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Executive Summary of the Thesis

Stable Minimal Hypersurfaces

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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1. Introduction

The thesis introduces the problem of minimal hypersurfaces and the mathematical tools needed to approach it, then it is presented and proved a flatness condition for minimal stable hypersurfaces and from the results in [1] an extension of [5, Theorem 2] in the flat ambient case is proven. In order to prepare the reader for the calculations a rapid introduction to vector bundles and sections is presented. The focus is on the definition of connection in the particular vector bundle $\mathcal{L}(\otimes^m TM, V)$, which arises from a repetitive application of the connection, but also choosing $V = \mathbb{R}$ corresponds to the tensor bundle. With this connection, differentiation is introduced in general vector bundles and it is possible to introduce the Hessian and the Laplacian of sections.

2. Hypersurfaces

The most intuitive way to represent hypersurfaces is through immersed smooth submanifolds, alas this description does not include non-regular hypersurfaces. The most general treatment of hypersurfaces of any co-dimension that includes irregularities and folding is given by the integer multiplicity currents. With this description, any hypersurface given by a countable union of C^1 hypersurfaces is represented. The general n-currents in $U \subset \mathbb{R}^{n+k}$ denoted $\mathscr{D}_n(U)$ are the dual space of smooth compact differential n-forms denoted $\mathscr{D}^n(U)$. The space of currents is incredibly huge, being an extension of classical distributions, and for it to be useful some restrictions must be introduced. Notice that $\mathscr{D}_0(U)$ identifies the classical distributions, furthermore, the structure theorem of distributions is valid also for currents and the concept of order can be thence extended. For the representation of hypersurfaces, the first restriction is to consider only zeroth order currents, that is the dual of continuous differential n-forms, the operator norm as dual is called mass and geometrically is the area of the hypersurface it is supported on. The mass $\mathbf{M}_W(T)$ of the current T with respect to the compact set $W \Subset U$ is

$$\mathbf{M}_W(T) = \sup_{\substack{|\omega|\leq 1 \\ \omega \in \mathscr{D}^n(U) \\ \operatorname{supp}(\omega)\Subset W}} T(\omega) \tag{1}$$

with $|\omega| = \sup_x \langle \omega, \omega \rangle^{1/2}$. The general Riesz Theorem regarding dual of continuous maps is applicable holding a representation of zeroth order currents through a Radon measure μ_T and an n-vector $\vec{T}$ as

$$T(\omega) = \int_{R^{n+k}} \langle \omega(x), \vec{T}(x) \rangle \; d\mu_T(x). \quad (2)$$

The last step to arrive at the integer multiplicity currents is to consider only zeroth order currents for which there exist an n-rectifiable subset $M \subset U$, a positive integer-valued $\mathcal{H}^n$-integrable function $\theta(x)$, and an $\mathcal{H}^n$-measurable function $\xi : M \to \bigwedge_n(\mathbb{R}^{n+k})$ such that for $\mathcal{H}^n$ almost everywhere $\xi(x) = \tau_1 \wedge ... \wedge \tau_n$ with $\{\tau_i\}$ orthonormal basis for $T_x M$, and

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \, \theta(x) \; d\mathcal{H}^n(x) \quad (3)$$

These currents are fully determined by the associated set M, the function θ, and the orientation ξ, hence it is convenient the notation $T = \tau(M, \theta, \xi)$, from which is evident that these types of currents have an associated integer n-varifold $V = v(M, \theta)$ on which they introduce an orientation. The definition of the boundary of a current is necessary to introduce the compactness result, extending the Stokes Theorem the boundary $\partial T \in \mathscr{D}_{n-1}(U)$ of $T \in \mathscr{D}_n(U)$ is defined as

$$\partial T(\omega) = T(d\omega) \quad \forall \omega \in \mathscr{D}^{n-1}(U). \quad (4)$$

The compactness result for integer multiplicity mentioned requires the sequence to have a bounded mass of both the currents and their boundary, under this bound it is possible to extract a subsequence converging to an integer multiplicity current. From this compactness result the existence of currents minimizing the mass among the competitors with fixed integer multiplicity boundary, is given by the application of the direct method. A compactness result holds also for these minimizing currents.
When considering only co-dimension one currents which are the boundary of a set, another analogous description may be used through the Caccioppoli sets or sets of finite perimeter, whose boundary defines a hypersurface. The definition of such sets corresponds to the requirement of having the characteristic function in the set of Bounded Variation functions. Hence all the nice properties of BV functions can be leveraged.

3. Variations

Computing the first variation of the area functional when the hypersurface S is variated through the vector space X is obtained

$$\delta A = \int_S \mathrm{div} X = -\int_S \langle X, \boldsymbol{H} \rangle \quad (5)$$

where the divergence is in S and not in the entire ambient space, and $\boldsymbol{H}$ represents the generalized mean curvature vector. The integration is abstract so to include also the n-varifolds (hence currents) in which case the integration represents the duality with the associated measure and the second equality defines $\boldsymbol{H}$. From (5) holds the statement that minimal hypersurfaces (i.e. critical points of area functional) have vanishing mean curvature.
Computing the second variation in the setting of a submanifold S of co-dimension one embedded in a general manifold M, where the variation is given by the vector field $X = u\nu$ with ν normal vector, it holds

$$\delta^2 \mathcal{A} = \int_S |\nabla u|^2 - (|A|^2 + Ric(\nu, \nu)) u^2 \; dV_g \quad (6)$$

where Ric is the Ricci tensor of the ambient space, and A represents the second fundamental form.

4. Bernstein Theorem

The most important result in the topic of minimal hypersurfaces is the Bernstein Theorem

Theorem 4.1 (Bernstein Theorem)**.** *Minimizing hypersurfaces in the whole $\mathbb{R}^{n+1}$ are hyperplane for $n < 7$. A counter-example for ambient dimensions $2m$ with $m \geq 4$ is given by the minimizing Simons Cone S^{2m} which gives counterexamples also in dimensions $2m + 1$ as $S^{2m} \times \mathbb{R}$.*

To prove the theorem the minimizing hypersurface S is blown-up and blown-down resulting in again two minimizing hypersurfaces, the monotonicity equation and the continuity of the area functional for minimizing sequences imply that the two blown hypersurfaces are cones. The regularity result regarding the measure of the singular set of minimizing hypersurfaces concludes the proof, in fact

$$\mathcal{H}^{n-7+\alpha}(\mathrm{sing}(S)) = 0 \quad \forall \alpha > 0 \quad (7)$$

proved in [4, Theorem 28.1] using a specific variation in the second variation formula and a dimension-reducing argument. The blown hypersurfaces for $n < 7$ are non-singular cones, hence hyperplanes, implying that the hypersurface itself is also flat.

The discussion on stable minimal hypersurfaces will be limited to smooth co-dimension one embedded submanifolds, hence it is important that the counter-example of the Bernstein Theorem can be extended also in the smooth case. The smoothing of minimizing cones with singularity in the origin can be found in [3], the proof, which uses the framework of currents, is discussed. This result implies that also when only smooth competitors are considered, for $n \geq 7$ there are non-stable minimal hypersurfaces.

5. Stability

Stability is discussed for smooth, complete, connected, and orientable embedded minimal submanifolds denoted with S. The stability condition (6) corresponds to the requirement of positivity of the differential, symmetric, and strongly elliptic operator

$$L_S = -\Delta - V \tag{8}$$

with $V = |A|^2 + Ric(\nu,\nu)$. The spectrum of L_S is discrete and diverging to infinity, being the inverse of a symmetric compact operator, hence stability coincides with the requirements of having a positive first eigenvalue. The behavior of the eigenvalues under contraction is given by the Morse Index Theorem [6].

Theorem 5.1. *The eigenvalues of L_S in $g_t(\Omega)$ where g_t is a contraction are strictly increasing functions of t. Furthermore, there exists an ε for which if the area of the contraction is less than ε then all the eigenvalues are positive.*

Hence as the area of the domain tends to zero the negative eigenvalues one by one passes through zero and become positive, implying that small enough patches are stable. Also given that the eigenvalues are *strictly* increasing in t implies that non-compact stable minimal hypersurfaces are strictly stable. Thanks to strict stability the following theorem is proved.

Theorem 5.2. *If the stable minimal hypersurface S considered is non-compact then there exists a function $u > 0$ on S such that $Lu = 0$.*

To prove it non-compactness gives strict stability, hence zero is never an eigenvalue, and thanks to Fredholm's alternative the boundary value problem

$$\begin{cases} L_S v = 0 \\ v = 1 \end{cases} \tag{9}$$

has a unique strictly positive solution. By solving this problem on domains that exhaust S and renormalizing the solutions, it is shown the existence of a function $u > 0$ in S such that $L_S = 0$ as the limit of these renormalizations. This function is fundamental for one of the rigidity results presented later.

6. Rigidity of Bernstein Theorem

The following results are for smooth, complete, connected, and orientable manifolds embedded in $\mathbb{R}^{n+1}$ and $n \geq 2$. The first result shows that a bound with the first eigenvalue of the Laplacian on the second form is sufficient for stability and flatness.

Theorem 6.1. *Let M be a minimal smooth, complete, connected, and orientable submanifold embedded in $\mathbb{R}^{n+1}$ with $n < 6$. Suppose that for any ball $B(p,r)$ $p \in M$ it holds the bound $|A|^2(x) \leq \lambda_1(-\Delta)$ $\forall x \in B(p,r)$, where $\lambda_1(-\Delta)$ is the first eigenvalue of $-\Delta$ in $B(p,r)$. Then the hypersurface is stable and flat.*

To prove this result it is first needed a bound from below for the Ricci curvature by the second form. Gauss Equation for submanifolds of flat space with vanishing mean curvature becomes

$$Ric = -A^2$$

and thanks again to zero mean curvature it holds the improved bound

$$Ric = -A^2 \geq -\frac{n-1}{n}|A|^2 \tag{10}$$

it is now sufficient to apply the bound on $\lambda_1(-\Delta)$ valid on balls found in [2, Theorem 5.2]. In fact for any ε it is possible to choose a radius r of the ball for which the bound becomes

$$\lambda_1(-\Delta) \leq \frac{(n-1)^2 \max_{B_r} |A|^2}{4n} + \varepsilon \tag{11}$$

this is incompatible with the supposed bound for non-zero second form as

$$\max_{B_r} |A|^2 \leq \frac{(n-1)^2 \max_{B_r} |A|^2}{4n} + \varepsilon \tag{12}$$

is absurd if $\frac{(n-1)^2}{4n} < 1$, that is for $n < 6$. Concluding the proof.

The results in [1] are repurposed as an extension to stable hypersurfaces of Theorem 2 in [5] in the case of flat ambient space. This extension will hold as a corollary the flatness of stable minimal hypersurfaces in $\mathbb{R}^4$ and $\mathbb{R}^3$.

Theorem 6.2. *Denoting with u the function of Theorem (5.2), let $f = [2\beta - k(q-n)]\log(u)$ with $k > 0$, $q + \delta \in [4, 4+\sqrt{8/n}]$ for $\delta > 0$ small enough and $\beta > 0$ satisfying $|\beta - 1| < \sqrt{2 - \frac{q-2}{4(q-4+2/n)}}$. Introducing the conformal metric $\tilde{g} = u^{2k}g$. Suppose that the conformal metric is complete and*

$$\lim_{R\to\infty} \frac{1}{R^{q+\delta}} \int_{B^{\tilde{g}}_{2R}} e^{-f}\, dV_{\tilde{g}} = 0 \tag{13}$$

then M is totally geodesic (being restricted to flat ambient space it is flat).

The proof is given by showing that $\forall \psi \in C^\infty_0(M)$ it holds the integral estimate

$$\int_M |A|^{q+\delta} u^{-2\beta-k\delta}\psi^{q+\delta}\, dV_g \le C\int_M u^{-2\beta-k\delta}|\nabla\psi|^{q+\delta}\, dV_g$$

that is [1, Lemma 2.5] with the added parameter β, and corresponds with [5, Theorem 1] weighted. To conclude choosing for ψ the cutoff function for the conformal metric ball $B^{\tilde{g}}_{2R}(x_0)$, which is compact thanks to the completeness of $\tilde{g}$, results in

$$\int_M |A|^{q+\delta} u^{-2\beta-k\delta}\eta^{q+\delta}\, dV_g \le \frac{C}{R^{q+\delta}}\int_{B^{\tilde{g}}_{2R}(x_0)} e^{-f}\, dV_{\tilde{g}}$$

concluding the proof.

Corollary 6.1. *For $n = 2$ and $n = 3$, smooth stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ are flat, hence minimizing.*

The proof is given by satisfying the requirements of Theorem (6.2). In fact, the completeness of the conformal metric is proved in [1], together with a lower bound for the 2-Bakry-Emery-Ricci tensor for the case $n = 3$. Hence using the weighted Bishop-Gromov volume comparison for $n = 3$ and the classical volume comparison for $n = 2$ the limit in Theorem (6.2) is satisfied proving flatness.

7. Conclusions

Considering the problem of whether smooth stable minimal hypersurfaces are flat, hence coincident with the minimizing ones, it has been shown that for $n \ge 7$ the answer is no from the smoothing of singular cones. The corollary of Theorem (6.2) gives a positive answer for $n = 2$ and $n = 3$, while for $n = 4$ and $n = 5$, it is still yet to prove the completeness of the conformal metric and an adequate volume comparison. Regarding $n = 6$ the theorem cannot prove flatness due to the bound on q. Also, Theorem (6.1) gives flatness for $n < 6$ under the condition $|A|^2(x) \le \lambda_1(-\Delta)$ which, if shown to be true a priori for stable hypersurfaces, would close the problem apart from $n = 6$. Both theorems share the limit at $n = 6$, this could certainly be a coincidence, but may also be a hint that the rigidity result does not hold for $n = 6$.

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