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SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

Normalized solutions for the fractional nonlinear Schrödinger equation

Tesi di Laurea Magistrale in
Mathematical Engineering - Ingegneria Matematica

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Academic Year: 2021-22

Abstract

Starting from the seminal contribution given by JeanJean in [18] in the local context, we prove existence of ground states for the nonlinear eigenvalue system

$$(-\Delta)^k u - g(u) = \lambda u \quad \text{in } \mathbb{R}^N,\ N \geq 2, k \in (0,1),\ \lambda \in \mathbb{R},$$

having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 = c^2.$$

In the equation, $(-\Delta)^k$ is the fractional Laplacian in $\mathbb{R}^N$ of order $k \in (0,1)$, which is the prototype of the integrodifferential operators of order $2k$ and is a natural nonlocal analogue of the standard Laplacian.

This work handles L^2-*supercritical* nonlinearities for g and looks for critical points for the *energy functional* constrained to the L^2 sphere

$$S_c := \left\{ u \in H^k(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^2 = c^2 \right\},$$

relying on a version of the min-max method valid on curved geometries. As far as we know, the strategy we adopt was never explored in a nonlocal context.

Keywords: fractional Laplacian, fractional Sobolev Spaces, ground state, L^2-*supercritical*, min-max approach, nonlinear eigenvalue system, Palais-Smale sequence, prescribed mass.

Abstract in lingua italiana

A partire dal contributo fornito da JeanJean in [18] in contesto locale, dimostriamo l'esistenza di soluzioni ad energia minima per il sistema agli autovalori non lineare

$$(-\Delta)^k u - g(u) = \lambda u \quad \text{in } \mathbb{R}^N,\ N \geq 2, k \in (0,1),\ \lambda \in \mathbb{R},$$

con massa fissata

$$\int_{\mathbb{R}^N} |u|^2 = c^2.$$

Nell'equazione, $(-\Delta)^k$ rappresenta il Laplaciano frazionario in $\mathbb{R}^N$ di ordine $k \in (0,1)$, che è il prototipo degli operatori integrodifferenziali di ordine $2k$ e costituisce l'estensione naturale in contesto nonlocale del Laplaciano standard.

In questo lavoro si affrontano nonlinearità L^2-*supercritiche* per g e si ricercano punti critici per il *funzionale energia* vincolato alla sfera L^2

$$S_c := \left\{ u \in H^k(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^2 = c^2 \right\},$$

basandosi su una versione del metodo min-max valida su geometrie curve. Per quanto ne sappiamo, la strategia che proponiamo non è mai stata adottata in contesto non locale.

Parole chiave: energia minima, Laplaciano frazionario, L^2-*supercritico*, massa fissata, metodo di min-max, sequenza di Palis-Smale, sistema agli autovalori, spazi di Sobolev frazionari.

Contents

Symbol and formalism

Throughout this paragraph we are going to fix some notations and formalisms.

If N denotes the space dimension, we can set the multi-index $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + ... + \alpha_N$ and refer to the usual notations for derivatives of order α

$$\partial^\alpha u = \frac{\partial^{|\alpha|} u}{(\partial x_1)^{\alpha_1} ... (\partial x_N)^{\alpha_N}}.$$

Then, we can create the following countable family of semi-norms

$$p_n(u) = \sup_{|\alpha| \le n} \sup_{x \in \mathbb{R}^N} (1 + |x|^2)^{\frac{n}{2}} |\partial^\alpha u(x)|, \quad n \in \mathbb{N} \cup \{0\}. \tag{1}$$

According to these notations, we define formally the space $\mathcal{S}(\mathbb{R}^N)$, namely the space of rapidly decaying function as

$$\mathcal{S}(\mathbb{R}^N) := \left\{ u \in C^\infty(\mathbb{R}^N) : \ p_n(u) < \infty \ \forall n \in \mathbb{N} \cup \{0\} \right\}.$$

Moreover, denoting by $\text{supp}(u)$ the support of u, we will also use the following space:

$$D(\mathbb{R}^N) := \left\{ u \in C^\infty(\mathbb{R}^N), \ \text{supp}(u) \text{ is compact} \right\}.$$

We will use $\mathcal{F}u$ and $\mathcal{F}^{-1}u$ to denote, respectively, the Fourier transform of u and its inverse.

$$\mathcal{F}u(\xi) := \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi} \, dx$$
$$\mathcal{F}^{-1}u(x) := \int_{\mathbb{R}^N} \mathcal{F}u(\xi) \, e^{ix\cdot\xi} \, d\xi.$$

For what concerns norms, we use $|\cdot|$ to indicate the euclidean norm (in dimension one it reduces to the modulus), while in $L^p(\Omega)$ we write

$$\|u\|_p = \int_\Omega |u(x)|^p \, dx.$$

We will specify the domain through the notation $\|u\|_{L^p(\Omega)}$ just in case the domain of integration is not immediate.
In $L^2(\Omega)$, the scalar product will be denoted by

$$\langle u, v\rangle_2 := \int_\Omega u(x)v(x)\,dx.$$

Moreover, when dealing with the dual space of the fractional Sobolev space $H^k(\mathbb{R}^N)$, $k \in (0,1)$ we are going to denote the norm on this space by $\|\cdot\|_*$.

1 | The fractional Sobolev spaces and the fractional Laplacian

In this first part of the thesis, we are going to analyze some basic properties concerning the fractional Laplacian operator and the fractional Sobolev spaces. With regard to the fractional Laplacian we are going to offer two different definitions, the first via the Cauchy principal value, while the second via the Fourier transform, shedding light on its intrinsic connection with the fractional Sobolev spaces. Then, the main consequences of the nonlocal nature of this operator will be inspected, paying particular attention to the fractional heat equation (together with its probabilistic interpretation) and to the nonlocal formulation of the maximum principles and of the Harnack inequality. Finally, aiming at providing concreteness to the whole chapter, we conclude offering a practical example of k-harmonic function in one dimension, $k \in (0, 1)$.

1.1. Introduction

This paragraph is intended to introduce the reader to the fractional Sobolev spaces and to the fractional Laplacian operator. In particular, we are going to expose the principal analytical features of both these topics, offering formal and well-posed definitions as usually offered in literature. For a more in-depth discussion on this topic, we refer to [6], [7], [9], [13] and [16].

Let Ω be an open set in $\mathbb{R}^N$, k real in $(0, 1)$ and p integer in $[1, +\infty)$. We define the fractional Sobolev space $W^{k,p}(\Omega)$ as follows:

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+k}} \in L^p(\Omega \times \Omega) \right\}. \tag{1.1}$$

We endow this space with the following norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} \, dxdy \right)^{\frac{1}{p}},$$

where the term

$$\lfloor u \rfloor_{k,p} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} \, dxdy \right)^{\frac{1}{p}} \tag{1.2}$$

is the so-called Gagliardo seminorm of u.

We now state a proposition that sheds lights on a parallelism matching these spaces with the classical $W^{k,p}(\Omega)$, k being an integer.

Proposition 1.1. *Let $p \in [1, +\infty)$, $0 < k \leq k' < 1$, Ω open set in $\mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ a measurable function. Then $W^{k',p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$, namely*

$$\exists C > 0 : \ \|u\|_{W^{k,p}(\Omega)} \leq C \|u\|_{W^{k',p}(\Omega)},$$

where C depends on N, k and p.

Proof. We firstly notice that

$$|u(x) - u(y)|^p \leq (|u(x)| - |u(y)|)^p \leq (2 \ \max(|u(x)|, |u(y)|))^p \leq 2^p(|u(x)|^p + |u(y)|^p)$$

and that

$$\int_\Omega \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{|u(x)|^p}{|x-y|^{N+kp}} \, dxdy \leq \int_\Omega \left(\int_{\Omega \cap \{z \geq 1\}} \frac{1}{|z|^{N+kp}} \, dz \right) |u(x)|^p \, dx$$
$$\leq \int_\Omega \left(\int_{\{z \geq 1\}} \frac{1}{|z|^{N+kp}} \, dz \right) |u(x)|^p \, dx = C_{N,k,p} \|u\|_p^p,$$

where $z = x - y$ and we have used the integrability of $|z|^{-(N+kp)}$ for $|z| \geq 1$. Exploiting the two previous inequalities we can write

$$\int_\Omega \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} \, dxdy \leq C_{N,k,p} 2^p \|u\|_p^p, \tag{1.3}$$

while, on the other hand, it is trivial that $0 < k \leq k'$ implies

$$\int_\Omega \int_{\Omega \cap \{|x-y| \leq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+kp}} \, dxdy \leq \int_\Omega \int_{\Omega \cap \{|x-y| \leq 1\}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+k'p}} \, dxdy. \tag{1.4}$$

Combining finally (1.3) and (1.4) we come up with

$$\begin{aligned}\|u\|^p_{W^{k,p}(\Omega)} &\leq (C_{N,k,p}2^p+1)\|u\|_p^p + \int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{N+k'p}}\,dxdy \\ &\leq C_{N,k,p}\|u\|^p_{W_{k',p}(\Omega)}\end{aligned}$$

and the proof is complete. □

At this point, we would like to denote the spaces $W^{k,p}$ as intermediate spaces between L^p and $W^{1,p}$. In order to get these results we shall show $W^{1,p}$ as a continuous limit of $W^{k,p}$ as $k \uparrow 1$, namely we should show that, for any domain Ω and function f in $W^{1,p}(\Omega)$, we have

$$\lim_{k\uparrow 1} \lfloor f \rfloor^p_{k,p} = \lim_{k\uparrow 1} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+kp}}\,dxdy = C_{N,p}\int_\Omega |\nabla f|^p,$$

but this is not true. What we are going to prove, instead, is that

$$\lim_{k\uparrow 1} \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+kp}}\,dxdy = +\infty,$$

while

$$\lim_{k\uparrow 1}\,(1-k) \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{N+kp}}\,dxdy = C_{N,p}\int_\Omega |\nabla f|^p.$$

We start for simplicity with $\Omega = \mathbb{R}^N$ and $1 < p < \infty$. By [7, Proposition IX.3], we know that if $f \in W^{1,p}(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} |f(x+h)-f(x)|^p\,dx \leq |h|^p \int_{\mathbb{R}^N} |\nabla f|^p\,dx \tag{1.5}$$

for every h in $\mathbb{R}^N$. On the other hand, $f \in L^p(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |f(x+h)-f(x)|^p\,dx \leq C|h|^p \quad \text{as } h \to 0 \tag{1.6}$$

imply $f \in W^{1,p}(\mathbb{R}^N)$.

Now we consider a sequence of radial mollifiers $(\rho_\epsilon)_{\epsilon>0}$, namely

$$\rho_\epsilon \in L^1_{loc}(0,+\infty), \quad \rho_\epsilon \geq 0$$

$$\int_0^\infty \rho_\epsilon(r) r^{N-1}\, dr = 1 \qquad \forall \epsilon > 0$$

$$\lim_{\epsilon\to 0} \int_\delta^\infty \rho_\epsilon(r) r^{N-1}\, dr = 0 \qquad \forall \delta > 0.$$

Exploiting these hypotheses on (ρ_ϵ), we can write for any $f \in W^{1,p}(\mathbb{R}^N)$

$$\iint_{\mathbb{R}^{2N}} \frac{|f(x+h)-f(x)|^p}{|h|^p} \rho_\epsilon(|h|)\, dhdx \leq \int_{\mathbb{R}^N} |\nabla f(x)|^p\, dx \int_{\mathbb{R}^N} \rho_\epsilon(|h|)\, dh \leq C \tag{1.7}$$

as $\epsilon \to 0$, where we have used (1.5) and

$$\int_{\mathbb{R}^N} \rho_\epsilon(|h|)\, dh \leq |\partial B_1| \int_0^\infty \rho_\epsilon(r) r^{N-1}\, dr = |\partial B_1|,$$

with $|\partial B_1|$ denoting the measure of the surface of the ball of radius 1 in dimension N. By a change of variables in (1.7), we get

$$\iint_{\mathbb{R}^{2N}} \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\epsilon(|x-y|)\, dxdy \leq C \quad \text{as } \epsilon \to 0. \tag{1.8}$$

As a consequence, $f \in W^{1,p}(\mathbb{R}^N)$ implies equation (1.8). Nevertheless, the core point of the discussion is that (1.8) gives a complete characterization of the space $W^{1,p}(\mathbb{R}^N)$, which is sharper then (1.6), as the next theorem points out.

Theorem 1.1. *Assume $f \in L^p(\mathbb{R}^N)$ and that f satisfies (1.8) with $p > 1$. Let $(\rho_\epsilon)_\epsilon$ be the sequence of radial mollifiers as described above. Then $f \in W^{1,p}(\mathbb{R}^N)$ and*

$$\lim_{\epsilon\to 0} \iint_{\mathbb{R}^{2N}} \frac{|f(x)-f(y)|^p}{|x-y|^p} \rho_\epsilon(|x-y|)\, dxdy = C \int_{\mathbb{R}^N} |\nabla f|^p,$$

where C depends only on N and p.

Since the proof of this theorem is rather long and technical, we decided to show it in the Appendix.

At this point we decide to focus on the following choice for ρ_ϵ:

$$\rho_\epsilon(r) = \begin{cases} \frac{\epsilon}{r^{N-\epsilon}} & 0 < r < 1 \\ 0 & r > 1. \end{cases} \tag{1.9}$$

Thus, we have this immediate corollary.

Corollary 1.2. *Assume that $f \in L^p(\mathbb{R}^N)$ is such that*

$$\epsilon \iint_{\mathbb{R}^{2N}} \frac{|f(x) - f(y)|^p}{|x - y|^{N+p-\epsilon}} \, dxdy \leq C \quad \text{as } \epsilon \to 0,$$

then $f \in W^{1,p}(\mathbb{R}^N)$ and

$$\lim_{\epsilon \to 0} \epsilon \iint_{\mathbb{R}^{2N}} \frac{|f(x) - f(y)|^p}{|x - y|^{N+p-\epsilon}} \, dxdy = C \int_{\mathbb{R}^N} |\nabla f|^p.$$

Proof. The proof is trivial since it is an immediate application of Theorem 1.1 to $(\rho_\epsilon)_\epsilon$ as defined in (1.9). This choice for $(\rho_\epsilon)_\epsilon$ works since it is a sequence of radial mollifiers. □

Now, we set

$$\epsilon = p(1 - k) \implies \lim_{\epsilon \to 0} \epsilon = \lim_{k \uparrow 1} p(1 - k)$$

and finally obtain

$$\lim_{k \uparrow 1} (1 - k) \iint_{\mathbb{R}^{2N}} \frac{|f(x) - f(y)|^p}{|x - y|^{N+pk}} \, dxdy = \frac{C}{p} \int_{\mathbb{R}^N} |\nabla f|^p. \tag{1.10}$$

By equation (1.10) we have straightforwardly that the usual Gagliardo seminorm diverges for $k \uparrow 1$ and that, in order to recover some continuity between the spaces $W^{k,p}(\mathbb{R}^N)$, $1 < p < \infty$, and $W^{1,p}(\mathbb{R}^N)$ we should use the norm

$$\|f\|^p_{W^{k,p}(\mathbb{R}^N)} = \|f\|^p_p + (1 - k)\lfloor f \rfloor^p_{k,p}.$$

Finally, we should prove these same results for bounded domains and for $p = 1$. These proofs require some more technicalities and will not be covered here. For a complete presentation of the topic we refer to [9, Section 2-3].

It is worth noticing that, from now on, we will only focus on the fractional Sobolev spaces $W^{k,2}(\mathbb{R}^N)$ with $k \in (0, 1)$, that are usually denoted by $H^k(\mathbb{R}^N)$. Even the Gagliardo seminorm $\lfloor u \rfloor_{k,2}$ will be denoted, for the sake of simplicity, by $\lfloor u \rfloor_k$.

The spaces $H^k(\mathbb{R}^N)$ indeed, in addition to being Hilbert and not just Banach, are strictly related to the fractional Laplacian operator, whose definition constitutes the next step of this paragraph.
To define the fractional Laplacian, denoted by $(-\Delta)^k$, $k \in (0,1)$, we start considering a function $u \in \mathcal{S}(\mathbb{R}^N)$ and set

$$(-\Delta)^k u(x) := C_{N,k}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2k}}\, dy = C_{N,k} \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2k}}\, dy, \tag{1.11}$$

for any $k \in (0,1)$, where $B_\epsilon(x)$ denotes the the N-dimensional ball of radius ϵ centered in x. Here $P.V.$ denotes the Cauchy principal value, while $C_{N,k}$ a dimensional constant depending on N and k, defined by

$$C_{N,k} = \left(\int_{\mathbb{R}^N} \frac{1-\cos(x_1)}{|x|^{N+2k}}\, dx \right)^{-1}, \tag{1.12}$$

where $x_1 = x \cdot e_1$ and e_1 denotes the first direction in $\mathbb{R}^N$. The reason under the choice of this constant will be clear later (see Proposition 1.2).

We claim that, for $u \in \mathcal{S}(\mathbb{R}^N)$, this operator is defined pointwise in $\mathbb{R}^N$ and indeed, looking at equation (1.11), we notice that outside a neighbourhood of x the integral converges, since

$$\int_{|x-y|>1} \frac{u(x)-u(y)}{|x-y|^{N+2k}}\, dy \leq 2\|u\|_\infty \int_{|x-y|>1} \frac{1}{|x-y|^{N+2k}}\, dy < \infty$$

and $\|u\|_\infty$ is well defined thanks to $u \in \mathcal{S}(\mathbb{R}^N)$. As a consequence, we are only left to analyze a possible singularity centred near x, that would justify the use of the principal value. Studying the integral in a ball of radius 1, we obtain:

$$\int_{|x-y|\leq 1} \frac{u(x)-u(y)}{|x-y|^{N+2k}}\, dy \leq C \int_{|x-y|\leq 1} |x-y|^{1-N-2k}\, dy = C \int_{|y|\leq 1} |y|^{1-N-2k}\, dy = C \int_0^1 \rho^{-2k}\, d\rho.$$

Thanks to this inequality we can infer that, for $k \in (0, 1/2)$, Definition (1.11) is well posed. On the contrary, if we now exploit the Taylor expansion for u centred in x

$$u(x+y) - u(x) = \nabla u(x) y + o(|y|),$$

we immediately see that a singularity in x appears in integral (1.11) if $k \in [1/2, 1)$.

Thus, the next step consists in proving that the Cauchy principal value for the fractional Laplacian is well defined for this range of k. We start noticing that

$$-(-\Delta)^k u(x) = C_{N,k}\ P.V. \int_{\mathbb{R}^N} \frac{u(y)-u(x)}{|x-y|^{N+2k}}\, dy \tag{1.13}$$
$$= C_{N,k}\ P.V. \int_{\mathbb{R}^N} \frac{u(x+\tilde{y})-u(x)}{|\tilde{y}|^{N+2k}}\, d\tilde{y},$$

by the simple change of variable $\tilde{y} = y - x$. Then, to complete this step, we have to prove that the right hand side of (1.13) is well defined. In this aim, we shall pass through the fact that

$$P.V. \int_{|y|\leq 1} \frac{\nabla u(x) y}{|y|^{N+2k}}\, dy = 0, \tag{1.14}$$

which is true is true since, if we fix $0 < \epsilon < 1$, we retrieve that, being the integrand an odd function of y,

$$\int_{\epsilon\leq|y|\leq 1} \frac{\nabla u(x) y}{|y|^{N+2k}}\, dy = 0 \quad \forall \epsilon \implies (1.14).$$

Finally, the combination of (1.13) and (1.14) gives us

$$-(-\Delta)^k u(x) = C_{N,k}\ P.V. \int_{\mathbb{R}^N} \frac{u(x+y)-u(x)-\nabla u(x)y}{|y|^{N+2k}}\, dy.$$

Relying again on the Taylor expansion of u, we can write

$$u(x+y) = u(x) + \nabla u(x) y + \frac{1}{2} y^T H_u(x) y + o(|y|^2)$$
$$\leq u(x) + \nabla u(x) y + \|H_u(x)\|_\infty \frac{|y|^2}{2} + o(|y|^2),$$

where $H_u(x)$ denotes the Hessian matrix of u evaluated at point x and $\|H_u(x)\|_\infty$ is well defined thanks to the regularity hypotheses on u. It is now sufficient to insert this expansion inside the equation of $-(-\Delta)^k$ to obtain that the Cauchy principal value in (1.11) is well defined, since

$$\frac{u(x+y)-u(x)-\nabla u(x)y}{|y|^{N+2k}} \leq \frac{\|H_u(x)\|_\infty}{2} \frac{|y|^2}{|y|^{N+2k}} + \frac{o(|y|^2)}{|y|^{N+2k}},$$

which is clearly integrable in a neighbourhood of 0.

We now offer an equivalent definition of the fractional Laplacian operator, which will be often used in the next chapters due to the absence of the principal value that makes it more manageable.

Theorem 1.3. *Let $(-\Delta)^k$ be the fractional Laplacian operator, $k \in (0,1)$. Then, for any $u \in \mathcal{S}(\mathbb{R}^N)$,*

$$(-\Delta)^k u(x) = -\frac{C_{N,k}}{2}\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2k}}\,dy \quad \forall x \in \mathbb{R}^N. \tag{1.15}$$

Proof. We start the proof relying on equation (1.13) and getting trivially

$$P.V.\int_{\mathbb{R}^N} \frac{u(x+y)-u(x)}{|y|^{N+2k}}\,dy = P.V.\int_{\mathbb{R}^N} \frac{u(x-\tilde{y})-u(x)}{|\tilde{y}|^{N+2k}}\,d\tilde{y},$$

changing the variable $\tilde{y} = -y$. Thus, we obtained that

$$(-\Delta)^k u(x) = -\frac{C_{N,k}}{2} P.V.\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2k}}\,dy$$

and we are just left to show that the principal value is useless. In this aim we recall again the Taylor expansion centred in x for u with increment, respectively, $+y$ and $-y$:

$$\begin{cases} u(x+y) \leq u(x) + \nabla u(x)y + \|H_u(x)\|_\infty \dfrac{|y|^2}{2} + o(|y|^2) \\ u(x-y) \leq u(x) - \nabla u(x)y + \|H_u(x)\|_\infty \dfrac{|y|^2}{2} + o(|y|^2). \end{cases}$$

Therefore we deduce that

$$\int_{|y|\leq 1} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2k}}\,dy \leq \int_{|y|\leq 1} \frac{\|H_u(x)\|_\infty |y|^2 + o(|y|^2)}{|y|^{N+2k}}\,dy < \infty,$$

then the singularity at 0 is removed and the Cauchy principal value can be canceled. □

Remark 1.4. *We invite the reader to notice that $u \in \mathcal{S}(\mathbb{R}^N)$ is just a sufficient condition in order to make $(-\Delta)^k u$ defined pointwise. If, indeed, we fix a constant c and set $u(x) \equiv c$, we obtain that trivially $u \notin \mathcal{S}(\mathbb{R}^N)$ but*

$$(-\Delta)^k u(x) = 0 \quad \forall x \in \mathbb{R}^N.$$

1.2. Definition via the Fourier transform

In this paragraph, we decided to analyze the fractional Laplacian from a different perspective via the Fourier transform. Through this transform, we reveal $(-\Delta)^k$ as a pseudo-differential operator of multiplier ξ^{2k}, gaining also powerful regularity results. Moreover, we are going to explicit the strict relation matching our operator with the fractional Sobolev spaces $H^k(\mathbb{R}^N)$, of which we will propose an alternative definition. For a more in-depth discussion on this topic, we refer to [10] and [13].

In order to apply the Fourier transform we ask for $(-\Delta)^k u \in L^1(\mathbb{R}^N)$ and, recalling that this operator is defined pointwise for any $u \in \mathcal{S}(\mathbb{R}^N)$, it will be enough to show that it vanishes at infinity fast enough. In this aim, we state the following theorem.

Theorem 1.5. *Let $u \in \mathcal{S}(\mathbb{R}^N)$. Then, for every $x \in \mathbb{R}^N$, $|x| > 1$ it holds*

$$|(-\Delta)^k u(x)| \leq C_{u,N,k}|x|^{-(N+2k)},$$

where $C_{u,N,k}$ depends on $C_{N,k}$ as defined in (1.12) *and on $p_N(u)$, $p_{N+2}(u)$, defined in* (1).

Proof. Up to normalization constants, if we take Definition (1.15) of the fractional Laplacian, we can separate the integral as follows

$$(-\Delta)^k u(x) \approx \int_{\mathbb{R}^N} \dots dy = \int_{|y|<\frac{|x|}{2}} \dots dy + \int_{|y|\geq \frac{|x|}{2}} \dots dy. \tag{1.16}$$

We start focusing on the first integral, exploiting the Taylor's formula with the Lagrange form of the reminder, that allows us to find t^*, $t^{**} \in (0,1)$ such that

$$2u(x) - u(x+y) - u(x-y) = -\frac{1}{2}y^T H_u(y^*)y - \frac{1}{2}y^T H_u(y^{**})y,$$

where $y^* = x + t^*y$ and $y^{**} = x + t^{**}y$. As a consequence, since both $H_u(y^*)$ and $H_u(y^{**})$ are bounded in $L^\infty(\mathbb{R}^N)$, we can write

$$\left|\int_{|y|<\frac{|x|}{2}} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2k}}\, dy\right| \leq \frac{1}{2}\int_{B_{1/2}(x)} \frac{\|H_u(y^*)\|_\infty + \|H_u(y^{**})\|_\infty}{|y|^{N+2k}}|y|^2\, dy$$
$$\leq C\, p_{N+2}(u) \int_{B_{1/2}(x)} \frac{|y|^{2-(N+2k)}}{(1+|x|^2)^{\frac{N+2}{2}}}\, dy \leq C|x|^{-N-2}\, p_{N+2}(u)|x|^{2-2k} = C_{u,N,k}|x|^{-(N+2k)}.$$

For what concerns the second integral of equation (1.16), the next estimate holds

$$\left|\int_{|y|\geq\frac{|x|}{2}} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2k}}\, dy\right| \leq 2\int_{|y|\geq\frac{|x|}{2}} \frac{|u(x+y)| + |u(x)|}{|y|^{N+2k}}\, dy.$$

We have that both

$$\int_{|y|\geq\frac{|x|}{2}} \frac{|u(x+y)|}{|y|^{N+2k}}\,dy \leq \frac{2^{N+2k}}{|x|^{N+2k}} \int_{|y|\geq\frac{|x|}{2}} |u(x+y)|\,dy \leq \frac{C\|u\|_{L^1(\mathbb{R}^N)}}{|x|^{N+2k}}$$

and

$$\int_{|y|\geq\frac{|x|}{2}} \frac{|u(x)|}{|y|^{N+2k}}\,dy \leq p_N(u) \int_{|y|\geq\frac{|x|}{2}} \frac{dy}{(1+|x|^2)^{\frac{N}{2}}|y|^{N+2k}} \leq \frac{C\,p_N(u)}{|x|^{N+2k}}.$$

The proof is completed. □

We let the reader notice that this theorem also shows that the fractional Laplacian decays at infinity as a kernel of the type $|x|^{-(N+2k)}$.

Once proved that $(-\Delta)^k u \in L^1(\mathbb{R}^N)$ for any $u \in \mathcal{S}(\mathbb{R}^N)$, we are ready to prove that the fractional Laplacian can be viewed as a pseudo-differential operator of multiplier $|\xi|^{2k}$. Explicitly, once considered the operator as set in (1.15), its multiplier is defined to be a function $\mathcal{M} : \mathbb{R}^N \to \mathbb{R}$ such that

$$(-\Delta)^k u = \mathcal{F}^{-1}(\mathcal{M} \cdot \mathcal{F}u).$$

Thus, we want to prove that

$$\mathcal{M}(\xi) = |\xi|^{2k}$$

and we rely on the following proposition.

Proposition 1.2. *For any $u \in \mathcal{S}(\mathbb{R}^N)$ it holds*

$$(-\Delta)^k u = \mathcal{F}^{-1}(|\xi|^{2k}\mathcal{F}u) \quad \forall \xi \in \mathbb{R}^N.$$

Proof. We start applying the Fourier transform in the variable x in (1.15), obtaining

$$\begin{aligned}
\mathcal{M}(\xi)\mathcal{F}u(\xi) &= -\frac{C_{N,k}}{2}\int_{\mathbb{R}^N} \frac{\mathcal{F}(u(x+y)+u(x-y)-2u(x))}{|y|^{N+2k}}\,dy \\
&= C_{N,k}\left(\int_{\mathbb{R}^N} \frac{1-(e^{i\xi\cdot y}+e^{-i\xi\cdot y})/2}{|y|^{N+2k}}\,dy\right)\mathcal{F}u(\xi) \\
&= C_{N,k}\left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot y)}{|y|^{N+2k}}\,dy\right)\mathcal{F}u(\xi), \qquad (1.17)
\end{aligned}$$

where the last equality comes from the famous Euler identity

$$\frac{e^{i\phi}+e^{-i\phi}}{2} = \cos(\phi).$$

We immediately notice that, in a neighbourhood of the origin, it holds

$$\frac{1-\cos(\xi\cdot y)}{|y|^{N+2k}} \sim |y|^{2-(N+2k)},$$

which is clearly integrable; thus the integral in (1.17) is finite and positive.
Moreover, in order to complete the proof we would like to explicit that integral as a function of $C_{N,k}$ and $|\xi|$; in this aim, then, we denote

$$I(\xi) = \int_{\mathbb{R}^N} \frac{1-\cos(\xi\cdot y)}{|y|^{N+2k}}\, dy$$

and try to show that it is rotationally invariant, namely

$$I(\xi) = I(|\xi|e_1),$$

e_1 denoting the first direction vector in $\mathbb{R}^N$. If in dimension 2 it is trivial that $I(\xi) = I(-\xi)$, in dimension $N > 2$ we are required to set a generic rotation matrix R such that $\xi = R|\xi|e_1$ and to compute

$$\begin{aligned} I(\xi) &= \int_{\mathbb{R}^N} \frac{1-\cos(R|\xi|e_1\cdot y)}{|y|^{N+2k}}\, dy = \int_{\mathbb{R}^N} \frac{1-\cos(|\xi|e_1\cdot R^T y)}{|y|^{N+2k}}\, dy \\ &= \int_{\mathbb{R}^N} \frac{1-\cos(|\xi|e_1\cdot \tilde{y})}{|\tilde{y}|^{N+2k}}\, d\tilde{y} = I(|\xi|e_1), \end{aligned}$$

where we have used $\tilde{y} = R^T y,$. We are finally able to compute

$$\begin{aligned} I(|\xi|e_1) &= \int_{\mathbb{R}^N} \frac{1-\cos(e_1\cdot |\xi|y)}{|y|^{N+2k}}\, dy \\ &= \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta\cdot e_1)}{|\zeta|^{N+2k}}\, d\zeta\right) |\xi|^{2k} = C_{N,k}^{-1}|\xi|^{2k}, \end{aligned}$$

exploiting the change of variable $\zeta = |\xi|y$ and Definition (1.12). Thus, if we finally consider (1.17) we can conclude

$$\mathcal{M}(\xi)\mathcal{F}u(\xi) = C_{N,k} I(\xi)\mathcal{F}u(\xi) = |\xi|^{2k}\mathcal{F}u(\xi)$$

and the proof is completed. □

The next step of our analysis wants to make clear the relation between the fractional Laplacian operator $(-\Delta)^k$ and the fractional Sobolev spaces $H^k(\mathbb{R}^N)$ and we start stating and important proposition concerning the Gagliardo seminorm.

Proposition 1.3. *Let $k \in (0,1)$. For any $u \in H^k(\mathbb{R}^N)$*

$$\lfloor u \rfloor_k^2 = 2\, C_{N,k}^{-1} \int_{\mathbb{R}^N} |\xi|^{2k} |\mathcal{F}u(\xi)|^2 \, d\xi,$$

where $C_{N,k}$ is defined in (1.12).

Proof. By direct calculations:

$$\begin{aligned}\lfloor u \rfloor_k^2 &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\, dxdy = \iint_{\mathbb{R}^{2N}} \left| \frac{u(z+y)-u(y)}{|z|^{\frac{N}{2}+k}} \right|^2 dxdy \\ &= \int_{\mathbb{R}^N} \left\| \frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{N}{2}+k}} \right\|_2^2 dz = \int_{\mathbb{R}^N} \left\| \mathcal{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{N}{2}+k}} \right) \right\|_2^2 dz,\end{aligned}$$

where the last equality is justified by the Plancherel formula. If we now follow the same passages used in Proposition 1.2, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \left\| \mathcal{F}\left(\frac{u(z+\cdot)-u(\cdot)}{|z|^{\frac{N}{2}+k}} \right) \right\|_2^2 dz &= \int_{\mathbb{R}^N} \left(2 \int_{\mathbb{R}^N} \frac{1-\cos(\xi \cdot z)}{|z|^{N+2k}} dz \right) |\mathcal{F}u(\xi)|^2 \, d\xi \\ &= 2\, C_{N,k}^{-1} \int_{\mathbb{R}^N} |\xi|^{2k} |\mathcal{F}u(\xi)|^2 \, d\xi.\end{aligned}$$

□

Remark 1.6. *Proposition 1.3 implicitly proves that $H^k(\mathbb{R}^N)$ admits the alternative definition with respect to (1.1):*

$$H^k(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1+|\xi|^{2k}) |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \right\}.$$

Indeed we just claim

$$\int_{\mathbb{R}^N} (1+|\xi|^{2k}) |\mathcal{F}u(\xi)|^2 \, d\xi < \infty \iff \int_{\mathbb{R}^N} |\xi|^{2k} |\mathcal{F}u(\xi)|^2 \, d\xi < \infty,$$

since $u \in H^k(\mathbb{R}^N)$ implies $u \in L^2(\mathbb{R}^N)$ which, in turn, implies $\mathcal{F}u \in L^2(\mathbb{R}^N)$. This definition moreover, unlike the one in (1.1), admits a continuous match between $W^{k,p}$ and $W^{1,p}$ as $k \uparrow 1$, without the need for a norm "correction".

We are finally able to state the connection relating $(-\Delta)^k$ to the fractional Sobolev spaces $H^k(\mathbb{R}^N)$.

Proposition 1.4. *Let $k \in (0,1)$ and let $u \in H^k(\mathbb{R}^N)$. Then*

$$\lfloor u \rfloor_k^2 = 2C_{N,k}^{-1} \|(-\Delta)^{\frac{k}{2}} u\|_2^2,$$

where $C_{N,k}$ is defined in (1.12).

Proof. Relying on Proposition 1.2 and on Proposition 1.3, we obtain

$$\|(-\Delta)^{\frac{k}{2}}\|_2^2 = \|\mathcal{F}(-\Delta)^{\frac{k}{2}} u\|_2^2 = \||\xi|^k \mathcal{F} u\|_2^2 = \frac{1}{2} C_{N,k} \lfloor u \rfloor_k^2.$$

□

1.3. Random Walk with long jumps

In this section, we propose an interesting interpretation of the fractional Laplacian operator, describing the evolution of a probabilistic process that arises from a particle moving randomly in the space $\mathbb{R}^N$, N being the space dimension. This particle will be subject to a probability law that allows long jumps and the peculiarity of its motion will allow us to strictly relate this phenomenon to the nonlocal effects induced by the fractional Laplacian, to the point of inferring the heat equation driving the fractional diffusion. For a more in-depth discussion on this topic, we refer to [32] and [10].

We start modelling a discrete random walk for the particle; in this aim, we define $h > 0$, $t > 0$, respectively the space and time steps and consider both a lattice $h\mathbb{Z}^N$ to discretize space and $\tau\mathbb{Z}$ to discretize time.
Now, we reason as follows: at each time step τ, the particle, which is supposed to be in a generic point x_0, randomly selects a point $x \in h\mathbb{Z}^N$ according to a probability law P; then, it moves from x_0 to x.
The probability law P is defined in order to let the whole process be homogeneous and isotropic, meaning that it does not depend on either position or direction of displacement, but only on the distance travelled. As a consequence, at each time step, the probability of moving from a starting point x_0 to a new one $y = x_0 + hx \in h\mathbb{Z}^N$, $x \in \mathbb{Z}^N$, which is $h|x|$ away from x_0, is

$$P(|x_0 - y|) = P(|x|h) = c_k \frac{1}{|x|^{N+2k}} \mathbb{1}_{[x \neq 0]},$$

where $k \in (0,1)$ and the constant c_k is set in order to normalize P to be a probability measure. Namely, if we set a system where the origin coincides with x_0, we ask for

$$\sum_{x\in\mathbb{Z}^N\setminus\{0\}} P(|x|h) = c_k \sum_{x\in\mathbb{Z}^N\setminus\{0\}} \frac{1}{|x|^{N+2k}} = 1, \tag{1.18}$$

where xh represents the coordinate vector of each grid point in $h\mathbb{Z}^N$. Then, we obtain

$$c_k = \left(\sum_{x\in\mathbb{Z}^N\setminus\{0\}} \frac{1}{|x|^{N+2k}} \right)^{-1}.$$

We stress that, in our model, the particle is not supposed to stand still, since $P(0) = 0$ and that the polynomial tails of the distribution lead to a small probability of long jumps. Moreover, it is worth noticing that in a standard diffusion process the probability law avoids long jumps, imposing $P(jh) = 0 \ \forall j \neq 1$.

At this point, we call $u(x,t)$ the probability for the particle to be at $x \in h\mathbb{Z}^N$ at time $t \in \tau\mathbb{Z}$. Thus, the quantity $u(x, t+\tau)$ can be calculated summing, on all $y \in \mathbb{Z}^N$, the probability for the particle to be at $x + yh$ at time t times the probability of travelling from $x + yh$ to x. In formulae, we obtain

$$u(x, t+\tau) = c_k \sum_{y\in\mathbb{Z}^N\setminus\{0\}} \frac{u(x+yh, t)}{|y|^{N+2k}}. \tag{1.19}$$

If we subtract $u(x,\tau)$ from (1.19) and make use of (1.18) to infer that

$$u(x,t) = c_k \sum_{y\in\mathbb{Z}^N\setminus\{0\}} \frac{u(x,t)}{|y|^{N+2k}},$$

we finally gather

$$u(x, t+\tau) - u(x,t) = c_k \sum_{y\in\mathbb{Z}^N\setminus\{0\}} \frac{u(x+yh, t) - u(x,t)}{|y|^{N+2k}}. \tag{1.20}$$

The next step requires us to approximate the first order derivative in time of $u(x,t)$ applying a forward difference discretization, which takes the form

$$\partial_t u(x,t) \approx \frac{u(x, t+\tau) - u(x,t)}{\tau}. \tag{1.21}$$

Indeed, combining (1.20) and (1.21), we can fix the incremental time step $\tau = h^{2k}$ in order to retrieve the following formula:

$$\partial_t u(x,t) \approx c_k \sum_{y\in\mathbb{Z}^N\setminus\{0\}} \frac{u(x+yh,t)-u(x,t)}{|yh|^{N+2k}} h^N.$$

We recall here, for the reader's convenience, that if we consider a continuous function $f:\mathbb{R}\to\mathbb{R}$ and we divide a generic interval $[a,b]$ according to the partition $(x_j)_j$ into n subintervals

$$(x_{j-1},x_j) = (a+(j-1)h, a+jh) \quad h=\frac{b-a}{n},\ j=1,...,n,$$

the following approximation regarding the area under the graph of f holds:

$$\sum_{j=1}^{n} f(a+jh)h \approx \int_a^b f(x)\,dx. \tag{1.22}$$

Generalizing, then, (1.22) to dimension $N>1$, we can claim

$$c_k \sum_{y\in\mathbb{Z}^N\setminus\{0\}} \frac{u(x+yh,t)-u(x,t)}{|yh|^{N+2k}} h^N \approx c_k \int_{\mathbb{R}^N} \frac{u(x+y,t)-u(x,t)}{|y|^{N+2k}}\,dy,$$

where the cancellation of $y=0$ from the sum is mirrored by the singularity of the integrand at that specific value for y.
It is now clear that the choice $\tau=h^{2k}$ has provided us with nice asymptotic results since, exploiting the previous approximation, we have obtained:

$$\partial_t u + C(-\Delta)^k u = 0,$$

for a suitable $C>0$.

1.4. Maximum principles and nonlocal effects

In this section, we are going to give some practical clarifications on the concept of nonlocality, as it is associated to the fractional Laplacian operator. In order to make this concept clear, we considered a good idea to carry on an explicit comparison between this operator and the classical Laplacian, whose local nature is well known in literature.
In particular, we will firstly examine two specific examples where the application of either one or the other operator will lead us to dramatically different conclusions, enlightening

in this way the nonlocal nature of the fractional Laplacian. Then, we will state and prove appropriate versions of the weak and strong maximum principles for $(-\Delta)^k$, different from the ones valid for sub and superharmonic functions. Finally, we are going to analyze why the classical version of the Harnack inequality for harmonic functions (see B.9 from the Appendix) fails to hold in a fractional setting. For a more in-depth discussion on this topic, we refer to [20], [21] and [28].

Our first example considers a generic function $u \in D(B_2)$ such that

$$u(x) = \begin{cases} 1 & x \in B_1 \\ 0 & x \notin B_2, \end{cases}$$

while, in $B_2 \backslash B_1$, $u(x)$ connects smoothly 1 to 0 and it holds $0 \leq u(x) \leq 1$, $\forall x \in B_2 \backslash B_1$. If we pick $\bar{x} \notin B_3$, for which it trivially holds that $-\Delta u(\bar{x}) = 0$, we aim at computing $(-\Delta)^k u(\bar{x})$ and showing that it is different from 0.
We start making $(-\Delta)^k u(\bar{x})$ explicit:

$$\begin{aligned} -(-\Delta)^k u(\bar{x}) &= C_{N,k}\, P.V. \int_{\mathbb{R}^N} \frac{u(y) - u(\bar{x})}{|y - \bar{x}|^{N+2k}}\, dy \\ &= C_{N,k} \int_{\mathbb{R}^N} \frac{u(y)}{|y - \bar{x}|^{N+2k}}\, dy, \end{aligned}$$

where the principal value in the first line disappeared since it is useless; outside B_2 indeed, the function vanishes while, inside, it holds that $|y - \bar{x}| > 1$. Then, since the integrand differs from 0 just in B_2, we keep computing:

$$\begin{aligned} -(-\Delta)^k u(\bar{x}) &= C_{N,k} \int_{B_2} \frac{u(y)}{|y - \bar{x}|^{N+2k}}\, dy \\ &\geq C_{N,k} \int_{B_1} \frac{1}{|y - \bar{x}|^{N+2k}}\, dy \\ &\geq C_{N,k} \int_{B_1} \frac{dy}{(1 + |\bar{x}|)^{N+2k}}, \end{aligned}$$

where the last inequality comes naturally if we expand

$$|y - \bar{x}| \leq |y| + |\bar{x}| \leq 1 + |\bar{x}|.$$

Finally, we have shown that

$$-(-\Delta)^k u(\bar{x}) \geq \frac{|B_1|}{(1 + |\bar{x}|)^{N+2k}} > 0,$$

where $|B_1|$ denotes the volume of the N-dimensional sphere of radius 1.

In the second example, we want to discuss how the nonlocal properties of the fractional Laplacian affect the wavefront spread dynamic associated to the following system:

$$\begin{cases} u_t + (-\Delta)^k u = au & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = \delta(0), \end{cases} \tag{1.23}$$

where $a > 0$ and $\delta(0)$ represents the Dirac delta centred in 0. In particular, we will study that system both for $s = 1$ and $s = \frac{1}{2}$ in order to explore, respectively, the local and nonlocal case.
We will proceed as follows: once computed the solution, by means of the Fourier transform, we will consider one of its contour lines and study the properties of its evolution in the time-space dimension.
We start asking for $u(x,t) = e^{at}v(x,t)$ and write system (1.23) in terms of v:

$$\begin{cases} v_t + (-\Delta)^k v = 0 \\ v(0) = \delta(0). \end{cases} \tag{1.24}$$

Now, defining $\hat{v}(t,\xi) = \mathcal{F}[v(t,\cdot)](\xi)$ the Fourier transform of v, we apply the Fourier transform to (1.24), that results in

$$\begin{cases} \hat{v}_t + |\xi|^{2k}\hat{v} = 0 \\ \hat{v}(0) = 1. \end{cases}$$

The solution to the system is

$$\hat{v}(\xi,t) = e^{-|\xi|^{2k}t}.$$

The first scenario we examine is the one of the Laplacian, with $k = 1$ and we immediately notice that:

$$\hat{v}(\xi,t) = e^{-|\xi|^2 t} \iff v(x,t) = Ce^{-\frac{|x|^2}{4t}}t^{-\frac{N}{2}} \iff u(x,t) = Ce^{at-\frac{|x|^2}{4t}}t^{-\frac{N}{2}}.$$

At this point, asking for $u(x,t)$ to be constant means asking for

$$e^{at-\frac{|x|^2}{4t}} = Ct^{\frac{N}{2}} \implies |x|^2 = 4at^2 - 2Nt\ln t + Ct,$$

which finally becomes

$$x = 2\sqrt{a}t(1 + o(1)) \quad \text{as } t \to \infty. \tag{1.25}$$

Now we examine the scenario where the fractional Laplacian appears, with $k = \frac{1}{2}$. Again, the following chain of co-implications holds

$$\hat{v}(\xi, t) = e^{-|\xi| t} \iff v(x,t) = \frac{1}{t(1+\frac{|x|^2}{t^2})^{\frac{N+1}{2}}} \iff u(x,t) = \frac{e^{at}}{t(1+\frac{|x|^2}{t^2})^{\frac{N+1}{2}}}.$$

Asking for the solution to be constant leads us to:

$$\left(1+\frac{|x|^2}{t^2}\right)^{\frac{N+1}{2}} = C\frac{e^{at}}{t} \implies |x|^2 = Ce^{\frac{2at}{N+1}} t^{\frac{2N}{N+1}} - t^2$$

and, finally, to

$$x = Ce^{\frac{at}{N+1}} t^{\frac{N}{N+1}} (1+o(1)) \quad \text{as } t \to \infty. \tag{1.26}$$

Thus, we have proved that, if we consider system (1.23) with $k = 1$, the wavefront of the solution evolves linearly in time, while, studying (1.23) with $k = \frac{1}{2}$, the nonlocal effects of the fractional Laplacian play a central role and they are responsible for the wavefront travel in space to be faster, in particular exponentially in time.

At this point, we carry on our study stating the weak and strong maximum principles valid for the fractional Laplacian. We stress here that, differently from the Laplacian case, here the "boundary values" for u must be prescribed globally in $\mathbb{R}^N \backslash \Omega$ and not just on $\partial\Omega$.

Theorem 1.7 (Weak maximum principle). *Assume that $u \in C^\infty(\mathbb{R}^N) \cup L^\infty(\mathbb{R}^N)$ and that $\Omega \subset \mathbb{R}^N$ is a bounded domain. Then, the weak maximum principle holds for $(-\Delta)^k$, namely*

$$\begin{cases} (-\Delta)^k u(x) \geq 0 & x \in \Omega \\ u(x) \geq 0 & x \in \mathbb{R}^N \backslash \Omega \end{cases} \implies u(x) \geq 0 \textit{ in } \Omega.$$

Proof. We will prove the theorem by contradiction, assuming that $u(x) \ngeq 0$ in Ω. As a consequence, we can consider $x_m = \operatorname{argmin}_{x \in \mathbb{R}^N} u(x)$, such that $x_m \in \Omega$ and

$$-\delta := u(x_m) = \min_{\bar{\Omega}} u(x) < 0. \tag{1.27}$$

Thus, the following inequality hold

$$(-\Delta)^k u(x_m) = \frac{C_{N,k}}{2}\int_{\mathbb{R}^N} \frac{2u(x_m)-u(x_m+z)-u(x_m-z)}{|z|^{N+2k}}\,dz$$
$$\leq \frac{C_{N,k}}{2}\int_{B} \frac{2u(x_m)-u(x_m+z)-u(x_m-z)}{|z|^{N+2k}}\,dz,$$

where $B := \{z \in \mathbb{R}^N : (x_m+z) \notin \Omega,\ (x_m-z)\notin \Omega\}$ and the inequality is due to $2u(x_m)-u(x_m+z)-u(x_m-z) \leq 0\ \forall z \in \mathbb{R}^N$ by definition of x_m. Moreover, exploiting $z \in B$ and (1.27), we state

$$(-\Delta)^k u(x_m) \leq \frac{C_{N,k}}{2}\int_B \frac{-2\delta}{|z|^{N+2k}}\,dz < 0. \tag{1.28}$$

Finally, assuming $u \ngeq 0$ in Ω, we have concluded that

$$\begin{cases}(-\Delta)^k u(x_m) \geq 0\\ (-\Delta)^k u(x_m) < 0,\end{cases}$$

where the first line comes from data while the second from (1.28), and the contradiction is immediate.
An alternative and more direct proof exists and notices that $2u(x_m)-u(x_m+z)-u(x_m-z) \leq 0\ \forall z \in \mathbb{R}^N$ and $(-\Delta)^k u(x_m) \geq 0$ respectively imply

$$\begin{cases}\dfrac{C_{N,k}}{2}\displaystyle\int_{\mathbb{R}^N} \frac{2u(x_m)-u(x_m+z)-u(x_m-z)}{|z|^{N+2k}}\,dz \leq 0\\ \dfrac{C_{N,k}}{2}\displaystyle\int_{\mathbb{R}^N} \frac{2u(x_m)-u(x_m+z)-u(x_m-z)}{|z|^{N+2k}}\,dz \geq 0.\end{cases} \tag{1.29}$$

From (1.29) we infer that

$$2u(x_m)-u(x_m+z)-u(x_m-z) \equiv 0 \implies u \equiv -\delta \text{ in } \mathbb{R}^N$$

and we fall into a clear contradiction if we consider, by data, that $u(x) \geq 0$ in $\mathbb{R}^N \backslash \Omega$. □

Theorem 1.8 (Strong maximum principle). *Assume that $u \in C^\infty(\mathbb{R}^N) \cup L^\infty(\mathbb{R}^N)$ and that $\Omega \subset \mathbb{R}^N$ is a bounded and connected domain. Then, the strong maximum principle holds for $(-\Delta)^k$, namely*

$$\begin{cases}(-\Delta)^k u(x) \geq 0 & x \in \Omega\\ u(x) \geq 0 & x \in \mathbb{R}^N\backslash\Omega\end{cases} \implies u(x) > 0 \text{ in } \Omega \text{ or } u \equiv 0.$$

Proof. We already know from Theorem 1.7 that $u \geq 0$ in $\mathbb{R}^N$. If there exists $x_0 \in \Omega$, $u(x_0) = 0$, it holds that

$$0 \leq \int_{\mathbb{R}^N} \frac{u(x_0) - u(y)}{|x_0 - y|^{N+2k}}\, dy = - \int_{\mathbb{R}^N} \frac{u(y)}{|x_0 - y|^{N+2k}}\, dy.$$

The only chance not to fall into contradiction is that $u \equiv 0$ since, if it were not true, the last integral would be strictly negative, contradicting the data information $(-\Delta)^k u(x_0) \geq 0$. □

Finally, the last part of this section aims at showing that the classical Harnack inequality for harmonic functions (see B.9 from the Appendix) can be false if we try to rigidly shift its frame to a fractional setting. Indeed, if we consider Corollary B.10, we deduce that for a non-negative harmonic function in B_1, its minimum and its maximum are comparable in B_r, r in $(0,1)$. Nevertheless it is not always the case when speaking of k-harmonic functions, as the following theorem points out.

Theorem 1.9. *Let $k \in (0,1)$, $R > 0$. Then, there exists a function $u \in L^\infty(\mathbb{R}^N) \cup C^2(B_R)$ such that*

$$\begin{cases} (-\Delta)^k u(x) = 0 & \textit{for } x \in B_R \\ u(x) > 0 & \textit{for } x \in B_R \setminus \{0\} \\ |u(x)| \leq 1 & \textit{for } x \in \mathbb{R}^N \end{cases}$$

and $u(0) = 0$.

Relying on Theorem 1.9, if we fix $R = 1$ and consider any $r \in (0,1)$, it is no more true that

$$u(x) \leq c_r u(y) \quad \forall x, y \in B_r.$$

Indeed, if we consider $y = 0$, since $u(0) = 0$ we should have that

$$u(x) \leq 0 \quad \forall x \in B_r,$$

which clearly contradicts the datum $u(x) > 0$ for any $x \in B_r \setminus \{0\}$.

Proof of Theorem 1.9. We are going to prove the theorem for dimension $N = 2$. The same idea and method can be applied then for any dimension N. In this proof we are going to construct two functions, $g : \mathbb{R}^2 \setminus B_R \to \mathbb{R}$ and $\tilde{u} : B_R \to \mathbb{R}$, where $\tilde{u}(y)$ will be computed by means of the Poisson formula for k-harmonic functions over balls and, at the end,

$$u(y) = \tilde{u}(y)\mathbb{1}_{[y\in B_R]} + g(y)\mathbb{1}_{[y\notin B_R]} \quad \forall y \in \mathbb{R}^2$$

will be the desired function.
We choose $g(y)$ defined as follows

$$g(y) = \begin{cases} 1 & R < |y| < S \\ -1 & S < |y| < T \\ 0 & T < |y|, \end{cases}$$

where $S > R$ and $T > S$ will be explicitly set later. Now, we focus on $\tilde{u}(y)$ that, exploiting the Poisson formula on B_R, is

$$\begin{aligned} \tilde{u}(y) &= \mathcal{C}_k (R^2 - |y|^2)^k \int_{\mathbb{R}^2\setminus B_R} g(x) \left|R^2 - |x|^2\right|^{-k} |x-y|^{-2}\, dx \\ &= \mathcal{C}_k (R^2 - |y|^2)^k \Bigg[\int_{B_S\setminus B_R} \left|R^2 - |x|^2\right|^{-k} |x-y|^{-2}\, dx \\ &\qquad - \int_{B_T\setminus B_S} \left|R^2 - |x|^2\right|^{-k} |x-y|^{-2}\, dx \Bigg] \\ &=: I_{R,S}(y) - I_{S,T}(y) \quad \forall y \in B_R, \end{aligned}$$

with $\mathcal{C}_k = \frac{\sin(\pi k)}{\pi^2}$. Now we define

$$A(y) := \int_{B_T\setminus B_S} \left|R^2 - |x|^2\right|^{-k} |x-y|^{-2}\, dx$$

and we study it in polar coordinates, remembering that

$$\begin{aligned} |x-y|^2 &= (x_1 - y_1)^2 + (x_2 - y_2)^2 = (r\cos(\theta) - y_1)^2 + (r\sin(\theta) - y_2)^2 \\ &= r^2\cos^2(\theta) + |y_1|^2 - 2r\cos(\theta)y_1 + r^2\sin^2(\theta) + |y_2|^2 - 2r\sin(\theta)y_2 \\ &= r^2 + |y|^2 - 2r|y| \left(\frac{y_1}{|y|}\cos(\theta) + \frac{y_2}{|y|}\sin(\theta) \right) \end{aligned}$$

and we obtain

$$A(y) = \int_S^T \frac{r}{|R^2 - r^2|^k} \left(\int_0^{2\pi} \frac{d\theta}{r^2 + |y|^2 - 2r|y|\left(\frac{y_1}{|y|}\cos(\theta) + \frac{y_2}{|y|}\sin(\theta)\right)} \right) dr.$$

We assume now $y \neq 0$ and select $\gamma \in [0, 2\pi]$ such that $\cos(\gamma) = \frac{y_1}{|y|}$; it holds that

$$\sin^2(\gamma) = 1 - \cos^2(\gamma) = 1 - \frac{y_1^2}{|y|^2} = \left(\frac{y_2}{|y|}\right)^2$$

and, as a consequence, if $\sin(\gamma) = 0$ there exists just one possible γ such that $\cos(\gamma) = \frac{y_1}{|y|}$; if instead $\sin(\gamma) \neq 0$, we chose γ such that $\sin(\gamma) = \frac{y_2}{|y|}$. Then, we define $\phi = \gamma - \theta$ and, remembering

$$\cos(\gamma - \theta) = \cos(\gamma) cos(\theta) + \sin(\gamma)\sin(\theta),$$

we have

$$A(y) = \int_S^T \frac{r}{|R^2 - r^2|^k} \left(\int_{\gamma - 2\pi}^{\gamma} \frac{d\phi}{r^2 + |y|^2 - 2r|y|\cos(\phi)} \right) dr.$$

At this point we set $t|y| = r$, $dt|y| = dr$ and write

$$\begin{aligned} A(y) &= \int_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t|y|}{|R^2 - t^2|y|^2|^k} \left(\int_{\gamma - 2\pi}^{\gamma} \frac{d\phi}{|y|^2 \left(t^2 + 1 - 2t\cos(\phi)\right)} \right) |y|\, dt \\ &= \int_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\left(t^2|y|^2 - R^2\right)^k} \left(\int_0^{2\pi} \frac{d\phi}{t^2 + 1 - 2t\cos(\phi)} \right) dt, \end{aligned} \tag{1.30}$$

where we have exploited $R < S \leq r$ to get rid of the modulus and the periodicity of the integrand with respect to ϕ to relabel the extrema of the inner integral. Moreover, it should be noticed that $R < S \leq r$ also implies that $t > 1$.
It is possible to provide an explicit solution for the inner integral of (1.30), namely, setting $\delta = \frac{2t}{1+t^2}$, $\delta < 1$, we have

$$B(t) := \int_0^{2\pi} \frac{d\phi}{(t^2 + 1) - (2t\cos(\phi))} = \frac{1}{1 + t^2} \int_0^{2\pi} \frac{d\phi}{1 - \delta\cos(\phi)}.$$

Indeed, we can make use of Lemma B.2 to infere

$$B(t) = \frac{2\pi}{\sqrt{1 - \delta^2}(1 + t^2)} = \frac{2\pi}{(1 + t^2)} \left(1 - \frac{4t^2}{(1 + t^2)^2}\right)^{-\frac{1}{2}} = \frac{2\pi}{|1 - t^2|} = \frac{2\pi}{t^2 - 1},$$

where we have used the definition of δ and $t > 1$.
Therefore, we came up with

$$A(y) = \int_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\left(t^2|y|^2 - R^2\right)^k} \frac{2\pi}{(t^2 - 1)}\, dt = \frac{2\pi}{|y|^{2k}} \int_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\left(t^2 - \rho^2\right)^k (t^2 - 1)}\, dt,$$

using $\rho = \frac{R}{|y|}$. Thus, we finally get

$$I_{S,T}(y) = 2\pi \mathcal{C}_k (\rho^2 - 1)^k \int_{\frac{S}{|y|}}^{\frac{T}{|y|}} \frac{t}{\left(t^2 - \rho^2\right)^k \left(t^2 - 1\right)} \, dt.$$

If we put $\alpha = t^2 - \rho^2$, $t = \sqrt{\alpha + \rho^2}$, $dt = \frac{d\alpha}{2\sqrt{\alpha+\rho^2}}$ we can develop $I_{S,T}$ in the following way:

$$\begin{aligned} I_{S,T}(y) &= 2\pi \mathcal{C}_k (\rho^2 - 1)^k \int_{\frac{S^2-R^2}{|y|^2}}^{\frac{T^2-R^2}{|y|^2}} \frac{\sqrt{\alpha + \rho^2}}{\alpha^k (\alpha + \rho^2 - 1)} \frac{d\alpha}{2\sqrt{\alpha + \rho^2}} \\ &= \pi \mathcal{C}_k (\rho^2 - 1)^k \int_{\frac{S^2-R^2}{|y|^2}}^{\frac{T^2-R^2}{|y|^2}} \frac{d\alpha}{\alpha^k (\alpha + \rho^2 - 1)}. \end{aligned}$$

If then we apply the substitution $\alpha = \tau(\rho^2 - 1)$, we get

$$\begin{aligned} I_{S,T}(y) &= \pi \mathcal{C}_k (\rho^2 - 1)^k \int_{\frac{S^2-R^2}{R^2-|y|^2}}^{\frac{T^2-R^2}{R^2-|y|^2}} \frac{(\rho^2 - 1)}{\tau^k (\rho^2 - 1)^{1+k} (1 + \tau)} \, d\tau \\ &= \pi \mathcal{C}_k \int_{\frac{S^2-R^2}{R^2-|y|^2}}^{\frac{T^2-R^2}{R^2-|y|^2}} \frac{d\tau}{\tau^k (1 + \tau)}. \end{aligned}$$

The case with $y = 0$ is pretty similar, indeed, again relying on the polar coordinates, we obtain

$$\begin{aligned} I_{S,T}(0) &= \mathcal{C}_k R^{2k} \int_{B_T \setminus B_S} \left| R^2 - |x|^2 \right|^{-k} |x|^{-2} \, dx = 2\pi \mathcal{C}_k R^{2k} \int_S^T \frac{dr}{r(r^2 - R^2)^k} \\ &= 2\pi \mathcal{C}_k \int_S^T \frac{dr}{r\left(\frac{r^2}{R^2} - 1\right)^k} \overset{v=\frac{r}{R}}{=} 2\pi \mathcal{C}_k \int_{\frac{S}{R}}^{\frac{T}{R}} \frac{dv}{v(v^2 - 1)^k} \\ &\overset{\omega=v^2-1}{=} \pi \mathcal{C}_k \int_{\left(\frac{S}{R}\right)^2+1}^{\left(\frac{T}{R}\right)^2+1} \frac{1}{\omega^k \sqrt{\omega + 1}} \frac{d\omega}{\sqrt{\omega + 1}} = \pi \mathcal{C}_k \int_{\left(\frac{S}{R}\right)^2+1}^{\left(\frac{T}{R}\right)^2+1} \frac{d\omega}{\omega^k (\omega + 1)}. \end{aligned}$$

Collecting all the results we have obtained we finally end up with

$$\tilde{u}(y) = \pi \mathcal{C}_k \left(\int_0^{\frac{S^2-R^2}{R^2-|y|^2}} \frac{d\tau}{\tau^k (1 + \tau)} - \int_{\frac{S^2-R^2}{R^2-|y|^2}}^{\frac{T^2-R^2}{R^2-|y|^2}} \frac{d\tau}{\tau^k (1 + \tau)} \right).$$

To make all these terms more manageable we define $E = \frac{S^2}{R^2} - 1$, $F = \frac{T^2}{R^2} - 1$, $q = \frac{R^2}{R^2-|y|^2}$

and $J(a,b)=\int_a^b \frac{d\tau}{\tau^k(1+\tau)}$ in such a way that

$$\tilde{u}(y)=\pi\mathcal{C}_k\begin{cases}J(0,qE)-J(qE,qF) & y\neq 0\\ J(0,E)-J(E,F) & y=0.\end{cases}$$

Therefore, our aim will be finding $E^*>0, F^*>0$ such that

$$J(0,qE)-J(qE,qF)=\begin{cases}=0 & \text{if } q=1\\ >0 & \text{if } q>1.\end{cases}$$

We start studying the function $q\mapsto J(0,qE)-J(qE,qF)$, $q\geq 1$, and by definition we get

$$\frac{d}{dq}[J(0,qE)-J(qE,qF)]=\frac{2E}{(qE)^k(1+qE)}-\frac{F}{(qF)^k(1+qF)}. \tag{1.31}$$

Then we see that (1.31) is positive if and only if

$$E^{k-1}-2F^{k-1}\leq q(2F^k-E^k);$$

In particular, the function $q\mapsto J(0,qE)-J(qE,qF)$, $q\geq 1$ is non-decreasing under $F>E$ if

$$E^k(1+E^{-1})\leq 2F^k(1+F^{-1}). \tag{1.32}$$

To be ready to complete the proof we need a last result:

$$\begin{cases}k\leq\frac{1}{2} \implies & J(0,+\infty)\geq 2J(0,1)\\ k>\frac{1}{2} \implies & J(0,+\infty)\leq 2J(0,1).\end{cases}$$

We just prove the first case, the second can be proved in the same way:

$$\begin{aligned}J(1,+\infty)&=\int_1^\infty\frac{d\tau}{\tau^k(1+\tau)}\geq\int_1^\infty\frac{d\tau}{\sqrt{\tau}(1+\tau)}\\ &\overset{\theta=\frac{1}{\tau}}{=}\int_0^1\frac{\sqrt{\theta}}{\left(1+\frac{1}{\theta}\right)}\frac{d\theta}{\theta^2}=\int_0^1\frac{d\theta}{\sqrt{\theta}(1+\theta)}\\ &\geq\int_0^1\frac{d\theta}{\theta^k(1+\theta)}=J(0,1).\end{aligned}$$

Then, it is trivial that

$$J(0,\infty)=J(0,1)+J(1,\infty)\geq 2J(0,1).$$

Now we are ready to find the explicit values for S and T, studying the two different cases, $k > \frac{1}{2}$ and $k \leq \frac{1}{2}$.
Starting with $k > \frac{1}{2}$, we fix $F > 1$. Since $J(0, x)$ is non decreasing in x, $J(0, 0) = 0$ and $J(0, F) < 2J(0, 1)$ we can rely on the intermediate value theorem and say that there exists $E < 1$ such that $J(0, F) = 2J(0, E)$, meaning that $J(0, E) = J(E, F)$. Moreover we notice that for $E \in (0, 1]$ the left hand side of (1.32) is non increasing in E, while for $F > 1$ its right hand side goes to $+\infty$. Then we can fix $F^* > 0$ large enough such that (1.32) is satisfied and $E^* \in [E, 1]$ such that $J(0, E^*) = J(E^*, F^*)$.
If, instead, $k \leq \frac{1}{2}$ we can proceed as follows. Since (1.32) is valid for $E = 1$ and $F > 1$, we fix $E^* = 1$. Then, exploiting $J(0, +\infty) \geq 2J(0, 1)$, we know that there exists some $F^* > 1$ satisfying $J(0, E^*) = J(E^*, F^*)$.
Finally, if we set $S = R\sqrt{E^* + 1}$ and $T = R\sqrt{F^* + 1}$, g takes an explicit form and, as a consequence,

$$u(y) = \tilde{u}(y)\mathbb{1}_{[y \in B_R]} + g(y)\mathbb{1}_{[y \notin B_R]} \quad \forall y \in \mathbb{R}^2$$

is well defined. Moreover, $(-\Delta)^k u(x) = 0$ for any $x \in B_R$ and $|u(x)| \leq 1$ thanks to the weak maximum principle. The proof is complete. □

1.5. A k-harmonic function

Once we have dealt with maximum principles and k-harmonic functions, in the next lines we would like to present an explicit example of a k-harmonic function in one dimension. This function is $x_k^+ = \max\{x, 0\}^k$, $k \in (0, 1)$ and it satisfies the following theorem. For a more in-depth discussion on this topic, we refer to [10].

Theorem 1.10. *Let $k \in (0, 1)$ and $w_k(x) = x_k^+$. Then w_k satisfies*

$$(-\Delta)^k w_k(x) = \begin{cases} -c_k|x|^{-k} & x < 0 \\ 0 & x > 0, \end{cases}$$

with c_k constant depending on k.

In particular, we are going to offer an alternative probabilistic interpretation of the fractional Laplacian, arising from a payoff approach and we will exploit this different point of view in order to justify Theorem 1.10. At the end of the paragraph, then, this theorem will be formally proved.

We start describing how the payoff approach works. This model aims at describing the expected payoff received by a particle moving over the real line and earning money depending on the points it moves to. For what concerns the discretization of the domain and the probabilistic law of its motion, we rely on the same assumptions of Section 1.3. In this case, however, we define a domain $\Omega \subset \mathbb{R}$ and, for any x in Ω, $u(x)$ represents the expected payoff received starting from position x. Reaching some point y outside Ω, means earning $u_0(y)$, with u_0 defined a-priori. Therefore, $u(x)$ can be computed summing, on all $y \in \mathbb{Z}$, the probability for the particle to reach $x + yh$ times the expected payoff $u(x + yh)$; namely

$$u(x) = c_k \sum_{y\in\mathbb{Z}\setminus\{0\}} \frac{u(x+yh)}{|y|^{N+2k}} = c_k \sum_{y\in\mathbb{Z}\setminus\{0\}} \frac{u(x-yh)}{|y|^{N+2k}}.$$

Thus, if we recall equation (1.18), we can trivially infer

$$0 = c_k \sum_{y\in\mathbb{Z}\setminus\{0\}} \frac{u(x+yh) + u(x-yh) - 2u(x)}{|y|^{N+2k}}.$$

Passing to the limit as in Section 1.3 we notice that this model describes the following system

$$\begin{cases} (-\Delta)^k u = 0 & x \in \Omega \\ u = u_0 & x \notin \Omega. \end{cases}$$

Now, we want to make use of the previous approach in order to give an heuristic interpretation of Theorem 1.10. In particular we justify the fact that

$$\begin{cases} (-\Delta)^k w_k = 0 & 0 < x < 1 \\ w_k = 0 & x < 0 \\ w_k = x_k^+ & x > 1 \end{cases} \tag{1.33}$$

admits as solution $w_k(x) = x_k^+$ and we will show this before for $k = 1$ and then for $k \in (0, 1)$.

We start fixing $k = 1$ and discretizing the real line with $h = 1/2$; then, we try to deduce the value of w_1 in $x = 1/2$ in terms of expected payoff. Since $k = 1$, our random walk does not allow any jump and, with the same probability, we can either reach $x = 0$ or $x = 1$. Since $w_1(0) = 0$ and $w_1(1) = 1$, our expected earn from $x = 1/2$ is the average between the two, namely $w_1(1/2) = 1/2$. If we keep discretizing with $h = 1/4$, we can follow the same procedure, considering $w_1(0)$ and $w_1(1/2)$ and we obtain $w_1(1/4) = 1/4$. Repeating these passages over and over, we find that the linear function is harmonic, which is true,

even if not surprising.
If instead we consider $k \in (0,1)$ the random walk we are asking for admits long jumps; then, starting from $x = 1/2$, with $h = 1/2$ we have the same probability to finish in $A := \{1/2 - y/2;\ y \in \mathbb{N}\backslash\{0\}\}$ or in $B := \{1/2 + y/2;\ y \in \mathbb{N}\backslash\{0\}\}$. If the expected payoff in A is equal to 0 by data, the one in B is strictly bigger then 1, since $w_k(1/2 + y/2) > 1$ for any $y > 1$, again by data. As a consequence, we expect a payoff $w_k(1/2) > 1/2$. Even in this case, we can repeat the procedure and notice that our solution w satisfies a concavity property. Therefore, it makes sense for $w_k = x_k^+ \mathbb{1}_{[x>0]}$ to be the solution to system (1.33).

We now move on to a formal proof for Theorem 1.10. This will come through two preliminary lemmas.

Lemma 1.11. *Let $\omega_k = x_k^+$. Then*

$$(-\Delta)^k \omega_k(1) = 0.$$

Proof. Since $t \mapsto (1+t)^k + (1-t)^k - 2$ is even, it holds that

$$\int_{-1}^{1} \frac{(1+t)^k + (1-t)^k - 2}{|t|^{1+2k}}\, dt = 2\int_{0}^{1} \frac{(1+t)^k + (1-t)^k - 2}{|t|^{1+2k}}\, dt.$$

Moreover, by the change of variable $t = -\tilde{t}$ we have

$$\int_{-\infty}^{-1} \frac{(1-t)^k - 2}{|t|^{1+2k}}\, dt = \int_{1}^{\infty} \frac{(1+\tilde{t})^k - 2}{|\tilde{t}|^{1+2k}}\, d\tilde{t};$$

then we can claim

$$\begin{aligned}
(-\Delta)^k \omega_k(1) &= \int_{-\infty}^{\infty} \frac{\omega_k(1+t) + \omega_k(1-t) - 2\omega_k(1)}{|t|^{1+2k}}\, dt \\
&= \int_{-\infty}^{-1} \frac{(1-t)^k - 2}{|t|^{1+2k}}\, dt + 2\int_{0}^{1} \frac{(1+t)^k + (1-t)^k - 2}{|t|^{1+2k}}\, dt \\
&\quad + \int_{1}^{\infty} \frac{(1+t)^k - 2}{|t|^{1+2k}}\, dt \\
&= 2\left[\int_{1}^{\infty} \frac{(1+t)^k - 2}{|t|^{1+2k}}\, dt + \int_{0}^{1} \frac{(1+t)^k + (1-t)^k - 2}{|t|^{1+2k}}\, dt\right].
\end{aligned}$$

If now we integrate by parts $t \mapsto t^{-1-2k}$ we get

$$\int_{1}^{\infty} t^{-1-2k}\, dt = -\frac{1}{2k}\left|t^{-2k}\right|_{1}^{\infty} = \frac{1}{2k}$$

and we can conclude by

$$(-\Delta)^k \omega_k(1) = 2\left[\int_1^\infty \frac{(1+t)^k}{|t|^{1+2k}}\,dt + \int_0^1 \frac{(1+t)^k + (1-t)^k - 2}{|t|^{1+2k}}\,dt - \frac{1}{k}\right] = 0,$$

where we have used Lemma B.3 for the last equality. □

Lemma 1.12. *Let* $\omega_k = x_k^+$*. Then*

$$-(-\Delta)^k \omega_k(-1) > 0.$$

Proof. The proof is immediate since

$$\omega_k(-1+t) + \omega_k(-1-t) - 2\omega_k(-1) = (-1+t)_+^k + (-1-t)_+^k \geq 0$$

and is not identically null. □

Proof of Theorem 1.10. We start defining $\sigma \in \{-1; 1\}$ to denote the sign of some $x \in \mathbb{R}$. Then we see that

$$\int_{-\infty}^{\infty} \frac{\omega_k(\sigma(1+t)) + \omega_k(\sigma(1-t)) - 2\omega_k(\sigma)}{|t|^{1+2k}}\,dt = \int_{-\infty}^{\infty} \frac{\omega_k(\sigma+t) + \omega_k(\sigma-t) - 2\omega_k(\sigma)}{|t|^{1+2k}}\,dt,$$

since, if $\sigma = 1$ is obvious, while, when $\sigma = -1$ we can put $t = -\tilde{t}$ and we see

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\omega_k(\sigma(1+t)) + \omega_k(\sigma(1-t)) - 2\omega_k(\sigma)}{|t|^{1+2k}}\,dt \\
&= \int_{-\infty}^{\infty} \frac{\omega_k(-1-t) + \omega_k(-1+t) - 2\omega_k(-1)}{|t|^{1+2k}}\,dt \\
&= \int_{-\infty}^{\infty} \frac{\omega_k(-1+\tilde{t}) + \omega_k(-1-\tilde{t}) - 2\omega_k(-1)}{|\tilde{t}|^{1+2k}}\,d\tilde{t} = \int_{-\infty}^{\infty} \frac{\omega_k(\sigma+\tilde{t}) + \omega_k(\sigma-\tilde{t}) - 2\omega_k(\sigma)}{|\tilde{t}|^{1+2k}}\,d\tilde{t}.
\end{aligned}$$

At this point, since it holds that for any $r \in \mathbb{R}$

$$\omega_k(|x|r) = (|x|r)_+^k = |x|^k r_+^k = |x|^k \omega_k(r),$$

we can state the following:

$$\omega_k(xr) = \omega_k(\sigma|x|r) = |x|^k \omega_k(\sigma r).$$

Finally, we end up with

$$\int_{-\infty}^{\infty} \frac{\omega_k(x+y)+\omega_k(x-y)-2\omega_k(x)}{|y|^{1+2k}}\,dy = \int_{-\infty}^{\infty} \frac{\omega_k(x(1+t))+\omega_k(x(1-t))-2\omega_k(x)}{|x|^{2k}|t|^{1+2k}}\,dt$$
$$= |x|^{-k}\int_{-\infty}^{\infty} \frac{\omega_k(\sigma(1+t))+\omega_k(\sigma(1-t))-2\omega_k(\sigma)}{|t|^{1+2k}}\,dt$$
$$= |x|^{-k}\int_{-\infty}^{\infty} \frac{\omega_k(\sigma+t)+\omega_k(\sigma-t)-2\omega_k(\sigma)}{|t|^{1+2k}}\,dt.$$

As a consequence, we can observe that

$$(-\Delta)^k\omega_k(x) = \begin{cases} |x|^{-k}(-\Delta)^k\omega_k(-1) & \text{if } x<0 \\ |x|^{-k}(-\Delta)^k\omega_k(1) & \text{if } x>0. \end{cases}$$

Relying now on Lemma 1.11 and Lemma 1.12, thesis is proved, defining

$$c_k = -(-\Delta)^k\omega_k(-1).$$

□

2 | Existence theorem for nonlocal Schrödinger type equations

2.1. Introduction

We fix $N \geq 2$, N being the space dimension and consider the following system:

$$\begin{cases} (-\Delta)^k u + \lambda u = |u|^{p-2}u & \text{in } \mathbb{R}^N \\ u \in H^k(\mathbb{R}^N),\ p \in (2, 2_k^*), \end{cases} \tag{2.1}$$

where $k \in (0,1)$ and $2_k^* = \frac{2N}{N-2k}$.
Consider now the following problem:

Problem 2.1. *To find a couple* $(u_c, \lambda_c) \in (H^k(\mathbb{R}^N) \times \mathbb{R})$, $u_c(x)$ *being a weak solution to (2.1) for* $\lambda = \lambda_c$, *such that* $\|u_c\|_2^2 = c^2$.

A classical solution to our system would be some function $u \in H^k(\mathbb{R}^N)$ solving (2.1) pointwise and vanishing at infinity. To this end, we could for example look for $u \in C_{loc}^{2k+\epsilon}$. However, Problem 2.1 asks for a weak solution to our system, thus in order to deal with a well posed problem, we have to analyze its formal definition and to define a correct variational framework.

Theorem 2.2. *The weak formulation of system (2.1) can be read as follows: look for a function* $u \in H^k(\mathbb{R}^N)$ *such that*

$$\frac{C_{N,k}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\, dxdy + \lambda \int_{\mathbb{R}^N} uv = \int_{\mathbb{R}^N} |u|^{p-2}uv, \tag{2.2}$$

for any $v \in H^k(\mathbb{R}^N)$ *and* $p \in (2, 2_k^*)$.

Proof. We suppose that $u \in H^k(\mathbb{R}^N)$ is a classical solution to equation (2.1), multiply the equation by $v \in D(\mathbb{R}^N)$ and integrate. We retrieve that

$$\begin{aligned}\int_{\mathbb{R}^N} (-\Delta)^k u(x)v(x)\,dx &= \int_{\mathbb{R}^N} \left[C_{N,k} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2k}}\,dy \right] v(x)\,dx \\ &= C_{N,k} \lim_{\epsilon\to 0} \iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2k}} v(x)\,dxdy.\end{aligned}$$

Then, we can exploit the domain symmetry in the following way:

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2k}} v(x)\,dxdy = \int_{\mathbb{R}^N} \left[\int_{\{|x-y|\geq\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2k}} v(x)\,dx \right] dy \\ &= \int_{\mathbb{R}^N} \left[\int_{\{|y-x|\geq\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2k}} v(x)\,dy \right] dx = \int_{\mathbb{R}^N} \left[\int_{\{|x-y|\geq\epsilon\}} \frac{u(y)-u(x)}{|x-y|^{N+2k}} v(y)\,dx \right] dy \\ &= \iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{u(y)-u(x)}{|x-y|^{N+2k}} v(y)\,dxdy.\end{aligned}$$

This chain of equalities is true; indeed, on one side, by $u \in H^k(\mathbb{R}^N)$ and $v \in D(\mathbb{R}^N)$ we can use Hölder, obtaining

$$\iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{|u(x)-u(y)|}{|x-y|^{N+2k}} |v(x)|\,dxdy \leq \|v\|_\infty \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|}{|x-y|^{N+2k}}\,dxdy \leq \infty.$$

Thus, the Fubini theorem can be applied and switching the order of integration is allowed. On the other side, then, if we define

$$f(x,y) = \frac{u(x)-u(y)}{|x-y|^{N+2k}},$$

we notice that $f(x,y) = -f(y,x)$. Moreover, if we consider that our double integral fixes x and integrates over y taking values in the complementary space of $B_\epsilon(x)$, we obtain the aforementioned result. Exploiting the previous equality we easily obtain

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2k}} v(x)\,dxdy \\ &\qquad = \frac{1}{2} \iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy\end{aligned}$$

and we are just left to show that

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy \\ &\qquad = \lim_{\epsilon\to 0} \iint_{\mathbb{R}^{2N}\cap\{|x-y|\geq\epsilon\}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy\end{aligned}$$

and this is possible exploiting the fact that both u, v are asked to be at least in $H^k(\mathbb{R}^N)$, then

$$\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))}{|x-y|^{\frac{N}{2}+k}}\frac{(v(x)-v(y))}{|x-y|^{\frac{N}{2}+k}}\,dxdy$$
$$\leq \lfloor u \rfloor_k \lfloor v \rfloor_k.$$

As a consequence, the integral over the whole $\mathbb{R}^{2N}$ is well defined and we are allowed to pass to the limit.
The thesis comes straightforwardly, noticing that $D(\mathbb{R}^N)$ is dense in $H^k(\mathbb{R}^N)$. □

Definition 2.3. *We define weak solution to system (2.1), a function $u \in H^k(\mathbb{R}^N)$ satisfying (2.2).*

Once we have clarified the meaning of weak solution, we are ready to solve Problem 2.1. Indeed, even if it can be proved that, for any $\lambda > 0$ fixed, a weak and positive $u \in H^k(\mathbb{R}^N)$ solving our original system exists (as shown in [14]), we are interested in a particular family of solutions, namely $u \in H^k(\mathbb{R}^N)$ with fixed *mass* $\|u\|_2^2 = c^2$.

2.2. The rescaling argument approach

Problem 2.1 can be solved applying a rescaling argument.
To apply this argument, we start considering a weak solution to system (2.1) for $\lambda = 1$; such a function, as already noticed, exists and we call it u. Then, we set $c_0^2 = \int_{\mathbb{R}^N} u^2$ and $w_{\alpha,q}(x) = \alpha^q u(\alpha x)$. The method aims at finding both $\alpha \in \mathbb{R}$ and $q > 0$ depending on c, such that $w_{\alpha,q}(x)$ solves Problem 2.1 for some $\lambda_c > 0$. Since, as just said, both α and q depend on c, we will make this dependence explicit denoting $w_{\alpha,q}(x)$ by $w_c(x)$. At the end, $(w_c(x), \lambda_c(\alpha))$ will be a solution to Problem 2.1. To avoid heavy notations, in the next calculations, we will denote $w_c(x)$ just as $w(x)$.

The first step consists in studying $(-\Delta)^k w(x)$:

$$\begin{aligned}(-\Delta)^k w(x) &= \frac{C_{N,k}}{2}\int_{\mathbb{R}^N} \frac{w(x)-w(x-y)-w(x+y)}{|y|^{N+2k}}\,dy \\ &= \frac{C_{N,k}}{2}\alpha^q\int_{\mathbb{R}^N} \frac{u(\alpha x)-u(\alpha x-\alpha y)-u(\alpha x+\alpha y)}{|y|^{N+2k}}\,dy \\ &= \frac{C_{N,k}}{2}\alpha^q\int_{\mathbb{R}^N} \frac{u(\alpha x)-u(\alpha x-\tilde{y})-u(\alpha x+\tilde{y})}{|\tilde{y}|^{N+2k}\alpha^{-(N+2k)}}\alpha^{-N}\,d\tilde{y} \\ &= \alpha^{q+2k}(-\Delta)^k u(\alpha x),\end{aligned}$$

where we have used $\tilde{y} = \alpha y$ and $d\tilde{y} = \alpha^N dy$. We can proceed to say

$$\begin{aligned}(-\Delta)^k w(x) &= \alpha^{q+2k}(-\Delta)^k u(\alpha x)\\ &= \alpha^{q+2k}\left(-u(\alpha x) + \frac{\alpha^{q(p-1)}}{\alpha^{q(p-1)}}|u(\alpha x)|^{p-2}u(\alpha x)\right)\\ &= -\alpha^{2k}w(x) + \frac{\alpha^{q+2k}}{\alpha^{q(p-1)}}|w(x)|^{p-2}w(x)\\ &= -\alpha^{2k}w(x) + \alpha^{2q+2k-pq}|w(x)|^{p-2}w(x).\end{aligned}$$

Now we ask for $\alpha^{2q+2k-pq} = 1$ and $\int_{\mathbb{R}^N} w^2 = c^2$, where

$$\int_{\mathbb{R}^N} w^2 = \alpha^{2q}\int_{\mathbb{R}^N} u^2(\alpha x) = \alpha^{2q-N}c_0^2,$$

so that the system becomes

$$\begin{cases} 2q + 2k - pq = 0\\ \alpha^{\frac{4k}{p-2}-N}c_0^2 = c^2.\end{cases}$$

This system admits one solution under the hypothesis $\frac{4k}{p-2} - N \neq 0$ and, in particular, it is solved by $q = \frac{2k}{p-2}$, $\alpha = (\frac{c}{c_0})^{\frac{2(p-2)}{4k-N(p-2)}}$ and then

$$\begin{cases} w_c(x) = \alpha^q u(\alpha x) = \left(\dfrac{c}{c_0}\right)^{\frac{4k}{4k-N(p-2)}} u\left(\left(\dfrac{c}{c_0}\right)^{\frac{2(p-2)}{4k-N(p-2)}} x\right)\\ \lambda_c = \alpha^{2k} = \left(\dfrac{c}{c_0}\right)^{\frac{4k(p-2)}{4k-N(p-2)}}.\end{cases}$$

It is clear that (w_c, λ_c) solves our problem.
If instead $\frac{4k}{p-2} - N = 0$, we are in the so called L^2-*critical* case, namely $p = \frac{4k}{N} + 2$. In this scenario, we notice that $c = c_0$ $\forall\alpha$, meaning that, even if we allow the free parameter λ to vary, the mass is still constant. In this scenario Problem 2.1 is solved just in case $c = c_0$ and admits infinite solutions of the form $(\alpha^q u(\alpha x), \alpha^{2k})$ $\forall\alpha \in \mathbb{R}$.

To briefly recap these lines, we recall that solving Problem 2.1 through a rescaling argument approach, coincides with searching for solutions to system (2.1) without the mass constraint. However, this possibility is strictly connected to the homogeneity of the second term in (2.1).

2.3. The variational approach

In this section, we decide to face the same system (2.1), adopting a variational approach, which requires the introduction of an energy functional whose critical points coincide with the solutions to our original problem. The geometry of the functional will be a core aspect in our analysis, since it is the geometrical structure itself to define the nature of the critical points under investigation and to determine, as a consequence, the method to be used in order to show its existence.

Before starting the discussion, we recall here an important preliminary result, whose proof can be found in Appendix B.

Theorem 2.4. *Let k be in $(0,1)$ and N be greater then $2k$. There exists a positive constant $C(N,k)$ such that, for any measurable and compactly supported function $f:\mathbb{R}^N\to\mathbb{R}$, it holds*

$$\|f\|^2_{2^*_k}\le C(N,k)\lfloor f\rceil^2_k. \tag{2.3}$$

Moreover, the embedding $H^k(\mathbb{R}^N)\hookrightarrow L^p(\mathbb{R}^N)$ is continuous $\forall p\in\ [2,2^*_k]$.

We refer the reader to [10, Theorem 3.2.1] for a more general result.

We now define the energy functional $E: H^k(\mathbb{R}^N)\to\mathbb{R}$

$$E(u):=\frac{C_{N,k}}{4}\lfloor u\rceil^2_k-\frac{1}{p}\|u\|^p_p,\quad p\in(2,2^*_k), \tag{2.4}$$

where $C_{N,k}$ is defined in (1.12).

Lemma 2.5. *Let $k\in(0,1)$ and $A: H^k(\mathbb{R}^N)\to\mathbb{R}$ be the functional defined by*

$$A(u)=\lfloor u\rceil^2_k.$$

Then A is a C^1 functional with

$$\langle dA_u,v\rangle=2\iint_{\mathbb{R}^{2N}}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy, \tag{2.5}$$

for any $u,v\in H^k(\mathbb{R}^N)$.

Proof. We start proving that, for any $u\in H^k(\mathbb{R}^N)$, $dA_u\in(H^k(\mathbb{R}^N))^*$, where dA_u is defined in (2.5). Its linearity is trivial and we verify definition of continuity as follows:

$$\begin{aligned}\|dA_u\|_* &= \sup_{\|v\|_{H^k}=1} |\langle dA_u, v\rangle| \leq 2 \sup_{\|v\|_{H^k}=1} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|}{|x-y|^{N/2+k}} \frac{|v(x)-v(y)|}{|x-y|^{N/2+k}}\,dxdy \\ &\leq 2 \sup_{\|v\|_{H^k}=1} \lfloor v\rfloor_k \lfloor u\rfloor_k \leq 2 \sup_{\|v\|_{H^k}=1} \|v\|_{H^k} \lfloor u\rfloor_k < \infty.\end{aligned}$$

At this point, we would like to show that

$$\lim_{\|v\|_{H^k}\to 0} \frac{|A(u+v)-A(u)-\langle dA_u, v\rangle|}{\|v\|_{H^k}} = 0.$$

Directs calculations yield to

$$\begin{aligned}\lim_{\|v\|_{H^k}\to 0} \frac{|\lfloor u+v\rfloor_k^2 - \lfloor u\rfloor_k^2 - \langle dA_u, v\rangle|}{\|v\|_{H^k}} &= \lim_{\|v\|_{H^k}\to 0} \frac{1}{\|v\|_{H^k}} \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2k}}\,dxdy \\ &= \lim_{\|v\|_{H^k}\to 0} \frac{\lfloor v\rfloor_k^2}{\|v\|_{H^k}}.\end{aligned}$$

This limit exists and equals 0, indeed

$$0 \leq \lim_{\|v\|_{H^k}\to 0} \frac{\lfloor v\rfloor_k^2}{\|v\|_{H^k}} \leq \lim_{\|v\|_{H^k}\to 0} \frac{\lfloor v\rfloor_k^2}{\lfloor v\rfloor_k} = \lim_{\|v\|_{H^k}\to 0} \lfloor v\rfloor_k = 0.$$

The proof is complete. □

Lemma 2.6. *Let $k \in (0,1)$ and $J : H^k(\mathbb{R}^N) \to \mathbb{R}$ be the functional defined by*

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \quad p \in (2, 2_k^*).$$

Then J is a C^1 functional with

$$\langle dJ_u, v\rangle = \int_{\mathbb{R}^N} |u|^{p-2} uv, \tag{2.6}$$

for any $u, v \in H^k(\mathbb{R}^N)$.

Proof. We start the proof showing that the Gateaux derivative for J exists and coincides with (2.6). The first step consists in proving that, for any $u \in H^k(\mathbb{R}^N)$, $dJ_u \in (H^k(\mathbb{R}^N))^*$.

We call $t := 2_k^*(p-1)/(2_k^*-1)$ and apply Hölder with $\alpha = \frac{2_k^*}{2_k^*-1}$ and $\beta := 2_k^*$ as follows

$$\begin{aligned}\|dJ_u\|_* = \sup_{\|v\|_{H^k}=1} |\langle dJ_u, v\rangle| &\leq \sup_{\|v\|_{H^k}=1} \int_{\mathbb{R}^N} |u|^{p-1} v \leq \sup_{\|v\|_{H^k}=1} \left(\int_{\mathbb{R}^N} |u|^t\right)^{1/\alpha} \|v\|_{2_k^*} \\ &\leq \sup_{\|v\|_{H^k}=1} C \left(\int_{\mathbb{R}^N} |u|^t\right)^{1/\alpha} \|v\|_{H^k},\end{aligned}$$

where we have used (2.3) and the density of $\mathcal{D}(\mathbb{R}^N)$ in $H^k(\mathbb{R}^N)$. We remark that the integral in the parenthesis is finite since

$$2 < \frac{2N}{N+2k} = \frac{2_k^*(2-1)}{2_k^*-1} < t < 2_k^* \quad \forall p \in (2, 2_k^*),\ \forall N \geq 2$$

and by Theorem 2.4 we have that $H^k(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous $\forall p \in\ [2, 2_k^*]$. We shall now obtain that

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \langle dJ_u, v\rangle,$$

with dJ_u as in (2.6). By the Lagrange theorem we know that there exists some $\theta \in \mathbb{R}$, $|\theta| \leq |t| \leq 1$ such that

$$\left|\frac{|u+tv|^p - |u|^p}{t}\right| \leq p|u+\theta v|^{p-2}(u+\theta v)v$$

and, as a consequence, dominated convergence can be applied again with exponents α and β, in order to obtain

$$\lim_{t\to 0} \frac{J(u+tv) - J(u)}{t} = \frac{1}{p}\int_{\mathbb{R}^N} \lim_{t\to 0}\left(\frac{|u+tv|^p - |u|^p}{t}\right). \tag{2.7}$$

Since the limit at the right hand side of (2.7) defines a classical derivative, we infer that J is the Gateaux derivative we are looking for.

At this point we would like to show that $J' : u \in H^k(\mathbb{R}^N) \mapsto J'_u \in (H^k(\mathbb{R}^N))^*$ is continuous. In this aim, we consider $(u_n)_n \subset H^k(\mathbb{R}^N)$ such that $u_k \to u$ for some $u \in H^k(\mathbb{R}^N)$ and we shall prove that

$$\lim_{n\to\infty} \|J'_{u_n} - J_u\|_* \to 0, \tag{2.8}$$

where

$$\|J'_{u_n} - J_u\|_* = \sup\left\{|\left(J'_{u_n} - J_u\right)v|,\ v \in H^k(\mathbb{R}^N), \|v\|_{H^k} = 1\right\}.$$

We notice that

$$\left|\left(J'_{u_n} - J'_u\right) v\right| = \left|\int_{\mathbb{R}^N} \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right) v\right| \le \left(\int_{\mathbb{R}^N} \left(u_n^{p-2}u_n - |u|^{p-2}u\right)^\alpha\right)^{1/\alpha} \|v\|_{2_k^*}$$
$$\le C\left(\int_{\mathbb{R}^N} \left(u_n^{p-2}u_n - |u|^{p-2}u\right)^\alpha\right)^{1/\alpha} \|v\|_{H^k},$$

where we have used Theorem 2.4. If we are able to show that the integral in the parenthesis vanishes as $n \to \infty$, we have proved the continuity in the dual space of J'. Exploiting the continuous embedding of $H^k(\mathbb{R}^N)$ in $L^p(\mathbb{R}^N)$, $p \in (2, 2_k^*)$, we can rely on the inverse dominated convergence theorem ([8, Theorem 4.9]) to obtain that, up to a subsequence, there exists a function $h \in L^p(\mathbb{R}^N)$ such that

$$|u_n(x)| \le h(x) \text{ a.e. in } \mathbb{R}^N, \ \forall n.$$

Thus we retrieve

$$\int_{\mathbb{R}^N} \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)^\alpha \le C\int_{\mathbb{R}^N} |u_n|^\alpha + |u|^\alpha \le C\int_{\mathbb{R}^N} |h|^\alpha + |u|^\alpha < \infty.$$

Moreover, since, again up to subsequence, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$, and $f : x \in \mathbb{R}^N \mapsto |u(x)|^{p-2}u(x)$ is continuous, then (2.8) is true. Finally, we want to show (2.8) for the whole sequence $(u_n)_n$ and not just for a subsequence. By contradiction we assume that

$$\exists n_j \to \infty, \ \bar{\epsilon} > 0 \text{ such that } \|J'_{u_{n_j}} J'_u\|_* \ge \bar{\epsilon} \quad \forall n_j. \tag{2.9}$$

Since, instead, we know that $u_{n_j} \to u$ in $H^k(\mathbb{R}^N)$ we can repeat the aforementioned reason in order to show a subsequence on n_j contradicting (2.9). Exploiting finally the famous total differentiation lemma, since we have shown that J is Gateaux differentiable in $H^k(\mathbb{R}^N)$ and that J' is continuous for any $u \in H^k(\mathbb{R}^N)$, we obtain that J is Fréchet differentiable and J' coincides with the Fréchet derivative. The theorem is proved. □

Theorem 2.7. *Let E be the functional defined in* (2.4). *Then $E \in C^1(H^k(\mathbb{R}^N))$.*

Proof. This proof is a trivial corollary of Lemma 2.5 and Lemma 2.6. □

Since E is differentiable with continuous derivatives, we can adopt a variational approach. In particular, if we define the $L^2(\mathbb{R}^N)$ sphere

$$S_c := \{v \in H^k(\mathbb{R}^N) : \int_{\mathbb{R}^N} |v|^2 = c^2\},$$

we aim at showing that solutions to our original system coincide with critical points for the energy functional E constrained to S_c. In particular, if we set $K : H^k(\mathbb{R}^N) \to \mathbb{R}$, $K(v) = \frac{\|v\|^2}{2} - \frac{c^2}{2}$, we are ready to prove this result through the following theorem.

Theorem 2.8. *$u \in H^k(\mathbb{R}^N)$ weakly solves the problem if and only if u is a constrained critical point for E on S_c.*

Proof. We start the proof defining the Lagrangian:

$$L(v, \lambda) = E(v) + \lambda K(v)$$

and we would like to show that a function u is a critical point for E constrained to S_c if and only if u is a free critical point for the Lagrangian. In other words, if we define the tangent space to u in S_c (see Definition (A.1) for a general introduction to tangent spaces) as

$$T_u := \left\{ v \in H^k(\mathbb{R}^N) : \langle u, v \rangle_2 = 0 \right\},$$

we are going to prove that

$$\langle dE_u, v \rangle = 0 \quad \forall v \in T_u \iff \exists \lambda : \langle dE_u + \lambda\, dK_u, \phi \rangle = 0 \quad \forall \phi \in H^k(\mathbb{R}^N). \tag{2.10}$$

We start focusing on the left-to-right implication and we notice that, for any $\phi \in H^k(\mathbb{R}^N)$, we can define $\tilde{\phi} \in T_u$

$$\tilde{\phi} = \phi - \frac{\langle \phi, u \rangle_2}{\|u\|_2^2} u.$$

From this definition, remembering $\langle dK_u, \phi \rangle = \langle \phi, u \rangle_2$ by definition of K, we can reason as follows:

$$\begin{aligned} 0 = \langle dE_u, \tilde{\phi} \rangle &= \langle dE_u, \phi \rangle - \langle dE_u, u \rangle \frac{\langle \phi, u \rangle_2}{\|u\|_2^2} \\ &= \langle dE_u, \phi \rangle + \left(-\frac{\langle dE_u, u \rangle}{\|u\|_2^2} \right) \langle dK_u, \phi \rangle \quad \forall \phi \in H^k(\mathbb{R}^N). \end{aligned}$$

Then thesis comes immediately fixing λ as the quantity in the parentheses and noticing that it just depends on u.

For what concerns the right-to-left implication in (2.10), it is proved by

$$\langle dE_u, v \rangle = -\lambda \int_{\mathbb{R}^N} uv \quad \forall v \in H^k(\mathbb{R}^N) \implies \langle dE_u, v \rangle = 0 \quad \forall v \in T_u$$

To complete the proof then, it is now sufficient to explicit the meaning for (u, λ_c) to be a free critical point for $L(v,\lambda)$ and this actually means solving

$$\begin{cases} \frac{\partial L}{\partial v}(u,\lambda_c) \equiv 0 \\ \frac{\partial L}{\partial \lambda}(u,\lambda_c) \equiv 0, \end{cases}$$

that explicitly becomes, for any $v \in H^k(\mathbb{R}^N)$,

$$\begin{cases} \frac{C_{N,k}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy + \lambda_c \int_{\mathbb{R}^N} uv = \int_{\mathbb{R}^N} |u|^{p-2}uv \\ \|u\|_{L^2(\mathbb{R}^N)} = c. \end{cases}$$

It is clear that this problem coincides with the weak formulation of the original system. □

What we are left to study now, is the nature of the critical points of E. Depending on the value of p indeed, E can be bounded from below or not. If E is bounded from below, we can adopt minimization techniques while, if not, we must rely on min-max methods. To study the geometry of the functional, we exploit both Theorem B.4 with $q = 2$ and $r = 2_k^*$ and Theorem 2.4, that still holds true for our weak solution since $D(\mathbb{R}^N)$ is dense in $H^k(\mathbb{R}^N)$. The combination of these two theorems leads us to the fractional Gagliardo-Niremberg inequality:

Lemma 2.9 (Fractional Gagliardo-Niremberg inequality)**.** *Let N be the space dimension, $p \in (2, 2_k^*)$ and $u \in H^k(\mathbb{R}^N)$, with $k \in (0,1)$. Then*

$$\|u\|_p \le C\|u\|_{2_k^*}^{\theta}\|u\|_2^{1-\theta} \le C\lfloor u\rfloor_k^{\theta}\|u\|_2^{1-\theta},$$

with $\theta = \frac{(p-2)N}{2pk}$.

Proof. The proof is direct if we make use of Theorem B.4 with $q = 2$, $r = 2_k^*$ and of Theorem 2.4. □

Lemma 2.9 allows us to bound from below our functional $E(u)$ on S_c in this way:

$$E(u) \ge \frac{C_{N,k}}{4}\lfloor u\rfloor_k^2 - C\lfloor u\rfloor_k^{p\theta}. \tag{2.11}$$

At this point, we can look for conditions on p that ensure $\inf_{S_c} E(u) > -\infty$ and these are

$$p\theta < 2 \iff \frac{(p-2)N}{2k} < 2 \iff p < \frac{4k}{N} + 2.$$

On the other side, let $p \in (\frac{4k}{N}+2, 2_k^*)$ and set $H(u,s) : (S_c \times R) \to S_c$, where $H(u,s)(x) = e^{\frac{sN}{2}} u(e^s x)$. Its values are in S_c since

$$\begin{aligned}\|H(u,s)\|_2^2 &= e^{sN} \int_{\mathbb{R}^N} |u(e^s x)|^2 \, dx \\ &= e^{sN} \int_{\mathbb{R}^N} |u(y)|^2 e^{-sN} \, dy \ = \|u\|_2^2\end{aligned}$$

and

$$\begin{aligned}\lfloor H(u,s) \rfloor_k^2 &= \iint_{\mathbb{R}^{2N}} \frac{|H(u,s)(x) - H(u,s)(y)|^2}{|x-y|^{N+2k}} \, dxdy \\ &= \iint_{\mathbb{R}^{2N}} e^{sN} \frac{|u(e^s x) - u(e^s y)|^2}{|x-y|^{N+2k}} \, dxdy \\ &= e^{2ks} \iint_{\mathbb{R}^{2N}} \frac{|u(z) - u(\omega)|^2}{|z-\omega|^{N+2k}} \, dzd\omega \\ &= e^{2ks} \lfloor u \rfloor_k^2.\end{aligned}$$

If then, we fix $u \in S_c$ and study $E(H(u,s))$, with s varying in $\mathbb{R}$, we obtain

$$\begin{aligned}E(H(u,s)) &= \frac{C_{N,k}\, e^{2ks}}{4} \lfloor u \rfloor_k^2 - \frac{1}{p} \int_{\mathbb{R}^N} e^{\frac{sNp}{2}} u^p(e^s x) \, dx \\ &= \frac{C_{N,k}\, e^{2ks}}{4} \lfloor u \rfloor_k^2 - \frac{e^{\frac{sNp}{2} - sN}}{p} \|u\|_p^p.\end{aligned}$$

Exploiting Theorem 2.4, we know that, if $u \in S_c$, then $u \in L^p(\mathbb{R}^N)$, since $p \in (2, 2_k^*)$ by definition of E. Since $p > \frac{4k}{N} + 2$, we obtain

$$\frac{sNp}{2} - sN > \left(2 + \frac{4k}{N}\right) \frac{sN}{2} - sN = 2ks$$

and it is immediate that $E(H(u,s)) \to -\infty$ as $s \to \infty$.

Therefore, we can recap what we showed in this paragraph through the following scheme:

$$\begin{aligned}
&2 < p < \frac{4k}{N} + 2 && \implies \inf_{S_c} E > -\infty \longrightarrow \text{minimization techniques} \\
&\frac{4k}{N} + 2 < p < 2_k^* && \implies \inf_{S_c} E = -\infty \implies \text{min-max methods.}
\end{aligned}$$

We just remark that, in the first line, we used an arrow and not an implication symbol since, in order to use minimization techniques, we should prove that the infimum for E is effectively reached, but we will not address this topic here.

If finally $p = \frac{4k}{N} + 2$, meaning that $p\theta = 2$, then $\inf_{S_c} E$ depends on c. In particular, in this case we want to name T the product of the constants coming from Theorem 2.4 and Theorem B.4, in order to make the mass value c appear explicitly. Then, equation (2.11) takes the following form

$$E(u) \geq \frac{C_{N,k}}{4} \lfloor u \rfloor_k^2 - Tc^{p(1-\theta)} \lfloor u \rfloor_k^{p\theta},$$

that, with $p\theta = 2$, becomes

$$E(u) \geq \left(\frac{C_{N,k}}{4} - Tc^{p-2} \right) \lfloor u \rfloor_k^2.$$

From this inequality it is trivial that we can guarantee that E is bounded from below only for small masses and, more precisely, for $c \leq (\frac{C_{N,k}}{2T})^{\frac{1}{p-2}}$.

2.4. Problem dissertation and resolution

In the introduction of this chapter we considered the homogeneous forcing term $|u|^{p-2}u$, with $p \in (2, 2_k^*)$. Then, analyzing the variational approach, we proved how $p \in (\frac{4k+2N}{N}, 2_k^*)$ implies that the functional E is not bounded from below. From now on, we will consider the unique case of p in $(\frac{4k+2N}{N}, 2_k^*)$, but we propose to face a more general case, where the forcing term is no more homogeneous. As a consequence, the rescaling technique previously adopted becomes unsuitable in order to solve problems of the type of (2.1) and we have to search for new methods.
In particular, throughout this section, we will face a nonlinear eigenvalue system of the form

$$(-\Delta)^k u(x) - g(u(x)) = \lambda u(x) \qquad x \in \mathbb{R}^N,\ N \geq 2,\ k \in (0,1),\ \lambda \in \mathbb{R} \tag{2.12}$$

with $u \in S_c$ and where the function g is asked to satisfy the following hypotheses:
(H1) $g : \mathbb{R} \to \mathbb{R}$

(H2) there exists $(\alpha, \beta) \in (\mathbb{R} \times \mathbb{R})$ satisfying

$$\begin{cases} \frac{4k+2N}{N} < \alpha \leq \beta < 2_k^* & N \geq 3 \\ \frac{4k+2N}{N} < \alpha \leq \beta & N = 2, \end{cases}$$

such that $\alpha G(s) \leq g(s)s \leq \beta G(s)$, with $G(s) := \int_0^s g(\tau)$.

(H3) if we define $\tilde{G}(s) = g(s)s - 2G(s)$, we ask for the existence of $\tilde{G}'(s)$ such that

$$\tilde{G}'(s)s > \frac{2N+4k}{N}\tilde{G}(s).$$

An example of admissible g is of the type:

$$g(u) = \sum_{i=1}^{m} \alpha_i |u|^{p_i-2}u \qquad p_i \in \left(\frac{4k+2N}{N}, 2_k^*\right) \ \forall i, \tag{2.13}$$

for some $m \in \mathbb{N}$.

We now state the main theorem we are going to prove in the remainder of this thesis:

Theorem 2.10. *There exists a couple $(u_c, \lambda_c) \in (H_r^k(\mathbb{R}^N) \times \mathbb{R})$ weakly solving (2.12), such that $\|u_c\|_2^2 = c^2$.*

Here $H_r^k(\mathbb{R}^N)$ denotes the space of radially symmetric function in $H^k(\mathbb{R}^N)$. We will firstly work in $H^k(\mathbb{R}^N)$ and just at a later stage, for reasons that will be specified, we will restrict our domain to $H_r^k(\mathbb{R}^N)$.

Similarly to the previous case, the weak formulation of system (2.12) set in $H^k(\mathbb{R}^N)$ reads as follows: look for a function $u \in H^k(\mathbb{R}^N)$ such that

$$\frac{C_{N,k}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy + \lambda\int_{\mathbb{R}^N} u(x)v(x)\,dx = \int_{\mathbb{R}^N} g(u(x))v(x)dx \tag{2.14}$$

for any $v \in H^k(\mathbb{R}^N)$.

If we consider g as in (2.13), the energy functional $F(u) : H^k(\mathbb{R}^N) \to \mathbb{R}$ used in the variational approach is

$$F(u) = \frac{C_{N,k}}{4}\lfloor u \rfloor_k^2 - \sum_{i=1}^{m} \frac{\alpha_i}{p_i}\|u\|_{p_i}^{p_i} \qquad p_i \in \left(\frac{4k+2N}{N}, 2_k^*\right) \ \forall i.$$

Let now pass to a general description of the strategy we are going to adopt in order to prove Theorem 2.10. Firstly, as in the motivational example, solutions to equation (2.12) are critical points for F constrained to the L^2 sphere S_c. Thus, again, the nature of its critical points depends on the geometrical properties of the functional, which change as the values of p_i change. It holds true that, if $0 < p_i < \frac{4k+2N}{N}$ $\forall i$, we have $\inf_{S_c} F > -\infty$ and the problem can be solved proving that the infimum for F is reached.

In this study, instead, we focus on nonlinearities of the kind

$$\frac{4k+2N}{N} < p < 2_k^* \text{ if } N \geq 3\,,\ p > \frac{4k+2N}{N} \text{ if } N = 2,$$

since, if this is the case, the geometry for F changes, with $\inf_{S_c} F = -\infty$. Thus, it is no more possible to look for a minimum for F on S_c. Intuition leads us to think that, if F owns some kind of mountain pass geometrical structure on S_c, we can look for its critical points at some level $\gamma(c)$.
As a consequence, at first, we analyze the geometry of this functional and prove that exist $u_1, u_2 \in S_c$, satisfying

$$\gamma(c) \equiv \inf_{g\in\Gamma(c)} \max_{s\in[0,1]} F(g(s)) > \max\{F(u_1), F(u_2)\},$$

with

$$\Gamma(c) = \{g \in C([0,1], S_c),\ g(0) = u_1, g(1) = u_2\}.$$

Relying on these information on F, we could apply a standard version of the mountain pass theorem on $F|_{S_c}$ to guarantee the existence of a Palais-Smale sequence at level $\gamma(c)$ (see [1]). However, since we work on S_c, standard methods used to prove boundedness of a Palais-Smale sequence in superlinear problems, do not work. To overcome this issue, we introduce the auxiliary functional $\tilde{F}(u,s) : (H^k(\mathbb{R}^N) \times \mathbb{R}) \to \mathbb{R}$

$$\tilde{F}(u,s) = \frac{C_{N,k}\, e^{2ks}}{4} \lfloor u \rfloor_k^2 - e^{-\,sN} \int_{\mathbb{R}^N} G(e^{\frac{sN}{2}} u(x))\, dx$$

with $G(t) = \int_0^t g(\tau)\, d\tau$ and we show that $\tilde{F}$ on $(S_c \times \mathbb{R})$ has the analogue geometrical properties as F on S_c. Through these information, we extract a Palais-Smale sequence $(u_n, s_n)_n \subset (S_c \times \mathbb{R})$ for $\tilde{F}$. The fact that $\frac{\partial}{\partial s}\tilde{F}(u_n, s_n) \to 0$ and that $\tilde{F}(u_n, s_n)$ is bounded, imply that $(v_n)_n := (H(u_n, s_n))_n$ is bounded in $H^k(\mathbb{R}^N)$. The fact that $\frac{\partial}{\partial s}\tilde{F}(u_n, s_n)|_{(S_c\times\mathbb{R})} \to 0$, impies that $(v_n)_n$ is a bounded Palais-Smale sequence for F on S_c at level $\gamma(c)$. Namely, $(v_n)_n$ is such that

$$\begin{cases} F(v_n) \to \gamma(c) \\ F'|_{S_c}(v_n) \to 0, \end{cases}$$

where $F'|_{S_c}$ denotes the constrained differential of F on S_c.
At this point we wish to prove that $v_n \to v$ in $H^k(\mathbb{R}^N)$, for some $v \in H^k(\mathbb{R}^N)$. The last difficulty we face is the unboundedness of the domain, that causes an apparent lack of compactness. The invariance under the group of translation in $\mathbb{R}^N$ prevents the embeddings

$H^k(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ from being compact for any p. To overcome this obstacle, instead of working in $H^k(\mathbb{R}^N)$, we frame our variational procedure into the subspace $H^k_r(\mathbb{R}^N)$. Exploiting Theorem B.6, we will recover $H^k_r(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N)$, for any $2 < p < 2^*_k$ if $N \geq 3$ or $p > 2$ if $N = 2$. Once compactness is recovered, we proceed to prove that $(v_n)_n$ converges in $H^k_r(\mathbb{R}^N)$.

2.4.1. Existence of solutions with prescribed norms

In this section we assume $c > 0$ to be fixed and recall, for the reader's convenience, the problem we are investigating, which is

$$(-\Delta)^k u(x) - g(u(x)) = \lambda u(x) \qquad x \in \mathbb{R}^N,\ N \geq 2,\ k \in (0,1),\ \lambda \in \mathbb{R},$$

under the constraint

$$\int_{\mathbb{R}^N} |u|^2 = c^2.$$

We start listing some properties of G that come up from our hypotheses.

Lemma 2.11. *From (H1) and (H2), it follows that, for all $t \in \mathbb{R}$ and $s \geq 0$*

$$\begin{cases} s^\beta G(t) \leq G(ts) \leq s^\alpha G(t) & \text{if } s \leq 1 \\ s^\alpha G(t) \leq G(ts) \leq s^\beta G(t) & \text{if } s \geq 1. \end{cases} \tag{2.15}$$

Proof. We define the following function

$$\phi_t(s) = s^\beta G(t) - G(ts) \quad \forall t \in \mathbb{R}$$

and stress that $\phi_t(1) = 0$. Studying its derivative, we obtain

$$\begin{aligned} \phi_t'(s) &= \beta s^{\beta-1} G(t) - g(ts)t = \frac{\beta}{s}\left[s^\beta G(t) - g(ts)\frac{ts}{\beta}\right] \\ &\geq \frac{\beta}{s}[s^\beta G(t) - G(ts)] = \frac{\beta}{s}\phi_t(s). \end{aligned}$$

This implies that $(s^{-\beta}\phi_t(s))' > 0$ and, as a consequence, that

$$\begin{cases} s \leq 1 \implies s^\beta G(t) - G(ts) \leq 0 & \implies G(ts) \geq s^\beta G(t) \quad \forall t \in \mathbb{R} \\ s \geq 1 \implies s^\beta G(t) - G(ts) \geq 0 & \implies G(ts) \leq s^\beta G(t) \quad \forall t \in \mathbb{R}. \end{cases}$$

The reverse inequality for α comes considering $\psi_t(s) = s^\alpha G(t) - G(ts)$ and noticing that $(s^{-\alpha}\psi_t(s))' < 0$. □

Lemma 2.12. *$G(s)$ is even and $G(v) \geq 0$ for any $v \in S_c$.*

Proof. $G(s)$ is trivially even, since it is the primitive of an odd function. To prove positivity, exploit $G(s) = G(-s)$ to take a generic $v \in S_c$, $v \geq 0$ and write v as follows:

$$v = \mathbb{1}_{[v\leq 1]} + \mathbb{1}_{[v\geq 1]} = v_1 + v_2.$$

It is clear that

$$G(v) = G(v_1) + G(v_2).$$

Exploiting $G(v) = G(1 \cdot v)$ and (2.15) we obtain

$$G(v) \geq G(1)[v_1^\beta + v_2^\alpha] \geq 0,$$

since $v_1 \geq 0$, $v_2 \geq 0$ and $G(1) > 0$. The positivity of $G(1)$ can be checked exploiting (2.15) with $t = 1$ and e.g. $s = 2$, that give us

$$2^\alpha G(1) \leq 2^\beta G(1).$$

This inequality, recalling $\alpha < \beta$, can be true only if $G(1) > 0$. □

Lemma 2.13. *From (H1), (H2) and from the definition of $\tilde{G}$, we get for any $s \in \mathbb{R}$*

$$\begin{cases} \frac{1}{\beta-2}\tilde{G}(s) \leq G(s) \leq \frac{1}{\alpha-2}\tilde{G}(s) \\ \frac{\beta}{\beta-2}\tilde{G}(s) \leq g(s)s \leq \frac{\alpha}{\alpha-2}\tilde{G}(s). \end{cases} \tag{2.16}$$

Proof. We are just going to prove the inequalities with β, the ones with α can be retrieved in the same way. Consider that

$$\frac{1}{\beta-2}\tilde{G}(s) = \frac{1}{\beta-2}g(s)s - \frac{2}{\beta-2}G(s) \leq \frac{\beta}{\beta-2}G(s) - \frac{2}{\beta-2}G(s) = G(s).$$

Concerning the second inequality, we write

$$\frac{\beta}{\beta-2}\tilde{G}(s) = \frac{\beta}{\beta-2}g(s)s - \frac{2\beta}{\beta-2}G(s) \leq \frac{\beta}{\beta-2}g(s)s - \frac{2}{\beta-2}g(s)s = g(s)s.$$

□

From now on, we will call $H = L^2(\mathbb{R}^N)$ while $E = H^k(\mathbb{R}^N)$, where we use

$$\langle u, v\rangle_H := \int_{\mathbb{R}^N} u(x)v(x)\,dx,$$

$$\langle u, v\rangle_E := \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\,dxdy + \langle u, v\rangle_H$$

as scalar product. Moreover, we consider the spaces $(H, \langle\cdot,\cdot\rangle_H)$ and $(E, \langle\cdot,\cdot\rangle_E)$ equipped with the induced norms

$$\|u\|_H^2 := \int_{\mathbb{R}^N} |u(x)|^2\,dx \qquad \|u\|_E^2 := \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dxdy + \|u\|_H^2.$$

Thus, the main definitions we need, adapted to these new formalisms, are :

1. $S_c := \{u \in E,\ \|u\|_H = c\}$

2. $F : E \to \mathbb{R}$
$$F(u) = \frac{C_{N,k}}{4}\lfloor u\rfloor_k^2 - \int_{\mathbb{R}^N} G(u)$$

3. $H : E \times \mathbb{R} \to E$
$$H(u,s)(x) = e^{\frac{sN}{2}} u(e^s x)$$

4. $\tilde{F} : E \times \mathbb{R} \to \mathbb{R}$
$$\tilde{F}(u,s) = F(H(u,s)).$$

Recalling that $\lfloor H(u,s)\rfloor_k^2 = e^{2ks}\lfloor u\rfloor_k^2$ and $\|H(u,s)\|_2 = \|u\|_2$, we can explicitly write

$$\tilde{F}(u,s) = \frac{C_{N,k}\,e^{2ks}}{4}\lfloor u\rfloor_k^2 - e^{-\,sN}\int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(x))\,dx. \tag{2.17}$$

Finally, we have already noticed that $H(u,s)$ is a transformation from $(S_c \times \mathbb{R})$ to S_c.

Before closing this section, we state and prove a lemma that allows us to express $\tilde{F}$ as function of $H(u,s)$.

Lemma 2.14. *If we set $v = H(u,s)$, we have*

$$\tilde{F}(u,s) = \frac{C_{N,k}}{4}\lfloor v \rfloor_k^2 - \int_{\mathbb{R}^N} G(v).$$

Proof. Starting from equation (2.17), we can write

$$\begin{aligned}\frac{e^{2ks}}{4}\lfloor u \rfloor_k^2 &= \frac{e^{2ks}}{4}\iint_{\mathbb{R}^{2N}} \frac{|u(z)-u(\omega)|^2}{|z-\omega|^{N+2k}}\,dzd\omega \\ &= \frac{1}{4}\int_{\mathbb{R}^N} \frac{|H(u,s)(x)-H(u,s)(y)|^2}{|x-y|^{N+2k}}\,dxdy = \frac{1}{4}\lfloor v \rfloor_k^2\end{aligned}$$

and

$$e^{-sN}\int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(x))\,dx = \int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(e^s x))\,dx = \int_{\mathbb{R}^N} G(v).$$

□

2.4.2. The mountain pass geometrical structure

In this section we proceed to prove that $\tilde{F}(u,s)$ possesses a mountain pass geometrical structure on $(S_c \times \mathbb{R})$.

In this aim, we start the discussion stating two preliminary lemmas.

Lemma 2.15. *Under (H1) and (H2), if $u \in S_c$, then*

(a) $\lfloor H(u,s) \rfloor_k \to 0$, $\tilde{F}(u,s) \to 0$ *as* $s \to -\infty$

(b) $\lfloor H(u,s) \rfloor_k \to +\infty$, $\tilde{F}(u,s) \to -\infty$ *as* $s \to +\infty$.

Proof. To handle *(a)*, we consider $s < 0$ so that $e^{\frac{sN}{2}} < 1$ and apply (2.15) to obtain

$$\begin{aligned}|\tilde{F}(u,s)| &\le \frac{C_{N,k}\,e^{2ks}}{4}\lfloor u \rfloor_k^2 + e^{-\,sN}\int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(x))\,dx \\ &\le \frac{C_{N,k}\,e^{2ks}}{4}\lfloor u \rfloor_k^2 + e^{sN\frac{\alpha-2}{2}}\int_{\mathbb{R}^N} G(u(x))\,dx,\end{aligned}$$

where we use *(H2)* to state

$$sN\left(\frac{\alpha-2}{2}\right) > sN\left(\frac{2k}{N}\right) = 2sk. \tag{2.18}$$

This proves that $|\tilde{F}(u,s)| \to 0$ as $s \to -\infty$.
At the same time, also $\lfloor H(u,s)\rfloor_k^2 = e^{2ks}\lfloor u\rfloor_k^2 \to 0$ as $s \to -\infty$ and *(a)* is proved.
For what concerns point *(b)*, we fix $s > 0$ so that $e^{\frac{sN}{2}} > 1$ to infer

$$\tilde{F}(u,s) \leq \frac{C_{N,k}\, e^{2ks}}{4}\lfloor u\rfloor_k^2 - e^{sN\frac{\alpha-2}{2}}\int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(x))\,dx,$$

where we have used (2.15).
Exploiting again (2.18), we immediately gather $\tilde{F}(u,s) \to -\infty$ as $s \to +\infty$. Finally, $\lfloor H(u,s)\rfloor_k^2 = e^{2ks}\lfloor u\rfloor_k^2 \to +\infty$ as $s \to +\infty$ and point *(b)* is proved. □

Lemma 2.16. *Under (H1) and (H2), there exists $T(c) > 0$ such that, setting*

$$\begin{cases} A = \{u \in S_c \,:\, \lfloor u\rfloor_k^2 \leq T(c)\} \\ B = \{u \in S_c \,:\, \lfloor u\rfloor_k^2 = 2T(c)\}, \end{cases}$$

we have

$$0 \;<\; \sup_{u\in A} F(u) \;<\; \inf_{u\in B} F(u).$$

Proof. Our first aim is to estimate $G(v)$ from above.
To achieve it, we consider that for any $u \in S_c$, we have

$$u = \mathbb{1}_{[u\leq 1]}u + \mathbb{1}_{[u\geq 1]}u =: u_1 + u_2$$

and the following chain of inequalities, that rely on (2.15), holds:

$$\begin{aligned}\int_{\mathbb{R}^N} G(u) &= \int_{\mathbb{R}^N} G(u_1) + \int_{\mathbb{R}^N} G(u_2) = \int_{\mathbb{R}^N} G(u_1\cdot 1) + \int_{\mathbb{R}^N} G(u_2\cdot 1)\\ &\leq G(1)\int_{\mathbb{R}^N}\left(|u_1|^\alpha + |u_2|^\beta\right) \leq G(1)\int_{\mathbb{R}^N}\left(|u|^\alpha + |u|^\beta\right)\\ &= G(1)(\|u\|_\alpha^\alpha + \|u\|_\beta^\beta).\end{aligned}$$

At this point, relying on Lemma 2.9, we consider α as defined in (H2) and write

$$\|u\|_\alpha \leq C\lfloor u\rfloor_k^\theta \|u\|_2^{1-\theta}, \qquad \theta = \frac{(\alpha-2)N}{2\alpha k}.$$

Moreover, exploiting $u \in S_c$, we obtain $\|u\|_\alpha \leq C\lfloor u\rfloor_k^{\frac{(\alpha-2)N}{2\alpha k}}$. This leads straightforwardly to

$$\int_{\mathbb{R}^N} G(u) \leq C\left(\lfloor u \rfloor_k^{\frac{(\alpha-2)N}{2k}} + \lfloor u \rfloor_k^{\frac{(\beta-2)N}{2k}} \right).$$

Moreover, we use $\beta \geq \alpha$ to get that, for $\lfloor u \rfloor_k$ small enough, it is still true that

$$\int_{\mathbb{R}^N} G(u) \leq C \lfloor u \rfloor_k^{\frac{(\alpha-2)N}{2k}}.$$

At this point, we have bounded $G(u)$ from above and we can proceed to prove the theorem through this strategy.

If we fix $T > 0$ and suppose that $u, v \in S_c$, $\lfloor u \rfloor_k^2 = T$, $\lfloor v \rfloor_k^2 = 2T$, it holds for T small enough

$$\begin{aligned} F(v) - F(u) &= \frac{C_{N,k}}{4}(\lfloor v \rfloor_k^2 - \lfloor u \rfloor_k^2) - \int_{\mathbb{R}^N} G(v) + \int_{\mathbb{R}^N} G(u) \\ &\geq \frac{C_{N,k}}{4}T - \int_{\mathbb{R}^N} G(v) \geq \frac{C_{N,k}}{4}T - CT^{\frac{N(\alpha-2)}{4k}} \\ &> \frac{C_{N,k}}{8}T. \end{aligned}$$

We stress that, for the previous inequalities, we have used

$$\frac{\alpha-2}{4} > \left(\frac{4k+2N}{N} - 2 \right) \frac{1}{4} = \frac{4k}{N} \frac{1}{4} = \frac{k}{N},$$

meaning that $\frac{N(\alpha-2)}{4k} > 1$ and hence, that $T^{\frac{N(\alpha-2)}{4k}}$ goes to 0 faster than T, as $T \to 0$. So, we have proved that $\sup_{u\in A} F(u) < \inf_{u\in B} F(u)$, but we still miss $0 < \sup_{u\in A} F(u)$. To complete the proof, we consider $\tilde{u} \in A$ such that $\lfloor \tilde{u} \rfloor_k^2 = T$, T small. Thus, it satisfies

$$\begin{aligned} F(\tilde{u}) &= \frac{C_{N,k}}{4} \lfloor \tilde{u} \rfloor_k^2 - \int_{\mathbb{R}^N} G(\tilde{u}) \\ &\geq \frac{C_{N,k}}{4}T - CT^{\frac{(\alpha-2)N}{4k}} \\ &> 0 \end{aligned}$$

and the thesis is proved. □

We are now ready to state the proposition that ensures the existence of a mountain pass geometrical structure for $\tilde{F}(u,s)$ on $(S_c \times \mathbb{R})$.

Proposition 2.1. *Assume (H1) and (H2) and let $T = T(c)$ be fixed as in Lemma 2.16. There exist u_1, $u_2 \in S_c$ such that:*

(a) $\lfloor u_1 \rfloor_k^2 \leq T$ *(i.e. $u_1 \in A$)*

(b) $\lfloor u_2 \rfloor_k^2 > 2T$

(c) $F(u_1) > 0 \geq F(u_2)$.

Moreover, defining

$$\tilde{\Gamma}(c) := \{\tilde{h} \in C([0,1], S_c \times \mathbb{R}),\ \tilde{h}(0) = (u_1, 0),\ \tilde{h}(1) = (u_2, 0)\}$$
$$\tilde{\gamma}(c) := \inf_{\tilde{h} \in \tilde{\Gamma}(c)} \max_{t \in [0,1]} \tilde{F}(\tilde{h}(t)),$$

we have, with $\tilde{\gamma}_0(c) := \max\{\tilde{F}(u_1, 0), \tilde{F}(u_2, 0)\}$, *that*

$$\tilde{\gamma}(c) > \tilde{\gamma}_0(c).$$

Proof. Combining both Lemma 2.15 and Lemma 2.16, we can reason as follows, starting from a generic $u \in S_c$.
By Lemma 2.15, we can find $s < 0$, with $|s|$ large enough such that, defining $\bar{u} := H(u, s)$, $\bar{u}$ belongs to A, since $\lfloor \bar{u} \rfloor_k$ approaches 0 as s approaches $-\infty$. In particular, since $\bar{u} \in A$, A is non empty. Thanks to Lemma 2.16 then, being $\sup_{u \in A} F(u)$ strictly positive, there exists $u_1 \in A$ with $F(u_1) > 0$.
Studying instead $H(u, s)$ for s big enough, it is trivial by Lemma 2.15 that there exists u_2, with $\lfloor u_2 \rfloor_k^2 > 2T$ and $F(u_2) < 0$.
So, we have proved points *(a), (b), (c)*, while we miss the last statement of the theorem and we start setting

$$\Gamma(c) := \{h \in C([0,1], S_c),\ h(0) = u_1,\ h(1) = u_2\}$$
$$\gamma(c) := \inf_{h \in \Gamma(c)} \max_{t \in [0,1]} F(h(t)).$$

We can study $\gamma(c)$ and say that, for any $h(t) \in \Gamma(c)$, by continuity there exists $\tilde{t}$ such that $\lfloor h \rfloor_k^2 = 2T$, namely $h(\tilde{t}) \in B$; thus, thanks to Lemma 2.16, we have that

$$F(h(\tilde{t})) > F(h(0)) = F(u_1).$$

This leads us to $F(h(\tilde{t})) > \max\{F(u_1), F(u_2)\} = F(u_1)$ for any $h(t) \in \Gamma(c)$ and, exploiting Lemma 2.16, we can infer that

$$\gamma(c) \geq \inf_{h \in \Gamma(c)} F(h(\tilde{t})) > \sup_A F \geq \max\{F(u_1), F(u_2)\}. \tag{2.19}$$

Moreover, we recall the following chain of equalities:

$$\begin{cases} F(u_1) = F(H(u_1,0)) = \tilde{F}(u_1,0) \\ F(u_2) = F(H(u_2,0)) = \tilde{F}(u_2,0). \end{cases} \tag{2.20}$$

Therefore, combining (2.19) and (2.20) we deduce that $\tilde{\gamma}_0(c) = \max\{F(u_1), F(u_2)\}$ and $\gamma(c) > \tilde{\gamma}_0(c)$. Thus, if we are able to gather $\tilde{\gamma}(c) = \gamma(c)$, then we have completed the proof.

We want to prove the equality $\tilde{\gamma}(c) = \gamma(c)$, showing that both $\tilde{\gamma}(c) \geq \gamma(c)$ and $\tilde{\gamma}(c) \leq \gamma(c)$; this is true since

$$\begin{array}{llll} \forall\, \tilde{h}(t) = (u,s)(t) & \exists h(t) = H((u,s)(t)) \,: & F(h(t)) = \tilde{F}(\tilde{h}(t)) & \Longrightarrow\ \tilde{\gamma}(c) \geq \gamma(c) \\ \forall\, h(t) = u(t) & \exists \tilde{h}(t) = (u,0)(t) \,: & \tilde{F}(\tilde{h}(t)) = F(h(t)) & \Longrightarrow\ \tilde{\gamma}(c) \leq \gamma(c). \end{array}$$

□

2.4.3. The min-max approach

In the previous section, we have shown how $\tilde{F}(u,s)$ possesses a mountain pass geometrical structure on $(S_c \times \mathbb{R})$ and we would like to exploit this information in order to rely on some version of the mountain pass theorem. In this way, we could guarantee the existence of a Palais-Smale sequence at level $\gamma(c)$. Nevertheless the standard hypotheses at the basis of this theorem do not apply in our framework since the sphere S_c is not a Banach space. In fact, in this section, we will state and prove a more general result, the min-max theorem, valid also on differential manifolds. In order to do this, we will implicitly refer to some basic notions of differential geometry, whose formal definitions and statements are collected in Section A of the appendix.

Before starting our analysis we state here an important definition.

Definition 2.17. *Let B be a closed subset of X. A class F of compact subsets of X is an homotopy-stable family with boundary B if*

(a) every set in F contains B

(b) for any set A in F and any $\eta \in C([0,1] \times X, X)$ that satisfies $\eta(t,x) = x$ for all (t,x) in $(\{0\} \times X) \cup ([0,1] \times B)$, we have $\eta(\{1\} \times A) \in F$.

Our first step consists in the statement of an important deformation lemma valid on Finsler manifolds; this lemma will be crucial in the formulation of the main theorem of this section.

Theorem 2.18 (Deformation lemma)**.** *Let ϕ be a C^1 functional on a completed connected C^1 - Finsler manifold X and let B and C be two closed and disjoint subsets of X. Assume that C is compact and that $\|d\phi_x\| > 2\epsilon > 0$ for every $x \in C$. Then, for each $k > 1$, there exists a positive and continuous function g on X and a deformation α in $C([0,1] \times X; X)$ such that, for some $t_0 > 0$, the following holds for every $t \in [0, t_0)$:*

i. $\alpha(t,x) = x$ *for every* $x \in B$

ii. $\rho(\alpha(t,x), x) \leq kt$ *for every* $x \in X$

iii. $\phi(\alpha(t,x)) - \phi(x) \leq -\epsilon g(x) t$ *for every* $x \in X$

iv. $g(x) = 1$ *for every* $x \in C$.

Proof. We fix $k > 1$ and use Definition (A.3) and Theorem A.8 from the appendix to find, for any $x_i \in C$, a neighbourhood U_i of x_i, a chart $f_i : U_i \to T_{x_i}(X)$ such that

$$\frac{1}{k}\|\cdot\|_x \leq \|\cdot\|_{x_i} \leq k\|\cdot\|_x \quad \text{for all } x \in U_i \tag{2.21}$$

and

$$\left\langle (\phi \circ f_i^{-1})'(y), \frac{v_i}{\|v_i\|} \right\rangle \geq \frac{1}{2}\|d\phi(x)\| \quad \text{for all } y \in f_i(U_i). \tag{2.22}$$

For each U_i we set $V_i \subset U_i$, V_i open neighbourhood of x_i, such that, for some $\delta_i > 0$, we have

$$B(V_i, \delta_i) \subset U_i \text{ and } B(f_i(V_i), \delta_i) \subset f_i(U_i),$$

denoting with $B(J, \delta)$ the δ-neighbourhood of the set J in the appropriate metric.
We notice that, since C is compact, it is possible to select a finite covering $(V_i)_{i=1}^m$ and a suitable partition of unity $(\chi_i)_{i=1}^m$ conditioned to $(V_i)_{i=1}^m$.
We now set a continuous function $l : X \to [0,1]$:

$$l(x) = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \in (X \setminus \bigcup_{i=1}^m V_i) \cup B. \end{cases}$$

If we name $\delta_0 = \min\{\delta_1, \delta_2, ..., \delta_m\}$ and $t_0 = \frac{\delta_0}{1+k^2}$, we can start setting the continuous deformation $\alpha_0(t,x) = x$ and define, by induction on j, $1 \le j \le m$, the deformations

$$\alpha_j(t,x) = \begin{cases} f_j^{-1}\left(f_j(\alpha_{j-1}(t,x)) - tl(x)\chi_j(x)\frac{v_i}{\|v_i\|}\right) & \text{if } \alpha_{j-1}(t,x) \in U_j \\ \alpha_{j-1}(t,x) & \text{otherwise.} \end{cases}$$

We will now prove by induction the following properties, valid for $t \in (0,t_0)$:

$$f_j(\alpha_{j-1}(t,x)) - tl(x)\chi(x)\frac{v_i}{\|v_i\|} \in f_j(U_j) \quad \text{if } \alpha_{j-1}(t,x) \in U_j \tag{2.23}$$

$$\rho(\alpha_{j-1}(t,x), \alpha_j(t,x)) \le kl(x)\chi_j(x)t \tag{2.24}$$

$$\phi(\alpha_j(t,x)) - \phi(\alpha_{j-1}(t,x)) \le -\epsilon l(x)\chi_j(x)t. \tag{2.25}$$

We start with $j = 1$ and recall that

$$\alpha_1(t,x) = \begin{cases} f_1^{-1}\left(f_1(x) - tl(x)\chi_1(x)\frac{v_1}{\|v_1\|}\right) & \text{if } x \in U_1 \\ x & \text{otherwise.} \end{cases}$$

Property (2.23) is true indeed, assuming $x \in U_1$, if it holds that $\chi_1(x) \neq 0$, then $x \in V_1$ but $\|tl(x)\chi_1(x)\frac{v_1}{\|v_1\|}\| \le t \le \delta_0$ and, as a consequence

$$f_1(x) - tl(x)\chi_1(x)\frac{v_1}{\|v_1\|} \in B(f_1(V_1), \delta_1) \subset f_1(U_1).$$

If instead $\chi_1(x) = 0$ then, obviously, since by hypothesis $x \in U_1$, property (2.23) is true. For what concerns property (2.24) with $j = 1$, we must study $\rho(x, \alpha_1(t,x))$ and we consider the path $\sigma_1(s) = \alpha_1(s,x)$, $s \in [0,t]$ that connects $x = \sigma_1(0)$ to $\alpha_1(t,x) = \sigma_1(t)$. We have that

$$\begin{aligned} \rho(x, \alpha_1(t,x)) &\le \int_0^t \|\sigma_1'(s)\|\, ds \le k \int_0^t \left\|\frac{d}{ds} f_1(\sigma_1(s))\right\|_{x_1} ds \\ &\le k\, \|f_1(\sigma_1(t)) - f_1(\sigma_1(0))\|_{x_1} = k \left\|\left(f_1(x) - tl(x)\chi_1(x)\frac{v_1}{\|v_1\|}\right) - f_1(x)\right\|_{x_1} \\ &= kl(x)\chi_1(x)t. \end{aligned}$$

Finally, again with $j = 1$, we miss to show (2.25) and, in this aim, we notice that if $x \notin U_1$, then $\alpha_0(t,x) = \alpha_1(t,x) = x$ and (2.25) is trivially satisfied; if instead $x \in U_1$, we can apply the mean value theorem with $\theta \in [0,1]$ and write

$$\begin{aligned}\phi(\alpha_1(t,x)) - \phi(x) &= \phi(\alpha_1(t,x)) - \phi \circ f_1^{-1}(f_1(x)) \\ &= \phi \circ f_1^{-1}\left(f_1(x) - tl(x)\chi_1(x)\frac{v_1}{\|v_1\|}\right) - \phi \circ f_1^{-1}(f_1(x)) \\ &= -tl(x)\chi_1(x)\left\langle (\phi \circ f_1^{-1})'\left(f_1(x) - \theta tl(x)\chi_1(x)\frac{v_1}{\|v_1\|}\right), \frac{v_1}{\|v_1\|}\right\rangle.\end{aligned}$$

It now follows from equation (2.22) that $x \in U_1$ implies that $\phi(\alpha_1(t,x)) - \phi(x) \leq -\epsilon l(x)\chi_1(x)t$ and (2.25) is proved.
Now, we proceed with our proof by induction assuming that (2.23), (2.24), (2.25) are verified up to α_{j-1} and prove them for $\alpha_j(t,x)$.
We start with property (2.23) and, before going on, we stress that, (2.24) being valid up to α_{j-1}, leads to

$$\begin{aligned}\rho(x,\alpha_{j-1}(t,x)) &= \rho(\alpha_0(t,x),\alpha_{j-1}(t,x)) && (2.26)\\ &\leq \sum_{i=1}^{j-1}\rho(\alpha_{i-1}(t,x),\alpha_i(t,x)) \\ &\leq ktl(x)\sum_{i=1}^{j-1}\chi_i(x) \leq kt. && (2.27)\end{aligned}$$

It is worth noticing that, if $x \notin \bigcup_{i=1}^{j-1} U_i$, then $\chi_i(x) = 0$ for any $1 < i < j-1$, leading us to $\sum_{i=1}^{j-1}\chi_i(x) = 0$ and, as a consequence, to $\rho(x,\alpha_{j-1}(t,x)) = 0$.

Thus, from (2.26) and from $t_0 < \frac{\delta_0}{1+k^2}$, we infer that $\rho(x,\alpha_{j-1}(t,x)) \leq \frac{\delta_0}{2}$. This information allows us to say that, for any $x \in$ supp(χ_j) and $\alpha_{j-1}(t,x) \in U_j$, it is true the following:

$$\rho(x,\alpha_{j-1}(t,x)) = \inf\{L(\sigma);\ \sigma \text{ joining } x \text{ to } \alpha_{j-1}(t,x) \text{ and } \sigma \subset U_j\}. \qquad (2.28)$$

Indeed, if we suppose that previous proposition is false, we fall into contradiction because we would have to choose a path σ leaving U_j and force, in this way, the existence of a point $\tilde{x} \notin U_j$, $\tilde{x} \in \sigma$ such that

$$\rho(x,\alpha_{j-1}(t,x)) \geq \rho(x,\tilde{x}) \geq \delta_0,$$

ending to $L(\sigma) \geq \delta_0$.

If now we consider a generic σ joining x to $\alpha_{j-1}(t,x)$, $\sigma \subset U_j$ and study $L(\sigma)$ we obtain

$$\begin{aligned} L(\sigma) = \int_a^b \|\sigma'(s)\|\, ds &\geq \frac{1}{k}\int_a^b \left\| \frac{d}{ds} f_j(\sigma(s))\right\|_{x_j} ds \geq \frac{1}{k}\left\| \int_a^b \frac{d}{ds} f_j(\sigma(s))\, ds \right\|_{x_j} \\ &= \frac{1}{k}\left\| f_j(\sigma(b)) - f_j(\sigma(a))\right\|_{x_j} \\ &= \frac{1}{k}\left\| f_j(\alpha_{j-1}(t,x)) - f_j(x)\right\|_{x_j}. \end{aligned}$$

Since, as stated, this inequality is valid for any path in U_j joining that two points, it is valid also for the infimum of L on that paths, that is, by (2.28), the definition of $\rho(x, \alpha_{j-1})$ and we finally get

$$\left\| f_j(\alpha_{j-1}(t,x)) - f_j(x)\right\|_{x_j} \leq k\rho(x, \alpha_{j-1}(t,x)) \leq k^2 t.$$

Hence

$$\left\| f_j(\alpha_{j-1}(t,x)) - f_j(x) - tl(x)\chi_j(x)\frac{v_j}{\|v_j\|}\right\|_{x_j} \leq k^2 t + t \leq \delta_0.$$

We have proved that $f_j(\alpha_{j-1}(t,x)) - tl(x)\chi(x)\frac{v_i}{\|v_i\|} \in f_j(U_j)$ if both $\alpha_{j-1}(t,x) \in U_j$ and $x \in \operatorname{supp}(\chi_j)$ and, as a consequence, (2.23) is proved.
The next step consists in proving (2.24) and we just consider the path $\sigma_j(s)$, for $0 \leq s \leq t$, such that $\sigma_j(0) = \alpha_{j-1}(t,x)$ and $\sigma_j(t) = \alpha_j(t,x)$, defining it as

$$\sigma_j(s) = f_j^{-1}\left(f_j(\alpha_{j-1}(t,x)) - sl(x)\chi_j(x)\frac{v_j}{\|v_j\|}\right).$$

It is straightforward then

$$\begin{aligned} \rho(\alpha_{j-1}(t,x), \alpha_j(t,x)) \leq L(\sigma_j) &\leq \int_0^t \|\sigma_j(s)'\|\, ds \\ &\leq k\|f_j(\alpha_j(t,x)) - f_j(\alpha_{j-1}(t,x))\|_{x_j} \\ &\leq kl(x)\chi_j(x)t, \end{aligned}$$

where the last inequality holds since, for any $x \in U_j$, we have that $\sigma_j \subset U_j$ by definition and $\alpha_{j-1}(t,x) = x\ \forall x \in U_j$.
We finally have that also (2.25) is true, exploiting the mean value theorem between $\alpha_{j-1}(t,x)$ and $\alpha_j(t,x)$ and recalling that $\alpha_{j-1}(t,x) = x\ \forall x \in U_j$. The induction is complete.
We are now ready to set $\alpha(t,x) = \alpha_m(t,x)$ and $g(x) = l(x)\sum_{i=1}^m \chi_j(x)$. We immediately see that (i) is satisfied by definition of $\chi_m(x)$, (ii) is true thanks to (2.26) with $j-1 = m$,

(iii) comes from

$$\begin{aligned}\phi(\alpha_m(t,x)) - \phi(\alpha_0(t,x)) &= \sum_{j=1}^{m} \phi(\alpha_j(t,x)) - \phi(\alpha_{j-1}(t,x)) \\ &\leq -\epsilon t l(x) \sum_{j=1}^{m} \chi_j(x) = -\epsilon g(x) t,\end{aligned}$$

whereas the definition of $(\chi_j)_j$ partition of unity and of $l(x)$ lead immediately to (iv). $\square$

Theorem 2.19. *Let ϕ be a C^1 functional on a completed connected C^1 Finsler manifold X and consider an homotopy-stable family F of compact subsets of X with a closed boundary B. Name c= $c(\phi, F) := \inf_{A\in F} \max_{x\in A} \phi(x)$ and suppose that*

$$\sup\ \phi(B) < c. \tag{2.29}$$

Then, for any min-maxing sequence A_n for ϕ, meaning $\lim_n \max_{A_n} \phi = c$, there exists $(x_n)_n \in X$ such that:

(a) $\lim_n\ \phi(x_n) = c$

(b) $\lim_n \|d\phi_{x_n}\| = 0$

(c) $\lim_n \inf_{y_n\in A_n} \rho(x_n, y_n) = 0.$

Proof. We start considering a set $\tilde{A} \in F$ such that:

$$c \leq \sup\ \phi(\tilde{A}) < c + \epsilon^2.$$

Then let L be the subspace of $C([0,1] \times X;\ X)$, consisting of all continuous deformations η satisfying:

$$\begin{aligned}&\eta(t,x) = x\ \text{ for all } (t,x) \in (\{0\} \times X) \cup ([0,1] \times B)\\ &\sup\left(\rho(\eta(t,x),x);\ (t,x) \in ([0,1] \times X)\right) \leq \infty.\end{aligned}$$

If we equip L with the following metric:

$$\delta(\eta,\eta') = \sup(\rho(\eta,\eta'),\ (t,x) \in ([0,1] \times X)) \quad \forall \eta,\eta' \in L,$$

the space L becomes a complete metric space.

We define now the functional $I:\ L \to \mathbb{R}$, by $I(\eta) = \sup_{x\in\tilde{A}}(\phi(\eta(1,x)))$ and notice that $\tilde{\eta}(t,x) = x$ for any $(t,x) \in ([0,1] \times X)$ is the identity in L.

Before proceeding, we stress that F being homotopy-stable inside X and $\tilde{A} \in F$, imply that $\eta(1, \tilde{A}) \in F \ \ \forall \eta \in L$ and leads to

$$\{I(\eta),\ \eta \in L\} = \{\sup \phi(\eta(1, \tilde{A})),\ \eta \in L\} \subseteq \{\sup \phi(A),\ A \in F\}. \tag{2.30}$$

In other words, we could say that, naming $A_\eta := \eta(1, \tilde{A})$, the fact that F is homotopy-stable inside X and $\tilde{A} \in F$, imply that $\{A_\eta,\ \eta \in L\} \subseteq F$.
Exploiting this inclusion and the definition of I, we write

$$I(\tilde{\eta}) = \sup_{x \in \tilde{A}} \phi(x) < c + \epsilon^2 \leq \inf_{\eta \in L} I(\eta) + \epsilon^2.$$

The last inequality allows us to apply the variational Ekeland's principle's Corollary B.8, with $\epsilon = \epsilon^2$ and $\lambda = \epsilon$, to retrieve that there exists $\eta_0 \in L$ such that:

$$I(\eta_0) \leq I(\tilde{\eta}) \tag{2.31}$$

$$\delta(\eta_0, \tilde{\eta}) \leq \epsilon \tag{2.32}$$

$$I(\eta) \geq I(\eta_0) - \epsilon\delta(\eta, \eta_0) \ \text{ for all } \eta \text{ in } L. \tag{2.33}$$

If we set $C := \{x \in \eta_0(1, \tilde{A});\ \phi(x) = I(\eta_0)\}$ and make use of hypothesis (2.29) together with (2.30), we can state

$$\phi(C) \geq c > \phi(B),$$

which means $C \cap B = \emptyset$.
We now claim the following proposition: there exists $x_\epsilon \in C$ such that $\|d\phi_{x_\epsilon}\| \leq 4\epsilon$.
We want to prove this proposition by contradiction, relying on the deformation lemma 2.18 applied with $1 < k < 2$. We start noticing that, if we suppose $\|d\phi_{x_\epsilon}\| \geq 4\epsilon$ and recall $C \cap B = \emptyset$, the hypotheses of the deformation lemma are verified and, as a consequence, we obtain $\alpha(t, x)$ satisfying the conclusions of that lemma, together with a suitable function g and a time $t_0 > 0$.
If we fix $0 < \lambda < t_0$, we can construct the deformation $\eta_\lambda(t, x) := \alpha(t\lambda, \eta_0(t, x))$ and prove

that it belongs to L since

$$\begin{aligned}
&\eta_\lambda(t,x) \in C([0,1]\times X))\\
&\eta_\lambda(t,x) = \alpha(0,\eta_0(0,x)) = x \quad \forall (t,x) \in (\{0\}\times X)\\
&\eta_\lambda(t,x) = \alpha(t\lambda,x) = x \quad \forall (t,x) \in ([0,1]\times B),
\end{aligned}$$

where in the first line we used that α is continuous $\forall (t,x) \in ([0,t_0]\times X)$, in the second we exploit point *(ii)* of Lemma 2.18 and $\eta_0 \in L$, while in the third we rely on point *(i)* of Lemma 2.18 and again on $\eta_0 \in L$, that implies $\eta_0(t,x) = x \;\; \forall x \in B$. Now, from point *(ii)* of Lemma 2.18 and $t < 1$, we retrieve that

$$\rho(\alpha(\lambda t,\eta_0(t,x)),\eta_0(t,x)) \le k\lambda t < k\lambda \quad \forall (t,x) \in ([0,1]\times X)$$

and this allows us to write $d(\eta_\lambda,\eta_0) < kt$.
Finally, combining this result with equation (2.33), we have, since $\eta_\lambda \in L$

$$I(\eta_\lambda) \ge I(\eta_0) - \epsilon\delta(\eta_\lambda,\eta_0) \ge I(\eta_0) - \epsilon k\lambda \ge \phi(\eta_0(1,x)) - \epsilon k\lambda \quad \forall x \in \tilde{A}.$$

Exploiting compactness of F, we can say that $\sup \phi(\eta_\lambda(1,\tilde{A}))$ is a supremum of ϕ, which is continuous, on the compact set $\eta_\lambda(1,\tilde{A}) \in F$ and, as a consequence, is a maximum and there exists $x_\lambda \in \tilde{A}$ such that $\phi(\eta_\lambda(1,x_\lambda)) = I(\eta_\lambda)$. Then, we can explicit the equation as follows:

$$\phi(\eta_\lambda(1,x_\lambda)) - \phi(\eta_0(1,x)) \ge -\epsilon k\lambda \quad \forall x \in \tilde{A}. \tag{2.34}$$

But, exploiting *(iii)* and $\phi(\eta_\lambda(1,x_\lambda)) = \phi(\alpha(\lambda,\eta_0(1,x_\lambda)))$, we obtain

$$\phi(\eta_\lambda(1,x_\lambda)) - \phi(\eta_0(1,x_\lambda)) \le -\epsilon\lambda g(\eta_0(1,x_\lambda)). \tag{2.35}$$

If we now combine (2.34) with $x = x_\lambda$ and (2.35), we retrieve the following expression

$$-\epsilon k\lambda \le \phi(\eta_\lambda(1,x_\lambda)) - \phi(\eta_0(1,x_\lambda)) \le -\epsilon\lambda g(\eta_0(1,x_\lambda)),$$

that can be simplified into

$$g(\eta_0(1,x_\lambda)) \le \frac{k}{2} \quad \text{for any } k > 1. \tag{2.36}$$

We want now to let $\lambda \to 0$, from which it is trivial $\lim_{\lambda\to 0}\eta_\lambda(t,x) = \eta_0(t,x)$ and we define x_0 any cluster point of x_λ as $\lambda \to 0$. It must be noticed that, if we study (2.34) with

$\lambda \to 0$, we have

$$\phi(\eta_0(1, x_0)) \geq \phi(\eta_0(1, x)) \quad \text{for any } x \in \tilde{A},$$

meaning that $\eta_0(1, x_0) = I(\eta_0)$ and so $x_0 \in C$.
Since $x_0 \in C$, *(iv)* ensures us that $g(\eta_0(1, x_0)) = 1$ and the contradiction is clear: choose, as said, $1 < k < 2$, e.g $k = \frac{3}{2}$ and compare this last equality with (2.36) as $\lambda \to 0$. It becomes:

$$\begin{cases} g(\eta_0(1, x_0)) \leq \frac{3}{4} \\ g(\eta_0(1, x_0)) = 1, \end{cases}$$

that is a clear contradiction.
Now that we have proved by contradiction that there exists $x_\epsilon \in C$ such that $\|d\phi_{x_\epsilon}\| \leq 4\epsilon$, which corresponds to point *(b)* of the theorem, we still miss to show that both $c < \phi(x_\epsilon) < c + \epsilon^2$ and $\delta(x_\epsilon, \tilde{A}) < \epsilon$. But they are trivial, indeed, exploiting (2.31), by the Ekeland's principle we obtain

$$\phi(x_\epsilon) = I(\eta_0) \leq I(\tilde{\eta}) \leq c + \epsilon^2$$

and, moreover, $c \leq I(\eta_0)$ is true by definition of c.
Finally, taking into account (2.32), we have the following chain of inequalities:

$$\begin{aligned} \inf_{y \in \tilde{A}} \rho(x_\epsilon, y) &\leq \sup_{y \in \tilde{A}} \rho(\eta_0(1, y), y) \leq \sup_{t \in [0,1], x \in X} \rho(\eta_0(t, x), x) \\ &= \sup_{t \in [0,1], x \in X} \rho(\eta_0(t, x), \tilde{\eta}(t, x)) = \delta(\eta_0, \tilde{\eta}) \\ &\leq \epsilon. \end{aligned}$$

The proof of the theorem is complete. □

2.4.4. Existence of a bounded Palais-Smale sequence

In this chapter, we want to prove that there exists a Palais-Smale sequence for F restricted to S_c at level $\gamma(c)$ and that it is bounded in E.

We start naming $\mathbf{E} := (E \times \mathbb{R})$ and equipping it with the scalar product

$$\langle \cdot, \cdot \rangle_{\mathbf{E}} = \langle \cdot, \cdot \rangle_E + \langle \cdot, \cdot \rangle_{\mathbb{R}},$$

associated with its induced norm

$$\| \cdot \|^2_{\mathbf{E}} = \| \cdot \|^2_E + \| \cdot \|^2_{\mathbb{R}}.$$

For convenience, moreover, we state here two definitions valid, respectively, $\forall v \in E$ and $\forall (u,s) \in \mathbf{E}$:

$$\begin{cases} T_v := \{z \in E: \ \langle v, z\rangle_H = 0\} \\ \tilde{T}(u,s) := \{(z_1, z_2) \in \mathbf{E}, \ \langle u, z_1\rangle_H = 0\}, \end{cases}$$

where T_v is the tangent space to v with respect to the manifold S_c, while $\tilde{T}(u,s)$ is the tangent space to (u,s) with respect to $(S_c \times \mathbb{R})$.
Before continuing the discussion, we now state two lemmas that will be important throughout this section.

Lemma 2.20. *Let $(v_n)_n$ be a bounded sequence in E and g a function satisfying (H1) and (H2); then it holds:*

1. $\int_{\mathbb{R}^N} g(v_n)z \to \int_{\mathbb{R}^N} g(v)z \quad \forall z \in E$
2. $\int_{\mathbb{R}^N} g(v_n)v_n \to \int_{\mathbb{R}^N} g(v)v$
3. $\int_{\mathbb{R}^N} g(v)v_n \to \int_{\mathbb{R}^N} g(v)v.$

Proof. In this proof we are going to apply the Corollary B.5, since E is Hilbert and $(v_n)_n$ is bounded in E by hypothesis . We will name v the limit in E of v_n and $h(x)$ the dominating function.
If we are able to prove

$$\lim \int_{\mathbb{R}^N} g(v_n)z = \int_{\mathbb{R}^N} g(\lim v_n)z \quad \forall z \in E,$$

then, since g is continuous and $v_n \to v$ a.e, then point (1) holds.
We start relying on *(H2)* to see

$$\int_{\mathbb{R}^N} g(v_n)z \leq \beta \int_{\mathbb{R}^N} \frac{G(v_n)}{v_n} z.$$

Now we consider $v_{n_1} = \mathbb{1}_{[|v_n(x)| \leq 1]} v_n$ and $v_{n_2} = \mathbb{1}_{[|v_n(x)| \geq 1]} v_n$ so that

$$v_n(x) = v_{n1}(x) + v_{n2}(x).$$

Moreover, being G even and linear, it is true that

$$G(v_n) = G(v_{n1} + v_{n2}) = G(v_{n1}) + G(v_{n2}) = G(|v_{n1}|) + G(|v_{n2}|).$$

We can use (H2) to say that

$$G(v_n) \leq G(1)(v_{n1}^{\alpha} + v_{n2}^{\beta}) \quad \forall\, 2 + \frac{4k}{N} < \alpha \leq \beta < 2_k^*.$$

If we now develop calculations, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{G(v_n)}{v_n} z &\leq G(1) \int_{\mathbb{R}^N} (v_{n1}^{\alpha-1} + v_{n2}^{\beta-1}) z \\ &\leq G(1) \int_{\mathbb{R}^N} (h^{\alpha-1} + h^{\beta-1}) z \\ &= \int_{\mathbb{R}^N} f, \end{aligned}$$

where $f := G(1)(h^{\alpha-1} + h^{\beta-1})z$. Moreover, $f \in L^1(\mathbb{R}^N)$ indeed, applying Hölder with $p = \frac{\alpha}{\alpha-1}$, $q = \alpha$ and $p = \frac{\beta}{\beta-1}$, $q = \beta$, we retrieve

$$\int_{\mathbb{R}^N} f \leq G(1) \max\left\{ \|z\|_{L^\alpha}^{\frac{1}{\alpha}}, \|z\|_{L^\beta}^{\frac{1}{\beta}} \right\} \left[\left(\int_{\mathbb{R}^N} h^\alpha \right)^{\frac{\alpha-1}{\alpha}} + \left(\int_{\mathbb{R}^N} h^\beta \right)^{\frac{\beta-1}{\beta}} \right],$$

that is well defined quantity thanks to Lemma B.
At this point, it is possible to apply dominated convergence on $\tilde{f} = \beta f$ and (1) is proved.
To prove point (2), we an follow the same reasoning to apply dominated convergence and thesis comes from

$$|g(v_n)v_n - g(v)v| \leq |g(v_n)v_n - g(v_n)v| + |g(v_n)v - g(v)v| \leq |g(v_n)||v_n - v| + |v||g(v_n) - g(v)|,$$

where $|v_n - v| \to 0$ and $|g(v_n) - g(v)| \to 0$, thanks to Lemma B and to the continuity of g.
Finally, point (3) comes straight forward applying dominated convergence in the same way. □

Lemma 2.21. $\tilde{F} \in C^1(\boldsymbol{E}; \mathbb{R})$.

Proof. We want to show that $\tilde{F}$ admits continuous partial derivatives. In particular, we will look for $d\tilde{F}(u) : E \to \mathbb{R}$, $d\tilde{F}(s) : \mathbb{R} \to \mathbb{R}$ linear and continuous such that

$$\begin{aligned} \lim_{\|v\|_E \to 0} \frac{|\tilde{F}(u+v,s) - \tilde{F}(u,s) - \langle d\tilde{F}_u, v\rangle|}{\|v\|_E} &= 0 \;\forall s \in \mathbb{R} \\ \lim_{|h| \to 0} \frac{|\tilde{F}(u,s+h) - \tilde{F}(u,s) - \langle d\tilde{F}_s, h\rangle|}{|h|} &= 0 \;\forall u \in E. \end{aligned}$$

To compute $d\tilde{F}(u)$, we should rely on (2.17) and perform the same calculations as for Theorem 2.7 to retrieve:

$$\langle d\tilde{F}(u), v\rangle = \frac{C_{N,k}\, e^{2ks}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\, dxdy - e^{-\frac{sN}{2}} \int_{\mathbb{R}^N} g(e^{\frac{sN}{2}}u)v.$$

Now, if we notice that $u_n \to u$ in E implies that $(e^{\frac{sN}{2}}u_n)_n$ is bounded, applying Lemma 2.20 together with Lemma 2.21 we can conclude that $\|u_n - u\|_E \to 0$ and we are finally led to $\|d\tilde{F}(u_n) - d\tilde{F}(u)\|_* \to 0$.
Focusing then on $d\tilde{F}(s)$, we see that it is a function of real variable with values in $\mathbb{R}$, then the derivative with respect to s can be computed trivially as

$$\langle d\tilde{F}(s), h\rangle = h\frac{C_{N,k}\, e^{2ks}}{2} k\lfloor u\rceil_k^2 + he^{-sN} N \int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u) - h\frac{N}{2}e^{-\frac{sN}{2}} \int_{\mathbb{R}^N} g(e^{\frac{sN}{2}}u)u.$$

It should be clear that $|s_n - s| \to 0$ implies $e^{\frac{s_nN}{2}}u \to e^{\frac{sN}{2}}u$ in E so that, if we recall (H2) to state $G(e^{\frac{s_nN}{2}}u) \le \alpha^{-1} g(e^{\frac{s_nN}{2}}u)e^{\frac{s_nN}{2}}u$, we can exploit again Lemma 2.20 to obtain that $|s_n - s| \to 0$ implies $\|d\tilde{F}(s_n) - d\tilde{F}(s)\|_* \to 0$.
Finally, the proof is complete remembering

$$\langle d\tilde{F}(u,s), (v,h)\rangle = \langle d\tilde{F}(u), v\rangle + \langle d\tilde{F}(s), h\rangle.$$

□

The following part of this section will be devoted to make explicit that hypotheses of Theorem 2.19 are valid in this context. It is possible to make the following analogies with the notations used in Section 2.4.3:

$$\begin{cases} X = (S_c \times \mathbb{R}) \\ \phi = \tilde{F}|_{(S_c\times\mathbb{R})} \to \mathbb{R} \\ F = \Theta(c) := \{\tilde{h}[0,1] \subset (S_c \times \mathbb{R});\ \forall \tilde{h} \in \tilde{\Gamma}(c)\}. \end{cases}$$

We can apply Dini's implicit function theorem to ensure that X admits a Finsler structure, indeed $(S_c \times \mathbb{R})$ is a regular hypersurface, meaning that it is a Riemannian manifold and, as a consequence, admits a Finsler structure (see [22] for further details).
We also notice that ϕ is the restriction of $\tilde{F}$ to $(S_c \times \mathbb{R})$ and we can exploit Lemma 2.21 to gather that $\phi \in C^1$.

Lemma 2.22. $\Theta(c)$ *is a family of compact sets.*

Proof. We want to prove that $A := \tilde{h}[0,1]$, $A \in \Theta(c)$ is compact for any $\tilde{h} \in \tilde{\Gamma}(c)$. Let $(x_n)_n$ be a sequence in A and we ask if there exists a subsequence $(x_{n_k})_{n_k}$ such that $(x_{n_k})_{n_k} \to \bar{x} \in A$. This is true, since $x_{n_k} = \tilde{h}(t_{n_k})$, $t_{n_k} \in [0,1]$ and thanks to Bolzano–Weierstrass theorem $t_{n_k} \to \bar{t} \in [0,1]$. The lemma is proved relying on the continuity of $\tilde{h}$, setting $\bar{x} = \tilde{h}(\bar{t})$. □

Lemma 2.23. *$\Theta(c)$ is an homotopy-stable family of compact sets in X, with boundary*

$$B := \{(u_1, 0),\ (u_2, 0)\}.$$

Proof. To prove this lemma, we immediately see that any $\tilde{h}(c) \in \tilde{\Gamma}(c)$ contains the boundary by definition. Then, we notice that any continuous deformation $\eta(t,x)$, $t \in [0,1], x \in X$ that is identity at $t = 0$ in X and in B for $0 \le t \le 1$, if applied to any element of F, at $t = 1$ results in a continuous curve connecting $(u_1, 0)$ to $(u_2, 0)$, which, as a consequence, is still an element of $\Theta(c)$. □

Then, if we set $\zeta(c) := \inf_{\theta \in \Theta(c)} \max_{x \in \theta} \tilde{F}(x)$, exploiting the trivial equality $\zeta(c) = \tilde{\gamma}(c)$, together with Proposition 2.1, we notice that $\sup \tilde{F}(B) < \zeta(c)$. As a consequence Theorem 2.19 can be applied in order to enunciate the following proposition.

Proposition 2.2. *Assume (H1) and (H2) hold and let $(g_n)_n \subset \tilde{\Gamma}(c)$, such that*

$$\max_{t \in [0,1]} \tilde{F}(g_n(t)) \le \tilde{\gamma}(c) + \frac{1}{n}.$$

Then, there exists a sequence $(u_n, s_n)_n \subset (S_c \times \mathbb{R})$ such that

(a) $\tilde{F}(u_n, s_n) \in [\tilde{\gamma}(c) - \frac{1}{n}, \tilde{\gamma}(c) + \frac{1}{n}]$

(b) $min_{t \in [0,1]} \|(u_n, s_n) - g_n(t)\|_{\boldsymbol{E}} \le \frac{1}{\sqrt{n}}$

(c) $\left\| \tilde{F}'|_{(S_c \times \mathbb{R})}(u_n, s_n) \right\|_* \le \frac{2}{\sqrt{n}}$,

where point (c) can be reformulated as

$$|\langle \tilde{F}'(u_n, s_n), z \rangle_{\boldsymbol{E}^* \times \boldsymbol{E}}| \le \frac{2}{\sqrt{n}} \|z\|_{\boldsymbol{E}} \quad \textit{for all } z \in \tilde{T}(u_n, s_n).$$

Proof. This proof is a corollary of Theorem 2.19, since we have already shown that its hypotheses are satisfied. It has just to be recalled that Theorem 2.19 would ensure for the existence of $(A_n)_n \subset \Theta(c)$ such that

$$\sup_{A_n} \tilde{F}(A_n) \leq \tilde{\gamma}(c) + \frac{1}{n}.$$

It is possible, however, to notice that any $A \in \Theta(c)$ represents the image of a curve $\tilde{h}$ in $(S_c \times \mathbb{R})$, then to any $(A_n)_n \subset \Theta(c)$ corresponds a $(g_n)_n \subset \tilde{\Gamma}(c)$ defining g_n such that $A_n = g_n[0,1]$ for any n. As a consequence taking $\sup_{A_n} \tilde{F}$ is the same as considering the $\sup_{t\in[0,1]} \tilde{F}(g_n(t))$. Then, we also recall that the distance on $(S_c \times \mathbb{R})$ is induced by the norm $\|\cdot\|_{\mathbf{E}}$ and has no more to be generally defined as ρ. This leads to the following equality:

$$\begin{aligned}\inf_{y_n \in A_n} \rho((u_n, s_n), y_n) &= \inf_{y_n \in g_n[0,1]} \rho((u_n, s_n), y_n) \\ &= \min_{y_n \in g_n[0,1]} \|(u_n, s_n), y_n\|_{\mathbf{E}} = \min_{t\in[0,1]} \|(u_n, s_n), g_n(t)\|_{\mathbf{E}}.\end{aligned}$$

□

We shall use these results in order show that there exists a Palais-Smale sequence $(v_n)_n \in S_c$ for $F|_{S_c}$, which is to say

$$\begin{cases} F(v_n) \to \gamma(c) \\ \|F'|_{S_c}(v_n)\|_* \to 0. \end{cases} \tag{2.37}$$

At this stage of the discussion, we are going to list four lemmas, that will be used frequently in this section.

Lemma 2.24. *The following formula holds:*

$$\begin{aligned}\langle \tilde{F}'(u,s), (v,0)\rangle = \frac{C_{N,k}\, e^{2ks}}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2k}}\, dxdy \\ - e^{-\frac{sN}{2}} \int_{\mathbb{R}^N} g(e^{\frac{sN}{2}} u(x)) v(x)\, dx.\end{aligned}$$

Proof. This lemma can be seen as a corollary of Lemma 2.21, isolating the partial derivative with respect to u, without any increment in the s direction. □

Lemma 2.25. *Setting $\partial_s \tilde{F}(u,s) := \langle \tilde{F}'(u,s), (0,1)\rangle$ and $v = H(u,s)$ we have that*

$$\partial_s \tilde{F}(u,s) = \frac{C_{N,k}\, k}{2} \lfloor v \rfloor_k^2 + N \int_{\mathbb{R}^N} G(v) - \frac{N}{2} \int_{\mathbb{R}^N} g(v) v.$$

Proof. Taking equation (2.17), we study $\partial_s\left(\frac{e^{2ks}}{4}\lfloor u\rfloor_k^2\right)$:

$$\begin{aligned}
\partial_s\left(\frac{e^{2ks}}{4}\lfloor u\rfloor_k^2\right) &= \frac{e^{2ks}}{2}\lfloor u\rfloor_k^2 = \frac{ke^{2ks}}{2}\iint_{\mathbb{R}^{2N}}\frac{|u(z)-u(\omega)|^2}{|z-\omega|^{N+2k}}\,dzd\omega\\
&= \frac{ke^{2ks}}{2}\iint_{\mathbb{R}^{2N}}\frac{e^{2sN}|u(e^sx)-u(e^sy)|^2}{|x-y|^{N+2k}e^{sN+2sk}}\,dxdy\\
&= \frac{ke^{2ks}}{2}\iint_{\mathbb{R}^{2N}}\frac{|e^{\frac{sN}{2}}u(e^sx)-e^{\frac{sN}{2}}u(e^sy)|^2}{|x-y|^{N+2k}e^{2sk}}\,dxdy\\
&= \frac{k}{2}\iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^2}{|x-y|^{N+2k}}\,dxdy\\
&= \frac{k}{2}\lfloor v\rfloor_k^2.
\end{aligned}$$

Finally, we consider that

$$\begin{aligned}
&\partial_s\left(e^{-\,sN}\int_{\mathbb{R}^N}G(e^{\frac{sN}{2}}u(x))\,dx\right)\\
&\quad= \partial_s(e^{-\,sN})\int_{\mathbb{R}^N}G(e^{\frac{sN}{2}}u(x))\,dx + e^{-\,sN}\partial_s\left(\int_{\mathbb{R}^N}G(e^{\frac{sN}{2}}u(x))\,dx\right)\\
&\quad= -N\int_{\mathbb{R}^N}G(v)+\frac{N}{2}\int_{\mathbb{R}^N}g(v)v.
\end{aligned}$$

□

Lemma 2.26. *Let $s\in\mathbb{R}$ be fixed. If we set $h\in E$, $\tilde{h}(x)=e^{-\frac{sN}{2}}h(e^{-s}x)$ and $v=H(u,s)$, it holds*

$$(\tilde{h},0)\in\tilde{T}(u,s) \iff h\in T_v.$$

Proof.

$$\begin{aligned}
\int_{\mathbb{R}^N}\tilde{h}u &= \int_{\mathbb{R}^N}e^{-\frac{sN}{2}}h(e^{-s}x)u(x)\,dx = e^{-\frac{sN}{2}}\int_{\mathbb{R}^N}h(y)u(e^sy)e^{sN}\,dy\\
&= \int_{\mathbb{R}^N}h(x)\left(e^{\frac{sN}{2}}u(e^sx)\right)\,dx = \int_{\mathbb{R}^N}hv
\end{aligned}$$

and then $\int_{\mathbb{R}^N}\tilde{h}u=0 \iff \int_{\mathbb{R}^N}hv=0$.

□

Lemma 2.27. *Setting $h\in E$, $\tilde{h}=e^{-\frac{sN}{2}}h(e^{-s}x)$, then*

$$\|(\tilde{h},0)\|_{\boldsymbol{E}}^2=\|h\|_2^2+e^{-2sk}\lfloor h\rfloor_k^2.$$

Proof. This proof is just a matter of calculating the following quantities:

$$\|\tilde{h}\|_2^2 = \int_{\mathbb{R}^N} e^{-sN}|h(e^{-s}x)|^2\,dx = \int_{\mathbb{R}^N} |h(y)|^2\,dy = \|h\|_2^2$$

and

$$\begin{aligned}\lfloor\tilde{h}\rfloor_k^2 &= \iint_{\mathbb{R}^{2N}} \frac{e^{-sN}|h(e^{-s}x)-h(e^{-s}y)|^2}{|x-y|^{N+2k}}\,dxdy\\ &= \iint_{\mathbb{R}^{2N}} \frac{e^{-sN}|h(z)-h(\omega)|^2}{e^{sN+2sk}|z-\omega|^{N+2k}}e^{2sN}\,dzd\omega = e^{-2sk}\lfloor h\rfloor_k^2.\end{aligned}$$

□

Exploiting these results and naming $v_n = H(u_n, s_n)$, with $(u_n, s_n)_n$ defined in Proposition 2.2, our first step consists in showing that $\lfloor v_n\rfloor_k$ and $\int_{\mathbb{R}^N} G(v_n)$ are bounded.

In this aim, we exploit Proposition 2.2 point *(c)* to retrieve that, since $(0,1) \in \tilde{T}(u_n, s_n)\ \forall n$, then $\partial_s\tilde{F}(u_n, s_n) \to 0$, where $\partial_s\tilde{F}(u_n, s_n)$ has been defined in Lemma 2.25. We also get, from point *(a)* of the same proposition, that $\tilde{F}(u_n, s_n)$ is bounded, obtaining

$$|N\tilde{F}(u_n, s_n) + \partial_s\tilde{F}(u_n, s_n)| \leq C \quad \forall n.$$

Now, if we expand calculations and make use of Lemmas 2.14 and 2.25, we get

$$\begin{aligned}|N\tilde{F}(u_n, s_n) + \partial_s\tilde{F}(u_n, s_n)| &= \frac{N+2k}{4}C_{N,k}\lfloor v_n\rfloor_k^2 - \frac{N}{2}\int_{\mathbb{R}^N} g(v_n)v_n\\ &\leq \frac{N+2k}{4}C_{N,k}\lfloor v_n\rfloor_k^2 - \frac{N\alpha}{2}\int_{\mathbb{R}^N} G(v_n)\end{aligned}$$

and we deduce that

$$(N+2k)\frac{C_{N,k}}{4}\lfloor v_n\rfloor_k^2 - \frac{N\alpha}{2}\int_{\mathbb{R}^N} G(v_n) \geq -C. \tag{2.38}$$

Since $F(v_n) = \tilde{F}(u_n, s_n)$ is bounded, meaning that $\frac{C_{N,k}}{4}\lfloor v_n\rfloor_k^2 - \int_{\mathbb{R}^N} G(v_n) < C$, we have the following:

$$\frac{C_{N,k}}{4}\lfloor v_n\rfloor_k^2 < C + \int_{\mathbb{R}^N} G(v_n). \tag{2.39}$$

Combining then equations (2.38) and (2.39), it arises

$$\left(N + 2k - \frac{N\alpha}{2}\right)\int_{\mathbb{R}^N} G(v_n) \geq -C.$$

If we are able to show that $0 \geq \left(N + 2k - \frac{N\alpha}{2}\right) \displaystyle\int_{\mathbb{R}^N} G(v_n) \geq -C$, then we have proved the boundedness of $\int_{\mathbb{R}^N} G(v_n)$ and indeed

$$\left(N + 2k - \frac{N\alpha}{2}\right) < \left(N + 2k - \frac{N}{2}\left(\frac{4k + 2N}{N}\right)\right) = N + 2k - N - 2k = 0.$$

As a consequence, we are finally allowed to write

$$\exists C > 0 \ : \qquad 0 \leq \int_{\mathbb{R}^N} G(v_n) \leq C \quad \forall n, \tag{2.40}$$

which, exploiting equation (2.39), gives

$$\exists C > 0 \ : \qquad 0 \leq \lfloor v_n \rfloor_k^2 \leq C \quad \forall n \tag{2.41}$$

and so step one is completed.
Summarizing our results, we are now ready to enounce the following two theorems.

Theorem 2.28. *$(v_n)_n$ is bounded in E.*

Proof. The proof is trivial if we combine $v_n \in S_c$ and equation (2.41). □

Theorem 2.29. *These two points are true:*

(a) $F(v_n) \in [\gamma(c) - \frac{1}{n}\, ,\, \gamma(c) + \frac{1}{n}]$

(b) $\|F'|_{S_c}(v_n)\|_* \leq \frac{4}{\sqrt{n}}$, *meaning* $|\langle F'(v_n), z\rangle_{E^* \times E}| \leq \frac{4}{\sqrt{n}} \|z\|_E \quad \forall z \in T_{v_n}$.

Proof. Point *(a)* is immediate, since $F(v_n) = F(H(u_n, s_n))$ and $\gamma(c) = \tilde{\gamma}(c)$. For what concerns point *(b)* instead, we set $h_n \in T_{v_n}$, h_n generic and consider

$$\begin{aligned}\langle F'(v_n), h_n\rangle &= \frac{C_{N,k}}{2}\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(h_n(x)-h_n(y))}{|x-y|^{N+2k}}\,dxdy \\ &\quad - \int_{\mathbb{R}^N} g(v_n(x))h_n(x)\,dx \\ &= \frac{C_{N,k}\,e^{\frac{s_nN}{2}}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(e^{s_n}x)-u(e^{s_n}y))(h_n(x)-h_n(y))}{|x-y|^{N+2k}}\,dxdy \\ &\quad - \int_{\mathbb{R}^N} g(v_n(x))h_n(x)dx \\ &= \frac{C_{N,k}\,e^{\frac{s_nN}{2}}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(z)-u(\omega))(h_n(e^{-s_n}z)-h_n(e^{-s_n}\omega))}{|z-\omega|^{N+2k}e^{-s_nN-2s_nk}}e^{-2s_nN}\,dzd\omega \\ &\quad - \int_{\mathbb{R}^N} g(e^{\frac{s_nN}{2}}u(e^{s_n}x))h_n(x)\,dx \\ &= \frac{C_{N,k}\,e^{-\frac{s_nN}{2}+2s_nk}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(z)-u(\omega))(h_n(e^{-s_n}z)-h_n(e^{-s_n}\omega))}{|z-\omega|^{N+2k}}\,dzd\omega \\ &\quad - e^{-s_nN}\int_{\mathbb{R}^N} g(e^{\frac{s_nN}{2}}u(y))h_n(e^{-s_n}y)\,dy,\end{aligned}$$

that finally becomes

$$\begin{aligned}\langle F'(v_n), h_n\rangle = \frac{C_{N,k}\,e^{2s_nk}}{2}\iint_{\mathbb{R}^{2N}} &\frac{(u(z)-u(\omega))[e^{-\frac{s_nN}{2}}(h_n(e^{-s_n}z)-h_n(e^{-s_n}\omega))]}{|z-\omega|^{N+2k}}\,dzd\omega \\ - e^{-\frac{s_nN}{2}}\int_{\mathbb{R}^N} &g(e^{\frac{s_nN}{2}}u(y))[e^{-\frac{s_nN}{2}}h_n(e^{-s_n}y)]\,dy.\end{aligned}$$

If we now define $\tilde{h}_n(x) = e^{-\frac{s_nN}{2}}h_n(e^{-s_n}x)$, we can state

$$\begin{aligned}\langle F'(v_n), h_n\rangle_{E^*\times E} \;=\; &\frac{C_{N,k}\,e^{2s_nk}}{2}\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\tilde{h}_n(x)-\tilde{h}_n(y))}{|x-y|^{N+2k}}\,dxdy \\ &- e^{-\frac{s_nN}{2}}\int_{\mathbb{R}^N} g(e^{\frac{s_nN}{2}}u(x))\tilde{h}_n(x)\,dx\end{aligned}$$

and we can exploit Lemma 2.24 to recognize

$$\langle F'(v_n), h_n\rangle \;=\; \langle \tilde{F}'(u_n, s_n), (\tilde{h}_n, 0)\rangle.$$

Since then Lemma 2.26 holds true, we can rely on Proposition 2.2, point *(c)*, to write

$$|\langle F'(v_n), h_n\rangle| = |\langle \tilde{F}'(u_n, s_n), (\tilde{h}_n, 0)\rangle| \le \frac{2}{\sqrt{n}}\|(\tilde{h}_n, 0)\|_{\mathbf{E}}.$$

At this point, we would like to explicit $\|(\tilde{h}_n, 0)\|_{\mathbf{E}}$ as a function of $\|h_n\|_E$ and, to do this, we have to show firstly that $s_n \to 0$ as $n \to \infty$.
In this aim, we notice that Proposition 2.2 does not loose of generality if, instead of a generic $(g_n)_n \in \tilde{\Gamma}(c)$, as minimizing sequence we consider $\tilde{g}_n = (H(g_n), 0)$, because

$$\tilde{F}(\tilde{g}_n) = \tilde{F}((H(g_n), 0)) = F(H(g_n)) = \tilde{F}(g_n),$$

where we exploited that $H((H(g_n), 0)) = H(g_n)$. As a consequence, $\max_{t\in[0,1]} \tilde{F}(g_n(t)) = \max_{t\in[0,1]} \tilde{F}(\tilde{g}_n(t))$ and Proposition 2.2 does not loose of generality.
According to point *(b)* of that proposition, then, we have :

$$\sqrt{s_n} \leq \min_{t\in[0,1]} \left(\|u_n - H(\tilde{g}_n(t))\|_k^2 + |s_n - 0|\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}}$$

and we have shown that $s_n \to 0$ as $n \to \infty$.
Now, we rely on Lemma 2.27, ensuring that $\|(\tilde{h}_n, 0)\|_{\mathbf{E}}^2 = \|h_n\|_2^2 + e^{-2s_n k} \lfloor h_n \rfloor_k^2$ and ask for n large enough such that $e^{-2s_n k} < 4$, meaning that

$$\|(\tilde{h}_n, 0)\|_{\mathbf{E}}^2 \leq 4\|h_n\|_E^2.$$

Finally, we can conclude the proof writing :

$$|\langle F'(v_n), h_n\rangle| \leq \frac{2}{\sqrt{n}}\|(\tilde{h}_n, 0)\|_{\mathbf{E}} \leq \frac{4}{\sqrt{n}}\|h_n\|_E.$$

□

2.4.5. Further properties of the Palais-Smale sequence

At this stage of the analysis, we just miss to prove the convergence of $(v_n)_n$ in E, asking for some compact embeddings of E into spaces L^p. Since it is possible to show that these embeddings are not compact in unbounded domains, due to translation invariance, we will restrict our domain to $H_r^k(\mathbb{R}^N)$, which, we recall, denotes the space of radially symmetric function in E. We remark that this choice is possible since our problem is invariant under rotations. Therefore, thanks to Theorem B.6, compactness can be recovered. Moreover, it is clear that the variational procedure employed so far does not change if we work in the subspace $H_r^k(\mathbb{R}^N)$. From now on, then, $E = H_r^k(\mathbb{R}^N)$.

Proposition 2.3. *There exists $(\lambda_n)_n \subset \mathbb{R}$ such that, up to a subsequence:*

(a) $\displaystyle\int_{\mathbb{R}^N} G(v_n) \to C,\ C > 0$

(b) $(-\Delta)^k v_n - \lambda_n v_n - g(v_n) \to 0$ *in* E^*

(c) $\lambda_n \to \lambda_c < 0.$

Proof. We start proving *(a)*, recalling Lemma 2.25, namely that $\partial_s \tilde{F}(u_n, s_n) \to 0$ means

$$\lim_n \left[\frac{C_{N,k}\, k}{2} \lfloor v_n \rfloor_k^2 - \frac{N}{2} \left(\int_{\mathbb{R}^N} g(v_n) v_n - 2 \int_{\mathbb{R}^N} G(v_n) \right) \right] = 0.$$

Moreover, since $g(v_n)v_n \le \beta G(v_n)$, we obtain

$$\int_{\mathbb{R}^N} g(v_n) v_n - 2 \int_{\mathbb{R}^N} G(v_n) \le (\beta - 2) \int_{\mathbb{R}^N} G(v_n)$$

and we finally infer that

$$\lim_n \left[\frac{C_{N,k}\, k}{2} \lfloor v_n \rfloor_k^2 - \frac{N(\beta-2)}{2} \int_{\mathbb{R}^N} G(v_n) \right] \le 0.$$

Expanding calculation we have

$$\frac{N\beta}{2} - N = \frac{2kN}{N - 2k} > 0,$$

so that, exploiting (2.40), we can write

$$0 \le \frac{C_{N,k}\, k}{2} \lim_n \lfloor v_n \rfloor_k^2 \le \frac{N(\beta-2)}{2} \lim_n \int_{\mathbb{R}^N} G(v_n).$$

As a consequence if, by contradiction, $\int_{\mathbb{R}^N} G(v_n) \to 0$ then also $\lfloor v_n \rfloor_k^2 \to 0$, leading to $F(v_n) \to 0$, which is clearly false since we know $F(v_n) \to \gamma(c) > 0$.
Now, we are going to prove *(b)*, recalling that $T_{v_n} = \{z \in E : \langle z, v_n \rangle_H = 0\}$.
We firstly define the orthogonal projector in H to T_{v_n}, $P_{v_n} : H \to T_{v_n}$, so that

$$\begin{aligned} z &= P_{v_n} z + z_2 \qquad \forall z \in H \\ P_{v_n} z &= z - \langle z, v_n \rangle_H \frac{v_n}{\|v_n\|_H^2} \\ z_2 &= \langle z, v_n \rangle_H \frac{v_n}{\|v_n\|_H^2}. \end{aligned}$$

We notice that, if $z \in E$, then $P_{v_n} z \in T_{v_n}$ since

$$\langle P_{v_n} z, v_n \rangle_H = \langle z, v_n \rangle_H - \langle z, v_n \rangle_H \frac{\langle v_n, v_n \rangle_H}{\|v_n\|_H^2} = 0.$$

In particular, from now on, we will consider P_{v_n} operating only on element of E and, for any $z \in E$, we can name $z_1 := P_{v_n} z$ and get that $z = z_1 + z_2$. It is clear that $\langle z_1, z_2 \rangle_H = 0$, meaning $z_1 \perp z_2$ in H. We now proceed to notice that, for any $z \in E$,

$$\begin{aligned}
&{}_{E^*}\langle F'(v_n), z - \frac{v_n}{\|v_n\|_H^2} \langle v_n, z \rangle_H \rangle_E \\
&= \langle F'(v_n),\ z_1 + z_2 \rangle - \frac{\langle F'(v_n), v_n \rangle}{\|v_n\|_H^2} \langle v_n, z \rangle_H \\
&= \langle F'(v_n),\ z_1 \rangle + \langle F'(v_n), v_n \rangle \frac{\langle v_n, z \rangle_H}{\|v_n\|_H^2} - \langle F'(v_n), v_n \rangle \frac{\langle v_n, z \rangle_H}{\|v_n\|_H^2} \\
&= \langle F'(v_n),\ z_1 \rangle
\end{aligned}$$

and remembering that $\langle F'(v_n),\ z_1 \rangle \leq \frac{4}{\sqrt{n}} \|z_1\|_E^2$, we get immediately

$${}_{E^*}\langle F'(v_n), z - \frac{v_n}{\|v_n\|_H^2} \langle v_n, z \rangle_H \rangle_E \leq \frac{4}{\sqrt{n}} \|z_1\|_E^2.$$

Since we want to explicit the previous inequality as a function of $\|z\|_E^2$, we are just left to control from above $\|z_1\|_E^2$ using $\|z\|_E^2$ and indeed

$$\begin{cases} \|z_1\|_E \leq \|z\|_E + (\|v_n\|_H^{-1}) \|v_n\|_E) \|z\|_E \\ \|v_n\|_H = c,\ (\|v_n\|_E)_n \text{ bounded,} \end{cases}$$

imply that $\|z_1\|_E \leq C \|z\|_E$ and allow us to state

$${}_{E^*}\langle F'(v_n), z - \frac{v_n}{\|v_n\|_H^2} \langle v_n, z \rangle_H \rangle_E \leq \frac{C}{\sqrt{n}} \|z\|_E \quad \forall z \in E.$$

To end the proof of point *(b)*, we would like to develop the operator at the left hand side of the previous equation, from now on named $\phi(v_n) : E \to \mathbb{R}$, since we have just proved $\|\phi(v_n)\|_{E^*} \to 0$. In this aim we recall that

$$\langle F'(v_n), z \rangle = \frac{C_{N,k}}{2} \iint_{\mathbb{R}^{2N}} \frac{(v_n(x) - v_n(y))(z(x) - z(y))}{|x - y|^{N+2k}}\, dxdy - \int_{\mathbb{R}^N} g(v_n(x)) z(x)\, dx.$$

Hence, we finally obtain, for any $z \in E$:

$$\langle \phi(v_n), z\rangle = \frac{C_{N,k}}{2}\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(z(x)-z(y))}{|x-y|^{N+2k}}\,dxdy - \int_{\mathbb{R}^N} g(v_n(x))z(x)\,dx$$
$$- \frac{\langle v_n, z\rangle_H}{\|v_n\|_H^2}\left[\frac{C_{N,k}}{2}\iint_{\mathbb{R}^{2N}} \frac{|(v_n(x)-v_n(y))|^2}{|x-y|^{N+2k}}\,dxdy - \int_{\mathbb{R}^N} g(v_n(x))v_n(x)\,dx\right].$$

Naming then

$$\lambda_n = \|v_n\|_H^{-2}\left[\frac{C_{N,k}}{2}\lfloor v_n\rfloor_k^2 - \langle g(v_n), v_n\rangle_H\right], \tag{2.42}$$

we get this final formula:

$$\begin{aligned}\langle \phi(v_n), z\rangle &= \frac{C_{N,k}}{2}\iint_{\mathbb{R}^{2N}} \frac{(v_n(x)-v_n(y))(z(x)-z(y))}{|x-y|^{N+2k}}\,dxdy - \langle g(v_n), z\rangle_H - \lambda_n\langle v_n, z\rangle_H \\ &= {}_{E^*}\langle(-\Delta)^k v_n - \lambda_n v_n - g(v_n),\ z\rangle_E\end{aligned}$$

for any $z \in E$, that proves point *(b)*.
In order to show point *(c)*, we start recalling that $\|v_n\|_H^2\lambda_n = \left[\frac{C_{N,k}}{2}\lfloor v_n\rfloor_k^2 - \langle g(v_n), v_n\rangle_H\right]$ and, from this equality, we retrieve

$$\begin{aligned}k\|v_n\|_H^2\lambda_n &= \frac{C_{N,k}\,k}{2}\lfloor v_n\rfloor_k^2 - \int_{\mathbb{R}^N} g(v_n)v_n \\ &= \left(\frac{C_{N,k}\,k}{2}\lfloor v_n\rfloor_k^2 + N\int_{\mathbb{R}^N} G(v_n) - \frac{N}{2}\int_{\mathbb{R}^N} g(v_n)v_n\right) \\ &\quad - N\int_{\mathbb{R}^N} G(v_n) + \frac{N-2}{2}\int_{\mathbb{R}^N} g(v_n)v_n,\end{aligned}$$

where the term in the parenthesis coincides with $\partial_s\tilde{F}(u_n, s_n)$, that vanishes as $n \to 0$. Exploiting (H2) then, we can state

$$\begin{aligned}k\|v_n\|_H^2\lambda_n &\le \left(\beta\frac{N-2}{2} - N\right)\int_{\mathbb{R}^N} G(v_n) \\ k\|v_n\|_H^2\lambda_n &\ge \left(\alpha\frac{N-2}{2} - N\right)\int_{\mathbb{R}^N} G(v_n),\end{aligned}$$

where, again from (H2)

$$\begin{aligned}&\beta\frac{N-2}{2} - N \le \frac{N(N-2)}{N-2k} - N = \frac{2N(k-1)}{N-2k} = C_1 < 0 \\ &\alpha\frac{N-2}{2} - N \ge C_2 \qquad \text{for some } C_2 < 0.\end{aligned}$$

As a consequence, we are now able to bound $(\lambda_n)_n$ away from 0, both from below and above, since we already know that $\int_{\mathbb{R}^N} G(v_n)$ and $\|v_n\|_H^2$ are positive and bounded away from 0. Therefore, setting $\tilde{C}_2 = C_2(k\|v_n\|_H^2)^{-1} < 0$ and $\tilde{C}_1 = C_1(k\|v_n\|_H^2)^{-1} < 0$, we can write

$$\tilde{C}_2 \int_{\mathbb{R}^N} G(v_n) \leq \lambda_n \leq \tilde{C}_1 \int_{\mathbb{R}^N} G(v_n).$$

Thus, it is true that, at least up to a subsequence, $\lambda_n \to \lambda_c < 0$. □

2.4.6. Convergence of the Palais-Smale sequence for $N>1$

Exploiting these results, it is possible to show that the Palais-Smale sequence $(v_n)_n$ converges in E.
In particular, we start stressing that Theorem 2.28 has an important implication on v_n, since it allows us to exploit Corollary B.5, from which we will borrow the notation of the limit function v. We now rely on Proposition 2.3 point **(b)**, taking the same notation $\phi(v_n)$ used in the proof, in order to claim that

$$\begin{cases} \phi(v_n)[v_n - v] \to 0 \\ \phi(v)[v_n - v] \to 0, \end{cases} \tag{2.43}$$

where the first line exploits $\|\phi(v_n)\|_* \to 0$ and

$$|\phi(v_n)[v_n - v]| \leq \|\phi(v_n)\|_* \|v_n - v\|_E \leq C\|\phi(v_n)\|_* \to 0,$$

thanks to Theorem 2.28, while the second holds true combining $v_n \rightharpoonup v$ in E with $\phi(v) \in E^*$. System (2.43) implies $(\phi(v_n) - \phi(v))\,[v_n - v] \to 0$, so that

$$\frac{C_{N,k}}{2} \lfloor v_n - v \rceil_k^2 - \lambda_n \int_{\mathbb{R}^N} (v_n - v)^2 - \int_{\mathbb{R}^N} (g(v_n) - g(v))(v_n - v) = o(1)$$

and thanks to Lemma 2.20 it holds that

$$\frac{C_{N,k}}{2} \lfloor v_n - v \rceil_k^2 - \lambda_n \int_{\mathbb{R}^N} (v_n - v)^2 = o(1). \tag{2.44}$$

To end the analysis, we need to refer to a basic analytical result saying that if $(a_n)_n$, $(b_n)_n$ are two numerical sequences such that $(a_n)_n$ is bounded and $(b_n)_n$ is infinitesimal, then it results that

$$\lim_n a_n b_n = 0.$$

If we remember that $(v_n)_n \subset S_c$, then we can infer that $v_n - v$ is bounded in L^2, since, to be specific, $\|v_n - v\|_2^2 \le 2c^2$.
Relying on the previous information, we can therefore set

$$a_n = \|v_n - v\|_2^2, \quad b_n = \lambda_n - \lambda_c$$

and the information we retrieve is

$$(\lambda_n - \lambda_c) \int_{\mathbb{R}^N} (v_n - v)^2 \to 0. \tag{2.45}$$

Finally, making use of (2.45) in equation (2.44), we obtain the following:

$$\frac{C_{N,k}}{2} \lfloor v_n - v \rfloor_k^2 - \lambda_c \int_{\mathbb{R}^N} (v_n - v)^2 = o(1).$$

Lemma 2.30. *Let $\langle \cdot, \cdot \rangle$ be the usual scalar product in $H^k(\mathbb{R}^N)$.*
Then $\langle \cdot, \cdot \rangle_c$ is an equivalent scalar product in $H^k(\mathbb{R}^N)$, where

$$\langle f, g \rangle_c := \frac{C_{N,k}}{2} \iint_{\mathbb{R}^{2N}} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{N+2k}} \, dxdy \; - \; \lambda_c \int_{\mathbb{R}^N} f(x) g(x) \, dx,$$

for any $f, g \in H^k(\mathbb{R}^N)$.

Proof. It is trivial for $\langle \cdot, \cdot \rangle_c$ to be positive, symmetric and linear with respect to the first argument. If, finally, we define $C_1 := \min\{1, \frac{C_{N,k}}{2}, |\lambda_c|\}$, $C_2 := \max\{1, \frac{C_{N,k}}{2}, |\lambda_c|\}$, then

$$C_1 \langle f, f \rangle \; \le \; \langle f, f \rangle_c \; \le \; C_2 \langle f, f \rangle \quad \forall f \in H^k(\mathbb{R}^N).$$

□

Making use of Lemma 2.30, we retrieve that $\langle v_n - v, v_n - v \rangle_c \to 0$ in $H_r^k(\mathbb{R}^N)$ implies that $\langle v_n - v, v_n - v \rangle \to 0$ in $H_r^k(\mathbb{R}^N)$, meaning that

$$v_n \to v \;\text{ in }\; H_r^k(\mathbb{R}^N).$$

Theorem 2.10 is finally proved.

2.4.7. Additional characterization of γ(c)

In the following lines, once that the main theorem of this chapter has been proved, we proceed by drawing an important variational characterization of $\gamma(c)$. In particular, we are going to prove the following theorem.

Theorem 2.31. *It results that*

$$\gamma(c) = \inf_{u \in \mathcal{W}(c)} F(u), \tag{2.46}$$

with

$$\mathcal{W}(c) := \{u \in S_c,\ F|'_{S_c}(u) = 0\}.$$

This characterization will allow us to describe the solution we have found as a *ground state*, namely a function minimizing the energy functional F among the set $\mathcal{W}(c)$ of all possible solutions to Problem 2.1.

At first, we present a nonlocal version of the celebrated Pohozaev identity. The proof can be found in [12, Proposition 4.1].

Lemma 2.32. *Let $u \in H^k(\mathbb{R}^N)$ be a weak solution of*

$$(-\Delta)^k u = f(u) \quad \text{in } \mathbb{R}^N,$$

with $N \geq 2$, $k \in (0,1)$ and $f : \mathbb{R} \to \mathbb{R}$ continuous function such that $f(0) = 0$. Moreover, define $F(t) = \int_0^t f(s)\, ds$. Then, u satisfies

$$C_{N,k}\,(N-2k)\lfloor u \rfloor_k^2 = 4N \int_{\mathbb{R}^N} F(u).$$

Proposition 2.4. *Assume that (H1) and (H2) hold true. Then, each weak solution $(u, \lambda_c) \in (S_c \times \mathbb{R})$ to (2.12) belongs to the set*

$$V_c = \left\{ u \in S_c,\ C_{N,k} \lfloor u \rfloor_k^2 = \frac{N}{k} \int_{\mathbb{R}^N} \tilde{G}(u) \right\}.$$

Proof. We apply Proposition 2.4 to $f(u) = \lambda u + g(u)$. Thus, we consider u solution to (2.12) and develop calculations as follows:

$$
\begin{aligned}
&(N-2k)\lfloor u\rceil_k^2\\
&\quad = \frac{4N}{C_{N,k}}\int_{\mathbb{R}^N} F(u(x))\,dx = \frac{4N}{C_{N,k}}\left\{\int_{\mathbb{R}^N}\left[\int_0^{u(x)} \lambda s\,ds + g(s)\,ds\right]\,dx\right\}\\
&\quad = \frac{4N}{C_{N,k}}\left\{\int_{\mathbb{R}^N}\left[\lambda\frac{u^2}{2}+G(u)\right]\right\} = \frac{4N}{C_{N,k}}\left\{\frac{\lambda}{2}\|u\|_H^2 + \int_{\mathbb{R}^N} G(u)\right\}\\
&\quad = \frac{N}{C_{N,k}}\left\{C_{N,k}\lfloor u\rceil_k^2 - 2\int_{\mathbb{R}^N} g(u)u\right\} + \frac{4N}{C_{N,k}}\int_{\mathbb{R}^N} G(u)\\
&\quad = \frac{N}{C_{N,k}}\left\{C_{N,k}\lfloor u\rceil_k^2 - 2\int_{\mathbb{R}^N}[g(u)u-2G(u)]\right\}\\
&\quad = \frac{N}{C_{N,k}}\left\{C_{N,k}\lfloor u\rceil_k^2 - 2\int_{\mathbb{R}^N}\tilde{G}(u)\right\},
\end{aligned}
$$

where we have used the definition of λ in (2.42). The lemma is straightforwardly proved. □

Lemma 2.33. *Let (H1) and (H2) be true and fix $T(c)$ as given in Lemma 2.16. Then,*

$$
\begin{aligned}
A &= \{u\in S_c,\ \lfloor u\rceil_k^2 \le T(c)\}\ \textit{and}\\
C &= \{u\in S_c,\ \lfloor u\rceil_k^2 \ge 2T(c),\ F(u)\le 0\}
\end{aligned}
$$

are arc-connected.

Proof. To prove this lemma, we start defining a function $h(u,v,s,t): E\times E\times\mathbb{R}\times[0,\frac{\pi}{2}]\to E$, with

$$h(u,v,s,t)(x) = \cos(t)H(u,s)(x) + \sin(t)H(u,s)(x).$$

Now, we fix two distinct points u_1, u_2 in S_c, that satisfy $\langle u_1,u_2\rangle_H \neq -c^2$ and $\lfloor u_1\rceil_k^2 = \lfloor u_1\rceil_k^2 = 2d^2C_{N,k}^{-1}$. Direct calculations prove the following identities:

$$\|H(u_1,s)\|_H = \|H(u_2,s)\|_H = c \qquad \forall s\in\mathbb{R}$$

$$\lfloor H(u_1,s)\rceil_k^2 = \lfloor H(u_2,s)\rceil_k^2 = e^{2ks}2d^2\,C_{N,k}^{-1} \qquad \forall s\in\mathbb{R}$$

$$\langle H(u_1,s),H(u_2,s)\rangle_H = \langle u_1,u_2\rangle_H \qquad \forall s\in\mathbb{R}$$

$$\left\langle(-\Delta)^{\frac{k}{2}}H(u_1,s),(-\Delta)^{\frac{k}{2}}H(u_2,s)\right\rangle_H = e^{2ks}\left\langle(-\Delta)^{\frac{k}{2}}u_1,(-\Delta)^{\frac{k}{2}}u_2\right\rangle_H \qquad \forall s\in\mathbb{R}$$

The first three identities are trivial and have already been proved, whereas the fourth one is proved in Lemma B.11. Therefore, we gain

$$\|h(u_1,u_2,s,t)\|_H^2 = c^2 + \sin(2t)\langle u_1,u_2\rangle_H$$

and we want to compute $\lfloor h(u_1,u_2,s,t)\rfloor_k^2$ developing the calculations in this way:

$$\begin{aligned}
&\lfloor h(u_1,u_2,s,t)\rfloor_k^2\\
&= \iint_{\mathbb{R}^{2N}} \frac{|\cos(t)(H(u_1,s)(x)-H(u_1,s)(y))+\sin(t)(H(u_2,s)(x)-H(u_2,s)(y))|^2}{|x-y|^{N+2k}}\,dxdy\\
&= \cos^2(t)\iint_{\mathbb{R}^{2N}} \frac{|H(u_1,s)(x)-H(u_1,s)(y)|^2}{|x-y|^{N+2k}}\,dxdy\\
&\quad+\sin^2(t)\iint_{\mathbb{R}^{2N}} \frac{|H(u_2,s)(x)-H(u_2,s)(y)|}{|x-y|^{N+2k}}\,dxdy\\
&\quad+\sin(2t)\iint_{\mathbb{R}^{2N}} \frac{(H(u_1,s)(x)-H(u_1,s)(y))\,(H(u_2,s)(x)-H(u_2,s)(y))}{|x-y|^{N+2k}}\,dxdy\\
&= e^{2ks}2d^2\,C_{N,k}^{-1}+\sin(2t)2\,C_{N,k}^{-1}\int_{\mathbb{R}^N}(-\Delta)^k H(u_1,s)(x)H(u_2,s)(x)\,dx\\
&= e^{2ks}2\,C_{N,k}^{-1}\left\{d^2+\sin(2t)\left\langle(-\Delta)^{\frac{k}{2}}u_1,(-\Delta)^{\frac{k}{2}}u_2\right\rangle_H\right\}.
\end{aligned}$$

Then, we deduce that there exist two positive constants $a(u_1,u_2)>0$, $b(u_1,u_2)>0$ such that, for all $s\in\mathbb{R}$ and $t\in[0,\pi/2]$,

$$a \le \|h(u_1,u_2,s,t)\|_H^2 \le 2c^2$$

$$e^{2ks}2C_{N,k}^{-1}b \le \lfloor h(u_1,u_2,s,t)\rfloor_k^2 \le e^{2ks}4d^2C_{N,k}^{-1}.$$

Thus, we want to define $\hat{h}(u,v,s,t): E\times E\times\mathbb{R}\times[0,\pi/2]\to S_c$, where

$$\hat{h}(u,v,s,t) = c\,\frac{h(u,v,s,t)}{\|h(u,v,s,t)\|_H}.$$

We immediately notice that, for any u,v in S_c such that $\langle u,v\rangle_H \neq -c^2$ and $\lfloor u\rfloor_k^2 = \lfloor v\rfloor_k^2 = 2d^2C_{N,k}^{-1}$,

$$\frac{C_{N,k}^{-1}\,e^{2ks}b}{c^2} \le \lfloor \hat{h}(u,v,s,t)\rfloor_k^2 \le \frac{e^{2ks}4d^2\,C_{N,k}^{-1}}{a} \tag{2.47}$$

and that

$$\int_{\mathbb{R}^N} G(\hat{h}(u,v,s,t)(x))\,dx \ge Ce^{\frac{sN}{2}(\alpha-2)}. \tag{2.48}$$

Indeed, we can write

$$
\begin{aligned}
&\int_{\mathbb{R}^N} G(\hat{h}(u,v,s,t)(x))\,dx\\
&= c\left[\int_{\mathbb{R}^N} G\left(\cos(t)\frac{H(u,s)}{\|h(u,v,s,t)\|_H}\right)+\int_{\mathbb{R}^N}\left(\sin(t)\frac{H(v,s)}{\|h(u,v,s,t)\|_H}\right)\right]\\
&\geq C\int_{\mathbb{R}^N} G(H(u,s))+G(H(v,s)) = C\int_{\mathbb{R}^N}\left[G(e^{\frac{sN}{2}}u(e^s x))+G(e^{\frac{sN}{2}}v(e^s x))\right]\,dx\\
&\geq CG(1)e^{\frac{sN\alpha}{2}}\int_{\mathbb{R}^N}\left(|u(e^s x)|^\alpha+|v(e^s x)|^\alpha\right)\,dx = CG(1)e^{\frac{sN\alpha}{2}}e^{-sN} = Ce^{\frac{sN}{2}(\alpha-2)}.
\end{aligned}
$$

We are now ready to prove that A is arc-connected. We start fixing v_1, v_2 in A such that $\lfloor v_1\rfloor_k^2 = \lfloor v_2\rfloor_k^2 = 2d^2 C_{N,k}^{-1} < T(c)$ and $\langle v_1, v_2\rangle_H \neq -c^2$. Then, we notice that

$$
\frac{e^{2ks}2}{a} < 1 \iff s < \frac{1}{2k}\ln\frac{a}{2}
$$

and we fix

$$
s_0 = -\left|\frac{1}{2k}\ln\frac{a}{2}\right|.
$$

In this way, we can ensure that $\hat{h}(v_1, v_2, s_0, t) \in A$ $\forall t \in \left[0, \frac{\pi}{2}\right]$, we can construct the connection $\Gamma_1 : [0, 2|s_0|+1] \to S_c$ defined in the following way

$$
\Gamma_1(r) = \begin{cases} h(v_1, 0, -r, 0) & 0 \leq r \leq |s_0| \\ \hat{h}(v_1, v_2, s_0, r-|s_0|) & |s_0| \leq r \leq |s_0|+1 \\ h(0, v_2, r-(2|s_0|+1), \pi/2) & |s_0|+1 \leq r \leq 2|s_0|+1. \end{cases}
$$

It is immediate that $\Gamma(0) = v_1$ and $\Gamma(1) = v_2$. Moreover, $\Gamma_1(r) \in A$; if $0 \leq r \leq |s_0|$ indeed, we obtain

$$
h(v_1, 0, -r, 0)(x) = H(v_1, -r)(x) \in S_c,
$$

since $v_1 \in S_c$ by hypothesis. Moreover,

$$
\lfloor H(v_1, -r)\rfloor_k^2 = e^{2ks}d^2 2\,C_{N,k}^{-1} < T(c) \implies h(v_1, 0, -r, 0) \in A.
$$

For the same reasons, if $|s_0|+1 \leq r \leq 2|s_0|+1$, $\Gamma_1(r) \in A$. Obviously, $\hat{h}(v_1, v_2, s_0, r-s_0) \in A$ by definition of s_0.

If instead $\lfloor v_1\rfloor_k^2 \neq \lfloor v_2\rfloor_k^2 = 2d^2\,C_{N,k}^{-1}$, we can proceed as follows. Consider $H(v_1, s)$, with $s \in (0, s_1)$,

$$
s_1 = \ln(\lfloor v_2\rfloor_k^{\frac{1}{k}}) + \frac{1}{2k}\ln(\lfloor v_1\rfloor_k^{-2}),
$$

such that
$$H(v_1,0)=v_1 \text{ and } \lfloor H(v_1,s_1)\rfloor_k^2 = e^{2ks_1} = \lfloor v_2\rfloor_k^2.$$
In this way, exploiting the continuity of $H(v_1,\cdot)$, we can firstly join v_1 to $H(v_1,s_1)$ (without exiting A) and then join $H(v_1,s_1)$ to v_2 relying on the aforementioned path.
If finally $\langle v_1,v_2\rangle = -c^2$, we can introduce $v_3\in A$ and construct a new path that joins v_1 to v_3 and v_3 to v_2.
The proof that C is arc-connected is almost analogous to the previous one. Let $v_1,v_2\in C$ be two arbitrary points such that $\lfloor v_1\rfloor_k^2 = \lfloor v_2\rfloor_k^2 = 2d^2\,C_{N,k}^{-1}$ and $\langle v_1,v_2\rangle_H \neq -c^2$. If we remember that
$$F(u)=\frac{C_{N,k}}{4}\lfloor u\rfloor_k^2 - \int_{\mathbb{R}^N} G(u)$$
and inequality (2.48), we immediately get that there exists some $s_0>0$ that guarantees
$$\lfloor \hat{h}(v_1,v_2,s,t)\rfloor_k^2 \geq 2T(c) \text{ and } F(\hat{h}(v_1,v_2,s,t))\leq 0.$$
As a consequence, we can define $\Gamma_2:[0,2s_0+1]\to S_c$ as
$$\Gamma_2(r)=\begin{cases} h(v_1,0,r,0) & 0\leq r\leq s_0\\ \hat{h}(v_1,v_2,s_0,r-s_0) & s_0\leq r\leq s_0+1\\ h(0,v_2,r-(s_0+1),\pi/2) & s_0+1\leq r\leq 2s_0+1.\end{cases}$$
Clearly $\Gamma_2(0)=v_1$ and $\Gamma_2(1)=v_2$. Following the same reasoning as before, we can show that C is arc-connected. □

Corollary 2.34. *For any $v_1\in A$, $v_2\in C$, it holds that*
$$\gamma(c)=\inf_{g\in\Gamma_{(v_1,v_2)}}\max_{s\in[0,1]} F(g(s)),$$
where
$$\Gamma_{(v_1,v_2)}=\{g\in C([0,1],S_c), g(0)=v_1, g(1)=v_2\}$$
and $\gamma(c)$ has been originally defined in Proposition 2.1.

Proof. By Proposition 2.1, we know that there exist $u_1\in A$, $u_2\in C$ such that
$$\gamma(c)=\inf_{h\in\Gamma(c)}\max_{t\in(0,1)} F(h(t)),$$

with

$$\Gamma(c) = \{h \in C([0,1], S_c), h(0) = u_1, h(1) = u_2\}\,;$$

exploiting now the fact that A and C are arc-connected, we reason as follows.
We start consider a generic path $g \in \Gamma_{(v_1,v_2)}$ connecting v_1 to v_2; then there exists a path $h \in \Gamma(c)$ joining u_1 to v_1, v_1 to v_2 and v_2 to u_2. The generality of g implies

$$\inf_{h\in\Gamma(c)} \max_{t\in[0,1]} F(h(t)) \leq \inf_{g\in\Gamma_{(v_1,v_2)}} \max_{s\in(0,1)} F(g(s)).$$

On the other hand reasoning in the opposite way, we obtain

$$\inf_{g\in\Gamma_{(v_1,v_2)}} \max_{s\in(0,1)} F(g(s)) \leq \inf_{h\in\Gamma(c)} \max_{t\in[0,1]} F(h(t)).$$

□

Lemma 2.35. *Assume that (H1), (H2), (H3) hold true and fix an arbitrary point $u \in S_c$. Then, the function $f_u : \mathbb{R} \to \mathbb{R}$ such that*

$$f_u(s) = F(H(u,s))$$

has just one maximum point at $s(u) \in \mathbb{R}$, such that $H(u,s(u)) \in V_c$.

Proof. Before starting the proof we recall equation 2.25:

$$f_u'(s) = \frac{C_{N,k}\,k}{2}\lfloor v\rfloor_k^2 + N\int_{\mathbb{R}^N} G(v) - \frac{N}{2}\int_{\mathbb{R}^N} g(v)v,$$

where $v = H(u,s)$. We shall now prove that there exists some s_0 such that $f_u'(s_0) = 0$. We start noticing that

$$f_u'(0) = \frac{k}{2}\lfloor v\rfloor_k^2 > 0.$$

Then, since by (H2) we know that $g(v)v \geq \alpha G(v)$, we write

$$\begin{aligned}
f_u'(s) &\leq \frac{C_{N,k}\,k}{2}\lfloor v\rfloor_k^2 + N\int_{\mathbb{R}^N} G(v) - \frac{N\alpha}{2}\int_{\mathbb{R}^N} G(v) \leq \frac{C_{N,k}\,k}{2}\lfloor v\rfloor_k^2 - N\left(\frac{\alpha}{2}-1\right)\int_{\mathbb{R}^N} G(v)\\
&= 2k\,\frac{C_{N,k}}{4}\lfloor v\rfloor_k^2 - N\left(\frac{\alpha}{2}-1\right)\int_{\mathbb{R}^N} G(v) \leq 2k\left(\frac{C_{N,k}}{4}\lfloor v\rfloor_k^2 - \int_{\mathbb{R}^N} G(v)\right)\\
&= 2k\tilde{F}(H(u,s)).
\end{aligned}$$

By Lemma 2.15 we know that there exists at least one $s_0 > 0$ such that $f_u'(s_0) = 0$.

Moreover, by

$$f_u'(s) = \frac{C_{N,k}\,k}{2}\lfloor v\rfloor_k^2 - \frac{N}{2}\int_{\mathbb{R}^N}\tilde{G}(v),$$

we obtain that $f_u'(s_0) = 0$ implies $H(u,s_0) \in V_c$.

Now we make clear the following equations:

$$\begin{aligned}\int_{\mathbb{R}^N} G(v) &= \int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(e^s x))\,dx = e^{-sN}\int_{\mathbb{R}^N} G(e^{\frac{sN}{2}}u(x))\,dx\\ \int_{\mathbb{R}^N} g(v)v &= \int_{\mathbb{R}^N} g(e^{\frac{sN}{2}}u(e^s x))e^{\frac{sN}{2}}u(e^s x)\,dx = e^{-\frac{sN}{2}}\int_{\mathbb{R}^N} g(e^{\frac{sN}{2}}u(x))u(x)\,dx.\end{aligned}$$

Then, we deduce that

$$\frac{\partial}{\partial s}\int_{\mathbb{R}^N} G(v) = -N\int_{\mathbb{R}^N} G(v) + \frac{N}{2}\int_{\mathbb{R}^N} g(v)v$$

and that

$$\begin{aligned}\frac{\partial}{\partial s}\int_{\mathbb{R}^N} g(v)v &= -\frac{N}{2}\int_{\mathbb{R}^N} g(v)v + \frac{N}{2}e^{-\frac{sN}{2}}\int_{\mathbb{R}^N} g'(e^{\frac{sN}{2}}u(x))e^{\frac{sN}{2}}u^2(x)\,dx\\ &= -\frac{N}{2}\int_{\mathbb{R}^N} g(v)v + \frac{N}{2}\int_{\mathbb{R}^N} g'(v)v^2.\end{aligned}$$

Moreover, we have that

$$\begin{aligned}\frac{C_{N,k}\,k}{2}\frac{\partial}{\partial s}\lfloor v\rfloor_k^2 &= \frac{C_{N,k}\,k}{2}\frac{\partial}{\partial s}\left(\lfloor H(u,s)\rfloor_k^2\right) = \frac{C_{N,k}\,k}{2}\frac{\partial}{\partial s}(e^{2ks}\lfloor u\rfloor_k^2)\\ &= 2k\,C_{N,k}\frac{\partial}{\partial s}\left(\frac{e^{2ks}}{4}\lfloor u\rfloor_k^2\right) = k^2\,C_{N,k}\lfloor v\rfloor_k^2.\end{aligned}$$

Combining all together we obtain that, naming $v = H(u,s_0)$,

$$\begin{aligned}f_u''(s_0) &= k^2\,C_{N,k}\lfloor v\rfloor_k^2 - N^2\int_{\mathbb{R}^N} G(v) + \frac{N^2}{2}\int_{\mathbb{R}^N} g(v)v\\ &\quad + \frac{N^2}{4}\int_{\mathbb{R}^N} g(v)v - \frac{N^2}{4}\int_{\mathbb{R}^N} g'(v)v^2\\ &= kN\left[\int_{\mathbb{R}^N} g(v)v - \int_{\mathbb{R}^N} 2G(v)\right] - \frac{N^2}{2}\int_{\mathbb{R}^N} 2G(v) + \frac{N^2}{2}\int_{\mathbb{R}^N} g(v)v\\ &\quad + \frac{N^2}{4}\int_{\mathbb{R}^N} g(v)v - \frac{N^2}{4}\int_{\mathbb{R}^N} g'(v)v^2\\ &= Nk\int_{\mathbb{R}^N}\tilde{G}(v) + \frac{N^2}{2}\int_{\mathbb{R}^N}\tilde{G}(v) - \frac{N^2}{4}\int_{\mathbb{R}^N}\tilde{G}'(v)v\\ &= \frac{N}{2}\left[(N+2k)\int_{\mathbb{R}^N}\tilde{G}(H(u,s_0)(x))\,dx - \frac{N}{2}\int_{\mathbb{R}^N}\tilde{G}'(H(u,s_0)(x))H(u,s_0)(x)\,dx\right],\end{aligned}$$

where we have used the fact that $f'_u(s_0) = 0$. Now we are ready to show that $f''_u(s_0) < 0$, which holds true since

$$\begin{aligned} f''_u(s_0) &< \frac{N}{2}\left[(N+2k)\int_{\mathbb{R}^N}\tilde{G}(H(u,s_0)(x))\,dx - \frac{N}{2}\int_{\mathbb{R}^N}\frac{2N+4k}{N}\tilde{G}(H(u,s_0)(x))\,dx\right] \\ &= \frac{N}{2}\left[(N+2k)-(N+2k)\right]\int_{\mathbb{R}^N}\tilde{G}(H(u,s_0)(x))\,dx \\ &= 0. \end{aligned}$$

As a consequence, we have shown that each possible critical point for f_u has negative second derivative. This is enough to infer that s_0 is the unique critical point and, in particular, that it is a maximum. □

Lemma 2.36. *If (H1), (H2), (H3) hold, then we can claim that*

$$\gamma(c) = \inf_{u\in V_c} F(u).$$

Proof. This proof relies on a contradiction argument, supposing that there exists some $v \in V_c$ such that $F(v) < \gamma(c)$. Then we define the map $T_v : \mathbb{R} \to S_c$ such that

$$T_v(s) = H(v, s).$$

Exploiting Lemma 2.15 we obtain that $\exists s_0 > 0$ such that $T_v(-s_0) \in A$ and $T_v(s_0) \in C$. If then we ask for $\tilde{T}_v : [0,1] \to S_c$ to be the path defined by

$$\tilde{T}_v(s) = H(v, (2s-1)s_0),$$

where $\tilde{T}_v(0) = T_v(-s_0)$ and $\tilde{T}_v(1) = T_v(s_0)$. By Lemma 2.35 we know that $F(\tilde{T}_v(s))$ reaches its unique maximum in $s = 1/2$, since $v \in V_c$ by hypothesis. This information, combined with Corollary 2.34, means that

$$\gamma(c) \le F(\tilde{T}_v(1/2)) = F(v). \tag{2.49}$$

This is a clear contradiction. □

Proof of Theorem 2.31. By Lemma 2.4 we know that the set of all weak solutions to (2.12), denoted by $\mathcal{W}(c)$ is a subset of V_c. Thus, relying on Lemma 2.36, we can write

$$\gamma(c) = \inf_{u\in V_c} F(u) \le \inf_{u\in \mathcal{W}(c)} F(u).$$

Moreover, by characterization (2.37) of our mountain pass solution, we have that $\gamma(c)$ is the value achieved by at least one element of $\mathcal{W}(c)$. As a consequence, we also get

$$\gamma(c) \geq \inf_{u \in \mathcal{W}(c)} F(u).$$

Combining these two inequalities, equation (2.46) is immediate. $\square$

Bibliography

[1] A. Ambrosetti, P. Rabinowitz Dual variational methods in critical point theory and applications, *J. Funct. Anal.*14, 349-381 (1973).

[2] A. Ambrosetti, M. Struwe Existence of steady vortex rings in an ideal fluid, *Arch. Ration. Mech. Anal.*108, No. 2, 97-109 (1989).

[3] H. Berestycki, T. Cazenave. Instabilité des etats stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires, *C. R. Acad. Sci., Paris, Sér. I* 293, 489-492 (1981).

[4] H. Berestycki, P.L. Lions. Nonlinear scalar field equations. I: Existence of a ground state, *Arch. Ration. Mech. Anal.* 82, 313-345 (1983).

[5] H. Berestycki, P.L. Lions. Nonlinear scalar field equations. II: Existence of infinitely many solutions, *Arch. Ration. Mech. Anal.* 82, 347-375 (1983).

[6] J. Bourgain, H. Brezis, P. Mironescu . Another look at Sobolev spaces, Menaldi, José Luis (ed.) et al. *Optimal control and partial differential equations.* In honour of Professor Alain Bensoussan's 60th birthday. Proceedings of the conference, Paris, France, December 4, 2000. Amsterdam: IOS Press; Tokyo: Ohmsha (ISBN 1-58603-096-5; 4-274-90412-1). 439-455 (2001).

[7] H. Brezis. Analyse fonctionnelle. Théorie et applications, *Collection Mathématiques Appliquées pour la Maîtrise. Paris etc.* Masson. XIV, 233 p. FF 125.00 (1983).

[8] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, New York, NY: Springer (2011; Zbl 1220.46002).

[9] H. Brezis. How to recognize constant functions. Connections with Sobolev spaces, *Russ. Math. Surv.* 57, No. 4, 693-708 (2002); translation from Usp. Mat. Nauk 57, No. 4, 59-74 (2002).

[10] C. Bucur, E. Valdinoci. *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana 20. Cham: Springer; Bologna: UMI (ISBN 978-3-319-28738-6/pbk; 978-3-319-28739-3/ebook). xii, 155 p. (2016).

[11] T. Cazenave, P.L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations, *Commun. Math. Phys.* 85, 549-561 (1982).

[12] H. Chang, Z.-Q. Wang. Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, *Nonlinearity* 26, No. 2, 479-494 (2013).

[13] E. Di Nezza, G. Palatucci, V. Enrico. Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*,136, No. 5, 521-573 (2012)

[14] S. Dipierro, G. Palatucci, V. Enrico. Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, *Matematiche*,68, No. 1, 201-216 (2013).

[15] A. Ferrero, F. Gazzola, M. Zanotti. *elementi di ANALISI SUPERIORE per la FISICA e l'INGEGNERIA*, Esculapio, 3rd edition (2015).

[16] N. Garofalo. Fractional thoughts, Danielli, Donatella (ed.) et al., New developments in the analysis of nonlocal operators. AMS special session, University of St. Thomas, Minneapolis, MN, USA, October 28–30, 2016. Providence, RI: American Mathematical Society (AMS). *Contemp. Math.*, 723, 1-135 (2019).

[17] Ghoussoub. *Duality and Perturbation Methods in Critical Point Theory*, Cambridge: Cambridge University Press (1993; Zbl 0790.58002).

[18] L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal., Theory Methods Appl.* 28, No. 10, 1633-1659 (1997).

[19] J. Dávila, M. del Pino, S. Dipierro, E. Valdinoci. Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum, *Anal. PDE* 8, No. 5, 1165-1235 (2015).

[20] M. Kassmann. A new formulation of Harnack's inequality for nonlocal operators, *C. R., Math., Acad. Sci. Paris* 349, No. 11-12, 637-640 (2011).

[21] M. Kassmann. The classical Harnack inequality fails for nonlocal operators, *Preprint No. 360*, Sonderforschungsbereich 611, Universität Bonn, 2007.

[22] S. Lang. *Fundamentals of differential geometry*, Corr. 2nd printing. Graduate Texts in Mathematics. 191. New York, NY: Springer. xvii, 535 p. (2001).

[23] N. Laskin. Fractional quantum mechanics, Hackensack, NJ: *World Scientific*, xv, 341 p. (2018) .

[24] J.M. Lee. Manifolds and differential geometry, *Graduate Studies in Mathematics*,

107. Providence, RI: American Mathematical Society (AMS) (ISBN 978-0-8218-4815-9/hbk). xiv, 671 p. (2009).

[25] P.L Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*1, 109-145 (1984).

[26] P.L Lions. The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*1, 223-283 (1984).

[27] R. S. Palais. Lusternik-Schnirelman theory on Banach manifolds, *Topology* 5, 115-132 (1966).

[28] X. Ros-Oton. *Nonlocal elliptic equations in bounded domains: a survey*, arXiv:1504.04099.

[29] S. Salsa. *Partial differential equations. Methods, models and applications. 3rd edition.* Unitext 97. La Matematica per il 3+2. Milan: Springer. xvi, 671 p. (2016).

[30] N. Soave. Normalized ground states for the NLS equation with combined nonlinearities, *J. Differ. Equations.*, 269, No. 9, 6941-6987 (2020).

[31] N. Soave. Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, *J. Funct. Anal.* , 279, No. 6, Article ID 108610, 42 p. (2020).

[32] E. Valdinoci. From the long jump random walk to the fractional Laplacian, *Bol. Soc. Esp. Mat. Apl., S$\vec{e}$MA* 49, 33-44 (2009).

A | Basic notions of differential geometry

In this section we propose to expound some basic notions of differential geometry which will be often referred to throughout the whole thesis. For a more in-depth discussion on this topic, we refer to [5], [17, Chapter 3], [22], [24] and [27] .

Let X be a differential manifold, $x \in X$ and $\phi : U \to \mathbb{R}^N$ a coordinate chart, with U open subset of X containing x.
If we consider $\gamma_1, \gamma_2 : (-1, 1) \to X$ such that $\gamma_1(0) = \gamma_2(0) = 0$, then we can compose the chart with the curves into $\phi \circ \gamma_1,\ \phi \circ \gamma_2 : (-1, 1) \to \mathbb{R}^N$.
We say that γ_1, γ_2 are differentiable if their relative composition with ϕ is differentiable, in the ordinary sense.
Moreover, if we impose an equivalence relation among all differentiable curves

$$\gamma_1 \equiv \gamma_2 \iff \frac{d}{dt}(\phi \circ \gamma_1)(t)\Big|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_2)(t)\Big|_{t=0},$$

the equivalence classes of such curves are named tangent vectors of X in x.

Definition A.1. *The set of all tangent vectors at x is known as tangent space of X in x and goes under the symbol $T_x(X)$.*

We notice that, to introduce any kind of vector space operations on $T_x(X)$, we may be able to transfer $\mathbb{R}^N$ to $T_x(X)$ and we require again a chart $\phi : U \to \mathbb{R}^N$ to define the map: $\psi_x : T_x(X) \to \mathbb{R}^N$

$$\psi_x(\gamma) := \frac{d}{dt}[(\phi \circ \gamma)(t)],$$

for any γ tangent vectors, $\gamma \in T_x(X)$.
In particular, it is true that both this construction does not depend on the coordinate chart chose and that ψ_x is bijective, allowing us to pass from $\mathbb{R}^N$ to the tangent space.
Once we have stated the previous definition, the next comes natural, again under the hypothesis of X being a differential manifold.

Definition A.2. *We call tangent bundle of X a manifold T(X) assembling all tangent vectors in X. More precisely, T(X) is defined as a disjoint union of all the tangent spaces of X.*

For the reader's convenience, we recall here the explicit definition of disjoint union, used to the define the tangent bundle T(X):

$$\begin{aligned} T(X) &= \sqcup_{x\in X} T_x(X) \\ &= \cup_{x\in X} (\{x\} \times T_x(X)) \\ &= \cup_{x\in X} \{(x,y)|\, y \in T_x(X)\} \\ &= \{(x,y)|\, x \in X, y \in T_x(X)\}, \end{aligned}$$

where $T_x(x)$ is the tangent space to X at point x.

Definition A.3. *Let X a C^1-Banach manifold, $T_x(X)$ and $T(X)$ as previously defined. We define a Finsler structure on $T(X)$ a continuous function $\|\cdot\| : T(X) \to [0,+\infty)$, such that*

(a) for any $x \in X$, the restriction of $\|\cdot\|$ to $T_x(X)$, named $\|\cdot\|_x$, is a norm on the latter

(b) for any $x_0 \in X$, $k > 1$, there exists a trivializing neighbourhood U of x_0, such that

$$\frac{1}{k}\|\cdot\|_x \leq \|\cdot\|_{x_0} \leq k\|\cdot\|_x \quad \forall x \in U.$$

Consistently with the topology of X, we can proceed to set a Finsler metric $\rho : X\times X \to \mathbb{R}$ on each connected component of the manifold. In particular, if we define $\sigma : [a,b] \to X$ a C^1 path in X and $L(\sigma) = \int_a^b \|\dot{\sigma}(t)\|\, dt$ its length, then we can set

$$\rho(x,y) := \inf L(\sigma), \tag{A.1}$$

over all σ joining x, y and for all x, y in the same X connected component.
Moreover, if we consider $\phi \in C^1(X,\mathbb{R})$, its differential at point $x \in X$ is an element of the cotangent space of X, meaning that it is a functional $d\phi_x \in T_x(X)^*$. We recall here that $d\phi_x$ is the linear functional satisfying:

$$\lim_{\|h\|_x\to 0} \frac{|\phi(x+h) - \phi(x) - \langle d\phi_x, h\rangle|}{\|h\|_x} = 0 \quad \forall h \in T_x(X).$$

We give here a useful definition in order to introduce the next topic of this discussion, i.e.

the construction of a pseudo-gradient vector field for a generic $\phi \in C^1(X, \mathbb{R})$ at the set of its regular points R,

$$R := \{x \in X : d\phi_x \neq 0\}.$$

Definition A.4. *Let X be a topological space; we define partition of unity a set I of continuous functions χ defined on X with values in $[0,1]$, such that, for every point $x \in X$*

- *there exists at least one neighbourhood of x where all but a finite number of functions in I are null*
- $\sum_{\chi \in I} \chi(x) = 1 \ \forall x \in X$.

In particular, it can be shown that for any open cover $(V_i)_{i\in K}$ of the topological space considered, there exists a partition of unity $(\chi_i)_{i\in K}$ indexed over the same set K and such that $\text{supp}(\chi_i) \subseteq V_i$; in this case we say that $(\chi_i)_{i\in K}$ is conditioned to $(V_i)_{i\in K}$.

Before closing this paragraph we decided to deal with another topic: the pseudo-gradient vector and of pseudo-gradient vector field.
We start with their formal definitions.

Definition A.5. *Let X be a Finsler manifold and let $\phi : X \to \mathbb{R}$ be differentiable at some $x \in X$. Then we call $v_x \in T_x(X)$ pseudo-gradient vector for ϕ at x if*

(a) $\|v_x\| \leq 2\|d\phi_x\|_*$

(b) $\langle d\phi_x, v_x\rangle \geq \|d\phi_x\|_*^2$.

Definition A.6. *Let X be a Finsler manifold and let $\phi : X \to \mathbb{R}$ be differentiable at each point of $S \subseteq X$. Let $V : S \to T(S)$, $V(x) = v_x$ be a C^k-vector field in S. Then V, is called a C^k pseudo-gradient vector field for ϕ on S if, for any $x \in S$, v_x is a pseudo-gradient vector for ϕ at x.*

Looking at Definition (A.5), we immediately notice that, if x is a critical point for ϕ, namely $d\phi_x = 0$, then only $v_x = 0$, $v_x \in T_x(X)$ is a pseudo-gradient vector for ϕ in x. If, instead, x is not a critical point we can find a non-trivial v_x as follows.
We start setting $w_x \in T_x(X)$ such that $\|w\| = 1$ and $\langle d\phi_x, w_x\rangle \geq (1-\epsilon)\|d\phi_x\|_*$. This choice of w_x is possible starting by the definition of norm for $d\phi_x$ in $T_x(X)^*$

$$\|d\phi_x\|_* = \sup_{\|y\|=1} \langle d\phi_x, y\rangle \quad y \in T_x(X)$$

and selecting, as a consequence, a maximizing sequence $(y_n)_n$ for $\langle d\phi_x, y_n\rangle$.

Now we set $v_x \in T_x(X)$ as follows:

$$v_x = \frac{1+\delta}{1-\epsilon}\|d\phi_x\|_* w_x, \quad \delta > 0,\ \frac{1+\delta}{1-\epsilon} \leq 2 \iff 0 < \delta < 1 - 2\epsilon.$$

It is clear that point *(a)* from Definition (A.5) is satisfied; point *(b)* is satisfied too, since

$$\begin{aligned}\langle d\phi_x, v_x\rangle &= \langle d\phi_x, \frac{1+\delta}{1-\epsilon}\|d\phi_x\|_* w_x\rangle \\ &= \frac{1+\delta}{1-\epsilon}\|d\phi_x\|_* \langle d\phi_x, w_x\rangle \\ &\geq (1+\delta)\|d\phi_x\|_*^2.\end{aligned}$$

As a consequence, we can allow $\|v_x\|$ to be as close as we wish to $\|d\phi_x\|_*$, having still point *(b)* satisfied and v_x is a pseudo-gradient vector for ϕ at point x.

Lemma A.7. *If X is a Finsler manifold and $\phi : X \to R$ is differentiable at point x, the set of pseudo-gradient vectors for ϕ at x is a convex subset of $T_x(X)$.*

Proof. Let $\{v_i,\ i = 1, ..., n\}$, $n \in \mathbb{N}$ be a set of pseudo-gradient vectors for ϕ at x. We want to show that

$$v =: \sum_{i=1}^{n} \lambda_i v_i, \quad \lambda_i > 0,\ \sum_{i=1}^{n} \lambda_i = 1,$$

is again a pseudo-gradient vectors for ϕ at x. Indeed, it holds true that

$$\begin{aligned}\|v\| &= \|\lambda_1 v_1 + ... + \lambda_n v_n\| \\ &\leq \lambda_1\|v_1\| + ... + \lambda_n\|v_n\| \\ &\leq 2\sum_{i=1}^{n} \lambda_i \|d\phi_x\|_* = 2\|d\phi_x\|_*.\end{aligned}$$

Moreover, we also have that

$$\begin{aligned}\langle d\phi_x, v\rangle &= \sum_{i=1}^{n} \lambda_i \langle d\phi_x, v_i\rangle \\ &\geq \sum_{i=1}^{n} \lambda_i \|d\phi_x\|_*^2 = \|d\phi_x\|_*^2.\end{aligned}$$

□

Proposition A.1. *Let X be a C^{s+1} Finsler manifold, $s \geq 0$ and let $\phi : X \to \mathbb{R}$ be a C^1 functional. If we ask for $x \in X$ not to be a critical point for ϕ, there exists an open* *neighbourhood U centred in x and a C^s pseudo-gradient vector field for ϕ in U.*

Proof. We already showed how to construct $v_x \in T_x(X)$ pseudo-gradient vector for ϕ at x. Now we extend v_x to be a C^s constant vector field, equal to v_x in a neighbourhood N of x. We finally define the set U

$$U := \{q \in N : \|v_q\| \leq 2\|d\phi_q\|_*,\ \langle d\phi_q, v_q\rangle \geq \|d\phi_q\|_*^2\}.$$

Exploiting the continuity in N of $\|d\phi\|_*$, $\|v\|$, $\langle d\phi, v\rangle$, U is open. □

Relying on the previous proposition, we are now ready to ensure the existence of a pseudo-gradient vector field for ϕ at the set of its regular points R through the next theorem.

Theorem A.8. *Let X be a C^2 Finsler manifold and let $\phi : X \to \mathbb{R}$ be a C^1 functional. Then, there exists a C^1 pseudo-gradient vector field for ϕ in R.*

Proof. For any x in R, we set a neighbourhood U_x of x satisfying Proposition A.1 and we recover, as a consequence, a C^1 pseudo-gradient vector field V_x for ϕ in U_x. Moreover, since we already noticed that the definition of the Finsler metric (A.1) is consistent with the topology of X and makes X metrizable, then R is paracompact. Thus, since $(U_x)_{x\in R}$ is an open cover of R we can find a finite subcover of R, $(U_x)_{x\in B}$. Then, recalling the aforementioned properties of partitions of unity, there exists a partition of unity $(\chi_x)_{x\in B}$ subordinated to $(U_x)_{x\in R}$ such that $\text{supp}(\chi_x) \subseteq U_x$. Finally, we have that

$$V = \sum_{x\in B} \chi_x V_x$$

is a C^1 vector field and, thanks to Lemma A.7, it is the pseudo-gradient vector field we are looking for. □

B | Proofs and useful results

Lemma B.1. *Let $g \in C^2(\mathbb{R}^N)$ and $(\rho_\epsilon)_\epsilon$ sequence of radial mollifiers as described in Subsection 1.1 such that*

$$\iint_{\mathbb{R}^{2N}} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_\epsilon(|x-y|)\,dxdy \leq C \quad \text{as } \epsilon \to 0. \tag{B.1}$$

Then, it holds that

$$\lim_{\epsilon\to 0}\iint_{\mathbb{R}^{2N}} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_\epsilon(|x-y|)\,dxdy = K_{p,N}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx.$$

with the constant

$$K_{p,N} = \int_{\partial B_1}|\sigma\cdot e|^p\,d\sigma \quad e\in\partial B_1.$$

Proof. Let K be a compact subset of $\mathbb{R}^N$. For $x\in K$ and $|h|\leq 1$ we have that

$$|g(x+h)-g(x)-h\nabla g(x)| \leq C_k|h|^2,$$

where $C_K = \max_{x\in K}\{\|H_g(x)\|_\infty\}$ which exists and is well defined since K is compact and $g\in C^2(K)$. This inequality implies

$$|h\nabla g(x)| \leq |g(x+h)-g(x)| + C_K|h|^2$$

and we can finally retrieve, for any $\theta>0$, that

$$|h\nabla g(x)|^p \leq (1+\theta)^p|g(x+h)-g(x)|^p + C_{\theta,K}|h|^{2p}. \tag{B.2}$$

Inequality (B.2) is not immediate but comes from the following reasoning, under $a>0$,

$b > 0$ and $\theta > 0$ fixed:

$$\begin{aligned}(a+b)^p \le 2^{p-1}(a^p+b^p) &\le (1+\theta)a^p + \frac{|2^{p-1}-(1+\theta)|\,a^p}{b^p}b^p + 2^{p-1}b^p \\ &\le (1+\theta)a^p + 2\max\left\{\frac{|2^{p-1}-(1+\theta)|\,a^p}{b^p}, 2^{p-1}\right\} b^p \\ &= (1+\theta)a^p + Cb^p.\end{aligned}$$

Inserting this result in (B.2) we obtain

$$\begin{aligned}\int_K \int_{|h|\le 1} \frac{|h\nabla g(x)|^p}{|h|^p}\rho_\epsilon(|h|)\,dhdx \le (1+\theta)\int_K\int_{|h|\le 1}\frac{|g(x+h)-g(x)|^p}{|h|^p}\rho_\epsilon(|h|)\,dhdx \\ + C_{\theta,K}|K|\int_{|h|\le 1}|h|^p\rho_\epsilon(|h|)\,dh,\end{aligned} \tag{B.3}$$

where $|K|$ denotes the dimension of the set K. Since it is obvious that

$$\lim_{\epsilon\to 0}\int_{|h|\le 1}|h|^p\rho_\epsilon(|h|)\,dh = 0,$$

we can study (B.3) as $\epsilon \to 0$. If we recall that, for any vector $V \in \mathbb{R}^N$, it holds

$$\int_{|h|\le 1}\frac{|(h\cdot V)|^p}{|h|^p}\rho_\epsilon(|h|)\,dh = K_{p,N}|V|^p\int_0^1 \rho_\epsilon(r)r^{N-1}\,dr,$$

we immediately find

$$K_{p,N}\int_K |\nabla g(x)|^p\,dx \le (1+\theta)\int_K\int_{|h|\le 1}\frac{|g(x+h)-g(x)|^p}{|h|^p}\rho_\epsilon(|h|)\,dhdx. \tag{B.4}$$

Moreover, (B.4) is true for any K compact set and $\theta > 0$ and we know, by hypothesis, that the double integral at the right hand side is well defined on any real domain. As a consequence, we can write

$$K_{p,N}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx \le \liminf_{\epsilon\to 0}\iint_{\mathbb{R}^{2N}}\frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_\epsilon(|h|)\,dxdy. \tag{B.5}$$

Conversely, if we consider $g \in C_0^2(\mathbb{R}^N)$ we obtain again

$$|g(x+h)-g(x)| \le |h\nabla g(x)| + C'|h|^2 \quad \forall x \in \mathbb{R}^N,\ \forall h \in \mathbb{R}^N.$$

Thus, we infer that

$$|g(x+h)-g(x)|^p \leq (1+\theta)|h\nabla g(x)|^p + C'_\theta |h|^{2p}.$$

Multiplying the last inequality by $\rho_\epsilon(|h|)/|h|^p$ and integrating over the set $\{(x,h) \in \mathbb{R}^{2N} : x \text{ or } x+h \in \text{ supp } g\}$ we conclude

$$\iint_{\mathbb{R}^{2N}} \frac{|g(x+h)-g(x)|^p}{|h|^p}\rho_\epsilon(|h|)\,dh \leq \\ (1+\theta)\int_{\mathbb{R}^N} K_{p,N}|\nabla g(x)|^p\,dx + 2C'_\theta|\text{supp } g|\int_{\mathbb{R}^N}|h|^p\rho_\epsilon(|h|)\,dh.$$

Asking for both $\epsilon \to 0$ and $\theta \to 0$, we finally get

$$\limsup_{\epsilon\to 0}\iint_{\mathbb{R}^{2N}} \frac{|g(x+h)-g(x)|^p}{|x-y|^p}\rho_\epsilon(|h|)\,dh \leq K_{p,N}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx. \tag{B.6}$$

We can conclude the proof combining (B.5) and (B.6) to gather

$$\lim_{\epsilon\to 0}\iint_{\mathbb{R}^{2N}} \frac{|g(x)-g(y)|^p}{|x-y|^p}\rho_\epsilon(|x-y|)\,dxdy = K_{p,N}\int_{\mathbb{R}^N}|\nabla g(x)|^p\,dx.$$

Just notice that the proof is completed since $C_0^2(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. □

Proof of Theorem 1.1. We start the proof considering any sequence of smooth mollifiers (γ_δ) and setting

$$f_\delta = \gamma_\delta * f.$$

We immediately notice that (1.8) is verified by f_δ with the same constant C, namely

$$\iint_{\mathbb{R}^{2N}} \frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p}\rho_\epsilon(|x-y|)\,dxdy \leq C, \tag{B.7}$$

thanks to stability of (1.8) with respect to translations and convex combinations. Moreover, since $f_\delta \in C^2(\mathbb{R}^N)$, hypotheses of Lemma B.1 are satisfied and we retrieve

$$\lim_{\epsilon\to 0}\iint_{\mathbb{R}^{2N}} \frac{|f_\delta(x)-f_\delta(y)|^p}{|x-y|^p}\rho_\epsilon(|x-y|)\,dxdy = K_{p,N}\int_{\mathbb{R}^N}|\nabla f_\delta(x)|^p\,dx.$$

Both the theses of the theorem come straightforwardly if we consider the limit as $\delta \to 0$. For what concerns $f \in W^{1,p}(\mathbb{R}^N)$ it is true since we have showed that

$$\int_{\mathbb{R}^N}|\nabla f_\delta(x)|^p\,dx \leq \frac{C}{K_{p,N}}.$$

□

Lemma B.2. *Let $\delta < 1$, then*

$$\int_0^{2\pi} \frac{1}{1-\delta\cos(\phi)}\,d\phi = \frac{2\pi}{\sqrt{1-\delta^2}}.$$

Proof. To solve this integral we rely on the Weierstrass substitution $t = \tan\left(\frac{\phi}{2}\right)$ and notice

$$\cos(\phi) = \cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right) = \frac{\cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)}{\cos^2\left(\frac{\phi}{2}\right) + \sin^2\left(\frac{\phi}{2}\right)} = \frac{1-\tan^2\left(\frac{\phi}{2}\right)}{1+\tan^2\left(\frac{\phi}{2}\right)} = \frac{1-t^2}{1+t^2}.$$

By $\phi = \arccos\left(\frac{1-t^2}{1+t^2}\right)$ and by $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$ we obtain

$$\begin{aligned} d\phi &= \frac{1}{\sqrt{1-\left(\frac{1-t^2}{1+t^2}\right)^2}}\frac{4t}{(1+t^2)^2}\,dt = \frac{4t}{(1+t^2)\sqrt{4t^2}}\,dt \\ &= \frac{2}{1+t^2}\,dt. \end{aligned}$$

Moreover, before starting with the computations, we notice that $\frac{1}{1-\delta\cos(\phi)}$ in $[0, 2\pi]$ is symmetric with respect to π. Then, naming $a = \frac{1}{\delta}$, we write

$$\begin{aligned} \int_0^{2\pi} \frac{1}{1-\delta\cos(\phi)}\,d\phi &= \frac{2}{\delta}\int_0^{\pi} \frac{d\phi}{a-\cos(\phi)} = \frac{2}{\delta}\int_0^{+\infty} \frac{1}{\left(a-\frac{1-t^2}{1+t^2}\right)}\frac{2}{(1+t^2)}\,dt \\ &= \frac{4}{\delta}\int_0^{+\infty} \frac{dt}{t^2(a+1)+(a-1)} = \frac{4}{\delta(a+1)}\int_0^{+\infty} \frac{dt}{\left(\frac{a-1}{a+1}+t^2\right)}. \end{aligned}$$

Now we study

$$\int \frac{1}{A^2+t^2}\,dt$$

and exploit the substitution $t = A\tan(\theta)$, $dt = A\sec^2\theta d\theta$ to infer

$$\begin{aligned} \int \frac{1}{A^2+t^2}\,dt &= \int \frac{1}{A^2+(A\tan(\theta))^2}A\sec^2(\theta)\,d\theta = \frac{1}{A}\int \frac{1}{1+\tan^2(\theta)}\sec^2(\theta)\,d\theta \\ &= \frac{1}{A}\int 1 d\theta = \frac{1}{A}\theta + c = \frac{1}{A}\tan^{-1}\left(\frac{t}{A}\right) + c. \end{aligned}$$

Then, we can solve

$$\frac{4}{\delta(a+1)}\int_0^{+\infty}\frac{dt}{\left(\sqrt{\frac{a-1}{a+1}}\right)^2+t^2}=\frac{4}{\delta(a+1)}\frac{\sqrt{a+1}}{\sqrt{a-1}}\left|\tan^{-1}\left(\frac{\sqrt{a+1}}{\sqrt{a-1}}t\right)\right|_0^{\infty}$$
$$=\frac{2\pi}{\sqrt{a^2-1}}\frac{1}{\delta}=\frac{2\pi}{\sqrt{\frac{1}{\delta^2}-1}}=\frac{2\pi}{\sqrt{1-\delta^2}}.$$

□

Lemma B.3. *For any $k\in(0,1)$, we have that*

$$\int_0^1\frac{(1+t)^k+(1-t)^k-2}{t^{1+2k}}\,dt+\int_1^{\infty}\frac{(1+t)^k}{t^{1+2k}}\,dt=\frac{1}{k}.$$

Proof. If we fix $\epsilon>0$, we can integrate by parts

$$\int_\epsilon^1\frac{(1+t)^k+(1-t)^k-2}{t^{1+2k}}\,dt$$
$$=\left|\frac{2-(1+t)^k-(1-t)^k}{2k}t^{-2k}\right|_\epsilon^1+\frac{1}{2}\int_\epsilon^1\frac{(1+t)^{k-1}-(1-t)^{k-1}}{t^{2k}}\,dt$$
$$=\frac{1}{2k}\left(\frac{(1+\epsilon)^k+(1-\epsilon)^k-2}{\epsilon^{2k}}+2-2^k\right)+\frac{1}{2}\int_\epsilon^1\frac{(1+t)^{k-1}-(1-t)^{k-1}}{t^{2k}}\,dt.$$

Now we focus on the following Taylor expansions centred in $\epsilon=0$

$$\begin{cases}(1+\epsilon)^k &=1+k\epsilon-\frac{1}{2}k(1-k)\epsilon^2+o(\epsilon^2)\\ (1-\epsilon)^k &=1-k\epsilon-\frac{1}{2}k(1-k)\epsilon^2+o(\epsilon^2).\end{cases}$$

This allows us to write

$$\frac{(1+\epsilon)^k+(1-\epsilon)^k}{\epsilon^{2k}}\sim\epsilon^{2(1-k)}+o(\epsilon^{2(1-k)}),$$

that vanishes as $\epsilon\to0$. Therefore we have

$$\int_\epsilon^1\frac{(1+t)^k+(1-t)^k-2}{t^{1+2k}}\,dt=\frac{1}{2k}(o(1)+2-2^k)$$
$$+\frac{1}{2}\left(\int_\epsilon^1\frac{(1+t)^{k-1}}{t^{2k}}\,dt-\int_\epsilon^1\frac{(1-t)^{k-1}}{t^{2k}}\,dt\right).$$

If now we consider the change of variables, $t = \frac{\tilde{t}}{1+\tilde{t}}$, $dt = \frac{1}{(1+\tilde{t})^2} d\tilde{t}$ it is immediate

$$\int_\epsilon^1 \frac{(1-t)^{k-1}}{t^{2k}} dt = \int_{\frac{\epsilon}{1-\epsilon}}^\infty \frac{(1-\frac{\tilde{t}}{1+\tilde{t}})^{k-1}}{\tilde{t}^{2k}} \frac{(1+\tilde{t})^{2k}}{(1+\tilde{t})^2} d\tilde{t} = \int_{\frac{\epsilon}{1-\epsilon}}^\infty \frac{(1+\tilde{t})^{k-1}}{\tilde{t}^{2k}} d\tilde{t}.$$

Summing up what we have shown till now and assuming $\frac{\epsilon}{1-\epsilon} < 1$, such that $\epsilon < \frac{1}{2}$, we obtain

$$\begin{aligned}\int_\epsilon^1 \frac{(1+t)^k + (1-t)^k - 2}{t^{1+2k}} dt &= \frac{1}{2k}(o(1) + 2 - 2^k) \\ &\quad + \frac{1}{2}\left(\int_\epsilon^1 \frac{(1+t)^{k-1}}{t^{2k}} dt - \int_{\frac{\epsilon}{1-\epsilon}}^\infty \frac{(1+t)^{k-1}}{t^{2k}} dt \right) \\ &= \frac{1}{2k}(o(1) + 2 - 2^k) \\ &\quad + \frac{1}{2}\left(\int_\epsilon^{\frac{\epsilon}{1-\epsilon}} \frac{(1+t)^{k-1}}{t^{2k}} dt - \int_1^\infty \frac{(1+t)^{k-1}}{t^{2k}} dt \right).\end{aligned}$$

Then we notice the following

$$\int_\epsilon^{\frac{\epsilon}{1-\epsilon}} \frac{(1+t)^{k-1}}{t^{2k}} dt \le \frac{(1+\epsilon)^{k-1}}{\epsilon^{2k}} \left(\frac{\epsilon}{1-\epsilon} - \epsilon \right) = \frac{(1+\epsilon)^{k-1}\epsilon^{2(1-k)}}{1-\epsilon},$$

vanishing as $\epsilon \to 0$. Passing to the limit we can write

$$\int_0^1 \frac{(1+t)^k + (1-t)^k - 2}{t^{1+2k}} dt = \frac{2-2^k}{2k} - \frac{1}{2}\int_1^\infty \frac{(1+t)^{k-1}}{t^{2k}} dt.$$

If we integrate by parts the last element of the previous equation we get

$$\int_1^\infty \frac{(1+t)^{k-1}}{t^{2k}} dt = \left| t^{-2k} \frac{(1+t)^k}{k} \right|_1^\infty + 2\int_1^\infty t^{-(1+2k)}(1+t)^k \, dt,$$

so that we finally retrieve

$$\begin{aligned}\int_0^1 \frac{(1+t)^k + (1-t)^k - 2}{t^{1+2k}} dt &= \frac{2-2^k}{2k} + \frac{2^k}{2k} - \int_1^\infty t^{-(1+2k)}(1+t)^k \, dt \\ &= \frac{1}{k} - \int_1^\infty \frac{(1+t)^k}{t^{1+2k}} dt\end{aligned}$$

and the theorem is proved. □

Theorem B.4. *Suppose that a function* $f \in L^q(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, *for* $1 \le q \le \infty$, $1 \le r \le$

∞, $q \le r$.
Then $f \in L^p(\mathbb{R}^N)$ for any $q \le p \le r$ and, more precisely :

$$\|f\|_p^p \le \|f\|_q^{\frac{q(r-p)}{p(r-q)}} \|f\|_r^{\frac{r(p-q)}{p(r-q)}}.$$

Proof. We call

$$\alpha = \frac{q(r-p)}{(r-q)}, \qquad \beta = \frac{r(p-q)}{(r-q)},$$

in such a way that $\alpha + \beta = p$ and $\int_{\mathbb{R}^N} |u|^p = \int_{\mathbb{R}^N} |u|^\alpha |u|^\beta$.
Now, applying Hölder with

$$l = \frac{r-q}{r-p}, \qquad m = \frac{r-q}{p-q},$$

the theorem is proved. □

Corollary B.5. *Let $(v_n)_n$ be a bounded sequence in $H^k_r(\mathbb{R}^N)$. Then, there exist v, h in $H^k_r(\mathbb{R}^N)$ such that, up to a subsequence:*

(a) $v_n \rightharpoonup v$ in E

(b) $v_n \to v$ in L^p $\forall\, 2 < p < 2^*_k$

(c) $v_n \to v$ a.e.

(d) $\exists\, h(x) \in L^p$: $|v_n(x)| \le h(x)$ $\forall n$, $\forall\, 2 < p < 2^*_k$.

Proof. Point *(a)* follows directly from the Banach-Alaoglu theorem, noticing that $H^k_r(\mathbb{R}^N)$ is Hilbert and then both Banach and reflexive. Point *(b)* is a consequence of *(a)* and of Theorem B.6. Finally point *(c)* and *(d)* are famous results in literature, nevertheless, for *(d)* we refer to [8, Theorem 4.9]. □

Proof of Theorem 2.4. We start fixing $r > 0$, $\alpha > 0$ and $x \in \mathbb{R}^N$. Thus, for any $y \in \mathbb{R}^N$

$$|u(x)| \le |u(x) - u(y)| + |u(y)|$$

and, if we integrate over $B_r(x)$ we get

$$\begin{aligned} |B_r||u(x)| &\le \int_{B_r(x)} |u(x) - u(y)|\, dy + \int_{B_r(x)} |u(y)|\, dy \\ &\le r^\alpha \int_{B_r(x)} \frac{|u(x) - u(y)|}{|x-y|^\alpha}\, dy + \int_{B_r(x)} |u(y)|\, dy, \end{aligned}$$

with B_r denoting the volume of the N-dimensional sphere of radius r. At this point we fix $\alpha = \frac{N+2k}{2}$ and apply Hölder to both the terms in the previous sum, respectively with $p = q = 2$ and $p = 2_k^*$, $q = \frac{2N}{N+2k}$.

$$\begin{aligned}
|B_r||u(x)| &\leq r^{\frac{N+2k}{2}} \int_{B_r(x)} \frac{|u(x)-u(y)|}{|x-y|^{\frac{N+2k}{2}}}\,dy + \int_{B_r(x)} |u(y)|\,dy \\
&\leq r^{\frac{N+2k}{2}} \left(\int_{B_r(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dy \right)^{\frac{1}{2}} |B_r|^{\frac{1}{2}} + \left(\int_{B_r(x)} |u(y)|^{2_k^*}\,dy \right)^{(2_k^*)^{-1}} |B_r|^{\frac{N+2k}{2N}} \\
&\leq C \left\{ r^{N+k} \left(\int_{B_r(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dy \right)^{\frac{1}{2}} + r^{\frac{N+2k}{2}} \left(\int_{B_r(x)} |u(y)|^{2_k^*}\,dy \right)^{(2_k^*)^{-1}} \right\},
\end{aligned}$$

where we have exploited $|B_r| \sim r^N$. Now we divide by r^N to obtain

$$|u(x)| \leq C\, r^k \left\{ \left(\int_{B_r(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dy \right)^{\frac{1}{2}} + r^{-\frac{N}{2}} \left(\int_{B_r(x)} |u(y)|^{2_k^*}\,dy \right)^{(2_k^*)^{-1}} \right\}.$$

At this point we denote as

$$\begin{aligned}
\alpha &:= \int_{B_r(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dy \\
\beta &:= \int_{B_r(x)} |u(y)|^{2_k^*}\,dy
\end{aligned}$$

and rise everything to the power 2_k^*,

$$|u(x)|^{2_k^*} \leq C\, r^{2_k^* k} \left\{ \alpha^{\frac{1}{2}} + r^{-\frac{N}{2}} \beta^{(2_k^*)^{-1}} \right\}^{2_k^*}.$$

At this point we fix $r := \beta^{\frac{N-2k}{N^2}} \alpha^{-\frac{1}{N}}$ such that

$$\begin{aligned}
r^{-\frac{N}{2}} \beta^{(2_k^*)^{-1}} &= \beta^{-\frac{N-2k}{2N}} \alpha^{\frac{1}{2}} \beta^{(2_k^*)^{-1}} = \alpha^{\frac{1}{2}} \\
r^{2_k^* k} &= \beta^{\frac{2k}{N}} \alpha^{-\frac{2k}{N-2k}}.
\end{aligned}$$

Therefore, the following inequalities hold

$$\begin{aligned}
|u(x)|^{2_k^*} &\leq C\, \beta^{\frac{2k}{N}} \alpha^{-\frac{2k}{N-2k}} \alpha^{\frac{2_k^*}{2}} = C\, \beta^{\frac{2k}{N}} \alpha \\
&= C \left(\int_{B_r(x)} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dy \right) \|u\|_{2_k^*}^{\frac{4k}{N-2k}}.
\end{aligned}$$

Finally, if we integrate over x we get

$$\|u\|_{2_k^*}^{2_k^*-\frac{4k}{N-2k}} = \|u\|_{2_k^*}^{2\cdot 2_k^*} \leq C \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2k}}\,dxdy$$

and the first part of the theorem is proved.
We are left to show that the embedding $H^k(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous $\forall p \in \ [2, 2_k^*]$, but this is a direct consequence of Definition (1.1), since, combined with the result we have just proved, gives us that

$$u \in W^{k,2}(\mathbb{R}^N) \implies u \in L^2(\mathbb{R}^N) \cap L^{2_k^*}(\mathbb{R}^N)$$

and, recalling Theorem B.4, our proof is complete. □

Theorem B.6. *For $1 < q < 2_k^*$, the embedding*

$$H_r^k(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$$

is compact for any $k > 0$.

Proof. To prove the theorem, we want to define a bounded sequence $(u_n)_n$ in $H_r^k(\mathbb{R}^N)$ and show that it converges strongly in $L^q(\mathbb{R}^N)$.
According to famous results of harmonic analysis, we start defining the classical potential spaces $H^{k,2}(\mathbb{R}^N)$, where

$$H^{k,2}(\mathbb{R}^N) = \{u = (I-\Delta)^{-k/2} f, \text{ with } f \in L^2(\mathbb{R}^N)\}.$$

We recall here, that the fractional power $(I-\Delta)^{-k/2}$ can be defined by means of the Fourier transform

$$(I-\Delta)^{-k/2} f = \mathcal{F}^{-1}((1+|\omega|^2)^{-k/2}\mathcal{F}(f)) = G_k * f,$$

where

$$G_k(\omega) = \mathcal{F}^{-1}((1+|\omega|^2)^{-k/2}) = \frac{(2\sqrt{\pi})^{-N}}{\Gamma(k/2)} \int_0^\infty e^{-t} e^{\frac{-|x|^2}{4t}} t^{\frac{k-N}{2}} \frac{dt}{t} \tag{B.8}$$

is called the Bessel potential. At this point, we notice that

$$G_k(x) \simeq C \begin{cases} |x|^{k-N} & \text{if } |x| \leq 2 \\ e^{-\frac{|x|}{2}} & \text{if } |x| \geq 2 \end{cases}$$

constitutes the asymptotic approximation of the Bessel potential for $k < N$ and remember that for some f_n radial it holds

$$u_n = G_k * f_n.$$

Moreover, if we name M the constant of equiboundedness of the sequence $(u_n)_n$, i.e. $\|u_n\|_k \leq M \; \forall n$, then f_n is bounded in $L^2(\mathbb{R}^N)$ by the same constant. Since we know that any metric space is compact if and only if it is complete and totally bounded and that $H^k_r(\mathbb{R}^N)$ is complete, we proceed to show that $H^k_r(\mathbb{R}^N)$ is a totally bounded subset of $L^q(\mathbb{R}^N)$.

We rely on the Kolmogorov-Riesz theorem of compactness and check that its hypotheses are satisfied. These are the hypotheses:

(a) $(u_n)_n$ is bounded in $L^q(\mathbb{R}^N)$

(b) for every $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that

$$\int_{\mathbb{R}^N} |u_n(x+h) - u_n(x)|^q dx \leq \epsilon,$$

for $|h| \leq \delta(\epsilon)$

(c) for every $\epsilon > 0$, there exists some $R > 0$ such that:

$$\int_{|x|>R} |f(x)|^q dx \leq \epsilon^q.$$

Point *(a)* is trivial, exploiting the continuous embedding of $H^k(\mathbb{R}^N)$ in $L^q(\mathbb{R}^N)$.

Point *(b)*, instead, can be retrieved through the definition of $\tau_h u(x) = u(x+h)$ and noticing

$$\begin{aligned}\|\tau_h u - u\|_q &= \|\tau_h(G_k * f_n) - G_k * f_n\|_q \\ &= \|(\tau_h G_k - G_k) * f_n\|_q \leq \|(\tau_h G_k - G_k)\|_r \|f_n\|_2,\end{aligned}$$

where the last inequality makes use of Young's convolution inequality with $\frac{1}{r} = \frac{1}{q} + \frac{1}{2}$, that explicitly becomes

$$r = \frac{2q}{q+2}.$$

Since, however, $G_k \in L^q(\mathbb{R}^N)$, then we infer

$$\|(\tau_h G_k - G_k)\|_r < \frac{\epsilon}{M}.$$

We conclude, thanks to $\|f_n\|_2 \leq M \; \forall n$, point *(b)* stating

$$\|\tau_h u_n - u_n\|_q \leq \epsilon,$$

with $|h| < \delta$.
For what concerns point *(c)*, we finally notice that we have $0 < k < 1$ and $N > 2$. Moreover, we can set $c = q(N-1)(\frac{1}{2} - \frac{1}{q})$ and it holds $2 < q < \frac{2(N+c)}{N-2k}$, indeed developing calculation we obtain

$$\begin{aligned}\frac{2(N+c)}{N-2k} &= \frac{2\left(N + q(N-1)(\frac{1}{2} - \frac{1}{q})\right)}{N-2k} \\ &= \frac{2 + Nq - q}{N-2k} = \frac{2}{N-2k} + \frac{q(N-1)}{N-2k}.\end{aligned}$$

Then, $q < \frac{2}{N-2k} + \frac{q(N-1)}{N-2k}$ if and only if

$$\begin{aligned}q\left(1 - \frac{N-1}{N-2k}\right) &= q\left(\frac{1-2k}{N-2k}\right) \\ &\leq \frac{2}{N-2k}.\end{aligned}$$

As a consequence, $q < \frac{2}{N-2k} + \frac{q(N-1)}{N-2k} \iff k > \frac{1}{2} - \frac{1}{q}$, which is necessarily true since $q < 2_k^*$ and we are led to

$$q < \frac{2N}{N-2k} \iff \frac{N}{2} - k < \frac{N}{q} \iff k > N\left(\frac{1}{2} - \frac{1}{q}\right) > \left(\frac{1}{2} - \frac{1}{q}\right).$$

Therefore, we can state that

$$H_r^k(\mathbb{R}^N) \subset L^q(\mathbb{R}^N, |x|^{c-\epsilon}dx),$$

for $\epsilon > 0$ small and, fixing ϵ and $\tilde{c} = c - \epsilon$, we can apply the theorem to this case and we write

$$R^{\tilde{c}} \int_{|x|>R} |u_n|^q \leq \int_{|x|>R} |x|^{\tilde{c}} |u_n|^q \leq C\|u_n\|_k^q,$$

where for the first inequality we trivially exploit that $R^{\tilde{c}} < |x|^{\tilde{c}}$ for any $|x| > R$ and $R > 1$. □

Theorem B.7 (Ekeland's variational principle). *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous functional, bounded from below. Set $\epsilon > 0$ and $x_0 \in X$ such that $f(x_0) \neq \infty$. Then, there exists some $v \in X$ such that* *(a)* $f(v) \leq f(x_0) - \epsilon d(x_0, v)$

(b) $f(v) < f(x) + \epsilon d(x, v) \qquad \forall x \in X,\ x \neq v.$

Proof. We start defining $G_z(x) : X \to \mathbb{R}$

$$G_z(x) = f(x) + \epsilon d(x, z)$$

and the first step consists in proving that $G_z(x)$ is lower semicontinuous. If we are able to show that the distance function $d_z(\cdot) : X \to \mathbb{R}$, $d_z(x) = \|z - x\|$ is lower semicontinuous, then we gather the lower semicontinuity for G_z, since it would be the sum of two semicontinuous functions. Distance function is even uniformly continuous, indeed, fixing $\epsilon > 0$, we can put $\delta_\epsilon = \epsilon$ and notice that

$$\begin{aligned} \|x - y\| \leq \delta_\epsilon = \epsilon \implies |d_z(x) - d_z(y)| &= |\|x - z\| - \|y - z\|| \\ &\leq \|x - y\| \leq \epsilon. \end{aligned}$$

Defining now

$$F(x) := \{y \in X : \ G_x(y) \leq f(x)\} = \{y \in X : \ f(y) + \epsilon d(y, x) \leq f(x)\},$$

we want to highlight three important properties that this set possesses. The first one is its closure, indeed we have two cases:

$$\begin{cases} f(x) = \infty & \implies F(X) = X \\ f(x) = t \in \mathbb{R} & \implies \{y \in X : G_x(y) \leq t\}, t \in \mathbb{R}, \end{cases}$$

where X is closed by hypothesis and, in the second line, an equivalent definition of semi-continuity for G_x ensures that $\{y \in X : G_x(y) \leq t\}$ is closed.
The second property reads as $y \in F(x) \implies F(y) \subset F(x)$ and is true indeed, if we take a generic $z \in F(y)$ and recall

$$\begin{cases} z \in F(y) & \implies f(z) + \epsilon d(z, y) \leq f(y) \\ y \in F(x) & \implies f(y) + \epsilon d(y, x) \leq f(x), \end{cases}$$

we can combine these equalities into

$$f(z) + \epsilon d(x, z) \leq f(z) + \epsilon(d(z, y) + d(y, x)) \leq f(x)$$

and we obtain immediately that $z \in F(x)$.

Finally, the third one is that $F(x)$ is not empty, trivial since at least $x \in F(x)$.
At this point, we proceed to set

$$s_0 = \inf_{x \in F(x_0)} f(x), \quad x_1 \in F(x_0) : \; f(x_1) \leq s_0 + 2^{-1},$$

where we stress that $s_0 \in \mathbb{R}$, since f is bounded from below. Then, we define recursively $s_n = \inf_{x \in F(x_n)} f(x)$ and $x_{n+1} \in F(x_n)$ such that $f(x_{n+1}) \leq s_n + 2^{-(n+1)}$. Moreover, $x_{n+1} \in F(x_n)$ implies that $f(x_{n+1}) > s_n$ and the following chain holds

$$x_{n+1} \in F(x_n) \implies F(x_{n+1}) \subset F(x_n) \implies s_{n+1} \geq s_n.$$

Thus, combining together:

$$f(x_{n+2}) > s_{n+1}, \quad s_{n+1} > s_n,$$

we retrieve that $f(x_{n+2}) > s_n$.
Now we are ready to provide calculations and start explicating $x_{n+1} \in F(x_n)$, which becomes

$$f(x_{n+1}) + \epsilon d(x_n, x - n + 1) \leq f(x_n),$$

that finally provides

$$\epsilon d(x_n, x_{n+1}) \leq f(x_n) - f(x_{n+1}) \leq 2^{-n}.$$

Since then $d(x_n, x_{n+p}) \to 0 \, \forall p > 0$ as $n \to \infty$, we obtain that $(x_n)_n$ is a Cauchy sequence in $F(x_0)$, hence $(x_n)_n$ is a Cauchy sequence in a closed set.
As a consequence, there exists some $v \in F(x_0)$ such that $x_n \to v$ in X and moreover $v \in F(x_n) \, \forall n > 0$.
Considering now $F(v)$, with $v = \lim_n x_n$ and setting a generic $x \in F(v)$, we can exploit that $F(v)$ is not empty. We have that

$$F(x_0) \supset \; ... \; \supset F(x_{n-1}) \supset F(x_n) \supset F(x_{n+1}) \supset \; ... \; \supset F(v)$$

and we deduce that $x \in F(x_n) \, \forall n$ implying $\epsilon d(x, x_n) \leq 2^{-n}$ and as a consequence $x_n \to x$; but it holds that both

$$\begin{cases} x_n \to v \\ x_n \to x \end{cases}$$

and, by uniqueness of the limit, we have shown that $x = v$, or, more profoundly, we have shown that $F(v) = \{v\}$. Changing perspective, we have that $x \in F(v)^c \;\; \forall x \neq v$.

Expliciting $v \in F(x_0)$ and $x \in F(v)^c \ \forall x \neq v$, we get the theses of the theorem, respectively

$$\begin{aligned} f(v) &\leq f(x_0) - \epsilon d(x_0, v) \\ f(v) &< f(x) + \epsilon d(x, v) \quad \forall x \in X \ x \neq v. \end{aligned}$$

□

Corollary B.8. *Let (X, d) be a complete metric space and $f : X \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous functional, bounded from below.*
Set $\epsilon > 0$ and $x_0 \in X$, such that $f(x_0) \leq \epsilon + \inf_{x \in X} f(x)$.
Then, for any $\lambda > 0$, there exists $v \in X$ such that:

$$f(v) \leq f(x_0) \qquad \textit{and} \qquad d(x_0, v) \leq \lambda.$$

Moreover, for any $x \in X$, $x \neq v$ it holds

$$f(x) > f(v) - \frac{\epsilon}{\lambda} d(x, v).$$

Proof. Once chosen x_0 as in the hypotheses, we apply the Ekeland's principle fixing a generic $\tilde{\lambda}$ on x_0 and, from $f(v) \leq f(x_0) - \tilde{\lambda} d(x_0, v)$, we retrieve

$$f(v) \leq \inf_{x \in X} f(x) + \epsilon - \tilde{\lambda} d(x_0, v) \implies d(x_0, v) \leq \frac{\epsilon}{\tilde{\lambda}}.$$

Instead, from $f(v) \leq f(x) + \tilde{\lambda} d(x, v)$, for any $x \in X$, $x \neq v$ we have

$$f(x) \geq f(v) - \tilde{\lambda} d(x, v) \quad \forall x \neq v.$$

Theses come straightforwardly fixing $\tilde{\lambda} = \frac{\epsilon}{\lambda}$. □

Theorem B.9 (Harnack inequality)**.** *Let u an harmonic and non-negative function in $B_R \subset \mathbb{R}^N$ for some $R > 0$. Then, for any $x \in B_R$ it holds*

$$\frac{R^{N-2}(R - |x|)}{(R + |x|)^{N-1}} u(0) \leq u(x) \leq \frac{R^{N-2}(R + |x|)}{(R - |x|)^{N-1}} u(0).$$

Proof. To carry on this proof we will refer to the Poisson formula for harmonic functions and to the mean value property as treated in detail in [29, Chapter 3.3]. From the Poisson formula we have that

$$u(x) = \frac{R^2 - |x|^2}{\omega_N R} \int_{\partial B_R} \frac{u(\sigma)}{|\sigma - x|^N}\, d\sigma,$$

where ω_N denotes the surface area of the N-dimensional sphere. It is trival that $R - |x| \leq |\sigma - x| \leq R + |x|$ and that $R^2 - |x|^2 = (R + |x|)(R - |x|)$. Thus, we can compute

$$\begin{aligned} u(x) &\leq \frac{(R+|x|)}{\omega_N R}(R-|x|)^{1-N} \int_{\partial B_R} u(\sigma)\, d\sigma \\ &= \frac{(R+|x|)R^{N-2}}{\omega_N (R-|x|)^{N-1}} \left(\frac{1}{R^{N-1}} \int_{\partial B_R} u(\sigma)\, d\sigma \right) \\ &= \frac{(R+|x|)R^{N-2}}{(R-|x|)^{N-1}}\, u(0), \end{aligned}$$

where the last equality makes use of the mean value property. Analogously we have that

$$u(x) \geq \frac{R^{N-2}(R-|x|)}{(R-|x|)^{N-1}}\, u(0)$$

and the theorem is proved. □

Corollary B.10. *For any non-negative harmonic function $u : B_1 \to \mathbb{R}$, for any r in $(0, 1)$, there exists a constant $c_r > 0$ depending only on r such that*

$$u(x) \leq c_r\, u(y) \quad \forall x, y \in B_r.$$

Proof. Directly from Theorem B.9 we retrieve

$$\begin{aligned} \frac{u(x)}{u(y)} &\leq \left(\frac{1+|x|}{1+|y|} \right) \left(\frac{1-|y|}{1-|x|} \right)^{N-1} \\ &\leq \frac{1+r}{(1-r)^{N-1}} =: c_r, \end{aligned}$$

for any $x, y \in B_r$. Just notice that c does depend neither on x nor on y. □

Lemma B.11. *Let u_1, u_2 be two functions in S_c. Let $H(u, s)$ the map defined in Section 2.4.1. Then*

$$\left\langle (-\Delta)^{\frac{k}{2}} H(u_1, s), (-\Delta)^{\frac{k}{2}} H(u_2, s) \right\rangle_H = e^{2ks} \left\langle (-\Delta)^{\frac{k}{2}} u_1, (-\Delta)^{\frac{k}{2}} u_2 \right\rangle_H.$$

Proof. We start exploiting [16, Corollary 5.3] and [16, Lemma 5.4] to write

$$\begin{aligned}\left\langle(-\Delta)^{\frac{k}{2}}H(u_1,s),(-\Delta)^{\frac{k}{2}}H(u_2,s)\right\rangle_H &= \int_{\mathbb{R}^N}(-\Delta)^{\frac{k}{2}}(-\Delta)^{\frac{k}{2}}H(u_1,s)H(u_2,s)\\ &= \int_{\mathbb{R}^N}(-\Delta)^k H(u_1,s)H(u_2,s).\end{aligned}$$

We now rely to the rescaling result shown in Section 2.2, according to which

$$(-\Delta)^k(\alpha^q u(\alpha x)) = \alpha^{q+2k}(-\Delta)^k u(\alpha x).$$

Thus, we fix $\alpha = e^s$, $q = N/2$ and obtain

$$(-\Delta)^k H(u_1,s) = e^{\frac{sN}{2}+2ks}(-\Delta)^k u_1(e^s x).$$

We can now conclude the proof

$$\begin{aligned}&\int_{\mathbb{R}^N}(-\Delta)^k H(u_1,s)H(u_2,s) = e^{sN+2ks}\int_{\mathbb{R}^N}(-\Delta)^k u_1(e^s x)\,u_2(e^s x)\,dx\\ &= e^{2ks}\int_{\mathbb{R}^N}(-\Delta)^k u_1(x)\,u_2(x)\,dx = \left\langle(-\Delta)^{\frac{k}{2}}u_1,(-\Delta)^{\frac{k}{2}}u_2\right\rangle_H.\end{aligned}$$

□