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EXECUTIVE SUMMARY OF THE THESIS

On the long time behavior of the segregation limit for a competition-diffusion system with periodic boundary conditions

LAUREA MAGISTRALE IN MATHEMATICAL ENGINEERING - INGEGNERIA MATEMATICA

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1. Introduction

In this master thesis, we study a system of parabolic differential inequalities arising from the strong competition limit for the following semilinear parabolic system:

$$\frac{\partial v_i}{\partial t} - \Delta v_i = f_i(v_i) - \kappa \sum_{j \neq i} \beta_{ij} v_i v_j, \qquad (1)$$

where $i \in \{1, ..., M\}$, $\kappa, \beta_{ij} > 0$, and the interest is in non-negative solutions. This system is used in spatial ecology to model M diffusing and competing populations with density functions v_i in a bounded domain of $\mathbb{R}^n$. The term $f_i(v_i)$ models the internal dynamics of the species, and we will choose a logistic-type dynamics $f_i(v_i) = a_i v_i - v_i^2$, where coefficients $a_i > 0$ represent the internal growth rate. Moreover, the term $-v_i^2$ models intra-specific competition, and coefficients $\kappa\beta_{ij}$ represent inter-specific competition rates between populations i and j. The system (1) is an instance of the class of diffusive Lotka-Volterra models, which have a long history and are a paradigmatic model for interacting populations encompassing, besides competition, predator-prey dynamics and mutualism. An important feature of (1) is that it may model scenarios where the competition between species is asymmetric, i.e. $\beta_{ij} \neq \beta_{ji}$. A crucial question about (1) is the occurrence of patterns, namely inhomogeneous stationary or periodic positive solutions that are stable, and thus represent the possibility of coexistence of competing species. This subject has been extensively studied since the 1980s, even with more general internal dynamics and in various parameter regimes, but mainly in the case of two populations. In particular, a body of work focused on the singularly perturbed strong competition regime, for which $\kappa \to \infty$. In this context, segregation of species occurs, meaning that species tend to progressively concentrate on different parts of the domain as κ grows. In the limit, $u_i u_j \equiv 0$ for any $i \neq j$, and a free boundary problem arises for the interfaces between the sets $\{u_i > 0\}$. Although much could be said about the literature investigating this regime, we will only mention what is more relevant for our work. From a dynamical point of view, this regime is believed to exhibit simpler dynamics compared to the generic case. Indeed, Dancer and Zhang showed in [4] that for large but finite κ, under Dirichlet boundary conditions, every solution to the two-species system tends to a stationary state. One of the reasons for the preference for the study of the two population case in the literature is that a suitable combination of densities satisfies a scalar equation in the limit $\kappa \to \infty$ whose study sheds light on the dynamics of the system for finite κ. For more than two species, an approach via a scalar equation is in general not viable. However, the case of three species was investigated in the 1990s. More recently, Conti, Terracini, and Verzini showed in [3] that for M species and in arbitrary dimension, solutions to the stationary equation converge up to a subsequence to a vector valued function satisfying a set of $2M$ elliptic differential inequalities. Moreover, they proved the Hölder regularity of the segregated limit and studied the structure of the associated free boundary. In [2], Caffarelli, Karakhanyan and Lin improved the regularity results and extended the analysis to the parabolic case. Dancer, Wang and Zhang studied in [10] the dynamics in one dimension with many species. In the strong competition limit, the limiting profiles satisfy a system of parabolic differential inequalities analogous to the stationary one. The authors proved that under Dirichlet or Neumann boundary conditions and for every choice of the interaction parameters, solutions to the singular system converge to a stationary solution, and this analysis proves useful in extending the convergence result to system (1) with κ finite but large enough.

In this work, we study the same system of differential inequalities on the interval $(0,1)$ but with periodic boundary conditions. For every $i \in \{1, ..., M\}$:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} \leq a_i u_i - u_i^2, \\ \left(\dfrac{\partial}{\partial t} - \dfrac{\partial^2}{\partial x^2}\right)\left(u_i - \displaystyle\sum_{j \neq i} \dfrac{\beta_{ij}}{\beta_{ji}} u_j\right) \geq \\ \qquad \geq a_i u_i - u_i^2 - \displaystyle\sum_{j \neq i} \dfrac{\beta_{ij}}{\beta_{ji}} \left(a_j u_j - u_j^2\right), \\ u_i u_j \equiv 0 \qquad \forall \quad j \neq i, \\ u_i(0,t) = u_i(1,t). \end{cases} \tag{2}$$

Inequalities are intended in a weak sense in the space-time cylinder $(0,1) \times (0,+\infty)$ with non-negative test functions. Here we remark that the above system can be equivalently defined on the flat one-dimensional torus $\mathbb{T} \simeq \mathbb{R}/\mathbb{Z}$. We will adopt this point of view when more convenient. We complement (2) with the initial conditions:

$$u_i(x,0) = \phi_i(x) \quad \forall \quad i$$

where ϕ_i are periodic, Lipschitz continuous non-negative and non-trivial functions satisfying the segregation condition, and for which we assume that the set $\bigcup_i \{\phi_i > 0\}$ has a finite number of connected components. In his master's thesis [8], Spadaro considered the particular case where $a_i = a$, Initial conditions are rotations of a fixed non-negative profile supported on $\left(0, \frac{1}{M}\right)$, and $\frac{\beta_{i,i+1}}{\beta_{i+1,i}} = \frac{1}{\lambda}$ for some $\lambda > 0$. In particular, he proved that for every λ near 1 there exist ω near 0 and a non-negative density U supported on $\left(0, \frac{1}{M}\right)$ such that the rotating wave:

$$u_i(x,t) = U\left(x - \omega t - \frac{i}{M}\right),$$

is a solution to (2). Moreover, there is a bijection between the velocity ω and the parameter λ. Therefore, for generic interactions, the singular system with periodic boundary conditions necessarily exhibits a different behavior compared to the Dirichlet case, for which periodic solutions are ruled out in [10]. The main contribution of our work is to give conditions on the interactions to ensure that the evolution of a certain state converges to a stationary solution as $t \to +\infty$. For a solution $\mathbf{u}(x,t) = (u_1, ..., u_m)$ to (2) such that u_i remains definitively positive and $\{u_i > 0\}$ are connected sets, the condition reads as follows.

$$\prod_{i=1}^{m-1} \frac{\beta_{i,i+1}}{\beta_{i+1,i}} = \frac{\beta_{1,m}}{\beta_{m,1}}. \tag{3}$$

We refer to such interactions as cyclic. They generalize the case of symmetric ones (in which $\beta_{ij}/\beta_{ji} = 1$ for every i,j), and for our purposes the analysis can be carried out essentially in the same way. We remark that a similar discrimination between the nature of interactions has already been found to affect the free boundary of stationary states in two dimensions, see [9]. Moreover, the case treated in [8], consistently, does not fall under the cyclic case.

The present document is a summary of the thesis. Some results treated there are only cited, and the most important ones are stated in a compact way and without proof. The sections of this summary are modeled on the chapters of the thesis. In Chapter 2, we state some preliminary results concerning the well posedness of the initial-boundary value problem for (1) in the one-dimensional periodic case, and perform the asymptotic analysis to prove that the weak differential inequalities are satisfied by the limit. In Chapter 3, we study the set of stationary solutions to (2) for a generic choice of coefficients β_{ij} and show that continua of stationary solutions may occur for appropriate choices of the interaction parameters. In Chapter 4, we prove some general properties of solutions to (2) which enjoy some minimal regularity that is enjoyed by segregated configurations of (1). In Chapter 5, we prove the main result. In order to to this, the energy methods used in [10] are still applicable but not sufficient to prove convergence due to the degeneracy in the set of stationary solutions introduced by rotational symmetry of the equations. However, they allow us to prove that the evolution converges to the set of stationary solutions, a property that is often referred to as orbital stability. To prove the stronger property of converge to a single stationary solution, instead, we resort to the monotonicity of the number of zeros of certain one-dimensional linear parabolic equations. This is a fundamental tool in the study of long time behavior for semilinear parabolic equations in one space dimension. We will conclude with some open problems and directions of research.

2. Asymptotic analysis in the strong competition regime

In this chapter, we consider the system:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial^2 u_i}{\partial x^2} = a_i u_i - u_i^2 - \kappa \displaystyle\sum_{j \neq i} \beta_{ij} u_i u_j \\ u_i(x,0) = \phi_i(x), \end{cases} \tag{4}$$

for $(x,t) \in \mathbb{T} \times (0,T) =: Q_T$ and any $i \in \{1, ..., M\}$. First, we prove some a priori bounds for regular solutions to (4) in the L^∞ norm. In order to do this, we state and prove a comparison principle for semilinear parabolic systems of competitive type, which applies to our case. Moreover, we state the following well-posedness result, whose proof is based on semigroup theory and parabolic regularity theory.

Theorem 2.1. *Consider the initial value problem* (4) *and suppose that the initial conditions $\phi_i \in Lip(\mathbb{T})$. Then, there exists a unique solution $\mathbf{u} = (u_1, ..., u_M) \in \mathcal{C}^{2+2\delta, 1+\delta}(Q_\infty) \cap \mathcal{C}^{2\delta,\delta}(\overline{Q_\infty})$ for any $\delta \in [0, \frac{1}{2})$.*

We remark that through direct computation, it is possible to verify that the regular, non-negative solutions to (4) provided by Theorem 2.1 satisfy the differential inequalities (2) in a classical sense. We now consider the behavior of solutions to (4) in the limit $\kappa \to +\infty$. We denote by u_κ the solution to (4) with interaction strength κ, and carry out our analysis for $T < +\infty$. We use a compactness principle in Bochner spaces due to Aubin, Lions and Simon. We refer to [7] for the statement and proof. This theorem allows us to prove that u_κ strongly converges, up to a subsequence, in the space $L^2(0,T; L^2(\mathbb{T}))$. The main ingredients needed to recover compactness are integral estimates that are uniform with respect to κ. After proving the estimates, we show that the segregation condition holds for the limit, and that the differential inequalities in (2) pass to the limit when considered in a weak sense. The following theorem summarizes these results. We denote as H^1_{per} the (closed) subspace of $H^1(0,1)$ consisting of functions u such that $u(0) = u(1)$ (notice that the trace is well defined in $H^1(0,1)$).

Theorem 2.2. *Consider the family $\{\mathbf{u}^\kappa\}_{\kappa>0}$. There exists a subsequence $\{\kappa_n\}_{n\in\mathbb{N}}$ with $\kappa_n \to \infty$, and a function $\mathbf{u} = (u_i)_i \in L^2(0,T; H^1_{per})$ such that, for every i:*

- *$u_i^\kappa \to u_i$ strongly in $L^2(0,T; L^2(\mathbb{T}))$;*
- *$u_i u_j \equiv 0$ almost everywhere in Q_T.*
- *the functions u_i satisfy the weak formulation of differential inequalities* (2) *with non-negative test functions $\psi \in \mathcal{C}_c^\infty(\mathbb{T} \times (0,T))$ such that $\psi \geq 0$.*

We remark that the above convergence results are far from optimal. For what concerns the elliptic case, we refer to [3], where Hölder uniform bounds are derived via a blow up argument for a general number of species and for any dimension. In [2], Caffarelli and Lin proved uniform a-priori Hölder and Lipschitz bounds (in the parabolic distance) for solutions to system (1) with $f_i(v_i) \equiv 0$, yielding the same regularity for the limit. The theorem was extended in [10] to equations with logistic internal dynamics and Dirichlet boundary conditions. These results are essentially local in nature, and therefore they can be extended to the periodic case as well. Moreover, the results in this chapter extend to the case where $T = +\infty$, and because of the L^∞ uniform bounds, the local Lipschitz continuity provided in [2] is necessarily global. Consequently, In Chapters 4 and 5 we will carry out our study for globally Lipschitz solutions to (2).

3. Singular elliptic system

In this chapter, we study the structure of the set of stationary solutions to the singular elliptic system, for a solution $\mathbf{u} \in [H^1_{\text{per}}(0,1)]^M$. For every $i \in \{1, ..., M\}$:

$$\begin{cases} -\dfrac{d^2 u_i}{dx^2} \leq a_i u_i - u_i^2, \\ -\dfrac{d^2}{dx^2}\left(u_i - \displaystyle\sum_{j\neq i} \frac{\beta_{ij}}{\beta_{ji}} u_j\right) \geq \\ \qquad\qquad a_i u_i - u_i^2 - \displaystyle\sum_{j\neq i} \frac{\beta_{ij}}{\beta_{ji}} \left(a_j u_j - u_j^2\right), \\ u_i u_j \equiv 0 \qquad \forall \quad j \neq i, \\ u_i(0) = u_i(1). \end{cases} \tag{5}$$

Again, we intend inequalities in a weak sense, with non-negative periodic test functions $\varphi \in H^1_{\text{per}}(0,1)$. Given some $\mathbf{w} \in [H^1_{\text{per}}(0,1)]^M$, it is possible to consider the periodic extension $\mathbf{w}^{\text{per}}$ on $\mathbb{R}$. We define the translation of $\mathbf{w}$ by $c \in \mathbb{R}$ as:

$$\sigma_c \mathbf{w}(x) := \mathbf{w}^{\text{per}}(x - c) \qquad \forall \quad x \in [0,1].$$

The first observation is that the change in the topology of the domain induces a new symmetry in the set of solutions with respect to the Dirichlet case.

Proposition 3.1. *If $\mathbf{u} := (u_1, ..., u_M)$ is a solution to the problem (5), then any of its space-translations is a solution as well.*

We will indicate as m the number of components of $\mathbf{u}$ that are not identically zero in $[0,1]$. Moreover, for the remainder of this chapter, we will suppose that each set $\{u_i > 0\}$ has a single connected component and discuss the more general case where needed. The case $m = 1$ reduces to the periodic boundary value problem for the diffusive logistic equation, for which there always exists exactly one positive solution $u_i \equiv a_i$. In case $m > 1$, we have the following non-existence result.

Theorem 3.1. *Suppose that:*

$$\prod_{i=1}^{m-1} \frac{\beta_{i,i+1}}{\beta_{i+1,i}} \neq \frac{\beta_{1,m}}{\beta_{m,1}}. \tag{6}$$

Then, There exists no solution $u = (u_1, ..., u_m)$ to (5) with $2 \leq m \leq M$.

The same idea used in the above theorem can also be used under non-cyclic interactions to rule out solutions for which each support is not necessarily connected, but one has then to change condition (6) accordingly as each connected component belonged to a different density. Therefore, if no condition of the form (3) holds for any $H \leq M$, the stationary solutions with $m = 1$ are the only non-trivial ones. Instead, when interactions are cyclic, we have the following result.

Theorem 3.2. *Let $\mathbf{u} = (u_1, ..., u_m)$ be segregated densities with $2 \leq m \leq M$. suppose that each support is connected, and that (3) holds. Then u solves (5) if and only if it is a translation of a solution to the same system of inequalities with homogeneous Dirichlet boundary conditions.*

We observe that this result also extends to the generic case where supports are not connected, but then, for the equivalence to hold, it is necessary to additionally require for solutions to the Dirichlet problem that the leftmost connected component does not belong to the support of the same density as the rightmost one. The Dirichlet case has been studied in [10], where authors proved that there exists only a finite number of positive solutions. Therefore we are able to characterize the set of solutions to the periodic problem. In particular, the set of non-negative stationary solutions $\mathcal{S}$ has a finite number of connected components A that take one of the following forms.

- $A = \{\mathbf{0}\}$
- $A = \{a_i\}$
- $A = \{\sigma_c \mathbf{u} : \mathbf{u}$ is a proper solution to the Dirichlet problem, and $c \in \mathbb{R}\}$

4. Singular parabolic system

In this chapter, we prove some properties that hold for solutions $\mathbf{u}(x,t) = (u_1, ..., u_M)(x,t)$ to (2) defined on $[0,1] \times (0, +\infty)$ that are globally Lipschitz continuous in the parabolic distance (see the discussion at the end of Chapter 2). For simplicity, we will restrict the analysis to the case where the initial data ϕ_i are such that $\bigcup_i \{\phi_i > 0\}$ has a finite number of connected components. The properties we will prove were shown for the Dirichlet case in [10], and we will adapt the proofs to the case with periodic boundary conditions. Occasionally, we will rely on different techniques that simplify proofs in our case. We define the free boundary as $\mathcal{F}(\mathbf{u}) := \bigcup_{i=1}^{M} \partial\{u_i > 0\}$. The first result, concerning the cardinality of a time-slice of the free boundary, is the following:

Proposition 4.1. *For each $T > 0$, $\mathcal{F}(\mathbf{u}) \cap \{t = T\}$ contains only finitely many points.*

This is an application of a dimensional reduction principle, relying on a blow-up argument. Thanks to the above proposition, the function $m : (0, +\infty) \to \mathbb{N}$, which counts the total number of connected components in a time-slice of the free boundary, is well defined with values in $\mathbb{N}$:

$$m(T) = \#\{\text{connected components of } \cup_i (\{u_i > 0\} \cap \{t = T\})\}.$$

We now observe that, on each open set $\{u_i > 0\}$, thanks to condition $u_i u_j = 0$ for every $j \neq i$, u_i satisfies the logistic equation and is thus smooth on that set because of parabolic regularity theory. This consideration allows us to use the strong maximum principle inside $\{u_i > 0\}$ to prove that the number $m(t)$ is non-increasing as a function of time. Therefore, it is possible to define a finite number of times $T_1, ...T_K$, with $K \leq M$ and $T_K < T_{K-1} < ... < T_1$, as all the points where $m(t)$ changes value; we will refer to them as extinction times. On each of these time intervals, we will consider each connected component of $\bigcup_i \{u_i > 0\}$ as referring to a different density. This can always be done, at the expense of increasing the size M of the system and properly re-defining the parameters.

It is now convenient to introduce a reference domain for the space-time cylinder $\mathbb{T} \times (T_{k+1}, T_k)$. To this end, we consider the periodic extension $\mathbf{u}^{\text{per}}$ of $\mathbf{u}$ to $\mathbb{R} \times (T_{k+1}, T_k)$, and define the set $\Theta \subset \mathbb{R} \times (T_{k+1}, T_k)$ as the union of closures of the connected components of $\{u_i^{\text{per}} > 0\}$ which intersect $[0, 1)$ at $t = T_{k+1}$. Here, without loss of generality, we suppose that $u_i(0, T_{k+1}) = 0$ for every i. With a little abuse of notation, we will equivalently refer to $\mathbf{u}$ as the solution on $\mathbb{T} \times (T_{k+1}, T_k)$ or its representation on Θ given by $\mathbf{u}^{\text{per}}|_\Theta$ depending on the more convenient choice. This allows us to consider naturally a spatial ordering of points. At this point, we are able to define an ordering of components u_i such that u_i is adjacent to u_{i-1} and u_{i+1} for all times $t \in (T_{k+1}, T_k)$. Thanks to these conventions, for any $T \in (T_{i+1}, T_i)$ there exist $\alpha_0(T), ..., \alpha_{m-1}(T)$, $\beta_0(T), ..., \beta_{m-1}(T)$ such that:

$$\begin{cases} \alpha_i(T) < \beta_i(T) \leq \alpha_{i+1}(T) \\ \{u_i > 0\} \cap \{t = T\} = (\alpha_{i-1}(T), \beta_{i-1}(T)). \end{cases}$$

Next, we define the (space-time) multiplicity of points as follows:

$$\begin{aligned} n(x_0, t_0) := \#\{&i = 1, ..., M : \\ &\forall \epsilon > 0, \quad \{u_i > 0\} \cap B_\epsilon(x_0, t_0) \neq \emptyset\}, \end{aligned}$$

and prove that if $t \in (T_{k+1}, T_k)$, then $n(x, t) \leq 2$. Moreover, $n(x, t) = 2$ for points in the free boundary. This allows us to prove that $\alpha_{i+1}(t) = \beta_i(t)$ for any $t \in (T_{k+1}, T_k)$ and $i = 0, ..., m-1$, and $\beta_{m-1} = \alpha_0(t)$. Therefore, we can consider the points $\alpha_0(t), ..., \alpha_m(t)$ (where $\alpha_0(t) \equiv \alpha_m(t)$ when seen on $\mathbb{T}$) and $\{u_i > 0\} \cap \{t = T\} = (\alpha_{i-1}(T), \alpha_i(T))$. Thanks to the fact that $n(x, t) = 2$ for points in the free boundary, by exploiting the differential inequalities restricted to a region where $u_j \equiv 0$ for all j except $i, i+1$ we can prove the following transmission conditions across the free boundary, where the partial derivatives are intended in a unilateral sense:

$$\frac{\partial u_i}{\partial x}(\alpha_i(t), t) = -\frac{\beta_{i,i+1}}{\beta_{i+1,i}} \frac{\partial u_{i+1}}{\partial x}(\alpha_i(t), t).$$

Actually, for any $(x, t) \in \mathcal{F}(\mathbf{u}) \cap \{T_{k+1} < t < T_k\}$, lying between densities u_i and u_{i+1}, $u := u_i - \frac{\beta_{i,i+1}}{\beta_{i+1,i}} u_{i+1}$ is $\mathcal{C}^{2+2\delta,1+\delta}$ in a neighborhood of (x, t), for any $\delta \in [0, \frac{1}{2})$. This allows us to define a regular density u on $\mathbb{R} \times (T_{k+1}, T_k)$ as follows:

$$u(x,t) := \sum_{i \in \mathbb{Z}} (-1)^{i-1} \lambda^{(i)} u_{i \bmod m} \left(x - \frac{i - i \bmod m}{m}, t \right), \tag{7}$$

where coefficients $\lambda^{(i)}$ are defined as follows.

$$\lambda^{(i)} := \begin{cases} 1 & \text{for} \quad i = 1, \\ \lambda^{(i-1)} \dfrac{\beta_{(i-1) \bmod m, i \bmod m}}{\beta_{i \bmod m, (i-1) \bmod m}} & \text{for} \quad i \geq 2, \\ \lambda^{(i+1)} \dfrac{\beta_{i \bmod m, (i-1) \bmod m}}{\beta_{(i-1) \bmod m, i \bmod m}} & \text{for} \quad i \leq 0. \end{cases}$$

The sum over $\mathbb{Z}$ in (7) is well defined because for any (x,t) there is actually only one i that gives a non-zero contribution. Moreover, u is a classical solution in $\mathbb{R} \times (T_{k+1}, T_k)$ of the following equation:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = V(x,t)u, \tag{8}$$

where $V(x,t)$ is a bounded function. If we consider cyclic interactions, the periodic boundary conditions are obviously satisfied on $[0,1]$ if m is even and $[0,2]$ if m is odd. Hence, the restriction of u can be considered a classical solution to (8) with periodic boundary conditions in a proper interval. Otherwise, u is not periodic but it satisfies a Tychonoff-type growth condition, in the sense that there exists $C, \alpha > 0$ such that:

$$|u(x,t)| \leq Ce^{\alpha|x|^2} \qquad \forall (x,t) \in \mathbb{R} \times (T_{k+1}, T_k). \tag{9}$$

This allows us to prove the following theorem:

Proposition 4.2. *For every $t \in (T_{i+1}, T_i)$, there exist $\alpha_0(t), \alpha_1(t), ..., \alpha_m(t)$ such that:*

$$\{u_i > 0\} \cap \{t = T\} = (\alpha_{i-1}(T), \alpha_i(T))$$

for every $T \in (T_{i+1}, T_i)$. Moreover, $\alpha_i(t) \in \mathcal{C}^1((T_{k+1}, T_k))$.

The proof relies on a strong unique continuation property that holds for solutions to equation (8) which enjoy (9), and on the study of the zero set of the same equation, carried out in [1] by Angenent. While the use of the unique continuation property is the same as in [10], here we simplify the proof of regularity of curves by exploiting directly the results in [1]. Finally, we also extend a uniqueness result to the case of periodic boundary conditions and cyclic interactions.

5. Long time behavior: cyclic interactions

In this chapter, we will prove that under cyclic interactions, the evolution of admissible initial states converges to a stationary state. Since we are interested in the long time behavior of the system, we will consider globally Lipschitz solutions $\mathbf{u}$ in the interval $(T_1, +\infty)$. We will indicate by m the number $m(T_1)$, and make the hypothesis that the region of positivity for each density is connected. At the end of the chapter, we will discuss this requirement and how our study generalizes to other situations. We will make a great use of the density u associated to $\mathbf{u}$ via (7). First, we prove that we can define an energy in an analogous way to the Dirichlet case. This allows us to prove that every solution converges to the set of stationary solutions $\mathcal{S}$, and in particular to a single connected component of $\mathcal{S}$. This property is sometimes referred to as orbital stability. Notice that we are considering in $\mathcal{S}$ only the stationary solutions with m positive components whose support is connected. In the latter part of the chapter, we will prove that solutions actually converge to a stationary one. We define the energy as follows:

$$E[\mathbf{u}](t) := \sum_{i=1}^{m} \int_0^1 [\lambda^{(i)}]^2 \left\{ \frac{1}{2} \left| \frac{\partial u_i}{\partial x} \right|^2 - \frac{1}{2} a_i u_i^2 + \frac{1}{3} u_i^3 \right\} dx,$$

and prove the following crucial property:

Proposition 5.1 (Energy Identity)**.** *Let $\mathbf{u} = (u_i)_i$ be a globally Lipschitz solution to system (2) with cyclic interactions. Then, the following identity holds:*

$$\dot{E}[\mathbf{u}](t) = -\int_0^1 \sum_{i=1}^{m} [\lambda^{(i)}]^2 \left| \frac{\partial u_i}{\partial t} \right|^2 dx \leq 0 \qquad \forall t > 0. \tag{10}$$

The fact that no term involving $\dot{\alpha}(t)$ appears here is due to the assumptions on the set of coefficient β_{ij}, which in particular lead to their cancellation. Moreover, the derivative of the energy can be written only in terms of the function u defined in (7).

$$\dot{E}(t) = -\int_0^1 \left| \frac{\partial u}{\partial t} \right|^2 dx. \tag{11}$$

Given a solution $\mathbf{u}$ to system (2) defined for all times $t > T_1$, we define the ω-limit set of $\mathbf{u}$ as follows:

$$\begin{aligned} \omega[\mathbf{u}] := \{\mathbf{w} \in \mathcal{C}^0_{\text{per}}([0,1]) : \quad \exists \quad t_k \to +\infty, \\ \mathbf{u}(x, t_k) \to \mathbf{w} \quad \text{in} \quad \mathcal{C}^0_{\text{per}}([0,1])\}. \end{aligned}$$

It is a standard result in the abstract theory of dynamical systems that the ω-limit sets enjoy properties of non-emptiness, compactness, and connectedness. Even though we are not going to precisely recast our problem as an abstract dynamical system, we are able to prove in a standard way that these properties also hold in our situation. The energy identity (10) and (11) allows us to prove the following result:

Theorem 5.1. *Let* $\mathbf{u}$ *be a globally Lipschitz solution of* (2) *with cyclic interactions. Then:*

$$d(\mathbf{u}(t), \mathcal{S}) = \inf_{\mathbf{v}\in\mathcal{S}} \|\mathbf{u}(\cdot,t) - \mathbf{v}(\cdot)\|_\infty \to 0 \quad \text{as} \quad t \to \infty.$$

In particular, $\omega[\mathbf{u}] \subset \mathcal{S}$.

Moreover, by the connectedness of the ω-limit sets, $\omega[\mathbf{u}]$ is a subset of a connected component of $\mathcal{S}$, whose structure has been investigated in Chapter 3. Using (11), we also prove the following fact:

Proposition 5.2. *Let* u *be as in* (7), *built from a solution* $\mathbf{u}$ *of system* (2) *with cyclic interactions. Then:*

$$\lim_{t\to\infty} \sup_{x\in(0,1)} \left|\frac{\partial u}{\partial t}\right| = 0.$$

This excludes the possibility that the dynamics resembles asymptotically that of a periodic solution. However, differently from the Dirichlet case, the use of the energy identity is not sufficient to prove convergence to a stationary state. Instead, we use again the property of monotonicity of the zero number, as stated in [5] by Matano. This property is used in that paper to prove convergence to a stationary point for a class of semilinear parabolic equations on the one dimensional flat torus. The reason for the effectiveness of this property is that it ensures that the zero number behaves as a discrete-valued Lyapunov functional for the evolution. When rephrasing the equation for u as an autonomous one, we obtain a nonlocal dependence of the nonlinearity on u, and therefore it is not possible to directly apply the results of [5]. However, the monotonicity of the zero number is applied at the level of a linear equation of the form (8), so we are able to apply the property in our context and prove the following theorem:

Theorem 5.2. *Let* $\mathbf{u}$ *be a globally Lipschitz solution to system* (2) *with cyclic interactions. Then, there exists some* $\mathbf{v}$ *solution to* (5) *such that* $\omega[\mathbf{u}] = \{\mathbf{v}\}$.

For the case where some $\{u_i > 0\}$ has more than one connected component, analogous remarks hold as in the previous chapters. Namely, the above results hold if the cyclic condition is properly modified to account for all the connected components. Moreover, in this chapter we have only considered the evolution starting from the last extinction time. Ideally, we would like to be able to express conditions for convergence based only on the initial configuration. Since we are not able to know which extinctions occur in the interval $(0, T_1]$, we should formulate an hypothesis that accounts for all the extinctions that may possibly occur. In particular, convergence of an initial state $(\phi_1, ...\phi_m)$ (where each support is connected) is guaranteed if a cyclic condition holds for every subset of indices obtained by $\{1, ..., m\}$ by removing some of the indices.

6. Conclusions

In this thesis, we proved a convergence results for solutions of the system of differential inequalities that enjoy some minimal regularity properties. We believe that in the case of acyclic interactions, even if an energy identity of the kind above does not hold, the the property concerning the monotonicity in time of the number of zeros can prove to be useful in characterizing the possible structure of omega-limit sets. In contrast, the higher-dimensional case with multiple species needs to be treated with a different approach since the monotonicity of the zero number is a genuinely one-dimensional property. Moreover, in [6], rigidly rotating spirals were found to be solutions to the singular problem in two dimensions for a generic number of species, and with linear internal dynamics, also in the case of cyclic interactions. Another natural continuation of this work would be to study the extent to which the behavior of the singular parabolic system helps to understand that of system (1) with large but finite κ. In particular, the interest is in characterizing the possible structure of ω-limit sets for generic coefficients, and moreover in studying whether stationary states or periodic orbits involving more than one non-zero species can be stable, thus representing stationary or dynamic patterns of coexistence.

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