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Executive Summary of the Thesis

# A Dam Management Problem with Energy Production

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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## 1. Introduction

This thesis presents a stochastic optimal control problem for the management of a dam in the context of energy production. The problem is formulated as an optimal regular-switching problem, where the dam manager has to optimize the timing and volume of water release (to maximize electricity production and profits) and has to regulate the water level to meet safety requirements, avoiding flooding or energy shortages. Moreover, for hydroelectric production it is crucial that the turbines generate electricity at a fixed frequency to ensure uniformity across the power grid. This implicitly constraints the electric unit production to a on-off strategy, with more flexible output achieved by installing multiple units per dam. In our context, the control variables consist of the switching times that determine when the turbine's operational status changes (i.e., whether water flows through it or not), and the volume of water released from the dam during emergency situations. We have two distinct value functions: one represents the revenue generated from power production when water flows through the turbine, and the other represents the revenue when no water flows through it. The problem is addressed using the Dynamic Programming Principle (DPP) and the Hamilton-Jacobi-Bellman (HJB) equations. Since these traditional methods assume smooth value functions, we use the the viscosity solution framework to handle the lack of smoothness generated by the switching. This approach allows us to characterize the value function as the unique viscosity solution for the HJB system associated to our problem. The thesis follows the approach outlined in [1]. In particular we examine all the key steps, from the problem formulation to the development of the numerical solutions. We analyze the underlying theory, explicit all the formal steps and implement using MATLAB a numerical scheme to explore the numerical results presented in [1].

## 2. Dam Management Problem

### 2.1. Modeling the Water Level and the Electricity Price

To formulate the stochastic optimal control problem for the dam, we first need a model for the water level inside the dam. The water level in the dam basin depends on various factors. Water extraction for energy production decreases the level, while external inputs (such as rainfall and river inflows) increase it. Furthermore, when the water level exceeds a critical threshold, spillways are employed to release excess water. Here the objective is not energy generation, but to maintain a safe water level. Technically speaking, given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and a Brownian motion $W_t$, the water level dynamics when both the turbine and the spillways are closed is described by the stochastic process $H_t = \mu t + \sigma W_t$ (i.e., a drifted Brownian motion with drift $\mu$ and volatility $\sigma$). Using Tanaka's formula[1], the stochastic differential equation (SDE) governing the water level is:

$$dH_t = \mathbf{1}_{\{H_t>0\}}(\mu dt + \sigma dW_t) + dZ_t,$$

[1] $|W_t| = \int_0^t \operatorname{sgn}(W_n)\, dW_n + Z_t$, where *sgn* is the sign function and $L_t = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} |\{s \in [0,t] \mid W_s \in (-\varepsilon, +\varepsilon)\}|$.

where $Z$ is the half-local time of $H$ at 0. In the more general case with controlled inputs, the water level dynamics becomes:

$$dH_t = -I_t C(H_t)dt - \nu_t dt + \mathbf{1}_{\{H_t>0\}}(\mu dt + \sigma dW_t) + dZ_t, \quad (1)$$

where $I_t \in \{0,1\}$ is the turbine status at time $t$ (0 = closed, 1 = open), $\nu_t \in [0,+\infty)$ is the spillways status (i.e., the amount of water extracted without producing energy), and $C$ is a deterministic function describing the amount of water extracted in order to produce power, defined on $[h_0,+\infty)$ where $h_0$ is the level of the turbine. In the mathematical model for the dam dynamics, several structural assumptions are essential for defining the optimization problem. As the zero level of the reference frame is assumed to be the bottom of the reservoir, note that $h_0$ can be negative. Changing the turbine status implies a cost $\kappa$, and the turbine must close if the water level drops below minimum operating threshold $h_-$. Finally, a penalty, governed by a Lipschitz continuous function $\psi$, is applied when the water level exceeds a critical threshold $h_+$. The control variables of the problem can then be resumed to be the following: $(\tau_k)_{k\geq 1}$, denoting the switching times for the turbine status, and $\nu_t$, representing the status of the extra flow allowing to deplete the dam in emergency situations. Using physical reasoning, we can derive some of the terms in (1). In particular, the function $C$ is:

$$C(h) = Q\frac{v}{S} = Q\frac{\sqrt{2g\,(h-h_0)}}{S}, \quad (2)$$

where $g$ is the gravity constant, $Q$ is the penstock's open section, $S$ is the basin's surface, and $v$ is the water speed at the conduct entrance. Similarly, the spillover impact is defined as:

$$\nu_t = \beta_t\sqrt{2g\,(H_t-h_0)}, \quad (3)$$

where $\beta_t \in [0,\bar{\beta}]$ does not depend on $H_t$. Note that $\nu_t$ is proportional to the spillover pipe's cross-sectional area and to the square root of the free height $(H_t - h_0)$. Furthermore, the power produced, in terms of energy units, is given by:

$$\mathcal{E}_t = \sqrt{2}(1-\chi)Q_t[g(H_t-h_0)]^{3/2}, \quad (4)$$

where $\chi$ is a dispersion coefficient. As a crucial issue for power production is to keep the frequency of the alternate electric current constant, the open section area is regulated by servo-mechanism in such a way that water flowing per second $Q_t$ keeps constant the rotational frequency of the turbine. That implies that $\mathcal{E}_t$ could be considered constant, and will be simply denoted by $\mathcal{E}$.
We also introduce a model for the spot price of the power produced by the dam:

$$dP_t = P_t\,(\lambda dt + \gamma dB_t)\,, \quad (5)$$

where $B_t$ is a Brownian motion, correlated with $W_t$ through the correlation coefficient $\rho \in (-1,1)$, reflecting the impact of weather conditions on both water levels and other power plants such as solar and thermal plants.

## 2.2. Problem Formulation

From now on, we use as control variables the sequence of turbine switching times $(\tau_k)_{k\geq 1}$, and the regulation of free flow $\beta_t$. The sequence $(\tau_k)_{k\geq 1}$ represents non-decreasing $\mathbb{F}$-stopping times, at which the dam manager can switch the turbine status between closed and open. To track regime changes at each $\tau_k$, the turbine status $(I_t)$ assumes the form of a controlled Markov chain:

$$I_t = \frac{1}{2}\left[1-(-1)^{I_0+N_t}\right], \quad (6)$$

where $N_t$ counts the number of status switches by time $t$. The second control variable $\beta$ is an $\mathbb{F}$-predictable process. Letting $\mathcal{B}$ the set of such processes, $\beta \in \mathcal{B}$, the set of admissible controls is defined as:

$$\mathcal{A} = \{\alpha = (\beta, (\tau_k)_{k\geq 1})\}\,. \quad (7)$$

When the water level exceeds the safety threshold $\bar{h} > h_+$, the entire problem ends, and the dam becomes a nation-wide issue. The time horizon $T$ for the control problem is $T = \inf\{t>0 \mid H_t^\alpha \geq \bar{h}\}$.
The optimization problem in general form aims to maximize the total expected profit by selecting a control strategy $\alpha = (\beta,(\tau_k)_{k\geq 1}) \in \mathcal{A}$, given the initial state $(x,i) \in \mathbb{R}^n \times \mathbb{I}_m$ with $\mathbb{I}_m = \{1,....m\}$, thus the associated value function can be expressed as:

$$v_i(x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{i,x,\alpha}\left[\int_0^T e^{-rt} f\left(X_t^{x,i,\alpha},\alpha, I_t^{i,\alpha}\right) dt - \sum_{k=1}^{T} e^{-rt} g_{i_{k-1},i_k}\right], \quad (8)$$

where $r > 0$ is the discount rate and $X_t$ evolves according to the SDE:

$$dX_t = b(X_t,\alpha,I_t^{\alpha,i})\,dt + \sigma(X_t,\alpha,I_t^{\alpha,i})\,dW_t, \quad (9)$$

with $W_t$ being an $n$-dimensional Brownian motion and the coefficients $b$ and $\sigma$ satisfying a uniform Lipschitz condition with respect to $X_t$.
The general value function (8) can be expressed in terms of the parameters governing the dam and the optimization problem. The state variable is $X_t = (H_t, P_t)$. The function $f$ depends on the state $X_t$, the control $\alpha$, and the turbine status $I_t$, and it represents the revenue obtained (computed as the product of the energy produced $\mathcal{E}$ and the price $P_t$) and incorporates penalties through $\psi$ for water levels exceeding $h_+$. The cost function $g$ accounts for

the switching cost $k$. Making everything explicit, (8) becomes:

$$v_i(h,p) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{i,h,p,\alpha}\left[\int_0^{T-} e^{-rt}\mathcal{E}I_t P_t d - \int_0^{T-} e^{-rt}\kappa P_t dN_t - \int_0^{T-} e^{-rt}P_t\psi\left((H_t-h_+)^+\right)dt\right], \tag{10}$$

where $v_0$ is defined on $D^0 = [0,\bar{h}]\times\mathbb{R}^+$ and $v_1$ on $D^1 = [h_-,\bar{h}]\times\mathbb{R}^+$.

## 2.3. Hamilton-Jacobi-Bellman Equation

Once the value function has been defined, we can derive the associated HJB equations which play a central role in characterizing the optimal policy, as they provides a necessary condition for optimality by linking the value function to the dynamics of the system.
Integrating stochastic control theory with the optimal switching theory presented in [5], we derive the HJB equations in the form of variational inequalities for the stochastic optimal control problem with switching:

$$\min\left\{rv_i - \sup_{\alpha\in\mathcal{A}}\mathcal{L}^{(i,\alpha)}v_i - f_i, v_i - \max_{i\neq j}(v_j - g_{ij})\right\} = 0 \quad x\in\mathbb{R}^n, \quad i\in\mathbb{I}_m, \quad \alpha\in\mathcal{A}, \tag{11}$$

where $g_{ij}$ is the cost of switching from regime $i$ to $j$, $f$ is the revenue function and $\mathcal{L}$ is the Lagrangian differential operator of the underlying process of $X_t$, defined as:

$$\mathcal{L}^{(i,\alpha)}f = \left[b_i(x,a)\cdot D_x\phi + \frac{1}{2}\operatorname{tr}\left(\sigma_i\sigma_i'(x,a)D_x^2\phi\right)\right] \tag{12}$$

In our setting, the variational inequalities incorporate both the dynamics of the water level and electricity price, as well as the discrete switching between the two regimes, capturing the trade-offs between revenue generation, switching costs, and penalties. However, as the two Brownian motions $W_t$ and $B_t$ are correlated with a coefficient $\rho$, we cannot directly apply the previously developed theory. Instead, we have to perform a transformation, introducing the Brownian motion $\widehat{B}_t$ such that $B_t = \rho W_t + \sqrt{1-\rho^2}\widehat{B}_t$. This ensures that $\widehat{B}_t$ and $W_t$ are independent . Applying (11) to the dam model outlined above, we obtain:

$$\min\Big\{rv_i - \sup_{\beta\in[0,\bar{\beta}]}\mathcal{L}^{(i,\beta)}v_i + pf\left((h-h_+)^+\right) - i\mathcal{E}p, \quad (v_i - v_{1-i} + \kappa p)\mathbf{1}_{h\geq h_-} + (1-i)\mathbf{1}_{h<h_-}\Big\} = 0, \quad \text{on } \overset{\circ}{D^i},$$

where the Lagrangian differential operator $\mathcal{L}^{(i,b)}$ becomes:

$$\mathcal{L}^{(i,\beta)}\phi = [-i\frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{h-h_0} - \beta\sqrt{2g\,(h-h_0)^+} + \mu]\frac{\partial\phi}{\partial h} + \left(\frac{\sigma^2}{2}\frac{\partial^2\phi}{\partial h^2} + \sigma\gamma\rho p\frac{\partial^2\phi}{\partial h\partial p}\right)\mathbf{1}_{h>0} + \lambda p\frac{\partial\phi}{\partial p} + p^2\frac{\gamma^2}{2}\frac{\partial^2\phi}{\partial p^2}.$$

For any $p\in\mathbb{R}^+$, we also have the following boundary conditions:

$$v_0(\bar{h},p) = 0; \quad \frac{\partial v_0}{\partial h}(0,p) = 0,$$
$$v_1(\bar{h},p) = 0; \quad v_1(h_-,p) = v_0(h_-,p) - \kappa p.$$

## 2.4. Reduced Dimensionality Model

In our model the value functions depend on the price $p$ and the level $h$. To simplify, we reformulate the problem into a single-variable function $w_i$, streamlining computations and representation.

**Proposition 2.1.** *Let $\mathbb{P}_*$ be the probability measure equivalent to $\mathbb{P}$ with Radon-Nikodym derivative $Y_t = e^{\rho\gamma W_t - \frac{1}{2}\rho^2\gamma^2 t}$. For $i\in\{0,1\}$, we have $w_i(h) = \frac{v_i(p,h)}{p}$ on $D^i$, where we have set*

$$w_i(h) = \sup_{\alpha\in\mathcal{A}_W}\mathbb{E}_*^{i,h,\alpha}\left[\int_0^T e^{-(r-\lambda)t}\mathcal{E}I_t dt - \int_0^{T-} e^{-(r-\lambda)t}\kappa dN_t - \int_0^T e^{-(r-\lambda)t}f\left((H_t-h_+)^+\right)dt\right], \tag{13}$$

*where $\mathcal{A}_w$ is the restriction of the set of admissible controls to the stopping times with respect to the filtration $\mathbf{F}_W$, generated by the Brownian motion $W$, i.e. $\mathcal{A}_W = \{\alpha = (\beta,(\widehat{\tau}_k)_{k\in\mathbb{N}}) \mid \beta\in\mathcal{B}$ and $(\widehat{\tau}_k)_{k\in\mathbb{N}}$ is a non decreasing sequence of $\mathbf{F}_W$-stopping times $\}$.*

We now present two key properties of $w_i$, which are crucial for understanding the behavior and stability of the value function within the framework of the stochastic optimal control problem.

**Proposition 2.2.** *Assume $r>\lambda$, then the reduced value function $w_i$ is finite and $w_i \leq \mathcal{E}/(r-\lambda)$.*

**Proposition 2.3.** *For $i\in\{0,1\}$, $w_i$ is Lipschitz.*

### 2.5. Dynamic Programming Principle

While the HJB equation provides necessary conditions for optimality, the DPP offers a recursive framework to ensure that the policy remains optimal at each stage and decision point. For $\theta$ any $\mathbb{F}$-stopping time,

$$w_i(h) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{i,h,\alpha}\left[\int_0^{(\theta\wedge T)} e^{-(r-\lambda)t}\mathcal{E}I_t dt - \int_0^{(\theta\wedge T)} e^{-(r-\lambda)t} f\left((H_t - h_+)^+\right) dt - \int_0^{(\theta\wedge T)-} e^{-(r-\lambda)t}\kappa dN_t + e^{-(r-\lambda)(\theta\wedge T)} w_{I_{\theta\wedge T}}\left(H_{\theta\wedge T}\right)\right]. \tag{14}$$

The DPP essentially simplifies the task of finding an optimal control policy by taking switching into account and dividing it into smaller and more manageable subproblems. Specifically, at each potential stopping time $\theta$, the approach involves evaluating whether to continue with the current regime or switch to a new one. This decision is made by comparing the expected value of rewards and costs accumulated up to $\theta$ with the expected value of switching to a new regime and the future value of the state after $\theta$.

## 3. Viscosity Solution

Since our value functions are generally not sufficiently regular, traditional methods cannot be used to characterize the solution. Consequently, the viscosity solution framework is employed to address the problem, as it relaxes the smoothness hypothesis. This approach allows us to characterize the value functions as the unique viscosity solutions to the system of HJB equations associated with the stochastic control problem with switching.

**Proposition 3.1.** *Let $\{\varphi_i\}_{i=0;1} \in C^2$ on $[0,\bar{h}]$ such that $\varphi(\bar{h}) \geq 0$ and*

$$\begin{aligned} &0 \leq \min\{(r-\lambda)\varphi_0 - [(\mu+\sigma\rho\gamma)\varphi_0' + \\ &\bar{\beta}\sqrt{2g\,(h-h_0)}\,(-\varphi_0')^+ + \frac{\sigma^2}{2}\varphi_0''\Big] + f\left((h-h_+)^+\right); \\ &(\varphi_0 - \varphi_1 + \kappa)\,\mathbf{1}_{(h\geq h_-)} + \mathbf{1}_{(h<h_-)}\},\quad \text{on } (0,\bar{h}), \end{aligned}$$

$$\begin{aligned} &0 \leq \min\Big\{(r-\lambda)\varphi_1 - [(\mu+\sigma\rho\gamma + \\ &\frac{\mathcal{E}}{Sg(1-\chi)\,(h-h_0)}\Big)\varphi_1' + \bar{\beta}\sqrt{2g\,(h-h_0)}\,(-\varphi_1')^+ \\ &+\frac{\sigma^2}{2}\varphi_1''\Big] - \mathcal{E} + f\left((h-h_+)^+\right); \varphi_1 - \varphi_0 + \kappa\Big\} \\ &\text{on } \left(h_-,\bar{h}\right), \end{aligned} \tag{15}$$

*then we have $w_i(h) \leq \varphi_i(h)$, for all $h \in (0,\bar{h})$ and for $i = 0,1$.*

We then have the PDE characterization of the value functions $w_i(h)$.

**Theorem 3.1.** *The value functions $w_i(h), i = 0,1$, are continuous on $(0,\bar{h})$ and are the unique viscosity solutions on $(0,\bar{h})$ with boundary data $w_i(\bar{h}) = 0, w_0'(0) = 0$, to the system of variational inequalities (15).*

The proposition provides a criterion to verify the validity of candidate value functions, while the theorem ensures that these value functions are indeed the unique solutions to the stochastic control problem, characterizing them as viscosity solutions. These results are fundamental to ensure that the solutions found are well defined and unique.

## 4. Numerical Approximation

In section 2 we introduced the model analytically. However, since the model involves complex expressions that cannot be solved in closed form, we need to rely on numerical approximation techniques to solve it. To this end, we opt for a deterministic approach based on a finite difference scheme, which transforms the problem into a controlled Markov chain. This approach is commonly used in operational research, see for example [2, 4].

### 4.1. Formulation of the Approximation Problem

Using the discretization method proposed in [1], we carefully derive the numerical expression that allows us to explicitly compute the approximate solution. As first step, we can introduce a discretization grid, which is fundamental for the formulation of our numerical problem. We define $N \in \mathbb{N}^*$ as the number of iterations that are used to solve the problem iteratively, and $\delta > 0$ as the discretization step used to model the independent variable $h$ of the value functions $w_0$ and $w_1$. Consequently we define the space grid as $\mathcal{G}_\delta = \{0,\delta,2\delta,\ldots,\bar{h}-\delta,\bar{h}\}$ where 0 and $\bar{h}$ represent the domain of the HJB equations.
Now we apply a finite difference scheme, considering the following approximations:

$$\begin{aligned} \frac{\partial w}{\partial h}(h) &\approx \frac{w(h+\delta)-w(h)}{\delta}\mathbf{1}_{\mu_h^i\geq 0} \\ &\quad - \frac{w(h-\delta)-w(h)}{\delta}\mathbf{1}_{\mu_h^i<0}, \\ \frac{\partial^2 w}{\partial h^2}(h) &\approx \frac{w(h+\delta)+w(h-\delta)-2w(h)}{\delta^2}. \end{aligned} \tag{16}$$

For sake of readability we introduce the following quantities:

$$\begin{aligned} h_1 &= h+\delta, \\ h_2 &= h-\delta, \\ \mu_h^i &= \mu + \sigma\gamma\rho - i\frac{\mathcal{E}}{Sg(1-\chi)}\frac{1}{h-h_0} - \beta\sqrt{2g\,(h-h_0)}. \end{aligned}$$

Rewriting the operator $\mathcal{L}_\star^{(i,\beta)}$, using the introduced notations, applying a finite difference scheme and substituting it into the HJB equations, we obtain the final discrete formulation of the solution for both states $i \in \{0,1\}$:

$$w_i(h) = \max\left\{ \max_{\beta\in\{0,\bar\beta\}} \frac{\sum_{j=1}^2 p_j w_i(h_j) + G_h^i \Delta t^\delta}{1+(r-\lambda)\Delta t^\delta}, \; w_{1-i} - \kappa \right\} \text{ on } \mathcal{G}_\delta \backslash \left\{0,\delta,\dots,h_- - \delta, \bar h\right\}, \tag{17}$$

$$w_0(h) = \max_{\beta\in\{0,\bar\beta\}} \frac{\sum_{j=1}^2 p_j w_0(h_j) + G_h^0 \Delta t^\delta}{1+(r-\lambda)\Delta t^\delta} \quad \text{on} \quad \{\delta, 2\delta, \dots, h_- - \delta\},$$

where:

$$p_1 = \frac{\sigma^2/2 + \delta\left(\mu_h^i\right)^+}{Q^\delta}, \quad p_2 := \frac{\sigma^2/2 + \delta\left(\mu_h^i\right)^-}{Q^\delta},$$
$$G_h^i = i\mathcal{E} - \psi((h-h_+)^+), \quad Q^\delta = \left|\mu_h^i\right|\delta + \sigma^2,$$
$$\Delta t^\delta = \frac{\delta^2}{Q^\delta}.$$

## 4.2. Numerical Scheme and Numerical Results

In order to explicitly compute the approximated solution of the discrete problem derived in (17), we introduce, for $i \in \{0,1\}$ and $n \in \mathbb{N}$, the following iterative scheme:

$$w_i^{(n+1)}(h) = \max\left\{ \max_{\beta\in\{0,\bar\beta\}} \frac{\sum_{j=1}^2 p_j w_i^{(n)}(h_j) + G_h^i \Delta t^\delta}{1+(r-\lambda)\Delta t^\delta}, \; w_{1-i}^{(n)} - \kappa \right\} \text{ on } \mathcal{G}_\delta \backslash \left\{0,\delta,\dots,h_- - \delta, \bar h\right\}, \tag{18}$$

$$w_0^{(n+1)}(h) = \max_{\beta\in\{0,\bar\beta\}} \frac{\sum_{j=1}^2 p_j w_0^{(n)}(h_j) + G_h^0 \Delta t^\delta}{1+(r-\lambda)\Delta t^\delta} \quad \text{on} \quad \{\delta, 2\delta, \dots, h_- - \delta\},$$

$$w_i^{(0)} \equiv 0,$$
$$w_0^{(n)}(\bar h) = 0, \qquad \frac{\partial w_0^{(n)}}{\partial h}(0) = 0,$$
$$w_1^{(n)}(\bar h) = 0, \qquad w_1^{(n)}(h_-) = w_0(h_-) - \kappa.$$

At this point, we have all the necessary components to implement (18). Algorithm 1 presents the pseudo code of the numerical scheme, which has been implemented in MATLAB. To simplify the notation, we introduce the following quantity:

$$V^{(n)}(i,h,\beta) = \frac{\sum_{j=1}^2 p_j w_i^{(n)}(h_j) + G_h^i \Delta t^\delta}{1+(r-\lambda)\Delta t^\delta}. \tag{19}$$

Algorithm 1 Iterative Scheme

1: Initialize parameters, and set discretization grid $\delta = \bar h/M$, $h_{val} = [\delta : \delta : \bar h - \delta]$, $h_{val\_1} = [0 : \delta : h_- - \delta]$, $h_{val\_2} = [h_- : \delta : \bar h - \delta]$
2: Initialize $w_0$ and $w_1$ to zero matrices of dimension $M \times N$
3: Initialization of Dirichlet and Neumann boundary conditions
4: Iterative Scheme on $\{\delta, 2\delta, \dots, h_- - \delta\}$
5: **for** $n = 2$ to $N$ **do**
6:  **for** $z = 2$ to $M/2$ **do**
7:  $h = h_{val\_1}[z]$
8:  $m_1 = max\left\{V^{(n)}(0,h,0), V^{(n)}(0,h,\bar\beta)\right\}$
9:  Update $w_0[\frac{h\cdot M}{\bar h}, n]$ with $m_1$
10:  **end for**
11:  Iterative Scheme on $\mathcal{G}_\delta \backslash \left\{0,\delta,\dots,h_- - \delta, \bar h\right\}$
12:  Case of value function $w_1$
13:  **for** $z = 2$ to $M/2$ **do**
14:  $h = h_{val\_2}[z]$
15:  $m_1 = max\left\{V^{(n)}(1,h,0), V^{(n)}(1,h,\bar\beta)\right\}$
16:  $m_2 = max\left\{m_1, w_0[\frac{h\cdot M}{\bar h}, n-1] - k\right\}$
17:  Update $w_1[\frac{h\cdot M}{\bar h}, n]$ with $m_2$
18:  **end for**
19:  Case of value function $w_0$
20:  **for** $z = 2$ to $M/2$ **do**
21:  $h = h_{val\_2}[z]$
22:  $m_1 = max\left\{V^{(n)}(1,h,0), V^{(n)}(1,h,\bar\beta)\right\}$
23:  $m_2 = max\left\{m_1, w_1[\frac{h\cdot M}{\bar h}, n-1] - k\right\}$
24:  Update $w_0[\frac{h\cdot M}{\bar h}, n]$ with $m_2$
25:  **end for**
26:  Set Dirichlet and Neumann boundary conditions
27: **end for**

All the involved parameters are summarized in Table 1, together with a brief description and the chosen value for the base scenario of the application of the iterative scheme. Furthermore, the penalization function $\psi$ is assumed to be quadratic with a coefficient of $10^{-3}$.

| Param | Description | Value |
|---|---|---|
| $\bar{h}$ | max. water level | 100 |
| $h_+$ | critical water level | 80 |
| $h_-$ | min. water level | 50 |
| $\bar{\beta}$ | max. flow of the spillover | 0.01 |
| $\mathcal{E}$ | energy production | 50 |
| $h_0$ | turbine position | -1 |
| $\chi$ | standard waste of a turbine | 0.05 |
| $\kappa$ | switch cost | 0.1 |
| $\mu$ | expected river flow level | 0.1 |
| $\sigma$ | volatility of river flow level | 0.05 |
| $\lambda$ | return of price model | 0.1 |
| $\gamma$ | volatility of price model | 0.05 |
| $r$ | discount rate | 0.2 |
| $\rho$ | Brownian motions correlation | 0.04 |
| g | gravity acceleration | 9.81 |
| S | renormalised surface | 1 |

Table 1: Parameters in the base scenario

The above implementation allows to illustrate the main results of the numerical approach with some graphical examples, also studying the behavior of the value functions when changing various model parameters.
Figure 1 and Figure 2 show the value functions $w_0$ and $w_1$ in the two operational states of the dam (closed and open turbine) as functions of the water level $h$ inside the basin. In Figure 1 we can observe that $w_0$ remains constant at zero until $h_- = 50$. This behavior is reasonable, since for values of $h$ below this threshold no electricity can be generated, consequently leading to zero profit. After that, both $w_0$ and $w_1$ start increasing, indicating that the dam is operating under optimal conditions. This trend continues until the water level exceeds a critical value $h^+ = 80$. At this point the penalty function is introduced, as an incentive to keep the water level under control (below $\bar{h} = 100$). Consequently, the spillways are opened, leading to a decrease in the water level as water starts flowing out of the reservoir.

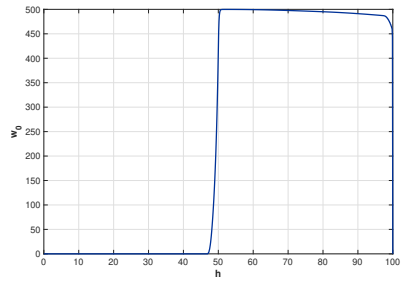


Figure 1: Value function $w_0$

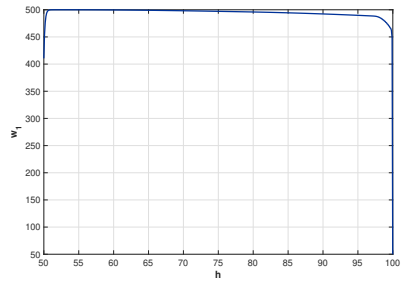


Figure 2: Value function $w_1$

Figure 3 illustrates the behavior of $w_0$ for different values of the electricity revenue $\mathcal{E}$. We can observe that, as the revenue from electricity increases, the value function also increases accordingly. This occurs because higher revenue makes electricity generation more valuable, which is reflected in the value function's increase. However, when the electricity revenue rises to 100, the value function stops being flat and continues to increase, almost up to the maximum water level $\bar{h}$. This result highlights the importance for a fair penalty system to incentivize the dam manager to keep the water level in a reasonable range, avoiding dangerous proximity to the maximum limit. When the value function is relatively flat over a large interval within $[h^-, h^+]$ and far from $\bar{h}$, it indicates that the manager is operating in a safe and controlled range, minimizing risks while still optimizing for revenue.

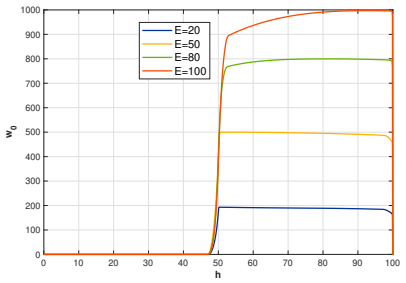


Figure 3: $w_0$ for different values of $\mathcal{E}$

We now focus on the penalization term more in detail. In our model, we assume that this term is quadratic and governed by two key parameters: the

coefficient of the quadratic term, which determines the strength of the penalty, and the critical water level $h^+$, at which the penalization mechanism is implemented. Figure 4 and Figure 5 zoom in on the interval $[h^+, \bar{h}]$ , allowing a closer comparison of how the value function changes with different values of parameters. The key observation is that adjusting the minimum height $h^+$, which marks the threshold where penalties begin to apply, has a more significant impact on the value function than changes to the quadratic term of the penalty.

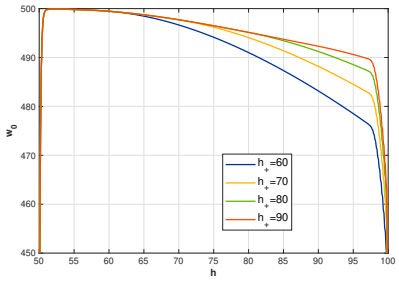


Figure 4: $w_0$ for different critical water level $h_+$

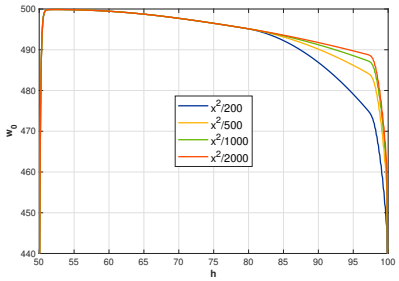


Figure 5: $w_0$ for different coefficients of the penalty function.

## 5. Conclusions

In the current transition to a low-carbon economy, one of the most prominent issues is that the most commonly used renewable energy sources, such photovoltaic and wind, are non-dispatchable. In this context, dam management plays a crucial role in terms of reducing reliance on coal and oil for ecological reasons and addressing the instability of solar and wind energy production. Additionally, the price of electricity has substantial economic impacts [3].
This thesis contributes to dam management by determining the optimal strategy for turbine operation in hydro-power production, and it focuses on the physical aspects of generation while incorporating regulatory constraints. Our findings show that, under reasonable assumptions, fluctuations in electricity prices do not impact the optimal policy, which is mainly determined by the water level in the dam. As previously discussed, one of the key advantages of hydropower is its dispatchability. This characteristic opens up additional considerations on the possibility to store the energy produced when it is available to use it during periods when it is not. Future research could build upon the analysis presented in [3] by expanding our model to include multiple interconnected dam-based plants, allowing not only for the discharge of water to produce electricity but also for the pumping of water into the higher reservoir. This would enable optimal use of energy based on fluctuations of the electricity prices. Additionally, climate change and increased agricultural demand are expected to impact water levels in artificial lakes, which may also be considered for more accurate modeling.

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