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ANALYSIS OF SOME FLUID-STRUCTURE INTERACTION PROBLEMS IN CHANNELS

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Abstract

This thesis is concerned with the analysis of some fluid-structure interaction problems in channels. The physical motivations mostly stem from the phenomenon of wind interacting with suspension bridges. In particular, our attention focuses on the instabilities which might affect the deck, which is the most sensitive part of the structure. Beside inducing on the deck some static effects, such as *lift* and *drag* forces, the wind generates dynamical instabilities, among which we count *vortex-induced vibrations*, *buffeting*, *one-degree of freedom instability* and *flutter*. This type of instabilities occurs in general during the interaction between a fluid and a structure, whenever the fluid's dynamic loading excites the natural modes of the structure.

While the motivations come from physics, the nature of our analysis is essentially theoretical. Starting from some models apt to describe the desired phenomena, we investigate their purely analytical properties.

First, we establish some existence and uniqueness results. Beside serving as a preliminary step, these results are interesting per se, since we treat the case of fluid-structure problems with non-homogeneous boundary conditions, still partially unexplored in the existing literature, both in a stationary framework and in a full evolutionary fluid-structure interaction framework. In particular, in the static case we study the connection between the multiplicity of solutions generating under large enough data and the apperance of forces acting on the fixed obstacle. In the dynamic case, we adapt some existing techniques for well-posedness to the non-homogeneous case, also dealing with the issue of collisions.

Then, we dig into the longterm dynamics of fluid-structure interaction problems. In this way, we directly approach through a theoretical strategy fluid-structure instabilities. In this context, we will use notions from the theory of infinite-dimensional dynamical systems, like the one of global attractor, showing how it can be extended to the field of fluid-structure interaction. The purely theoretical description of the long time behaviour is partially combined with a numerical investigation, aiming at enriching the picture.

Contents

CHAPTER 1

Introduction

Naturally arising from the description of any physical phenomenon implying the reciprocal action between a fluid and a solid, fluid-structure interaction (FSI) as a purely mathematical research field already exhibits remarkable difficulties. The very setup required for the study of this kind of phenomenon is complex from the beginning. If FSI problems have been widely studied from the numerical and experimental point of view (see, for instance, [46, 102] and the literature therein), their rigorous mathematical analysis is quite a recent subject and only in the last decades ad hoc techniques have been made available to the mathematical community.

Rather than considering a fluid-body system at equilibrium, whose description began with Archimedes' principle, we are concerned with the evolution of a viscous fluid interacting with a structure that is free to move, which results into a coupled problem. Independently of the nature of the fluid and of the structure, which might be respectively compressible or incompressibile and rigid or elastic, the coupling between the two media expresses in three aspects. First, a kinematic condition imposes the fluid velocity to match with the structure velocity at the interface. The second transmission condition appears in the equation governing the motion of the body, translating the action-reaction principle. Finally, in the most interesting cases where the displacements of the structure are not negligible, the fluid domain depends on time through the solution itself, which makes the variation of the domain part of the unknowns.

In this thesis, we will be concerned with a Newtonian, viscous, incompressible fluid governed by the Navier-Stokes equations interacting with a rigid body. We are interested in aerodynamics modelling. In particular, the physical motivations mostly lie in the phenomenon of wind interacting with suspension bridges. We will consider problems which are set on channels, thus representing either wind tunnels or the atmospheric boundary layer surrounding the bridge.

Since we will discuss ideal models, we could interpret them in the light of other physical frameworks. In this introduction we focus on the mathematical overview of the problem, leaving the precise presentation of the physics behind the considered models in the next chapters. The wind blowing on a bridge gives rise to a miscellaneous lump of effects, producing both static and dynamics actions.

The introduction has a double purpose. First, we retrace to the best of our knowledge a brief state of the art of the mathematical study of the motion of a rigid solid inside a viscous fluid, concerning well-posedness and long time behaviour. Specifically, we highlight which are the difficulties as well as the open problems concerning both aspects. Then, we present the structure of the thesis, focusing on our contributions to the subject.

1.1 State of the art

Well-posedness Concerning well-posedness, standard methods ([57, 134]) are not of direct application due to the fluid domain depending on time. Thus, all efforts are devoted to conceiveing either new techniques or different ways to apply standard techniques.

A germinal paper is the one by D. Serre [124], which followed a previous work by H. Weinberger [137], where the author proves existence for the falling rigid body problem inside a viscous incompressible fluid occupying $\mathbb{R}^3$ (see also [61]). The author invokes a simple change of coordinates, by considering a reference frame attached to the obstacle so that the fluid domain consequently loses its dependence on time.

However, it is only after 2000 that we witness a breakthrough in the development of the theoretical results concerning fluid-structure problems. In [31], Conca, San Martin and Tucksnak adopt the same change of coordinates used in [124] to prove existence of weak solutions for a problem describing the interaction of a spherical body with a viscous incompressible fluid inside a bounded domain of $\mathbb{R}^3$, but, since dealing with a bounded domain, the change of reference frame is combined with a *penalty method*, first devised by Fujita and Sauer [56] to solve the Navier-Stokes equations inside a domain moving with a prescribed law. Besides that given in [31], several definitions of weak solutions have been adopted, as well as several techniques to find such solutions. The methods introduced in [41, 42, 81, 87] are all based on approximating procedures. In [41, 42] the authors build a sequence of suitable regularized problems, while in [87] they approximate the solid bodies by very viscous fluids. Finally, the solutions in [81] are approximated by time discretized problems. We emphasize that all of these results have been shown under the hypothesis that no-collision occurs between the obstacle and the boundary of the fluid domain.

Moving to the framework of strong solutions, a pioneering result can be found in [79]. The strategy, anticipated in [78], consists in several steps and allows to prove existence and uniqueness, still assuming the absence of collisions. After rewriting the fluid equations in Lagrangian coordinates, the authors prove, through the contraction mapping principle, the existence of a solution for a *given* velocity of the obstacle and small enough time. This gives rise to a new velocity at the interface, which generates a second fixed point argument allowing to recouple the problem. A fixed point argument is also used in [130] to obtain existence and uniqueness of strong solutions for a fluid-rigid body system in a bounded domain. Here, the author adopts a change of variables inspired by [93], by which he is able to set the equations on a fixed domain. Solutions are global-in-time provided that the rigid body does not touch the boundary of the channel. For what concerns strong solutions, we also mention the results in [34, 62, 131], where the fluid is assumed to fill the whole space. Finally, the no-slip case is treated in [5].

Collisions are a delicate issue in the context of FSI problems, especially because different phenomena occur in the context of strong or weak solutions. The above considerations suggest that collisions are strictly related to well-posedness in the context of FSI problems, affecting the possibility to obtain solutions with a global-in-time character. Within the realm of weak solutions, no general results are available on whether collisions do occur in finite time or not. However, as shown in [119], weak solutions exist globally even if collisions occur in finite time, because they are characterized by vanishing relative velocity and acceleration; this result holds under regularity hypothesis on the boundary of the body and on the fluid domain (see also [128]). In some particular 2D cases, where the contact surfaces are regular enough, collisions are outright excluded, as it was indipendently shown in [83] and [84] (see also [128]). The method suggested in [84] is based on the construction of a particular test function depending on the inverse of the distance between the contact surfaces, which is multiplied by the fluid equations. Suitable estimates are then obtained to some bounds in terms of the inverse of the distance, which prevent collisions. An extension of [84] to the 3D case was given in [85]. However, this no-collision result appears physically unrealistic both at the macroscopic and microscopic scales, as pointed out in [72, 73] where it is suggested that the flaw lies in the modelling. Among the most common explanations to the paradox we find *roughness*, which should be included to recover the occurrence of collisions, and the *no-slip* condition, which is no longer valid when the interactions between the solids are of the order of micro- or nanoscale (see [88]). Consequently, in order to let the fluid slip on the boundary of the solids, in literature different boundary conditions have been explored, such as the Navier slip boundary conditions (see [25, 74, 75, 109, 136]) or, very recently, the Tresca boundary conditions (see [86]). While the Navier conditions render the effect of the fluid slipping whatever the size of the shear on the boundaries might be, the Tresca conditions impose the fluid to stick to the solid interface until a shear-rate threshold is reached after which purely Navier conditions appear.

Besides deciding on their global-in-time character, collisions also influence uniqueness of solutions. Again, we need to distinguish between weak and strong solutions. Starovoitov [129] was able to construct at least two weak solutions admitting collision in finite time; for the first solution the body goes away from the boundary after the collision, while in the second solution the body and the boundary remain in contact. Hence, in order to guarantee uniqueness of weak solutions, one needs to exclude collisions. In particular, while uniqueness of strong solutions is a known fact both in two and three-dimensions, and both for no-slip case and the slip case (see [5, 34, 79, 130]), in [76] the authors proved a uniqueness result for weak solutions for a two-dimensional fluid-rigid body system, and in [19] the author proved it for the slip case. In both papers, the strategy is based on the application of a suitable change of variables allowing to compare two different solutions, which a priori would be defined on different domains due to the time-dependency of the fluid domain.

Although many aspects concerning well-posedness have been investigated in the existing literature, it is important to mention that these results have been obtained under the assumption of homogenous boundary conditions at the boundary of the domain occupied by the fluid and the body. In all of the existing cases, the motion of fluid is driven by initial conditions and/or external forces. Although adding further difficulties, assuming that the motion is driven by boundary conditions is more physical. Moreover, it fits for the purpose to represent phenomena concerning the action of the wind over bridges. Indeed, already in the case of a stationary fluid impinging a fixed obstacle, one can study the static effect exerted by the average wind to the structure; provided that the non-homogeneous boundary conditions driving the flow as well as the domain satisfy some symmetry assumptions (see [54, 55, 98, 108]), in [68] the authors were able to prove in fact that the appearance of a *lift force* acting on the obstacle is connected to the non-uniqueness of solutions to the corresponding stationary problem.

The general procedure to deal with non-homogeneous boundary conditions implies lifting them by building a solenoidal extension, which is nothing but a divergence-free function carrying along the whole domain the flux imposed by the boundary conditions. After subtracting this extension to the original solution of the problem, one is led to cope with an homogenous problem, where the effect of the boundary conditions appears in some extra terms. In order to let this function effectively act as a flux-carrier, it is fundamental to guarantee that there exists a separation strip between the rigid body and the boundary of the surrounding domain, whose width enters in the construction of the solenoidal extension. This is always guaranteed if the obstacle is fixed, but as soon as the obstacle is free to move as in the case of a full FSI problem, one needs to prove a no-collision result. However, the solenoidal extension itself appears in the estimates needed to infer a control on the distance between the contact surfaces, generating a "looping" problem.

Long time behaviour The natural step after exploring well-posedness is to look at the long time behaviour of the fluid-body system. Concerning this topic, several questions can be addressed, belonging to different ways to understand the problem, all borrowed from the theory of dynamical systems. Here, we are interested in three main aspects.

Preliminarily, continuous dependence on the initial data must be established. It is well-known that, by considering the same initial-value problem, this result also yields uniqueness. If in the theory of the Navier-Stokes equations in domains with fixed boundaries continuous dependence on the initial data is a standard result (see for instance [100]), in the case of variable domains, as for FSI systems, the problem is by far non-trivial, since the two solutions are defined in different domains and an energy identity of perturbations cannot be obtained by taking the difference between the two weak formulations. This issue is explored in [80] (which preceeds [76], where a result of uniqueness of weak solutions for a fluid-body system is obtained). In this paper, the authors prove a result of continuous dependence on the initial data for a FSI problem, where the structure is supposed to be a viscoelastic thin body, having in mind cardiovascular applications. The method consists in applying a suitable coordinate transformations so as to write both solutions in the same time-dependent domain, thus preserving the problem in Eulerian coordinates.

A significant information on the asymptotic behaviour of the fluid-structure system can be gained by establishing whether the rigid body stabilizes around some position in the domain or not. The issue is well understood in the classical theory of the Navier-Stokes equations (see for instance [114]), but not many results are available for a moving body inside a Navier-Stokes fluid. It is worth mentioning the result by Fereisl and Nečasová [49], where the authors consider the simplest situation of a rigid ball in a viscous fluid occupying a two-dimensional bounded domain. Introducing the effect of gravity, it is shown that the rigid body approaches in the asymptotic regime a static state corresponding to the contact with the bottom boundary of the container. When the fluid fills the whole plane $\mathbb{R}^2$, a result has been obtained in [47], where the authors obtain some time-decay estimates for the solutions to a Navier-Stokes fluid-rigid disk system which suggest that the trajectory of the mass centre of the ball is possibly unbounded in the long term. The same problem is studied in $\mathbb{R}^3$ in [48], where it is proven that the rigid body eventually stabilizes around some position at infinite times.

In order to fully reinterpret the problem in the spirit of dynamical systems theory, so as to draw notions from a further branch of mathematics and characterize the system to a greater extent, one might view the solution to the fluid-body system as a trajectory in an appropriate phace space. Well-posedness immediately proves the existence of a map yielding from all initial data the solution at any positive time. One then usually investigates the existence of small (in a suitable sense) subsets of the phase space able to confine the longterm dynamics, namely to substantially reduce the degrees of freedom of the system. To this end, the most effective tool in the theory of infinite-dimensional dynamical systems is the notion of *global attractor*, which is a compact subset of the phase space to which all the solutions to the problem eventually approach. But since in this context the fluid domain and the phase space for the solutions are time-dependent, the very definition of such an object introduces a major difficulty: there is no way to describe the solutions in terms of a semigroup, and not even in terms of a process [27].

For this reason, the existence of a global attractor for the whole fluid-structure-interaction problem with a time-dependent fluid domain has not been treated in literature, and, to the best of our knowledge, only partial results are known. On the one hand, a part of the literature is devoted to the study of the longtime behaviour of fluid-plate interaction models, see, e.g., [28–30] with a *fixed* fluid domain. On the other hand, the longterm dynamics of the Navier-Stokes equations set on time-varying domains has been studied only when the motion of the domain is prescribed and sufficiently smooth, see [126]; this allows to reformulate the problem on a fixed domain by a coordinate transformations and to apply the techniques for non-autonomous systems, see [22, 23, 95, 106].

1.2 Structure of the thesis

The thesis is organized as follows.

In **Chapter 2** we present the general characteristics of the FSI models later considered in Chapter 4-5-6. The first model is a modification of the one introduced in [15], while the second model is a natural extension of the first one, considered in [16].

In **Chapter 3**, we study the static effects that the wind produces when interacting with suspension bridges. We consider the behaviour of a fluid governed by the steady Navier-Stokes equations impinging a fixed obstacle in a three-dimensional bounded domain, assuming that at the boundary the flow is of *Poiseuille* type. This problem is not to be considered a FSI problem in the sense given above. Nonetheless, this configuration allows to predict the appearance of a *lift force* acting on the obstacle. Indeed, in a symmetric configuration, it can be proven that the occurrence of lift forces is a consequence of non-uniqueness of solutions to the considered steady problem: obtaining an explicit threshold on the incoming flow ensuring uniqueness gives the threshold for the appearance of lift. This requires building an explicit solenoidal extension of the prescribed Poiseuille flow and bounding some embedding and cutoff constants. The chapter is based on [66].

In **Chapter 4**, we proceed to the description of the dynamic effects induced by the wind on suspension bridges, focusing on the *flutter* phenomenon. We consider a FSI problem for a Poiseuille flow through a *bounded* two-dimensional channel containing a rectangular rigid body, which is only free to move in the vertical direction. As explained in Section 4.1 this model is suitable to represent the first stage of the flutter phenomenon as in a wind tunnel experiment. We deal for the first time with the treatment of non-homogeneous boundary conditions in the context of FSI problems, and we prove the existence and uniqueness of a global-in-time weak solution, ensuring the absence of collisions by introducing a restoring force in the equation governing the motion of the body. The chapter is based on [111].

In **Chapter 5**, the critical stage of the flutter phenomenon is studied, by extending the FSI model analyzed in Chapter 4 to an *unbounded* two-dimensional channel. We consider the full coupled vertical-torsional motion of the body and we prove a global-in-time existence result for weak solutions, up to collision. The problem is treated by introducing a new technique based on a double-approximating procedure, which requires building a sequence of strong solutions for a penalized problem. The chapter is based on [16].

Chapter 6 is concerned with the longterm dynamics of the FSI problem considered in Chapter 4. In this way, we study in mathematical terms the practical problem of which permanent state will be observed after a transient period in a wind tunnel experiment, where an air flow interacts with the deck of a bridge. As previously mentioned in Section 1.1, the fluid domain being part of the unknowns prevents from borrowing notions from the classical theory of semigroups and processes. The main contribution of the chapter is in fact the extension of the notion of *global attractor* to the setting of FSI problems, and the proof of its existence and regularity. Furthermore, when the prescribed inflow is sufficiently small, we are able to prove that the solution to the FSI evolution system converges to the unique stationary solution, corresponding to a perfectly symmetric configuration. This gives an explicit characterization of the attractor in a particular case. To complete the study, we intervene through numerical simulations giving a qualitative description of the attractor for any intensity of the incoming flow. The chapter is based on [65, 112].

In **Chapter 7**, we draw the concluding remarks and provide some future perspectives to our study.

CHAPTER 2

The Fluid-Structure Interaction models

In this chapter, we collect the main features of the two fluid-structure interaction models studied in Chapters 4-5-6, to avoid repetitions in the sequel. Here, we only present general characteristics of the models to be considered later. Chapters 4 and 6 deal with the same model. Any specific hypothesis, depending on the goal of the study to be tackled in a given chapter, will be detailed in this one.

In both models, we consider a fluid-structure interaction problem for a Poiseuille flow through a 2D channel containing an obstacle, but in the first model the channel is bounded while in the second one it is unbounded. The obstacle is a rigid body and it is free to interact driven by the action of the fluid flow, smooth elastic restoring forces and viscous damping forces.

We denote by B a general rigid body immersed in a channel. According to the chapter, B can be either a rectangular or an elliptical rigid body, and the channel can be either unbounded or bounded. In particular, let $\delta < 1$ and $L \gg \delta$, $I \gg L$. In Chapter 4-6, we have that

$$B = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; |x_1| \leq d \wedge |x_2| \leq \delta \right\}$$

has a rectangular geometry and it is immersed in

$$\mathcal{R} = (-I, I) \times (-L, L), \tag{2.0.1}$$

while in Chapter 5

$$B = \left\{ (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \frac{x_1^2}{d^2} + \frac{x_2^2}{\delta^2} \leq 1 \right\}$$

has an elliptical geometry and it is immersed in

$$A = \mathbb{R} \times (-L, L)\,. \tag{2.0.2}$$

The reasons for these choices will be precised in each of the aforementioned chapter. Without loss of generality, we always take $d = 1$ so that d is the reference length unit. We denote by $\Gamma_{\mathcal{R}} = \partial\mathcal{R}$ the boundary of the channel $\mathcal{R}$ in (2.0.1). The upper and lower boundaries of the channel A in (2.0.2) are instead given by $\Gamma_A = \mathbb{R} \times \{-L, L\}$. The parameters h and θ respectively denote the vertical displacement of the barycenter of the rigid body and its rotation from the equilibrium line $x_2 = 0$, see Figure 2.1.

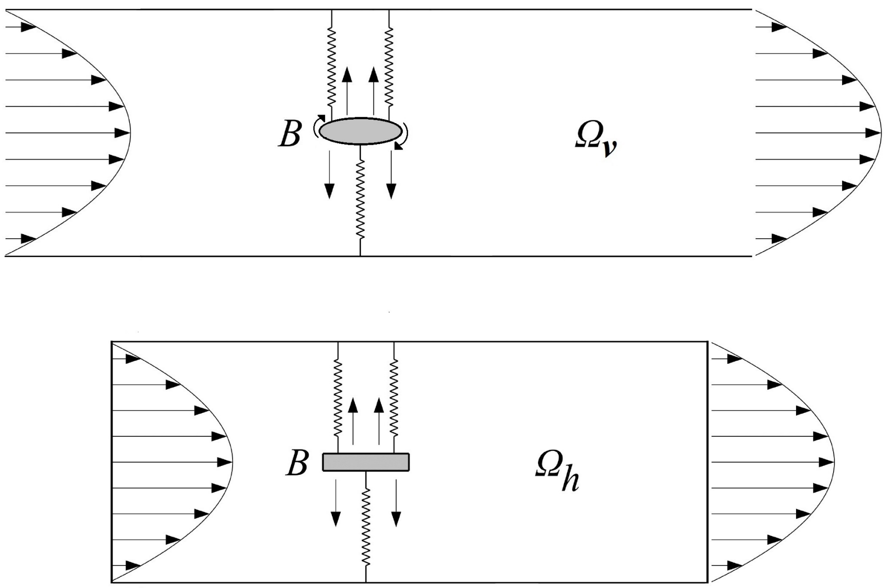


Figure 2.1: *Above: the unbounded channel with the vertically moving and rotating elliptical obstacle B. Below: the bounded channel with the vertically moving rectangular obstacle B.*

Thus,

$$B_v = Q(\theta)B + h\,\widehat{e}_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} B + h\,\widehat{e}_2 \qquad \forall\,(h,\theta) \in A_{d,\delta} \tag{2.0.3}$$

tracks the position of the body after the vertical translation and rotation. Here, $A_{d,\delta}$ denotes the set of admissible values for (h, θ), that is

$$A_{d,\delta} = \left\{ (h,\theta) \in \mathbb{R}^2 \;\middle|\; |\theta| < \frac{\pi}{2} \quad \text{and} \quad |h| + d|\sin\theta| + \delta\cos\theta < L \right\}, \tag{2.0.4}$$

which excludes the possibility of collisions between the obstacle and the boundaries of the channel. Whenever $Q(\theta) \equiv \mathbf{I}$, being $\mathbf{I}$ the 2×2-identity matrix, the obstacle is only free to move in the vertical direction and

$$B_v = B_h = B + h\,\widehat{e}_2 \qquad \forall\,|h| < L - \delta.$$

Due to the motion of the rigid body, the domain occupied by the fluid is variable in time and is given by

$$\Omega_v(t) = A \setminus B_v(t)\,, \qquad \text{where } h = h(t) \text{ and } \theta = \theta(t) \text{ in (2.0.3) are functions of time.}$$

Accordingly, when $B_v = B_h$, then we will write

$$\Omega_v(t) = \Omega_{h(t)} = \mathcal{R} \setminus B_{h(t)}, \tag{2.0.5}$$

see Figure 2.1. For simplicity, in the sequel we will sometimes omit emphasizing the dependence on $t \in (0,T)$ and, with an abuse of notation, we will denote through a Cartesian product the non-cylindrical space-time domain given by

$$\Omega_v \times (0,T) = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R}_+ \,|\, x \in \Omega_v(t)\,,\ \ t \in (0,T)\}.$$

The same notation is used when $\Omega_v = \Omega_h$.

Let us denote by v_P a stationary Poiseuille flow on the channels with a prescribed pressure drop $p_0 > 0$, characterized by the function $v_0 = v_0(x_2)$ that solves

$$\begin{cases} v_0''(x_2) = -\dfrac{p_0}{\mu} \qquad \forall x_2 \in (-L, L)\,, \\ v_0(-L) = v_0(L) = 0\,, \end{cases}$$

so that

$$v_P(x_2) = v_0(x_2)\widehat{e}_1 = \lambda(L^2 - x_2^2)\widehat{e}_1 \qquad \forall x_2 \in [-L, L]\,, \tag{2.0.6}$$

with

$$\lambda = \frac{p_0}{2\mu}$$

regulating the intensity of the flow. We notice that v_P and the associated pressure $\pi_P(x_1,x_2) = -p_0\, x_1$, for every $(x_1,x_2) \in A$ and every $(x_1,x_2) \in \mathcal{R}$ satisfy the steady-state Navier-Stokes equations

$$-\mu\,\Delta v_P + (v_P \cdot \nabla)v_P + \nabla \pi_P = 0\,, \qquad \nabla \cdot v_P = 0$$

since $(v_P \cdot \nabla)v_P \equiv 0$ both in A and $\mathcal{R}$.

Let $m > 0$ and $\mathcal{I} > 0$ respectively be the mass of the body B and its moment of inertia. Moreover, we denote by $\beta_1 > 0$ and $\beta_2 > 0$ the damping coefficients associated to the viscous forces acting in the vertical and angular directions. We suppose, without loss of generality, that the density of the fluid is $\rho = 1$.

If the obstacle is only allowed to move in the vertical direction in the *bounded* channel $\mathcal{R}$, and if at the inlet and outlet section of the channel the velocity field reproduces the stationary Poiseuille flow v_P, we are lead to consider the **first** fluid-structure interaction evolution problem on the time-interval, which will be extensively studied in Chapter 4-6:

$$\begin{aligned} & u_t = \mu\,\Delta u - (u\cdot\nabla)u - \nabla p, \quad \nabla\cdot u = 0 \quad \text{in } \ \Omega_h \times (0,T)\,, \\ & u = v_P(x_2) \quad \text{on } \ \Gamma_{\mathcal{R}} \times (0,T)\,, \\ & u = h'\,\widehat{e}_2 \quad \text{on } \ \partial B_h \times (0,T)\,, \\ & mh'' + f(h) = -\widehat{e}_2 \cdot \int_{\partial B_h} \mathcal{T}(u,p)\widehat{n}\, d\sigma \quad \text{in } \ (0,T)\,, \end{aligned} \tag{2.0.7}$$

to which we associate the initial conditions

$$h(0) = h_0, \quad h'(0) = k_0, \quad u(x,0) = u_0(x) \quad \text{in} \quad \Omega_{h_0} = \Omega_{h(0)}\,,$$

for some $h_0, k_0 \in \mathbb{R}$. Here $u : \Omega_h \times (0,T) \longrightarrow \mathbb{R}^2$ and $p : \Omega_h \times (0,T) \longrightarrow \mathbb{R}$ are, respectively, the velocity vector field and the scalar pressure, while $\widehat{n}$ denotes the outward normal to $\partial\Omega_h$, thus directed towards the interior of ∂B_h. By compatibility, the initial data is supposed to satisfy the conditions

$$\begin{cases} \text{div}(u_0) = 0 \quad \text{in } \Omega_{h_0}\,, \\ u_0 = v_P(x_2) \quad \text{on } \Gamma_{\mathcal{R}} \times (0,T)\,, \\ u_0 = k_0\,\widehat{e}_2 \quad \text{on } \partial B_{h_0} \times (0,T), \end{cases} \tag{2.0.8}$$

where $B_{h_0} = B_{h(0)}$. The motion of the body is governed by the ordinary differential equation in $(2.0.7)_4$, where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an elastic restoring force, and $\mathcal{T}(u,p)$ is the strain tensor of the fluid flow. More precisely,

$$\mathcal{T}(u,p) = -p\mathbf{I} + 2\mu D(u) \quad \text{with} \quad D(u) = \frac{\nabla u + \nabla^\top u}{2}\,,$$

being $\mathbf{I}$ the 2×2-identity matrix, so that the right-hand sides of the ODE $(2.0.7)_4$ expresses the lift force exerted by the fluid on the body (see [68]). Further assumptions on f are collected in the paragraph below.

In the **second** problem, which will be studied in Chapter 5, we extend the previous model by assuming that B is also free to rotate around a pin located at its center of mass, inside the *unbounded* channel A. Moreover, some viscous damping forces describing the dissipation mechanism of the structure are also considered. At infinity, the velocity field of the fluid reproduces the prescribed Poiseuille flow. The fluid-structure interaction evolution problem on the time interval is then described by:

$$\begin{aligned} & u_t = \mu\,\Delta u - (u \cdot \nabla)u - \nabla p, \quad \nabla \cdot u = 0 \quad \text{in } \Omega_v \times (0,T)\,, \\ & \lim_{|x_1| \to \infty} u(x_1,x_2,t) = v_P(x_2) \quad \forall x_2 \in [-L,L]\,,\ t \in [0,T]\,, \qquad u = 0 \quad \text{on } \Gamma \times (0,T)\,, \\ & u = h'\,\widehat{e}_2 + \theta'(x - h\widehat{e}_2)^\perp \quad \text{on } \partial B_v \times (0,T)\,, \\ & mh'' + \beta_1\,h' + F_1(h,\theta) = -\widehat{e}_2 \cdot \int_{\partial B_v} \mathcal{T}(u,p)\widehat{n}\,d\sigma \quad \text{in } (0,T)\,, \\ & \mathcal{J}\theta'' + \beta_2\,\theta' + F_2(h,\theta) = -\int_{\partial B_v} (x - h\widehat{e}_2)^\perp \cdot \mathcal{T}(u,p)\widehat{n}\,d\sigma \quad \text{in } (0,T)\,. \end{aligned} \tag{2.0.9}$$

to which we associate the initial conditions

$$h(0) = 0, \quad h'(0) = h_0, \quad \theta(0) = 0, \quad \theta'(0) = \theta_0, \quad u(x,0) = u_0(x) \quad \text{in} \quad \Omega_0 = \Omega_v(0)\,, \tag{2.0.10}$$

for some $h_0, \theta_0 \in \mathbb{R}$. Notice that, for the sake of simplicity, we took the initial position $h(0)$ and rotation $\theta(0)$ of the obstacle equal to zero, but all the computations on the model can be easily generalized to a different case. Again, $u : \Omega_v \times (0,T) \longrightarrow \mathbb{R}^2$ and $p : \Omega_v \times (0,T) \longrightarrow \mathbb{R}$ are, respectively, the velocity vector field and the scalar pressure, while $\widehat{n}$ denotes the outward normal to $\partial\Omega_v$, thus directed towards the interior of ∂B_v. Again, by compatibility, the initial data satisfies

$$\begin{cases} \operatorname{div}(u_0) = 0 \quad \text{in } \Omega_0\,, \\ \lim\limits_{|x_1|\to\infty} u_0(x_1,x_2) = v_P(x_2) \quad \forall x_2 \in [-L,L]\,, \qquad u_0 = 0 \quad \text{on } \Gamma_A \times (0,T)\,, \\ u_0 = h_0\,\widehat{e}_2 + \theta_0 x^\perp \quad \text{on } \partial B_0 \times (0,T), \end{cases} \tag{2.0.11}$$

where $B_0 = B_v(0) = B$. The motion of the body is in this case governed by the ordinary differential equations in $(2.0.9)_4$-$(2.0.9)_5$, where the right-hand sides express, respectively, the lift force and the torque exerted by the fluid on the body. The forces $F_1, F_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ in the ODEs $(2.0.9)_4$-$(2.0.9)_5$ represent smooth elastic restoring forces, which are better characterized in the following paragraph.

Assumptions on the restoring forces The motion of the body in problem (2.0.7) is governed by $(2.0.7)_4$, where we assume that

$$f \in \mathcal{C}^1(-L+\delta, L-\delta;\mathbb{R}) \quad \text{is s.t.} \quad f(0)=0 \quad \text{and} \quad f'(h)>0\ \forall\, h \in (-L+\delta, L-\delta). \tag{2.0.12}$$

Thus, we can think about f as the derivative of some (positive) potential $F \in \mathcal{C}^2(-L+\delta, L-\delta;\mathbb{R}_+)$ given by

$$F(h) = \int_0^h f(s)\,ds. \tag{2.0.13}$$

From (2.0.12), it follows that $f(h)h > 0$ for all $h \neq 0$ and that there exists ρ such that $f'(h) > \rho > 0$ for all h. Hence, given (2.0.13), we obtain

$$f(h)\,h \geq F(h) \geq \frac{\rho}{2}h^2. \tag{2.0.14}$$

In problem (2.0.9), the body can also rotate, thus its motion is driven by the action of two elastic restoring forces in $(2.0.9)_4$-$(2.0.9)_5$. In particular, the restoring force F_1 and F_2 might be thought as a generalization to the extended model in (2.0.9) of the force f. We can think about these forces as the derivatives, with respect to the first variable and the second variable accordingly, of some (positive) potential $F \in \mathcal{C}^2(A_{d,\delta};\mathbb{R}_+)$. In particular, we assume that

$$\begin{cases} \dfrac{\partial F}{\partial h}(0,\theta) = \dfrac{\partial F}{\partial \theta}(h,0) = 0\,, \quad \dfrac{\partial^2 F}{\partial h^2}(0,0) > 0\,, \quad \dfrac{\partial^2 F}{\partial \theta^2}(0,0) > 0\,, \\ h\dfrac{\partial F}{\partial h}(h,\theta) > 0 \quad \text{if} \quad h \neq 0\,, \qquad \theta\dfrac{\partial F}{\partial \theta}(h,\theta) > 0 \quad \text{if} \quad \theta \neq 0. \end{cases} \tag{2.0.15}$$

We can take $F(0,0) = 0$ and, from (2.0.15), we obtain that the mixed second derivatives vanish when $h = \theta = 0$, thus

$$F(h,\theta) = \frac{\partial^2 F}{\partial h^2}(0,0)\frac{h^2}{2} + \frac{\partial^2 F}{\partial \theta^2}(0,0)\frac{\theta^2}{2} + o(\|(h,\theta)\|^2) \quad \text{as} \quad (h,\theta) \longrightarrow (0,0).$$

CHAPTER 3

An explicit threshold for the appearance of lift on the deck of a bridge

In the present chapter, we set up the analytical framework for studying the threshold for the appearance of a lift force exerted by a viscous steady fluid (the wind) on the deck of a bridge. We model this interaction as in a wind tunnel experiment, where at the inlet and outlet sections the velocity field of the fluid has a Poiseuille flow profile. Since in a symmetric configuration the appearance of lift forces is a consequence of non-uniqueness of solutions, we compute an explicit threshold on the incoming flow ensuring uniqueness.

3.1 The model

The *lift force* is the component of the total force exerted by the fluid over an obstacle which is perpendicular to the stream, see (e.g.) the Introduction in [2] and an updated state-of-the art in [68]. Since the airplane flight is based on lift, improving the lift characteristics of aircrafts is highly desirable. Instead, if the fluid interacts with an obstacle representing a structure in civil engineering (e.g., a bridge or a skyscraper), the lift is an unpleasant factor of instability which needs to be avoided. In order to evaluate the lift force exerted on a designed structure, engineers usually exploit wind tunnel tests. We set up our model in the same context; we intend to compute an explicit threshold for the appearance of lift on the deck of a scaled bridge. The experiment is illustrated in Figure 3.1. In the left picture we sketch the wind tunnel with a bridge within it while the right picture is taken during a wind tunnel experiment at Politecnico di Milano: the appearance of vortices around the plate (deck) generates a lift due to the asymmetry of the vortex shedding.

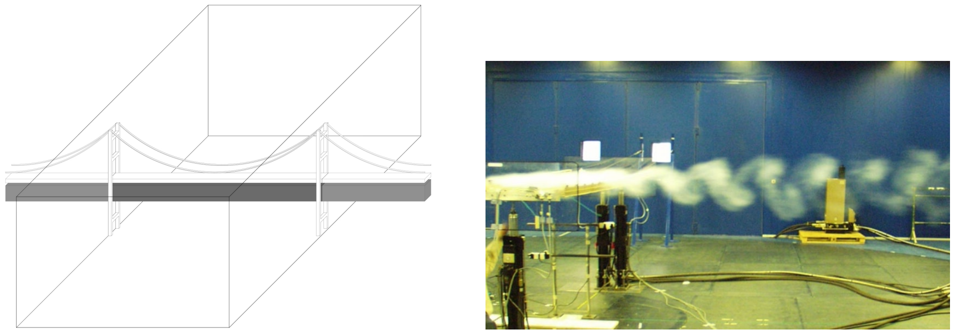


Figure 3.1: *Left: sketch of a bridge within a wind tunnel. Right: wind tunnel experiment at Politecnico di Milano.*

Denoting by Ω the 3D (non simply connected) domain consisting of a right parallelepiped (the wind tunnel) crossed by the plate, the fluid flow is assumed to be governed by the steady Navier-Stokes equations

$$-\mu\,\Delta u + (u\cdot\nabla)\,u + \nabla p = 0 \qquad \nabla\cdot u = 0 \qquad \text{in } \Omega, \tag{3.1.1}$$

where $u:\Omega\to\mathbb{R}^3$ is the unknown velocity vector field, $p:\Omega\to\mathbb{R}$ is the scalar pressure, μ is the coefficient of kinematic viscosity. We emphasize that we do not consider the action of any external force, in agreement with the experimental set-up in a wind tunnel where the flow is driven only by the inflow conditions. These conditions usually reproduce a *Poiseuille flow* profile, which we indicate with q, at the inlet and outlet sections. This leads to the following non-homogeneous boundary conditions, that we associate to (3.1.1),

$$u = q \quad \text{on } \partial T \qquad u = 0 \quad \text{on } \partial K; \tag{3.1.2}$$

here ∂T represents the boundary of the parallelepiped (tube) while ∂K represents boundary of the crossing plate. The velocity profile for the Poiseuille flow through a rectangular section was first derived by Boussinesq [18] and it correctly reproduces the (imposed) inflow and outflow conditions in a wind tunnel, if the latter is sufficiently long with respect to the characteristic length of the bridge (see, e.g. [17, Figure 11]), which justifies assuming the reorganization of the flow past the obstacle for sufficiently low Reynolds numbers.

Analyzing the well-posedness of (3.1.1)-(3.1.2), in order to obtain *explicit* bounds for the uniqueness of its solutions, is the main purpose of the present chapter; see Section 3.2 that contains the main result of the chapter, Theorem 3.2.2. It turns out that, under symmetry assumptions on both the domain Ω and the boundary conditions (3.1.2), the obstacle K may suffer the action of a lift force exerted by the fluid only in presence of multiplicity of solutions. In Section 3.2 we also provide quantitative bounds on the Poiseuille flow for the occurrence of lift on the deck of some bridge models which were tested in the wind tunnel at Politecnico di Milano.

The subsequent sections are devoted to the proof of Theorem 3.2.2, which is organized in several steps. Section 3.3 is devoted to computing the bounds for the quantities that appear in the estimates needed for uniqueness. We then construct a suitable solenoidal extension of the Poiseuille flow q, which overcomes the presence of non-homogeneous boundary conditions in the problem and of the obstacle K pulled out from the domain. This enables us to obtain the sought estimates in Section 3.3.2. Then, in Section 3.3.3 we derive bounds for the Sobolev constants involved in the problem. Finally, in Section 3.4 we conclude the proof of Theorem 3.2.2: we prove existence and uniqueness for solutions of (3.1.1)-(3.1.2) and give a bound for uniqueness, which is fully explicit in view of the information derived in the previous sections. Proposition 3.4.5 aggregates all considerations to give an explicit expression for the bound which induces the appearance of a lift force over the obstacle.

Nowadays, computers are extremely precise but the importance of having *explicit theoretical bounds* remains unchanged. In the case of a suspension bridge subject to the wind, several different thresholds need to be compared in order to understand which phenomenon first triggers the instability; besides the appearance of the lift force (as in the present chapter), one is also interested in thresholds for hangers slackening [67], and in the appearance of the so-called aerodynamic flutter [3]. The exact (or, at least, the explicit) value of the thresholds in general problems from mathematical physics is well-explained in the celebrated monograph by Pólya-Szegö [117], in particular for problems related to the electrostatic capacity, to the torsional rigidity, and to the principal frequency of a body; several further geometric inequalities are contained in the monograph. The techniques vary from symmetrization methods to a priori bounds and functional inequalities. These tools are also used in shape optimization problems [82] and in equimeasurable rearrangements of real-valued functions [132], both in calculus of variations and in partial differential equations. And variational problems within PDE's, such as the ∞-Laplacian, turn out to be extremely powerful in bounding solenoidal extensions for non-homogeneous boundary value problems in Navier-Stokes equations [51]. In this chapter we derive bounds for some Sobolev embedding constants in a non simply connected 3D domain, a topic that is already quite involved in 2D domains [69]. Moreover, we need a precise bound on the solenoidal extension which is used to get rid of the non-homogeneous boundary condition. Bounds for solenoidal extensions are also needed in different areas of mathematical physics: a whole bunch of inequalities arises both in fluid mechanics and elasticity [13,32,53,90,97], and they are all linked to each other. Our approach and bounds may also be fruitfully employed for these problems.

3.2 Appearance of the lift

We consider a steady fluid filling a three-dimensional cylindrical domain T which contains an obstacle K

$$\begin{aligned} T = (-L, L) \times \omega, \quad \omega = (-1, 1) \times (-d, d), \\ K = (-l, l) \times (-1, 1) \times (-h, h), \qquad \Omega = T \setminus \bar{K}. \end{aligned} \tag{3.2.1}$$

The cross section of the cylinder is $\omega = (-1, 1) \times (-d, d)$, L is the length of the cylinder and the obstacle K represents the deck of a bridge ($l < L$, $h < d$). The region of the flow is the domain Ω, see Figure 3.2 for two lateral views of Ω.

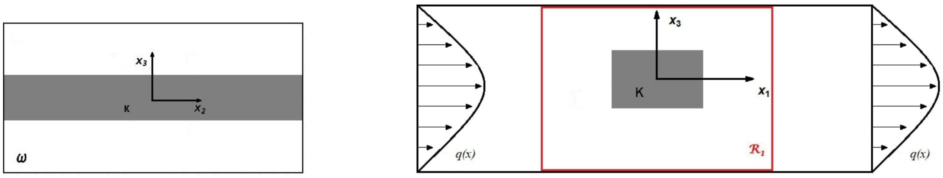


Figure 3.2: *Left: rectangular cross-section ω of the cylinder T. Right: Poiseuille inflow-outflow.*

The cylinder T (the space occupied by the wind tunnel) has a rectangular cross-section ω, see the left picture in Figure 3.2. This is the usual shape of a wind tunnel. The obstacle K has the section $(-l, l) \times (-h, h)$ on the plane x_1x_3, while the x_2-coordinate is confined in $(-1, 1)$. Note that $\Omega = T \setminus \bar{K}$ *is not* simply connected. Around the cross-section of the obstacle we construct a "technical rectangle" $\mathcal{R}_1$ where the cut-off function will be supported.

At the inlet and outlet sections of the cylinder the flow is of Poiseuille-type, namely a unidirectional flow along the axis of the channel and defined on the rectangular cross-section ω; to this end, we define the function, for all $(x_2, x_3) \in \omega$,

$$g(x_2, x_3) = \Big[1 - \frac{x_3^2}{d^2} + 4\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha_k^3} \frac{\cosh(\frac{\alpha_k x_2}{d})}{\cosh(\frac{\alpha_k}{d})} \cos(\frac{\alpha_k x_3}{d})\Big], \tag{3.2.2}$$

where $\alpha_k = (2k-1)\frac{\pi}{2}$ $(k = 1, 2...)$. In the boundary conditions (3.1.2), q is the profile of a Poiseuille flow, see the right picture in Figure 3.2. More precisely, we take

$$q(x) = \{v_1(x_2, x_3), 0, 0\} \qquad \text{with} \quad v_1(x_2, x_3) = k_p \frac{g}{\|\nabla g\|_{L^2(\omega)}} \tag{3.2.3}$$

so that

$$\|\nabla q\|_{L^2(\omega)} = k_p\,,$$

see Figure 3.3 for the plot; hence, the magnitude of the inflow is measured by the parameter $k_p = -\frac{1}{2\mu}\frac{\partial P}{\partial x_1} d^2 > 0$, the flow itself being driven by a (constant and negative) pressure drop $\frac{\partial P}{\partial x_1} < 0$.

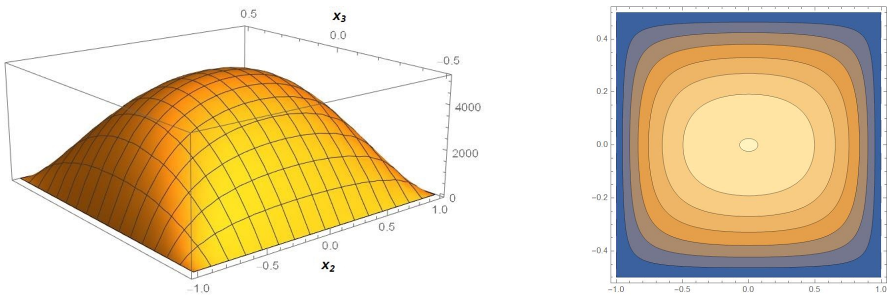


Figure 3.3: *Profile of the Poiseuille flow through a rectangular parallelepiped, together with its velocity contours. The rectangular cross section is $(-1, 1) \times (-0.5, 0.5)$, the value of the parameter k_p is chosen to be $k_p \approx 0.84 \cdot 10^5$*

We consider the space of vector fields vanishing only on the boundary of the obstacle

$$H^1_*(\Omega) = \{v \in H^1(\Omega)\,|v = 0 \text{ on } \partial K\},$$

and two functional spaces of solenoidal vector fields

$$\begin{aligned} V_*(\Omega) &= \{\phi \in H^1_*(\Omega)\,|\,\nabla\cdot\boldsymbol{\phi} = 0 \text{ in } \Omega\}, \\ V(\Omega) &= \{\phi \in H^1_0(\Omega)\,|\,\nabla\cdot\boldsymbol{\phi} = 0 \text{ in } \Omega\}. \end{aligned} \tag{3.2.4}$$

Note that if $u \in V_*(\Omega)$ satisfies (3.1.2), then its trace $u|_{\partial\Omega}$ is continuous. Then we introduce the standard trilinear form

$$\psi\,(u, v, w) = \int_\Omega (u\cdot\nabla)v\cdot w, \tag{3.2.5}$$

which is continuous in $H^1_*(\Omega) \times H^1_*(\Omega) \times H^1_*(\Omega)$, see e.g. [60, Lemma IX.1.1]). These tools enable us to define weak solutions of (3.1.1)-(3.1.2).

Definition 3.2.1. Let Ω be as in (3.2.1). Given q as in (3.2.3), so that $q \in W^{1,\infty}(\partial T)$, a vector field $u : \Omega \to \mathbb{R}^3$ is called a weak solution to (3.1.1)-(3.1.2) if $u \in V_*(\Omega)$ satisfies (3.1.2) in the trace sense and

$$\mu(\nabla u, \nabla\phi)_{L^2(\Omega)} + \psi(u, u, \phi) = 0 \qquad \forall\,\phi \in V(\Omega). \tag{3.2.6}$$

Let us now define rigorously what is meant by *lift force* in this context. The stress tensor of an incompressible viscous fluid, whose velocity and pressure fields obey to the three-dimensional Navier-Stokes equations (3.1.1), is expressed through the following 3×3 matrix (see [101, Chapter 2])

$$\mathbf{T} = -p\mathbf{I} + \mu[\nabla u + (\nabla u)^T], \tag{3.2.7}$$

which combines the action of both the pressure p and the shear forces. In (3.2.7), $\mathbf{I}$ is the 3×3 - identity matrix. Hence, according to (3.2.1), the force exerted by the fluid over the obstacle K is

$$F_K = -\int_{\partial K} \mathbf{T}\cdot\hat{n}\,ds$$

where $\hat{n}$ is the outward unit normal to Ω, therefore directed towards the interior of K. But since we merely deal with weak solutions of (3.1.1)-(3.1.2), we need to weaken this definition and, as in [68, Definition 3.3], to redefine F_K by:

$$F_K = -\left\langle \mathbf{T}\cdot\hat{n}, 1\right\rangle_{\partial K}, \tag{3.2.8}$$

where $\langle\cdot,\cdot\rangle_{\partial K}$ is the duality between $W^{-\frac{2}{3},\frac{3}{2}}(\partial K)$ and $W^{\frac{2}{3},3}(\partial K)$. Accordingly, since the inflow velocity (3.2.3) only has the first component, if $\hat{k}$ denotes the unit vector along x_3, then the lift force exerted by the fluid on the obstacle K is

$$\mathcal{L}_K = F_K\cdot\hat{k}. \tag{3.2.9}$$

We can now state our main result :

Theorem 3.2.2. *Let Ω be as in* (3.2.1) *and q as in* (3.2.3)*, so that $q \in W^{1,\infty}(\partial T)$. For any $k_p > 0$, there exists a weak solution $(u, p) \in V_*(\Omega) \times L^2(\Omega)$ of* (3.1.1)-(3.1.2)*. Moreover, there exists $\bar{k}_p = \bar{k}_p(\mu, L, d, l, h)$ such that, if*

$$0 < k_p < \bar{k}_p(\mu, L, d, l, h) \tag{3.2.10}$$

then the weak solution is unique. Hence, in order to observe a lift force over the obstacle, it must be $k_p > \bar{k}_p$.

Theorem 3.2.2 deserves several **important comments** that explain its possible applications. The first statement does not come unexpected, existence and uniqueness for a non-homogeneous problem such as (3.1.1)-(3.1.2) usually hold under smallness assumptions on the data. The main novelty of Theorem 3.2.2 is the second statement since it allows to explicitly compute a threshold of stability for the obstacle K in terms of the flux at the inlet and outlet sections of the wind tunnel. This is possible because we are considering a symmetric framework, both for the domain and the boundary condition. We can give a quantitative form to $\bar{k}_p$, after determining its dependence on the physical parameters μ, L, d, l, h; the explicit form of $\bar{k}_p$ is given in (3.4.8), (see also Proposition 3.4.5). We emphasize that our purpose is not to determine the optimal (largest) value of $\bar{k}_p$; instead, we aim to provide an effective method to obtain an explicit expression for $\bar{k}_p$ yielding a *quantitative sufficient condition* for uniqueness of solutions of (3.1.1)-(3.1.2).

We now determine some numerical values of $\bar{k}_p$ computed through the final formula (3.4.8). We take real geometrical data from the online database [94] by referring to few experiments which took place in the wind tunnel (GVPM) at Politecnico di Milano. The coefficient of kinematic viscosity is chosen to be the one of air $\mu = 1.5 \cdot 10^{-5}$. The first data are taken from the model of the Izmit bay bridge in Turkey, a 1:30 sectional model. The second data come from another Turkish bridge: the model of the Third Bosphorus bridge. Finally, we took data from the model of the Talavera de la Reina Cable-stayed bridge near Toledo, in Spain. All parameters are made dimensionless with respect to the characteristic length of the problem, half of the channel's width, coinciding with half of the obstacle's length. The results are summarized in the following table.

bridge model	L	d	l	h	$\bar{k}_p \times 10^6$
Izmit bay	5	0.555	0.072	0.011	6.242
Third Bosphorus	5	0.555	0.084	0.008	6.245
Talavera de la Reina Cable-stayed	5	0.555	0.041	0.003	6.258

From a theoretical point of view, we consider the height d, the size of the bridge and the viscosity μ as fixed data for the problem, and we discuss the dependence of $\bar{k}_p$ on L (the length of the wind tunnel). We give here some qualitative properties on the behaviour of $\bar{k}_p$, derived from its explicit form given in (3.4.8). As L diminishes, $\bar{k}_p$ grows, by making condition (3.2.10) less restrictive. This is as expected since a short channel does not let the velocity of the fluid deviate from the field prescribed at the inlet and outlet sections; the unique solution would tend to resemble the imposed Poiseuille flow $q(x)$, also fairly close to the obstacle. On the other hand, as L increases, $\bar{k}_p = \bar{k}_p(L)$ diminishes and tends to an horizontal asympote when

$L \to \infty$. In other words, provided that we impose a sufficiently weak flux at the inlet and outlet sections, uniqueness is guaranteed even for an arbitrarily long channel (see also Remark 3.3.7).

Although L could tend to infinity, we do not consider an infinitely long channel. This would not be physically meaningful since our problem models an experimental test in a wind tunnel and we would also snag on a mathematical issue; existence of solutions for (3.1.1)-(3.1.2) would not be guaranteed for any value of the parameter k_p (as we have when $L < +\infty$), but only for sufficiently small values. Indeed, the problem would resemble the so-called Leray's problem (see [60, XII, Introduction]), for which the question around unconditional existence of solutions is still an open issue.

Finally, we briefly discuss the regularity of the solutions of (3.1.1)-(3.1.2).

Remark 3.2.3. Weak solutions of (3.1.1)-(3.1.2) are smooth in the interior of the domain Ω, defined as in (3.2.1), see for instance [60, Theorem IX.5.1]. Regularity up to the boundary is by far more difficult; although the obstacle K has a Lipschitz boundary, it generates non-convex corners within Ω. The H^2-regularity can be obtained for a convex polyhedron-like domain: see [35–37], where the authors precisely considers this type of domain and [99], where regularity of the Navier-Stokes system in three-dimensional domains with conic points is studied. However, arbitrary domains of polyhedral types which may possess reentrant corners, as in the case that we are considering, do not allow to consider solutions exhibiting better regularity than the minimal $H^1(\Omega)$, see for instance [103].

3.3 Explicit bounds

3.3.1 Determination of the solenoidal extension

Given g as in (3.2.2), let b_2 and b_3 be the following functions, defined over ω:

$$b_2(x_2,x_3) = -\frac{k_p}{3\,\|\nabla g\|_{L^2(\omega)}} x_3 \Big[2 - \frac{x_3^2}{d^2} + 6\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha_k^3} \frac{\cosh(\frac{\alpha_k x_2}{d})}{\cosh(\frac{\alpha_k}{d})} \cos(\frac{\alpha_k x_3}{d})\Big] \tag{3.3.1}$$

$$b_3(x_2,x_3) = \frac{k_p}{3\,\|\nabla g\|_{L^2(\omega)}} \Big\{x_2 + 6\sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha_k^4} \frac{\sinh(\frac{\alpha_k x_2}{d})}{\cosh(\frac{\alpha_k}{d})} \Big[x_3\alpha_k \sin(\frac{\alpha_k x_3}{d}) + d\cos(\frac{\alpha_k x_3}{d})\Big]\Big\}. \tag{3.3.2}$$

Observe that

$$\frac{\partial}{\partial x_2} b_3(x_2,x_3) - \frac{\partial}{\partial x_3} b_2(x_2,x_3) = v_1(x_2,x_3),$$

where $v_1(x_2,x_3)$ describes the velocity profile of the Poiseuille flow in (3.2.3). Hence, if we define the vector field

$$b(x) = \{0, b_2(x_2,x_3), b_3(x_2,x_3)\}, \tag{3.3.3}$$

we obtain that $\nabla \times (b(x)) = q(x)$, where $q(x)$ is as in (3.2.3).

We aim to build a function $a(x)$ that plays the role of a "flux carrier", *i.e.* a smooth solenoidal extension of the prescribed velocity field at the inlet and outlet sections, vanishing on $\partial\Omega$. We seek a function $a(x)$ equal to $q(x)$ far away from the obstacle and equal to zero in a neighbourhood of the obstacle. Hence, following the classical procedure by Ladyzhenskaya [100], we take

$$a(x) = \nabla \times \big(b(x)\,\theta(x)\big)\,, \tag{3.3.4}$$

where b is as in (3.3.3) and $\theta(x)$ is a $\mathcal{C}^1$ cut-off function equal to 1 at all points of Ω far away from ∂K and to 0 near ∂K (we shall later specify what we mean by "near"). Clearly $a \in H^1(\Omega)$, a is solenoidal and it vanishes close to the obstacle whereas it coincides with q far away from it. In order to give the explicit expression of the solenoidal extension $a(x)$, from (3.3.4), we need to determine both $\theta(x)$ and $b(x)$.

We first proceed in building the cut-off function $\theta(x)$, merely depending on x_1 and x_3, whose profile is "specular" to a function supported in the rectangular region

$$\mathcal{R}_1 = \{(x_1, x_3) \in (-l-\alpha, l+\alpha) \times (-d, d)\} \tag{3.3.5}$$

that fully invades the domain Ω in the x_3-direction but not in the x_1-direction. In fact, the parameter α satisfies $0 < \alpha < L - l$, is independent of L and is chosen so as to optimize the estimates for unique solvability of (3.1.1)-(3.1.2) while the same trick in the x_3-direction does not help because we numerically saw that this would not lead to better estimates. The rectangle $\mathcal{R}_1$, which contains the cross-section of the obstacle K (see the right picture in Figure 3.2), enables us to partition the domain Ω in (3.2.1) as follows

$$\begin{gathered}\Omega = \bigcup_{i=0}^{2} \Omega_i, \quad \Omega_0 = \mathcal{R}_1 \times (-1,1) \setminus \bar{K} \\ \Omega_1 = \{x \in \mathbb{R}^3 : x_1 < -l-\alpha,\ (x_2, x_3) \in \omega\}, \\ \Omega_2 = \{x \in \mathbb{R}^3 : x_1 > l+\alpha,\ (x_2, x_3) \in \omega\}.\end{gathered} \tag{3.3.6}$$

We consider the functions

$$\theta_1(x_1) = \begin{cases} 1 & \text{if } x_1 < l \\ 0 & \text{if } x_1 > l+\alpha \\ \phi_1(x_1) & \text{otherwise} \end{cases}, \qquad \theta_2(x_3) = \begin{cases} 1 & \text{if } x_3 < h \\ 0 & \text{if } x_3 > d \\ \phi_2(x_3) & \text{otherwise} \end{cases},$$

with

$$\begin{aligned}\phi_1(x_1) &= \frac{2\,x_1^3}{\alpha^3} - \frac{3x_1^2\,(\alpha+2l)}{\alpha^3} + \frac{6x_1\,(l^2+\alpha l)}{\alpha^3} - \frac{-\alpha^3+2l^3+3\alpha\, l^2}{\alpha^3}, \\ \phi_2(x_3) &= -\frac{{x_3}^2}{(d-h)^2} + \frac{2h\,x_3}{(d-h)^2} + \frac{d(d-2h)}{(d-h)^2}.\end{aligned}$$

Then we take

$$\theta(x_1, x_3) = 1 - \theta_1(x_1)\theta_1(-x_1)\theta_2(x_3)\theta_2(-x_3). \tag{3.3.7}$$

This function is represented in Figure 3.4.

The above construction enables us to state:

Proposition 3.3.1. *Let $\Omega \subset \mathbb{R}^3$ be as in* (3.2.1) *and $q(x)$ as in* (3.2.3). *Let $a(x) = \nabla \times \big(b(x)\theta(x)\big)$, where $b(x)$ is as in* (3.3.3) *and $\theta(x)$ as in* (3.3.7). *Then, the vector field $a(x) \in H^1(\Omega)$ is such that*

$$\nabla \cdot a(x) = 0 \qquad a(x) = q(x) \quad \textit{in } \Omega_i \qquad a(x) = 0 \quad \textit{on } \partial K. \tag{3.3.8}$$

with Ω_i, $i = 1, 2$ defined in (3.3.6).

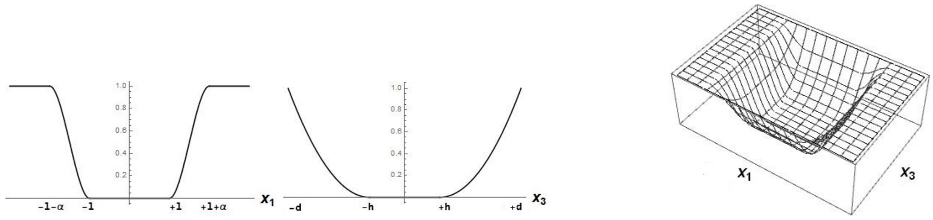


Figure 3.4: *Left: restriction of θ to the x_1-axis and to the x_3-axis. Right: three-dimensional representation of θ.*

3.3.2 Bounds for the solenoidal extension

The aim of this subsection is to provide quantitative estimates for suitable norms of the solenoidal extension a, defined in Proposition 3.3.1, which will play a role in the uniqueness bound for solutions of (3.1.1)-(3.1.2). This bound will involve the L^4-norm of a and the L^2-norm of its gradient in the region Ω_0 defined by the partition (3.3.6); these are the quantities that we intend to estimate here.

To this end, we first prove some technical inequalities that involve the so-called Apéry constant [8]:

$$\zeta(3) = \frac{8}{7} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^3} \approx 1.202. \tag{3.3.9}$$

The next lemmas provide some estimates for the functions that we have introduced so far.

Lemma 3.3.2. *Let b_2 and b_3 be as in* (3.3.1), (3.3.2)*. Then,*

$$|b_2(x_2,x_3)|^2 \le \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}} \left(|2x_3 - \frac{x_3^3}{d^2}| + |x_3|\frac{42}{\pi^3}\,\zeta(3) \right)^2,$$

$$|b_3(x_2,x_3)|^2 \le \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}} \left(|x_2| + |x_3|\frac{42}{\pi^3}\,\zeta(3) + d \right)^2,$$

$$\left|\frac{\partial b_2(x_2,x_3)}{\partial x_2}\right|^2 \le \frac{k_p^2}{d^2\,\|\nabla g\|^2_{L^2(\omega)}}\, x_3^2,$$

$$\left|\frac{\partial b_2(x_2,x_3)}{\partial x_3}\right|^2 \le \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}} \left(2 + 3\frac{x_3^2}{d^2} + \frac{42}{\pi^3}\,\zeta(3) + \frac{3|x_3|}{d} \right)^2,$$

$$\left|\frac{\partial b_3(x_2,x_3)}{\partial x_2}\right|^2 \le \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}} \left(1 + \frac{42}{\pi^3}\,\zeta(3) + \frac{3|x_3|}{d} \right)^2,$$

$$\left|\frac{\partial b_3(x_2,x_3)}{\partial x_3}\right|^2 \le \frac{k_p^2}{9\,\|\nabla g\|^2_{L^2(\omega)}} \left(\frac{3|x_3|}{d} \right)^2.$$

The bounds in Lemma 3.3.2 are obtained with some computations, by using (3.3.9) and the convergence of the series

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8},\qquad \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}=\frac{\pi^4}{96}.$$

Then, we provide some bounds for the cut-off function.

Lemma 3.3.3. *Let θ be as in* (3.3.7). *Then, given the partition* (3.3.6)*, in the region Ω_0 it holds that*

$$|\theta(x_1,x_3)|\le 1,$$

$$\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\le\frac{36(x_1-l)^2(d-x_3)^2(d-2h+x_3)^2(l-x_1+\alpha)^2}{\alpha^6(d-h)^4},$$

$$\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\le\frac{4(x_3-h)^2(l-x_1+\alpha)^4(2x_1-2l+\alpha)^2}{\alpha^6(d-h)^4},$$

$$\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\le\frac{36(d-x_3)^2(d-2h+x_3)^2(2x_1-2l-\alpha)^2}{\alpha^6(d-h)^4},$$

$$\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\right|^2\le\frac{4(2x_1-2l+\alpha)^2(l-x_1+\alpha)^4}{\alpha^6(d-h)^4},$$

$$\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3}\right|^2\le\frac{144(x_1-l)^2(x_3-h)^2(l-x_1+\alpha)^2}{\alpha^6(d-h)^4}.$$

Now we are ready to proceed. To begin with, we seek an upper bound for the L^4-norm of the solenoidal extension. We remark that all the integrals that we will encounter are well-defined, as we are considering smooth bounded functions over bounded domains.

Proposition 3.3.4. *Let $a=a(x)$ be the function defined in Proposition 3.3.1. Let Ω_0 be defined as in* (3.3.6)*. Let the constants δ_i, $i=1,..,24$ be defined as in the Appendix. Then*

$$\|a\|_{L^4(\Omega_0)}\le\Lambda_1 k_p,\qquad \|\nabla a\|_{L^2(\Omega_0)}\le\Lambda_2 k_p,$$

where Λ_1 and Λ_2 are defined by

$$\Lambda_1=\frac{\sqrt[4]{8}}{\|\nabla g\|_{L^2(\omega)}}\left\{\left[\delta_1+\delta_2+\delta_3+\left(\sqrt[4]{\delta_4}+\sqrt[4]{\delta_5}\right)^4+\left(\sqrt{2\delta_6}+\sqrt{2\delta_7}\right)^2+\left(\sqrt{2\delta_8}+\sqrt{2\delta_9}\right)^2\right]^{1/4}\right\},$$

$$\Lambda_2=\frac{\sqrt{8}}{\|\nabla g\|_{L^2(\omega)}}\left\{\left[\delta_{10}+\delta_{11}+\left(\sqrt{\delta_{12}}+\sqrt{\delta_{13}}\right)^2+\delta_{14}+\delta_{15}+\left(\sqrt{\delta_{16}}+\sqrt{\delta_{17}}\right)^2+\left(\sqrt{\delta_{18}}+\sqrt{\delta_{17}}\right)^2+\left(\sqrt{\delta_{19}}+\sqrt{\delta_{20}}\right)^2+\left(\sqrt{\delta_{21}}+\sqrt{\delta_{22}}+\sqrt{\delta_{23}}+\sqrt{\delta_{24}}\right)^2\right]^{1/2}\right\}.$$

Proof. The curl of the vector field

$$b(x)\theta(x) = \{0, b_2(x_2,x_3)\theta(x_1,x_3), b_3(x_2,x_3)\theta(x_1,x_3)\}$$

reads

$$\nabla \times (b(x_2,x_3)\theta(x_1,x_3)) = \Big\{ \theta(x_1,x_3)v_1(x_2,x_3) - b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}, \\ - b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}, b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\Big\}.$$

The L^4-norm of this quantity involves both the square of each of the three components and the corresponding double products, as follows:

$$\begin{aligned}\|a\|^4_{L^4(\Omega_0)} = &\|\nabla\times(b(x)\,\theta(x,\delta))\|^4_{L^4(\Omega_0)} = \int_{\Omega_0}\left|\nabla\times(b(x_2,x_3)\,\theta(x_1,x_3))\right|^4 dx \\ &= \int_{\Omega_0}\Bigg(\left|\theta(x_1,x_3)v_1(x_2,x_3) - b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2 \\ &\quad + \left|b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 + \left|b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\Bigg)^2 dx.\end{aligned}$$

The trinomial expansion gives six terms, that we estimate separately. The simplest terms can be estimated using Lemmas 3.3.2 and 3.3.3 (their use is hidden in the computation of the constants δ_i, but it does not appear explicitly here). Notice also that in the computation of the integrals we exploited the evenness of the function and the symmetries of the domain of integration:

$$\int_{\Omega_0}\left|b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 dx \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\,\delta_2,$$

$$\int_{\Omega_0}\left|b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 dx \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\,\delta_1,$$

$$\int_{\Omega_0} 2\left|b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left|b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx \le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\,\delta_3,$$

while the remaining terms are estimated after an intermediate step which exploits the Minkowski inequality.

$$\begin{aligned}&\int_{\Omega_0}\left|\theta(x_1,x_3)v_1(x_2,x_3) - b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^4 dx \\ &\le \left[\left(\int_{\Omega_0}\left|\theta(x_1,x_3)v_1(x_2,x_3)\right|^4 dx\right)^{1/4} + \left(\int_{\Omega_0} b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^4 dx\right)^{1/4}\right]^4 \\ &\le \frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\left(\sqrt[4]{\delta_4}+\sqrt[4]{\delta_5}\right)^4,\end{aligned}$$

$$\int_{\Omega_0} 2\left|\theta(x_1,x_3)v_1(x_2,x_3)-b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left|b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx$$
$$\leq 2\Bigg[\left(\int_{\Omega_0}\left|\theta(x_1,x_3)v_1(x_2,x_3)b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx\right)^{1/2}$$
$$+\left(\int_{\Omega_0}\left|b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}b_3(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx\right)^{1/2}\Bigg]^2$$
$$\leq 2\cdot\frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\left(\sqrt{\delta_6}+\sqrt{\delta_7}\right)^2,$$
$$\int_{\Omega_0} 2\left|\theta(x_1,x_3)v_1(x_2,x_3)-b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left|b_2(x_2,x_3)\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx$$
$$\leq 2\cdot\frac{8\,k_p^4}{\|\nabla g\|^4_{L^2(\omega)}}\left(\sqrt{\delta_8}+\sqrt{\delta_9}\right)^2.$$

By combining these estimates, we obtain the L^4-bound.

The upper bound for the Dirichlet norm of a is obtained through the very same procedure, even though it turns out to be slightly more elaborate, since the gradient of the vector field $\nabla\times(b\,\theta)$ returns the following 3×3 matrix

$$\nabla(\nabla\times(b(x)\,\theta(x)))=\Bigg\{\nabla\left[\theta(x_1,x_3)v_1(x_2,x_3)-b_2(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_3}\right],$$
$$\nabla\left[-b_3(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_1}\right],\nabla\left[b_2(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_1}\right]\Bigg\}^T.$$

In order not to burden the writing of equations, we rewrite this term as

$$\nabla(\nabla\times(b(x)\,\theta(x)))=\Big\{\nabla A,\nabla B,\nabla C\Big\}^T.$$

Thus we obtain

$$\begin{aligned}\|\nabla a\|^2_{L^2(\Omega_0)}&=\|\nabla(\nabla\times(b(x)\theta(x)))\|^2_{L^2(\Omega_0)}\\&=\|\nabla A\|^2_{L^2(\Omega_0)}+\|\nabla B\|^2_{L^2(\Omega_0)}+\|\nabla C\|^2_{L^2(\Omega_0)}.\end{aligned}$$

We start estimating the second and third integral, which are slightly simpler. Analogously to what has been done before, we obtain, after having exploiting the Minkowski inequality and Lemmas 3.3.2 and 3.3.3,

$$\|\nabla B\|^2_{L^2(\Omega_0)}=\int_{\Omega_0}\left|b_3(x_2,x_3)\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2 dx+\int_{\Omega_0}\left|\frac{\partial b_3(x_2,x_3)}{\partial x_2}\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx$$
$$+\int_{\Omega_0}\left|-\frac{\partial b_3(x_2,x_3)}{\partial x_3}\frac{\partial\theta(x_1,x_3)}{\partial x_1}-b_3(x_2,x_3)\frac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3}\right|^2 dx$$
$$\leq\frac{8\,k_p^2}{\|\nabla g\|^2_{L^2(\omega)}}\left[\delta_{10}+\delta_{11}+(\sqrt{\delta_{12}}+\sqrt{\delta_{13}})^2\right],$$

$$\begin{aligned}\|\nabla C\|^2_{L^2(\Omega_0)} &= \int_{\Omega_0}\left|b_2(x_2,x_3)\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2 dx + \int_{\Omega_0}\left|\frac{\partial b_2(x_2,x_3)}{\partial x_2}\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 dx\\ &+ \int_{\Omega_0}\left|\frac{\partial b_2(x_2,x_3)}{\partial x_3}\frac{\partial\theta(x_1,x_3)}{\partial x_1} + b_2(x_2,x_3)\frac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3}\right|^2 dx\\ &\le \frac{8\,k_p^2}{\|\nabla g\|^2_{L^2(\omega)}}\left[\delta_{14}+\delta_{15}+(\sqrt{\delta_{16}}+\sqrt{\delta_{17}})^2\right].\end{aligned}$$

For what concerns the third integral, we obtain

$$\begin{aligned}\|\nabla A\|^2_{L^2(\Omega_0)} &= \int_{\Omega_0}\left|v_1(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_1} - b_2(x_2,x_3)\tfrac{\partial^2\theta(x_1,x_3)}{\partial x_1\partial x_3}\right|^2 dx\\ &+ \int_{\Omega_0}\left|\theta(x_1,x_3)\tfrac{\partial v_1(x_2,x_3)}{\partial x_2} - \tfrac{\partial b_2(x_2,x_3)}{\partial x_2}\tfrac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2 dx\\ &+ \int_{\Omega_0}\Big|\theta(x_1,x_3)\tfrac{\partial v_1(x_2,x_3)}{\partial x_3} + v_1(x_2,x_3)\tfrac{\partial\theta(x_1,x_3)}{\partial x_3}\\ &\qquad - \tfrac{\partial b_2(x_2,x_3)}{\partial x_3}\tfrac{\partial\theta(x_1,x_3)}{\partial x_3} - b_2(x_2,x_3)\tfrac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\Big|^2 dx\\ &\le \frac{8\,k_p^2}{\|\nabla g\|^2_{L^2(\omega)}}\Big[(\sqrt{\delta_{18}}+\sqrt{\delta_{17}})^2\\ &\quad + (\sqrt{\delta_{19}}+\sqrt{\delta_{20}})^2 + (\sqrt{\delta_{21}}+\sqrt{\delta_{22}}+\sqrt{\delta_{23}}+\sqrt{\delta_{24}})^2\Big].\end{aligned}$$

Thus, if we blend these bounds, we obtain the claimed estimate for the Dirichlet norm of a. □

3.3.3 Bounds for the Sobolev embedding constants

We preliminarily remark that a slight modification of the procedure developed in [68, Theorem 2.2] yields, for all $u \in H^1_0(\omega)$,

$$\sigma_*\,\|u\|^2_{L^4(\omega)} \le \|\nabla u\|^2_{L^2(\omega)} \quad \text{with} \quad \sigma_* = \sqrt{3}\left(\frac{\pi}{2}\right)^{3/2}\frac{\sqrt{1+d^2}}{d}. \tag{3.3.10}$$

Note that σ_* provides a lower bound for the Sobolev constant σ_0 of the embedding $H^1_0(\omega) \subset L^4(\omega)$ in the 2D-rectangle ω, see (3.2.1), defined by

$$\sigma_0 = \min_{v\in H^1_0(\omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\omega)}}{\|v\|^2_{L^4(\omega)}}. \tag{3.3.11}$$

This section is devoted to computing an explicit lower bound $\mathcal{S}_*$ for the Sobolev constant

$$\mathcal{S}_0 = \min_{v\in H^1_0(\Omega)\setminus\{0\}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^4(\Omega)}} \tag{3.3.12}$$

for the (compact) embedding $H_0^1(\Omega) \subset L^4(\Omega)$, appearing in the estimate ensuring uniqueness for (3.1.1)-(3.1.2). A more significant modification of [68, Theorem 2.3] allows to find a constant Γ^*, as small as possible, satisfying

$$\|v\|^2_{L^4(\Omega)} \leq \Gamma^* \|\nabla v\|^2_{L^2(\Omega)} \qquad \forall\, v \in H_0^1(\Omega),$$

so that $\mathcal{S}_0 \geq \mathcal{S}_* = 1/\Gamma^*$. We emphasize that Γ^* is sought as small as possible, in order to obtain less restrictive conditions ensuring uniqueness of the solutions of (3.1.1). In order to obtain an explicit form for $\mathcal{S}_*$, we need the following bound for the Poincaré constant.

Lemma 3.3.5. *Let Ω be the domain in* (3.2.1). *For any scalar function $w \in H_0^1(\Omega)$, one has*

$$\|w\|_{L^2(\Omega)} \leq \min\left\{\frac{2}{\pi}\frac{Ld}{(d^2+L^2+L^2d^2)^{1/2}}, \sqrt[3]{\frac{6\,(Ld-lh)}{\pi^4}}\right\} \|\nabla w\|_{L^2(\Omega)}. \tag{3.3.13}$$

Proof. We start by proving the first bound in (3.3.13). The least eigenvalue $\lambda_1 > 0$ of $-\Delta$ in T under homogeneous Dirichlet boundary conditions is given by

$$\lambda_1 = \frac{\pi^2}{4\,L^2} + \frac{\pi^2}{4\,d^2} + \frac{\pi^2}{4} = \frac{\pi^2}{4}\frac{d^2+L^2+L^2d^2}{L^2d^2},$$

as it is associated with the eigenfunction $\cos(\frac{\pi x_1}{2L})\cos(\frac{\pi x_2}{2})\cos(\frac{\pi x_3}{2d})$. Hence, the Poincaré constant inequality returns

$$\|w\|^2_{L^2(T)} \leq \frac{4}{\pi^2}\frac{L^2d^2}{d^2+L^2+L^2d^2}\|\nabla w\|^2_{L^2(T)} \qquad \forall\, w \in H_0^1(T).$$

Since any function in $H_0^1(\Omega)$ can be extended by 0 in K, thereby becoming a function in $H_0^1(T)$, this inequality proves the first bound in (3.3.13).

For the second bound in (3.3.13) we invoke the Faber-Krahn inequality (see [117]) which states that

$$\min_{w\in H_0^1(\Omega)} \frac{\|\nabla w\|_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)}} \geq \min_{w\in H_0^1(\Omega^*)} \frac{\|\nabla w\|_{L^2(\Omega^*)}}{\|w\|_{L^2(\Omega^*)}},$$

where Ω^* is a ball having the same volume as Ω. In order to compute the right-hand side in this inequality, we recall that the Poincaré constant constant in the unit sphere is given by π, which corresponds to the first zero of the spherical Bessel function of order 0, $\frac{\sin x}{x}$. Then, in the ball Ω^*, it holds that

$$\min_{w\in H_0^1(\Omega^*)} \frac{\|\nabla w\|_{L^2(\Omega^*)}}{\|w\|_{L^2(\Omega^*)}} = \frac{\pi}{R},$$

where R is the radius of this ball

$$R = \sqrt[3]{\frac{6\,(Ld-lh)}{\pi}},$$

and we used the fact that $|\Omega| = |T| - |K| = 8\,(Ld - lh)$. Therefore we obtain the second bound in (3.3.13). □

Note that equality between the two upper bounds in (3.3.13) occurs whenever

$$\frac{4\pi}{3} = (Ld - lh)\left(1 + \frac{1}{L^2} + \frac{1}{d^2}\right)^{3/2};$$

therefore which bound is better depends on the relative size of the obstacle K within T. Now, we are ready to prove:

Proposition 3.3.6. *For any $v \in H_0^1(\Omega)$, one has*

$$\|v\|_{L^4(\Omega)}^2 \leq \min\left\{\frac{Ld}{\pi^3(d^2+L^2+L^2d^2)^{1/2}}, \sqrt[3]{\frac{3\,(Ld-lh)}{4\,\pi^{10}}}\right\}^{1/2} \|\nabla v\|_{L^2(\Omega)}^2 \tag{3.3.14}$$

This inequality holds both for scalar functions and vector-valued functions.

Proof. We first prove (3.3.14) for scalar functions w for which del Pino-Dolbeault [38, Theorem 1] obtained the following optimal Gagliardo-Nirenberg inequality in $\mathbb{R}^3$:

$$\|w\|_{L^4(T)}^2 \leq \frac{1}{2^{1/3}\,\pi^{2/3}}\,\|\nabla w\|_{L^2(T)}\|w\|_{L^3(T)} \quad \forall w \in H_0^1(T). \tag{3.3.15}$$

Here we exploit the fact that functions in $H_0^1(T)$ can be extended by zero outside T becoming functions defined on the whole space $\mathbb{R}^3$. To get rid of the L^3-norm, we use the Hölder inequality

$$\|w\|_{L^3(T)}^3 = \int_T |w||w|^2\,dx \leq \|w\|_{L^2(T)}\|w\|_{L^4(T)}^2$$

which, combined with (3.3.15), gives

$$\|w\|_{L^4(T)}^2 \leq \frac{1}{\sqrt{2}\,\pi}\,\|\nabla w\|_{L^2(T)}^{3/2}\,\|w\|_{L^2(T)}^{1/2}. \tag{3.3.16}$$

Next, we estimate the term $\|w\|_{L^2(T)}^{1/2}$ through Lemma 3.3.5. Using (3.3.13) within (3.3.17) leads to (3.3.14) (for scalar functions) since $H_0^1(\Omega) \subset H_0^1(T)$.

Once we have obtained (3.3.14) for scalar functions, we claim that it also holds for vector-valued functions. Indeed, let $v = (v_1, v_2, v_3) \in H_0^1(\Omega)$; then, applying the Minkowski inequality, we can consider the L^4-norm of each component of this function individually and we use (3.3.14) as follows

$$\begin{aligned}
\|v\|_{L^4(\Omega)}^4 &\leq \left(\|v_1\|_{L^4(\Omega)}^2 + \|v_2\|_{L^4(\Omega)}^2 + \|v_3\|_{L^4(\Omega)}^2\right)^2 \\
&\leq \min\left\{\frac{Ld}{\pi^3(d^2+L^2+L^2d^2)^{1/2}}, \sqrt[3]{\frac{3\,(Ld-lh)}{4\,\pi^{10}}}\right\} \\
&\quad \times \left(\|\nabla v_1\|_{L^2(\Omega)}^2 + \|\nabla v_2\|_{L^2(\Omega)}^2 + \|\nabla v_3\|_{L^2(\Omega)}^2\right)^2 \\
&= \min\left\{\frac{Ld}{\pi^3(d^2+L^2+L^2d^2)^{1/2}}, \sqrt[3]{\frac{3\,(Ld-lh)}{4\,\pi^{10}}}\right\} \|\nabla v\|_{L^2(\Omega)}^4.
\end{aligned} \tag{3.3.17}$$

This proves (3.3.14) also for vector fields in $H_0^1(\Omega)$. □

We point out that (3.3.17) significantly improves the usual interpolation inequalities in fluid mechanics [60, (II.3.10)]. Proposition 3.3.6 yields the lower bound for the Sobolev constant $\mathcal{S}_0$, defined in (3.3.12), which reads as

$$\mathcal{S}_0 \geq \mathcal{S}_* = \max\left\{\pi^3\left(1+\frac{1}{L^2}+\frac{1}{d^2}\right)^{1/2}, \sqrt[3]{\frac{4\,\pi^{10}}{3\,(Ld-lh)}}\right\}^{1/2}. \tag{3.3.18}$$

Remark 3.3.7. Once we have fixed the height d of the wind tunnel and the size of the obstacle K by choosing l and h, there exists a critical threshold L^* where the two bounds within the maximum in (3.3.18) coincide. If $L < L^*$ then $\mathcal{S}_*(L)$ equals the second bound in (3.3.18) while if $L > L^*$ then $\mathcal{S}_*(L)$ equals the first bound in (3.3.18) and, therefore,

$$\lim_{L\to\infty} \mathcal{S}_*(L) = \pi^{3/2}\left(1+\frac{1}{d^2}\right)^{1/4}.$$

The existence of an horizontal asympote for the function $\mathcal{S}_*(L)$ is **particularly significant** in terms of uniqueness of solutions for the problem (3.1.1)-(3.1.2). Indeed, $\mathcal{S}_*$ is part of the expression of $\bar{k}_p$ which determines the condition for uniqueness of solutions (see Theorem 3.2.2) and is the only parameter in this expression depending on L; however, since $\mathcal{S}_*$ loses this dependence by virtue of the presence of such an asymptote, we infer that L does not play a direct role in terms of uniqueness of the solution. Figure 3.5 shows the behaviour of the map $L \mapsto \mathcal{S}_*(L)$ with the parameters from the model of the Izmit bay bridge ($d = 0.555$, $h = 0.011$, $l = 0.072$), see Section 3.2; in this case, $L^* \approx 0.0014$.

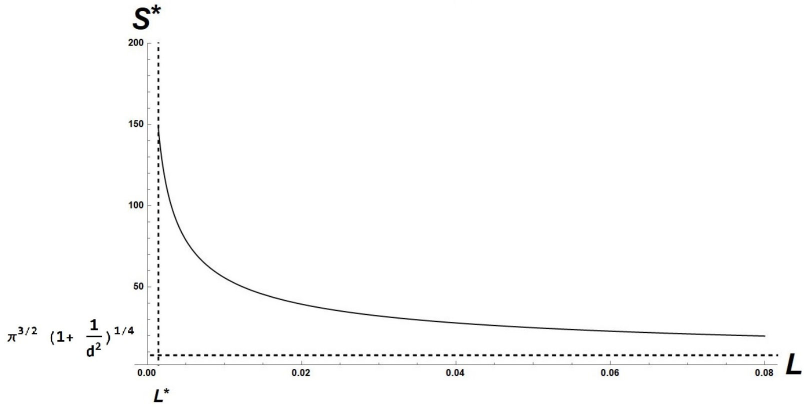


Figure 3.5: *Graph of the function $L \mapsto \mathcal{S}_*(L)$ with the parameters from the model of the Izmit bay bridge.*

3.4 Proof of Theorem 3.2.2

3.4.1 Existence and uniqueness

The idea of the proof is quite standard but, for our purposes, it is mandatory to fully report it since we need to emphasize the role played by each of the constants appearing in the a priori estimates.

In order to prove existence of a weak (or *generalized*) solution of (3.1.1)-(3.1.2) we look for velocity fields of the form

$$u = \hat{u} + s \tag{3.4.1}$$

where s is a sufficiently smooth *general solenoidal extension* of the prescribed velocity at the boundary q, which reproduces a Poiseuille-flow, while $\hat{u}$ (weakly, see Definition 3.2.1) solves the following problem:

$$\begin{gathered} -\mu\,\Delta\hat{u} + (\hat{u}\cdot\nabla)\,\hat{u} + (\hat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\hat{u} + \nabla p = f := \mu\,\Delta s - (s\cdot\nabla)\,s, \\ \nabla\cdot\hat{u} = 0 \quad \text{in } \Omega, \quad \hat{u} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.4.2}$$

It is well-known (see for instance [60, Theorem IX.4.1]) that the existence of a solution follows once we find a uniform bound on $\|\nabla\hat{u}\|_{L^2(\Omega)}$ depending only on the data. On the other hand, uniqueness of solutions relies on some *a priori* bound from the data, this is why we need the following statement.

Lemma 3.4.1. *Let* Ω *be as in* (3.2.1) *and* $q \in W^{1,\infty}(\partial T)$ *as in* (3.2.3). *Let* $\hat{u}$ *be a weak solution of* (3.4.2) *defined as in* (3.4.1). *If* $\mathcal{S}_*$ *is as in* (3.3.18)*,* σ_* *as in* (3.3.10)*, the constants* Λ_i *(*$i = 1, 2$*) as in Proposition 3.3.4, and if*

$$k_p < \frac{\mu\,\sigma_*}{\mathcal{S}_* + \Lambda_2\sigma_*}, \tag{3.4.3}$$

then the following a priori estimate holds:

$$\|\nabla\hat{u}\|_{L^2(\Omega)} \le \frac{\mu\,\Lambda_2 k_p + \Lambda_1 \frac{\Lambda_2}{\sqrt{\mathcal{S}_*}} k_p^2}{\mu - \frac{k_p}{\sigma_*} - \frac{k_p\Lambda_2}{\mathcal{S}_*}}. \tag{3.4.4}$$

Proof. Consider (3.4.2) where we substitute s with the *specific* solenoidal extension a from Proposition 3.3.1. Multiply (3.4.2) by $\hat{u}$ and integrate by parts over Ω and, recalling that $\hat{u} = 0$ on $\partial\Omega$, obtain

$$\mu\|\nabla\hat{u}\|^2_{L^2(\Omega)} + \psi\,(\hat{u},\hat{u},\hat{u}) + \psi\,(\hat{u},a,\hat{u}) + \psi\,(a,\hat{u},\hat{u}) = \langle\, f,\hat{u}\rangle$$

with

$$\langle\, f,\hat{u}\rangle = -\mu(\nabla a,\nabla\hat{u})_{L^2(\Omega)} - \psi\,(a,a,\hat{u})$$

and ψ as in (3.2.5). The properties of ψ guarantee that the second and fourth terms on the left-hand side vanish:

$$\mu\|\nabla\hat{u}\|^2_{L^2(\Omega)} + \psi\,(\hat{u},a,\hat{u}) = \langle\, f,\hat{u}\rangle = -\mu(\nabla a,\nabla\hat{u})_{L^2(\Omega)} - \psi\,(a,a,\hat{u}). \tag{3.4.5}$$

For the right-hand side of this equation we first exploit the partition (3.3.6)

$$(\nabla a,\nabla\hat{u})_{L^2(\Omega)} = \int_{\Omega_0}\nabla a : \nabla\hat{u}\,dx + \int_{\Omega_1}\nabla a : \nabla\hat{u}\,dx + \int_{\Omega_2}\nabla a : \nabla\hat{u}\,dx$$

and we remark that

$$\int_{\Omega_1} \nabla q : \nabla \hat{u}\, dx = \int_{\Omega_1} \Delta q \cdot \hat{u}\, dx$$
$$= -2\frac{k_p}{d^2\,\|\nabla g\|_{L^2(\omega)}}\int_{-L}^{-l-\alpha}\left[\int_\omega \hat{u}\cdot\hat{n}\,dx_2\,dx_3\right]dx_1 = 0$$

as $\hat{u}$ carries no flux being divergence-free on Ω. For the same reason, also the integral over Ω_2 vanishes and we obtain

$$\left|\mu(\nabla a, \nabla\hat{u})_{L^2(\Omega)}\right| = \mu\left|\int_{\Omega_0}\nabla a : \nabla\hat{u}\,dx\right| \le \mu\|\nabla a\|_{L^2(\Omega_0)}\|\nabla\hat{u}\|_{L^2(\Omega_0)}$$
$$\le \mu\,\Lambda_2 k_p\|\nabla\hat{u}\|_{L^2(\Omega)},$$

where we used the bound in Ω_0 given in Proposition 3.3.4. Since $a = q$ in $\Omega_1\cup\Omega_2$ by Proposition 3.3.1 and since $(q\cdot\nabla)q \equiv 0$, we have that

$$\left|\psi\,(a,a,\hat{u})\right| = \left|\int_{\Omega_0}(a\cdot\nabla)a\cdot\hat{u}\,dx\right| \le \|a\|_{L^4(\Omega_0)}\|\nabla a\|_{L^2(\Omega_0)}\|\hat{u}\|_{L^4(\Omega_0)}\,.$$

Using again Proposition 3.3.4 and collecting terms we may then bound the right hand side of (3.4.5) as

$$|\langle f,\hat{u}\rangle| \le \left(\mu\,\|\nabla a\|_{L^2(\Omega_0)} + \|a\|_{L^4(\Omega_0)}\frac{\|\nabla a\|_{L^2(\Omega_0)}}{\sqrt{\mathcal{S}_0}}\right)\|\nabla\hat{u}\|_{L^2(\Omega)}$$
$$\le \left(\mu\,\Lambda_2 k_p + \frac{\Lambda_1\Lambda_2}{\sqrt{\mathcal{S}_*}}k_p^2\right)\|\nabla\hat{u}\|_{L^2(\Omega)}$$

where we also used (3.3.18).

On the other hand, since a obeys (3.3.8),

$$\begin{aligned}|\psi\,(\hat{u},a,\hat{u})| &\le \left|\int_{\Omega_1}(\hat{u}\cdot\nabla)q\cdot\,\hat{u}\,dx + \int_{\Omega_2}(\hat{u}\cdot\nabla)q\cdot\,\hat{u}\,dx + \int_{\Omega_0}(\hat{u}\cdot\nabla)a\cdot\hat{u}\,dx\right| \\ &\le \left(\frac{k_p}{\sigma_0} + \frac{\|\nabla a\|_{L^2(\Omega_0)}}{\mathcal{S}_0}\right)\|\nabla\hat{u}\|^2_{L^2(\Omega)},\end{aligned} \tag{3.4.6}$$

where we used the Hölder inequality together with the Sobolev inequalities in (3.3.11) and (3.3.12) as follows:

$$\left|\int_{\Omega_1}(\hat{u}\cdot\nabla)q\cdot\,\hat{u}\,dx\right| \le \int_{-L}^{-l-\alpha}\|\hat{u}\|_{L^4(\omega)}\|\nabla q\|_{L^2(\omega)}\|\hat{u}\|_{L^4(\omega)}\,dx_1$$
$$\le k_p\int_{-L}^{-l-\alpha}\frac{\|\nabla_{x'}\hat{u}\|^2_{L^2(\omega)}}{\sigma_0}\,dx_1 \le \frac{k_p}{\sigma_0}\|\nabla\hat{u}\|^2_{L^2(\Omega_1)},$$

with $x' = (x_2, x_3)$, while ∇ indicates the gradient with respect to the three variables (x_1, x_2, x_3). The integral over Ω_2 can be treated analogously. Finally, by exploiting the inequality $\|\nabla\hat{u}\|^2_{L^2(\Omega_1)}+$

$\|\nabla \hat{u}\|^2_{L^2(\Omega_2)} \le \|\nabla \hat{u}\|^2_{L^2(\Omega)}$, we see that the left-hand side of (3.4.5) can be lower bounded by

$$\begin{aligned} -\frac{k_p}{\sigma_0}\|\nabla \hat{u}\|^2_{L^2(\Omega)} - \frac{\Lambda_2\, k_p}{\mathcal{S}_0}\|\nabla \hat{u}\|^2_{L^2(\Omega)} &\le -\frac{k_p}{\sigma_0}\|\nabla \hat{u}\|^2_{L^2(\Omega)} \\ &\quad - \frac{\|\nabla a\|_{L^2(\Omega_0)}}{\mathcal{S}_0}\|\nabla \hat{u}\|^2_{L^2(\Omega)} \\ &\le \psi\,(\hat{u}, a, \hat{u}), \end{aligned}$$

where we used again the bounds in Proposition 3.3.4. At last, we exploit the inequalities (3.3.10) and (3.3.18) to obtain

$$-\frac{k_p}{\sigma_*}\|\nabla \hat{u}\|^2_{L^2(\Omega)} - \frac{\Lambda_2\, k_p}{\mathcal{S}_*}\|\nabla \hat{u}\|^2_{L^2(\Omega)} \le \psi\,(\hat{u}, a, \hat{u}). \tag{3.4.7}$$

By plugging (3.4.7) into (3.4.5) and dividing by $\|\nabla \hat{u}\|_{L^2(\Omega)}$, we obtain the bound (3.4.4), provided that (3.4.3) holds. □

We are now in position to prove the following statement, whose major result is the form of the quantitative bound for uniqueness of solutions of (3.1.1)-(3.1.2); this bound will be used for the overall conclusion in Proposition 3.4.5 below.

Proposition 3.4.2. *Let* Ω *be as in* (3.2.1) *and* $q \in W^{1,\infty}(\partial T)$ *as in* (3.2.3)*. Then, there exists at least one weak solution* u *to problem* (3.1.1)-(3.1.2) *with corresponding* $p \in L^2(\Omega)$*.*

Moreover, if $\mathcal{S}_*$ *is as in* (3.3.18)*,* σ_* *as in* (3.3.10)*, the constants* Λ_i *(*$i = 1, 2$*) as in Proposition 3.3.4, and*

$$k_p < \bar{k}_p := \mu\sigma_* \frac{2\mathcal{S}_* + \sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_* - \sigma_*\sqrt{(\sqrt{\mathcal{S}_*}\Lambda_1 + 2\Lambda_2)^2 + \frac{4\mathcal{S}_*\Lambda_2}{\sigma_*}}}{2\mathcal{S}_* + 2\sqrt{\mathcal{S}_*}\Lambda_1\sigma_* + 2\Lambda_2\sigma_*}. \tag{3.4.8}$$

then the weak solution is unique.

Proof. Existence of u satisfying (3.2.6) follows from [60, Theorem IX.4.1], provided that we have an a priori bound on $\|\nabla \hat{u}\|_{L^2(\Omega)}$, where $\hat{u}$ solves (3.4.1)-(3.4.2) in a weak sense. Multiply (3.4.2) by $\hat{u}$ and integrate by parts over Ω: the two terms $\psi\,(\cdot, \hat{u}, \hat{u})$ vanish and we bound the right-hand side through the Hölder inequality and (3.3.12):

$$\mu\|\nabla \hat{u}\|^2_{L^2(\Omega)} + \psi(\hat{u}, s, \hat{u}) \le \left(\mu\|\nabla s\|_{L^2(\Omega)} + \|s\|_{L^4(\Omega)}\frac{\|\nabla s\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}}\right)\|\nabla \hat{u}\|_{L^2(\Omega)}. \tag{3.4.9}$$

We draw attention to the fact that s is here a sufficiently smooth *general* solenoidal extension of q; that is why its norms in (3.4.9) live on the whole domain Ω, rather than on a component of the partition (3.3.6).

The flux of q is null across the two connected components of the boundary $\partial\Omega = \partial K \cup \partial T$, i.e.

$$\int_{\partial T} q \cdot \hat{n} = \int_{\partial K} q \cdot \hat{n} = 0.$$

Hence, in view of [60, Lemma IX.4.2], there exists a Hopf extension [89], namely for any $\eta > 0$ there exists a solenoidal extension s satisfying

$$|\psi(v,s,v)| \leq \eta \, \|\nabla v\|^2_{L^2(\Omega)} \qquad \forall v \in V(\Omega).$$

By choosing $\eta < \mu$ and plugging this bound into (3.4.9), we obtain the desired a priori bound on $\hat{u}$ ensuring existence of u satisfying (3.2.6) for any given value of $\mu > 0$. The existence of a pressure field $p \in L^2(\Omega)$ corresponding to the weak solution u follows, for instance, from [60, Lemma IX.1.2].

We now turn to uniqueness. Let us suppose that u_0 and u_1 are two weak solutions of (3.1.1)-(3.1.2). Define $w = u_0 - u_1$; it satisfies the following identity

$$\mu(\nabla w, \nabla \phi)_{L^2(\Omega)} + \psi\,(u_0, w, \phi) + \psi\,(w, u_1, \phi) = 0 \qquad \forall\, \phi \in V(\Omega)$$

where $V(\Omega)$ is defined in (3.2.4). Since $w \in V(\Omega)$, we may substitute ϕ with it and obtain

$$\mu \|\nabla w\|^2_{L^2(\Omega)} = -\psi\,(w, u_1, w).$$

Then, we obtain an upper bound for the right-hand side. If we define $u_1 = \hat{u}_1 + a$, where a is the *specific* solenoidal extension built in Proposition 3.3.1 (not the above Hopf extension), we can divide this member in two terms:

$$-\psi\,(w, u_1, w) = \psi\,(w, w, u_1) = \psi\,(w, w, \hat{u}_1 + a) = \psi\,(w, w, \hat{u}_1) + \psi\,(w, w, a). \tag{3.4.10}$$

For the first term, by applying the Hölder inequality, the Sobolev inequality in Ω and finally the lower bound for $\mathcal{S}_0$ in (3.3.12), labelled as $\mathcal{S}_*$, we deduce that

$$\begin{aligned} |\psi\,(w, w, \hat{u}_1)| &\leq \|w\|_{L^4(\Omega)} \|\nabla w\|_{L^2(\Omega)} \|\hat{u}_1\|_{L^4(\Omega)} \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_0}} \|\hat{u}_1\|_{L^4(\Omega)} \\ &\leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\mathcal{S}_0} \|\nabla \hat{u}_1\|_{L^2(\Omega)} \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\mathcal{S}_*} \|\nabla \hat{u}_1\|_{L^2(\Omega)} \ . \end{aligned}$$

The second term in (3.4.10) can be treated similarly to (3.4.6), after using the property of the trilinear form ψ and both the lower bounds (3.3.10) and (3.3.18):

$$\begin{aligned} |\psi\,(w, w, a)| = |\psi\,(w, a, w)| &\leq \left(\frac{k_p}{\sigma_0} + \frac{\|a\|_{L^4(\Omega_0)}}{\sqrt{\mathcal{S}_0}} \right) \|\nabla w\|^2_{L^2(\Omega)} \\ &\leq \left(\frac{k_p}{\sigma_*} + \frac{\|a\|_{L^4(\Omega_0)}}{\sqrt{\mathcal{S}_*}} \right) \|\nabla w\|^2_{L^2(\Omega)} \,. \end{aligned}$$

By combining these bounds and using the result of Proposition 3.3.4 we infer

$$\begin{aligned} \mu \|\nabla w\|^2_{L^2(\Omega)} &\leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}} \Big(\frac{\|\nabla \hat{u}_1\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}} + \frac{k_p}{\sigma_*} \sqrt{\mathcal{S}_*} + \|a\|_{L^4(\Omega_0)} \Big) \\ &\leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}} \Big(\frac{\|\nabla \hat{u}_1\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_*}} + \frac{k_p}{\sigma_*} \sqrt{\mathcal{S}_*} + \Lambda_1 \, k_p \Big) \,. \end{aligned}$$

Then, provided that (3.4.3) holds, we insert the a priori bound (3.4.4) for the gradient of $\hat{u}_1$ and we obtain

$$\mu\|\nabla w\|^2_{L^2(\Omega)} \leq \|\nabla w\|^2_{L^2(\Omega)} \frac{-k_p^2\left(\frac{\mathcal{S}_*}{\sigma_*^2}+\frac{\sqrt{\mathcal{S}_*}\Lambda_1}{\sigma_*}+\frac{\Lambda_2}{\sigma_*}\right)+k_p\left(\sqrt{\mathcal{S}_*}\Lambda_1\mu+\Lambda_2\mu+\frac{\mathcal{S}_*\mu}{\sigma_*}\right)}{\mathcal{S}_*\,\mu-\frac{k_p}{\sigma_*}\mathcal{S}_*-k_p\Lambda_2},$$

which implies $w = 0$ if the following condition holds:

$$-k_p^2\left(\frac{\mathcal{S}_*}{\mu\sigma_*^2}+\frac{\sqrt{\mathcal{S}_*}\Lambda_1}{\mu\sigma_*}+\frac{\Lambda_2}{\mu\sigma_*}\right)+k_p\left(\sqrt{\mathcal{S}_*}\Lambda_1+2\Lambda_2+\frac{2\mathcal{S}_*}{\sigma_*}\right)<\mu\,\mathcal{S}_*.$$

This is a condition of negativity on a concave parabola as a function of k_p, which crosses the vertical axis in $-\mu\mathcal{S}_0$. Hence, it is fulfilled if k_p is less than the smallest between the two roots of the second-order polynomial, which reads as (3.4.8). Some tedious computations show that the right-hand side of inequality (3.4.8) is smaller than the right-hand side of inequality (3.4.3), thus (3.4.8) implies (3.4.3): this proves uniqueness. □

Remark 3.4.3. Notice that $\bar{k}_p > 0$ since the denominator is strictly positive as well as the same can be easily check to go for the numerator, and it can be rewritten as

$$\bar{k}_p = \mu\sigma_*\left(1-\frac{\sqrt{\mathcal{S}_*}\Lambda_1\sigma_*+\sigma_*\sqrt{(\sqrt{\mathcal{S}_*}\Lambda_1+2\Lambda_2)^2+\frac{4\mathcal{S}_*\Lambda_2}{\sigma_*}}}{2\mathcal{S}_*+2\sqrt{\mathcal{S}_*}\Lambda_1\sigma_*+2\Lambda_2\sigma_*}\right).$$

3.4.2 Threshold for the appearance of the lift

Before stating the main result of this section, in the spirit of [68] we recall the following implications for solutions of (3.1.1)-(3.1.2)

$$\text{uniqueness} \implies \text{symmetry} \implies \text{no } \textit{lift} \text{ exerted over } K.$$

It is a well-know experimental fact that lift vanishes whenever the obstacle is symmetric with respect to the angle of attack of the fluid; see e.g. [40, Figure 7.21]. We state here a small variant of [68, Proposition 4.1]:

Proposition 3.4.4. *Let Ω be as in* (3.2.1) *and $q \in W^{1,\infty}(\partial T)$ be as in* (3.2.3). *Let $u = (u_1, u_2, u_3) \in V_*(\Omega)$ be a weak solution of problem* (3.1.1)-(3.1.2). *Let $\mathcal{S}_*$ be as in* (3.3.18), *σ_* as in* (3.3.10)*, the constants Λ_i, $i = 1, 2$ as in Proposition 3.3.4. Then also $w = (w_1, w_2, w_3) \in V_*(\Omega)$ defined by*

$$w_1(x_1,x_2,x_3) = u_1(x_1,x_2,-x_3) \qquad w_2(x_1,x_2,x_3) = -u_2(x_1,x_2,-x_3)$$
$$w_3(x_1,x_2,x_3) = u_3(x_1,x_2,-x_3)$$

for a.e. $(x_1, x_2, x_3) \in \Omega$, solves (3.1.1)-(3.1.2) *in a weak sense. Moreover, if* (3.4.8) *is valid, the weak solution of* (3.1.1)-(3.1.2) *is unique and it satisfies the symmetry property*

$$u_1(x_1,x_2,x_3) = u_1(x_1,x_2,-x_3) \qquad u_2(x_1,x_2,x_3) = -u_2(x_1,x_2,-x_3)$$
$$u_3(x_1,x_2,x_3) = u_3(x_1,x_2,-x_3)$$

for a.e. $(x_1, x_2, x_3) \in \Omega$

Proposition 3.4.4 stands because Ω is symmetric with respect to all three axes x_1, x_2, x_3 and because the boundary datum q is x_3-even in its first component v_1 and null in its other components. Proposition 3.4.4 then shows that uniqueness implies symmetry. Then, [68, Theorem 3.7] shows that symmetry implies no lift exerted on the obstacle K. We recall that we adopt a generalized definition for the lift force, (3.2.9), since we are considering weak solutions.

In view of [68, Theorem 3.7], we can state the following proposition, which embodies the explicit realisation of the purpose of this chapter.

Proposition 3.4.5. *Let* Ω *be as in* (3.2.1) *and* $q \in W^{1,\infty}(\partial T)$ *as in* (3.2.3). *Let* $\delta_i,\ i = 1, 2, .., 24$ *be reported in the Appendix, where we also emphatized the dependence on the parameter* α*, used to define the rectangle* $\mathcal{R}_1$ *in* (3.3.5). *Let* F_K *be the total force exerted by the fluid over the obstacle* K*, given in* (3.2.8). *For any* $k_p \geq 0$*, there exists a weak solution* $(u, p) \in V_*(\Omega) \times L^2(\Omega)$ *of* (3.1.1)-(3.1.2).

Moreover, given the constants Λ_1 *and* Λ_2 *in Proposition 3.3.4, given* σ_* *as in* (3.3.10) *and* $\mathcal{S}_*$ *as in* (3.3.18)*, if the parameter* k_p*, regulating the inlet and outlet flow, is such that* $k_p < \bar{k}_p$ *(see* (3.4.8)*), then the weak solution is unique and the fluid exerts no lift on the obstacle* K*, that is*

$$\left\langle \{-p\mathbf{I} + \mu[\nabla u + (\nabla u)^T]\} \cdot \hat{n}, 1 \right\rangle_{\partial K} = 0$$

It remains to show how to compute the constants $\delta_i,\ i = 1, 2, .., 24$, depending on α (defining the region $\mathcal{R}_1$ in (3.3.5)), which come from the explicit estimates for the norms of the solenoidal extension $a(x)$ given in Proposition 3.3.4.

We computed the δ_i's with the software *Mathematica*, that was also used to compute the value of α maximizing $\bar{k}_p$, once we know the structural parameters of the problem, in particular for the table in Section 3.2. Since the computations are unpleasant, we give their explicit value in the Appendix.

CHAPTER 4

Well-posedness of a FSI problem in a Poiseuille flow: vertical motion

In the next two chapters, we analyze the well-posedness of the two fluid-structure interaction models presented in Chapter 2. Chapter 4 treats the well-posedness of problem (2.0.7), while Chapter 4 treats that of problem (2.0.9). In the present chapter, after a general introduction to the purpose of the study, we enter into the details of problem (2.0.7), which models the interaction between the cross-section of the deck of a suspension bridge and the wind in the first phase of the so-called *flutter* phenomenon. In particular, we obtain a global-in-time existence and uniqueness result for problem (2.0.7).

4.1 The flutter instability

Suspension bridges may experience several types of instability phenomena, which affect more or less critically each component of the structure. Among all components, the deck is the most sensitive part. When analyzing the dynamic response of the bridge to the wind from the engineering point of view, we observe that the bridge may suffer from a variety of problems: one degree and two degrees of freedom instability, buffeting and vortex shedding. Two degrees of freedom instability, also known as *flutter* instability, occurs when the vertical and the torsional motion of the deck synchronize so that aerodynamic forces introduce energy into the system. However, this only occurs after reaching a critical value of the incoming wind velocity (see [46], [64], [120]). Thus, the vertical and torsional displacements are decoupled in a regime of small oscillations.

In order to properly study flutter instability, one is lead to consider the two different models presented in Chapter 2 so as to reproduce the two aforementioned regimes.

We set up the **first** model precisely under the hypothesis of a small flux for the incoming flow field. We consider the interaction between an obstacle and a fluid in a 2D bounded channel, where the flow is of *Poiseuille* type at the inlet and outlet sections; we analyze a fluid-structure problem by allowing the obstacle to move in a vertical translation. Physically, this models the interaction between the wind and the deck of a bridge in a wind tunnel experiment.

In a regime of strong incoming flux, it is necessary to consider the full coupled vertical-torsional motion, which is taken into account in the **second** model. Also, here the channel is unbounded and the Poiseuille flow is imposed at infinity so as to model the experiment at consideration in a real-life framework, where the wind interacts with the bridge in the atmosphere.

4.2 Hypotheses and well-posedness

We refer to Chapter 2 for the notation and the rigorous formulation of problem (2.0.7).

To the purpose of studying the well-posedness of problem (2.0.7), we will assume that the restoring force f in $(2.0.7)_4$ satisfies some further conditions besides those given in Chapter 2, given later in (4.2.1), which translate the assumption of f being a *strong force*, preventing the obstacle from colliding with the boundary of the channel $\Gamma_{\mathcal{R}}$. From the physical view point indeed, f resumes the action of three kind of forces acting on the deck of a suspension bridge, as also explained in [15, Introduction]: the upward restoring force due to the elastic action of both the hangers and the sustaining cables, the weight of the deck acting downwards and the elastic resistance to deformations of the whole deck, preventing the obstacle to go too far from its equilibrium horizontal position (see also [64]). Our model indeed loses its physical meaning in case of collisions; on the other hand, collisions would be purely virtual in the physical framework at consideration, because we can not expect the bridge to be subjected to very large deformations. This justifies the limit in the assumption in (4.2.1); we emphasize that, in this context, we are not interested in choosing the optimal growth hypothesis on f ensuring the absence of collisions.

In [15], the authors prove that for the stationary version of (2.0.7) the equilibrium position of the obstacle is perfectly symmetric under smallness assumption on the imposed flow rate magnitude of the Poiseuille flow; as a matter of fact, they are able to prove their result with an assumption on f weaker than (4.2.1) (their result holds in an unbounded channel, but it can be easily extended to a bounded domain). The main purpose of this chapter is to prove the existence and the uniqueness of a weak solution for the fluid-structure interaction evolution problem (2.0.7). In order to prove existence we adopt a penalty method devised in a paper by Fujita and Sauer, see [56]; this technique was later exploited by Conca, San Martin and Tucsnak in [31] in order to prove existence of solutions for a boundary problem modelling the motion of a rigid ball in a viscous fluid occupying a bounded domain (see Chapter 1). Among the several aforementioned techniques, we choose to apply the method by [31], because it enables to work with a non-global weak formulation, in the sense that the terms concerning the fluid sub-problem and those concerning the rigid body sub-problem of (2.0.7), although coupled, remain distinguished. On the other hand, both the method introduced in [41] and [87, 119] would require working with global quantities and a global weak formulation, in the sense that the integrals would be defined on the whole domain $\mathcal{R}$; in this context, the approach by [31] is more convenient because of the presence of the strong force f in the ordinary differential equation governing the motion of the obstacle, which makes it non-trivial to define a global weak form for the original coupled frame. The main difference with the problem considered in [31] lies in the fact that the domain occupied by both the rigid body and the fluid in (2.0.7) is an channel where a non-zero velocity field is imposed at the inlet and outlet; this requires building a solenoidal extension of the Poiseuille flow. Moreover, besides the less affecting difference of a rectangular shaped obstacle, the introduction of the *strong force f* in the ordinary differential equation governing the motion of the obstacle in (2.0.7) allows to obtain solutions with a global character in time, as well as uniqueness of such solutions: indeed, this force prevents the obstacle from colliding with the boundary of the channel.

The main result of this chapter reads as

Theorem 4.2.1. *Assume that $f(h) \in \mathcal{C}^1(-L+\delta, L-\delta;\mathbb{R})$ satisfies the assumptions given in Chapter 2 and*

$$\begin{aligned} &\exists\, r>0 \qquad \textit{s.t.} \qquad f'(h)>0\,\forall\, h\in(-L+\delta, L-\delta),\\ &\lim_{|h|\to L-\delta} |f(h)|\,\frac{1}{\exp\frac{1}{(L-\delta-|h|)^{4+r}}} = +\infty. \end{aligned} \tag{4.2.1}$$

Moreover, let $|h_0|<L-\delta$ and u_0 satisfying $(2.0.8)_1$-$(2.0.8)_2$ *be such that $u_0\cdot\widehat{n} = k_0\widehat{e}_2\cdot\widehat{n}$ on ∂B_{h_0}, with $k_0\in\mathbb{R}$. Then, problem* (2.0.7) *admits a unique weak solution (u,h), defined in a suitable sense, for any $T>0$. Moreover the solution (u,h) satisfies an energy estimate.*

Theorem 4.2.1 deserves some **comments**. First, we emphasize that, since (2.0.7) is a fluid-structure interaction problem, one needs to give a suitable definition of weak solutions which takes into account the presence of the moving obstacle. Because of such difficulty, we adopt a definition of weak solutions which is a compromise between the one given in [63] and the one in [31]; weak solutions are defined in Definition 4.3.3 after transforming problem (2.0.7) into an equivalent problem, as we shall see in Section 4.3. Finally, the global-in-time property of solutions is ensured because of the energy estimate associated to problem (2.0.7), which guarantees that no collision occurs between the obstacle and the boundary of the channel. Such energy estimate will be made explicit in Theorem 4.3.6 for the equivalent problem.

The first part of the chapter is devoted to proving Theorem 4.2.1 and it is organized as follows. In Section 4.3 we present some notations and preliminary results which are essential to apply the penalty method and are useful throughout the chapter. Then we reformulate problem (2.0.7) in a reference frame attached to the obstacle; this produces the equivalent problem, (4.3.9)-(4.3.10). At the end of Section 4.3, Theorem 4.3.6 states the existence and uniqueness of solutions for the equivalent problem (4.3.9)-(4.3.10). Thus, Theorem 4.2.1 is proven once we develop the proof of such statement, which is addressed in the subsequent sections. In Section 4.4, we introduce an auxiliary problem, at the core of the penalty method, for which we prove existence of solutions with the Faedo-Galerkin procedure. In Section 4.5, after some additional results, we conclude the proof of the existence part of Theorem 4.3.6 and, consequently, the existence part of the main result of the chapter, Theorem 4.2.1. Section 4.6 is devoted to proving uniqueness of solutions to problem (4.3.9)-(4.3.10), which concludes the proof of Theorem 4.3.6 and thus also of Theorem 4.2.1. The main difficulty of proving uniqueness is that we cannot simply take the difference between two weak solutions of problem (4.3.9)-(4.3.10) because, since the fluid domain has moving boundaries, those solutions are not defined on the same domain, thus we will adopt a suitable change of variables to solve such issue, following the work in [76].

4.3 An equivalent formulation

Problem (2.0.7) is set in a two-dimensional bounded channel with a prescribed non-zero velocity field at the inlet and outlet (the Pouseuille flow in (2.0.6)). We begin by capturing the flow for large values of $|x_1|$: we construct a suitable extension of the Poiseuille velocity profile by using similar arguments to the classical procedure by Ladyzhenskaya [100] (see also [6]). For this, we divide the channel $\mathcal{R}$ as

$$\mathcal{R} = \bigcup_{i=0}^{2} \mathcal{R}_i \,,$$

where

$$\mathcal{R}_0 = \mathcal{R}\cap([-3,3]\times\mathbb{R}), \qquad \mathcal{R}_1 = \mathcal{R}\cap((-\infty,-3)\times\mathbb{R}), \qquad \mathcal{R}_2 = \mathcal{R}\cap((3,\infty)\times\mathbb{R})\,, \quad (4.3.1)$$

and then prove the following result:

Lemma 4.3.1. *For every $\varepsilon_0 > 0$ there exists a vector field $s = s_{\varepsilon_0} \in W^{2,\infty}(\mathcal{R}) \cap H^2(\mathcal{R})$ such that*

$$\nabla\cdot s = 0 \quad \text{in } \mathcal{R}\,, \qquad s=(0,0) \quad \text{in} \quad [-2,2]\times[-L+\varepsilon_0, L-\varepsilon_0]\,, \qquad s=(0,0) \quad \text{on} \quad \partial\mathcal{R}\,,$$
$$s = v_P \quad \text{in } \mathcal{R}_1\cup\mathcal{R}_2\,, \quad \text{supp}(s) = \mathcal{R}\setminus((-2,2)\times(-L+\varepsilon_0, L-\varepsilon_0))\,, \tag{4.3.2}$$

and there hold the estimates

$$\|\nabla s\|_{L^\infty(\mathcal{R})} \le \frac{c_1}{\varepsilon_0^2}, \qquad \|\nabla s\|_{L^2(\mathcal{R}_0)} \le \frac{c_2}{\varepsilon_0^2}, \tag{4.3.3}$$

where $c_1, c_2 > 0$ depend on L and λ.

Proof. Let $b : [-L, L] \longrightarrow \mathbb{R}$ the function defined by

$$b(x_2) = \frac{p_0 L^3}{2\mu}\left[\frac{x_2}{L} - \frac{1}{3}\left(\frac{x_2}{L}\right)^3\right] \qquad \forall x_2 \in [-L, L]\,,$$

so that

$$b'(x_2) = v_P(x_2) \qquad \forall x_2 \in (-L, L)\,.$$

As in [70, 127] we take a smooth cutoff function $\zeta_1 \in W^{2,\infty}(\mathbb{R})$ acting on the horizontal direction such that

$$\zeta_1(x_1) = \begin{cases} 1 & \text{if } |x_1| \le 2 \\ 0 & \text{if } |x_1| > 3\,, \end{cases} \qquad \text{supp}(\zeta_1) = [-2,2]\,. \tag{4.3.4}$$

Then, let $\varepsilon_0 \in (0, L)$. We take a cutoff function $\zeta_{\varepsilon_0} \in W^{2,\infty}(\mathbb{R})$ acting in the vertical direction such that

$$\zeta_{\varepsilon_0}(x_2) = \begin{cases} 1 & \text{if } |x_2| \leq L - \dfrac{\varepsilon_0}{2} \\ 0 & \text{if } |x_2| > L - \dfrac{\varepsilon_0}{4}, \end{cases} \qquad \operatorname{supp}(\zeta_{\varepsilon_0}) = \left[-L + \frac{\varepsilon_0}{4}, L - \frac{\varepsilon_0}{4}\right]. \tag{4.3.5}$$

Furthermore, there exists a constant $C > 0$ (independent of ε_0) such that

$$\|\zeta_{\varepsilon_0}\|_{L^\infty(\mathbb{R})} = 1\,, \qquad \|\zeta'_{\varepsilon_0}\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\varepsilon_0}\,, \qquad \|\zeta''_{\varepsilon_0}\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\varepsilon_0^2}\,.$$

Then we write

$$Z_{\varepsilon_0}(x) = 1 - \zeta_1(x_1)\,\zeta_{\varepsilon_0}(x_2) \qquad \forall x \in \mathcal{R}\,,$$

and define the vector field

$$s(x) = \left(\frac{\partial}{\partial x_2}\big(b(x_2)Z_{\varepsilon_0}(x)\big), -\frac{\partial}{\partial x_1}\big(b(x_2)Z_{\varepsilon_0}(x)\big)\right) \qquad \forall x \in \mathcal{R}\,.$$

In view of (4.3.4)-(4.3.5) we have that $s \in W^{2,\infty}(\mathcal{R}) \cap H^2(\mathcal{R})$ and that it verifies the properties in (4.3.2)-(4.3.3). ☐

As a first step of the penalty method implemented in [31], we change problem (2.0.7) into an equivalent problem by considering a frame attached to the rigid body, whose origin coincides with its center of mass. Thus we set

$$y = x - h(t)\,\widehat{e}_2, \tag{4.3.6}$$

and we denote

$$\begin{aligned} &v(y,t) = u(y + h(t)\,\widehat{e}_2, t), \quad \mathfrak{q}(y,t) = p(y + h(t)\,\widehat{e}_2, t), \\ &\mathcal{T}(v, \mathfrak{q}) = -\mathfrak{q}\mathbf{I} + \mu[\nabla v + (\nabla v)^T] \\ &\tilde{\Omega}(t) = \Omega_{h(t)} - h(t)\widehat{e}_2, \quad \mathcal{R}_{h(t)} = \mathcal{R} - h(t)\widehat{e}_2, \quad \tilde{\Gamma}_{\mathcal{R}}(t) = \Gamma_{\mathcal{R}} - h(t)\widehat{e}_2, \\ &B = B_{h_0} = B_{h(t)} - h(t)\widehat{e}_2. \end{aligned} \tag{4.3.7}$$

The domain of the fluid in the new reference frame $\tilde{\Omega}$ shall also be partitioned

$$\begin{aligned} &\tilde{\Omega}(t) = \bigcup_{i=0}^{2} \tilde{\Omega}_i(t), \qquad \tilde{\Omega}_0(t) = \tilde{\Omega}(t) \cap \{-3 \leq y_1 \leq 3\}, \\ &\tilde{\Omega}_1(t) = \tilde{\Omega}(t) \cap \{y_1 < -3\}, \qquad \tilde{\Omega}_2(t) = \tilde{\Omega}(t) \cap \{y_1 > 3\}. \end{aligned} \tag{4.3.8}$$

We emphasize that the obstacle is now fixed, while the domain occupied by both the fluid and the rigid body, $\mathcal{R}_{h(t)} = B \cup \partial B \cup \tilde{\Omega}(t)$, changes with time. Then, we notice that

$$\nabla_y v = \nabla_x u, \quad \operatorname{div}_y v = \operatorname{div}_x u, \quad \Delta_y v = \Delta_x u, \quad v_t = u_t + (h'\widehat{e}_2 \cdot \nabla_y)v.$$

Thus, we obtain the following problem (see also [58]):

$$
\begin{aligned}
&v_t = \mu\,\Delta v - (v\cdot\nabla)\,v - \nabla\mathfrak{q} + (h'(t)\widehat{e}_2\cdot\nabla)\,v \quad (y,t)\in\tilde{\Omega}(t)\times(0,T),\\
&\mathrm{div}\,v = 0 \quad (y,t)\in\tilde{\Omega}(t)\times(0,T)\\
&v = \tilde{v}_P(y) := \lambda(L^2-(y_2+h(t))^2)\,\widehat{e}_1 \quad (y,t)\in\tilde{\Gamma}_{\mathcal{R}}(t)\times(0,T),\\
&v = h'(t)\widehat{e}_2 \quad (y,t)\in\partial B\times(0,T),\\
&v(y,0) = v_0(y) \quad y\in\tilde{\Omega}(0)
\end{aligned}
\tag{4.3.9}
$$

where $v_0(y) = u_0(y + h_0\widehat{e}_2)$ with u_0 as in $(2.0.8)_1$-$(2.0.8)_2$. Notice that all derivatives appearing in this problem are now taken with respect to the new variable $y = (y_1, y_2)$. The Poiseuille flow is obtained by transporting v_p as in (2.0.7) to the new reference frame:

$$\tilde{v}_P(y) = v_P(y + h(t)\widehat{e}_2).$$

Its associated pressure in the new reference frame is

$$\tilde{\pi}_P = \pi_P(y + h(t)\widehat{e}_2).$$

The motion of the rectangular body is governed by:

$$m\,h''(t) + f(h(t)) = -\widehat{e}_2\cdot\int_{\partial B}\mathcal{T}(v,\mathfrak{q})\widehat{n} \qquad t\in(0,T). \tag{4.3.10}$$

The original problem (2.0.7) is equivalent to (4.3.9)-(4.3.10), because we simply adopted a change in the system of coordinates. Thus Theorem 4.2.1 is proven if we prove existence of solutions to (4.3.9)-(4.3.10). We look for solutions to the problem (4.3.9)-(4.3.10) of the form

$$v = \widehat{v} + a, \qquad \mathfrak{q} = \mathfrak{p} + \tilde{\pi}_P$$

where a is the solenoidal extension of the Poiseuille flow in the new reference frame, strongly depending on h and on the choice of ε_0 in Lemma 4.3.1:

$$a(y) = a_{h(t)}(y;\varepsilon_0) = s(y + h(t)\,\widehat{e}_2). \tag{4.3.11}$$

The function a enjoys the same properties of s stated in Lemma 4.3.1 once we substituted v_p and $\mathcal{R}$ with their counterparts in the new reference frame, $\tilde{v}_P$ and $\mathcal{R}_h$, and we partitioned $\mathcal{R}_h$ similarly to what we did in (4.3.1). Assume that

$$h_0\in[-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}]$$

where $\widehat{\varepsilon} > 0$ small is arbitrarily fixed. Then, the pair $(\widehat{v},\mathfrak{p})$ solves the following problem:

$$
\begin{aligned}
&\widehat{v}_t - \mu\,\Delta\widehat{v} + (\widehat{v}\cdot\nabla)\,\widehat{v} + \nabla\mathfrak{p} - (h'(t)\widehat{e}_2\cdot\nabla)\,\widehat{v} - (h'(t)\widehat{e}_2\cdot\nabla)\,a\\
&\quad + (\widehat{v}\cdot\nabla)\,a + (a\cdot\nabla)\,\widehat{v} = \widehat{g} \quad (y,t)\in\tilde{\Omega}(t)\times(0,T),\\
&\mathrm{div}\,\widehat{v} = 0 \quad (y,t)\in\tilde{\Omega}(t)\times(0,T),\\
&\widehat{v} = 0 \quad (y,t)\in\tilde{\Gamma}_{\mathcal{R}}(t)\times(0,T), \qquad \widehat{v} = h'(t)\widehat{e}_2 \quad (y,t)\in\partial B\times(0,T),\\
&\widehat{v}(y,0) = \widehat{v}_0(y) = v_0 - a_{h_0}(y;\widehat{\varepsilon}) \quad y\in\tilde{\Omega}(0),
\end{aligned}
\tag{4.3.12}
$$

where
$$\widehat{g} := \mu\,\Delta(a - \tilde{v}_P) - (a \cdot \nabla)\,a. \tag{4.3.13}$$
Notice that $\widehat{v}_0 \in L^2(\tilde{\Omega}(0))$ is such that $\widehat{v}_0 \cdot \widehat{n} = k_0 \widehat{e}_2 \cdot \widehat{n}$ on ∂B. We also point out that $\text{supp}(\widehat{g}) \in \mathcal{R}_h \setminus \{|y_1| < 2 \wedge |y_2| < L - \varepsilon_0 - h\}$. The vertical translation of the obstacle h responds to
$$m\,h''(t) + f(h(t)) = -\widehat{e}_2 \cdot \int_{\partial B} \mathcal{T}(\widehat{v} + a, \mathfrak{p})\widehat{n} \quad t \in (0, T), \tag{4.3.14}$$
with initial conditions $h(0) = h_0$, $h'(0) = k_0$.

As already mentioned, in order to prove our main result we make use of a procedure which is similar to the one adopted in [31]: problem (4.3.12)-(4.3.14) is set in a region with moving boundaries, $\tilde{\Omega}(t)$, which makes it impossible to apply the Faedo-Galerkin approximation with the standard functional spaces of hydrodynamic evolutionary problems. The idea exploited in [31] is that of a penalty method and it was first elaborated in the paper by Fujita and Sauer (see [56]). The crucial idea of the method implies introducing an auxiliary fixed domain $\tilde{\mathcal{R}}$ given by:
$$\tilde{\mathcal{R}} = \mathcal{R} - \mathcal{R} = \{x - y \,|\, x \in \mathcal{R}, y \in \mathcal{R}\}, \tag{4.3.15}$$
such that $\tilde{\Omega}(t) \subset \mathcal{R}_{h(t)} \subset \tilde{\mathcal{R}}$ (see Figure 4.1 for the new configuration). Actually, $\mathcal{R}$ can be chosen as
$$\tilde{\mathcal{R}} = (-2I, 2I) \times (-2L + \delta, 2L - \delta)\,.$$
Notice that the vertical motion of $\tilde{\Omega}(t)$ inside $\tilde{\mathcal{R}}$ is confined: dist $(\partial\tilde{\mathcal{R}}, \tilde{\Gamma}_{\mathcal{R}}(t)) \geq \delta$. This is an obvious consequence of impenetrability of bodies but we will later prove that this inequality actually holds strictly, thus the obstacle never collides with the boundary of the channel. Inside the auxiliary fixed domain $\tilde{\mathcal{R}}$, we can naturally extend the velocity field $\tilde{v}_P(y)$ outside $\mathcal{R}_{h(t)}$ for every $h(t)$ through its definition in (4.3.9), see Figure 4.1.

We report three facts, that we will later use. First, an estimate for the L^2-norm of the gradient of $\tilde{v}_P$ in each one-dimensional section of the domain $\tilde{\mathcal{R}}$:
$$\|\nabla \tilde{v}_P\|_{L^2(-2L+\delta,\, 2L-\delta)} \leq \lambda\,\tilde{\xi} \quad \text{with } \tilde{\xi} = \sqrt{\frac{8}{3}\,(2L - \delta)\,(7L^2 - 10\,L\,\delta + 4\,\delta^2)}. \tag{4.3.16}$$
Then, we give the following partition of $\tilde{\mathcal{R}}$:
$$\begin{aligned} &\tilde{\mathcal{R}} = \bigcup_{i=0}^{2} \tilde{\mathcal{R}}_i, \quad \tilde{\mathcal{R}}_0 = \tilde{\mathcal{R}} \cap \{-3 \leq y_1 \leq 3\}, \quad \tilde{\mathcal{R}}_1 = \tilde{\mathcal{R}} \cap \{y_1 < -3\}, \\ &\tilde{\mathcal{R}}_2 = \tilde{\mathcal{R}} \cap \{y_1 > 3\}. \end{aligned} \tag{4.3.17}$$
Finally, we comment on the property of f in (4.3.14) being a strong force. The condition (4.2.1) may be interpreted in the spirit of [77]. In [77], the author considers systems of the general type $x'' + \nabla V(x) = 0$, with $x \in \mathbb{R}^n$, where the potential $V(x)$ associated to a conservative dynamical system, such as a n-body system, is assumed to be $\mathcal{C}^2$ everywhere except at a closed non empty set S at which it has infinitely deep wells, i.e. $V(x) \to -\infty$ as $x \to S$; then, the system is said to satisfy the *strong force* condition if and only if there exists a neighborhood N of S and

$U \in \mathcal{C}^2$ such that $U(x) \to -\infty$ as $x \to S$ and $-V(x) \geq |\nabla U(x)|^2$ for all $x \in N \setminus S$. In a n-body system the singularities correspond to collisions of the masses and the strong force condition allows to avoid them (see also [11]). In our case, in the terminology of [77], f plays the role of V' and the ODE in (4.3.14) satisfies the strong force assumption, with the singularity exhibited at $|h| = L - \delta$.

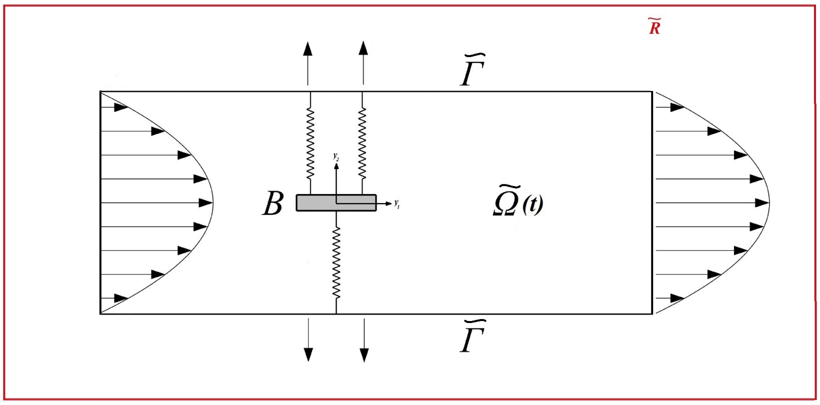


Figure 4.1: *The channel, after the change of variables, moves in the fixed red region* $\tilde{\mathcal{R}}$*.*

Now, we seek a rigorous definition of weak solutions to (4.3.9)-(4.3.10). We introduce some classical functional spaces from mathematical fluid dynamics (see [134] for instance):

$$\begin{aligned} \mathcal{V}(\tilde{\mathcal{R}}) &= \{v \in \mathcal{D}(\tilde{\mathcal{R}}) \,|\, \text{div}\, v = 0\}, \\ H(\tilde{\mathcal{R}}) &= \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\cdot\|_{L^2(\tilde{\mathcal{R}})}, \\ V(\tilde{\mathcal{R}}) &= \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\nabla \cdot\|_{L^2(\tilde{\mathcal{R}})}. \end{aligned}$$

We emphasize that since $\tilde{\mathcal{R}}$ is bounded, there holds the Poincaré inequality, which makes $H_0^1(\tilde{\mathcal{R}})$ an Hilbert space with respect to the scalar product $(u, v)_{H_0^1(\tilde{\mathcal{R}})} = (\nabla u, \nabla v)_{L^2(\tilde{\mathcal{R}})}$. The presence of the obstacle B requires introducing some further spaces:

$$\begin{aligned} \mathcal{W}(\tilde{\mathcal{R}}) &= \{(v, l) \in \mathcal{V}(\tilde{\mathcal{R}}) \times \mathbb{R} \,|\, v|_B = l\,\widehat{e}_2\}, \\ \mathbb{H}(\tilde{\mathcal{R}}) &= \text{closure of } \mathcal{W} \text{ in } L^2(\tilde{\mathcal{R}}) \times \mathbb{R}, \\ \mathbb{V}(\tilde{\mathcal{R}}) &= \text{closure of } \mathcal{W} \text{ in } H_0^1(\tilde{\mathcal{R}}) \times \mathbb{R} \end{aligned}$$

to which we associate the scalar products

$$\begin{aligned} \langle (v_1, \ell_1), (v_2, \ell_2) \rangle_{\mathbb{H}(\tilde{\mathcal{R}})} &= \int_{\tilde{\mathcal{R}} \setminus B} v_1 \cdot v_2 \, dy + m \ell_1 \ell_2, \\ \langle (v_1, \ell_1), (v_2, \ell_2) \rangle_{\mathbb{V}(\tilde{\mathcal{R}})} &= 2 \int_{\tilde{\mathcal{R}} \setminus B} D(v_1) \cdot D(v_2) \, dy + m \ell_1 \ell_2. \end{aligned}$$

Finally, the weak formulation of problem (4.3.9)-(4.3.10) will exploit the following spaces, which may be defined for any given function $h(t)$. In particular, for every t

$$\begin{aligned} \mathcal{W}_{h(t)} &= \{(v,\ell) \in \mathcal{W}(\tilde{\mathcal{R}}) \,|\, \operatorname{supp} v \in \mathcal{R}_{h(t)}\}, \\ \mathbb{H}_{h(t)} &= \text{closure of } \mathcal{W}_{h(t)} \text{ in } L^2(\tilde{\mathcal{R}}) \times \mathbb{R}, \\ \mathbb{V}_{h(t)} &= \text{closure of } \mathcal{W}_{h(t)} \text{ in } H^1_0(\tilde{\mathcal{R}}) \times \mathbb{R}. \end{aligned} \tag{4.3.18}$$

Then, we introduce the standard trilinear form:

$$\psi(u,v,w) = \int_{\tilde{\Omega}} (u \cdot \nabla)\, v \cdot w. \tag{4.3.19}$$

We are ready to state and prove the following proposition. Let us denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V and V'.

Proposition 4.3.2. *Let the couple (v,h) be a classical solution to* (4.3.9)-(4.3.10) *such that $|h(t)| \le L - \delta - \varepsilon_0$ for all $t \in [0,T]$ for some $\varepsilon_0 > 0$. Then, building the extension $a = a_h(y;\varepsilon_0)$ in* (4.3.11) *choosing the same ε_0, the function $\widehat{v} = v - a$ satisfies*

$$\begin{aligned} &-\int_0^T \{(\widehat{v},\phi_t)_{L^2(\tilde{\Omega}(t))} + m\, h'\ell' - f(h)\,\ell\} + 2\mu \int_0^T (D(\widehat{v}), D(\phi))_{L^2(\tilde{\Omega}(t))} \\ &+ \int_0^T \{\psi(\widehat{v},\widehat{v},\phi) + \psi(\widehat{v},a,\phi) + \psi(a,\widehat{v},\phi) - \psi(h'\widehat{e}_2,a,\phi) \\ &- \psi(h'\widehat{e}_2,\widehat{v},\phi)\} = \int_0^T \langle \widehat{g},\phi\rangle + m\,k_0\,\ell(0) + (\widehat{v}_0,\phi(0))_{L^2(\tilde{\Omega}(0))} \end{aligned} \tag{4.3.20}$$

for every $(\phi,\ell) \in \mathcal{C}^1([0,T];\mathbb{V}_{h(t)})$ such that $\phi(\cdot,T) = \ell(T) = 0$, with

$$\widehat{g} := \mu\,\Delta(a - \tilde{v}_P) - (a \cdot \nabla)\, a.$$

.

Proof. Consider the problem satisfied by $(\widehat{v},h)$, (4.3.12)-(4.3.14). In order to obtain (4.3.20), we choose a test couple $(\phi,\ell) \in \mathcal{C}^1([0,T],\mathbb{V}_{h(t)})$ such that $\phi(\cdot,T) = \ell(T) = 0$. We multiply the first equation in (4.3.12) by ϕ and integrate by parts on $\tilde{Q}_T = \tilde{\Omega}(t) \times [0,T]$. All terms may be treated in a standard manner (see, e.g., [57]). Though, a particular attention must be devoted to the diffusive and pressure terms. Indeed, we temporally move the term $\mu\Delta a$ appearing in $\widehat{g}$

in (4.3.12) on the left-hand side and we get:

$$
\begin{aligned}
&\int_0^T (-\mu\Delta\widehat{v} - \mu\Delta a + \nabla\mathfrak{p}, \phi)_{L^2(\tilde{\Omega}(t))} = \int_0^T (\mathrm{div}\mathcal{T}(\widehat{v}+a,\mathfrak{p}), \phi)_{L^2(\tilde{\Omega}(t))} \\
&= -\int_0^T \int_{\partial B} (\mathcal{T}\widehat{n})\cdot\phi + \int_0^T \int_{\tilde{\Omega}(t)} \mathcal{T} : \nabla\phi \\
&= -\int_0^T \widehat{e}_2 \cdot \int_{\partial B} (\mathcal{T}(\widehat{v}+a,\mathfrak{p})\widehat{n})\,\ell + \int_0^T \int_{\tilde{\Omega}(t)} \mathcal{T}(\widehat{v}+a,\mathfrak{p}) : \nabla\phi \\
&= \int_0^T (m\,h'' + f(h))\,\ell + 2\mu \int_0^T (D(\widehat{v}), D(\phi))_{L^2(\tilde{\Omega}(t))} \\
&+ 2\mu \int_0^T (D(a), D(\phi))_{L^2(\tilde{\Omega}(t))} \\
&= -\int_0^T mh'\ell' + \int_0^T f(h)\ell + 2\mu \int_0^T (D(\widehat{v}), D(\phi))_{L^2(\tilde{\Omega}(t))} \\
&+ 2\mu \int_0^T (D(a), D(\phi))_{L^2(\tilde{\Omega}(t))} - m\,k_0\ell(0).
\end{aligned}
$$

Thus, given ψ in (4.3.19) and

$$\langle\widehat{g}, \phi\rangle = 2\mu(D(a), D(\phi))_{L^2(\tilde{\Omega}(t))} - \psi(a, a, \phi).$$

we obtain the weak formulation (4.3.20). □

These tools enable us to define weak solutions of (4.3.9)-(4.3.10). This definition is a compromise between the definition given in [31, Definition 1] and the one given in [63, Definition 3.1], although the extension in [63] is constructed so as to isolate the obstacle.

Definition 4.3.3. A couple (v, h) is called a weak solution of (4.3.9)-(4.3.10) with initial data (v_0, h_0, k_0) if, given $\widehat{v} = v - a$, where $a = a_h$ is the extension in (4.3.11) depending on some $\varepsilon_0 = \varepsilon_0(v_0, h_0, k_0, T) > 0$, it satisfies the following requirements:

$$
\begin{aligned}
& h \in W^{1,\infty}(0,T;\mathbb{R}) \cap \mathcal{C}([0,T];[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0]), \\
& (\widehat{v}, h') \in L^2(0,T;\mathbb{V}_{h(t)}) \cap L^\infty(0,T;\mathbb{H}_{h(t)}), \\
& (\widehat{v}, h) \text{ satisfies (4.3.20) for every } (\phi,\ell) \in \mathcal{C}^1([0,T];\mathbb{V}_{h(t)}) \\
& \text{such that } \phi(\cdot,T) = \ell(T) = 0.
\end{aligned}
\tag{4.3.21}
$$

Remark 4.3.4. The requirement $h \in \mathcal{C}([0,T];[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0])$ ensures that no collision occurs between the obstacle and the boundary of the channel as there exists a separation strip of size $\varepsilon_0 > 0$ for all $t \in [0,T]$. This makes the definition of weak solution consistent, since it also allows to build the solenoidal extension $a = a_h$ in (4.3.11) precisely by choosing such $\varepsilon_0 > 0$. As already mentioned, we will prove that this requirement is satisfied by making use of the strong force assumption satisfied by f, given in (4.2.1). On the other hand, it is worth mentioning that, one could prove the no-collisions result without adding the strong force assumption, at least in the case of an obstacle purely translating in the vertical direction, because the contact surfaces are of the class $\mathcal{C}^\infty$ (see [73, 128]). However, as soon as one allows the obstacle to rotate, this result does not hold anymore because the contact surfaces are merely Lipschitz continuous.

Finally, we provide an estimate on the norm of $\widehat{g}$, defined as in (4.3.13), on the auxiliary domain $\tilde{\mathcal{R}}$, which we will exploit later.

Lemma 4.3.5. *For any* $\varepsilon_0 > 0$, *define* $\widehat{g} = \widehat{g}_{\varepsilon_0} \in L^\infty(\tilde{\mathcal{R}}) \cap L^2(\tilde{\mathcal{R}})$ *as in* (4.3.13). *Then* $\widehat{g} \in L^2(\tilde{\mathcal{R}})$ *and the following estimate holds:*

$$\|\widehat{g}\|_{L^2(\tilde{\mathcal{R}})} \leq \mu\|\Delta a\|_{L^2(\tilde{\mathcal{R}}_0)} + \|a\|_{L^4(\tilde{\mathcal{R}}_0)}\|\nabla a\|_{L^2(\tilde{\mathcal{R}}_0)}.$$

Proof. Multiply $\widehat{g}$ by a divergence-free vector field $\varphi \in H_0^1(\tilde{\mathcal{R}})$ and integrate by parts over $\tilde{\mathcal{R}}$, so that

$$\int_{\tilde{\mathcal{R}}} \widehat{g} \cdot \varphi \, dx = -\mu \int_{\tilde{\mathcal{R}}} \nabla(a - \tilde{v}_P) \cdot \nabla\varphi \, dx - \int_{\tilde{\mathcal{R}}} (a \cdot \nabla)a \cdot \varphi \, dx. \tag{4.3.22}$$

We follow [66, Lemma 4.1] and we exploit the partition (4.3.17) together with the properties of a. In particular, since $a = \tilde{v}_P$ in $\mathcal{R}_1 \cup \mathcal{R}_2$, the first term in (4.3.22) corresponds to:

$$\int_{\tilde{\mathcal{R}}} \nabla(a - \tilde{v}_P) \cdot \nabla\varphi \, dx = \int_{\tilde{\mathcal{R}}_0} \nabla a : \nabla\varphi \, dx.$$

Since $(\tilde{v}_P \cdot \nabla)\tilde{v}_P \equiv 0$ in $\tilde{\mathcal{R}}$, we also have

$$\int_{\tilde{\mathcal{R}}_1} (a \cdot \nabla)a \cdot \varphi \, dx = 0\,,$$

and in a similar way,

$$\int_{\tilde{\mathcal{R}}_2} (a \cdot \nabla)a \cdot \varphi \, dx = 0\,.$$

Since $a \in H^2(\tilde{\mathcal{R}})$ by applying the previous results and Hölder's inequality, we obtain

$$\Big| \int_{\tilde{\mathcal{R}}} \nabla a : \nabla\varphi \, dx \Big| = \left| \int_{\tilde{\mathcal{R}}_0} \Delta a \cdot \varphi \, dx \right| \leq \|\Delta a\|_{L^2(\tilde{\mathcal{R}}_0)} \|\varphi\|_{L^2(\tilde{\mathcal{R}}_0)}\,, \tag{4.3.23}$$

and also

$$\left| \int_{\tilde{\mathcal{R}}} (a \cdot \nabla)a \cdot \varphi \, dx \right| = \left| \int_{\tilde{\mathcal{R}}_0} (a \cdot \nabla)a \cdot \varphi \, dx \right| \leq \|a\|_{L^4(\tilde{\mathcal{R}}_0)} \|\nabla a\|_{L^2(\tilde{\mathcal{R}}_0)} \|\varphi\|_{L^2(\tilde{\mathcal{R}}_0)}\,. \tag{4.3.24}$$

From (4.3.13)-(4.3.23)-(4.3.24) we thus obtain

$$\left| \int_{\tilde{\mathcal{R}}} \widehat{g} \cdot \varphi \, dx \right| \leq \Big(\mu\|\Delta a\|_{L^2(\tilde{\mathcal{R}}_0)} + \|a\|_{L^4(\tilde{\mathcal{R}}_0)} \|\nabla a\|_{L^2(\tilde{\mathcal{R}}_0)} \Big) \|\varphi\|_{L^2(\tilde{\mathcal{R}})}$$

for every divergence-free field $\varphi \in H_0^1(\tilde{\mathcal{R}})$, from which the thesis of the lemma follows. □

The following theorem states existence and uniqueness of weak solutions to problem (4.3.9)-(4.3.10).

Theorem 4.3.6. *Let $\tilde{\Omega}$ be as in* (4.3.7). *Let $f \in \mathcal{C}^1(-L+\delta, L-\delta;\mathbb{R})$ satisfy conditions* (4.2.1). *Let $h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}]$, for some $\widehat{\varepsilon} > 0$ small arbitrarily fixed, and $(v_0 - a_{h_0}(y,\widehat{\varepsilon}), k_0) \in \mathbb{H}_{h_0}$, where a_h is as in* (4.3.11). *Then, problem* (4.3.9)-(4.3.10) *admits a unique weak solution (v,h), defined as in Definition 4.3.3, for any $T>0$. Moreover, let*

$$F(h) = \int_0^h f(s)\,ds \tag{4.3.25}$$

and let $\varepsilon_0 = \varepsilon_0(v_0,h_0,k_0,T) > 0$ be such that $|h(t)| \leq L-\delta-\varepsilon_0$ for all $t \in [0,T]$. Then, given $\widehat{v} = v - a$, where $a = a_h(y,\varepsilon_0)$, the pair $(\widehat{v}, h')$ is almost everywhere equal to a function continuous from $[0,T]$ into $\mathbb{H}_{h(t)}$, and $(\widehat{v}, h)$ satisfies the following energy estimate:

$$\begin{aligned}
&\|\widehat{v}(t)\|^2_{L^2(\tilde{\Omega}(t))} + m\|h'(t)\|^2_{L^\infty(0,T;\mathbb{R})} + 2\,F(h(t)) + 4\mu\int_0^t \|D(\widehat{v}(s))\|^2_{L^2(\tilde{\Omega}(s))} ds \\
&\leq \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2F(h_0) + \frac{4}{\mu}\int_0^T \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) \\
&\quad \times \int_0^T \alpha(s)\,\exp\left[\frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right)s\right] ds,
\end{aligned} \tag{4.3.26}$$

where $\alpha(s)$ is defined as

$$\alpha(s) = \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2\,F(h_0) + \frac{4}{\mu}\int_0^s \|\widehat{g}(\tau)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)}\,d\tau. \tag{4.3.27}$$

Since the original problem (2.0.7) is equivalent to problem (4.3.9)-(4.3.10), the proof of Theorem 4.2.1 is completed if we prove Theorem 4.3.6. We emphasize that Theorem 4.3.6 has a stronger statement than Theorem 4.2.1, because it guarantees the continuity of the (unique) solution in a suitable sense. The proof of the existence part is developed in Section 4.4 and Section 4.5 and, as previously mentioned, it takes advantage of a penalty method. Uniqueness is proven in Section 4.6: there, we will consider two weak solutions of problem (4.3.9)-(4.3.10), (v_1,h_1) and (v_2,h_2), in the sense of Definition 4.3.3. Given the extensions of the Poiseuille flow $a_1 = a_{h_1}$ and $a_2 = a_{h_2}$ defined as in (4.3.11), one can put $\widehat{v}_1 = v_1 - a_1$ and $\widehat{v}_2 = v_2 - a_2$ and do the computations on $(\widehat{v}_1, h_1)$ and $(\widehat{v}_2, h_2)$. As we already pointed out in the introduction, we are not allowed to take the difference between the equation in weak form satisfied by the two solutions, (4.3.20), because $\widehat{v}_1$ and $\widehat{v}_2$ are not defined on the same domain: the definition of the functional spaces, (4.3.18), depends on h_1, h_2, and the weak formulation (4.3.20) is set on two different domains, $\tilde{\Omega}^1(t)$ and $\tilde{\Omega}^2(t)$. To solve such issue, we follow the procedure devised in [76]; here, the authors build a map ψ_t projecting $\tilde{\Omega}^2(t)$ on $\tilde{\Omega}^1(t)$, which allows to define a change of variables, so that they can introduce a solenoidal velocity vector field $\widehat{\mathfrak{v}}_2$, the pullback of $\widehat{v}_2$ by such map, on $\tilde{\Omega}^1(t)$. As a consequence, one can define

$$w := \widehat{v}_1 - \widehat{\mathfrak{v}}_2, \qquad \widehat{h} := h_1 - h_2. \tag{4.3.28}$$

Then, one proceeds standardly, obtaining the equation satisfied by $(w, \widehat{h})$, providing some proper estimates, and finally one can conclude by applying a Grönwall's inequality.

4.4 The penalized problem

After changing problem (2.0.7) into the equivalent problem (4.3.12)-(4.3.14), the second step of the penalty method exactly implies penalizing (4.3.20), which is the weak formulation of problem (4.3.12)-(4.3.14). We denote by E_h the complementary domain of $\mathcal{R}_h$ in $\tilde{\mathcal{R}}$, $E_h = \tilde{\mathcal{R}} \setminus \mathcal{R}_h$ and we introduce its characteristic function χ_{E_h}. We emphasize that the functions belonging to $\mathcal{W}(\tilde{\mathcal{R}})$ differ from those belonging to $\mathcal{W}_h$ precisely because their support might be also in E_h. The penalty method eliminates the difficulty induced by the time dependent domain by allowing to solve the problem in the fixed domain $\tilde{\mathcal{R}}$, so that classical methods can be applied, by introducing a penalization term which takes care of the remainder in E_h.

We extend $\widehat{v}_0$ by zero outside $\mathcal{R}_{h(0)}$, while $a(y)$ is naturally extended outside $\mathcal{R}_{h(t)}$ to the whole fixed domain $\tilde{\mathcal{R}}$ through its definition (4.3.11) for every $t \in [0, T]$ and we solve the following problem:

Let $n \geq 1$ be fixed. Find

$$h \in W^{1,\infty}(0, T; \mathbb{R}) \cap \mathcal{C}([0, T]; [-L + \delta + \varepsilon_0, L - \delta - \varepsilon_0])$$

$$(\widehat{v}, h') \in L^2(0, T; \mathbb{V}(\tilde{\mathcal{R}})) \cap L^\infty(0, T; \mathbb{H}(\tilde{\mathcal{R}})),$$

for some $\varepsilon_0 = \varepsilon_0(\widehat{v}_0, h_0, k_0, T) > 0$, satisfying

$$\begin{aligned} &- \int_0^T \{(\widehat{v}, \phi_t)_{L^2(\tilde{\mathcal{R}}\setminus B)} + m\, h'\ell' - f(h)\, \ell\} + 2\mu \int_0^T (D(\widehat{v}), D(\phi))_{L^2(\tilde{\mathcal{R}}\setminus B)} \\ &+ \int_0^T \{\psi(\widehat{v}, \widehat{v}, \phi) + \psi(\widehat{v}, a, \phi) + \psi(a, \widehat{v}, \phi) - \psi(h'\widehat{e}_2, a, \phi) - \psi(h'\widehat{e}_2, \widehat{v}, \phi)\} \\ &+ n \int_0^T (\chi_{E_h} \widehat{v}, \phi)_{L^2(\tilde{\mathcal{R}})} = \int_0^T \langle \widehat{g}, \phi \rangle + m\, k_0 \ell(0) + (\widehat{v}_0, \phi(0))_{L^2(\tilde{\mathcal{R}}\setminus B)}, \\ &\forall\, (\phi, \ell) \in \mathcal{C}^1([0, T], \mathbb{V}(\tilde{\mathcal{R}})) \text{ such that } \phi(\cdot, T) = \ell(T) = 0. \end{aligned} \tag{4.4.1}$$

We remark that the trilinear forms in (4.4.1) are defined as in (4.3.19), where the integral is now on $\tilde{\mathcal{R}} \setminus B$, but with a little abuse of notation we still use ψ as a label function. The existence of a solution to the penalized problem is proven in the following proposition:

Proposition 4.4.1. *Let $\tilde{\mathcal{R}}$ be a fixed domain defined by* (4.3.15)*, partitioned as in* (4.3.17)*. Let $f \in \mathcal{C}^1(-L+\delta, L-\delta; \mathbb{R})$ satisfy conditions* (4.2.1)*. Assume that $h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}]$ for some $\widehat{\varepsilon} > 0$ small arbitrarily fixed, and $(\widehat{v}_0, k_0) \in \mathbb{H}(\tilde{\mathcal{R}})$. Then, there exists at least one solution*

$(\widehat{v}, h)$ to problem (4.4.1) such that $|h(t)| \leq L - \delta - \varepsilon_0$ for some $\varepsilon_0 = \varepsilon_0(\widehat{v}_0, h_0, k_0, T) > 0$. This solution is global in time and, moreover, $(\widehat{v}, h')$ is almost everywhere equal to a function continuous from $[0, T]$ into $\mathbb{H}(\tilde{\mathcal{R}})$; furthermore, given $F(h)$ as in (4.3.25), it satisfies the energy estimate:

$$\begin{aligned}
&\|\widehat{v}(t)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m\|h'(t)\|^2_{L^\infty(0,T;\mathbb{R})} + 2F(h(t)) + 4\mu\int_0^t \|D(\widehat{v}(s))\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)}\,ds + 2\,n\int_0^t\int_{E_h}|\widehat{v}(s)|^2\,ds \\
&\leq \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2F(h_0) + \frac{4}{\mu}\int_0^T \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\cdot\max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) \\
&\quad \times \int_0^T \alpha(s)\,\exp\left[\frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\cdot\max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right)s\right]ds
\end{aligned} \tag{4.4.2}$$

with

$$\alpha(s) = \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2\,F(h_0) + \frac{4}{\mu}\int_0^s \|\widehat{g}(\tau)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)}\,d\tau.$$

Proof. Since the domain $\tilde{\mathcal{R}}$ is fixed, we can apply the Faedo-Galerkin procedure. The space $\mathbb{V}(\tilde{\mathcal{R}})$ is a separable Hilbert space, and $\mathcal{W}(\tilde{\mathcal{R}})$ is dense in $\mathbb{V}(\tilde{\mathcal{R}})$, hence we can choose a sequence given by a countable set of couples $\{(w_i, 1)\}_{i=1}^\infty$ belonging to $\mathcal{W}(\tilde{\mathcal{R}})$ to be a basis in $\mathbb{V}(\tilde{\mathcal{R}})$, orthonormal in $\mathbb{H}(\tilde{\mathcal{R}})$. For each $N \geq 1$ we construct an approximate solution

$$\begin{pmatrix}\widehat{v}_N \\ H_N\end{pmatrix} = \sum_{i=1}^N c_{iN}(t)\begin{pmatrix} w_i \\ 1\end{pmatrix},$$

where the coefficients c_{iN} are determined by the following first-order integro-ordinary differential system

$$\begin{aligned}
&\sum_{i=1}^N (w_i, w_j)_{L^2(\tilde{\mathcal{R}}\setminus B)} c'_{iN}(t) + m\sum_{i=1}^N c'_{iN}(t) + f(h_N(t)) \\
&+ 2\mu\sum_{i=1}^N (D(w_i), D(w_j))_{L^2(\tilde{\mathcal{R}}\setminus B)} c_{iN}(t) + \sum_{i,l=1}^k \psi(w_i, w_l, w_j)\, c_{iN}(t) c_{lN}(t) \\
&+ \sum_{i=1}^N \psi(w_i, a, w_j)\, c_{iN}(t) + \sum_{l=1}^N \psi(a, w_l, w_j)\, c_{lN}(t) - \sum_{i=1}^N \psi(\widehat{e}_2, a, w_j)\, c_{iN}(t) \\
&- \sum_{i,l=1}^k \psi(\widehat{e}_2, w_l, w_j)\, c_{iN}(t) c_{lN}(t) + \sum_{i=1}^N n(\chi_{E_{h_N}} w_i, w_j)_{L^2(\tilde{\mathcal{R}})} c_{iN}(t) \\
&= \langle \widehat{g}, w_j\rangle \qquad j = 1, ..., N
\end{aligned} \tag{4.4.3}$$

$$h_N(t) = h_0 + \sum_{i=1}^N \int_0^t c_{iN}(s)ds, \qquad c_{iN}(0) = \text{the } i^{th} \text{ component of } \widehat{v}_{0,N},$$

$$H_N(0) = k_{0,N}.$$

The initial conditions are given by the orthogonal projections in $\mathbb{H}(\tilde{\mathcal{R}})$ of $(\widehat{v}_0, k_0)$ onto the space spanned by $\{(w_i, 1)\}_{i=1}^N$, which we call $(\widehat{v}_{0,N}, k_{0,N})$. This system has a solution defined on some interval $[0, t_N]$, provided that there exists $\varepsilon_0 > 0$ such that

$$|h_N(t)| \leq L - \delta - \varepsilon_0 \qquad \forall\, t \in [0, t_N]. \tag{4.4.4}$$

In this way, we are allowed to build the function a as in (4.3.11), by choosing the same ε_0. We shall see that this condition is indeed always guaranteed for any $t \in [0, T]$ with $T > 0$, because of the presence of f in (4.3.10). We will prove this claim at the end of the proof of this proposition.

Our aim is finding an apriori estimate for the approximate solution $(\widehat{v}_N, H_N)$. To this end, we multiply (4.4.3) by $c_{jN}(t)$ and add the equations for $j = 1, ..., N$:

$$\begin{aligned}
&(\widehat{v}'_N, \widehat{v}_N)_{L^2(\tilde{\mathcal{R}}\setminus B)} + mH'_N H_N + f(h_N)H_N + 2\mu\, \|D(\widehat{v}_N)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} \\
&+ \psi(\widehat{v}_N, a, \widehat{v}_N) - \psi(H_N\widehat{e}_2, a, \widehat{v}_N) + n\|\widehat{v}_N\|^2_{L^2(E_{H_N})} = \langle \widehat{g}, \widehat{v}_N\rangle
\end{aligned}$$

which we rewrite as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|H_N|^2 + 2\,F(h_N)\right) + 2\mu\,\|D(\widehat{v}_N)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} \\
&+ n\|\widehat{v}_N\|^2_{L^2(E_{H_N})} = \langle \widehat{g}, \widehat{v}_N\rangle - \psi(\widehat{v}_N, a, \widehat{v}_N) + \psi(H_N\widehat{e}_2, a, \widehat{v}_N),
\end{aligned} \tag{4.4.5}$$

where F is defined in (4.3.25). Our aim is finding an a priori estimate for the approximate solution $(\widehat{v}_N, H_N)$. To this end, we start by estimating the right-hand side of (4.4.5). The first trilinear form may be bounded exploiting the Hölder inequality and the Young inequality

$$\begin{aligned}
\psi(\widehat{v}_N, a, \widehat{v}_N) &\leq \|\nabla a\|_{L^\infty(\tilde{\mathcal{R}}\setminus B)}\|\widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} \\
&\leq \frac{(4L-2\delta)}{\pi}\|\nabla a\|_{L^\infty(\tilde{\mathcal{R}}\setminus B)}\|\widehat{v}_N\|_{L^2(\tilde{\mathcal{R}}\setminus B)}\|\nabla\widehat{v}_N\|_{L^2(\tilde{\mathcal{R}}\setminus B)} \\
&\leq \frac{2}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}\|\widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + \frac{\mu}{8}\|\nabla\widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)},
\end{aligned}$$

where we used the fact that the Poincaré constant in the domain $\tilde{\mathcal{R}}$ is $\pi^2/(4L-2\delta)^2$. For what concerns the second trilinear form, we exploit again the definition of a, as we did when proving Lemma 4.3.5. Since $a(y)$ is equal to 0 on the obstacle B, we write

$$\begin{aligned}
\psi(H_N\widehat{e}_2, a, \widehat{v}_N) &= \int_{\tilde{\mathcal{R}}\setminus B}(H_N\widehat{e}_2\cdot\nabla)a\cdot\widehat{v}_N = \int_{\tilde{\mathcal{R}}}(H_N\widehat{e}_2\cdot\nabla)a\cdot\widehat{v}_N \\
&= \int_{-2I}^{+2I}\left(\int_{-2L+\delta}^{2L-\delta}(H_N\widehat{e}_2\cdot\nabla)a\cdot\widehat{v}_N\,dy_2\right)dy_1.
\end{aligned}$$

Then

$$\begin{aligned}\psi(H_N\widehat{e}_2, a, \widehat{v}_N) \leq \Bigg| \int_{-2I}^{-3} \left(\int_{-2L+\delta}^{2L-\delta} (H_N\widehat{e}_2 \cdot \nabla)\tilde{v}_P \cdot \widehat{v}_N \, dy_2 \right) dy_1 \\ + \int_{-3}^{+3} \left(\int_{-2L+\delta}^{2L-\delta} (H_N\widehat{e}_2 \cdot \nabla) a \cdot \widehat{v}_N \, dy_2 \right) dy_1 \\ + \int_{+3}^{2I} \left(\int_{-2L+\delta}^{2L-\delta} (H_N\widehat{e}_2 \cdot \nabla)\tilde{v}_P \cdot \widehat{v}_N \, dy_2 \right) dy_1 \Bigg|.\end{aligned} \tag{4.4.6}$$

Consider the partition (4.3.17). The first term is treated as follows, by using (4.3.16):

$$\begin{aligned}&\left| \int_{-2I}^{-3} \left(\int_{-2L+\delta}^{2L-\delta} (H_N\widehat{e}_2 \cdot \nabla)\tilde{v}_P \cdot \widehat{v}_N \, dy_2 \right) dy_1 \right| \\ &\leq |H_N| \int_{-2I}^{-3} \left(\|\nabla \tilde{v}_P\|_{L^2(-2L+\delta, 2L-\delta)} \|\widehat{v}_N\|_{L^2(-2L+\delta, 2L-\delta)} \right) dy_1 \\ &\leq |H_N| \lambda \tilde{\xi} \int_{-2I}^{-3} \|\widehat{v}_N\|_{L^2(-2L+\delta, 2L-\delta)} \, dy_1 \\ &\leq |H_N| \lambda \tilde{\xi} \, \|\widehat{v}_N\|_{L^2(\tilde{\mathcal{R}}_1)} \leq \frac{4L-2\delta}{\pi} |H_N| \lambda \tilde{\xi} \, \|\nabla \widehat{v}_N\|_{L^2(\tilde{\mathcal{R}}_1)}.\end{aligned}$$

The third integral in (4.4.6) can be treated analogously, so that we obtain:

$$\left| \int_{+3}^{2I} \left(\int_{-2L+\delta}^{2L-\delta} (H_N\widehat{e}_2 \cdot \nabla)\tilde{v}_P \cdot \widehat{v}_N \, dy_2 \right) dy_1 \right| \leq \frac{4L-2\delta}{\pi} |H_N| \lambda \tilde{\xi} \, \|\nabla \widehat{v}_N\|_{L^2(\tilde{\mathcal{R}}_2)}.$$

For what concerns the term in the region $\tilde{\mathcal{R}}_0 = [-3, 3] \times [-2L+\delta, 2L-\delta]$:

$$\int_{-3}^{+3} \left(\int_{-2L+\delta}^{2L-\delta} (H_N\widehat{e}_2 \cdot \nabla) a \cdot \widehat{v}_N dy_2 \right) dy_1 \leq \frac{4L-2\delta}{\pi} |H_N| \|\nabla a\|_{L^2(\tilde{\mathcal{R}}_0)} \|\nabla \widehat{v}_N\|_{L^2(\tilde{\mathcal{R}} \setminus B)}.$$

Thus, we finally obtain that:

$$\begin{aligned}\psi(H_N\widehat{e}_2, a, \widehat{v}_N) &\leq \frac{4L-2\delta}{\pi} |H_N| \|\nabla a\|_{L^2(\tilde{\mathcal{R}}_0)} \|\nabla \widehat{v}_N\|_{L^2(\tilde{\mathcal{R}} \setminus B)} \\ &\quad + \frac{4L-2\delta}{\pi} |H_N| \lambda \tilde{\xi} \, \|\nabla \widehat{v}_N\|_{L^2(\tilde{\mathcal{R}} \setminus B)} \\ &\leq \frac{2}{\mu} \frac{(4L-2\delta)^2}{\pi^2} |H_N|^2 (\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2 \tilde{\xi}^2) \\ &\quad + \frac{\mu}{4} \|\nabla \widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}} \setminus B)}\end{aligned}$$

where we used the Young inequality and the fact that

$$\|\nabla \widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}_1)} + \|\nabla \widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}_2)} \leq \|\nabla \widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}} \setminus B)}.$$

Then, we apply again the Young inequality, after the Schwarz inequality, to provide a bound for the first term on the right-hand side in (4.4.5):

$$|\langle \widehat{g}, \widehat{v}_N\rangle| \leq \|\widehat{g}\|_{V'(\tilde{\mathcal{R}}\setminus B)}\|\nabla \widehat{v}_N\|_{L^2(\tilde{\mathcal{R}}\setminus B)} \leq \frac{2}{\mu}\|\widehat{g}\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} + \frac{\mu}{8}\|\nabla \widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)},$$

since Lemma 4.3.5 guarantees that $\widehat{g} \in V'(\tilde{\mathcal{R}} \setminus B)$. Thus, by reordering equation (4.4.5), once we have plugged these estimates and used the following fact

$$\int_{\tilde{\mathcal{R}}\setminus B} |\nabla \widehat{v}_N|^2\, dy \leq \int_{\tilde{\mathcal{R}}} |\nabla \widehat{v}_N|^2\, dy = 2\int_{\tilde{\mathcal{R}}} |D(\widehat{v}_N)|^2\, dy = 2\int_{\tilde{\mathcal{R}}\setminus B} |D(\widehat{v}_N)|^2\, dy,$$

since $\widehat{v}_N$ is a divergence free vector field vanishing on $\partial\tilde{\mathcal{R}}$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\|\widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|H_N|^2 + 2F(h_N)\right) + \mu\|\nabla \widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + 2n\|\widehat{v}_N\|^2_{L^2(E_{H_N})} \\
&\leq \frac{4}{\mu}\|\widehat{g}\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}\|\widehat{v}_N\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}|H_N|^2(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2).
\end{aligned} \tag{4.4.7}$$

From (4.4.7), by integrating between 0 and t, we deduce that

$$\begin{aligned}
&\|\widehat{v}_N(t)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|H_N(t)|^2 + 2F(h_N(t)) + 2\mu\int_0^t \|\nabla \widehat{v}_N(s)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)}\, ds \\
&+ 2n\int_0^t\int_{E_{H_N}} |\widehat{v}_N(s)|^2\, ds \\
&\leq \|\widehat{v}_{0,N}\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_{0,N}|^2 + 2F(h_0) \\
&\quad + \frac{4}{\mu}\int_0^t \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}\int_0^t \|\widehat{v}_N(s)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\int_0^t |H_N(s)|^2 ds \\
&\leq \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2\,F(h_0) \\
&\quad + \frac{4}{\mu}\int_0^t \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}\int_0^t \|\widehat{v}_N(s)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{m\pi^2}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\int_0^t m|H_N(s)|^2 ds \\
&\leq \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2F(h_0) + \frac{4}{\mu}\int_0^t \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2}\cdot\max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) \\
&\quad \times \int_0^t \left(\|\widehat{v}_N(s)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|H_N(s)|^2\right) ds.
\end{aligned}$$

Since $F(h_N) \geq 0$ by (2.0.14), invoking Grönwall's lemma, we obtain for any instant $t \in [0,T]$:

$$\begin{aligned}
&\|\widehat{v}_N(t)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|H_N(t)|^2 \\
&\leq \alpha(t) \exp\left[\frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2} \cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) t\right],
\end{aligned}$$

with

$$\alpha(t) = \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2\,F(h_0) + \frac{4}{\mu}\int_0^t \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)}\, ds.$$

Thus, we finally get

$$\begin{aligned}
&\|\widehat{v}_N(t)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|H_N(t)|^2 + 2\,F(h_N(t)) + 2\,\mu \int_0^t \|\nabla\widehat{v}_N(s)\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)}\, ds \\
&+2\,n\int_0^t\int_{E_{H_N}} |\widehat{v}_N(s)|^2\, ds \\
&\leq \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2F(h_0) + \frac{4}{\mu}\int_0^t \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2} \cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) \\
&\quad \times \int_0^t \alpha(s)\, \exp\left[\frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2} \cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) s\right] ds \\
&\leq \|\widehat{v}_0\|^2_{L^2(\tilde{\mathcal{R}}\setminus B)} + m|k_0|^2 + 2F(h_0) + \frac{4}{\mu}\int_0^T \|\widehat{g}(s)\|^2_{V'(\tilde{\mathcal{R}}\setminus B)} ds \\
&\quad + \frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2} \cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) \\
&\quad \times \int_0^T \alpha(s)\, \exp\left[\frac{4}{\mu}\frac{(4L-2\delta)^2}{\pi^2} \cdot \max\left(\|\nabla a\|^2_{L^\infty(\tilde{\mathcal{R}}\setminus B)}, \frac{1}{m}(\|\nabla a\|^2_{L^2(\tilde{\mathcal{R}}_0)} + \lambda^2\tilde{\xi}^2)\right) s\right] ds
\end{aligned} \tag{4.4.8}$$

for all $t \in [0,T]$, where the right hand-side is bounded if $T > 0$. In particular, the solution exists globally in time provided that condition (4.4.4) holds, which we still need to prove; such condition in the original inertial reference system translates the condition of absence of collisions occurring between the rectangular obstacle and the boundary of the channel.

Estimate (4.4.8) implies the existence of a couple

$$(\widehat{v}, h') \in L^\infty(0,T;\mathbb{H}(\tilde{\mathcal{R}})) \cap L^2(0,T;\mathbb{V}(\tilde{\mathcal{R}}))$$

and of a subsequence, which we denote by $(\widehat{v}_N, h'_N)$, such that, as $N \to \infty$:

$$\begin{aligned}
&(\widehat{v}_N, h'_N) \rightharpoonup (\widehat{v}, h') \qquad \text{in}\, L^2(0,T;\mathbb{V}(\tilde{\mathcal{R}})), \\
&(\widehat{v}_N, h'_N) \overset{*}{\rightharpoonup} (\widehat{v}, h') \qquad \text{in}\, L^\infty(0,T;\mathbb{H}(\tilde{\mathcal{R}})), \\
&h_N \to h \qquad \text{in}\, \mathcal{C}([0,T];\mathbb{R}),
\end{aligned} \tag{4.4.9}$$

the latter holding because of the compact embedding of $W^{1,\infty}(0,T;\mathbb{R})$ into $\mathcal{C}([0,T];\mathbb{R})$. Moreover, through classical methods (see [31, Section 3] and [134, Chapter 3, Section 3]), in view of (4.4.8) together with the fact that $f(h_N)$ can be thought to be a bounded function as long as the rigid obstacle does not touch the boundary of the channel, one can prove a further convergence property (up to the extraction of a subsequence):

$$(\widehat{v}_N, h'_N) \to (\widehat{v}, h') \qquad \text{in } L^2(0,T;\mathbb{H}(\mathcal{R}))\,. \tag{4.4.10}$$

By [31, Lemma 1] we also have that

$$\chi_{E_{h_N}\cap\mathcal{O}} \to \chi_{E_h\cap\mathcal{O}} \qquad \text{in } L^p(0,T;L^p(\mathcal{R}))\,. \tag{4.4.11}$$

The convergence results in (4.4.9) together with (4.4.10)-(4.4.11) enable us to pass to the limit in the system satisfied by $(\widehat{v}_N, h'_N)$, which is

$$\begin{aligned}
&(\widehat{v}'_N, w_j)_{L^2(\tilde{\mathcal{R}}\setminus B)} + mh'_N + f(h'_N) + 2\mu\,(D(\widehat{v}_N), D(w_j))_{L^2(\tilde{\mathcal{R}}\setminus B)} \\
&+ \psi(\widehat{v}_N, \widehat{v}_N, w_j) \,+ \psi(\widehat{v}_N, a, w_j) + \psi(a, \widehat{v}_N, w_j) \\
&- \psi(h'_N\widehat{e}_2, a, w_j) - \psi(h'_N\widehat{e}_2, \widehat{v}_N, w_j) + n(\chi_{E_{h_N}}\widehat{v}_N, w_j)_{L^2(\tilde{\mathcal{R}})} = \langle \widehat{g}, w_j\rangle \\
&\qquad\qquad \widehat{v}_N(0) = \widehat{v}_{0,N}, \qquad h'_N(0) = k_{0,N},
\end{aligned}$$

with $j = 1, ..., N$ and to obtain that $(\widehat{v}, h)$ in (4.4.9) satisfies (4.4.1), as well as $\widehat{v}(0) = \widehat{v}_0$ in the distributional sense, by exploiting classical arguments (see for instance [134, Chapter 3, Section 3]). To prove that $(\widehat{v}, h')$ is almost everywhere equal to a function continuous from $[0,T]$ into $\mathbb{H}(\tilde{\mathcal{R}})$, one can easily proceed as in [134, Chapter 3, Theorem 3.1] to obtain that $(\widehat{v}', h'') \in L^2(0,T;\mathbb{V}'(\tilde{\mathcal{R}}))$ and then apply [134, Chapter 3, Lemma 1.2], by exploiting the fact that $\{\mathbb{V}(\tilde{\mathcal{R}}), \mathbb{H}(\tilde{\mathcal{R}}), \mathbb{V}'(\tilde{\mathcal{R}})\}$ is an Hilbert triplet. Moreover, the function $\widehat{v}$ obeys the energy inequality (4.4.2), which is a natural consequence of the convergence results that we have just proved.

The proof of Proposition 4.4.1 is complete once we prove the following lemma, which, combined with the last convergence result in (4.4.9), allows to conclude on the global-in-time character of the solution.

Lemma 4.4.2. *For all $T > 0$, there exists $\varepsilon_0 = \varepsilon_0(\widehat{v}_0, h_0, k_0, T) > 0$ such that*

$$|h_N(t)| \le L - \delta - \varepsilon_0 \qquad \forall\, t \in [0,T].$$

Proof. By contradiction, let us suppose that there exists $T > 0$ such that $h_N(\bar{t}) = L - \delta$ for some $\bar{t} \in [0,T]$. The same procedure can be applied if $h_N = -L + \delta$. Since $h_N \in \mathcal{C}([0,T];\mathbb{R})$, for any $\eta > 0$ small there exist $\varepsilon_1 > \varepsilon_2 > 0$ such that

$$h_N(t) \in [L - \delta - 2\eta, L - \delta - \eta], \qquad \forall\, t \in (\bar{t} - \varepsilon_1, \bar{t} - \varepsilon_2).$$

Taking into account the conditions satisfied by f reported in (4.2.1), one can take η small enough so that there exists a constant $\bar{C} > 0$ such that

$$\begin{aligned}
F(h_N) &> \int_{L-\delta-2\eta}^{L-\delta-\eta} f(s)ds > \int_{L-\delta-2\eta}^{L-\delta-\eta} \bar{C}\exp\frac{1}{(L-\delta-s)^{4+r}}\,ds \\
&= \int_{\eta}^{2\eta} \bar{C}\exp\frac{1}{\tau^{4+r}}\,d\tau \ge \bar{C}\eta\,\exp\frac{1}{(2\eta)^{4+r}}.
\end{aligned} \tag{4.4.12}$$

The solenoidal extension a in (4.3.11) is now built taking $\varepsilon_0 = \eta$, thus

$$a(y) = a_{h_N}(y;\eta).$$

From the energy estimate (4.4.8) (which is uniform with respect to N), (4.4.12) and the estimates (4.3.3), we obtain that there exist a positive constant C_0, depending on the initial data $h_0, k_0, \widehat{v}_0$, and two positive constants C_1, C_2 depending on the geometrical and physical parameters, i.e. $m, \lambda, L, \delta, \mu$, such that

$$2\,\bar{C}\eta\,\exp\frac{1}{(2\eta)^{4+r}} < C_0 + \frac{C_1}{\eta^4}T + \frac{C_2}{\eta^4}\int_0^T (C_0 + \frac{C_1}{\eta^4}s)\,\exp(\frac{C_2}{\eta^4}s)\,ds. \tag{4.4.13}$$

Integrating by parts the right-hand side of (4.4.13), we obtain that

$$2\,\bar{C}\eta\,\exp\frac{1}{(2\eta)^{4+r}} < \frac{C_1}{C_2} + C_0\exp\frac{C_2T}{\eta^4} - \frac{C_1}{C_2}\exp\frac{C_2T}{\eta^4} + \frac{C_1}{\eta^4} + \frac{C_1}{\eta^4}T\exp\frac{C_2T}{\eta^4}.$$

This immediately yields a contradiction, since $r > 0$. From the above argument, for all $T > 0$ it must be that $|h_N(t)| < L - \delta$ when $t \in [0,T]$. Since $h_N \in \mathcal{C}([0,T];\mathbb{R})$ and $[0,T]$ is compact, the thesis of the lemma follows. □

□

4.5 Existence

We already mentioned how proving Theorem 4.2.1 is equivalent to proving Theorem 4.3.6. The idea of the proof of the first part of Theorem 4.3.6 is exploiting the result of existence for the penalized problem, given in Proposition 4.4.1. Indeed, the next step of the penalty method implies passing to the limit in (4.4.1) with respect to n. Again, we will follow the procedure by [31], highlighting the differences when it is necessary. Let us label the weak solution to (4.4.1) making the dependence on n explicit as $(\widehat{v}_n, h_n)$; of course E_h also depends on n, thus it may be relabelled as E_{h_n} This solution satisfies the energy estimate (4.4.2), where we should also make explicit the dependence on n. Let us state and prove two consequences of this fact.

The first consequence is natural: there exists a subsequence, which we denote again by $(\widehat{v}_n, h'_n)$ such that

$$\begin{aligned} &(\widehat{v}_n, h'_n) \rightharpoonup (\widehat{v}, h') \qquad \text{in } L^2(0,T;\mathbb{V}(\tilde{\mathcal{R}})),\\ &(\widehat{v}_n, h'_n) \overset{*}{\rightharpoonup} (\widehat{v}, h') \qquad \text{in } L^\infty(0,T;\mathbb{H}(\tilde{\mathcal{R}})),\\ &h_n \to h \qquad \text{in } \mathcal{C}([0,T];\mathbb{R}), \end{aligned} \tag{4.5.1}$$

where the latter is due to the compact embedding of $W^{1,\infty}(0,T;\mathbb{R})$ onto $\mathcal{C}([0,T];\mathbb{R})$.

The second consequence is proven in the following lemma.

Lemma 4.5.1. *The sequence $\widehat{v}_n$ satisfies*

$$\lim_{n\to\infty}\int_0^T\int_{E_h}|\widehat{v}_n|^2\,dy\,ds = 0. \tag{4.5.2}$$

Proof. From (4.4.2), we obtain that

$$\lim_{n\to\infty}\int_0^T\int_{E_{h_n}}|\widehat{v}_n|^2\,dy\,ds=0. \tag{4.5.3}$$

Following [31], we write

$$\begin{aligned}\int_0^T\int_{E_h}|\widehat{v}_n|^2\,dy\,ds\le\int_0^T&\int_{E_{h_n}}|\widehat{v}_n|^2\,dy\,ds\\&+\int_0^T\int_{\{E_h\setminus E_{h_n}\}}|\widehat{v}_n|^2\,dy\,ds.\end{aligned} \tag{4.5.4}$$

Then, we use the Hölder inequality:

$$\begin{aligned}\int_0^T\int_{\{E_h\setminus E_{h_n}\}}|\widehat{v}_n|^2\,dy\,ds&=\int_0^T\int_{\tilde{\mathcal{R}}}\chi_{\{E_h\setminus E_{h_n}\}}|\widehat{v}_n|^2\,dy\,ds\\&\le\int_0^T\|\chi_{\{E_h\setminus E_{h_n}\}}\|_{L^2(\tilde{\mathcal{R}})}\|\widehat{v}_n\|^2_{L^4(\tilde{\mathcal{R}})}\to 0\\&\text{as } n\to\infty,\end{aligned} \tag{4.5.5}$$

where we use the result in [31, Lemma 1] to infer that $\chi_{\{E_h\setminus E_{h_n}\}}\to 0$ in $L^p(\tilde{\mathcal{R}}\times[0,T])$ strongly $\forall\, p\in[1,\infty)$ and that $\widehat{v}_n\in L^4(\tilde{\mathcal{R}}\times[0,T])$ as it is proven in [134, Lemma 3.3]. If we combine (4.5.3), (4.5.5), (4.5.4), we obtain the sought result (4.5.2). □

Now, we introduce, for any $\eta>0$

$$Q_{h(t)}=\{(y,t)\in\tilde{\mathcal{R}}\times[0,T]\,|\,y\in\mathcal{R}_{h(t)}\}.$$

In order to develop the proof of Theorem 4.3.6, we need to prove an auxiliary result, i.e. that $\widehat{v}_n$ is relatively compact in $L^2(Q)$, which implies the existence of a subsequence, still labelled as $\widehat{v}_n$, satisfying the following strong convergence result

$$\widehat{v}_n\to\widehat{v}\qquad\text{in } L^2(Q). \tag{4.5.6}$$

For the moment, let (4.5.6) be true. We postpone the proof of this result at the end.

We can now proceed to the proof of Theorem 4.3.6. The objective is proving that the limit $(\widehat{v},h)$ in (4.5.1) is a solution to the original problem, thus it satisfies (4.3.21).

We start by proving that $(4.3.21)_3$ is satisfied. In other words, we pass to the limit in (4.4.1). Take $(\phi,\ell)\in\mathcal{C}^1([0,T];\mathcal{W}_{h(t)})$. Since there holds the convergence result in $(4.5.1)_3$, there exists $n_0\in\mathbb{N}$ such that

$$(\phi,\ell)\in\mathcal{C}^1([0,T];\mathcal{W}_{h_n(t)})\qquad\forall n\ge n_0,$$

and the penalty term in (4.4.1) vanishes for all $n\ge n_0$, by applying Lemma 4.5.1. The convergence results (4.5.1), (4.5.6) together with the density of $\mathcal{W}_{h(t)}$ in $\mathbb{V}_{h(t)}$ prove that $(\widehat{v},h)$ also satisfies (4.3.20) for $T>0$.

The results of convergence (4.5.1) prove that $(\widehat{v}, h) \in L^2(0,T;\mathbb{V}(\tilde{\mathcal{R}})) \cap L^\infty(0,T;\mathbb{H}(\tilde{\mathcal{R}}))$. By contradiction, let us suppose that supp$(\widehat{v})$ invades E_h. Then, we can find $K \subseteq \tilde{\mathcal{R}}$ such that supp$(\widehat{v}) \subset K$. However, from (4.5.6) and Lemma 4.5.1, we get a contradiction; thus in particular we obtain that $(\widehat{v}, h)$ satisfies $(4.3.21)_1$- $(4.3.21)_2$. Moreover, with analogous considerations as those that we did when proving Proposition 4.4.1, we obtain the continuity property of $(\widehat{v}, h)$ expressed in the statement of Theorem 4.3.6. Finally, the energy estimate (4.3.26) simply follows from taking the limit as $n \to \infty$ in (4.4.2); that explains the expression for $\alpha(s)$ in (4.3.27). As a consequence of (4.4.2), we also obtain the global character in time of the solution and the existence of some $\varepsilon_0 > 0$ such that $|h(t)| \leq L - \delta - \varepsilon_0$ for all $t \in [0, T]$, by proceeding precisely as in Lemma 4.4.2.

To conclude the proof, we show how to prove (4.5.6). The procedure implemented in [31] in order to prove (4.5.6) implies exploiting an Aubin-Lions type lemma: more precisely, one wants to apply [56, Lemma 4.6]. However, first we need to build the structure to apply such lemma. Thus, we introduce an open bounded set $D \subset \mathbb{R}^2$ with Lipschitz boundary such that $\bar{B} \subset D$, and the function spaces

$$M(D) = \{(v, l) \in L^2(\bar{D}) \times \mathbb{R} \,|\, \mathrm{div}\, v = 0, v|_B = l\, \widehat{e}_2\},$$
$$Z(D) = \{(v, l) \in H(D) \times \mathbb{R} \,|\, v|_B = l\, \widehat{e}_2\} \subset M(D),$$

where $H(D)$ can be characterized as follows since D is a Lipschitz open bounded set (see [134, Theorem 1.4])

$$H(D) = \{v \in L^2(D) \,|\, \mathrm{div}\, v = 0\,,\, \gamma_\nu u = 0 \,\mathrm{on}\, \partial D\}.$$

We associate to both $Z(D)$ and $M(D)$ the following scalar product:

$$((v_1, v_2)) = \int_{D\setminus B} v_1 \cdot v_2 \, dy + l_1 l_2,$$

which makes both of them Hilbert spaces. Then, we define the projector $P(D)$,

$$P(D) : M(D) \to Z(D).$$

We will now prove two technical lemmas, starting by a useful property of such projector.

Lemma 4.5.2. *There exists a positive constant C such that*

$$\|v - P(D)v\|_{L^2(D)} \leq C\|v\|_{L^2(\partial D)} \qquad \forall\, v \in M(D) \cap H^1(D).$$

Proof. We follow step by step the proof of [31, Lemma 4]. We emphasize that even if D is merely a Lipschitz domain this does not compromise the validity of the proof. □

Then we prove the second technical lemma needed to guarantee (4.5.6).

Lemma 4.5.3. *Let D be defined as above. Then we choose $0 < \alpha < \beta < T$, so that $D \times (\alpha, \beta) \subseteq Q$; let U_n be the restriction of $\widehat{v}_n$ to $D \times (\alpha, \beta)$. Then $P(D)U_n$ is strongly convergent in $L^2(D \times (\alpha, \beta))$.*

Proof. We introduce an auxiliary function space:

$$F(D) = \{(v, l) \in H^1_0(D) \times \mathbb{R} \,|\, \operatorname{div} v = 0, v|_B = l\,\widehat{e}_2\}.$$

Any element in $F(D)$ can be extended by 0 in $\tilde{\mathcal{R}} \setminus B$, so we can consider $F(D) \subseteq \mathbb{V}(\tilde{\mathcal{R}})$. We pick as a test function ϕ in (4.4.1) $\phi = a(t)\varphi$, with $\varphi \in F(D)$ and $a(t) \in \mathcal{D}(\alpha, \beta)$ to obtain

$$\begin{aligned} &\frac{d}{dt}\{(U_n, \varphi)_{L^2(D\setminus B)} + m\, h_n'\, l\} + f(h_n)\, l = {}_{F(D)'}\langle g_n, \varphi\rangle_{F(D)} \qquad \forall\, \varphi \in F(D) \\ &\frac{d}{dt}((U_n, \varphi)) = {}_{F(D)'}\langle g_n, \varphi\rangle_{F(D)} - f(h_n)\, l \qquad \forall\, \varphi \in F(D) \end{aligned} \tag{4.5.7}$$

where

$$\begin{aligned} {}_{F(D)'}\langle g_n, \varphi\rangle_{\,F(D)} = &- 2\mu(D(U_n), D(\varphi))_{L^2(D)} - \psi(U_n, U_n, \varphi) - \psi(U_n, a, \varphi) - \psi(a, U_n, \varphi) \\ &+ \psi(h_n'\widehat{e}_2, a, \varphi) + \psi(h_n'\widehat{e}_2, U_n, \varphi) - \langle \widehat{g}, \varphi\rangle, \end{aligned}$$

provided that n is large enough. Since $F(D) \subseteq Z(D)$, we may rewrite (4.5.7) as

$$\frac{d}{dt}((P(D)U_n, \varphi)) = {}_{F(D)'}\langle g_n, \varphi\rangle_{F(D)} - f(h_n)\, l \qquad \forall\, \varphi \in F(D). \tag{4.5.8}$$

From (4.5.8), we have that

$$|\frac{d}{dt}((P(D)U_n, \varphi))| = |{}_{F(D)'}\langle g_n, \varphi\rangle_{F(D)}| + |f(h_n)\, l| \qquad \forall\, \varphi \in F(D). \tag{4.5.9}$$

Then, we can prove that the right-hand side of (4.5.9) can be bounded. Indeed, by exploiting classical estimates (see [134, Chapter 3, Section 3.3]), Lemma 4.3.5, the energy estimate (4.3.26), we obtain that

$$|{}_{F(D)'}\langle g_n, \varphi\rangle_{F(D)}| \leq M_1 \|\nabla\varphi\|_{L^2(D)}, \quad \forall\, n \geq 1, \tag{4.5.10}$$

where M_1 is a constant depending on T, μ, λ, the geometry of the problem and the initial conditions. Also

$$|f(h_n)\, l| \leq M_2 |l|, \quad \forall\, n \geq 1, \tag{4.5.11}$$

because $f(h_n)$ can be thought to be a bounded function as long as the rigid obstacle does not touch the boundary of the channel, which has been proven. Bounds (4.5.10) and (4.5.11) imply that

$$\|\frac{d}{dt}P(D)U_n\|_{L^2(0,T;[F(D)]')} \leq M_1 + M_2, \quad \forall\, n \geq 1.$$

This inequality, together with the fact that U_n is a bounded set in $L^2(0, T; Z(D) \cap H^1(D))$ because of the energy estimate (4.4.2), and the compact inclusions $Z(D) \cap H^1(D) \subset Z(D) \subset [F(D)]'$, allow to apply [56, Lemma 4.6] so as to obtain that $P(D)U_n$ forms a compact set in $L^2(D \times (\alpha, \beta))$. □

Finally, one can give the following lemma:

Lemma 4.5.4. *The sequence $\widehat{v}_n$ is relatively compact in $L^2(Q)$, thus there holds* (4.5.6).

Proof. In order to prove this lemma, one can follow precisely the procedure given in [31, Theorem 3], once we declare the following notations: given $s \in \mathbb{N}$, define $\alpha_0, \alpha_1, ..., \alpha_s$, real numbers, and the sets $\Omega_1, \Omega_2, ..., \Omega_s$ such that

$$0 = \alpha_0 < \alpha_1 < ... < \alpha_s = T$$

$$\bar{\Omega}_i \times]\alpha_{i-1}, \alpha_i[\subset Q, \qquad Q \setminus (\Omega_i \times]\alpha_{i-1}, \alpha_i[) \subset Q_{i,\eta}.$$

Then we denote

$$I_\eta = \bigcup_{i=1}^{s} \Omega_i \times]\alpha_{i-1}, \alpha_i[, \qquad Q_\eta = Q \setminus I_\eta.$$

With the help of such definitions, we follow step by step the proof of [31, Theorem 3], which is divided in two parts. The first part uses Lemma 4.5.3, while the second part aims at exploiting the classical compactness result by Kolmogorov, [96, Chapter 3, Section 11.3, Theorem 3]. □

4.6 Uniqueness

We begin by stating the following regularity property on a solution given by Theorem 4.3.6.

Lemma 4.6.1. *Let* (v, h) *be a weak solution to problem* (4.3.9)-(4.3.10) *in the sense of Definition 4.3.3. Then, given* $\widehat{v} = v - a$*, where* $a = a_h$ *is the extension defined as in* (4.3.11)*, there holds*

$$\begin{gathered} t\,\widehat{v} \in L^{4/3}(0,T; W^{2,4/3}(\tilde{\Omega}(t))), \qquad t\,\partial_t\widehat{v} \in L^{4/3}(0,T; L^{4/3}(\tilde{\Omega}(t)))\,, \\ t\,\nabla\mathfrak{p} \in L^{4/3}(0,T; L^{4/3}(\tilde{\Omega}(t)))\,. \end{gathered} \tag{4.6.1}$$

Moreover, one can estimate the trilinear form as follows, for any $w \in H^1_0(\tilde{\Omega}(t))$*:*

$$|\,\psi(w,\widehat{v},w)| \leq 2^{1/2}\|w\|_{L^2(\tilde{\Omega}(t))}\|\nabla w\|_{L^2(\tilde{\Omega}(t))}\|\nabla\widehat{v}\|_{L^2(\tilde{\Omega}(t))}. \tag{4.6.2}$$

Proof. The proof of (4.6.1) follows the proof of [76, Proposition 3], up to some slight modification. We report here the main steps. We start by deducing, from a classical interpolation argument [134, Chapter 3, Lemma 3.3] and suitable Sobolev embeddings, that

$$\widehat{v} \in L^{4/3}(0,T; L^{4/3}(\tilde{\Omega}(t)))\,, \qquad (\widehat{v}\cdot\nabla)\widehat{v} \in L^{4/3}(0,T; L^{4/3}(\tilde{\Omega}(t)))\,.$$

The same interpolation argument together with the Hölder inequality can be used to deduce (4.6.2). For any $t \in [0,T]$ and any $y \in \tilde{\Omega}(t)$, let us undo the change of variables in (4.3.6) and introduce

$$\widehat{u}(x,t) = \widehat{v}(x - h(t)\,\widehat{e}_2, t)\,, \quad \widehat{p} = \mathfrak{p}(x - h(t)\,\widehat{e}_2, t) \quad \forall\,(x,t) \in \Omega_{h(t)} \times [0,T]\,. \tag{4.6.3}$$

Obviously, $\widehat{u}$ inherits the properties satisfied by $\widehat{v}$. In particular, it satisfies

$$\begin{aligned} &-\int_0^T \{(\widehat{u},\phi_t)_{L^2(\tilde{\Omega}(t))} + m\,h'\ell' - f(h)\,\ell\} + 2\mu\int_0^T (D(\widehat{u}), D(\phi))_{L^2(\tilde{\Omega}(t))} \\ &+ \int_0^T \{\psi(\widehat{u},\widehat{u},\phi) + \psi(\widehat{v},s,\phi) + \psi(s,\widehat{v},\phi)\} = \int_0^T \langle\widehat{g},\phi\rangle + m\,k_0\,\ell(0) + (\widehat{u}_0,\phi(0))_{L^2(\tilde{\Omega}(0))} \end{aligned} \tag{4.6.4}$$

for every $\phi \in \mathcal{C}^1([0,T]; H_0^1(\mathcal{R}))$ such that $\phi(\cdot,t)|_{B_{h(t)}} = \ell(t)\,\widehat{e}_2$, with $\ell(t) \in \mathbb{R}$ and $\phi(\cdot,T) = \ell(T) = 0$, and

$$\widehat{g} := \mu\,\Delta(s - v_P) - (s\cdot\nabla)\,s.$$

Equality (4.6.4) corresponds to (4.3.20) once we undo the change of variables in (4.3.6). The second step to prove 4.6.1 implies introducing the following auxiliary linear system with unknown (U,H):

$$\begin{aligned} \frac{\partial U}{\partial t} - \mu\,\Delta U + \nabla Q = f \qquad & \text{for}\, x \in \Omega_{h(t)}\,, \\ \operatorname{div} U = 0 \qquad & \text{for}\, x \in \Omega_{h(t)}\,, \\ U = H'\hat{e}_2 \qquad & \text{for}\, x \in \partial B_{h(t)}\,, \\ U = 0 \qquad & \text{for}\, x \in \partial\Gamma_{\mathcal{R}}\,, \\ m\,H''(t) = -\hat{e}_2\cdot\int_{\partial B_{h(t)}} \mathcal{T}(U,Q)\widehat{n}\,d\sigma + m\,f_1\,, & \end{aligned} \tag{4.6.5}$$

where f and f_1 are given source terms, and $\Omega_{h(t)}$ and $B_{h(t)}$ are prescribed and not unknown. In particular, they are associated to $(\widehat{u},h)$, where $\widehat{u}$ is known and it is as in (4.6.3). Following [76, Definition 2], we say that, given $f \in L^{4/3}((0,T)\times\Omega_{h(t)})$, $f_1 \in L^{4/3}(0,T;\mathbb{R})$ and $\varepsilon_0 > 0$, then

$$(U,H) \in [L^2(0,T;H_0^1(\Omega_{h(t)})) \cap L^\infty(0,T;L^2(\Omega_{h(t)}))] \times \mathcal{C}([0,T];[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0])$$

is a weak solution to (4.6.5), with vanishing initial data and source term f, f_1, if U is divergence free, $U(0) = 0$, $H(0) = 0$ and

$$\int_{\Omega_{h(t)}} \frac{\partial U}{\partial t}\,\phi\,dx + 2\mu\int_{\mathcal{R}} DU : D\phi\,dx + m(H'' - f_1)\ell_\phi = \int_{\Omega_{h(t)}} f\,\phi\,dx$$

for all $\phi \in \mathcal{C}_0^\infty([0,T]\times\mathcal{R};\mathbb{R}^2)$ such that $\phi(\cdot,t)|_{B_{h(t)}} = \ell(t)\widehat{e}_2$, with $\ell(t) \in \mathbb{R}$. The third step implies showing that weak solutions to (4.6.5) in the sense given above are unique. This can be done precisely as in [76, Lemma 8], by taking the difference between two weak solutions, which is allowed because the fluid domain is in this case prescribed, thus identical for the two solutions. Then, by [76, Lemma 4] and [71, Theorem 4.1] we know that problem (4.6.5) has a unique strong solution with vanishing initial data belonging to

$$\begin{aligned} U \in L^{4/3}(0,T;W^{2,4/3}(\Omega_{h(t)}))\,, \qquad \partial_t U \in L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)})), \qquad H \in W^{2,4/3}(0,T;\mathbb{R})\,, \\ \nabla Q \in L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))\,, \end{aligned} \tag{4.6.6}$$

and such that

$$\begin{aligned} &\|U\|_{L^{4/3}(0,T;W^{2,4/3}(\Omega_{h(t)}))} + \|\partial_t U\|_{L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))} + \|\nabla Q\|_{L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))} \\ &\le C\left(\|f\|_{L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))} + \|f_1\|_{L^{4/3}(0,T;\mathbb{R})}\right)\,, \end{aligned} \tag{4.6.7}$$

where C depends on the geometry of the rigid body and on T. Through some integration by parts, it can be shown that any strong solution to (4.6.5) is also a weak solution to (4.6.5). The last step implies showing that, given

$$U := t\widehat{u}\,, \qquad H := th\,, \qquad Q := t\widehat{p}\,, \tag{4.6.8}$$

then $(t\widehat{u}, th)$ is a weak solution to (4.6.5) in the sense given above, with source term

$$\begin{aligned} f &:= \widehat{u} - t(\widehat{u}\cdot\nabla)\widehat{u} - t(\widehat{u}\cdot\nabla)s - t(s\cdot\nabla)\widehat{u} + t\widehat{g} \in L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))\,,\\ f_1 &:= h' + t\frac{f(h)}{m} \in L^{4/3}(0,T;\mathbb{R})\,. \end{aligned}$$

Finally, we conclude by the properties of weak-strong uniqueness satisfied by the solutions to (4.6.5), that the solution (4.6.8) must be strong, thus it satisfies the regulary in (4.6.6) and the estimate in (4.6.7), which yields the desired result, up to undoing the change of variables in (4.6.3) . □

Let us consider two weak solutions of the equivalent problem (4.3.9)-(4.3.10), (v_1, h_1) and (v_2, h_2), in the sense of Definition 4.3.3, with the same initial conditions, where v_1 is defined on the fluid domain $\tilde{\Omega}^1(t)$ while v_2 is defined on $\tilde{\Omega}^2(t)$. Let $\varepsilon_0 > 0$ be such that

$$\min_{t\in[0,T]}(\mathrm{dist}(B, \partial\tilde{\Omega}^i)) \geq \varepsilon_0$$

for $i = 1, 2$; the existence of such ε_0 comes from Theorem 4.3.6. Finally, let $\zeta(y_1, y_2)$ be a smooth cutoff function equal to 0 in a $\varepsilon_0/4$ neighbourhood of ∂B and to 1 for any (t, y) such that dist$(y, \partial B) \geq \varepsilon_0/2$. Then, for each of the two solutions, we define a solenoidal velocity vector field $V_i : [0,T]\times\tilde{\Omega}^i \to \mathbb{R}^2$ as

$$V_i(t,y) := \{-y_1 h_i' \partial_{y_2}\zeta, \zeta h_i' + y_1 h_i' \partial_{y_1}\zeta\}. \tag{4.6.9}$$

Notice that

$$V_i(t,y) = \begin{cases} 0 & \text{if } \mathrm{dist}(y,\partial B) \geq \varepsilon_0/2\\ -h_i'\widehat{e}_2 & \text{if } \mathrm{dist}(y,\partial B) \leq \varepsilon_0/4. \end{cases}$$

We introduce a further domain $\tilde{\Omega}^0$, which serves as reference configuration, corresponding to the initial condition (v_0, h_0). Then, we build the deformation mappings of such domain respectively into $\tilde{\Omega}^1$ and $\tilde{\Omega}^2$, $X_i : [0,T]\times\tilde{\Omega}^0 \to \tilde{\Omega}^i(t)$, $i = 1, 2$ as the flow associated to (4.6.9):

$$\begin{cases} \frac{\partial}{\partial t}X_i(t,y) = V_i(t, X_i(t,y))\\ X_i(0,y) = y. \end{cases}$$

Notice that, since $\nabla\cdot V_i = 0$, X_i is volume preserving. More precisely, taking $y = (y_1, y_2) \in \tilde{\Omega}^0$,

$$X_i(t,y_1,y_2) = \begin{cases} (y_1, y_2 + h_0 - h_i(t)) & \text{if } \mathrm{dist}(y,\partial B) \geq \varepsilon_0/2\\ (y_1, y_2) & \text{if } \mathrm{dist}(y,\partial B) \leq \varepsilon_0/4. \end{cases}$$

The mapping X_i is a smooth function of V_i. In particular, for some $C > 0$

$$\|\partial_t{}^j X_i(t,y)\|_{\mathcal{C}^k(\bar{\tilde{\Omega}}_i)} \leq C|h_i^{(j)}| \qquad \forall\, j = 0, 1, \qquad \forall\, k \in \mathbb{N}. \tag{4.6.10}$$

For each $t \in [0,T]$ we define the volume preserving diffeomorphisms

$$\begin{aligned} \psi_t : \tilde{\Omega}^2(t) &\longrightarrow \tilde{\Omega}^1(t) \\ y &\longmapsto \psi_t(y) = X_1(t, X_2^{-1}(t,y)) \\ \varphi_t = \psi_t^{-1} : \tilde{\Omega}^1(t) &\longrightarrow \tilde{\Omega}^2(t) \\ y &\longmapsto \varphi_t(y) = X_2(t, X_1^{-1}(t,y)). \end{aligned} \tag{4.6.11}$$

Thus for any $y = (y_1, y_2)$ such that $\mathrm{dist}(y, \partial B) \geq \varepsilon_0/2$

$$\begin{aligned} \psi_t(y_1,y_2) &= (y_1, y_2 + h_2(t) - h_1(t)), \\ \varphi_t(y_1,y_2) &= (y_1, y_2 + h_1(t) - h_2(t)). \end{aligned}$$

Given the extensions of the Poiseuille flow associated to each of the two solutions, $a_1 = a_{h_1}$ and $a_2 = a_{h_2}$, defined as in (4.3.11), we put $\widehat{v}_1 = v_1 - a_1$ and $\widehat{v}_2 = v_2 - a_2$. For any given $y = (y_1, y_2) \in \tilde{\Omega}^1(t)$, we introduce the function

$$\widehat{\mathfrak{v}}_2 = \nabla\psi_t(y) \cdot \widehat{v}_2(t, \varphi_t(y)),$$

the pullback of $\widehat{v}_2$ by map φ_t in (4.6.11). Since $a_2 = 0$ near the obstacle B, because of (4.3.11) and the properties of s as in Lemma 4.3.1, we obtain that the pullback of a_2 corresponds to

$$\mathfrak{a}_2 = \nabla\psi_t(y) \cdot a_2(\varphi_t(y)) = a_2(y_1, y_2 + h_1 - h_2) = s(y_1, y_2 + h_1 - h_2 + h_2) = a_1,$$

which implies that the solenoidal extension a_1 and $\mathfrak{a}_2$ are equal after the change of variables. Thus, from now on, $a_1 = \mathfrak{a}_2 = a$. We remark that $\widehat{\mathfrak{v}}_2$ mantains the property of being solenoidal since φ_t is volume preserving. We also define

$$\pi_2 = \mathfrak{p}_2(t, \varphi_t(y))\,.$$

The weak formulation satisfied by $(\widehat{v}_1, h_1)$ can be obtained from (4.3.20), after rewriting the equation by integrating by parts the two first terms. For a.e. $t \in [0,T]$, there holds, for every $(\phi(t), \ell(t)) \in \mathbb{V}_{h_1}$ such that $\phi(\cdot, T) = \ell(T) = 0$,

$$\begin{aligned} &\langle \partial_t \widehat{v}_1(t), \phi(t)\rangle + m\, h_1''(t)\,\ell(t) + f(h_1(t))\,\ell(t) + 2\mu(D(\widehat{v}_1(t)), D(\phi(t)))_{L^2(\tilde{\Omega}^1(t))} \\ &+ \psi(\widehat{v}_1(t), \widehat{v}_1(t), \phi(t)) + \psi(\widehat{v}_1(t), a, \phi(t)) + \psi(a, \widehat{v}_1(t), \phi(t)) \\ &- \psi(h_1'(t)\widehat{e}_2, \widehat{v}_1(t), \phi) - \psi(h_1'(t)\widehat{e}_2, a, \phi(t)) = \langle \widehat{g}, \phi(t)\rangle. \end{aligned}$$

We refer to [76, Section 3.2] for the explicit computation of the partial derivatives of $\widehat{v}_2$ in terms of those of $\widehat{\mathfrak{v}}_2$, so as to obtain that the equation satisfied by $\mathfrak{v}_2$ reads as

$$\begin{aligned} &\langle \partial_t \widehat{\mathfrak{v}}_2, \phi\rangle + m\, h_2''\,\ell + f(h_2)\,\ell + 2\mu\,(D(\widehat{\mathfrak{v}}_2), D(\phi))_{L^2(\tilde{\Omega}^1(t))} + \psi(\widehat{\mathfrak{v}}_2, \widehat{\mathfrak{v}}_2, \phi) \\ &\quad + \psi(\widehat{\mathfrak{v}}_2, a, \phi) + \psi(a, \widehat{\mathfrak{v}}_2, \phi) - \psi(h_2'\widehat{e}_2, \widehat{\mathfrak{v}}_2, \phi) - \psi(h_2'\widehat{e}_2, a, \phi) = \langle \widehat{g}, \phi\rangle - \langle \mathfrak{f}, \phi\rangle, \end{aligned}$$

for every $(\phi(t), \ell(t)) \in \mathbb{V}_{h_1}$ such that $\phi(\cdot, T) = \ell(T) = 0$, where, using Einstein's summation convention and omitting the index t in ψ_t and φ_t,

$$\begin{aligned} \mathfrak{f}^i = &+ (\partial_k\varphi^i - \delta_{ik})\partial_t \widehat{\mathfrak{v}}_2^k + \partial_k\varphi^i \partial_l \widehat{\mathfrak{v}}_2^k(\partial_t\psi^l) + (\partial_k\partial_t\varphi^i)\widehat{\mathfrak{v}}_2^k + (\partial^2_{kl}\varphi^i)(\partial_t\psi^l)\widehat{\mathfrak{v}}_2^k \\ &+ \widehat{\mathfrak{v}}_2^l \partial_l \widehat{\mathfrak{v}}_2^k(\partial_k\varphi^i - \delta_{ik}) + (\partial^2_{lk}\varphi^i)\widehat{\mathfrak{v}}_2^l\widehat{\mathfrak{v}}_2^k + \partial_k\pi_2(\partial_i\psi^k - \delta_{ik}) - \partial_j\psi^m(\partial^2_{mk}\varphi^i)\partial_l\widehat{\mathfrak{v}}_2^k\partial_j\psi^l \\ &- (\partial_k\varphi^i\partial_j\psi^m\partial_j\psi^l - \delta_{ik}\delta_{jm}\delta_{jl})\partial^2_{ml}\widehat{\mathfrak{v}}_2^k - \partial_k\varphi^i\partial_l\widehat{\mathfrak{v}}_2^k(\partial^2_{jj}\psi^l) \\ &- \partial_j\psi^m(\partial^3_{mlk}\varphi^i)\partial_j\psi^l\widehat{\mathfrak{v}}_2^k - (\partial^2_{lk}\varphi^i)\partial^2_{jj}\psi^l\widehat{\mathfrak{v}}_2^k - (\partial^2_{lk}\varphi^i)\partial_j\psi^l\partial_j\psi^m\partial_m\widehat{\mathfrak{v}}_2^k. \end{aligned}$$

Now, let $(w, \widehat{h})$ be defined as in (4.3.28). Then, taking the difference of the weak formulations satisfied by $\widehat{v}_1$ and $\mathfrak{v}_2$, one has

$$\begin{aligned}&\langle \partial_t w, \phi\rangle + m\,\widehat{h}''\,\ell + [f(h_1)-f(h_2)]\,\ell + 2\mu(D(w), D(\phi))_{L^2(\tilde\Omega^1(t))}\\&+\psi(\widehat{v}_1, w,\phi)+\psi(w,\widehat{\mathfrak{v}}_2,\phi)+\psi(w,a,\phi)+\psi(a,w,\phi)-\psi(h_1'\widehat{e}_2, w,\phi)\\&-\psi(\widehat{h}'\widehat{e}_2,\widehat{\mathfrak{v}}_2,\phi)-\psi(\widehat{h}'\widehat{e}_2,a,\phi)=\langle \mathfrak{f},\phi\rangle.\end{aligned}$$

Then we take $(\phi, \ell) = (w, h')$ and we obtain

$$\begin{aligned}&\langle \partial_t w, w\rangle + m\,\widehat{h}''\,\widehat{h}' + [f(h_1)-f(h_2)]\,\widehat{h}' + 2\mu\|D(w)\|^2_{L^2(\tilde\Omega^1(t))} = \langle \mathfrak{f}, w\rangle\\&-\psi(w,\widehat{\mathfrak{v}}_2,w)-\psi(w,a,w)+\psi(\widehat{h}'\widehat{e}_2,a,w).\end{aligned}$$

Thus, using [134, Chapter 3, Lemma 1.2] and that $(w', \widehat{h}'') \in L^2(0,T;\mathbb{V}'_{h_1})$ (from the properties of weak solutions to problem (4.3.9)-(4.3.10)), there holds

$$\begin{aligned}&\frac{d}{dt}\left\{\|w\|^2_{L^2(\tilde\Omega^1)} + m\,|\widehat{h}'|^2 + 2\int_{h_2}^{h_1} f(s)\,ds\right\} + 4\,\mu\|D(w)\|^2_{L^2(\tilde\Omega^1)} = 2\langle \mathfrak{f},\phi\rangle\\&-2\,\psi(w,\widehat{\mathfrak{v}}_2,w)-2\,\psi(w,a,w)+2\,\psi(\widehat{h}'\widehat{e}_2,a,w).\end{aligned} \tag{4.6.12}$$

Now, we estimate the right hand side of the above inequality, starting from the trilinear forms. For what concerns the second term, we exploit Lemma 4.6.1 and the Young inequality:

$$\begin{aligned}|2\,\psi(w,\widehat{\mathfrak{v}}_2,w)| &\le 2\|w\|^2_{L^4(\tilde\Omega^1(t))}\|\nabla\widehat{\mathfrak{v}}_2\|_{L^2(\tilde\Omega^1(t))}\\&\le 2^{3/2}\|w\|_{L^2(\tilde\Omega^1(t))}\|\nabla w\|_{L^2(\tilde\Omega^1(t))}\|\nabla \mathfrak{v}_2\|_{L^2(\tilde\Omega^1(t))}\\&\le \frac{10}{\mu}\|w\|^2_{L^2(\tilde\Omega^1(t))}\|\nabla\widehat{\mathfrak{v}}_2\|^2_{L^2(\tilde\Omega^1(t))}+\frac{\mu}{5}\|\nabla w\|^2_{L^2(\tilde\Omega^1(t))}\end{aligned}$$

The third term can be estimated analogously to what we did in the proof of Proposition 4.4.1, through the Hölder inequality, the Poincaré inequality in the domain $\tilde\Omega^1(t)$ and the Young inequality:

$$\begin{aligned}|2\,\psi(w,a,w)| &\le \frac{4L}{\pi}\|\nabla a\|_{L^\infty(\tilde\Omega^1(t))}\|\nabla w\|_{L^2(\tilde\Omega^1(t))}\|w\|_{L^2(\tilde\Omega^1(t))}\\&\le \frac{5}{4\mu}\frac{16L^2}{\pi^2}\|\nabla a\|^2_{L^\infty(\tilde\Omega^1(t))}\|w\|^2_{L^2(\tilde\Omega^1(t))}+\frac{\mu}{5}\|\nabla w\|^2_{L^2(\tilde\Omega^1(t))}.\end{aligned}$$

Then, we consider the domain $\tilde\Omega^1(t)$ to be partitioned as in (4.3.8) and we recall that the function a enjoys the same properties of s stated in Lemma 4.3.1 once we substituted v_p and $\mathcal{R}_h$ with $\tilde v_P$ as in (4.3.9) (where, instead of h, we consider h_1) and $\mathcal{R}_{h_1}$. The last term on the right hand side of (4.6.12) is bounded following the reasoning developed in the proof of Proposition 4.4.1 for the terms in (4.4.6). Given

$$\|\nabla \tilde v_P\|_{L^2(-L,L)} = \lambda\,\xi \qquad \text{with } \xi = \sqrt{\frac{8L^3}{3}},$$

one can write

$$|2\,\psi(\widehat{h}'\widehat{e}_2, a, w)| \leq \frac{5}{2\mu}\frac{4L^2}{\pi^2}|\widehat{h}'|^2(\|\nabla a\|^2_{L^2(\tilde{\Omega}^1_0(t))} + \lambda^2\xi^2) + \frac{2\mu}{5}\|\nabla w\|^2_{L^2(\tilde{\Omega}^1(t))}\,.$$

In order to estimate the first term on the right-hand side of equation (4.6.12), following [76], we divide $\mathfrak{f}$ into pieces

$$\mathfrak{f} = \mathfrak{f}_1 + \mathfrak{f}_2 + \mathfrak{f}_3 + \mathfrak{f}_4 + \mathfrak{f}_5$$

with

$$\begin{aligned}
\mathfrak{f}_1 :=& (\partial_k\partial_t\varphi^i)\widehat{\mathfrak{v}}_2^k + (\partial^2_{kl}\varphi^i)(\partial_t\psi^l)\widehat{\mathfrak{v}}_2^k - \sum_j\Big[\partial_j\psi^m(\partial^3_{mlk}\varphi^i)\partial_j\psi^l\widehat{\mathfrak{v}}_2^k + (\partial^2_{lk}\varphi^i)\partial^2_{jj}\psi^l\widehat{\mathfrak{v}}_2^k\Big],\\
\mathfrak{f}_2 :=& \partial_k\varphi^i\partial_l\widehat{\mathfrak{v}}_2^k(\partial_t\psi^l) - \sum_j\Big[\partial_j\psi^m(\partial^2_{mk}\varphi^i)\partial_l\widehat{\mathfrak{v}}_2^k\partial_j\psi^l + \partial_k\varphi^i\partial_l\widehat{\mathfrak{v}}_2^k(\partial^2_{jj}\psi^l)\\
&+ (\partial^2_{lk}\varphi^i)\partial_j\psi^l\partial_j\psi^m\partial_m\widehat{\mathfrak{v}}_2^k\Big],\\
\mathfrak{f}_3 :=& (\partial^2_{lk}\varphi^i)\widehat{\mathfrak{v}}_2^l\widehat{\mathfrak{v}}_2^k, \quad \mathfrak{f}_4 := \widehat{\mathfrak{v}}_2^l\partial_l\widehat{\mathfrak{v}}_2^k(\partial_k\varphi^i - \delta_{ik}),\\
\mathfrak{f}_5 :=& (\partial_k\varphi^i - \delta_{ik})\partial_t\widehat{\mathfrak{v}}_2^k + \partial_k\pi_2(\partial_i\psi^k - \delta_{ik}) - \sum_j(\partial_k\varphi^i\partial_j\psi^m\partial_j\psi^l - \delta_{ik}\delta_{jm}\delta_{jl})\partial^2_{ml}\widehat{\mathfrak{v}}_2^k.
\end{aligned}$$

We have the following estimates, where we use (4.6.10).

- Concerning the first three terms:

$$\begin{aligned}
&\Big|\int_0^t\int_{\tilde{\Omega}^1(s)}\mathfrak{f}_1\cdot w\,dy\,ds\Big|\\
&\leq C\|\widehat{\mathfrak{v}}_2\|_{L^\infty(0,T;L^2(\tilde{\Omega}^1))}\int_0^t\Big(\max_{[0,s]}\|w(\cdot,s)\|_{L^2(\tilde{\Omega}^1(s))}\max_{[0,s]}|(\widehat{h}(s),\widehat{h}'(s))|\Big)\,ds\\
&\leq C\|\widehat{\mathfrak{v}}_2\|_{L^\infty(0,T;L^2(\tilde{\Omega}^1))}\int_0^t\Big(\max_{[0,s]}\|w(\cdot,s)\|^2_{L^2(\tilde{\Omega}^1(s))}\\
&\qquad+\max_{[0,s]}|(\widehat{h}(s),\widehat{h}'(s))|^2\Big)\,ds,
\end{aligned}$$

$$\begin{aligned}
&\Big|\int_0^t\int_{\tilde{\Omega}^1(s)}\mathfrak{f}_2\cdot w\,dy\,ds\Big|\\
&\leq C\int_0^t\|\nabla\widehat{\mathfrak{v}}_2(\cdot,s)\|_{L^2(\tilde{\Omega}^1(s))}\Big(\max_{[0,s]}\|w(\cdot,s)\|_{L^2(\tilde{\Omega}^1(s))}\max_{[0,s]}|(\widehat{h}(s),\widehat{h}'(s))|\Big)\,ds\\
&\leq C\int_0^t\|\nabla\widehat{\mathfrak{v}}_2(\cdot,s)\|_{L^2(\tilde{\Omega}^1(s))}\Big(\max_{[0,s]}\|w(\cdot,s)\|^2_{L^2(\tilde{\Omega}^1(s))}\\
&\qquad+\max_{[0,s]}|(\widehat{h}(s),\widehat{h}'(s))|^2\Big)\,ds,
\end{aligned}$$

$$
\begin{aligned}
&\Big|\int_0^t \int_{\tilde{\Omega}^1(s)} \mathfrak{f}_3 \cdot w \, dy \, ds\Big| \\
&\leq C \int_0^t \|\widehat{\mathfrak{v}}_2(\cdot, s)\|^2_{L^4(\tilde{\Omega}^1(a))} \|w(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \max_{[0,s]} |\widehat{h}(s)| \, ds \\
&\leq C \|\widehat{\mathfrak{v}}_2\|_{L^\infty(0,T;L^2(\tilde{\Omega}^1))} \int_0^t \|\nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \max_{[0,s]} \|w(\cdot, s)\|_{L^2(\tilde{\Omega}^1)} \\
&\quad \times \max_{[0,s]} |\widehat{h}(s)| \, ds \\
&\leq C \|\widehat{\mathfrak{v}}_2\|_{L^\infty(0,T;L^2(\tilde{\Omega}^1))} \int_0^t \|\nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \Big(\max_{[0,s]} \|w(\cdot, s)\|^2_{L^2(\tilde{\Omega}^1(s))} \\
&\quad + \max_{[0,s]} |\widehat{h}(s)|^2 \Big) \, ds \\
&\leq C \|\widehat{\mathfrak{v}}_2\|_{L^\infty(0,T;L^2(\tilde{\Omega}^1))} \int_0^t \|\nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \Big(\max_{[0,s]} \|w(\cdot, s)\|^2_{L^2(\tilde{\Omega}^1(s))} \\
&\quad + \max_{[0,s]} |\widehat{h}(s), \widehat{h}'(s)|^2 \Big) \, ds.
\end{aligned}
$$

- For the fourth and fifth terms, following [76], thanks to Lemma 4.6.1, we have

$$
\begin{aligned}
&\Big|\int_0^t \int_{\tilde{\Omega}^1(s)} \mathfrak{f}_4 \cdot w \, dy \, ds\Big| \\
&\leq C \int_0^t \|\widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^4(\tilde{\Omega}^1(s))} \|t \, \nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^4(\tilde{\Omega}^1(s))} \|\frac{1}{t}(\partial_k \varphi^i - \delta_{ik})\|_{L^\infty(\tilde{\Omega}^1(s))} \\
&\quad \times \|w(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \, ds \\
&\leq C \int_0^t \|\widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^4(\tilde{\Omega}^1(s))} \|t \, \nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^4(\tilde{\Omega}^1(s))} \max_{[0,s]} |\widehat{h}'(s)| \\
&\quad \times \|w(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \, ds \\
&\leq C \int_0^t \|\nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|^{1/2}_{L^2(\tilde{\Omega}^1(s))} \|\widehat{\mathfrak{v}}_2(\cdot, s)\|^{1/2}_{L^2(\tilde{\Omega}^1(s))} \|t \, \nabla \widehat{\mathfrak{v}}_2(\cdot, s)\|_{L^4(\tilde{\Omega}^1(s))} \max_{[0,s]} |\widehat{h}'(s)| \\
&\quad \times \|w(\cdot, s)\|_{L^2(\tilde{\Omega}^1(s))} \, ds \, .
\end{aligned}
$$

Next we notice that

$$
b_1(t) := \|\nabla \widehat{\mathfrak{v}}_2(\cdot, t)\|^{1/2}_{L^2(\tilde{\Omega}^1(t))} \|t \, \nabla \widehat{\mathfrak{v}}_2(\cdot, t)\|_{L^4(\tilde{\Omega}^1(t))} \in L^1(0, T) \, ,
$$

due to the Hölder inequality with exponent $p = 4$ and $q = 4/3$. Hence we obtain

$$
\begin{aligned}
&\Big|\int_0^t \int_{\tilde{\Omega}^1(s)} \mathfrak{f}_4 \cdot w \, dy \, ds\Big| \\
&\leq C \|\widehat{\mathfrak{v}}_2\|^{1/2}_{L^\infty(0,T;L^2(\tilde{\Omega}^1(t)))} \int_0^t b_1(s) \Big(\max_{[0,s]} \|w(\cdot, s)\|^2_{L^2(\tilde{\Omega}^1(s))} + \max_{[0,s]} |\widehat{h}(s), \widehat{h}'(s)|^2 \Big) \, ds.
\end{aligned}
$$

Next, we introduce

$$b_2(t) := \|t\partial_t \widehat{\mathfrak{v}}_2^k\|_{L^{4/3}(\tilde{\Omega}^1(t))} + \|\partial_k \pi_2\|_{L^{4/3}(\tilde{\Omega}^1(t))} + \|t\widehat{\mathfrak{v}}_2^k\|_{W^{2,4/3}(\tilde{\Omega}^1(t))} \in L^{4/3}(0,T)\,.$$

We deduce that

$$\left| \int_0^t \int_{\tilde{\Omega}^1(s))} \mathfrak{f}_5 \cdot w \, dy \, ds \right| \le C \int_0^t b_2(s) \max_{[0,s]} |(\widehat{h}(s), \widehat{h}'(s))| \|w(\cdot,s)\|_{L^4(\tilde{\Omega}_1(s))} \, ds\,.$$

Next, we apply the Young inequality twice as follows

$$\begin{aligned}
\left| \int_0^t \int_{\tilde{\Omega}^1(s)} \mathfrak{f}_5 \cdot w \, dy \, ds \right| &\le C \int_0^t b_2(s)^{2/3} \|w(\cdot,s)\|^2_{L^4(\tilde{\Omega}_1(s))} \, ds \\
&\quad + C \int_0^t b_2(s)^{4/3} \max_{[0,s]} |(\widehat{h}(s), \widehat{h}'(s))|^2 \, ds \\
&\le C \int_0^t b_2(s)^{4/3} \|w(\cdot,s)\|^2_{L^2(\tilde{\Omega}_1(s))} \, ds \\
&\quad + C \int_0^t b_2(s)^{4/3} \max_{[0,s]} |(\widehat{h}(s), \widehat{h}'(s))|^2 \, ds \\
&\quad + \frac{2\mu}{5} \int_0^t \|\nabla w(\cdot,s)\|^2_{L^2(\tilde{\Omega}_1(s))} \, ds\,,
\end{aligned}$$

where $b_2(t)^{4/3} \in L^1(0,T)$.

If we set

$$\begin{aligned}
A(t) := &\|\widehat{\mathfrak{v}}_2\|_{L^\infty(0,T;L^2(\tilde{\Omega}^1(t)))} (1 + \|\nabla \widehat{\mathfrak{v}}_2(\cdot,t)\|_{L^2(\tilde{\Omega}^1(t))}) \\
&+ \|\widehat{\mathfrak{v}}_2\|^{1/2}_{L^\infty(0,T;L^2(\tilde{\Omega}^1(t)))} b_1(t) + b_2(t)^{4/3} \in L^1(0,T),
\end{aligned}$$

we obtain

$$\begin{aligned}
\left| \int_0^t \int_{\tilde{\Omega}^1(s)} \mathfrak{f} \cdot w \, dy \, ds \right| \le\; & C \int_0^t A(s) \Big(\max_{[0,s]} \|w(\cdot,s)\|^2_{L^2(\tilde{\Omega}^1(s))} \\
&+ \max_{[0,s]} |(\widehat{h}(s), \widehat{h}'(s))|^2 \Big) \, ds \\
&+ \frac{2\mu}{5} \int_0^t \|\nabla w(\cdot,s)\|^2_{L^2(\tilde{\Omega}^1(s))} \, ds.
\end{aligned}$$

Then, given $A(t)$ as above, we define

$$\begin{aligned}
\bar{A}(t) &:= A(t) + \frac{10}{\mu} \|\nabla \widehat{\mathfrak{v}}_2(\cdot,t)\|^2_{L^2(\tilde{\Omega}^1(t))} + \frac{5}{4\mu} \frac{16L^2}{\pi^2} \|\nabla a\|^2_{L^\infty(\tilde{\Omega}^1(t))}, \\
\bar{B}(t) &:= A(t) + \frac{5}{2\mu} \frac{4L^2}{\pi^2} (\|\nabla a\|^2_{L^2(\tilde{\Omega}^1_0(t))} + \lambda^2 \xi^2).
\end{aligned}$$

We reorder (4.6.12) once we plugged the above estimates, considering the above definitions and using that

$$\int_{\tilde{\Omega}^1} |\nabla w|^2 \, dy \leq \int_{A_{h_1}} |\nabla w|^2 \, dy = 2\int_{A_{h_1}} |D(w)|^2 \, dy = 2\int_{\tilde{\Omega}^1} |D(w)|^2 \, dy,$$

since w is a divergence free vector field vanishing on ∂A_{h_1}. Thus, integrating between 0 and t we obtain, since $w(0) = 0 = \widehat{h}'(0)$,

$$\begin{aligned} &\|w(t)\|^2_{L^2(\tilde{\Omega}^1(t))} + m|\widehat{h}'(t)|^2 + 2\int_{h_2(t)}^{h_1(t)} f(s)\, ds \\ &\leq C\int_0^t \left(\bar{A}(s) \max_{[0,s]} \|w(\cdot, s)\|^2_{L^2(\tilde{\Omega}^1(s))} + \bar{B}(s) \max_{[0,s]} |(\widehat{h}(s), \widehat{h}'(s))|^2 \right) ds. \end{aligned} \tag{4.6.13}$$

Then we set

$$\mathcal{D}(t) = \bar{A}(t) + \bar{B}(t)$$

and from (4.6.13), we infer

$$\begin{aligned} \|w(t)\|^2_{L^2(\tilde{\Omega}^1(t))} + m|\widehat{h}'(t)|^2 \leq \int_0^t C\,\mathcal{D}(s) \Big(&\max_{[0,s]} \|w(\cdot, s)\|^2_{L^2(\tilde{\Omega}^1(s))} \\ &+ \max_{[0,s]} |(\widehat{h}(s), \widehat{h}'(s))|^2 \Big) \, ds. \end{aligned}$$

As in [76], we notice that

$$\frac{d}{dt}|\widehat{h}|^2 \leq C(|\widehat{h}'|^2 + |\widehat{h}|^2). \tag{4.6.14}$$

Using $\mathcal{D}(t) \in L^1(0,T)$ and Grönwall's lemma, we conclude that

$$\|w(t)\|^2_{L^2(\tilde{\Omega}^1(t))} + m|\widehat{h}'(t)|^2 = 0,$$

which, if one uses (4.6.14), implies finishing the proof.

CHAPTER 5

Well-posedness of a FSI problem in a Poiseuille flow: full motion

In the present chapter, we treat problem (2.0.9) which allows to model the second phase of the *flutter* phenomenon, when the vertical and torsional motion of the deck of a suspension bridge synchronize under the action of the wind (see the introduction in Chapter 4). In particular, we obtain a global-in-time (up to collision) existence result for problem (2.0.9).

5.1 Global existence (up to collision) of weak solutions

We refer to the introduction in Chapter 4 for a general presentation of the *flutter* phenomenon, and to Chapter 2 for the notation and rigorous formulation of problem (2.0.9). Problem (2.0.9), analyzed in the present chapter, represents an extension of problem (2.0.7) which takes into account a full coupled vertical-torsional motion of the obstacle in an unbounded channel. This allows to model the interaction between the deck of a suspension bridge and the wind in a regime of strong oscillations, as in real-life experiment, where the vertical and torsional displacements are lead to be coupled under the action of the areodynamic forces. In this context, in analogy to problem (2.0.7), the restoring forces F_1 and F_2 driving the motion of the obstacle in $(2.0.9)_4$-$(2.0.9)_5$ can be seen as forces resuming the elastic upward action of both the hangers and the substaining cables on the deck, the downward action of the weight of the deck and the elastic resistance to bending and stretching of the whole deck.

The main purpose of this chapter is to prove the global-in-time (up to collision) existence of weak solutions for the fluid-structure interaction evolution problem (2.0.9). In order to build weak solutions, we exploit a penalization technique by allowing the obstacle to also move in the horizontal direction and penalizing its motion in such a direction. This produces the penalized problem (5.3.1)-(5.3.2)-(5.3.3) such that, in the limit, we recover a purely vertical-torsional oscillation of the obstacle. Actually, our method is based on a double-limit procedure, since the initial data of the penalized problem will be approximated by more regular data allowing to obtain a sequence of strong solutions to the penalized problem. By letting simultaneously the penalization index go to infinity and the sequence of initial data converge to less regular initial data, we obtain a weak solution to the original problem. The construction of strong solutions follows the method in [33, 130, 131], which is based on a change of variables to make the fluid domain time-independent, and a fixed point procedure. Using strong solutions allows to circumvent the difficulty of working with a global weak formulation, which would be for instance required when obtaining directly a weak solution through the methods in [41, 87, 119]. However, due to the coupling with rotation, the change of variables needed to work in a fixed domain and obtain strong solutions is not of immediate application to the original problem (2.0.9), where the obstacle has a purely vertical translation. This is why we add an intermediate step by introducing the penalized problem.

The main result of this chapter reads as

Theorem 5.1.1. *Assume that $F_1(h,\theta), F_2(h,\theta) \in \mathcal{C}^1(A_{d,\delta};\mathbb{R})$ satisfy the assumptions given in Chapter 2, with $A_{d,\delta}$ as in* (2.0.4). *Moreover, let u_0 satisfying* $(2.0.11)_1$-$(2.0.11)_2$ *be such that $u_0 \cdot \widehat{n} = (h_0\widehat{e}_2 + \theta_0 x^\perp)\cdot\widehat{n}$ on ∂B, with $h_0, \theta_0 \in \mathbb{R}$. Then, there exists at least one weak solution (u, h, θ) to problem* (2.0.9). *Moreover, the following alternative holds*

(1) $T = \infty$;

(2) $T < \infty$ and $\lim_{t\to T}((k(t), h(t)), \theta(t)) \notin A_{d,\delta}$.

This chapter is devoted to the proof of Theorem 5.1.1. We emphasize that the statement of Theorem 5.1.1 has to be understood as a global-in-time existence result up to collision; indeed, the set $A_{d,\delta}$ in (2.0.4) is the set of admissible values for (h,θ), which excludes the possibility of collisions between the obstacle and the boundary of the channel.

Chapter 5 is organized as follows. Section 5.2 is devoted to presenting the preliminary notions needed to define in a suitable way a weak solution to problem (2.0.9). In particular, we introduce a solenoidal extension for the Poiseuille flow v_P, by which we reformulate the original problem (2.0.9) into an equivalent problem with fully homogeneous boundary conditions, problem (5.2.3)-(5.2.4)-(5.2.5). In Section 5.3, we introduced the penalized problem (5.3.1)-(5.3.2)-(5.3.3) and we prove the existence of a global-in-time strong solution to this problem. This requires transforming (5.3.1)-(5.3.2)-(5.3.3) via a change of variables to a fixed domain, investigating a linearized problem associated to the transformed penalized problem by means of a semigroup approach, and then applying a fixed point procedure. This only gives local-in-time existence of a strong solution to problem (5.3.1)-(5.3.2)-(5.3.3); the global-in-time character is then obtained through some suitable a priori estimates in Theorem 5.3.12. Finally, in Section 5.4, we conclude the proof of Theorem 5.1.1, by proving the existence of at least one weak solution to the equivalent problem (5.2.3)-(5.2.4)-(5.2.5) in Theorem 5.4.1. As already mentioned, the proof is based on a diagonal argument, that implies approximating the initial data in (5.3.3) by more regular data and passing to the limit in the penalized problem.

Remark 5.1.2. One might wonder why the obstacle B considered in Chapter 5 has an elliptical shape, whereas in Chapter 4 it was a rectangle. The explanation partially lies in Remark 4.3.4. As soon as the contact surfaces are of the class $\mathcal{C}^\infty$, one is in the position to extend the result in Theorem 5.1.1 and prove a no-collision result without using any additional assumptions on the restoring forces F_1 and F_2.

Remark 5.1.3. We observe that we do not expect the penalty method used in the case of translation in a bounded domain (see Chapter 4) to work also in this case, because the same technique will generate an infinite energy term. For the purpose of clarifying this observation, we consider a simplification of problem (2.0.9), by assuming that the obstacle B is only free to rotate around a fixed pin placed at its center of mass and that any translation is absent. If we aim at using the penalty method seen in Chapter 4, we must introduce the change of variables allowing to write the equations of motion in a frame attached to B, whose coordinates are labelled as (y_1, y_2):

$$Q(\theta)\, y = x, \tag{5.1.1}$$

where $Q(\theta)$ is defined as in (2.0.3). The fluid-structure interaction evolution problem in the new rotating reference frame is obtained consequently, similarly to what is done in [58, 124]. Let us just highlight that the Poiseuille flow at infinity after the change of variables (5.1.1) assumes the following expression, for each value of θ:

$$\tilde{v}_P(y) = \Big(\cos[\theta\, \lambda\big(L^2 - (y_1 \sin\theta + y_2 \cos\theta)^2\big)], -\sin[\theta\, \lambda\big(L^2 - (y_1 \sin\theta + y_2 \cos\theta)^2\big)] \Big). \tag{5.1.2}$$

The auxiliary fixed domain that we built in Section 4.3 then corresponds to $\mathbb{R}^2$; indeed, the channel is unbounded and it may cover the whole plane when rotating. The velocity field at infinity $\tilde{v}_P$ defined in (5.1.2) in the new reference frame is then extended outside the rotating channel to the whole $\mathbb{R}^2$ if one wants to apply the penalty method, as we did in Section 4.3. Thus, the Poiseuille flow is extended to be a parabolic branch diverging to infinity. This implies that the solenoidal extension would capture a flow to which it is associated an infinite energy, thus compromising the possibility of proving the existence of a weak solution to the penalized problem and, consequently, to the original problem, through the penalty method of Chapter 4. This is the main justification why we analyze (2.0.9) with a different technique in the present chapter.

5.2 Some technical tools

5.2.1 Functional spaces

In the sequel, we will be lead to consider the case where the body B is characterized by both a vertical and an horizontal translation, thus denoting

$$B_f = B_v + k\, \widehat{e}_1, \qquad k \in \mathbb{R},$$

the final position of B, after a full translation and rotation. Then, we define

$$\begin{aligned}\mathcal{H}(B_f) =&\{(v,\ell,\alpha)\in L^2(A)\times\mathbb{R}^2\times\mathbb{R}\mid \nabla\cdot v=0 \text{ in } A, \quad v\cdot\widehat{n}=0 \text{ on } \partial A,\\ &v=\ell+\alpha\,(x_1-k,x_2-h)^\perp \text{ in } B_f\},\\ \mathcal{V}(B_f) =&\{(v,\ell,\alpha)\in H^1_0(A)\times\mathbb{R}^2\times\mathbb{R}\mid \nabla\cdot v=0 \text{ in } A,\\ &v=\ell+\alpha\,(x_1-k,x_2-h)^\perp \text{ in } B_f\}\,.\end{aligned}$$

In order to define a weak solution to problem (2.0.9), we introduce two closed subspaces of respectively $\mathcal{H}(B_f)$ and $\mathcal{V}(B_f)$, namely

$$\begin{aligned}\mathcal{H}_v(B_v) =&\{(v,\ell,\alpha)\in L^2(A)\times\mathbb{R}\times\mathbb{R}\mid \nabla\cdot v=0 \text{ in } A, \quad v\cdot\widehat{n}=0 \text{ on } \partial A,\\ &v=\ell\,\widehat{e}_2+\alpha\,(x_1,x_2-h)^\perp \text{ in } B_v\},\\ \mathcal{V}_v(B_v) =&\{(v,\ell,\alpha)\in H^1_0(A)\times\mathbb{R}\times\mathbb{R}\mid \nabla\cdot v=0 \text{ in } A,\\ &v=\ell\,\widehat{e}_2+\alpha\,(x_1,x_2-h)^\perp \text{ in } B_v\}\,.\end{aligned}$$

We define an inner product in $L^2(A)$ by

$$\langle\phi_1,\phi_2\rangle_{L^2(A)}=\int_{\Omega_f}\phi_1\cdot\phi_2\,dx+\int_{B_f}\rho\,\phi_1\cdot\phi_2\,dx \qquad \forall\,\phi_1,\phi_2\in L^2(A)\,,$$

where $\rho>0$ is the density of the body and $\Omega_f=A\backslash B_f$. This product induces a norm equivalent to the usual norm in $L^2(A)$. Moreover, we notice that if $\phi_1,\phi_2\in L^2(A)$ are such that

$$\phi_i(x_1,x_2)=\begin{cases}u_i(x_1,x_2) & \forall\,(x_1,x_2)\in\Omega_f\\ \ell_i+\alpha_i\,(x_1-k,x_2-h)^\perp & \forall\,(x_1,x_2)\in B_f\end{cases}$$

for some $u_i\in L^2(A)$, $\ell_i\in\mathbb{R}^2$ and $\alpha_i\in\mathbb{R}$, $i\in\{1,2\}$, then

$$\langle\phi_1,\phi_2\rangle_{L^2(A)}=\int_{\Omega_f}u_1\cdot u_2\,dx+m\,\ell_1\cdot\ell_2+\mathcal{J}\,\alpha_1\alpha_2\,.$$

Since the Poincaré inequality holds in $\mathcal{V}(B_f)$, we endow $\mathcal{H}(B_f)$ and $\mathcal{V}(B_f)$ with the scalar products

$$\begin{aligned}\langle z_1,z_2\rangle_{\mathcal{H}(B_f)}&=\int_{\Omega_f}u_1\cdot u_2\,dx+m\,\ell_1\cdot\ell_2+\mathcal{J}\,\alpha_1\alpha_2\,,\\ \langle z_1,z_2\rangle_{\mathcal{V}(B_f)}&=\int_{\Omega_f}\nabla u_1:\nabla u_2\,dx+m\,\ell_1\cdot\ell_2+\mathcal{J}\,\alpha_1\alpha_2\,,\end{aligned}\tag{5.2.1}$$

where $z_i=(u_i,\ell_i,\alpha_i)$, $i\in\{1,2\}$. We call $\|\cdot\|_{\mathcal{H}(B_f)}$, $\|\cdot\|_{\mathcal{V}(B_f)}$ the norms induced by the scalar products in (5.2.1). The integral in the second formula in (5.2.1) can be defined on the whole channel A; indeed, $\nabla u_1=\nabla u_2=0$ on B_f, since any element of $\mathcal{V}(B_f)$ is a rigid motion on

B_f. Recalling that $D(\cdot)$ denotes the symmetric part of the gradient, for all $u_1, u_2 \in H_0^1(A)$ with $\nabla \cdot u_1 = \nabla \cdot u_2 = 0$ we have

$$2 \int_A D(u_1) : D(u_2)\, dx = \int_A \nabla u_1 : \nabla u_2\, dx\,.$$

If $k, h, \theta : [0,T] \longrightarrow \mathbb{R}$ are functions of time such that $(h(t), \theta(t)) \in A_{d,\delta}$ for every $t \in [0,T]$, we define the spaces

$$L^p(0,T;\mathcal{V}(B_f(t))) = \Big\{ f : [0,T] \to \mathcal{V}(B_f(t)) \ \text{ s.t.} \qquad \|f\|^p_{L^p(0,T;\mathcal{V}(B_f(t)))} = \int_0^T \|f(\tau)\|^p_{\mathcal{V}(B_f(t))}\, d\tau < +\infty \Big\}$$

for $1 \le p < \infty$, and also

$$L^\infty(0,T;\mathcal{H}(B_f(t))) = \Big\{ f : [0,T] \to \mathcal{H}(B_f(t)) \ \text{ s.t.} \qquad \|f\|_{L^\infty(0,T;\mathcal{H}(B_f(t)))} = \operatorname*{ess\,sup}_{\tau\in[0,T]} \|f(\tau)\|_{\mathcal{H}(B_f(t))} < +\infty \Big\},$$

and analogously for

$$L^p(0,T;\mathcal{V}(B_v(t))), \qquad L^\infty(0,T;\mathcal{H}(B_v(t))).$$

In order to define further functional spaces which will be used in the sequel, we suppose that, given $T > 0$, there exists a diffeomorphism $\varphi : \Omega_0 \times [0,T] \longrightarrow \Omega_f(t)$ of class $\mathcal{C}^\infty$ such that its derivatives

$$(y,t) \in \Omega_0 \times (0,T) \longrightarrow \frac{\partial^{i+j+1}\varphi}{\partial t\, \partial y_1^i\, y_2^j}(y,t) \qquad \forall i, j \in \mathbb{N}\,,$$

exist, are continuous and compactly supported in Ω_0. Furthermore, for any function $g : \Omega_f(t) \times [0,T] \longrightarrow \mathbb{R}^2$, we denote by $g_\varphi : \Omega_0 \times [0,T] \longrightarrow \mathbb{R}^2$ the mapping $g_\varphi(y,t) = g(\varphi(y,t),t)$, for every $y \in \Omega_0$ and $t \ge 0$. Then, we can define the following spaces in the time-dependent domain:

$$\begin{aligned}
L^2(0,T;H^2(\Omega_f(t))) &= \{u : \Omega_0 \times [0,T] \longrightarrow \mathbb{R}^2 \mid u_\varphi \in L^2(0,T;H^2(\Omega_0))\}\\
H^1(0,T;L^2(\Omega_f(t))) &= \{u : \Omega_0 \times [0,T] \longrightarrow \mathbb{R}^2 \mid u_\varphi \in H^1(0,T;L^2(\Omega_0))\}\\
\mathcal{C}([0,T];H_0^1(\Omega_f(t))) &= \{u : \Omega_0 \times [0,T] \longrightarrow \mathbb{R}^2 \mid u_\varphi \in \mathcal{C}([0,T];H_0^1(\Omega_0))\}\\
L^2(0,T;\widehat{H}^1(\Omega_f(t))) &= \{u : \Omega_0 \times [0,T] \longrightarrow \mathbb{R}^2 \mid u_\varphi \in L^2(0,T;\widehat{H}^1(\Omega_0))\}\,,
\end{aligned}$$

where we have defined

$$\widehat{H}^1(\Omega_0) = \{p \in L^2_{\text{loc}}(\Omega_0) \mid \nabla p \in L^2(\Omega_0)\}\,.$$

5.2.2 Construction of a solenoidal flux carrier

Since problem (2.0.9) is set in a two-dimensional unbounded channel with a prescribed non-zero velocity field at infinity, we construct a solenoidal extension of the Poiseuille velocity profile at infinity by following precisely the same procedure of Lemma 4.3.1 in Chapter 4. The following result holds:

Lemma 5.2.1. *For every $\varepsilon > 0$ there exists a vector field $s = s_\varepsilon \in W^{2,\infty}(A) \cap H^2_{loc}(A)$ such that*

$$\nabla \cdot s = 0 \quad \textit{in } A\,, \qquad s = (0,0) \quad \textit{in} \quad [-2,2]\times[-L, L-\varepsilon]\,, \qquad s=(0,0) \quad \textit{on} \quad \partial A\,,$$
$$s = v_P \quad \textit{in } A_1 \cup A_2\,, \quad \operatorname{supp}(s) = A \setminus ((-2,2)\times(-L, L-\varepsilon))\,.$$

Proof. See the proof of Lemma 4.3.1 in Chapter 4. It sufficies to change the support of the cutoff function acting in the vertical direction in (4.3.5). We emphasize that the fact that A is an unbounded domain only changes the regularity of the function. □

Let $\varepsilon \in (0, L-\delta)$ and define the vector field $s = s_\varepsilon \in W^{2,\infty}(A) \cap H^2_{\text{loc}}(A)$ as in Lemma 5.2.1. Introduce

$$\widehat{g} = \mu\,\Delta(s - v_P) - (s\cdot\nabla)s\,, \tag{5.2.2}$$

so that $\widehat{g} \in L^\infty(A) \cap L^2_{\text{loc}}(A)$. In fact, by following step-by-step the proof of Lemma 4.3.5, we have

Lemma 5.2.2. *For any $\varepsilon \in (0, L-\delta)$, define $\widehat{g} = \widehat{g}_\varepsilon \in L^\infty(A) \cap L^2_{loc}(A)$ as in* (5.2.2)*. Then $\widehat{g} \in L^2(A)$ and the following estimate holds:*

$$\|\widehat{g}\|_{L^2(A)} \le \mu\|\Delta s\|_{L^2(A_0)} + \|s\|_{L^4(A_0)}\|\nabla s\|_{L^2(A_0)}.$$

5.2.3 Definition of weak solution

Given $\varepsilon \in (0, L-\delta)$, let $s = s_\varepsilon$ be as in Lemma 5.2.1. Moreover, we define a subset $A_{v,\varepsilon} \subset A_{d,\delta}$ as

$$A_{v,\varepsilon} = \left\{ (h,\theta) \in \mathbb{R}\times\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \;\middle|\; d\cos\theta - \delta|\sin\theta| \le 2 \quad \text{and} \right.$$
$$\left. |h| + d|\sin\theta| + \delta\cos\theta \le L - \varepsilon \right\},$$

so that

$$(h,\theta) \in A_{v,\varepsilon} \qquad \Longrightarrow \qquad B_v \subset [-2,2]\times[-L+\varepsilon, L-\varepsilon]\,.$$

We look for solutions to problem (2.0.9) in the form $u = \widehat{u} + s$ and $p = \widehat{p} + \pi_p$ where $(\widehat{u}, \widehat{p})$ solves the problem:

$$\begin{aligned}
&\widehat{u}_t - \mu\,\Delta\widehat{u} + (\widehat{u}\cdot\nabla)\,\widehat{u} + \nabla\widehat{p} + (\widehat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\widehat{u} = \widehat{g}, \quad \text{div}(\widehat{u}) = 0 \qquad \text{in } \Omega_v \times (0,T)\,,\\
&\lim_{|x_1|\to\infty} \widehat{u}(x_1,x_2,t) = 0 \quad \forall x_2 \in [-L,L]\,,\ t\in[0,T]\,, \qquad \widehat{u} = 0 \quad \text{on } \Gamma_A \times (0,T)\,,\\
&\widehat{u} = h'\,\widehat{e}_2 + \theta'(x - h\widehat{e}_2)^\perp \quad \text{on } \partial B_v \times (0,T).
\end{aligned} \tag{5.2.3}$$

According to $(2.0.9)_4$-$(2.0.9)_5$, the vertical translation and the rotation of the obstacle B respond to

$$\begin{aligned} &m\,h'' + \beta_1\,h' + F_1(h,\theta) = -\widehat{e}_2 \cdot \int_{\partial B_v} \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \quad \text{in } (0,T)\,,\\ &\mathcal{J}\,\theta'' + \beta_2\,\theta' + F_2(h,\theta) = -\int_{\partial B_v} (x - h\widehat{e}_2)^{\perp} \cdot \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \quad \text{in } (0,T), \end{aligned} \tag{5.2.4}$$

with the initial conditions

$$h(0)=0, \quad h'(0)=h_0, \quad \theta(0)=0, \quad \theta'(0)=\theta_0, \quad \widehat{u}_0(x)=u_0(x)-s(x) \quad \text{in} \quad \Omega_0=\Omega_v(0)\,, \tag{5.2.5}$$

for some $h_0,\theta_0 \in \mathbb{R}$. The properties of s given in Lemma 5.2.1 imply that $\widehat{u}_0$ must verify

$$\begin{cases} \nabla \cdot \widehat{u}_0 = 0 \quad \text{in } \Omega_0\,, \\ \lim\limits_{|x_1|\to\infty} \widehat{u}_0(x_1,x_2) = 0 \quad \forall x_2 \in [-L,L]\,, \qquad \widehat{u}_0 = 0 \quad \text{on } \Gamma_A \times (0,T)\,, \\ \widehat{u}_0 = k_0\,\widehat{e}_2 + \theta_0 x^{\perp} \quad \text{on } \partial B_0 \times (0,T). \end{cases}$$

Now, let $(\widehat{u},h,\theta)$ be a smooth solution to problem (5.2.3)-(5.2.4). Taking

$$(\phi,\ell,\alpha) \in \mathcal{C}^1([0,T];\mathcal{V}_v(B_v(t)))$$

such that $\phi(T)=\ell(T)=\alpha(T)=0$, we multiply $(5.2.3)_1$ by ϕ and integrate over space and time to obtain

$$\int_0^T \int_{\Omega_v(t)} [\widehat{u}_t - \mu\,\Delta\widehat{u} + (\widehat{u}\cdot\nabla)\,\widehat{u} + \nabla\widehat{p} + (\widehat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\widehat{u}]\cdot\phi\,dx\,dt = \int_0^T \int_{\Omega_v(t)} \widehat{g}\cdot\phi\,dx\,dt\,. \tag{5.2.6}$$

Since $\phi(t) \in \mathcal{V}(B_v(t))$ for a.e. $t \in [0,T]$, an integration by parts and the boundary conditions in (5.2.4) yield

$$
\begin{aligned}
&\int_0^T \int_{\Omega_v(t)} (-\mu\, \Delta \widehat{u} + \nabla \widehat{p}) \cdot \phi \, dx\, dt = - \int_0^T \int_{\Omega_v(t)} \mathrm{div}(\mathcal{T}(\widehat{u},\widehat{p})) \cdot \phi \, dx\, dt \\
&= \int_0^T \int_{\Omega_v(t)} \mathcal{T}(\widehat{u},\widehat{p}) : \nabla \phi \, dx\, dt - \int_0^T \int_{\partial B_v(t)} \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \cdot \phi \, dx\, dt \\
&= 2\mu \int_0^T \int_{\Omega_v(t)} D(\widehat{u}) : D(\phi)\, dx\, dt - \int_0^T \int_{\partial B_v(t)} \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \cdot \phi \, dx\, dt \\
&= 2\mu \int_0^T \int_{\Omega_v(t)} D(\widehat{u}) : D(\phi)\, dx\, dt - \ell\, \widehat{e}_2 \cdot \int_0^T \int_{\partial B_v(t)} \mathcal{T}(\widehat{u},\widehat{p})\widehat{n}\, d\sigma\, dt \\
&\quad - \alpha \int_0^T \int_{\partial B_v(t)} (x - h\widehat{e}_2)^{\perp} \cdot \mathcal{T}(\widehat{u},\widehat{p})\widehat{n}\, d\sigma\, dt \\
&= 2\mu \int_0^T \int_{\Omega_v(t)} D(\widehat{u}) : D(\phi)\, dx\, dt + \int_0^T \ell(m\, h'' + \beta_1\, h' + F_1(h,\theta))\, dt \\
&\quad + \int_0^T \alpha(\mathcal{J}\, \theta'' + \beta_2\, \theta' + F_2(h,\theta))\, dt \\
&= 2\mu \int_0^T \int_{\Omega_v(t)} D(\widehat{u}) : D(\phi)\, dx\, dt + \int_0^T (-m\, h'\, \ell' + \beta_1\, h'\, \ell + F_1(h,\theta)\, \ell)\, dt - m\, h_0\, \ell(0) \\
&\quad + \int_0^T (-\mathcal{J}\, \theta'\, \alpha' + \beta_2\, \theta'\, \alpha + F_2(h,\theta)\, \alpha)\, dt - J\, \theta_0\, \alpha(0)\,.
\end{aligned}
\tag{5.2.7}
$$

In a similar way,

$$
\int_0^T \int_{\Omega_v(t)} \widehat{u}_t \cdot \phi \, dx\, dt = - \int_0^T \int_{\Omega_v(t)} \widehat{u} \cdot \phi_t \, dx\, dt - \widehat{u}_0 \cdot \phi(0)\,. \tag{5.2.8}
$$

By inserting (5.2.7)-(5.2.8) into (5.2.6) we deduce

$$
\begin{aligned}
& -\int_0^T \left(\int_{\Omega_v(t)} \widehat{u}\cdot\phi_t\,dx + m\,h'\,\ell' - \beta_1\,h'\,\ell - F_1(h,\theta)\,\ell + \mathcal{J}\,\theta'\,\alpha' - \beta_2\,\theta'\,\alpha - F_2(h,\theta)\,\alpha \right) dt \\
& + 2\mu \int_0^T \int_{\Omega_v(t)} D(\widehat{u}) : D(\phi)\,dx\,dt + \int_0^T \int_{\Omega_v(t)} [(\widehat{u}\cdot\nabla)\,\widehat{u}\cdot\phi + (\widehat{u}\cdot\nabla)\,s\cdot\phi \\
& + (s\cdot\nabla)\,\widehat{u}\cdot\phi]\,dx\,dt \\
= {} & \widehat{u}_0\cdot\phi(0) + m\,h_0\,\ell(0) + J\,\theta_0\,\alpha(0) + \int_0^T \int_{\Omega_v(t)} \widehat{g}\cdot\phi\,dx\,dt
\end{aligned}
\tag{5.2.9}
$$

We are now ready to give a definition of weak solution to (5.2.3)-(5.2.4)-(5.2.5), which is equivalent to the original problem (2.0.9).

Definition 5.2.3. Let $T > 0$. Given $(\widehat{u}_0, h_0, \theta_0) \in \mathcal{H}_v(B_v(0))$ and $s = s_\varepsilon$ as in Lemma 5.2.1 depending on some $\varepsilon \in (0, L-\delta)$, we say that a triplet $(\widehat{u}, h, \theta)$ is a weak solution to problem (5.2.3)-(5.2.4)-(5.2.5) if

$$
\begin{aligned}
& (h,\theta) \in W^{1,\infty}(0,T;\mathbb{R}^2) \cap \mathcal{C}([0,T]; A_{v,\varepsilon_0})\,, \\
& (\widehat{u}, h', \theta') \in L^2(0,T;\mathcal{V}_v(B_v(t))) \cap L^\infty(0,T;\mathcal{H}_v(B_v(t)))\,, \\
& (\widehat{u}, h, \theta) \text{ satisfies (5.2.9) for every } (\phi,\ell,\alpha) \in \mathcal{C}^1([0,T];\mathcal{V}_v(B_v(t))) \\
& \text{such that } \phi(\cdot,T) = \ell(T) = \alpha(T) = 0\,.
\end{aligned}
$$

5.3 The penalized problem

In order to solve problem (5.2.3)-(5.2.4) we will adopt a *penalization* technique, in the sense that we will assume that the obstacle is free to move in all directions in the plane, by penalizing its motion in the horizontal direction so that in the limit we will recover a purely vertical motion. To this purpose, we introduce a set of admissible values for the three coordinates of the full motion of B in the plane, that is

$$
\begin{aligned}
A_{f,\varepsilon} = & \Big\{ (h,k,\theta) \in \mathbb{R}\times\mathbb{R}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \;\Big|\; |k| + d\cos\theta - \delta|\sin\theta| \le 2 \quad \text{and} \\
& |h| + d|\sin\theta| + \delta\cos\theta \le L - \varepsilon \Big\}\,,
\end{aligned}
$$

so that

$$
(k,h,\theta) \in A_{f,\varepsilon} \qquad \Longrightarrow \qquad B_f \subset [-2,2]\times[-L+\varepsilon, L-\varepsilon]\,.
$$

Let $n \geq 1$ be a fixed integer. We introduce the following penalized problem:

$$\begin{aligned}
&\widehat{u}_t - \mu\,\Delta\widehat{u} + (\widehat{u}\cdot\nabla)\,\widehat{u} + \nabla\widehat{p} + (\widehat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\widehat{u} = \widehat{g}, \quad \mathrm{div}(\widehat{u}) = 0 \qquad \text{in } \Omega_f \times (0,T)\,,\\
&\lim_{|x_1|\to\infty} \widehat{u}(x_1,x_2,t) = 0 \quad \forall x_2 \in [-L,L]\,,\ t \in [0,T]\,, \qquad \widehat{u} = 0 \quad \text{on } \Gamma_A \times (0,T)\,,\\
&\widehat{u} = (k',h') + \theta'(x_1-k, x_2-h)^{\perp} \quad \text{on } \partial B_f \times (0,T).
\end{aligned} \tag{5.3.1}$$

In this case, the motion of the obstacle is governed by

$$\begin{aligned}
&m\,k'' + \beta_1\,k' + n\,k = -\widehat{e}_1 \cdot \int_{\partial B_f} \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \quad \text{in } (0,T)\,,\\
&m\,h'' + \beta_1\,h' + F_1(h,\theta) = -\widehat{e}_2 \cdot \int_{\partial B_f} \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \quad \text{in } (0,T)\,,\\
&\mathcal{J}\,\theta'' + \beta_2\,\theta' + F_2(h,\theta) = -\int_{\partial B_f} (x_1-k, x_2-h)^{\perp} \cdot \mathcal{T}(\widehat{u},\widehat{p})\widehat{n} \quad \text{in } (0,T),
\end{aligned} \tag{5.3.2}$$

with the initial conditions

$$\begin{aligned}
&k(0) = 0, \quad k'(0) = k_0, \quad h(0) = 0, \quad h'(0) = h_0, \quad \theta(0) = 0, \quad \theta'(0) = \theta_0,\\
&\widehat{u}_0(x) = u_0(x) - s(x) \quad \text{in} \quad \Omega_0\,,
\end{aligned} \tag{5.3.3}$$

for some $k_0, h_0, \theta_0 \in \mathbb{R}$. The properties of s given in Lemma 5.2.1 imply that, in this case, $\widehat{u}_0$ must verify

$$\begin{cases}
\nabla\cdot\widehat{u}_0 = 0 \quad \text{in } \Omega_0\,,\\
\lim_{|x_1|\to\infty} \widehat{u}_0(x_1,x_2) = 0 \quad \forall x_2 \in [-L,L]\,, \qquad \widehat{u}_0 = 0 \quad \text{on } \Gamma_A \times (0,T)\,,\\
\widehat{u}_0 = (k_0,h_0) + \theta_0(x_1-k_0, x_2-h_0)^{\perp} \quad \text{on } \partial B_0 \times (0,T).
\end{cases}$$

Since we will deal with both weak and strong solutions of the penalized problem (5.3.1)-(5.3.2)-(5.3.3), such definitions are given below.

Definition 5.3.1. Let $T > 0$. Given $(\widehat{u}_0, (k_0,h_0), \theta_0) \in \mathcal{H}(B_f(0))$ and $s = s_\varepsilon$ as in Lemma 5.2.1 depending on some $\varepsilon \in (0, L-\delta)$, we say that a quadruplet $(\widehat{u}, k, h, \theta)$ is a **weak solution** of (5.3.1)-(5.3.2)-(5.3.3) if

$$\begin{aligned}
&((k,h),\theta) \in W^{1,\infty}(0,T;\mathbb{R}^2\times\mathbb{R}) \cap \mathcal{C}([0,T];A_{f,\varepsilon_0})\,,\\
&(\widehat{u},(k',h'),\theta') \in L^2(0,T;\mathcal{V}(B_f(t))) \cap L^\infty(0,T;\mathcal{H}(B_f(t)))\,,
\end{aligned}$$

and also $(\widehat{u}, k, h, \theta)$ satisfies the identity

$$
\begin{aligned}
&-\int_0^T \bigg(\int_{\Omega_f(t)} \widehat{u} \cdot \phi_t \, dx + m\, k'\, \ell_1' - \beta_1\, k'\, \ell_1 - n\, k\, \ell_1 + m\, h'\, \ell_2' - \beta_1\, h'\, \ell_2 - F_1(h,\theta)\, \ell_2 \\
&+ \mathcal{J}\, \theta'\, \alpha' - \beta_2\, \theta'\, \alpha - F_2(h,\theta)\, \alpha \bigg) dt + 2\mu \int_0^T \int_{\Omega_f(t)} D(\widehat{u}) : D(\phi)\, dx\, dt \\
&\quad + \int_0^T \int_{\Omega_f(t)} [(\widehat{u} \cdot \nabla)\, \widehat{u} \cdot \phi + (\widehat{u} \cdot \nabla)\, s \cdot \phi + (s \cdot \nabla)\, \widehat{u} \cdot \phi]\, dx\, dt \\
&= \widehat{u}_0 \cdot \phi(0) + m\, k_0\, \ell_1(0) + m\, h_0\, \ell_2(0) + J\, \theta_0\, \alpha(0) + \int_0^T \int_{\Omega_f(t)} \widehat{g} \cdot \phi \, dx\, dt
\end{aligned}
$$

for every $(\phi, (\ell_1, \ell_2), \alpha) \in \mathcal{C}^1([0,T]; \mathcal{V}(B_f(t)))$ such that $\phi(T) = \ell_1(T) = \ell_2(T) = \alpha(T) = 0$.

Definition 5.3.2. Let $T > 0$. Given $(\widehat{u}_0, (k_0, h_0), \theta_0) \in \mathcal{V}(B_f(0))$ and $s = s_\varepsilon$ as in Lemma 5.2.1 depending on some $\varepsilon \in (0, L - \delta)$, a quintuplet $(\widehat{u}, p, k, h, \theta)$ such that

$$
\begin{aligned}
&\widehat{u} \in L^2(0,T; H^2(\Omega_f(t))) \cap H^1(0,T; L^2(\Omega_f(t))) \cap \mathcal{C}([0,T]; H_0^1(\Omega_f(t))) \\
&\widehat{p} \in L^2(0,T; \widehat{H}^1(\Omega_f(t))), \quad (k,h) \in H^2(0,T;\mathbb{R}^2), \quad \theta \in H^2(0,T;\mathbb{R}),
\end{aligned}
$$

and $(k(t), h(t), \theta(t)) \in A_{f,\varepsilon_0}$ for every $t \in [0,T]$, is called a **strong solution** of (5.3.1)-(5.3.2)-(5.3.3) if $(5.3.1)_1$ is satisfied almost everywhere in $\Omega_f(t) \times (0,T)$, $(5.3.1)_3$ is satisfied in the trace sense and (5.3.2) is satisfied almost everywhere in $(0,T)$. Moreover, the initial conditions (5.3.3) are attained for every $t \in [0,T]$.

5.3.1 The transformed equations

We now transform problem (5.3.1)-(5.3.2)-(5.3.3). We preliminary state a proposition allowing to define a change of variables associated to the rigid motion of the obstacle in problem (5.3.1)-(5.3.2)-(5.3.3) in order to be able to set the problem on a fixed domain and eventually also to compare different solutions; indeed since (5.3.1)-(5.3.2)-(5.3.3) is set on a time-dependent fluid domain, different solutions are defined on different domains. This change of variables depends on time t through k, h and θ; it was first introduced by Takahashi [130, Section 4.1], inspired by Inoue-Wakimoto [93]. For every $\varepsilon \in (0, L/2)$ we write

$$
\mathcal{O}_\varepsilon = \left(-\frac{3}{2}, \frac{3}{2}\right) \times (-L + 2\varepsilon, L - 2\varepsilon) \quad \text{and} \quad \mathcal{A}_\varepsilon = A \setminus ([-2,2] \times [-L+\varepsilon, L-\varepsilon]) .
$$

Note that, on one hand,

$$
\mathcal{O}_\varepsilon \subset \{x \in A \,|\, \mathrm{dist}(x, \partial A) > 2\varepsilon\} \ .
$$

On the other hand, if s is the solenoidal extension of Lemma 5.2.1 with such ε, we have that $\mathrm{supp}(s) \subset \mathcal{A}_\varepsilon$, so that

$$
s \equiv 0 \qquad \text{on} \qquad A \setminus \mathcal{A}_\varepsilon. \tag{5.3.4}
$$

In [130, Section 4.1], the following result is proved:

Proposition 5.3.3. *Consider a fixed pair $(k,h) \in H^2(0,T;\mathbb{R}^2)$ and a fixed function $\theta \in H^2(0,T;\mathbb{R})$ such that $k(0) = h(0) = \theta(0) = 0$. For every $t \in [0,T]$ there exist two volume-preserving diffeomorphisms*

$$\psi(t,\cdot) : \Omega_f(t) \longrightarrow \Omega_0 \quad \text{and} \quad \varphi(t,\cdot) : \Omega_0 \longrightarrow \Omega_f(t) \tag{5.3.5}$$

satisfying, for all $\varepsilon \in (0, L/2)$ and $t \in [0,T]$, the following properties:

$$\psi(t,x) = \begin{cases} Q(\theta(t))^\top (x_1 - k(t), x_2 - h(t)) & \text{if } x \in \mathcal{O}_\varepsilon \\ x & \text{if } x \in \mathcal{A}_\varepsilon\,, \end{cases}$$

and

$$\varphi(t,y) = \begin{cases} Q(\theta(t))y + (k(t), h(t)) & \text{if } y \in \mathcal{O}_\varepsilon \\ y & \text{if } y \in \mathcal{A}_\varepsilon\,. \end{cases}$$

More precisely, we have that $\psi \in \mathcal{C}^1(0,T;\mathcal{C}^\infty(\Omega_f(t)))$ and $\varphi \in \mathcal{C}^1(0,T;\mathcal{C}^\infty(\Omega_0))$. In particular, for some constant $C > 0$ that depends on ℓ, L and ε, there hold

$$\|\partial_t^j \psi(t,\cdot)\|_{\mathcal{C}^\ell(\Omega_f(t))} \leq C\left(|k^{(j)}(t)| + |h^{(j)}(t)| + |\theta^{(j)}(t)|\right) \qquad \forall j \in \{0,1\},\ \ell \in \mathbb{N}\,.$$

and

$$\|\partial_t^j \varphi(t,\cdot)\|_{\mathcal{C}^\ell(\Omega_0)} \leq C\left(|k^{(j)}(t)| + |h^{(j)}(t)| + |\theta^{(j)}(t)|\right) \qquad \forall j \in \{0,1\},\ \ell \in \mathbb{N}\,.$$

Proof. We start by noticing that $(k,h) \in \mathcal{C}^1([0,T];\mathbb{R}^2)$ and $\theta \in \mathcal{C}^1([0,T];\mathbb{R})$ due to a classical Sobolev embedding theorem. Let $\zeta \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$ be a smooth cutoff function equal to 0 in $\mathcal{A}_\varepsilon$ and equal to 1 in $\mathcal{O}_\varepsilon$. Define the rigid motion associated to (k,h) and θ as

$$V(x,t) = k'(t)\widehat{e}_1 + h'(t)\widehat{e}_2 + \theta'(t)(x - h(t)\,\widehat{e}_2)^\perp = V_1(x_2,t)\widehat{e}_1 + V_2(x_1,t)\widehat{e}_2 \quad \forall (x,t) \in \mathbb{R}^2 \times [0,T]\,,$$

where

$$V_1(x_2,t) = k'(t) - (x_2 - h(t))\theta'(t) \quad \text{and} \quad V_2(x_1,t) = x_1\theta'(t) + h'(t) \qquad \forall (x,t) \in \mathbb{R}^2 \times [0,T]\,.$$

Notice that $V(\cdot,t) \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ for every $t \in [0,T]$ and $V(x,\cdot) \in H^1(0,T;\mathbb{R}^2)$ for every $x \in \mathbb{R}^2$. We introduce now the stream function associated to the rigid velocity field V as

$$w(x,t) = -\int_0^{x_1} V_2(s,t)\,ds + \int_0^{x_2} V_1(s,t)\,ds = k'(t)x_2 - h'(t)x_1 - \frac{\theta'(t)}{2}(x_1^2 + x_2^2 - 2x_2 h(t)),$$

for all $(x,t) \in \mathbb{R}^2 \times [0,T]$. We then define the solenoidal vector field $\Lambda : \mathbb{R}^2 \times [0,T] \longrightarrow \mathbb{R}^2$ as

$$\Lambda(x,t) = \left(-w(x,t)\frac{\partial \zeta}{\partial x_2}(x) + \zeta(x)V_1(x_2,t),\, w(x,t)\frac{\partial \zeta}{\partial x_1}(x) + \zeta(x)V_2(x_1,t)\right) \forall (x,t) \in \mathbb{R}^2 \times [0,T]\,,$$

so that $\Lambda \in \mathcal{C}(\mathbb{R}^2 \times [0,T];\mathbb{R}^2)$, $\Lambda(\cdot,t) \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ for every $t \in [0,T]$, $\Lambda(x,\cdot) \in H^1(0,T;\mathbb{R}^2)$ for every $x \in \mathbb{R}^2$ and

$$\Lambda(x,t) = \begin{cases} 0 & \text{if } x \in \mathcal{A}_\varepsilon, \\ V(x,t) & \text{if } x \in \mathcal{O}_\varepsilon \end{cases} \qquad \forall t \in [0,T].$$

Then, the deformation map $\varphi : \Omega_0 \times [0,T] \longrightarrow \Omega_f(t)$ from Ω_0 into $\Omega_f(t)$ is defined as the flow associated to the vector field Λ, that is, the unique solution to the initial-value problem:

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(y,t) = \Lambda(\varphi(y,t),t) \qquad \forall (y,t) \in \Omega_0 \times (0,T)\,, \\ \varphi(y,0) = y \qquad \forall y \in \Omega_0\,, \end{cases}$$

see [130, Lemma 4.2] for further details.

We have that $\varphi(x,\cdot) \in \mathcal{C}^1([0,T];A)$ for every $x \in \Omega_0$ and that, for every $t \in [0,T]$, $\varphi(\cdot,t) \in \mathcal{C}^\infty(\Omega_0;\Omega_f(t))$ is a diffeomorphism. In particular, for some constant $C > 0$ that depends on L and ε, there holds

$$\|\partial_t^j \varphi(t,\cdot)\|_{\mathcal{C}^\ell(\bar{\Omega}_0)} \leq C\left(|k^{(j)}(t)| + |h^{(j)}(t)| + |\theta^{(j)}(t)|\right) \qquad \forall j \in \{0,1\},\ \ell \in \mathbb{N}\,.$$

Furthermore, since Λ is divergence-free in $\mathbb{R}^2$, φ is volume-preserving, meaning that the determinant of the Jacobian matrix of $\varphi(\cdot,t)$ is constant and equal to 1 in $\mathbb{R}^2$, for every $t \in [0,T]$. Finally, we define $\psi : \mathbb{R}^+ \times \Omega_{h(t),\theta(t)} \longrightarrow \Omega_0$ by $\psi = \varphi^{-1}$ in the space variables. Similarly, there holds

$$\|\partial_t^j \psi(t,\cdot)\|_{\mathcal{C}^\ell(\bar{\Omega}_{h(t),\theta(t)})} \leq C\left(|k^{(j)}(t)| + |h^{(j)}(t)| + |\theta^{(j)}(t)|\right) \qquad \forall j \in \{0,1\},\ \ell \in \mathbb{N}\,.$$

for some constant $C > 0$ that depends on L, ε and ℓ. Obviously, ψ has the same regularity properties of φ, so in particular, $\psi \in \mathcal{C}^1(0,T;\mathcal{C}^k(\Omega_f(t)))$ for any $k \in \mathbb{N}$. ☐

Through Proposition 5.3.3, we can construct (we use the Einstein convention)

$$g_{ij} = \frac{\partial \varphi^k}{\partial y^i}\frac{\partial \varphi^k}{\partial y^j}, \qquad g^{ij} = \frac{\partial \psi^i}{\partial x^k}\frac{\partial \psi^j}{\partial x^k}, \qquad \Gamma_{A}{}^{i}_{kj} = g^{ir}\left(\frac{\partial g_{kr}}{\partial y^j} + \frac{\partial g_{jr}}{\partial y^k} - \frac{\partial g_{kj}}{\partial y^r}\right) = \frac{\partial \psi^i}{\partial x^r}\frac{\partial^2 \varphi^r}{\partial y^k \partial y^i},$$

where g_{ij} defines a metric on $\mathbb{R}^2$ since det$(\nabla_y \varphi) \equiv 1$.

Now, let $(\widehat{u}, p, k, h, \theta)$ a sufficiently smooth solution of problem (5.3.1)-(5.3.2)-(5.3.3). Call (now the space variable is y)

$$v(y,t) = \nabla\psi(\varphi(y,t),t)\,\widehat{u}(\varphi(y,t),t) \qquad \forall\,(y,t) \in \Omega_0 \times [0,T]\,, \tag{5.3.6}$$

the *pullback* of $\widehat{u}$ by φ, and set

$$q(y,t) = \widehat{p}(\varphi(y,t),t) \qquad \forall\,(y,t) \in \Omega_0 \times [0,T]\,. \tag{5.3.7}$$

We follow the procedure in [130, Paragraph 4.2] to transform problem (5.3.1)-(5.3.2)-(5.3.3) in the cylindrical domain $\Omega_0 \times (0,T)$. Thanks to (5.3.4), for each term involving s, the maps ψ and φ correspond to the identity. Furthermore, as in [130, Proposition 4.6], it can be proved that

$$\int_{\partial B_f(t)} \mathcal{T}(u,p)\widehat{n}\, d\sigma(x) = Q(t)\left(\int_{\partial B_0} \mathcal{T}(v,q)\widehat{n}\, d\sigma(y)\right) \qquad \forall t \in [0,T]\,,$$

$$\int_{\partial B_f(t)} (x_1 - k, x_2 - h)^\perp \cdot \mathcal{T}(u,p)\widehat{n}\, d\sigma(x) = \int_{\partial B_0} y^\perp \cdot \mathcal{T}(v,q)\widehat{n}\, d\sigma(y) \qquad \forall t \in [0,T]\,.$$

Thus, we obtain the following problem with variable coefficients in the new unknown (v, q, k, h, θ):

$$
\begin{aligned}
& v_t + \mathcal{M}v - \mu\,\mathcal{L}v + \mathcal{N}v + (v\cdot\nabla)\,s + (s\cdot\nabla)\,v + \mathcal{G}q = \widehat{g}, \quad \nabla\cdot v = 0 \qquad \text{in } \Omega_0\times(0,T)\,,\\
& \lim_{|y_1|\to\infty} v(y_1,y_2,t) = 0 \quad \forall y_2\in[-L,L]\,,\ t\in[0,T]\,, \qquad v=0 \quad \text{on } \Gamma_A\times(0,T)\,,\\
& v = Q(\theta)^\top(k',h') + \theta'\,y^\perp \quad \text{on } \partial B_0\times(0,T)\,,\\
& m\,k'' + \beta_1\,k' + n\,k = -\,\widehat{e}_1\cdot Q(\theta)\int_{\partial B_0}\mathcal{T}(v,q)\widehat{n}\,d\sigma(y) \ \text{ in } (0,T)\\
& m\,h'' + \beta_1\,h' + F_1(h,\theta) = -\,\widehat{e}_2\cdot Q(\theta)\int_{\partial B_0}\mathcal{T}(v,q)\,d\sigma(y)\widehat{n} \ \text{ in } (0,T)\\
& \mathcal{J}\,\theta'' + \beta_2\,\theta' + F_2(h,\theta) = \widehat{e}_3\cdot\int_{\partial B_0} y^\perp\cdot(\mathcal{T}(v,q)\widehat{n})\,d\sigma(y) \ \text{ in } (0,T)\,.
\end{aligned}
\tag{5.3.8}
$$

with initial conditions

$$
\begin{aligned}
& v(y,0)=\widehat{u}_0, \qquad k(0)=0, \qquad k'(0)=k_0, \qquad h(0)=0, \qquad h'(0)=h_0,\\
& \theta(0)=0, \qquad \theta'(0)=\theta_0.
\end{aligned}
\tag{5.3.9}
$$

The operators $\mathcal{M}, \mathcal{L}, \mathcal{N}$ appearing in (5.3.8) are defined here below (the exponent i stands for the i-th component, and we use the Einstein notation).

$$
\begin{aligned}
(\mathcal{M}v)^i &= \partial_r v^i\partial_t\psi^r + \partial_k\psi^i(\partial_k\partial_t\varphi^i)v^k + \partial_k\psi^i\partial^2_{kr}\varphi^i\partial_t\psi^r v^k,\\
(\mathcal{L}v)^i &= \partial_k\psi^i\partial_j\psi^m(\partial^2_{mk}\varphi^i)\partial_r v^k\partial_j\psi^r + \partial_j\psi^m\partial^2_{mr}v^i\partial_j\psi^r + \partial_r v^i(\partial^2_{jj}\psi^r)\\
&\quad + \partial_k\psi^i\partial_j\psi^m(\partial^3_{mrk}\varphi^i)\partial_j\psi^r v^k + \partial_k\psi^i(\partial^2_{rk}\varphi^i)\partial^2_{jj}\psi^r v^k + \partial_k\psi^i(\partial^2_{rk}\varphi^i)\partial_j\psi^r\partial_j\psi^m\partial_m v^k,\\
(\mathcal{N}v)^i &= v^r\partial_r v^i + \partial_k\psi^i v^r(\partial^2_{rk}\varphi^i)v^k,\\
(\mathcal{G}q)^i &= g^{ij}\partial_j q.
\end{aligned}
$$

Remark 5.3.4. Note that:
- $(\partial_t + \mathcal{M})v$ corresponds to the original time derivative $\partial_t\widehat{u}$;
- $\mathcal{L}v$ corresponds to $\Delta\widehat{u}$;
- $\mathcal{N}v$ corresponds to $(\widehat{u}\cdot\nabla)\widehat{u}$;
- $\mathcal{G}q$ corresponds to ∇p.

In particular, in $\mathcal{A}_\varepsilon$ these operators coincide with the original ones; the same is true in $\mathcal{O}_\varepsilon$, except for

$$(\partial_t + \mathcal{M})v = (\partial_t - (k',h')\cdot\nabla - \theta' y^\perp\cdot\nabla)v + \theta' v^\perp\,.$$

The first equation in (5.3.8) can be rewritten as

$$v_t - \mu\,\Delta v + (v\cdot\nabla)\,v + \nabla q + (v\cdot\nabla)\,s + (s\cdot\nabla)\,v = \widehat{g} + \mathcal{F}(v,k,h,\theta,q),$$

where
$$\mathcal{F}(v,k,h,\theta,q) = \mu(\mathcal{L}-\Delta)v - \mathcal{M}v - (\mathcal{N}v - (v\cdot\nabla)v) - (\mathcal{G}-\nabla)q.$$
Observe that
$$\mathcal{F}(v,h,\theta,q) = \begin{cases} 0 & \text{in} \quad \mathcal{A}_\varepsilon \\ ((k',h')\cdot\nabla + \theta' y^\perp\cdot\nabla)v - \theta' v^\perp & \text{in} \quad \bar{\mathcal{O}}_\varepsilon, \end{cases}$$

thus $\mathcal{F}$ has compact support in Ω_0. The introduction of the maps ψ and φ allows to remove the dependence on time from the fluid domain, with a consequent strengthening of the coupling between the equations governing the motion of the fluid and the one governing the motion of the obstacle. Such a strengthening appears in the fictitious force $\mathcal{F} = \mathcal{F}(v,k,h,\theta,q)$, where the dependence on h and θ is hidden in ψ and φ.

The following proposition, which can be proven as Proposition 4.5 and 4.6 in [130], guarantees that the system of equations (5.3.8)-(5.3.9) is equivalent to the original penalized problem (5.3.1)-(5.3.2)-(5.3.3).

Proposition 5.3.5. *A quintuplet (u,p,k,h,θ) is a strong solution to* (5.3.1)-(5.3.2)-(5.3.3) *in $[0,T]$ in the sense of Definition 5.3.2 if and only if the quintuplet (v,q,k,h,θ), where v and q are defined by* (5.3.6)-(5.3.7)*, satisfies*
$$\begin{aligned} &v \in L^2(0,T;H^2(\Omega_0)) \cap H^1(0,T;L^2(\Omega_0)) \cap \mathcal{C}([0,T];H^1(\Omega_0)) \\ &q \in L^2(0,T;\widehat{H}^1(\Omega_0)), \quad (k,h) \in H^2(0,T;\mathbb{R}^2), \quad \theta \in H^2(0,T;\mathbb{R}), \end{aligned}$$

together with (5.3.8)-(5.3.9) *in an almost-everywhere sense both in space and in time.*

5.3.2 The linearized problem

The operators in (5.3.8) depend on the solution v through the map $\varphi(t,\cdot)$, thus they are highly nonlinear. However, as they correspond to the identity when $t=0$, for small times we can think about them as small perturbations, collecting them in a source term. The same can be said about the restoring forces F_1 and F_2.

Notice that the equations $(5.3.8)_4$-$(5.3.8)_5$ can be jointly written in vector form as
$$m\,\eta'' + \beta_1\,\eta' + R_n(\eta,\theta) = -Q(\theta)\int_{\partial B_0} \mathcal{T}(v,q)\widehat{n}\,d\sigma(y) \ \text{ in } \ (0,T)\,,$$

where $\eta = (k,h)$ and $R_n(\eta,\theta) = (n\,k, F_1(h,\theta))$ is the restoring forces vector. Let the function $\Lambda : [0,T] \longrightarrow \mathbb{R}^2$ be
$$\Lambda(t) = \int_0^t Q(\theta(\tau))^\top \eta'(\tau)\,d\tau \qquad \forall\, t\in[0,T]\,,$$

so that
$$\Lambda'(t) = Q(\theta(t))^\top\eta'(t) \quad \text{and} \quad \Lambda''(t) = Q(\theta(t))^\top\eta''(t) + \theta'(t)\Lambda'(t)^\perp \qquad \forall\, t\in[0,T]\,,$$

and also

$$\eta(t) = (k(t), h(t)) = \int_0^t Q(\theta(\tau))\Lambda'(\tau)\, d\tau \qquad \forall\, t \in [0,T]\,.$$

Then, problem (5.3.8) may be re-written as

$$\begin{aligned}
& v_t - \mu\, \Delta v + \nabla q = f, \qquad \text{div}(v) = 0 \quad \text{in } \Omega_0 \times (0,T)\,, \\
& \lim_{|y_1| \longrightarrow \infty} v(y_1, y_2) = 0 \quad \forall y_2 \in [-L, L]\,, \qquad v = 0 \quad \text{on } \Gamma_A \times (0,T)\,, \\
& v = \Lambda' + \theta' y^{\perp} \quad \text{on } \partial B_0 \times (0,T)\,, \\
& m\, \Lambda'' + \beta_1\, \Lambda' = - \int_{\partial B_0} \mathcal{T}(v,q)\widehat{n}\, d\sigma(y) + f_m \quad \text{in } (0,T), \\
& \mathcal{J}\, \theta'' + \beta_2\, \theta' = - \int_{\partial B_0} y^{\perp} \cdot \mathcal{T}(v,q)\widehat{n}\, d\sigma(y) + f_{\mathcal{J}} \quad \text{in } (0,T).
\end{aligned} \tag{5.3.10}$$

with initial conditions

$$v(y,0) = \widehat{u}_0\,, \qquad \Lambda(0) = 0\,, \qquad \Lambda'(0) = (k_0, h_0)\,, \qquad \theta(0) = 0\,, \qquad \theta'(0) = \theta_0\,,$$

where

$$\begin{cases}
f = \widehat{g} + \mu(\mathcal{L} - \Delta)v - \mathcal{M}v - \mathcal{N}v - (\mathcal{G} - \nabla)q - (v \cdot \nabla)\, s - (s \cdot \nabla)\, v, \\
f_m = -Q^{\top}(\theta)\, R_n(\Lambda', \theta) + m\, \theta' \Lambda'^{\perp}, \\
f_{\mathcal{J}} = -F_2(\Lambda', \theta).
\end{cases}$$

Problem (5.3.10) can be linearized by thinking of f, f_m and f_θ as prescribed given forces. In particular, suppose that

$$f \in L^2(0,T; L^2(\Omega_0))\,, \qquad f_m \in L^2(0,T; \mathbb{R}^2), \quad \text{and} \quad f_{\mathcal{J}} \in L^2(0,T; \mathbb{R}).$$

From Section 5.2.1, we recall the spaces

$$\begin{aligned}
\mathcal{H}(B_0) =& \{(v, \ell, \alpha) \in L^2(A) \times \mathbb{R}^2 \times \mathbb{R} \mid \text{div}(v) = 0 \text{ in } A, \quad v \cdot \widehat{n} = 0 \text{ on } \partial A, \\
& v = \ell + \alpha\, y^{\perp} \text{ in } B_0\} \\
\mathcal{V}(B_0) =& \{(v, \ell, \alpha) \in H_0^1(A) \times \mathbb{R}^2 \times \mathbb{R} \mid \text{div}(v) = 0 \text{ in } A, \quad v = \ell + \alpha\, y^{\perp} \text{ in } B_0\}\,.
\end{aligned}$$

As in [33, 130, 131], the linearized problem is solved by exploiting a semi-group approach. For this, let us define the set

$$D(S) = \{z \in \mathcal{V}(B_0) \mid v|_{\Omega_0} \in H^2(\Omega_0)\}\,, \tag{5.3.11}$$

which is a dense linear subspace of $\mathcal{V}$, and the (unbounded) linear operator $\mathcal{S} : D(S) \longrightarrow L^2(A)$ by

$$\mathcal{S}(z) = \begin{cases}
-\mu\, \Delta v \quad \text{in } \Omega_0 \\
\dfrac{1}{m}\left(2\mu \displaystyle\int_{\partial B_0} D(v)\widehat{n} d\sigma(y) + \beta_1\, \ell\right) + \dfrac{1}{\mathcal{J}}\left(2\mu \displaystyle\int_{\partial B_0} y^{\perp} \cdot D(v)\widehat{n} d\sigma(y) + \beta_2 \alpha\right) y^{\perp} \text{ in } B_0\,,
\end{cases}$$

for any $z = (v, \ell, \alpha) \in D(S)$. Then, if $\mathbb{P}$ denotes the orthogonal projection from $L^2(A)\times\mathbb{R}^2\times\mathbb{R}$ onto $\mathcal{H}(B_0)$, we define

$$S(z) = \mathbb{P}(\mathcal{S}(z)) \qquad \forall z \in D(S)\,,$$

and prove the following result:

Proposition 5.3.6. *The unbounded linear operator $S : D(S) \longrightarrow \mathcal{H}(B_0)$ is self-adjoint and positive. Furthermore, there exists a constant $C > 0$ such that*

$$\|u\|_{H^2(\Omega_0)} \leq C\|(I + S)u\|_{\mathcal{H}(B_0)} \qquad \forall z = (u, \ell, \alpha) \in D(S)\,. \tag{5.3.12}$$

Proof. We start by proving that S is symmetric and positive. Take $z_1 = (v_1, \ell_1, \alpha_1)$, $z_2 = (v_2, \ell_2, \alpha_2) \in D(S)$, so that

$$\begin{aligned}\langle S(z_1), z_2\rangle_{\mathcal{H}(B_0)} &= \langle S(z_1), z_2\rangle_{L^2(A)} = \langle \mathcal{S}(z_1), z_2\rangle_{L^2(A)}\\ &= \int_{\Omega_0} -\mu\,\Delta v_1\cdot v_2 + 2\,\mu\bigg(\int_{\partial B_0} D(v_1)\,\widehat{n}\,d\sigma(y)\bigg)\cdot \ell_2 + \beta_1\,\ell_1\cdot\ell_2\\ &\quad + 2\,\mu\bigg(\int_{\partial B_0} y^{\perp}\cdot D(v_1)\,\widehat{n}\,d\sigma(y)\bigg)\alpha_2 + \beta_2\,\alpha_1\alpha_2.\end{aligned}$$

Notice that

$$\Delta v_1\cdot v_2 = 2\,\mathrm{div}(D(v_1))\cdot v_2 = 2\,\mathrm{div}(D(v_1)v_2) - 2\,D(v_1) : D(v_2),$$

and thus we obtain

$$\begin{aligned}\langle S(z_1), z_2\rangle_{\mathcal{H}(B_0)} =&- 2\mu\int_{\Omega_0}\mathrm{div}(D(v_1)v_2)\,dy + 2\mu\int_{\Omega_0} D(v_1) : D(v_2)\,dy\\ &+ 2\,\mu\bigg(\int_{\partial B_0} D(v_1)\widehat{n}\,d\sigma(y)\bigg)\cdot\ell_2 + \beta_1\,\ell_1\cdot\ell_2\\ &+ 2\,\mu\bigg(\int_{\partial B_0} y^{\perp}\cdot D(v_1)\widehat{n}\,d\sigma(y)\bigg)\alpha_2 + \beta_2\,\alpha_1\alpha_2,\end{aligned}$$

which, after integration by parts of the first term, yields

$$\begin{aligned}\langle S(z_1), z_2\rangle_{\mathcal{H}(B_0)} =&- 2\mu\int_{\partial B_0}(D(v_1)v_2)\cdot\widehat{n}\,dy + 2\mu\int_{\Omega_0} D(v_1) : D(v_2)\,dy\\ &+ 2\,\mu\bigg(\int_{\partial B_0} D(v_1)\widehat{n}\,d\sigma(y)\bigg)\cdot\ell_2\\ &+ \beta_1\,\ell_1\cdot\ell_2 + 2\,\mu\bigg(\int_{\partial B_0} y^{\perp}\cdot D(v_1)\widehat{n}\,d\sigma(y)\bigg)\alpha_2 + \beta_2\,\alpha_1\alpha_2.\end{aligned}$$

By the properties of the elements belonging to $D(S)$ in (5.3.11), we obtain that

$$\begin{aligned}\langle S(z_1), z_2\rangle_{\mathcal{H}(B_0)} = & -2\mu \int_{\partial B_0} (D(v_1)(\ell_2 + \alpha_2 y^\perp)) \cdot \widehat{n}\, d\sigma(y) \\ & + 2\mu \int_{\Omega_0} D(v_1) : D(v_2)\, dy + 2\,\mu\bigg(\int_{\partial B_0} D(v_1)\widehat{n}\, d\sigma(y)\bigg) \cdot \ell_2 \\ & + \beta_1\, \ell_1 \cdot \ell_2 + 2\,\mu\bigg(\int_{\partial B_0} y^\perp \cdot D(v_1)\widehat{n}\, d\sigma(y)\bigg)\alpha_2 + \beta_2\, \alpha_1\alpha_2\end{aligned}$$

so that, for all $z_1, z_2 \in D(S)$,

$$\langle S(z_1), z_2\rangle_{\mathcal{H}(B_0)} = 2\mu \int_{\Omega_0} D(v_1) : D(v_2)\, dy + \beta_1\, \ell_1 \cdot \ell_2 + \beta_2\, \alpha_1\alpha_2 = \langle S(z_2), z_1\rangle_{\mathcal{H}(B_0)}\,, \quad (5.3.13)$$

thus S is symmetric. Moreover, for all $z \in D(S)$, there holds

$$\langle S(z), z\rangle_{\mathcal{H}(B_0)} = 2\mu \int_{\Omega_0} |D(z)|^2\, dy + \beta_1\, |\ell|^2 + \beta_2\, \theta^2 \geq 0\,,$$

which gives that S is positive. The next step is to prove that S is not only symmetric but self-adjoint; it sufficies to prove that $I + S : D(S) \longrightarrow \mathcal{H}(B_0)$ is onto. For any $f_0 \in \mathcal{H}(B_0)$, we need to show that there exists $z = (u, \ell_z, \alpha_z) \in D(S)$ such that

$$(I + S)z = f_0$$

or, equivalently, that

$$\langle z, w\rangle_{\mathcal{H}(B_0)} + \langle S(z), w\rangle_{\mathcal{H}(B_0)} = \langle f_0, w\rangle_{\mathcal{H}(B_0)} \qquad \forall\, w \in \mathcal{H}(B_0).$$

Since $\mathcal{V}(B_0)$ is dense in $\mathcal{H}(B_0)$, by (5.3.13), this is equal to prove

$$a(z, w) = \langle z, w\rangle_{\mathcal{H}(B_0)} + 2\mu \int_{\Omega_0} D(u) : D(v)\, dy + \beta_1\, \ell_z \cdot \ell_w + \beta_2\, \alpha_z\alpha_w = \langle f_0, w\rangle_{\mathcal{H}(B_0)} \quad (5.3.14)$$

for all $w = (v, \ell_w, \alpha_w) \in \mathcal{V}(B_0)$. By noticing that the bilinear form $a : \mathcal{V}(B_0) \times \mathcal{V}(B_0) \longrightarrow \mathbb{R}$ is continuous as well as coercive on the Hilbert space $\mathcal{V}(B_0)$, and that the mapping

$$w \longmapsto \langle f_0, w\rangle_{\mathcal{H}(B_0)}$$

defines a linear and continous functional on $\mathcal{V}(B_0)$, by Lax-Milgram Theorem we obtain that there exists a unique $z = (u, \ell_z, \alpha_z) \in \mathcal{V}(B_0)$ satisfying (5.3.14). Next, we choose $w \in \mathcal{V}(B_0)$ such that

$$w = \begin{cases} \phi & \text{in } \ \Omega_0 \\ 0 & \text{in } \ B_0 \end{cases},$$

for some divergence-free $\phi \in \mathcal{C}_0^\infty(\Omega_0)$. Then, from [60, Lemma IV.1.1], we have that there exists a $p \in L^2_{loc}(\Omega_0)$ such that

$$u - \mu\Delta u + \nabla p = f_0 \quad \text{in } \ H^{-1}(\Omega_0)\,.$$

Moreover, as $z \in \mathcal{V}(B_0)$, by definition of such space, we have that u is solenoidal and it is such that

$$u = \ell_z + \alpha_z\, y^\perp \qquad \text{in } \ B_0.$$

As a consequence, the triplet (u, ℓ_z, α_z) is a weak solution of the following Stokes-type problem:

$$\begin{cases} u - \mu\Delta u + \nabla p = f_0\,, \quad \nabla\cdot u = 0 \qquad \text{in } \ \Omega_0\,, \\ \lim\limits_{|y_1|\longrightarrow\infty} u(y_1,y_2) = 0 \quad \forall y_2 \in [-L,L]\,, \quad u = 0 \quad \text{on } \ \Gamma_A\,, \\ u = \ell_z + \alpha_z\, y^\perp \qquad \text{on } \ \partial B_0\,. \end{cases} \tag{5.3.15}$$

So far we have proved that there exists $z \in \mathcal{V}(B_0)$ satisfying (5.3.14). Thus, in order to show that $I+S : D(S) \longrightarrow \mathcal{H}(B_0)$ is onto, we must prove that $z \in D(S)$. We extend the boundary data in (5.3.15)$_3$ to the interior of B_0, by considering a cut-off function $\zeta \in \mathcal{C}^\infty(A;\mathbb{R})$ such that

$$\zeta \equiv 1 \quad \text{on } \ \overline{B}_0\,, \qquad \text{supp}(\zeta) = \{x \in A \mid |x| \leq 2\}\,.$$

We then set

$$\psi(y) = \zeta(y)(\ell_z + \alpha_z\, y^\perp) \qquad \forall y \in \Omega_0\,,$$

so that $\psi \in H^2(\Omega_0)$ and

$$\int_{\partial B_0} \psi \cdot \widehat{n}\, d\sigma(y) = 0\,.$$

Let us denote by $C > 0$ a generic constant which might change during the computations. By combining (through a localization argument) the well known results concerning the regularity of solutions of the Stokes equations in smooth bounded (see [24]) and unbounded domains (see [60, Theorem VI.1.3] or [107]), we then obtain that $u \in H^2(\Omega_0)$ and

$$\|u\|_{H^2(\Omega_0)} \leq C(\|f_0\|_{L^2(\Omega_0)} + \|\psi\|_{H^2(\Omega_0)}). \tag{5.3.16}$$

Thus, $z \in D(S)$, which, as already mentioned, implies that $(I+S)$ is onto. By choosing $w = z$ in (5.3.14), we obtain

$$\|z\|^2_{\mathcal{H}(B_0)} \leq \|f_0\|^2_{\mathcal{H}(B_0)},$$

by which, through (5.2.1), we obtain

$$m|\ell_z|^2 + \mathcal{J}\alpha_z^2 \leq \|f_0\|^2_{\mathcal{H}(B_0)},$$

which implies

$$\|\psi\|_{H^2(\Omega_0)} \leq C\|f_0\|_{\mathcal{H}(B_0)}.$$

Then, from (5.3.16), we obtain

$$\|u\|_{H^2(\Omega_0)} \leq C\|f_0\|_{\mathcal{H}(B_0)},$$

which yields the sought estimate (5.3.12). □

Next, we want to apply [131, Proposition 3.3] and prove well-posedness of the linearized problem.

Theorem 5.3.7. *Let $T > 0$. Suppose that $(\widehat{u}_0, (k_0, h_0), \theta_0) \in \mathcal{V}(B_0)$ and*

$$f \in L^2(0, T; L^2(\Omega_0)), \qquad f_m \in L^2(0, T; \mathbb{R}^2), \qquad f_{\mathcal{J}} \in L^2(0, T; \mathbb{R}).$$

Then the linear problem (5.3.10) *admits a unique strong solution* (v, q, Λ, θ) *in* $[0, T]$*, that is*

$$\begin{aligned} &v \in L^2(0, T; H^2(\Omega_0)) \cap H^1(0, T; L^2(\Omega_0)) \cap \mathcal{C}([0, T]; H^1(\Omega_0)) \\ &q \in L^2(0, T; H^1(\Omega_0)), \quad \Lambda \in H^2(0, T; \mathbb{R}^2), \quad \theta \in H^2(0, T; \mathbb{R}). \end{aligned}$$

Moreover, there exists a positive constant $K > 0$ such that

$$\begin{aligned} &\|v\|_{L^2(0,T;H^2(\Omega_0))} + \|v\|_{L^\infty(0,T;H^1(\Omega_0))} + \|v\|_{H^1(0,T;L^2(\Omega_0))} + \|\nabla q\|_{L^2(0,T;L^2(\Omega_0))} + \|\Lambda'\|_{H^1(0,T;\mathbb{R}^2)} + \|\theta'\|_{H^1(0,T;\mathbb{R})} \\ &\leq C_0 \left(\|\nabla \widehat{u}_0\|_{L^2(\Omega_0)} + |k_0| + |h_0| + |\theta_0| + \|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})} \right). \end{aligned} \tag{5.3.17}$$

The constant C_0 depends on Ω_0, μ, m, $\mathcal{J}$, β_1, β_2 and T, and it is non-decreasing with respect to T.

Proof. The proof follows closely [131, Corollary 4.3]. We observe that, by (5.3.13), the graph's norm of $D(S^{1/2})$ is

$$\|z\|^2_{D(S^{1/2})} = \langle z, z\rangle_{\mathcal{H}(B_0)} + \langle S(z), z\rangle_{\mathcal{H}(B_0)} = \langle z, z\rangle_{\mathcal{H}(B_0)} + 2\|D(v)\|^2_{L^2(A)} + \beta_1 |\ell|^2 + \beta_2 \alpha^2,$$

thus

$$D(S^{1/2}) = \mathcal{V}(B_0).$$

Notice that $z_0 = (\hat{u}_0, (k_0, h_0), \theta_0) \in D(S^{1/2})$. We extend f to the whole of A by setting, almost everywhere in $[0, T]$,

$$f_*(y, t) = \begin{cases} f(y, t) & \text{if } y \in \Omega_0 \\ \dfrac{f_m(t)}{m} + \dfrac{f_{\mathcal{J}}(t)}{\mathcal{J}} y^\perp & \text{if } y \in B_0\,, \end{cases}$$

so that $(f_*, f_m, f_{\mathcal{J}}) \in L^2(0, T; L^2(A))$ and

$$\|f_*\|^2_{L^2(0,T;L^2(A))} = \|f\|^2_{L^2(0,T;L^2(\Omega_0))} + \frac{1}{m}\|f_m\|^2_{L^2(0,T;\mathbb{R}^2)} + \frac{1}{\mathcal{J}}\|f_{\mathcal{J}}\|^2_{L^2(0,T;\mathbb{R})}\,. \tag{5.3.18}$$

Consider now the abstract Cauchy problem

$$\begin{cases} z' + Sz = \mathbb{P}(f_*) \quad \text{in } (0, T)\,, \\ z(0) = z_0\,. \end{cases} \tag{5.3.19}$$

Due to Proposition 5.3.6, we can apply [131, Proposition 3.3] so as to deduce that problem (5.3.19) admits a unique solution

$$z \in L^2(0, T; D(S)) \cap \mathcal{C}([0, T]; \mathcal{V}(B_0)) \cap H^1(0, T; \mathcal{H}(B_0))$$

that satisfies the estimate

$$\|z\|_{L^2(0,T;D(S))} + \|z\|_{L^\infty(0,T;\mathcal{V}(B_0))} + \|z\|_{H^1(0,T;\mathcal{H}(B_0))} \leq C_0 \left(\|z_0\|_{\mathcal{V}(B_0)} + \|f_*\|_{L^2(0,T;L^2(A))} \right), \tag{5.3.20}$$

where $C_0 > 0$ is a constant depending on the operator S and on T, and it is non-decreasing with respect to T. As $z \in H^1(0,T;\mathcal{H}(B_0))$, there exist (unique functions) $v \in H^1(0,T;L^2(A))$, $V \in H^1(0,T;\mathbb{R}^2)$ and $\omega \in H^1(0,T;\mathbb{R})$ such that $z = (v, V, \omega)$ and

$$v(y,t) = V(t) + \omega(t)\, y^\perp \qquad \forall\, y \in B_0\,,\ t \in [0,T]\,.$$

Then, denoting by

$$\Lambda(t) = \int_0^T V(\tau)\, d\tau, \qquad \theta(t) = \int_0^T \omega(\tau)\, d\tau, \qquad \forall t \in [0,T]\,,$$

we have that $\Lambda \in H^1(0,T;\mathbb{R}^2)$, $\theta \in H^2(0,T;\mathbb{R})$ and $v(y,t) = \Lambda'(t) + \theta'(t)y^\perp$ for all $y \in B_0$ and $t \in [0,T]$. Consider $w = (u,\ell,\alpha) \in \mathcal{H}(B_0)$ and multiply (5.3.19)$_1$ with w to obtain

$$\langle z'(t), w\rangle_{\mathcal{H}(B_0)} + \langle S(z(t)), w\rangle_{\mathcal{H}(B_0)} = \langle \mathbb{P}(f_*), w\rangle_{\mathcal{H}(B_0)} = \langle f_*, w\rangle_{\mathcal{H}(B_0)}\,,$$

which is equal to

$$\langle z'(t), w\rangle_{\mathcal{H}(B_0)} + \langle \mathcal{S}(z(t)), w\rangle_{\mathcal{H}(B_0)} = \langle f_*, w\rangle_{\mathcal{H}(B_0)} \qquad \text{for a.e. } t \in [0,T]\,. \tag{5.3.21}$$

From the definition of the operator $\mathcal{S}$, and since $z \in L^2(0,T;D(S))$, we rewrite (5.3.21) as

$$\begin{aligned}&\int_{\Omega_0} v'(t)\cdot u\, dy + m\,\Lambda''(t)\cdot \ell + \mathcal{J}\theta''(t)\,\alpha + \int_{\Omega_0} (-\mu\,\Delta v)\cdot u\, dy + 2\,\mu\bigg(\int_{\partial B_0} D(v)\,\widehat{n}\, d\sigma(y)\bigg)\cdot \ell \\ &+ \beta_1\Lambda'(t)\cdot\ell + 2\,\mu\bigg(\int_{\partial B_0} y^\perp\cdot D(v)\,\widehat{n}\, d\sigma(y)\bigg)\alpha + \beta_2\theta'(t)\,\alpha = \int_{\Omega_0} f(t)\cdot u\, dy + f_m\cdot \ell + f_{\mathcal{J}}\,\alpha\end{aligned} \tag{5.3.22}$$

for all $w = (u,\ell,\alpha) \in \mathcal{H}(B_0)$. In particular, we can choose

$$w = \begin{cases}\phi & \text{in } \Omega_0\\ 0 & \text{in } B_0\end{cases},$$

with $\phi \in \mathcal{C}_c^\infty(\Omega_0)$ such that $\nabla\cdot\phi = 0$ in Ω_0, so as to obtain

$$\int_{\Omega_0} v'(t)\cdot\phi\, dy + \int_{\Omega_0} (-\mu\,\Delta v(t))\cdot\phi\, dy = \int_{\Omega_0} f(t)\cdot\phi\, dy$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega_0)$ such that $\nabla\cdot\phi = 0$ in Ω_0. Thus, by [60, Lemma XIII.1.1] we deduce the existence of a unique $q \in L^2(0,T;\widehat{H}^1(\Omega_0))$ such that

$$v'(t) - \mu\Delta v(t) + \nabla q(t) = f(t) \quad \text{a.e. in } \Omega_0\,,\ \text{ for a.e. } t \in [0,T]\,. \tag{5.3.23}$$

From (5.3.23) and (5.3.20) we can easily derive the estimate

$$\|\nabla q\|_{L^2(0,T;L^2(\Omega_0))} \le C_0\left(\|z_0\|_{\mathcal{V}(B_0)} + \|f\|_{L^2(0,T;L^2(A))}\right)\,. \tag{5.3.24}$$

Notice that (5.3.23) is, almost everywhere in $[0,T]$, an equality between elements of $\mathcal{H}(B_0)$. Thus, it implies that

$$\int_{\Omega_0} (v'(t) - \mu\Delta v(t) + \nabla q(t) - f(t))\cdot u = 0 \qquad \forall\, w = (u,\ell,\alpha) \in \mathcal{H}(B_0)\,.$$

After an integration by parts and in view of (5.3.22) we obtain

$$\begin{aligned} &m\,\Lambda''(t)\cdot\ell + \mathcal{J}\theta''(t)\,\alpha + \beta_1\Lambda'(t)\cdot\ell + \beta_2\theta'(t)\,\alpha - f_m\cdot\ell - f_{\mathcal{J}}\,\alpha \\ &= \int_{\partial B_0} q(t)u\cdot\widehat{n}\,d\sigma - 2\,\mu\bigg(\int_{\partial B_0} D(v)\,\widehat{n}\,d\sigma(y)\bigg)\cdot\ell - 2\,\mu\bigg(\int_{\partial B_0} y^\perp\cdot D(v)\,\widehat{n}\,d\sigma(y)\bigg)\alpha \end{aligned}$$

for all $w=(u,\ell,\alpha)\in\mathcal{H}(B_0)$. By testing with $w=(u,\ell,0)$, for arbitrary $\ell\in\mathbb{R}^2$ and $w=(u,0,1)$ we recover respectively $(5.3.10)_4$ and $(5.3.10)_5$. In order to state the uniqueness of (v,q,Λ,θ), we observe that any strong solution (5.3.10) must necessarily satisfy the abstract Cauchy problem (5.3.19), which has a unique solution in virtue of Proposition 5.3.6 and [131, Proposition 3.3]. Finally, notice that now the estimate (5.3.20) implies that

$$\begin{aligned} &\|v\|_{L^2(0,T;H^2(\Omega_0))} + \|v\|_{L^\infty(0,T;H^1(\Omega_0))} + \|v\|_{H^1(0,T;L^2(\Omega_0))} + \|\Lambda'\|_{H^1(0,T;\mathbb{R}^2)} + \|\theta'\|_{H^1(0,T;\mathbb{R})} \\ &\leq C_0\left(\|\nabla\widehat{u}_0\|_{L^2(\Omega_0)} + |k_0| + |h_0| + |\theta_0| + \|f_*\|_{L^2(0,T;L^2(A))}\right)\,, \end{aligned}$$

which, combined with (5.3.18) and (5.3.24), yields (5.3.17). □

5.3.3 Global existence (up to collision) and uniqueness of strong solutions

The main purpose of this section is to prove a global-in-time (up to collision) existence and uniqueness result for the penalized system (5.3.1)-(5.3.2)-(5.3.3). From the well-posedness of the linear problem (5.3.10) we learn that there exists a mapping $\mathfrak{F}$ from

$$L^2(0,T;L^2(\Omega_0))\times L^2(0,T;\mathbb{R}^2)\times L^2(0,T;\mathbb{R}) \tag{5.3.25}$$

into itself, defined by

$$\mathfrak{F}\begin{pmatrix} f\\ f_m\\ f_{\mathcal{J}}\end{pmatrix} = \begin{pmatrix}\mathfrak{F}_1(f,f_m,f_{\mathcal{J}})\\ \mathfrak{F}_2(f,f_m,f_{\mathcal{J}})\\ \mathfrak{F}_3(f,f_m,f_{\mathcal{J}})\end{pmatrix} =$$

$$\begin{pmatrix} \widehat{g} + \mu(\mathcal{L}-\Delta)v - \mathcal{M}v - \mathcal{N}v - (\mathcal{G}-\nabla)q - (v\cdot\nabla)\,s - (s\cdot\nabla)\,v \\ -Q^\top(\theta)\,R_n(\Lambda',\theta) + m\,\theta'\Lambda'^\perp \\ -F_2(\Lambda',\theta)\end{pmatrix}, \tag{5.3.26}$$

where (v,q,Λ,θ) is the unique strong solution to (5.3.10). Let $T>0$ and define, for any $R>0$, the ball of radius R (centered at the origin) in the space (5.3.25), that is

$$\begin{aligned} B_R(0) = &\{(f,f_m,f_{\mathcal{J}})\in L^2(0,T;L^2(\Omega_0))\times L^2(0,T;\mathbb{R}^2)\times L^2(0,T;\mathbb{R}) \text{ s.t.} \\ &\|(f,f_m,f_{\mathcal{J}})\|_{L^2(0,T;L^2(\Omega_0))\times L^2(0,T;\mathbb{R}^2)\times L^2(0,T;\mathbb{R})} \leq R\}. \end{aligned}$$

We want to show that the map $\mathfrak{F}$ possesses a unique fixed point in a two-steps procedure.

The *first step* consists of proving that, provided that T is small enough and R is large enough, then $\mathfrak{F}$ maps $B_R(0)$ into itself. Following [130], let us denote in the sequel K_0, C_0, D_0, C as four positive quantities, respectively, satisfying the following requirements:

(A_1) K_0 is a function of $k_0,\theta_0,\|\nabla\widehat{u}_0\|_{L^2(\Omega_0)},T$ and R, non-decreasing with respect to T, R, $\|\nabla\hat{u}_0\|_{L^2(\Omega_0)}$, $|k_0|$, $|\theta_0|$.

(A_2) C_0 is a function of k_0, θ_0, $\|\nabla \hat{u}_0\|_{L^2(\Omega_0)}$ and T, non-decreasing with respect to T, $\|\nabla \widehat{u}_0\|_{L^2(\Omega_0)}$, $|k_0|$, $|\theta_0|$.

(A_3) D_0 is a function of k_0, θ_0 and $\|\nabla \hat{u}_0\|_{L^2(\Omega_0)}$, non-decreasing with respect to $\|\nabla \widehat{u}_0\|_{L^2(\Omega_0)}$, $|k_0|$ and $|\theta_0|$.

(A_4) C is a constant independent of k_0, θ_0, $\|\nabla \widehat{u}_0\|_{L^2(\Omega_0)}$, T and R.

Then, the following result can be easily proven:

Lemma 5.3.8. *Let $\mathfrak{F}$ be defined as in* (5.3.26). *Suppose that $(f, f_m, f_{\mathcal{J}}) \in B_R(0)$. Then, there exist K_0, C_0, D_0, C with the properties respectively stated in $(A_1) - (A_2) - (A_3) - (A_4)$ such that*

$$\begin{aligned}
&\|\mathfrak{F}_1(f, f_m, f_{\mathcal{J}})\|_{L^2(0,T;L^2(\Omega_0))} \leq \|\widehat{g}\|_{L^2(A)}T^{1/2} + C_0(\|\nabla s\|_{L^\infty(A)} + \|s\|_{L^\infty(A)})(D_0 + R) + K_0 T^{1/10},\\
&\|\mathfrak{F}_2(f, f_m, f_{\mathcal{J}})\|_{L^2(0,T;\mathbb{R}^2)} \leq CC_0T^{1/2}(D_0 + R)\left(n\, T^{1/2} + D_0 + R\right) ,\\
&\|\mathfrak{F}_3(f, f_m, f_{\mathcal{J}})\|_{L^2(0,T;\mathbb{R})} \leq CC_0T(D_0 + R).
\end{aligned} \tag{5.3.27}$$

Proof. From Lemma 5.2.2, we have that

$$\|\widehat{g}\|_{L^2(0,T;L^2(\Omega_0))} = \|\widehat{g}\|_{L^2(\Omega_0)}T^{1/2}. \tag{5.3.28}$$

Then, by the properties of the function s in Lemma 5.2.1, estimate (5.3.17), we infer that

$$\begin{aligned}
\|(v\cdot\nabla)s\|_{L^2(0,T;L^2(\Omega_0))} &\leq \|\nabla s\|_{L^\infty(A)}\|v\|_{L^2(0,T;L^2(\Omega_0))} = \|\nabla s\|_{L^\infty(A)}\left(\int_0^T \|v\|^2_{L^2(\Omega_0)}\right)^{1/2}\\
&\leq \|\nabla s\|_{L^\infty(A)}\|v\|_{L^2(0,T;H^2(\Omega_0))}\\
&\leq \|\nabla s\|_{L^\infty(A)}C_0(\|\nabla\widehat{u}_0\|_{L^2(\Omega_0)} + |k_0| + |h_0| + |\theta_0|\\
&\qquad + \|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})})\\
&\leq \|\nabla s\|_{L^\infty(A)}C_0(D_0 + R)
\end{aligned} \tag{5.3.29}$$

and

$$\begin{aligned}
\|(s\cdot\nabla)v\|_{L^2(0,T;L^2(\Omega_0))} &\leq \|s\|_{L^\infty(A)}\|\nabla v\|_{L^2(0,T;L^2(\Omega_0))} = \|s\|_{L^\infty(A)}\left(\int_0^T \|\nabla v\|^2_{L^2(\Omega_0)}\right)^{1/2}\\
&\leq \|s\|_{L^\infty(A)}\|v\|_{L^2(0,T;H^2(\Omega_0))}\\
&\leq \|s\|_{L^\infty(A)}C_0(\|\nabla\widehat{u}_0\|_{L^2(\Omega_0)} + |k_0| + |h_0| + |\theta_0|\\
&\qquad + \|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})})\\
&\leq \|s\|_{L^\infty(A)}C_0(D_0 + R).
\end{aligned} \tag{5.3.30}$$

From [130, Corollary 6.9], the remaining terms in $\mathfrak{F}_1$ can be bounded as

$$\|\mu(\mathcal{L} - \Delta)v - \mathcal{M}v - \mathcal{N}v - (\mathcal{G} - \nabla)q\|_{L^2(0,T;L^2(\Omega_0))} \leq K_0\, T^{1/10}. \tag{5.3.31}$$

By collecting the inequalities in (5.3.28)-(5.3.29)-(5.3.30)-(5.3.31), we obtain $(5.3.27)_1$.

Since for small times the restoring forces F_1 and F_2 are of class $\mathcal{C}^1$ and noticing that $\|Q\|_{L^\infty(0,T;\mathbb{R}^4)} = 1$ we have, as a consequence of Jensen's inequality,

$$\begin{aligned} |R_n(\Lambda'(t),\theta(t))|^2 &\leq n^2 \left(\int_0^t |Q(\theta(\tau))\Lambda'(\tau)|\, d\tau \right)^2 + |F_1(\Lambda'(t),\theta(t))|^2 \\ &\leq n^2\, t \int_0^t |\Lambda'(\tau)|^2\, d\tau + C \left(\int_0^t |\Lambda'(\tau)|\, d\tau + |\theta(t)| \right)^2 \\ &\leq n^2\, t \int_0^t |\Lambda'(\tau)|^2\, d\tau + C \left(t \int_0^t |\Lambda'(\tau)|^2\, d\tau + |\theta(t)|^2 \right) \qquad \forall t \in [0,T]\,. \end{aligned} \tag{5.3.32}$$

Then, using the embedding $H^1(0,T;\mathbb{R}^2) \subset L^\infty(0,T;\mathbb{R}^2)$, by estimate (5.3.17) we have

$$\begin{aligned} \|Q^\top R_n(\Lambda',\theta)\|_{L^2(0,T;\mathbb{R}^2)} &\leq \left(\int_0^T \left[n^2\, t\, \|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + C\, t\, \|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + C|\theta(t)|^2 \right] dt \right)^{1/2} \\ &\leq C \left[T^2(n^2+1)\|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + T\, \|\theta\|^2_{L^\infty(0,T;\mathbb{R})} \right]^{1/2} \\ &\leq C\,T \left[(n^2+1)\|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\theta'\|^2_{L^2(0,T;\mathbb{R})} \right]^{1/2} \\ &\leq n\, C\, C_0 T(D_0+R)\,, \end{aligned} \tag{5.3.33}$$

and

$$\|m\,\theta'\Lambda'^\perp\|_{L^2(0,T;\mathbb{R}^2)} \leq C\, C_0 T^{1/2}(D_0+R)^2\,. \tag{5.3.34}$$

A combination of (5.3.33)-(5.3.34)-(5.3.35) yields $(5.3.27)_2$. Using the same reasoning, we infer that

$$\|F_2(\Lambda',\theta)\|_{L^2(0,T;\mathbb{R})} \leq C\, C_0 T(D_0+R)\,, \tag{5.3.35}$$

and thus we obtain $(5.3.27)_3$. □

From Lemma 5.3.8, it is then clear that

$$\mathfrak{F}(B_R(0)) \subset B_R(0)\,, \tag{5.3.36}$$

provided that R is sufficiently large (which depends on ε and the initial data) and T is sufficiently small.

Once R has been fixed to be large enough so that (5.3.48) holds, the *second step* consists in proving that, for T small, the mapping $\mathfrak{F} : B_R(0) \longrightarrow B_R(0)$ is a contraction. We consider $(f_1, f_{m,1}, f_{\mathcal{J},1})$, $(f_2, f_{m,2}, f_{\mathcal{J},2}) \in B_R(0)$, as well as the corresponding strong solutions to (5.3.10), with the same initial data, given by $(v_1, q_1, \Lambda_1, \theta_1)$ and $(v_2, q_2, \Lambda_2, \theta_2)$. A first simple consequence of Theorem 5.3.7, specifically of estimate (5.3.17), is that the size (in a suitable sense) of each solution is bounded from above by $C_0(D_0+R)$.

We denote the differences

$$f = f_1 - f_2\,, \qquad f_m = f_{m,1} - f_{m,2}\,, \qquad f_{\mathcal{J}} = f_{\mathcal{J},1} - f_{\mathcal{J},2}\,,$$
$$v = v_1 - v_2\,, \qquad q = q_1 - q_2\,, \qquad \Lambda = \Lambda_1 - \Lambda_2\,, \qquad \theta = \theta_1 - \theta_2\,,$$

so that (v, q, Λ, θ) satisfies

$$\begin{cases} v_t - \mu\,\Delta v + \nabla q = f, \;\; \mathrm{div}(v) = 0 \quad \text{in } \Omega_0 \times (0,T)\,, \\ \lim\limits_{|y_1|\longrightarrow\infty} v(y_1,y_2) = 0 \quad \forall y_2 \in [-L,L]\,, \qquad v = 0 \quad \text{on } \Gamma_A \times (0,T)\,, \\ v = \Lambda' + \theta' y^{\perp} \quad \text{on } \partial B_0 \times (0,T)\,, \\ m\,\Lambda'' + \beta_1\,\Lambda' = -\displaystyle\int_{\partial B_0} \mathcal{T}(v,q)\widehat{n} + f_m \quad \text{in } (0,T), \\ \mathcal{J}\,\theta'' + \beta_2\,\theta' = -\displaystyle\int_{\partial B_0} y^{\perp} \cdot \mathcal{T}(v,q)\widehat{n} + f_{\mathcal{J}} \quad \text{in } (0,T), \end{cases}$$

with zero initial conditions. From Theorem 5.3.7, we have that

$$\begin{aligned} &\|v\|_{L^2(0,T;H^2(\Omega_0))} + \|v\|_{L^\infty(0,T;H^1(\Omega_0))} + \|v\|_{H^1(0,T;L^2(\Omega_0))} + \|\nabla q\|_{L^2(0,T;L^2(\Omega_0))} + \|\Lambda'\|_{H^1(0,T;\mathbb{R}^2)} + \|\theta'\|_{H^1(0,T;\mathbb{R})} \\ &\leq C_0(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}). \end{aligned} \tag{5.3.37}$$

Notice that each solution $(v_i, q_i, \Lambda_i, \theta_i)$, $i \in \{1,2\}$, induces the differential operators $\mathcal{L}_i$, $\mathcal{M}_i$, $\mathcal{N}_i$ and $\mathcal{G}_i$. According to which of the two solutions is considered, we ought add subscripts to all quantities involved in the expression of

$$\mathfrak{F}\begin{pmatrix} f \\ f_m \\ f_{\mathcal{J}} \end{pmatrix}$$

given in (5.3.26), say $\mathcal{L} = \mathcal{L}_1 - \mathcal{L}_2$, $\mathcal{M} = \mathcal{M}_1 - \mathcal{M}_2$ and $\mathcal{G} = \mathcal{G}_1 - \mathcal{G}_2$. We notice that

$$\mathfrak{F}\begin{pmatrix} f_1 \\ f_{m,1} \\ f_{\mathcal{J},1} \end{pmatrix} - \mathfrak{F}\begin{pmatrix} f_2 \\ f_{m,2} \\ f_{\mathcal{J},2} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{12}(v,\Lambda,\theta,q) - (v\cdot\nabla)\,s - (s\cdot\nabla)\,v, \\ m\,\theta_1'\Lambda_1'^{\perp} - m\,\theta_2'\Lambda_2'^{\perp} + Q(\theta_2)^{\top} R_n(\Lambda_2',\theta_2) - Q(\theta_1)^{\top} R_n(\Lambda_1',\theta_1) \\ -F_2(\Lambda_1',\theta_1) + F_2(\Lambda_2',\theta_2) \end{pmatrix},$$

where

$$\mathcal{F}_{12}(v,H,\theta,q) = \mu(\mathcal{L}_1 - \Delta)v + \mu\,\mathcal{L}v_2 - \mathcal{M}_1 v - \mathcal{M}v_2 - \mathcal{N}_1 v_1 + \mathcal{N}_2 v_2 - (\mathcal{G}_1 - \nabla)q + \mathcal{G}q_2.$$

In order to prove that the mapping $\mathfrak{F}$ is a contraction in $B_R(0)$, in the following lemmas we give component-wise estimates, in view of (5.3.26).

Lemma 5.3.9. *Let* $(f_1, f_{m,1}, f_{\mathcal{J},1}), (f_2, f_{m,2}, f_{\mathcal{J},2}) \in B_R(0)$*. There exist* K_0, C_0 *with the properties stated in* (A_1)*-*(A_2) *such that*

$$\begin{aligned} &\|\mathfrak{F}_1(f_1, f_{m,1}, f_{\mathcal{J},1}) - \mathfrak{F}_1(f_2, f_{m,2}, f_{\mathcal{J},2})\|_{L^2(0,T;L^2(\Omega_0))} \leq C_0\left(K_0\,T^{1/10} + \|\nabla s\|_{L^\infty(A)} + \|s\|_{L^\infty(A)}\right) \\ &\quad \times \left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right)\,. \end{aligned}$$

Proof. From [130, Corollary 6.16] and (5.3.37), we easily infer the following estimates:

$$\|\mathcal{F}_{12}(v,H,\theta,q)\|_{L^2(0,T;L^2(\Omega_0))} \le C_0 K_0\, T^{1/10}\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right)$$

and

$$\begin{aligned}&\|(v\cdot\nabla)\, s + (s\cdot\nabla)\, v\|_{L^2(0,T;L^2(\Omega_0))} \le C_0\left(\|\nabla s\|_{L^\infty(A)} + \|s\|_{L^\infty(A)}\right)\\ &\quad\times\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right).\end{aligned}$$

□

Lemma 5.3.10. *Let* $(f_1, f_{m,1}, f_{\mathcal{J},1}), (f_2, f_{m,2}, f_{\mathcal{J},2}) \in B_R(0)$*. There exists* $C_0, D_0, C > 0$ *with the properties stated in* $(A_2) - (A_3) - (A_4)$ *such that*

$$\begin{aligned}&\|\mathfrak{F}_2(f_1, f_{m,1}, f_{\mathcal{J},1}) - \mathfrak{F}_2(f_2, f_{m,2}, f_{\mathcal{J},2})\|_{L^2(0,T;\mathbb{R}^2)} \le n\, C\, C_0\, T\,(D_0+R)\\ &\quad\times\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right).\end{aligned} \tag{5.3.38}$$

Proof. We start by noticing that

$$m\,\theta_1'\Lambda_1'^{\perp} - m\,\theta_2'\Lambda_2'^{\perp} = m\,\theta'\Lambda_1'^{\perp} + m\,\theta_2'\Lambda'^{\perp}.$$

Therefore, in view of the embedding $H^1(0,T;\mathbb{R}^2) \subset \mathcal{C}([0,T];\mathbb{R}^2)$ we can estimate the difference as follows:

$$\begin{aligned}\|m\,\theta_1'\Lambda_1'^{\perp} - m\,\theta_2'\Lambda_2'^{\perp}\|_{L^2(0,T;\mathbb{R}^2)} &= \|m\,\theta'\Lambda_1'^{\perp} + m\,\theta_2'\Lambda'^{\perp}\|_{L^2(0,T;\mathbb{R}^2)}\\ &\le m\left(\|\Lambda_1'\|_{L^\infty(0,T;\mathbb{R}^2)}\|\theta'\|_{L^2(0,T;\mathbb{R}^2)} + \|\theta_2'\|_{L^\infty(0,T;\mathbb{R}^2)}\|H'\|_{L^2(0,T;\mathbb{R}^2)}\right)\\ &\le C\left(\|\Lambda_1'\|_{H^1(0,T;\mathbb{R}^2)}\|\theta'\|_{L^2(0,T;\mathbb{R}^2)} + \|\theta_2'\|_{H^1(0,T;\mathbb{R}^2)}\|\Lambda'\|_{L^2(0,T;\mathbb{R}^2)}\right)\\ &\le C\,C_0(D_0+R)\left(\|\theta'\|_{L^2(0,T;\mathbb{R}^2)} + \|\Lambda'\|_{L^2(0,T;\mathbb{R}^2)}\right)\\ &\le C\,C_0(D_0+R)(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}).\end{aligned} \tag{5.3.39}$$

In a similar way, we have

$$\begin{aligned}Q(\theta_2)^\top R_n(\Lambda_2',\theta_2) - Q(\theta_1)^\top R_n(\Lambda_1',\theta_1) = &-Q(\theta_1)^\top[R_n(\Lambda_1',\theta_1) - R_n(\Lambda_2',\theta_2)]\\ &-[Q(\theta_1)^\top - Q(\theta_2)^\top]\,R_n(\Lambda_2',\theta_2).\end{aligned} \tag{5.3.40}$$

In order to estimate the right-hand side of (5.3.40), we start noticing that

$$\|Q(\theta_1) - Q(\theta_2)\|_{L^2(0,T;\mathbb{R}^4)} = \|Q(\theta_1)^\top - Q(\theta_2)^\top\|_{L^2(0,T;\mathbb{R}^4)} \le C\,T^{1/2}\|\theta'\|_{L^\infty(0,T;\mathbb{R})}, \tag{5.3.41}$$

and also, from (5.3.32),

$$\|R_n(\Lambda_i',\theta_i)\|_{L^\infty(0,T;\mathbb{R}^2)} \le n\,C\,C_0\,T^{1/2}\,(D_0+R) \qquad i\in\{1,2\}. \tag{5.3.42}$$

Therefore, by (5.3.41) and arguing similarly to (5.3.32),

$$
\begin{aligned}
&\|Q(\theta_1)^\top[R_n(\Lambda_1',\theta_1)-R_n(\Lambda_2',\theta_2)]\|_{L^2(0,T;\mathbb{R}^2)} \le T\,\|Q(\theta_1)^\top\|_{L^\infty(0,T;\mathbb{R}^4)}\|R_n(\Lambda_1',\theta_1)-R_n(\Lambda_2',\theta_2)\|_{L^\infty(0,T;\mathbb{R}^2)}\\
&\le \left[C\,T\,(n^2+1)\left(\int_0^T |Q(\theta_1(\tau))\Lambda_1'(\tau)-Q(\theta_1(\tau))\Lambda_1'(\tau)|\,d\tau\right)^2 + C\,T\|\theta\|^2_{L^\infty(0,T;\mathbb{R})}\right]^{1/2}\\
&\le \left[C\,T\,(n^2+1)\left(\int_0^T |Q(\theta_1(\tau))\Lambda'(\tau)|\,d\tau + \int_0^T |Q(\theta_1(\tau))-Q(\theta_2(\tau))||\Lambda_2'(\tau)|\,d\tau\right)^2 + C\,T\|\theta\|^2_{L^\infty(0,T;\mathbb{R})}\right]^{1/2}\\
&\le \left[C\,T\,(n^2+1)\left(T\|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + \|Q(\theta_1(\tau))-Q(\theta_2(\tau))\|^2_{L^2(0,T;\mathbb{R}^4)}\|\Lambda_2'\|^2_{L^2(0,T;\mathbb{R}^2)}\right) + C\,T^2\|\theta'\|^2_{L^2(0,T;\mathbb{R})}\right]^{1/2}\\
&\le \left[C\,T^2\,(n^2+1)\left(\|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\theta'\|^2_{L^\infty(0,T;\mathbb{R})}\|\Lambda_2'\|^2_{L^2(0,T;\mathbb{R}^2)}\right) + C\,T^2\|\theta'\|^2_{L^2(0,T;\mathbb{R})}\right]^{1/2}\\
&\le \left[C\,T^2\,(n^2+1)\left(\|\Lambda'\|^2_{L^2(0,T;\mathbb{R}^2)} + \|\theta'\|^2_{H^1(0,T;\mathbb{R})}\|\Lambda_2'\|^2_{L^2(0,T;\mathbb{R}^2)}\right) + C\,T^2\|\theta'\|^2_{L^2(0,T;\mathbb{R})}\right]^{1/2}\\
&\le n\,C\,C_0\,T\,(D_0+R)\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right)
\end{aligned}
\tag{5.3.43}
$$

and also, (5.3.41)-(5.3.42) imply that

$$
\begin{aligned}
&\|(Q(\theta_1)^\top - Q(\theta_2)^\top)\,R_n(\Lambda_2',\theta_2)\|_{L^2(0,T;\mathbb{R}^2)} \le \|R_n(\Lambda_2',\theta_2)\|_{L^\infty(0,T;\mathbb{R}^2)}\|Q(\theta_1)^\top - Q(\theta_2)^\top\|_{L^2(0,T;\mathbb{R}^4)}\\
&\le n\,C\,C_0\,T\,(D_0+R)\|\theta'\|_{L^\infty(0,T;\mathbb{R})} \le n\,C\,C_0\,T\,(D_0+R)\|\theta'\|_{H^1(0,T;\mathbb{R})}\\
&\le n\,C\,C_0\,T\,(D_0+R)\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right)\,.
\end{aligned}
\tag{5.3.44}
$$

Combining (5.3.43)-(5.3.44) we infer

$$
\begin{aligned}
&\|Q(\theta_2)^\top\,R_n(\Lambda_2',\theta_2) - Q(\theta_1)^\top\,R_n(\Lambda_1',\theta_1)\|_{L^2(0,T;\mathbb{R}^2)}\\
&\le n\,C\,C_0\,T\,(D_0+R)\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right)\,.
\end{aligned}
$$

This last inequality and (5.3.39) yield (5.3.45). □

Finally, by proceeding as in (5.3.43) one can prove

Lemma 5.3.11. *Let $(f_1, f_{m,1}, f_{\mathcal{J},1}), (f_2, f_{m,2}, f_{\mathcal{J},2}) \in B_R(0)$. There exists $C_0, D_0, C > 0$ with the properties stated in $(A_2)-(A_3)-(A_4)$ such that*

$$
\begin{aligned}
&\|\mathfrak{F}_3(f_1, f_{m,1}, f_{\mathcal{J},1}) - \mathfrak{F}_3(f_2, f_{m,2}, f_{\mathcal{J},2})\|_{L^2(0,T;\mathbb{R})} \le C\,C_0\,T\,(D_0+R)\\
&\quad\times\left(\|f\|_{L^2(0,T;L^2(\Omega_0))} + \|f_m\|_{L^2(0,T;\mathbb{R}^2)} + \|f_{\mathcal{J}}\|_{L^2(0,T;\mathbb{R})}\right)\,.
\end{aligned}
\tag{5.3.45}
$$

The main result of this section reads:

Theorem 5.3.12. *Given $\varepsilon \in (0, L/2)$, $(\widehat{u}_0, (k_0, h_0), \theta_0) \in \mathcal{V}(B_f(0))$ and $s = s_\varepsilon$ as in Lemma 5.2.1, there exists $T_1 > 0$ (depending on ε and $(\widehat{u}_0, k_0, h_0, \theta_0)$) such that the penalized problem* (5.3.1)-(5.3.2)-(5.3.3) *admits a unique strong solution $(\widehat{u}, \widehat{p}, k, h, \theta)$ such that*

$$
\widehat{u} \in L^2(0, T_1; H^2(\Omega_f(t))) \,\cap\, H^1(0, T_1; L^2(\Omega_f(t))) \,\cap\, \mathcal{C}([0, T_1]; H^1_0(\Omega_f(t)))
$$

$$
\widehat{p} \in L^2(0, T_1; \widehat{H}^1(\Omega_f(t))), \quad (k, h) \in H^2(0, T_1; \mathbb{R}^2), \quad \theta \in H^2(0, T_1; \mathbb{R}),
$$

and

$$(k(t), h(t), \theta(t)) \in A_{f,\varepsilon} \qquad \forall\, t \in [0, T_1]\,. \tag{5.3.46}$$

Moreover, the solution $(\widehat{u}, \widehat{p}, k, h, \theta)$ *satisfies the energy estimate*

$$\begin{aligned} & E_n(t) + \int_0^t \left[\mu \|\nabla \widehat{u}(s)\|^2_{L^2(\Omega_f(s))} + 2\beta_1 |\eta'(s)|^2 + 2\beta_2 |\theta'(s)|^2 \right] ds \\ & \leq \frac{L^2\, T_1}{\mu} \|\widehat{g}\|^2_{L^2(A)} + E_n(0) + \int_0^{T_1} \exp\left(2\tau \|\nabla s\|_{L^\infty(A)}\right) \left(\frac{L^2\, \tau}{\mu} \|\widehat{g}\|^2_{L^2(A)} + E_n(0) \right) d\tau \end{aligned} \tag{5.3.47}$$

for all $t \in [0, T_1]$ *, where* E_n *is energy associated to the penalized system, that is,*

$$E_n(t) = \|\widehat{u}(t)\|^2_{L^2(\Omega_f(t))} + m|\eta'(t)|^2 + \mathcal{J}|\theta'(t)|^2 + 2F(\theta(t), h(t)) + n\, k(t)^2 \qquad \forall\, t \in [0, T_1]\,.$$

Proof. From Lemma 5.3.8, it is clear that

$$\mathfrak{F}(B_R(0)) \subset B_R(0)\,, \tag{5.3.48}$$

provided that R is sufficiently large (which depends on ε and the initial data) and the se T is sufficiently small. A combination of Lemmas 5.3.9-5.3.10-5.3.11 then allows us to conclude that $\mathfrak{F}$ is a contraction in $B_R(0)$ provided that T_0 is small enough so that

$$C_0 \left(K_0\, T_0^{1/10} + \|\nabla s\|_{L^\infty(A)} + \|s\|_{L^\infty(A)} \right) + n\, C\, C_0\, T_0\, (D_0 + R) + C\, C_0\, T_0\, (D_0 + R) < 1\,.$$

Notice that, except for $n \geq 1$, all the terms on the left-hand of the previous inequality depend on ε, $(\widehat{u}_0, k_0, h_0, \theta_0)$ and the prescribed magnitude λ for the Poiseuille flow. Therefore, the mapping $\mathfrak{F}$ possesses a unique fixed point which, by definition of $\mathfrak{F}$ in (5.3.26), corresponds to the unique strong solution to the penalized problem (5.3.1)-(5.3.2)-(5.3.3) in $[0, T_0]$.

In order to prove (5.3.47), we multiply $(5.3.1)_1$ by $\widehat{u}(t)$. Integrating by parts over $\Omega_f(t)$ term by term and applying the Reynolds Transport Theorem, we obtain

$$\int_{\Omega_f(t)} \widehat{u}_t(t) \cdot \widehat{u}(t)\, dx = \frac{1}{2} \frac{d}{dt} \int_{\Omega_f(t)} |\widehat{u}(t)|^2\, dx - \frac{1}{2} \int_{\partial B_f(t)} (\widehat{u}(t) \cdot \widehat{n}) |\widehat{u}(t)|^2\, d\sigma\,, \tag{5.3.49}$$

$$\int_{\Omega_f(t)} (\widehat{u}(t) \cdot \nabla) \widehat{u}(t) \cdot \widehat{u}(t)\, dx = \frac{1}{2} \int_{\partial B_f(t)} (\widehat{u}(t) \cdot \widehat{n}) |\widehat{u}(t)|^2\, d\sigma\,, \qquad \int_{\Omega_f(t)} (s \cdot \nabla) \widehat{u}(t) \cdot \widehat{u}(t)\, dx = 0\,, \tag{5.3.50}$$

$$\int_{\Omega_f(t)} [-\mu \Delta \widehat{u}(t) + \nabla \widehat{p}(t)] \cdot \widehat{u}(t)\, dx = 2\mu \int_{\Omega_f(t)} |D(\widehat{u}(t))|^2\, dx - \int_{\partial B_f(t)} \widehat{u}(t) \cdot \mathcal{T}(\widehat{u}(t), p(t)) \widehat{n}\, d\sigma\,. \tag{5.3.51}$$

By inserting (5.3.1)$_3$-(5.3.2) into (5.3.51), we infer

$$
\begin{aligned}
&\int_{\Omega_f(t)} [-\mu\Delta\widehat{u}(t) + \nabla\widehat{p}(t)]\cdot\widehat{u}(t)\,dx \\
&= 2\mu\int_{\Omega_f(t)} |D(\widehat{u}(t))|^2\,dx - \int_{\partial B_f(t)} [\eta'(t)+\theta(t)'(x-\eta(t))^{\perp}]\cdot\mathcal{T}(\widehat{u}(t),p(t))\widehat{n}\,d\sigma\,) \\
&= 2\mu\int_{\Omega_f(t)} |D(\widehat{u}(t))|^2\,dx + m\,\eta''(t)\cdot\eta'(t) + \beta_1\,|\eta(t)|^2 + R_n(\eta(t),\theta(t))\cdot\eta'(t) \\
&\quad + \mathcal{J}\theta''(t)\theta'(t) + \beta_2\,|\theta'(t)|^2 + F_2(\theta(t),h(t))\theta'(t) \\
&= \mu\int_{\Omega_f(t)} |\nabla\widehat{u}(t)|^2\,dx + \frac{1}{2}\frac{d}{dt}\left(m|\eta'(t)|^2 + \mathcal{J}|\theta'(t)|^2 + 2F(\theta(t),h(t)) + n\,k(t)^2\right) \\
&\quad + \beta_1|\eta'(t)|^2 + \beta_2|\theta'(t)|^2\,.
\end{aligned}
$$

By combining this last identity with (5.3.49)-(5.3.50)-(5.3.51), we estimate as follows

$$
\begin{aligned}
&\frac{1}{2}\frac{dE_n}{dt}(t) + \mu\|\nabla\widehat{u}(t)\|^2_{L^2(\Omega_f(t))} + \beta_1|\eta'(t)|^2 + \beta_2|\theta'(t)|^2 \\
&= \int_{\Omega_f(t)} \widehat{g}\cdot\widehat{u}(t)\,dx - \int_{\Omega_f(t)} (\widehat{u}(t)\cdot\nabla)s\cdot\widehat{u}(t)\,dx \\
&\le \|\widehat{g}\|_{L^2(\Omega_f(t))}\|\widehat{u}(t)\|_{L^2(\Omega_f(t))} + \|\nabla s\|_{L^\infty(\Omega_f(t))}\|\widehat{u}(t)\|^2_{L^2(\Omega_f(t))} \\
&\le \frac{L^2}{2\mu}\|\widehat{g}\|^2_{L^2(\Omega_f(t))} + \frac{\mu}{2}\|\nabla\widehat{u}(t)\|^2_{L^2(\Omega_f(t))} + \|\nabla s\|_{L^\infty(\Omega_f(t))}\|\widehat{u}(t)\|^2_{L^2(\Omega_f(t))}\,.
\end{aligned}
\tag{5.3.52}
$$

Next, from (5.3.55) we get

$$
\frac{dE_n}{dt}(t)+\mu\|\nabla\widehat{u}(t)\|^2_{L^2(\Omega_f(t))}+2\beta_1|\eta'(t)|^2+2\beta_2|\theta'(t)|^2 \le \frac{L^2}{\mu}\|\widehat{g}\|^2_{L^2(\Omega_f(t))}+2\|\nabla s\|_{L^\infty(\Omega_f(t))}E_n(t)\,. \tag{5.3.53}
$$

An application of the Grönwall lemma yields

$$
E_n(t) \le \exp\left(2t\|\nabla s\|_{L^\infty(\Omega_f(t))}\right)\left(E_n(0) + \frac{L^2\,t}{\mu}\|\widehat{g}\|^2_{L^2(\Omega_f(t))}\right) \qquad \forall\, t\in[0,T_0]\,, \tag{5.3.54}
$$

and then, after integrating the inequality (5.3.53) in $[0,t]$ and using (5.3.54), we end up with (5.3.47).

To obtain the global-in-time existence (until the obstacle leaves the admissible set, i.e., $(k(T_*),h(T_*),\theta(T_*)) \notin A_{f,\varepsilon}$ for some $T_*>0$) of a strong solution $(\widehat{u},\widehat{p},k,h,\theta)$ to problem (5.3.1)-(5.3.2)-(5.3.3), we observe that, from (5.3.47), a time $T_1\ge T_0$ depending on ε, λ and $(\widehat{u}_0,k_0,h_0,\theta_0)$ can be chosen in such a way that

$$
(k(t),h(t),\theta(t)) \in A_{f,\frac{\varepsilon}{2}} \qquad \forall\, t\in[0,T_1]\,.
$$

In order to then extend the solution, already defined in $[0, T_0]$, to the interval $[0, T_1]$, we need estimates on $\|\nabla\widehat{u}\|_{L^2(\Omega_f(t))}$. We proceed similarly to [33, Section 4], introducing some auxiliary functions. Let $\zeta \in \mathcal{C}^\infty(A;\mathbb{R})$ be a function with compact support such that $\zeta \equiv 1$ in neighborhood of $\overline{B}_f(0)$, and set

$$\widehat{\zeta}(x,t) = \zeta\left(Q(\theta(t))^\top (x - \eta(t))\right) \qquad \forall (x,t) \in A \times [0, T_1]\,,$$

where we remind $\eta = (k, h)$. We then define the rigid velocity field associated to (η, θ) as

$$V(x,t) = \eta'(t) + \theta'(t)(x - \eta(t))^\perp \qquad \forall (x,t) \in \mathbb{R}^2 \times [0, T_1]\,,$$

and its associated stream function

$$\begin{aligned} w(x,t) &= -\int_0^{x_1} V_2(s,t)\,ds + \int_0^{x_2} V_1(s,t)\,ds \\ &= -h'(t)\,x_1 + k'(t)\,x_2 - \frac{\theta'(t)}{2}\left(x_1^2 + x_2^2 - 2x_1 k(t) - 2x_2 h(t)\right) \end{aligned} \tag{5.3.55}$$

for all $(x,t) \in \mathbb{R}^2 \times [0, T_1]$. The solenoidal vector field $\widehat{\Lambda}$ given by

$$\widehat{\Lambda}(x,t) = \left(-w(x,t)\frac{\partial\widehat{\zeta}}{\partial x_2}(x) + \widehat{\zeta}(x)V_1(x_2,t), w(x,t)\frac{\partial\widehat{\zeta}}{\partial x_1}(x) + \widehat{\zeta}(x)V_2(x_1,t)\right)$$

for all $(x,t) \in \mathbb{R}^2 \times [0, T_1]$, satisfies the properties reported in [33, Section 4]. In particular,

$$\|\widehat{\Lambda}\|_{W^{2,\infty}(\Omega_f(t))} \le C\left(|h'(t)| + |\theta'(t)|\right) \tag{5.3.56}$$

where C is a positive constant depending on the geometry. The following useful identity can be proven by exactly following the procedure in [33, Lemma 4.3]:

$$\begin{aligned} &-\int_{\Omega_f} \mathrm{div}\mathcal{T}(\widehat{u},\widehat{p}) \cdot \left(\widehat{u}_t + (\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}\right)\,dx = \mu\frac{d}{dt}\int_{\Omega_f} |D\widehat{u}|^2\,dx + m\,|\eta''|^2 \\ &+ \beta_1\,\eta''\cdot\eta' + \eta''\cdot R_n(\eta,\theta) \quad + \mathcal{J}\,|\theta''|^2 + \beta_2\,\theta''\theta' + \theta'' F_2(h,\theta) - m\,\left[\theta'(\eta')^\perp\right]\cdot\eta'' \\ &- \beta_1\left[\theta'(\eta')^\perp\right]\cdot\eta' - \left[\theta'(\eta')^\perp\right]\cdot R_n(\eta,\theta) + 2\mu\int_{\Omega_f} (D\widehat{u}) : ((\nabla\widehat{u})(\nabla\widehat{\Lambda}) - D((\widehat{u}\cdot\nabla)\widehat{\Lambda}))\,dx\,. \end{aligned} \tag{5.3.57}$$

We take the inner product of the equation $(5.3.1)_1$ with

$$\widehat{u}_t + (\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}$$

and, by using (5.3.57), we obtain

$$
\begin{aligned}
&\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \mu\frac{d}{dt}\int_{\Omega_f}|D\widehat{u}|^2\,dx + m\,|\eta''|^2 + \mathcal{J}\,|\theta''|^2 + \beta_1\frac{d}{dt}\frac{|\eta'|^2}{2} + \beta_2\frac{d}{dt}\frac{|\theta'|^2}{2} = \\
&+ m\,\left[\theta'(\eta')^\perp\right]\cdot\eta'' + \beta_1\left[\theta'(\eta')^\perp\right]\cdot\eta' + \left[\theta'(\eta')^\perp\right]\cdot R_n(\eta,\theta) - \eta''\cdot R_n(\eta,\theta) - \theta'' F_2(h,\theta) \\
&- \int_{\Omega_f}\widehat{u}_t\cdot\left[(\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]\,dx - \int_{\Omega_f}(\widehat{u}\cdot\nabla)\widehat{u}\cdot\left[\widehat{u}_t + (\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]\,dx \\
&- \int_{\Omega_f}(\widehat{u}\cdot\nabla)s\cdot\left[\widehat{u}_t + (\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]\,dx - \int_{\Omega_f}(s\cdot\nabla)\widehat{u}\cdot\left[\widehat{u}_t + (\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]dx \\
&+ \int_{\Omega_f}\widehat{g}\cdot\left[\widehat{u}_t + (\widehat{\Lambda}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]\,dx - 2\mu\int_{\Omega_f}(D\widehat{u}) : ((\nabla\widehat{u})(\nabla\widehat{\Lambda}) - D((\widehat{u}\cdot\nabla)\widehat{\Lambda}))\,dx\ .
\end{aligned}
\tag{5.3.58}
$$

At this point, some estimates for the terms of the right-hand side of (5.3.58) are needed, by exploiting in a suitable way the Hölder, the Young and the Poincaré inequalities, together with the properties of the solenoidal extension s and of the mapping $\widehat{\Lambda}$ given in (5.3.56). By recalling that, as long as (5.3.46) holds, the restoring forces F_1 and F_2 are of class $\mathcal{C}_1$, we get the two following inequalities

$$
\left|m\left[\theta'(\eta')^\perp\right]\cdot\eta''\right| \leq \frac{m}{4}|\eta''|^2 + m|\theta'(\eta')^\perp|^2 \leq \frac{m}{4}|\eta''|^2 + \frac{m}{2}(|\theta'|^4 + |\eta'|^4)\,,
$$

and

$$
\begin{aligned}
|\eta''\cdot R_n(\eta,\theta)| &\leq \frac{m}{4}|\eta''|^2 + \frac{1}{2m}|R_n(\eta,\theta)|^2 \leq \frac{m}{4}|\eta''|^2 + \frac{1}{2m}|R_n(\eta,\theta)|^2 \\
&\leq \frac{m}{4}|\eta''|^2 + \frac{1}{2m}(n^2|k|^2 + |F_1(h,\theta)|^2) \leq \frac{m}{4}|\eta''|^2 + \frac{1}{2m}\left[n^2|k|^2 + C(|h|^2 + |\theta|^2)\right]\ .
\end{aligned}
$$

Similarly

$$
\left|\beta_1\left[\theta'(\eta')^\perp\right]\cdot\eta'\right| \leq \frac{\beta_1}{2}|\eta'|^2 + \frac{\beta_1}{2}|\theta'(\eta')^\perp|^2 \leq \frac{\beta_1}{2}|\eta'|^2 + \frac{\beta_1}{2}(|\theta'|^4 + |\eta'|^4)
$$

and

$$
|\theta'' F_2(h,\theta)| \leq \frac{\mathcal{J}}{2}|\theta''|^2 + \frac{1}{2\mathcal{J}}|F_2(h,\theta)|^2 \leq \frac{\mathcal{J}}{2}|\theta''|^2 + \frac{C}{2\mathcal{J}}(|h|^2 + |\theta|^2)\,.
$$

Next, since the Poincaré constant for the strip $\mathbb{R} \times (-L, L)$ is $\pi^2/4L^2$, we have

$$\begin{aligned}
&\left| -\int_{\Omega_f} \widehat{u}_t \cdot (\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\, dx \right| \leq \|\widehat{u}_t\|_{L^2(\Omega_f)} \|(\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\|_{L^2(\Omega_f)} \\
&\leq \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{5}{2}\|(\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\|^2_{L^2(\Omega_f)} \\
&\leq \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{5}{2}\|(\widehat{\Lambda} \cdot \nabla)\widehat{u}\|^2_{L^2(\Omega_f)} + \frac{5}{2}\|(\widehat{u} \cdot \nabla)\widehat{\Lambda}\|^2_{L^2(\Omega_f)} \\
&\leq \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{5}{2}\|\widehat{\Lambda}\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + \frac{5}{2}\|\nabla\widehat{\Lambda}\|^2_{L^\infty(\Omega_f)}\frac{4L^2}{\pi^2}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} \\
&\leq \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{5}{2}\max\left(1, \frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2 + |\theta'|^2)\right] .
\end{aligned}$$

For what concerns the trilinear terms, proceeding in a similar way yields

$$\begin{aligned}
&\left| -\int_{\Omega_f} (\widehat{u} \cdot \nabla)\widehat{u} \cdot \left[\widehat{u}_t + (\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\right] \right| \\
&\leq \|(\widehat{u} \cdot \nabla)\widehat{u}\|_{L^2(\Omega_f)}\|\widehat{u}_t + (\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\|_{L^2(\Omega_f)} \\
&\leq \frac{5}{2}\|(\widehat{u} \cdot \nabla)\widehat{u}\|^2_{L^2(\Omega_f)} + \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{1}{10}\|(\widehat{\Lambda} \cdot \nabla)\widehat{u}\|^2_{L^2(\Omega_f)} \\
&\quad + \frac{1}{10}\|(\widehat{u} \cdot \nabla)\widehat{\Lambda}\|^2_{L^2(\Omega_f)} \\
&\leq \frac{5}{2}\|(\widehat{u} \cdot \nabla)\widehat{u}\|^2_{L^2(\Omega_f)} + \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} \\
&\quad + \frac{1}{10}\max\left(1, \frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2 + |\theta'|^2)\right] ,
\end{aligned}$$

and

$$\begin{aligned}
&\left| -\int_{\Omega_f} (\widehat{u} \cdot \nabla)s \cdot \left[\widehat{u}_t + (\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\right] \right| \\
&\leq \|(\widehat{u} \cdot \nabla)s\|_{L^2(\Omega_f)}\|\widehat{u}_t + (\widehat{\Lambda} \cdot \nabla)\widehat{u} - (\widehat{u} \cdot \nabla)\widehat{\Lambda}\|_{L^2(\Omega_f)} \\
&\leq \frac{5}{2}\|(\widehat{u} \cdot \nabla)s\|^2_{L^2(\Omega_f)} + \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} \\
&\quad + \frac{1}{10}\max\left(1, \frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2 + |\theta'|^2)\right] \\
&\leq \frac{5}{2}\|\nabla s\|^2_{L^\infty(\Omega_f)}\frac{4L^2}{\pi^2}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + \frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} \\
&\quad + \frac{1}{10}\max\left(1, \frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2 + |\theta'|^2)\right] ,
\end{aligned}$$

as well as

$$\begin{aligned}
&\left|-\int_{\Omega_f}(s\cdot\nabla)\widehat{u}\cdot\left[\widehat{u}_t+(\widehat{\Lambda}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]\right|\\
&\le \|(s\cdot\nabla)\widehat{u}\|_{L^2(\Omega_f)}\|\widehat{u}_t+(\widehat{\Lambda}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{\Lambda}\|_{L^2(\Omega_f)}\\
&\le \frac{5}{2}\|(s\cdot\nabla)\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)}\\
&\quad+\frac{1}{10}\max\left(1,\frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2+|\theta'|^2)\right]\\
&\le \frac{5}{2}\|s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)}\\
&\quad+\frac{1}{10}\max\left(1,\frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2+|\theta'|^2)\right]\ .
\end{aligned}$$

The last two terms can be treated using the same arguments to obtain

$$\begin{aligned}
&\left|\int_{\Omega_f}\widehat{g}\cdot\left[\widehat{u}_t+(\widehat{\Lambda}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{\Lambda}\right]\right|\le\|\widehat{g}\|_{L^2(\Omega_f)}\|\widehat{u}_t+(\widehat{\Lambda}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{\Lambda}\|_{L^2(\Omega_f)}\\
&\le\frac{5}{2}\|\widehat{g}\|^2_{L^2(\Omega_f)}+\frac{1}{10}\|\widehat{u}_t\|^2_{L^2(\Omega_f)}+\frac{1}{10}\max\left(1,\frac{4L^2}{\pi^2}\right)\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left[C(|\eta'|^2+|\theta'|^2)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left|2\mu\int_{\Omega_f}(D\widehat{u}):((\nabla\widehat{u})(\nabla\widehat{\Lambda})-D((\widehat{u}\cdot\nabla)\widehat{\Lambda}))\,dx\right|\le2\mu\|D\widehat{u}\|_{L^2(\Omega_f)}\|(\nabla\widehat{u})(\nabla\widehat{\Lambda})-D((\widehat{u}\cdot\nabla)\widehat{\Lambda})\|_{L^2(\Omega_f)}\\
&\le\mu^2\|D\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{2}\|(\nabla\widehat{u})(\nabla\widehat{\Lambda})-D((\widehat{u}\cdot\nabla)\widehat{\Lambda})\|^2_{L^2(\Omega_f)}\\
&\le\mu^2\|D\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{2}\|(\nabla\widehat{u})(\nabla\widehat{\Lambda})\|^2_{L^2(\Omega_f)}+\frac{1}{2}\|D((\widehat{u}\cdot\nabla)\widehat{\Lambda})\|^2_{L^2(\Omega_f)}\\
&\le\mu^2\|D\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{2}\|\nabla\widehat{\Lambda}\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|_{L^2(\Omega_f)}\\
&\quad+\frac{1}{2}\||D\widehat{u}||\nabla\widehat{\Lambda}|+|\widehat{u}||D(\nabla\widehat{\Lambda})|\|^2_{L^2(\Omega_f)}\\
&\le\mu^2\|D\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{2}\|\nabla\widehat{\Lambda}\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|_{L^2(\Omega_f)}\\
&\quad+\frac{1}{2}\|D\widehat{u}\|_{L^2(\Omega_f)}\|\nabla\widehat{\Lambda}\|_{L^\infty(\Omega_f)}+\|\widehat{u}\|_{L^2(\Omega_f)}\|D(\nabla\widehat{\Lambda})\|^2_{L^\infty(\Omega_f)}\\
&\le\mu^2\|D\widehat{u}\|^2_{L^2(\Omega_f)}+\frac{1}{2}(1+\frac{4L^2}{\pi^2})\|\nabla\widehat{u}\|_{L^2(\Omega_f)}\left[C(|\eta'|^2+|\theta'|^2)\right]\\
&\quad+\frac{1}{2}\|D\widehat{u}\|_{L^2(\Omega_f)}\left[C(|\eta'|^2+|\theta'|^2)\right]\ .
\end{aligned}$$

Combining the above inequalities and plugging them into (5.3.58), we obtain that for a.e. $t \in [0, T_1]$

$$\frac{1}{2}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \mu\frac{d}{dt}\int_{\Omega_f} |D\widehat{u}|^2\,dx + \frac{m}{2}\,|\eta''|^2 + \frac{\mathcal{J}}{2}\,|\theta''|^2 + \beta_1\frac{d}{dt}\frac{|\eta'|^2}{2} + \beta_2\frac{d}{dt}\frac{|\theta'|^2}{2} \leq$$

$$\frac{m}{2}(|\theta'|^4 + |\eta'|^4) + \frac{1}{2m}\left[n^2|k|^2 + C(|h|^2 + |\theta|^2)\right] + \frac{\beta_1}{2}|\eta'|^2 + \frac{\beta_1}{2}(|\theta'|^4 + |\eta'|^4) + \frac{C}{2\mathcal{J}}(|h|^2 + |\theta|^2)$$
$$+ \frac{5}{2}\|(\widehat{u}\cdot\nabla)\widehat{u}\|^2_{L^2(\Omega_f)} + \frac{5}{2}\|\widehat{g}\|^2_{L^2(\Omega_f)}$$
$$+ \|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left\{\frac{29}{10}\max\left(1, \frac{4L^2}{\pi^2}\right)\left[C(|\eta'|^2 + |\theta'|^2)\right] + \frac{1}{2}(1 + \frac{4L^2}{\pi^2})\left[C(|\eta'|^2 + |\theta'|^2)\right]\right.$$
$$\left. + \frac{5}{2}\|\nabla s\|^2_{L^\infty(\Omega_f)}\frac{4L^2}{\pi^2} + \frac{5}{2}\|s\|^2_{L^\infty(\Omega_f)}\right\} + \|D\widehat{u}\|_{L^2(\Omega_f)}\left[C(|\eta'|^2 + |\theta'|^2) + \mu^2\right]\,. \tag{5.3.59}$$

Proceeding step by step as in [33, Theorem 1.2], taking into account for the extra terms, we arrive to an estimate for the term $(\widehat{u}\cdot\nabla)\widehat{u}$. For any $\nu > 0$, there exists a positive constant C_ν depending on B and μ such that

$$\|(\widehat{u}\cdot\nabla)\widehat{u}\|^2_{L^2(\Omega_f)} \leq C_\nu\left[\|\widehat{u}\|_{L^2(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}(\|\widehat{u}\|_{L^2(\Omega_f)} + \|\nabla\widehat{u}\|_{L^2(\Omega_f)})\right.$$
$$+ \|\widehat{u}\|^2_{L^2(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}(\|\widehat{u}\|_{L^2(\Omega_f)} + \|\nabla\widehat{u}\|_{L^2(\Omega_f)})^2 + \|\widehat{g}\|^2_{L^2(\Omega_f)} + \frac{4L^2}{\pi^2}\|\nabla s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}$$
$$\left. + \|s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + C(|\eta'|^2 + |\theta'|^2)\right] + \nu\|\widehat{u}_t\|^2_{L^2(\Omega_f)}\,. \tag{5.3.60}$$

Therefore, by replacing (5.3.60) inside (5.3.59) and choosing ν sufficiently small, we infer that for a.e. $t \in [0, T_1]$

$$\frac{1}{4}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{\mu}{2}\frac{d}{dt}\int_{\Omega_f} |\nabla\widehat{u}|^2\,dx + \frac{m}{2}\,|\eta''|^2 + \frac{\mathcal{J}}{2}\,|\theta''|^2 + \beta_1\frac{d}{dt}\frac{|\eta'|^2}{2} + \beta_2\frac{d}{dt}\frac{|\theta'|^2}{2} \leq$$

$$\frac{m}{2}(|\theta'|^4 + |\eta'|^4) + \frac{1}{2m}\left[n^2|k|^2 + C(|h|^2 + |\theta|^2)\right] + \frac{\beta_1}{2}|\eta'|^2 + \frac{\beta_1}{2}(|\theta'|^4 + |\eta'|^4) + \frac{C}{2\mathcal{J}}(|h|^2 + |\theta|^2)$$
$$+ \frac{5}{2}C\Big[\|\widehat{u}\|_{L^2(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}(\|\widehat{u}\|_{L^2(\Omega_f)} + \|\nabla\widehat{u}\|_{L^2(\Omega_f)}) + \|\widehat{u}\|^2_{L^2(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}(\|\widehat{u}\|_{L^2(\Omega_f)} + \|\nabla\widehat{u}\|_{L^2(\Omega_f)})^2$$
$$+ \|\widehat{g}\|^2_{L^2(\Omega_f)} + \frac{4L^2}{\pi^2}\|\nabla s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + \|s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + C(|\eta'|^2 + |\theta'|^2)\Big] + \frac{5}{2}\|\widehat{g}\|^2_{L^2(\Omega_f)}$$
$$+ \|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left\{\frac{29}{10}\max\left(1, \frac{4L^2}{\pi^2}\right)\left[C(|\eta'|^2 + |\theta'|^2)\right] + \frac{1}{2}(1 + \frac{4L^2}{\pi^2})\left[C(|\eta'|^2 + |\theta'|^2)\right] + \frac{5}{2}\|\nabla s\|^2_{L^\infty(\Omega_f)}\frac{4L^2}{\pi^2} + \frac{5}{2}\|s\|^2_{L^\infty(\Omega_f)}\right\}$$
$$+ \frac{1}{2}\|\nabla\widehat{u}\|_{L^2(\Omega_f)}\left[C(|\eta'|^2 + |\theta'|^2) + \mu^2\right]\,.$$

Let us define

$$K_n(t) = \frac{L^2\,t}{\mu}\|\widehat{g}\|^2_{L^2(A)} + E_n(0) + \int_0^t \exp\left(2\tau\|\nabla s\|_{L^\infty(A)}\right)\left(\frac{L^2\,\tau}{\mu}\|\widehat{g}\|^2_{L^2(A)} + E_n(0)\right)\,d\tau\,,$$

Then, combining the above inequality and (5.3.47) yields

$$\frac{1}{4}\|\widehat{u}_t\|^2_{L^2(\Omega_f)} + \frac{\mu}{2}\frac{d}{dt}\int_{\Omega_f}|\nabla\widehat{u}|^2\,dx + \frac{m}{2}|\eta''|^2 + \frac{\mathcal{J}}{2}|\theta''|^2 + \beta_1\frac{d}{dt}\frac{|\eta'|^2}{2} + \beta_2\frac{d}{dt}\frac{|\theta'|^2}{2} \leq$$

$$\begin{aligned}
&mK_n^2 + \frac{1}{2m}\left[n^2|k|^2 + C(|h|^2+|\theta|^2)\right] + \frac{\beta_1}{2}K_n + \beta_1 K_n^2 + \frac{C}{2\mathcal{J}}(|h|^2+|\theta|^2)\\
&+\frac{5}{2}C\Big[\|\widehat{u}\|_{L^2(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}(\|\widehat{u}\|_{L^2(\Omega_f)}+\|\nabla\widehat{u}\|_{L^2(\Omega_f)}) + \|\widehat{u}\|^2_{L^2(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}(\|\widehat{u}\|_{L^2(\Omega_f)}+\|\nabla\widehat{u}\|_{L^2(\Omega_f)})^2\\
&+\|\widehat{g}\|^2_{L^2(\Omega_f)} + \frac{4L^2}{\pi^2}\|\nabla s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + \|s\|^2_{L^\infty(\Omega_f)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)} + CK_n\Big] + \frac{5}{2}\|\widehat{g}\|^2_{L^2(\Omega_f)}\\
&+\|\nabla\widehat{u}\|^2_{L^2(\Omega_f)}\left\{\frac{29}{10}\max\left(1,\frac{4L^2}{\pi^2}\right)CK_n + \frac{1}{2}\left(1+\frac{4L^2}{\pi^2}\right)K_n + \frac{5}{2}\frac{4L^2}{\pi^2}\|\nabla s\|^2_{L^\infty(\Omega_f)} + \frac{5}{2}\|s\|^2_{L^\infty(\Omega_f)} + \frac{1}{2}(CK_n+\mu^2)\right\}.
\end{aligned} \tag{5.3.61}$$

Hence, integrating (5.3.61) with respect to t, and applying (5.3.47), we have that for all $t \in [0,T_1]$

$$\begin{aligned}
&\frac{1}{4}\int_0^t\|\widehat{u}_t(s)\|^2_{L^2(\Omega_f(s))}ds + \frac{\mu}{2}\|\nabla\widehat{u}(t)\|^2_{L^2(\Omega_f(t))} + \frac{m}{2}\int_0^t|\eta''(s)|^2ds + \frac{\mathcal{J}}{2}\int_0^t|\theta''(s)|^2ds\\
&+\frac{\beta_1}{2}|\eta'(t)|^2 + \frac{\beta_2}{2}|\theta'(t)|^2 \leq K_2 + K_3\int_0^t\left(\frac{\mu}{2}\|\nabla\widehat{u}(s)\|^2_{L^2(\Omega_f(s))}\right)\left(\frac{2}{\mu}\|\nabla\widehat{u}(s)\|^2_{L^2(\Omega_f(s)}\right)ds\,,
\end{aligned} \tag{5.3.62}$$

where

$$\begin{aligned}
K_2 =&\frac{\mu}{2}\|\nabla\widehat{u}_0\|^2_{L^2(\Omega_f)} + \frac{\beta_1}{2}(|k_0|^2+|h_0|^2) + \frac{\beta_2}{2}|\theta_0|^2 + (m+\beta_1)K_n^2(T_1)T_1 + \frac{\beta_1}{2}K_n(T_1)T_1 + \left(\frac{1}{2m}+\frac{C}{2\mathcal{J}}\right)K_1(T_1)\frac{T_1^2}{2}\\
&+\frac{5}{2}(C+1)\|\widehat{g}\|^2_{L^2(\Omega_f)} + \frac{5}{2}CK_n(T_1)T_1 + K_n(T_1)\Big\{\frac{29}{10}\max\left(1,\frac{4L^2}{\pi^2}\right)CK_n(T_1) + \frac{1}{2}\left(1+\frac{4L^2}{\pi^2}\right)K_n(T_1)\\
&+\frac{5}{2}\frac{4L^2}{\pi^2}\|\nabla s\|^2_{L^\infty(\Omega_f)} + \frac{5}{2}\|s\|^2_{L^\infty(\Omega_f)} + \frac{1}{2}(CK_n(T_1)+\mu^2)\Big\},
\end{aligned}$$

and

$$K_3 = C\frac{5L}{\pi}\left(\frac{2L}{\pi}+1\right) + K_1(T_1)\left(1+\frac{2L}{\pi}\right).$$

By applying the Grönwall Lemma to (5.3.62) and using again (5.3.47), we get that for all $t \in [0,T_1]$

$$\begin{aligned}
&\frac{1}{4}\int_0^t\|\widehat{u}_t(s)\|^2_{L^2(\Omega_f(s))}ds + \frac{\mu}{2}\|\nabla\widehat{u}(t)\|^2_{L^2(\Omega_f(t))} + \frac{m}{2}\int_0^t|\eta''(s)|^2ds + \frac{\mathcal{J}}{2}\int_0^t|\theta''(s)|^2ds\\
&+\frac{\beta_1}{2}|\eta'(t)|^2 + \frac{\beta_2}{2}|\theta'(t)|^2 \leq K_2\exp\left\{K_3\int_0^{T_1}\frac{2}{\mu}\|\nabla\widehat{u}(s)\|^2_{L^2(\Omega_f(s))}\,ds\right\} \leq K_2\frac{2}{\mu^2}\exp\left\{K_3K_n(T_1)\right\}.
\end{aligned} \tag{5.3.63}$$

The estimate (5.3.63) shows that the mapping

$$t \mapsto \|\nabla\widehat{u}(t)\|_{L^2(\Omega_f(t))}$$

is bounded on $[0,T_1]$. □

The global-in-time (up to a possible contact) strong solution obtained in Theorem 5.3.12 is also a weak solution to problem (5.3.1)-(5.3.2)-(5.3.3) in the sense of Definition 5.3.1. Thus, an immediate consequence of Theorem 5.3.12, which will play a fundamental role in the sequel of the chapter, is now given.

Corollary 5.3.13. *The unique strong solution* $(\widehat{u}, k, h, \theta)$ *to problem* (5.3.1)-(5.3.2)-(5.3.3) *given by Theorem 5.3.12 is a weak solution to problem* (5.3.1)-(5.3.2)-(5.3.3) *in* $[0, T_1]$ *with initial datum* $(\widehat{u}_0, (k_0, h_0), \theta_0) \in \mathcal{V}(B_f(0))$. *Precisely, we have that*

$$((k, h), \theta) \in W^{1,\infty}(0, T_1; \mathbb{R}^2 \times \mathbb{R}) \cap \mathcal{C}([0, T_1]; A_{f,\varepsilon}),$$
$$(\widehat{u}, (k', h'), \theta') \in L^2(0, T_1; \mathcal{V}(B_f(t))) \cap L^\infty(0, T_1; \mathcal{H}(B_f(t)))$$

satisfies the weak formulation given in Definition 5.3.1 and the energy estimate (5.3.47). *Moreover, we have the alternative*

(1) $T_1 = +\infty$*;*

(2) $T_1 < \infty$ *and* $\lim_{t \to T_1}((k(t), h(t)), \theta(t)) \notin A_{f,\varepsilon}$.

5.4 Weak solutions

The purpose of this section is to obtain a global-in-time (up to collision) weak solution to the problem (5.2.3)-(5.2.4)-(5.2.5), in the sense of Definition 5.2.3. This concludes the proof of Theorem 5.1.1, because problem (5.2.3)-(5.2.4)-(5.2.5) is equivalent to the original one, problem (2.0.9). Our method is based on a diagonal argument, since the inital data in (5.2.5) is approximated by more regular data belonging to $\mathcal{V}(B_f(0))$ and we also let the penalization index n go to infinity. The main result of this section reads

Theorem 5.4.1. *Assuming* $(\widehat{u}_0, h_0, \theta_0) \in \mathcal{H}_v(B_v(0))$, *there exists at least one weak solution* $(\widehat{u}, h, \theta)$ *to problem* (5.2.3)-(5.2.4)-(5.2.5). *Moreover, the following alternative holds*

(1) $T = \infty$*;*

(2) $T < \infty$ *and* $\lim_{t \to T}((k(t), h(t)), \theta(t)) \notin A_{d,\delta}$.

Proof. Given $\varepsilon \in (0, L/2)$, consider an inital datum $(\widehat{u}_0, h_0, \theta_0) \in \mathcal{H}_v(B_v(0))$. Then, there exists a sequence

$$\{(\widehat{u}_0^j, h_0^j, \theta_0^j)\}_{j \in \mathbb{N}} \subset \mathcal{V}_v(B_v(0))$$

such that

$$\widehat{u}_0^j \longrightarrow \widehat{u}_0 \quad \text{in} \quad L^2(A) \quad \text{as} \quad j \to \infty, \qquad \lim_{j \to \infty} (h_0^j, \theta_0^j) = (h_0, \theta_0) \quad \text{in} \quad \mathbb{R}^2.$$

Notice that

$$\{\widehat{z}_0^j\}_{j \in \mathbb{N}} = \{(\widehat{u}_0^j, (0, h_0^j), \theta_0^j)\}_{j \in \mathbb{N}} \subset \mathcal{V}(B_f(0))$$

is such that

$$z_0^j \longrightarrow z_0 = (\widehat{u}_0, (0, h_0), \theta_0) \quad \text{in} \quad \mathcal{H}(B_f(0)) \quad \text{as} \quad j \to \infty.$$

In view of Corollary 5.3.13, for every $j,n \in \mathbb{N}$, there exists $T_1^{j,n} > 0$ (depending on ε and $(\widehat{u}_0^j, k_0^j, h_0^j, \theta_0^j)$) such that the penalized problem (5.3.1)-(5.3.2)-(5.3.3) admits a weak solution $z_{j,n} = (\widehat{u}_{j,n}, k_{j,n}, h_{j,n}, \theta_{j,n})$ such that

$$((k_{j,n}, h_{j,n}), \theta_{j,n}) \in W^{1,\infty}(0, T_1^{j,n}; \mathbb{R}^2 \times \mathbb{R}) \cap \mathcal{C}([0, T_1^{j,n}); A_{f,\varepsilon}),$$
$$(\widehat{u}_{j,n}, (k'_{j,n}, h'_{j,n}), \theta'_{j,n}) \in L^2(0, T_1^{j,n}; \mathcal{V}(B_f^{j,n}(t))) \cap L^\infty(0, T_1^{j,n}; \mathcal{H}(B_f^{j,n}(t)))$$

with initial data $z_0^j \in \mathcal{V}(B_f(0))$. Moreover, the solution $(\widehat{u}_{j,n}, k_{j,n}, h_{j,n}, \theta_{j,n})$ satisfies

$$(k_{j,n}(t), h_{j,n}(t), \theta_{j,n}(t)) = \int_0^t (k'_{j,n}(\tau), h'_{j,n}(\tau), \theta'_{j,n}(\tau))\, d\tau \qquad \forall t \in [0, T_1^{j,n}], \tag{5.4.1}$$

and also the energy estimate

$$\begin{aligned} &E^{j,n}(t) + \int_0^t \left[\mu \|\nabla \widehat{u}_{j,n}(s)\|^2_{L^2(\Omega_f^{j,n}(s))} + \beta_1 |\eta'_{j,n}(s)|^2 + \beta_2 |\theta'_{j,n}(s)|^2\right] ds \\ &\leq \frac{L^2\, T_1^{j,n}}{\mu} \|\widehat{g}\|^2_{L^2(A)} + E^{j,n}(0) + \int_0^{T_1^{j,n}} \exp\left(2\tau \|\nabla s\|_{L^\infty(A)}\right) \left(\frac{L^2\,\tau}{\mu} \|\widehat{g}\|^2_{L^2(A)} + E^{j,n}(0)\right) d\tau \end{aligned}$$

forall $t \in [0, T_1^{j,n}]$, where $\eta_{j,n} = (k_{j,n}, h_{j,n})$, for every $j,n \in \mathbb{N}$, and

$$E^{j,n}(t) = \|\widehat{u}_{j,n}(t)\|^2_{L^2(\Omega_f^{j,n}(t))} + m|\eta'_{j,n}(t)|^2 + \mathcal{J}|\theta'_{j,n}(t)|^2 + 2F(\theta_{j,n}(t), h_{j,n}(t)) + n\, k_{j,n}(t)^2$$

for all $t \in [0, T_1^{j,n}]$, $\quad j,n \in \mathbb{N}$. Notice that, since $\{z_0^j\}_{j\in\mathbb{N}}$ converges to z_0 in $\mathcal{H}(B_f(0))$ and $F(0,0) = 0$, we have

$$\lim_{j\to\infty} E^{j,n}(0) = \|\widehat{u}_0\|^2_{L^2(\Omega_f(0))} + m h_0^2 + \mathcal{J}|\theta_0|^2,$$

and therefore, there exists $j_0 \in \mathbb{N}$ such that

$$E^{j,n}(0) \leq C_0 = 2\left(\|\widehat{u}_0\|^2_{L^2(\Omega_f(0))} + m h_0^2 + \mathcal{J}|\theta_0|^2\right) \forall j \geq j_0\,, n \in \mathbb{N}.$$

Since $T_1^{j,n}$ depends on the initial data through the approximated initial energy $E_{j,n}(0)$, we deduce that there exists $T > 0$, depending only on ε, such that

$$\begin{aligned} &((k_{j,n}, h_{j,n}), \theta_{j,n}) \in W^{1,\infty}([0,T]; A_{f,\varepsilon}), \\ &(\widehat{u}_{j,n}, (k'_{j,n}, h'_{j,n}), \theta'_{j,n}) \in L^2(0,T; \mathcal{V}(B_f(t))) \cap L^\infty(0,T; \mathcal{H}(B_f(t))), \end{aligned}$$

are uniformly bounded with respect to $j,n \in \mathbb{N}$ with

$$\begin{aligned} &E^{j,n}(t) + \int_0^t \left[\mu \|\nabla \widehat{u}_{j,n}(s)\|^2_{L^2(\Omega_f^{j,n}(s))} + \beta_1 |\eta'_{j,n}(s)|^2 + \beta_2 |\theta'_{j,n}(s)|^2\right] ds \\ &\leq \frac{L^2\, T}{\mu} \|\widehat{g}\|^2_{L^2(A)} + C_0 + \int_0^T \exp\left(2\tau \|\nabla s\|_{L^\infty(A)}\right) \left(\frac{L^2\,\tau}{\mu} \|\widehat{g}\|^2_{L^2(A)} + C_0\right) d\tau \qquad \forall t \in [0,T]. \end{aligned} \tag{5.4.2}$$

We define, for all $j,n\in\mathbb{N}$, the density function $\rho_{j,n}\in\mathcal{C}([0,T];L^\infty(A))$ by

$$\rho_{j,n}(t)=\chi_{\Omega_f^{j,n}(t)}+\rho\chi_{B_f^{j,n}(t)}\qquad\forall t\in[0,T]\,,\tag{5.4.3}$$

so that the approximated energy is equivalent to

$$E^{j,n}(t)=\int_A\rho_j(t)|\widehat{u}_{j,n}(t)|^2\,dx+2F(\theta_{j,n}(t),h_{j,n}(t))+n\,k_{j,n}(t)^2\qquad\forall t\in[0,T]\,.$$

Therefore, in view of the compact embedding $W^{1,\infty}(0,T;\mathbb{R}^2\times\mathbb{R})\subset\mathcal{C}([0,T];\mathbb{R}^2\times\mathbb{R})$, from (5.4.1) and a diagonal argument, we deduce that there exist

$$((k,h),\theta)\in\mathcal{C}([0,T];A_{f,\varepsilon})\,,\quad \widehat{u}\in L^2(0,T;H_0^1(A))\cap L^\infty(0,T;L^2(A))$$

and subsequences (still denoted in the same way) such that

$$\begin{aligned}
&((k_{j,n},h_{j,n}),\theta_j)\to((k,h),\theta)\quad\text{in}\quad\mathcal{C}([0,T];A_{f,\varepsilon})\,,\\
&\rho_{j,n}\to\rho\quad\text{in}\quad\mathcal{C}([0,T];L^\infty(A))\,,\\
&\widehat{u}_{j,n}\rightharpoonup\widehat{u}\quad\text{in}\quad L^2(0,T;H_0^1(A)),\\
&\widehat{u}_{j,n}\overset{*}{\rightharpoonup}\widehat{u}\quad\text{in}\quad L^\infty(0,T;L^2(A))\,,
\end{aligned}\tag{5.4.4}$$

as $j,n\to\infty$. From (5.4.2) we easily deduce that $k\equiv0$. In fact we have that $(h,\theta)\in H^1(0,T;\mathbb{R}^2)$ and

$$((k'_{j,n},h'_{j,n}),\theta'_{j,n})\overset{*}{\rightharpoonup}((0,h'),\theta')\quad\text{in}\quad L^\infty(0,T;\mathbb{R}^2\times\mathbb{R})\quad\text{as}\ j,n\to\infty\,.\tag{5.4.5}$$

To see this, take any $\phi\in\mathcal{C}_c^\infty(0,T;\mathbb{R})$, so that

$$\int_0^T h(t)\phi'(t)\,dt=\lim_{n\to\infty}\lim_{j\to\infty}\int_0^T h_{j,n}(t)\phi'(t)\,dt=-\lim_{n\to\infty}\lim_{j\to\infty}\int_0^T h'_{j,n}(t)\phi(t)\,dt\,.$$

From (5.4.2) we find a constant $C>0$, independent of j and n, such that

$$\left|\int_0^T h'_{j,n}(t)\phi(t)\,dt\right|\le C\|\phi\|_{L^2(0,T)}\qquad\forall j,n\in\mathbb{N}\,,$$

thus implying

$$\left|\int_0^T h(t)\phi'(t)\,dt\right|\le C\|\phi\|_{L^2(0,T)}\qquad\forall\phi\in\mathcal{C}_c^\infty(0,T;\mathbb{R})\,.$$

The same argument can be applied to θ, so that $(h,\theta)\in H^1(0,T;\mathbb{R}^2)$. In order to prove (5.4.5) we start by noticing that $h(0)=0$, in view of $(5.4.4)_1$. Therefore

$$h_{j,n}(t)=\int_0^t h'_{j,n}(\tau)\,d\tau\qquad\text{and}\qquad h(t)=\int_0^t h'(\tau)\,d\tau\qquad\forall t\in[0,T]\,,\ \ j,n\in\mathbb{N}\,,$$

so that, again in view of $(5.4.4)_1$,

$$\lim_{n\to\infty}\lim_{j\to\infty}\int_0^t h'_{j,n}(\tau)\,d\tau = \int_0^t h'(\tau)\,d\tau \qquad \forall t\in[0,T]\,. \tag{5.4.6}$$

Since $(h'_{j,n})_{j,n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;\mathbb{R})$, there exists $\widetilde{h}\in L^\infty(0,T;\mathbb{R})$ such that $h'_{j,n}\overset{*}{\rightharpoonup}\widetilde{h}$ in $L^\infty(0,T;\mathbb{R})$:

$$\lim_{n\to\infty}\lim_{j\to\infty}\int_0^T h'_{j,n}(t)\phi(t)\,dt = \int_0^T \widetilde{h}(t)\phi(t)\,dt \qquad \forall \phi\in L^1(0,T;\mathbb{R})\,.$$

In particular, the above convergence implies that

$$\lim_{n\to\infty}\lim_{j\to\infty}\int_0^t h'_{j,n}(\tau)\,d\tau = \int_0^t \widetilde{h}(\tau)\,d\tau \qquad \forall t\in[0,T]\,. \tag{5.4.7}$$

From (5.4.6)-(5.4.7) we deduce that $h'=\widetilde{h}$ a.e. in $[0,T]$, so that (5.4.5) follows. We emphasize that

$$\rho(t)=\chi_{\Omega_v(t)}+\rho\chi_{B_v(t)} \quad \text{with} \quad B_v(t)=Q(\theta(t))B+(0,h(t)) \qquad \forall t\in[0,T]\,.$$

Taking $\phi\in\mathcal{C}_c^\infty((0,T)\times A)$ such that $\nabla\cdot\phi=0$ in A and $\phi=\ell\widehat{e}_2+\alpha\,(x_1,x_2-h)^\perp$ in a neighborhood of $B_v(t)$ for some $(\ell,\alpha)\in\mathcal{C}_c^\infty(0,T;\mathbb{R}^2)$, we can multiply $(5.3.1)_1$ by ϕ for $j,n\in\mathbb{N}$ sufficiently large, say $j\geq j_*$ and $n\geq n_*$, so as to obtain

$$\begin{aligned}
&-\int_0^T\left(\int_A \rho_{j,n}\,\widehat{u}_{j,n}\cdot\phi_t\,dx-\beta_1 h'_{j,n}\ell+m\,h'_{j,n}\,\ell'-F_1(h_{j,n},\theta_{j,n})\,\ell-\beta_2\,\theta'_{j,n}\,\alpha-F_2(h_{j,n},\theta_{j,n})\,\alpha\right)dt\\
&+2\mu\int_0^T\int_A D(\widehat{u}_{j,n}):D(\phi)\,dx\,dt+\int_0^T\int_{\Omega_f^{j,n}(t)}[(\widehat{u}_{j,n}\cdot\nabla)\,\widehat{u}_{j,n}\cdot\phi+(\widehat{u}_{j,n}\cdot\nabla)\,s\cdot\phi\\
&+(s\cdot\nabla)\,\widehat{u}_{j,n}\cdot\phi]\,dx\,dt=\int_0^T\int_{\Omega_f^{j,n}(t)}\widehat{g}\cdot\phi\,dx\,dt.
\end{aligned} \tag{5.4.8}$$

In order to pass to the limit in the above inequality, the weak convergences in (5.4.4)-(5.4.5) are not enough. That is why the next paragraph is devoted to proving L^2-compactness.

We closely follow the method devised in [81], later exploited in [85]. This method is based on the application of Friedrichs Lemma [60, Lemma II.5.2]: for every bounded set $\mathcal{O}\subset A$, for any $\gamma>0$, we can choose $I=I(\gamma,\mathcal{O})\in\mathbb{N}$ and a sequence of functions $\{\psi_k\}_{k=1}^I\subset L^\infty(\mathcal{O};\mathbb{R}^2)$ such that

$$\begin{aligned}
\|\widehat{u}_{j,n}-\widehat{u}\|^2_{L^2(0,T;L^2(\mathcal{O}))}\leq \sum_{k=1}^I\int_0^T\left(\int_{\mathcal{O}}\rho\,(\widehat{u}_{j,n}(t)-\widehat{u}(t))\cdot\psi_k\,dx\right)^2 dt\\
+\gamma\|\nabla\widehat{u}_{j,n}-\nabla\widehat{u}\|^2_{L^2(0,T;L^2(\mathcal{O}))}.
\end{aligned} \tag{5.4.9}$$

Since $\widehat{u}_{j,n}$ is uniformily bounded in $L^2(0,T;H_0^1(A))$, the crucial idea is to prove that the first term on the right-hand side of (5.4.9) vanishes as $j,n \to \infty$, which immediately implies that

$$\lim_{n\to\infty}\lim_{j\to\infty} \|\widehat{u}_{j,n} - \widehat{u}\|^2_{L^2(0,T;L^2(\mathcal{O}))} = 0\,. \tag{5.4.10}$$

This is enough to pass to the limit in (5.4.8), where we take $\mathcal{O} = \text{supp}(\phi)$. By a density argument, we then conclude that $(\widehat{u},k,h,\theta)$ is a weak solution of (5.2.3)-(5.2.4)-(5.2.5) in the sense of Definition 5.2.3. Set then

$$I_{j,n} = \int_0^T \left(\int_{\mathcal{O}} \rho\,(\widehat{u}_{j,n}(t) - \widehat{u}(t))\cdot\psi\,dx\right)^2 dt \qquad \forall j \geq j_*\,,\ \ n \geq n_*\,.$$

All efforts will be devoted to prove that, for any $\psi \in L^\infty(\mathcal{O})$, there exists a subsequence, denoted by $\widehat{u}_{j,n}$ such that

$$\lim_{j,n\to\infty} I_{j,n} = 0\,. \tag{5.4.11}$$

In order to prove (5.4.11), we introduce

$$\bar{I}_{j,n} = \int_0^T \left(\int_{\mathcal{O}} \rho\,(\bar{u}_{j,n}(t) - \widehat{u}(t))\cdot\psi\,dx\right)^2 dt\,,$$

where $\bar{u}_{j,n}$ is a sequence of functions, to be determined later, which is rigid in $B_v(t)$. Notice that

$$|I_{j,n}| \leq 2\left|\bar{I}_{j,n}\right| + 2\|\rho\,(\widehat{u}_{j,n} - \bar{u}_{j,n})\cdot\psi\|^2_{L^2(0,T;L^2(\mathcal{O}))} \qquad \forall j \geq j_*\,,\ \ n \geq n_*\,. \tag{5.4.12}$$

To the purpose of proving that the first term on the right-hand side of (5.4.12) goes to 0, given $N \in \mathbb{N}$, we split $[0,T]$ into N intervals $[t_{i-1},t_i]$, with $\Delta t = t_i - t_{i-1} = T/N$, for $i \in \{1,...,N\}$. Then, given a sufficiently small $\bar{\varepsilon} < \varepsilon$, we "thicken" the elliptical rigid body by denoting

$$B^{\bar{\varepsilon}} = \left\{(x_1,x_2)\in\mathbb{R}^2 \;\middle|\; \frac{x_1^2}{(d+\bar{\varepsilon})^2} + \frac{x_2^2}{(\delta+\bar{\varepsilon})^2} \leq 1\right\},$$

and introducing, for any $t \in [0,T]$,

$$B_v^{\bar{\varepsilon}}(t) = Q(\theta(t))B^{\bar{\varepsilon}} + h(t)\,\widehat{e}_2\,.$$

Given $i \in \{1,...,N\}$, we consider an orthonormal basis $(e_M^{i,\bar{\varepsilon}})_{M\in\mathbb{N}} \subset \mathcal{V}_v(B_v^{\bar{\varepsilon}}(t_i))$, where, without loss of generality, we can assume that each $e_M^{i,\bar{\varepsilon}}$ has compact support in A. For every $M,r\in\mathbb{N}$ we define the piecewise linear function in time $\phi_{M,r}^{\bar{\varepsilon}} : [0,T]\times A \longrightarrow \mathbb{R}^2$ by

$$\phi_{M,r}^{\bar{\varepsilon}}(t,x) = e_M^{i-1,\bar{\varepsilon}}(x) + \frac{t-t_i}{\Delta t}(e_r^{i,\bar{\varepsilon}}(x) - e_M^{i-1,\bar{\varepsilon}}(x)) \qquad \forall (t,x)\in[0,T]\times A\,,$$

for $i \in \{1,...,N\}$ and $t \in [t_{i-1},t_i]$. Since (h,θ) are uniformly continuous in $[0,T]$, we can assure that $D(\phi_{M,r}^{\bar{\varepsilon}}(t)) = 0$ in $B_v^{\bar{\varepsilon}/2}(t)$ for every $t\in[0,T]$, provided that N is sufficiently large (i.e., Δt is sufficiently small). In particular, we can deduce that $\phi_{M,r}^{\bar{\varepsilon}} \in \mathcal{C}([0,T];\mathcal{V}_v(B_v(t))$

for every $M, r \in \mathbb{N}$, so that they will be employed to approximate functions belonging to $L^2(0,T;\mathcal{H}_v(B_v(t))$. Due to $(5.4.4)_1$, there exist $j_1 = j_1(\bar{\varepsilon}), n_1 = n_1(\bar{\varepsilon}) \in \mathbb{N}$ such that

$$D(\phi^{\bar{\varepsilon}}_{M,r}(t)) = 0 \quad \text{in} \quad B^{j,n}_f(t) \qquad \forall t \in [0,T]\,, \quad \forall j \geq j_1\,,\, n \geq n_1\,,$$

for every $M, r \in \mathbb{N}$. Taking $j \geq j_1$ and $n \geq n_1$, we can multiply $(5.3.1)_1$ by $\phi^{\bar{\varepsilon}}_{M,r}$ with

$$\phi^{\bar{\varepsilon}}_{M,r}(t) = V^{\bar{\varepsilon}}_{M,r}(t)\,\widehat{e}_2 + \alpha^{\bar{\varepsilon}}_{M,r}(t)(x_1 - k_{j,n}(t), x_2 - h_{j,n}(t))^{\perp} \qquad \forall (x_1,x_2) \in B^{j,n}_f(t)\,.$$

Then, after integrating by parts in $\Omega^{j,n}_f(t)$ and using the Reynolds Transport Theorem we obtain, for all $t \in [0,T]$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\Omega^{j,n}_f(t)} \widehat{u}_{j,n}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx = \int_{\Omega^{j,n}_f(t)} \partial_t\phi^{\bar{\varepsilon}}_{M,r}\cdot\widehat{u}_{j,n}dx + \int_{\Omega^{j,n}_f(t)} (\widehat{u}^j\cdot\nabla)\phi^{\bar{\varepsilon}}_{M,r}\cdot\widehat{u}_{j,n}\,dx\\
&+\int_{\Omega^{j,n}_f(t)} [(\widehat{u}_{j,n}\cdot\nabla)s + (s\cdot\nabla)\widehat{u}_{j,n}]\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx - 2\mu\int_{\Omega^{j,n}_f(t)} D(\widehat{u}_{j,n}) : D(\phi^{\bar{\varepsilon}}_{M,r})\,dx\\
&- V^{\bar{\varepsilon}}_{M,r}\left(m\,h''_{j,n} + \beta_2 h'_{j,n} + F_1(h_{j,n},\theta_{j,n})\right) - \alpha^{\bar{\varepsilon}}_{M,r}\left(\mathcal{J}\theta''_{j,n} + \beta_2\theta'_{j,n} + F_2(h_{j,n},\theta_{j,n})\right)\\
&+\int_{\Omega^{j,n}_f(t)} \widehat{g}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx
\end{aligned}$$

which, by using the density function $\rho_{j,n}$ in (5.4.3), can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\int_A \rho_{j,n}\,\widehat{u}_{j,n}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx = \int_{\Omega^{j,n}_f(t)} \rho_{j,n}\,\partial_t\phi^{\bar{\varepsilon}}_{M,r}\cdot\widehat{u}_{j,n}\,dx + \int_{\Omega^{j,n}_f(t)} (\widehat{u}_{j,n}\cdot\nabla)\phi^{\bar{\varepsilon}}_{M,r}\cdot\widehat{u}^j\,dx\\
&+\int_{\Omega^{j,n}_f(t)} [(\widehat{u}_{j,n}\cdot\nabla)s + (s\cdot\nabla)\widehat{u}_{j,n}]\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx - 2\mu\int_{\Omega^{j,n}_f(t)} D(\widehat{u}_{j,n}) : D(\phi^{\bar{\varepsilon}}_{M,r})\,dx\\
&- V^{\bar{\varepsilon}}_{M,r}\left(\beta_2 h'_{j,n} + F_1(h_{j,n},\theta_{j,n})\right) - \alpha_{\phi^{\bar{\varepsilon}}}\left(\beta_2\theta'_{j,n} + F_2(h_{j,n},\theta_{j,n})\right)\\
&+\int_{\Omega^{j,n}_f(t)} \widehat{g}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx\,.
\end{aligned}$$

Fix $M, r \in \mathbb{N}$. In view of (5.4.2) and following [81, Section 4], we can prove that there exists a constant $C = C(M,r) > 0$ independent of j, n such that

$$\left[\int_A \rho_{j,n}(t)\,\widehat{u}_{j,n}(t)\cdot\phi^{\bar{\varepsilon}}_{M,r}(t)\,dx\right]_{t=t_i}^{t=t_i+\Delta t} \leq C(M,r)(\Delta t)^a \quad \text{for some } \ a \in (0,1)\,.$$

Hence, the sequence

$$\left\{\int_A \rho_{j,n}\,\widehat{u}_{j,n}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx\right\}_{j\geq j_1, n\geq n_1} : [0,T] \to \mathbb{R}$$

is bounded and equicontinuous, thus, by the Ascoli-Arzelá Theorem and applying the diagonal Cantor procedure, we obtain that there exists a subsequence (denoted in the same way) such that

$$\int_A \rho_{j,n}\,\widehat{u}_{j,n}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx \to \int_A \rho\,\widehat{u}\cdot\phi^{\bar{\varepsilon}}_{M,r}\,dx \qquad \text{in} \quad \mathcal{C}([0,T];\mathbb{R})\,,$$

for every $M, r \in \mathbb{N}$. Since $\widehat{u}_{j,n}$ is uniformly bounded in $L^\infty(0,T;L^2(A)) \cap L^2(0,T;H_0^1(A))$, from $(5.4.4)_2$ we also infer

$$\int_A \rho\, \widehat{u}_{j,n} \cdot \phi^{\bar{\varepsilon}}_{M,r}\, dx \to \int_A \rho\, \widehat{u} \cdot \phi^{\bar{\varepsilon}}_{M,r}\, dx \qquad \text{in} \quad \mathcal{C}([0,T];\mathbb{R}),$$

By invoking the density results contained in [81, Lemma 4.1, Lemma 4.2] we further deduce that

$$\int_A \rho\, \widehat{u}_{j,n} \cdot \phi\, dx \to \int_A \rho\, \widehat{u} \cdot \phi\, dx \qquad \text{in} \quad \mathcal{C}([0,T];\mathbb{R}), \qquad \forall \phi \in \mathcal{C}([0,T];\mathcal{H}_v(B_v(t))),$$

thus guaranteeing that $\widehat{u}$ attains the initial conditions, and also

$$\int_A \rho\, \widehat{u}_{j,n} \cdot \phi\, dx \to \int_A \rho\, \widehat{u} \cdot \phi\, dx \qquad \text{in} \quad L^2(0,T;\mathbb{R}), \qquad \forall \phi \in L^2([0,T];\mathcal{H}_v(B_v(t)). \tag{5.4.13}$$

Next we move to define $\overline{u}_{j,n}$. Given $t \in [0,T]$, we introduce as in (5.3.5) the functions $\varphi(t,\cdot) : \Omega_0 \longrightarrow \Omega_v(t)$ and $\varphi_j(t,\cdot) : \Omega_0 \longrightarrow \Omega_f^{j,n}(t)$ which satisfy, for every $\varepsilon \in (0, L/2)$, the properties

$$\varphi(t,z) = \begin{cases} Q(\theta(t))z + h(t)\widehat{e}_2 & \text{if } z \in \mathcal{O}_\varepsilon \\ z & \text{if } z \in \mathcal{A}_\varepsilon\,. \end{cases}$$

and

$$\varphi_{j,n}(t,z) = \begin{cases} Q(\theta_{j,n}(t))z + (k_{j,n}(t), h_{j,n}(t)) & \text{if } z \in \mathcal{O}_\varepsilon \\ z & \text{if } z \in \mathcal{A}_\varepsilon\,. \end{cases}$$

With this we can introduce the functions $X_j(t,\cdot) : \Omega_f^{j,n}(t) \longrightarrow \Omega_v(t)$ and $Y_j(t,\cdot) : \Omega_v(t) \longrightarrow \Omega_f^{j,n}(t)$ respectively by

$$\begin{aligned} X_{j,n}(t,x) &= \varphi(t, \varphi_{j,n}^{-1}(t,x)) \qquad \forall (t,x) \in [0,T] \times \Omega_f^{j,n}(t)\,, \\ Y_{j,n}(t,y) &= \varphi_{j,n}(t, \varphi^{-1}(t,y)) \qquad \forall (t,y) \in [0,T] \times \Omega_v(t)\,. \end{aligned}$$

In particular, notice that

$$X_{j,n}(t,x) = \begin{cases} Q(\theta(t) - \theta_{j,n}(t))[x - (k_{j,n}(t), h_{j,n}(t))] + h(t)\widehat{e}_2) & \text{if } x \in \mathcal{O}_\varepsilon \\ x & \text{if } x \in \mathcal{A}_\varepsilon\,. \end{cases}$$

and

$$Y_j(t,y) = \begin{cases} Q(\theta_{j,n}(t) - \theta(t))(y - h(t)\widehat{e}_2) + (k_{j,n}(t), h_{j,n}(t)) & \text{if } y \in \mathcal{O}_\varepsilon \\ y & \text{if } y \in \mathcal{A}_\varepsilon\,. \end{cases}$$

We finally set

$$\overline{u}_{j,n}(t,y) = \nabla X_{j,n}(t, Y_j(t,y))\, \widehat{u}_{j,n}(t, Y_{j,n}(t,y)) \qquad \forall (t,y) \in [0,T] \times \Omega_v(t)\,.$$

A simple computation shows that

$$\overline{u}_{j,n}(t,y) = Q(\theta(t)-\theta_{j,n}(t))(k'_{j,n}(t),h'_{j,n}(t))+\theta'_{j,n}(t)(y_1,y_2-h(t))^{\perp} \qquad \forall (t,y)\in[0,T]\times B_v(t)\,,$$

so that

$$(\overline{u}_{j,n}, Q(\theta-\theta_{j,n})(k'_{j,n},h'_{j,n}),\theta'_{j,n}) \in L^2(0,T;\mathcal{V}(B_v(t)))\cap L^\infty(0,T;\mathcal{H}(B_v(t)))\,.$$

Then, for every $j\geq j_1$ and $n\geq n_1$ we have

$$\begin{aligned}
&\|\bar{u}_{j,n}-\widehat{u}_{j,n}\|^2_{L^2(0,T;L^2(A))} = \int_0^T\int_{A\setminus\mathcal{A}_\varepsilon} |\nabla X_{j,n}(t,Y_{j,n}(t,y))\widehat{u}_{j,n}(t,Y_{j,n}(t,y)) - \widehat{u}_j(t,y)|^2\,dy\,dt\\
&\leq 2\int_0^T\int_{A\setminus\mathcal{A}_\varepsilon} |\nabla X_{j,n}(t,Y_{j,n}(t,y))\widehat{u}_{j,n}(t,Y_{j,n}(t,y)) - \nabla X_{j,n}(t,Y_{j,n}(t,y))\widehat{u}_{j,n}(t,y)|^2\,dy\,dt\\
&\quad + 2\int_0^T\int_{A\setminus\mathcal{A}_\varepsilon} |\nabla X_{j,n}(t,y)\widehat{u}_{j,n}(t,Y_{j,n}(t,y)) - \widehat{u}_{j,n}(t,y)|^2\,dy\,dt\\
&\leq 2\|\nabla X_{j,n}\|^2_{L^\infty(0,T;L^\infty(A))}\int_0^T\int_{A\setminus\mathcal{A}_\varepsilon} |\widehat{u}_{j,n}(t,Y_{j,n}(t,y)) - \widehat{u}_{j,n}(t,y)|^2\,dy\,dt\\
&\quad + 2T\|\widehat{u}_{j,n}\|^2_{L^\infty(0,T;L^2(A))}\|\nabla X_{j,n}-\mathbf{I}\|^2_{L^\infty(0,T;L^\infty(A))}\\
&\leq 2\|\nabla X_{j,n}\|^2_{L^\infty(0,T;L^\infty(A))}\|\nabla\widehat{u}_{j,n}\|^2_{L^2(0,T;L^2(A)}\|Y_j-\mathrm{id}_A\|^2_{L^\infty(0,T;L^\infty(A))}\\
&\quad + 2T\|\widehat{u}_{j,n}\|^2_{L^\infty(0,T;L^2(A))}\|\nabla X_{j,n}-\mathbf{I}\|^2_{L^\infty(0,T;L^\infty(A))}
\end{aligned} \tag{5.4.14}$$

At this point, we may apply [76, Corollary 1] to estimate

$$\begin{aligned}
&\|Y_{j,n}-\mathrm{id}_A\|_{L^\infty(0,T;L^\infty(A))} \leq C\left(\|k_{j,n}\|_{L^\infty(0,T;\mathbb{R})} + \|h_{j,n}-h\|_{L^\infty(0,T;\mathbb{R})} + \|\theta_{j,n}-\theta\|_{L^\infty(0,T;\mathbb{R})}\right)\\
&\|\nabla X_{j,n}-\mathbf{I}\|_{L^\infty(0,T;L^\infty(A))} \leq C\left(\|k_{j,n}\|_{L^\infty(0,T;\mathbb{R})} + \|h_{j,n}-h\|_{L^\infty(0,T;\mathbb{R})} + \|\theta_{j,n}-\theta\|_{L^\infty(0,T;\mathbb{R})}\right)
\end{aligned}\,,$$

for all $j,n\in\mathbb{N}$, for some constant $C>0$ independent of j and n. From $(5.4.4)_1$ we then get

$$\lim_{j,n\to\infty}\|Y_{j,n}-\mathrm{id}_A\|_{L^\infty(0,T;L^\infty(A))} = \lim_{j,n\to\infty}\|\nabla X_{j,n}-\mathbf{I}\|_{L^\infty(0,T;L^\infty(A))} = 0\,, \tag{5.4.15}$$

which also proves that $(\nabla X_{j,n})_{j,n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;L^\infty(A))$. By plugging (5.4.15) into (5.4.14) and recalling that $(\widehat{u}_{j,n})_{j,n\in\mathbb{N}}$ is uniformly bounded in

$$L^2(0,T;H^1_0(A))\cap L^\infty(0,T;L^2(A)),$$

we finally conclude that

$$\lim_{j,n\to\infty}\|\bar{u}_{j,n}-\widehat{u}_{j,n}\|_{L^2(0,T;L^2(A))} = 0\,.$$

We are now in the position to prove (5.4.11) for $\psi\in L^\infty(\mathcal{O})$. Given $t\in[0,T]$ we denote by $P_{(h(t),\theta(t))}$ the orthogonal projection from $L^2(A,\rho\,dx)$ onto $\mathcal{H}_v(B_v(t))$ and we introduce

$$\widetilde{\psi}(t,x) = P_{(h(t),\theta(t))}\psi(x)\,.$$

Thus, we have that

$$\widetilde{\psi} \in L^{\infty}(0,T;\mathcal{H}_v(B_v))$$

and, by the properties of the projection,

$$\bar{I}_{j,n} = \int_0^T \left(\int_{\mathcal{O}} \rho \left(\bar{u}_{j,n}(t) - \widehat{u}(t) \right) \cdot \widetilde{\psi} \, dx \right)^2 dt \, .$$

We set

$$\tilde{I}_{j,n} = \int_0^T \left(\int_{\mathcal{O}} \rho \left(\widehat{u}_{j,n}(t) - \widehat{u}(t) \right) \cdot \widetilde{\psi} \, dx \right)^2 dt \, ,$$

and we notice that

$$\left| \bar{I}_{j,n} \right| \leq 2 |\tilde{I}_{j,n}| + 2 \| \rho \left(\widehat{u}_{j,n} - \bar{u}_{j,n} \right) \cdot \psi \|^2_{L^2(0,T;L^2(\mathcal{O}))} \, . \tag{5.4.16}$$

We can choose in (5.4.13) the function $\phi = \widetilde{\psi}$ to obtain that there exists a subsequence (still denoted by $\widehat{u}_{j,n}$) such that $\tilde{I}_{j,n} \to 0$ as $j, n \to \infty$. Finally, a combination of (5.4.12)-(5.4.15)-(5.4.16) allows to conclude that (5.4.10) holds. □

CHAPTER 6

Attractors for a FSI problem in a time-dependent phase space

The present chapter is concerned with the longterm dynamics of problem (2.0.7), that is a fluid-structure interaction problem describing a Poiseuille inflow through a 2D bounded channel containing a rectangular obstacle. Physically, this models the interaction between the wind and the deck of a bridge in a wind tunnel experiment, as time goes to infinity. We are able to extend the notion of global attractor to this particular setting, where the solution operator associated to the system acts on a time-dependent phase space, and prove its existence and regularity. Moreover, when the inflow is sufficiently small, we give an explicit analytical characterization of the attractor. Finally, we numerically simulate the case to give a qualitative description of the attractor for any value of the inflow.

6.1 Longterm dynamics of fluid-structure interaction problems

We study the longterm dynamics of a coupled system describing the motion of a fluid in a 2D channel with a rectangular obstacle. We aim at modelling the interaction between the cross-section of the deck of a suspension bridge and the wind as in a wind tunnel experiment where, at the inlet and outlet sections, the velocity field of the fluid has a Poiseuille flow profile. See Figure 3.1 on the right for a picture taken during a wind tunnel experiment held at Politecnico di Milano. The asymmetry of the vortex shedding generates a lift force on the plate (deck) (see Chapter 3 for a study of the lift phenomenon).

Our analysis is performed on the two-dimensional fluid-structure interaction problem (2.0.7) introduced in Chapter 2 whose well-posedness has been later established in Chapter 4. The reason for this choice is that we aim at modelling the framework of a wind tunnel (Figure 3.1), where the longtime behaviour may also be studied experimentally. This is clearly impossible in actual bridges where the wind has a random action on the structure.

Nowadays, the understanding of the dynamic response to turbulent wind is a crucial issue in the design of long-span bridges. As already mentioned in the Introduction in Chapter 4, numerous phenomena affect suspension bridges, like vortex-induced oscillations, buffeting and flutter instabilities (see [46, 64, 125]). Specifically, flutter instability occurs at high wind velocities ($> 70\ m/s$ for long-span bridges) at which the vertical and torsional motion of the deck synchronize, leading the deck to oscillate with growing amplitudes and eventually causing the bridge to collapse [3, 4, 52]. In engineering terms, the flutter instability onset can be predicted with different approaches, i.e. experimental, numerical or hybrid methods [43]. Experimental methods rely on wind tunnel testing of full-bridge aeroelastic models [10] while numerical approaches typically involve Computational Fluid Dynamics (CFD) simulations. Hybrid methods are widely employed due to the implementation complexity of aeroelastic models, and they usually combine numerical models of the structure with experimentally identified aerodynamic coefficients [9, 44, 45]. On the other hand, Scanlan's linearized theory constitutes the most widely used approach to estimate the flutter limit (see [120, 121, 125]). In the 2D case, a major contribution is also due to Theodersen's inviscid flat-plate theory [135] and the approximate formula by Selberg [123]. These studies are used as a modelling framework to select which experimental coefficients are to be evaluated during wind tunnel tests, namely the so-called *flutter derivatives*.

The crucial (physical and engineering) issue to prevent structural and areodynamic instabilities translates into having a nice behaviour of the body-fluid as time goes to infinity. In mathematical terms, this can be described by looking for the attractor of the dynamical system at consideration, which is the natural mathematical object in the theory of infinite-dimensional system capturing the asymptotic observed nonstationary flow. Indeed, it is the smallest subset of the phase space to which the trajectories of the system converge in the long term. However, due to fact that that the fluid domain and the phase space are time-dependent, one cannot apply neither the theory of semigroups nor that of processes. We refer to Section 1.1 in Chapter 1 for a more extended discussion on the issue and all related references.

Accordingly, one of the main purposes of the present chapter is to extend the notion of global attractor to cover the case of maps lacking the concatenation property (typical of semigroups or processes), referred to in this chapter as semiflows. This allows us to circumvent the main obstruction, leading to a proper definition of global attractor apt to describe the asymptotics of our fluid-structure interaction problem acting on a time-dependent phase space. With this notion at hand, we are able to study the dissipativity properties of (2.0.7), showing that in the longterm it indeed admits an attractor. As we will see, this is a compact subset of the (variable-in-time) phase space to which all the solutions (u, h) of (2.0.7) eventually approach. In this respect, the first step is to characterize explicitly the attractor in some particular situation: we will show that if the inflow v_p is sufficiently small, then the attractor reduces to the unique stationary solution of (2.0.7). To provide a description of the global attractor in all other situations and, thus, create an explicit link between the analytical and experimental framework, we then intervene through numerical simulations, by which we are able to capture some orbit of the dynamical system at consideration. In particular, we simulate the static and dynamic behaviour of the cross-section of the deck immersed in a fluid flow at different conditions.

The chapter is organized in two parts, one devoted to the purely analytical study and the second one to the numerical study. In Section 6.2 we introduce the main tools for the analysis of (2.0.7), and we recall some results about the well-posedness and the existence and uniqueness of equilibrium solutions, the latter holding under smallness assumptions on the flow. In Section 6.3 we explain why the classical approach does not apply and, particularly, why the description of the dynamics of (2.0.7) in terms of semigroups or processes seems to be out of reach. In Section 6.4 we show that, in case of uniqueness, the equilibrium solution is stable. In Section 6.5 we define what we mean by *semiflow*, and we introduce a time-dependent map which enables us to transform (2.0.7), which is set in the time-dependent domain (2.0.5), into a different problem in a *fixed* domain. In Section 6.6 we state and prove our final result on the existence of a global attractor for (2.0.7). Then we proceed to the second part of the chapter, whose main purpose is giving an explicit characterization of the global attractor for any range of the incoming flow, through numerical simulations. As far as we are aware, this is the first time that numerics is used to improve the knowledge on the structure of the attractor associated to a fluid dynamical system. In Section 6.7, the numerical strategy adopted is described. Section 6.8 reports and discusses the numerical results obtained in terms of the main objective of the second part of chapter.

6.2 Weak solutions and well-posedness

6.2.1 Assumptions on the restoring force

To the purpose of guaranteing the well-posedness of problem (2.0.7), we assume that the restoring force f in $(2.0.7)_4$ satisfies some further conditions besides those given in Chapter 2. In particular, as in Chapter 4, in order to prevent collisions, we require that f is a *strong force*, that is

$$\exists\, r>0 \qquad \text{s.t.} \qquad \lim_{|h|\to L-\delta} |f(h)|\, \exp\left\{ -\frac{1}{(L-\delta-|h|)^{4+r}} \right\} = +\infty. \tag{6.2.1}$$

From a mathematical point of view, (6.2.1) may be probably weakened but, for our purposes, it is not essential to determine the minimal growth condition for f as $|h| \to L-\delta$. The simplest example of function f satisfying (6.2.1) is

$$f(h) = h \exp \frac{1}{(L-\delta-|h|)^{4+q}},$$

where $q>0$ (in this case $r=\frac{q}{2}$). Nevertheless, since the restoring force for the deck of a bridge also involves gravity, the function f may not be odd. Then, we define the function $M:[0,+\infty)\to(-L+\delta, L-\delta)$ as

$$M(y) := \sup\big\{|s| : F(s) \le y\big\},$$

where F is as in (2.0.13). Observe that M is a continuous increasing function with $M(0)=0$. Moreover,

$$\forall\, C\ge 0 \qquad F(h)\le C \quad \Longrightarrow \quad |h|\le M(C). \tag{6.2.2}$$

6.2.2 Definition of a solenoidal extension

In order to be able to capture the non-homogeneous boundary condition in (2.0.7), we build a solenoidal extension for the Poiseuille flow, by combining some results from [20, 60, 68]. We need an H^2-solenoidal extension (and not merely H^1) because we need some additional regularity to study the dissipation properties of our system. An alternative construction to that used in Chapter 4-5 is here proposed. Let $\varepsilon_0 \in (0, L-\delta)$ and consider the "smoothened rectangle"

$$A_{\varepsilon_0} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \,;\, |x_2| < L - \varepsilon_0, |x_1| < 2 + \sqrt[4]{(L-\varepsilon_0)^4 - x_2^4} \right\},$$

so that $\partial A_{\varepsilon_0} \in \mathcal{C}^3$. Then, we take the non-simply connected domain

$$\Sigma_{\varepsilon_0} = \mathcal{R} \setminus \bar{A}_{\varepsilon_0}, \qquad \partial\Sigma_{\varepsilon_0} = \partial\mathcal{R} \cup \partial A_{\varepsilon_0}, \tag{6.2.3}$$

and we state

Lemma 6.2.1. *Let Σ_{ε_0} be as in* (6.2.3). *Then, for any $\eta > 0$, there exists a solenoidal vector field $s = s_{\varepsilon_0} = s_{\varepsilon_0}(\eta)$, depending on ε_0 and η, such that*

$$\begin{aligned} & s \in H^2(\mathcal{R}) \cap L^\infty(\mathcal{R}), \qquad s = v_p \text{ on } \partial\mathcal{R}, \qquad s = 0 \text{ on } \partial A_{\varepsilon_0}, \\ & \left| \int_{\mathcal{R}} (u \cdot \nabla) s \cdot u \right| \leq \eta \|\nabla u\|^2_{L^2(\mathcal{R})} \quad \forall\, u \in H^1_0(\mathcal{R}). \end{aligned} \tag{6.2.4}$$

Moreover, there exist $c_1, c_2, c_3 > 0$, depending on $\mathcal{R}$ and λ, such that

$$\|s\|_{L^2(\mathcal{R})} \leq c_1 \varepsilon_0 e^{2/\varepsilon_0}, \qquad \|\nabla s\|_{L^2(\mathcal{R})} \leq c_2 \varepsilon_0 e^{4/\varepsilon_0}, \qquad \|\Delta s\|_{L^2(\mathcal{R})} \leq c_3 \varepsilon_0 e^{6/\varepsilon_0}. \tag{6.2.5}$$

Proof. Consider the Stokes system

$$\begin{cases} -\Delta v + \nabla q = 0 & \text{in } \Sigma_{\varepsilon_0}, \\ \operatorname{div} v = 0 & \text{in } \Sigma_{\varepsilon_0}, \\ v = v_p & \text{on } \partial\mathcal{R}, \\ v = 0 & \text{on } \partial A_{\varepsilon_0}. \end{cases} \tag{6.2.6}$$

By [60, Theorem IV.1.1], there exists a unique weak solution $(v, q) \in H^1(\Sigma_{\varepsilon_0}) \times L^2(\Sigma_{\varepsilon_0})$ to (6.2.6) such that

$$\|\nabla v\|_{L^2(\Sigma_{\varepsilon_0})} + \|q\|_{L^2(\Sigma_{\varepsilon_0})} \leq c\,\lambda,$$

for some c depending on ε_0. Although $\partial\Sigma_{\varepsilon_0}$ is not globally of class $\mathcal{C}^2$ (it contains the corners of $\mathcal{R}$), we may proceed as in [68, Theorem 3.3] to infer that the regularity of the solution can be improved to $(v, q) \in H^2(\Sigma_{\varepsilon_0}) \times H^1(\Sigma_{\varepsilon_0})$. Then, we follow the idea of [20], which is inspired by [104], that is, we localize the solution of (6.2.6) in an ε-neighborhood of Σ_{ε_0}. More precisely, let $v = (v_1, v_2)$ be the solution to (6.2.6), fix $x_0 \in \Sigma_{\varepsilon_0}$ and let

$$g(x) = \int_{x_0}^{x} (v_1\, dx_2 - v_2\, dx_1) \qquad \forall\, x \in \Sigma_{\varepsilon_0}$$

be the stream function associated to v (see also [60, Lemma IX.4.1]). As a consequence,

$$v_1 = \frac{\partial g}{\partial x_2}, \qquad v_2 = -\frac{\partial g}{\partial x_1}, \qquad g \in H^3(\Sigma_{\varepsilon_0}). \tag{6.2.7}$$

For any sufficiently small $\varepsilon > 0$, let $\psi_\varepsilon \in \mathcal{C}^\infty(\bar{\Sigma}_{\varepsilon_0})$ be the cut-off function defined in [60, Lemma III.6.2] so that

$$\begin{aligned} &|\psi_\varepsilon(x)| \le 1 \text{ for all } x \in \Sigma_{\varepsilon_0}, \qquad |\psi_\varepsilon(x)| = 1 \text{ if } \delta < \gamma^2(\varepsilon)/2k_1, \qquad |\psi_\varepsilon(x)| = 0 \text{ if } \delta \ge 2\gamma(\varepsilon), \\ &|\nabla\psi_\varepsilon(x)| \le k_2\varepsilon/\delta(x) \text{ for all } x \in \Sigma_{\varepsilon_0}, \qquad |D^\alpha\psi_\varepsilon(x)| \le k_3\varepsilon/\delta^{|\alpha|}(x) \text{ for } |\alpha| \in \{2,3\}, \end{aligned} \tag{6.2.8}$$

where $\delta = \text{dist}(x, \partial\Sigma_{\varepsilon_0})$, $\gamma(\varepsilon) = \exp(-1/\varepsilon)$ while $k_1, k_2, k_3 > 0$ are some constants. We set

$$s = \left(\frac{\partial}{\partial x_2}(g\,\psi_\varepsilon), -\frac{\partial}{\partial x_1}(g\,\psi_\varepsilon)\right) \quad \text{in } \Sigma_{\varepsilon_0}, \qquad s = 0 \quad \text{in } \mathcal{R} \setminus \Sigma_{\varepsilon_0}.$$

From (6.2.7), we see that s satisfies the first two properties in (6.2.4). Moreover, we can proceed as in [60, Lemma IX.4.2] to find ε so as to obtain the third property in (6.2.4). Finally, the estimates in (6.2.8) imply that

$$|\nabla\psi_\varepsilon(x)| \le k_4\varepsilon e^{2/\varepsilon}; \qquad |D^2\psi_\varepsilon(x)| \le k_5\varepsilon e^{4/\varepsilon}; \qquad |D^3\psi_\varepsilon(x)| \le k_6\varepsilon e^{6/\varepsilon},$$

for all x such that $\gamma^2(\varepsilon)/2k_1 < \text{dist}(x, \partial\Sigma_{\varepsilon_0}) \le 2\gamma(\varepsilon)$, for some constants $k_4, k_5, k_6 > 0$. This gives (6.2.5). □

6.2.3 Steady states

We denote by (u_s, h_s) the steady solutions to problem (2.0.7), namely the solutions to

$$\begin{aligned} &-\mu\,\Delta u_s + (u_s\cdot\nabla)u_s + \nabla p_s = 0, \quad \text{div}\, u_s = 0 \qquad \text{in } \Omega_{h_s}, \\ &u_s = \lambda(L^2 - x_2^2)\widehat{e}_1 \quad \text{on } \partial\mathcal{R}, \qquad u_s = 0 \quad \text{on } \partial B_{h_s}, \end{aligned} \tag{6.2.9}$$

together with the static fluid-structure interaction condition

$$f(h_s) = -\widehat{e}_2 \cdot \int_{\partial B_{h_s}} \mathcal{T}(u_s, p_s)\cdot\widehat{n}. \tag{6.2.10}$$

Weak solutions $(u_s, h_s) \in H^1(\Omega_{h_s}) \times (-L+\delta, L-\delta)$ to (6.2.9)-(6.2.10) represent equilibrium positions of the body, for a given flow regime of the fluid. In the following theorem we provide a well-posedness result for (6.2.9)-(6.2.10).

Theorem 6.2.2. *Assume that f satisfies* (2.0.12) *and* (6.2.1). *For any $\lambda > 0$ the problem* (6.2.9)-(6.2.10) *admits a weak solution. Furthermore, there exists $\lambda_s > 0$ such that if $\lambda < \lambda_s$ the problem* (6.2.9)-(6.2.10) *admits a unique weak solution $(u_s, h_s) \in H^1(\Omega_{h_s}) \times (-L+\delta, L-\delta)$ given by $(u_\lambda, 0)$, that is u_s, with $h_s = 0$. Moreover, there exists $C(\lambda) > 0$, with $C(\lambda) \to 0$ as $\lambda \to 0$, such that*

$$\|\nabla u_\lambda\|_{L^2(\Omega_0)} \le C(\lambda).$$

Proof. By symmetry of the problem one can assume that B_{h_s} entirely lies above the horizontal line $x_2 = -L+\delta+\tau$ where $\tau > 0$ and $-L+\delta+\tau < 0$. Given the "smoothened rectangle"

$$
\begin{aligned}
A_\tau =&(-2,2)\times(-L+\delta+\tau, L)\\
&\cup\{(x_1,x_2)\in\mathbb{R}^2\,|\,(x_1-2)^4+(x_2-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4<(L-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4, x_1\geq 2, x_2\leq \tfrac{\delta}{2}+\tfrac{\tau}{2}\}\\
&\cup\{(x_1,x_2)\in\mathbb{R}^2\,|\,(x_1+2)^4+(x_2-\tfrac{\delta}{2}-\tfrac{\tau}{2})^2<(L-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4, x_1\leq -2, x_2\leq \tfrac{\delta}{2}+\tfrac{\tau}{2}\}\\
&\cup\{(x_1,x_2)\in\mathbb{R}^2\,|\,(x_1-2-2L+\delta+\tau)^4+(x_2-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4<(L-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4, x_1\geq 2, x_2> \tfrac{\delta}{2}+\tfrac{\tau}{2}\}\\
&\cup\{(x_1,x_2)\in\mathbb{R}^2\,|\,(x_1+2+2L-\delta-\tau)^4+(x_2-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4<(L-\tfrac{\delta}{2}-\tfrac{\tau}{2})^4, x_1\leq -2, x_2> \tfrac{\delta}{2}+\tfrac{\tau}{2}\},
\end{aligned}
$$

take the domain

$$\Sigma=\mathcal{R}\backslash\bar{A}_\tau, \qquad \partial\Sigma=\partial\Sigma_1\cup\partial\Sigma_2=\big(\partial\mathcal{R}\backslash\partial A_\tau\big)\cup\big(\partial A_\tau\backslash\{-2\leq x_1\leq 2\wedge x_2=L\}\big). \tag{6.2.11}$$

Then, by Lemma 6.2.1, in which we replace Σ_{ε_0} with Σ as in (6.2.11), there exists a function s satisfying (6.2.4).

The proof of existence is similar to the proof of [15, Theorem 1], with some modifications; see also [66]. We define

$$u_s=\widehat{u}_s+s.$$

Clearly $\widehat{u}_s$ will depend on the particular s chosen, but when we get rid of the solenoidal extension by undoing the change of unknown, we go back to the solution to the original problem u_s. Then, we take as weak formulation of (6.2.9)-(6.2.10)

$$
\begin{aligned}
&\mu\int_{\Omega_{h_s}}\nabla\widehat{u}_s\cdot\nabla\phi+\int_{\Omega_{h_s}}(\widehat{u}_s\cdot\nabla)\widehat{u}_s\cdot\phi+\int_{\Omega_{h_s}}(\widehat{u}_s\cdot\nabla)s\cdot\phi+\int_{\Omega_{h_s}}(s\cdot\nabla)\widehat{u}_s\cdot\phi=\\
&\int_{\Omega_{h_s}}(s\cdot\nabla)s\cdot\phi+\int_{\Omega_{h_s}}\mu\Delta s\cdot\phi
\end{aligned}
\tag{6.2.12}
$$

for any solenoidal test function $\phi\in\mathcal{C}^\infty_{\mathcal{C}}(\mathcal{R})$. We notice that the existence of a weak solution of (6.2.9)-(6.2.10) follows once we find an apriori bound on $\|\nabla\widehat{u}_s\|_{L^2(\Omega_{h_s})}$ (see for instance [60, Theorem IX.4.1]). Take $\phi=\widehat{u}$ in (6.2.12). After using the fact that

$$\int_{\Omega_{h_s}}(\widehat{u}_s\cdot\nabla)\widehat{u}_s\cdot\widehat{u}_s=\int_{\Omega_{h_s}}(s\cdot\nabla)\widehat{u}_s\cdot\widehat{u}_s=0,$$

we obtain

$$\mu\|\nabla\widehat{u}_s\|^2_{L^2(\Omega_{h_s})}+\int_{\Omega_{h_s}}(\widehat{u}_s\cdot\nabla)s\cdot\widehat{u}_s=\int_{\Omega_{h_s}}(s\cdot\nabla)s\cdot\phi+\int_{\Omega_{h_s}}\mu\Delta s\cdot\widehat{u}_s. \tag{6.2.13}$$

The terms on the right-hand side of (6.2.13) can then be bounded as

$$\int_{\Omega_{h_s}}(s\cdot\nabla)s\cdot\widehat{u}_s\leq\|s\|_{L^4(\Omega_{h_s})}\|\nabla s\|_{L^2(\Omega_{h_s})}\|\widehat{u}_s\|_{L^4(\Omega_{h_s})}\leq C\|s\|_{L^4(\Omega_{h_s})}\|\nabla s\|_{L^2(\Omega_{h_s})}\|\nabla\widehat{u}_s\|_{L^2(\Omega_{h_s})}$$

where C is the embedding constant for $H^1_0(\Omega_{h_s})\subset L^4(\Omega_{h_s})$, and

$$\int_{\Omega_{h_s}}\mu\Delta s\cdot\widehat{u}_s=-\mu\int_{\Omega_{h_s}}\nabla s\cdot\nabla\widehat{u}_s\leq\mu\|\nabla s\|_{L^2(\Omega_{h_s})}\|\nabla\widehat{u}_s\|_{L^2(\Omega_{h_s})}.$$

Finally, by exploiting the third property in (6.2.4) and taking η sufficiently small we obtain the desired uniform bound on $\|\nabla\widehat{u}_s\|_{L^2(\Omega_{h_s})}$, which guarantees existence of weak solutions for any value of the parameter λ.

By introducing the (different) *specific* solenoidal extension s used in [15, Theorem 1], here defined on a bounded domain, we obtain uniqueness of the solution of (6.2.12) and its specific form $(u_s, h_s) = (u_\lambda, 0)$ in a similar way. □

In [15, Theorem 1], the authors impose a bound both on the Poiseuille flow rate λ and on the Reynolds number $Re = cV/\mu$, where V is a reference speed and $c > 0$ a real constant. In the statement of Theorem 6.2.2, we joined those two bounds in a unique condition on λ by choosing as reference speed in the Reynolds number precisely the velocity of the Poiseuille flow at the outlets of the channel. As expected, Theorem 6.2.2 guarantees that the equilibrium position is unique and symmetric, at least for small flow rate of the incoming Poiseuille flow.

To develop our analysis in the subsequent sections, we rewrite problem (6.2.9)-(6.2.10) in an equivalent form. For a given $\varepsilon_0 \in (0, L-\delta)$, let

$$s = s_{\varepsilon_0} = s_{\varepsilon_0}(\eta) \tag{6.2.14}$$

be the function obtained through Lemma 6.2.1. The unique solution $(u_s, h_s) = (u_\lambda, 0)$ to problem (6.2.9), may be rewritten as

$$(u_s, h_s) = (\widehat{u}_\lambda + s, 0).$$

Denoting by

$$\widehat{g} := \mu\,\Delta s - (s\cdot\nabla)\,s, \tag{6.2.15}$$

we have that $(\widehat{u}_s, h_s) = (\widehat{u}_\lambda, 0)$ satisfies in a weak sense

$$\begin{aligned} &-\mu\,\Delta\widehat{u}_\lambda + (\widehat{u}_\lambda\cdot\nabla)\,\widehat{u}_\lambda + \nabla p_\lambda + (\widehat{u}_\lambda\cdot\nabla)\,s + (s\cdot\nabla)\,\widehat{u}_\lambda = \widehat{g}, \qquad \operatorname{div}\widehat{u}_\lambda = 0 \qquad \text{in } \Omega_0,\\ &\widehat{u}_\lambda = 0 \quad \text{on } \partial\mathcal{R}, \qquad \widehat{u}_\lambda = 0 \quad \text{on } \partial B_0, \end{aligned} \tag{6.2.16}$$

and

$$0 = f(0) = -\widehat{e}_2\cdot\int_{\partial B_0}\mathcal{T}(\widehat{u}_\lambda + s, p_\lambda)\cdot\widehat{n}. \tag{6.2.17}$$

Finally, we prove a property of the function $\widehat{g}$ in (6.2.15).

Lemma 6.2.3. *Let $\widehat{g}$ be as in* (6.2.15)*, $s = s_{\varepsilon_0}$ as in Lemma 6.2.1 for some $\varepsilon_0 \in (0, L-\delta)$. Then $\widehat{g} \in L^2(\Omega_h)$ and*

$$\|\widehat{g}\|_{L^2(\Omega_h)} \le \mu\|\Delta s\|_{L^2(\Omega_h)} + \|s\|_{L^4(\Omega_h)}\|\nabla s\|_{L^4(\Omega_h)}.$$

Proof. Multiply $\widehat{g}$ by $\varphi \in \mathcal{C}_c^\infty(\mathcal{R})$ and integrate by parts over Ω_h. We obtain

$$\int_{\Omega_h}\widehat{g}\cdot\varphi\,dx = \mu\int_{\Omega_h}\Delta s\cdot\varphi\,dx - \int_{\Omega_h}(s\cdot\nabla)s\cdot\varphi.$$

We bound the two terms on the right-hand side through the Hölder inequality and we get

$$\left|\int_{\Omega_h}\widehat{g}\cdot\varphi\,dx\right| \le \Big(\mu\|\Delta s\|_{L^2(\Omega_h)} + \|s\|_{L^4(\Omega_h)}\|\nabla s\|_{L^4(\Omega_h)}\Big)\|\varphi\|_{L^2(\Omega_h)} \qquad \forall\,\varphi \in \mathcal{C}_c^\infty(\mathcal{R}),$$

from which the thesis of the lemma follows. □

6.2.4 Weak solutions to the evolution problem

The notions and the results contained in this section as well as those contained in the subsequent section are analogous to those seen in Chapter 4. However, since in Chapter 4, we treated problem (2.0.7) after imposing a change of reference frame, we report here some results directly concerning problem (2.0.7) in its original formulation. Moving to the evolution problem (2.0.7), we assume that

$$u_0 \in L^2(\Omega_{h_0}), \qquad h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}],$$

where $\widehat{\varepsilon} \in (0, L-\delta)$ is arbitrarily fixed. If $\widehat{g}$ is as in (6.2.15), the solutions to the problem (2.0.7) have the form

$$u = \widehat{u} + s,$$

where $s = s_{\varepsilon_0}$ is as in (6.2.14). Again, we point out that $\widehat{u}$ depends on the choice of the solenoidal extension s built through Lemma 6.2.1, but, by undoing the change of variables, one recovers the solution to the original problem. Hence,

$$\text{the solution to the original problem (2.0.7) does not depend on the solenoidal extension.} \tag{6.2.18}$$

We have that $\widehat{u}$ solves the problem:

$$\begin{aligned}
&\widehat{u}_t - \mu\,\Delta\widehat{u} + (\widehat{u}\cdot\nabla)\,\widehat{u} + \nabla p + (\widehat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\widehat{u} = \widehat{g}, \quad \operatorname{div}\widehat{u} = 0 \qquad \text{in } \Omega_h\times(0,T),\\
&\widehat{u} = 0 \quad \text{on } \partial\mathcal{R}\times(0,T), \qquad \widehat{u} = h'\widehat{e}_2 \quad \text{on } \partial B_h\times(0,T),\\
&\widehat{u}(x,0) = \widehat{u}_0(x) = u_0(x) - s_{\widehat{\varepsilon}}(x) \quad \text{for a.e. } x\in\Omega_{h_0}.
\end{aligned} \tag{6.2.19}$$

According to (2.0.7), the vertical translation of the obstacle h responds to

$$m\,h'' + f(h) = -\widehat{e}_2\cdot\int_{\partial B_h} \mathcal{T}(\widehat{u}+s,p)\cdot\widehat{n} \qquad \text{in } (0,T), \tag{6.2.20}$$

with some initial conditions $h(0) = h_0$, $h'(0) = k_0$. Notice that $\widehat{u}_0 \in L^2(\Omega_{h_0})$ is such that $\widehat{u}_0\cdot\widehat{n} = k_0\widehat{e}_2\cdot\widehat{n}$ on ∂B_{h_0}. It is worthwhile to emphasize that the knowledge of $h'(t)$ allows to reconstruct the position of the body:

$$B_{h(t)} = B + h(t)\widehat{e}_2, \qquad \text{with} \qquad h(t) = h_0 + \int_0^t h'(\tau)\,d\tau. \tag{6.2.21}$$

We now recall the classical functional spaces from fluid mechanics (see, e.g., [60, 134]):

$$\begin{aligned}
\mathcal{V}(\mathcal{R}) &= \{v\in\mathcal{C}_c^\infty(\mathcal{R})\,|\,\operatorname{div} v = 0 \text{ in } \mathcal{R}\},\\
H(\mathcal{R}) &= \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\cdot\|_{L^2(\mathcal{R})},,\\
V(\mathcal{R}) &= \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\nabla\cdot\|_{L^2(\mathcal{R})}.
\end{aligned}$$

Next, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between V and V'. Finally, we introduce the product spaces

$$\mathbb{H}(\mathcal{R}) = H(\mathcal{R})\times\mathbb{R}, \qquad \mathbb{V}(\mathcal{R}) = V(\mathcal{R})\times\mathbb{R}.$$

To define a weak solution to problem (6.2.19)-(6.2.20), for every $h \in (-L+\delta, L-\delta)$ we introduce the closed subspaces $\mathcal{H}_h \subset \mathbb{H}$ and $\mathcal{H}^1_h \subset \mathbb{V}$ of *compatible pairs*

$$\mathcal{H}_h = \{z = (u,\ell) \in \mathbb{H}(\mathcal{R}) \,|\, u_{B_h} = \ell\, \widehat{e}_2\}, \qquad \mathcal{H}^1_h = \{z = (u,\ell) \in \mathbb{V}(\mathcal{R}) \,|\, u_{B_h} = \ell\, \widehat{e}_2\}, \tag{6.2.22}$$

endowed with the scalar products

$$\langle z_1, z_2\rangle_{\mathcal{H}_h} = \int_{\Omega_h} u_1 \cdot u_2 \, dx + m\, \ell_1 \ell_2, \qquad \langle z_1, z_2\rangle_{\mathcal{H}^1_h} = \int_{\Omega_h} \nabla u_1 : \nabla u_2 \, dx + m\, \ell_1 \ell_2, \tag{6.2.23}$$

where $z_i = (u_i, \ell_i)$, and m is the mass of the body as in (6.2.20). We call $\|\cdot\|_{\mathcal{H}_h}, \|\cdot\|_{\mathcal{H}^1_h}$ the norms induced by the scalar products in (6.2.23), and we denote by $\mathcal{H}^{-1}_h$ the dual of $\mathcal{H}^1_h$. The integral in the second formula in (6.2.23) can be defined on the whole channel $\mathcal{R}$; indeed, $\nabla u_1 = \nabla u_2 = 0$ on B_h, since any element of $\mathcal{H}^1_h$ is a rigid motion on B_h. Recalling that $D(\cdot)$ denotes the symmetric part of the gradient, for all $u_1, u_2 \in V$

$$2 \int_{\mathcal{R}} D(u_1) : D(u_2) \, dx = \int_{\mathcal{R}} \nabla u_1 : \nabla u_2 \, dx. \tag{6.2.24}$$

If $h = h(t)$ is a function from $[0,T]$ to $(-L+\delta, L-\delta)$, we define the following spaces:

$$L^p(0,T;\mathcal{H}^1_{h(t)}) = \left\{ f : [0,T] \to \mathcal{H}^1_{h(t)} \quad \text{s.t.} \quad \|f\|^p_{L^p(0,T;\mathcal{H}^1_{h(t)})} = \int_0^T \|f(t)\|^p_{\mathcal{H}^1_{h(t)}} \, dt < +\infty \right\}$$

for $1 \le p < \infty$, and

$$L^\infty(0,T;\mathcal{H}_{h(t)}) = \left\{ f : [0,T] \to \mathcal{H}_{h(t)} \quad \text{s.t.} \quad \|f\|_{L^\infty(0,T;\mathcal{H}_{h(t)})} = \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_{\mathcal{H}_{h(t)}} < +\infty \right\}.$$

We can now define weak solutions to (6.2.19)-(6.2.20).

Definition 6.2.4. A pair $(\widehat{u}, h)$ is called a weak solution of (6.2.19)-(6.2.20) with initial data $(\widehat{u}_0, h_0, k_0)$ if there exists $\varepsilon_0 = \varepsilon_0(\widehat{u}_0, h_0, k_0, T) \in (0, L-\delta)$ such that, for $s = s_{\varepsilon_0}$ as in (6.2.14),

$$h \in W^{1,\infty}(0,T;[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0]), \qquad (\widehat{u}, h') \in L^2(0,T;\mathcal{H}^1_{h(t)}) \cap L^\infty(0,T;\mathcal{H}_{h(t)}),$$

$$(\widehat{u}_t, h'') \in L^2(0,T;\mathcal{H}^{-1}_{h(t)}),$$

and the pair $(\widehat{u}(t), h(t))$ verifies, for any $(\phi(t), l(t)) \in \mathcal{H}^1_{h(t)}$ and almost every $t \ge 0$,

$$\begin{aligned} \langle \widehat{u}_t(t), \phi(t)\rangle &+ m\, h''(t)\ell(t) + f(h(t))\,\ell(t) + \mu \int_{\mathcal{R}} \nabla \widehat{u}(t) : \nabla \phi(t) \, dx \\ + \int_{\Omega_h} (\widehat{u}(t)\cdot\nabla)\widehat{u}(t)\cdot\phi(t)\, dx &+ \int_{\Omega_h} (\widehat{u}(t)\cdot\nabla)s\cdot\phi(t)\, dx + \int_{\Omega_h} (s\cdot\nabla)\widehat{u}(t)\cdot\phi(t)\, dx \\ = \int_{\Omega_h} \widehat{g}\cdot\phi(t)\, dx, & \end{aligned} \tag{6.2.25}$$

and $\widehat{u}(0) = \widehat{u}_0$, $h(0) = h_0$, $h'(0) = k_0$.

Remark 6.2.5. The requirement $h \in W^{1,\infty}(0,T;[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0])$ makes Definition 6.2.4 consistent: it ensures that no collision occurs between the obstacle and the boundary of the channel because there exists a separation strip of size $\varepsilon_0 \in (0, L-\delta)$ for all times, by which one is allowed to build the solenoidal extension s as in (6.2.14) precisely by choosing such an ε_0. As the no-collision result is a non trivial issue, it will be recalled explicitly in Corollary 6.2.8. Also, we point out that the test functions depend on time and on the solution of the problem itself.

Let us show that any classical solution to (6.2.19)-(6.2.20) is a weak solution according to Definition 6.2.4. Incidentally, this also confirms (6.2.18).

Proposition 6.2.6. *Let $\widehat{g}$ be as in* (6.2.15). *If a pair $(\widehat{u}, h)$ is a classical solution to* (6.2.19)-(6.2.20) *such that $|h(t)| \leq L-\delta-\varepsilon_0$ for all $t \in [0,T]$ for some $\varepsilon_0 \in (0, L-\delta)$, then it satisfies* (6.2.25) *for all $t \in [0,T]$ and for every pair of test functions $(\phi(t), l(t)) \in \mathcal{H}^1_{h(t)}$.*

Proof. In order to obtain (6.2.25), we choose a test pair $(\phi(t), l(t)) \in \mathbb{V}(\mathcal{R})$ such that $\phi(t)|_{B_{h(t)}} = l(t)\widehat{e}_2$. We multiply the first equation in (6.2.19) by ϕ and integrate by parts over Ω_h. All terms may be treated in a standard manner (see, e.g., [57]). Though, a particular attention must be devoted to the diffusive and pressure terms. We temporarily move the term $\mu\Delta s$ appearing in $\widehat{g}$ in (6.2.19) on the left-hand side and we get, recalling (6.2.24),

$$\begin{aligned}
\langle -\mu\Delta\widehat{u} - \mu\Delta s + \nabla p, \phi\rangle = \langle \mathrm{div}\mathcal{T}(\widehat{u}+s,p), \phi\rangle &= -\int_{\partial B_h} (\mathcal{T}\cdot\widehat{n})\cdot\phi + \int_{\Omega_h} \mathcal{T} : \nabla\phi \\
&= -\widehat{e}_2 \cdot \int_{\partial B_h} (\mathcal{T}(\widehat{u}+s,p)\cdot\widehat{n})\, l + \int_{\Omega_h} \mathcal{T}(\widehat{u}+s,p) : \nabla\phi \\
&= (m\, h'' + f(h))\, l + \mu\int_{\Omega_h} \nabla\widehat{u} : \nabla\phi + \mu\int_{\Omega_h} \nabla s : \nabla\phi \\
&= (m\, h'' + f(h))\, l + \mu\int_{\mathcal{R}} \nabla\widehat{u} : \nabla\phi + \mu\int_{\Omega_h} \nabla s : \nabla\phi,
\end{aligned}$$

where the last equality holds because $\nabla\phi = 0$ on B_h, since ϕ is a rigid motion on B_h. Thus, given

$$\int_{\Omega_h} \widehat{g}\cdot\phi = -\mu\int_{\Omega_h} \nabla s : \nabla\phi - \int_{\Omega_h} (s\cdot\nabla)s\cdot\phi = \mu\int_{\Omega_h} \Delta s\cdot\phi - \int_{\Omega_h} (s\cdot\nabla)s\cdot\phi$$

we obtain the weak formulation (6.2.25). □

6.2.5 Well-posedness of the evolution problem

We provide the following well-posedness result for (6.2.19)-(6.2.20).

Theorem 6.2.7. *Let $\widehat{\varepsilon} \in (0, L-\delta)$ be fixed. For any initial data $h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}]$, $(\widehat{u}_0, k_0)$ in the space $\mathcal{H}_{h_0}$ and every $T > 0$, there exists a unique weak solution $(\widehat{u}, h)$ to* (6.2.19)-(6.2.20) *for some $\varepsilon_0 = \varepsilon_0(h_0, \widehat{u}_0, k_0, T) \in (0, L-\delta)$. Moreover $(\widehat{u}, h')$ is equal almost* *everywhere to a continuous function from* $[0,T]$ *to* $\mathcal{H}_h$ *and it satisfies the following energy estimate, for every* $t \geq 0$

$$\|\widehat{u}(t)\|^2_{L^2(\Omega_{h(t)})}+mh'(t)^2+2F(h(t))+\mu\int_0^t\|\nabla\widehat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}d\tau \leq \|\widehat{u}_0\|^2_{L^2(\Omega_{h_0})}+m{k_0}^2+2F(h_0)$$
$$+\frac{2}{\mu}\frac{4L^2}{\pi^2}\int_0^t\|\widehat{g}\|^2_{L^2(\Omega_h)}d\tau. \tag{6.2.26}$$

Proof. The result can be proven following the procedure implemented in Chapter 4, up to using the solenoidal extension $s = s_{\varepsilon_0}$ in (6.2.14). In order to prove (6.2.26), it sufficies to take $(\phi, l) = (\widehat{u}, h')$ in (6.2.25). We obtain

$$\langle \widehat{u}_t, \widehat{u}\rangle + m\, h''h' + f(h)h' + \mu\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} = \int_{\Omega_h}\widehat{g}\cdot\widehat{u} - \int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot\widehat{u}. \tag{6.2.27}$$

We seek bounds for the two terms on the right-hand side. The first term is bounded by Lemma 6.2.3, the Hölder inequality, the Young inequality and the Poincaré inequality (here the Poincaré constant is $4L^2/\pi^2$). This gives

$$\left|\int_{\Omega_h}\widehat{g}\cdot\widehat{u}\right| \leq \frac{1}{\mu}\frac{4L^2}{\pi^2}\|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{\mu}{4}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.$$

In order to bound the second term we exploit the third property in (6.2.4), with $\eta = \mu/4$,

$$\left|\int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot\widehat{u}\right| \leq \frac{\mu}{4}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.$$

Thus from (6.2.27) and the above bounds we obtain

$$\frac{d}{dt}\Big(\|\widehat{u}\|^2_{L^2(\Omega_h)} + mh'^2 + 2F(h)\Big) + \mu\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \leq \frac{2}{\mu}\frac{4L^2}{\pi^2}\|\widehat{g}\|^2_{L^2(\Omega_h)},$$

where $F(h)$ is defined as in (2.0.13). Integrating on $(0,t)$, we infer (6.2.26). □

In the sequel, we will use the following consequence of the energy estimate (6.2.26). It reads as

Corollary 6.2.8. *For all* $T > 0$*, there exists* $\varepsilon_0 = \varepsilon_0(h_0, \widehat{u}_0, k_0, T) \in (0, L-\delta)$ *such that*

$$|h(t)| \leq L - \delta - \varepsilon_0 \qquad \forall\, t \in [0,T]. \tag{6.2.28}$$

Moreover, ε_0 *decreases as* $|h_0|, \|\widehat{u}_0\|_{L^2(\Omega_0)}, |k_0|$ *and* T *increase. In particular, the solution to* (2.0.7) *is global in time.*

Proof. To prove (6.2.28), we can proceed as in [111, Lemma 3.2]. By contradiction, if the solution of (2.0.7) was not global in time, then a collision would occur at some finite time $t = T$. But, according to (6.2.18), the collision of the solution is independent of the solenoidal extension and we reach a contradiction by taking $s = s_{\varepsilon_0}$ with $\varepsilon_0 = \varepsilon_0(T)$ ensuring (6.2.28). □

6.3 Dissipation of the solution operator

From Theorem 6.2.7 we learn that, for every $\widehat{\varepsilon} > 0$ small and every $h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}]$, problem (6.2.19)-(6.2.20) generates an operator

$$U(t) : \mathcal{H}_{h_0} \longrightarrow \mathcal{H}_{h(t)},$$

defined by the rule

$$z_0 = (\widehat{u}_0, k_0) \longmapsto U(t)z_0 = (\widehat{u}(t), h'(t)), \tag{6.3.1}$$

where, reconstructing h as in (6.2.21), the pair $(\widehat{u}(t), h(t))$ is the unique weak solution at time t to problem (6.2.19)-(6.2.20) with initial data

$$\widehat{u}(0) = \widehat{u}_0, \quad h(0) = h_0, \quad h'(0) = k_0.$$

Remark 6.3.1. It is worth noting that, although not explicitly written so not to burden the notation, the operator $U(t)$ depends on the particular h_0 chosen. Besides, it acts between different spaces; but this reflects the nature of the fluid-structure interaction problem (6.2.19)-(6.2.20), where the functional framework is influenced by the evolution itself.

We begin our analysis by introducing the proper notion of dissipation for the dynamical system under consideration.

Definition 6.3.2. We call $R_0 > 0$ a *zero-order absorbing radius* if, for any $R > 0$ and $\widehat{\varepsilon} > 0$, there exists $t_0 = t_0(R, \widehat{\varepsilon})$, called *entering time*, such that, for every

$$h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}] \qquad \text{and} \qquad \|z_0\|_{\mathcal{H}_{h_0}} \leq R$$

it follows that

$$\|U(t)z_0\|_{\mathcal{H}_{h(t)}} = \Big[\|\widehat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2\Big]^{\frac{1}{2}} \leq R_0 \qquad \forall\, t \geq t_0.$$

We call $R_1 > 0$ a *first-order absorbing radius* if, under the same assumptions, there exists $t_1 = t_1(R, \widehat{\varepsilon})$, such that

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} = \Big[\|\nabla\widehat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2\Big]^{\frac{1}{2}} \leq R_1 \qquad \forall\, t \geq t_1.$$

We now address the dissipation properties of the solution operator $U(t)$ in terms of zero-order and first-order absorptions.

Theorem 6.3.3. *There exists a universal constant $R_0 = R_0(\lambda, L, \delta, d, m, \mu) > 0$ with the following property: for any $R > 0$ and any $\widehat{\varepsilon} \in (0, L-\delta)$, there is an entering time $t_0 = t_0(R, \widehat{\varepsilon})$ such that*

$$\|U(t)z_0\|_{\mathcal{H}_{h(t)}} \leq R_0 \qquad \forall\, t \geq t_0,$$

whenever

$$h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}] \qquad \textit{and} \qquad \|z_0\|_{\mathcal{H}_{h_0}} \leq R.$$

In compliance with Definition 6.3.2, the constant R_0 is a zero-order absorbing radius. Moreover, $R_0 \to 0$ as $\lambda \to 0$.

Proof. Define a solenoidal vector field $w \in C^\infty(\mathcal{R}) \cap H_0^1(\mathcal{R})$ such that $w = h\,\widehat{e}_2$ in B_h in the following way:

$$w(x,t) = h(t)\left[-\frac{\partial}{\partial x_2}(\zeta(x,t)\,x_1), \frac{\partial}{\partial x_1}(\zeta(x,t)\;x_1)\right] \qquad \forall\, x \in \mathcal{R}\,, \qquad \forall\, t \geq 0\;,$$

where ζ is a $\mathcal{C}^\infty$ cut-off function equal to 1 in a small neighbourhood of $B_{h(t)}$ and equal to 0 outside a larger neighbourhood. The following estimates hold

$$\|w\|_{L^2(\mathcal{R})} \leq a_1|h|\,, \qquad \|\nabla w\|_{L^2(\mathcal{R})} \leq a_2|h|\,, \qquad \|\nabla w\|_{L^\infty(\mathcal{R})} \leq a_3|h|\,, \tag{6.3.2}$$

where a_1, a_2 and a_3 are constants depending on the cut-off function ζ. We observe that the function ζ depends on the width of the separation strip between the obstacle and the channel. Hence, for all $t \in (0,T)$,

$$w(x,t) = w_{\varepsilon_0}(x,t)\,, \tag{6.3.3}$$

where ε_0 is given by Corollary 6.2.8. Our aim is to explore what happens for any $t \geq 0$. Given $F(h)$ as in (2.0.13), we introduce the energy functional:

$$E(t) = \|\widehat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2 + 2F(h(t)),$$

and, for $\omega \in (0,1)$ to be fixed later, its perturbation

$$E_\omega(t) = \|\widehat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2 + 2F(h(t)) + \omega\, h(t)\, h'(t) + \omega \int_{\Omega_{h(t)}} \widehat{u}(t)\cdot w(t)\, dx.$$

From the Young inequality, we have

$$\omega|h\,h'| + \omega\left|\int_{\Omega_h} \widehat{u}\cdot w\, dx\right| \leq \frac{\omega}{2\rho}h'^2 + \left(\frac{\omega\rho}{2} + \frac{c_1\omega^2}{2}\right)h^2 + \frac{1}{2}\|\widehat{u}\|^2_{L^2(\Omega_h)},$$

with ρ as in (2.0.14). Hence, we draw the bounds

$$c_1\, E \leq E_\omega \leq c_2\, E, \tag{6.3.4}$$

for some $c_2 \geq c_1 > 0$, provided that ω is small enough. So far, we have used an arbitrary ε_0 to build $s = s_{\varepsilon_0}$ in (6.2.14), but in view of (6.2.18), ε_0 may be modified. To this end, we claim that

$$\exists\, \varepsilon_1 = \varepsilon_1(\lambda) > 0 \quad \text{and} \quad t_0 = t_0(R, \widehat{\varepsilon}) \qquad \text{s.t.} \qquad |h(t)| \leq L - \delta - \varepsilon_1 \qquad \forall\, t \geq t_0\,. \tag{6.3.5}$$

By contradiction, suppose that the above statement does not hold. Then, by Corollary 6.2.8, this implies that

$$\limsup_{t\to\infty} h(t) = L - \delta$$

and/or similarly for liminf $= -L + \delta$. Then, since $h \in \mathcal{C}^0(\mathbb{R}_+)$, for all $\varepsilon > 0$ there exist two divergent sequences $\{t_1^n\}$ and $\{t_2^n\}$ such that

$$h(t) \geq L - \delta - \varepsilon \qquad \forall t \in \bigcup_{n=1}^{\infty}[t_1^n, t_2^n]\,.$$

By (6.2.1), we can take $\varepsilon > 0$ small enough so that there exists $c > 0$ such that

$$\begin{aligned} F\big(h(t)\big) &= \int_0^{h(t)} f(s)ds > \int_{L-\delta-2\varepsilon}^{L-\delta-\varepsilon} f(s)ds > c\int_{L-\delta-2\varepsilon}^{L-\delta-\varepsilon} \exp\frac{1}{(L-\delta-s)^{4+r}}ds \\ &= c\int_{\varepsilon}^{2\varepsilon} \exp\frac{1}{\tau^{4+r}}d\tau \geq c\varepsilon\exp\frac{1}{\varepsilon^{4+r}} \end{aligned} \tag{6.3.6}$$

whenever $t \in \cup_n[t_1^n, t_2^n]$. By (6.2.18), we can replace the solenoidal extension $s = s_{\varepsilon_0}$ to the solenoidal extension $s = s_\varepsilon$. Taking $(\phi, l) = (\widehat{u} + w, h' + \omega h) \in \mathcal{H}_h^1$ as a pair of test functions in (6.2.25), by omitting (t) we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E_\omega - \omega h'^2 + \omega f(h)h + \mu\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} &= \int_{\Omega_h} \widehat{g}\cdot\widehat{u}\,dx - \int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot\widehat{u}\,dx \\ &\quad - \mu\,\omega\int_{\Omega_h}\nabla\widehat{u} : \nabla w\,dx + \omega\int_{\Omega_h}\widehat{g}\cdot w\,dx \\ &\quad - \omega\int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot w\,dx + \omega\int_{\Omega_h}(\widehat{u}\cdot\nabla)w\cdot\widehat{u}\,dx. \end{aligned} \tag{6.3.7}$$

We proceed to bound each term on the right hand-side of (6.3.7). We control the first term in the right-hand side of (6.3.7) by exploiting the Hölder inequality, Lemma 6.2.3 above, the Young inequality, and the Poincaré inequality (here the Poincaré constant is $4L^2/\pi^2$). This gives

$$\int_{\Omega_h}\widehat{g}\cdot\widehat{u}\,dx \leq \|\widehat{g}\|_{L^2(\Omega_h)}\|\widehat{u}\|_{L^2(\Omega_h)} \leq \frac{3}{2\mu}\frac{4L^2}{\pi^2}\|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{\mu}{6}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.$$

Similarly for the fourth term

$$\omega\int_{\Omega_h}\widehat{g}\cdot w\,dx \leq \omega\|\widehat{g}\|_{L^2(\Omega_h)}\|w\|_{L^2(\Omega_h)} \leq \frac{\mu}{3}\frac{4L^2}{\pi^2}\|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{3}{4\mu}\omega^2\|\nabla w\|^2_{L^2(\mathcal{R})}.$$

Concerning the second term in the right-hand side, we make use of the third property in (6.2.4) with $\eta = \mu/3$, to get

$$-\int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot\widehat{u}\,dx \leq \frac{\mu}{3}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.$$

The third term is bounded through the the Hölder inequality and the the Young inequality as

$$-\mu\omega\int_{\Omega_h}\nabla\widehat{u} : \nabla w\,dx \leq \frac{\mu}{6}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{3}{2\mu}\omega^2\|\nabla w\|^2_{L^2(\mathcal{R})}.$$

Again, for the fifth term, we exploit the third property in (6.2.4) with $\eta = 1$, and the Young inequality. We obtain

$$-\omega\int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot w\,dx \leq \omega\eta\|\nabla\widehat{u}\|_{L^2(\Omega_h)}\|\nabla w\|_{L^2(\Omega_h)} \leq \frac{\mu}{3}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \omega^2\frac{3}{4\mu}\|\nabla w\|^2_{L^2(\mathcal{R})}.$$

Finally, the sixth term requires exploing $w \in \mathcal{C}^\infty(\mathcal{R})$ to deduce

$$\omega \int_{\Omega_h} (\widehat{u} \cdot \nabla) w \cdot \widehat{u}\, dx \leq \omega \frac{4L^2}{\pi^2} \|\nabla w\|_{L^\infty(\mathcal{R})} \|\nabla \widehat{u}\|^2_{L^2(\Omega_h)} .$$

At this point, by (6.4.6) and using the fact that $|h| \leq L - \delta - \varepsilon_1 < L - \delta$, we set

$$\mu - \omega \frac{4L^2}{\pi^2} \|\nabla w\|_{L^\infty(\mathcal{R})} > \mu - a_3 \omega |h| \frac{4L^2}{\pi^2} \geq \mu - a_3 \omega\, (L - \delta)\, \frac{4L^2}{\pi^2} := 3\nu > 0$$

if ω is small enough. Inserting all above inequalities in (6.3.7) and, recalling that from (2.0.14) $f(h)h \geq F(h)$, we arrive at

$$\frac{d}{dt} E_\omega - 2\omega h'^2 + \omega F(h) + \omega F(h) + 3\nu \|\nabla \widehat{u}\|^2_{L^2(\Omega_h)} \leq \frac{9 + 2\mu^2}{6\mu} \frac{4L^2}{\pi^2} \|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{3}{\mu} \omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})}.$$

We apply the following trace inequality, through which we extract a damping term for the obstacle B_h:

$$\|\nabla \widehat{u}\|_{L^2(\Omega_h)} \geq c \|h' \widehat{e}_2\|_{L^2(\partial B_h)} = c |\partial B_h| |h'|,$$

for some positive constant c. Moreover we use (6.4.6) to deduce that

$$\omega F(h) - \frac{3}{\mu} \omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})} \geq (\omega \frac{\rho}{2} - a_2 \frac{\mu}{3} \omega^2)\, h^2 \geq c\, h^2 ,$$

where c is a positive constant, if ω is small enough. Thus,

$$\frac{d}{dt} E_\omega + (c\nu |\partial B_h|^2 - 2\omega) h'^2 + \omega F(h) + 2\nu \|\nabla \widehat{u}\|^2_{L^2(\Omega_h)} \leq \frac{9 + 2\mu^2}{6\mu} \frac{4L^2}{\pi^2} \|\widehat{g}\|^2_{L^2(\Omega_h)}.$$

where $c\nu |\partial B_h|^2 - 2\omega > 0$ if ω is small enough. Finally, applying the Poincaré inequality in the left-hand side, we find

$$\frac{d}{dt} E_\omega + (c\nu |\partial B_h|^2 - 2\omega) h'^2 + \omega F(h) + \frac{\nu \pi}{4L^2} \|\widehat{u}\|^2_{L^2(\Omega_h)} + \nu \|\nabla \widehat{u}\|^2_{L^2(\Omega_h)} \leq \frac{9 + 2\mu^2}{6\mu} \frac{4L^2}{\pi^2} \|\widehat{g}\|^2_{L^2(\Omega_h)}.$$

Defining

$$\beta = \min \left(\frac{c\nu |\partial B_h|^2 - 2\omega}{m}, \frac{\omega}{2}, \frac{\nu \pi}{4L^2} \right) > 0,$$

we end up with

$$\frac{d}{dt} E_\omega + \beta E + \nu \|\nabla \widehat{u}\|^2_{L^2(\Omega_h)} \leq \kappa, \tag{6.3.8}$$

having set

$$\kappa = \frac{9 + 2\mu^2}{6\mu} \frac{4L^2}{\pi^2} \|\widehat{g}\|^2_{L^2(\Omega_h)}.$$

Then, renaming β / c_2 as β, we infer from (6.3.4) that

$$\frac{d}{dt} E_\omega + \beta E_\omega \leq \kappa.$$

The Gronwall Lemma yields

$$E_\omega(t) \le E_\omega(0)e^{-\beta t} + \frac{\kappa}{\beta}.$$

Inequalities (6.3.4) imply that

$$\|U(t)z_0\|^2_{\mathcal{H}_{h(t)}} \le E(t) \le \frac{c_2}{c_1}\, E(0)e^{-\beta t} + \frac{\kappa}{\beta c_1}. \tag{6.3.9}$$

Therefore,

$$\limsup_{t\to\infty} \|U(t)z_0\|_{\mathcal{H}_{h(t)}} \le \sqrt{\frac{\kappa}{\beta c_1}}. \tag{6.3.10}$$

Then, by using (6.3.6)-(6.3.9) and Lemma 6.2.3,

$$2\,c\,\varepsilon\, \exp\frac{1}{\varepsilon^{4+r}} \le 2F\big(h(t)\big) \le E(t) \le \frac{c_2}{c_1}\, E(0)e^{-\beta t} + \frac{\kappa}{\beta c_1} \le \frac{c_2}{c_1}\, E(0)e^{-\beta t} + \frac{1}{\beta c_1} c_3\varepsilon \exp\frac{12}{\varepsilon}\,. \tag{6.3.11}$$

Choose $\varepsilon > 0$ small enough in such a way that

$$c\,\varepsilon\, \exp\frac{1}{\varepsilon^{4+r}} > \frac{1}{\beta c_1} c_3\varepsilon \exp\frac{12}{\varepsilon}\,.$$

Then take $t_0 = t_0(R, \widehat{\varepsilon})$ such that

$$\frac{c_2}{c_1} E(0)e^{-\beta t} < \frac{1}{\beta c_1} c_3\varepsilon \exp\frac{12}{\varepsilon} \qquad \forall\, t \ge t_0\,.$$

With these two choices, we contradict (6.3.11) and we prove (6.3.5).

We modify once more the solenoidal extension s in (6.2.14) and the function w in (6.3.3) by taking $s = s_{\varepsilon_1}$, $w = w_{\varepsilon_1}$, with ε_1 given by (6.3.5). This is allowed thanks to (6.2.18). With this choice, we reach again (6.3.10). Given

$$\widehat{R} = \sqrt{\frac{\kappa}{\beta c_1}}\,,$$

this tells us the balls in $\mathcal{H}_h$ of radius $R_0 > \widehat{R}$ are absorbing, namely they contain the dynamics of (6.2.19)-(6.2.20) for t large. This translates into the existence of the zero-order absorbing radius R_0 in the sense of Definition 6.3.2. Summarizing, for any $h_0 \in [-L+\delta+\widehat{\varepsilon}, L-\delta-\widehat{\varepsilon}]$, given $z_0 = (\widehat{u}_0, h_0)$ such that $\|z_0\|_{\mathcal{H}_{h_0}} \le R$, we have

$$\|U(t)z_0\|_{\mathcal{H}_h} \le \sqrt{E(t)}\, \le R_0, \qquad \forall t \ge t_0 = t_0(R, \widehat{\varepsilon}) := \frac{1}{\beta}\log\left(\min\left\{1, \tfrac{c_2}{c_1}\tfrac{E(0)}{R_0^2-\widehat{R}^2}\right\}\right). \tag{6.3.12}$$

Finally, by Lemma 6.2.3, $\kappa \to 0$ as $\lambda \to 0$. This completes the proof of Theorem 6.3.3.

□

Note that 6.3.5 allows us to improve the conclusion of Corollary 6.2.8 on the separation strip between the obstacle and the boundary of the channel. Accordingly, throughout the whole section, we take s as in (6.2.14) by choosing the value ε_1 given (6.3.5) instead of ε_0, i.e., we take $s = s_{\varepsilon_1}$. A further consequence of Theorem 6.3.3 is the existence of a suitable dissipation integral for the solution $(\widehat{u}, h)$ to problem (6.2.19)-(6.2.20).

Corollary 6.3.4. *Let $R > 0$ and $\widehat{\varepsilon} > 0$ small be arbitrarily given, t_0 as in* (6.3.12) *and R_0 as in Theorem 6.3.3. Assume that $h_0 \in [-L + \delta + \widehat{\varepsilon}, L - \delta - \widehat{\varepsilon}]$ and $\|z_0\|_{\mathcal{H}_{h_0}} \leq R$. There exists $D = D(R_0) > 0$ such that*

$$\int_t^{t+1} \|\nabla \widehat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\, d\tau \leq D \qquad \forall\, t \geq t_0.$$

Moreover, $D \to 0$ as $\lambda \to 0$.

Proof. Integrating inequality (6.3.8) on the time-interval $(t, t+1)$, and exploiting (6.3.4), we get

$$\int_t^{t+1} \nu \|\nabla \widehat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\, d\tau \leq \kappa + E_\omega(t) \leq \kappa + c_2\, E(t).$$

Hence, in light of (6.3.12), for $t \geq t_0$ we are led to

$$\int_t^{t+1} \|\nabla \widehat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\, d\tau \leq \frac{1}{\nu}\kappa + \frac{c_2}{\nu} R_0^2.$$

Since we know that $\kappa, R_0 \to 0$ as $\lambda \to 0$, we are done. □

We have now all the ingredients to proceed to show the existence of the first-order absorbing radius, ensuring dissipativity of higher-order.

Theorem 6.3.5. *There exists a universal constant $R_1 = R_1(\lambda, L, \delta, d, m, \mu) > 0$ with the following property: for any $R > 0$, and any $\widehat{\varepsilon} > 0$ small, it follows that*

$$\|U(t) z_0\|_{\mathcal{H}^1_{h(t)}} \leq R_1 \qquad \forall t \geq t_0 + 1,$$

whenever

$$h_0 \in [-L + \delta + \widehat{\varepsilon}, L - \delta - \widehat{\varepsilon}] \qquad \text{and} \qquad \|z_0\|_{\mathcal{H}_{h_0}} \leq R,$$

where t_0 is given by (6.3.12). *In compliance with Definition 6.3.2, the constant R_1 is a first-order absorbing radius with entering time $t_1 = t_0 + 1$. Moreover, $R_1 \to 0$ as $\lambda \to 0$.*

The proof of the theorem will make use of the uniform Gronwall lemma, that we recall for the sake of convenience (see [133, §III.2, Lemma 1.1])

Lemma 6.3.6. *Let f_1, f_2, f_3 be three positive locally integrable functions on $(t_0, +\infty)$ such that f_3' is locally integrable on $(t_0, +\infty)$, and which satisfy*

$$\frac{df_3}{dt} \leq f_1 f_3 + f_2, \qquad \int_t^{t+1} f_1(s)\, ds \leq a_1, \qquad \int_t^{t+1} f_2(s)\, ds \leq a_2, \qquad \int_t^{t+1} f_3(s)\, ds \leq a_3$$

for $t \geq t_0$, where a_1, a_2, a_3 are positive constants. Then

$$f_3(t+1) \leq (a_3 + a_2) \exp(a_1) \qquad \forall\, t \geq t_0.$$

Proof of Theorem 6.3.5. Let $(\widehat{u},h)$ be the unique weak solution to (6.2.19)-(6.2.20), and $p \in L^2_{\rm loc}(\Omega_h)$ the associated pressure field (which can be standardly obtained for instance by applying [57, Theorem 2.1]). Then, we denote

$$\mathcal{S}\widehat{u} = -\nabla \cdot \mathcal{T}(\widehat{u},p) = -\mu\,\Delta\widehat{u} + \nabla p.$$

As a consequence, after an integration by parts, (6.2.25) is equivalent to

$$\begin{aligned}&\int_{\Omega_h} \widehat{u}_t \cdot \phi\,dx + \int_{\Omega_h} \mathcal{S}\widehat{u}\cdot\phi\,dx + \int_{\Omega_h} (\widehat{u}\cdot\nabla)\widehat{u}\cdot\phi\,dx + \int_{\Omega_h} (\widehat{u}\cdot\nabla)s\cdot\phi\,dx \\ &+ \int_{\Omega_h} (s\cdot\nabla)\widehat{u}\cdot\phi\,dx = \int_{\Omega_h} \widehat{g}\cdot\phi\,dx.\end{aligned} \tag{6.3.13}$$

We proceed similarly to [33] by starting to define some auxiliary functions. Consider ζ a smooth cut-off function with compact support such that $\zeta = 1$ in a neighborhood of $\bar{B}_{h_0}$, and set $\widehat{\zeta}(t,x_1,x_2) = \zeta(x_1, x_2 - h(t) + h_0)$. Then, we define the map $\widehat{V}(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\widehat{V}(t,x_1,x_2) = \nabla \times \{0, 0, -\widehat{\zeta}(t,x_1,x_2)\, x_1 h'(t)\}.$$

We notice that $\widehat{V}(t,\cdot) \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ for all $t \geq 0$, $\widehat{V}(\cdot,x) \in L^\infty(0,T;\mathbb{R}^2)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$, and

$$\|\widehat{V}(t,\cdot)\|_{W^{1,\infty}(\Omega_{h(t)})} \leq C|h'(t)| \tag{6.3.14}$$

for some $C > 0$. At this stage, one proceeds formally (see [57, 133]) and assumes that $\mathcal{S}\widehat{u}, \widehat{u}_t \in L^2(\Omega_h)$. We choose as test function ϕ in (6.3.13) the following function

$$\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V},$$

so as to obtain

$$\begin{aligned}&\int_{\Omega_h} \widehat{u}_t \cdot \Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx + \int_{\Omega_h} \mathcal{S}\widehat{u}\cdot\Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx = \\ &-\int_{\Omega_h} (\widehat{u}\cdot\nabla)\,\widehat{u}\cdot\Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx + \int_{\Omega_h} (\widehat{u}\cdot\nabla)\,s\cdot\Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]dx \\ &+ \int_{\Omega_h} (s\cdot\nabla)\,\widehat{u}\cdot\Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx + \int_{\Omega_h} \widehat{g}\cdot\Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx.\end{aligned} \tag{6.3.15}$$

Arguing as in [33, Lemma 4.3], we find

$$\begin{aligned}\int_{\Omega_h} \widehat{u}_t \cdot \Big[\mathcal{S}\widehat{u} + (\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx &= \mu\frac{d}{dt}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + m\,h''^2 + f(h)h'' \\ &+ 2\mu\int_{\Omega_h} (D\widehat{u}) : \Big[(\nabla\widehat{u})(\nabla\widehat{V}) - D((\widehat{u}\cdot\nabla)\widehat{V})\Big]\,dx.\end{aligned}$$

Thus, by plugging the above equality into (6.3.15), we obtain

$$\mu\frac{d}{dt}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + m\,h''^2 + \|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)} = -\int_{\Omega_h}\mathcal{S}\widehat{u}\cdot\Big[(\widehat{V}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx - f(h)h''$$
$$-2\mu\int_{\Omega_h}(D\widehat{u}):\Big[(\nabla\widehat{u})(\nabla\widehat{V})-D((\widehat{u}\cdot\nabla)\widehat{V})\Big]\,dx - \int_{\Omega_h}(\widehat{u}\cdot\nabla)\,\widehat{u}\cdot\Big[\mathcal{S}\widehat{u}+(\widehat{V}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{V}\Big]dx$$
$$-\int_{\Omega_h}(\widehat{u}\cdot\nabla)\,s\cdot\Big[\mathcal{S}\widehat{u}+(\widehat{V}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx - \int_{\Omega_h}(s\cdot\nabla)\,\widehat{u}\cdot\Big[\mathcal{S}\widehat{u}+(\widehat{V}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{V}\Big]dx$$
$$+\int_{\Omega_h}\widehat{g}\cdot\Big[\mathcal{S}\widehat{u}+(\widehat{V}\cdot\nabla)\widehat{u}-(\widehat{u}\cdot\nabla)\widehat{V}\Big]\,dx. \tag{6.3.16}$$

At this point, some estimates for the terms of the right-hand side of (6.3.16) are needed, by exploiting in a suitable way the Hölder, the Young and the Poincaré inequalities, together with the properties of the solenoidal extension s. We have the two following inequalities

$$f(h)h'' \le \frac{m}{2}h''^2 + \frac{1}{2m}|f(h)|^2,$$

and

$$\left|\int_{\Omega_h}\widehat{g}\cdot\mathcal{S}\widehat{u}\right| \le \|\widehat{g}\|_{L^2(\Omega_h)}\|\mathcal{S}\widehat{u}\|_{L^2(\Omega_h)} \le \frac{7}{4}\|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{1}{7}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)}.$$

Using the renowned Ladyzhenskaya inequality (see [133, p.108, (2.16)]),

$$\left|\int_{\Omega_h}(\widehat{u}\cdot\nabla)\widehat{u}\cdot\mathcal{S}\widehat{u}\right| \le c_1\|\widehat{u}\|^{1/2}_{L^2(\Omega_h)}\|\nabla\widehat{u}\|_{L^2(\Omega_h)}\|\mathcal{S}\widehat{u}\|^{3/2}_{L^2(\Omega_h)} \le \frac{1}{14}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)} + c_2\|\widehat{u}\|^2_{L^2(\Omega_h)}\|\nabla\widehat{u}\|^4_{L^2(\Omega_h)},$$

where c_1 and c_2 are two strictly positive constants. For the remaining terms, we argue in a similar manner, finding

$$\left|\int_{\Omega_h}(\widehat{u}\cdot\nabla)s\cdot\mathcal{S}\widehat{u}\right| \le \|\nabla s\|_{L^4(\Omega_h)}\|\widehat{u}\|_{L^4(\Omega_h)}\|\mathcal{S}\widehat{u}\|_{L^2(\Omega_h)} \le C\frac{7}{4}\|\nabla s\|^2_{L^4(\Omega_h)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{1}{7}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)},$$

where $C > 0$ depends on the constant describing the Sobolev embedding $H^1_0(\Omega_h) \subset L^2(\Omega_h)$, and, exploiting again the Ladyzhenskaya inequality,

$$\left|\int_{\Omega_h}(s\cdot\nabla)\widehat{u}\cdot\mathcal{S}\widehat{u}\right| \le \|s\|^{1/2}_{L^2(\Omega_h)}\|\nabla\widehat{u}\|_{L^2(\Omega_h)}\|\mathcal{S}\widehat{u}\|^{3/2}_{L^2(\Omega_h)} \le c_3\|s\|^2_{L^2(\Omega_h)}\|\nabla\widehat{u}\|^4_{L^2(\Omega_h)} + \frac{1}{14}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)},$$

for some strictly positive constant $c_3 > 0$. For all terms involving the map $\widehat{V}(x,t)$, we exploit

(6.3.14). Thus, we have

$$\begin{aligned}
&\left| \int_{\Omega_h} \mathcal{S}\widehat{u} \cdot \left[(\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V} \right] \right| \\
&\leq \|\widehat{V}\|_{W^{1,\infty}(\Omega_h)} \|\mathcal{S}\widehat{u}\|_{L^2(\Omega_h)} \|\nabla\widehat{u}\|_{L^2(\Omega_h)} + c_4 \|\widehat{V}\|_{W^{1,\infty}(\Omega_h)} \|\mathcal{S}\widehat{u}\|_{L^2(\Omega_h)} \|\nabla\widehat{u}\|_{L^2(\Omega_h)} \\
&\leq c_4 \|\widehat{V}\|^2_{W^{1,\infty}(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{2}{7}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)} \leq c_4\, {h'}^2 \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{2}{7}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)},
\end{aligned}$$

where $c_4 > 0$ changes from line to line, and it depends on the Poincaré constant. Then,

$$\left| \int_{\Omega_h} (D\widehat{u}) : \left[(\nabla\widehat{u})(\nabla\widehat{V}) - D((\widehat{u}\cdot\nabla)\widehat{V}) \right] \right| \leq c_5 \|\widehat{V}\|_{W^{1,\infty}(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \leq c_5 |h'| \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)},$$

for some strictly positive constant $c_5 > 0$. Let c_6, c_7, c_8, c_9 be some strictly positive constant that might change from line to line. Arguing by the Ladyzhenskaya inequality again, we obtain

$$\begin{aligned}
&\left| \int_{\Omega_h} (\widehat{u}\cdot\nabla)\,\widehat{u} \cdot \left[(\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V} \right] \right| \\
&\leq c_1 \|\widehat{V}\|_{W^{1,\infty}(\Omega_h)} \|\widehat{u}\|^{1/2}_{L^2(\Omega_h)} \|\mathcal{S}\widehat{u}\|^{1/2}_{L^2(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \\
&\quad + c_1 \|\widehat{V}\|_{W^{1,\infty}(\Omega_h)} \|\widehat{u}\|^{3/2}_{L^2(\Omega_h)} \|\mathcal{S}\widehat{u}\|^{1/2}_{L^2(\Omega_h)} \|\nabla\widehat{u}\|_{L^2(\Omega_h)} \\
&\leq c_6 \|\widehat{u}\|^2_{L^2(\Omega_h)} \|\widehat{V}\|^4_{W^{1,\infty}(\Omega_h)} \left(1 + \|\widehat{u}\|^4_{L^2(\Omega_h)}\right) + \frac{2}{7}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)} \\
&\quad + \frac{1}{7}\left(1 + \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}\right) \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \\
&\leq \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \left(c_6 {h'}^4 + c_6 \|\widehat{u}\|^4_{L^2(\Omega_h)} + c_6 \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \right) + \frac{2}{7}\|\mathcal{S}\widehat{u}\|^2_{L^2(\Omega_h)}.
\end{aligned}$$

Then, by similar arguments, we have

$$\begin{aligned}
\left| \int_{\Omega_h} (\widehat{u}\cdot\nabla)\, s \cdot \left[(\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V} \right] \right| &\leq c_7 \|\nabla s\|_{L^4(\Omega_h)} \|\widehat{V}\|_{W^{1,\infty}(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \\
&\leq c_7 \|\nabla s\|_{L^4(\Omega_h)} |h'| \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}
\end{aligned}$$

and

$$\left| \int_{\Omega_h} (s\cdot\nabla)\,\widehat{u} \cdot \left[(\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V} \right] \right| \leq c_8 \|s\|_{L^\infty(\Omega_h)} |h'| \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.$$

For what concerns the last term, we have

$$\left| \int_{\Omega_h} \widehat{g} \cdot \left[(\widehat{V}\cdot\nabla)\widehat{u} - (\widehat{u}\cdot\nabla)\widehat{V} \right] \right| \leq c_9 |h'| \|\widehat{g}\|_{L^2(\Omega_h)} \|\nabla\widehat{u}\|_{L^2(\Omega_h)} \leq \frac{7}{4}\|\widehat{g}\|^2_{L^2(\Omega_h)} + c_9 {h'}^2 \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.$$

Collecting all together, and dividing by μ, we finally get

$$\begin{aligned}
\frac{d}{dt}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} &\leq \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \Big(\frac{c_2}{\mu}\|\widehat{u}\|^2_{L^2(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{c_3}{\mu}\|s\|^2_{L^2(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + C\frac{7}{4\mu}\|\nabla s\|^2_{L^4(\Omega_h)} \\
&+ \frac{c_4}{\mu}{h'}^2 + \frac{c_5}{\mu}|h'| + \frac{c_6}{\mu}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{c_6}{\mu}\|\widehat{u}\|^4_{L^2(\Omega_h)} + \frac{c_6}{\mu}{h'}^4 + c_7\|\nabla s\|_{L^4(\Omega_h)}|h'| + \frac{c_8}{\mu}\|s\|_{L^\infty(\Omega_h)}|h'| + \frac{c_9}{\mu}|h'|^2 \Big) \\
&+ \frac{7}{4\mu}\|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{1}{2m\mu}|f(h)|^2.
\end{aligned}$$

We are now in a position to apply Lemma 6.3.6 with the choice

$$\begin{aligned}
f_1 &= \frac{c_2}{\mu}\|\widehat{u}\|^2_{L^2(\Omega_h)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{c_3}{\mu}\|s\|^2_{L^2(\Omega_h)}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + C\frac{7}{4\mu}\|\nabla s\|^2_{L^4(\Omega_h)} + \frac{c_4}{\mu}h'^2 + \frac{c_5}{\mu}|h'| \\
&\quad + \frac{c_6}{\mu}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \frac{c_6}{\mu}\|\widehat{u}\|^4_{L^2(\Omega_h)} + \frac{c_6}{\mu}h'^4 + \frac{c_7}{\mu}\|\nabla s\|_{L^4(\Omega_h)}|h'| + \frac{c_8}{\mu}\|s\|_{L^\infty(\Omega_h)}|h'| + \frac{c_9}{\mu}|h'|^2, \\
f_2 &= \frac{7}{4\mu}\|\widehat{g}\|^2_{L^2(\Omega_h)} + \frac{1}{2m\mu}|f(h)|^2, \\
f_3 &= \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}.
\end{aligned}$$

Indeed, since $t \geq t_0$ with t_0 as in (6.3.12), from (6.3.12) we have that $2F(h) \leq R_0^2$, so that by (6.2.2)

$$|h| \leq \frac{M(R_0^2)}{2},$$

providing in turn a uniform bound for $|f(h)|$. Therefore, with reference to Lemma 6.3.6, denoting by Q a generic increasing positive function, and exploiting (6.3.12) and Corollary 6.3.4, we draw the estimates

$$\int_t^{t+1} f_1(s)\,ds \leq a_1, \qquad \int_t^{t+1} f_2(s)\,ds \leq a_2, \qquad \int_t^{t+1} f_3(s)\,ds \leq a_3 \qquad \text{for } t \geq t_0,$$

with

$$\begin{cases}
a_1 = \frac{c_2}{\mu}R_0^2\,a_3 + \frac{c_3}{\mu}\|s\|^2_{L^2(\Omega_h)}a_3 + C\frac{7}{4\mu}\|\nabla s\|^2_{L^4(\Omega_h)} + \frac{c_4}{\mu}R_0^2 + \frac{c_5}{\mu}R_0 + \frac{c_6}{\mu}a_3 + \frac{2c_6}{\mu}R_0^4 \\
\qquad + \frac{c_7}{\mu}\|\nabla s\|_{L^4(\Omega_h)}R_0 + \frac{c_8}{\mu}\|s\|_{L^\infty(\Omega_h)}R_0 + \frac{c_9}{\mu}R_0^2, \\
a_2 = \frac{7}{2\mu}\|\widehat{g}\|^2_{L^2(\Omega_h)} + Q(R_0), \\
a_3 = D(R_0).
\end{cases}$$

The conclusion is

$$\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} \leq (a_3 + a_2)\exp(a_1) \qquad \forall t\, \geq t_0 + 1.$$

The last step is to add mh'^2 to both sides of the inequality above, which allows us to reconstruct the norm of the norm of the solution, to wit,

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} \leq \sqrt{(a_3 + a_2)\exp(a_1) + R_0^2} \qquad \forall t\, \geq t_0 + 1.$$

Here, we leaned on the estimate $mh'^2 \leq R_0^2$, coming from (6.3.12). By calling R_1 the right-hand side, the proof is finished. □

Remark 6.3.7. Due to the compact embedding

$$\mathcal{H}^1_{h(t)} \Subset \mathcal{H}_{h(t)},$$

which holds true for all t, the closed ball $\mathcal{B}_1(t)$ of radius R_1 in $\mathcal{H}^1_{h(t)}$ is compact in $\mathcal{H}_{h(t)}$. Theorem 6.3.5 tells that

$$U(t)\mathcal{B} \subset \mathcal{B}_1(t) \qquad \forall\, t \geq t_0 + 1,$$

where $\mathcal{B}$ is the ball of radius R in $\mathcal{H}_{h_0}$, for $R > 0$ arbitrarily given. This shows that the solution operator $U(t)$ defined by the rule (6.3.1) is not only dissipative in the sense of Definition 6.3.2, but it has also a regularizing effect.

6.4 Stability of the unique steady state

In this section, we investigate the convergence of the solutions of (2.0.7) to the unique steady state, if $\lambda < \lambda_s$, see Theorem 6.2.2. In particular, we study the convergence of the solutions of (2.0.7) to those of (6.2.9)-(6.2.10), in terms of the convergence of the solutions of (6.2.19)-(6.2.20) to the solution of (6.2.16)-(6.2.17).

Theorem 6.4.1. *Let $R > 0$ be arbitrarily fixed and λ_s as in Theorem 6.2.2. There exists $\lambda_1 = \lambda_1(R) \in (0, \lambda_s)$ such that if $\lambda < \lambda_1$, the weak solution $(\widehat{u}, h)$ of problem* (6.2.19)-(6.2.20)*, with initial position of the obstacle $h_0 = 0$ and initial velocities $z_0 = (\widehat{u}_0, k_0) \in \mathcal{H}_0$ such that $\|z_0\|_{\mathcal{H}_0} \leq R$, converges to the solution $(\widehat{u}_\lambda, 0)$ of* (6.2.16)-(6.2.17) *in $\mathcal{H}_0$ as $t \to \infty$.*

In order to prove Theorem 6.4.1, we preliminarily state and prove a basic proposition allowing to define a change of variables associated to the rigid motion of the obstacle in problem (6.2.19)-(6.2.20) in order to be able to compare different solutions; indeed since (6.2.19)-(6.2.20) is set on a time-dependent fluid domain, different solutions are defined on different domains. This change of variables depends on time t through h; it was first introduced by Takahashi ([130, Section 4.1]), inspired by Inoue and Wakimoto ([93]). We denote for all $\varepsilon > 0$

$$\mathcal{O}_\varepsilon = \left\{x \in R : \text{dist}(x, \Gamma) \geq 2\varepsilon \,\wedge\, |x_1| < \tfrac{3}{2}\right\}, \qquad \mathcal{A}_\varepsilon = \left\{x \in R : \text{dist}(x, \Gamma) \leq \varepsilon \,\vee\, |x_1| > 2\right\},$$

see Figure 6.1 for a representation. Note that, if we choose $\varepsilon_0 = \varepsilon$, any function s defined in Lemma 6.2.1 is such that

$$s(x) \equiv 0 \qquad \text{on} \qquad \mathcal{R} \setminus \mathcal{A}_\varepsilon. \tag{6.4.1}$$

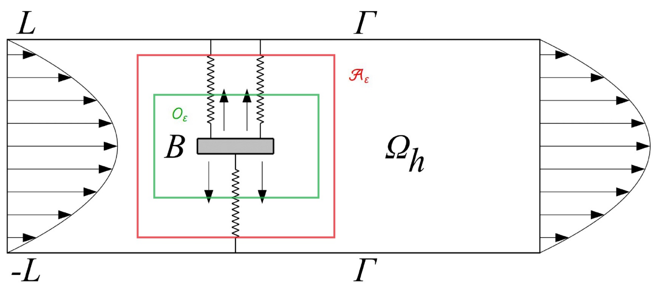


Figure 6.1: *The subsets $\mathcal{O}_\varepsilon$, $\mathcal{A}_\varepsilon \subset \mathcal{R}$.*

Proposition 6.4.2. *Consider a fixed $h \in W^{1,\infty}(0, T; (-L + \delta, L - \delta))$ with $h(0) = h_0$. For every $t \in [0, T]$ there exists a volume preserving diffeomorphism*

$$\psi(t, \cdot) : \Omega_{h(t)} \longrightarrow \Omega_{h_0}$$

satisfying, for all $\varepsilon > 0$, the following properties:

$$\psi(t,x_1,x_2) = (x_1, x_2 + h(t) - h_0) \quad \forall\, x = (x_1,x_2) \in \mathcal{O}_\varepsilon,$$
$$\psi(t,x_1,x_2) = (x_1,x_2) \quad \forall\, x = (x_1,x_2) \in \mathcal{A}_\varepsilon.$$

Proof. Let $\zeta(x_1,x_2)$ be a smooth cutoff function equal to 0 in $\mathcal{A}_\varepsilon$ and equal to 1 in $\mathcal{O}_\varepsilon$. Then, we define the solenoidal vector field $V : \mathbb{R}^+ \times \Omega_{h(t)} \to \mathbb{R}^2$ as

$$V(t,x) = \nabla \times \{0,0,-\zeta(x_1,x_2)x_1 h'\}.$$

Notice that

$$V(t,x) = \begin{cases} 0 & \text{in } \mathcal{A}_\varepsilon, \\ h'\widehat{e}_2 & \text{in } \mathcal{O}_\varepsilon, \end{cases} \tag{6.4.2}$$

and $V(\cdot,t) \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ for all $t \geq 0$, $V(x,\cdot) \in L^\infty(0,T;\mathbb{R}^2)$ for all $x \in \mathbb{R}^2$. Then we build the deformation mapping of $\Omega_{h(t)}$ into Ω_{h_0}, $\psi : \mathbb{R}^+ \times \Omega_{h(t)} \to \Omega_{h_0}$, as the flow associated to (6.4.2):

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,x) = V(t,\psi(t,x)), \\ \psi(0,x) = x. \end{cases}$$

Since $\nabla \cdot V = 0$, ψ is volume preserving and $\det(\frac{\partial \psi_i}{\partial x_j})_{i,j} = 1$ for all $t \geq 0$. The mapping ψ is a smooth function of V. In particular, for some $C > 0$,

$$\|\partial_t^j \psi(t,\cdot)\|_{\mathcal{C}^k(\bar{\Omega}_{h(t)})} \leq C|h^j(t)| \qquad \forall j = 0,1, \forall\, k = 0,1,2.$$

Notice that $\psi \in W^{1,\infty}(0,T;\mathcal{C}^k(\Omega_{h(t)}))$ for any $k = 0,1,2$. □

Through Proposition 6.4.2, we define $\varphi : \mathbb{R}^+ \times \Omega_{h_0} \longrightarrow \Omega_{h(t)}$ by $\varphi = \psi^{-1}$ in the space variables to be the volume preserving diffeomorphism such that, for any $y = (y_1,y_2) \in \mathcal{O}_\varepsilon$,

$$\varphi(t,y_1,y_2) = (y_1, y_2 + h_0 - h(t))$$

and, for any $y = (y_1,y_2) \in \mathcal{A}_\varepsilon$, $\varphi(t,y_1,y_2) = (y_1,y_2)$. There holds

$$\|\partial_t^j \varphi(t,\cdot)\|_{\mathcal{C}^k(\bar{\Omega}_{h_0})} \leq C|h^j(t)| \qquad \forall j = 0,1, \forall\, k = 0,1,2.$$

Obviously $\varphi \in W^{1,\infty}(0,T;\mathcal{C}^k(\Omega_{h_0}))$ for any $k = 0,1,2$. We can now give the

Proof of Theorem 6.4.1. Let $\varepsilon_1 > 0$ be given by (6.3.5) and take $s = s_{\varepsilon_1}$ from Lemma 6.2.1. Multiply the fluid equation in (6.2.16) by a function $\phi \in V(\mathcal{R})$ such that $\phi|_{B_0} = \widehat{e}_2$ and integrate over Ω_0. After integration by parts it comes

$$\mu\int_{\Omega_0} \nabla\widehat{u}_\lambda : \nabla\phi + \int_{\partial\Omega_0} (-\mu\frac{\partial \widehat{u}_\lambda}{\partial \widehat{n}} + p_s\widehat{n})\cdot\phi + \int_{\Omega_0} (\widehat{u}_\lambda\cdot\nabla)\widehat{u}_\lambda\cdot\phi + \int_{\Omega_0} (\widehat{u}_\lambda\cdot\nabla)s\cdot\phi + \int_{\Omega_0} (s\cdot\nabla)\widehat{u}_\lambda\cdot\phi = \int_{\Omega_0} \widehat{g}\cdot\phi.$$

Next, by the coupling condition in (6.2.17), we have that

$$\int_{\partial\Omega_0} (-\mu\frac{\partial \widehat{u}_\lambda}{\partial \widehat{n}} + p\widehat{n}) \cdot \phi = -\widehat{e}_2 \cdot \int_{\partial B_0} (\mathcal{T}(\widehat{u}_\lambda, p_s) \cdot \widehat{n}) = 0.$$

Consequently, we obtain that, for any $\phi \in V(\mathcal{R})$ such that $\phi|_{B_0} = \widehat{e}_2$,

$$\mu \int_{\Omega_0} \nabla \widehat{u}_\lambda : \nabla \phi + \int_{\Omega_0} (\widehat{u}_\lambda \cdot \nabla)\widehat{u}_\lambda \cdot \phi + \int_{\Omega_0} (\widehat{u}_\lambda \cdot \nabla) s \cdot \phi + \int_{\Omega_0} (s \cdot \nabla)\widehat{u}_\lambda \cdot \phi = \int_{\Omega_0} \widehat{g} \cdot \phi. \quad (6.4.3)$$

Notice that (6.4.3) does not see the value of ϕ on B_0, thus we could have taken $\phi|_{B_0} = c\,\widehat{e}_2$ with $c \in \mathbb{R}$. Then, let $(\widehat{u}, h)$ be the unique solution to problem (6.2.19)-(6.2.20) given by Theorem 6.2.7, with $h_0 = 0$ and some initial velocities $z_0 = (\widehat{u}_0, k_0) \in \mathcal{H}_0$ such that $\|z_0\|_{\mathcal{H}_0} \leq R$, for any arbitrary $R > 0$. In order to be able to subtract the weak formulation satisfied by $(\widehat{u}, h)$ and that satisfied by $(\widehat{u}_s, h_s) = (\widehat{u}_\lambda, 0)$, we need to properly map $\widehat{u}(t)$ from $\Omega_{h(t)}$ to Ω_0 for every $t > 0$. We follow [76, 111]. From (6.3.5), we infer that $h \in W^{1,\infty}(0, T; [-L + \delta - \varepsilon_1, L - \delta - \varepsilon_1])$. Thus, we can build ψ as in Proposition 6.4.2 with $h_0 = 0$ and $\varepsilon = \varepsilon_1$; we also define $\varphi = \psi^{-1}$. We introduce

$$v(y,t) = \nabla\psi(\varphi(t,y),t) \cdot \widehat{u}(\varphi(t,y),t) \qquad y \in \Omega_0,$$

to be the pullback of $\widehat{u}$ by φ, and we set $q(t,y) = p(t, \varphi(y,t))$. We refer to [76, Section 3.2] (see also [111, Section 5]) for the explicit computation of the partial derivatives of v in terms of those of $\widehat{u}$, so that the equation satisfied by v reads

$$\begin{aligned} &\langle \partial_t v(t), \phi \rangle + m h''(t) l(t) + f(h(t)) l(t) + \mu \int_{\Omega_0} \nabla v(t) : \nabla \phi + \int_{\Omega_0} (v(t) \cdot \nabla) v(t) \cdot \phi \\ &+ \int_{\Omega_0} (v(t) \cdot \nabla) s \cdot \phi + \int_{\Omega_0} (s \cdot \nabla) v(t) \cdot \phi = \int_{\Omega_0} \widehat{g} \cdot \phi - \int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t)) \cdot \phi, \end{aligned} \quad (6.4.4)$$

for any test pair $(\phi, l) \in \mathcal{H}^1_0$ where, using Einstein's summation convention,

$$\begin{aligned} \mathfrak{f}^i = &+ (\partial_k \varphi^i - \delta_{ik}) \partial_t v^k + \partial_k \varphi^i \partial_l v^k (\partial_t \psi^l) + (\partial_k \partial_t \varphi^i) v^k + (\partial^2_{kl} \varphi^i)(\partial_t \psi^l) v^k \\ &+ v^l \partial_l v^k (\partial_k \varphi^i - \delta_{ik}) + (\partial^2_{lk} \varphi^i) v^l v^k + \partial_k q (\partial_i \psi^k - \delta_{ik}) + \mu \Big[- \partial_j \psi^m (\partial^2_{mk} \varphi^i) \partial_l v^k \partial_j \psi^l \\ &- (\partial_k \varphi^i \partial_j \psi^m \partial_j \psi^l - \delta_{ik} \delta_{jm} \delta_{jl}) \partial^2_{ml} v^k - \partial_k \varphi^i \partial_l v^k (\partial^2_{jj} \psi^l) \\ &- \partial_j \psi^m (\partial^3_{mlk} \varphi^i) \partial_j \psi^l v^k - (\partial^2_{lk} \varphi^i) \partial^2_{jj} \psi^l v^k - (\partial^2_{lk} \varphi^i) \partial_j \psi^l \partial_j \psi^m \partial_m v^k \Big]. \end{aligned} \quad (6.4.5)$$

Then set $w(t) = v(t) - \widehat{u}_\lambda$ and subtract (6.4.3) from (6.4.4) to obtain:

$$\begin{aligned} &\langle \partial_t w(t), \phi \rangle + m h''(t) l + f(h(t)) l + \mu \int_{\Omega_0} \nabla w(t) : \nabla \phi + \int_{\Omega_0} (v(t) \cdot \nabla) w(t) \cdot \phi \\ &+ \int_{\Omega_0} (w(t) \cdot \nabla) \widehat{u}_\lambda \cdot \phi + \int_{\Omega_0} (w(t) \cdot \nabla) s \cdot \phi + \int_{\Omega_0} (s \cdot \nabla) w(t) \cdot \phi \\ &= - \int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t)) \cdot \phi. \end{aligned}$$

We follow the same reasoning of the proof of Theorem 6.3.3. We define

$$z(x,t) = h(t) \left[\frac{\partial}{\partial x_2}(-\zeta(x,t)\, x_1), -\frac{\partial}{\partial x_1}(\zeta(x,t)\, x_1) \right] \qquad \forall x \in \mathcal{R}\,, \qquad \forall\, t \geq 0,$$

where ζ is a $\mathcal{C}^\infty$ cut-off function equal to 1 in a small neighbourhood of the obstacle B_0 and equal to 0 outside a larger neighbourhood. We observe that

$$z \in C^\infty(\mathcal{R}) \cap H_0^1(\mathcal{R}), \qquad \text{div} z = 0\,, \qquad z = h\,\widehat{e}_2 \quad \text{in} \quad B_h\,.$$

The following estimates hold:

$$\|z\|_{L^2(\mathcal{R})} \le a_1|h|\,, \qquad \|\nabla z\|_{L^2(\mathcal{R})} \le a_2|h|\,, \qquad \|\nabla z\|_{L^\infty(\mathcal{R})} \le a_3|h|\,, \tag{6.4.6}$$

where a_1, a_2 and a_3 are constants depending on the cut-off function ζ. We introduce

$$E(t) = \|w(t)\|^2_{L^2(\Omega_0)} + mh'(t)^2 + 2F(h(t)),$$

and, for $\omega \in (0,1)$ to be fixed later, its perturbation

$$E_\omega(t) = \|w(t)\|^2_{L^2(\Omega_0)} + mh'(t)^2 + 2F(h(t)) + \omega\, h(t)\, h'(t) + \omega \int_{\Omega_0} w \cdot z.$$

Such functionals satisfy (6.3.4), provided thath ω is small enough. Then, choosing $(\phi, l) = (w + \omega z, h' + \omega h)$, we infer

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_\omega - \omega h'^2 + \frac{\omega}{2} f(h)h + \frac{\omega}{2} f(h)h + \mu\|\nabla w\|^2_{L^2(\Omega_0)} = &- \int_{\Omega_0} \mathfrak{f}(v,h,q)\cdot w - \int_{\Omega_0} (w\cdot\nabla)\widehat{u}_\lambda \cdot w \\ &- \int_{\Omega_0} (w\cdot\nabla)s\cdot w \\ &- \mu\omega \int_{\Omega_h} \nabla w : \nabla z\, dx - \omega \int_{\Omega_h} (w\cdot\nabla)s\cdot z\, dx \\ &+ \omega \int_{\Omega_h} (w\cdot\nabla) z \cdot w\, dx.\end{aligned} \tag{6.4.7}$$

Next, we estimate the right-hand side of (6.4.7). For the last two terms on the first line, we exploit [68, (2.26)], the Poincaré inequality and the properties of s defined as in Lemma 6.2.1. We obtain

$$\begin{aligned}\int_{\Omega_0} (w\cdot\nabla)\widehat{u}_\lambda \cdot w \le \|w\|^2_{L^4(\Omega_0)} \|\nabla \widehat{u}_\lambda\|_{L^2(\Omega_0)} &\le \left(\frac{2}{3\pi}\right)^{1/2} \|\nabla w\|_{L^2(\Omega_0)} \|w\|_{L^2(\Omega_0)} \|\nabla \widehat{u}_\lambda\|_{L^2(\Omega_0)} \\ &\le \frac{L}{\sqrt{3}}\left(\frac{2}{\pi}\right)^{3/2} \|\nabla w\|^2_{L^2(\Omega_0)} \|\nabla \widehat{u}_\lambda\|_{L^2(\Omega_0)},\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_0} (w\cdot\nabla)s\cdot w \le \|\nabla s\|_{L^4(\Omega_0)} \|w\|_{L^4(\Omega_0)} \|w\|_{L^2(\Omega_0)} &\le \left(\frac{2}{3\pi}\right)^{1/4} \|\nabla s\|_{L^4(\Omega_0)} \|\nabla w\|^{1/2}_{L^2(\Omega_0)} \|w\|^{3/2}_{L^2(\Omega_0)} \\ &\le \left(\frac{2L}{\pi}\right)^{3/2} \left(\frac{2}{3\pi}\right)^{1/4} \|\nabla s\|_{L^4(\Omega_0)} \|\nabla w\|^2_{L^2(\Omega_0)}.\end{aligned}$$

With the same arguments of those in Theorem 6.3.3, we estimate the terms involving the function z as follows

$$-\mu\omega \int_{\Omega_h} \nabla w : \nabla z \, dx \leq \frac{\mu}{8} \|\nabla w\|^2_{L^2(\mathcal{R})} + \frac{2}{\mu}\omega^2 \|\nabla z\|^2_{L^2(\mathcal{R})} \,,$$

and

$$-\omega \int_{\Omega_h} (w \cdot \nabla) s \cdot z \, dx \leq \frac{\mu}{8} \|\nabla w\|^2_{L^2(\mathcal{R})} + \frac{2}{\mu}\omega^2 \|\nabla z\|^2_{L^2(\mathcal{R})} \,,$$

and, finally,

$$\omega \int_{\Omega_h} (w \cdot \nabla) z \cdot w \, dx \leq \omega \frac{4L^2}{\pi^2} \|\nabla z\|_{L^\infty(\mathcal{R})} \|\nabla w\|^2_{L^2(\mathcal{R})} \,.$$

As already mentioned, (6.2.19)-(6.2.20) corresponds to the problem treated in Chapter 4, but here we do not need to change the reference frame. Reproducing the arguments in Lemma 4.6.1 in Chapter 4 we find that

$$t\, v \in L^{4/3}(0,T; W^{2,4/3}(\Omega_0)), \qquad t\, \partial_t v \in L^{4/3}(0,T; L^{4/3}(\Omega_0)) \,,$$
$$t\, \nabla q \in L^{4/3}(0,T; L^{4/3}(\Omega_0)) \,,$$

and

$$\|tv\|_{L^{4/3}(0,T;W^{2,4/3}(\Omega_0))} + \|t\partial_t v\|_{L^{4/3}(0,T;L^{4/3}(\Omega_0))} + \|t\nabla q\|_{L^{4/3}(0,T;L^{4/3}(\Omega_0))}$$
$$\leq C \left(\|f\|_{L^{4/3}(0,T;L^{4/3}(\Omega_0))} + \|f_1\|_{L^{4/3}(0,T;\mathbb{R})} \right) \,,$$

where

$$f := v - t(v\cdot\nabla)v - t(v\cdot\nabla)s - t(s\cdot\nabla)v + t\widehat{g} \in L^{4/3}(0,T; L^{4/3}(\Omega_0)) \,,$$
$$f_1 := h' + t\frac{f(h)}{m} \in L^{4/3}(0,T;\mathbb{R}) \,.$$

This regularity, through the steps in Section 4.6 in Chapter 4 allows to infer the existence of a function

$$A(t) \in L^1(0,T) \,, \qquad A(t) = Q(\|v\|_{L^2(\Omega_0)}, \|\nabla v(t)\|_{L^2(\Omega_0)}, \|\widehat{g}\|_{L^\infty(\mathcal{R})}, \|s\|_{L^\infty(\mathcal{R})}, \|\nabla s\|_{L^\infty(\mathcal{R})}) \,,$$

with Q being positive and increasing with respect to its variables, such that

$$\int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t)) \cdot w(t) \leq A(t) \big(\|w(t)\|^2_{L^2(\Omega_0)} + m\, h'(t)^2 \big) + \frac{\mu}{8} \|\nabla w(t)\|^2_{L^2(\Omega_0)}.$$

Let ω be small enough so that

$$\frac{5\mu}{8} - \omega \|\nabla z\|_{L^\infty(\mathcal{R})} \frac{4L^2}{\pi^2} > \frac{5\mu}{8} - a_3\omega(L-\delta)\frac{4L^2}{\pi^2} := \nu > 0 \,,$$

and, by using (2.0.14),

$$\frac{\omega}{2} f(h)h - \omega^2 \frac{4}{\mu} \|\nabla z\|^2_{L^2(\mathcal{R})} \geq \frac{\omega}{2} F(h) - \omega^2 \frac{4}{\mu} \|\nabla z\|^2_{L^2(\mathcal{R})} = (\rho\frac{\omega}{4} - \frac{4}{\mu} a_2 \omega^2) h^2 \geq ch^2 \,,$$

for some c positive. After using the Poincaré inequality and the following trace inequality

$$\|\nabla w\|_{L^2(\Omega_0)} \geq c\|h'\widehat{e}_2\|_{L^2(\partial B_0)} = c|\partial B_0||h'|,$$

through which we extract a damping term for the obstacle B_h from $\frac{\mu}{2}\|\nabla w\|^2_{L^2(\Omega_0)}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E_\omega(t) + \frac{\omega}{2}f(h(t))h(t) + \nu\|\nabla w(t)\|^2_{L^2(\Omega_0)} + (\frac{\mu}{2}c|\partial B_0|^2 - \omega)h'(t)^2 \\ \leq& \frac{L}{\sqrt{3}}\left(\frac{2}{\pi}\right)^{3/2}\|\nabla w(t)\|^2_{L^2(\Omega_0)}\|\nabla \widehat{u}_\lambda\|_{L^2(\mathcal{R})} + \frac{L^{3/2}}{3^{1/4}}\left(\frac{2}{\pi}\right)^{7/4}\|\nabla s\|_{L^4(\mathcal{R})}\|\nabla w(t)\|^2_{L^2(\Omega_0)} \\ &+A(t)\left(\frac{4L^2}{\pi^2}\|\nabla w(t)\|^2_{L^2(\Omega_0)} + m\,h'(t)^2\right),\end{aligned} \tag{6.4.8}$$

with $\frac{\mu}{2}c|\partial B_0|^2 - \omega > 0$ provided that ω is small enough. From Theorem 6.2.2 we have that, if $\lambda < \lambda_s$ (the threshold for uniqueness of solutions for the stationary problem), there exists $C_1 = C_1(\lambda) > 0$ such that $C_1(\lambda) \to 0$ as $\lambda \to 0$ and

$$\|\nabla \widehat{u}_\lambda\|_{L^2(\Omega_0)} \leq C_1(\lambda).$$

On the other hand, from Theorem 6.3.3 and Theorem 6.3.5,

$$A(t) \to 0 \qquad \text{as} \qquad \lambda \to 0 \qquad \forall\, t \geq t_0 + 1,$$

with t_0 as in (6.3.12). Thus, with t sufficiently large, every term on the right-hand side of (6.4.8) multiplying $\|\nabla w(t)\|^2_{L^2(\Omega_0)}$ and $h'(t)^2$ tends to 0 as $\lambda_s \to 0$. As a consequence, one can find $\lambda_1 > 0$ such that, if $\lambda < \lambda_1$, there exist $c_3, c_4 > 0$ such that

$$\frac{1}{2}\frac{d}{dt}E_\omega + \frac{\omega}{2}f(h)h + c_3\|w\|^2_{L^2(\Omega_0)} + c_4 h'^2 \leq 0,$$

where we have used implicitely the Poincaré inequality. Multiply by 2, use (2.0.14) and the definition of E to obtain

$$\frac{d}{dt}E_\omega + \gamma E \leq 0,$$

with $\gamma = \min\{4\omega, 2c_3, \frac{2c_4}{m}\} > 0$. Then, renaming γ/c_2 as γ, we find from (6.3.4) that $\frac{d}{dt}E_\omega + \gamma E_\omega \leq 0$, which, together with (6.3.4), implies that $c_1E(t) \leq c_2E(0)e^{-\gamma t}$. Thus there exists $c > 0$ such that

$$\|v(t) - \widehat{u}_\lambda\|^2_{L^2(\Omega_0)} + mh'(t)^2 + 2F(h(t)) \leq c\big(\|\widehat{u}_0 - \widehat{u}_s\|^2_{L^2(\Omega_0)} + mk_0{}^2\big)e^{-\gamma t}, \tag{6.4.9}$$

since $F(0) = 0$. From (6.4.9), the convergence of the solutions in $\mathcal{H}_0$ is established as $t \to \infty$, which proves the claim. □

6.5 The dynamical system approach

6.5.1 Semiflow vs semigroup

We now want to revisit the results of Section 6.3 within the framework of infinite-dimensional dynamical systems, where the solution is viewed as a trajectory in a suitable phase space. Let us begin with the abstract definition of a strongly continuous semiflow.

Definition 6.5.1. Let $(\mathcal{X}, d)$ be a complete metric space. A family of one-parameter maps $S(t) : \mathcal{X} \to \mathcal{X}$ is called a *(strongly continuous) semiflow* on $\mathcal{X}$ if

(i) $S(0) = \mathrm{id}_{\mathcal{X}}$ (the identity map in $\mathcal{X}$);

(ii) the map $t \mapsto S(t)x$ is continuous for all $x \in \mathcal{X}$;

(iii) the map $x \mapsto S(t)x$ is continuous for all $t \geq 0$.

If in addition the concatenation property holds, that is, $S(t+\tau) = S(t)S(\tau)$ for all $t, \tau \geq 0$, then $S(t)$ is called a *strongly continuous semigroup* (see, e.g., [133]).

Obviously, a semigroup would simplify the analysis of the dynamics but, as already mentioned, the evolution maps $S(t)$ of (6.2.19)-(6.2.20) do not satisfy the concatenation property. The main reason relies on peculiarity of (6.2.19)-(6.2.20), where at each time step the pair $(\widehat{u}(t), h'(t))$ belongs to a different functional space $\mathcal{H}_{h(t)}$, for the domain of fluid $\Omega_{h(t)}$ depends itself on the solution. The key idea to overcome this difficulty, for a given initial position h_0 of the obstacle, is to map at every time t the cylindrical domain $\Omega_{h(t)} \times (0, T)$ onto $\Omega_{h_0} \times (0, T)$, via a suitable change of variables.

Throughout the whole section, let then

$$h_0 \in (-L + \delta, L - \delta)$$

be fixed, and denote

$$\mathcal{H} = \{z = (v, l) \in \mathbb{H}(\mathcal{R}) \,|\, v_{B_{h_0}} = l\widehat{e}_2\}, \qquad \mathcal{H}^1 = \{z = (v, l) \in \mathbb{V}(\mathcal{R}) \,|\, v_{B_{h_0}} = l\widehat{e}_2\},$$

to which we associate the norms

$$\|z\|^2_{\mathcal{H}} = \int_{\Omega_{h_0}} |v|^2 \, dx + m\, l^2, \qquad \|z\|^2_{\mathcal{H}^1} = \int_{\Omega_{h_0}} |\nabla v|^2 \, dx + m\, l^2,$$

where $z = (v, l)$, and m is the mass of the body as in (6.2.20). The spaces $\mathcal{H}$ and $\mathcal{H}^1$ are exactly the ones defined in (6.2.22), where the dependence of h_0 is dropped, since the position h_0 of the obstacle is now fixed. In particular, we have the compact embedding $\mathcal{H}^1 \Subset \mathcal{H}$. Given $z_0 = (\widehat{u}_0, k_0) \in \mathcal{H}$, we consider the solution operator $U(t) : \mathcal{H} \to \mathcal{H}_{h(t)}$ of Section 6.3, recalling that $h(t)$ is the second component of the weak solution to (6.2.19)-(6.2.20) with initial data $(\widehat{u}_0, h_0, k_0)$. Hence,

$$U(t)z_0 = (\widehat{u}(t), h'(t)).$$

Let $\varepsilon_0 \in (0, L - \delta)$ be such that

$$\min_{t \in [0,T]} \mathrm{dist}(\partial B_{h(t)}, \Gamma) \geq \varepsilon_0. \tag{6.5.1}$$

The existence of such an ε_0 comes from Corollary 6.2.8. For this $h(t)$, we can build for any $t > 0$ the map $\psi(t, \cdot)$ of Proposition 6.4.2, where we take $\varepsilon = \varepsilon_0$, and define its inverse with respect to the space variables, that we denote by $\varphi(t, \cdot) = \psi^{-1}(t, \cdot)$. In order to recast our results in the semiflow language, the main ingredient is the introduction of the family of maps, depending on $h(t)$,

$$\Phi_t : \mathcal{H}_{h(t)} \to \mathcal{H},$$

given by

$$\Phi_t(\widehat{u}(t), h'(t)) = (\nabla\psi(\varphi(t,y),t) \cdot \widehat{u}(\varphi(t,y),t), h'(t)),$$

whose properties will be described later in this section. Then, we define the one-parameter family of operators

$$S(t) : \mathcal{H} \to \mathcal{H},$$

acting by the rule

$$z_0 \mapsto S(t)z_0 = \Phi_t(U(t)z_0). \tag{6.5.2}$$

Roughly speaking, what we do is to think the obstacle as fixed during the whole evolution. Accordingly, the variable h loses its physical meaning, since it does not represent any longer the position of the obstacle, but its effects appear inside the equation through the map Φ_t.

Theorem 6.5.2. *The map $S(t)$ fulfills the semiflow axioms* (i)-(iii) *of Definition 6.5.1 on the complete metric space $\mathcal{H}$, endowed with the distance induced by the norm $\|\cdot\|_{\mathcal{H}}$.*

As it will be clear, it is however false that $S(t)$ is a semigroup. The proof of Theorem 6.5.2 is carried out in the remaining of the section. In particular, in the next Subsection 6.5.2, we state and prove some preliminary results in order to properly characterize the action of the map Φ_t on problem (6.2.19)-(6.2.20). The conclusion of the proof will be given in Subsection 6.5.3, by verifying the semiflow properties of Definition 6.5.1.

6.5.2 Properties of the map Φ_t

Recalling thath $h_0 \in (-L+\delta, L-\delta)$ has been fixed once for all, throughout this subsection, we consider a given $\varepsilon > 0$ and a given function

$$h \in W^{1,\infty}(0,T;[-L+\delta+\varepsilon, L-\delta-\varepsilon]) \qquad \forall\, T > 0 \tag{6.5.3}$$

such that

$$h(0) = h_0, \qquad B_{h(t)} \subset \mathcal{O}_\varepsilon \qquad \forall\, t \in [0,T].$$

With this choice, let $s = s_\varepsilon$ be a function obtained through Lemma 6.2.1. Moreover, we can build the volume preserving diffeomorphism ψ of Proposition 6.4.2, along with its inverse $\varphi = \psi^{-1}$. Then, we denote

$$g_{ij} = \frac{\partial\varphi^k}{\partial y^i}\frac{\partial\varphi^k}{\partial y^j}, \qquad g^{ij} = \frac{\partial\psi^i}{\partial x^k}\frac{\partial\psi^j}{\partial x^k}, \qquad \Gamma^i_{kj} = g^{il}\left(\frac{\partial g_{kl}}{\partial y^j} + \frac{\partial g_{jl}}{\partial y^k} - \frac{\partial g_{kj}}{\partial y^l}\right) = \frac{\partial\psi^i}{\partial x^l}\frac{\partial^2\varphi^l}{\partial y^k \partial y^i}, \tag{6.5.4}$$

where g_{ij} defines a metric on $\mathbb{R}^2$ since $\det(\frac{\partial\psi_i}{\partial x_j})_{i,j} = 1$. Call (now the space variable is y)

$$v(y,t) = \nabla\psi(\varphi(t,y),t) \cdot \widehat{u}(\varphi(t,y),t) \qquad y \in \Omega_{h_0}, \tag{6.5.5}$$

the pullback of $\widehat{u}$ by φ, and set

$$q(y,t) = p(\varphi(y,t),t).$$

We follow the procedure in [130, paragraph 4.2] to transform the Navier-Stokes equation (6.2.19) in the cylindrical domain $\Omega_{h_0} \times (0,T)$. Thanks to (6.4.1), for each term involving s, the maps ψ and φ correspond to the identity. Thus, we obtain the (weak) problem with variable coefficients in the new unknown v (at this stage, the function $h(t)$ is prescribed)

$$\begin{aligned}
&v_t + \mathcal{M}v - \mu\,\mathcal{L}v + \mathcal{N}v + (v\cdot\nabla)\,s + (s\cdot\nabla)\,v + \mathcal{G}q = \widehat{g} \qquad \text{in } \Omega_{h_0}\times(0,T)\\
&\operatorname{div} v = 0 \qquad \text{in } \Omega_{h_0}\times(0,T)\\
&v = 0 \qquad \text{on } \Gamma\times(0,T)\\
&v = h'\widehat{e}_2 \qquad \text{on } \partial B_h\times(0,T)\\
&\lim_{|y_1|\to\infty} v(y_1,y_2) = 0\\
&v(0) = \widehat{u}_0.
\end{aligned} \tag{6.5.6}$$

The operators $\mathcal{M},\mathcal{L},\mathcal{N}$ appearing in (6.5.6) are defined here below (the exponent i stands for the i-th component, and we use the Einstein notation).

$$\begin{aligned}
(\mathcal{M}v)^i =&\partial_l v^i\partial_t\psi^l + \partial_k\psi^i(\partial_k\partial_t\varphi^i)v^k + \partial_k\psi^i\partial^2_{kl}\varphi^i\partial_t\psi^l v^k,\\
(\mathcal{L}v)^i =&\partial_k\psi^i\partial_j\psi^m(\partial^2_{mk}\varphi^i)\partial_l v^k\partial_j\psi^l + \partial_j\psi^m\partial^2_{ml}v^i\partial_j\psi^l + \partial_l v^i(\partial^2_{jj}\psi^l) + \partial_k\psi^i\partial_j\psi^m(\partial^3_{mlk}\varphi^i)\partial_j\psi^l v^k\\
&+ \partial_k\psi^i(\partial^2_{lk}\varphi^i)\partial^2_{jj}\psi^l v^k + \partial_k\psi^i(\partial^2_{lk}\varphi^i)\partial_j\psi^l\partial_j\psi^m\partial_m v^k,\\
(\mathcal{N}v)^i =&v^l\partial_l v^i + \partial_k\psi^i v^l(\partial^2_{lk}\varphi^i)v^k,\\
(\mathcal{G}q)^i =&g^{ij}\partial_j q.
\end{aligned} \tag{6.5.7}$$

Remark 6.5.3. Note that:

- $(\partial_t + \mathcal{M})v$ corresponds to the original time derivative $\widehat{u}_t$;
- $\mathcal{L}v$ corresponds to $\Delta\widehat{u}$;
- $\mathcal{N}v$ corresponds to $(\widehat{u}\cdot\nabla)\widehat{u}$;
- $\mathcal{G}q$ corresponds to ∇p.

In particular, in $\mathcal{A}_\varepsilon$ these operators coincide with the original ones; the same is true in in $\mathcal{O}_\varepsilon$, except for

$$(\partial_t + \mathcal{M})v = (\partial_t - h'\widehat{e}_2\cdot\nabla)v.$$

The first equation in (6.5.6) can be rewritten as

$$v_t - \mu\,\Delta v + (v\cdot\nabla)\,v + \nabla q + (v\cdot\nabla)\,s + (s\cdot\nabla)\,v = \widehat{g} + \mathcal{F}(v,h,q),$$

where

$$\mathcal{F}(v,h,q) = \mu(\mathcal{L}-\Delta)v - \mathcal{M}v - (\mathcal{N}v - (v\cdot\nabla)v) - (\mathcal{G}-\nabla)q.$$

Observe that

$$\mathcal{F}(v,h,q) = \begin{cases} 0 & \text{in } \mathcal{A}_\varepsilon\\ h'\widehat{e}_2\cdot\nabla v & \text{in } \bar{\mathcal{O}}_\varepsilon,\end{cases}$$

thus $\mathcal{F}$ has compact support in Ω_{h_0}. The introduction of the maps ψ and φ allows to remove the dependence on time from the fluid domain, with a consequent strengthening of the coupling between the equations governing the motion of the fluid and the one governing the motion of the obstacle. Such a strengthening appears in the fictitious force $\mathcal{F} = \mathcal{F}(v, h, q)$, where the dependence on h is hidden in ψ and φ. This renders the dynamics structurally non-autonomous, and this is the reason why we do not end up with a semigroup.

Remark 6.5.4. If $h(t)$ is not just any prescribed function, but it is exactly the second component of the weak solution to (6.2.19)-(6.2.20) with initial data $(\widehat{u}_0, h_0, k_0)$, and $\varepsilon = \varepsilon_0$ with ε_0 as in (6.5.1), we have the equivalence between (6.5.6) and the original equation (6.2.19), in terms of *strong solutions*. This was proven in [130, Propositions 4.5, 4.6], which in turn refers to [93, Theorem 2.5]

Here, we are are interested in the construction of *weak solutions*. To this aim, leaning on some ideas of [105], we introduce for any fixed $t > 0$ the scalar products

$$\langle v_1, v_2\rangle_t = \int_{\Omega_{h_0}} g_{ij}(y,t) v_1^i(y) v_2^j(y)\, dy, \qquad \langle D_g v_1, D_g v_2\rangle_t = \int_{\Omega_{h_0}} g_{ij}(y,t) g^{kl}(y,t) \nabla_k v_1^i \nabla_l v_2^j\, dy, \tag{6.5.8}$$

where

$$\nabla_k v^i = \frac{\partial v^i}{\partial y^k} + \Gamma^i_{kj} v^j,$$

and we denote by

$$\|v\|_t^2 = \langle v, v\rangle_t \qquad \text{and} \qquad \|D_g v\|_t^2 = \langle D_g v, D_g v\rangle_t$$

the induced (square) norms. We emphasize that the scalar products in (6.5.8) explicitely depend on the choice of the function h in (6.5.3), that for the moment is understood to be given. Under the change of variables induced by φ, for any $t \geq 0$ we have the equalities

$$\langle v_1, v_2\rangle_t = \int_{\Omega_{h(t)}} \widehat{u}_1 \cdot \widehat{u}_2\, dx, \qquad \langle D_g v_1, D_g v_2\rangle_t = \int_{\Omega_{h(t)}} \nabla \widehat{u}_1 : \nabla \widehat{u}_2\, dx.$$

Moreover, since g_{ij} is a positive definite invertible matrix and the spatial derivatives of $\varphi(\cdot, t)$ are bounded functions (see also [93, Section 3]), there exist $C_1, C_2 > 0$ (depending on $T > 0$), such that, for any fixed $t \in [0, T]$,

$$C_1 \|v\|_{L^2(\Omega_{h_0})} \leq \|v\|_t \leq C_2 \|v\|_{L^2(\Omega_{h_0})}. \tag{6.5.9}$$

Analogously, there exist two positive constants C_3 and C_4 such that

$$C_3 \|\nabla v\|_{L^2(\Omega_{h_0})} \leq \|D_g v\|_t \leq C_4 \|\nabla v\|_{L^2(\Omega_{h_0})}. \tag{6.5.10}$$

This allows us to introduce the norms on $\mathcal{H}$ and $\mathcal{H}^1$

$$|z|_{t,\mathcal{H}} = \sqrt{\|v\|_t^2 + m\, l^2}\,, \qquad |z|_{t,\mathcal{H}^1} = \sqrt{\|D_g v\|_t^2 + m\, l^2}\,,$$

equivalent to the original ones. Again, we point out that such an equivalence is uniform for a fixed $T > 0$. Now we give the rigorous definition of a weak solution to problem (6.5.6).

Definition 6.5.5. Let the given function h comply with (6.5.3). A function v is a weak solution to (6.5.6), with initial value $v(0) = \widehat{u}_0$, if

$$(v, h') \in L^2(0,T;\mathcal{H}^1) \cap L^\infty(0,T;\mathcal{H}), \qquad (\partial_t v, h'') \in L^2(0,T;\mathcal{H}^{-1}),$$

and, for any pair $(\tilde{\phi}, l) \in \mathcal{H}^1$ and almost every $t \geq 0$,

$$\begin{aligned}&\langle \partial_t v(t), \tilde{\phi}\rangle_t + \langle \mathcal{M} v(t), \tilde{\phi}\rangle_t + m\, h''(t) l + f(h(t))\, l - \mu \langle \mathcal{L} v(t), \tilde{\phi}\rangle_t + \langle \mathcal{N} v(t), \tilde{\phi}\rangle_t \\ &+ \int_{\Omega_{h_0}} (v(t)\cdot\nabla)\, s \cdot \tilde{\phi}\, dy + \int_{\Omega_{h_0}} (s\cdot\nabla)\, v(t)\cdot\tilde{\phi}\, dy = \int_{\Omega_{h_0}} \widehat{g}\cdot\tilde{\phi}\, dy.\end{aligned} \tag{6.5.11}$$

We are ready to prove the equivalence between problem (6.5.6) and the original problem (6.2.19)-(6.2.20) in terms of weak solutions.

Proposition 6.5.6. *Let $(\widehat{u}, h)$ be the weak solution to problem* (6.2.19)-(6.2.20) *with initial data $(\widehat{u}_0, h_0, k_0)$, and let v be a weak solution to* (6.5.6) *with the same h and initial datum $\widehat{u}_0$, in the sense of Definition 6.5.5. Then $\widehat{u}$ and v are related by* (6.5.5)*, that is,*

$$v(\cdot,t) = \nabla\psi(\varphi(t,\cdot),t)\cdot\widehat{u}(\varphi(t,\cdot),t).$$

Proof. The proposition is proven by establishing a correspondence among each term in (6.5.11) and in (6.2.25). The function h is now the second component of the weak solution $(\widehat{u}, h)$ to problem (6.2.19)-(6.2.20) with initial data $(\widehat{u}_0, h_0, k_0)$, and ε_0 is as in (6.5.1). Then, we can build the map ψ of Proposition 6.4.2, where we take $\varepsilon = \varepsilon_0$, and define its inverse with respect to the space variables, $\varphi = \psi^{-1}$. From (6.5.5), we obtain that

$$\widehat{u}(x,t) = \nabla\varphi(\psi(t,x),t)\cdot v(\psi(t,x),t). \tag{6.5.12}$$

Concerning the test function ϕ and $\tilde{\phi}$ appearing in the two definitions of solution, applying the change of variable we produce a bijection $\phi \leftrightarrow \tilde{\phi}$ given by

$$\phi(x,t) = \nabla\varphi(\psi(t,x),t)\cdot\tilde{\phi}(\psi(t,x)). \tag{6.5.13}$$

Indeed, as ψ and φ are volume preserving, we do not lose the divergence-free property of the functions (see for instance [93, Proposition 2.4]). Thus, by plugging (6.5.12)-(6.5.13) into (6.2.25), after integrating by parts and using the fact that $y = \psi(t,x) \in \Omega_{h_0}$, we obtain

$$\begin{aligned}&\int_{\Omega_{h_0}} \partial_t[\nabla\varphi(y,t)\cdot v(y,t)]\cdot\nabla\varphi(y,t)\cdot\tilde{\phi}(y)\, dy + m\, h''(t) l(t) + f(h(t))\, l(t) \\ &- \mu\int_{\Omega_{h_0}} \Delta[\varphi(y,t)\cdot v(y,t)]\cdot\nabla\varphi(y,t)\cdot\tilde{\phi}(y)\, dy + \int_{\Omega_{h_0}} [(v(y,t)\cdot\nabla)\, s\cdot\tilde{\phi}(y) + (s\cdot\nabla)\, v(y,t)\cdot\tilde{\phi}(y)]\, dy \\ &+ \int_{\Omega_{h_0}} (\nabla\varphi(y,t)\cdot v(y,t)\cdot\nabla)\,[\nabla\varphi(y,t)\cdot v(y,t)]\cdot\nabla\varphi(y,t)\cdot\tilde{\phi}(y)\, dy = \int_{\Omega_{h_0}} \widehat{g}\cdot\tilde{\phi}(y)\, dy.\end{aligned} \tag{6.5.14}$$

We remark that in the equality above we have used the properties of a function s of Lemma 6.2.1, which is nonzero whenever φ, ψ are the identity, together with the function $\widehat{g}$ of (6.2.15). From (6.5.7), we have that

$$\begin{aligned}\partial_t[\nabla\varphi(y,t)\cdot v(y,t)] &= \partial_k\varphi^i\partial_t v^k + \partial_k\varphi^i\partial_l v^k\partial_t\psi^l + (\partial_k\partial_t\varphi^i)v^k + \partial^2_{kl}\varphi^i\partial_t\psi^l v^k \\ &= \partial_k\varphi^i\partial_t v^k + \partial_k\varphi^i(\mathcal{M}v)^i,\end{aligned}$$

$$\Delta[\varphi(y,t)\cdot v(y,t)] = \partial_j\psi^m(\partial^2_{mk}\varphi^i)\partial_l v^k\partial_j\psi^l + \partial_k\varphi^i\partial_j\psi^m\partial^2_{ml}v^k\partial_i\psi^l + \partial_k\varphi^i\partial_l v^k(\partial^2_{jj}\psi^l) +$$
$$\partial_j\psi^m(\partial^3_{mlk}\varphi^i)\partial_j\psi^l v^k + (\partial^2_{lk}\varphi^i)\partial^2_{jj}\psi^l v^k + (\partial^2_{lk}\varphi^i)\partial_j\psi^l\partial_j\psi^m\partial_m\widehat{v}^k = \partial_k\varphi^i(\mathcal{L}v)^i,$$
$$(\nabla\varphi(y,t)\cdot v(y,t)\cdot\nabla)\,[\nabla\varphi(y,t)\cdot v(y,t)] = \partial_k\varphi^i v^l\partial_l v^k + v^l(\partial^2_{lk}\varphi^i)v^k = \partial_k\varphi^i(\mathcal{N}v)^i.$$

Thus, through the definition of the scalar products in (6.5.8), we obtain that (6.5.14) is equivalent to (6.5.11), which completes the proof. □

6.5.3 Proof of Theorem 6.5.2

On account of (6.5.5), we rewrite the map Φ_t as

$$\Phi_t(\widehat{u}(t), h'(t)) = (v(t), h'(t)),$$

where now v is defined by choosing the function $h(t)$ to be the second component of the weak solution to (6.2.19)-(6.2.20) with initial data $(\widehat{u}_0, h_0, k_0)$, and $\varepsilon = \varepsilon_0$, with ε_0 as in (6.5.1). Then, point (i) of Definition 6.5.1 follows directly from the properties of ψ and φ. Point (ii) is a consequence of Theorem 6.2.7 and Proposition 6.5.6, from which we learn that (v, h') is equal almost everywhere to a continuous function from $[0,T]$ to $\mathcal{H}$ with respect to the norm $|\cdot|_{t,\mathcal{H}}$. By the equivalence relation between the norms given in (6.5.9), this implies the continuity with respect to $\|\cdot\|_{\mathcal{H}}$ as well. The next proposition proves point (iii).

Proposition 6.5.7. *Let $R > 0$ be arbitrarily fixed, and let $n = 1,2$. For any pair of initial velocities*

$$z_{0,n} = (\widehat{u}_{0,n}, k_{0,n}) \in \mathcal{H} \qquad \text{such that} \qquad \|z_{0,n}\|_{\mathcal{H}} \le R,$$

the estimate

$$\|S(t)z_{0,1} - S(t)z_{0,2}\|_{\mathcal{H}} \le K\|z_{0,1} - z_{0,2}\|_{\mathcal{H}}$$

holds for every $t \in [0,T]$, for some positive constant $K = K(R,T)$.

Proof. Let $z_{0,n} = (\widehat{u}_{0,n}, k_{0,n}) \in \mathcal{H}$ be such that $\|z_{0,n}\|_{\mathcal{H}} \le R$. Setting further $h_n(0) = h_0$, there exists a unique weak solution $(\widehat{u}_n, h_n)$ to problem (6.2.19)-(6.2.20). From Corollary 6.2.8, there is ε_0, depending on R and T, such that $B_{h_n} \subset \mathcal{O}_{\varepsilon_0}$. Thus, through Lemma 6.2.1 we can build $s = s_{\varepsilon_0}$ as well as ψ_n as in Proposition 6.4.2 and $\varphi_n = \psi_n^{-1}$, where the subscript $n = 1,2$ depends on whether we consider h_1 or h_2. In order to estimate the distance between $S(t)z_{0,1} = (v_1, h_1)$ and $S(t)z_{0,2} = (v_2, h_2)$ in terms of the distance between $z_{0,1}$ and $z_{0,2}$, we exploit again the result and the procedure implemented in Section 4.6 in Chapter 4, where it is proven the uniqueness for solutions to problem (6.2.19)-(6.2.20). In order to make the proof of the theorem self-contained, let us briefly describe the procedure: we introduce the two maps

$$\mathsf{F} : \mathbb{R}^+ \times \Omega_{h_2(t)} \longrightarrow \Omega_{h_1(t)} \qquad \text{and} \qquad \mathsf{G} : \mathbb{R}^+ \times \Omega_{h_1(t)} \longrightarrow \Omega_{h_2(t)},$$

defined as

$$\mathsf{F} = \varphi_1(t, \psi_2(t,x)) \qquad \text{and} \qquad \mathsf{G} = \varphi_2(t, \psi_1(t,x)).$$

This is possible since $h_1(0) = h_2(0) = h_0$. Following Section 4.6 in Chapter 4 (see also [111, Section 5]), let

$$\widehat{\mathfrak{u}}_2(x,t) = \nabla\mathsf{F}(\mathsf{G}(x),t)\cdot\widehat{u}_2(\mathsf{G}(x),t) \qquad x \in \Omega_{h_1(t)}$$

be the pullback of $\widehat{u}_2$ by G. Next, we call

$$w = \widehat{u}_1 - \widehat{\mathfrak{u}}_2, \qquad h = h_1 - h_2,$$

and we take the difference between the weak formulation satisfied by $(\widehat{u}_1, h_1)$ and that satisfied by $(\widehat{\mathfrak{u}}_2, h_2)$. We obtain

$$\langle \partial_t w, \phi\rangle + m\,h''l + [f(h_1) - f(h_2)]\,l + \mu\int_{\mathcal{R}} \nabla w : \nabla\phi + \int_{\Omega_{h_1}} (\widehat{u}_1\cdot\nabla)w\cdot\phi + \int_{\Omega_{h_1}} (w\cdot\nabla)\widehat{\mathfrak{u}}_2\cdot\phi$$
$$+ \int_{\Omega_{h_1}} (s\cdot\nabla)w\cdot\phi + \int_{\Omega_{h_1}} (w\cdot\nabla)s\cdot\phi = \int_{\Omega_{h_1}} \mathfrak{f}\cdot\phi,$$

where the expression of $\mathfrak{f}$ reads as in (6.4.5), once we substitute v with $\widehat{\mathfrak{u}}_2$, Ω_0 with Ω_{h_1}, φ with G and ψ with F. Then, we take $(\phi, l) = (w, h')$ and we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2_{L^2(\Omega_{h_1})} + mh'^2 + 2\int_{h_2}^{h_1} f(s)\,ds\right) + \mu\|\nabla w\|^2_{L^2(\Omega_{h_1})} = -\int_{\Omega_{h_1}} (w\cdot\nabla)\widehat{\mathfrak{u}}_2\cdot w$$
$$- \int_{\Omega_{h_1}} (w\cdot\nabla)s\cdot w$$
$$+ \int_{\Omega_{h_1}} \mathfrak{f}\cdot\phi. \tag{6.5.15}$$

Next, we estimate each term on the right-hand side. The first two term can be bounded by suitably exploiting the Hölder inequality, [68, (2.26)], the Poincaré inequality and the Young inequality. We obtain

$$\left|\int_{\Omega_{h_1}} (w\cdot\nabla)\widehat{\mathfrak{u}}_2\cdot w\right| \le \|w\|^2_{L^4(\Omega_{h_1})}\|\nabla\widehat{\mathfrak{u}}_2\|_{L^2(\Omega_{h_1})} \le \left(\frac{2}{3\pi}\right)^{1/2}\|w\|_{L^2(\Omega_{h_1})}\|\nabla w\|_{L^2(\Omega_{h_1})}\|\nabla\widehat{\mathfrak{u}}_2\|_{L^2(\Omega_{h_1})}$$
$$\le \frac{2}{3\pi\mu}\|\nabla\widehat{\mathfrak{u}}_2\|^2_{L^2(\Omega_{h_1})}\|w\|^2_{L^2(\Omega_{h_1})} + \frac{\mu}{4}\|\nabla w\|^2_{L^2(\Omega_{h_1})},$$

and

$$\left|\int_{\Omega_{h_1}} (w\cdot\nabla)s\cdot w\right| \le \|w\|^2_{L^4(\Omega_{h_1})}\|\nabla s\|_{L^2(\Omega_{h_1})} \le \left(\frac{2}{3\pi}\right)^{1/2}\|w\|_{L^2(\Omega_{h_1})}\|\nabla w\|_{L^2(\Omega_{h_1})}\|\nabla s\|_{L^2(\Omega_{h_1})}$$
$$\le \frac{2}{3\pi\mu}\|\nabla s\|^2_{L^2(\Omega_{h_1})}\|w\|^2_{L^2(\Omega_{h_1})} + \frac{\mu}{4}\|\nabla w\|^2_{L^2(\Omega_{h_1})}.$$

For what concerns the last term, analogously to what we did in Section 6.4, we can proceed step by step as in 4.6 in Chapter 4 to obtain the existence of a function

$$A = A(t) = Q(\|\widehat{\mathfrak{u}}_2(t)\|_{L^2(\Omega_{h_1})}, \|\nabla\widehat{\mathfrak{u}}_2(t)\|_{L^2(\Omega_{h_1})}, \|\widehat{g}\|_{L^\infty(\mathcal{R})}, \|s\|_{L^\infty(\mathcal{R})}, \|\nabla s\|_{L^\infty(\mathcal{R})}) \in L^1(0,T),$$

the L^1-bound depending only on R, such that

$$\int_{\Omega_{h_1}} \mathfrak{f}\cdot\phi \le A(\|w\|^2_{L^2(\Omega_{h_1})} + mh'^2) + \frac{\mu}{2}\|\nabla w\|^2_{L^2(\Omega_{h_1})},$$

Then, calling

$$\Lambda(t) = A(t) + \frac{2}{3\pi\mu}\|\nabla s\|^2_{L^2(\Omega_{h_1(t)})} + \frac{2}{3\pi\mu}\|\nabla\widehat{\mathfrak{u}}_2(t)\|^2_{L^2(\Omega_{h_1(t)})},$$

and by inserting all the above inequalities in (6.5.15), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2_{L^2(\Omega_{h_1})} + mh'^2 + 2\int_{h_2}^{h_1} f(s)\,ds\right) \le \Lambda(\|w\|^2_{L^2(\Omega_{h_1})} + mh'^2).$$

Moreover, defining the functions

$$\begin{aligned}\Theta(t) = \|w(x,t)\|^2_{L^2(\Omega_{h_1(t)})} + mh'(t)^2 &= \|\widehat{u}_1(x,t) - \nabla\mathsf{F}(\mathsf{G}(x),t)\cdot\widehat{u}_2(\mathsf{G}(x),t)\|^2_{L^2(\Omega_{h_1(t)})}\\ &\quad + m(h_1'(t) - h_2'(t))^2,\end{aligned}$$

and observing that

$$\Theta(0) = \|\widehat{u}_{0,1} - \widehat{u}_{0,2}\|^2_{L^2(\Omega_{h_0})} + m(k_{0,1} - k_{0,2})^2,$$

we obtain

$$\Theta(t) \le \Theta(0) + \int_0^t 2\,\Lambda(\tau)\Theta(\tau)\,d\tau \qquad \forall\, t\in[0,T].$$

The Gronwall Lemma (integral form) then gives

$$\Theta(t) \le K\Theta(0) \qquad \forall\, t\in[0,T], \tag{6.5.16}$$

having set $K = \exp\left[\int_0^T 2\,\Lambda(\tau)\,d\tau\right]$. The final step is to rewrite (6.5.16) on Ω_{h_0} by applying the coordinate transformation $x = \varphi_1(t,y)$. Given

$$v_1(y,t) = \nabla\psi_1(\varphi_1(t,y),t)\cdot\widehat{u}_1(\varphi_1(t,y),t) \qquad \text{and} \qquad v_2(y,t) = \nabla\psi_2(\varphi_2(t,y),t)\cdot\widehat{u}_2(\varphi_2(t,y),t),$$

for all $t\in[0,T]$ we obtain

$$\begin{aligned}&\|\nabla\varphi_1(t,y)\cdot v_1(y,t) - \nabla\varphi_1(t,y)\cdot v_2(y,t)\|^2_{L^2(\Omega_{h_0})} + m(h_1'(t) - h_2'(t))^2\\ &\le K\Big(\|\widehat{u}_{0,1} - \widehat{u}_{0,2}\|^2_{L^2(\Omega_{h_0})} + m(k_{0,1} - k_{0,2})^2\Big),\end{aligned}$$

which in turn can be rewritten as

$$\|v_1(t) - v_2(t)\|^2_t + m(h_1'(t) - h_2'(t))^2 \le K\Big(\|\widehat{u}_{0,1} - \widehat{u}_{0,2}\|^2_{L^2(\Omega_{h_0})} + m(k_{0,1} - k_{0,2})^2\Big).$$

Note that the norm $\|\cdot\|_t$ above is constructed by taking φ_1 in (6.5.4). Therefore, recalling the definition of the norm $|\cdot|_{t,\mathcal{H}}$, we arrive at

$$|S(t)z_{0,1} - S(t)z_{0,2}|_{t,\mathcal{H}} \le K\|z_{0,1} - z_{0,2}\|_{\mathcal{H}} \qquad \forall\, t\in[0,T].$$

The desired conclusion follows by applying (6.5.9), up to redefining the constant K. $\square$

6.6 The global attractor of the semiflow

The further step is to translate the dissipative features of our system in the semiflow language. Let us begin by recalling some classical notions (see, e.g., [12, 26, 133]). In what follows, $S(t)$ is a strongly continuous semiflow acting on a complete metric space $(\mathcal{X}, d)$.

Definition 6.6.1. A set $\mathcal{B}_0 \subset \mathcal{X}$ is called an *absorbing set* for $S(t)$ if for every bounded set $\mathcal{B} \subset \mathcal{X}$ there exists an *entering time* $t_{\mathcal{B}} \geq 0$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0 \qquad \forall\, t \geq t_{\mathcal{B}}.$$

The existence of a bounded absorbing set witnesses the dissipative character of a semiflow, since the dynamics is eventually confined in a bounded subset of the phase space. And indeed, in the recent literature, the definition of a dissipative semiflow is exactly the one of a semiflow possessing a bounded absorbing set. Nonetheless, in spite of its boundedness, an absorbing set can be to some extent a very large object. For instance, if $\mathcal{X}$ is a (closed) subset of a Banach space, an absorbing set might share the same dimension of the whole space (think to a ball). For this reason, one would like to exhibit a stronger form of dissipation. The natural way to do that is to invoke compactness, since this is the correct notion to translate the fact that the dynamics loses degrees of freedom. Accordingly, the strategy is to look for the existence of compact sets, hence meager in the space, able to attract (in a suitable sense) all the trajectories of the semiflow in the longterm. This attraction property is expressed in terms of Hausdorff semidistance in $\mathcal{X}$: given two (nonempty) sets $\mathcal{B}, \mathcal{C} \subset X$, their Hausdorff semidistance is defined as

$$\boldsymbol{\delta}(\mathcal{B}, \mathcal{C}) = \sup_{x \in \mathcal{B}} d(x, \mathcal{C}) = \sup_{x \in \mathcal{B}} \inf_{y \in \mathcal{C}} d(x, y).$$

In a completely equivalent manner, we can write

$$\boldsymbol{\delta}(\mathcal{B}, \mathcal{C}) = \inf \big\{ \varepsilon > 0 : \, \mathcal{B} \subset \mathcal{O}_\varepsilon(\mathcal{C}) \big\},$$

where $\mathcal{O}_\varepsilon(\mathcal{C}) = \bigcup_{y \in \mathcal{C}} \big\{ x \in \mathcal{X} : \, d(x, y) < \varepsilon \big\}$ is the ε-neighborhood of $\mathcal{C}$.

Definition 6.6.2. A set $\mathcal{K} \subset \mathcal{X}$ is called an *attracting set* for $S(t)$ if, for every bounded set $\mathcal{B} \subset \mathcal{X}$,

$$\lim_{t \to \infty} \boldsymbol{\delta}(S(t)\mathcal{B}, \mathcal{K}) = 0.$$

Whenever there exists a compact attracting set the semiflow is said to be *asymptotically compact*.

Remark 6.6.3. Clearly, an absorbing set is in particular an attracting set. It is also apparent that if the semiflow is asymptotically compact, then it is dissipative, in the sense that it possesses a bounded absorbing set.

Once the existence of a compact absorbing set is established, one might ask if there is the best possible one among those sets. This leads to our last definition.

Definition 6.6.4. The *global attractor* $\mathcal{A}$ of $S(t)$ is the smallest compact attracting set.

In the literature, the notion of global attractor is usually given in the context of semigroups, and not just semiflows (see, e.g., [12, 133]). In particular, with the only exception of [26], the classical definition differs from the one given above since, besides the attraction property, one requires also the invariance, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. Unfortunately, when dealing with semiflows (and not semigroups), the invariance seems to be out of reach. Nonetheless, our definition makes perfectly sense. The only problem is the existence of such a set. To this aim, we state the following result.

Theorem 6.6.5. *An asymptotically compact semiflow possesses the global attractor in the sense of Definition 6.6.4.*

Proof. The idea of the proof is somehow already contained in [26], although in that paper $S(t)$ is a semigroup. In fact, the theorem remains true if $S(t)$ is a one-parameter selfmap of $\mathcal{X}$, without requiring any of the axioms (i)-(iii) of Definition 6.5.1. Consider the family of sets

$$\mathbb{K} = \big\{\mathcal{K} \subset \mathcal{X} : \mathcal{K} \text{ is compact and attracting}\big\},$$

which, due to the hypothesis, is nonempty. Besides, let $\mathfrak{C}$ be the collection of all possible sequences of the form

$$y_n = S(t_n)x_n,$$

where x_n is a bounded sequence in $\mathcal{X}$ and $t_n \to \infty$. For any $y_n \in \mathfrak{C}$ we denote

$$\mathfrak{L}(y_n) = \big\{w \in \mathcal{X} : y_n \to w \text{ up to a subsequence}\big\}.$$

Note that $\mathfrak{L}(y_n) \neq \emptyset$. Indeed, let $\mathcal{K} \in \mathbb{K}$. Then there exists $w_n \in \mathcal{K}$ such that

$$d(y_n, w_n) \to 0.$$

Invoking the compactness of $\mathcal{K}$, there is $w \in \mathcal{K}$ and a subsequence w_{n_i} converging to w. Hence,

$$d(y_{n_i}, w) \leq d(y_{n_i}, w_{n_i}) + d(w_{n_i}, w) \to 0.$$

Finally, define the set

$$\mathcal{A}^\star = \bigcup_{y_n \in \mathfrak{C}} \mathfrak{L}(y_n).$$

We claim that $\mathcal{A}^\star$ is attracting: if not, there exist a bounded set $\mathcal{B} \subset \mathcal{X}$, a sequence $t_n \to \infty$ and $\varepsilon > 0$ such that

$$\boldsymbol{\delta}(S(t_n)\mathcal{B}, \mathcal{A}^\star) \geq 2\varepsilon.$$

From the definition of Hausdorff semidistance, this implies the existence of a sequence $x_n \in \mathcal{B}$, hence bounded, for which

$$d(S(t_n)x_n, \mathcal{A}^\star) \geq \varepsilon.$$

But, as we saw, $y_n = S(t_n)x_n$ has limit points, which belong to $\mathcal{A}^\star$ by construction. This yields the claim. It is also apparent that $\mathcal{A}^\star$ is contained in any closed attracting set. Accordingly, the set

$$\mathcal{A} = \overline{\mathcal{A}^\star} \quad \text{(closure in } \mathcal{X})$$

is the smallest element of $\mathbb{K}$. An equivalent way to define $\mathcal{A}$ is to put

$$\mathcal{A} = \bigcap_{\mathcal{K}\in\mathbb{K}} \mathcal{K},$$

noting that the (compact) sets $\mathcal{K} \in \mathbb{K}$ fulfill the finite intersection property, for they all contain $\mathcal{A}^\star$. □

We can now go back to our particular semiflow $S(t)$ on $\mathcal{H}$ associated to problem (6.2.19)-(6.2.20), and defined in (6.5.2). The main result of this section reads as follows.

Theorem 6.6.6. *The semiflow $S(t) : \mathcal{H} \to \mathcal{H}$ possesses the global attractor.*

Before entering the details of the proof, let us recall once again that we are working under the hypothesis that $h_0 \in (-L+\delta, L-\delta)$ is fixed. Theorem 6.6.6 makes use of the technical results of Section 6.5.2. In view of the definition of the semiflow $S(t)$, the function $h(t)$ will always be the second component of the weak solution to (6.2.19)-(6.2.20) with initial data $(\widehat{u}_0, h_0, k_0)$. Our purpose is to investigate the longtime behaviour of $S(t)z_0$ as $z_0 = (\widehat{u}_0, k_0)$ is allowed to run in a bounded set of $\mathcal{H}$. To this aim, we need to improve (and make uniform) the equivalence relations between the norms given in (6.5.9) and (6.5.10). This can be done by exploiting the dissipation properties of the solution operator $U(t)$ of Section 6.3.

Proof of Theorem 6.6.6. In the light of Theorem 6.6.5, all we need to do is showing that $S(t)$ is asymptotically compact. In fact, we will obtain a stronger result, namely, the existence of a compact absorbing set $\mathcal{B}_1 \subset \mathcal{H}$. Indeed, given a bounded set $\mathcal{B} \subset \mathcal{H}$, we know from Theorem 6.3.3 and Theorem 6.3.5 that there exist two universal constants $R_0, R_1 > 0$ and two entering times

$$t_0 = t_0(\mathcal{B}) \qquad \text{and} \qquad t_1 = t_1(\mathcal{B}) = t_0 + 1$$

such that, for every $z_0 \in \mathcal{B}$,

$$\|U(t)z_0\|_{\mathcal{H}_{h(t)}} = \sqrt{\|\widehat{u}(t)\|_{L^2(\Omega_{h(t)})} + mh'(t)^2} \le R_0 \qquad \forall\, t \ge t_0, \tag{6.6.1}$$

and

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} = \sqrt{\|\nabla\widehat{u}(t)\|_{L^2(\Omega_{h(t)})} + mh'(t)^2} \le R_1 \qquad \forall\, t \ge t_1. \tag{6.6.2}$$

Inequality (6.6.1), together with (6.3.5), imply the existence of a constant $C = C(R_0)$ such that

$$\|h\|_{W^{1,\infty}(t_0,\infty;\mathbb{R})} \le C.$$

Thus, for every $t \ge t_0$, relations (6.5.9)-(6.5.10) improve into

$$C_1\|v\|_{L^2(\Omega_{h_0})} \le \|v\|_t \le C_2\|v\|_{L^2(\Omega_{h_0})}, \tag{6.6.3}$$

$$C_3\|\nabla v\|_{L^2(\Omega_{h_0})} \le \|D_g v\|_t \le C_4\|\nabla v\|_{L^2(\Omega_{h_0})}, \tag{6.6.4}$$

where now the constants C_1, C_2, C_3, C_4 depend only on R_0 (and on t_0). Invoking the coordinate transformation φ, we have the equality

$$\|\nabla\widehat{u}(t)\|_{L^2(\Omega_{h(t)})} = \|D_g v(t)\|_t.$$

Looking at (6.6.2) and to the definition of $|\cdot|_{t,\mathcal{H}^1}$, this yields

$$|S(t)\mathcal{B}|_{t,\mathcal{H}^1} = \|U(t)\mathcal{B}\|_{\mathcal{H}^1_{h(t)}} \leq R_1 \qquad \forall\, t \geq t_1.$$

Hence, taking $t \geq t_1 > t_0$, from the (uniform) equivalence of the norms established in (6.6.3) and (6.6.4), up to redefining the universal constant R_1, we conclude that

$$\|S(t)\mathcal{B}\|_{\mathcal{H}^1} \leq R_1 \qquad \forall\, t \geq t_1.$$

This means that the ball $\mathcal{B}_1$ of $\mathcal{H}_1$ of radius R_1, which is compact in $\mathcal{H}$ in view of the compact embedding $\mathcal{H}^1 \Subset \mathcal{H}$, is an absorbing set for $S(t)$. □

6.7 Simulation strategy

In this section, the numerical settings and schemes adopted for the simulations of (2.0.7) are described. The analyses were performed in two stages, labeled as *static* and *dynamic* cases. During the static stage, a Poiseuille flow impinges the fixed obstacle at different flow rates. The static case was used both as a starting point to validate the chosen numerical setting and to obtain a description of the flow field at different values of the inlet velocity magnitude. As for the dynamic stage, the deck cross-section interacts with the fluid through a vertical translation. Specifically, the global attractor associated with the dynamical system is characterized by describing the behaviour of the velocity field of the fluid and the position of the obstacle in the long term.

Following the numerical strategy presented in [39], the simulations were carried out considering an aspect ratio for the obstacle d/δ equal to 5. As shown in Figure 6.2, the computational domain was characterized by an along wind length and a height respectively equal to $30\,d$ and $14.2\,d$. Moreover, the windward edge of the obstacle was located at a distance $8\,d$ from the inlet.

The same mesh was adopted for both the static and the dynamic case. Specifically, the fluid domain was divided into seven regions with different refinements. Close to the boundaries of the obstacle, the mesh size is $2.7 \times 10^{-3}\,d$, resulting in a dimensionless wall distance $y^+ < 5$ almost everywhere, and it is coarsened up to a maximum of $0.35\,d$ in the wake. Overall, the mesh consists of 50000 cells, see Figure 6.2. A RANS-based modelling technique was employed, adopting the well-known ($k - \omega$ SST) turbulence model. This closure model is one of the most commonly exploited in the context of RANS simulations [39] since it exhibits a more reliable behaviour when predicting separating flows with respect to other turbulence models. Second-order schemes were selected for all quantities both in space and time and a PISO scheme was used for the velocity-pressure coupling. At the inlet of the computational domain, a parabolic velocity profile was imposed and Dirichlet boundary conditions were set for both the turbulence kinetic energy k and the specific dissipation rate ω. Moreover, Neumann boundary conditions were imposed for the pressure. The inlet turbulence intensity was set equal to 1% with a value of the turbulence viscosity ratio $\nu/\nu_t = 2$, where ν_t is the turbulence kinematic viscosity. It is worth pointing out that the $k - \omega$ SST turbulence model is quite stable with respect to the inlet boundaries [39] and thus, with this approach, it is not possible to faithfully reproduce the experimental wind tunnel inlet conditions. A no-slip condition was imposed on the top and bottom sides of the domain, to keep the parabolic profile unchanged up to the windward edge of the obstacle. Finally, wall functions were employed for the near-wall treatment of the obstacle. The time step was chosen so as to guarantee a value of the Courant number lower than 1 almost everywhere, with a maximum of approximately 2 in very small areas around the corners. The simulations were performed with the open-source software OpenFOAM [1]. The next paragraphs treat the specific settings for the static and dynamic simulations.

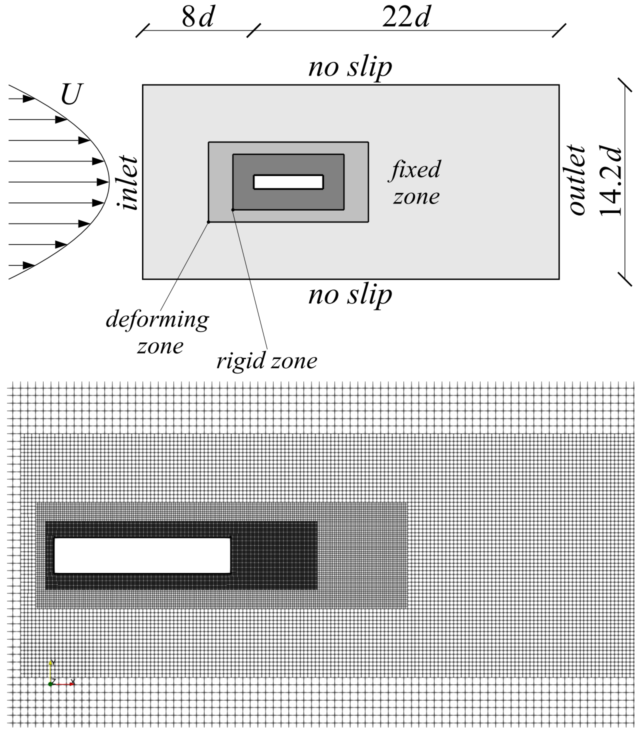


Figure 6.2: *The figure above depicts a sketch of the computational domain while the figure below reports a close up of the mesh structure adopted near the obstacle.*

6.7.1 Static case

Let $U = \lambda L^2$ be the maximum velocity of the parabolic profile at the inlet. We define the Reynolds number as

$$Re = \frac{U\delta}{\nu},$$

and thus, given that $U = \lambda L^2$, we consider the variations of Re rather than the variations of λ. Another meaningful dimensionless parameter is the so-called Strouhal number, defined as

$$St = \frac{f_w \delta}{U},$$

where f_w is the vortex shedding frequency, as well as the Scruton number

$$Sc = \frac{2\pi m \xi}{\rho \delta^2},$$

where m and ξ are respectively the linear mass and the damping ratio of the system while ρ is the density of the fluid. Let the dimensionless time t^* be defined as

$$t^* = \frac{t\,U}{\delta}.$$

Figure 6.4 shows the time histories, obtained through the simulations, of the dimensionless drag C_D, lift C_L and moment C_M for $U = 1$, corresponding to $Re \approx 1.3 \times 10^4$. As expected, the lift coefficient oscillates around zero. Indeed, it is well-known both from a theoretical and an experimental point of view (see [66, 68, 110]) that no lift is exerted on a bluff body immersed in a viscous fluid whenever its cross-section is symmetric with respect to the angle of attack. Table 6.1 compares the time-averaged values of the aerodynamic coefficients and the Strouhal number with the experimental ones taken from the literature [122]. It is worth pointing out that, in this specific context, if the Reynolds number is not sufficiently large, the values of the moment are not available, see [122, Section 3]. Figure 6.3 depicts the comparison between the numerical values of the Strouhal number, as a function of Re, and the experimental data taken from [122]. A reasonable agreement between the numerical and the experimental results allows validating the chosen setting.

	CL	CD	CM	St
Experimental	0.08 (rms)	1.03	——	0.11
Numerical	-0.0013	1.004	-1.519×10^{-4}	0.118

Table 6.1: *Comparison between the numerical and experimental time-averaged aerodynamic coefficients (experimental data taken from [122]).*

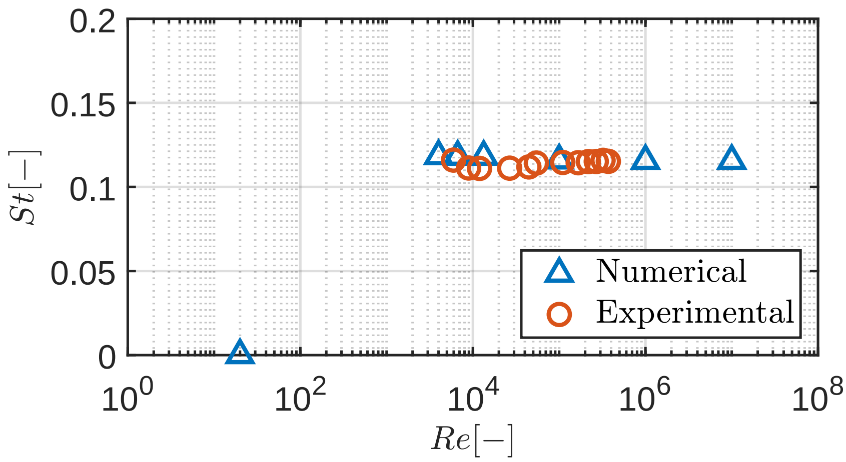


Figure 6.3: *Numerical and experimental [122] Strouhal number as a function of Re.*

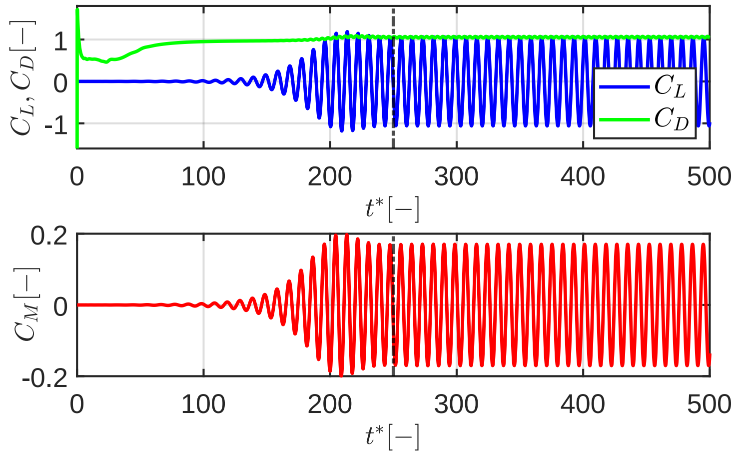


Figure 6.4: *Time traces of the numerical lift, drag and moment. The figures report just a portion of the time histories. The dotted black line indicates the end of the transient stage.*

6.7.2 Dynamic case

As previously mentioned, in the dynamic case the obstacle is allowed to move in the vertical direction. Following the literature [21, 39, 92, 113], the mesh is organized in three regions (see Figure 6.2): a boundary layer rigid motion region, a deforming region and a non-deforming region. The first is defined close to the rigid body profile so that all the mesh nodes identically move through a rigid translation. The mesh in the deforming region is modified by the body motion at each time step while the non-deforming zone is fixed.

Before discussing the results, we conclude with a remark on the relation between the model considered to obtain the analytical result in the first part of the chapter (all sections before Section 6.7) and the model adopted for the computational analyses.

Remark 6.7.1. The numerical model differs from the analytical one presented in Section 6.1 in two aspects. Specifically, in the numerical model the obstacle is subjected to a linear elastic force while, in the analytical one, $f(h)$ satisfies conditions (6.2.1). The introduction of the restoring force $f(h)$ satisfying (6.2.1) in problem (2.0.7) comes from the necessity to avoid collisions, by which one is able to prove Theorems 6.4.1, 6.2.7 and Theorem 6.6.6. A posteriori, we observed that, in the numerical framework, collisions are prevented by the computational setting itself and thus, it makes sense to substitute $f(h)$ with its first order approximation $f(h) \sim -kh$, where $k > 0$. The second difference with respect to problem (2.0.7) lies in the fact that a viscous damping force is also driving the motion of the obstacle; this only contributes increasing to the dissipative character of the system, and thus increasing the stable character of the simulations, while not influencing any of the proofs of the theoretical results. All the analytical results presented in Section the first part of the chapter still hold for the numerical problem.

6.8 Numerical results and discussion

In this section, the numerical results are presented, giving a general discussion and an explicit characterization of the global attractor associated with the fluid-structure interaction system.

As for the static simulation, a qualitative description of the flow topology varying Re is depicted in Figures 6.5, 6.6 and 6.7. The instantaneous flow fields and streamlines all correspond to the non-dimensional time $t^* = t^*_{regime} + 500$, where t^*_{regime} has been identified through the time history of the aerodynamic coefficients. For $Re = 30$, the streamlines are characterized by an upstream-downstream symmetry. In particular, it is possible to observe the presence of two symmetrical eddies in a closed re-circulation zone in the wake of the obstacle and, thus, the flow can be considered steady.

A different regime is observed at $Re \approx 1.3 \times 10^4$, see Figures 6.5 and 6.7. The flow becomes unsteady and time-periodic, and we can observe the appearance of the Von Kármán wake, typically characterizing the vortex shedding phenomenon. From a mathematical point of view, this implies the occurrence of a so-called ***Hopf*** unsteady bifurcation, see [59] for a rigorous description of the phenomenon. The same regime is observed also for $Re = 10^5, 10^6, 10^7$, see Figure 6.6 and 6.7. In all cases, the vortexes shed from the obstacle with a specific frequency. This phenomenon can be also interpreted in the light of Figure 6.3. Specifically, at sufficiently high Re, the Strouhal number is non-zero, suggesting the presence of time-periodic flow mechanisms. The fact that the flow never becomes unstable, even when considering very high Re, is probably due to the usage of a RANS-based model. Indeed, it is well-known from the state-of-the-art [39] that RANS approaches, although being computationally convenient, mostly lose capturing the unstable features of the flow. RANS-based models assume a stationary turbulent regime, which prevents from sizing highly irregular velocity fluctuations. As a consequence, although capturing the average changes in topology, we do not observe the generation of a turbulent wake as observed in [14, 91].

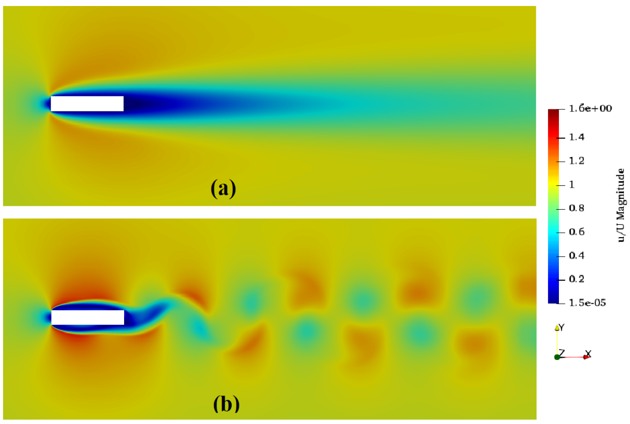


Figure 6.5: *Instantaneous flow field at* $t^* = t_{regime} + 500$ *for different Reynolds number (a)* $Re = 30$*; (b)* $Re \approx 1.3 \times 10^4$.

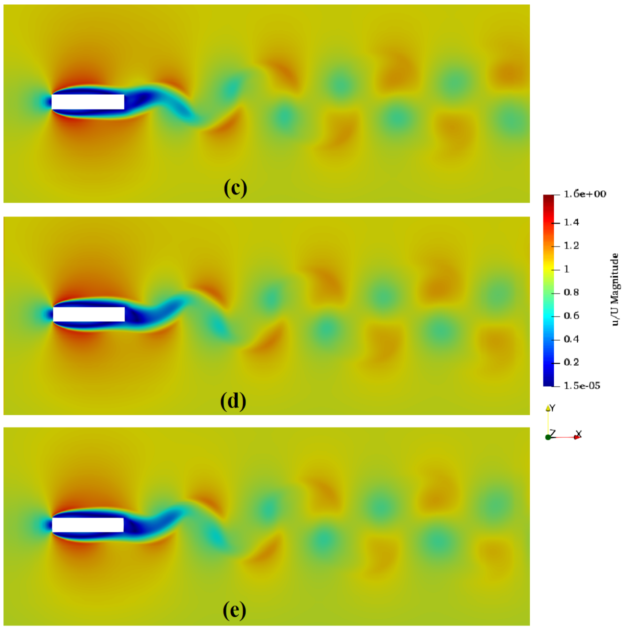


Figure 6.6: *Instantaneous flow field at* $t^* = t_{regime} + 500$ *for different Reynolds number (c)* $Re = 10^5$*; (d)* $Re = 10^6$*; (e)* $Re = 10^7$.

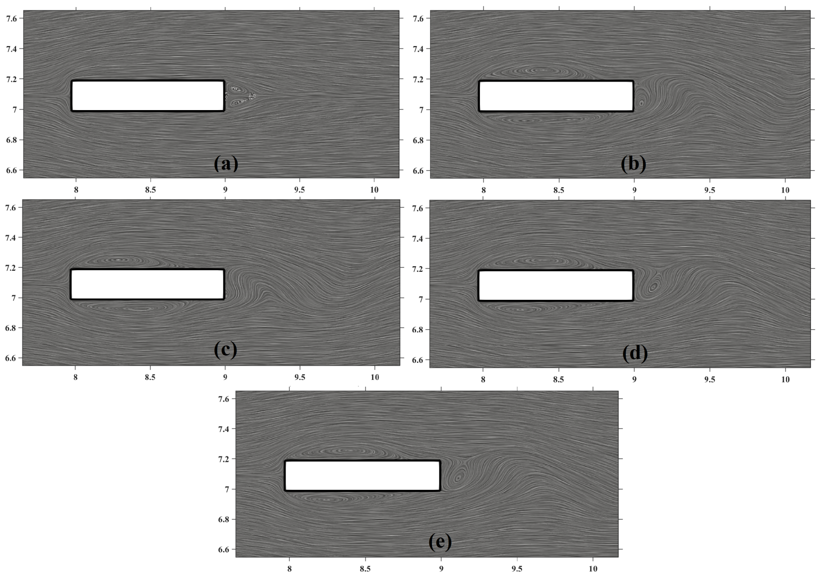


Figure 6.7: *Instantaneous streamlines obtained with the LIC technique at* $t^* = t_{regime} + 500$ *for different Reynolds number (a)* $Re = 30$*; (b)* $Re \approx 1.3 \times 10^4$*; (c)* $Re = 10^5$*; (d)* $Re = 10^6$*; (e)* $Re = 10^7$.

Regarding the dynamic case, as observed in the static simulations, increasing Re the flow pattern makes a transition from a fully symmetrical and steady configuration to an unsteady periodical one, where it exhibits a vortex shedding phenomenon. This naturally influences the dynamics of the body, leading to vortex-induced vibrations whose one of the most relevant features is the so-called *lock-in* effect. Figure 6.8 depicts a time history of the dimensionless vertical displacement, obtained imposing a maximum inlet velocity equal to $1.15m/s$, which leads to a Re approximately equal to 10^4. In particular, we observe that the obstacle oscillates around the equilibrium position $h_{eq} = 0$.

As soon as the vortex shedding frequency becomes equal to the natural frequency of the structure, a resonance phenomenon occurs and the body may be subjected to large amplitude oscillations. If f_n is the natural frequency of the body, for the 1-DOF system at consideration, the onset of resonance occurs when

$$U = U_{St} = \frac{f_n \delta}{St}.$$

Defining the reduced velocity $U^* = \frac{U}{f_n \delta}$, the vortex shedding synchronizes with the natural frequency of the structure at $U^* = \frac{1}{St}$. Subsequently, the motion of the obstacle starts contributing to the coherence of the flow eddies, capturing the shedding frequency by locking it to the natural frequency of the structure and spreading the lock-in effect on a range of velocities around U_{St}.

In Figure 6.9(a), it is reported the maximum non-dimensional amplitude of oscillation at regime as a function of the reduced velocity ($h_{eq} = 0$ is the position of the body given by the unique stationary solution, see Theorem 6.4.1). Figure 6.9(b) shows the non-dimensional vortex shedding frequency versus the reduced velocity. Both plots are built considering two regimes for the Reynolds number Re and increasing the stiffness of the system accordingly. It is possible to observe the synchronization between the vortex shedding and the natural frequency of the system for a specific range of reduced velocities around the predicted value $U^* = 1/St$, inducing the body to reach its maximum vibration amplitude. Furthermore, increasing the stiffness of the system, the Scruton number Sc decreases from 256.4 to 7.0 and, as expected, the maximum non-dimensional amplitude of oscillation increases.

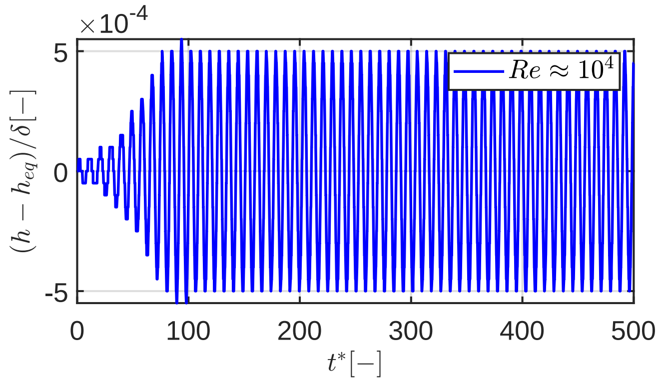
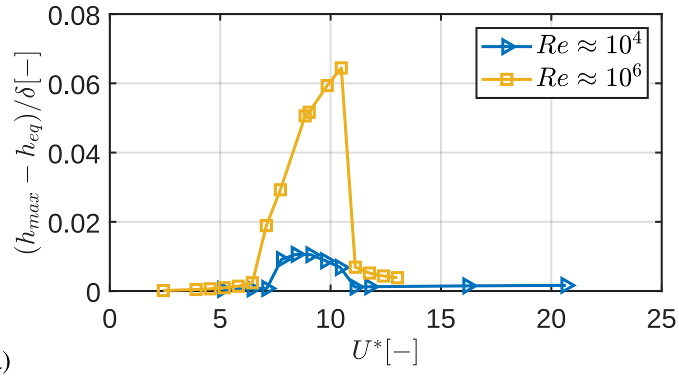
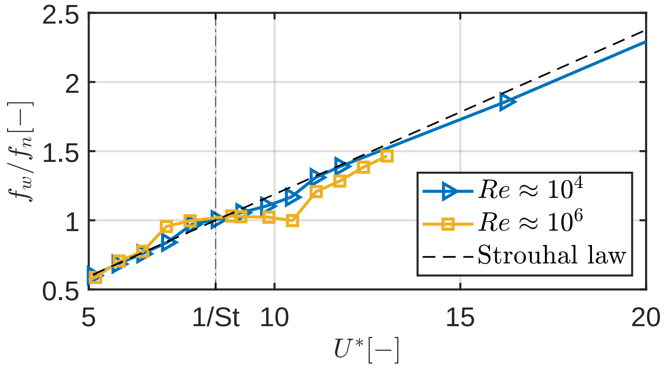


Figure 6.8: *Time history of the structural motion for* $Re \approx 10^4$.



(a)



(b)

Figure 6.9: *Figure (a): non-dimensional amplitudes of oscillation of body versus the reduced velocity U^*. Figure (b): vortex shedding frequency over the natural frequency of the system as a function of the reduced velocity U^*.*

In the last paragraph of this section, a reinterpretation of the previous results is given in terms of the main purpose of the chapter, i.e. giving an explicit characterization of the global attractor associated with the dynamical fluid-structure system. In view of the numerical observations that we have described, we can identify two (three) different regimes (subregimes) as a function of λ:

- If λ is sufficiently small, specifically lower than a critical value $\lambda < \lambda_c$, the attractor reduces to a one-point set in the phase space, corresponding to the unique symmetrical stationary solution $(u_s, h_s) = (u_\lambda, 0)$ given by Theorem 6.4.1 and thus, the dimension of the attractor is 0. The flow topology corresponding to this solution is reported in Figures 6.5(a) and 6.7(a). This is in accordance with the result already proven in Theorem 6.4.1 and thus, $\lambda_c = \lambda_1$.

- As soon as $\lambda > \lambda_c$, a Hopf bifurcation occurs and, after a short transient motion, the dynamics of the problem becomes time-periodic. In other words, given $y(\cdot) = (u(\cdot), h(\cdot))$ the unique solution to (2.0.7), we have that $y(\cdot)$ converges to a non-symmetrical time-periodic solution $\varphi(\cdot)$ to problem (2.0.7). The global attractor reduces in this case to the orbit of $\varphi(\cdot)$ and therefore, it corresponds to a closed curve. However, we can identify two subregimes, $\lambda_c < \lambda_{c_1} < \lambda < \lambda_{c_2}$ and $\lambda_c < \lambda < \lambda_{c_1} \vee \lambda_{c_2} < \lambda$, where the time-periodic solution to which $y(\cdot)$ converges drastically changes, at least from a quantitative point of view. Indeed, according to the value of λ, we are inside or outside what we have previously described as the lock-in range. As shown in Figure 6.9, the structure motion inside the lock-in range may reach large amplitudes of vibration. As a result, although the global attractor still corresponds to the orbit of a time-periodic solution, i.e. to a closed curve, this orbit might deform consistently to the value of λ. The threshold values λ_{c_1} and λ_{c_2} can be defined numerically, as it was done in Figure 6.9 for two Re regimes.

In principle, we might expect that the structure of the global attractor further complicates as λ increases. Specifically, this would correspond to a chaotic structure for the flow, exhibiting random variations of velocity and pressure in time and space. However, as already mentioned, due to the numerical methods adopted, we are able to catch only the average stable features of the flow.

CHAPTER 7

Conclusions and future developments

This work was devoted to the analysis of some fluid-structure interaction problems in channels, with a double purpose. On the one hand, we gave a mathematical characterization of some phenomena concerning areodynamics, in particular those arising when the wind interacts with suspension bridges. On the other hand, our investigation was driven mostly by mathematical motivations, which lead us to consider some partially unexplored aspects in the analysis of fluid-structure problems.

We studied the case of a fixed obstacle impinged by a steady Poiseuille flow. Exploiting the relation induced by symmetry between non-uniqueness of solutions and the lift force, already invoked in [68] for a two-dimensional problem, we obtained analytically an explicit threshold on the prescribed inflow predicting the apperance of the lift force on the obstacle, modelling in this context the deck of a bridge. This required dealing with non-homogeneous boundary conditions and to construct a suitable solenoidal extension. A future research line concerns the generalization of this result to different boundary conditions and shapes of the obstacle, possibly breaking the symmetry of the problem as well as investigating the apperance of other solicitations, such as the torque. Some preliminary steps in this direction have already been made in the two-dimensional case in [15].

We imposed a Poiseuille inflow and outflow also in the second problem, aiming to model the dynamic effect of wind over suspension bridges. In this problem, where the full fluid-structure interaction takes place, an unsteady flow in an bounded two-dimensional channel hits a rectangular obstacle that is free to move vertically. We obtained global well-posedness for this problem, ensuring the absence of collisions thanks to a restoring force. A more involved version of the same model was obtained including a second-degree of freedom in the motion of the obstacle, i.e. rotation, and making the channel unbounded. Since in this case we took the obstacle to have a smooth boundary, the obtained result of existence up to collision can be extended by proving no-contact without using any restoring force. This will require extending the techniques by Hillairet et al. [72, 84] to the case of a rotating non-spherical obstacle. Furthermore, with the purpose of representing more accurately the atmospheric boundary layer in which bridges are immersed, different boundary conditions could be imposed to the general model, such as a Couette flow.

Besides well-posedness, this work treated the long time behaviour of fluid-structure interaction problems. Our main contribution concerns the extension of the notion of global attractor to problems where the fluid domain depends on time in an unknown fashion. Since we restricted our analysis to a bounded domain, it would be interesting to investigate if and how this notion changes when the domain is unbounded. A further understanding of the long-term dynamics of fluid-structure problems could be given by studying the finite dimensionality of the attractor, as done for the classical Navier-Stokes system [133].

Using the obtained theoretical results, we also presented numerical simulations of the flow past a rectangular obstacle. Two different simulation strategies were adopted: the static one considers the obstacle fixed while the dynamic case allows its vertical translation. In order to give an explicit qualitative characterization of the global attractor associated with the problem, different Reynolds number regimes were investigated, detecting significant variations of the system dynamics. The most natural developments of our numerical work would be to perform Large Eddies Simulations (LES) of the problem, as well as including the torsional degree of freedom in the motion of the obstacle. Specifically, LES are computationally very expensive but they would allow to capture some turbulent features of the flow and, thus, have a more precise knowledge on the characterization of the structure of the global attractor.

CHAPTER 8

Appendix

With reference to Chapter 3, in the sequel, we report the values of the constants δ_i's used in Proposition 3.3.4. Notice that the domain of integration has been reduced exploiting the fact that the cut-off function $\theta(x)$ in (3.3.7) is equal to 0 in the region $I = \{(x_1, x_3) \in (-l, l) \times (-h, h)\}$.

Explicit values of the constants δ_i's

$$
\begin{aligned}
\delta_1 = & \int_l^{\alpha+l}\int_0^1\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 (d + x_2 + \tfrac{42\zeta(3)}{\pi^3}x_3)^4 dx_3dx_2dx_1 \\
& + \int_l^{\alpha+l}\int_0^1\int_0^h \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 (d + x_2 + \tfrac{42\zeta(3)}{\pi^3}x_3)^4 dx_3dx_2dx_1 = \\
& \frac{8d^5}{70945875\pi^{12}\alpha^3(d-h)^8}(130d^6(466754400h^2\zeta(3)^4+1764\pi^6(4554h^2-2145h+128)\zeta(3)^2 \\
& +1386\pi^9(282h^2-187h+21)\zeta(3)11\pi^{12}(1044h^2-837h+128)+222264\pi^3(418h-103)h\zeta(3)^3) \\
& -715d^5(74680704h^3\zeta(3)^4+148176\pi^3(104h-57)h^2\zeta(3)^3+3528\pi^6(396h^2-414h+65)h\zeta(3)^2 \\
& +21\pi^9(3456h^3-5076h^2+1496h-63)\zeta(3)+2\pi^{12}(1176h^3-2088h^2+837h-64))+143d^4(124467840h^4\zeta(3)^4 \\
& +2963520\pi^3(9h-13)h^3\zeta(3)^3+105840\pi^6(24h^2-66h+23)h^2\zeta(3)^2 \\
& +210\pi^9(672h^3-2592h^2+1692h-187)h\zeta(3)+\pi^{12}(5040h^4-23520h^3+20880h^2-4185h+128)) \\
& +429\pi^3d^3h(4445280h^3\zeta(3)^3+70560\pi^3(12h-11)h^2\zeta(3)^2+630\pi^6(112h^2-192h+47)h\zeta(3) \\
& +\pi^9(3360h^3-7840h^2+3480h-279))+3432\pi^6d^2h^2(35280h^2\zeta(3)^2+\pi^6(420h^2-490h+87) \\
& +420\pi^3(14h-9)h\zeta(3))-65d^7(476089488h\zeta(3)^4+111132\pi^3(824h-77)\zeta(3)^3 \\
& +10584\pi^6(715h-128)\zeta(3)^2+462\pi^9(748h-189)\zeta(3)+11\pi^{12}(837h-256))+20d^8(189189\pi^9\zeta(3) \\
& +4402944\pi^6\zeta(3)^2+55621566\pi^3\zeta(3)^3+298722816\zeta(3)^4+4576\pi^{12}) \\
& +48048\pi^9dh^3(105h\zeta(3)+\pi^3(15h-7))+144144\pi^{12}h^4)
\end{aligned}
$$

$$
\begin{aligned}
\delta_2 = & \int_l^{\alpha+l}\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 \left(2x_3 - \tfrac{x_3^3}{d^2} + x_3\tfrac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3dx_1 \\
& + \int_l^{\alpha+l}\int_0^h \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4 \left(2x_3 - \tfrac{x_3^3}{d^2} + x_3\tfrac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3dx_1 = \\
& \frac{4c^9}{4583103525\pi^{12}\alpha^3(d-h)^8}(36d^2h^2(58924852\pi^9\zeta(3)+2151459072\pi^6\zeta(3)^2+35193676896\pi^3\zeta(3)^3 \\
& +217766858400\zeta(3)^4+609689\pi^{12})-d^3h(1054256952\pi^9\zeta(3)+38811729096\pi^6\zeta(3)^2+640378146240\pi^3\zeta(3)^3 \\
& +3998199520224\zeta(3)^4+10822633\pi^{12})+1792d^4(110664\pi^9\zeta(3)+4108104\pi^6\zeta(3)^2+68372640\pi^3\zeta(3)^3 \\
& +430747632\zeta(3)^4+1127\pi^{12})-152dh^3(12599244\pi^9\zeta(3)+456297408\pi^6\zeta(3)^2+7400798496\pi^3\zeta(3)^3 \\
& +45387197856\zeta(3)^4+131377\pi^{12})+152h^4(4306848\pi^9\zeta(3)+154738080\pi^6\zeta(3)^2+2488764096\pi^3\zeta(3)^3 \\
& +15129065952\zeta(3)^4+45251\pi^{12}))
\end{aligned}
$$

$$
\begin{aligned}
\delta_3 =& 2\int_{l}^{\alpha+l}\int_{0}^{1}\int_{h}^{d}\frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4\left(x_2+x_3\frac{42}{\pi^3}\,\zeta(3)+d\right)^2 \\
&\times\left(2x_3-\frac{x_3^3}{d^2}+x_3\frac{42}{\pi^3}\ \zeta(3)\right)^2 dx_2dx_3dx_1 \\
&+2\int_{l}^{\alpha+l}\int_{0}^{1}\int_{0}^{h}\frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^4\left(x_2+x_3\frac{42}{\pi^3}\,\zeta(3)+d\right)^2 \\
&\times\left(2x_3-\frac{x_3^3}{d^2}+x_3\frac{42}{\pi^3}\ \zeta(3)\right)^2 dx_2dx_3dx_1 =
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2d^7}{723647925\pi^{12}\alpha^3(d-h)^8}(136d^4(36406843200h^2\zeta(3)^4+1764\pi^6(264006h^2-69591h+1664)\zeta(3)^2 \\
&+42\pi^9(363726h^2-135987h+5888)\zeta(3)+\pi^{12}(193554h^2-89571h+5248) \\
&+666792\pi^3(9846h-1339)h\zeta(3)^3)-136d^3h(32038022016h^2\zeta(3)^4+1764\pi^6(244308h^2 \\
&-144612h+9295)\zeta(3)^2+42\pi^9(345480h^2-288567h+33202)\zeta(3)+\pi^{12}(188292h^2-193554h+29857) \\
&+2889432\pi^3(2048h-627)h\zeta(3)^3)+816d^2h^2(1779890112h^2\zeta(3)^4+1764\pi^6(14304h^2 \\
&-22542h+3289)\zeta(3)^2+84\pi^9(10387h^2-22975h+5928)\zeta(3)+\pi^{12}(11596h^2-31382h+10753) \\
&+1926288\pi^3(175h-143)h\zeta(3)^3)-51d^5(49513306752h\zeta(3)^4+444528\pi^3(19632h-1001)\zeta(3)^3 \\
&+7056\pi^6(85551h-8476)\zeta(3)^2+147\pi^9(131328h-18545)\zeta(3)+8\pi^{12}(29857h-5248)) \\
&+6d^6(29527827\pi^9\zeta(3)+946266048\pi^6\zeta(3)^2+14013152496\pi^3\zeta(3)^3+81252605952\zeta(3)^4+356864\pi^{12}) \\
&+7072\pi^3dh^3(11002068h\zeta(3)^3+58212\pi^3(28h-11)\zeta(3)^2+63\pi^6(1283h-880)\zeta(3)+\pi^9(1338h-1207)) \\
&+14144\pi^6h^4(10164\pi^3\zeta(3)+116424\zeta(3)^2+223\pi^6))
\end{aligned}
$$

$$
\delta_4 =(l+\alpha)\int_{h}^{d}\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^4 dx_3+\alpha\int_{0}^{h}\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^4 dx_3 =
$$

$$
\begin{aligned}
&\frac{(d-h)(2\alpha+l)}{315\pi^{12}c^8}(-\pi^3d^6h^2(4788\pi^6\zeta(3)+43218\pi^3\zeta(3)^2+144060\zeta(3)^3+187\pi^9)+\pi^6d^5h^3(4032\pi^3\zeta(3) \\
&+18522\zeta(3)^2+233\pi^6)+\pi^6d^4h^4(4032\pi^3\zeta(3)+18522\zeta(3)^2+233\pi^6)-5\pi^9d^3h^5(252\zeta(3) \\
&+29\pi^3)-5\pi^9d^2h^6(252\zeta(3)+29\pi^3)-\pi^3d^7h(4788\pi^6\zeta(3)+43218\pi^3\zeta(3)^2+144060\zeta(3)^3 \\
&+187\pi^9)+d^8(4032\pi^9\zeta(3)+49392\pi^6\zeta(3)^2+288120\pi^3\zeta(3)^3+756315\zeta(3)^4+128\pi^{12}) \\
&+35\pi^{12}dh^7+35\pi^{12}h^8)
\end{aligned}
$$

$$\delta_5 = \int_l^{\alpha+l}\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^4 \left(2x_3 - \frac{x_3^3}{d^2} + x_3\frac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3dx_1$$
$$+\int_0^l\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^4 \left(2x_3 - \frac{x_3^3}{d^2} + x_3\frac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3dx_1 =$$
$$\frac{4(\alpha+l)^9}{44349279975\pi^{12}}(191\alpha^4+2160\alpha^2l^2-2200\alpha l^3+880l^4$$
$$-1004\alpha^3l)(-3d^{10}h^2(1770720\pi^9\zeta(3)-71971200\pi^6\zeta(3)^2-5239503360\pi^3\zeta(3)^3$$
$$-75645329760\zeta(3)^4+37687\pi^{12})+d^9h^3(-11743872\pi^9\zeta(3)-300839616\pi^6\zeta(3)^2$$
$$+654937920\pi^3\zeta(3)^3+75645329760\zeta(3)^4-124511\pi^{12})-d^8h^4(7174272\pi^9\zeta(3)$$
$$+283566528\pi^6\zeta(3)^2+3012714432\pi^3\zeta(3)^3-15129065952\zeta(3)^4+52723\pi^{12})$$
$$+6\pi^3d^7h^5(-148512\pi^6\zeta(3)-17393040\pi^3\zeta(3)^2-327468960\zeta(3)^3+2453\pi^9)$$
$$+18\pi^3d^6h^6(118048\pi^6\zeta(3)+1079568\pi^3\zeta(3)^2-21831264\zeta(3)^3+1971\pi^9)$$
$$+126\pi^6d^5h^7(9520\pi^3\zeta(3)+199920\zeta(3)^2+99\pi^6)-6\pi^6d^4h^8(5712\pi^3\zeta(3)$$
$$-839664\zeta(3)^2+1021\pi^6)-15\pi^9d^3h^9(11424\zeta(3)+313\pi^3)-21\pi^9d^2h^{10}(1632\zeta(3)+7\pi^3)$$
$$+d^{11}h(25715424\pi^9\zeta(3)+1793522304\pi^6\zeta(3)^2+51347132928\pi^3\zeta(3)^3$$
$$+529517308320\zeta(3)^4+114463\pi^{12})+28d^{12}(3544296\pi^9\zeta(3)+182626920\pi^6\zeta(3)^2$$
$$+4257096480\pi^3\zeta(3)^3+37822664880\zeta(3)^4+e26291\pi^{12})$$
$$+495\pi^{12}dh^{11}+99\pi^{12}h^{12})\alpha^{12}d^8(d-h)^3$$

$$\delta_6 = \int_l^{\alpha+l}\int_0^1\int_h^d \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(x_2 + \frac{x_3^2}{d^2} + x_3\frac{42}{\pi^3}\ \zeta(3) + d\right)^2$$
$$\times\left(1-\frac{x_3^2}{d^2} + \frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_2dx_3dx_1$$
$$+\int_l^{\alpha+l}\int_0^1\int_0^h \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2 \left(x_2 + \frac{x_3^2}{d^2} + x_3\frac{42}{\pi^3}\ \zeta(3) + d\right)^2$$
$$\times\left(1-\frac{x_3^2}{d^2} + \frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_2dx_3dx_1 =$$
$$\frac{d^3}{155925\pi^{12}\alpha(d-h)^4}(11d^2(21781872h^2\zeta(3)^4+294\pi^6(3264h^2-1893h+56)\zeta(3)^2$$
$$+126\pi^9(534h^2-495h+32)\zeta(3)+\pi^{12}(2088h^2-2511h+256)+7044\pi^3(190h-49)h\zeta(3)^3)$$
$$-33d^3(10890936h\zeta(3)^4+6174\pi^3(544h-35)\zeta(3)^3+1029\pi^6(422h-61)\zeta(3)^2+42\pi^9(682h-159)\zeta(3)$$
$$+\pi^{12}(837h-256))+6d^4(51282\pi^9\zeta(3)+825699\pi^6\zeta(3)^2+6723486\pi^3\zeta(3)^3+22819104\zeta(3)^4$$
$$+1408\pi^{12})+33\pi^3dh(432180h\zeta(3)^3+686\pi^3(204h-25)\zeta(3)^2+14\pi^6(1179h-308)\zeta(3)$$
$$+3\pi^9(232h-93))+264\pi^6h^2(441\pi^3\zeta(3)+1715\zeta(3)^2+29\pi^6))$$

$$\delta_7 = \int_{l}^{\alpha+l} \int_{0}^{1} \int_{h}^{d} \frac{1}{81} \left| \frac{\partial\theta(x_1,x_3)}{\partial x_1} \right|^2 \left| \frac{\partial\theta(x_1,x_3)}{\partial x_3} \right|^2 \left(x_2 + x_3 \frac{42}{\pi^3}\ \zeta(3) + d \right)^2$$
$$\times \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3) \right)^2 dx_2 dx_3 dx_1 =$$
$$\frac{29}{842836995 d^4 (d-h) \alpha \pi^{12}} ((42dh^5\pi^6((35+41h)\pi^6+3h(1099+834h)\pi^3\zeta(3)$$
$$+105840h^2\zeta(3)^2)+14h^6\pi^6(41\pi^6+3753h\pi^3\zeta(3)+117936h^2\zeta(3)^2)+2d^3h^3\pi^3((-4193$$
$$-3777h+2205h^2)\pi^9-42(2834+12081h+4284h^2)\pi^6\zeta(3)-26460h(403+714h)\pi^3\zeta(3)^2-326728080h^2\zeta(3)^3)$$
$$+2d^2h^4\pi^3((-1259+2205h+861h^2)\pi^9+42(-1222-2142h+3297h^2)\pi^6\zeta(3)-111132h(39$$
$$+20h)\pi^3\zeta(3)^2-127579536h^2\zeta(3)^3)+24d^8(1388\pi^{12}+146328\pi^9\zeta(3)+5976873\pi^6\zeta(3)^2$$
$$+111928446\pi^3\zeta(3)^3+809040960\zeta(3)^4)+3d^7((11104+10973h)\pi^{12}+168(5252+8587h)\pi^9\zeta(3)$$
$$+8820(2717+7829h)\pi^6\zeta(3)^2+518616(429+2804h)\pi^3\zeta(3)^3+11488381632h\zeta(3)^4)$$
$$+d^4h^2((511-25158h-7554h^2)\pi^12-21(-10712+44721h+81984h^2)\pi^9\zeta(3)-1764(-3289-4836h$$
$$+30816h^2)\pi^6\zeta(3)^2+111132h(3861+296h)\pi^3\zeta(3)^3+11407477536h^2\zeta(3)^4)$$
$$+d^5h((10973+1533h-25158h^2)\pi^{12}-63(-10712-20775h+18478h^2)\pi^9\zeta(3)+1764(5863+38103h$$
$$+15588h^2)\pi^6\zeta(3)^2+333396h(2717+6198h)\pi^3\zeta(3)^3+26455639392h^2\zeta(3)^4)$$
$$+d^6((11104+32919h+1533h^2)\pi^12+42(14144+75624h+46257h^2)\pi^9\zeta(3)+1764(4576$$
$$+56862h+84063h^2)\pi^6\zeta(3)^2+666792h(1573+5947h)\pi^3\zeta(3)^3+36406843200h^2\zeta(3)^4)))$$

$$\delta_8 = \int_{l}^{\alpha+l} \int_{h}^{d} \frac{1}{9} \left| \frac{\partial\theta(x_1,x_3)}{\partial x_1} \right|^2 \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3) \right)^2 \left(1 - \frac{x_3^2}{d^2} + \frac{7}{\pi^3}\ \zeta(3) \right)^2 dx_3 dx_1$$
$$+ \int_{l}^{\alpha+l} \int_{0}^{h} \frac{1}{9} \left| \frac{\partial\theta(x_1,x_3)}{\partial x_1} \right|^2 \left(2x_3 - \frac{x_3^3}{d^2} + x_3 \frac{42}{\pi^3}\ \zeta(3) \right)^2 \left(1 - \frac{x_3^2}{d^2} + \frac{7}{\pi^3}\ \zeta(3) \right)^2 dx_3 dx_1 =$$
$$\frac{d^5}{675675\pi^{12}\alpha(d-h)^4} (16d^2(42980\pi^9\zeta(3)+1007097\pi^6\zeta(3)^2$$
$$+10005996\pi^3\zeta(3)^3+37081044\zeta(3)^4+656\pi^{12})-dh(1931202\pi^9\zeta(3)$$
$$+44515471\pi^6\zeta(3)^2+432612180\pi^3\zeta(3)^3+1557403848\zeta(3)^4+29857\pi^{12})$$
$$+2h^2(687050\pi^9\zeta(3)+15590575\pi^6\zeta(3)^2+148324176\pi^3\zeta(3)^3+519134616\zeta(3)^4$$
$$+10753\pi^{12}))$$

$$\delta_9 = \int_l^{\alpha+l}\int_h^d \frac{1}{81}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(2x_3 - \frac{x_3^3}{d^2} + x_3\frac{42}{\pi^3}\ \zeta(3)\right)^4 dx_3dx_1 =$$

$$\frac{29}{90745449795\pi^{12}\alpha d^8(d-h)}(26132023008d^8\zeta(3)^4\cdot(80d^4+142d^3h+150d^2h^2$$
$$+109dh^3+47h^4)+95721696\pi^3d^6\zeta(3)^3(3040d^6+5095d^5h+4845d^4h^2 2718d^3h^3$$
$$+74d^2h^4-1470dh^5-574h^6)+13674528\pi^6d^4\zeta(3)^2(1128d^8+1775d^7h+1477d^6h^2$$
$$+493d^5h^3-553d^4h^4-966d^3h^5-210d^2h^6+210dh^7+78h^8)+1596\pi^9d^2\zeta(3)(231104d^{10}$$
$$+339379d^9h+236757d^8h^2-788d^7h^3-219196d^6h^4-256410d^5h^5+3402d^4h^6+124740d^3h^7$$
$$+31788d^2h^8-17655dh^9-6369h^{10})+\pi^{12}(3354976d^{12}+4567786d^{11}h+2486970d^{10}h^2$$
$$-1418721d^9h^3-4509979d^8h^4-4150776d^7h^5+1393616d^6h^6+3466134d^5h^7$$
$$+408114d^4h^8-1033230d^3h^9-280038d^2h^{10}+110253dh^{11}+39039h^{12}))$$

$$\delta_{10} = \int_l^{\alpha+l}\int_0^1\int_h^d\frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\left(x_2 + x_3\frac{42}{\pi^3}\ \zeta(3) + d\right)^2 dx_2dx_3dx_1$$
$$+\int_l^{\alpha+l}\int_0^1\int_0^h\frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\left(x_2 + x_3\frac{42}{\pi^3}\ \zeta(3) + d\right)^2 dx_2dx_3dx_1 =$$

$$\frac{4d^3}{135\pi^6\alpha^3}(d^2(10584h^2\zeta(3)^2+\pi^6(60h^2-75h+8)+126\pi^3(10h-7)h\zeta(3))$$
$$-3d^3(5292h\zeta(3)^2+21\pi^3(28h-5)\zeta(3)+\pi^6(25h-8))+6d^4(105\pi^3\zeta(3)+1008\zeta(3)^2$$
$$+4\pi^6)+5\pi^3dh(126h\zeta(3)+\pi^3(12h-5))+20\pi^6h^2)(d-h)^4$$

$$\delta_{11} = \int_l^{\alpha+l}\int_h^d\frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(1+\frac{42}{\pi^3}\ \zeta(3)+\frac{3\,x_3}{d}\right)^2 dx_3dx_1$$
$$+\int_l^{\alpha+l}\int_0^h\frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(1+\frac{42}{\pi^3}\ \zeta(3)+\frac{3\,x_3}{d}\right)^2 dx_3dx_1 =$$

$$\frac{2d^3}{1575\pi^6\alpha(d-h)^4}(d^2(9114\pi^3\zeta(3)+98784\zeta(3)^2+233\pi^6)-14dh(1932\pi^3\zeta(3)$$
$$+22050\zeta(3)^2+47\pi^6)+28h^2(735\pi^3\zeta(3)+8820\zeta(3)^2+17\pi^6))$$

$$\delta_{12} = \int_l^{\alpha+l}\int_h^d\frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(\frac{3\,x_3}{d}\right)^2 dx_3dx_1 + \int_l^{\alpha+l}\int_0^h\frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(\frac{3\,x_3}{d}\right)^2 dx_3dx_1 =$$
$$\frac{2d^3(8d^2-21dh+14h^2)}{175(d-h)^4\alpha}$$

$$\delta_{13} = \int_l^{\alpha+l}\int_0^1\int_h^d\frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1x_3}\right|^2\left(x_2 + x_3\frac{42}{\pi^3}\ \zeta(3) + d\right)^2 dx_2dx_3dx_1 =$$

$$\frac{4}{675\pi^6\alpha(d-h)}\Big[10(1+3d+3d^2)\pi^6 + 315(1+2d)(3d+h)\pi^3\zeta(3)$$
$$+ 5292(6d^2+3dh+h^2)\zeta(3)^2\Big]$$

$$\delta_{14} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1$$
$$+\int_l^{\alpha+l}\int_0^h \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1^2}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{2d^5}{10395\pi^6\alpha^3(d-h)^4}(48d^2(1540\pi^3\zeta(3)+19404\zeta(3)^2+31\pi^6)$$
$$-11dh(17892\pi^3\zeta(3)+222264\zeta(3)^2+365\pi^6)+22h^2(6048\pi^3\zeta(3)+74088\zeta(3)^2+125\pi^6))$$

$$\delta_{15} = \int_l^{\alpha+l}\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\frac{x_3^2}{d^2}\,dx_3dx_1 + \int_l^{\alpha+l}\int_0^h \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\frac{x_3^2}{d^2}\,dx_3dx_1 =$$
$$\frac{2d^3(8d^2-21dh+14h^2)}{175(d-h)^4\alpha}$$

$$\delta_{16} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1$$
$$+\int_l^{\alpha+l}\int_0^h \frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{d^3}{3150\pi^6\alpha(d-h)^4}(d^2(63336\pi^3\zeta(3)+395136\zeta(3)^2+2819\pi^6)$$
$$-2dh(94080\pi^3\zeta(3)+617400\zeta(3)^2+3971\pi^6)+16h^2(8967\pi^3\zeta(3)+61740\zeta(3)^2+359\pi^6))$$

$$\delta_{17} = \int_l^{\alpha+l}\int_h^d \frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_1 x_3}\right|^2\left(2x_3 + x_3\frac{42}{\pi^3}\ \zeta(3) - \frac{x_3^2}{d^2}\right)^2 dx_3dx_1 =$$
$$\frac{2}{4725\pi^6\alpha d^4(d-h)}\Big[74088d^4\zeta(3)^2\big[6d^2+3dh+h^2\big)+1008\pi^3d^2\zeta(3)(27d^4$$
$$+11d^3h+d^2h^2-3dh^3-h^4)+\pi^6(428d^6+129d^5h-45d^4h^2-94d^3h^3-18d^2h^4+15dh^5+5h^6)\Big]$$

$$\delta_{18} = \int_l^{\alpha+l}\int_h^d \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1$$
$$+\int_l^{\alpha+l}\int_0^h \left|\frac{\partial\theta(x_1,x_3)}{\partial x_1}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 =$$
$$\frac{c^3}{525\pi^6\alpha(d-h)^4}(16d^2(252\pi^3\zeta(3)+1029\zeta(3)^2+16\pi^6)-3dh(4312\pi^3\zeta(3)$$
$$+17150\zeta(3)^2+279\pi^6)+24h^2(441\pi^3\zeta(3)+1715\zeta(3)^2+29\pi^6))$$

$$\delta_{19} = (l+\alpha)\int_h^d \frac{1}{d^2}4\,dx_3 + \alpha\int_0^h \frac{1}{d^2}4\ =$$
$$4(l+\alpha)\,\frac{d-h}{d^2} + 4\alpha\frac{h}{d^2}$$

$$\delta_{20} = \int_{l}^{\alpha+l}\int_{h}^{d}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2 \frac{x_3^2}{d^2}\,dx_3dx_1 + \int_{0}^{l}\int_{h}^{d}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2 \frac{x_3^2}{d^2}\,dx_3dx_1 = \frac{26\alpha\left(6d^2+3dh+h^2\right)}{525d^2(d-h)}$$

$$\delta_{21} = (l+\alpha)\int_{h}^{d}\frac{1}{d^2}\left(2\frac{x_3}{d^2}+2\right)^2 dx_3 + \alpha\int_{0}^{h}\frac{1}{d^2}\left(2\frac{x_3}{d^2}+2\right)^2 dx_3 = \frac{4}{3c^6}(-3c^2h^2l - h^3l + c^3(l+\alpha) + 3c^5(l+\alpha) + 3c^4(l-hl+\alpha))$$

$$\delta_{22} = \int_{l}^{\alpha+l}\int_{h}^{d}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 + \int_{0}^{l}\int_{h}^{d}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(1-\frac{x_3^2}{d^2}+\frac{7}{\pi^3}\ \zeta(3)\right)^2 dx_3dx_1 = \frac{4(\alpha+l)^5(13\alpha^2+20l^2-30\alpha l)}{3675\pi^6\alpha^6d^4(d-h)}(-\pi^3d^2h^2(49\zeta(3)+\pi^3) -\pi^3d^3h(147\zeta(3)+11\pi^3)+d^4(196\pi^3\zeta(3)+1715\zeta(3)^2+8\pi^6)+3\pi^6dh^3+\pi^6h^4)$$

$$\delta_{23} = \int_{l}^{\alpha+l}\int_{h}^{d}\frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 + \int_{0}^{l}\int_{h}^{d}\frac{1}{9}\left|\frac{\partial\theta(x_1,x_3)}{\partial x_3}\right|^2\left(2+\frac{42}{\pi^3}\ \zeta(3)+\frac{3x_3}{d}+\frac{3x_3^2}{d^2}\right)^2 dx_3dx_1 = \frac{2(\alpha+l)^5}{33075\pi^6\alpha^6c^4(c-h)}(13\alpha^2+20l^2-30\alpha l)(12\pi^3c^2h^2(147\zeta(3) +37\pi^3)+3\pi^3c^3h(3234\zeta(3)+403\pi^3)+2c^4(17787\pi^3\zeta(3)+61740\zeta(3)^2+1346\pi^6) +117\pi^6ch^3+18\pi^6h^4)$$

$$\delta_{24} = \int_{l}^{\alpha+l}\int_{h}^{d}\frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\right|^2\left(2x_3+x_3\frac{42}{\pi^3}\ \zeta(3)-\frac{x_3^2}{d^2}\right)^2 dx_3dx_1 + \int_{0}^{l}\int_{h}^{d}\frac{1}{9}\left|\frac{\partial^2\theta(x_1,x_3)}{\partial x_3^2}\right|^2\left(2x_3+x_3\frac{42}{\pi^3}\ \zeta(3)-\frac{x_3^2}{d^2}\right)^2 dx_3dx_1 = \frac{4(\alpha+l)^5}{33075\pi^6\alpha^6d^4(d-h)^3}(13\alpha^2+20l^2-30\alpha l)(d^4h^2(4116\pi^3\zeta(3) +61740\zeta(3)^2+71\pi^6)-3\pi^3d^3h^3(588\zeta(3)+23\pi^3)-3\pi^3d^2h^4(588\zeta(3)+23\pi^3) +d^5h(4116\pi^3\zeta(3)+61740\zeta(3)^2+71\pi^6)+d^6(4116\pi^3\zeta(3)+61740\zeta(3)^2+71\pi^6) +15\pi^6dh^5+15\pi^6h^6)$$

Bibliography

[1] OpenCFD. *OpenFOAM Programmer's Guide, Springer*, 2004.

[2] D. Acheson. *Elementary Fluid Dynamics*. Oxford University Press, 1990.

[3] T.J.A. Agar. The analysis of aerodynamic flutter of suspension bridges. *Computers & Structures 30(3), 593-600*, 1988.

[4] T.J.A. Agar. Aerodynamic flutter analysis of suspension bridges by a modal technique. *Engineering Structures*, 11:75–82, 1989.

[5] H. Al Baba, N. V. Chemetov, S. Necasova, and B. Muha. Strong solution in L^2 framework for fluid-rigid body interaction problem. mixed case. *Topol. Methods Nonlinear Anal.*, 52, 2018.

[6] C. J. Amick. Steady solutions of the Navier-Stokes equations in unbounded channels and pipes. *Ann. Sc. Norm. Sup. Pisa 4, 473-513*, 1977.

[7] C. J. Amick. Existence of solutions to the nonhomogeneous steady Navier-Stokes equations. *Indiana Univ. Math. J. 33(6), 817-830*, 1984.

[8] R. Apéry. Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Société Mathématique de France, Asterisque 61, 11-13*, 1979.

[9] T. Argentini, G. Diana, D. Rocchi, and C. Somaschini. A case-study of double multi-modal bridge flutter: Experimental result and numerical analysis. *Journal of Wind Engineering and Industrial Aerodynamics*, 151:25–36, 2016.

[10] T. Argentini, D. Rocchi, and C. Somaschini. Effect of the low-frequency turbulence on the aeroelastic response of a long-span bridge in wind tunnel. *Journal of Wind Engineering and Industrial Aerodynamics*, 197:104072, 2020.

[11] G. Arioli, F. Gazzola, and S. Terracini. Minimization properties of Hill's orbits and applications to some N-body problems. *Ann. Inst. Henri Poincaré, Analyse non Linéaire*, 18, 2000.

[12] A.V. Babin and M.I. Vishik. *Attractors of Evolutions Equations*, volume 25. Studies in Mathematics and its Applications, North-Holland Publishing Co., Amsterdam, 1992.

[13] I. Babuška and A.K. Aziz. Survey lectures on the mathematical foundations of the finite element method. In *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, pages 1–359. Academic Press, 1972.

[14] S. Berrone, V. Garbero, and M. Marro. Numerical simulation of low-reynolds number flows past rectangular cylinders based on adaptive finite element and finite volume methods. *Computers and Fluids*, 2011.

[15] D. Bonheure, G.P Galdi, and F. Gazzola. Equilibrium configuration of a rectangular obstacle immersed in a channel flow, revised version. *Comptes Rendus Acad. Sci. Paris*, 2021.

[16] D. Bonheure, M. Hillairet, C. Patriarca, and G. Sperone. Flutter motion of a rigid body in an unbounded channel under the action of a poiseuille flow. *In preparation*, 2022.

[17] G. Bordogna, S. Muggiasca, S. Giappino, M. Belloli, J.A Keuning, and R.H.M Huijsmans. The effects of the aerodynamic interaction on the performance of two Flettner rotors. *J. Wind. Eng. Ind. Aerodyn. 196, 104024*, 2020.

[18] J. Boussinesq. Mémoire sur l'influence des frottements dans les mouvements réguliers des fluides. *Journal de Mathématique Pures et Appliquées 13.2, 377-424*, 1868.

[19] M. Bravin. Energy equality and uniqueness of weak solutions of a "viscous incompressible fluid + rigid body" system with Navier slip-with-friction conditions in a 2d bounded domain. *J. Math. Fluid Mech.*, 21, 2019.

[20] R.M. Brown, P.A. Perry, and Z. Shen. On the Dimension of the Attractor for the Non-Homogeneous Navier-Stokes Equations in Non-Smooth Domains. *Indiana University Mathematics Journal*, 49:81–112, 2000.

[21] F. Brusiani, S. de Miranda, L. Patruno, F. Ubertini, and P. Vaona. On the evaluation of bridge deck flutter derivatives using RANS turbulence models. *J. Wind Eng. Ind. Aerodyn. 119*, 39-47, 2013.

[22] T. Caraballo, G. Lukaszewicz, and J. Real. Pullback attractors for asymptotically compact non-autonomous dynamical systems. *Nonlinear Anal.*, 64:484–498, 2006.

[23] T. Caraballo, G. Lukaszewicz, and J. Real. Pullback attractors for non-autonomous 2D-Navier-Stokes equations in some unbounded domains. *C.R. Math. Acad. Sci. Paris*, 342:263–268, 2006.

[24] L. Cattabriga. Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rendiconti del Seminario Matematico della Universita di Padova 31*, 308-340, 1961.

[25] N.V. Chemetov and S. Necasova. The motion of the rigid body in the viscous fluid including collisions. Global solvability result. *Nonlinear Anal. Real World Appl.*, 34:416-445, 2017.

[26] V. Chepyzhov, M. Conti, and V. Pata. A minimal approach to the theory of global attractors. *Discrete an Continuous Dynamical Systems*, 32(8):2079–2088, 2012.

[27] V. Chepyzhov and M.I. Vishik. *Attractors for equations of mathematical physics*. American Mathematical Society Colloquium Publications, 49. American Mathematical Society, Providence, RI, 2002., 2002.

[28] I. Chueshov, I. Lasiecka, and J. Webster. Flow-plate interactions: well-posedness and long-time behaviour. *Discrete Contin. Dynam. Systems*, 7:925–965, 2014.

[29] I. Chueshov and I. Ryzhkova. A global attractor for a fluid-plate interaction model. *Comm. Pure Appl. Math.*, 12:1635–1656, 2013.

[30] I. Chueshov and I. Ryzhkova. On the interaction of an elastic wall with a Poiseuille-type flow. *Ukrainian Mathematical Journal*, 65:158–177, 2013.

[31] C. Conca, J.A. San Martin, and M. Tucsnak. Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Communications in Partial Differential Equations*, 25, 2000.

[32] M. Costabel and M. Dauge. On the inequalities of Babuška-Aziz, Friedrichs and Horgan-Payne. *Archive for Rational Mechanics and Analysis*, 217:873–898, 2015.

[33] C. Cumsille and T. Takahashi. Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. *Czechoslovak Mathematical Journal*, 58:961–992, 2008.

[34] P. Cumsille and T. Takahashi. Well-posedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid. *Czechoslovak Math. J.*, 58, 2008.

[35] M. Dauge. Stationary Stokes and Navier-Stokes systems on two- and three-dimensional domains with corners. part I: Linearized equations. *SIAM J. Math. Anal. 20, 74-97*, 1989.

[36] C. De Coster, S. Nicaise, and G. Sweers. Solving the biharmonic dirichlet problem on domains with corners: Biharmonic dirichlet problem on domains with corners. *Mathematische Nachrichten*, 288, 11 2014.

[37] C. De Coster, S. Nicaise, and G. Sweers. Comparing variational methods for the hinged kirchhoff plate with corners. *Mathematische Nachrichten*, 10 2019.

[38] M. del Pino and J. Dolbeault. Best constants for Gagliardo-Nirenberg inequalities and application to nonlinear diffusions. *Journal de Mathematique Pures et Appliquées 81, 847-875*, 2002.

[39] S. De Miranda, L. Patruno, F. Ubertini, and G. Vairo. On the identification of flutter derivatives of bridge decks via rans turbulence models: Benchmarking on rectangular prisms. *Engineering Structures*, 76:359–370, 2014.

[40] J. P. Den Hartog. *Elementary Fluid Dynamics*. Dover Publ, New York, 1934.

[41] B. Desjardins and M. Esteban. Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rational Mech. Anal.*, 146, 1999.

[42] B. Desjardins and M. Esteban. On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations*, 25:1399-1413, 2000.

[43] G. Diana, D. Rocchi, and M. Belloli. Wind tunnel: a fundamental tool for long-span bridge design. *Structure and Infrastructure Engineering*, 11(4):533–555, 2015.

[44] G. Diana, S. Stoyanoff, K. Aas-Jakobsen, A. Allsop, M. Andersen, T. Argentini, M. C. Montoya, S. Hernández, J. Á. Jurado, H. Katsuchi, I. Kavrakov, H.-K. Kim, G. Larose, A. Larsen, G. Morgenthal, O. Øiseth, S. Omarini, D. Rocchi, M. Svendsen and T. Wu. IABSE Task Group 3.1 Benchmark Results. Part 1: Numerical Analysis of a Two-Degree-of-Freedom Bridge Deck Section Based on Analytical Aerodynamics. *Structural Engineering International 30, 401-410*, 2020.

[45] G. Diana, S. Stoyanoff, K. Aas-Jakobsen, A. Allsop, M. Andersen, T. Argentini, M. C. Montoya, S. Hernández, J. Á. Jurado, H. Katsuchi, I. Kavrakov, H.-K. Kim, G. Larose, A. Larsen, G. Morgenthal, O. Øiseth, S. Omarini, D. Rocchi, M. Svendsen and T. Wu. IABSE Task Group 3.1 Benchmark Results. Part 2: Numerical Analysis of a Three-Degree-of-Freedom Bridge Deck Section Based on Experimental Aerodynamics. *Structural Engineering International 30, 411-420*, 2020.

[46] E.H. Dowell. *A Modern Course in Aeroelasticity, 5th ed.* Springer, 2015.

[47] S. Ervedoza, M. Hillairet and L. Lacave. Long-time behaviour for the two-dimensional motion of a disk in a viscous fluid. *Communications in Mathematical Physics 329*, 325-382, 2014.

[48] S. Ervedoza, D. Maity and M. Tucsnak. Large time behaviour for the motion of a solid in a viscous incompressible fluid. *Mathematische Annalen*, 2021.

[49] E. Feireisl and S. Nečasová . On the long-time behaviour of a rigid body immersed in a viscous fluid. *Applicable Analysis*, 90:1, 59-66, 2011.

[50] L. Formaggia, F. Nobile, A. Quarteroni and Alessandro Veneziani. Multiscale modelling of the circulatory system: a preliminary analysis. *Comput. Vis. Sci.,* 2(2-3): 75-83, 1999.

[51] I. Fragalà, F. Gazzola, and G. Sperone. Solenoidal extensions in domains with obstacles: explicit bounds and applications to navier-stokes equations. *Calc. Var. 59:196*, 2020.

[52] J.B. Frandsen. Numerical bridge deck studies using finite elements. part i: flutter. *Journal of Fluids and Structures*, 19:171–191, 2004.

[53] K.O. Friedrichs. On the boundary-value problems of the theory of elasticity and Korn's inequality. *Annals of Mathematics*, 48:441–471, 1947.

[54] H. Fujita. On stationary solutions to Navier-Stokes equation in symmetric plane domains under general outflow condition. *Pitman Research Notes in Mathematics Series*, 16-30, 1998.

[55] H. Fujita and N. Morimoto. A remark on the existence of steady Navier-Stokes flows in a certain two-dimensional infinite channel. *Tokyo J. Math.* 25(2), 307-321, 2002.

[56] H. Fujita and N. Sauer. On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries. *Journal of the Faculty of Science. Section I A*, 17, 01 1970.

[57] G.P. Galdi. An introduction to the Navier-Stokes Initial-Boundary Value Problem. *Fundamental Direction in Mathematical Fluid Mechanics, G. P. Galdi, J. H. Heywood, and R. Rannacher, eds., Adv. Math. Fluid Mech., Birkhäuser, Basel*, pages 1–70, 2000.

[58] G.P. Galdi. On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications. *Handbook of Mathematical Fluid Dynamics, Volume I, Edited by S. Friedlander and D. Serre, Elsevier*, pages 653–791, 2002.

[59] G. P. Galdi. *Navier–Stokes equations: A Mathematical Analysis*. Springer New York, 2009.

[60] G.P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer, 2011.

[61] G.P. Galdi, J.G. Heywood, and Y. Shibata. On the global existence and convergence to steady state of Navier-Stokes flow past an obstacle that is started from rest. *Arch. Rational Mech. Anal.* 138:307-319, 1997.

[62] G.P. Galdi and A.L. Silvestre. Strong solutions to the problem of motion of a rigid body in a Navier-Stokes liquid under the action of prescribed forces and torques. *Nonlinear Problems in Mathematical Physics and Related Topics I, Kluwer/Plenum, New York, 1*, 2002, 121-144.

[63] G.P. Galdi and A.L. Silvestre. Existence of time-periodic solutions to the Navier-Stokes equations around a moving body. *Pacific Journal of Mathematics*, 2006.

[64] F. Gazzola. *Mathematical Models for Suspension Bridges - Nonlinear Structural Instability*. Springer-Verlag, 2015.

[65] F. Gazzola, V. Pata, and C. Patriarca. Attractors for a fluid-structure interaction problem in a time-dependent phase space. *Preprint*, 2022.

[66] F. Gazzola and C. Patriarca. An explicit threshold for the appearance of lift on the deck of a bridge. *J. Math. Fluid Mech. 24:9*, 2022.

[67] F. Gazzola and G. Sperone. Thresholds for hanger slackening and cable shortening in the melan equation for suspension bridges. *Nonlin. Anal. Real World Appl. 39, 520-536*, 2018.

[68] F. Gazzola and G. Sperone. Steady Navier-Stokes equations in planar domains with obstacle and explicit bounds for its unique solvability. *Arch. Rat. Mech. Anal. 238, 1283-1347*, 2020.

[69] F. Gazzola and G. Sperone. Bounds for Sobolev embedding constants in non-simply connected planar domains. *In: Geometric Properties for Parabolic and Elliptic PDE's. Editors: V. Ferone, P. Salani, F. Takahashi, K. Tatsuki, Springer INdAM Series*, 2021.

[70] F. Gazzola, G. Sperone, and T. Weth. A connection between symmetry breaking for sobolev minimizers and stationary Navier-Stokes flows past a circular obstacle. *Applied Mathematics & Optimization*, 85, 2022.

[71] M. Geissert, K. Götze, and M. Hieber. Lp-theory for strong solutions to fluid-rigid body interaction in Newtonian and generalized Newtonian fluids. *Trans. Am. Math. Soc.* 365(3), 1393-1439, 2013.

[72] D. Gérard-Varet and M. Hillairet. Regularity issues in the problem of fluid structure interaction. *Arch. Rational Mech. Anal.*, 195, 2010.

[73] D. Gérard-Varet and M. Hillairet. Computation of the drag force on a sphere close to a wall: the roughness issue. *ESAIM:M2AN*, 46, 2012.

[74] D. Gérard-Varet and M. Hillairet. Existence of weak solutions up to collision for viscous fluid-solid systems with slip. *Comm. Pure Appl. Math.*, 67(12):2022-2075, 2014.

[75] D. Gérard-Varet and M. Hillairet. The influence of boundary conditions on the contact problem in a 3D Navier-Stokes flow. *J. Math. Pures Appl.*, (9) 103(1):1-38, 2015.

[76] O. Glass and F. Sueur. Uniqueness results for weak solutions of two-dimensional fluid-solid systems. *Arch. Rational Mech. Anal.*, 218, 2015.

[77] W.B. Gordon. Conservative dynamical systems involving strong forces. *Trans. Amer. Math. Soc.*, 204, 1975.

[78] C. Grandmont and Y. Maday. Existence de solutions d'un problème de couplage fluid-structure bidimensionnel instationnaire. *C.R. Acad. Sci. Paris*, SÃ©rie I, 525-530, 1998.

[79] C. Grandmont and Y. Maday. Existence for an unsteady fluid-structure interaction problem. *M2AN Math. Model. Numer. Anal*, 24, 2000.

[80] G. Guidoboni, M. Guidorzi and M. Padula. Continuous Dependence on Initial Data in Fluid-Structure Motions. *J. Math. Fluid Mech.*, 14, 1-32. 2010.

[81] M. D. Gunzburger, H-C. Lee, and G. A. Seregin. Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimension. *Journal of Mathematical Fluid Mechanics*, 2:219–266, 2000.

[82] A. Henrot and M. Pierre. *Shape variation and optimization, a geometric analysis*. Mathematics & Applications, 48. Springer, Berlin, xii+334 pp., 2005.

[83] T.I. Hesla. Collisions of Smooth Bodies in Viscous Fluids: a Mathematical Investigation. *PhD thesis, University of Minnesota*, 146, 2004.

[84] M. Hillairet. Lack of collision between solid bodies in 2D incompressible viscous flow. *Commun. Partial Differ. Equi.*, 32, 2007.

[85] M. Hillairet and T. Takahashi. Collisions in 3D fluid structure interactions problems. *SIAM Journal on Mathematical Analysis*, 40:2451–2477, 2009.

[86] M. Hillairet and T. Takahashi. Existence of contacts for the motion of a rigid body into a viscous incompressible fluid with the Tresca boundary conditions. *Tunisian J. Math.*, 3(3):447-468, 2021.

[87] K.-H. Hoffmann and V.N. Starovoitov. On a motion of a solid body in a viscous fluid. two-dimensional case. *Adv. Math. Sci. Appl.*, 9, 1999.

[88] L. Hocking. The effect of slip on the motion of a sphere close to a wall and of two adjacent spheres. *J. Eng. Mech.*, 7, 207-221, 1973.

[89] E. Hopf. On non-linear partial differential equations. *Lecture Series of the Symposium on Partial Differential Equations, Berkeley, 1-31*, 1957.

[90] C.O. Horgan and L.E. Payne. On inequalities of Korn, Friedrichs and Babuška-Aziz. *Archive for Rational Mechanics and Analysis*, 82:165–179, 1983.

[91] K. Hourigan, M. C. Thompson, and B. T. Tan. Self-sustained oscillations in flows around long blunt plates. *Journal of Fluids and Structures*, 15:387–398, 2001.

[92] L. Huang, H. Liao, B. Wang, and Y. Li. Numerical simulation for aerodynamic derivatives of bridge deck. *Simulation Modelling Practice and Theory*, pages 719–729, 2009.

[93] A. Inoue and M. Wakimoto. On existence of solutions of the Navier-Stokes equation in a time dependent domain. *Journal of the Faculty of Science. University of Tokyo. Section IA. Mathematics*, 24:303–319, 1977.

[94] N. Janberg. Structurae. `https://structurae.net/en/structures/bridges`. Online; accessed 1 April 2020.

[95] P.E. Kloeden, J. Real, and C. Sun. Pullback attractors for a semilinear heat equation on time-varying domains. *J. Differential Equations*, 246:4702–4730, 2009.

[96] A.N. Kolmogorov, and S.V. Fomin. *Introductory real analysis*, Dover Publications, 1970.

[97] A. Korn. Über die Cosserat'schen Funktionentripel und ihre Anwendung in der Elastizitätstheorie. *Acta Mathematica*, 32:81–96, 1909.

[98] M.V. Korobkov, K. Pileckas, and R. Russo. Solutions of Leray's problem for stationary Navier-Stokes equations in plane and axially symmetric spatial domains. *Ann. Math.* 181, 769-807, 2015.

[99] V.A. Kozlov, V.G. Maz'ya, and C. Schwab. On singularities of solutions to the Dirichlet problem of hydrodynamics near the vertex of a cone. *J. Reine Angew. Math., 65-98*, 1994.

[100] O. Ladyzhensakaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, 1963.

[101] L. Landau and E. Lifshitz. *Fluid Mechanics, Theoretical Physics Volume 6*. Pergamon Press, 1987.

[102] E. Livne. Future of Airplane Aeroelasticity. *J. of Aircraft*, 40, 2003, 1066-1092.

[103] V.G. Maz'ya and B. A. Plamenevskii. The first boundary value problem for the classical equations of mathematical physics on piecewise smooth domains. *Part I: Z. Anal. Anwendungen 2, 335-359, Part 2: Z. Anal. Anwendungen 2, 523-551*, 1983.

[104] A. Miranville and X. Wang. Upper bound on the dimension of the attractor for nonhomogeneous Navier-Stokes equations. *Discrete Contin. Dynam. Systems*, 2:95–110, 1996.

[105] T. Miyakawa and Y. Teramoto. Existence and periodicity of weak solutions of the Navier-Stokes equations in a time dependent domain. *Hiroshima Math. J.*, 12:513–528, 1982.

[106] I. Moise, R. Rosa, and X. Wang. Attractors for noncompact nonautonomous systems via energy equations. *Discrete Cont. Dynam. Syst.*, 10:473–496, 2004.

[107] H. Morimoto. A remark on the pressure for the Navier-Stokes flows in 2D straight channel with an obstacle. *Math. Methods Appl. Sci. 27*, 891-906, 2004.

[108] H. Morimoto. A remark on the existence of 2-D Navier-Stokes flow in bounded symmetric domain under general outflow conditions. *J. Math. Fluid Mech. 9(3)*, 411-418, 2007.

[109] C.L.M.H. Navier. Mémoire sur les lois de mouvements des fluides. *Mem. Acad. Sci. Inst. Fr. 2*, 389-440.

[110] M.P. Païdoussis, S.J. Price, and E. De Langre. *Attractors of Evolutions Equations*. Cambridge University Press, 2011.

[111] C. Patriarca. Existence and uniqueness result for a fluid–structure–interaction evolution problem in an unbounded 2D channel. *Nonlinear Differential Equations and Applications NoDEA*, 29(4):1–38, 2022.

[112] C. Patriarca, F. Calamelli, P. Schito, T. Argentini, and D. Rocchi. A numerical characterization of the attractor for a fluid-structure interaction problem. *Special Volume "Interactions between elasticity and fluid mechanics", EMS Series in Industrial and Applied Mathematics 3 (ECR)*, 2023.

[113] L. Patruno. Accuracy of numerically evaluated flutter derivatives of bridge deck sections using rans: Effects on the flutter onset velocity. *Engineering Structures*, pages 49–65, 2015.

[114] L.E. Payne. On the stability of solutions of the Navier-Stokes equations and convergence to steady state. *SIAM J. Appl. MATH.* 15, 2, 1967.

[115] C.S. Peskin. Numerical analysis of blood flow in the heart. *JCP*, 25:220-252, 1977.

[116] K. Pileckas. The Navier-Stokes system in domains with cylindrical outlets to infinity. Leray's problem. *Handbook of Mathematical Fluid Dynamics, Volume IV, Edited by S. Friedlander and D. Serre, Elsevier*, 4:445–643, 2007.

[117] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics*. Princeton University Press, 1951.

[118] G. Rappitsch, K. Perktold and E. Pernkopf. Numerical modelling of shear-dependent mass transfer in large arteries. *Int. J. Numer. Methods Fluids,* 25(7): 847-857, 1997.

[119] J. A. San Martin, V.N. Starovoitov, and M. Tucsnak. Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rational Mech. Anal.*, 161, 2002.

[120] R.H. Scanlan. The action of flexible bridges under wind, I: flutter theory. *Journal of Sound and Vibration*, 60, 1978.

[121] R.H. Scanlan and J.J. Tomko. Airfoil and bridge deck flutter derivatives. *ASCE Journal of Engineering Mechanics Division*, pages 1717–1737, 1971.

[122] G. Schewe. Reynolds-number-effects in flow around a rectangular cylinder with aspect ratio 1:5. *Journal of Fluids and Structures*, pages 15–26, 2013.

[123] A. Selberg. Oscillation and aerodynamic stability of suspension bridges. *Technical Report Acta Polytechnica, Scandinavica Cil3*, 1961.

[124] D. Serre. Chûte Libre d'un Solide dans un Fluide Visqueux Incompressible. Existence. *Japan J. Appl. Math.*, 4, 1987.

[125] E. Simiu and R.H. Scanlan. *Wind Effects on Structure*. Wiley-Interscience Publication, 1966.

[126] X. Song, C. Sun, and L. Yang. Pullback attractors for 2d Navier-Stokes equations on time-varying domains. *Nonlinear Anal. Real World Appl.*, 45:437–460, 2019.

[127] G. Sperone. On the steady motion of Navier-Stokes flows past a fixed obstacle in a three-dimensional channel under mixed boundary conditions. *Annali di Matematica Pura ed Applicata (1923-) 200, 1961-1985*, 2021.

[128] V.N. Starovoitov. Behavior of a Rigid Body in an Incompressible Viscous Fluid near a Boundary. *Internat. Ser. Numer. Math.*, 147, 2002.

[129] V.N. Starovoitov. Nonuniqueness of a solution to the problem on motion of a rigid body in a viscous incompressible fluid. *J. Math. Sci.*, 130, 2005.

[130] T. Takahashi. Analysis of strong solutions for the equations modelling the motion of a rigid-fluid system in a bounded domain. *Adv. Differential Equations*, 8, 2003.

[131] T. Takahashi and M. Tucsnak. Global strong solutions for the two-dimensional motion of an infinite cylinder in a viscous fluid. *Journal of Mathematical Fluid Mechanics*, 6:53–77, 2004.

[132] G. Talenti. The art of rearranging. *Milan J. Math. 84, 105-157*, 2016.

[133] R. Temam. *Infinite-Dimensional Systems in Mechanics and Physics*. Springer, 1997.

[134] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, 2001.

[135] T. Theodorsen. General theory of aerodynamic instability and the mechanism of flutter. *Technical Report 496, NACA*, 1935.

[136] C. Wang. Strong solutions for the fluid-solid systems in a 2-D domain. *Asymptot. Anal.*, 89(3-4):263-306, 2014.

[137] H. Weinberger. On the steady fall of a body in a Navier-Stokes fluid. *Proc. Symposia Pure Math. Vol.* XXIII, AMS, 1973.