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On the long time behavior of the segregation limit for a competition-diffusion system with periodic boundary conditions

Tesi di Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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Abstract

We study the strong competition limit for an evolutive Lotka-Volterra competition-diffusion system arising in spatial modeling of populations. As the interaction strength increases, populations tend to accumulate on separate domains, and in the limit they segregate. Therefore, a free boundary problem arises for the interfaces between these regions. Moreover, the limiting profiles satisfy a system of parabolic differential inequalities.

In this thesis, we focus on the one-dimensional case with periodic boundary conditions, and extend to this setting the free boundary analysis that has already been carried out for the one-dimensional Dirichlet case. Differently from that situation, where convergence to a stationary state holds for every initial segregated configuration, the long term dynamics in the strong competition limit depends in the periodic setting on the pattern of interactions between the populations. In particular, we identify a condition on the interactions that ensures that convergence to a stationary state holds. The analysis is complicated by the degeneracy in the set of stationary solutions introduced by the rotational invariance of the domain. To prove our result, we combine energy methods, regularity techniques and a property of linear parabolic equations in one dimension, namely the monotonicity of the number of zeros of solutions.

Keywords: Semilinear parabolic systems, Singular perturbation analysis, Long time behavior, Free boundary problems, Mathematical Ecology.

Abstract in lingua italiana

Studiamo il limite di competizione forte per un sistema di competizione-diffusione di tipo Lotka-Volterra evolutivo con applicazioni in dinamica delle popolazioni. al crescere dell'intensità delle interazioni, le popolazioni tendono a concentrarsi su domini separati e, al limite, vanno incontro a segregazione. Di conseguenza, emerge un problema a frontiera libera per le interfacce di tali regioni. Inoltre, Le densità limite soddisfano un sistema di disuguaglianze differenziali paraboliche. In questa tesi, ci concentriamo sul caso unidimensionale con condizioni al contorno periodiche. L'analisi della frontiera libera, già effettuata nel caso Dirichlet, è estesa a questo contesto. Per quanto riguarda il comportamento per tempi lunghi, è noto che nel caso Dirichlet ogni configurazione iniziale segregata converge a uno stato stazionario per qualsiasi scelta dei parametri di competizione. Nel caso periodico, invece, la dinamica asintotica nel limite di competizione forte dipende dalla natura della competizione. In particolare, identifichiamo una condizione sulle interazioni che assicura la convergenza a uno stato stazionario. L'analisi è complicata dalla degenerazione dell'insieme delle soluzioni stazionarie indotta dall'invarianza per rotazioni del dominio. Per provare il risultato di convergenza, combiniamo stime dell'energia, tecniche di regolarità e una proprietà delle equazioni paraboliche lineari in una dimensione spaziale, che consiste nella monotonia del numero di zeri delle soluzioni.

Parole chiave: Sistemi parabolici semilineari, Perturbazione singolare, Comportamento per tempi lunghi, Problemi a frontiera libera, Ecologia Matematica

Contents

Introduction

In this thesis, we study the long time behavior of solutions to a system of differential inequalities arising in the singular limit of the Lotka-Volterra system for competing and diffusing populations. In what follows, we will briefly discuss the relevance of the system from a modeling point of view, with a focus on spatial ecology, and introduce some relevant mathematical questions related to it. The Lotka-Volterra system of ODEs was historically one of the first models for interacting populations, introduced independently by Lotka and Volterra to describe respectively oscillating chemical reactions and predator-prey dynamics. We consider it in the following form:

$$\frac{dP_i}{dt} = \left(a_i - \beta_{ii} P_i - \sum_{j \neq i} \beta_{ij} P_j \right) P_i. \tag{1}$$

Here $P_i = P_i(t)$ represents, for $i = 1, ..., M$, the total population of species i at time t. The coefficients a_i, typically positive, are called internal growth rates and are related to the reproduction process of population i. Coefficients β_{ii} are also usually positive, and represent in that case the intraspecific competition, or "crowding" effects that progressively limit growth of population i as the density increases. Overall, the term $a_i P_i - \beta_{ii} P_i^2$ in (1) models the internal dynamics of population i, i.e. its behavior in absence of other species, and is of logistic type. Coefficients β_{ij} for $j \neq i$ model instead the interactions between species.

One of the strengths of (1) is that it can model a range of interactions of different nature, such as competition, mutualism, or predator-prey dynamics, based on the signs of coefficients β_{ij}. A distinctive feature of the interaction term $\beta_{ij} P_i P_j$ is that it can be interpreted as a mass action law, analogously to the mass action principles in chemistry. In particular, the shape of the interaction terms reflect an underlying modeling hypotheses that the probability of effective interaction between individuals of populations i and j is linearly proportional through coefficient β_{ij} to the product of population densities $P_i P_j$. There are several other possibilities of choice for interaction terms and internal dynamics, for which we refer to [3], [20] and references therein.

A major limitation of model (1) is that it does not allow to take into account spatial effects. Instead, there is strong empirical evidence that such effects play an important role in determining the dynamics of populations. Moreover, some biological phenomena such as those regarding habitat invasions are intrinsically referred to a spatial modeling scenario. For a more detailed discussion and for different approaches used to include spatial information in models, we refer again to [3]. The approach in which we are interested was first introduced in mathematical ecology by Skellam [25] and relies on reaction-diffusion equations. It consists in considering space-dependent population densities $u_i(x,t)$ and adding a diffusion term to the set of equations modeling population dynamics. The assumption underlying this modification is that each individual in the population moves randomly in the prescribed domain. If individuals of the i-th population move by brownian motion, the diffusion term corresponds to the laplacian of the i-th density $\Delta u_i(x,t)$. One could generalize this hypothesis by including bias effects in the random motion, leading to nonlinear diffusion, or the possibility of random jumps, leading to nonlocal diffusion operators such as the fractional laplacian. There are several methods to make precise the derivation of reaction diffusion equations as macroscopic limits of systems described by the behavior of single individuals. The rigorous mathematical treatment of this problem, however, can become quite involved, and we will not go into any detail since that kind of analysis is completely different from the one we are concerned with in this work.

In our analysis, we will focus on the competition regime, implying $\beta_{ij} > 0$ for any $i \neq j$. Given a bounded domain $\Omega \subset \mathbb{R}^n$, the competition-diffusion Lotka-Volterra system of equations reads as follow.

$$\frac{\partial v_i}{\partial t} - d_i \Delta v_i = f_i(v_i) - \kappa \sum_{j \neq i} \beta_{ij} v_i v_j \qquad \forall \quad i = 1, ..., M, \quad (x,t) \in \Omega \times (0,T), \tag{2}$$

where we added the parameter $\kappa > 0$ controlling the global strength of interactions, and the coefficients d_i, controlling the diffusivity of each population. We consider for the moment a generic internal dynamics $f_i(s)$. System (2) has of course to be complemented with initial and boundary conditions, such as homogeneous Dirichlet or Neumann conditions, which are often referred to in spatial ecology as "lethal" or "no-flux" boundaries.

There are several mathematical questions related to competition diffusion models. On a bounded domain, one of the most important is the range of parameters and geometry of Ω that supports pattern formation, namely inhomogeneous stationary or periodic positive solutions that are stable. In our context, the existence of patterns represents the possibility of coexistence of competing species. This subject has been extensively studied since the 1980s. For what concerns the dependence on the shape of the domain, it was proven in [13] by Kishimoto and Weinberger that convex domains with Neumann boundary conditions do not support stationary coexistence patterns for two species. On the other hand, Matano and Mimura constructed in [19] examples of non-convex domains which support pattern formation under appropriate choices of the diffusion and interaction coefficients.

A vast body of work also focused on the singularly perturbed strong competition regime, for which $\kappa \to +\infty$. In this context, segregation of species occurs, meaning that species tend to progressively concentrate on different parts of the domain as κ grows. In the limit, $u_i u_j \equiv 0$ for any $i \neq j$, and a free boundary problem arises for the interfaces between the sets $\{u_i > 0\}$. Although much could be said about the literature investigating this regime, we will only mention what is more relevant for our work. From a dynamical point of view, in the strong competition regime, i.e. for large values of κ, system (2) is believed to exhibit simpler dynamics compared to the generic case. Indeed, Dancer and Zhang showed in [9] that for large but finite κ, under Dirichlet boundary conditions, every solution to the two-species system converges to a stationary state. The case of three species has been less studied, although existence of positive solutions for some parameter ranges was investigated by Dancer and Du in [7] and [8] with topological index techniques, with the aim of extending a similar analysis that they carried out for two species in [6].

More recently, for what concerns the singular problem, Conti, Terracini, and Verzini showed in [5] that for M species and in arbitrary dimension, solutions to the stationary equation converge, up to a subsequence, to a vector valued function satisfying a set of $2M$ elliptic differential inequalities. Moreover, they proved the Hölder regularity of the segregated limit through a blow up argument, and studied the structure of the associated free boundary. Caffarelli, Karakhanyan and Lin improved in [2] the regularity results to obtain Lipschitz regularity, and they also extended the analysis to the parabolic case. These results will be discussed further in Chapter 1. Dancer, Wang and Zhang studied in [28] the dynamics in a one dimensional domain for many species. In the strong competition limit, the limit solution satisfies a set of parabolic differential inequalities analogous to the stationary one. The authors proved that, under Dirichlet or Neumann boundary conditions and for every choice of the interaction parameters, solutions to the singular system converge to a stationary solution. Importantly, this result proves useful in the analysis of system (2) with κ finite but large enough, and allows to reach the same conclusion about its long time behavior.

In this work, we study the same system of differential inequalities as [28] on the reference interval $(0, 1)$ but with periodic boundary conditions. For every $i \in \{1, ..., M\}$:

$$
\begin{cases}
\dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} \leq a_i u_i - u_i^2, \\
\left(\dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}\right)\left(u_i - \displaystyle\sum_{j\neq i} \dfrac{\beta_{ij}}{\beta_{ji}} u_j\right) \geq a_i u_i - u_i^2 - \displaystyle\sum_{j\neq i} \dfrac{\beta_{ij}}{\beta_{ji}} \left(a_j u_j - u_j^2\right), \\
u_i u_j \equiv 0 \qquad \forall \quad j \neq i, \\
u_i(0,t) = u_i(1,t).
\end{cases} \tag{3}
$$

Inequalities are intended in a weak sense, that will be specified in chapters 1 and 3, in the space-time cylinder $(0,1)\times(0,+\infty)$ with non-negative test functions. Here we remark that we will often consider the above system as formulated on the one dimensional flat torus $\mathbb{T}^1 = \mathbb{T}$, which is defined as the quotient space $\mathbb{R}/\mathbb{Z}$. This space can also be represented with the interval $[0,1)$, or $[0,1]$ where points $x=0$ and $x=1$ are identified. These choices for the domain are equivalent when appropriate solution and test spaces are considered, and we will deal from time to time with the more convenient formulation. Moreover, as in the Dirichlet case, the above system is solved, up to a subsequence, by the limit of solutions to the competition-diffusion Lotka-Volterra system with periodic boundary conditions as $\kappa \to \infty$. We complement (3) with the initial conditions:

$$
u_i(x,0) = \phi_i(x) \quad \text{for every } i,
$$

where ϕ_i are periodic, Lipschitz continuous non-negative and non-trivial functions satisfying the segregation condition:

$$
\phi_i(x)\phi(x) \equiv 0 \qquad\qquad \text{for every } i,j : i \neq j.
$$

In his master's thesis [26], Spadaro considered the particular case where the data of the problem satisfy the following conditions:

$$
\begin{cases}
\phi_i(x) = \Phi\left(x - \dfrac{1}{M}\right) & \text{for every } x \in [0,1] \text{ and } i = 1,\dots,M, \\
a_i = a & \text{for every } i = 1,\dots,M, \\
\dfrac{\beta_{i,i+1}}{\beta_{i+1,i}} = \dfrac{1}{\lambda} & \text{for some } \lambda > 0 \text{ and for every } i = 1,\dots,M,
\end{cases}
$$

where Φ is a non-negative, Lipschitz continuous density supported on $\left(0, \frac{1}{M}\right)$, and in the third hypothesis, by convention, we intend that $i = M+1$ is equivalent to $i = 1$. Notice that under these hypothesis, the rotational symmetry is preserved under the evolution, and it is possible to reduce the problem to a scalar one. In [26], it was proven that for every λ near 1 there exist ω near 0 and a non-negative density U supported on $\left(0, \frac{1}{M}\right)$ such that the rotating wave:

$$u_i(x,t) = U\left(x - \omega t - \frac{i}{M}\right),$$

is a solution to (3). Moreover, there is a bijection between the velocity ω and the parameter λ. Therefore, for generic interactions, the singular system with periodic boundary conditions necessarily exhibits a different behavior compared to the Dirichlet case, for which periodic solutions are ruled out in [28]. The main contribution of our work is to give conditions on the interactions to ensure that the evolution of a certain state converges to a stationary solution. For a solution $\mathbf{u}(x,t) = (u_1, ..., u_M)$ to (3) such that u_i remains definitively positive and $\{u_i > 0\}$ are connected sets, the condition reads as follows.

$$\prod_{i=1}^{M-1} \frac{\beta_{i,i+1}}{\beta_{i+1,i}} = \frac{\beta_{1,M}}{\beta_{M,1}}. \tag{4}$$

We refer to such interactions as cyclic. They generalize the case of symmetric ones (in which $\beta_{ij}/\beta_{ji} = 1$ for every i,j), and for our purposes the analysis can be carried out essentially in the same way. We remark that a similar discrimination between the nature of interactions has already been found to affect the free boundary of stationary states in two dimensions, see [27]. Moreover, the case treated in [26], consistently, does not fall under the cyclic case.

The thesis is structured as follows. In Chapter 1, after introducing the relevant spaces of functions, we state some preliminary results concerning the well posedness of the initial-boundary value problem for (2) in the one-dimensional periodic case. Moreover, we perform the asymptotic analysis to prove that the weak differential inequalities are satisfied in the limit, and we show that the limit profiles satisfy the segregation conditions. Even though this is a known fact in literature, it is hard for us to find references for the rigorous proof of these properties for the parabolic problem. We partially rely on [11], but the treatment can be simplified in our case, as will be specified later. In Chapter 2, we study the set of stationary solutions to (3) for a generic choice of coefficients β_{ij} and show that continua of stationary solutions may occur if certain conditions related to (4) are satisfied by the interaction parameters. In Chapter 3 , we prove some general properties of solutions to (3) which enjoy some minimal regularity that is enjoyed by limiting configurations of (2). In particular, we prove additional regularity, characterize the extinction times, and prove regularity of the time evolution of the free boundary. The results of this chapter are extensions to the periodic case of results contained in [28], although our treatment differs in some aspects. Finally, in Chapter 4, we prove the main result. In order to to this, the energy methods used in [28] are still applicable but not sufficient to prove convergence due to the degeneracy in the set of stationary solutions introduced by the rotational equivariance of problem (3). Indeed, those methods only allow us to prove that the distance between trajectories and the set of stationary solutions approaches zero as $t \to +\infty$, a property that is often referred to as orbital stability. Instead, we resort to the monotonicity of the number of zeros of certain linear parabolic equations, a fundamental tool in the study of long time behavior for semilinear parabolic equations in one space dimension, to prove convergence to a stationary state. We will conclude with some open problems and possible directions of research.

1 | Asymptotic analysis in the strong competition regime

In this chapter, we consider the Lotka-Volterra competition-diffusion system with periodic boundary conditions and perform the asymptotic analysis as the competition strength $\kappa \to +\infty$. We will consider the following system of semilinear parabolic equations:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} = a_i u_i - u_i^2 - \kappa \displaystyle\sum_{j \neq i} \beta_{ij} u_i u_j & i \in \{1, ..., M\}, \quad (x,t) \in \mathbb{T} \times (0,T) \\ u_i(x,0) = \phi_i(x) & i \in \{1, ..., M\}, \quad x \in \mathbb{T}, \end{cases} \tag{1.1}$$

where $a_i > 0, \quad \kappa > 0, \quad \beta_{ij} > 0$ and the initial conditions are non-negative, non-trivial Lipschitz functions on $\mathbb{T}$ which are assumed to satisfy the segregation condition:

$$\phi_i(x)\phi_j(x) = 0 \qquad\qquad \forall \quad x \in \mathbb{T}.$$

We will use the notation $Q_T = \mathbb{T} \times (0,T)$, where we also allow $T = +\infty$. The case with two competing species in arbitrary dimension was treated in [11], where Hilhorst, Martin and Mimura considered as well the PDE-ODE limit with vanishing diffusivity for one of the two species. Moreover, they imposed inhomogeneous Dirichlet conditions for one of the species, and Neumann homogeneous conditions for the other one. Our analysis is inspired by that work, although different in some elements that make the analysis simpler in our case.

We now introduce some spaces of functions that will be relevant in this and the following chapters. The space $H^1(0,1)$ is defined in the usual way:

$$H^1(0,1) := \left\{ u \in L^2(0,1) : \quad \frac{\partial u}{\partial x} \in L^2(0,1) \right\}, \tag{1.2}$$

and is a separable Hilbert space with the norm:

$$\|u\|_{H^1} := \int_0^1 \left[u^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right] dx. \tag{1.3}$$

Notice that for functions $u \in H^1(0,1)$, the evaluation in a point $x_0 \in [0,1]$ is well-defined because of the embedding $H^1(0,1) \hookrightarrow C^0([0,1])$, or because of the theory of traces which also generalizes to functions defined on higher dimensional domains. Therefore, we can define:

$$\begin{aligned} H^1_{\text{per}}(0,1) &:= \left\{ u \in L^2(0,1) : \frac{\partial u}{\partial x} \in L^2(0,1), \quad u(0) = u(1) \right\} \\ &= \left\{ u|_{(0,1)} : u \in H^1_{\text{loc}}(\mathbb{R}), \quad u(x+1) = u(x) \text{ for every } x \right\}. \end{aligned}$$

Here, $u \in H^1_{\text{loc}}$ if $u \in H^1(B)$ for any bounded open set $B \subset \mathbb{R}$. $H^1_{\text{per}}(0,1)$ is a closed subspace of $H^1(0,1)$ and a separable Hilbert space with the norm (1.3). Notice that it is also possible to define the space $H^1(\mathbb{T})$ in the same way as (1.2) but with $\mathbb{T}$ instead of the interval $(0,1)$, or equivalently by using the Fourier transform on the circumference. With these definitions, we have $H^1_{\text{per}}(0,1) \simeq H^1(\mathbb{T})$, where $\simeq$ represents an isometric isomorphism. We will denote, for example, as $[H^1(0,1)]^K$ the space of $\mathbb{R}^K$-valued functions for which each component belongs to $H^1(0,1)$. In a similar way, given $k \in \mathbb{N} \cup \{+\infty\}$, we will consider the spaces:

$$\mathcal{C}^k_{\text{per}}([0,1]) := \{\varphi \in \mathcal{C}^k([0,1]) : \quad \frac{\partial^j \varphi}{\partial x^j}(0) = \frac{\partial^j \varphi}{\partial x^j}(1) \quad \forall \quad 0 \le j \le k\}.$$

Also in this case, we can define the spaces $\mathcal{C}^k(\mathbb{T})$ and have $\mathcal{C}^k_{\text{per}}([0,1]) \simeq \mathcal{C}^k(\mathbb{T})$. Notice that another equivalent definition of $H^1_{\text{per}}(0,1)$ is the following:

$$H^1_{\text{per}}(0,1) := \overline{\mathcal{C}^\infty_{\text{per}}([0,1])}^{H^1},$$

where $\mathcal{C}^\infty_{\text{per}}([0,1])$ is seen as a subspace of $H^1(0,1)$ and the apex means that the closure is taken in the norm (1.3). Moreover, with a conventional notation, we will write c as a subscript for spaces of differentiable functions to identify the subspace of functions that are compactly supported, meaning that the set $\overline{\{\varphi \neq 0\}}$ is compact in the domain of definition of the functions. We will also consider spaces of functions depending both on space and time. In particular, we will make use of parabolic Hölder spaces, that represent the right environment to carry out the parabolic regularity theory in Hölder spaces. Our reference for this discussion is [14]. We define the parabolic distance ρ between points $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2)$ as follows:

$$\rho(z_1, z_2) := |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}.$$

We now fix a constant $\delta \in [0, \frac{1}{2}]$. If u is a function $u : Q_T \to \mathbb{R}$, we define the following seminorms:

$$[u]_{\mathcal{C}^{2\delta,\delta}(Q_T)} := \sup_{z_1 \neq z_2 \in Q_T} \frac{|u(z_1) - u(z_2)|}{\rho^{2\delta}(z_1, z_2)}$$
$$[u]_{\mathcal{C}^{2+2\delta,1+\delta}(Q_T)} := [u_t]_{\mathcal{C}^{2\delta,\delta}(Q_T)} + [u_{xx}]_{\mathcal{C}^{2\delta,\delta}(Q_T)}.$$

We can now define the following norms:

$$\|u\|_{\mathcal{C}^{2\delta,\delta}(Q_T)} := \|u\|_{L^\infty(Q_T)} + [u]_{\mathcal{C}^{2\delta,\delta}(Q_T)} \tag{1.4}$$

$$\begin{aligned} \|u\|_{\mathcal{C}^{2+2\delta,1+\delta}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \|u_t\|_{L^\infty(Q_T)} + \\ + \|u_x\|_{L^\infty(Q_T)} + \|u_{xx}\|_{L^\infty(Q_T)} + [u]_{\mathcal{C}^{2+2\delta,1+\delta}(Q_T)}. \end{aligned} \tag{1.5}$$

We define spaces $\mathcal{C}^{2\delta,\delta}(Q_T)$ and $\mathcal{C}^{2+2\delta,1+\delta}(Q_T)$ as the spaces of functions for which, respectively, (1.4) and (1.5) are finite. We remark that they are Banach spaces. Moreover, for any $\eta < \delta$, $\mathcal{C}^{2\eta,\eta}(Q_T) \hookrightarrow \mathcal{C}^{2\delta,\delta}(Q_T)$ with compact embedding, and the same holds for spaces $\mathcal{C}^{2+2\delta,1+\delta}(Q_T)$. The proofs of these compactness theorems are essentially based on the Ascoli-Arzelà theorem. We will also consider the case where $T = +\infty$. In this case, we will indicate for example as $\mathcal{C}^{2,1}(Q_\infty)$ the class of functions $u : Q_\infty \to \mathbb{R}$ belonging to $\mathcal{C}^{2,1}(Q_T)$ for any $T > 0$.

This chapter is structured as follows. In the first section, we will prove some a priori bounds on classical solutions to the system via a comparison principle for competitive systems, and then state the well-posedness of system (1.1) for finite κ. In the second section, we will perform the asymptotic analysis in a weak setting, and prove that the limit profiles segregate and satisfy a system of differential inequalities. The aim is to be as self contained as possible, but we will necessarily omit the proofs of some theorems, such as the well posedness of (1.1) and the a priori uniform Lipschitz bounds.

1.1. Well posedness

A priori bounds on solutions in the $L^\infty(Q_T)$ norm will play an important role throughout the chapter. While for semilinear equations they can often be directly derived via the usual comparison principle, for semilinear parabolic systems, one has to either adapt the formulation of the comparison principle, and in particular strengthen the notion of subsolutions and supersolutions, or to consider stronger hypotheses on the nonlinearity, such as, for example, quasi-monotonicity. However, in the context of Lotka-Volterra systems, quasi-monotonicity is typically satisfied by cooperative systems, which is the opposite situation with respect to the one we are considering. A detailed discussion of comparison results can be found in the first chapter of the book [16] by Leung, for general quasilinear and space-time inhomogeneous parabolic systems and Robin boundary conditions. Since we are not interested in pursuing such general cases, which of course introduce a certain level of unnecessary complications, we will state and prove a comparison principle valid for semilinear, homogeneous competitive systems, and use it to derive the desired L^∞ bounds. This will also allow us to consider the domain $\mathbb{T}$ explicitly, which is something not usual to find in the literature. We will deal with functions in the class $\mathcal{C}^{2,1}(Q_T) \cap \mathcal{C}^0(\overline{Q_T})$, with $T < +\infty$. However, we will see that the arguments that used are also valid in Q_∞.

Theorem 1.1 (Comparison principle). *Let* $\mathbf{v}, \mathbf{u}, \mathbf{w} \in \mathcal{C}^{2,1}(Q_T) \cap \mathcal{C}^0(\overline{Q_T})$ *such that:*

$$0 \leq v_i(x,t) \leq w_i(x,t) \qquad \text{in} \quad \overline{Q_T}, \quad \forall \quad i \in \{1, ..., M\} \tag{1.6}$$

$$v_i(x,0) \leq u_i(x,0) \leq w_i(x,0) \qquad \text{in} \quad \mathbb{T}, \quad \forall \quad i \in \{1, ..., M\} \tag{1.7}$$

Let $\mathbf{f}(\mathbf{s}) = \mathbf{f}(s_1, ..., s_M)$ *be a Locally Lipschitz function* $\mathbf{f} : \mathbb{R}^M \to \mathbb{R}^M$ *such that:*

$$f_i(\mathbf{s}) \leq f_i(\mathbf{s}') \quad \text{for every} \quad \mathbf{s}, \mathbf{s}' \quad s.t.: \quad s_j \geq s'_j \geq 0 \quad \forall \quad j \neq i, \quad s_i = s'_i \tag{1.8}$$

Moreover, suppose that, for every $i \in \{1, ..., M\}$*:*

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} = f_i(\mathbf{u}) & \text{in} \quad Q_T \\ \dfrac{\partial v_i}{\partial t} - \dfrac{\partial^2 v_i}{\partial x^2} \leq f_i(w_1, ..., w_{i-1}, v_i, w_{i+1}, ..., w_M) & \text{in} \quad Q_T \\ \dfrac{\partial w_i}{\partial t} - \dfrac{\partial^2 w_i}{\partial x^2} \geq f_i(v_1, ..., v_{i-1}, w_i, v_{i+1}, ..., v_M) & \text{in} \quad Q_T \end{cases} \tag{1.9}$$

Then:

$$v_i(x,t) \leq u_i(x,t) \leq w_i(x,t) \qquad \text{in} \quad \overline{Q_T}, \quad \forall \quad i \in \{1, ..., M\}$$

Proof. By the continuity properties of $\mathbf{v}, \mathbf{w}$ in $\overline{Q_T}$, we define the following subsets:

$$
\begin{aligned}
&R_i := \left[\min_{(x,t)\in\overline{Q_T}} v_i(x,t),\ \max_{(x,t)\in\overline{Q_T}} w_i(x,t)\right] && \forall \quad i \in \{1,...,M\}\\
&R := R_1 \times ... \times R_M\\
&R_i^\delta := \left[\min_{(x,t)\in\overline{Q_T}} v_i(x,t) - \delta,\ \max_{(x,t)\in\overline{Q_T}} w_i(x,t) + \delta\right] && \forall \quad i \in \{1,...,M\}\\
&R^\delta := R_1^\delta \times ... \times R_M^\delta
\end{aligned}
$$

for a small positive constant $\delta > 0$. Notice that, by local Lipschitz continuity of $\mathbf{f}$, there exists a constant $L > 0$ such that for every $i \in \{1,...,M\}$:

$$
|f_i(\mathbf{s}_1) - f_i(\mathbf{s}_2)| \leq L\sqrt{\sum_{j=1}^{M}({s_{1,j}}^2 - {s_{2,j}}^2)} \qquad \forall \quad i, \forall \quad \mathbf{s}_1, \mathbf{s}_2 \in R^\delta \tag{1.10}
$$

where we used the notation $\mathbf{s}_1 = (s_{1,1}, ..., s_{1,M})$. We remark here that for the present proof to be valid, we are only concerned with the validity of (1.10), so that the Lipschitz regularity of $\mathbf{f}$ on larger sets is not actually necessary. Now for any $i = 1, ..., M$ and for $0 < \epsilon < \frac{\delta}{2}$, define:

$$
u_i^{\pm\epsilon}(x,t) := u(x,t) \pm (1 + 3Lt)\epsilon \qquad \forall \quad (x,t) \in \overline{Q_T}
$$

By assumption, for any $i = 1, ..., M$, we have:

$$
v_i(x,0) < u_i^{+\epsilon}(x,0), \quad u_i^{-\epsilon}(x,0) < w_i(x,0) \qquad \forall \quad x \in \mathbb{T}.
$$

Suppose by contradiction that one of the inequalities:

$$
v_i(x,t) < u_i^{+\epsilon}(x,t), \quad u_i^{-\epsilon}(x,t) < w_i(x,t)
$$

fails at some point in $\mathbb{T} \times (0,\tau)$ for $\tau = \min\left\{T, \frac{1}{3L}\right\}$. Let $(x_0, t_0) \in \mathbb{T} \times (0,\tau)$ such that t_0 is the minimal time such that the inequality fails. At (x_0, t_0), either $v_j = u_j^{+\epsilon}$ or $w_j = u_j^{-\epsilon}$ for some $j \in \{1,...,M\}$. Suppose the first case holds; the latter can be treated in an analogous way. Then we must have that $\frac{\partial}{\partial t}(v_j - u_j^{+\epsilon})(x_0, t_0) \geq 0$. Moreover:

$$
\frac{\partial^2}{\partial x^2}(v_j - u_j^{+\epsilon})(x_0, t_0) = \frac{\partial^2}{\partial x^2}(v_j - u_j)(x_0, t_0) \leq 0, \tag{1.11}
$$

since we have that $(v_j - u_j^{+\epsilon})(x_0, t_0) = 0$ and $(v_j - u_j^{+\epsilon})(x, t_0) \leq 0$ for any x in a neighborhood of x_0. Consider now the vector $\tilde{\mathbf{v}}$ defined as follows:

$$\tilde{v}_k = \begin{cases} \min\{u_k^{+\epsilon}(x_0,t_0), w_k(x_0,t_0)\} & \text{for} \quad k \neq j \\ u_j^{+\epsilon}(x_0,t_0) & \text{for} \quad k = j \end{cases}$$

and notice that $\tilde{\mathbf{v}} \in R \subset R^\delta$. By the regularity of functions $\mathbf{u}$ and $\mathbf{v}$, we can evaluate the equation in (x_0, t_0).

$$\begin{aligned} &\frac{\partial}{\partial t}(v_j - u_j^{+\epsilon})(x_0,t_0) = \frac{\partial}{\partial t}(v_j - u_j)(x_0,t_0) - 3\epsilon L \leq \\ &\leq \frac{\partial^2}{\partial x^2}(v_j - u_j)(x_0,t_0) + f_j(w_1, ..., v_j, ..., w_M)(x_0,t_0) - f_j(\mathbf{u})(x_0,t_0) - 3\epsilon L \leq \\ &\leq f_j(\tilde{\mathbf{v}}) - f_j(\mathbf{u})(x_0,t_0) - 3\epsilon L \end{aligned} \tag{1.12}$$

where we used assumption (1.9) in the first inequality. In the second one, we used (1.11) and property (1.8) of $\mathbf{f}$, which applies because of the definition of $\tilde{\mathbf{v}}$. Now notice also that $\mathbf{u}(x_0,t_0) \in R^\delta$, since:

$$v_j(x_0,t_0) - u_j(x_0,t_0) = u_j^{+\epsilon}(x_0,t_0) - u_j(x_0,t_0) = [1 + 3Lt_0]\epsilon \leq 2\epsilon < \delta$$

Therefore, we can apply the estimate (1.10) to derive:

$$[f_j(\tilde{\mathbf{v}}) - f_j(\mathbf{u})](x_0,t_0) - 3\epsilon L \leq L\epsilon(1 + 3Lt_0) - 3\epsilon L < 2L\epsilon - 3L\epsilon = -L\epsilon < 0 \tag{1.13}$$

where we used the fact that, by definition, $t_0 \leq \frac{1}{3L}$. Together, equations (1.12) and (1.13) imply that

$$\frac{\partial}{\partial t}(v_j - u_j^{+\epsilon})(x_0,t_0) < 0$$

But this is in contradiction with $\frac{\partial}{\partial t}(v_j - u_j^{+\epsilon})(x_0,t_0) \geq 0$. By an analogous argument with $u_j^{-\epsilon}$ and w_j, we deduce that:

$$v_i(x,t) < u_i^{+\epsilon}(x,t), \quad u_i^{-\epsilon}(x,t) < w_i(x,t) \qquad \forall \quad (x,t) \in \mathbb{T} \times [0,\tau]. \tag{1.14}$$

By iterating the above argument on $\mathbb{T} \times (\tau, T)$, it is possible to prove that inequalities are satisfied for $(x,t) \in [\tau, \min\{T, \tau + \frac{1}{3L}\}]$. Since the constant L is fixed, we have eventually that inequalities (1.14) hold on $\mathbb{T} \times [0,T]$. Now passing to the limit as $\epsilon \to 0$, we get the thesis. $\square$

Notice that when dealing with solutions defined on Q_∞, the same theorem can be proven for functions $\mathbf{v}, \mathbf{w} \in \mathcal{C}^{2,1}(Q_\infty) \cap \mathcal{C}^0(\overline{Q_\infty})$ under the additional hypothesis that they are bounded on Q_∞ (Notice that this was already guaranteed by the continuity up to the boundary of $\mathbf{v}, \mathbf{w}$, in the case $T < +\infty$). Indeed, the same arguments used in the above proof can be applied in the new context on any finite time interval of the form $[\tau, \tau + T]$, with $T > 0$ fixed and $\tau > 0$. We are now able to prove the a priori bounds for solutions to (1.1).

Proposition 1.2 (L^∞ bounds)**.** *Let* $\mathbf{u} \in \mathcal{C}^{2,1}(Q_\infty) \cap \mathcal{C}^0(\overline{Q_\infty})$ *be a solution to* (1.1) *with* $u_i(x, 0) = \phi_i(x) \geq 0$ *for any* $x \in \mathbb{T}$*. Then:*

$$0 \leq u_i(x,t) \leq \max\left\{a_i, \quad \max_{x\in\mathbb{T}} \phi_i(x)\right\} =: A_i \qquad\qquad \forall \quad (x,t) \in Q_\infty$$

Proof. By direct computations, it is easy to verify that the nonlinearity

$$f_i(\mathbf{s}) = a_i s_i - s_i^2 - \kappa \sum_{j\neq i} \beta_{ij} s_i s_j$$

in (1.1) satisfies the hypotheses of theorem 1.1. Moreover, we define:

$$\begin{aligned} v_i &= 0 && \forall \quad i \in \{1, ..., M\} \\ w_i &= \max\left\{a_i, \max_{x\in\mathbb{T}} \phi_i(x)\right\} && \forall \quad i \in \{1, ..., M\} \end{aligned}$$

Again, functions $\mathbf{v}, \mathbf{u}, \mathbf{w}$ satisfy hypotheses (1.6), (1.7), (1.9), and $\mathbf{v}, \mathbf{w}$ are smooth and bounded. Therefore, we can directly apply 1.1 to obtain the desired bounds. $\square$

We now consider the well posedness of the initial value problem (1.1). Since this is not the focus of the present work, we will limit ourselves to state a theorem concerning existence, uniqueness and regularity of solutions. The theorem can be proved by resorting to semigroup theory and parabolic regularity theory, for which the choice of periodic boundary conditions does not introduce essentially new difficulties with respect to the Neumann or Dirichlet case. We cite the book [17] by Lunardi for detailed discussion of theorems of this type.

Theorem 1.3 (Well posedness)**.** *Consider the initial value problem* (1.1) *and suppose that the initial conditions* $\phi_i \in Lip(\mathbb{T})$*. Then, there exists a unique solution* $\mathbf{u} = (u_1, ..., u_M) \in \mathcal{C}^{2+2\delta,1+\delta}(Q_\infty) \cap \mathcal{C}^{2\delta,\delta}(\overline{Q_\infty})$ *for any* $\delta \in [0, \frac{1}{2})$.

Moreover, the solution satisfies the bounds of Proposition 1.2, so that non-negativity is preserved in time for any non-negative initial condition. Throughout all this work, we will consider segregated initial conditions, i.e. such that:

$$\phi_i \phi_j \equiv 0 \qquad \forall \quad x \in \mathbb{T}, \quad \forall \quad i, j \in \{1, ..., M\} \quad s.t. \quad i \neq j$$

We already anticipated that for every finite κ, we are able to prove that the solutions to (1.1) satisfy a remarkable set of $2M$ differential inequalities involving the following functions.

$$\hat{u}_i = u_i - \sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} u_j$$

Proposition 1.4. *Let* $\mathbf{u}$ *be the solution to system* (1.1) *provided by Theorem 1.3. Then* u_i *solves the following system of differential inequalities:*

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} \leq a_i u_i - u_i^2 \\ \dfrac{\partial \hat{u}_i}{\partial t} - \dfrac{\partial^2 \hat{u}_i}{\partial x^2} \geq a_i u_i - u_i^2 - \displaystyle\sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} (a_j u_j - u_j^2) \end{cases} \tag{1.15}$$

for any $i \in \{1, ..., M\}$ *and* $(x, t) \in \mathbb{T} \times (0, T)$.

Proof. The proof is based on direct computations. □

We remark that, since κ does not occur in (1.15), the system is a natural candidate to characterize the singular limit of solutions to (1.1) as $\kappa \to +\infty$.

1.2. Asymptotic analysis

The main purpose of this section is to show that as $\kappa \to \infty$, solutions converge in a suitable norm up to a subsequence, and the limit satisfies the segregation properties and a weak formulation of the differential inequalities (1.15). To this aim, we will consider solutions to (1.1) in spaces of weakly differentiable functions, because it is easier to prove uniform bounds in their norms. At the end of this chapter, we will however also discuss Hölder regularity of the limit, which will play an important role in this work. We will consider solutions only on a finite time-interval $[0, T]$ and perform the asymptotic analysis for a fixed $T > 0$. To pass to the limit in weak formulations of linear PDEs, weak convergence is usually enough, and therefore, if one is working with a reflexive Banach space, it is sufficient to show uniform boundedness of sequences thanks to the Banach-Alaoglu theorem. However, since we are considering a nonlinear integral formulation, we need strong convergence to pass to the limit, and therefore it is needed to resort to an appropriate compactness result. In [11], $L^2(Q_T)$ convergence is shown relying on the Kolmogorov-Riesz-Frechet theorem. In our setting, the framework of Bochner spaces interacts more nicely with the choice of periodic boundary conditions and simplifies the arguments. We recall that if X is a Banach space, the Bochner space $L^p(0,T;X)$ is defined as follows:

$$L^p(0,T;X) := \{v : (0,T) \to X \quad \text{measurable} \quad s.t. : \quad \int_0^T \|v(t)\|_X^p dt < +\infty\}$$

and is still a Banach space with the norm:

$$\|v\|_{L^p(0,T;X)} := \left(\int_0^T \|v(t)\|_X^p dt \right)^{\frac{1}{p}}$$

In particular, we will apply the following compactness principle:

Theorem 1.5 (Aubin-Lions-Simon, [23, Section 3]). *Let X, Y be Banach spaces, with compact embedding $Y \hookrightarrow X$, and let $1 \leq p \leq \infty$. Consider a family $\mathcal{F} \subset X$ such that:*

- *$\mathcal{F}$ is bounded in $L^p(0,T;Y)$*
- *There exists a function $\delta(h)$ with $\lim_{h\to 0} \delta(h) = 0$ such that, for every $f \in \mathcal{F}$:*

$$\int_0^{T-h} \|f(t+h) - f(t)\|_B^p dt \leq \delta(h) \tag{1.16}$$

Then, $\mathcal{F}$ is relatively compact in $L^p(0,T;X)$

We refer to [24] for a discussion on compactness principles in Bochner spaces, and to [23] for a proof of the above theorem. We will apply this result in the setting $p = 2$, $Y = H^1_{\text{per}}$ and $X = L^2(\mathbb{T})$. H^1_{per} is a separable Hilbert Space, compactly embedded in $L^2(\mathbb{T})$. It will now be useful to explicitly denote with $\mathbf{u}^\kappa$ the solution, provided by Theorem 1.3, to (1.1) with interaction strength κ. Of course, $u_i^\kappa \in L^2(0,T;H^1_{\text{per}})$ for any $\kappa > 0$, and we will prove that the bounds needed to apply Theorem 1.5 hold uniformly with respect to κ. In doing this, we will often need to deal with integration by parts in the space variable. We remark here, once and for all, that since we are considering solutions that are twice differentiable in space in the 1-torus $\mathbb{T}$, boundary terms will always vanish. Moreover, we will make use of the constants A_i defined by:

$$
\begin{aligned}
A_i &:= \max\left\{a_i, \quad \max_{x\in\mathbb{T}} \phi_i(x)\right\},\\
A &:= \max_{i=1,\dots,M} A_i.
\end{aligned}
$$

For later use, we will also define for any h small enough and for any function $w \in L^2(0,T;L^2(\mathbb{T}))$ the function $\tau_h w(x,t) \in L^2(0,T;L^2(\mathbb{T}))$ such that:

$$
\tau_h w(x,t) := w(x,t+h).
$$

Proposition 1.2 allows to derive easily $L^2(Q_T)$-bounds for u_i^κ which are independent of κ:

$$
\int_{Q_T} |u_i^\kappa|^2 \leq TA^2
$$

Moreover, we prove the following proposition.

Proposition 1.6. *There exist $C_1, C_2 > 0$ such that, for every $i \in \{1, ..., M\}$ and for every $\kappa > 0$:*

$$
\kappa \sum_{j\neq i} \int_{Q_T} \beta_{ij} u_i^\kappa u_j^\kappa \leq C_1 \tag{1.17}
$$

$$
\int_{Q_T} \left|\frac{\partial u_i^\kappa}{\partial x}\right|^2 \leq C_2 \tag{1.18}
$$

Proof. Integrating the equation for u_i^κ over Q_T yields, because of periodic boundary conditions:

$$
\begin{aligned}
&\kappa \sum_{j\neq i} \int_{Q_T} \beta_{ij} u_i^\kappa u_j^\kappa = \int_{Q_T} a_i u_i^\kappa - u_i^{\kappa 2} - \int_{\mathbb{T}} u_i^\kappa(x,T)dx + \int_{\mathbb{T}} \phi_i \leq\\
&\leq T(A + A^2) + 2A =: C_2
\end{aligned}
$$

To derive an estimate for the gradient, we multiply the i-th equation for u_i^κ and integrate by parts with respect to the space variable, obtaining the following.

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|u_i^\kappa|^2+\int_{\mathbb{T}}|u_{i,x}^\kappa|^2+\kappa\sum_{j\neq i}\int_{\mathbb{T}}\beta_{ij}u_i^{\kappa 2}u_j^\kappa=\int_{\mathbb{T}}a_iu_i^{\kappa 2}-u_i^{\kappa 3}\leq 2A^3$$

After integrating in time between 0 and T, we get:

$$\int_{Q_T}|u_{i,x}^\kappa|^2+\kappa\sum_{j\neq i}\int_{Q_T}\beta_{ij}u_i^{\kappa 2}u_j^\kappa\leq 2A^3T+\frac{1}{2}\int_{\mathbb{T}}|u_i^\kappa|^2(\cdot,0)-\frac{1}{2}\int_{\mathbb{T}}|u_i^\kappa|^2(\cdot,T)\leq$$
$$\leq 2A^3T+A^2=:C_2$$

Since the second term on the left hand side is positive, we can discard it and obtain:

$$\int_{Q_T}|u_{i,x}^\kappa|^2\leq C_2$$

□

we remark that (1.18) represents the (uniform in κ) bound in $L^2(0,T;H^1_{\mathrm{per}})$ for the family of functions $\{u^\kappa\}_{\kappa>0}$. We now prove estimate (1.16) .

Proposition 1.7. *There exists a constant $C_3>0$ such that, for each $h>0$ sufficiently small:*

$$\int_0^{T-h}\int_{\mathbb{T}}|\tau_hu_i^\kappa-u_i^\kappa|^2<C_3h$$

for every $i\in\{1,...,M\}$ and $\kappa>0$.

Proof.
$$\begin{aligned}&\int_0^{T-h}\int_{\mathbb{T}}|\tau_hu_i^\kappa-u_i^\kappa|^2(x,t)dxdt=\\&=\int_0^{T-h}\int_{\mathbb{T}}[\tau_hu_i^\kappa-u_i^\kappa](x,t)\left(\int_t^{t+h}\partial_tu_i^\kappa(x,s)ds\right)dxdt=\\&=\int_0^{T-h}\int_{\mathbb{T}}[\tau_hu_i^\kappa-u_i^\kappa](x,t)\left(\int_0^{h}\partial_tu_i^\kappa(x,t+s)ds\right)dxdt=\\&=I_1+I_2+I_3\end{aligned}$$

where:

$$
\begin{aligned}
I_1 &= \int_0^h \int_0^{T-h} \int_{\mathbb{T}} [\tau_h u_i^\kappa - u_i^\kappa](x,t) \cdot \frac{\partial^2 u_i^\kappa}{\partial x^2}(x,t+s) dxdtds \\
I_2 &= \int_0^h \int_0^{T-h} \int_{\mathbb{T}} [\tau_h u_i^\kappa - u_i^\kappa](x,t) \cdot [a_i u_i^\kappa - u_i^{\kappa 2}](x,t+s) dxdtds \\
I_3 &= \int_0^h \int_0^{T-h} \int_{\mathbb{T}} [\tau_h u_i^\kappa - u_i^\kappa](x,t) \cdot [\kappa u_i^\kappa \sum_{j\neq i} \beta_{ij} u_j^\kappa](x,t+s) dxdtds
\end{aligned}
$$

by using the equation for u_i^κ. We first consider the term I_1:

$$
\begin{aligned}
I_1 &= -\int_0^h \int_0^{T-h} \int_{\mathbb{T}} \frac{\partial}{\partial x}[\tau_h u_i^\kappa - u_i^\kappa](x,t) \cdot \frac{\partial u_i^\kappa}{\partial x}(x,t+s) dxdtds = \\
&= -\int_0^h \int_0^{T-h} \int_{\mathbb{T}} \frac{\partial \tau_h u_i^\kappa}{\partial x}(x,t) \cdot \frac{\partial u_i^\kappa}{\partial x}(x,t+s) dxdtds + \\
&+ \int_0^h \int_0^{T-h} \int_{\mathbb{T}} \frac{\partial u_i^\kappa}{\partial x}(x,t) \cdot \frac{\partial u_i^\kappa}{\partial x}(x,t+s) dxdtds =
\end{aligned}
$$

We consider the first term (Notice that $\frac{\partial}{\partial x}$ and τ_h commute), and apply Cauchy-Schwartz inequality:

$$
\begin{aligned}
&-\int_0^h \int_0^{T-h} \int_{\mathbb{T}} \frac{\partial \tau_h u_i^\kappa}{\partial x}(x,t) \cdot \frac{\partial \tau_s u_i^\kappa}{\partial x}(x,t) dxdtds \leq \\
&\leq \int_0^h \left(\int_0^{T-h} \int_{\mathbb{T}} \left| \frac{\partial \tau_h u_i^\kappa}{\partial x} \right|^2 (x,t) \right)^{\frac{1}{2}} \cdot \left(\int_0^{T-h} \int_{\mathbb{T}} \left| \frac{\partial \tau_s u_i^\kappa}{\partial x} \right|^2 (x,t) \right)^{\frac{1}{2}} ds \leq \\
&\leq \int_0^h \left(\int_0^{T-h} \int_{\mathbb{T}} \tau_h \left| \frac{\partial u_i^\kappa}{\partial x} \right|^2 (x,t) \right)^{\frac{1}{2}} \cdot \left(\int_0^{T-h} \int_{\mathbb{T}} \tau_s \left| \frac{\partial u_i^\kappa}{\partial x} \right|^2 (x,t) \right)^{\frac{1}{2}} ds \leq \\
&\leq \int_0^h \left(\int_0^{T} \int_{\mathbb{T}} \left| \frac{\partial u_i^\kappa}{\partial x} \right|^2 (x,t) \right)^{\frac{1}{2}} \cdot \left(\int_0^{T} \int_{\mathbb{T}} \left| \frac{\partial u_i^\kappa}{\partial x} \right|^2 (x,t) \right)^{\frac{1}{2}} ds \leq h \cdot \int_{Q_T} \left| \frac{\partial u_i^\kappa}{\partial x} \right|^2 \leq C_2 h
\end{aligned}
$$

In a similar way, relying on the L^∞ bounds in proposition 1.2, that are independent of κ, we can treat the terms I_2, I_3, to get:

$$
\begin{aligned}
I_2 &\leq \tilde{I}_1 h \\
I_2 &\leq \tilde{I}_2 h,
\end{aligned}
$$

For some positive constants $\tilde{C}_1, \tilde{C}_2$ which do not depend on κ. Therefore, we get:

$$\int_0^{T-h}\int_{\mathbb{T}}|\tau_h u_i^\kappa - u_i^\kappa|^2(x,t)dxdt \le (C_2+\tilde{I}_1+\tilde{I}_2)h =: C_3 h$$

And the proposition is proved. □

We now introduce a suitable space of test functions:

$$V := \{\psi \in \mathcal{C}_c^\infty(\mathbb{T}\times(0,T)) : \quad \psi \ge 0\}, \tag{1.19}$$

and prove the main theorem of this section.

Theorem 1.8. *Consider the family $\{\mathbf{u}^\kappa\}_{\kappa>0}$. There exists a subsequence $\{\kappa_n\}_{n\in\mathbb{N}}$ with $\kappa_n \to \infty$, and a function $\mathbf{u} \in L^2(0,T;H^1_{per})$ such that, for every i:*

- *$u_i^\kappa \to u_i$ strongly in $L^2(0,T;L^2)$;*
- *$u_iu_j \equiv 0$ almost everywhere in Q_T.*
- *the functions u_i and $\hat{u}_i := u_i - \sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}u_j$ satisfy the following weak differential inequalities:*

$$\begin{cases} -\int_0^T\int_{\mathbb{T}} u_i\psi_t + \int_0^T\int_{\mathbb{T}} u_{i,x}\psi_x \le \int_0^T\int_{\mathbb{T}}(a_iu_i - u_i^2)\psi \\ -\int_0^T\int_{\mathbb{T}} \hat{u}_i\psi_t + \int_0^T\int_{\mathbb{T}} \hat{u}_{i,x}\psi_x \ge \int_0^T\int_{\mathbb{T}}\left(a_iu_i - u_i^2 - \sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}(a_ju_j - u_j^2)\right)\psi \end{cases} \tag{1.20}$$

  for every non-negative test function $\psi \in V$.

Proof. Thanks to the uniform bounds proven in Propositions 1.6 and 1.7, The strong convergence of u_i^κ is a result of the application of Theorem 1.5. Moreover, the limit u_i is in $L^2(0,T;H^1_{\text{per}})$ because of bound (1.18), which ensures weak convergence in the $L^2(0,T;H^1_{\text{per}})$ topology.

We now consider the segregation condition. because of the strong convergence in $L^2(0,T;L^2)$, $u_i^\kappa u_j^\kappa \to u_iu_j$ a.e. in Q_T, up to a subsequence. Now the bound (1.17) implies:

$$\sum_{j\neq i}\int_{Q_T}\beta_{ij}u_i^\kappa u_j^\kappa \to 0.$$

Therefore, $u_i^\kappa u_j^\kappa \to 0$, so we must have $u_iu_j \equiv 0$ a.e. in Q_T.

For what concerns item 3, the weak formulation is clearly satisfied for any finite κ because of Proposition 1.4. We need to show that all the terms pass to the limit, for any fixed $\psi \geq 0$ in the test space. We consider the terms in the first inequality (1.20); the second one can be treated in the same way.

- $-\int_0^T \int_{\mathbb{T}} u_i^\kappa \psi_t \to -\int_0^T \int_{\mathbb{T}} u_i \psi_t$ because of the strong $L^2(Q_T)$ convergence; the same holds for the corresponding term involving $\hat{u}_i^\kappa$ in the second inequality.
- $\int_0^T \int_{\mathbb{T}} u_{i,x}^\kappa \psi_x \to \int_0^T \int_{\mathbb{T}} u_{i,x} \psi_x$ because of the weak convergence in $L^2(0,T;H^1_{\mathrm{per}})$; the same holds for the corresponding term involving $\hat{u}_i^\kappa$ in the second inequality.
- For the term $\int_0^T \int_{\mathbb{T}} (a_i u_i - u_i^2)\psi$, the strong L^2 convergence is still sufficient.

This concludes the proof of the theorem. □

Notice that in case a uniqueness theorem is available for the singular problem (1.20) in the space H^1_{per}, we have that each converging subsequence of the family $\{u_\kappa\}_{\kappa>0}$ must have the same limit for $\kappa_n \to +\infty$. Therefore, strong L^2 convergence would actually hold for any sequence u_{κ_n} where $\kappa_n \to +\infty$, and not just up to a subsequence. However, we will not take care of investigating the uniqueness of solutions for the singular parabolic system in the most general setting, since this is not really relevant for the purposes of our work. Nevertheless, a uniqueness theorem for the case of cyclic interactions will be provided at the end of Chapter 3, so that the above arguments can be applied.

The analysis in this chapter was based on spaces of integrable functions because it is more straightforward to obtain the necessary uniform estimates. However, the results we showed are far from optimal, and in particular the Hölder and Lipschitz continuity of the segregated limit for competition-diffusion systems has been widely studied. For what concerns the elliptic case, we refer to [5], where Hölder uniform bounds are derived via a blow up argument. Their result is valid in any dimension, and for an arbitrary number of species. In [2], a different technique is used to prove uniform bounds in the Hölder norm (in the parabolic distance), and the result is improved in the sense that Lipschitz uniform bounds are shown as well. Moreover, the analysis is extended to parabolic systems without internal reaction, but with an arbitrary number of species and in a general dimension. In [28], The authors show that the same results hold when one is considering a logistic internal reaction, as in our case. Since the proof of uniform bounds in Hölder and Lipschitz parabolic norms relies on essentially local arguments, and they do not involve boundary conditions, they are valid also in our setting of periodic boundary conditions.

Theorem 1.9 (Caffarelli-Karakhanyan-Lin)**.** *The function* $\mathbf{u}$ *is locally Lipschitz continuous in the parabolic distance.*

However, since the focus of the present work is on the long time behavior of the singular system, rather than on the asymptotic analysis, and the proofs are rather involved, we won't delve into the detail of the proof. Nevertheless, thanks to Theorem 1.9, we can always assume that the singular, segregated limit $\mathbf{u}$ obtained by Theorem 1.8 is a Lipschitz continuous function in the parabolic distance on Q_T.

At last, we remark the fact that we have only considered the case Q_T so far, but for the long time behavior of $\mathbf{u}$ we need of course to consider solutions defined on the whole $\mathbb{T} \times (0, \infty)$. However, we showed in Section 1.1 that solutions to (1.1) are defined globally in time, and uniformly bounded in the L^∞ norm. Moreover, Lipschitz bounds are local in nature, which means that they are not necessarily uniform in time a priori, but they turn to be global thanks to the L^∞ uniform bounds. In particular, in the asymptotic analysis, we can deal with bounded, globally Lipschitz functions on $\mathbb{T} \times [0, +\infty)$. The segregation property and the weak formulation of differential inequalities, stated in Theorem 1.8, extend to solutions defined globally in $\mathbb{T} \times (0, +\infty)$, as we will specify in Chapter 3, which is dedicated to the analysis of the singular parabolic system.

2 | Analysis of the singular elliptic system

The aim of this chapter is to study the structure of the set of solutions to the stationary differential inequalities associated with the singular parabolic system in the periodic case. The results in this chapter show the difference between the Dirichlet and periodic case, and form the basis upon which we will build the analysis of the long time behavior of solutions. Notice that our focus, in this perspective, is on the possible structure that may arise, such as continua of solutions, and not on providing precise existence results based on the values of the coefficients. Let $a_i > 0$ for every $i = 1, ..., M$ and $\beta_{ij} > 0$ for every $i, j = 1, ..., M$. We consider the following system of differential inequalities:

$$\begin{cases} -\dfrac{d^2 u_i}{dx^2} \leq a_i u_i - u_i^2 & \forall \quad i, \quad \forall \quad x \in (0,1), \\ -\dfrac{d^2}{dx^2}\left(u_i - \displaystyle\sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} u_j\right) \geq a_i u_i - u_i^2 - \displaystyle\sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} \left(a_j u_j - u_j^2\right) & \forall \quad i, \quad \forall \quad x \in (0,1), \\ u_i u_j \equiv 0 & \forall \quad j \neq i, \quad x \in (0,1), \\ u_i(0) = u_i(1) & \forall \quad i. \end{cases} \tag{2.1}$$

The inequalities are intended in the weak sense, for non-negative test functions $\varphi \in H^1_{\text{per}}(0,1) = \overline{C^\infty_{\text{per}}([0,1])}^{H^1(0,1)}$, where:

$$C^\infty_{\text{per}}([0,1]) := \left\{ \varphi \in \mathcal{C}^\infty([0,1]) : \quad \frac{\partial^k \varphi}{\partial x^k}(0) = \frac{\partial^k \varphi}{\partial x^k}(1) \quad \forall \quad k \geq 0 \right\}.$$

Therefore, for example, the first inequality in (2.1) is meant in the following sense for u_i:

$$\int_0^1 \frac{du_i}{dx}\frac{d\varphi}{dx} dx \leq \int_0^1 (a_i u_i - u_i^2)\varphi dx \qquad \forall \quad \varphi \geq 0, \quad \varphi \in H^1_{\text{per}}(0,1).$$

Notice that since the dimension of the domain is one, the embedding $H^1_{\text{per}}(0,1) \hookrightarrow \mathcal{C}([0,1])$ is continuous. In this chapter, we will often write u_i' for $\frac{du_i}{dx}$. We are only interested in positive solutions, by which we mean non-trivial and non-negative solutions. For a solution to the periodic problem (2.1), we mean a vector-valued function $\mathbf{u} = (u_1, ..., u_M)$ in $[H^1_{\text{per}}(0,1)]^M$ satisfying the differential inequalities in the sense specified above. However, we remark that $\mathcal{C}^1_{\text{per}}([0,1]) := \{\varphi \in \mathcal{C}^1([0,1]) : \quad \varphi(0) = \varphi(1), \quad \frac{\partial \varphi}{\partial x}(0) = \frac{\partial \varphi}{\partial x}(1)\}$ or $C^\infty_{\text{per}}([0,1])$ are equivalent choices for the test spaces. Sometimes, when dealing with solutions for which only $m \leq M$ components are positive, we will simply indicate such solutions as $\mathbf{u} = (u_1, ..., u_m)$ by implicitly considering a re-ordering of the indices and a restriction of the set of coefficients. This can be done by considering $[0,1]$ as the reference interval. We will also consider the Dirichlet problem for the above differential inequalities. In this case, we call a solution a function in $[H^1_0(0,1)]^M$ that satisfies the inequalities weakly with non-negative tests in $H^1_0(0,1)$. Here we mention the spaces $C^1_c((0,1))$, $C^\infty_c((0,1))$ of differentiable and smooth functions compactly supported in $(0,1)$ as an equivalent choice for the test space. In this chapter, we will also sometimes refer to the "restriction" of the above differential inequalities to a subdomain of $[0,1]$. With this term, for the periodic problem, we mean the differential inequalities integrated on the subset against test functions $\varphi \geq 0, \quad \varphi \in H^1(0,1)$.

The case with homogeneous Dirichlet boundary conditions has already been studied in [28], where the authors prove that, for any choice of the interaction coefficients β_{ij}, there is a finite set of solutions such that at least one component is positive somewhere in the domain $(0,1)$. In our case, we observe that for any choice of coefficients β_{ij}, thanks to the periodic boundary conditions, the set of solutions is translation-invariant. Notice that this is by itself sufficient to exclude the possibility that it has finite cardinality unless it only contains constant functions. We now define space-translations for $c \in \mathbb{R}$. Given some $\mathbf{u} \in H^1_{\text{per}}(0,1)$, it is possible to consider the periodic extension $\mathbf{u}^{\text{per}}$ on $\mathbb{R}$. We define the translation of $\mathbf{u}$ by c as:

$$\sigma_c \mathbf{u}(x) := \mathbf{u}^{\text{per}}(x - c) \qquad\qquad \forall \quad x \in [0,1].$$

Proposition 2.1 (Translational invariance of solutions). *If $\mathbf{u} := (u_1, ..., u_M)$ is a solution to the problem* (2.1)*, then any of its space-translations is a solution as well.*

Now it is easy to notice that solutions with $m = 1$ reduce to a scalar boundary value problem.

Proposition 2.2. *Let $i \in \{1, ..., M\}$. $\mathbf{u}$ is a positive solution to* (2.1) *such that $u_j \equiv 0$ for every $j \neq i$ if and only $u_i = a_i$.*

Proof. since $u_j \equiv 0$ on $(0,1)$ for any $j \neq i$, the combination of the first two inequalities in (2.1) gives the logistic equation for u_i, which also satisfies the periodic boundary conditions. Therefore:

$$\begin{cases} -\dfrac{d^2 u_i}{dx^2} = a_i u_i - u_i^2 \quad \forall \quad x \in (0,1), \\ u_i(0) = u_i(1). \end{cases} \tag{2.2}$$

This problem always admits a positive solution $u_i = a_i$. consider a generic positive solution $u \geq 0$, $u \not\equiv 0$. By elliptic regularity, it is actually smooth on $[0,1]$, and it satisfies (2.2) in a classical sense. Moreover, since the interval is compact, u admits a local maximum x_M and a local minimum x_m, with $-\frac{d^2u}{dx^2}(x_M) \geq 0$ and $-\frac{d^2u}{dx^2}(x_m) \leq 0$. suppose that $u(x_m) = 0$. Then, $\frac{du}{dx}(x_m) = 0$. However, by the Cauchy uniqueness theorem (notice that $u \equiv 0$ solves (2.2)) this would imply $u \equiv 0$. Therefore, for a nontrivial solution, $u(x_M) \geq u(x_m) > 0$. By evaluating the equation in these two points, we get:

$$\begin{cases} 0 \leq -\dfrac{d^2u}{dx^2}(x_M) = u(x_M)\,(a_i - u(x_M)) \quad \Longrightarrow \quad u(x_M) \leq a_i, \\ 0 \geq -\dfrac{d^2u}{dx^2}(x_m) = u(x_m)\,(a_i - u(x_m)) \quad \Longrightarrow \quad u(x_m) \geq a_i. \end{cases}$$

Therefore, the only possibility is that $u(x_M) = u(x_m) = a_i$, that implies $u \equiv a_i$. □

From now on, let $m \geq 2$ for our solution $\mathbf{u}$. For any of these solutions, there exists a translation which satisfies the Dirichlet boundary problem.

Proposition 2.3. *Suppose* $\mathbf{u}$ *is a solution to* (2.1) *such that* $m \geq 2$. *Then, there exists a translation of* $\mathbf{u}$ *solving the associated Dirichlet boundary problem.*

Proof. Since $m \geq 2$, there must be at least one point $x_0 \in [0,1)$ where all the densities vanish simultaneously, meaning that $u_i(x_0) = 0$ for every $i \in \{1, ..., m\}$. By the translation invariance of solutions, the differential inequalities are still satisfied, and the Dirichlet boundary conditions are also satisfied. □

Now, let us consider some basic facts proven in [28] that are useful in our situation. First, we recall the following results about the diffusive logistic equation with homogeneous Dirichlet boundary conditions:

Proposition 2.4. *Let* $a, L > 0$. *For any* $L > \frac{\pi}{2\sqrt{a}}$, *there exists a unique positive solution to:*

$$\begin{cases} -\dfrac{d^2u}{dx^2} = au - u^2 \quad in \quad (0, L), \\ u(0) = u(L) = 0. \end{cases} \tag{2.3}$$

For any $L \leq \frac{\pi}{2\sqrt{a}}$, there exists no positive solution. Moreover, the following quantity is constant for any $x \in [0, L]$:

$$C(x) := \frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{2}au^2 + \frac{1}{3}u^3 = const \qquad for \quad x \in [0, L], \tag{2.4}$$

and it is monotone increasing when seen as a function of L.

The above proposition plays an essential role in [28] for the study of solutions to (2.1) with Dirichlet boundary conditions. We will now summarize the main properties that are important in our treatment. Notice that, for solutions of (2.1), we cannot exclude a-priori the fact that $\bigcup_{i=1}^m \{u_i > 0\}$ has infinitely many connected components. However, this is precisely what the first point of the following proposition deals with.

Proposition 2.5. *Let $\mathbf{u}$ be a solution to (2.1) with Dirichlet boundary conditions, and fix some $i \in \{1, ..., m\}$. Then:*

- *There exists some $l \in \mathbb{N}$ such that $\{u_i > 0\} = \bigcup_{j=1}^l (a_j, b_j)$ for some choice of points $0 \leq a_1 < b_1 < ... < a_l < b_l \leq 1$.*
- *$\frac{du_i}{dx}(a_j) = -\frac{du_i}{dx}(b_j) \neq 0$ for any $j = 1, ..., l$.*
- *for any interval (b, c) where $u_i > 0$ and $u(b) = 0$, there exists some index $k \neq i$ such that $u_k > 0$ in some interval (a, b) and $\frac{du_k}{dx}(b) = -\frac{\beta_{ki}}{\beta_{ik}}\frac{du_i}{dx}(b)$.*

Proof. Since $H_0^1(0, 1) \subset \mathcal{C}([0, 1])$ in dimension 1, u_i is continuous, so $\{u_i > 0\}$ is an open set. Therefore, it is a countable union of disjoint open intervals: $\{u_i > 0\} = \bigcup_{j\in\mathbb{N}}(a_j, b_j)$. Notice that, on $\{u_i > 0\}$, u_i is a positive solution to the logistic equation with Dirichlet boundary conditions. Therefore, by Proposition 2.4, the length of each interval (a_j, b_j) must be larger than $\frac{\pi}{2\sqrt{a}}$. Thus, there must be only a finite number of terms in the sum. Moreover, suppose $b_j = a_{j+1}$ for some j. Then, $u_i(b_j) = u_i(a_{j+1}) = 0$, but u_i solves the logistic equation on (a_j, a_{j+1}) so it must be positive in b_j, giving a contradiction. Thus $b_j < a_{j+1}$ and this proves the first item.

Now, by the conservation of quantity (2.4) we get:

$$\left(\frac{du_i}{dx}\right)^2(a_i) = \left(\frac{du_i}{dx}\right)^2(a_{i+1}),$$

and the two derivatives have opposite sign. Therefore item 2 is proved.

To prove item three, suppose by contradiction that on a set (a,b) every u_j vanishes identically. Then again u_i must satisfy the logistic equation with homogeneous boundary conditions on (a,c) and must be positive there. thus, some u_j must be positive on (a,c). Here we remark that since connected components of u_i cannot be arbitrary small for any i, by taking a sufficiently close to b we can ensure there is only one index j such that u_j is positive on (a,b). Consider now the second inequalities in (2.1) for i and j. When restricted to (a,c), they read as follows:

$$\begin{cases} -\dfrac{d^2}{dx^2}\left(u_i - \dfrac{\beta_{ij}}{\beta_{ji}}u_j\right) \geq a_i u_i - u_i^2 - \dfrac{\beta_{ij}}{\beta_{ji}}\left(a_j u_j - u_j^2\right) \quad \text{on} \quad (a,c), \\ -\dfrac{d^2}{dx^2}\left(u_j - \dfrac{\beta_{ji}}{\beta_{ij}}u_j\right) \geq a_j u_j - u_j^2 - \dfrac{\beta_{ji}}{\beta_{ij}}\left(a_i u_i - u_i^2\right) \quad \text{on} \quad (a,c). \end{cases}$$

It is now possible to multiply the second inequality by $-\frac{\beta_{ij}}{\beta_{ji}}$ to get that the following boundary value problem is satisfied on (a,c) by $u := u_i - \frac{\beta_{ij}}{\beta_{ji}}u_j$:

$$\begin{cases} -\dfrac{d^2}{dx^2}u = a_i u_i - u_i^2 - \dfrac{\beta_{ij}}{\beta_{ji}}\left(a_j u_j - u_j^2\right) \quad \text{on} \quad (a,c), \\ u(a) = u(c) = 0. \end{cases}$$

Since the right hand side is a Lipschitz function, by elliptic regularity theory it is possible to show that that u is smooth on (a,c), and in particular at $x = b$. Therefore, we have that $\frac{du_k}{dx}(b) = -\frac{\beta_{ki}}{\beta_{ik}}\frac{du_i}{dx}(b)$, and this ends the proof of the proposition. □

Notice that, thanks to Proposition 2.3, The above proposition can be easily applied to derive the same properties for periodic solutions. In the periodic case, unlike the Dirichlet one, the set of solutions differs substantially depending on the choice of interaction coefficients β_{ij}. To be more precise, for a periodic solution $\mathbf{u} = (u_1, ..., u_m)$, let us define l_i the (finite) number of connected components of the set $\{u_i > 0\}$. Notice that, for brevity, we will often refer to the sets $\{u_i > 0\}$, which are open since u_i is continuous, as "support of u_i", regardless of the fact that this name is usually reserved for $\overline{\{u_i \neq 0\}}$. We now define the function $\mathbf{v} = (v_1, ..., v_n)(x)$, with $n = \sum_{i=1}^m l_i \geq m$, by giving to each connected component of $\bigcup_{i=1}^m \{u_i > 0\}$ the order in which it appears on $[0,1]$. Suppose that the k-th connected component A_k is a subset of $\{u_i > 0\}$. Then, we define:

$$v_k = \begin{cases} u_i & x \in A_k, \\ 0 & x \notin A_k. \end{cases}$$

In this way, each density v_k has a connected support, and we can more explicitly keep track of the number of interfaces occurring on the domain for function $\mathbf{u}$. Notice that the definition of $\mathbf{v}$ allows us to naturally introduce the map $\sigma : \{1, ..., n\} \to \{1, ..., m\}$ such that $\sigma(k) = i$ if and only if the support of v_k is a subset of $\{u_i > 0\}$. Let us now consider the following condition:

$$\prod_{i=1}^{n-1} \frac{\beta_{\sigma(i),\sigma(i+1)}}{\beta_{\sigma(i+1),\sigma(i)}} = \frac{\beta_{\sigma(1),\sigma(n)}}{\beta_{\sigma(n),\sigma(1)}}. \tag{2.5}$$

We will say that the interactions are cyclic with respect to $\mathbf{u}$ if condition (2.5) holds with the map σ associated to $\mathbf{u}$. We will call the interactions acyclic otherwise. First, we show a non-existence results for positive solutions that do not satisfy the cyclic condition.

Theorem 2.6. *[Non-existence of solutions, acyclic case] Let $\mathbf{u} = (u_1, ..., u_m)$ be a positive solution to (2.1) with $2 \leq m \leq M$, and consider the map σ defined as above. Then:*

$$\prod_{i=1}^{m-1} \frac{\beta_{\sigma(i),\sigma(i+1)}}{\beta_{\sigma(i+1),\sigma(i)}} = \frac{\beta_{\sigma(1),\sigma(n)}}{\beta_{\sigma(n),\sigma(1)}}.$$

Proof. We will consider the function $\mathbf{v}$ associated to $\mathbf{u}$. Without loss of generality, consider the translation of $\mathbf{v}$, still denoted in the same way, such that $v_1 = 0$ at $x = 0$. We will refer to zero points lying between densities v_i and v_{i+1} as a_i, with the convention that $n + 1 = 1$ for indices i. It is possible to derive, similarly to Proposition 2.5, the following transmission condition for derivatives:

$$\frac{dv_i}{dx}(a_i) = -\frac{\beta_{\sigma(i),\sigma(i+1)}}{\beta_{\sigma(i+1),\sigma(i)}} \frac{dv_{i+1}}{dx}(a_i),$$

for any $i = 1, ..., n - 1$. However, due to proposition 2.1, this condition must hold for $i = n$ as well:

$$\frac{dv_n}{dx}(a_n) = -\frac{\beta_{\sigma(n),\sigma(1)}}{\beta_{\sigma(1),\sigma(n)}} \frac{dv_1}{dx}(a_0).$$

Moreover, by the second item in (2.5) we have:

$$\frac{dv_i}{dx}(a_i) = -\frac{dv_i}{dx}(a_{i+1}).$$

Therefore, we can write $\frac{dv_1}{dx}(a_0)$ in the following ways:

$$\begin{cases} \dfrac{du_1}{dx}(a_0) = -\left(\prod_{i=1}^{n-1} \dfrac{\beta_{\sigma(i),\sigma(i)+1}}{\beta_{\sigma(i)+1,\sigma(i)}}\right) \dfrac{du_n}{dx}(a_n), \\ \dfrac{du_1}{dx}(a_0) = -\dfrac{\beta_{\sigma(1)\sigma(n)}}{\beta_{\sigma(n),\sigma(1)}} \dfrac{du_n}{dx}(a_n). \end{cases}$$

Therefore, either (2.5) holds, or $\frac{du_i}{dx}(a_i) = 0$ for every i, but that would contradict the positivity of the solution. □

Therefore, when no condition of the form (2.5) holds, the only non-zero solutions to the periodic problem are those with exactly one non-zero component, i.e. $m = 1$. We now consider the case where the cyclic condition holds, and we remark that for solutions to the Dirichlet problem, $\sigma(k) \neq \sigma(k+1)$ for $k = 1, ..., n-1$ by Proposition 2.5.

Theorem 2.7 (Characterization of solutions, cyclic case)**.** *Let* $\mathbf{u} = (u_1, ..., u_m)$ *be segregated densities with* $2 \leq m \leq M$*. Let* σ *be its associated map, and suppose that* (2.5) *holds. Then,* u *solves* (2.1) *if and only if*

- $\mathbf{u}$ *is the translation of a solution to the same system of inequalities with homogeneous Dirichlet boundary conditions*
- $\sigma(1) \neq \sigma(n)$

Proof. Consider the density $\mathbf{v}$ associated with $\mathbf{u}$. One implication is trivial: if $\mathbf{u}$ is a solution to (2.1), then by Proposition 2.3 there exists a translation which is a Dirichlet solution. Moreover, $\sigma(1) \neq \sigma(m)$, because otherwise, by the maximum principle applied in a neighborhood of the point x between v_1 and v_n, we would have that $u_{\sigma(1)}(x) > 0$, a contradiction.

Suppose now that $\mathbf{u} = (u_i)_i$ is a solution to the Dirichlet problem. We remark that it is clearly also an element of $H^1_{\text{per}}(0,1)$, since $H^1_0(0,1) \subset H^1_{\text{per}}(0,1)$. Fix $i \in \{1, ..., m\}$. We want to prove that the weak formulation of the differential inequalities is also satisfied for any test function in $\mathcal{C}^1_{\text{per}}([0,1])$. By Proposition 2.5, $\{u_i > 0\} = \bigcup_{j=1}^{l}(a_j, b_j)$. Moreover, by standard elliptic regularity, u_i is smooth on each of those open intervals. Therefore, we can apply integration by parts and get that for $\varphi \in \mathcal{C}^1_{\text{per}}([0,1])$, with $\varphi \geq 0$:

$$\int_0^1 u_i'\varphi' dx = \sum_{j=1}^{l} \left[u_i(b_j)\varphi(b_j) - u_i(a_j)\varphi(a_j) - \int_{a_j}^{b_j} u_i''\varphi dx \right] \leq -\sum_{j=1}^{l} \int_{a_j}^{b_j} u_i''\varphi dx = \int_0^1 u_i''\varphi dx.$$

Indeed: $u_i'(b_j) \leq 0$ and $u_i'(b_j) \geq 0$ for every $j \in \{1, ..., l\}$. Now consider for any small $\epsilon > 0$ the following subsets of $[0,1]$: $K_\epsilon := [\epsilon, 1-\epsilon]$, $U_\epsilon := (\frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}) \supset K_\epsilon$. For every $\epsilon > 0$, let η_1^ϵ be the bump function associated to K_ϵ, U_ϵ, and $\eta_2^\epsilon := 1 - \eta_1^\epsilon$. Notice that $\varphi\eta_1^\epsilon \in \mathcal{C}^1_c((0,1))$, So that since $\mathbf{u}$ is a weak Dirichlet solution we have:

$$\int_0^1 u_i''\varphi dx = \int_0^1 u_i''\varphi\eta_1^\epsilon dx + \int_0^1 u_i''\varphi\eta_2^\epsilon dx \leq \int_0^1 (a_i u_i - u_i^2)\varphi\eta_1^\epsilon dx + \int_0^1 u_i''\varphi\eta_2^\epsilon dx.$$

Now for any $\epsilon > 0$, for example, $(a_i u_i - u_i^2)\varphi$ is an integrable majorant for the first term, and $u_i''\varphi$ is for the second one. Moreover:

$$\begin{cases} \eta_1^\epsilon \varphi \to \varphi, \\ \eta_2^\epsilon \varphi \to 0. \end{cases}$$

Therefore, we obtain the following inequality for any $\varphi \in \mathcal{C}^1_{\text{per}}((0,1))$:

$$\int_0^1 u_i''\varphi dx \leq \int_0^1 (a_i u_i - u_i^2)\varphi dx.$$

We now prove the second inequality is also satisfied for periodic test functions, if $\mathbf{u}$ is a solution to the Dirichlet problem. We consider the map σ and the function $\mathbf{v}$ associated to $\mathbf{u}$. Again, we let $\varphi \in \mathcal{C}^1_{\text{per}}([0,1])$, and we will denote as $a_0, ..., a_n$ the zeroes between the connected supports of densities v_{k-1} and v_k. Fixing some $i = 1, ..., m$ and letting the parameter k range in $1, ..., n$, we have:

$$\begin{aligned} \int_0^1 \left(u_i - \sum_{j\neq i} \frac{\beta_{ij}}{\beta_{ji}} u_j \right)' \varphi' dx &= \int_{\{u_i > 0\}} u_i'\varphi' dx - \sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} \int_{\{u_i > 0\}} u_j'\varphi' dx = \\ = \sum_{k:\sigma(k)=i} \left[\int_{a_{k-1}}^{a_k} v_k''\varphi - [v_k'\varphi]_{a_{k-1}}^{a_k} \right] &- \sum_{j\neq i} \frac{\beta_{ij}}{\beta_{ji}} \sum_{k:\sigma(k)=j} \left[\int_{a_{k-1}}^{a_k} v_k''\varphi - [v_k'\varphi]_{a_{k-1}}^{a_k} \right] = \\ = \int_0^1 \left(u_i - \sum_{j\neq i} \frac{\beta_{ij}}{\beta_{ji}} u_j \right)'' \varphi dx &+ \left[-\sum_{k:\sigma(k)=i} [v_k'\varphi]_{a_{k-1}}^{a_k} + \sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} \sum_{k:\sigma(k)=j} [v_k'\varphi]_{a_{k-1}}^{a_k} \right]. \end{aligned}$$

We now aim at showing that the term:

$$-\sum_{k:\sigma(k)=i}[v_k'\varphi]_{a_{k-1}}^{a_k}+\sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}\sum_{k:\sigma(k)=j}[v_k'\varphi]_{a_{k-1}}^{a_k} \tag{2.6}$$

is nonnegative. We now define the following coefficients:

$$\lambda_k^{\pm}=\begin{cases}\mp 1 & \text{if} \quad \sigma(k)=i,\\ \pm\dfrac{\beta_{i,\sigma(k)}}{\beta_{\sigma(k),i}} & \text{if} \quad \sigma(k)\neq i.\end{cases}$$

By re-ordering the terms of the sum, we can express the term (2.6) as:

$$[\lambda_1^- v_1'(a_0)+\lambda_n^+ v_n'(a_n)]\varphi(a_0)+\sum_{k=1}^{n-1}[\lambda_{k-1}^+ v_{k-1}'(a_k)+\lambda_k^- v_k'(a_k)]\varphi(a_k). \tag{2.7}$$

Notice that every evaluation of φ is nonnegative, and we used that $\varphi(a_0)=\varphi(a_n)$. For each term in the sum there are two possibilities: either one between k and $k+1$ is such that $\sigma=i$, or both have $\sigma\neq i$. In the first case, the term vanishes; in the latter, the signs combine so that it is nonnegative. The term outside the sum can be treated in the same way because, by hypothesis, $\sigma(1)\neq\sigma(k)$. Therefore, we have:

$$\int_0^1\left(u_i-\sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}u_j\right)'\varphi' dx\geq\int_0^1\left(u_i-\sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}u_j\right)''\varphi dx.$$

Now, arguing with bump functions as before, we are able to show that:

$$\int_0^1\left(u_i-\sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}u_j\right)'\varphi' dx\geq\int_0^1\left[a_iu_i-u_i^2-\sum_{j\neq i}\frac{\beta_{ij}}{\beta_{ji}}\left(a_ju_j-u_j^2\right)\right]\varphi dx.$$

which completes the proof. □

We remark that the existence for the Dirichlet problem is reduced in [28] to existence of solutions to an algebraic problem. Thanks to the above theorem, to ensure existence for the periodic problem, it is necessary to add to the previous algebraic problem the condition that $\sigma(k)=\sigma(i)$. However, in order to perform the analysis of the long time behavior of solution to the parabolic singular system, we are not really interested in defining precisely the structure of the set of stationary solutions $\mathcal{S}$, but rather its qualitative features. By

Proposition 2.2, $\mathcal{S}$ includes the set of solutions to (2.2), which is clearly translation-invariant. In [28], the authors prove that for the Dirichlet problem, there exists $\tilde{M} \in \mathbb{N}$ such that:

- for any m with $1 \leq m \leq \tilde{M}$ there exists at most one solution $\mathbf{u}_m = (u_1, ..., u_m)$, up to permutation of the supports, such that each component is not identically zero
- for any m with $\tilde{M} \leq m$, there exists no positive solution.

This ensures that there exists only a finite number of solutions. However, we remark that not every possible permutation necessarily leads to a new solution for the Dirichlet problem. By proposition 2.7, for every solution to the Dirichlet problem $\mathbf{u}$ there exists a connected component A of $\mathcal{S}_{sym}$ such that:

$$\Gamma := \{\mathbf{u}(\cdot - c) : \quad c \in [0, 1]\} \subset A$$

Moreover, we can prove that the opposite inclusion holds because of the fact that the Dirichlet problem for the same equation has only finitely many solutions. We only consider solutions $\mathbf{u}$ for which $m > 1$. Suppose by contradiction that there exists an element $\mathbf{v} \in A \setminus \Gamma$ that is not a translation of $\mathbf{u}$. Then $\mathbf{v}$ is joined to $\mathbf{u}$ by a continuous curve $\gamma \subset A$. Therefore, we can construct a sequence $\mathbf{v}_n \to \mathbf{u}$ such that $\mathbf{v}_n \in A \setminus \Gamma$ for every $n \geq 1$. consider now the sets $\Gamma_n := \{\sigma_c \mathbf{v} : \quad c \in [0, 1]\} \subset A$ and set as convention $\Gamma_0 = \Gamma$. Now any pair of sets in the family $\{\Gamma, \Gamma_1, ...\Gamma_2\}$ is either disjoint or coincident. Indeed, consider for example the pair Γ, Γ_1: suppose $\delta := \inf_{\mathbf{u} \in \Gamma, \mathbf{v_1} \in \Gamma_1} \|\mathbf{u} - \mathbf{v_1}\|_{H^1_{\text{per}}(0,1)} = 0$. The distance must be equal to $\|\mathbf{u}(\cdot - a) - \mathbf{v}(\cdot - b)\|$ for two $a, b \in [0, 1]$. But then $\mathbf{v}$ would be a translation of $\mathbf{u}$, so $\delta > 0$. However, since $m > 1$, for any set Γ there exists a representative in $H^1_0(0, 1)$. as shown before, this is a solution to the Dirichlet problem. Therefore, there is only a finite number of indices i such that the corresponding sets Γ_i are disjoint. Hence, we must have $\Gamma_j = \Gamma_0$ for infinitely many indices j, which contradicts the fact that $\mathbf{v}_n \subset A \setminus \Gamma$. Therefore, we must have $A \setminus \Gamma = \emptyset$.

By the previous discussion, we can conclude that the set of stationary solutions to (2.1) consists of a finite number of connected components A that take one of the following forms:

- $A = \{0\}$
- $A = \{a_i\}$ for some $i \in \{1, ..., M\}$
- $A = \Gamma = \{\mathbf{u}(\cdot - c) : \quad c \in [0, 1]\} \subset A\}$ for some $\mathbf{u}$ solution to the Dirichlet boundary problem such that $\sigma(1) \neq \sigma(n)$.

3 | Analysis of the singular parabolic system

In this chapter we study the dynamics of the singular parabolic system. First, we extend some useful properties of solutions, concerning extinction times and regularity of the free boundary, to the case of periodic boundary conditions, with general assumptions on coefficients a_i, β_{ij}. We will, however, occasionally remark some differences occurring between the cases with cyclic and acyclic interactions. These properties were proved for the Dirichlet boundary condition case in [28], and extension does not usually require much effort. However, our treatment differs from that in [28] for some aspects, especially concerning space-time multiplicity of points and a simplified proof of the time-regularity of the boundary evolution. The system under investigation is the following:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} \leq a_i u_i - u_i^2 & \forall x \in (0,1), \quad t > 0, \\ \left(\dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}\right)\left(u_i - \displaystyle\sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} u_j\right) \geq & \\ \qquad\qquad \geq a_i u_i - u_i^2 - \displaystyle\sum_{j \neq i} \frac{\beta_{ij}}{\beta_{ji}} \left(a_j u_j - u_j^2\right) & \forall x \in (0,1), \quad t > 0, \\ u_i u_j \equiv 0 & \forall \quad j \neq i, \quad x \in (0,1), \quad t > 0, \\ u_i(x,0) = \phi_i(x) & \forall \quad x \in (0,1), \\ u_i(0,t) = u_i(1,t) & \forall \quad t > 0. \end{cases} \tag{3.1}$$

All the indices take values in $1, \ldots, M$. The coefficients a_i, β_{ij} are as in (2.1), and the initial conditions $\phi_i(x)$ are nontrivial, nonnegative Lipschitz continuous functions such that $\phi_i(x)\phi_j(x) \equiv 0 \quad \forall \quad i \neq j$. We will often refer to these functions as admissible initial states. Notice that existence of weak solutions to the system (3.1) is guaranteed by considering solutions to the associated competition diffusion system (1.1) for finite κ and taking the limit as $\kappa \to \infty$, as done in Section 1.2. Moreover, solutions arising as singular limits of the competition-diffusion system are defined globally in time and they are globally Lipschitz continuous by the discussion at the end of the same section. Therefore, we will carry out the analysis based on this regularity hypothesis. As in the previous chapters, we consider the differential inequalities to be intended in a weak sense. As test space, we consider the space:

$$V := \{\psi \in \mathcal{C}_c^\infty(\mathbb{T} \times (0, +\infty)) : \quad \psi \geq 0\}, \tag{3.2}$$

which only differs from the choice (1.19) made in section 1.2 for the fact the fact that we are now considering $T = +\infty$. Therefore, for example, the first inequality of (3.1) for the density u_i reads as follows:

$$-\int_0^\infty \int_{\mathbb{T}} u_i \frac{\partial \psi}{\partial t} + \int_0^\infty \int_{\mathbb{T}} \frac{\partial u_i}{\partial x}\frac{\partial \psi}{\partial x} \leq \int_0^\infty \int_{\mathbb{T}} (a_i u_i - u_i^2)\psi \qquad \forall \quad \psi \in V. \tag{3.3}$$

3.1. General properties of the free boundary

For a solution $\mathbf{u}$, we define the free boundary as $\mathcal{F}(\mathbf{u}) := \cup_{i=1}^M \partial\{u_i > 0\}$. Notice that, with respect to the previous chapter, we now intend $\{u_i > 0\}$ as a subset of $\mathbb{T} \times [0, +\infty)$, according to the fact that u_i is now a function of both space and time. Since u_i is continuous, moreover, the set $\{u_i > 0\}$ is open. The following result is based on a blow up argument. The idea is to construct for each i a blow up sequence that converges up to subsequences to a function v_i satisfying parabolic differential inequalities and which is homogeneous in the sense that there exists a positive integer d such that $v_i(\lambda x, \lambda^2 t) = \lambda^d v_i(x,t)$ for any $\lambda > 0$. This allows to derive a contradiction with an estimate of the Hausdorff dimension of the nodal set of solutions to the parabolic inequalities satisfied by v_i. This argument is completely local in nature and, therefore, it extends naturally to our setting. For the proof, we refer to [28, Proposition 3.2].

Proposition 3.1. *For each $T > 0$, $\mathcal{F}(\mathbf{u}) \cap \{t = T\}$ contains only finitely many points.*

We remark that the above proposition holds even when we consider an initial state with infinitely many connected components, and this allows to rule out the possibility that the number of connected components blows up in finite time. At this point, the function $m : (0, +\infty) \to \mathbb{N}$, which counts the total number of connected components in a time-slice of the free boundary, is well defined with values in $\mathbb{N}$:

$$m(T) = \#\{\text{connected components of } \cup_i (\{u_i > 0\} \cap \{t = T\})\}.$$

Indeed, an infinite value for $m(T)$ would contradict proposition 3.1. We now observe that, on each open set $\{u_i > 0\}$, thanks to condition $u_i u_j = 0$ for every $j \neq i$, u_i satisfies the logistic equation. Therefore, by the standard theory of parabolic regularity, since the logistic nonlinearity is smooth, each u_i is actually smooth on its support. In what follows, we outline the argument. We will refer to the book [15] (and i particular Chapter III) by Ladyzhenskaya, Solonnikov and Ural'ceva, where parabolic regularity theory is treated in great generality and detail.

Proposition 3.2. *Let* $\mathbf{u}$ *be a solution to the system* (3.1). *Then, for each* i, $u_i \in C^\infty(\{u_i > 0\})$ *and solves the logistic equation on* $\{u_i > 0\}$.

Proof. Since $\{u_i > 0\}$ is an open set, for any $(x_0, t_0) \in \{u_i > 0\}$ there exist $\epsilon, \delta > 0$ such that the small cylinder $Q_{\epsilon,\delta} = (x_0 - \epsilon, x_0 + \epsilon) \times (t_0 - \delta, t_0 + \delta)$ is included in $\{u_i > 0\}$. Since u_i is positive there, $u_j \equiv 0$ for any $j \neq i$. Therefore, the first and second equations of system (3.1) become, thanks to their weak formulation:

$$\begin{cases} \displaystyle\int_{Q_{\epsilon,\delta}} -u_i\varphi_t + u_{i,x}\varphi_x \leq \int_{Q_{\epsilon,\delta}} (a_i u_i - u_i^2)\varphi \quad \forall \varphi \in C^\infty(\overline{Q_{\epsilon,\delta}}), \quad \varphi \geq 0 \\ \displaystyle\int_{Q_{\epsilon,\delta}} -u_i\varphi_t + u_{i,x}\varphi_x \geq \int_{Q_{\epsilon,\delta}} (a_i u_i - u_i^2)\varphi \quad \forall \varphi \in C^\infty(\overline{Q_{\epsilon,\delta}}), \quad \varphi \geq 0. \end{cases}$$

Now, since any $\varphi \in C^\infty(\overline{Q_{\epsilon,\delta}})$ can be written as a sum of two functions $\varphi_1 \geq 0, \varphi_2 \leq 0$, each belonging to $C^\infty(\overline{Q_{\epsilon,\delta}})$, by combining the two inequalities we get the following:

$$\int_{Q_{\epsilon,\delta}} -u_i\varphi_t + u_{i,x}\varphi_x = \int_{Q_{\epsilon,\delta}} (a_i u_i - u_i^2)\varphi \quad \forall \varphi \in C^\infty(\overline{Q_{\epsilon,\delta}}). \tag{3.4}$$

We know by hypothesis that u_i is Lipschitz continuous on $\mathbb{T} \times (0, +\infty)$, and so in particular has the same regularity in $Q_{\epsilon,\delta}$. As standard in regularity theory, we will consider the right hand side in (3.4) as a given function of (x,t), i.e. $f_i(x,t) = a_i u_i - u_i^2$. Notice it is still Lipschitz continuous in $Q_{\epsilon,\delta}$. Now by assumption, u_i belongs to the space:

$$L^2([t_0 - \delta, t_0 + \delta]; H^1(x_0 - \epsilon, x_0 + \epsilon)) \cap \mathcal{C}([t_0 - \delta, t_0 + \delta]; L^2(x_0 - \epsilon, x_0 + \epsilon)).$$

Moreover, if we define:

$$\mathcal{T} := \left\{ \varphi \in L^2(Q_{\epsilon,\delta}) : \int_{Q_{\epsilon,\delta}} \varphi^2 + \varphi_x^2 + \varphi_t^2 < +\infty, \quad \varphi = 0 \text{ on } \partial Q_{\epsilon,\delta} \right\},$$

we have that the weak formulation (3.4) is satisfied for every test function $\varphi \in \mathcal{T}$, by a density argument. Therefore, u_i is a weak solution to the equation:

$$u_{i,t} - u_{i,xx} = f_i \qquad\qquad \text{in } Q_{\epsilon,\delta},$$

in the sense used in [15]. In particular, we refer to Chapter III.1 for the notion of generalized solution that we are using. We are thus able to apply Theorem 2.1 in Chapter III.12 of [15], to derive that $u_i \in \mathcal{C}^{2+2\delta,1+\delta}(Q_{\epsilon,\delta})$ for any $\delta \in [0, \frac{1}{2})$.

This result relies on a classical procedure in regularity theory. First, estimates for Hölder norms of solutions to weak formulations with respect to Hölder norms of the datum f_i are derived under additional regularity hypotheses on u_i. Then, these estimates are applied to a family of regularizations of u_i which solve a regularized problem, and converge to u_i. The uniformity of the estimates allows then to conclude that the same regularity holds for the original solution u_i. For the precise argument we refer again to Chapter III.12 of [15].

To derive higher regularity for u_i, we can iterate the procedure by noticing that the $\mathcal{C}^{2+2\delta,1+\delta}$ regularity of u_i reflects on that of $f_i(x,t) = a_i u_i - u_i^2$, and this in turn allows one to derive further regularity for u_i. By iterating this process, and by the generality of (x_0, t_0), it is possible to deduce that $u_i \in C^\infty(\{u_i > 0\})$. □

We now state and prove a property that roughly asserts that any connected component of $\{u_i > 0\} \cap \{t = T\}$ must intersect any previous time slice.

Proposition 3.3. *For every $T_1 > T_2 > 0$, each connected component of $\{u_i > 0\} \cap \{t < T_1\}$ must intersect with $\{t = T_2\}$.*

Proof. Since $\{u_i > 0\}$ is an open set, it is locally path connected, and the connected components coincide with the path connected ones. Assume by contradiction that there exists a connected component A of $\{u_i > 0\}$ such that $A \cap \{t = T_2\} = \emptyset$. Now consider the following restriction $\hat{u}_i : \mathbb{T} \times [T_2, T_1] \to \mathbb{R}$ of u_i to A:

$$\hat{u}_i := \begin{cases} u_i & (x,t) \in A \cap (T_2 < t \leq T_1), \\ 0 & \qquad\qquad\text{otherwise.} \end{cases}$$

Such a function is smooth on $A \cap \{T_2 < t < T_1\}$ because of Proposition 3.2, and it is continuous on $\overline{A \cap (T_2 < t < T_1)}$. Now, let $C > a_i$ and define $v(x,t) := e^{-Ct}\hat{u}_i(x,t)$. Then, $v > 0 \iff \hat{u}_i > 0$. Moreover, v is also smooth on the open set where $\hat{u}_i > 0$, and continuous up to its boundary. Therefore, it must reach its maximum in a point (x_0, t_0) where $v_i > 0$, and in particular either in $A \cap \{T_2 < t < T_1\}$ or in $A \cap \{t = T_1\}$. Moreover:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = e^{-Ct}\left((a_i - C)\hat{u}_i - \hat{u}_i^2\right) < 0 \qquad \text{on } A \cap \{T_2 < t < T_1\}. \tag{3.5}$$

In particular, there always exists a small cylinder $Q_{\epsilon,\delta} = (x_0 - \epsilon, x_0 + \epsilon) \times (t_0 - \delta, t_0) \subset A \cap \{T_2 < t < T_1\}$ such that (x_0, t_0) does not belong to $\partial_p Q_{\epsilon,\delta}$. Indeed, even if $(x_0, t_0) \in A \cap \{t = T_1\}$, by the positivity and continuity of v there must exist some $\epsilon > 0$ such that $(x_0 - \epsilon, x_0 + \epsilon) \subset A \cap \{t = T_1\}$. By applying the strong maximum principle to v in $Q_{\epsilon,\delta}$, we obtain that v is constant there, contradicting (3.5). Therefore, there cannot exist a connected component A such that $A \cap \{t = T_2\} = \emptyset$.

□

To prove that m is decreasing, it is still necessary to prove that one initial connected component cannot split into two. This is done in the following proposition.

Proposition 3.4. *Suppose that $\{u_i > 0\} \cap \{t = T_1\}$ is an interval for each i. For every $T > T_1$, $\{u_i > 0\} \cap \{t = T\}$ is either empty or still an interval.*

Proof. By contradiction, suppose there exists $T > T_1$ such that $\{u_i > 0\} \cap \{t = T\}$ has two connected components $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ with $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$. We remark that the case $\beta_1 = \alpha_2$ cannot occur because of a simple application of the maximum principle. Moreover, without loss of generality, $(\beta_1, \alpha_2) \cap \{u_i > 0\} = \emptyset$ (This depends on the fact that m(T) is finite). By Proposition 3.3, there exist two continuous paths $\gamma_1(t), \gamma_2(t)$ joining some point in $\{u_i > 0\} \cap \{t = T_1\}$ with two points belonging to $(\alpha_1, \beta_1) \times \{T\}$ and $(\alpha_2, \beta_2) \times \{T\}$, respectively. Similarly to the elliptic case, there exists $j \neq i$ such that $((\beta_1, \alpha_2) \times \{T\}) \cap \{u_j > 0\} \neq \emptyset$. Indeed, if we suppose by contradiction that $u_j \equiv 0$ on the set delimited by the curves $\gamma_1(t), \gamma_2(t)$ for every $j \neq i$, we have that u_i solves the logistic equation in the same set. but then, by the maximum principle, it must be zero everywhere. Therefore, $u_j > 0$ on some open set. However, that set is contained in a connected component that cannot be connected to any point in $\{t = T_1\}$, giving a contradiction with proposition (3.3).

□

We remark that the hypothesis made in the above proposition that $\{u_i > 0\} \cap \{t = T_1\}$ is an interval for each i does not lead to any loss of generality: since $m(T)$ is finite for any positive time, It is possible to consider any distinct connected component of $\{u_i > 0\}$ as a new density in the system. Thus, at the expense of increasing the size of the system, and properly defining the parameters for the new densities, we can always assume that $\{u_i > 0\} \cap \{t = T_1\}$ is an interval for each i. At this point, we notice that the quantity $m(T)$ is unaffected by this convention. By Proposition 3.4, it is clear that $m(T)$ is a non-increasing function of time, and therefore we can define a sequence of times $\{T_k\}_{k\in\mathbb{N}}$ such that $T_{k+1} < T_k$ for any $k \geq 1$, as all the points where $m(T)$ changes value; we will refer to them as extinction times. since $m(T)$ is finite for any positive time T, the sequence can only accumulate at zero. Moreover, the set is countable only if the set $\bigcup_{i=1}^{M}\{\phi_i > 0\}$ has infinitely many connected components, and has finite cardinality otherwise. We also define by convention $T_0 := +\infty$. Since $m(T)$ is constant for any $T \in (T_{k+1}, T_k)$, we will refer to it simply as m when we are restricting our analysis to a specific interval of the form (T_{k+1}, T_k).

It is now convenient to introduce a reference domain for the space-time cylinder $\mathbb{T} \times (T_{k+1}, T_k)$. One possibility is to represent it with $[0, 1) \times (T_{k+1}, T_k)$, but it will be more convenient for us to make a choice which is compatible with the dynamics. To this end, we consider the periodic extension $\mathbf{u}^{\text{per}}$ of $\mathbf{u}$ to $\mathbb{R} \times (T_{k+1}, T_k)$, and define the set $\Theta \subset \mathbb{R} \times (T_{k+1}, T_k)$ as follows:

$$\Theta := \bigcup_{i=1}^{m} \overline{\{\text{connected components of } \{u_i^{\text{per}} > 0\} \text{ intersecting } [0,1) \text{ at } t = T_{k+1}\}}. \tag{3.6}$$

Here, without loss of generality, we suppose that the identification of $[0, 1)$ with $\mathbb{T}$ is chosen in such a way that $u_i(0, T_{k+1}) = 0$ for every i. This assumption will be implicitly considered in what follows. With a little abuse of notation, we will equivalently refer to $\mathbf{u}$ as the solution on $\mathbb{T}\times(T_{k+1}, T_k)$ or its representation on Θ given by $\mathbf{u}^{\text{per}}|_\Theta$ depending on the more convenient choice. This allows us to consider naturally a spatial ordering of points. As a consequence of Proposition 3.3, one can prove that connected components of the supports of two different densities cannot interweave, meaning that if $\{u_i > 0\} \cup \{t = T\}$ lies on the left or on the right of $\{u_j > 0\}\cup\{t = T\}$ for some $T \in (T_{k+1}, T_k)$, then the same must hold for any other $t \in (T_{k+1}, T_k)$. Therefore, in any interval (T_{k+1}, T_k), where m is constant, it is possible to re-order the indices of densities in such a way that the support of u_i is adjacent to those of u_{i-1} and u_{i+1}. Finally, we see that for any $T \in (T_{i+1}, T_i)$ there exist $\alpha_0(T), ..., \alpha_{m-1}(T)$, $\beta_0(T), ..., \beta_{m-1}(T)$ such that:

$$\begin{cases} \alpha_i(T) < \beta_i(T) \leq \alpha_{i+1}(T), \\ \{u_i > 0\} \cap \{t = T\} = (\alpha_{i-1}(T), \beta_{i-1}(T)). \end{cases}$$

Moreover, in this chapter we will occasionally refer to the cyclic condition on the interval (T_{k+1}, T_k). Thanks to the above assumptions, since each set $\{u_i > 0\}$ is connected in $\mathbb{T} \times (T_{k+1}, T_k)$, The cyclic condition is simply:

$$\prod_{i=1}^{m-1} \frac{\beta_{i,i+1}}{\beta_{i+1,i}} = \frac{\beta_{1m}}{\beta_{m1}}.$$

We will also make use of the quantity:

$$\tilde{\beta} := \frac{\beta_{m1}}{\beta_{1m}} \prod_{i=1}^{m-1} \frac{\beta_{i,i+1}}{\beta_{i+1,i}}, \tag{3.7}$$

in order to state in a more compact way the cyclic condition as $\tilde{\beta} = 1$. From now on, we will consider all the conventions introduced above. Now, we will state and prove some results concerning multiplicity of points which are not explicitly considered in [28]. First, we define the following quantity for every $(x_0, t_0) \in \mathbb{T} \times [0, +\infty)$:

$$n(x_0, t_0) := \#\{i = 1, ..., m \quad s.t. : \forall \epsilon > 0, \{u_i > 0\} \cap B_\epsilon(x_0, t_0) \neq \emptyset\},$$

which is often referred to as the multiplicity of points of the free boundary. For example, we refer to [5] for their definition in a multi-dimensional elliptic context, and notice that with respect to that definition, we are considering in some sense a space-time multiplicity. We observe that if $n(x_0, t_0) \geq 2$, by the continuity of functions u_i we must have $u_i(x_0, t_0) = 0$ for every $i \in \{1, ..., m\}$. Moreover, we have the following proposition, which ensures that for all points (x_0, t_0) with $t_0 \in (T_{k+1}, T_k)$, $n(x_0, t_0) \leq 2$. The obstruction is fundamentally of topological type.

Proposition 3.5. *Let $t_0 \in (T_{k+1}, T_k)$ for some k. Then, for every $x_0 \in \mathbb{T}$, $n(x_0, t_0) \leq 2$.*

Proof. Suppose, by contradiction, that there exists a point (x_0, t_0) with $n(x_0, t_0) \geq 3$. As noted before, we must have $u(x_0, t_0) = 0$. Therefore, the point lies in some interval of the form $(\beta_{i-1}(t_0), \alpha_i(t_0))$ for some $i \in \{1, ..., m\}$. Now by Proposition 3.4 and since no extinction occurs by hypothesis in the time interval (T_{k+1}, T_k), there exist two continuous paths $\gamma_{i-1}(t)$ and $\gamma_i(t)$ in $\{u_{i-1} > 0\}$ and $\{u_i > 0\}$ respectively joining points in $\{t = T_{k+1}\}$ to points in $\{t = T_k - \epsilon\}$, for any small $\epsilon > 0$. Moreover:

$$\alpha_{i-1}(t) < \gamma_{i-1}(t) < \gamma_i(t) < \beta_i(t) \qquad \text{for every } t \in (T_{k+1}, T_k). \tag{3.8}$$

Now (x_0, t_0) belongs to the open set:

$$\Sigma := \{(x,t) \in \mathbb{T} \times (T_{k+1}, T_k - \epsilon) : \quad \gamma_{i-1}(t) < x < \gamma_i(t)\}.$$

but because of (3.8), for any $j \neq i, i+1$, u_j must vanish identically on Σ, which is in contradiction with $n(x_0,t_0) \geq 3$. Therefore, we must have $n(x_0,t_0) \geq 2$. $\square$

Moreover, by the maximum principle, it is possible to prove, with an argument similar to that of Proposition (3.3), that there cannot exist points (x_0,t_0) such that $n(x_0,t_0) = 0$. Therefore, for any $(x,t) \in \mathcal{F}(\mathbf{u})$ such that $t \in (T_{k+1}, T_k)$, we must have $n(x,t) = 2$. Indeed, we know by Proposition 3.5 and the discussion before that $1 \leq n(x,t) \leq 2$, but if $n(x,t)$ was equal to one, we would be able to show existence of points $(\overline{x}, \overline{t})$ with $n(\overline{x}, \overline{t}) = 0$.

3.2. Transmission conditions

We already remarked in the previous section that each density is smooth on its own support. Now we show that it is also possible to derive a transmission condition for points of the free boundary. Those points have multiplicity equal to two, and by the convention mentioned in the last paragraph, they lie at the interface between two densities u_i, u_{i+1} for some $i \in \{1, ..., M\}$.

Proposition 3.6. *For any $(x,t) \in \mathcal{F}(\mathbf{u}) \cap \{T_{k+1} < t < T_k\}$:*

$$u := u_i - \frac{\beta_{i,i+1}}{\beta_{i+1,i}} u_{i+1} \quad \textit{is } C^{2+2\delta, 1+\delta} \textit{ in a neighborhood of } (x,t)$$

for any $\delta \in [0, \frac{1}{2})$.

Proof. Since each point $(x_0,t_0) \in \mathcal{F}(\mathbf{u}) \cap \{T_{k+1} < t < T_k\}$ has multiplicity equal to 2, there exists a small cylinder $Q_{\epsilon,\delta} = (x_0 - \epsilon, x_0 + \epsilon) \times (t_0 - \delta, t_0 + \delta)$ and two indices i, j such that $u_h \equiv 0$ in $Q_{\epsilon,\delta}$ for any $h \neq i, j$. Moreover, according to the convention adopted, we can take $j = i+1$. Now, we consider the localization to $Q_{\epsilon,\delta}$ of the second inequalities of (3.1) for i and $i+1$:

$$\begin{cases} \displaystyle\int_{Q_{\epsilon,\delta}} -u_i\varphi_t + u_{i,x}\varphi_x - \frac{\beta_{i,i+1}}{\beta_{i+1,i}}\left(u_{i+1}\varphi_t + u_{i+1,x}\varphi_x\right) \geq \\ \qquad\qquad \displaystyle\geq \int_{Q_{\epsilon,\delta}} (a_iu_i - u_i^2 - \frac{\beta_{i,i+1}}{\beta_{i+1,i}}(a_{i+1}u_{i+1} - u_{i+1}^2))\varphi \quad \forall\varphi \in C^\infty(\overline{Q_{\epsilon,\delta}}), \quad \varphi \geq 0, \\ \displaystyle\int_{Q_{\epsilon,\delta}} -u_{i+1}\varphi_t + u_{i+1,x}\varphi_x - \frac{\beta_{i+1,i}}{\beta_{i,i+1}}\left(u_i\varphi_t + u_{i,x}\varphi_x\right) \geq \\ \qquad\qquad \displaystyle\geq \int_{Q_{\epsilon,\delta}} (a_{i+1}u_{i+1} - u_{i+1}^2 - \frac{\beta_{i+1,i}}{\beta_{i,i+1}}(a_iu_i - u_i^2))\varphi \quad \forall\varphi \in C^\infty(\overline{Q_{\epsilon,\delta}}), \quad \varphi \geq 0. \end{cases} \tag{3.9}$$

By changing the sign of the second inequality and multiplying by $\frac{\beta_{i,i+1}}{\beta_{i+1,i}}$, we get:

$$\int_{Q_{\epsilon,\delta}} -u_i\varphi_t + u_{i,x}\varphi_x - \frac{\beta_{i,i+1}}{\beta_{i+1,i}}\left(u_{i+1}\varphi_t + u_{i+1,x}\varphi_x\right) \leq$$
$$\leq \int_{Q_{\epsilon,\delta}} (a_iu_i - u_i^2 - \frac{\beta_{i,i+1}}{\beta_{i+1,i}}(a_{i+1}u_{i+1} - u_{i+1}^2))\varphi \quad \forall\varphi \in C^\infty(\overline{Q_{\epsilon,\delta}}), \quad \varphi \geq 0.$$

Combining the above inequality with the first one in (3.9), we have that the equation is satisfied for any non-negative test function in $C^\infty(\overline{Q_{\epsilon,\delta}})$, but then it is trivial to show that it is also satisfied for any sign-changing test function. Therefore, the function $\hat{u} := u_i - \frac{\beta_{i,i+1}}{\beta_{i+1,i}}u_{i+1}$ is a weak solution on $Q_{\epsilon,\delta}$ of the following equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\hat{u} = a_iu_i - u_i^2 - \frac{\beta_{i,i+1}}{\beta_{i+1,i}}(a_{i+1}u_{i+1} - u_{i+1}^2).$$

Since the right hand side is Lipschitz, we can apply the same argument as before to derive that $\hat{u}$ is $\mathcal{C}^{2+2\delta,1+\delta}$ in a neighborhood of (x_0, t_0). In particular, we have the following transmission condition, where the partial derivatives are intended in a unilateral sense:

$$\frac{\partial u_i}{\partial x}(\alpha_i(t), t) = -\frac{\beta_{i,i+1}}{\beta_{i+1,i}}\frac{\partial u_{i+1}}{\partial x}(\alpha_i(t), t) \qquad\qquad \forall \quad t \in (T_{k+1}, T_k).$$

$\square$

Thanks to the previous result, we can construct a regular, sign-changing density $u : \mathbb{R} \times (T_{k+1}, T_k) \to \mathbb{R}$ as a linear combination of the components u_i. For two integers

$n \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}$, we will use the notation: $n \bmod m := n - m\lfloor \frac{n}{m} \rfloor$. Let us first define the following aggregated coefficients recursively for any $i \in \mathbb{Z}$:

$$\beta^{(i)} := \begin{cases} 1 & \text{for} \quad i = 1, \\ \beta^{(i-1)} \dfrac{\beta_{(i-1) \bmod m, i \bmod m}}{\beta_{i \bmod m, (i-1) \bmod m}} & \text{for} \quad i \geq 2, \\ \beta^{(i+1)} \dfrac{\beta_{i \bmod m, (i-1) \bmod m}}{\beta_{(i-1) \bmod m, i \bmod m}} & \text{for} \quad i \leq 0, \end{cases} \tag{3.10}$$

with the convention that 0 and m are equivalent indices for β_{ij}, so that, for example, we intend β_{01} as β_{m1}. Notice that $\beta^{(m+1)} = \tilde{\beta}$. We can now define a density u as follows:

$$u(x,t) := \sum_{i \in \mathbb{Z}} (-1)^{i-1} \beta^{(i)} u_{i \bmod m} \left(x - \frac{i \bmod m}{m}, t \right), \tag{3.11}$$

where the sum over $\mathbb{Z}$ is well defined because for any (x,t) there is actually only one i giving a nonzero contribution. It is clear from the previous discussion that $u \in \mathcal{C}^{2,1}(\mathbb{R} \times (T_{k+1}, T_k))$. Moreover, in the case of cyclic interactions, the function u is periodic with period one if m is even, two if m is odd. In particular, it is bounded. However, in the general case, it grows exponentially as $x \to \pm\infty$ and tends to zero as $x \to \mp\infty$, depending on the sign of $\tilde{\beta} - 1$. Nevertheless, it satisfies a Tychonoff-type growth condition, in the sense that there exists $C, \alpha > 0$ such that:

$$|u(x,t)| \leq C e^{\alpha |x|^2} \qquad \forall \quad (x,t) \in \mathbb{R} \times (T_{k+1}, T_k). \tag{3.12}$$

In all cases, u also satisfies in the classical sense the equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = V(x,t) u \qquad \text{for } (x,t) \in \mathbb{R} \times (T_{i+1}, T_i), \tag{3.13}$$

where:

$$V(x,t) := \sum_{i \in \mathbb{Z}} (-1)^{i-1} \left[a_i - u_{i \bmod m} \left(x - \frac{i \bmod m}{m}, t \right) \right].$$

Notice that $V(x,t)$ is a bounded function since the functions u_i are bounded. Moreover, if we consider cyclic interactions, the periodic boundary conditions are obviously satisfied on $[0,1]$ if m is even and $[0,2]$ if m is odd. Hence, considering the case m even, a proper restriction of u can be considered a classical $\mathcal{C}^{2+2\delta,1+\delta}([0,1]\times(T_{k+1},T_k))$ solution to (3.13) with periodic boundary conditions, or equivalently, a solution to the equation on the 1-torus $\mathbb{T}$ with regularity $\mathcal{C}^{2+2\delta,1+\delta}(\mathbb{T}\times(T_{k+1},T_k))$.

3.3. Regularity of the free boundary

In this section, we introduce a parametrization $\alpha_i(t)$ of the free boundary and prove a result concerning its regularity in time. With respect to the paper [28], the proof involves the same ideas, but we slightly change it to get all at once the differentiability of the parameterization $\alpha_i(t)$, by exploiting the results in [1]. In that paper, Angenant investigated the behavior of the zero set of one-dimensional parabolic equations on $\mathbb{R}$, such as the equation (3.13) that is solved by our density u. Although the theory of monotonicity of the zero number was substantially developed in earlier works, Angenent's paper is generally regarded as the definitive reference for the modern formulation and proof. In this regard, we mention that a further generalization of his results is provided by Chen in [4]. In what follows, we give the statements of theorems and propositions that will be necessary to study the regularity of the free boundary. We consider solutions w of the following equation, with $S_2>S_1>0$:

$$\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial x^2}+q(x,t)w=0 \qquad \text{in} \quad \mathbb{R}\times(S_1,S_2), \tag{3.14}$$

with $q(x,t)\in L^\infty(\mathbb{R}\times(S_1,S_2))$ and such that the bound (3.12) is satisfied by w; By regularity theory, w, $\frac{\partial w}{\partial x}$ are then locally Hölder continuous functions for any exponent $\delta<1$, so in particular it is possible to evaluate both functions pointwise. We now define the following set, and state some properties.

$$\mathcal{Z}:=\{(x,t)\in\mathbb{R}\times(S_1,S_2): w(x,t)=0\}.$$

Proposition 3.7 (Angenent, [1, Lemma 5.5]). *Let $(x_0,t_0)\in\mathcal{Z}$ be given. Then, there exists at least one continuous curve γ defined on $(0,t_0]$ such that $\gamma(t_0)=x_0$.*

Theorem 3.8 (Angenent, [1, Theorem B]). *If at (x_0,t_0) both w, $\frac{\partial w}{\partial x}$ vanish, then there is a neighborhood $Q_{\epsilon,\delta}=[x_0-\epsilon,x_0+\epsilon]\times[t_0-\delta,t_0+\delta]$ such that:*

- *$w(x_0\pm\epsilon,t)\neq 0$ for $t\in[t_0-\delta,t_0+\delta]$.*
- *$w(\cdot,t+\delta)$ has at most one zero for $x\in[x_0-\epsilon,x_0+\epsilon]$.*

- *$w(\cdot, t-\delta)$ has at leas two zeroes for $x \in [x_0 - \epsilon, x_0 + \epsilon]$.*

Moreover, we will make use of a strong unique continuation theorem, proven in [4] for systems of parabolic inequalities in arbitrary dimension. We will state it in a form that is adapted to our context. In particular, the regularity needed for the result to hold is actually lower and involves local Sobolev spaces.

Theorem 3.9 (Strong Unique Continuation Theorem). *Let w be a solution to (3.14) with $q(x,t) \in L^\infty(\mathbb{R} \times (S_1, S_2))$. Suppose moreover that $w \in \mathcal{C}^{2,1}(\mathbb{R})$ and it satisfies the bound (3.12).*

Then, w vanishes identically in $\mathbb{R} \times (S_1, S_2)$, provided there exists $(x_0, t_0) \in \mathbb{R} \times (S_1, S_2)$ such that:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^m} \int_{|x-x_0|<\epsilon} |u(x,t_0)| dx = 0 \qquad \text{for any } m > 0.$$

Notice that, in particular, Theorem 3.9 ensures that w must vanish on the full set $\mathbb{R} \times (S_1, S_2)$ whenever it vanishes identically on some open interval of $\mathbb{R}$ for some $t \in (S_1, S_2)$. We can now state and prove the main result of this section, for a solution $\mathbf{u}$ to (3.1) as in the previous sections:

Proposition 3.10. *There exist $\alpha_0(t), ..., \alpha_m(t)$ with support in Θ such that:*

$$\{u_i > 0\} \cap \{t = T\} = (\alpha_{i-1}(T), \alpha_i(T)) \qquad \forall \quad T \in (T_{i+1}, T_i),$$

with $\alpha_0 + 1 \equiv \alpha_m$. Moreover, $\alpha_i(t) \in \mathcal{C}^1((T_{k+1}, T_k))$.

Proof. We will now consider $\mathbf{u}$ as represented on the set Θ defined in (3.6). From proposition (3.4), we already know that for each $T \in (T_{i+1}, T_i)$ there exist $\alpha_0(T), ..., \alpha_{m-1}(T)$, $\beta_0(T), ..., \beta_{m-1}(T)$ such that:

$$\begin{cases} \alpha_i(T) < \beta_i(T) \leq \alpha_{i+1}(T), \\ \{u_i > 0\} \cap \{t = T\} = (\alpha_{i-1}(T), \beta_{i-1}(T)). \end{cases}$$

Suppose by contradiction that there exist i, T such that $\beta_{i-1}(T) < \alpha_i(T)$. Then, by the definition of u, there exists some $(x_0, t_0) \in \mathbb{R} \times (T_{k+1}, T_k)$ such that u vanishes identically on an open set including (x_0, t_0). However, since Theorem 3.9 applies to u, this implies that $u \equiv 0$ in $\mathbb{R} \times (T_{k+1}, T_k)$. Even if the definition of the density and the precise form of the equation it solves change across extinction times, we can still apply the unique continuation theorem on (T_{k+2}, T_{k+1}) by using the fact that $u \equiv 0$ on $\mathbb{R} \times \{T_{k+1}\}$. This yields that the density defined on $\mathbb{R} \times (T_{k+2}, T_{k+1})$ vanishes identically. By iterating this argument backwards in time, we get a contradiction with the assumption that the initial condition is nontrivial. Therefore, we must have:

$$\beta_{i-1}(T) = \alpha_i(T) \qquad\qquad \text{for } i = 2, ..., m \text{ and } T \in (T_{k+1}, T_k).$$

Thus, we can identify the nodal set of u with the set:

$$\cup_i \{(\alpha_i(t), t) : \quad t \in (T_{k+1}, T_k)\}. \tag{3.15}$$

For any i, consider the sets $\{u_{i-1} > 0\}$ and $\{u_i > 0\}$. As in Proposition 3.4, there are paths in both open sets connecting $\{t = T_{k+1}\}$ to $\{t = T_k\}$. Now, for any T, $\alpha_i(T)$ is the unique zero with $t = T$ in the open set delimited by the two paths. By Proposition 3.7, for any t_0, there exists at least one continuous curve $\gamma(t)$ in $\mathcal{Z}$ with $\gamma(t_0) = \alpha_i(t_0)$ defined on $(0, t_0]$. However, because of the identification of $\mathcal{Z}$ with (3.15) and since for any $t < t_0$ the only zero is $\alpha_i(t)$, the two curves must coincide. hence, $\alpha_i(t) \in \mathcal{C}((T_{k+1}, T_k))$. Now this immediately implies that $\frac{\partial u}{\partial x}(\alpha_i(t), t) \neq 0$ for any $t \in (T_{k+1}, T_k)$, because otherwise, we would get a contradiction with item three in Theorem 3.8. Hence, a straightforward application of the implicit function theorem implies that $\alpha_i(t) \in \mathcal{C}^1((T_{k+1}, T_k))$.

□

Notice that, when considering the function $\mathbf{u}$ as defined on $\mathbb{T} \times (T_{k+1}, T_k)$, the curves $\alpha_0(t)$ and $\alpha_m(t)$ coincide.

3.4. A uniqueness result

As remarked at the beginning of this chapter, existence of solutions to system (3.1) is usually guaranteed by the construction of a converging limiting sequence made of solutions to the associated system of parabolic PDEs for $\kappa \to \infty$. Up to now, we have not provided any uniqueness result for the singular parabolic system, and such a property is irrelevant for the proof of all the above results. However, we here observe for the sake of completeness that the proof of uniqueness provided in [28] for Dirichlet boundary conditions can easily be extended to evolutions of initial data that satisfy the cyclic condition on the coefficients. We also remark that this proof only works under the hypothesis for the initial condition

ϕ that $\bigcup_{i=1}^m \{\phi_i > 0\}$ has a finite number of connected components, which however is considered to be reasonable from a modeling point of view. In the following discussion, we will again suppose, without loss of generality, that each set $\{\phi_i > 0\}$ is an interval. Notice that, since we cannot control the possible extinctions that may occur along the evolution of ϕ, to be able to apply in a straightforward way the local argument used in [28] to the entirety of the evolution, we will need to ask that not only the cyclic condition is satisfied for the ordered sequence of indices appearing in the initial datum, but also for each of its subsequences that preserve the original ordering, i.e. for every subsequence obtained by the original one by neglecting an arbitrary number of indices.

Proposition 3.11. *Consider system* (3.1)*, and suppose that the initial condition ϕ is such that $\bigcup_{i=1}^m \{\phi_i > 0\}$ has a finite number of connected components. Moreover, suppose that the cyclic condition holds in the sense specified above. Then, there is at most one solution.*

Proof. By the above discussion, it is sufficient to prove uniqueness in a local time-interval $[0, \epsilon)$. Suppose that there exist two local solutions $\mathbf{u}, \mathbf{v}$ to (3.1). By the discussion after proposition 3.4 and since $\bigcup_{i=1}^m \{\phi_i > 0\}$ has a finite number of connected components, there exists a time $\epsilon > 0$ such that the number of connected components of $\bigcup_{i=1}^m \{\phi_i > 0\}$ is equal to that of $\bigcup_{i=1}^m \{u_i > 0\}$ and $\bigcup_{i=1}^m \{v_i > 0\}$. Consider the functions u and v defined on $[0, \epsilon)$ via (3.11). Notice that $u - v$, $\frac{\partial^2}{\partial x^2}(u - v)$ belong to $L^1_{loc}(0, 1)$. Therefore, we can apply the Kato Inequality (see [12, Lemma A]) to derive the following:

$$\frac{\partial^2}{\partial x^2}|u - v| \geq \operatorname{sign}(u - v)\frac{\partial^2}{\partial x^2}(u - v) \qquad \text{in } [0, 1],$$

where the inequality is intended in distributional sense, and:

$$\operatorname{sign}(s) = \begin{cases} +1 & \text{if} \quad s > 0, \\ 0 & \text{if} \quad s = 0, \\ -1 & \text{if} \quad s < 0. \end{cases}$$

Moreover:

$$\frac{\partial}{\partial t}|u - v| = \operatorname{sign}(u - v)\frac{\partial}{\partial t}(u - v) \qquad \text{in } [0, 1].$$

Therefore, we have the following:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)|u - v| \leq \operatorname{sign}(u - v)\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(u - v).$$

Now, in a distributional sense, each function u_i satisfies the following equation on the torus for $t \in (0, \epsilon)$:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x^2}\right) u_i = a_i u_i - u_i^2 - \frac{\partial u_i}{\partial x}\delta_{\alpha_{i-1}(t)} + \frac{\partial u_i}{\partial x}\delta_{\alpha_i(t)}.$$

However, because of the transmission conditions, we have that the delta measures cancel out when considering an appropriate combination of components:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) \sum_{i=1}^{m} (-1)^{i-1} \beta^{(i)} u_i = \sum_{i=1}^{m} (-1)^{i-1} \beta^{(i)} (a_i u_i - u_i^2),$$

and the same holds for v_i. Therefore:

$$\begin{aligned}
&\operatorname{sign}(u - v) \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(u - v) = \\
&= \operatorname{sign}(u - v) \sum_{i=1}^{m} (-1)^{i-1} \beta^{(i)} [(a_i u_i - u_i^2) - (a_i v_i - v_i^2)] = \\
&= \operatorname{sign}(u - v) \sum_{i=1}^{m} (-1)^{i-1} \beta^{(i)} [a_i - (u_i + v_i)](u_i - v_i) = \\
&= \operatorname{sign}(u - v) \sum_{i=1}^{m} [a_i - (u_i + v_i)](u - v) = \\
&= \sum_{i=1}^{m} [a_i - (u_i + v_i)]|u - v| \leq \\
&\leq C|u - v|,
\end{aligned}$$

where C depends only on a_i, β_{ij}, ϕ_i. Therefore, we have:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)|u - v| \leq C|u - v|.$$

Moreover, $|u - v| \equiv 0$ for $t = 0$, so by applying the maximum principle we get that $|u - v| \equiv 0$ for $t \in (0, \epsilon]$, and consequently $u_i \equiv v_i$ for any i. $\square$

4 | Long time behavior of solutions: cyclic interactions

In this section, we study the long-time behavior of solutions to system (3.1) in the cyclic interactions case. We will now enumerate a series of working hypotheses that we will consider throughout this chapter. By the discussion in the previous chapter, we know that there always exist a time $T_1 > 0$ such that no extinction occurs in $(T_1, +\infty)$. Since we are interested in the long time behavior of the system, we will work without loss of generality with solutions $\mathbf{u}(x,t)$ defined on $\mathbb{T} \times [0, +\infty)$ such that no extinction occurs at any finite time. We will now denote $\mathbf{u}(x, 0)$ as $\mathbf{u}_0$ as it plays a different role with respect to the true initial conditions of our problem $\phi_i(x)$. Since the number of connected components $m(t)$ is constant in time, moreover, we will simply indicate it with m. We will also use the same convention as the previous chapters that each set $\{u_i > 0\}$ is connected for any $i \in \{1, ..., m\}$. Moreover, we recall that, in this context, the cyclic condition for a solution $\mathbf{u}$ reads as follow:

$$\prod_{i=1}^{m-1} \frac{\beta_{i,i+1}}{\beta_{i+1,i}} = \frac{\beta_{1m}}{\beta_{m1}}, \tag{4.1}$$

or, equivalently, $\tilde{\beta} = 1$ (see (3.7)). The last assumption of cyclic interactions is crucial for the method we use, because it guaranties the existence of an energy functional satisfying a dissipation inequality along trajectories, similarly to the Dirichlet case treated in [28]. Therefore, our set of hypotheses for a solution $\mathbf{u}$ to (3.1) is the following:

(H1) $\mathbf{u}$ is globally Lipschitz continuous in $\mathbb{T} \times [0, +\infty)$.

(H2) No extinction occurs in $[0, +\infty)$.

(H3) $\{u_i > 0\}$ is connected for any $i \in \{1, ..., m\}$.

(H4) $\tilde{\beta} = 1$.

Notice that, since for a generic admissible initial state $(\phi_i)_i$ we do not have any control on which precise extinctions may occur, we cannot decide wether a certain initial state will behave asymptotically as subject to cyclic or acyclic interactions, since extinctions may change the interaction patterns. However, there is a simple way to deal with this fact in the case that the total number of connected components of supports of the initial condition is finite. Indeed, it is sufficient to assume that the cyclic condition holds for the initial index set I and for any ordered subset of indices of it. This ensures that whatever extinction occurs, the interaction pattern between the densities that survive remains cyclic. We remark that this condition is not empty in general. Indeed, one can verify that the coefficients β_{ij} satisfy this condition if, for example, they are chosen in the form $\beta_{ij} = \frac{b_i}{b_j}$ where coefficients b_i are positive. The periodicity of the density u defined via (3.11) will play an important role in this chapter. Notice that to ensure this property for u, an appropriate choice of the interval should be made depending on the parity of m. Without loss of generality, we will consider in this chapter the case of even m, corresponding to the choice of interval $[0,1) \simeq \mathbb{T}$. However, all the arguments are exactly the same for an odd m, in which case one should choose $[0,2) \simeq \frac{\mathbb{R}}{2\mathbb{Z}} =: 2\mathbb{T}$.

The chapter is structured as follows. First, we prove that we can define an energy in an analogous way to the Dirichlet case. This allows us to prove that every solution converges to the set of stationary solutions $\mathcal{S}$, and in particular to a single connected component of $\mathcal{S}$. This property is sometimes referred to as orbital stability. In the second section of the chapter, we will prove that solutions actually converge to a stationary one.

4.1. Convergence to the set of stationary solutions

We define the energy as follows:

$$E[\mathbf{u}](t) := \sum_{i=1}^{m} \int_0^1 [\beta^{(i)}]^2 \left\{ \frac{1}{2} \left| \frac{\partial u_i}{\partial x} \right|^2 - \frac{1}{2} a_i u_i^2 + \frac{1}{3} u_i^3 \right\} dx,$$

where coefficients $\beta^{(i)}$ are defined as in (3.10). Notice that, because of the regularity of the solutions to system (3.1) and its free boundary, $E[\mathbf{u}](t)$ is well defined and differentiable over time. Moreover, we have the following important identity:

Proposition 4.1 (Energy Identity). *Let $\mathbf{u} = (u_i)_i$ be a solution to system (3.1) under hypotheses (H1-H4). Then, the following identity holds:*

$$\dot{E}[\mathbf{u}](t) = - \int_0^1 \sum_{i=1}^{m} [\beta^{(i)}]^2 \left| \frac{\partial u_i}{\partial t} \right|^2 dx \leq 0 \qquad \forall \quad t > 0 \tag{4.2}$$

Proof. For each i, consider the quantity:

$$\int_0^1 \left\{ \frac{1}{2}\left|\frac{\partial u_i}{\partial x}\right|^2 - \frac{1}{2}a_i u_i^2 + \frac{1}{3}u_i^3 \right\} dx = \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left\{ \frac{1}{2}\left|\frac{\partial u_i}{\partial x}\right|^2 - \frac{1}{2}a_i u_i^2 + \frac{1}{3}u_i^3 \right\} dx,$$

where in the right hand side we consider the functions u_i as defined on Θ (see the definition (3.6) and the discussion that follows). Due to the regularity of the free boundaries obtained in Proposition 3.10, we are able to apply the chain rule while differentiating the quantity with respect to time. Using the fact that $-\frac{1}{2}au_i^2 + \frac{1}{3}u_i^3 = 0$ on $x = \alpha_{i-1}(t)$ and $x = \alpha_i(t)$, we obtain the following:

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left\{ \frac{1}{2}\left|\frac{\partial u_i}{\partial x}\right|^2 - \frac{1}{2}a_i u_i^2 + \frac{1}{3}u_i^3 \right\} dx \right) \quad = \\ &= \quad \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left\{ \frac{\partial u_i}{\partial x}\frac{\partial^2 u_i}{\partial x \partial t} - a_i u_i \frac{\partial u_i}{\partial t} + u_i^2 \frac{\partial u_i}{\partial t} \right\} dx \quad + \\ &+ \quad \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_i(t), t)\right|^2 \dot{\alpha}_i(t) - \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_{i-1}(t), t)\right|^2 \dot{\alpha}_{i-1}(t). \end{aligned} \tag{4.3}$$

Now, we integrate by parts the first term in the integral, obtaining:

$$\begin{aligned} &\frac{d}{dt}\left(\int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left\{ \frac{1}{2}\left|\frac{\partial u_i}{\partial x}\right|^2 - \frac{1}{2}a_i u_i^2 + \frac{1}{3}u_i^3 \right\} dx \right) \quad = \\ &= \quad -\int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left\{ \frac{\partial^2 u_i}{\partial x^2}\frac{\partial u_i}{\partial t} - a_i u_i \frac{\partial u_i}{\partial t} + u_i^2 \frac{\partial u_i}{\partial t} \right\} dx \quad + \\ &+ \quad \frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial t}(\alpha_i(t), t) - \frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial t}(\alpha_{i-1}(t), t) \\ &+ \quad \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_i(t), t)\right|^2 \dot{\alpha}_i(t) - \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_{i-1}(t), t)\right|^2 \dot{\alpha}_{i-1}(t). \end{aligned}$$

First, we notice that:

$$\int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left\{ \frac{\partial^2 u_i}{\partial x^2}\frac{\partial u_i}{\partial t} - a_i u_i \frac{\partial u_i}{\partial t} + u_i^2 \frac{\partial u_i}{\partial t} \right\} dx \quad = \quad \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \left|\frac{\partial u_i}{\partial t}\right|^2 dx.$$

Since u_i satisfies the logistic equation on $(\alpha_{i-1}(t), \alpha_i(t))$. By Proposition 3.6, the function u defined by (3.11) is differentiable in a neighborhood of $(\alpha_i(t), t)$, so that we obtain:

$$0 = \frac{d}{dt}u(\alpha_i(t), t) = \frac{\partial u}{\partial t}(\alpha_i(t), t) + \dot{\alpha}_i(t)\frac{\partial u}{\partial x}(\alpha_i(t), t).$$

Therefore:

$$\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial t}(\alpha_i(t),t) = -\dot{\alpha}_i(t)\left|\frac{\partial u_i}{\partial x}\right|^2(\alpha_i(t),t),$$

where the fact it holds for u_i is because near $\alpha_i(t)$ only u_i and u_{i+1} are not identically zero. By applying the same argument to $\alpha_{i-1}(t)$, we finally get:

$$\begin{cases} \dfrac{\partial u_i}{\partial x}\dfrac{\partial u_i}{\partial t}(\alpha_i(t),t) = -\dot{\alpha}_i(t)\left|\dfrac{\partial u}{\partial x}(\alpha_i(t),t)\right|^2, \\ \dfrac{\partial u_i}{\partial x}\dfrac{\partial u_i}{\partial t}(\alpha_{i-1}(t),t) = -\dot{\alpha}_{i-1}(t)\left|\dfrac{\partial u}{\partial x}(\alpha_{i-1}(t),t)\right|^2. \end{cases} \tag{4.4}$$

Now, we substitute (4.4) into (4.3) and get:

$$\begin{aligned} \frac{d}{dt}\left(\int_{\alpha_{i-1}(t)}^{\alpha_i(t)}\left\{\frac{1}{2}\left|\frac{\partial u_i}{\partial x}\right|^2 - \frac{1}{2}a_i u_i^2 + \frac{1}{3}u_i^3\right\}dx\right) \quad &= \\ = \quad -\int_{\alpha_{i-1}(t)}^{\alpha_i(t)}\left|\frac{\partial u_i}{\partial t}\right|^2 dx \quad &+ \\ - \quad \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_i(t),t)\right|^2\dot{\alpha}_i(t) + \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_{i-1}(t),t)\right|^2\dot{\alpha}_{i-1}(t). \end{aligned}$$

Finally, multiplying the equation involving u_i by $[\beta^{(i)}]^2$ and summing over i, we get the following identity:

$$\begin{aligned} \dot{E}[\mathbf{u}](t) = -\sum_{i=1}^{m}\int_{\alpha_{i-1}(t)}^{\alpha_i(t)}\left|\frac{\partial u_i}{\partial t}\right|^2 dx \quad + \\ + \quad \sum_{i=1}^{m}[\beta^{(i)}]^2\left\{-\frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_i(t),t)\right|^2\dot{\alpha}_i(t) + \frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_{i-1}(t),t)\right|^2\dot{\alpha}_{i-1}(t)\right\}. \end{aligned} \tag{4.5}$$

By rearranging the terms of the sum in the second line, we get:

$$
\begin{aligned}
&\sum_{i=1}^{m}[\beta^{(i)}]^2\left\{-\frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_i(t),t)\right|^2\dot{\alpha}_i(t)+\frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_{i-1}(t),t)\right|^2\dot{\alpha}_{i-1}(t)\right\}=\\
&=\left(-\frac{1}{2}[\beta^{(m)}]^2\left|\frac{\partial u_m}{\partial x}(\alpha_m(t),t)\right|^2\dot{\alpha}_m(t)+\frac{1}{2}\left|\frac{\partial u_1}{\partial x}(\alpha_0(t),t)\right|^2\dot{\alpha}_0(t)\right)+\\
&+\sum_{i=1}^{m-1}[\beta^{(i)}]^2\left\{-\frac{1}{2}\left|\frac{\partial u_i}{\partial x}(\alpha_i(t),t)\right|^2\dot{\alpha}_i(t)+\frac{1}{2}\left(\frac{\beta_{i,i+1}}{\beta_{i+1,i}}\right)^2\left|\frac{\partial u_{i+1}}{\partial x}(\alpha_i(t),t)\right|^2\dot{\alpha}_i(t)\right\}=\\
&=0,
\end{aligned}
\tag{4.6}
$$

because of the transmission conditions, and the fact that $\tilde{\beta}=1$. In particular, notice that by periodicity $\left|\frac{\partial u_m}{\partial x}(\alpha_m(t),t)\right|^2\dot{\alpha}_m(t)=\left|\frac{\partial u_m}{\partial x}(\alpha_0(t),t)\right|^2\dot{\alpha}_0(t)$. This proves the proposition, since by combining (4.5) and (4.6), and by periodicity we get:

$$
\dot{E}[\mathbf{u}](t)=-\sum_{i=1}^{m}\int_{\alpha_{i-1}(t)}^{\alpha_i(t)}\left|\frac{\partial u_i}{\partial t}\right|^2dx=-\int_0^1\sum_{i=1}^{m}\left|\frac{\partial u_i}{\partial t}\right|^2dx.
$$

□

We remark here that it is possible to write the left hand side in the energy identity only depending on the function u defined via (3.11):

$$
-\int_0^1\sum_{i=1}^{m}[\beta^{(i)}]^2\left|\frac{\partial u_i}{\partial t}\right|^2dx=-\int_0^1\left|\frac{\partial u}{\partial t}\right|^2dx. \tag{4.7}
$$

In the context of dynamical systems arising from PDEs, a dissipation inequality of the kind $\dot{E}(t)\leq 0$, when available, is often crucial to study the long time behavior of orbits, and is related to the existence of a variational structure for the evolution equation. However, a rigorous treatment of this topic, and the reformulation of our problem in this general framework, is out of the scope of this thesis. Nevertheless, we will use the concept of ω-limit sets. Given an admissible initial configuration $\mathbf{u}_0$ for system (3.1) and the associated solution $\mathbf{u}[\mathbf{u}_0](x,t)$ defined for all times $t>0$, we define the ω-limit set of $\mathbf{u}_0$ following e.g. [10]:

$$
\omega[\mathbf{u}_0]:=\{\mathbf{w}\in\mathcal{C}^0_{\text{per}}([0,1]):\quad\exists\quad t_k\to+\infty,\quad\mathbf{u}[\mathbf{u}_0](x,t_k)\to\mathbf{w}\quad\text{in}\quad\mathcal{C}^0_{\text{per}}([0,1])\}
$$

We will often refer to ω-limit sets of a solution $\mathbf{u}$ as $\omega[\mathbf{u}]$, without necessarily refer to the initial condition. Later, we will focus on some rather general properties of ω-limit sets that are useful in our analysis. Now we state and prove a theorem that ensures that each global solution tends to the set of stationary solutions as time tends to infinity. We recall that, according to the definition given at the end of Chapter 2 on the elliptic singular problem, we refer to the set of stationary solutions as $\mathcal{S} \subset \mathcal{C}^0_{\text{per}}([0,1])$.

Theorem 4.2. *Let $\mathbf{u} = (u_i)_i$ be a solution to system* (3.1) *under hypotheses (H1-H4). Then:*

$$d(\mathbf{u}(t), \mathcal{S}) = \inf_{\mathbf{v}\in\mathcal{S}} \|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot)\|_\infty \to 0 \qquad \text{as} \quad t \to \infty. \tag{4.8}$$

In particular $\omega[\mathbf{u}] \subset \mathcal{S}$.

We remark that in the language of dynamical systems, (4.8) is equivalent to saying that $\mathcal{S}$ is the global attractor: it is compact, it attracts points, and it is invariant.

Proof. For any sequence $(t_k)_{k\in\mathbb{N}}$, $t_k \to \infty$, define:

$$u_{i,k}(x,t) := u_i(x, t + t_k) \qquad \forall \quad k \in \mathbb{N}, \quad \forall \quad i = 1, \dots, M.$$

Since u_i are Lipschitz continuous in space, uniformly in time, so are $u_{i,k}$, and in particular there exists a subsequence converging locally uniformly to some v_i which are uniformly Lipschitz continuous. This is due to Ascoli-Arzelà theorem. Moreover, $u_{i,k}$ are solutions to the system (3.1) for any $k \in \mathbb{N}$, and it is easy to show, by dominated convergence, that equations pass to the limit. Therefore, v_i is also a solution to the system (3.1). We must now prove that such a limit is a stationary state.

From the energy Identity (4.2), by integrating on (t, ∞) and passing to the limit as $t \to \infty$, we get:

$$\lim_{t\to\infty} \int_t^\infty \int_0^1 \sum_{i=1}^M [\beta^{(i)}]^2 \left|\frac{\partial u_i}{\partial t}\right|^2 = 0. \tag{4.9}$$

Now consider the quantity:

$$\begin{aligned} &- E[\mathbf{u}](t+t_k) + E[\mathbf{u}](t_k) = \int_{t_k}^{t+t_k} \int_0^1 \sum_{i=1}^M [\beta^{(i)}]^2 \left|\frac{\partial u_i}{\partial t}\right|^2 = \\ &= - \int_{t+t_k}^\infty \int_0^1 \sum_{i=1}^M [\beta^{(i)}]^2 \left|\frac{\partial u_i}{\partial t}\right|^2 + \int_{t_k}^\infty \int_0^1 \sum_{i=1}^M [\beta^{(i)}]^2 \left|\frac{\partial u_i}{\partial t}\right|^2 \to 0 \quad \text{as } k \to \infty, \quad \forall \quad t > 0, \end{aligned}$$

because both terms tend to zero by (4.9). Therefore, we have the following:

$$\begin{aligned} &E[\mathbf{u}_k](0) \leq E[\mathbf{u}_k](t) + \epsilon(k,t) \\ &\text{where} \quad \epsilon(k,t) \to 0 \quad \text{as} \quad k \to \infty, \quad \forall \quad t > 0. \end{aligned}$$

Moreover, the energy passes to the limit, in the sense that $\lim_{k\to\infty} E[\mathbf{u}_k](t) = E[\mathbf{v}](t)$ for every $t > 0$, and therefore:

$$E[\mathbf{v}](0) \leq E[\mathbf{v}](t) \quad \forall \quad t > 0.$$

On the other hand, we have seen that $\mathbf{v}$ is a Lipschitz continuous solution to system (3.1), therefore it enjoys the energy decreasing property

$$E[\mathbf{v}](0) \geq E[\mathbf{v}](t) \quad \forall \quad t > 0.$$

Therefore:

$$\begin{aligned} 0 = E[\mathbf{v}](0) - E[\mathbf{v}](t) = \int_0^t \int_0^1 \sum_{i=1}^M [\beta^{(i)}]^2 \left|\frac{\partial v_i}{\partial t}\right|^2 \qquad & \forall \quad t > 0 \\ \implies \frac{\partial v_i}{\partial t} = 0 \qquad & \forall \quad i, \forall \quad t > 0 \\ \implies \mathbf{v} \in \mathcal{S} \qquad & \end{aligned}$$

This shows that for any sequence $t_k \to \infty$, there exists a subsequence converging uniformly to some $\mathbf{v} \in \mathcal{S}$. Suppose now by contradiction that $d(\mathbf{u}, \mathcal{S}) \not\to 0$. Then,

$$\exists \quad (t_k)_{k\in\mathbb{N}} \quad \text{s.t.}: \quad \exists \quad \epsilon > 0, \quad \forall \quad T > 0 \quad \exists \quad k > T: \quad \inf_{\mathbf{v}\in\mathcal{S}} \|\mathbf{u}(\cdot, t_k) - \mathbf{v}(\cdot)\| \geq \epsilon.$$

But for what we just proved, we could extract from the above subsequence another one converging to some $\mathbf{v} \in \mathcal{S}$, deriving a contradiction. Therefore, $\omega - \text{limit}[\phi] \subset \mathcal{S}$ for every admissible initial state.

□

We now prove that the ω-limit set must be connected. This is a property that holds in quite general conditions for dynamical systems in metric spaces. We follow [10] for our proof.

Proposition 4.3. *For any $\mathbf{u} = (u_i)_i$ solution to system* (3.1) *under hypotheses (H1-H4), $\omega[\mathbf{u}]$ is nonempty, compact and connected.*

Proof. By the global Lipschitz continuity, because of Ascoli-Arzelà theorem, for any sequence $t_k \to \infty$, there exists a subsequence, still denoted by t_k, such that $\mathbf{u}(\cdot, t_k)$ converges in $\mathcal{C}^0_{\text{per}}([0,1])^m$. Therefore, $\{\mathbf{u}(\cdot,t), t \geq 0\}$ lies is a compact set in $\mathcal{C}^0_{\text{per}}([0,1])^m$. In the general theory of dynamical systems in metric spaces, this is sufficient to prove the compactness and connectedness of $\omega[\mathbf{u}]$. Indeed, the following characterization holds for an ω-limit set:

$$\omega[\mathbf{u}] = \bigcap_{\tau \geq 0} \overline{\{\mathbf{u}(\cdot,t) : t \geq 0\}}.$$

$\omega[\mathbf{u}]$ is therefore the intersection of a decreasing collection of nonempty compact sets. Therefore, it is nonempty and compact. Now suppose it is not connected. Then, there exist nonempty closed disjoint sets A, B such that $\omega[\mathbf{u}] = A \cup B$. But then both are compact and $d(A,B) := \inf_{(x,y)\in A\times B} \|x - y\|_{\mathcal{C}^0([0,1])^m}$ is positive. Let $3\delta := d(A,B)$. Now by definition of ω-limit set, there exist $t_k, t'_k \to \infty$ with $t_k < t'_k < t_{k+1}$ for any $k \geq 0$, and there exists some $n \geq 0$ such that:

$$\begin{cases} d(\mathbf{u}(\cdot,t_k), A) < \delta, \\ d(\mathbf{u}(\cdot,t'_k), B) < \delta, \end{cases} \qquad \text{for any } k \geq n.$$

By continuity, there exists a sequence $t''_k \to \infty$ with $t_k < t''_k < t'_k$ and $k \geq n$ such that:

$$d(\mathbf{u}(\cdot,t_k), A \cup B) > \delta. \tag{4.10}$$

Now $\{\mathbf{u}(\cdot,t_k) : k \geq n\}$ lies in a compact set, and any of its accumulation points $\mathbf{v}$ must lie in $\omega[\mathbf{u}]$. However, by (4.10), $d(\mathbf{v}, A\cup B) > \delta$, giving a contradiction. Therefore, $\omega[\mathbf{u}]$ must be connected. □

We now observe that the question of whether there is convergence to a certain state is still left open. Notice that in the Dirichlet case, as tackled in [28], the answer is simple at this point because $\mathcal{S}$ is a finite (and thus discrete) set, and $\omega - \text{limit} \subset \mathcal{S}$ is connected. This implies that the $\omega - \text{limit}$ must be a singleton. For what concerns the periodic case with symmetric interactions, we showed in Chapter 2 that the connected components of $\mathcal{S}$ are of the form:

$$\Gamma(\mathbf{v}) = \{\mathbf{v}(\cdot - c) : \quad a \in [0,1]\},$$

for some $\mathbf{v}$ solution to the stationary system (2.1). Therefore, we are still able to affirm that ω-limit $\subset \Gamma(\mathbf{v})$ for some $\mathbf{v}$, and so if two elements $\mathbf{v}_1, \mathbf{v}_2$ belong to the ω-limit set of the same trajectory, they must be a translation of each other. However, the argument used in [28] clearly fails in our case. In the following proposition, we prove a property deriving from the energy identity.

Proposition 4.4. *Let u be as in* (3.11)*, built from a solution $\mathbf{u} = (u_i)_i$ to system* (3.1) *under hypotheses (H1-H4). Then:*

$$\lim_{t\to\infty} \sup_{x\in(0,1)} \left|\frac{\partial u}{\partial t}\right| = 0. \tag{4.11}$$

.

Proof. Consider equation (3.13). As seen in the discussion in Section 3.2, $u \in \mathcal{C}^{2+\delta,1+\delta/2}(\mathbb{T}\times [0,+\infty))$, so that $w := \frac{\partial u}{\partial t} \in \mathcal{C}^{2+\delta,\delta/2}(\mathbb{T} \times [0,+\infty))$. In particular:

$$|w(x,t+h) - w(x,t)| \leq C\,|h|^{\frac{\delta}{2}} \qquad \forall \quad (x,t) \in [0,1] \times (0,+\infty), \quad \forall \quad h > 0. \tag{4.12}$$

Moreover, by the energy identity we know that:

$$\lim_{t\to\infty} \int_t^\infty \int_0^1 |w|^2 = 0. \tag{4.13}$$

Suppose that $\int_0^1 |w|^2 \nrightarrow 0$ as $t \to \infty$. Then, there exists $\epsilon > 0$, $t_k \to \infty$ such that:

$$\int_0^1 |w|^2(t_k) \geq \epsilon \qquad\qquad \forall \quad k \geq 0.$$

Notice that (4.12) implies, for $h = s - t$ sufficiently small:

$$|w(x,s)|^2 \geq \left|w(x,t) - C\,|s-t|^{\frac{\delta}{2}}\right|^2 \qquad \forall \quad x \in [0,1], \quad s > t > 0.$$

For every $k \geq 0$, consider now for some $h > 0$ sufficiently small the quantity:

$$
\begin{aligned}
&\int_{t_k}^{t_k+h}\int_0^1 |w(x,s)|^2 dxds \geq \int_{t_k}^{t_k+h}\int_0^1 \left|w(x,t_k) - C\,|s-t_k|^{\frac{\delta}{2}}\right|^2 dxds \geq \\
&\geq \int_{t_k}^{t_k+h}\int_0^1 |w(x,t_k)|^2 - C\,|s-t_k|^{\delta}\, dxds = \|w(x,t_k)\|^2\, h - Ch^{\delta+1} \geq \\
&\geq \epsilon h - Ch^{\delta+1} =: \mu > 0,
\end{aligned}
$$

for h small enough. However, this is in contradiction with (4.13). Hence, $\int_0^1 w^2 \to 0$ as $t \to \infty$. Moreover, w satisfies the following equation in the weak sense:

$$
\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) w = \tilde{V}(x) w \qquad \forall \quad x \in (0,1), \quad t > 0, \tag{4.14}
$$

where $V(x)$ is a bounded globally Lipschitz function. We consider the weak formulation for (4.14) with the choice of w itself as a test.

$$
\int_0^1 w_t w = \frac{1}{2}\frac{d}{dt}\int_0^1 w^2 = -\int_0^1 w_x^2 + \int_0^1 V(x) w^2. \tag{4.15}
$$

By neglecting the first term on the right and applying the Gronwall inequality, we get:

$$
\|u(t)\|_2^2 \leq e^{2\|V(x)\|_\infty (t-s)} \|u(s)\|_2^2 \qquad \forall \quad t > s > 0. \tag{4.16}
$$

Since $V(x,t)$ is bounded on $[0,1] \times (0,+\infty)$. Moreover, by integrating (4.15) in time between s and t, with $t > s > 0$, we obtain:

$$
\begin{aligned}
&\int_s^t\int_0^1 w_x^2 = -\frac{1}{2}\int_0^1 w^2(t) + \frac{1}{2}\int_0^1 w^2(s) + \int_s^t\int_0^1 V(x) w^2 \qquad \forall \quad t > s > 0,\\
&\int_s^t\int_0^1 w_x^2 \leq \frac{1}{2}\int_0^1 w^2(s) + 2(t-s)\,\|V(x)\|_\infty\, e^{2\|V(x)\|_\infty (t-s)} \int_0^1 w^2(s),\\
&\int_s^t\int_0^1 w_x^2 \leq \left[\frac{1}{2} + 2(t-s)\,\|V(x)\|_\infty\, e^{2\|V(x)\|_\infty (t-s)}\right] \int_0^1 w^2(s) \qquad \forall \quad t > s > 0,
\end{aligned}
$$

where we used (4.16) in the second line. By using (4.16) again and the Sobolev embedding, we conclude that for every $\delta > 0$, $t > 0$:

$$
\int_t^{t+\delta} \|w\|_\infty \leq C\left(\|V(x)\|_\infty, \delta\right) \|w\|_2 (t).
$$

Therefore, by letting $t \to \infty$ and knowing that convergence in the L^2 norm holds, we derive that for every $\delta > 0$:

$$\lim_{t\to\infty}\int_t^{t+\delta}\sup_{x\in[0,1]}\left|\frac{\partial u}{\partial t}\right| \le C\lim_{t\to\infty}\int_0^1\left|\frac{\partial u}{\partial t}\right|^2 = 0. \tag{4.17}$$

Taking advantage of the regularity in time of w with an analogous argument as before, it is finally possible to prove the thesis:

$$\lim_{t\to\infty}\sup_{x\in[0,1]}\left|\frac{\partial u}{\partial t}\right| = 0.$$

$\square$

The methods used in the previous proposition are not sufficient, however, to prove that the time-integrals on $(0,+\infty)$ of the L^∞ or L^1 norm of $\frac{\partial u}{\partial t}$ are finite. That would be sufficient to prove convergence to a stationary point; with the property that $\int_0^\infty\int_0^1|u_t|^2 \to 0$, we are only able to prove that if two elements belong to the ω-limit set, their visit is infinitely rare as $t\to+\infty$. Suppose that there exist $\mathbf{v}_1 \neq \mathbf{v}_2 \in \omega$-limit$[\mathbf{u}]$, for some $\mathbf{u}$ globally Lipschitz solution to system (3.1). We will work with the functions defined as in (3.11) for $\mathbf{u}$, $\mathbf{v}_1$, $\mathbf{v}_2$, and we will indicate them, respectively, with u, $\tilde{v}_1$, $\tilde{v}_2$. Note that all these functions are regular, as $\mathbf{v}_1$ and $\mathbf{v}_2$ solve the stationary problem associated to (3.1). By the definition of ω-limit, there exist two sequences $(t_k)_{k\in\mathbb{N}}, (s_k)_{k\in\mathbb{N}}$, both tending to infinity, and such that:

$$\begin{cases}\sup\limits_{x\in[0,1]}|u(x,t_k)-\tilde{v}_1(x)|\to 0,\\ \sup\limits_{x\in[0,1]}|u(x,s_k)-\tilde{v}_2(x)|\to 0,\\ \text{as } k\to\infty.\end{cases}$$

Moreover, by discarding some elements of the sequences t_k, s_k, we can suppose without loss of generality that:

$$t_k < s_k < t_{k+1} < s_{k+1} \qquad \forall \quad k \ge 0.$$

In this context, we can state the following proposition:

Proposition 4.5. *Let* $\mathbf{u} = (u_i)_i$ *be a solution to system* (3.1) *under hypotheses (H1-H4). Suppose that there exist* $\mathbf{v}_1 \neq \mathbf{v}_2 \in \omega[\mathbf{u}]$*, and let* $(t_k)_{k\in\mathbb{N}}$*,* $(s_k)_{k\in\mathbb{N}}$ *be defined as above.*

Then:

$$(s_k - t_k) \to \infty \qquad\qquad as \quad k \to \infty.$$

Proof. Since $\mathbf{v}_1, \neq \mathbf{v}_2$, $\sup_{x\in[0,1]} |\tilde{v}_1 - \tilde{v}_2| > 0$, and moreover, by the continuity in space of the functions, it is attained in some point $x_M \in [0,1]$. Now, for any arbitrary small $\delta > 0$, for k sufficiently large, by elementary estimates and convergence of the subsequences, we have the following:

$$|u(x_M, t_k) - u(x_M, s_k)| \geq |\tilde{v}_1(x_M) - \tilde{v}_2(x_M)| - 2\delta.$$

Since u is differentiable in time, we can apply Lagrange theorem to each interval (t_k, s_k) and derive that:

$$\forall \quad k \geq 0, \quad \exists \quad \zeta_k \in (t_k, s_k) \quad \text{s.t.}: \quad \frac{\partial u}{\partial t}(x_M, \zeta_k) = \frac{u(x_M, s_k) - u(x_M, t_k)}{s_k - t_k}.$$

Therefore, we have:

$$\left|\frac{\partial u}{\partial t}(x_M, \zeta_k)\right| \geq \frac{|u(x_M, s_k) - u(x_M, t_k)|}{s_k - t_k} \geq \frac{|\tilde{v}_1(x_M) - \tilde{v}_2(x_M)| - 2\delta}{s_k - t_k},$$

where the last numerator is positive for δ sufficiently small and does not depend on k. Therefore, since the left hand side tends to zero, we must have:

$$s_k - t_k \to +\infty \qquad\qquad \text{as} \quad k \to \infty.$$

$\square$

In particular, by the above proposition, there cannot exist periodic solutions, nor solutions tending to periodic functions of time as $t \to +\infty$. Further results require different techniques and will be addressed in the next section.

4.2. Convergence to a stationary solution

To prove convergence to a stationary solution, we will rely on the already mentioned fundamental property of one-dimensional linear parabolic equations, that concerns the monotonicity in time of the number of zeroes. This proves to be a useful tool in characterizing the generic dynamics of one-dimensional semilinear parabolic equations of the form:

$$v_t - v_{xx} = f(v, v_x), \tag{4.18}$$

since it ensures that the number of zeroes behaves as a discrete valued Lyapunov functional for the evolution of the dynamical system. For equation (4.18) on $\mathbb{T}$, with $f = f(q,p)$ is analytic, a Poincarè-Bendixson theorem was proven by Angenent and Fiedler in [22] for bounded solutions. The Poincarè-Bendixson theorem is a classical result which holds for generic two-dimensional systems of ODEs and ensures that the ω-limit set consists either of a stationary point or a periodic solution. Matano considered then in [18] the case where $f(q,p)$ is only twice differentiable in each variable. For bounded solutions v of (4.18), he proved that there exists $\tilde{c} \in \mathbb{R}$ and a profile $\varphi(x)$ solving the travelling wave equation associated to (4.18) with velocity $\tilde{c}$ such that:

$$\omega[v] \subset \{\varphi(\cdot + c) : c \in \mathbb{T}\}.$$

In particular, φ is a stationary solution in case $\tilde{c} = 0$. Notice that, in our case, an analogous characterization of ω-limit sets has been proved thanks to the analysis of the set of stationary solutions and energy methods (see the discussion after Proposition 4.3 in the last section). Most importantly for our analysis, Matano also proved that under the additional requirement that $f(q,p) = f(q,-p)$ for any $(q,p) \in \mathbb{R}^2$, $\omega[v]$ contains exactly one element, so that the solution v must converge to a stationary one. In our case, we cannot directly rephrase our problem as an equation of the form (4.18). However, we are able to apply the most important arguments in [18] to prove that convergence to a stationary state holds for the sign-changin density u defined in (3.11) and associated with solutions of (3.1). Notice that there exist various slightly different definitions of zero number, and we could as well adapt the results of [1] for the present analysis. However, for convenience, we will consider in what follows the conventions used by Matano in [18].

We recall that we restrict our analysis to solutions satisfying (H1-H4). We already remarked that, if $\tilde{\beta} = 1$, u is a classical solution to the equation:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u = V(x,t)u, \tag{4.19}$$

in $[0,1] \simeq \mathbb{T}$. Now we define for $c \in \mathbb{T}$ the space-reflection:

$$\rho_c u := u(2c - \cdot),$$

which clearly solves the same equation with $\rho_c V(x,t)$ instead of $V(x,t)$. Therefore, the difference $w := u - \rho_c u$ solves:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) w = W(x,t)w,$$

where $W(x,t) = (V - \rho_c V)(x,t)$ is still bounded on $\mathbb{T} \times (T_1, +\infty)$. Consider now the following quantity for any function $w \in C^1(\mathbb{T})$:

$$\mu(w) := \#\{ \text{ connected components of } \{x : w(x) \neq 0\}\},$$

with the convention that $\mu(w) = 0$ if and only if $\{x : w(x) \neq 0\}$ is empty. This implies that $\mu(w) = 0$ if and only if $w \equiv 0$ in $\mathbb{T}$. We also introduce the following class of functions:

$$\Sigma := \left\{ w \in \mathcal{C}^1(\mathbb{T}) : \quad \frac{\partial w}{\partial x}(x_0) \neq 0 \quad \forall \quad x_0 \in \mathbb{T} \quad \text{s.t.} \quad w(x_0) = 0 \right\}.$$

We are now ready to present the core result ensuring that convergence to a stationary solution holds in our case. As anticipated, we refer to [18, Lemma 3.3] for its formulation.

Theorem 4.6. *Let $w(x,t)$ be a continuous function on $\mathbb{T} \times (\tau_1, \tau_2)$ with $-\infty \leq \tau_1 < \tau_2 \leq +\infty$; Suppose $w \in \mathcal{C}^{2,1}(\mathbb{T} \times (\tau_1, \tau_2))$ and it satisfies the linear differential equation:*

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + q(x,t)\frac{\partial w}{\partial x} + r(x,t)w = 0,$$

where $q, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial t}$ and r are bounded on any compact subset of $\mathbb{T} \times (\tau_1, \tau_2)$. Then:

- *$\mu(w(\cdot,t)) < +\infty$ for any $t \in (\tau_1, \tau_2)$*
- *$\mu(w(\cdot,t))$ is monotone nonincreasing and right-continuous in t*
- *for any given $t_0 \in (\tau_1, \tau_2)$, $\lim_{t \uparrow t_0} \mu(w(\cdot,t)) = \mu(w(\cdot,t_0))$ if and only if either $w(\cdot,t) \in \Sigma$ or $w \equiv 0$ on $\mathbb{T} \times (\tau_1, \tau_2)$.*
- *if $w \not\equiv 0$ on $\mathbb{T} \times (\tau_1, \tau_2)$, then for any $\delta \in (\tau_1, \tau_2)$ there exists a finite set $A \in [\delta, \tau_2)$ such that $w(\cdot,t) \in \Sigma$ for any $t \in [\delta, \tau_2) \setminus A$.*

Now we can state and prove the following proposition:

Proposition 4.7. *Let u be the density associated via* (3.11) *with a solution of* (3.1) *satisfying (H1-H4). Then:*

$$\lim_{t\to\infty} \operatorname{sign}\left(\frac{\partial u}{\partial x}(x,t)\right)$$

exists.

Proof. For any fixed $a \in [0,1)$, with the notation introduced above:

$$\begin{cases} w(a,t) = 0, \\ \dfrac{\partial w}{\partial x}(a,t) = 2\dfrac{\partial u}{\partial x}(a,t). \end{cases} \tag{4.20}$$

Now, suppose by contradiction that $\operatorname{sign}(\frac{\partial u}{\partial x}(a,t))$ does not converge. Then, by (4.20), also $\operatorname{sign}(\frac{\partial w}{\partial x}(a,t))$ does not converge. In particular, since the sign function can only attain values in $\{-1,0,+1\}$, by the continuity of $\frac{\partial w}{\partial x}$ in time there must exist a sequence of times $t_k \to \infty$ such that $\frac{\partial w}{\partial x}(a,t_k) \to 0$. However, we know that $w(a,t) = 0$ for every $t > 0$, and therefore $w(\cdot,t_k)$ has non-simple zeroes for every k. Now by the last item of Theorem 4.6, $w \equiv 0$ for every (x,t). Therefore, $\operatorname{sign}(\frac{\partial u}{\partial x}(a,t)) = 0$ for any $t > 0$, contradicting the hypothesis that this quantity does not converge as $t \to \infty$. By the generic choice of a made at the beginning, the proposition is proved.

□

The last ingredient needed for the proof of convergence is the following, which relies on a standard argument.

Lemma 4.8. *Let* $\mathbf{u}$ *a solution of* (3.1) *satisfying (H1-H4), and suppose* $\mathbf{u}(\cdot,t_k) \to \mathbf{v}$ *uniformly on* $\mathbb{T}$*, for some sequence* $t_k \to +\infty$*. Let* u, v *be the densities associated with* $\mathbf{u}, \mathbf{v}$ *via* (3.11)*. Then:*

$$\frac{\partial u}{\partial x}(\cdot,t_k) \to \frac{\partial v}{\partial x}(\cdot) \qquad \text{as } t_k \to \infty,$$

uniformly in $\mathbb{T}$.

Proof. An immediate consequence of the assumptions is that $u(\cdot,t_k) \to v(\cdot)$ uniformly in $\mathbb{T}$. However, we know that u is a classical solution to (4.19), where $V(x,t)$ is uniformly bounded. We define the sequence of functions $u_k(x,t) := u(x, t + t_k)$ on the set $Q_\epsilon := \mathbb{T} \times (-\epsilon, \epsilon)$. Standard Schauder estimates give:

$$\|u\|_{\mathcal{C}^{2+2\delta,1+\delta}(Q_\epsilon)} \leq C[\|V\|_{L^\infty(Q_\infty)}\|u\|_{\mathcal{C}^{2\delta,\delta}(Q_\epsilon)} + \|u\|_{L^\infty(Q_\epsilon)}]$$

where the constant C does only depend on δ, ϵ and not on k (see for example [15] or [14]). Therefore, the sequence u_k is bounded in $\mathcal{C}^{2+2\delta,1+\delta}(Q_\epsilon)$, and in particular the sequence $u_k(\cdot, 0) = u(\cdot, t + t_k)$ is precompact in $\mathcal{C}^1(\mathbb{T})$. This implies that any subsequence strongly converges to a limit $w \in \mathcal{C}^1(\mathbb{T})$. However, since $\mathcal{C}^1(\mathbb{T})$ is continuously embedded in $\mathcal{C}^0(\mathbb{T})$, we must have $w = v$ for any subsequence, which implies that $\frac{\partial u}{\partial x}(\cdot, t_k) \to \frac{\partial v}{\partial x}(\cdot)$ uniformly in $\mathbb{T}$. □

We are now ready to prove the convergence to a stationary point. This is equivalent to proving that the ω-limit set reduces to a point.

Theorem 4.9. *Let* $\mathbf{u}$ *be solution to system* (3.1) *satisfying (H1-H4). Then, there exists some* $\mathbf{v}$ *solution to* (2.1) *such that* $\omega[\mathbf{u}] = \{\mathbf{v}\}$.

Proof. Let $\mathbf{v}_1 \in \omega[\mathbf{u}]$, and let u, v_1 be the regular densities associated with $\mathbf{u}, \mathbf{v}_1$. By Proposition 4.2, we know that $\mathbf{v}_1$ is a solution to (2.1), and by the discussion after proposition 4.3, we know that $\omega[\mathbf{u}] \subset \Gamma := \{\sigma_c \mathbf{v}_1 : c \in \mathbb{T}\}$. If v_1 is a constant function, then $\Gamma = \{v_1\}$ and the theorem is proved. Therefore, suppose v_1 is not a constant. Let $x_M \in \mathbb{T}$ be such that v_1 attains its maximum there. Then, by the regularity of v_1, $v_1'(x_M) = 0$. Moreover, $v_1''(x_M) \neq 0$. Indeed, suppose by contradiction that $v_1''(x_M) = 0$. $v_1(x_M) \neq 0$, and so there exists a connected component of the open set $\{x \in \mathbb{T} : \quad v_1(x) \neq 0\}$ which contains x_M. on that connected component, v_1 is smooth and solves:

$$-v_1'' = a_i v_1 - v_1^2,$$

for some $i \in \{1, ..., m\}$, with Dirichlet conditions on the boundary. We remark that, under the hypotheses of existence of such a function v_1, the operator $-v_1'' - a_i v_1$ enjoys the maximum principle. if $v_1''(x_M) = 0$, then $av_1(x_M) - v_1(x_M)^2 = 0$, so that either $v_1(x_M) = 0$ or $v_1(x_M) = a_i$. However, by the maximum principle, we would have in both cases that v_1 is constant on the open set, thus contradicting the hypothesis that v_1 is not a constant.

Now, suppose that $\omega[\mathbf{u}]$ contains an element $\mathbf{v}_2 \neq \mathbf{v}_1$, and call its associated density v_2. Then, By the connectedness of the ω-limit set, there is a non-degenerate arc segment

$\gamma \subset \mathbb{T}$ such that:

$$\{\sigma_c \mathbf{v}_1 : c \in \gamma\} \subset \omega[\mathbf{u}].$$

Let now b be an interior point of γ. Then $(\sigma_b v_1)'(x_M - b) = v_1'(x_M) = 0$ and $(\sigma_b v_1)''(x_M - b) \neq 0$. Therefore $\sigma_c v_1(x_M - b)$, being a non-constant function of c, must change sign as c varies in γ. Consequently, there exist functions ψ_1, ψ_2 associated to elements of $\omega[\mathbf{u}]$ such that $\psi_1'(x_M - b) > 0, \psi_2'(x_M - b) < 0$. Therefore, by Lemma 4.8, there are sequences $t_k, s_k \to \infty$ such that $\frac{\partial u}{\partial x}(x_M - b, t_k) \to \psi_1'(x_M - b)$, $\frac{\partial u}{\partial x}(x_M - b, s_k) \to \psi_2'(x_M - b)$, so that by proposition 4.7, $\psi_1'(x_M - b) > 0 \implies \psi_2'(x_M - b) \geq 0$, giving a contradiction. Therefore, $\omega[\mathbf{u}] = \{\mathbf{v}_1\}$. $\square$

5 | Conclusions and future developments

In this thesis, we proved a convergence results for solutions to the singular system associated with the evolutive Lotka-Volterra competition-diffusion model in one dimension and with periodic boundary conditions. The proof of our main result is based on the joint application of energy methods and arguments regarding the number of zeroes of solutions to linear parabolic equations in one spatial dimension. If the interactions do not satisfy the cyclic conditions, our technique used to prove convergence breaks down since no energy identity is available. However, we believe that even in the acyclic case, the property concerning the monotonicity in time of the number of zeros can prove to be useful in characterizing the possible structure of ω-limit sets.

On the other hand, the higher-dimensional case with multiple species needs to be treated with a different approach since the monotonicity of the zero number is a genuinely one-dimensional property. In [21], moreover, Salort, Terracini, Verzini and Zilio considered the singular parabolic system in a two dimensional rotationally invariant domain and with linear internal dynamics. In particular, they showed existence of solutions in the form of rigidly rotating spirals, also when the interactions between adjacent densities are of cyclic type. This shows that, in general, the higher dimensional case behaves differently from the case studied here of the one dimensional rotationally invariant domain $\mathbb{T}$.

Another natural continuation of this work would be to study the extent to which the behavior of the singular parabolic system helps to understand that of the Lotka-Volterra competition-diffusion system with large but finite κ. In particular, the interest is in characterizing the possible structure of ω-limit sets for generic coefficients, and in studying whether stationary states or periodic orbits involving more than one non-zero species can be stable, thus representing stationary or dynamic patterns of coexistence.

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