-\frac{1}{2}} G(t_n(\theta)), \end{aligned}$$

and then

$$\begin{aligned} G(t) &= \sum_n K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right) e^{ib(t_n(\theta) - t)} \Theta(t)^{\frac{1}{2}} \overline{\Theta(t_n(\theta))}^{\frac{1}{2}} G(t_n(\theta)) \\ &= \sum_n K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right) G(t_n(\theta)) \\ &= \sum_n \overline{f(t_n(\theta))} \frac{g(t_n(\theta)) - g(t)}{t_n(\theta) - t} f(t) G(t_n(\theta)). \end{aligned}$$

In particular, for the case $\theta = 0$, we observe that

$$\lim_{s \to \theta_n} g'(t_n)^{-\frac{1}{2}} g(t_n) = \sqrt{t'_n},$$

and then we obtain

$$G(t) = \sum_n (-1)^n e^{-ibt_n} \frac{\sqrt{t'_n}}{t_n - t} f(t) G(t_n).$$

Now, for $N \in \mathbb{Z}$, $N > 0$ we set

$$F_N(t) = \left(F(t) - \sum_{n=-N}^{N} K_{\mathcal{KN}(E)}\left(t_n(\theta), t\right) F(t_n(\theta)) \right) \in \mathcal{KM}(E),$$

$$\begin{aligned} G_N(t) &= \left(G(t) - \sum_{n=-N}^{N} K_{\mathcal{KN}(E)}\left(t_n(\theta), t\right) M(t) \frac{1}{M(t_n(\theta))} G(t_n(\theta)) \right) \\ &= M(t) F_N(t) \in \mathcal{V}(\Theta), \end{aligned}$$

where $M(t) = ie^{-ibt}\sqrt{\Theta(t)}$, and

$$\|F_N(t)\|_{\mathcal{KM}(E)} = \|G_N(t)\|_{\mathcal{V}(\Theta)}\,.$$

Given any interval $[\alpha, \beta] \subset \mathbb{R}$, by *Theorem* 6.5 we have

$$\lim_{N\to+\infty} \sup_{t\in[\alpha,\beta]} |F_N(t)| = 0,$$
$$\lim_{N\to+\infty} \|F_N\|_{\mathcal{KM}(E)} = 0.$$

Therefore, since $F_N(t) = \frac{G_N(t)}{M(t)}$ and $|M(t)| = 1$ for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned}\lim_{N\to+\infty} \sup_{t\in[\alpha,\beta]} |G_N(t)| &= \lim_{N\to+\infty} \sup_{t\in[\alpha,\beta]} |F_N(t)M(t)| \\ &= \lim_{N\to+\infty} \sup_{t\in[\alpha,\beta]} |F_N(t)| \\ &= 0,\end{aligned}$$

and

$$\lim_{N\to+\infty} \|G_N\|_{\mathcal{V}(\Theta)} = \lim_{N\to+\infty} \|F_N\|_{\mathcal{KM}(E)} = 0.$$

Hence the series converges in norm, and uniformly on the intervals of $\mathbb{R}$. □

Theorem 7.3. *Let $\Theta(z)$ be a meromorphic inner function, with spectral function $t(x)$. For $\theta \in [0, 1)$ and $n \in \mathbb{Z}$ set $t_n(\theta) = t(n + \theta)$. Then for every $\theta \in [0, 1)$ the set*

$$\{K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right)\}_n$$

is an orthonormal basis of $\mathcal{V}(\Theta)$.

Proof. Let $E(z)$ be a de Branges function of $\Theta(z)$, and let $\mathcal{KM}(E)$ be the correspondent Kempf-Martin space. Thanks to (7.8), for $\theta \in [0, 1)$ and $n, m \in \mathbb{Z}$ we have

$$\begin{aligned}&\left\langle K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right), K_{\mathcal{V}(\Theta)}\left(t_m(\theta), t\right)\right\rangle_{\mathcal{V}(\Theta)} \\ &= K_{\mathcal{V}(\Theta)}(t_n(\theta), t_m(\theta)) \\ &= e^{ibt_n(\theta)}\overline{\Theta(t_n(\theta)}^{\frac{1}{2}} K_{\mathcal{KM}(E)}(t_n(\theta), t_m(\theta))e^{-ibt_m(\theta)}\Theta(t_m(\theta))^{\frac{1}{2}} \\ &= e^{ibt_n(\theta)}\overline{\Theta(t_n(\theta)}^{\frac{1}{2}} \left\langle K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right), K_{\mathcal{KM}(E)}\left(t_m(\theta), t\right)\right\rangle_{\mathcal{V}(\Theta)} e^{-ibt_m(\theta)}\Theta(t_m(\theta))^{\frac{1}{2}}.\end{aligned}$$

Thanks to (6.24), for $m \neq n$ we obtain

$$\left\langle K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right), K_{\mathcal{V}(\Theta)}\left(t_m(\theta), t\right)\right\rangle_{\mathcal{V}(\Theta)} = 0.$$

Moreover, using (6.23), for $m = n$ we get

$$\begin{aligned}
&\left\langle K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right), K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right)\right\rangle_{\mathcal{V}(\Theta)} \\
&= e^{ibt_n(\theta)}\overline{\Theta(t_n(\theta)}^{\frac{1}{2}} \left\langle K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right), K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right)\right\rangle_{\mathcal{V}(\Theta)} e^{-ibt_n(\theta)}\Theta(t_n(\theta))^{\frac{1}{2}}. \\
&= \left\langle K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right), K_{\mathcal{KM}(E)}\left(t_n(\theta), t\right)\right\rangle_{\mathcal{V}(\Theta)} \\
&= 1.
\end{aligned}$$

Therefore we have shown that

$$\{K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right)\}_n$$

is an orthonormal set. Hence, thanks to *Theorem* 7.2, we obtain that the only function that is perpendicular to all the elements of the sequence $\{K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right)\}_n$ is the null vector, hence $\{K_{\mathcal{V}(\Theta)}\left(t_n(\theta), t\right)\}_n$ is a complete orthonormal sequence in $\mathcal{V}(\Theta)$, and therefore an orthonormal basis.

□

Now we set

$$\begin{aligned}
\Theta_0(z) &= e^{2ibz}, \\
\Theta_1(z) &= \gamma B(z),
\end{aligned}$$

so that $\Theta(z) = \Theta_0(z)\Theta_1(z)$, and we observe that $\Theta(z) = LCM(\Theta_0(z), \Theta_1(z))$. Setting $\tilde{\mathcal{K}}_n(\Theta) = \bigoplus_{n=1}^{\infty} \Theta_0^n \mathcal{K}\left(\Theta_0\right) \cap \left(\bigoplus_{m=1}^{n} \Theta_0^m \Theta_1 \mathcal{K}\left(\Theta_0\right)\right)^{\perp}$, thanks to (5.11) the space $\mathcal{V}(\Theta)$ can be represented as

$$\begin{aligned}
\mathcal{V}(\Theta) &= \frac{e^{-ibt}}{\sqrt{\tau'(t)}}\left(\mathcal{K}(\Theta_0) \oplus \left(\bigoplus_{n>0} \tilde{\mathcal{K}}_n(\Theta)\right)\right) \\
&= \frac{1}{\sqrt{\tau'(t)}}\left(\mathcal{PW}_b \oplus e^{-ibz}\left(\bigoplus_{n>0} \tilde{\mathcal{K}}_n(\Theta)\right)\right)
\end{aligned} \tag{7.8}$$

We observe that the sampling sequences of the spaces $\mathcal{V}(\Theta)$ are the same of the corresponding de Branges space $\mathcal{B}(E)$. Hence we can inherit also the definition of time-varying bandlimit given in *Section* 7.1. Therefore, for any $\alpha \in [0, 1)$, the functions of the space $\mathcal{V}(\Theta)$ can be rebuilt with samples taken on the sampling sequence $\{t_n(\alpha)\}_n$, where

$$t_n(\alpha) = t(n + \alpha),$$

and $t(x)$ is the spectral function of $\Theta(z)$. Now, given $F \in \mathcal{V}(\Theta)$, consider $F(x)$ and $F(t(x))$ for $x \in \mathbb{R}$. For any $\alpha \in [0, 1)$ $F(x)$ can be rebuilt on samples taken on the sampling sequence $\{t(n+\alpha)\}_n$, and then it is easy to see that $F(t(x))$ can be rebuilt on samples taken on the sampling sequence $\{n+\alpha\}_n$. Then, according to the definition of time-varying bandlimit, $F(t(x))$

has a time-constant bandlimit. We have obtained that replacing x with $t(x)$ in $F(x)$ we cancel the effect of the time-varying bandlimit, flattening it to a time-constant bandlimit. Since $F(t(\tau(x))) = F(x)$, the function $F(x)$ can be interpreted as the result of the application of a distortion $\tau(x)$ to the time domain of the time-constant bandlimit function $F(t(x))$.

Now we define the space

$$\tilde{\mathcal{V}}(\Theta) := \{F(t(x)), F \in \mathcal{V}(\Theta)\},$$

equipped with the scalar product and the norm of $\mathcal{L}^2(\mathbb{R})$. Given $F, G \in \mathcal{V}(\Theta)$ we observe that

$$\begin{aligned} \langle F(t(x)), G(t(x))\rangle_{\mathcal{L}^2(\mathbb{R})} &= \int_{-\infty}^{+\infty} F(t(x))\overline{G(t(x))}dx \\ &= \int_{-\infty}^{+\infty} F(t)\overline{G(t)}\tau'(t)dt \\ &= \langle F(t)\sqrt{\tau'(t)}, G(t)\sqrt{\tau'(t)}\rangle_{\mathcal{L}^2(\mathbb{R})}. \end{aligned} \tag{7.9}$$

By definition of $\mathcal{V}(\Theta)$ we have

$$F(t)\sqrt{\tau'(t)} \in e^{-ibt}\mathcal{K}(\Theta) = \mathcal{PW}_b \oplus e^{-ibt}\left(\bigoplus_{n>0} \tilde{\mathcal{K}}_n(\Theta)\right).$$

Hence $F(t)\sqrt{\tau'(t)}$ is the restriction on the real line of a function of $e^{-ibz}\mathcal{K}(\Theta)$.

We recall also that the scalar product and the norm in $\mathcal{PW}_b$ are the same of $\mathcal{L}^2(\mathbb{R})$. Since $e^{-i(2n-1)bz}\tilde{\mathcal{K}}_n(\Theta) \subseteq \mathcal{PW}_b$ and $|e^{-2inbx}| = 1$ for $x \in \mathbb{R}$, the scalar product and the norm of $\mathcal{L}^2(\mathbb{R})$ are a scalar product and a norm also for every space $e^{-ibz}\tilde{\mathcal{K}}_n(\Theta)$. Moreover we know that the space $e^{-ibz}\mathcal{K}(\Theta)$ is the closure of the subspace formed by all the possible finite sums of elements of $\mathcal{PW}_b$ and of $\{e^{-ibz}\tilde{\mathcal{K}}_n(\Theta)\}_{n>0}$, then the norm $\mathcal{L}^2(\mathbb{R})$ is a norm also for the whole space $e^{-ibz}\mathcal{K}(\Theta)$.

Therefore, by (7.9) we get that there exists a unitary isomorphism between the restriction on $\mathbb{R}$ of the space $e^{-ibz}\mathcal{K}(\Theta)$ and the space $\tilde{\mathcal{V}}(\Theta)$. We have obtained that canceling the distortion of the time-varying bandlimit to the functions of the space $\mathcal{V}(\Theta)$ we get the space of time-constant bandlimit functions $\tilde{\mathcal{V}}(\Theta)$, which is unitarily isomorphic to the space of bandlimited functions $e^{-ibz}\mathcal{K}(\Theta)$. Then we can interpret the space $\mathcal{V}(\Theta)$ as the space obtained applying a distorsion in the time domain to the space $e^{-ibz}\mathcal{K}(\Theta)$.

More is true. Indeed, recalling here the definitions of $\mathcal{L}^2_\Theta$ given in (5.7), the next theorem shows that the weighted Fourier transform $\mathcal{F}_{\sqrt{\tau'}}$, given by

$$\mathcal{F}_{\sqrt{\tau'}}(F)(z) := \int_{-\infty}^{+\infty} F(t)e^{-izt}\sqrt{\tau'(t)}dt,$$

induces an isomorphism between $\mathcal{V}(\Theta)$ and $\mathcal{L}^2_\Theta$.

Theorem 7.4. *Let $\Theta(z) = \gamma e^{2ibz} B(z)$ be a meromorphic inner function according to the representation given in* (2.3)*, with logarithmic residue* $2b > 0$ *and phase function $\tau(t)$. Let $F(t) \in \mathcal{V}(\Theta)$. Then the weighted Fourier transform*

$$\mathcal{F}_{\sqrt{\tau'}} : \mathcal{V}(\Theta) \to \mathcal{L}^2_\Theta, \qquad \mathcal{F}_{\sqrt{\tau'}}(F)(z) := \int_{-\infty}^{+\infty} F(t) e^{-izt} \sqrt{\tau'(t)} dt,$$

is a unitary isomorphism between $\mathcal{V}(\Theta)$ and $\mathcal{L}^2_\Theta$.

Proof. Let $E(z)$ be a de Branges function of $\Theta(z)$, and let $F(t) \in \mathcal{V}(\Theta)$ be such that $F(t) = \frac{e^{-ibt}}{E(t)\sqrt{\tau'(t)}} G(t)$, where $G(t) \in \mathcal{B}(E)$. We observe that

$$\begin{aligned}
\mathcal{F}_{\sqrt{\tau'}}(F)(s) &:= \int_{-\infty}^{+\infty} F(t) e^{-ist} \sqrt{\tau'(t)} dt \\
&= \int_{-\infty}^{+\infty} G(t) \frac{e^{-ibt}}{E(s)} \frac{1}{\sqrt{\tau'(t)}} e^{-ist} \sqrt{\tau'(t)} dt \\
&= \int_{-\infty}^{+\infty} G(t) \frac{e^{-ibt}}{E(s)} e^{-ist} dt \\
&= \tilde{\mathcal{F}}_E(G)(s).
\end{aligned}$$

Now the conclusion follows easily from *Theorem* 5.1. □

This theorem is, for spaces $\mathcal{V}(\Theta)$, analogous to what Paley-Wiener theorem is for the Paley-Wiener spaces. Moreover now we can associate to every time-varying bandlimit function $F(t) \in \mathcal{V}(\Theta)$ a concept of frequency representation, that is the frequency representation of the function obtained canceling the distortion of the time-varying bandlimit effect.

In conclusion, summarizing the previous observations, the spaces $\mathcal{V}(\Theta)$ are very interesting for sampling and reconstruction of time-varying bandlimit functions, for the following reasons:

1. For every meromorphic inner function $\Theta(z)$, there exists an isomorphic multiplier between every space $\mathcal{V}(\Theta)$ and the Kempf-Martin space $\mathcal{KM}(T)$ that has Livsic characteristic function $\Theta(z)$, and then the space $\mathcal{V}(\Theta)$ maintains many of the properties of the Kempf-Martin space $\mathcal{KM}(T)$. Indeed, for example, the functions of the space $\mathcal{V}(\Theta)$ can be rebuilt exactly with samples taken on the same sequences from which the functions of $\mathcal{KM}(T)$ can be rebuilt, and for every $\theta \in [0, 1)$, the sequence
$$\{K_{\mathcal{V}(\Theta)}(t_n(\theta), \cdot)\}_n$$
is an orthonormal basis for the space $\mathcal{V}(\Theta)$, as the sequence
$$\{K_{\mathcal{KM}(T)}(t_n(\theta), \cdot)\}_n$$
is an orthonormal basis for the space $\mathcal{KM}(T)$.

2. We can associate to every time-varying bandlimit function $F(t) \in \mathcal{V}(\Theta)$ a frequency representation, that is the frequency representation of the function obtained flattening the distortion of the time-varying bandlimit effect in $F(z)$. We define this frequency representation and the corresponding bandwidth as the *normalized frequency representation* and the *normalized bandwidth* of $F(t)$.

3. The normalized frequency representation of $F \in \mathcal{V}(\Theta)$ is obtained applying the weighted Fourier transform $\mathcal{F}_{\sqrt{\tau'}}$ to $F(z)$. This transform induces a unitary isomorphism between $\mathcal{V}(\Theta)$ and $\mathcal{L}^2_\Theta$, and hence it is, for the spaces $\mathcal{V}(\Theta)$, the analogous of what the Fourier transform is for the Paley-Wiener spaces.

4. Thanks to the previous point, the spaces $\mathcal{V}(\Theta)$ are, for time-varying bandlimit functions, analogous to what the Paley-Wiener spaces are for bandlimited functions.

5. Thanks to all these observations, the spaces $\mathcal{V}(\Theta)$ result to be more interpretable and controllable than other time-varying bandlimit spaces. Indeed, by (7.8), every function $F(t) \in \mathcal{V}(\Theta)$ can be written as

$$F(z) = \frac{1}{\sqrt{\tau'(t)}} \left(F_0(z) \left(\bigoplus_{n>0} F_n(t) \right) \right),$$

   for $F_0 \in \mathcal{PW}_b$ and $F_n(z) \in e^{-ibz}\tilde{\mathcal{K}}_n(\Theta) \subseteq e^{2inbz}\mathcal{PW}_b$. We recall that $\mathcal{F}(F_n) \in \mathcal{L}^2[b(2n-1), b(2n+1)]$. Then the normalized bandwidth of the function $F(z)$ is defined by the values of n for which $F_n(z) \neq 0$, while the shape of the signal, for every interval $[b(2n-1), b(2n+1)]$ in the normalized frequency domain, is controlled by $F_n(z)$.

For these reasons in the next sections we propose a generalization of the sampling method for time-varying bandlimit functions based on the reproducing kernel and the sampling formula of the spaces $\mathcal{V}(\Theta)$.

7.4 Generalized sampling method

The classical Shannon sampling method allows sampling and perfect reconstruction of bandlimited signals with time-constant bandlimits. However, it is clear that in real applications the effective bandwidth of a signal could vary in time. In this case, sampling a signal at a constant rate is clearly not optimally efficient, since choosing the highest needed sampling rate leads to wasteful redundancy, while taking a lower sampling rate causes loss of information.

To improve the sampling efficiency, we generalize the classical Shannon sampling method, and we propose a method which allows the samples to be taken only as often as necessary according to the behavior of the given signals (and hence to its time-varying bandlimit), and maintains the ability to perfectly and stably reconstruct the continuous signals from their discrete values on the set of sampling points.

Therefore, the goal of this section is to generalize the Shannon sampling theorem, following the sampling scheme of the Shannon method, but for time-varying bandlimit functions. As already pointed out, the bandlimit as a function of time is ill-defined since the bandlimit of a signal is simply time-independent. However, we can consider the bandlimit of a time-varying bandlimit function as the bandlimit of its normalized frequency representation defined in *Section* 7.2, that is to say the bandlimit of the function obtained flattening the distortion in the time domain introduced by the time-varying bandlimit. Our generalized sampling method is mainly based on the reconstruction properties of the spaces $\mathcal{V}(\Theta)$.

First of all we recap the behavior of the classical Shannon sampling method. Suppose to have a set of raw signals, i.e. a continuous function $F_{raw} : \mathbb{R} \to \mathbb{C}$. The Shannon sampling method consists of the following four steps.

1. Analyze the frequency of the raw signals of interest $F_{raw}(t)$ in order to choose a suitable bandlimit b.

2. Filter $F_{raw}(t)$ to obtain a bandlimited function $F(t)$ such that $\mathcal{F}(F)(s) = 0$ for $s \notin [-b, b]$:

$$F(t) = (PF_{raw})(t) = \int_{-\infty}^{+\infty} F_{raw}(s) K_{\mathcal{PW}_b}(t, s) ds, \tag{7.10}$$

   where

$$K_{\mathcal{PW}_b}(t, s) = \frac{b}{\pi} \operatorname{sinc}(b(t - s))$$

   is the reproducing kernel of the Paley-Wiener space $\mathcal{PW}_b$.

3. Store the samples $\{F(t_n)\}_n$ for $t_n = \frac{\pi}{b} n$.

4. Reconstruct $F(t)$ for all $t \in \mathbb{R}$ from the discrete samples using the Shannon sampling theorem:

$$F(t) = \sum_n K_{\mathcal{PW}_b}(t, t_n). \tag{7.11}$$

A given arbitrary raw signal $F_{raw}(t)$ generally hasn't bandlimit b, so we need to first pre-filter it in order to consider only the frequencies inside the interval $[-b, b]$. The bandlimit is chosen in step (1) so that the frequencies of $F_{raw}(t)$ outside the interval $[-b, b]$ are negligible.

In step (2), we approximate the raw signal $F_{raw}(t)$ with the filtered signal $F(t)$ such that $\mathcal{F}(F)(s) = 0$ for $s \notin [-b, b]$. In order to obtain such $F(t)$ it is sufficient to multiply the Fourier transform of $F_{raw}(t)$ with the rectangular function which is 1 in $[-b, b]$ and 0 elsewhere. Let $\hat{F}_{raw}(w)$ and $\hat{F}(w)$ denote the Fourier transform respectively of $F_{raw}(t)$ and $F(t)$. The filtering operation $F(t) = (PF_{raw})(t)$ in the frequency domain becomes

$$\hat{F}(w) = \hat{F}_{raw}(w)\,\mathrm{rect}\left(\frac{w}{2b}\right).$$

Thanks to the Fourier transform properties, in the time domain this is equivalent to the convolution of $\hat{F}_{raw}(t)$ with the function $\frac{b}{\pi} sinc(bt)$:

$$\begin{aligned} F(t) = (PF_{raw})(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{F}_{raw}(w)\,\mathrm{rect}\left(\frac{w}{2b}\right) dw \\ &= F_{raw}(t) * \left(\frac{b}{\pi}\,\mathrm{sinc}\,(bt)\right) \\ &= \int_{-\infty}^{+\infty} F_{raw}(s)\left(\frac{b}{\pi}\,\mathrm{sinc}(b(t-s))\right) ds. \end{aligned}$$

Hence $F(t)$ is obtained applying the scalar product of the Kempf-Martin space $\mathcal{PW}_b$ between $F_{raw}(t)$ and the reproducing kernel of $\mathcal{PW}_b$, even if $F_{raw}(t)$ generally is not in $\mathcal{PW}_b$.

The resulting signal $F(t)$ has support contained in $[-b, b]$. Therefore in steps (3) and (4) the sampling theorem is applied. The samples $\{F(t_n)\}_n$ are taken on a set of equidistant points such that $t_{n+1} - t_n = \frac{\pi}{b}$, and the continuous bandlimited signal $F(t)$ is perfectly reconstructed for all $t \in \mathbb{R}$ from these samples according to (7.11).

Now, let's introduce the scheme of the generalized sampling theory. It consists of the following four steps.

1. Analyze the frequency of the raw signals of interest $F_{raw}(t)$ in order to choose a suitable time-varying bandlimit space, which is specified by the normalized bandlimit pair $(\{t_n\}, \{t'_n\})$ and the corresponding meromorphic inner function $\Theta(z)$ according to *Theorem* 4.6, given by

$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) - i\pi}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + i\pi}.$$

2. Filter $F_{raw}(t)$ to obtain a function $F(t)$ with the desired time-varying bandlimit:

$$F(t) = (PF_{raw})(t) = \int_{-\infty}^{+\infty} F_{raw}(s) K_{\mathcal{V}(\Theta)}(t, s)\tau'(s) ds, \tag{7.12}$$

where

$$K_{\mathcal{V}(\Theta)}(t,s) = f(t)f(s)\sum_n \frac{t'_n}{(t_n-t)(t_n-s)}e^{-2ibt_n},$$

$$f(t) = (-1)^{\lfloor \tau(t) \rfloor} e^{-ibt}\left(\frac{g(t)-i\pi}{g'(t)(g(t)+i\pi)}\right)^{\frac{1}{2}},$$

$$g(t) = \sum_n t'_n \left(\frac{1}{t_n-t} - \frac{t_n}{t_n^2+1}\right).$$

3. Store the samples $\{F(t_n)\}_n$ on the chosen sampling sequence $\{t_n\}$.

4. Reconstruct $F(t)$ for all $t \in \mathbb{R}$ from the discrete samples using the same reconstruction formula of time-varying bandlimit spaces $\mathcal{V}(\Theta)$ (see (7.9)):

$$F(t) = \sum_n K_{\mathcal{V}(\Theta)}(t, t_n)\, F(t_n)\,, \tag{7.13}$$

where

$$K_{\mathcal{V}(\Theta)}(t,t_n) = \sum_n (-1)^n e^{-ibt_n}\frac{\sqrt{t'_n}}{t_n-t}f(t),$$

In step (1) we choose a suitable bandlimit pair, according to the frequency of the raw signals of interest $F_{raw}(t)$.

An arbitrary raw signal $F_{raw}(t)$ generally doesn't have the chosen time-varying bandlimit. Hence in step (2) we need to filter the raw signal $F_{raw}(t)$ in order to obtain the better approximation $F(t)$ of $F_{raw}(t)$ in the set of functions with the desired time-varying bandlimit, which are the functions that can rebuilt exactly by the sampling formula (7.13).

Working in analogy with the classical Shannon sampling method, we define the filter operator P as the scalar product of $\mathcal{V}(\Theta)$ between $F_{raw}(t)$ and the reproducing kernel of $\mathcal{V}(\Theta)$, even if $F_{raw}(t)$ generally is not in $\mathcal{V}(\Theta)$:

$$\begin{aligned} F(t) &= (PF_{raw})(t) \\ &= \left\langle F_{raw}(s), K_{\mathcal{V}(\Theta)}(t,s)\right\rangle_{\mathcal{V}(\Theta)} \\ &= \int_{-\infty}^{+\infty} F_{raw}(s)K_{\mathcal{V}(\Theta)}(t,s)\tau'(s)ds. \end{aligned}$$

By *Theorem* 7.3 we have that

$$\{K_{\mathcal{V}(\Theta)}(t_n,\cdot)\}_n$$

is an orthonormal basis of $\mathcal{V}(\Theta)$, and expanding $K_{\mathcal{V}(\Theta)}(t,s)$ in this basis, we get

$$\begin{aligned} K_{\mathcal{V}(\Theta)}(t,s) &= \sum_n K_{\mathcal{V}(\Theta)}(t_n(\theta),s)\, K_{\mathcal{V}(\Theta)}(t_n(\theta),t) \\ &= f(t)f(s)\sum_n \frac{t'_n}{(t_n-t)(t_n-s)}e^{-2ibt_n}. \end{aligned} \tag{7.14}$$

To be consistent with our purposes, the filter operator P need to satisfy the following two constraints:

- the resulting signal $F(t) = (PF_{raw})(t)$ must have the desired time-varying bandlimit, i.e.
$$F(t) = \sum_n K_{\mathcal{V}(\Theta)}(t, t_n)\, F(t_n);$$
- the operator P must be a projection, which means $P^2 = P$.

We see this in the following theorem.

Theorem 7.5. *Let $\Theta(z)$ be a meromorphic inner function, given by*
$$\Theta(z) = \frac{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) - i\pi}{\sum_n t'_n \left(\frac{1}{t_n - z} - \frac{t_n}{t_n^2+1}\right) + i\pi},$$
with phase function $\tau(t)$. Given a continuous function $G(t) : \mathbb{R} \to \mathbb{C}$, let the operator P be defined as
$$(PG)(t) = \int_{-\infty}^{+\infty} G(s) K_{\mathcal{V}(\Theta)}(t, s) \tau'(s) ds.$$
Take any $G(t)$ for which $(PG)(t)$ is finite for every $t \in \mathbb{R}$. Then

- *for the function $F(t) = (PG)(t)$ the following sampling formula holds:*
$$F(t) = \sum_n K_{\mathcal{V}(\Theta)}(t, t_n)\, F(t_n);$$
- $(P^2 G)(t) = (PG)(t)$.

Proof. By *Theorem* 7.3, for every $\theta \in [0, 1)$ we have that
$$\{K_{\mathcal{V}(\Theta)}(t_n(\theta), \cdot)\}_n$$
is an orthonormal basis of $\mathcal{V}(\Theta)$. Hence, expanding $K_{\mathcal{V}(\Theta)}(t, s)$ in this basis we get
$$K_{\mathcal{V}(\Theta)}(t, s) = \sum_n K_{\mathcal{V}(\Theta)}(t_n(\theta), s)\, K_{\mathcal{V}(\Theta)}(t_n(\theta), t). \tag{7.15}$$
Therefore we have
$$\begin{aligned} F(t) &= \int_{-\infty}^{+\infty} G(s) K_{\mathcal{V}(\Theta)}(t, s) \tau'(s) ds \\ &= \int_{-\infty}^{+\infty} G(s) \left(\sum_{n=-\infty}^{+\infty} K_{\mathcal{V}(\Theta)}(t_n, s)\, K_{\mathcal{V}(\Theta)}(t_n, t) \right) \tau'(s) ds \\ &= \sum_n K_{\mathcal{V}(\Theta)}(t_n, t) \left(\int_{-\infty}^{+\infty} G(s) K_{\mathcal{V}(\Theta)}(t_n, s) \tau'(s) ds \right) \\ &= \sum_n K_{\mathcal{V}(\Theta)}(t_n, t)\, (PG)(t_n) \\ &= \sum_n K_{\mathcal{V}(\Theta)}(t_n, t)\, F(t_n). \end{aligned} \tag{7.16}$$

Hence we have obtained that the resulting signal $F(t)$ has the desired time-varying bandlimit.

It remains to show that $(P^2G)(t) = (PG)(t)$. This is equivalent to show that, given $F(t) = (PG)(t)$, then $F(t) = (PF)(t)$. If $F(t)$ were in $\mathcal{V}(\Theta)$, $F(t) = (PF)(t)$ would be an easy consequence of the definition of the reproducing kernel $K_{\mathcal{V}(\Theta)}(t,s)$, but $F(t)$ is not necessarily in $\mathcal{V}(\Theta)$. Since $K_{\mathcal{V}(\Theta)}(t_n,\cdot) \in \mathcal{V}(\Theta)$ we observe that

$$\begin{aligned} PK_{\mathcal{V}(\Theta)}(t_n,t) &= \int_{-\infty}^{\infty} K_{\mathcal{V}(\Theta)}(t_n,s)\, K_{\mathcal{V}(\Theta)}(t,s)\tau'(x)dx \\ &= K_{\mathcal{V}(\Theta)}(t_n,t)\,. \end{aligned}$$

Then, by (7.16), for $F(t) = (PF_{raw})(t)$ we obtain

$$\begin{aligned} PF(t) &= \sum_n PK_{\mathcal{V}(\Theta)}(t_n,t)\, F(t_n) \\ &= \sum_n K_{\mathcal{V}(\Theta)}(t_n,t)\, F(t_n) \\ &= F(t), \end{aligned}$$

and we conclude that $(P^2G)(t) = (PG)(t)$. $\square$

Finally, in steps (3) and (4) the sampling formula for time-varying bandlimit functions is applied. The samples $\{F(t_n)\}_n$ are taken on the chosen sampling sequence $\{t_n\}_n$, and the signal $F(t)$ is perfectly reconstructed for all $t \in \mathbb{R}$ from these samples according to (7.13).

CHAPTER 8

Canonical Systems and de Branges Spaces

8.1 Canonical inverse problem

A canonical system is a differential equation of the form

$$J\frac{dY}{dx} = zH(x)Y, \tag{8.1}$$

where

- $H(x)$ is a function $(0, L) \to \mathrm{Mat}_2(\mathbb{R}), 0 < L \leq \infty$, such that $H(x) \geq 0$ a.e., and $H \in L^1(0, L')$ for all $L' < L$. Without loss of generality we can assume that $\mathrm{tr}(H(x)) = 1$;
- $Y = \begin{bmatrix} Y_+ \\ Y_- \end{bmatrix} \in \mathbb{C}^2$;
- $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$;
- $z \in \mathbb{C}$.

The parameter z in (8.1) is referred to as the spectral parameter. Let $\mathrm{tr}\, H(x) = 1$ a. e. and let $M(x, z)$ be a matrix solution to (8.1). Then

$$\|M(x, z)\| \leq e^{x|z|}\|M(0, z)\|,$$

where

$$\|M\| = \inf\{t \geq 0 : \|MY\|_{\mathbb{C}^2} \leq t\|Y\|_{\mathbb{C}^2} \text{ for all } Y \in \mathbb{C}^2\},$$

and

$$\|Y\|_{\mathbb{C}^2} = |Y_+|^2 + |Y_-|^2.$$

The solution that satisfies the boundary condition $M(0, z) = I$ is called the *fundamental solution*. When $L < \infty$ the fundamental solution at $x = L$ is called the *monodromy matrix*. A crucial aspect is that the determinant of a matrix solution to (8.1) does not depend on x, and in particular for the fundamental solution we have $\det M(x, z) = 1$ for all $x \leq L$. The fundamental solution is real entire in z for all $x \leq L$.

The *chain rule* states that, if $0 < a < b$, then the fundamental solutions satisfy $M(b, z) = N(b, z)M(a, z)$ where $N(x, z)$ is a solution of (8.1) on (a, b) with $N(a, z) = I$.

For more details see [46], p. 4-5. We recall also the following important and useful result in [46]:

Theorem 8.1 (Romanov)**.** *Given $a > 0$, the finite interval $I = (0, a)$ and a vector $e = \begin{bmatrix} e_+ \\ e_- \end{bmatrix} \in \mathbb{R}^2$ of unit norm, the monodromy matrix of the canonical system (H, a) with the Hamiltonian $H(x) = \begin{bmatrix} e_+^2 & e_+e_- \\ e_+e_- & e_-^2 \end{bmatrix}, x \in I$, is easily verified to be*

$$M(z) = I + zR = I + z \begin{bmatrix} ae_-e_+ & ae_-^2 \\ -ae_+^2 & -ae_-e_+ \end{bmatrix}$$

The matrix R obeys $R^2 = 0, R_{12} \geq 0, R_{21} \leq 0$. Conversely, for any nonzero matrix R satisfying these three properties there exists an $a > 0$ and $e \in \mathbb{R}^2, \|e\| = 1$, such that $I + zR$ is a monodromy matrix for the corresponding canonical system, given by $a = R_{12} - R_{21}, e_- = \sqrt{R_{12}/a}$, $e_+ = (\operatorname{sign} R_{11}) \sqrt{-R_{21}/a}$.

In [11] de Branges shows that if $Y(x, z)$ is the solution of (8.1), then $E_x(z) = Y_+(x, z) + iY_-(x, z)$ is a Hermite Biehler function of z for each $x \in (0, L)$. Given any $E(z)$, the problem of building $H(x)$ such that $E_L(z) = E(z)$ (and then $\Theta_L = \Theta$) is known as the *canonical inverse problem*.

An iterative algorithm to solve this problem was proposed by Romanov in [46], *Section* 7 (p. 37). The downside of this solution is that it is not explicit, and that neither the result of each iteration is explicit. We report here the algorithm proposed by Romanov, for completeness and to express it with the notations of this work.

Theorem 8.2 (Romanov). *Let $E(z)$ be a Hermite Biehler function having no real zeros and such that $E(0) = 1$. Let $\Theta(z)$ be defined by*

$$\Theta(z) = \begin{bmatrix} \Theta_+(z) \\ \Theta_-(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} E(z) + E^\#(z) \\ \frac{1}{i}\left(E(z) - E^\#(z)\right) \end{bmatrix}. \tag{8.2}$$

Let $t_j, j \geq 0, t_0 = 0$, be the set of zeroes of $\Theta_-(z)$ ordered by $|t_j| \leq |t_{j+1}|$,

$$\Theta_{N-}(z) = \dot{\Theta}_-(0) z \prod_{j=0}^{N-1} \left(1 - \frac{z}{t_j}\right)$$

$$\Theta_{N+}(z) = \left(\sum_{j=0}^{N} \frac{\Theta_+(t_j)}{\dot{\Theta}_-(t_j)} \frac{1}{z - t_j} + a + bz\right) \Theta_{N-}(z),$$

a and b being the constants in the linear term in the Nevanlinna representation of the Herglotz function $\frac{\Theta_+}{\Theta_-}$. Then $E_N(z) = \Theta_{N+}(z) + i\Theta_{N-}(z)$ is a Hermite Biehler polynomial function of degree $N \geq 1$ having no real zeros and such that $E(0) = 1$. Let (H_N, L_N) be the corresponding canonical system constructed using the algorithm for the polynomial case (see Sections 4.2, 4.4 in [46]). Let

$$L = \frac{1}{\pi} \left\| \frac{\Theta_+ - 1}{z} \right\|^2_{\mathcal{H}(E)} - \dot{\Theta}_-(0)$$

Define

$$\widetilde{\mathrm{H}}_N(x) = \begin{cases} \mathrm{H}_N\left(x - \max\{0, L - L_N\}\right), & x \geq \max\{0, L - L_N\} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & 0 \leq x \leq \max\{0, L - L_N\} \end{cases}$$

$$F_N(x) = \int_{l_N - h}^{l_N - h + x} \widetilde{\mathrm{H}}_N(s) \mathrm{d}s, l_N = \max\{L, L_N\}$$

Then, as $N \to +\infty$, F_N converges in $C(0, L)$ to a monotone non-decreasing function, F. The canonical system $(\Theta, L) := (F', L)$ is such such that $\Theta = \Theta_L$ and that there is no $\varepsilon > 0$ such that $\mathrm{H}(x) = \langle \cdot, e\rangle e, e = (0,1)^T$, for a.e. $x \in (0, \varepsilon)$.

8.2 Improved algorithm

In this section we propose an improvement to the Romanov algorithm, mainly based on the result of *Theorem* 8.3. The big advantage of the proposed solution is that, even if it is not yet explicit, the result of every iteration is explicit in terms of the result of the previous iteration.

Let $E(z)$ be a Hermite Biehler function having no real zeros, such that $E(0) = 1$, and let $\Theta(z)$ be defined by

$$\Theta(z) = \begin{bmatrix} \Theta_+(z) \\ \Theta_-(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} E(z) + E^{\#}(z) \\ \frac{1}{i}\left(E(z) - E^{\#}(z)\right) \end{bmatrix}.$$

Let $\{t_n\}_{n \geq 0}$ be the set of zeroes of $\Theta_-(z)$ ordered by $|t_n| \leq |t_{n+1}|$, with $t_0 = 0$. Using the canonical product for $\Theta_-(z)$, for each $N > 0$ we have

$$\begin{aligned} \Theta_-(z) &= \Theta_{N-}(z) e^{\alpha_N z} R_N(z), \\ \Theta_{N-}(z) &= \Theta'_-(0) z \prod_{n=1}^{N-1} \left(1 - \frac{z}{t_n}\right), \\ R_N(z) &= \prod_{n \geq N} \left(1 - \frac{z}{t_n}\right) e^{\frac{z}{t_n}}, \end{aligned}$$

where $\Theta'_-(0), \alpha_N \in \mathbb{R}$. Let

$$\frac{\Theta_+(z)}{\Theta_-(z)} = \sum_{n=0}^{\infty} \left(\frac{\mu_n}{t_n - z} - \frac{\mu_n t_n}{1 + t_n^2}\right) + b_0 + cz,$$

be the Nevanlinna representation of the Herglotz function $\frac{\Theta_+(z)}{\Theta_-(z)}$, with $\mu_n = -\frac{\Theta_+(t_n)}{\Theta'_-(t_n)} \geq 0 \ \forall n \geq 0$ (see *Section* 5 in [46], p. 23-24), so that

$$\begin{aligned} E(z) &= \Theta_+(z) + i\Theta_-(z) \\ &= \left(\sum_{n=0}^{\infty} \left(\frac{\mu_n}{t_n - z} - \frac{\mu_n t_n}{1 + t_n^2}\right) + b_0 + cz\right) \Theta_-(z). \end{aligned} \tag{8.3}$$

Define

$$b = b_0 - \sum_{n=0}^{\infty} \frac{\mu_n t_n}{1 + t_n^2},$$

and

$$\Theta_{N+}(z) = \left(\sum_{n=0}^{N-1} \frac{\mu_n}{t_n - z} + b + cz\right) \Theta_{N-}(z).$$

Then $\Theta_N(z) := \begin{bmatrix} \Theta_{N+}(z) \\ \Theta_{N-}(z) \end{bmatrix}$ is a polynomial, and

$$E_N(z) = \Theta_{N+}(z) + i\Theta_{N-}(z) = \left(\sum_{n=0}^{N-1} \frac{\mu_n}{t_n - z} + b + i + cz\right) \Theta_{N-}(z). \tag{8.4}$$

is a Hermite Biehler polynomial having no real zeroes, and $E_N(0) = 1$ since $E(0) = 1$ and hence $\mu_0 = 1/\Theta'_-(0)$. If $c = 0$, $E_N(z)$ is a polynomial of degree N:

$$E_N(z) = a_N z^N + a_{N-1} z^{N-1} + \ldots + a_1 z + a_0.$$

With a simple computation we get

$$
\begin{aligned}
a_N =& \Theta'_-(0)(b+i)(-1)^{N-1}\prod_{n=1}^{N-1}\frac{1}{t_n} \\
a_{N-1} =& \Theta'_-(0)\left((-1)^{N-1}\prod_{n=1}^{N-1}\frac{1}{t_n}\right)\left(-\sum_{n=0}^{N-1}\mu_n\right) \\
&+\Theta'_-(0)(b+i)\left((-1)^{N-1}\prod_{n=1}^{N-1}\frac{1}{t_n}\right)\left(1-\sum_{n=1}^{N-1}t_n\right) \\
=& a_N\left(\left(\sum_{n=0}^{N-1}\mu_n\right)\frac{-b+i}{b^2+1}+1-\sum_{n=1}^{N-1}t_n\right),
\end{aligned}
\tag{8.5}
$$

and hence we observe that

$$
\begin{aligned}
\Im(a_N) &\neq 0, \\
\Im(\overline{a}_N a_{N-1}) &= \frac{|a_N|^2\sum_{n=1}^{N-1}\mu_n}{b^2+1} > 0.
\end{aligned}
\tag{8.6}
$$

If $c \neq 0$, $E_N(z)$ is a polynomial of degree $N+1$:

$$E_N(z) = a_{N+1}z^{N+1} + a_N z^N + ... + a_1 z + a_0.$$

Similarly to above we get

$$
\begin{aligned}
a_{N+1} &= c\Theta'_-(0)(-1)^{N-1}\prod_{n=1}^{N-1}\frac{1}{t_n} \\
a_N &= a_{N+1}\left((b+i)+1-\sum_{n=1}^{N-1}t_n\right),
\end{aligned}
\tag{8.7}
$$

hence we observe that $\Im(a_{N+1}) \neq 0$ and

$$
\begin{aligned}
\Im(a_{N+1}) &\neq 0 \\
\Im(\overline{a}_{N+1}a_N) &= |a_{N+1}|^2 > 0.
\end{aligned}
\tag{8.8}
$$

Now, let $E_N(z)$ be a generic Hermite Biehler polynomial function of degree N having no real zeroes, such that $E_N(0)=1$, and given by

$$E_N(z) = a_N z^N + a_{N-1}z^{N-1} + ... + a_1 z + a_0.$$

Then, by definition, $\Theta_{N+}(z)$ and $\Theta_{N-}(z)$ must be polynomials, and hence

$$
\begin{aligned}
\Theta_{N-}(z) &= \Theta'_-(0)z\prod_{n=1}^{\hat{N}}\left(1-\frac{z}{t_n}\right), \\
\Theta_{N+}(z) &= \left(\sum_{n=0}^{\hat{N}}\frac{\mu_n}{t_n-z}+b+cz\right)\Theta_-(z),
\end{aligned}
$$

where $\hat{N} = N - 1$ if $c = 0$, or $\hat{N} = N - 2$ if $c > 0$. Then

$$E_N(z) = \Theta_{N+}(z) + i\Theta_{N-}(z)$$

has the same form in (8.4), and we the same argoments above we conclude that for $E_N(z)$ we have

$$\begin{aligned} \Im(a_N) &\neq 0, \\ \Im(\overline{a}_N a_{N-1}) &> 0. \end{aligned} \tag{8.9}$$

Theorem 8.3. *Let $E(z)$ be a polynomial Hermite Biehler function of degree $N \geq 1$ having no real zeros and such that $E(0) = 1$. Consider the sequences of vectors $\{\Theta_n(z)\}_{n=0,...N}$ and $\{S_n\}_{n=1,...N}$ and function $E_n(z)$ given by*

$$\begin{aligned} \Theta_0(z) &= \begin{bmatrix} \Theta_{0+}(z) \\ \Theta_{0-}(z) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} E(z) + E^{\#}(z) \\ \frac{1}{i}\left(E(z) - E^{\#}(z)\right) \end{bmatrix}, \\ \Theta_n(z) &= \begin{bmatrix} \Theta_{n+}(z) \\ \Theta_{n-}(z) \end{bmatrix} = (I + zS_n)\Theta_{n-1}(z) \qquad (n \geq 1) \\ S_n &= -\frac{1}{\Im(\overline{\alpha}_n \beta_n)} \begin{bmatrix} -\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2 \\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n) \end{bmatrix}, \\ E_n(z) &= \Theta_{n+}(z) + i\Theta_{n-}(z). \end{aligned} \tag{8.10}$$

where α_n and β_n are the coefficients of z^{N-n+1} and z^{N-n} in the $(N-n+1)$-degree polynomial $E_{n-1}(z)$.

Then

$$\Theta(z) = M_1(z)M_2(z)\dots M_N(z) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where

$$\begin{aligned} M_n(z) &= I - zS_n \\ &= I + \frac{z}{\Im(\overline{\alpha}_n \beta_n)} \begin{bmatrix} -\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2 \\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n) \end{bmatrix} \qquad (n = 1, ...N). \end{aligned}$$

Proof. Let

$$E(z) = a_N z^N + a_{N-1} z^{N-1} + ... + a_1 z + a_0,$$

be the polynomial representation of $E(z)$.

We define

$$\begin{aligned} E_\Phi(z) =& (b_1 z + b_0)E(z) + (c_1 z + c_0)E^{\#}(z) \\ =& (b_1 a_N + c_1 \overline{a}_N) z^{N+1} + (b_1 a_{N-1} + c_1 \overline{a}_{N-1} + b_0 a_N + c_0 \overline{a}_N) z^N \\ &+ P_{N-1}(z), \end{aligned}$$

where $b_1, b_0, c_1, c_0 \in \mathbb{C}$ and $P_{N-1}(z)$ is a polynomial of degree $N-1$. We want to choose b_0, b_1, c_0, c_1 so that $E_\Phi(z) = P_{N-1}(z)$ is a polynomial of degree $N-1$

and that $E_\Phi(0) = 1$. Hence b_0, b_1, c_0, c_1 must satisfy the following system of equations:

$$\begin{aligned} b_1 a_N + c_1 \overline{a}_N &= 0, \\ b_1 a_{N-1} + c_1 \overline{a}_{N-1} + b_0 a_N + c_0 \overline{a}_N &= 0, \\ b_0 + c_0 &= 1. \end{aligned} \tag{8.11}$$

Now we define

$$\begin{aligned} \Phi(z) &= \frac{1}{2} \begin{bmatrix} E_\Phi(z) + E^\#_\Phi(z) \\ \frac{1}{i}(E_\Phi(z) - E^\#_\Phi(z)) \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} k_{11}(z) & k_{12}(z) \\ k_{21}(z) & k_{22}(z) \end{bmatrix} \begin{bmatrix} E(z) + E^\#(z) \\ \frac{1}{i}(E(z) - E^\#(z)) \end{bmatrix} \\ &= K(z)\Theta(z), \end{aligned} \tag{8.12}$$

where

$$\begin{aligned} k_{11}(z) - ik_{12}(z) &= (b_1 + \overline{c}_1)z + (b_0 + \overline{c}_0), \\ k_{11}(z) + ik_{12}(z) &= (\overline{b}_1 + c_1)z + (\overline{b}_0 + c_0), \\ k_{21}(z) - ik_{22}(z) &= -i(b_1 - \overline{c}_1)z - i(b_0 - \overline{c}_0), \\ k_{21}(z) + ik_{22}(z) &= i(\overline{b}_1 - c_1)z + i(\overline{b}_0 - c_0), \end{aligned}$$

and then

$$\begin{aligned} k_{11}(z) &= \Re(b_1 + c_1)z + \Re(b_0 + c_0), \\ k_{12}(z) &= -\Im(b_1 - c_1)z - \Im(b_0 - c_0), \\ k_{21}(z) &= \Im(b_1 + c_1)z + \Im(b_0 + c_0), \\ k_{22}(z) &= \Re(b_1 - c_1)z + \Re(b_0 - c_0). \end{aligned}$$

Hence

$$\begin{aligned} K(z) &= \begin{bmatrix} \Re(b_0 + c_0) & -\Im(b_0 - c_0) \\ \Im(b_0 + c_0) & \Re(b_0 - c_0) \end{bmatrix} + z \begin{bmatrix} \Re(b_1 + c_1) & -\Im(b_1 - c_1) \\ \Im(b_1 + c_1) & \Re(b_1 - c_1) \end{bmatrix} \\ &= \Lambda_0 + z\Lambda_1. \end{aligned}$$

We observe

$$\begin{aligned} \det(\Lambda_0) &= |b_0|^2 - |c_0|^2, \\ \det(\Lambda_1) &= |b_1|^2 - |c_1|^2. \end{aligned}$$

Then Λ_0 is invertible if $|b_0| \neq |c_0|$. In this condition we have

$$\Lambda_0^{-1} = \frac{1}{|b_0|^2 - |c_0|^2} \begin{bmatrix} \Re(b_0 - c_0) & \Im(b_0 - c_0) \\ -\Im(b_0 + c_0) & \Re(b_0 + c_0) \end{bmatrix}$$

and

$$S = \Lambda_0^{-1}\Lambda_1 = \frac{1}{|b_0|^2 - |c_0|^2} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \tag{8.13}$$

where

$$\begin{aligned}
\gamma_{11} &= Re(b_0 - c_0)\Re(b_1 + c_1) + \Im(b_0 - c_0)\Im(b_1 + c_1),\\
\gamma_{12} &= -\Re(b_0 - c_0)\Im(b_1 - c_1) + \Im(b_0 - c_0)\Re(b_1 - c_1),\\
\gamma_{21} &= -\Im(b_0 + c_0)\Re(b_1 + c_1) + \Re(b_0 + c_0)\Im(b_1 + c_1),\\
\gamma_{22} &= \Im(b_0 + c_0)\Im(b_1 - c_1) + \Re(b_0 + c_0)\Re(b_1 - c_1).
\end{aligned}$$

Now we want to consider all the solutions of the system (8.11) for which Λ_0 is invertible. Observe that since $a_N \neq 0$, if $b_1 = 0$ or $c_1 = 0$ then $b_1 = c_1 = 0$ thanks to first equation of (8.11), and if $b_1 = c_1 = 0$ we obtain $|b_0| = |c_0|$ thanks to second equation of (8.11), and hence Λ_0 is not invertible. Then we need $b_1, c_1 \neq 0$.

We set $b_1 = \frac{h\overline{a}_N}{2}$, where $h = u + iv \in \mathbb{C} \setminus \{0\}$ (with $u, v \in \mathbb{R}$) is a parameter. Obviously as h varies on $\mathbb{C} \setminus \{0\}$, b_1 can take every value on $\mathbb{C} \setminus \{0\}$. Then solving the system (8.11) we obtain that all its solutions for which $b_1, c_1 \neq 0$ can be written as:

$$\begin{aligned}
b_1 &= \frac{h\overline{a}_N}{2},\\
c_1 &= -\frac{ha_N}{2},\\
b_0 &= \frac{-h\Im(\overline{a}_N a_{N-1}) + i\overline{a}_N}{2\Im(a_N)},\\
c_0 &= \frac{h\Im(\overline{a}_N a_{N-1}) - ia_N}{2\Im(a_N)},
\end{aligned} \tag{8.14}$$

as h varies on $\mathbb{C} \setminus \{0\}$.

We observe that with this choice, for b_1 and c_1 we have

$$\begin{aligned}
\Re(b_1 + c_1) &= v\Im(a_N),\\
\Im(b_1 + c_1) &= -u\Im(a_N),\\
\Re(b_1 - c_1) &= u\Re(a_N),\\
\Im(b_1 - c_1) &= v\Re(a_N),
\end{aligned}$$

while for b_0 and c_0 we obtain

$$\begin{aligned}
\Re(b_0 + c_0) &= 1,\\
\Im(b_0 + c_0) &= 0,\\
\Re(b_0 - c_0) &= -\frac{u\Im(\overline{a}_N a_{N-1})}{\Im(a_N)}\\
\Im(b_0 - c_0) &= \frac{\Re(a_N)}{\Im(a_N)} - \frac{v\Im(\overline{a}_N a_{N-1})}{\Im(a_N)},\\
|b_0|^2 - |c_0|^2 &= -\frac{u\Im(\overline{a}_N a_{N-1})}{\Im(a_N)}.
\end{aligned}$$

A straightforward calculation gives

$$S = \frac{1}{|b_0|^2 - |c_0|^2} \begin{bmatrix} -u\Re(a_N) & \frac{u\Re(a_N)^2}{\Im(a_N)} \\ -u\Im(a_N) & u\Re(a_N) \end{bmatrix},$$
$$= \frac{1}{\Im(\overline{a}_N a_{N-1})} \begin{bmatrix} \Im(a_N)\Re(a_N) & -\Re(a_N)^2 \\ \Im(a_N)^2 & -\Im(a_N)\Re(a_N) \end{bmatrix},$$

with

$$\det(S) = 0,$$
$$\mathrm{tr}(S) = 0.$$

It is very important to observe that S does not depend on h. Moreover we recall that by (8.9) we have $\Im(\overline{a}_N a_{N-1}) > 0$, and then we obtain that the matrix S has the form

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} \sigma & \sigma_1 \\ \sigma_2 & -\sigma \end{pmatrix}, \text{ with } \sigma^2 + \sigma_1\sigma_2 = 0, s_{21} \geq 0, s_{12} \leq 0.$$

We define $\Theta_1(z)$ as

$$\Theta_1(z) := \Lambda_0^{-1}\Phi(z) = (I + zS)\Theta(z). \tag{8.15}$$

We observe that

$$S^2 = 0,$$

then $(I + zS)(I - zS) = I$, and

$$I - zS = (I + zS)^{-1}.$$

Then

$$\Theta(z) = (I - zS)\Theta_1(z). \tag{8.16}$$

We observe that

$$E_1(z) = \Theta_{1+}(z) + i\Theta_{1-}(z)$$

is a polynomial of degree $N - 1$. Moreover we have $E_1(0) = 1$ since $\Theta_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. It is important to observe that $\Theta(z)$ and $E_1(z)$ do not depend on the parameter h.

Now we show that $E_1(z)$ is also a Hermite Biehler function without real zeros on the real line. Thanks to *Paragraph* 4.2 in [46] (p. 17, 18), we know that there exists $\hat{E}(z)$ such that:

1. $\hat{E}(z)$ is a Hermite Biehler polynomial of degree $N - 1$;

2. $\hat{E}(z) = (\hat{b}_1 z + \hat{b}_0)E(z) + (\hat{c}_1 z + \hat{c}_0)E^{\#}(z)$ for some $\hat{b}_1, \hat{b}_0, \hat{c}_1, \hat{c}_0 \in \mathbb{C}$, with $\hat{b}_1, \hat{c}_1 \neq 0$;

3. $\hat{E}(0) = 1$;

4. $\hat{E}(t) \neq 0\ \forall t \in \mathbb{R}$.

In particular $\hat{E}(z)$ is a polynomial of degree $N-1$ with $\hat{E}(0) = 1$, hence $\hat{b}_1, \hat{b}_0, \hat{c}_1, \hat{c}_0$ must satisfy system (8.11), and hence by (8.14) we have

$$\begin{aligned}
\hat{b}_1 &= \frac{\hat{h}\overline{a}_N}{2}, \\
\hat{c}_1 &= -\frac{\hat{h}a_N}{2}, \\
\hat{b}_0 &= \frac{-\hat{h}\Im(\overline{a}_N a_{N-1}) + i\overline{a}_N}{2\Im(a_N)}, \\
\hat{c}_0 &= \frac{\hat{h}\Im(\overline{a}_N a_{N-1}) - ia_N}{2\Im(a_N)},
\end{aligned}$$

for some $\hat{h} \in \mathbb{C} \setminus 0$. If we set $h = \hat{h}$ in (8.14) we get $E_\Phi(z) = \hat{E}(z)$, then recalling (8.12), (8.13), (8.15) and (8.16) we can apply *Lemma* 9 in [46] (p. 19), obtaining that $E_1(z) = \Theta_{1+}(z) + i\Theta_{1-}(z)$ is a polynomial Hermite Biehler function without real zeros on the real line.

We set

$$M(z) = I - zS = I + \frac{z}{\Im(\overline{a}_N a_{N-1})}\begin{bmatrix} -\Re(a_N)\Im(a_N) & \Re(a_N)^2 \\ -\Im(a_N)^2 & \Re(a_N)\Im(a_N) \end{bmatrix}, \quad (8.17)$$

so that

$$\Theta(z) = M(z)\Theta_1(z).$$

Now, setting for convenience $\Theta_0(z) = \Theta(z)$, we repeat the same approach iterating from $n = 1$ to $n = N-1$, calculating $\Theta_n(z)$ according to $\Theta_{n-1}(z)$. We can do this because, for each iteration n, the obtained polynomial $E_n(z)$ is a polynomial Hermite Biehler function having no real zeros and such that $E(0) = 1$, like $E(z)$. Let α_n and β_n be the coefficients of z^{N-n+1} and z^{N-n} in the $(N-n+1)$-degree polynomial $E_{n-1}(z)$, and let $h_n = u_n + iv_n$ be the value of the parameter h. Then proceeding like above and we obtain

$$\begin{aligned}
S_n &= -\frac{1}{\Im(\overline{\alpha}_n\beta_n)}\begin{bmatrix} -\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2 \\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n) \end{bmatrix}, \\
\Theta_n(z) &= (I + zS_n)\Theta_{n-1}(z), \\
E_n(z) &= \Theta_{n+}(z) + i\Theta_{n-}(z), \\
M_n(z) &= I - zS_n = I + z\frac{1}{\Im(\overline{\alpha}_n\beta_n)}\begin{bmatrix} -\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2 \\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n) \end{bmatrix}, \\
\Theta_{n-1}(z) &= M_n(z)\Theta_n(z).
\end{aligned}$$

For every iteration n, $\Theta_{n,+}(z)$, $\Theta_{n,-}(z)$ are polynomials of degree $(N-n)$ such that $\Theta_n(0) = \begin{bmatrix}1\\0\end{bmatrix}$. Moreover $E_n(z)$ is a Hermite Biehler polynomial such that $E_n(0) = 1$, hence it verifies (8.9). In the last step ($n = N$) we obtain $\Theta_N(z) = \begin{bmatrix}1\\0\end{bmatrix}$ since $\Theta(0) = \begin{bmatrix}1\\0\end{bmatrix}$ and $\Theta_N(z)$ must be a polynomial of degree 0, hence a constant. Thanks to this, at the end of the iterations we get

$$\Theta(z) = M_1(z)\dots M_N(z)\begin{bmatrix}1\\0\end{bmatrix}.$$

□

Corollary 8.4. *In the conditions of Theorem 8.3, consider the representation of $E_{n-1}(z)$ given by* (8.4)*:*

$$E_{n-1}(z) = \left(\sum_{k=0}^{\hat{N}} \frac{\mu_{k,n}}{t_k - z} + b_n + c_n z\right)\Theta_{(n-1)-}(z) + i\Theta_{(n-1)-}(z). \tag{8.18}$$

Then, if $c_n = 0$

$$M_n(z) = I + \frac{z}{\sum_{k=0}^{N-1}\mu_{k,n}}\begin{bmatrix}-b_n & b_n^2\\ -1 & b_n\end{bmatrix},$$

while if $c_n \neq 0$

$$M_n(z) = I + z\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}.$$

Proof. By *Theorem* 8.3, for $n = 1, \dots N$, we have

$$\begin{aligned} M_n(z) &= I - zS_n \\ &= I + \frac{z}{\Im(\overline{\alpha}_n\beta_n)}\begin{bmatrix}-\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2\\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n)\end{bmatrix}. \end{aligned} \tag{8.19}$$

If $c = 0$ in (8.18), thanks to (8.5) and (8.6) we have $\Im(\overline{\alpha}_n\beta_n) = \frac{|\alpha_n|^2\sum_{k=0}^{N-1}\mu_{k,n}}{b_n^2+1}$, and we obtain

$$\begin{aligned} M_n(z) &= I + z\frac{b_n^2+1}{|\alpha_n|^2\sum_{n=0}^{N-1}\mu_n}\begin{bmatrix}-\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2\\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n)\end{bmatrix} \\ &= I + z\frac{b_n^2+1}{\sum_{n=0}^{N-1}\mu_n}\begin{bmatrix}-\frac{b_n}{b_n^2+1} & \frac{b_n^2}{b_n^2+1}\\ -\frac{1}{b_n^2+1} & \frac{b_n}{b_n^2+1}\end{bmatrix} \\ &= I + \frac{z}{\sum_{n=0}^{N-1}\mu_n}\begin{bmatrix}-b_n & b_n^2\\ -1 & b_n\end{bmatrix}. \end{aligned}$$

If $c \neq 0$, thanks to (8.7) and (8.8) we have $\alpha_n \in \mathbb{R}$ and $\Im(\overline{\alpha}_n \beta_n) = \alpha_n^2$, and then we obtain

$$\begin{aligned} M_n(z) &= I + z\frac{1}{\alpha_n^2}\begin{bmatrix} -\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2 \\ -\Im(\alpha_n)^2 & \Re(\alpha_n)\Im(\alpha_n) \end{bmatrix} \\ &= I + z\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

□

Corollary 8.5. *In the conditions of Theorem 8.3, for* $n = 1, ..., N$, *let*

$$\begin{aligned} H_n(x) &= \frac{1}{|\alpha_n|^2}\begin{bmatrix} \Im(\alpha_n)^2 & -\Re(\alpha_n)\Im(\alpha_n) \\ -\Re(\alpha_n)\Im(\alpha_n) & \Re(\alpha_n)^2 \end{bmatrix}, \\ x_n &= -\frac{|\alpha_n|^2}{\Im(\overline{\alpha}_n\beta_n)} + x_{n-1}, \\ x_0 &= 0, \end{aligned}$$

and

$$\begin{aligned} H(x) &= H_n(x) \quad x_{n-1} \leq x < x_n, \\ L &= x_N. \end{aligned}$$

Then the canonical system (H, L) *is such that* $\Theta(z) = \Theta_L(z)$.

Proof. It is a straightforward consequence of the application of the *chain rule* to the result of *Theorem* 8.1. □

Theorem 8.6. *Let* $E(z)$ *be a Hermite Biehler function having no real zeros and such that* $E(0) = 1$. *Let* $\Theta(z)$ *be defined by*

$$\Theta(z) = \begin{bmatrix} \Theta_+(z) \\ \Theta_-(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} E(z) + E^\#(z) \\ \frac{1}{i}\left(E(z) - E^\#(z)\right) \end{bmatrix}. \tag{8.20}$$

Let $t_j, j \geq 0, t_0 = 0$, *be the set of zeroes of* $\Theta_-(z)$ *ordered by* $|t_j| \leq |t_{j+1}|$,

$$\begin{aligned} \Theta_{N-}(z) &= \dot{\Theta}_-(0)z\prod_{j=0}^{N-1}\left(1 - \frac{z}{t_j}\right) \\ \Theta_{N+}(z) &= \left(\sum_{j=0}^{N}\frac{\Theta_+(t_j)}{\dot{\Theta}_-(t_j)}\frac{1}{z - t_j} + a + bz\right)\Theta_{N-}(z), \end{aligned}$$

a and b being the constants in the linear term in the Nevanlinna representation of the Herglotz function $\frac{\Theta_+}{\Theta_-}$. *Then* $E_N(z) = \Theta_{N+}(z) + i\Theta_{N-}(z)$ *is a Hermite Biehler polynomial function of degree* $N \geq 1$ *having no real zeros* *and such that $E(0) = 1$. Let (H_N, L_N) be the corresponding canonical system constructed in Corollary 8.5, iterating on $n = 1, ..., N$:*

$$
\begin{aligned}
H_{N,n}(x) &= \frac{1}{|\alpha_{N,n}|^2} \begin{bmatrix} \Im(\alpha_{N,n})^2 & -\Re(\alpha_{N,n})\Im(\alpha_{N,n}) \\ -\Re(\alpha_{N,n})\Im(\alpha_{N,n}) & \Re(\alpha_{N,n})^2 \end{bmatrix}, \\
x_{N,n} &= -\frac{|\alpha_{N,n}|^2}{\Im(\overline{\alpha}_{N,n}\beta_{N,n})} + x_{N,n-1}, \\
x_{N,0} &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
H_N(x) &= H_{N,n}(x) \quad x_{N,n-1} \le x < x_{N,n}. \\
L_N &= x_{N,N}.
\end{aligned}
$$

Let

$$
L = \frac{1}{\pi} \left\| \frac{\Theta_+ - 1}{z} \right\|^2_{\mathcal{H}(E)} - \dot{\Theta}_-(0)
$$

Define

$$
\widetilde{\mathrm{H}}_N(x) = \begin{cases} \mathrm{H}_N\left(x - \max\{0, L - L_N\}\right), & x \ge \max\{0, L - L_N\} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & 0 \le x \le \max\{0, L - L_N\} \end{cases}
$$

$$
F_N(x) = \int_{l_N - h}^{l_N - h + x} \widetilde{\mathrm{H}}_N(s)\mathrm{d}s, l_N = \max\{L, L_N\}
$$

Then, as $N \to +\infty$, F_N converges in $C(0, L)$ to a monotone non-decreasing function, F. The canonical system $(\Theta, L) := (F', L)$ is such such that $\Theta = \Theta_L$ and that there is no $\varepsilon > 0$ such that $\mathrm{H}(x) = \langle \cdot, e \rangle e, e = (0, 1)^T$, for a.e. $x \in (0, \varepsilon)$.

Proof. The result is a simple consequence of *Corollary* 8.5 applied to the algorithm for solving the inverse problem in the regular case, described in *Theorem* 8.2. □

CHAPTER 9

Conclusions and future works

In the first part of this work we have shown that the functions of the Paley-Wiener spaces can be rebuilt exactly from many different families of non-uniform sampling sequences, with various types of constraints, very different from those already known. Furthermore, we have shown that some of these families of non-uniform sampling sequences can be very useful for real applications, since they allow to perfectly reconstruct a function with any given precision from any finite and large enough set of samples. However some details regarding these sampling sequences are not yet fully understood. Probably the most important yet unanswered questions concern the bandlimit pairs that satisfy property 5 in *Theorem* 4.14.

- Is it possible to obatain an explicit necessary and sufficient condition to know if a bandlimit pair satisfies this property?
- Given a sequence $\{t_n\}_n$, is it always possible to find a sequence $\{t'_n\}_n$ such that $(\{t_n\}_n, \{t'_n\}_n)$ is a bandlimit pair that verifies this property?
- Given a finite subsequence, is it always possible to build a bandlimit pair such that $\{t_n\}_n$ contains this subsequence and verifies this property?

Other interesting questions about the described sampling sequences are the following.

- Does exist a sequence $\{t_n\}_n$ that verifies the condition (4.37) in *Theorem* 4.18 and such that $\left|\frac{\pi}{a}n - t_n\right|$ doesn't necessarily have a limit as n goes to $\pm\infty$?

- Is it possible to give an alternative proof of the Paley-Wiener-Levinson theorem deriving it from 3.3, similarly to what we did for *Theorem* 4.18? In other words, is the Paley-Wiener-Levinson theorem a particular case of a more general theorem?

The answers to these questions would make the sampling theorems described in this work even more useful in real applications since they would allow to more easily derive the sequences that can be used to reconstruct a bandlimited function.

In the second part we have investigated the isomorphism between the Kempf-Martin spaces and the de Branges spaces and its consequences, as for example a necessary and sufficient condition for a function to belong the a Kempf-Martin space. Moreover we have also shown that all the results about the Kempf-Martin spaces can be obtained without the use of the theory of simple symmetric operators. An already known necessary and sufficient condition to establish if a function belongs to a de Branges spaces is based on the Weyl-Titchmarsh transform, but it is valid only for the de Branges spaces $\mathcal{B}(E)$ for which $\Theta(z) = \frac{E^{\#}(z)}{E(z)}$ is a Weyl-Titchmarsh meromorphic inner fucntion. A Weyl-Titchmarsh meromorphic inner function is defined as follows. Let $q(x)$ be a real locally integrable function on (a, b), and fix a selfadjoint boundary condition β at b. The Weyl-Titchmarsh m-function of $(q; b, \beta)$, evaluated at a, is defined by the formula

$$m(z) = \frac{u_z'(a)}{u_z(a)}, \quad z \in \mathbb{C},$$

where $u_z(x)$ is a non-trivial solution of the Schrodinger equation

$$-u_z''(x) + q(x)u_z(x) = zu_z(x), \qquad x \in (a, b),$$

satisfying the boundary condition β. It is well-known that $m(z)$ is a Herglotz function, and therefore we can define the corresponding meromorphic inner function $\Theta^a_{b,\beta}(z)$ according to (2.6), given by

$$\Theta^a_{b,\beta}(z) = \frac{m(z) - i}{m(z) + i}.$$

We call $\Theta^a_{b,\beta}(z)$ the Weyl-Titchmarsh inner function of $q(x)$. The problem with the Weyl-Titchmarsh inner functions is that, given a real locally integrable function $q(x)$, it is possible to build the corresponding Weyl-Titchmarsh inner function, but given a meromorphic inner function it is not yet known a method to establish if it is a Weyl-Titchmarsh inner function or not, and to eventually build the corresponding function $q(x)$. To solve this problem, it would be interesting to investigate in depth what relationship exists between the Weyl-Titchmarsh transform and the generalization of the Fourier Tranfsorm introduced in this work, in order to eventually exploit the isomorphism induced by this transform between a de Branges space and the corresponding space L_Θ.

In the thid part we have explained the concept of time-varying bandlimit for the Kempf-Martin spaces. Then we have introduced a new family of spaces of time-varying bandlimit functions, referred as spaces $\mathcal{V}(\Theta)$, which are compatible with an improved definition of the concept of time-varying bandlimit. At the end, we have presented a generalization of the Shannon sampling method for time-varying bandlimit functions. The sampling formulas presented for the spaces $\mathcal{V}(\Theta)$ and the derived generalized sampling method are based on the sampling sequences $\{t_n(\theta)\}_n$ of solutions of $\Theta(t) = e^{2\pi i\theta}$ for $t \in \mathbb{R}$. The same is true also for the Paley-Wiener spaces, indeed we know that for any $a > 0$ the space $\mathcal{PW}_a$ is associated to the meromorphic inner function $\Theta(z) = e^{2\pi iz}$ and the corresponding sampling sequences are given by $\{t_n(\theta)\}_n$ with $t_n = \frac{\pi}{a}(n+\theta)$. From the theorems of the first part of this work we know that the functions of the Paley-Wiener spaces can be rebuilt also from many other sequences, that satisfy different constraints, all derived from the theorems presented in Chapter 3. It would be very interesting use the same theorems to derive similar results for the spaces $\mathcal{V}(\Theta)$. In this way, the generalized sampling method would become much more flexible, since it would allow to reconstruct the functions of the spaces $\mathcal{V}(\Theta)$ not only starting from the sampling sequences defined by the function $\Theta(z)$, but also from their perturbations which satisfy some given constraints.

In the fourth part we have improved the algorithm to solve the canonical inverse problem, presenting an explicit formula for the solution of every iteration. A very important aspect of the inverse canonical problem is its connection with the Weyl-Titchmarsh's inner functions, which we briefly summarize here following the arguments proposed in [4]. Let $q(x)$ be a locally summable function on $(0, L)$, and consider the Schrodinger equation

$$-y_z''(x) + q(x)y_z(x) = zy_z(x). \tag{9.1}$$

Suppose $u_z(x)$ and $v_z(x)$ are the linearly independent solutions of this equation, satisfying some boundary condition α at 0. Then $u_0(x)$ and $v_0(x)$ are the solutions of $-y_z''(x) + q(x)y_z(x) = 0$. Let

$$H(x) = \begin{pmatrix} u_0^2(x) & u_0v_0(x) \\ u_0v_0(x) & v_0^2(x) \end{pmatrix}. \tag{9.2}$$

Then the Schrodinger equation (9.1) is equivalent to the canonical system

$$JY_z'(x) = zH(x)Y_z(x). \tag{9.3}$$

Indeed, if $y_z(x)$ solves Schrodinger equation (9.1) then

$$Y_z(x) = \begin{bmatrix} Y_z^+(x) \\ Y_z^-(x) \end{bmatrix} := \begin{bmatrix} u_0(x) & v_0(x) \\ u_0'(x) & v_0'(x) \end{bmatrix}^{-1} \begin{bmatrix} y_z(x) \\ y_z'(x) \end{bmatrix}$$

solves the canonical system (9.3). Moreover, a fundamental detail is that the Hermite Biehler function $E(z) = Y_z^+(L) + iY_z^-(L)$ results to be a de Branges function of the Weyl-Titchmarsh inner function of $q(x)$. Hence, given any meromorphic inner function $\Theta(z)$, solving the problem of finding the function $q(x)$ associated to $\Theta(z)$ would give also the solution of the canonical inverse problem for the Hamiltonians with the form given in (9.2). For this reason investigating the relationship between the Weyl-Titchmarsh transform and the generalization of the Fourier transform presented in this paper assumes an even greater importance than that already described above.

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