Politecnico di Milano
M. Sc. in Computational Science and Engineering
Master Thesis

Mathematical analysis and numerical modeling of a Cahn-Hilliard-Boussinesq system with logarithmic potential

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A mio nonno Aldo, matematico:

come avresti voluto, sono giunto al traguardo

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Abstract

This thesis aims at investigating some theoretical and numerical properties of the 2D Cahn-Hilliard-Boussinesq equations with logarithmic potential. They consist in the coupling of the Cahn-Hilliard equation for the concentration difference φ of two components of a binary system to the heat-conductive Boussinesq equations for the (volume averaged) fluid velocity $\mathbf{u}$ of the mixture and the temperature θ. The resulting system models the interactions between the thermodynamic transition and the hydrodynamic flow of a compressible binary mixture in a phase separation process. We first prove the existence of weak solutions, with standard boundary conditions for φ and $\mathbf{u}$ (i.e. no-flux and no-slip) and nonhomogeneous Dirichlet boundary conditions for θ. Then we establish the existence of more regular solutions. More precisely, the existence of a quasi-strong solution provided that the initial data for φ and $\mathbf{u}$ are more regular, and the existence of a strong solution if the initial temperature is enough regular as well. We also obtain some stability estimates in case of more regular initial data: a continuous dependence estimate on the initial data which yields, in particular, the weak-strong uniqueness and a stronger stability estimate for strong solutions. In particular, we also find the uniqueness of quasi-strong solutions. We then study the problem numerically by discretizing the equations. We first develop a numerical scheme that we prove to be mass-preserving and stable with respect to the total energy, under suitable conditions on the parameters. Furthermore, we exploit an adaptive timestep since the time scales are quite variable over time. Finally, we implement the proposed algorithm and we numerically simulate some scenarios with various initial temperature fields, by verifying the stability and mass-preserving properties of the scheme.

Sommario

L'obiettivo di questa tesi è quello di investigare alcune proprietà teoriche e analitiche delle equazioni di Cahn-Hilliard-Boussinesq in 2D con potenziale logaritmico. Queste consistono nell'accoppiamento dell'equazione di Cahn-Hilliard, per la differenza di concentrazione φ tra due componenti di un sistema binario, con le equazioni di Boussinesq per la conduzione del calore, per la velocità $\mathbf{u}$ (media sul volume) della miscela e per la temperatura θ. Il sistema risultante modellizza le interazioni tra transizione termodinamica e flusso idrodinamico di una miscela binaria comprimibile durante il processo di separazione di fase. Per prima cosa si dimostra l'esistenza di soluzioni deboli, con condizioni al bordo standard per φ e $\mathbf{u}$ (i.e. assenza di flusso e no-slip) e condizioni di Dirichlet non omogenee per θ. In seguito si stabilisce l'esistenza di soluzioni più regolari. Più precisamente, si prova l'esistenza di una soluzione quasi-forte, purché i dati iniziali per φ e $\mathbf{u}$ siano più regolari, e l'esistenza di una soluzione forte se anche la temperatura iniziale è sufficientemente regolare. Si ottengono anche delle stime di stabilità in caso di dati iniziali più regolari: una stima di dipendenza continua dai dati iniziali che garantisce, in particolare, l'unicità debole-forte e una stima di stabilità per soluzioni forti. In particolare, si mostra anche l'unicità delle soluzioni quasi-forti. Si studia poi numericamente il problema, discretizzando le equazioni: si sviluppa uno schema numerico che si dimostra conservare la massa ed essere stabile rispetto all'energia totale, sotto opportune condizioni sui parametri. Si sfrutta poi un passo temporale adattivo, poiché le scale temporali sono piuttosto variabili nel tempo. In conclusione, si implementa l'algoritmo proposto e si simulano numericamente vari scenari con diversi campi iniziali di temperatura, verificando la stabilità e la proprietà di conservazione della massa.

Contents

Introduction

In many scientific, engineering and industrial applications, for instance in hydrodynamics, the study of the evolution in time of incompressible binary mixtures and their interfacial dynamics plays an important role to understand the behavior of the systems. We can find many applications where the modeling of mixtures of different fluids is needed, such as phase separation, liquid crystals, image processing and, more recently, tumor growth (see, for instance, [19], [33], [35], [36], [38], [47], [48]).

One of the oldest approaches to multi-phase problems (i.e., problems with different components of a mixture) is the phase-field method. According to [42], as early as 1873, the work of Gibbs on thermodynamics already served as a foundation ([59]). The phase-field method works with diffuse interfaces, which means that the transition layer between the phases has a finite size. There is no tracking mechanism for the interface, but the phase state is included implicitly in the governing equations. The interface is associated with a smooth, but highly localized variation of the so-called phase-field variable.

In this introduction, we introduce the Cahn-Hilliard (CH) equation, which is probably the most known mathematical model, for phase separation and then we motivate and formulate the Cahn-Hilliard-Boussinesq equation (CHB), which is the coupling of CH to the heat-conductive Boussinesq equations, system whose analysis is the object of this thesis.

Consider a mixture of two incompatible substances A and B, which is homogeneously distributed and isothermal. Under certain circumstances, namely if the temperature is above a critical threshold T_c, this configuration is stable; however, if suddenly cooled down and kept at $\bar{T} < T_c$, the initially (macroscopically) homogeneous alloy evolves in a way such that A-rich and B-rich regions appear and grow.

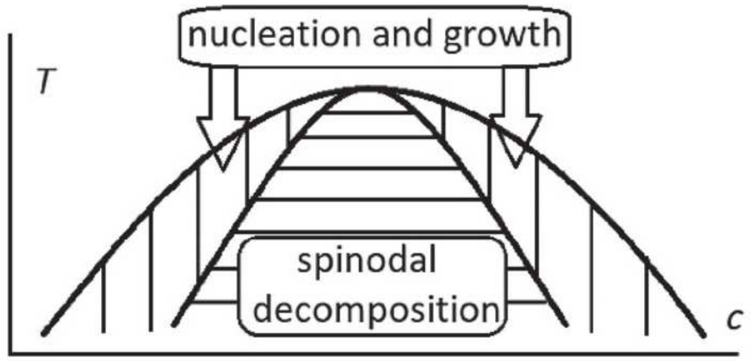


Figure 1: Phase diagram: on the x axis the concentration, on the y axis the temperature is represented, as taken from [84].

We can better describe what happens with the aid of the phase diagram in Figure 1, which is in good agreement with experimental evidence (see [12], [70]). On the x-axis the relative concentration of one of the two substances is represented, while temperature is on the y-axis. The state of the mixture is then efficiently described by the different locations on the graph relatively to the two represented curves.

The coexistence curve, which is the external curve in the graph, separates the diagram in regions where a homogeneous distribution is the only stable configuration (above the curve) and where heterogeneous mixtures are allowed (under the curve); on the points along the curve the mixed and unmixed states are in equilibrium with each other.

On the other hand, the spinodal curve, which is the inner curve in the graph, divides the area under the coexistence curve in regions where the mixed configuration is metastable (that is, stable with respect to small perturbations) and unstable. The distinction in these two cases is due to a difference in the free energy of the configuration: the central region is characterized by the spinodal decomposition phenomenon, which is the phenomenon we want to observe in the numerical simulations of this thesis. It occurs for phases that are thermodynamically unstable, and thus it is spontaneous. On the contrary nucleation happens in the metastable regions, but only if an external source is provided which makes it possible to get over a local maximum in the free energy (see, e.g., [94]).

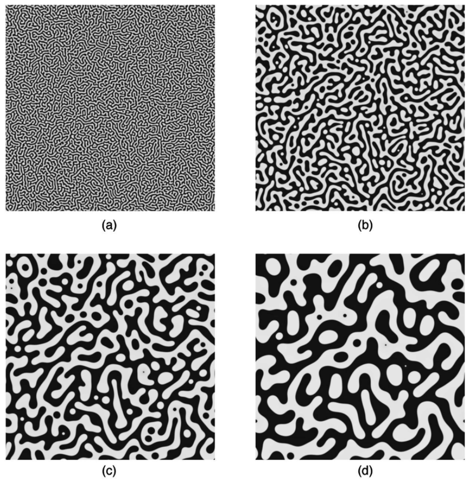


Figure 2: Spinodal decomposition in time, as taken from a numerical simulation in [26].

The evolution in the spinodal decomposition region is that of wave-like concentration fluctuations which in the end form zones of the two phases, with a subsequent coarsening; the process comes to an end when the concentration lays on the intersection of the spinodal curve with the line corresponding to the temperature $\bar{T}$ (see Figure 2). For more information about the process of *phase separation*, we refer the interested reader to, e.g., [63] and [91], and references therein.

The CH equation was introduced in [6] and [21] to model phase transitions in iron alloys and the thermodynamic forces driving phase separation, respectively. We now briefly present it. Let Ω be a bounded domain in $\mathbb{R}^n$, $n = 2, 3$, filled with a binary solution consisting of A and B atoms. We define their relative mass fraction (assumed to be non-uniform) as $\varphi_A(x)$ and $\varphi_B(x)$, with $\varphi_k : \Omega \to [0, 1]$, $k = A,\ B$ and $\varphi_A(x) + \varphi_B(x) \equiv 1$. Considering φ_A and relabeling it as φ, if the mixture is isothermal and the molar volume is uniform and independent on pressure, the system evolves in order to minimize the free energy functional

$$\mathcal{E}(\varphi) = \int_\Omega \Big(\frac{\alpha}{2} |\nabla \varphi|^2 + \Psi(\varphi) \Big)\, dx \tag{1}$$

where $\Psi(\varphi)$ is the Helmholtz free energy density

$$\Psi(\varphi) = 2k_B T_c \varphi(1-\varphi) + k_B T(\varphi \ln(\varphi) + (1-\varphi)\ln(1-\varphi))$$

with k_B as the Boltzman constant, T, T_c the temperature and the critical threshold, respectively. The phase separation process takes place when $T < T_c$, i.e. when Ψ is a double-well function. As described in [85], the term with α, a constant called capillary coefficient, was added in the definition of free energy in order to add concentration gradients to Ψ, which regularize the problem, otherwise ill-posed in the spinodal region, but it is also consequence of experimental evidence, since in the experiments there is an intermediate diffusive stripe, whose thickness is proportional to $\sqrt{\alpha}$. The capillary coefficient is assumed to be very small, so the first term in (1) is not negligible only where strong gradients of concentration, i.e., at interfaces, are present, as explained in [99]. In general for the mathematical treatment of this kind of equations, it is used the *order parameter*, $\varphi(x) = \varphi_A(x) - \varphi_B(x)$, such that $\varphi : \Omega \to [-1, 1]$, instead of the relative concentration. The values -1 and 1 represent the pure phases. Nevertheless, in the numerical analysis section we shall use the relative concentration, as done, e.g., in [64]. It can be shown that, with this substitution, up to a multiplicative constant which therefore does not change anything in the description of the problem, (1) holds unchanged whereas the function Ψ can be rewritten as

$$\Psi(s) = \frac{\bar{\alpha}}{2}((1+s)\ln(1+s) + (1-s)\ln(1-s)) - \frac{\alpha_0}{2}s^2 \quad \forall s \in [-1, 1] \tag{2}$$

with $\bar{\alpha}$ such that $0 < \bar{\alpha} < \alpha_0$, constants related to the temperature of the mixture, thus related to T and T_c.

The potential defined in this way is called *singular*, whereas many authors (see, e.g., [51]) considered a proper approximation, which avoids the fact that Ψ' is unbounded at the pure phases -1 and 1: namely, the significant potential is considered to be still a double-well, but with the two local minima coinciding with the pure phases. The most common choice is polynomial of even degree, like the case $\Psi(s) = \frac{1}{4}(s^2 - 1)^2$, which is compared with the logarithmic potential in Figure 3. However, in the case of polynomial potentials, it is worth recalling that it is not possible to guarantee the existence of physical solutions, that is, solutions for which $-1 \le \varphi(x,t) \le 1$.

Following again [85], we get a differential description of the phenomenon of the phase separation as

$$\partial_t \varphi + \operatorname{div} \mathbf{J} = 0 \ \text{ in } \Omega \times (0, T), \tag{3}$$

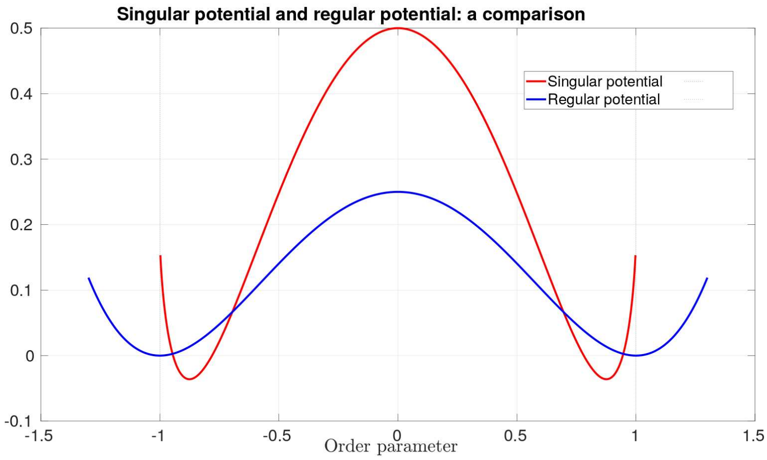


Figure 3: Logarithmic and polynomial potentials (the logarithmic potential is vertically translated to have a more clear graphical representation): we see that the polynomial potential gives rise to nonphysical solutions.

where φ is the order parameter and $\mathbf{J}$ is the diffusional flux given by Fick's law,

$$\mathbf{J} = -M(\varphi)\nabla \frac{\delta \mathcal{E}(\varphi)}{\delta \varphi} = -M(\varphi)\nabla(-\alpha \Delta \varphi + \Psi'(\varphi)),$$

where $\frac{\delta \mathcal{E}(\varphi)}{\delta \varphi}$ is the variational derivative of $\mathcal{E}(\varphi)$. The function $M(\varphi)$ is the mobility of the substances and in this thesis it will be considered as a unitary constant (see, for instance, [29] and [39] for an analysis of the case of non constant and degenerate, i.e., vanishing at the pure phases).

In order to simplify the representation, which otherwise must contain the bilaplacian operator, the equation (3) is usually written introducing the chemical potential μ, obtaining the complete CH equation:

$$\begin{cases} \partial_t \varphi = \text{div } (M(\varphi)\nabla \mu) & \text{in } \Omega \\ \mu = -\alpha \Delta \varphi + \Psi'(\varphi) & \text{in } \Omega. \end{cases} \tag{4}$$

with the initial condition φ_0 and two boundary conditions, since the system is of fourth order. In this thesis, as commonly done in the literature, the boundary conditions are the following:

$$\mathbf{n} \cdot M(\varphi)\nabla \mu = 0, \qquad \partial_n \varphi = 0 \qquad \text{on } \partial\Omega, \tag{5}$$

with $\mathbf{n}$ as the outer normal vector. These conditions are often used, since they guarantee mass conservation. Since in this thesis we only considered the case of constant mobility, for the sake of simplicity, from now on we will set $M(\varphi) = 1$.

As introduced in [2], a system describing the flow of two viscous incompressible Newtonian fluids of the same density but different viscosity can be described by means of the coupling of CH equation with a hydrodynamic model. Indeed, although it is assumed that the fluids are macroscopically immiscible, the model takes a partial mixing on a small length scale measured by the aforementioned parameter $\alpha > 0$. Therefore the classical sharp interface between both fluids is replaced by an interfacial region and an order parameter related to the concentration difference of both fluids is introduced, leading to the coupling with CH equation. The model goes back to [67] and is known as *model H*. In [66], the authors gave a continuum mechanical derivation based on the concept of microforces. These have been successfully used during last years to describe flows of two or more fluids beyond the occurrence of topological singularities of the separating interface (for example, coalescence or formation of droplets). We refer to [7] for a review on that topic.

This model leads to the so called incompressible Navier-Stokes-Cahn-Hilliard (NSCH) system

$$\begin{cases} \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = \Delta \mu \\ \mu = \ -\alpha \Delta \varphi + \Psi'(\varphi) \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \operatorname{div}(\nu(\varphi) \nabla \mathbf{u}) = \mu \nabla \varphi \\ \operatorname{div}\, \mathbf{u} = 0 \end{cases} \tag{6}$$

in $\Omega \times (0, T]$, subject to the boundary and initial conditions

$$\begin{cases} \mathbf{u} = \mathbf{0}, \quad \partial_{\mathbf{n}} \varphi = 0, \quad \partial_{\mathbf{n}} \mu = 0 \quad \text{on } \partial\Omega \times (0, T) \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad \varphi(\cdot, 0) = \varphi_0 \quad \text{in } \Omega. \end{cases} \tag{7}$$

Here, $\mathbf{u}$ is the volume averaged velocity, p the pressure, φ the order parameter related to the concentration of the fluids, Ψ is the double-well potential defined in (2), or a suitable smooth approximation, and $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a suitable bounded domain. Moreover, $\nu(\varphi) > 0$ is the viscosity of the mixture. We consider $T > 0$.

Now, assuming that ν_1 and ν_2 are the viscosities of the two homogeneous fluids, the viscosity of the mixture is modeled by the concentration dependent term $\nu = \nu(\varphi)$. In the unmatched viscosity case ($\nu_1 \neq \nu_2$), a typical form for ν is the linear combination (see, e.g., [77]):

$$\nu(z) = \nu_1 \frac{1+z}{2} + \nu_2 \frac{1-z}{2} \quad \forall z \in [-1,1] \tag{8}$$

The particular case $\nu_1 = \nu_2$ is called matched viscosity case, and ν is a positive constant.

In this thesis we consider a coupling of the CH equation with a different hydrodynamic model, which is represented by the heat-conductive Boussinesq equations. Such equations account for the presence of a further variable: the temperature θ.

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \operatorname{div}(\nu(\varphi, \theta)\nabla \mathbf{u}) = \theta \mathbf{e}_n \\ \operatorname{div} \, \mathbf{u} = 0 \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0, \end{cases} \tag{9}$$

in $\Omega \times (0,T]$, where $\kappa > 0$ is the thermal conductivity, possibly depending on θ. The kinematic viscosity ν could depend on the temperature itself and $\mathbf{e}_n = (0,0,1)$ if $n = 3$, and $\mathbf{e}_n = (0,1)$ if $n = 2$.

System (9) describes the motion of an incompressible two-phase flow subjected to convective heat transfer under the influence of gravitational force, which is closely related to the studies of 3D incompressible flows (see, e.g., [11] and [88]) and has been widely studied in the literature. We refer to [22], [24], [25], [72], [73] and the references therein for the Cauchy problem on the whole space, and to [14], [79], [100] and the references therein for initial-boundary value problems, where global existence and large time behavior of solutions to 2D Boussinesq equations with full or partial viscosity terms are investigated, whereas for a regularity analysis in bounded domains we recall [74].

In a bounded domain $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$, with smooth boundary $\partial\Omega$, the resulting coupled system reads as follows:

$$\begin{cases} \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = \Delta \mu \\ \mu = \ -\alpha \Delta \varphi + \Psi'(\varphi) \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \operatorname{div}(\nu(\varphi, \theta)\nabla \mathbf{u}) = \mu \nabla \varphi + \theta \mathbf{e}_n \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \operatorname{div}(\kappa(\theta)\nabla \theta) = 0 \\ \operatorname{div}\ \mathbf{u} = 0 \end{cases} \tag{10}$$

in $\Omega \times (0, T]$, equipped with the boundary conditions

$$\mathbf{u} = \mathbf{0} \quad \partial_{\mathbf{n}} \varphi = 0 \quad \partial_{\mathbf{n}} \mu = 0 \quad \theta = g(t) \quad \text{on } \partial\Omega \times (0, T) \tag{11}$$

being $\mathbf{n}$ the outward normal to $\partial\Omega$, and the initial conditions

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \varphi(0) = \varphi_0 \quad \mu(0) = \mu_0 \quad \theta(0) = \theta_0 \quad \text{in } \Omega. \tag{12}$$

where g is a sufficiently regular function defined on $\partial\Omega \times [0, T]$.
The main focus of this thesis is the analysis of this system, called Cahn-Hilliard-Boussinesq system (CHB), in a two dimensional bounded domain $\Omega \subset \mathbb{R}^2$.

We stress again that, as noticed in [86], hydrodynamic models like the CHB system play an important role in the mathematical study of multi-phase flows, since the applications of these systems cover a very wide range of physical objects, such as complicated phenomena in fluid mechanics involving phase transition, two-phase flow under shear through an order parameter formulation (see, e.g., [17]), tumor growth (see, e.g., [19],[33], [35] and [38]), cell sorting ([9]), and two phase flows in porous media (see, e.g., [30] and [75]).

We now give another interesting motivation to study system (10): indeed, apart from the physical relevance of the system itself, it can be regarded as a suitable approximation of the compressible Navier-Stokes-Cahn-Hilliard system, as obtained in [85] with a rigorous physical derivation.

In particular, this system reads: in $\Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$, $n = 2,3$

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho + \operatorname{div}(\mathbf{u}) = 0 \\ \rho \partial_t \mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \operatorname{div}(\nu(\varphi) D\mathbf{u}) - \nabla(\operatorname{div} \mathbf{u}) = -\operatorname{div}(\rho \nabla \varphi \otimes \nabla \varphi) + \rho \mathbf{g} \\ \rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla \varphi = \Delta \left(-\frac{\alpha}{\rho} \operatorname{div}(\rho \nabla \varphi) + \Psi'(\varphi) \right), \end{cases} \tag{13}$$

with suitable boundary and initial conditions. Here ρ is the fluid density, $\mathbf{u}$ is the mean velocity, φ the aforementioned order parameter, Ψ is the double-well potential, α is the capillary coefficient, and $\mathbf{g} = -g\mathbf{e}_n$ is the gravitational force.

We can now apply a variational method in order to obtain a simpler system of equations ([62]). We consider the stationary solution:

$$\rho^* = const \neq 0, \ \mathbf{u}^* = 0, \ \varphi^* = 0, \ p^*$$

and write the system for the perturbation

$$(\rho + \rho^*, \ \mathbf{u}, \ \varphi, \ p + p^*).$$

From the first equation in (13) we obtain

$$\partial_t(\rho + \rho^*) + \mathbf{u} \cdot \nabla(\rho + \rho^*) = -(\rho + \rho^*) \operatorname{div} \mathbf{u},$$

implying that

$$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = -\rho^* \operatorname{div} \mathbf{u} - \rho \operatorname{div} \mathbf{u}.$$

Now, since this equation holds for any $\rho^* = const \in \mathbb{R}^+$, we can decouple the following two contributions, the first one which is a first order equation in ρ, the second one which is a zero order equation in ρ and ρ^*:

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = -\rho \operatorname{div} \mathbf{u} \\ \operatorname{div} \mathbf{u} = 0 \end{cases}$$

and thus

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0 \\ \text{div } \mathbf{u} = 0. \end{cases} \tag{14}$$

From the second equation of (13), we deduce, after performing the perturbation argument, that

$$\begin{aligned} &(\rho + \rho^*)\partial_t \mathbf{u} + (\rho + \rho^*)(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla(p + p^*) - \text{div}(\nu(\varphi) D\mathbf{u}) - \nabla(\text{div } \mathbf{u}) \\ &= -\text{div}\left((\rho + \rho^*)\nabla\varphi \otimes \nabla\varphi\right) + (\rho + \rho^*)\mathbf{g}. \end{aligned}$$

We recall that, being $(\rho^*, \mathbf{u}^*, \varphi^*, p^*)$ a stationary solution of (13), it holds the hydrostatic balance

$$\nabla p^* = \rho^* \mathbf{g}.$$

Thus we get, remembering that we found, in (14), div $\mathbf{u} = 0$,

$$\begin{aligned} &(\rho + \rho^*)\partial_t \mathbf{u} + (\rho + \rho^*)(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \text{div}\left(\nu(\varphi) D\mathbf{u}\right) \\ &= -\rho^* \text{div}(\nabla\varphi \otimes \nabla\varphi) - \text{div}(\rho \nabla\varphi \otimes \nabla\varphi) + \rho \mathbf{g}. \end{aligned}$$

Dividing by ρ^* we reach

$$\begin{aligned} &\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{\rho}{\rho^*}\partial_t \mathbf{u} + \frac{\rho}{\rho^*}(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho^*}\nabla p - \text{div}\left(\frac{\nu(\varphi)}{\rho^*} D\mathbf{u}\right) \\ &= -\text{div}(\nabla\varphi \otimes \nabla\varphi) - \text{div}\left(\frac{\rho}{\rho^*}\nabla\varphi \otimes \nabla\varphi\right) + \frac{\rho}{\rho^*}\mathbf{g}. \end{aligned}$$

Since ρ^* is arbitrary, we can take it arbitrarily large, such that $\rho << \rho^*$, namely $\frac{\rho}{\rho^*} \approx 0$, and we can neglect all the terms with this coefficient in front, except the gravitational one, because it is linear and for an energy budget argument, finding:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho^*}\nabla p - \text{div}\left(\frac{\nu(\varphi)}{\rho^*} D\mathbf{u}\right) = -\text{div}(\nabla\varphi \otimes \nabla\varphi) + \frac{\rho}{\rho^*}\mathbf{g}. \tag{15}$$

In conclusion, from the third equation in (13), we obtain

$$(\rho + \rho^*)\partial_t \varphi + (\rho + \rho^*)\mathbf{u} \cdot \nabla\varphi = \Delta\left(-\frac{\alpha}{\rho + \rho^*}\text{div}\left((\rho + \rho^*)\nabla\varphi\right) + \Psi'(\varphi)\right).$$

Dividing by ρ^* we get

$$\left(1 + \frac{\rho}{\rho^*}\right)\partial_t \varphi + \left(1 + \frac{\rho}{\rho^*}\right)\mathbf{u} \cdot \nabla\varphi = \Delta\left(-\frac{\alpha}{\rho^*}\frac{1}{1 + \frac{\rho}{\rho^*}}\text{div}\left((1 + \frac{\rho}{\rho^*})\nabla\varphi\right) + \frac{1}{\rho^*}\Psi'(\varphi)\right)$$

where we exploited the fact that

$$-\frac{1}{\rho+\rho^*} = -\frac{1}{\rho^*}\frac{1}{1+\frac{\rho}{\rho^*}}$$

By using again that $\dfrac{\rho}{\rho^*} \approx 0$, we find

$$\partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = \Delta\left(-\frac{\alpha}{\rho^*}\mathrm{div}(\nabla\varphi) + \frac{1}{\rho^*}\Psi'(\varphi)\right),$$

namely

$$\partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = \Delta(-\frac{\alpha}{\rho^*}\Delta\varphi + \frac{1}{\rho^*}\Psi'(\varphi)). \tag{16}$$

Putting together equations (14), (15) and (16), we are then led to formulate an equivalent version of system (13), always with suitable boundary and initial conditions:

$$\begin{cases} \partial_t\rho + \mathbf{u}\cdot\nabla\rho = 0 \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \dfrac{1}{\rho^*}\nabla p - \mathrm{div}\left(\dfrac{\nu(\varphi)}{\rho^*}D\mathbf{u}\right) = -\mathrm{div}(\nabla\varphi\otimes\nabla\varphi) - \dfrac{\rho}{\rho^*}g\mathbf{e}_n \\ \partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = \Delta\left(-\dfrac{\alpha}{\rho^*}\Delta\varphi + \dfrac{1}{\rho^*}\Psi'(\varphi)\right) \\ \mathrm{div}\ \mathbf{u} = 0. \end{cases} \tag{17}$$

Comparing it to the CHB system (10), we can find that, up to multiplicative constants, irrelevant from the point of view of the mathematical analysis, considering θ to be the density ρ, we obtain the same system, for $\kappa = 0$, i.e. for vanishing thermal conductivity. Actually, the two systems differ for the term $-\mathrm{div}(\nabla\varphi\otimes\nabla\varphi)$, which substitutes the term $\mu\nabla\varphi$ in (10), with μ the chemical potential, but this is just an equivalent formulation, since we have

$$\mu\nabla\varphi = (-\alpha\Delta\varphi + \Psi'(\varphi))\nabla\varphi$$

and

$$\mu\nabla\varphi = \nabla\left(\frac{\alpha}{2}|\nabla\varphi|^2 + \Psi(\varphi)\right) - \alpha\ \mathrm{div}(\nabla\varphi\otimes\nabla\varphi)$$

and then the weak formulations of the systems are the same, as we can see by a simple integration by parts, taking into account the boundary conditions of (10). This is a further strong motivation to study the CHB system, because it can give information about different problems arising from different contexts, namely the solution to the system (17) could be seen as the limit of the solutions to the CHB system when $\kappa \to 0$ (considering the temperature as the density): analyzing the properties of the CHB system solutions could thus give important information also for this system.

We recall that the literature on the incompressible NSCH system is rather vast. For instance, the system has been widely studied in the case of a regular approximation of the logarithmic potential. In the matched viscosity case we refer the reader to [10], [17], [53], [54], [55] and [60] (see also [16], [23] and [57] for the analysis of similar systems). In the unmatched viscosity case, the author in [17] proved the global existence of weak solutions and the existence and uniqueness of strong solutions (global if $n = 2$, local if $n = 3$). The NSCH system with unmatched viscosities and logarithmic potential has been studied in [2], where existence of global weak (physical) solutions and existence and uniqueness of strong solutions (global if $n = 2$, local if $n = 3$) are shown (see [2], Theorem 1 and 2), and in [61], where in dimension two it is proven the uniqueness of weak (physical) solutions and the global existence and uniqueness of strong solutions under regular initial conditions, together with long time behavior properties, whereas in dimension three it is proven the local existence and uniqueness of strong solutions when the initial data are sufficiently regular. Finally, we refer to [1] for the existence of weak solutions for the corresponding compressible model of NSCH, like system (13).

On the contrary, not so many papers have been devoted so far to the analysis of the CHB system. In [101] the author proves, in two-dimensional bounded domains, the global existence and uniqueness of smooth solutions to problem (10) with smooth initial data $\mathbf{u}_0, \theta_0 \in H^3(\Omega)$ and $\varphi_0 \in H^5(\Omega)$, considering constant Dirichlet boundary conditions for the temperature and no-penetration boundary condition for the velocity ($\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega \times (0, T)$), since the fluid is considered inviscid ($\nu = 0$), together with a regular potential $\Psi \in C^6(\mathbb{R})$ (see [101], Theorem 1.1 for the details). The same author then studied in [102] the large time asymptotic behavior of the solutions, under the same hypotheses. In [46] the authors considered the vanishing limit for a 2D Cahn-Hilliard-Navier-Stokes system with a slip boundary condition, and in a similar way they considered the inviscid CHB system in [45], finding some blow-up criteria of smooth solutions for three dimensional bounded domains, proving that a smooth solution of the 3D CHB system with zero viscosity in a bounded domain breaks down if a certain norm of vorticity blows up at the same time. They always consider in the analysis the regular polynomial potential

$$\Psi(\varphi) = \frac{1}{4}(\varphi^2 - 1)^2, \tag{18}$$

where φ is the order parameter.

Few works dealing with the CHB system with nonzero viscosity are available in the literature, though: in [86] the existence and uniqueness of a weak solution $(\mathbf{u}, \varphi, \theta)$, with no-slip boundary conditions for velocity and homogeneous Neumann conditions for temperature, in two-dimensional bounded domains with smooth boundary, is established and proved, together with some further regularity properties when the initial data are sufficiently regular (namely when at least $\varphi_0 \in H^4(\Omega)$ together with $\mathbf{u}_0$ and θ_0 belonging to suitable spaces, see [86], Theorem 1.2) and again the analysis is performed in the case of the polynomial potential as in (18). In conclusion, in [44], vanishing thermal conductivity κ limit for the 2D CHB system in a bounded domain with no-slip boundary conditions for the velocity and homogeneous Dirichlet boundary conditions for the temperature is studied, always considering the potential (18) in the analysis. At this stage we note that so far some important issues are still unsolved, in particular the analysis of the CHB system with the physically relevant singular potential (2). No results about existence or uniqueness of weak solutions or strong solutions are available. Moreover the nonhomogeneous Dirichlet conditions for the temperature field has not been considered so far.

The aim of this work is to give an answer to the aforementioned open questions. Namely, our main results for the CHB system in a two-dimensional bounded domain with singular potential are the following:

(a) The existence of weak *physical* solutions in the unmatched viscosity case, with no-slip boundary conditions for the velocity and nonhomogeneous Dirichlet boundary conditions for temperature.

(b) The existence, in the matched viscosity case, of more regular solutions, namely of a quasi-strong solution (see Definition 1.2), and of a strong solution (see Definition 1.3), when the initial data are sufficiently regular.

(c) In the matched viscosity case, we obtain some stability estimates in different norms, depending on the regularity of the initial data, from which we obtain a weak-strong uniqueness result and, in particular, the uniqueness of the quasi-strong and strong solutions.

For what concerns the numerical analysis and approximation of the phase field model, in the literature we can find a large number of studies: we refer to [8], [39], [40], [41], [64], [87] and the references therein. About the NSCH system instead, we refer the reader to [27], [28], [49], [50], [71] and [95]: in particular stability and convergence analysis and numerical simulations are performed. We then cite [77] for multicomponent fluid flows, [76] for a multigrid approach applied to CH fluids and [15] for a study on the advective CH equation by means of Isogeometric Analysis. Nevertheless, to the best of our knowledge, results concerning and specifically addressing the numerical approximation of the CHB system are not available in literature yet. Here we propose a numerical scheme to address this not yet studied system by means of finite elements, based on an extension of the one employed for the only CH equation in [64]. Differently from the scheme in [64], in this scheme we consider also the velocity $\mathbf{u}$ and the temperature θ. We prove that the scheme is mass-preserving and energetically stable, under some conditions on the parameters κ and ν. The total energy is defined as

$$E = \frac{1}{2}||\mathbf{u}||^2 + \frac{1}{2}||\theta||^2 + \frac{\alpha}{2}||\nabla\varphi||^2 + \int_\Omega \Psi(\varphi)dx. \tag{19}$$

Energy stability means that the total energy of the system does not increase in time, as it is physically necessary, at least for the homogeneous Dirichlet boundary conditions for the temperature. In order to reduce the computation time, we also introduce an adaptive time step, which should exploit the different time scales characteristic of the CHB system. The time adaptivity does not change the properties of the scheme, since they do not depend on the size of the timestep. By means of the software FreeFem++ ([69]), we simulate five different cases, corresponding to five different initial conditions and verify the main properties of the scheme: conservation of mass and energy stability.

Thus, the plan of the thesis is the following:

- In Chapter 1 we introduce the functional spaces and the main assumptions on the system, leading to the definition of weak formulation of the problem. We then conclude with the definitions of quasi-strong and strong solutions.

- In Chapter 2 we state the theorems of existence of weak, quasi-strong and strong solutions, together with the stability estimates leading to the uniqueness of the quasi-strong and strong solutions.

- In Chapter 3 we give the proofs of the existence theorems stated in Chapter 2.

- In Chapter 4 we give the proofs of the stability estimates, and uniqueness theorems, stated in Chapter 2.

- In Chapter 5 we realize the numerical approximation of the CHB system in space, by means of Finite Elements Method, and in time. We perform the numerical analysis of this approximation, concentrating on its stability, in terms of total energy, and accuracy.

- In Chapter 6 we perform and discuss five simulations in order to verify the numerical properties highlighted in Chapter 5.

- "Conclusions and future work" contains some issues which are worth investigating but have not been explored in this thesis.

- Appendix A reports some basic tools from functional analysis used in the thesis. Appendix B is devoted to some results on three stationary problems which play a basic role in the proofs.

Chapter 1

Weak formulation and notions of solution

1.1 Functional setup

Here we introduce notation and the functional spaces which are needed to introduce the weak formulation of problem (10). Let Ω be a smooth bounded domain of $\mathbb{R}^2$.

- For the velocity field we set:

  $$\mathbf{H}_\sigma = \overline{\{\mathbf{u} \in C_0^\infty(\Omega)^2 : \operatorname{div}(\mathbf{u}) = 0\}}^{L^2(\Omega)^2} \qquad \mathbf{V}_\sigma = \overline{\{\mathbf{u} \in C_0^\infty(\Omega)^2 : \operatorname{div}(\mathbf{u}) = 0\}}^{H^1(\Omega)^2}$$

  In the sequel, we denote by $(\cdot,\cdot)$ and $||\cdot||$ the norm and the inner product, respectively, in $\mathbf{H}_\sigma$ and we consider in $\mathbf{V}_\sigma$, by means of Poincaré's inequality (A.1), the inner product $(\mathbf{u},\mathbf{v})_{\mathbf{V}_\sigma} = (\nabla\mathbf{u},\nabla\mathbf{v})$ and the norm $||\mathbf{v}||_{\mathbf{V}_\sigma} = ||\nabla\mathbf{v}||$.

- For the temperature field we define:

  $$H = L^2(\Omega), \qquad V_\theta = H_0^1(\Omega) \qquad V_\theta^2 = V_\theta \cap H^2(\Omega).$$

  We denote by $(\cdot,\cdot)$ and $||\cdot||$ also the norm and the inner product, respectively, in H.

- For the concentration field φ we set:

  $$V = H^1(\Omega), \quad V_2 = \{v \in H^2(\Omega) : \partial_\mathbf{n} v = 0 \text{ on } \partial\Omega\}.$$

We also denote by $(\cdot,\cdot)_1$ and $||\cdot||_1$ (or also $||\cdot||_V$) the inner product and the norm in V ($||v||_1^2 = ||v||^2 + ||\nabla v||^2$).

- For any $f \in L^1(\Omega)$ we define its spatial average $\bar{f} = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega f \, d\Omega$.
- We define, for any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in [H^1(\Omega)]^2$:

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, dx.$$

We take a slight generalization of the logarithmic potential Ψ, namely a quadratic perturbation of a singular (strictly) convex function in the closed interval [-1, 1].

More precisely, we consider

$$\Psi(s) = F(s) - \frac{\alpha_0}{2}s^2 \tag{1.1}$$

where the convex part F, extended by continuity at -1 and 1, belongs to $C([-1,1]) \cap C^3(-1,1)$ and fulfills

$$\lim_{s\to-1} F'(s) = -\infty \qquad \lim_{s\to 1} F'(s) = +\infty \qquad F''(s) \geq \bar{\alpha} \quad \forall\, s \in (-1,1),$$

namely we consider a double well potential (see the Introduction), assuming $\tilde{\alpha} = \alpha_0 - \bar{\alpha} > 0$. This means that

$$\Psi''(s) \geq -\tilde{\alpha} \qquad \forall s \in (-1,1). \tag{1.2}$$

We also extend $F(s) = +\infty$ for any $s \notin [-1,1]$.

Notice that the above assumptions imply that there exists $s_0 \in (-1,1)$ such that $F'(s_0) = 0$. Without loss of generality, we assume that $s_0 = 0$ and that $F(s_0) = 0$ as well. In particular, this entails that $F(s) \geq 0$ for any $s \in [-1,1]$. Moreover we require that F'' is convex and

$$F''(s) \leq Ce^{C|F'(s)|} \qquad \forall s \in (-1,1) \tag{1.3}$$

for some positive constant C. Also, we assume that there exists $\gamma \in (0,1)$ such that F'' is nondecreasing in $[1-\gamma, 1)$ and nonincreasing in $(-1, -1+\gamma]$. These hypotheses are fulfilled by the potential in (2), which is

$$\Psi(s) = \frac{\bar{\alpha}}{2}\left((1+s)\ln(1+s) + (1-s)\ln(1-s)\right) - \frac{\alpha_0}{2}s^2 \quad \forall s \in [-1,1] \tag{1.4}$$

with $\bar{\alpha}$ such that $0 < \bar{\alpha} < \alpha_0$.

We now consider the following properties for the kinematic viscosity ν and the thermal conductivity κ:
let $\nu : \mathbb{R}^2 \to \mathbb{R}$ and $\kappa : \mathbb{R} \to \mathbb{R}$ two globally Lipschitz functions, such that:

$$0 < \nu_* \leq \nu(z_1, z_2) \leq \nu^* \quad \forall\ (z_1, z_2) \in \mathbb{R}^2 \tag{1.5}$$

and

$$0 < k_* \leq \kappa(z) \leq k^* \quad \forall z \in \mathbb{R} \tag{1.6}$$

for some positive values $\nu_*,\ \nu^*,\ k_*$ and k^*.

We notice that the viscosity function (8) can be easily extended on the whole $\mathbb{R}$ in such way to comply (1.5).

We are now ready to define the weak formulation of the problem.

1.2 Weak formulation

We can now list the assumptions on the thermal conductivity κ, the kinematic viscosity ν, the boundary values and the initial conditions.

(H_1) κ and ν are globally Lipschitz functions fulfilling (1.5) and (1.6),

(H_2) the boundary value g satisfies $g \in L^4(0, T; H^{1/2}(\partial\Omega))$ and $\partial_t g \in L^2(0, T; H^{1/2}(\partial\Omega))$,

(H_3) $\varphi_0 \in V \cap L^\infty(\Omega)$ with $||\varphi_0||_{L^\infty(\Omega)} \leq 1$, $|\bar{\varphi}_0| < 1$,

(H_4) $\mathbf{u}_0 \in \mathbf{H}_\sigma$,

(H_5) $\theta_0 \in H$.

Definition 1.1. Weak solution
Let hypotheses (H_1)-(H_5) be satisfied. Given $T > 0$, a triple ($\mathbf{u}$, φ, θ) is a weak solution on [0,T] if

- $\mathbf{u} \in L^\infty(0, T; \mathbf{H}_\sigma) \cap L^2(0, T; \mathbf{V}_\sigma)$ and $\partial_t \mathbf{u} \in L^2(0, T; \mathbf{V}'_\sigma)$;

- $\varphi \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \subset L^2(0,T;V)$ and $\partial_t\varphi \in L^2(0,T;V')$, $\varphi \in L^\infty(\Omega \times (0,T))$ and $|\varphi(x,t)| < 1$ a.e. $(x,t) \in \Omega \times (0,T)$;

- $\theta \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $\theta = g$ a.e. on $\partial\Omega \times (0,T)$ in the sense of traces and $\partial_t\theta \in L^2(0,T;V'_\theta)$;

$$< \partial_t\mathbf{u}, \mathbf{w} > + b(\mathbf{u},\mathbf{u},\mathbf{w}) + (\nu(\varphi,\theta)\nabla\mathbf{u}, \nabla\mathbf{w}) = -(\varphi\nabla\mu, \mathbf{w}) + (\theta, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_\sigma \tag{1.7}$$

$$< \partial_t\varphi, v > + (\nabla\mu, \nabla v) + (\mathbf{u} \cdot \nabla\varphi, v) = 0 \quad \forall v \in V \tag{1.8}$$

$$< \partial_t\theta, \xi > + (\kappa(\theta)\nabla\theta, \nabla\xi) + (\mathbf{u} \cdot \nabla\theta, \xi) = 0 \quad \forall \xi \in V_\theta \tag{1.9}$$

for almost every $t \in (0,T)$;

- $\mu = -\alpha\Delta\varphi + \Psi'(\varphi)$ a.e. in $\Omega \times (0,T)$ with $\mu \in L^2(0,T;V)$;
- $\mathbf{u}(0) = \mathbf{u}_0 \qquad \varphi(0) = \varphi_0 \qquad \theta(0) = \theta_0$.

.

Remark 1.2.1. The initial conditions mean that, respectively, in L^2 norms,

$$\lim_{t\to 0} ||\mathbf{u}(t) - \mathbf{u}_0|| = 0 \qquad \lim_{t\to 0} ||\varphi(t) - \varphi_0|| = 0 \qquad \lim_{t\to 0} ||\theta(t) - \theta_0|| = 0. \tag{1.10}$$

Indeed, $\mathbf{u} \in C([0,T], \mathbf{H}_\sigma)$, $\varphi \in C([0,T], H)$ and $\theta \in C([0,T], H)$ by continuous embeddings of Lemma A.2.2.

Remark 1.2.2. Notice that any φ_0 in the class of admissible initial conditions has finite energy $\mathcal{E}(\varphi_0) < \infty$. Indeed, by $||\varphi_0||_{L^\infty(\Omega)} \leq 1$ we easily infer that $\Psi(\varphi_0) \in L^1(\Omega)$, where

$$\mathcal{E}(\varphi) = \int_\Omega (\frac{\alpha}{2}|\nabla\varphi|^2 + \Psi(\varphi))dx. \tag{1.11}$$

The assumption on the total mass $|\bar{\varphi}_0| < 1$, however, prevents the existence of the pure phases (i.e. $\varphi_0 \equiv 1$ or $\varphi_0 \equiv -1$). Besides, we notice that any solution satisfies the mass conservation property (by testing equation (1.8) against $v = 1$), namely

$$\overline{\varphi}(t) = \overline{\varphi}_0(t) \quad \forall t \geq 0.$$

Remark 1.2.3. As customary, the pressure term is dropped in the weak formulation. The pressure can be recovered (up to a constant) thanks to the classical de Rham's theorem (see [18], [93] or [97]): there exists, up to an additive constant, the pressure in $L^2(0,T,H)$ such that, in the distributional sense, given

$$\mathcal{S} = -\partial_t \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathrm{div}(\nu(\varphi,\theta)\nabla\mathbf{u}) + \mu\nabla\varphi + \theta\mathbf{e}_2 \in L^2(0,T;\mathbf{V}'_\sigma), \tag{1.12}$$

since as will be clear from the proof, all the terms belong to $L^2(0,T,\mathbf{V}'_\sigma)$, we have

$$\nabla p = \mathcal{S} \tag{1.13}$$

in the distributional sense, meaning that

$$(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\nabla\varphi - \theta\mathbf{e}_2, \chi) + (\nu(\varphi,\theta)\nabla\mathbf{u}, \nabla\chi) - (p, \mathrm{div}\chi) = 0 \quad \forall\ \chi \in C_0^\infty(\Omega)$$

due to the fact that

$$< S, \mathbf{v} >_{([H_0^1(\Omega)]^2)', [H_0^1(\Omega)]^2} = 0 \text{ for every } \mathbf{v} \in \mathbf{V}_\sigma,$$

that is for every $\mathbf{v} \in [H_0^1(\Omega)]^2$ such that div $\mathbf{v} = 0$.

Remark 1.2.4. Due to regularity estimates, since $\mu \in L^2(0,T;V)$ we deduce from the definition of μ itself and from (B.7) that $\varphi \in L^2(0,T;W^{2,p}(\Omega))$, where $2 \le p < \infty$.

1.3 More regular solutions

If we require more regularity on the initial data, we are able to define other two notions of solution: the first one is the quasi-strong solution.

Definition 1.2. Quasi-strong solution

A weak solution in the sense of Definition 1.1 is a quasi-strong solution if the CH equation and the NS system are satisfied almost everywhere and

- $\mathbf{u} \in L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{W}_\sigma) \cap H^1(0,T;\mathbf{H}_\sigma)$

- $\varphi \in L^\infty(0,T;W^{2,p}(\Omega)) \cap L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap H^1(0,T;V)$, with $2 \leq p < \infty$.

.

Remark 1.3.1. Since we have that $\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;V')$, we also get $\partial_n \mu = 0$ almost everywhere in $\partial\Omega \times (0,T)$.

Further regularity of the initial temperature leads to the notion of:

Definition 1.3. Strong solution

A weak solution in the sense of Definition 1.1 is a strong solution if it satisfies almost everywhere all the equations of the CHB system, and

- $\mathbf{u} \in L^\infty(0,T;\mathbf{V}_\sigma) \cap L^2(0,T;\mathbf{W}_\sigma) \cap H^1(0,T;\mathbf{H}_\sigma)$;

- $\varphi \in L^\infty(0,T;W^{2,p}(\Omega)) \cap L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap H^1(0,T;V)$ with $|\varphi(x,t)| < 1$ a.e. $(x,t) \in \Omega \times (0,T)$, where $2 \leq p < \infty$;

- $\theta \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$, $\theta = g$ a.e. on $\partial\Omega \times (0,T)$ in the sense of traces and $\partial_t \theta \in L^2(0,T;H)$;

with $\partial_n \mu = 0$ almost everywhere in $\partial\Omega \times (0,T)$.

In the following chapter we state the existence and uniqueness of solutions theorems, which are the main results of the analytical part of this thesis.

Chapter 2

Existence and stability estimates

2.1 Existence results

We can now state the existence theorems for weak and strong solutions, according to the regularity of the initial and boundary data. Under hypotheses (H_1)-(H_5) we can prove the existence of a weak solution to the problem.

Theorem 2.1.1. *Let hypotheses (H_1)-(H_5) be satisfied. Given $T > 0$, there exists a triple $(\mathbf{u}, \varphi, \theta)$ which is a weak solution on [0,T] according to Definition 1.1.*

Remark 2.1.2. We notice that the existence of a weak solution can also be obtained in the case of a bounded domain $\Omega \subset \mathbb{R}^3$, with slight changes in the functional setting of the time derivatives of velocity and temperature and in the proof of the estimates leading to the exhibition of a solution candidate.

We now consider stronger hypotheses on the initial conditions, namely the additional hypotheses are the following:

(I_1) κ and ν are positive constants,

(I_2) $\varphi_0 \in V_2$ with $||\varphi_0||_{L^\infty(\Omega)} \leq 1$ and $|\bar{\varphi}_0| < 1$,

(I_3) $\mu_0 = -\alpha\Delta\varphi_0 + \Psi'(\varphi_0) \in V$,

(I_4) $\mathbf{u}_0 \in \mathbf{V}_\sigma$.

In this case we can state the existence of a quasi-strong solution as in Definition 1.2, with additional regularity for the velocity field and the phase field.

Theorem 2.1.3. *Let (I_1)-(I_4) be fulfilled. Given $T > 0$, there exists a triple ($\boldsymbol{u}$, φ, θ), which is a quasi-strong solution on [0, T] according to Def. 1.2, such that*

- $\boldsymbol{u} \in L^\infty(0,T;\, \boldsymbol{V}_\sigma) \cap L^2(0,T;\, \boldsymbol{W}_\sigma) \cap H^1(0,T; \boldsymbol{H}_\sigma)$

- $\varphi \in L^\infty(0,T; W^{2,p}(\Omega)) \cap L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap H^1(0,T;V)$ *with* $|\varphi(x,t)| < 1$ *a.e.* $(x,t) \in \Omega \times (0,T)$*;*

- $\mu = -\alpha\Delta\varphi + \Psi'(\varphi)$ *a.e. in* $\Omega \times (0,T)$ *with* $\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;V')$*.*

- $\theta \in L^\infty(0,T;H) \cap L^2(0,T;V)$*,* $\theta = g$ *a.e. on* $\partial\Omega \times (0,T)$ *in the sense of traces and* $\partial_t\theta \in L^2(0,T;V_\theta')$*,*

where $2 \leq p < \infty$*.*

Remark 2.1.4. Due to the regularity of the solutions stated in the above theorem, since $\mu \in L^\infty(0,T;V)$ we deduce from the definition of μ itself and from (B.7) that

$$\varphi \in L^\infty(0,T;W^{2,p}(\Omega)), \text{ with } 2 \leq p < \infty.$$

Remark 2.1.5. By the regularity of the solutions, we also have that equations for velocity $\mathbf{u}$ and for φ also hold almost everywhere in $\Omega \times (0,T)$ and $\partial_{\mathbf{n}}\mu = 0$ almost everywhere on $\partial\Omega \times (0,T)$. Moreover there exists a pressure $\pi \in L^2(0,T,V)$ such that (1.13) also holds almost everywhere in $\Omega \times (0,T)$. Moreover, by the regularity for $\mathbf{u}$ and φ we have that the initial conditions are satisfied pointwise, $\mathbf{u}(\cdot,0) = \mathbf{u}_0$ and $\varphi(\cdot,0) = \varphi_0$ in Ω. Only the temperature θ is not regular enough to retrieve a strong solution according to Definition 1.3.

We now state the last theorem of existence of strong solutions. Since we look for $\theta \in H^2(\Omega)$, we ask for a more regular Dirichlet boundary datum: $g \in L^2(0,T;H^{3/2}(\partial\Omega)) \cap L^4(0,T;H^{1/2}(\partial\Omega))$, $\partial_t g \in L^2(0,T;H^{1/2}(\partial\Omega))$.

We also ask for a more regular initial datum, say $\theta_0 \in V$. Then the additional hypotheses to (H_1)-(H_5) and (I_1)-(I_4) are the following:

(J_1) The boundary value g satisfies $g \in L^2(0,T;H^{3/2}(\partial\Omega)) \cap L^4(0,T;H^{1/2}(\partial\Omega))$ and $\partial_t g \in L^2(0,T;H^{1/2}(\partial\Omega))$,

(J_2) $\theta_0 \in V$ and $\theta_0 = g(0)$ on $\partial\Omega$ in the sense of traces.

Theorem 2.1.6. *Let hypotheses (I_1)-(I_4) and (J_1)-(J_2) be fulfilled. Given $T > 0$, there exists a triple ($\boldsymbol{u}$, φ, θ) which is a strong solution on $[0,T]$ according to Def. 1.3, such that*

- $\boldsymbol{u} \in L^\infty(0,T;\, \boldsymbol{V}_\sigma) \cap L^2(0,T;\, \boldsymbol{W}_\sigma) \cap H^1(0,T;\boldsymbol{H}_\sigma)$;

- $\varphi \in L^\infty(0,T;W^{2,p}(\Omega)) \cap L^\infty(0,T;V) \cap L^4(0,T;V_2) \cap H^1(0,T;V)$ *with* $|\varphi(x,t)| < 1$ *a.e.* $(x,t) \in \Omega \times (0,T)$;

- $\theta \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$, $\theta = g$ *a.e. on* $\partial\Omega \times (0,T)$ *in the sense of traces and* $\partial_t\theta \in L^2(0,T;H)$;

- $\mu = -\alpha\Delta\varphi + \Psi'(\varphi)$ *a.e. in* $\Omega \times (0,T)$ *with* $\mu \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \cap H^1(0,T;V')$ *and* $\partial_n\mu = 0$ *a.e. on* $\partial\Omega \times (0,T)$.

where $2 \leq p < \infty$.

Remark 2.1.7. We recall that the strong solution satisfies the equations of problem (10) almost everywhere in $\Omega \times (0,T)$.

2.2 Stability estimates and uniqueness

We can now state some continuous dependence estimates, according to the regularity of the initial and boundary data, together with some uniqueness theorems, which are direct consequence of the aforementioned estimates. We start with a weak-strong uniqueness result, which leads to a stability estimate in the dual norms, if we start from a weak solution (Definition 1.1) and a strong solution (Definition 1.3). An immediate consequence is clearly the uniqueness of a strong solution.

Theorem 2.2.1. *Let* κ *and* ν *positive constants and let* $(\boldsymbol{u}_1, \varphi_1, \theta_1)$ *be a weak solution according to Def.1.1, with initial data* $\varphi_{01} \in V \cap L^\infty(\Omega)$ *with* $||\varphi_{01}||_{L^\infty(\Omega)} \leq 1$ *and* $|\bar{\varphi}_{01}| < 1$, $\boldsymbol{u}_{01} \in \boldsymbol{H}_\sigma$, $\theta_{01} \in H$, *and let* $(\boldsymbol{u}_2, \varphi_2, \theta_2)$ *be a strong solution according to Def. 1.3, with* $\varphi_{02} \in V_2 \cap L^\infty(\Omega)$, $||\varphi_{02}||_{L^\infty(\Omega)} \leq 1$ *and* $|\bar{\varphi}_{02}| < 1$, $\mu_{02} = -\alpha\Delta\varphi_{02} + \Psi'(\varphi_{02}) \in V$ *and* $\partial_{\boldsymbol{n}}\varphi_{02} = 0$ *on* $\partial\Omega$, $\boldsymbol{u}_{02} \in \boldsymbol{V}_\sigma$, $\theta_{02} \in V$ *and* $\theta_2 = g(t)$ *almost everywhere on* $\partial\Omega \times (0, T)$. *Define also the same Dirichlet boundary datum* $g \in L^2(0, T; H^{3/2}(\partial\Omega)) \cap L^4(0, T; H^{1/2}(\partial\Omega))$ *and* $\partial_t g \in L^2(0, T; H^{1/2}(\partial\Omega))$. *If* $\bar{\varphi}_{01} = \bar{\varphi}_{02}$, *then there exists a positive constant* C *depending on* T *and on the norms of the initial data such that*

$$
\begin{aligned}
&||\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)||_{\boldsymbol{V}'_\sigma} + ||\varphi_1(t) - \varphi_2(t)||_{H'} + ||\theta_1(t) - \theta_2(t)||_{\boldsymbol{V}'_\theta} \\
&\leq C||\boldsymbol{u}_{01} - \boldsymbol{u}_{02}||_{\boldsymbol{V}'_\sigma} + C||\varphi_{01} - \varphi_{02}||_{H'} + C||\theta_{01} - \theta_{02}||_{\boldsymbol{V}'_\theta} \qquad \forall t \in [0, T].
\end{aligned} \tag{2.1}
$$

Remark 2.2.2. We notice that if also the initial data coincide, we have $(\mathbf{u}_1, \varphi_1, \theta_1) = (\mathbf{u}_2, \varphi_2, \theta_2)$, implying that if the strong solution exists, it coincides with the weak one with the same initial and boundary data. In particular, this implies the uniqueness of the strong solution.

If we consider two quasi-strong solutions according to Definition 1.2, we find a stability estimate with respect to stronger norms than the previous case: in particular we strengthen the norms for the velocity field and the phase field, which are now the L^2 norms. Clearly, as an immediate consequence, we deduce the uniqueness of the quasi-strong solution.

Theorem 2.2.3. *Consider two sets of initial data* $(\boldsymbol{u}_{01}, \varphi_{01}, \theta_{01})$ *and* $(\boldsymbol{u}_{02}, \varphi_{02}, \theta_{02})$ *satisfying the assumptions* (I_2)*-*(I_4) *and denote by* $(\boldsymbol{u}_1, \varphi_1, \theta_1)$ *and* $(\boldsymbol{u}_2, \varphi_2, \theta_2)$ *the corresponding quasi-strong solutions, according to Definition 1.2. We have the continuous dependence es-*

timate

$$
\begin{aligned}
&||\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|| + ||\varphi_1(t) - \varphi_2(t)|| + ||\theta_1(t) - \theta_2(t)||_{V_\theta'} \\
&\leq C||\boldsymbol{u}_{01} - \boldsymbol{u}_{02}|| + C||\varphi_{01} - \varphi_{02}|| + C||\theta_{01} - \theta_{02}||_{V_\theta'} \quad \forall t \in [0,T],
\end{aligned} \tag{2.2}
$$

where C is a positive constant depending on T and on the norms of the initial data.

Remark 2.2.4. From this theorem we immediately deduce that the quasi-strong solution is unique.

As already noticed in Remark 2.2.2, the uniqueness of the strong solutions is a consequence of Theorem 2.2.1 (and obviously of Theorem 2.2.3), nevertheless we state the following theorem in order to show a continuous dependence estimate with respect to a stronger norm for the temperature, compared to the one presented in Theorem 2.2.3, namely the L^2 norm.

Theorem 2.2.5. *Consider two sets of initial data $(\boldsymbol{u}_{01}, \varphi_{01}, \theta_{01})$ and $(\boldsymbol{u}_{02}, \varphi_{02}, \theta_{02})$ satisfying the assumptions (I_1)-(I_4) and (J_1)-(J_2) and denote by $(\boldsymbol{u}_1, \varphi_1, \theta_1)$ and $(\boldsymbol{u}_2, \varphi_2, \theta_2)$ the corresponding strong solutions, according to Def. 1.3. We have the continuous dependence estimate*

$$
\begin{aligned}
&||\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)|| + ||\varphi_1(t) - \varphi_2(t)|| + ||\theta_1(t) - \theta_2(t)|| \\
&\leq C||\boldsymbol{u}_{01} - \boldsymbol{u}_{02}|| + C||\varphi_{01} - \varphi_{02}|| + C||\theta_{01} - \theta_{02}|| \quad \forall t \in [0,T],
\end{aligned} \tag{2.3}
$$

where C is a positive constant depending on T and on the norms of the initial data.

Chapter 3

Existence of a weak solution

In order to prove the existence theorems, we firstly consider the same weak form (without the condition of $|\varphi(x,t)| < 1$ almost everywhere $(x,t) \in \Omega \times (0,T)$, which will be required only for the final system), but with an approximation of the logarithmic potential Ψ, which will be called Ψ_λ, with $\lambda \in \mathbb{R}^+$, instead of Ψ and then show the convergence of the solutions of the approximated sequences to the desired solution of the original problem. Thus, before proving the theorems, we need to make explicit in the next section the construction of the approximants Ψ_λ.

Moreover, in another section we introduce the lifting operator technique for the case of the temperature θ, which is a standard way to be able to consider a solution with homogeneous Dirichlet boundary conditions, instead of nonhomogeneous conditions, which are more difficult to be treated directly.

3.1 Approximating the logarithmic potential Ψ

Let us recall some results in [56], and then in [34], concerning the existence of a sequence of regular functions F_λ which approximate the singular function F. First of all, for any $\lambda > 0$ we introduce the quadratic perturbation of F_λ by $\Psi_\lambda(s) = F_\lambda(s) - \frac{\alpha_0}{2}s^2$, which is the approximation on the potential as defined in (1.1).

Moreover, there exists a family $F_\lambda : \mathbb{R} \to \mathbb{R}$ $(\lambda > 0)$ such that $F_\lambda(0) = F'_\lambda(0) = 0$ and

1. F'_λ is Lipschitz on $\mathbb{R}$ with constant $1/\lambda$,

2. There exist $0 < \bar{\lambda} < \gamma \leq 1$, where γ has been defined in Section 1.1, and $\hat{C} > 0$ such that $F_\lambda(s) \geq \alpha_0 s^2 - \hat{C},\ \forall s \in \mathbb{R},\ \forall \lambda \in (0, \bar{\lambda}]$.

3. As $\lambda \to 0$, $F_\lambda(s) \to F(s)$ for all $s \in \mathbb{R}$, $|F'_\lambda(s)| \to |F'(s)|$ for $s \in (-1, 1)$ and F'_λ converges uniformly to F' on any (compact) set $[a, b] \subset (-1, 1)$. We have also that (see [56]) $|F'_\lambda(s)| \to +\infty$ for every $|s| \geq 1$. Moreover, we have

$$F_\lambda(s) \leq F(s) \quad \forall s \in [-1, 1]$$

and

$$|F'_\lambda(s)| \leq |F'(s)| \quad \forall s \in (-1, 1).$$

4. $F''_\lambda(s) \geq 0\ \forall s \in \mathbb{R}$

Some other properties of the approximations F_λ are the following:

- From property (2), we have that, $\forall \lambda \in (0, \bar{\lambda}]$ and $\forall s \in \mathbb{R}$

$$\Psi_\lambda(s) \geq \frac{\alpha_0}{2} s^2 - \hat{C} \geq -\hat{C}. \tag{3.1}$$

- From property (1) of F_λ and from the convexity of F_λ we deduce that:

$$F_\lambda(s) - F_\lambda(s_0) \leq (s - s_0) F'_\lambda(s) \leq |s - s_0|^2 \frac{1}{\lambda} \quad \forall s \in \mathbb{R},$$

which implies, since $s_0 = 0$, and $F_\lambda(0) = 0$, that

$$F_\lambda(s) \leq s F'_\lambda(s) \leq |s|^2 \frac{1}{\lambda} \quad \forall s \in \mathbb{R},$$

then we have the following property for Ψ_λ:

$$\Psi_\lambda(s) = F_\lambda(s) - \frac{\alpha_0}{2} s^2 \leq F_\lambda(s) \leq F_1 s^2 \quad \forall s \in \mathbb{R}, \tag{3.2}$$

where $F_1 = \dfrac{1}{\lambda}$.

- As proved in [52] and [56], there exists a positive constant C such that

$$\int_\Omega |F_\lambda'(\varphi)|dx \le C \mid \int_\Omega F_\lambda'(\varphi)(\varphi - \bar{\varphi})dx| + C, \tag{3.3}$$

  that holds for $C = C(\bar{\varphi}_0)$, independent of $\lambda \in (0, \bar{\lambda}]$, with the hypothesis that $\bar{\varphi}_0 \in (-1, 1)$.

- From property (4), we deduce that

$$\Psi_\lambda''(s) \ge -\alpha_0 \quad \forall s \in \mathbb{R}. \tag{3.4}$$

For the proofs of existence of more regular solutions, namely for the quasi-strong and the strong solution, we consider a slightly different approximation of the logarithmic potential (see [52]). In particular we define

$$F_\lambda(s) = \begin{cases} \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(1-\lambda)[s-(1-\lambda)]^j & \forall s \ge 1-\lambda \\ F(s) \qquad \forall s \in [-1+\lambda, 1-\lambda] \\ \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(-1+\lambda)[s-(-1+\lambda)]^j & \forall s \le -1+\lambda. \end{cases} \tag{3.5}$$

In this case, $\Psi_\lambda \in \mathcal{C}^2(\mathbb{R})$, and all the aforementioned properties hold, apart from (3.2), since F_λ' is not globally Lipschitz anymore. Moreover, property (3.4) holds for sufficiently small $\lambda > 0$.

3.2 The lifting operator

We analyze the case of nonhomogeneous boundary conditions for the temperature and we follow the method of the lift operator presented in [83] and also used in detail for example in [14]. We consider the problem

$$\begin{cases} -\Delta\theta_g(t) = 0 \text{ in } \Omega \text{ a.a. t } \in (0,T) \\ \theta_g(t) = g(t) \text{ on } \partial\Omega \text{ a.a. t } \in (0,T). \end{cases} \tag{3.6}$$

Thus it is well known that if Ω is at least $C^{0,1}$, as in our case, then $\theta_g(t) \in V$ for $g(t) \in H^{1/2}(\partial\Omega)$ and we have the estimate $||\theta_g(t)||_1 \le C||g(t)||_{1/2,\partial\Omega}$. When Ω is smooth, as in our case, we can apply a duality argument, as in [83], to conclude that $\theta_g(t) \in H^m(\Omega)$ for $g(t) \in H^{m-1/2}(\partial\Omega)$ for every $m \geq -1$. Also, the following estimate holds:

$$||\theta_g(t)||_m \leq C||g(t)||_{m-1/2,\partial\Omega} \quad \forall m \geq -1. \tag{3.7}$$

Moreover, $\partial_t\theta_g(t)$ is a solution of the Dirichlet problem

$$\begin{cases} -\Delta\partial_t\theta_g(t) = 0 \ \in \Omega \text{ a.a. t} \in (0,T) \\ \partial_t\theta_g(t) = \partial_t g(t) \text{ on } \partial\Omega \text{ a.a. t} \in (0,T). \end{cases}$$

Thus $\partial_t\theta_g(t)$ is a function in V and satisfies $||\partial_t\theta_g(t)||_1 \leq C||\partial_t g(t)||_{1/2,\partial\Omega}$ for almost any t in $(0,T)$. Analogously, we have $\partial_t\theta_g(t) \in H^m(\Omega)$ for $\partial_t g(t) \in H^{m-1/2}(\partial\Omega)$ for every $m \geq -1$. Also, the following estimates hold:

$$||\partial_t\theta_g(t)||_m \leq C||\partial_t g(t)||_{m-1/2,\partial\Omega} \quad \forall m \geq -1. \tag{3.8}$$

Then, from these estimates, it is easy to see that if the original boundary datum g satisfies $g \in L^p(0,T;H^{m-1/2}(\partial\Omega))$ for some $m \geq -1$ and some $p \in [1,\infty]$, and $\partial_t g \in L^q(0,T;H^{k-1/2}(\partial\Omega))$ for some $m \geq -1$ and some $q \in [1,\infty]$, then $\theta_g \in L^p(0,T;H^m(\Omega))$ and $\partial_t\theta_g \in L^q(0,T;H^k(\Omega))$. In the case analyzed in the previous sections, it is thus sufficient to consider the following regularity for the boundary value g:

$$g \in L^4(0,T;H^{1/2}(\partial\Omega)) \text{ and } \partial_t g \in L^2(0,T;H^{1/2}(\partial\Omega)), \tag{3.9}$$

implying at least that

$$\theta_g \in L^4(0,T;H^1(\Omega)) \text{ and } \partial_t\theta_g \in L^2(0,T;H). \tag{3.10}$$

3.3 Proof of Theorem 2.1.1

3.3.1 Galerkin approximations for the approximating problem

We can now prove that a weak solution exists. In order to do that, we firstly consider the same weak form but with Ψ_λ instead of Ψ (without the condition of $|\varphi(x,t)| < 1$ a.e. $(x,t) \in \Omega \times (0,T)$, which will be required only for the final system).

We say that $(\mathbf{u}_\lambda, \varphi_\lambda, \theta_\lambda)$ is a weak solution of the approximating problem if

- $\mathbf{u}_\lambda \in L^\infty(0,T;\mathbf{H}_\sigma) \cap L^2(0,T;\mathbf{V}_\sigma)$ and $\partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}'_\sigma)$;

- $\varphi_\lambda \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \subset L^2(0,T;V)$ and $\partial_t \varphi \in L^2(0,T;V')$;
  $\varphi \in L^\infty(\Omega \times (0,T))$ and $|\varphi(x,t)| < 1$ a.e. $(x,t) \in \Omega \times (0,T)$;

- $\theta_\lambda \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $\theta_\lambda = g$ on $\partial\Omega \times (0,T)$ in the sense of traces and $\partial_t \theta_\lambda \in L^2(0,T;V'_\theta)$;

$$< \partial_t \mathbf{u}_\lambda, \mathbf{w} > + b(\mathbf{u}_\lambda, \mathbf{u}_\lambda, \mathbf{w}) + (\nu(\varphi_\lambda, \theta_\lambda)\nabla \mathbf{u}_\lambda, \nabla \mathbf{w}) = -(\varphi_\lambda \nabla \mu_\lambda, \mathbf{w}) + (\theta_\lambda, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_\sigma \tag{3.11}$$

$$< \partial_t \varphi_\lambda, v > + (\nabla \mu, \nabla v) + (\mathbf{u}_\lambda \cdot \nabla \varphi_\lambda, v) = 0 \quad \forall v \in V \tag{3.12}$$

$$< \partial_t \theta_\lambda, \xi > + (\kappa(\theta_\lambda)\nabla \theta_\lambda, \nabla \xi) + (\mathbf{u}_\lambda \cdot \nabla \theta_\lambda, \xi) = 0 \quad \forall \xi \in V_\theta \tag{3.13}$$

  for almost every $t \in (0,T)$;

- $\mu_\lambda = -\alpha \Delta \varphi_\lambda + \Psi'_\lambda(\varphi_\lambda)$ a.e. in $\Omega \times (0,T)$ with $\mu \in L^2(0,T;V)$;

- $\mathbf{u}_\lambda(0) = \mathbf{u}_0 \qquad \varphi_\lambda(0) = \varphi_0 \qquad \theta_\lambda(0) = \theta_0$.

We then show the convergence in λ of the solutions of the approximated sequences to the desired solution of the original problem.
The proof will be carried out by means of a Faedo-Galerkin approximation scheme: we prove that the solution exists for the approximations and then we extract a converging subsequence showing that the limit is a solution of the original problem. Then, letting, up to subsequences, $\lambda \to 0$, we get the desired solution. For simplicity, from now on we omit the subscript λ on the variables.

First of all, we define the solution of the nonhomogeneous boundary Dirichlet problem as $\theta = \Theta + \theta_g$, where $\Theta(t)$ belongs to V_θ for almost any $t \in (0,T)$ and $\theta_g(t)$ is the harmonic extension of the boundary datum previously defined. We set $\Theta_0 = \theta_0 - \theta_g(0)$.

We then introduce the family $\{\mathbf{w}_j\}_{j\geqslant 1}$ of the eigenfunctions of the Stokes operator (see Appendix B.2 and, e.g., [32] and [97]) as a Galerkin base in $\mathbf{V}_\sigma$ (orthonormal in $\mathbf{H}_\sigma$ and orthogonal in $\mathbf{V}_\sigma$) and the family $\{\psi_j\}_{j\geqslant 1}$ of the eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions as a Galerkin base in V (orthonormal in H and orthogonal in V). In conclusion we introduce the family $\{v_j\}_{j\geqslant 1}$ of the eigenfunctions of the Laplace operator with homogeneous Dirichlet boundary conditions as a Galerkin base in V_θ (orthonormal in H and orthogonal in V_θ).

We define the n-dimensional subspaces $\mathbb{W}_n := Span(\mathbf{w}_1, \dots, \mathbf{w}_n)$, $\mathbb{Z}_n := Span(\psi_1, \dots, \psi_n)$ (we consider as $\psi_1 \equiv 1/\sqrt{|\Omega|}$, so that $\|\psi_1\| = 1$, since the first eigenspace, of the eigenvalue $\lambda_1 = 0$, is made of the constant functions), $\mathbb{V}_n := Span(v_1, \dots, v_n)$ and consider the orthogonal projectors on these subspaces in $\mathbf{H}_\sigma$ and H (i.e. with respect to L^2 norm), respectively, i.e. $P_n := P_{\mathbb{W}_n}$, $\tilde{P}_n := P_{\mathbb{Z}_n}$ and $\hat{P}_n := P_{\mathbb{V}_n}$. We then look for four functions of the form

$$\mathbf{u}_n(t) = \sum_{i=1}^n \hat{\alpha}_i(t)\mathbf{w}_i \in \mathbb{W}_n \qquad \varphi_n(t) = \sum_{i=1}^n \beta_i(t)\psi_i \in \mathbb{Z}_n \tag{3.14}$$

$$\mu_n(t) = \sum_{i=1}^n \gamma_i(t)\psi_i \in \mathbb{Z}_n \qquad \Theta_n(t) = \sum_{i=1}^n \delta_i(t)v_i \in \mathbb{V}_n, \tag{3.15}$$

where $\hat{\alpha}_i, \beta_i, \gamma_i, \delta_i$ are real valued functions of (we will see) C^1 class and $\theta_n = \Theta_n + \theta_g$, such that

- $\mathbf{u}_n(0) = P_n(\mathbf{u}_0),\ \varphi_n(0) = \tilde{P}_n(\varphi_0),\ \Theta_n(0) = \hat{P}_n(\Theta_0)$, which means

$$\mathbf{u}_n(0) = \sum_{i=1}^n \hat{\alpha}_i(0)\mathbf{w}_i \in \mathbb{W}_n \qquad \varphi_n(0) = \sum_{i=1}^n \beta_i(0)\psi_i \in \mathbb{Z}_n \qquad \Theta_n(0) = \sum_{i=1}^n \delta_i(0)v_i \in \mathbb{V}_n, \tag{3.16}$$

- $\mu_n = \tilde{P}_n(-\alpha\Delta\varphi_n + \Psi'_\lambda(\varphi_n)) = -\alpha\Delta\varphi_n + \tilde{P}_n(\Psi'_\lambda(\varphi_n))$;

$$(\partial_t\mathbf{u}_n, \mathbf{w}) + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{w}) + (\nu(\varphi_n, \theta_n)\nabla\mathbf{u}_n, \nabla\mathbf{w}) = -(\varphi_n\nabla\mu_n, \mathbf{w}) + (\Theta_n + \theta_g, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall\mathbf{w} \in \mathbb{W}_n \tag{3.17}$$

$$(\partial_t\varphi_n, v) + (\nabla\mu_n, \nabla v) + (\mathbf{u}_n \cdot \nabla\varphi_n, v) = 0 \qquad \forall v \in \mathbb{Z}_n \tag{3.18}$$

$$(\partial_t \Theta_n, \xi) + (\kappa(\theta_n)\nabla\Theta_n, \nabla\xi) + (\mathbf{u}_n \cdot \nabla\Theta_n, \xi) =$$

$$- < \partial_t \theta_g, \xi > -(\kappa(\theta_n)\nabla\theta_g, \nabla\xi) - (\mathbf{u} \cdot \nabla\theta_g, \xi) \quad \forall \xi \in \mathbb{V}_n \tag{3.19}$$

for every $t \in (0, T)$.

We notice that $\tilde{P}_n(-\alpha\Delta\varphi_n) = -\alpha\Delta\varphi_n$ because the linear operator $-\Delta$ commutes with the orthogonal projector $\tilde{P}_n$. Moreover we recall that, due to regularity theorems and since the domain is supposed to have a sufficiently smooth boundary, the aforementioned eigenfunctions are smooth functions, see [97], thus, for example, the duality in the time derivatives can be considered as an L^2 inner product.

Since the function $\Psi'_\lambda(s) = F'_\lambda(s) - \alpha_0 s$ is at least locally Lipschitz and κ, ν are globally Lipschitz, one can easily see that this system of equations is equivalent to a Cauchy problem for an ordinary differential equations system in the unknowns $\hat{\alpha}_i$, β_i, δ_i. The Cauchy-Lipschitz theorem ensures that this system has a unique solution into an interval $[0, t_n)$, $t_n > 0$: in fact, we have a system of equations for the unknowns $\hat{\alpha}^{(n)}(t) = [\hat{\alpha}_1(t), \ldots, \hat{\alpha}_n(t)]$, $\beta^{(n)}(t) = [\beta_1(t), \ldots, \beta_n(t)]$, $\delta^{(n)}(t) = [\delta_1(t), \ldots, \delta_n(t)]$, solving the system of ODE:

$$[\dot{\hat{\alpha}}^{(n)}(t), \dot{\beta}^{(n)}(t), \dot{\delta}^{(n)}(t)]^T = \mathcal{G}(\hat{\alpha}^{(n)}(t), \beta^{(n)}(t), \delta^{(n)}(t)).$$

with $\mathcal{G}$ a locally Lipschitz continuous function of $[\hat{\alpha}^{(n)}, \beta^{(n)}, \delta^{(n)}]^T$ and with the initial conditions $[\hat{\alpha}^{(n)}(0), \beta^{(n)}(0), \delta^{(n)}(0)]^T$ as shown in (3.16).

Then, the Cauchy–Lipschitz theorem entails the existence of a unique maximal solution $\hat{\alpha}^{(n)} \in C^1([0, t_n), \mathbb{R}^n)$, $\beta^{(n)} \in C^1([0, t_n), \mathbb{R}^n)$, $\delta^{(n)} \in C^1([0, t_n), \mathbb{R}^n)$.

We now derive some uniform estimates in order to guarantee that $t_n = +\infty$. First of all, we have the mass conservation property: from equation (3.18), considering $v \equiv 1$ as test function ($v \in \mathbb{Z}_n \quad \forall n \geq 1$) and integrating by parts the third term we get

$$\int_\Omega \partial_t \varphi_n = |\Omega| \frac{d\bar{\varphi}_n}{dt} = 0 \tag{3.20}$$

thus $\bar{\varphi}_n = const = \int_\Omega \tilde{P}_n(\varphi_0)/|\Omega| = \bar{\varphi}_0$ since the only component of the projection $\tilde{P}_n$, with respect to the basis, with nonzero mean, is the component with respect to ψ_1, so $\int_\Omega (\varphi_0, \psi_1)\psi_1 = \bar{\varphi}_0$. That is to say that $\bar{\varphi}_n$ is independent of n and t and depends only on the initial datum φ_0.

Now, we start from equation (3.18): with the classical procedure of applying the equation for each element of the basis of $\mathbb{Z}_n$, then multiplying each equation by γ_i and then summing up, we can use $\mu_n \in \mathbb{Z}_n$ as a test function, integrate by parts applying the boundary conditions and obtain:

$$(\partial_t \varphi_n, \mu_n(t)) + (\nabla \mu_n, \nabla \mu_n) - (\mathbf{u}_n \cdot \nabla \mu_n, \varphi_n) = 0. \tag{3.21}$$

Substituting the value for μ_n in the time derivative we obtain (by construction and orthonormality of the basis, for every $v = \sum_{i=1}^{n} a_i \psi_i$ we have $(\tilde{P}_n(\Psi'_\lambda(\varphi_n)), v) = \sum_{i=0}^{n} (\Psi_\lambda(\varphi_n)', a_i \psi_i) = (\Psi_\lambda(\varphi_n)', \sum_{i=0}^{n} a_i \psi_i) = (\Psi'_\lambda(\varphi_n), v)$):

$$\frac{d}{dt}(\frac{\alpha}{2}||\nabla \varphi_n||^2 + \int_\Omega \Psi_\lambda(\varphi_n)) + ||\nabla \mu_n||^2 - (\mathbf{u}_n \cdot \nabla \mu_n, \varphi_n) = 0. \tag{3.22}$$

From now on, for simplicity, we will omit the dependence of κ and ν from the variables φ_n and θ_n.

We analyze the equation (3.19) for the temperature: by the same argument as before, we can test the equation against $\xi = \Theta_n$ and, remembering property (1.6):

$$\frac{d}{dt}\frac{1}{2}||\Theta_n||^2 + \kappa_*||\nabla \Theta_n||^2 \leq - < \partial_t \theta_g, \Theta_n > -(\kappa \nabla \theta_g, \nabla \Theta_n) - (\mathbf{u}_n \cdot \nabla \theta_g, \Theta_n). \tag{3.23}$$

We then have, by Cauchy-Schwarz's and Poincaré's inequalities, the property of the lift operator and Young's inequality (we recall that $||\partial_t \theta_g|| \leq ||\partial_t \theta_g||_1 \leq C||\partial_t g(t)||_{1/2,\partial\Omega}$):

$$\begin{aligned} - < \partial_t \theta_g, \Theta_n > \leq C_0||\partial_t \theta_g|| \; ||\nabla \Theta_n|| &\leq C||\partial_t g(t)||_{1/2,\partial\Omega}||\nabla \Theta_n|| \\ &\leq \frac{k_*}{8}||\nabla \Theta_n||^2 + \tilde{C}||\partial_t g(t)||^2_{1/2,\partial\Omega}. \end{aligned}$$

Then we have, by the same properties (in particular, by Young's inequality, we make k_* appear and since we have $k_* > 0$ - see hypothesis (1.6) - the constant C is well defined)

$$\begin{aligned} -(\kappa \nabla \theta_g, \nabla \Theta_n) \leq \frac{k_*}{8}||\nabla \Theta_n||^2 + C||\nabla \theta_g||^2 &\leq \frac{k_*}{8}||\nabla \Theta_n||^2 + C||\theta_g||_1^2 \\ &\leq \frac{k_*}{8}||\nabla \Theta_n||^2 + \bar{C}||g(t)||^2_{1/2,\partial\Omega}. \end{aligned}$$

We recall that, for example by Lemma (A.1.4) together with Poincaré's inequality, we obtain that, given $\chi \in H_0^1(\Omega)$ (the same holds for the vectorial case $\chi \in [H_0^1(\Omega)]^2$):

$$||\chi||_{L^4(\Omega)} \leq C||\chi||^{1/2}||\nabla\chi||^{1/2}. \tag{3.24}$$

We can then apply the previous result together with the classical Sobolev embedding $V_\theta \hookrightarrow L^4(\Omega)$, the generalized Young's inequality for three terms and the lift operator's properties:

$$\begin{aligned} -(\mathbf{u}_n \cdot \nabla\theta_g, \Theta_n) &\leq ||\mathbf{u}_n||_{L^4(\Omega)}||\nabla\theta_g||\ ||\Theta_n||_{L^4(\Omega)} \\ &\leq ||\mathbf{u}_n||^{1/2}||\nabla\mathbf{u}_n||^{1/2}||\nabla\theta_g||\ ||\nabla\Theta_n|| \\ &\leq \frac{\nu_*}{2}||\nabla\mathbf{u}_n||^2 + \frac{k_*}{8}||\nabla\Theta_n||^2 + C||\mathbf{u}_n||^2||\nabla\theta_g||^4 \\ &\leq \frac{\nu_*}{2}||\nabla\mathbf{u}_n||^2 + \frac{k_*}{8}||\nabla\Theta_n||^2 + \bar{C}||\mathbf{u}_n||^2||g(t)||^4_{1/2,\partial\Omega}. \end{aligned}$$

Remark 3.3.1. *If we considered the homogeneous Dirichlet case, the proof of this fact simplifies a lot: we can use Θ_n as test function in the equation for temperature to get, applying the boundary conditions and property (1.6):*

$$\frac{d}{dt}\|\Theta_n\|^2 + 2k_*||\nabla\Theta_n||^2 \leq 0. \tag{3.25}$$

Now, we can apply Poincaré's inequality due to homogeneous Dirichlet boundary conditions: $||\Theta_n||^2 \leq C_0||\nabla\Theta_n||^2$, with $C_0 = C_0(\Omega)$, to get

$$\frac{d}{dt}||\Theta_n||^2 + \beta_0||\Theta_n||^2 \leq 0 \tag{3.26}$$

where $\beta_0 = 2\frac{k_*}{C_0}$. Then, applying Gronwall's inequality (Lemma A.1.7), we get

$$||\Theta_n||^2 \leq ||\Theta_n(0)||^2e^{-\beta_0 t} \leq ||\Theta_0||^2e^{-\beta_0 t} \leq ||\Theta_0||^2 \tag{3.27}$$

since $\hat{P}_n$ is an orthogonal projector, thus $||\hat{P}_n(\Theta_0)|| \leq ||\Theta_0||$. So, we can deduce that Θ_n is bounded in $L^\infty(0,T;H)$ for any $T \leq t_n$.
We then look for a uniform estimate for $\nabla\Theta_n$: starting again from equation (3.25), we multiply by $e^{\frac{\beta_0}{2}t}$, knowing that

$$\frac{d}{dt}e^{\frac{\beta_0}{2}t}||\Theta_n||^2 = \frac{\beta_0}{2}e^{\frac{\beta_0}{2}t}||\Theta_n||^2 + e^{\frac{\beta_0}{2}t}\frac{d}{dt}||\Theta_n||^2$$

we have, using the bound (3.27)

$$\frac{d}{dt}e^{\frac{\beta_0}{2}t}||\Theta_n||^2 + 2k_*\ e^{\frac{\beta_0}{2}t}||\nabla\Theta_n||^2 \leq \frac{\beta_0}{2}e^{\frac{\beta_0}{2}t}||\Theta_n||^2 \leq \frac{\beta_0}{2}e^{-\frac{\beta_0}{2}t}||\Theta_0||^2.$$

Integrating in time the previous inequality, we obtain, remembering that $||\hat{P}_n(\Theta_0)|| \leq ||\Theta_0||$

$$e^{\frac{\beta_0}{2}T}||\Theta_n(T)||^2 + 2k_*\int_0^T e^{\frac{\beta_0}{2}t}||\nabla\Theta_n||^2 \leq ||\Theta_0||^2 + ||\Theta_0||^2(1-e^{-\frac{\beta_0}{2}t}) \leq 2||\Theta_0||^2,$$

from which we get

$$\int_0^T ||\nabla\Theta_n||^2 \leq \int_0^T e^{\frac{\beta_0}{2}t}||\nabla\Theta_n||^2 \leq \frac{1}{k_*}||\Theta_0||^2 = \bar{C}^2$$

where $\bar{C}$ *is independent of* n*: we have, by Poincaré's inequality on* V_θ*:*

$$||\Theta_n||_{L^2(0,T;V_\theta)} \leq \bar{C}, \tag{3.28}$$

obtaining directly the uniform boundedness of the norm in $L^2(0,T;V_\theta)$*.*

Now we can consider equation (3.17): considering $\mathbf{u}_n$ as a test function, knowing that the trilinear form $b(\cdot,\cdot,\cdot)$ is antysimmetric and applying the boundary conditions, we get:

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}||\mathbf{u}_n||^2 + \nu_*||\nabla\mathbf{u}_n||^2 + (\mathbf{u}_n\cdot\nabla\mu_n,\varphi_n) &\leq \frac{d}{dt}\frac{1}{2}||\mathbf{u}_n||^2 + (\nu\nabla\mathbf{u}_n,\nabla\mathbf{u}_n) + (\mathbf{u}_n\cdot\nabla\mu_n,\varphi_n)\\ &= (\Theta_n,\mathbf{e}_2\cdot\mathbf{u}_n) + (\theta_g,\mathbf{e}_2\cdot\mathbf{u}_n). \end{aligned}\tag{3.29}$$

The terms $(\theta_g,\mathbf{e}_2\cdot\mathbf{u}_n)$ and $(\Theta_n,\mathbf{e}_2\cdot\mathbf{u}_n)$ can be easily estimated by means of Cauchy-Schwarz's and Young's inequalities and by lift operator's properties:

$$(\theta_g,\mathbf{e}_2\cdot\mathbf{u}_n) \leq ||\theta_g||\ ||\mathbf{u}_n|| \leq ||\theta_g||_1\ ||\mathbf{u}_n|| \leq \frac{1}{2}||\theta_g||_1^2 + \frac{1}{2}||\mathbf{u}_n||^2 \leq \frac{1}{2}||g(t)||^2_{1/2,\partial\Omega} + \frac{1}{2}||\mathbf{u}_n||^2,$$

whereas for the second one we get:

$$(\Theta_n,\mathbf{e}_2\cdot\mathbf{u}_n) \leq C_0||\nabla\Theta_n||\ ||\mathbf{u}_n|| \leq \frac{k_*}{8}||\nabla\Theta_n||^2 + C||\mathbf{u}_n||^2.$$

We can then sum up all the terms, obtaining the following inequality, having defined the total energy

$$E_n(t) = \frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega(\Psi_\lambda(\varphi_n)+\hat{C}), \tag{3.30}$$

where $\hat{C}$ is the positive constant such that (3.1) holds:

$$\frac{d}{dt}\{E_n(t)\} + ||\nabla\mu_n||^2 + \nu_*||\nabla\mathbf{u}_n||^2 + k_*||\nabla\Theta_n||^2 \leq \frac{k_*}{8}||\nabla\Theta_n||^2 + C||\mathbf{u}_n||^2 + \frac{1}{2}||g(t)||^2_{1/2,\partial\Omega}$$
$$+ \frac{1}{2}||\mathbf{u}_n||^2$$
$$+ \frac{k_*}{8}||\nabla\Theta_n||^2 + \tilde{C}||\partial_t g(t)||^2_{1/2,\partial\Omega}$$
$$+ \frac{k_*}{8}||\nabla\Theta_n||^2 + \bar{C}||g(t)||^2_{1/2,\partial\Omega}$$
$$+ \frac{k_*}{8}||\nabla\Theta_n||^2 + \frac{\nu_*}{2}||\nabla\mathbf{u}_n||^2$$
$$+ \bar{C}||\mathbf{u}_n||^2||g(t)||^4_{1/2,\partial\Omega}$$

then, we obtain, having set:

$$\mathcal{D}_n(t) = ||\nabla\mu_n||^2 + \frac{\nu_*}{2}||\nabla\mathbf{u}_n||^2 + \frac{k_*}{2}||\nabla\Theta_n||^2 \tag{3.31}$$

$$\frac{d}{dt}\{E_n(t)\} + \mathcal{D}_n(t) \leq \bar{C}||\mathbf{u}_n||^2 + \frac{1}{2}||g(t)||^2_{1/2,\partial\Omega} + \tilde{C}||\partial_t g(t)||^2_{1/2,\partial\Omega}$$
$$+ \bar{C}||g(t)||^2_{1/2,\partial\Omega} + \bar{C}||\mathbf{u}_n||^2||g(t)||^4_{1/2,\partial\Omega}.$$

In conclusion, changing the constants, since $\int_\Omega(\Psi_\lambda(\varphi_n) + \hat{C}) \geq 0$, we can get

$$\frac{d}{dt}E_n(t)+\mathcal{D}_n(t) \leq K_1(1+||g(t)||^4_{1/2,\partial\Omega})\, E_n(t)+K_2(1+||g(t)||^2_{1/2,\partial\Omega}+||\partial_t g(t)||^2_{1/2,\partial\Omega}). \tag{3.32}$$

Thus, due to the regularity hypothesis made on the boundary datum g, we have that $\mathcal{Q} = K_1(1 + ||g||^4_{1/2,\partial\Omega}) \in L^1(0,T)$
and also $\mathcal{R} = K_2(1 + ||g||^2_{1/2,\partial\Omega} + ||\partial_t g||^2_{1/2,\partial\Omega}) \in L^1(0,T)$, we can apply Gronwall's Lemma (A.1.7), since E_n is at least continuous in time: for any $t \in (0, t_n)$, for $t_n \leq T$:

$$E_n(t) \leq E_n(0)e^{\int_0^t \mathcal{Q}(r)} + \int_0^t e^{\int_s^t \mathcal{Q}(r)}\mathcal{R}(s)ds \leq e^{\int_0^T \mathcal{Q}(r)}(E_n(0) + \int_0^T \mathcal{R}(s)ds). \tag{3.33}$$

We have now to estimate the value of $E_n(0)$: remembering that $\Theta_0 = \theta_0 - \theta_g(0)$, we obtain

$$E_n(0) = \frac{1}{2}||P_n(\mathbf{u}_0)||^2 + \frac{1}{2}||\hat{P}_n(\Theta_0)||^2 + \frac{\alpha}{2}||\nabla\tilde{P}_n(\varphi_0)||^2 + \int_\Omega(\Psi_\lambda(\tilde{P}_n(\varphi_0)) + \hat{C}).$$

Since all the projections are orthogonal in the spaces $\mathbf{H}_\sigma$ and V respectively, we can apply $||P_n(\mathbf{u}_0)|| \leq ||\mathbf{u}_0||$, $||\hat{P}_n(\Theta_0)|| \leq ||\Theta_0||$ and $||\nabla \tilde{P}_n(\varphi_0)|| \leq ||\nabla \varphi_0||$. We also have, form (3.2), that

$$\int_\Omega \Psi_\lambda(\hat{P}_n(\varphi_0)) \leq \int_\Omega F_1(\hat{P}_n(\varphi_0))^2 = F_1||\hat{P}_n(\varphi_0)||^2 \leq F_1||\varphi_0||^2$$

again because $\hat{P}_n$ is an orthogonal projector.

We can conclude that

$$E_n(0) \leq \frac{1}{2}||\mathbf{u}_0||^2 + \frac{1}{2}||\Theta_0||^2 + \frac{\alpha}{2}||\nabla\varphi_0||^2 + F_1||\varphi_0||^2 + \hat{C}|\Omega|. \tag{3.34}$$

Thus we can say, defining as K_0 a generic constant depending on the initial data and λ, but not on t nor n, we obtain:

$$E_n(t) \leq \frac{1}{2}||\mathbf{u}_0||^2 + \frac{1}{2}||\Theta_0||^2 + \frac{\alpha}{2}||\nabla\varphi_0||^2 + F_1||\varphi_0||^2 + \hat{C}|\Omega| + k_0 = K_0 \tag{3.35}$$

which implies, since $\int_\Omega (\Psi_\lambda(\varphi_n) + \hat{C}) \geq 0$, that

$$\frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 \leq K_0.$$

Now, since from Poincaré's inequality (Lemma A.1) and from conservation of mass previously shown we get

$$||\varphi_n|| \leq ||\varphi_n - \bar{\varphi}_n|| + ||\bar{\varphi}_n|| \leq C_0||\nabla\varphi_n|| + ||\bar{\varphi}_0|| \leq C_0\sqrt{K_0} + ||\bar{\varphi}_0||. \tag{3.36}$$

In conclusion we have that, for a generic constant $\bar{C}$ independent of n and t:

$$||\varphi_n|| \leq \bar{C} \qquad ||u_n|| \leq \bar{C} \qquad ||\Theta_n|| \leq \bar{C}$$

Since we have that $||\varphi_n|| = |\beta^{(n)}(t)|$, $||u_n|| = |\alpha^{(n)}(t)|$ and $||\Theta_n|| = |\delta^{(n)}(t)|$, by means again of Gronwall's Lemma (see, e.g., [68] for this kind of arguments) we deduce that $t_n = +\infty$ for every $n \geq 1$, i.e. the problem (3.17)-(3.19) has a unique global in time solution.

Furthermore, for every $0 < T < +\infty$, we have that:

$$||\Theta_n||_{L^\infty(0,T;H)} \leq \bar{C}, \tag{3.37}$$

$$||\mathbf{u}_n||_{L^\infty(0,T;\mathbf{H}_\sigma)} \leq \bar{C}. \tag{3.38}$$

But since we have that

$$||\varphi_n||_V \leq ||\varphi_n - \bar{\varphi}_n||_V + ||\bar{\varphi}_n||_V = ||\varphi_n - \bar{\varphi}_n||_V + ||\bar{\varphi}_0||_V \leq \tilde{C}||\nabla\varphi_n - \nabla\bar{\varphi}_n|| + ||\bar{\varphi}_0||_V \leq \bar{C}$$

for some $\bar{C}$ independent of n and t, we deduce:

$$||\varphi_n||_{L^\infty(0,T;V)} \leq \bar{C}. \tag{3.39}$$

Also, from (3.32), integrating in time over (0,T) and applying the inequality (3.34) for $E_n(0)$, and since $E_n(t) \geq 0$, we obtain that

$$\int_0^T ||\nabla\mu_n||^2 + \frac{\nu_*}{2}\int_0^T ||\nabla\mathbf{u}_n||^2 + \frac{k_*}{2}\int_0^T ||\nabla\Theta_n||^2 \leq \tilde{C}.$$

Thus, due to Poincaré's inequality for velocity and temperature we also obtain, again for some constant $\bar{C}$ independent of n and T, that

$$||\mathbf{u}_n||_{L^2(0,T;\mathbf{V}_\sigma)} \leq \bar{C} \tag{3.40}$$

$$||\nabla\mu_n||_{L^2(0,T;H)} \leq \bar{C} \tag{3.41}$$

and

$$||\Theta_n||_{L^2(0,T;V_\theta)} \leq \bar{C} \tag{3.42}$$

for any $0 < T < +\infty$.

Coming back to the equation for μ_n, we can get a further estimate for φ_n, testing μ_n by $-\Delta\varphi_n$ and integrating by parts, using the boundary conditions:

$$(\nabla\mu_n, \nabla\varphi_n) = \alpha||\Delta\varphi_n||^2 + (\Psi''_\lambda(\varphi_n)\nabla\varphi_n, \nabla\varphi_n).$$

Then, remembering property (3.4) of Ψ_λ, we have:

$$\alpha||\Delta\varphi_n||^2 - \alpha_0||\nabla\varphi_n||^2 \leq (\nabla\mu_n, \nabla\varphi_n)$$

entailing, by Cauchy-Schwarz's and Young's inequality:

$$\alpha||\Delta\varphi_n||^2 \leq \alpha_0||\nabla\varphi_n||^2 + ||\nabla\mu_n||\ ||\varphi_n|| \leq k_1 \cdot (1 + ||\nabla\mu_n||)$$

where we have used the uniform-in-n bounds on $||\nabla\varphi_n||$ and $||\varphi_n||$: k_1 depends only on the initial data, but not on n. We can then integrate in time from 0 to T, after elevating to the square, getting, also using equation (A.2):

$$\alpha^2 \int_0^T ||\varphi_n - \bar{\varphi}_n||^4_{H^2(\Omega)} \leq C^4 \alpha^2 \int_0^T ||\Delta\varphi_n||^4 \leq 2k_1 T + 2k_1 \int_0^T ||\nabla\mu_n||^2 \leq C(T), \quad (3.43)$$

where $C(T)$ depends on T and initial data (due to (3.41)), but not on n. Thus we can say

$$||\varphi_n||_{L^4(0,T;V_2)} \leq ||\varphi_n - \bar{\varphi}_n||_{L^4(0,T;V_2)} + ||\bar{\varphi}_0||_{L^4(0,T;V_2)} \leq \bar{C}(T), \quad (3.44)$$

with $\bar{C}(T)$ dependent on initial data and T, but not on n. We then find an estimate for $\bar{\mu}_n$: multiplying the equation for μ_n by $\varphi_n - \bar{\varphi}_n$ ad integrating, after an integration by parts, applying boundary conditions, we get

$$(\mu_n, \varphi_n - \bar{\varphi}_n) = \alpha||\nabla\varphi_n||^2 + (F'_\lambda(\varphi_n), \varphi_n - \bar{\varphi}_n) - \alpha_0(\varphi_n, \varphi_n - \bar{\varphi}_n).$$

Then we obtain, since $(\bar{\mu}_n, \varphi_n - \bar{\varphi}_n) = 0$ and applying Poincaré's inequality (A.1) for zero-integral-mean functions and as usual Young's inequality:

$$\begin{aligned}(F'_\lambda(\varphi_n), \varphi_n - \bar{\varphi}_n) =& (\mu_n - \bar{\mu}_n, \varphi_n - \bar{\varphi}_n) - \alpha||\nabla\varphi_n||^2 + \alpha_0(\varphi_n, \varphi_n - \bar{\varphi}_n) \\ &\leq C_0^2(||\nabla\mu_n||\ ||\nabla\varphi_n||) - \alpha||\nabla\varphi_n||^2 + 2\alpha_0(||\varphi_n||^2 + ||\varphi_n - \bar{\varphi}_n||^2) \\ &\leq \tilde{C}(1 + ||\nabla\mu_n||)\end{aligned}$$

by the previous bounds on φ_n in $L^\infty(0, T, V)$ (and by Poincaré's inequality since $||\varphi_n - \bar{\varphi}_n|| \leq C_0||\nabla\varphi_n||$), where $\tilde{C}$ is independent of n. Considering now the mean value of μ_n, we obtain, since $(\Delta\varphi_n, 1) = 0$ and from property (3.3) of F'_λ:

$$\begin{aligned}|\bar{\mu}_n| &= |\frac{1}{|\Omega|}(\mu_n, 1)| = \frac{1}{|\Omega|}\int_\Omega |\Psi'_\lambda(\varphi_n)| \leq \frac{1}{|\Omega|}(\int_\Omega |F'_\lambda(\varphi_n)| + \alpha_0 \int_\Omega |\varphi_n|) \\ &\leq \frac{1}{|\Omega|}(\int_\Omega |F'_\lambda(\varphi_n)| + \alpha_0\sqrt{|\Omega|}\ ||\varphi_n||) \leq \frac{1}{|\Omega|}(C \cdot |\int_\Omega F'_\lambda(\varphi_n)(\varphi_n - \bar{\varphi}_n)| + C + \bar{C}) \\ &\leq \hat{C}(1 + ||\nabla\mu_n||)\end{aligned}$$

due to the previous bounds on F'_λ and φ_n. The previous estimates entail that, due to Poincaré's inequality (A.1) and Young's inequality:

$$||\mu_n||^2_{L^2(0,T;V)} \leq 2\int_0^T ||\mu_n - \bar{\mu}_n||^2_V + 2\int_0^T ||\bar{\mu}_n||^2_V$$
$$\leq 2C_0\int_0^T ||\nabla\mu_n||^2 + 4\hat{C}^2|\Omega|^2\int_0^T (1 + ||\nabla\mu_n||^2) \leq \bar{K}_0, \tag{3.45}$$

from the previous bound on $||\nabla\mu_n||$, we get that $\bar{K}_0$ is independent of n. From the equation for μ_n we can study, for further use, the following estimate (since $L^4(0,T;V_2) \hookrightarrow L^2(0,T;V_2)$ and $V \hookrightarrow H = L^2(\Omega)$ for μ_n):

$$\int_0^T ||F'_\lambda(\varphi_n)||^2 \leq C\int_0^T \{||\mu_n||^2 + ||\Delta\varphi_n||^2 + ||\varphi_n||^2\} \leq \bar{C} \tag{3.46}$$

with $\bar{C}$ independent of n, implying $F'_\lambda(\varphi_n) \in L^2(0,T;H)$.

We have now to address the time derivatives of the variables. We start from the temperature Θ_n: equation (3.19) can be rewritten as

$$\frac{d\Theta_n}{dt} + \hat{P}^*_n(\mathbf{u}_n\cdot\nabla\Theta_n + \bar{\mathcal{A}}(\Theta_n) + \frac{d\theta_g}{dt} + \mathbf{u}_n\cdot\nabla\theta_g + \bar{\mathcal{A}}(\theta_g)) = 0 \quad \text{in } V'_\theta \tag{3.47}$$

where $\hat{P}^*_n : \mathbb{V}'_n \to V'_\theta$ is the adjoint of the orthogonal projector $\hat{P}_n$: since $||\hat{P}_n||_{\mathcal{L}(V_\theta,\mathbb{V}_n)} \leq 1$, being a projector, also $||\hat{P}^*_n||_{\mathcal{L}(\mathbb{V}'_n,V'_\theta)} \leq 1$ for every $n \geq 1$, and the linear operator $\bar{\mathcal{A}} = \bar{\mathcal{A}}_{\varphi_n}$: $\mathbb{V}_n \to \mathbb{V}'_n$, such that $< \bar{\mathcal{A}}(\Theta_n), \xi >= (\kappa\nabla\Theta_n, \nabla\xi)$ for every $\xi \in \mathbb{V}_n$.

Then recalling hypothesis (1.6) for κ, we can start with

$$| < \bar{\mathcal{A}}(\Theta_n), \xi > | \leq k^*||\nabla\Theta_n||||\nabla\xi||.$$

So, by Poincaré's inequality, we can say

$$||\hat{P}^*_n(\bar{\mathcal{A}}(\Theta_n))||_{V'_\theta} \leq ||\bar{\mathcal{A}}(\Theta_n)||_{\mathbb{V}'_n} \leq k^*||\nabla\Theta_n||. \tag{3.48}$$

We now consider the transport term in the equation (3.47), applying boundary conditions: for every $\xi \in \mathbb{V}_n$, applying Holder's inequality

$$| < \mathbf{u}_n\cdot\nabla\Theta_n, \xi > | = |(\mathbf{u}_n\cdot\nabla\Theta_n, \xi)| = | - (\mathbf{u}_n\cdot\nabla\xi, \Theta_n)| \leq ||\mathbf{u}_n||_{L^4(\Omega)}||\Theta_n||_{L^4(\Omega)}||\nabla\xi||.$$

Thus, applying Cauchy-Schwarz and Young inequality and (A.1.4), we get

$$\begin{aligned}||\hat{P}_n^*(\mathbf{u}_n \cdot \nabla\Theta_n)||_{V_\theta'} \leq ||\mathbf{u}_n \cdot \nabla\Theta_n||_{\mathbb{V}_n'} \leq ||\mathbf{u}_n||_{L^4(\Omega)}||\Theta_n||_{L^4(\Omega)} &\leq \frac{1}{2}(||\mathbf{u}_n||^2_{L^4(\Omega)} + ||\Theta_n||^2_{L^4(\Omega)}) \\ &\leq \frac{C}{2}(||\mathbf{u}_n||\ ||\nabla\mathbf{u}_n|| + ||\mathbf{u}_n||^2 + ||\Theta_n||\ ||\nabla\Theta_n|| + ||\Theta_n||^2) \\ &\leq \bar{C}(1 + ||\nabla\mathbf{u}_n|| + ||\nabla\Theta_n||)\end{aligned}$$

where, due to uniform bounds (independent of time) on $\mathbf{u}_n$ and Θ_n (see (3.37 and (3.38)), $\bar{C}$ is a generic constant independent from n.

Furthermore we have

$$||\hat{P}_n^*(\bar{\mathcal{A}}(\theta_g))||_{V_\theta'} \leq ||\bar{\mathcal{A}}(\theta_g)||_{\mathbb{V}_n'} \leq k^*||\nabla\theta_g|| \leq C||g||_{1/2,\partial\Omega}. \tag{3.49}$$

We now consider the transport term in the equation (3.47): for every $\xi \in \mathbb{V}_n$, applying Holder's inequality and Sobolev embedding $V_\theta \hookrightarrow L^4(\Omega)$

$$| < \mathbf{u}_n \cdot \nabla\theta_g, \xi > | = |(\mathbf{u}_n \cdot \nabla\theta_g, \xi)| \leq ||\mathbf{u}_n||_{L^4(\Omega)}\ ||\nabla\theta_g||\ ||\xi||_{L^4(\Omega)} \leq C||\mathbf{u}_n||_{L^4(\Omega)}\ ||\nabla\theta_g||\ ||\nabla\xi||.$$

Thus, applying Cauchy-Schwarz's and Young's inequality and (3.24) and considering the lift operator's properties, we get

$$\begin{aligned}||\hat{P}_n^*(\mathbf{u}_n \cdot \nabla\theta_g)||_{V_\theta'} \leq ||\mathbf{u}_n \cdot \nabla\theta_g||_{\mathbb{V}_n'} \leq ||\mathbf{u}_n||_{L^4(\Omega)}||\nabla\theta_g|| &\leq ||\mathbf{u}_n||^{1/2}||\nabla\mathbf{u}_n||^{1/2}||\nabla\theta_g|| \\ &\leq \bar{C}(||\nabla\mathbf{u}_n|| + ||\nabla\theta_g||^2||\mathbf{u}_n||) \leq C(||\nabla\mathbf{u}_n|| + ||\nabla\theta_g||^2),\end{aligned}$$

since, by previous estimates, $||\mathbf{u}_n|| \leq C$.

The last extra term to be estimated is the time derivative of θ_g: since by Poincaré's inequality $| < \partial_t\theta_g, \xi > | \leq ||\partial_t\theta_g||\ ||\xi|| \leq C_0||\partial_t\theta_g||\ ||\nabla\xi||$ for every $\xi \in \mathbb{V}_n$,

$$||\hat{P}_n^*(\partial_t\theta_g)||_{V_\theta'} \leq ||\partial_t\theta_g||_{\mathbb{V}_n'} \leq C_0||\partial_t\theta_g(t)|| \leq C||\partial_t g||_{1/2,\partial\Omega}$$

by the regularity properties of the lift operator.

Considering also the other terms and the lift operator properties, we obtain the following estimate:

$$\begin{aligned}\left\|\frac{d\Theta_n}{dt}\right\|_{V_\theta'} &\leq \bar{C}(1 + ||\nabla\mathbf{u}_n|| + ||\nabla\Theta_n|| + ||\nabla\theta_g||^2 + ||\partial_t g||_{1/2,\partial\Omega} + ||g||_{1/2,\partial\Omega}) \\ &\leq \bar{C}(1 + ||\nabla\mathbf{u}_n|| + ||\nabla\Theta_n|| + ||g(t)||^2_{1/2,\partial\Omega} + ||\partial_t g||_{1/2,\partial\Omega} + ||g(t)||_{1/2,\partial\Omega})\end{aligned}$$

then, by Young's inequality we obtain

$$\int_0^T \left\|\frac{d\Theta_n}{dt}\right\|^2_{V'_\theta} \leq \tilde{C}\int_0^T (1+||\nabla \mathbf{u}_n||^2+||\nabla\Theta_n||^2+||g(t)||^4_{1/2,\partial\Omega}+||\partial_t g||^2_{1/2,\partial\Omega}+||g(t)||^2_{1/2,\partial\Omega})$$
$$\leq K(T)$$

since the right-hand side is bounded, indeed $g \in L^4(0,T,H^{1/2}(\partial\Omega)) \hookrightarrow L^2(0,T,H^{1/2}(\partial\Omega)))$ and $||\mathbf{u}_n||_{L^2(0,T;\mathbf{V}_\sigma)} \leq \bar{C}$. The constant $K(T)$ is independent of n and t.
Then we obtain that

$$\frac{d\Theta_n}{dt} \in \; L^2(0,T,V'_\theta). \tag{3.50}$$

We now pass to consider an estimate for the time derivative of φ_n. We can rewrite equation (3.18) as:

$$\frac{d\varphi_n}{dt} + \tilde{P}_n^*(\mathbf{u}_n\cdot\nabla\varphi_n + \tilde{\mathcal{A}}(\mu_n)) = 0 \quad \text{in } V' \tag{3.51}$$

with the same property as the previous case for the norm of the adjoint of the orthogonal projector. Starting from $\tilde{\mathcal{A}}(\mu_n) : \mathbb{Z}_n \to \mathbb{Z}'_n$ such that $< \tilde{\mathcal{A}}(\mu_n), \psi >= (\nabla\mu_n, \nabla\psi)$ for any $\psi \in \mathbb{Z}_n$:

$$| < \tilde{\mathcal{A}}(\mu_n), \psi > | \leq ||\nabla\mu_n||\;||\nabla\psi|| \leq ||\nabla\mu_n||\;||\psi||_{H^1},$$

thus we get

$$||\tilde{P}_n^*(\tilde{\mathcal{A}}(\mu_n))||_{V'} \leq ||\tilde{\mathcal{A}}(\mu_n)||_{\mathbb{Z}'_n} \leq ||\nabla\mu_n||. \tag{3.52}$$

We now consider the transport term: as before we have, applying the Sobolev embedding $V = H^1(\Omega) \hookrightarrow L^4(\Omega)$ and Poincaré's inequality for velocity field

$$| < \mathbf{u}_n\cdot\nabla\varphi_n, \psi > | = |(\mathbf{u}_n\cdot\nabla\varphi_n, \psi)| = | - (\mathbf{u}_n\cdot\nabla\psi, \varphi_n)| \leq ||\mathbf{u}_n||_{L^4(\Omega)}||\varphi_n||_{L^4(\Omega)}||\nabla\psi||$$
$$\leq C^2||\mathbf{u}_n||_V||\varphi_n||_V||\psi||_V \leq C^2\sqrt{C_0^2+1}||\nabla\mathbf{u}_n||\;||\varphi_n||_V||\psi||_V,$$

entailing

$$||\tilde{P}_n^*(\mathbf{u}_n\cdot\nabla\varphi_n)||_{V'} \leq ||\mathbf{u}_n\cdot\nabla\varphi_n||_{\mathbb{Z}'_n} \leq \hat{C}||\nabla\mathbf{u}_n||$$

by using the uniform bound (independent of time) on $||\varphi_n||_V$ obtained before: $\hat{C}$ is independent of n (and t as all the previous constants). So we can find a constraint for the aforementioned time derivative:

$$\left\|\frac{d\varphi_n}{dt}\right\|_{V'} \leq \hat{C}||\nabla \mathbf{u}_n|| + ||\nabla \mu_n||;$$

applying Young's inequality and integrating in time we get

$$\int_0^T \left\|\frac{d\varphi_n}{dt}\right\|_{V'}^2 \leq 2\hat{C}^2 \int_0^T ||\nabla \mathbf{u}_n||^2 + 2\int_0^T ||\nabla \mu_n||^2 \leq \bar{K}, \tag{3.53}$$

due to the estimates previously obtained for the terms in the right-hand side (as usual $\bar{K}$ depends only on initial data and at most T but not on n). So, for every $0 < T < +\infty$

$$\left\|\frac{d\varphi_n}{dt}\right\|_{L^2(0,T;V')} \leq \sqrt{\bar{K}}. \tag{3.54}$$

We conclude the estimates with the analysis of the time derivative of velocity field: equation (3.17) can be rewritten as

$$\frac{d\mathbf{u}_n}{dt} + P_n^*(\mathcal{B}(\mathbf{u}_n, \mathbf{u}_n) + \mathcal{A}(\mathbf{u}_n) + \varphi_n \nabla \mu_n - \Theta_n \mathbf{e}_2 - \theta_g \mathbf{e}_2) = 0 \quad \text{in } \mathbf{V}'_\sigma, \tag{3.55}$$

where $\mathcal{A} = \mathcal{A}_{\varphi_n,\theta_n} : \mathbb{W}_n \subset \mathbf{V}_\sigma \to \mathbb{W}'_n$ such that $< \mathcal{A}(\mathbf{u}_n), \mathbf{w} >= (\nu \nabla \mathbf{u}_n, \nabla \mathbf{w})$ for every $\mathbf{w} \in \mathbb{W}_n$ and $\mathcal{B} : \mathbb{W}_n \times \mathbb{W}_n \to \mathbb{W}'_n$, such that $< \mathcal{B}(\mathbf{u}_n, \mathbf{u}_n), \mathbf{w} >= b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{w})$ for every $\mathbf{w} \in \mathbb{W}_n$. Now, we have, recalling hypothesis (1.5) for ν,

$$| < \mathcal{A}(\mathbf{u}_n, \mathbf{w}) > | \leq \nu^* ||\nabla \mathbf{u}_n|| \; ||\nabla \mathbf{w}||$$

so that, by Poincaré's inequality and, as usual, the property of orthogonal projector P_n

$$||P_n^* \mathcal{A}(\mathbf{u}_n)||_{\mathbf{V}'_\sigma} \leq ||\mathcal{A}(\mathbf{u}_n)||_{\mathbb{W}'_n} \leq \nu^* ||\nabla \mathbf{u}_n||. \tag{3.56}$$

Also for $\mathcal{B}$ we obtain, from (A.6) and Poincaré's inequality (C_0 is the Poincaré's constant), for any $\mathbf{w} \in \mathbb{W}_n$

$$| < \mathcal{B}(\mathbf{u}_n, \mathbf{u}_n), \mathbf{w} > | \leq ||\mathbf{u}_n||^{\frac{1}{2}} ||\mathbf{u}_n||_1^{\frac{1}{2}} ||\mathbf{u}_n||^{\frac{1}{2}} ||\mathbf{u}_n||_1^{\frac{1}{2}} ||\mathbf{w}||_1 \leq C \; (C_0^2 + 1)^{\frac{3}{4}} \; ||\nabla \mathbf{u}_n|| \; ||\mathbf{u}_n|| \; ||\nabla \mathbf{w}||$$

so that always by Poincaré's inequality and from previous bounds on $\mathbf{u}_n$ in $L^\infty(0, T; \mathbf{H}_\sigma)$:

$$||P_n^*(\mathcal{B}(\mathbf{u}_n, \mathbf{u}_n))||_{\mathbf{V}'_\sigma} \leq ||\mathcal{B}(\mathbf{u}_n, \mathbf{u}_n)||_{\mathbf{W}'_n} \leq C \; (C_0^2 + 1)^{\frac{3}{4}} \; ||\mathbf{u}_n|| \; ||\nabla \mathbf{u}_n|| \leq \bar{C} \; ||\nabla \mathbf{u}_n||, \tag{3.57}$$

with $\bar{C}$ independent of t and n. Then, we have for any $\mathbf{w} \in \mathbf{W}_n$, using the Sobolev embedding $V \hookrightarrow L^4(\Omega)$ and Poincaré's inequality for $\mathbf{w}$,

$$|<\varphi_n \nabla \mu_n, \mathbf{w}>| = |(\varphi_n \nabla \mu_n, \mathbf{w})| \leq ||\varphi_n||_{L^4(\Omega)} ||\nabla \mu_n|| \; ||\mathbf{w}||_{L^4(\Omega)} \leq \tilde{C} ||\varphi_n||_V ||\nabla \mu_n|| \; ||\nabla \mathbf{w}||$$

so that, by the bound on φ_n in $L^\infty(0,T;V)$

$$||P_n^*(\varphi_n \nabla \mu_n)||_{\mathbf{V}'_\sigma} \leq ||\varphi_n \nabla \mu_n||_{\mathbb{W}'_n} \leq \tilde{C} ||\varphi_n||_V ||\nabla \mu_n|| \leq \bar{C} ||\nabla \mu_n|| \tag{3.58}$$

with $\bar{C}$ independent of n and t.

We are left to consider an estimate for the terms of temperature in (3.55): for every $\mathbf{w} \in \mathbb{W}_n$, by Poincaré's inequality

$$|<\Theta_n \mathbf{e}_2, \mathbf{w}>| = |(\Theta_n, \mathbf{e}_2 \cdot \mathbf{w})| \leq ||\Theta_n|| \; ||\mathbf{w}|| \leq C_0 \; ||\Theta_n|| \; ||\nabla \mathbf{w}||;$$

therefore we get, by the estimate on Θ_n in $L^\infty(0,T;H)$

$$||P_n^*(\Theta_n \mathbf{e}_2)||_{\mathbf{V}'_\sigma} \leq ||\Theta_n \mathbf{e}_2||_{\mathbb{W}'_n} \leq C_0 ||\Theta_n|| \leq \bar{C} \tag{3.59}$$

with $\bar{C}$ independent of t and n.

We have now to consider the last remaining term: for every $\mathbf{w} \in \mathbb{W}_n$, by Poincaré's inequality

$$|<-\theta_g \mathbf{e}_2, \mathbf{w}>| = |(\theta_g, \mathbf{e}_2 \cdot \mathbf{w})| \leq ||\theta_g|| \; ||\mathbf{w}|| \leq C_0 \; ||\theta_g|| \; ||\nabla \mathbf{w}||.$$

Therefore we get, using again the lift operator's properties:

$$||P_n^*(\theta_g \mathbf{e}_2)||_{\mathbf{V}'_\sigma} \leq ||\theta_g \mathbf{e}_2||_{\mathbb{W}'_n} \leq C_0 ||\theta_g|| \leq C ||\theta_g||_1 \leq \tilde{C} ||g(t)||_{1/2,\partial\Omega} \tag{3.60}$$

with $\bar{C}$ independent of t and n.

Since by assumption $g \in L^4(0,T,H^{1/2}(\partial\Omega)) \hookrightarrow L^2(0,T,H^{1/2}(\partial\Omega))$ we do not spoil the estimate on the derivative of the velocity and we can then conclude that

$$\left\| \frac{d\mathbf{u}_n}{dt} \right\|_{\mathbf{V}'_\sigma} \leq (\nu^* + \bar{C}) ||\nabla \mathbf{u}_n|| + \bar{C} ||\nabla \mu_n|| + \tilde{C} ||g(t)||_{1/2,\partial\Omega} + \bar{C}$$

which entails, by applying Young's inequality, integrating in time and using the previous obtained bounds, that:

$$\int_0^T \left\| \frac{d\mathbf{u}_n}{dt} \right\|^2_{\mathbf{V}'_\sigma} \leq K \Big(\int_0^T ||\nabla \mathbf{u}_n||^2 + \int_0^T ||\nabla \mu_n||^2 + \int_0^T ||g(t)||^2_{1/2,\partial\Omega} + T \Big) \leq \bar{K}$$

with $\bar{K} = \bar{K}(T)$ generic constant independent of n and t.

Thus we have:

$$\left\| \frac{d\mathbf{u}_n}{dt} \right\|_{L^2(0,T;\mathbf{V}'_\sigma)} \leq \sqrt{\bar{K}} \tag{3.61}$$

We are ready to do the last step to conclude the first part of the proof. From all the estimates above, which are independent of n, we deduce that the sequences $\mathbf{u}_n$, φ_n, μ_n, Θ_n are bounded in the corresponding spaces, independently of n: thus, by properties of reflexive spaces (apart from the cases $L^\infty(0,T;X)$, with X general Hilbert space, for which we apply the Banach-Alaoglu theorem) we obtain the following convergences, up to non relabeled subsequences: for any $0 \leq T < \infty$

$$\mathbf{u}_n \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;\mathbf{H}_\sigma) \tag{3.62}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{V}_\sigma) \tag{3.63}$$

$$\frac{d\mathbf{u}_n}{dt} \rightharpoonup \frac{d\mathbf{u}}{dt} \quad \text{in } L^2(0,T;\mathbf{V}'_\sigma) \tag{3.64}$$

$$\varphi_n \overset{*}{\rightharpoonup} \varphi \quad \text{in } L^\infty(0,T;H) \tag{3.65}$$

$$\varphi_n \rightharpoonup \varphi \quad \text{in } L^4(0,T;V_2) \tag{3.66}$$

$$\frac{d\varphi_n}{dt} \rightharpoonup \frac{d\varphi_n}{dt} \quad \text{in } L^2(0,T;V') \tag{3.67}$$

$$\mu_n \rightharpoonup \mu \quad \text{in } L^2(0,T;V) \tag{3.68}$$

$$\Theta_n \overset{*}{\rightharpoonup} \Theta \quad \text{in } L^\infty(0,T;H) \tag{3.69}$$

$$\Theta_n \rightharpoonup \Theta \quad \text{in } L^2(0,T;V_\theta) \tag{3.70}$$

$$\frac{d\Theta_n}{dt} \rightharpoonup \frac{d\Theta}{dt} \quad \text{in } L^2(0,T;V_\theta') \tag{3.71}$$

We recall that the weak convergence of time derivative to the derivative of the weak limit of the sequence of variables can be easily proven by means of integration by parts in time and due to the uniqueness of limit in distribution we get the result (see, e.g., [58]). Uniqueness of limit guarantees also that all the weak convergences in different spaces are to the same limit.

We notice that, defining $\theta = \Theta + \theta_g$, the solution $(\mathbf{u}, \varphi, \theta)$ has the required regularity, since, by the above convergences,

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}_\sigma) \cap L^2(0,T;\mathbf{V}_\sigma) \text{ and } \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}_\sigma'),$$

$$\varphi \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \text{ and } \partial_t \varphi \in L^2(0,T;V'),$$

$$\theta \in L^\infty(0,T;H) \cap L^2(0,T;V) \text{ and } \partial_t \theta \in L^2(0,T;V_\theta' + V'),$$

and the bounds on the corresponding norms are the same as for the approximating sequences, thus depending only on T and the initial data. We recall that $V_\theta' = H^{-1}(\Omega)$ and an element of V' can be seen, by restriction on V_θ, as an element of V_θ'; thus we have

$$\partial_t \theta \in L^2(0,T;V_\theta').$$

The fundamental step is now to derive strong convergences in order to pass to the limit for the nonlinear terms in the equations and show that the candidate solution is indeed a real weak solution to the problem: we exploit Theorem A.2.1: since $\mathbf{V}_\sigma \hookrightarrow \mathbf{H}_\sigma \equiv \mathbf{H}_\sigma' \hookrightarrow \mathbf{V}_\sigma'$ from the previous weak convergences, up to a non relabeled subsequence (furthermore, strong convergence implies convergence almost everywhere up to another subsequence), we get

$$\mathbf{u}_n \to \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{H}_\sigma) \text{ and a.e. in } \Omega \times (0,T). \tag{3.72}$$

Also, since $V \hookrightarrow H \equiv H' \hookrightarrow V'$ and since $L^4(0,T;V_2) \hookrightarrow L^2(0,T;V_2) \hookrightarrow L^2(0,T,V)$, a bounded sequence in $L^4(0,T;V_2)$ is also bounded in $L^2(0,T,V)$ and then we can deduce by compactness that

$$\varphi_n \to \varphi \quad \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega \times (0,T). \tag{3.73}$$

In conclusion, since $V_\theta \hookrightarrow H \equiv H' \hookrightarrow V_\theta'$ we deduce, by the same Theorem A.2.1, that

$$\Theta_n \to \Theta \quad \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega \times (0,T). \tag{3.74}$$

By standard argument we can now pass to the limit in the weak formulation of the problem; we start from equation (3.17): multiply the equation by $\omega \in C_0^\infty(0,T)$ and integrate in time between 0 and T. We obtain, fixing $n_k \leq n$

$$\int_0^T (< \dot{\mathbf{u}}_n, \mathbf{w} > + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{w}) + (\nu \nabla \mathbf{u}_n, \nabla \mathbf{w}) + (\varphi_n \nabla \mu_n, \mathbf{w}) - (\Theta_n + \theta_g, \mathbf{e}_2 \cdot \mathbf{w}))\omega(t) = 0$$
$$\forall \mathbf{w} \in \mathbb{W}_{n_k}.$$

Exploiting the convergences already shown, as done, e.g., in [18] and by the density of $\{\mathbb{W}_{n_k}\}_{n_k \geq 1}$ in $\mathbf{V}_\sigma$, equation (3.17) converges to the desired weak form:

$$\int_0^T (< \dot{\mathbf{u}}, \mathbf{w} > + b(\mathbf{u}, \mathbf{u}, \mathbf{w}) + (\nu \nabla \mathbf{u}, \nabla \mathbf{w}) + (\varphi \nabla \mu, \mathbf{w}) - (\theta, \mathbf{e}_2 \cdot \mathbf{w}))\omega(t) = 0$$
$$\forall \mathbf{w} \in \mathbf{V}_\sigma.$$

We now consider the equation (3.18): multiply the equation by $\chi \in C_0^\infty(0,T)$ and integrate in time between 0 and T, letting $n_k \leq n$. We obtain

$$\int_0^T (< \partial_t \varphi_n, v > + (\nabla \mu_n, \nabla v) + (\mathbf{u}_n \cdot \nabla \varphi_n, v))\chi(t) = 0 \quad \forall v \in \mathbb{Z}_{n_k}. \tag{3.75}$$

Again, by the density of $\{\mathbb{Z}_{n_k}\}_{n_k \geq 1}$ in V, as $n \to \infty$ the entire equation (3.75) converges as expected to

$$\int_0^T (< \partial_t \varphi, v > + (\nabla \mu, \nabla v) + (\mathbf{u} \cdot \nabla \varphi, v))\chi(t) = 0 \quad \forall v \in V. \tag{3.76}$$

In conclusion, we are left to consider the equation for the temperature, (3.19): multiply the equation by $\psi \in C_0^\infty(0,T)$ and integrate in time between 0 and T, letting $n_k \leq n$. We obtain

$$\int_0^T (< \partial_t \Theta_n, \xi > + (\kappa \nabla \Theta_n, \nabla \xi)$$
$$+ (\mathbf{u}_n \cdot \nabla \Theta_n, \xi) + < \partial_t \theta_g, \xi > + \kappa (\nabla \theta_g, \nabla \xi) + (\mathbf{u}_n \cdot \nabla \theta_g, \xi))\psi(t) = 0 \quad \forall \xi \in \mathbb{V}_{n_k}.$$

Defining $\theta = \Theta + \theta_g$, by the density of $\{\mathbb{V}_{n_k}\}_{n_k \geq 1}$ in V_θ, also this equation, as $n \to +\infty$, converges to

$$\int_0^T (< \partial_t \theta, \xi > + (\kappa \nabla \theta, \nabla \xi) + (\mathbf{u}_n \cdot \nabla \theta, \xi)) \psi(t) = 0 \quad \forall \xi \in V_\theta.$$

We recall that the convergence is possible due to the strong (and almost everywhere) convergences (3.74) and (3.73) to Θ, and thus $\theta = \Theta + \theta_g$, and of φ. Indeed, since κ and ν are globally Lipschitz functions, so they are continuous, we can reach the convergence $\nu(\varphi_n, \theta_n) \to \nu(\varphi, \theta)$ a.e. in $\Omega \times (0, T)$ and $\kappa(\theta_n) \to \kappa(\theta)$ a.e. in $\Omega \times (0, T)$ and then exploit, e.g., Lebesgue Theorem to reach the convergence of the integrals in the weak formulation. For example, in the case of κ we obtain, in the temperature equation,

$$\begin{aligned} &| \int_0^T (\kappa(\theta_n) \nabla \Theta_n, \nabla \xi) \Psi(t) - (\kappa(\theta) \nabla \Theta, \nabla \xi) \Psi(t) dt | \\ &\leq \int_0^T \int_\Omega |\kappa(\theta_n) \nabla \Theta_n \cdot \nabla \xi - \kappa(\theta_n) \nabla \Theta \cdot \nabla \xi| \, |\Psi(t)| dx dt \\ &+ \int_0^T \int_\Omega |\kappa(\theta_n) \nabla \Theta \cdot \nabla \xi - \kappa(\theta) \nabla \Theta \cdot \nabla \xi| \, |\Psi(t)| dx dt \end{aligned}$$

and the first term converges to zero since $\kappa(\theta_n) \leq k^*$ and $\Theta_n \rightharpoonup \Theta$, whereas the second one converges by Lebesgue Dominated Convergence Theorem. The same goes for the kinematic viscosity ν in the equation for velocity.

By the arbitrariness of ω, χ, ψ, since the space of functions of the kind $\Phi(t, x) = \sum_{k=1}^{N} \eta_k(t) \psi_k(x)$, where N is an integer, $\eta_k(t) \in C_0^\infty(0, T)$ and $\psi_k \in S$, with $S = \mathbf{V}_\sigma$, V, V_θ respectively, is dense in $C_0^0((0, T); S)$, thus in $L^2(0, T; S)$ (see [18], Chap. V, Secs. 1-2, or [43], Chap. 7, Sec. 7.1), we obtain that $\mathbf{u}, \varphi, \mu, \theta$ satisfy:

$$\begin{aligned} &\int_0^T < \partial_t \mathbf{u}, \mathbf{w} > dt + \int_0^T b(\mathbf{u}, \mathbf{u}, \mathbf{w}) + \int_0^T (\nu \nabla \mathbf{u}, \nabla \mathbf{w}) dt \\ &= - \int_0^T (\varphi \nabla \mu, \mathbf{w}) dt + \int_0^T (\theta, \mathbf{e}_2 \cdot \mathbf{w}) dt \quad \forall \mathbf{w} \in L^2(0, T; \mathbf{V}_\sigma) \end{aligned} \tag{3.77}$$

$$\int_0^T < \partial_t \varphi, v > dt + \int_0^T (\nabla \mu, \nabla v) dt + \int_0^T (\mathbf{u} \cdot \nabla \varphi, v) dt = 0 \quad \forall v \in L^2(0, T; V) \tag{3.78}$$

$$\int_0^T < \partial_t \theta, \xi > dt + \int_0^T (\kappa \nabla \theta, \nabla \xi) dt + \int_0^T (\mathbf{u} \cdot \nabla \theta, \xi) dt = 0 \quad \forall \xi \in L^2(0, T; V_\theta). \tag{3.79}$$

Then, by a standard argument (see, e.g., [18] or [93]), we obtain that $\mathbf{u}, \varphi, \mu, \theta$ satisfy:

$$< \partial_t \mathbf{u}, \mathbf{w} > + b(\mathbf{u}, \mathbf{u}, \mathbf{w}) + (\nu \nabla \mathbf{u}, \nabla \mathbf{w}) = -(\varphi \nabla \mu, \mathbf{w}) + (\theta, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_\sigma \tag{3.80}$$

$$< \partial_t \varphi, v > + (\nabla \mu, \nabla v) + (\mathbf{u} \cdot \nabla \varphi, v) = 0 \quad \forall v \in V \tag{3.81}$$

$$< \partial_t \theta, \xi > + (\kappa \nabla \theta, \nabla \xi) + (\mathbf{u} \cdot \nabla \theta, \xi) = 0 \quad \forall \xi \in V_\theta. \tag{3.82}$$

Since $\varphi \in L^2(0,T;V_2)$, we have from the equations, by integration by parts, that $\partial_{\mathbf{n}} \varphi \in H^{1/2}(\partial\Omega)$ and $\partial_{\mathbf{n}} \varphi = 0$ almost everywhere on $\partial\Omega \times (0,T)$.

We now consider the equation for μ: we multiply the equation by $\chi \in C_0^\infty(0,T)$, test against $v \in \mathbb{Z}_n$ and integrate by parts in time between 0 and T, obtaining

$$\int_0^T (\mu_n, v)\chi(t) = \int_0^T (\alpha \nabla \varphi_n, \nabla v)\chi(t) + \int_0^T (F'_\lambda(\varphi_n), v)\chi(t) - \int_0^T (\alpha_0 \varphi_n, v)\chi(t). \tag{3.83}$$

Since $\varphi_n \to \varphi$ almost everywhere on $\Omega \times (0,T)$, by continuity of F_λ we have also $F'_\lambda(\varphi_n) \to F'_\lambda(\varphi)$ almost everywhere on $\Omega \times (0,T)$. But from (3.46) we know that the sequence $F_\lambda(\varphi_n)$ is uniformly bounded in $L^2(0,T;H) \approx L^2(\Omega \times (0,T))$, therefore we can apply the weak Lebesgue Theorem A.1.8 to get that $F'_\lambda(\varphi_n) \rightharpoonup F'_\lambda(\varphi)$ in $L^2(0,T;H)$. We can then pass to the limit in the equation (3.83), as done above, and then by the arbitrariness of $\chi(t)$ and by density of $\{\mathbb{Z}_n\}_{n\geq 1}$ in H:

$$(\mu, v) = (-\alpha \Delta \varphi, v) + (F'_\lambda(\varphi), v) - (\alpha_0 \varphi, v) \quad \forall v \in H \quad \text{for a.a. } t \in (0,T) \tag{3.84}$$

where we could integrate by parts since $\varphi \in V_2$ for almost any $t \in (0,T)$ and then, since the equality holds $\forall v \in H$ and all the other terms belong at least to H for almost any $t \in (0,T)$,

$$\mu = -\alpha \Delta \varphi + \Psi'_\lambda(\varphi) \quad \text{a.e. in } \Omega \times (0,T). \tag{3.85}$$

In conclusion we are left to study the initial conditions.

First of all, due to the embeddings of Lions-Magenes Lemma A.2.2, we obtain that the initial conditions are reached in the strong L^2 sense:

$$\lim_{t\to 0} ||\mathbf{u}(t) - \mathbf{u}(0)|| = 0 \qquad \lim_{t\to 0} ||\varphi(t) - \varphi(0)|| = 0 \qquad \lim_{t\to 0} ||\theta(t) - \theta(0)|| = 0. \tag{3.86}$$

Indeed, $\mathbf{u} \in C([0,T],\mathbf{H}_\sigma)$, $\varphi \in C([0,T],H)$ and $\theta \in C([0,T],H)$ by continuous embeddings of the aforementioned lemma. We now study the value of such initial conditions: since the method is the same for the three conditions, we analyze only one of them, for example the one for temperature. Multiply equation (3.82) by $\chi(t) \in C^1([0,T])$ such that $\chi(0) = 1$ and $\chi(T) = 0$, integrating in time in (0,T) and apply integration by parts (Lemma A.2.2), we get:

$$\int_0^T \{-(\Theta(t) + \theta_g(t), \xi)\dot{\chi}(t) + (\kappa\nabla\Theta + \nabla\theta_g(t), \nabla\xi)\chi(t) + (\mathbf{u}\cdot(\nabla\Theta(t) + \nabla\theta_g(t)), \xi)\chi(t)\}$$
$$= (\Theta(0) + \theta_g(0), \xi) \qquad \forall \xi \in V_\theta.$$

We now recall that the formulation in (3.19) is equivalent to $\int_0^T (\partial_t\Theta_n + \partial_t\theta_g, \xi(t)) + (\kappa\nabla\Theta_n + \nabla\theta_g, \nabla\xi(t)) + (\mathbf{u}_n \cdot (\nabla\Theta_n + \nabla\theta_g), \xi(t)) = 0 \quad \forall \xi \in L^2(0,T;\mathbb{V}_n)$. Thus we obtain, from (3.19) applying the same procedure, with $\chi(t)\xi$ as test function, that

$$\int_0^T \{-(\Theta_n(t) + \theta_g(t), \xi)\dot{\chi}(t) + (\kappa\nabla\Theta_n(t) + \nabla\theta_g(t), \nabla\xi)\chi(t) + (\mathbf{u}\cdot(\nabla\Theta_n(t) + \nabla\theta_g(t)), \xi)\chi(t)\}$$
$$= (\hat{P}_n(\Theta_0) + \theta_g(0), \xi) \qquad \forall \xi \in \mathbb{V}_{n_k}.$$

As done before, we can pass to the limit in the previous equation, then by density of $\{\mathbb{V}_{n_k}\}_{n_k \geq 1}$ in V_θ we obtain

$$\int_0^T \{-(\Theta(t) + \theta_g(t), \xi)\dot{\chi}(t) + (\kappa\nabla\Theta + \nabla\theta_g(t), \nabla\xi)\chi(t) + (\mathbf{u}\cdot(\nabla\Theta(t) + \nabla\theta_g(t)), \xi)\chi(t)\}$$
$$= (\Theta_0 + \theta_g(0), \xi) \qquad \forall \xi \in V_\theta$$

since $\hat{P}_n$ is an orthogonal projector and thus:

$$(\hat{P}_n(\Theta_0), \xi) \to (\Theta_0, \xi) \qquad \text{for } n \to \infty.$$

Comparing the two equations, we obtain that $(\Theta_0, \xi) = (\Theta(0), \xi)$ $\forall \xi \in V_\theta$, which by density of V_θ in H implies that

$$(\Theta_0, \xi) = (\Theta(0), \xi) \qquad \forall \xi \in H.$$

Then, we get $\Theta(0) = \Theta_0$ a.e in Ω, i.e., since $\Theta(0) = \theta(0) - \theta_g(0) = \Theta_0 = \theta_0 - \theta_g(0)$,

$$\theta(0) = \theta_0 \text{ a.e. in } \Omega.$$

With an analogous argument, we get $\mathbf{u}(0) = \mathbf{u}_0$ and $\varphi(0) = \varphi_0$ almost everywhere in Ω, and this concludes the proof of the existence of a weak solution for any Ψ_λ, with $\lambda \in (0, \bar{\lambda}]$. From now on, we will call again $\mathbf{u}_\lambda, \varphi_\lambda, \mu_\lambda, \Theta_\lambda$ a weak solution to the problem with the substitution of Ψ with Ψ_λ, thus depending on λ.

3.3.2 Convergence to the original problem

Now we need to find further estimates in order to pass to the limit as λ goes to 0 (up to a subsequence). Analyzing the previous proof, we see that the only part in which there is a dependence on the value of λ is in the initial approximating energy $E_n(0)$ in equation (3.33), therefore, if we change the estimation of this term we can consider all the other bounds as valid independently of λ itself. Proceeding as already done, we reach, changing subscript n with λ: $E_\lambda(t) = \frac{1}{2}||\mathbf{u}_\lambda||^2 + \frac{1}{2}||\Theta_\lambda||^2 + \frac{\alpha}{2}||\nabla\varphi_\lambda||^2 + \int_\Omega (\Psi_\lambda(\varphi_\lambda) + \hat{C})$ and we have, exactly as before,

$$\begin{aligned}\frac{d}{dt}\{E_\lambda(t)\} + \mathcal{D}_\lambda(t) \leq \bar{C}||\mathbf{u}_\lambda||^2 + \frac{1}{2}||g(t)||^2_{1/2,\partial\Omega} + \tilde{C}||\partial_t g(t)||^2_{1/2,\partial\Omega} \\ + \bar{C}||g(t)||^2_{1/2,\partial\Omega} + \bar{C}||\mathbf{u}_\lambda||^2||g(t)||^4_{1/2,\partial\Omega},\end{aligned}$$

where we recall that

$$\mathcal{D}_\lambda(t) = ||\nabla\mu_\lambda||^2 + \frac{\nu_*}{2}||\nabla\mathbf{u}_\lambda||^2 + \frac{k_*}{2}||\nabla\Theta_\lambda||^2.$$

In conclusion, changing the constants, since $\int_\Omega (\Psi_\lambda(\varphi_\lambda) + \hat{C}) \geq 0$, we can get

$$\frac{d}{dt}E_\lambda(t)+\mathcal{D}_\lambda(t) \leq K_1(1+||g(t)||^4_{1/2,\partial\Omega})\, E_\lambda(t)+K_2(1+||g(t)||^2_{1/2,\partial\Omega}+||\partial_t g(t)||^2_{1/2,\partial\Omega}). \quad (3.87)$$

Thus, due to the regularity hypothesis made on the boundary datum g, we have that $\mathcal{Q} = K_1(1 + ||g||^4_{1/2,\partial\Omega}) \in L^1(0,T)$ and also $\mathcal{R} = K_2(1 + ||g||^2_{1/2,\partial\Omega} + ||\partial_t g||^2_{1/2,\partial\Omega}) \in L^1(0,T)$, we can apply Gronwall's Lemma (A.1.7), since E_λ is at least continuous in time: for any $t \in (0,T)$:

$$E_\lambda(t) \leq E_\lambda(0)e^{\int_0^t \mathcal{Q}(r)} + \int_0^t e^{\int_s^t \mathcal{Q}(r)}\mathcal{R}(s)ds \leq e^{\int_0^T \mathcal{Q}(r)}(E_\lambda(0) + \int_0^T \mathcal{R}(s)ds). \quad (3.88)$$

If we define the interface energy functional as $\mathcal{E}_\lambda(\varphi) = \dfrac{1}{2}||\nabla\varphi||^2 + \displaystyle\int_\Omega \Psi_\lambda(\varphi)$ and $\mathcal{E}(\varphi) = \dfrac{1}{2}||\nabla\varphi||^2 + \displaystyle\int_\Omega \Psi(\varphi)$, we obtain that $E_\lambda(0) = \mathcal{E}_\lambda(\varphi_0) + \dfrac{1}{2}||\mathbf{u}_0||^2 + \dfrac{1}{2}||\Theta_0||^2$. If we show that $\mathcal{E}_\lambda(\varphi_0) \leq \mathcal{E}(\varphi_0)$ we are done, since we have by hypothesis on the initial conditions that $\mathcal{E}(\varphi_0) < \infty$ (see Remark 1.2.2) and it does not depend on λ, clearly. From the property (3) Section 3.1 of F_λ, we know that $F_\lambda(s) \leq F(s)$, $\forall s \in \mathbb{R}$ (we recall that $F(s) = +\infty$ outside the interval $[-1, 1]$). Then we have that

$$\mathcal{E}_\lambda(\varphi_0) = \int_\Omega \{F_\lambda(\varphi_0) - \frac{1}{2}\alpha_0\varphi_0^2\}dx \leq \int_\Omega \{F(\varphi_0) - \frac{1}{2}\alpha_0\varphi_0^2\}dx = \mathcal{E}(\varphi_0)$$

as we needed. Thus, we have that

$$0 \leq E_\lambda(t) \leq \bar{C}_0 \tag{3.89}$$

where $\bar{C}_0$ depends only on initial conditions, but not on λ.

Since all the other estimates are still valid, by means of this new estimate, we deduce that the same convergences, up to subsequences, are valid: for any $0 \leq T < \infty$, as $\lambda \to 0$

$$\mathbf{u}_\lambda \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;\mathbf{H}_\sigma) \tag{3.90}$$

$$\mathbf{u}_\lambda \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{V}_\sigma) \tag{3.91}$$

$$\mathbf{u}_\lambda \to \mathbf{u} \quad \text{in } L^2(0,T;\mathbf{H}_\sigma) \text{ and a.e. in } \Omega \times (0,T) \tag{3.92}$$

$$\frac{d\mathbf{u}_\lambda}{dt} \rightharpoonup \frac{d\mathbf{u}}{dt} \quad \text{in } L^2(0,T;\mathbf{V}'_\sigma) \tag{3.93}$$

$$\varphi_\lambda \overset{*}{\rightharpoonup} \varphi \quad \text{in } L^\infty(0,T;H) \tag{3.94}$$

$$\varphi_\lambda \rightharpoonup \varphi \quad \text{in } L^4(0,T;V_2) \tag{3.95}$$

$$\varphi_\lambda \to \varphi \quad \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega \times (0,T) \tag{3.96}$$

$$\frac{d\varphi_\lambda}{dt} \rightharpoonup \frac{d\varphi_n}{dt} \quad \text{in } L^2(0,T;V') \tag{3.97}$$

$$\mu_\lambda \rightharpoonup \mu \quad \text{in } L^2(0,T;V) \tag{3.98}$$

$$\Theta_\lambda \overset{*}{\rightharpoonup} \Theta \quad \text{in } L^\infty(0,T;H) \tag{3.99}$$

$$\Theta_\lambda \rightharpoonup \Theta \quad \text{in } L^2(0,T;V_\theta) \tag{3.100}$$

$$\Theta_\lambda \to \Theta \quad \text{in } L^2(0,T;H) \text{ and a.e. in } \Omega \times (0,T) \tag{3.101}$$

$$\frac{d\Theta_\lambda}{dt} \rightharpoonup \frac{d\Theta}{dt} \quad \text{in } L^2(0,T;V_\theta'). \tag{3.102}$$

We notice again that, defining $\theta = \Theta + \theta_g$ the solution $(\mathbf{u}, \varphi, \theta)$ has the required regularity, since, by the above convergences,

$$\mathbf{u} \in L^\infty(0,T;\mathbf{H}_\sigma) \cap L^2(0,T;\mathbf{V}_\sigma) \text{ and } \partial_t \mathbf{u} \in L^2(0,T;\mathbf{V}_\sigma')$$

$$\varphi \in L^\infty(0,T;V) \cap L^4(0,T;V_2) \text{ and } \partial_t \varphi \in L^2(0,T;V'),$$

$$\theta \in L^\infty(0,T;H) \cap L^2(0,T;V) \text{ and } \partial_t \theta \in L^2(0,T;V_\theta' + V').$$

We recall that $V_\theta' = H^{-1}(\Omega)$ and an element of V' can be seen, by restriction on V_θ, as an element of V_θ'; thus we have

$$\partial_t \theta \in L^2(0,T;V_\theta').$$

Moreover, we get, with the same proof as for obtaining (3.46), that

$$\int_0^T ||F_\lambda'(\varphi_\lambda)||^2 \leq C \int_0^T \{||\mu_\lambda||^2 + ||\Delta\varphi_\lambda||^2 + ||\varphi_\lambda||^2\} \leq \bar{C}, \tag{3.103}$$

with $\bar{C}$ independent of λ, implying $F_\lambda'(\varphi_\lambda) \in L^2(0,T;L^2(\Omega))$.

We claim that the limit $(\mathbf{u}, \varphi, \Theta + \theta_g)$ is a weak solution of the initial problem with singular potential. This part is similar to [34]. The boundedness of φ can be proved by a standard argument as follows: for any fixed $\eta \in (0, 1/2)$ we can introduce the set $E^{\lambda}_{\eta} = \{(x,t) \in \Omega \times [0,T] : |\varphi_\lambda(x,t)| > 1-\eta\}$. From (3.103) we obtain, thanks to the monotonicity of F'_λ (by the convexity of F_λ), that, for some constant C independent of λ,

$$|E^{\lambda}_{\eta}| \le \frac{C}{min\{F'_\lambda(1-\eta), |F'_\lambda(\eta-1)|\}}.$$

By means of a double application of Fatou's lemma, remembering, by property (3), that $|F'_\lambda(s)| \to +\infty$ for every $|s| \ge 1$, we can pass to the limit as $\lambda \to 0$ and $\eta \to 0$, obtaining $|\{(x,t) \in \Omega \times [0,T] : |\varphi_\lambda(x,t)| \ge 1\}| = 0$, meaning that $\varphi \in L^\infty(\Omega \times (0,T))$ with $|\varphi(x,t)| < 1$ for almost any $(x,t) \in \Omega \times (0,T)$.

We now study the convergence for μ, in a similar way as done in the previous analysis. From the pointwise convergence of φ_λ, the previous property on its essential supremum, and the uniform convergence of F'_λ to F' on any compact set of (-1, 1), according to property (3), Section 3.1, we obtain:

$$\begin{aligned}|F'_\lambda(\varphi_\lambda) - F'(\varphi)| &\le |F'_\lambda(\varphi) - F'(\varphi)| + |F'_\lambda(\varphi_\lambda) - F'_\lambda(\varphi)| \\ &\le \sup_{s \in K \subset (-1,1)} |F'_\lambda(s) - F'(s)| + |F'_\lambda(\varphi_\lambda) - F'_\lambda(\varphi)| \to 0,\end{aligned}$$

where the second term in the right-hand side vanishes as $\lambda \to 0$ since F'_λ is continuous (property (1), Section 3.1), and the first one because we know that $|\varphi(x,t)| < 1$ almost everywhere in $\Omega \times (0,T)$. This means that exists a compact set $K \subset (-1,1)$ such that $|\varphi| \in K$ for every (x,t) in $\Omega \times (0,T)$ (up to redefinitions of φ on a zero Lebesgue measure set), and then we can apply the uniform convergence on K.

Therefore, this entails $F'_\lambda(\varphi_\lambda) \to F'(\varphi)$ almost everywhere on $\Omega \times (0,T)$. From weak Lebesgue theorem A.1.8 we then deduce that $F'_\lambda(\varphi_\lambda) \rightharpoonup F'(\varphi)$ in $L^2(0,T;L^2(\Omega))$. Then we can conclude as in the previous part of the proof, that

$$\mu = -\alpha\Delta\varphi + \Psi'(\varphi) \quad \text{a.e. in } \Omega \times (0,T). \tag{3.104}$$

In conclusion, extracting a subsequence $\lambda_k \to 0$, we can pass to the limit exactly as before for the equations for the velocity, the temperature and φ and also the initial conditions can be obtained in the same way. Therefore, we have concluded the proof of the existence of weak solutions of the problem in analysis. The regularity of the solutions follows directly from the weak convergences above. Finally, it is easily verified (by integration by parts) that $\partial_\nu \varphi = 0$ for almost every $(x,t) \in \partial\Omega \times (0,T)$, since $\varphi \in H^2(\Omega)$ for almost any $t \in (0,T)$. $\square$

3.4 Existence of more regular solutions

In this section we give the proofs of the existence of quasi-strong and strong solutions (Theorems 2.1.3 and 2.1.6, respectively).

3.4.1 Proof of theorem 2.1.3

The properties of regularity of the weak solution, already proven in the previous sections, are still valid (see Definition 1.1), but we need to come back again to Galerkin approximation, in order to prove further regularity estimates, which are needed to obtain a quasi-strong solution according to Definition 1.2. This approximating procedure is necessary in order to make the proof rigorous, since we need to exploit the regularity of the approximating functions and then pass to the limit to get the same estimates on the candidate solution. As explained in [61], we perform a cutoff procedure on the initial condition. The idea is to carry out a three level procedure: the Galerkin approximation, the approximation of the potential Ψ with Ψ_λ and the cut off procedure of the initial concentration. The solution will be obtained extracting a converging subsequence in all the three levels, showing that it converges to the solution of the original problem.

To perform the cutoff procedure, we introduce the globally Lipschitz function $h_k : \mathbb{R} \to \mathbb{R},\ k \in \mathbb{N}$, such that

$$h_k(z) = \begin{cases} -k, & z < -k, \\ z, & z \in [-k,k], \\ k, & z > k. \end{cases} \tag{3.105}$$

We define $\tilde{\mu}_{0,k} = h_k \circ \tilde{\mu}_0$, where $\tilde{\mu}_0 = -\alpha\Delta\varphi_0 + F'(\varphi_0) = \mu_0 + \alpha_0\varphi$. Since $\tilde{\mu}_0 \in V$, the result on compositions in Sobolev spaces [96] yelds $\tilde{\mu}_{0,k} \in V$, for any $k > 0$, and $\nabla\tilde{\mu}_{0,k} = \nabla\tilde{\mu}_0 \cdot \chi_{[-k.k]}(\tilde{\mu}_0)$, which in turn gives

$$||\tilde{\mu}_{0,k}||_1 \le ||\tilde{\mu}_0||_1. \tag{3.106}$$

For $k \in \mathbb{N}$ we consider the Neumann problem

$$\begin{cases} -\alpha\Delta\varphi_{0,k} + F'(\varphi_{0,k}) = \tilde{\mu}_{0,k} & \text{in } \Omega \\ \partial_{\mathbf{n}}\varphi_{0,k} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.107}$$

Thanks to Lemma B.1.1, there exists a unique solution to this problem such that $\varphi_{0,k} \in V_2$, $F'(\varphi_{0,k}) \in H$, which satisfies (3.107) almost everywhere in Ω and $\partial_{\mathbf{n}}\varphi_{0,k} = 0$ almost everywhere on $\partial\Omega$. In addition, by (B.3) and from (3.106) we get

$$||\varphi_{0,k}||_{V_2} \leq C(1 + ||\tilde{\mu}_0||). \tag{3.108}$$

Since $\tilde{\mu}_{0,k} \to \tilde{\mu}_0$ in H, Lemma B.1.1 also entails that $\varphi_{0,k} \to \varphi_0$ in V. As a consequence there exist an $\tilde{m} \in (0,1)$, independent of k, and $\bar{k}$ sufficiently large such that

$$||\varphi_{0,k}||_1 \leq 1 + ||\varphi_0||_1, \quad |\bar{\varphi}_{0,k}| \leq \tilde{m} < 1 \;\; \forall k > \bar{k}. \tag{3.109}$$

In addition, from Theorem B.1.2 with $f = \tilde{\mu}_{0,k}$, we obtain

$$||F'(\varphi_{0,k})||_{L^\infty(\Omega)} \leq ||\tilde{\mu}_{0,k}||_{L^\infty(\Omega)} \leq k.$$

In conclusion, since we know that F' goes to infinite if the argument is greater than or equal to 1, we can say that there exists $\delta = \delta(k) > 0$ such that

$$||\varphi_{0,k}||_{L^\infty(\Omega)} \leq 1 - \delta. \tag{3.110}$$

Now since F'' is continuous on $(-1,1)$, thus bounded on compact sets (see, e.g., [92]):

$$\nabla F'(\varphi_{0,k}) = F''(\varphi_{0,k})\nabla\varphi_{0,k} \in H.$$

Then, being $F'(\varphi_{0,k}) \in H$, we deduce that $F'(\varphi_{0,k}) \in V$. Due to $\alpha\Delta\varphi_{0,k} = -\tilde{\mu}_{0,k} + F'(\varphi_{0,k}) \in V$, we obtain that $\varphi_{0,k} \in H^3(\Omega)$. Finally, for any $\lambda \in (0,\lambda^*)$, where $\lambda^* = min\left\{\frac{1}{2}\delta(k), \bar{\lambda}\right\}$ ($\bar{\lambda}$ defined in the properties of F_λ, Section 3.1), since $F(z) = F_\lambda(z)$ for all $z \in [-1+\lambda, 1-\lambda]$, we infer from (3.110) that $-\alpha\Delta\varphi_{0,k} + F'_\lambda(\varphi_{0,k}) = \tilde{\mu}_{0,k}$, which entails

$$||-\alpha\Delta\varphi_{0,k} + F'_\lambda(\varphi_{0,k})||_1 \leq ||\tilde{\mu}_0||_1. \tag{3.111}$$

We now introduce the Galerkin approximation, having considered $\theta = \Theta + \theta_g$ (so $\Theta_0 = \theta_0 - \theta_g(0) \in H$), as already done in the previous case with weaker assumptions on initial conditions. With the same finite dimensional spaces introduced for the previous case, i.e., Theorem 2.1.1, we obtain the following problem depending on n, k, λ

- $\mathbf{u}^n_{k,\lambda}(0) = P_n(\mathbf{u}_0),\ \varphi^n_{k,\lambda}(0) = \tilde{P}_n(\varphi_{0,k}),\ \Theta^n_{k,\lambda}(0) = \hat{P}_n(\Theta_0)$

$$(\partial_t \mathbf{u}^n_{k,\lambda}, \mathbf{w}) + b(\mathbf{u}, \mathbf{u}^n_{k,\lambda}, \mathbf{w}) + \nu(\nabla \mathbf{u}^n_{k,\lambda}, \nabla \mathbf{w}) = \tag{3.112}$$

$$- (\varphi^n_{k,\lambda} \nabla \mu^n_{k,\lambda}, \mathbf{w}) + (\Theta^n_{k,\lambda}, \mathbf{e}_2 \cdot \mathbf{w}) + (\theta_g, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall \mathbf{w} \in \mathbb{W}_n$$

$$(\partial_t \varphi^n_{k,\lambda}, v) + (\nabla \mu^n_{k,\lambda}, \nabla v) + (\mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}, v) = 0 \quad \forall v \in \mathbb{Z}_n \tag{3.113}$$

$$(\partial_t \Theta^n_{k,\lambda}, \xi) + \kappa(\nabla \Theta^n_{k,\lambda}, \nabla \xi) + (\mathbf{u}^n_{k,\lambda} \cdot \nabla \Theta^n_{k,\lambda}, \xi) = \tag{3.114}$$

$$- < \partial_t \theta_g, \xi > - \kappa(\nabla \theta_g, \nabla \xi) - (\mathbf{u}^n_{k,\lambda} \cdot \nabla \theta_g, \xi) \quad \forall \xi \in \mathbb{V}_n$$

  for every $t \in (0, T)$.

- $\mu^n_{k,\lambda} = \tilde{P}_n(-\alpha \Delta \varphi^n_{k,\lambda} + \Psi'_\lambda(\varphi^n_{k,\lambda})) = -\alpha \Delta \varphi^n_{k,\lambda} + \tilde{P}_n(\Psi'_\lambda(\varphi^n_{k,\lambda}))$

Let us notice that the basis chosen for V is still a basis for $H^3(\Omega)$, then we have that

$$\varphi^n_{k,\lambda}(0) \to \varphi_{0,k} \quad \text{in } H^3(\Omega).$$

In turn, by the embedding $H^3(\Omega) \hookrightarrow L^\infty(\Omega)$, we get

$$\varphi^n_{k,\lambda}(0) \to \varphi_{0,k} \quad \text{in } L^\infty(\Omega)$$

Hence there exists $\bar{n} = \bar{n}(k)$, such that

$$||\varphi^n_{k,\lambda}(0)||_\infty \leq \frac{1}{2}\delta(k) + ||\varphi_{0,k}||_\infty \leq 1 - \frac{1}{2}\delta(k) \quad \forall n > \bar{n}. \tag{3.115}$$

For any $k > \tilde{k}$ we fix $\lambda \in (0, \lambda^*(k))$ and $n > \bar{n}(k)$. The existence of a solution and the first energy estimates are exactly the same as in the less regular case: the only difference is the choice of the approximations Ψ_λ (chosen to be (3.5)) and the choice of the initial condition for $\varphi^n_{k,\lambda}$: not a real problem due to the previous analysis. Indeed, the only difference in the proof already shown is in the term $E_n(0)$ (see (3.30)). This term is now estimated in the following way: introducing again the approximated energy

$$\tilde{E}_\lambda(\mathbf{v}, \psi, \theta) = E_n(t) - \hat{C} = \frac{1}{2}||\mathbf{v}||^2 + \frac{\alpha}{2}||\nabla \psi||^2 + \frac{1}{2}||\theta||^2 + \int_\Omega \Psi_\lambda(\psi) dx.$$

Since $\Psi_\lambda(z) \leq \Psi(z) \quad \forall z \in [-1,1]$ and from (3.115) we have that, being the essential supremum of $\varphi^n_{k,\lambda}(0)$ bounded in a compact set contained in $[-1,1]$, since the function Ψ is bounded on $[-1.1]$, $\Psi_\lambda(\varphi^n_{k,\lambda}(0)) \leq \Psi(\varphi^n_{k,\lambda}(0)) \leq K$ (here we do not exploit the property of being globally Lipschitz, since this choice of approximations Ψ_λ does not enjoy this property). Thus we have, by the well known properties of the orthogonal projectors, and from (3.109):

$$
\begin{aligned}
\tilde{E}_\lambda(\mathbf{u}^n_{k,\lambda}(0), \varphi^n_{k,\lambda}(0), \Theta^n_{k,\lambda}(0)) &= \frac{1}{2}||P_n(\mathbf{u}_0)||^2 + \frac{\alpha}{2}||\nabla \tilde{P}_n(\varphi_{0,k})||^2 \\
&+ \frac{1}{2}||\hat{P}_n(\Theta_0)||^2 + \int_\Omega \Psi_\lambda(\varphi^n_{k,\lambda}(0))dx \\
&\leq \frac{1}{2}||\mathbf{u}_0||^2 + \frac{\alpha}{2}||\nabla \varphi^n_{k,\lambda}||^2 + \frac{1}{2}||\Theta_0||^2 + K|\Omega| \\
&\leq \frac{1}{2}||\mathbf{u}_0||^2 + \frac{\alpha}{2}||\varphi_0||_1^2 + \frac{1}{2}||\Theta_0||^2 + K_1 \leq \bar{K}.
\end{aligned}
$$

where $\bar{K}$ does not depend on n, λ, k. After this technical passage, the first part of the proof goes identical, producing the following results (the constants defined here do not depend on n, λ, k).

$$||\Theta^n_{k,\lambda}||_{L^\infty(0,T;H)} \leq \bar{C} \tag{3.116}$$

$$||\mathbf{u}^n_{k,\lambda}||_{L^\infty(0,T;\mathbf{H}_\sigma)} \leq \bar{C} \tag{3.117}$$

$$||\mathbf{u}^n_{k,\lambda}||_{L^2(0,T;\mathbf{V}_\sigma)} \leq \bar{C} \tag{3.118}$$

$$||\varphi^n_{k,\lambda}||_{L^\infty(0,T;V)} \leq \bar{C} \tag{3.119}$$

$$||\Theta^n_{k,\lambda}||_{L^2(0,T;V_\theta)} \leq \bar{C} \tag{3.120}$$

$$\alpha||\Delta\varphi_n||^2 \leq \alpha_0||\nabla\varphi_n||^2 + ||\nabla\mu_n|| \; ||\varphi_n|| \leq k_1 \cdot (1 + ||\nabla\mu_n||) \tag{3.121}$$

entailing

$$||\varphi^n_{k,\lambda}||_{L^4(0,T;V_2)} \leq \bar{C}. \tag{3.122}$$

Then we have

$$||\mu^n_{k,\lambda}||_{L^2(0,T;V)} \leq \bar{C} \tag{3.123}$$

$$||\mu^n_{k,\lambda}||_1 \leq C(1 + ||\nabla\mu^n_{k,\lambda}||) \tag{3.124}$$

$$\left\| \frac{d\Theta^n_{k,\lambda}}{dt} \right\|_{L^2(0,T;V'_\theta)} \le \bar{C} \tag{3.125}$$

$$\left\| \frac{d\varphi^n_{k,\lambda}}{dt} \right\|_{V'} \le \tilde{C}||\nabla \mathbf{u}^n_{k,\lambda}|| + ||\nabla \mu^n_{k,\lambda}||. \tag{3.126}$$

Then

$$\left\| \frac{d\varphi^n_{k,\lambda}}{dt} \right\|_{L^2(0,T;V')} \le \bar{C} \tag{3.127}$$

$$\left\| \frac{d\mathbf{u}^n_{k,\lambda}}{dt} \right\|_{L^2(0,T;\mathbf{V}'_\sigma)} \le \bar{C} \tag{3.128}$$

where $\bar{C} = \bar{C}(T)$ for any $0 < T < +\infty$.

We now pass to analyze higher order energy estimates: taking $v = \partial_t \mu^n_{k,\lambda}$, we obtain

$$\frac{1}{2}\frac{d}{dt}||\nabla \mu^n_{k,\lambda}||^2 + (\partial_t \mu^n_{k,\lambda}, \partial_t \varphi^n_{k,\lambda}) + (\partial_t \mu^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}) = 0. \tag{3.129}$$

Since $\bar{\varphi}^n_{k,\lambda}$ is constant in time, by the regularity of the eigenfunctions we have $\overline{\partial_t \varphi^n_{k,\lambda}} = \partial_t \bar{\varphi}^n_{k,\lambda} = 0$, then $\partial_t \varphi^n_{k,\lambda} \in V_0$ and thus $\partial_t \varphi^n_{k,\lambda} \in V'_0$ for all $t \in [0,T]$, then we can use the equivalent norm on V'_0, $||\cdot||_*$ (see the Appendix, (B.18)).

Thus we get by Cauchy-Schwartz's and Young's inequality:

$$\begin{aligned}
\alpha_0||\partial_t \varphi^n_{k,\lambda}||^2 = \alpha_0(\nabla \partial_t \varphi^n_{k,\lambda}, \nabla \bar{A_0}^{-1} \partial_t \varphi^n_{k,\lambda}) &\le \alpha_0 ||\nabla \partial_t \varphi^n_{k,\lambda}||\ ||\nabla \bar{A_0}^{-1} \partial_t \varphi^n_{k,\lambda}|| \\
&\le \frac{\alpha}{2}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + \frac{\alpha_0^2}{2\alpha}||\nabla \bar{A_0}^{-1} \partial_t \varphi^n_{k,\lambda}||^2 \\
&= \frac{\alpha}{2}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + \frac{\alpha_0^2}{2\alpha}||\partial_t \varphi^n_{k,\lambda}||^2_*.
\end{aligned}$$

Then, from property (3.4) of Ψ_λ, we deduce

$$\begin{aligned}
(\partial_t \mu^n_{k,\lambda}, \partial_t \varphi^n_{k,\lambda}) &= \alpha||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + (\Psi''_\lambda(\varphi^n_{k,\lambda})\partial_t \varphi^n_{k,\lambda}, \partial_t \varphi^n_{k,\lambda}) \\
&\ge \alpha||\nabla \partial_t \varphi^n_{k,\lambda}||^2 - \alpha_0||\partial_t \varphi^n_{k,\lambda}||^2 \\
&\ge \alpha||\nabla \partial_t \varphi^n_{k,\lambda}||^2 - \frac{\alpha}{2}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 - \frac{\alpha_0^2}{2\alpha}||\partial_t \varphi^n_{k,\lambda}||^2_* \\
&= \frac{\alpha}{2}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 - K||\partial_t \varphi^n_{k,\lambda}||^2_*.
\end{aligned}$$

Moreover we observe:

$$(\partial_t \mu^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}) = \frac{d}{dt}[(\mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}, \mu^n_{k,\lambda})] - (\partial_t \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}, \mu^n_{k,\lambda}) - (\mathbf{u}^n_{k,\lambda} \cdot \nabla \partial_t \varphi^n_{k,\lambda}, \mu^n_{k,\lambda}). \tag{3.130}$$

From Holder's inequality, the embedding of $V \hookrightarrow L^6(\Omega)$ and (3.124), we then have:

$$
\begin{aligned}
(\mu^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda} \cdot \nabla \partial_t \varphi^n_{k,\lambda}) &\leq ||\mu^n_{k,\lambda}||_{L^6(\Omega)} \, ||\mathbf{u}^n_{k,\lambda}||_{L^3(\Omega)} \, ||\nabla \partial_t \varphi^n_{k,\lambda}|| \\
&\leq \frac{\alpha}{4} ||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + C||\mu^n_{k,\lambda}||^2_{L^6(\Omega)} \, ||\mathbf{u}^n_{k,\lambda}||^2_{L^3(\Omega)} \\
&\leq \frac{\alpha}{4} ||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + C(1 + ||\mu^n_{k,\lambda}||^2)||\mathbf{u}^n_{k,\lambda}||^2_{L^3(\Omega)}.
\end{aligned}
$$

Since on V_0' the norm $|| \cdot ||_*$ is equivalent to the natural norm and since $|| \cdot ||_{V_0'} \leq || \cdot ||_{V'}$, because $V_0 \subset V$, due to (3.126) we also have that

$$
||\partial_t \varphi^n_{k,\lambda}||_* \leq \bar{C}(||\nabla \mathbf{u}^n_{k,\lambda}|| + ||\nabla \mu^n_{k,\lambda}||). \tag{3.131}
$$

The overall result, summing up (3.129 and (3.130), is the following:

$$
\begin{aligned}
\frac{d}{dt}\{(\mu^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}) + \frac{1}{2}||\nabla \mu^n_{k,\lambda}||^2\} &= (\partial_t \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}, \mu^n_{k,\lambda}) \\
&+ (\mathbf{u}^n_{k,\lambda} \cdot \nabla \partial_t \varphi^n_{k,\lambda}, \mu^n_{k,\lambda}) - (\partial_t \mu^n_{k,\lambda}, \partial_t \varphi^n_{k,\lambda}) \\
&\leq \frac{\alpha}{4}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + C(1 + ||\mu^n_{k,\lambda}||^2)||\mathbf{u}^n_{k,\lambda}||^2_{L^3(\Omega)} \\
&- \frac{\alpha}{2}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 + K||\partial_t \varphi^n_{k,\lambda}||^2_* + (\partial_t \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}, \mu^n_{k,\lambda}).
\end{aligned}
$$

Then we obtain, by (3.131) and Young's inequality for $||\partial_t \varphi^n_{k,\lambda}||^2_*$:

$$
\begin{aligned}
\frac{d}{dt}\{(\mu^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}) + \frac{1}{2}||\nabla \mu^n_{k,\lambda}||^2\} + \frac{\alpha}{4}||\nabla \partial_t \varphi^n_{k,\lambda}||^2 \\
\leq C_2(1 + ||\mathbf{u}^n_{k,\lambda}||^2_{L^3(\Omega)})(1 + ||\nabla \mu^n_{k,\lambda}||^2 + ||\nabla \mathbf{u}^n_{k,\lambda}||^2) \\
+ (\partial_t \mathbf{u}^n_{k,\lambda} \cdot \nabla \varphi^n_{k,\lambda}, \mu^n_{k,\lambda}).
\end{aligned} \tag{3.132}
$$

Now, taking $\mathbf{w} = \partial_t \mathbf{u}^n_{k,\lambda}$ in the equation for the velocity we obtain

$$
\begin{aligned}
||\partial_t \mathbf{u}^n_{k,\lambda}||^2 + b(\mathbf{u}^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda}, \partial_t \mathbf{u}^n_{k,\lambda}) + \nu(\nabla \mathbf{u}^n_{k,\lambda}, \nabla \partial_t \mathbf{u}^n_{k,\lambda}) \\
= (\mu^n_{k,\lambda} \nabla \varphi^n_{k,\lambda}, \partial_t \mathbf{u}^n_{k,\lambda}) + (\Theta^n_{k,\lambda}, \mathbf{e}_2 \cdot \partial_t \mathbf{u}) + (\theta_g, \mathbf{e}_2 \cdot \partial_t \mathbf{u}).
\end{aligned} \tag{3.133}
$$

Now, we have, by (3.24) applied to both $\mathbf{u}^n_{k,\lambda}$ and $\nabla \mathbf{u}^n_{k,\lambda}$ the Sobolev embedding $\mathbf{V}_\sigma \hookrightarrow$

$[L^4(\Omega)]^2$ and then (B.10):

$$
\begin{aligned}
|b(\mathbf{u}^n_{k,\lambda},\mathbf{u}^n_{k,\lambda},\partial_t\mathbf{u}^n_{k,\lambda})| &\le ||\mathbf{u}^n_{k,\lambda}||_{L^4(\Omega)}||\nabla\mathbf{u}^n_{k,\lambda}||_{L^4(\Omega)}\ ||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le C||\mathbf{u}^n_{k,\lambda}||^{1/2}||\nabla\mathbf{u}^n_{k,\lambda}||^{1/2}||\nabla\mathbf{u}^n_{k,\lambda}||^{1/2}||\nabla\mathbf{u}^n_{k,\lambda}||_1^{1/2}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le C||\mathbf{u}^n_{k,\lambda}||^{1/2}||\nabla\mathbf{u}^n_{k,\lambda}||\ ||\mathbf{u}^n_{k,\lambda}||^{1/2}_{H^2(\Omega)}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le \bar{C}||\mathbf{u}^n_{k,\lambda}||^{1/2}||\nabla\mathbf{u}^n_{k,\lambda}||\ ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^{1/2}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le \tilde{C}||\nabla\mathbf{u}^n_{k,\lambda}||\ ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^{1/2}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + C(||\nabla\mathbf{u}^n_{k,\lambda}||^4 + ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2),
\end{aligned}
$$

where in the last two passages we exploited (3.117) and the generalized Young's inequality. We then analyze, integrating by parts and applying Cauchy-Schwartz's and Young's inequality and then again (B.10):

$$
\begin{aligned}
-\nu(\nabla\mathbf{u}^n_{k,\lambda},\nabla\partial_t\mathbf{u}^n_{k,\lambda}) = \nu(\Delta\mathbf{u}^n_{k,\lambda},\partial_t\mathbf{u}^n_{k,\lambda}) &\le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + C||\mathbf{u}^n_{k,\lambda}||^2_{H^2(\Omega)} \\
&\le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + \bar{C}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2.
\end{aligned}
$$

Then, by Holder's inequality, the Sobolev embedding $V \hookrightarrow L^6(\Omega)$ and $V_2 \hookrightarrow W^{1,3}(\Omega)$ and the estimate (3.124), we have:

$$
\begin{aligned}
(\mu^n_{k,\lambda}\nabla\varphi^n_{k,\lambda},\partial_t\mathbf{u}^n_{k,\lambda}) &\le ||\mu^n_{k,\lambda}||_{L^6(\Omega)}||\nabla\varphi||_{L^3(\Omega)}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le C||\mu^n_{k,\lambda}||_1\ ||\varphi^n_{k,\lambda}||_{H^2(\Omega)}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le \bar{C}(1+||\nabla\mu^n_{k,\lambda}||)\ ||\varphi^n_{k,\lambda}||_{H^2(\Omega)}||\partial_t\mathbf{u}^n_{k,\lambda}|| \\
&\le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + K||\varphi^n_{k,\lambda}||^2_{H^2(\Omega)}(1+||\nabla\mu^n_{k,\lambda}||^2).
\end{aligned}
\tag{3.134}
$$

We are now left with the terms in temperature: by Cauchy-Schwartz's inequality and Young's inequality we get, since we know (3.116):

$$
|(\Theta^n_{k,\lambda},\mathbf{e}_2\cdot\partial_t\mathbf{u}^n_{k,\lambda})| \le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + C||\Theta^n_{k,\lambda}||^2 \le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + K. \tag{3.135}
$$

Analogously we obtain, by the regularity of the lift operator:

$$
|(\theta_g,\mathbf{e}_2\cdot\partial_t\mathbf{u}^n_{k,\lambda})| \le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + C||\theta_g||^2 \le \frac{1}{6}||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + C||g(t)||^2_{1/2,\partial\Omega}. \tag{3.136}
$$

To sum up, we can estimate:

$$
\begin{aligned}
||\partial_t \mathbf{u}^n_{k,\lambda}||^2 \leq & \frac{1}{6}||\partial_t \mathbf{u}^n_{k,\lambda}||^2 + C(||\nabla \mathbf{u}^n_{k,\lambda}||^4 + ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2) \\
& + \frac{1}{6}||\partial_t \mathbf{u}^n_{k,\lambda}||^2 + \bar{C}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 \\
& + \frac{1}{6}||\partial_t \mathbf{u}^n_{k,\lambda}||^2 + K||\varphi^n_{k,\lambda}||^2_{H^2(\Omega)}(1 + ||\nabla \mu^n_{k,\lambda}||^2) \\
& + \frac{1}{6}||\partial_t \mathbf{u}^n_{k,\lambda}||^2 + K \\
& + \frac{1}{6}||\partial_t \mathbf{u}^n_{k,\lambda}||^2 + C||g(t)||^2_{1/2,\partial\Omega},
\end{aligned}
$$

implying, that

$$
||\partial_t \mathbf{u}^n_{k,\lambda}||^2 \leq \hat{C}\{||\nabla \mathbf{u}^n_{k,\lambda}||^4 + ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + ||\varphi^n_{k,\lambda}||^2_{H^2(\Omega)}(1 + ||\nabla \mu^n_{k,\lambda}||^2) + ||g(t)||^2_{1/2,\partial\Omega} + 1\}. \tag{3.137}
$$

Let now $\mathbf{w} = \mathbf{A}\mathbf{u}^n_{k,\lambda}$ in the same equation for the velocity. Since we know that $\mathbf{A}\mathbf{u}^n_{k,\lambda} \in L^2(0, T, \mathbf{H}_\sigma)$ (see [97]): there exists $p^n_{k,\lambda} \in L^2(0, T; V)$ such that $-\Delta \mathbf{u}^n_{k,\lambda} + \nabla p^n_{k,\lambda} = \mathbf{A}\mathbf{u}^n_{k,\lambda}$ almost everywhere in $\Omega \times (0, T)$. Then, since $(\partial_t \mathbf{u}^n_{k,\lambda}, \nabla p^n_{k,\lambda}) = 0$, being the velocity divergence free,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\nabla \mathbf{u}^n_{k,\lambda}||^2 + b(\mathbf{u}^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda}) - \nu(\Delta \mathbf{u}^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda}) = & (\mu^n_{k,\lambda}\nabla\varphi^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda}) \\
& + (\Theta^n_{k,\lambda}, \mathbf{e}_2 \cdot \mathbf{u}^n_{k,\lambda}) + (\theta_g, \mathbf{e}_2 \cdot \mathbf{u}^n_{k,\lambda}).
\end{aligned}
$$

But, since $-\Delta \mathbf{u}^n_{k,\lambda} + \nabla p^n_{k,\lambda} = \mathbf{A}\mathbf{u}^n_{k,\lambda}$ and $(-\nabla p^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda}) = 0$ (to see this, it is enough to integrate by parts and exploit $\mathbf{A}\mathbf{u}^n_{k,\lambda} \in \mathbf{H}_\sigma$ for almost any t in $(0, T)$), we obtain

$$
(-\nu\Delta \mathbf{u}^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda}) = \nu||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2.
$$

For the transport term we apply the same inequalities as the previous case, to reach (3.137) obtaining

$$
\begin{aligned}
|b(\mathbf{u}^n_{k,\lambda}, \mathbf{u}^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda})| &\leq ||\mathbf{u}^n_{k,\lambda}||_{L^4(\Omega)} ||\nabla \mathbf{u}^n_{k,\lambda}||_{L^4(\Omega)} \; ||\mathbf{A}\mathbf{u}^n_{k,\lambda}|| \\
&\leq C||\mathbf{u}^n_{k,\lambda}||^{1/2} ||\nabla \mathbf{u}^n_{k,\lambda}||^{1/2} ||\nabla \mathbf{u}^n_{k,\lambda}||^{1/2} ||\nabla \mathbf{u}^n_{k,\lambda}||_1^{1/2} ||\mathbf{A}\mathbf{u}^n_{k,\lambda}|| \\
&\leq C||\mathbf{u}^n_{k,\lambda}||^{1/2} ||\nabla \mathbf{u}^n_{k,\lambda}|| \; ||\mathbf{u}^n_{k,\lambda}||^{1/2}_{H^2(\Omega)} ||\mathbf{A}\mathbf{u}^n_{k,\lambda}|| \\
&\leq \bar{C}||\mathbf{u}^n_{k,\lambda}||^{1/2} ||\nabla \mathbf{u}^n_{k,\lambda}|| \; ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^{3/2} \\
&\leq \tilde{C}||\nabla \mathbf{u}^n_{k,\lambda}|| \; ||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^{3/2} \\
&\leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + C||\nabla \mathbf{u}^n_{k,\lambda}||^4.
\end{aligned}
$$

Proceeding in the analysis, as done in the case of (3.134):

$$
\begin{aligned}
(\mu^n_{k,\lambda} \nabla \varphi^n_{k,\lambda}, \mathbf{A}\mathbf{u}^n_{k,\lambda}) &\leq ||\mu^n_{k,\lambda}||_{L^6(\Omega)} ||\nabla \varphi||_{L^3(\Omega)} ||\mathbf{A}\mathbf{u}^n_{k,\lambda}|| \\
&\leq C||\mu^n_{k,\lambda}||_1 \; ||\varphi^n_{k,\lambda}||_{H^2(\Omega)} ||\mathbf{A}\mathbf{u}^n_{k,\lambda}|| \\
&\leq \bar{C}(1 + ||\nabla \mu^n_{k,\lambda}||) \; ||\varphi^n_{k,\lambda}||_{H^2(\Omega)} ||\mathbf{A}\mathbf{u}^n_{k,\lambda}|| \\
&\leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + K||\varphi^n_{k,\lambda}||^2_{H^2(\Omega)}(1 + ||\nabla \mu^n_{k,\lambda}||^2).
\end{aligned}
$$

Again in the case of the terms for temperature we repeat the same analysis as in (3.135) and (3.136):

$$
|(\Theta^n_{k,\lambda}, \mathbf{e}_2 \cdot \mathbf{A}\mathbf{u}^n_{k,\lambda})| \leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + C||\Theta^n_{k,\lambda}||^2 \leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + K. \tag{3.138}
$$

Analogously we obtain, by the regularity of the lift operator:

$$
|(\theta_g, \mathbf{e}_2 \cdot \mathbf{A}\mathbf{u}^n_{k,\lambda})| \leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + C||\theta_g||^2 \leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + C||g(t)||^2_{1/2,\partial\Omega}. \tag{3.139}
$$

To sum up we can estimate:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\nabla \mathbf{u}^n_{k,\lambda}||^2 + \nu||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 &\leq \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + C||\nabla \mathbf{u}^n_{k,\lambda}||^4 \\
&\quad + \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + K||\varphi^n_{k,\lambda}||^2_{H^2(\Omega)}(1 + ||\nabla \mu^n_{k,\lambda}||^2) \\
&\quad + \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + K \\
&\quad + \frac{\nu}{8}||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + C||g(t)||^2_{1/2,\partial\Omega},
\end{aligned}
$$

$$\frac{1}{2}\frac{d}{dt}||\nabla \mathbf{u}_{k,\lambda}^n||^2+\frac{\nu}{2}||\mathbf{A}\mathbf{u}_{k,\lambda}^n||^2 \le \bar{C}(||\nabla \mathbf{u}_{k,\lambda}^n||^4+||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2(1+||\nabla \mu_{k,\lambda}^n||^2)+||g(t)||_{1/2,\partial\Omega}^2+1). \tag{3.140}$$

If we then multiply (3.137) by $\bar{\omega} = \dfrac{\nu}{4\hat{C}}$ and then sum up together with (3.140):

$$\frac{1}{2}\frac{d}{dt}||\nabla \mathbf{u}_{k,\lambda}^n||^2+\frac{\nu}{2}||\mathbf{A}\mathbf{u}_{k,\lambda}^n||^2+\bar{\omega}||\partial_t\mathbf{u}_{k,\lambda}^n||^2 \le \tilde{K}(||\nabla \mathbf{u}_{k,\lambda}^n||^4+||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2(1+||\nabla \mu_{k,\lambda}^n||^2)$$
$$+||g(t)||_{1/2,\partial\Omega}^2+1).$$

If we add equation (3.132) and we set

$$\Lambda(t) = (\mu_{k,\lambda}^n, \mathbf{u}_{k,\lambda}^n\cdot\nabla\varphi_{k,\lambda}^n)+\frac{1}{2}||\nabla\mu_{k,\lambda}^n||^2+\frac{1}{2}||\nabla \mathbf{u}_{k,\lambda}^n||^2,$$

then we get

$$\begin{aligned}\frac{d}{dt}\Lambda(t)+\frac{\nu}{2}||\mathbf{A}\mathbf{u}_{k,\lambda}^n||^2+\bar{\omega}||\partial_t\mathbf{u}_{k,\lambda}^n||^2+\frac{\alpha}{4}||\nabla\partial_t\varphi_{k,\lambda}^n||^2 &\\ \le C_2(1+||\mathbf{u}_{k,\lambda}^n||_{[L^3(\Omega)]^2}^2)&(1+||\nabla\mu_{k,\lambda}^n||^2+||\nabla \mathbf{u}_{k,\lambda}^n||^2)\\ +(\partial_t\mathbf{u}_{k,\lambda}^n\cdot\nabla\varphi_{k,\lambda}^n,\mu_{k,\lambda}^n)&\\ \tilde{K}(||\nabla \mathbf{u}_{k,\lambda}^n||^4+||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2&(1+||\nabla \mu_{k,\lambda}^n||^2)\\ +||g(t)||_{1/2,\partial\Omega}^2+1).&\end{aligned} \tag{3.141}$$

We are now left to estimate one last term: we have by Holder's inequality, the Sobolev embedding $V \hookrightarrow L^6(\Omega)$ and $V_2 \hookrightarrow W^{1,3}(\Omega)$ and the estimate (3.124):

$$\begin{aligned}(\partial_t\mathbf{u}_{k,\lambda}^n\cdot\nabla\varphi_{k,\lambda}^n,\mu_{k,\lambda}^n) &\le ||\partial_t\mathbf{u}_{k,\lambda}^n||\ ||\nabla\varphi_{k,\lambda}^n||_{L^3(\Omega)}||\mu_{k,\lambda}^n||_{L^6(\Omega)}\\ &\le \frac{\bar{\omega}}{2}||\partial_t\mathbf{u}_{k,\lambda}^n||^2+C||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2||\mu_{k,\lambda}^n||_1^2\\ &\le \frac{\bar{\omega}}{2}||\partial_t\mathbf{u}_{k,\lambda}^n||^2+K||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2(1+||\nabla\mu_{k,\lambda}^n||^2).\end{aligned}$$

Then we reach from (3.141):

$$\begin{aligned}\frac{d}{dt}\Lambda(t)+\frac{\nu}{2}||\mathbf{A}\mathbf{u}_{k,\lambda}^n||^2+\frac{\bar{\omega}}{2}||\partial_t\mathbf{u}_{k,\lambda}^n||^2+\frac{\alpha}{4}||\nabla\partial_t\varphi_{k,\lambda}^n||^2 &\\ \le C_2(1+||\mathbf{u}_{k,\lambda}^n||_{L^3(\Omega)}^2)&(1+||\nabla\mu_{k,\lambda}^n||^2+||\nabla \mathbf{u}_{k,\lambda}^n||^2)\\ +\tilde{K}(||\nabla \mathbf{u}_{k,\lambda}^n||^4+||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2&(1+||\nabla \mu_{k,\lambda}^n||^2)\\ +||g(t)||_{1/2,\partial\Omega}^2+1).&\end{aligned} \tag{3.142}$$

We now show that Λ is bounded from below: we have, since $\mathbf{u}_{k,\lambda}^n \in L^\infty(0,T;\mathbf{H}_\sigma)$ and using (3.24), the Sobolev embedding $V \hookrightarrow L^4(\Omega)$, (3.124) and generalized Young's inequality:

$$\begin{aligned}(\mathbf{u}_{k,\lambda}^n \cdot \nabla\varphi_{k,\lambda}^n, \mu_{k,\lambda}^n) &\leq ||\mathbf{u}_{k,\lambda}^n||_{L^4(\Omega)}||\nabla\varphi_{k,\lambda}^n||\ ||\mu_{k,\lambda}^n||_{L^4(\Omega)} \\ &\leq C||\mathbf{u}_{k,\lambda}^n||^{1/2}||\nabla\mathbf{u}_{k,\lambda}^n||^{1/2}||\mu_{k,\lambda}^n||_1 \\ &\leq \bar{C}||\mathbf{u}_{k,\lambda}^n||^{1/2}||\nabla\mathbf{u}_{k,\lambda}^n||^{1/2}(1+||\nabla\mu_{k,\lambda}^n||) \\ &\leq \frac{1}{4}||\nabla\mathbf{u}_{k,\lambda}^n||^2 + \frac{1}{4}||\nabla\mu_{k,\lambda}^n||^2 + C'.\end{aligned}$$

Then we have

$$-(\mathbf{u}_{k,\lambda}^n \cdot \nabla\varphi_{k,\lambda}^n, \mu_{k,\lambda}^n) \leq |(\mathbf{u}_{k,\lambda}^n \cdot \nabla\varphi_{k,\lambda}^n, \mu_{k,\lambda}^n)| \leq \frac{1}{4}||\nabla\mathbf{u}_{k,\lambda}^n||^2 + \frac{1}{4}||\nabla\mu_{k,\lambda}^n||^2 + C',$$

obtaining

$$(\mathbf{u}_{k,\lambda}^n \cdot \nabla\varphi_{k,\lambda}^n, \mu_{k,\lambda}^n) \geq -\frac{1}{4}||\nabla\mathbf{u}_{k,\lambda}^n||^2 - \frac{1}{4}||\nabla\mu_{k,\lambda}^n||^2 - C'.$$

We now get

$$\Lambda(t) \geq \frac{1}{4}||\nabla\mathbf{u}_{k,\lambda}^n||^2 + \frac{1}{4}||\nabla\mu_{k,\lambda}^n||^2 - C'$$

and in the end, defining $\tilde{\Lambda}(t) = \Lambda(t) + C'$ we obtain

$$\tilde{\Lambda}(t) \geq \frac{1}{4}||\nabla\mathbf{u}_{k,\lambda}^n||^2 + \frac{1}{4}||\nabla\mu_{k,\lambda}^n||^2 \geq 0.$$

Then, from the previous estimates, it easily seen that

$$\tilde{\Lambda}(t) \leq \tilde{C}(1 + ||\nabla\mu_{k,\lambda}^n||^2 + ||\nabla\mathbf{u}_{k,\lambda}^n||^2).$$

Since

$$\tilde{\Lambda}(t)^2 \geq \frac{1}{16}(||\nabla\mathbf{u}_{k,\lambda}^n||^4 + 2||\nabla\mathbf{u}_{k,\lambda}^n||^2||\nabla\mu_{k,\lambda}^n||^2 + ||\nabla\mu_{k,\lambda}^n||^4)$$

and since, by previous estimate (3.121) and Young's inequality,

$$||\varphi_{k,\lambda}^n||_{H^2(\Omega)}^2 \leq C(1 + ||\nabla\mu_{k,\lambda}^n||) \leq C(1 + ||\nabla\mu_{k,\lambda}^n||)^2 \leq C''(1 + ||\nabla\mu_{k,\lambda}^n||^2),$$

we are able to deduce, from estimate 3.142, $(\frac{d}{dt}\Lambda(t) = \frac{d}{dt}\tilde{\Lambda}(t))$, exploiting the Sobolev embedding $[H^1(\Omega)]^2 \hookrightarrow [L^3(\Omega)]^2$ together with Poincaré's inequality for the term $||\mathbf{u}_{k,\lambda}^n||_{[L^3(\Omega)]^2}^2$, that

$$\frac{d}{dt}\tilde{\Lambda}(t) \leq K(1 + ||g(t)||_{1/2,\partial\Omega}^2 + \tilde{\Lambda}(t)^2). \tag{3.143}$$

Moreover, $\tilde{\Lambda} \in L^1(0,T)$, since

$$\int_0^T \tilde{\Lambda}(t) \leq \tilde{C}T + \int_0^T ||\nabla \mu_{k,\lambda}^n||^2 + \int_0^T ||\nabla \mathbf{u}_{k,\lambda}^n||^2 \leq \bar{C} < \infty \tag{3.144}$$

where the last term is due to (3.123) and (3.118). We can then apply Gronwall's Lemma A.1.7 to (3.143) with $a = K\tilde{\Lambda} \in L^1(0,T)$ and $b = K(1 + ||g||^2_{1/2,\partial\Omega}) \in L^1(0,T)$ obtaining

$$\begin{aligned} \tilde{\Lambda}(t) &\leq \tilde{\Lambda}(0)e^{K\int_0^t \tilde{\Lambda}} ds + \int_0^t e^{K\int_s^t \tilde{\Lambda}} K(1 + ||g(t)||^2_{1/2,\partial\Omega}) ds \\ &\leq e^{K\bar{C}}[\tilde{\Lambda}(0) + KT + K||g||^2_{L^2(0,T;H^{1/2}(\partial\Omega))}]. \end{aligned} \tag{3.145}$$

Since $g \in L^2(0,T;H^{1/2}(\partial\Omega))$ we are only left to estimate $\tilde{\Lambda}(0)$: since the projectors are orthogonal, by Holder's inequality and Sobolev embeddings $\mathbf{V}_\sigma \hookrightarrow [L^3(\Omega)]^2$ and $V \hookrightarrow L^6(\Omega)$:

$$\begin{aligned} \tilde{\Lambda}(0) &= (\mu_{k,\lambda}^n(0), \mathbf{u}_{k,\lambda}^n(0) \cdot \nabla \varphi_{k,\lambda}^n(0)) + \frac{1}{2}||\nabla \mu_{k,\lambda}^n(0)||^2 + \frac{1}{2}||\nabla \mathbf{u}_{k,\lambda}^n(0)||^2 + C' \\ &\leq (\mu_{k,\lambda}^n(0), P_n(\mathbf{u}_0) \cdot \nabla \tilde{P}_n(\varphi_{0,k})) + \frac{1}{2}||\nabla \mu_{k,\lambda}^n(0)||^2 + \frac{1}{2}||\nabla P_n(\mathbf{u}_0)||^2 + C' \\ &\leq ||P_n(\mathbf{u}_0)||_{L^3(\Omega)}||\mu_{k,\lambda}^n(0)||_{L^6(\Omega)}||\nabla \tilde{P}_n(\varphi_{0,k})|| + \frac{1}{2}||\nabla \mu_{k,\lambda}^n(0)||^2 + \frac{1}{2}||\nabla \mathbf{u}_0||^2 + C' \\ &\leq ||\nabla P_n(\mathbf{u}_0)||\ ||\mu_{k,\lambda}^n(0)||_1\ ||\nabla \varphi_{0,k}|| + \frac{1}{2}||\nabla \mu_{k,\lambda}^n(0)||^2 + \frac{1}{2}||\nabla \mathbf{u}_0||^2 + C' \\ &\leq ||\nabla \mathbf{u}_0||\ ||\mu_{k,\lambda}^n(0)||_1\ ||\nabla \varphi_{0,k}|| + \frac{1}{2}||\nabla \mu_{k,\lambda}^n(0)||^2 + \frac{1}{2}||\nabla \mathbf{u}_0||^2 + C'; \end{aligned}$$

now from (3.109) and by Young's inequality we get

$$\begin{aligned} ||\nabla \mathbf{u}_0||\ ||\mu_{k,\lambda}^n(0)||_1\ ||\nabla \varphi_{0,k}|| &\leq ||\nabla \mathbf{u}_0||\ ||\mu_{k,\lambda}^n(0)||_1(1 + ||\varphi_0||_1) \\ &\leq \frac{1}{2}||\nabla \mathbf{u}_0||^2 + C\ ||\mu_{k,\lambda}^n(0)||_1^2\ (1 + ||\varphi_0||_1^2). \end{aligned}$$

The only term to be handled is then the last one in the previous inequality. Indeed we have, by the orthogonality of the projector $\tilde{P}_n$, by definition of Ψ_λ and triangular inequality, using definition $\tilde{\mu}_{0,k} = -\alpha\Delta\varphi_{0,k} + F'_\lambda(\varphi_{0,k})$ with $||\tilde{\mu}_{0,k}||_1 \leq ||\tilde{\mu}_0||_1$ (see (3.106))

$$\begin{aligned} ||\mu_{k,\lambda}^n(0)||_1 &= ||\tilde{P}_n(-\alpha\Delta\varphi_{k,\lambda}^n(0) + \Psi'_\lambda(\varphi_{k,\lambda}^n(0)))||_1 \\ &\leq ||-\alpha\Delta\varphi_{k,\lambda}^n(0) + \Psi'_\lambda(\varphi_{k,\lambda}^n(0))||_1 \\ &\leq ||-\alpha\Delta\varphi_{k,\lambda}^n(0) + F'_\lambda(\varphi_{k,\lambda}^n(0))||_1 + \alpha_0||\varphi_{k,\lambda}^n(0)||_1 \\ &\leq ||-\alpha\Delta\varphi_{k,\lambda}^n(0) + F'_\lambda(\varphi_{k,\lambda}^n(0)) + \alpha\Delta\varphi_{0,k} - F'_\lambda(\varphi_{0,k})||_1 + ||\tilde{\mu}_{0,k}||_1 + \alpha_0||\varphi_{k,\lambda}^n(0)||_1 \\ &\leq ||\varphi_{k,\lambda}^n(0) - \varphi_{0,k}||_{H^3(\Omega)} + ||F'_\lambda(\varphi_{k,\lambda}^n(0)) - F'_\lambda(\varphi_{0,k})||_1 + C\ (||\tilde{\mu}_0||_1 + ||\varphi_0||_1). \end{aligned}$$

Since $\varphi^n_{k,\lambda}(0) = \tilde{P}_n(\varphi_{0,k})$, we know that $\varphi^n_{k,\lambda}(0) \to \varphi_{0,k}$ in $H^3(\Omega)$ as $n \to \infty$, thus the first term in the right-hand side is bounded and $||\varphi^n_{k,\lambda}(0)||_{H^3(\Omega)} \leq C \quad \forall n \geq 0$. For the second one we have

$$||F'_\lambda(\varphi^n_{k,\lambda}(0)) - F'_\lambda(\varphi_{0,k})||_1 \leq ||F'_\lambda(\varphi^n_{k,\lambda}(0)) - F'_\lambda(\varphi_{0,k})|| + ||\nabla(F'_\lambda(\varphi^n_{k,\lambda}(0)) - F'_\lambda(\varphi_{0,k}))||.$$

The second term in the right-hand side (and in a similar way the first term) can be estimated as follows, exploiting the fact that F_λ and its first and second derivatives coincide with F on $[-1+\lambda^*, 1-\lambda^*]$:

$$\begin{aligned}
||\nabla(F'_\lambda(\varphi^n_{k,\lambda}(0)) - F'_\lambda(\varphi_{0,k}))|| &\leq ||F''_\lambda(\varphi^n_{k,\lambda}(0))\nabla\varphi^n_{k,\lambda}(0) - F''_\lambda(\varphi_{0,k})\nabla\varphi_{0,k}|| \\
&\leq ||F''_\lambda(\varphi_{0,k})\nabla(\varphi^n_{k,\lambda}(0) - \varphi_{0,k})|| \\
&+ ||(F''_\lambda(\varphi^n_{k,\lambda}(0)) - F''_\lambda(\varphi_{0,k}))\nabla\varphi_{0,k}|| \\
&\leq C(\max_{z\in[-1+\lambda^*,1-\lambda^*]} |F''(z)| \\
&+ \max_{z\in[-1+\lambda^*,1-\lambda^*]} |F'''(z)|)||\varphi^n_{k,\lambda}(0) - \varphi_{0,k}||_1.
\end{aligned}$$

The quantity between brackets is finite because $F \in \mathcal{C}^3(-1,1)$ and it depends only on λ^* and thus only on k. We recall that, even though the maxima of F'' and F''' could explode as $k \to +\infty$, if $\lambda^* \to 0$, the presence of the norm $||\varphi^n_{k,\lambda}(0) - \varphi_{0,k}||_1$, which goes to zero as $n \to +\infty$ (we have convergence in $H^3(\Omega)$ and thus in V), allows the possibility of choosing a sufficiently large n so that the estimated difference $||\nabla(F'_\lambda(\varphi^n_{k,\lambda}(0)) - F'_\lambda(\varphi_{0,k}))||$ is arbitrarily small for every k.

In conclusion, we can infer that for any fixed $k > \tilde{k}$, $\lambda \in (0, \lambda^*)$ and $n > \bar{n}$ (where $\bar{n}$ is sufficiently large, as already noticed):

$$\tilde{\Lambda}(0) \leq C(\mathbf{u}_0, \varphi_0, \Theta_0) < \infty$$

In view of (3.145) we deduce that

$$\sup_{t\in[0,T]} ||\nabla\mathbf{u}^n_{k,\lambda}|| + \sup_{t\in[0,T]} ||\nabla\mu^n_{k,\lambda}|| \leq C_1 \tag{3.146}$$

where $C_1 > 0$ depends on T and on initial data, but not on k, λ, n. Moreover an integration in time of the ODE for $\Lambda(t)$, leads to

$$\int_0^T (||\mathbf{A}\mathbf{u}^n_{k,\lambda}||^2 + ||\partial_t\mathbf{u}^n_{k,\lambda}||^2 + ||\nabla\partial_t\varphi^n_{k,\lambda}||^2)dt \leq C_2 \tag{3.147}$$

where $C_2 > 0$ depends on T and on initial data, but not on $k,\ \lambda,\ n$. Then, we have, besides the other uniform estimates obtained from the weaker case (which are already enough to pass to the limit, as we have seen), we have also that

- $\mathbf{u}^n_{k,\lambda}$ is uniformly bounded in $L^\infty(0,T;\mathbf{V}_\sigma)\cap L^2(0,T;\mathbf{W}_\sigma)\cap H^1(0,T;\mathbf{H}_\sigma)$
- $\varphi^n_{k,\lambda}$ is uniformly bounded in $L^\infty(0,T;V)\cap L^4(0,T;V_2)\cap H^1(0,T;V)$
- $\mu^n_{k,\lambda}$ is uniformly bounded in $L^\infty(0,T;V)$.

We notice that, defining $\theta=\Theta+\theta_g$, the solution $(\mathbf{u},\varphi,\mu,\theta)$ has the required regularity, since, by the above convergences,

$$\mathbf{u}\in L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma) \text{ and } \partial_t\mathbf{u}\in L^2(0,T;\mathbf{V}_\sigma'),$$

$$\varphi\in L^\infty(0,T;V)\cap L^4(0,T;V_2) \text{ and } \partial_t\varphi\in L^2(0,T;V'),$$

$$\theta\in L^\infty(0,T;H)\cap L^2(0,T;V) \text{ and } \partial_t\theta\in L^2(0,T;V_\theta'+V'),$$

and the bounds on the corresponding norms are the same as for the approximating sequences, thus depending only on T and the initial data. We recall that $V_\theta'=H^{-1}(\Omega)$ and an element of V' can be seen, by restriction on V_θ, as an element of V_θ'; thus we have

$$\partial_t\theta\in L^2(0,T;V_\theta').$$

By standard methods and by the uniqueness of weak limits, we deduce that the candidate solution is indeed a solution to the weak problem, passing to the limit in $k,\ \lambda$ and n, up to subsequences. Clearly, by the same argument, (for λ small enough, F_λ is convex, see [52], Lemma 2) as for the existence of weak solutions (Section 3.3.2), we deduce that $|\varphi(x,t)|<1$ almost everywhere in $\Omega\times(0,T)$.

By the regularity for $\mathbf{u}$ and φ we have also that the initial conditions are reached in the pointwise sense, $\mathbf{u}(\cdot,0)=\mathbf{u}_0$ and $\varphi(\cdot,0)=\varphi_0$ in Ω. We can now deduce the last regularity issues.

Since $\partial_t\varphi+\mathbf{u}\cdot\nabla\varphi\in L^2(0,T;V)$, we infer from classical regularity theory of homogeneous Neumann operator that $\mu\in L^2(0,T;H^3(\Omega))$, $\partial_{\mathbf{n}}\mu=0$ almost everywhere on

$\partial\Omega \times (0,T)$ and $\partial_t\varphi + \mathbf{u}\cdot\nabla\varphi = \Delta\mu$ holds almost everywhere in $\Omega\times(0,T)$. Finally we can recover the pressure, arguing as in [97], since we have (we can use the global θ, since by the regularity of Θ we deduce its regularity, due to $\theta = \Theta + \theta_g$)

$$\mathbf{f} = \mu\nabla\varphi - \partial_t\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \theta\mathbf{e}_2 \in L^2(0,T;[L^2(\Omega)]^2);$$

then we deduce that the pressure π satisfies

$$\int_0^T ||\pi||^2_{W^{1,2}} \leq K \int_0^T ||\mathbf{f}||^2 < \infty,$$

therefore $\pi \in L^2(0,T;V)$ and $\nabla\pi = \mathbf{f}$ a.e. in $\Omega\times(0,T)$.

From the regularity $\mu \in L^\infty(0,T;V)$, Theorem B.1.2 implies that $\varphi \in L^\infty(0,T;W^{2,p}(\Omega))$ and $F'(\varphi) \in L^\infty(0,T;L^p(\Omega))$ for any $2 \leq p < \infty$. Furthermore, thanks to the growth condition (1.3) we deduce, by Theorem B.1.2, that $F''(\varphi) \in L^\infty(0,T;L^p(\Omega))$ for any $p \in (2,\infty)$.

In conclusion, arguing as in [61] with the incremental quotients we deduce that $\partial_t\mu$ exists and belongs to $L^2(0,T;V')$.

Thus by Lemma A.2.2 we deduce that $\mu \in C([0,T];H)$. □

3.4.2 Proof of Theorem 2.1.6

To prove the existence of strong solutions, most of the proof is already done in the previous section. Indeed the hypotheses on the initial data and on boundary datum g fulfill the ones needed in the aforementioned theorem.

Passing again through the Galerkin setting, we can reproduce the proof of that theorem, adding further higher-order bounds for the temperature approximation, and then passing to the limit in k, λ (extracting, as usual, a subsequence $\lambda_j \to 0$) and n, up to subsequences, in the same way. In particular, we add the following estimates.

We recall that $\Theta_0 = \theta_0 - \theta_g(0)$, therefore $\Theta_0 \in V_\theta$.

Let $\xi = \partial_t\Theta^n_{k,\lambda}$ as a test function in the equation for the temperature:

$$\begin{aligned}\frac{k}{2}\frac{d}{dt}||\nabla\Theta^n_{k,\lambda}||^2 + ||\partial_t\Theta^n_{k,\lambda}||^2 = &-(\mathbf{u}^n_{k,\lambda}\cdot\nabla\Theta^n_{k,\lambda}, \partial_t\Theta^n_{k,\lambda}) - (\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g, \partial_t\Theta^n_{k,\lambda})\\ &+\kappa(\Delta\theta_g, \partial_t\Theta^n_{k,\lambda}) - (\partial_t\theta_g, \partial_t\Theta^n_{k,\lambda}). \qquad (3.148)\end{aligned}$$

Now, $\Theta^n_{k,\lambda}$ solves the following elliptic problem:

$$\begin{cases} -\kappa\Delta\Theta^n_{k,\lambda} = f & \text{a.e. in } \Omega\times(0,T) \\ \Theta^n_{k,\lambda} = 0 & \text{on } \partial\Omega, \end{cases}$$

where f summarizes all the terms present in the equation for the temperature. We can then apply elliptic regularity estimates (see e.g.[20]) and obtain that

$$||\Theta^n_{k,\lambda}||_{H^2(\Omega)} \le C||f||,$$

meaning, by Young's inequality, that

$$||\Theta^n_{k,\lambda}||^2_{H^2(\Omega)} \le C(||\partial_t\Theta^n_{k,\lambda}||^2 + ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\Theta^n_{k,\lambda}||^2 + ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g||^2 + ||\Delta\theta_g||^2 + ||\partial_t\theta_g||^2). \quad (3.149)$$

By Holder's inequality, a standard Sobolev embedding and from the uniform bound on $\nabla\mathbf{u}^n_{k,\lambda}$, we have (cf. proof of Theorem 2.1.3):

$$||\mathbf{u}^n_{k,\lambda}\cdot\nabla\Theta^n_{k,\lambda}||^2 \le ||\mathbf{u}^n_{k,\lambda}||^2_{[L^4(\Omega)]^2}||\nabla\Theta^n_{k,\lambda}||^2_{L^4(\Omega)} \le C||\nabla\mathbf{u}^n_{k,\lambda}||^2||\nabla\Theta^n_{k,\lambda}||^2_{L^4(\Omega)} \le K||\nabla\Theta^n_{k,\lambda}||^2_{L^4(\Omega)}. \quad (3.150)$$

From Lemma A.1.4, in a similar way as done in [101] and [102], we deduce by Young's inequality, for a fixed $\delta > 0$

$$K||\nabla\Theta^n_{k,\lambda}||^2_{L^4(\Omega)} \le KC(||\nabla\Theta^n_{k,\lambda}||\ ||D^2\Theta^n_{k,\lambda}|| + ||\nabla\Theta^n_{k,\lambda}||^2) \le \delta||\Theta^n_{k,\lambda}||^2_{H^2(\Omega)} + \tilde{C}||\nabla\Theta^n_{k,\lambda}||^2. \quad (3.151)$$

From this estimate we then deduce, choosing $\delta = \frac{1}{2}$, that

$$\begin{aligned} &||\Theta^n_{k,\lambda}||^2_{H^2(\Omega)} \\ &\le \bar{C}(||\partial_t\Theta^n_{k,\lambda}||^2 + ||\nabla\Theta^n_{k,\lambda}||^2 + ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g||^2 + ||\Delta\theta_g||^2 + ||\partial_t\theta_g||^2) + \frac{1}{2}||\Theta^n_{k,\lambda}||^2_{H^2(\Omega)}. \end{aligned}$$

We are left to consider only one term, which is estimated by means of Holder's inequality, Sobolev embedding and estimate A.1.4 (which implies $||\nabla\theta_g||_{L^4(\Omega)} \le C||\theta_g||_{H^2(\Omega)}$) and from the uniform estimate (3.146) on $\nabla\mathbf{u}^n_{k,\lambda}$, which was derived in Theorem 2.1.3 and is still valid in this context:

$$||\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g||^2 \le ||\mathbf{u}^n_{k,\lambda}||^2_{[L^4(\Omega)]^2}||\nabla\theta_g||^2_{L^4(\Omega)} \le ||\nabla\mathbf{u}^n_{k,\lambda}||^2||\theta_g||^2_{H^2(\Omega)} \le C||\theta_g||^2_{H^2(\Omega)}.$$

We are ready to use the regularity of the lift operator, to get

$$\begin{aligned}||\Theta^n_{k,\lambda}||^2_{H^2(\Omega)} &\le 2\bar{C}(||\partial_t\Theta^n_{k,\lambda}||^2 + ||\nabla\Theta^n_{k,\lambda}||^2 + ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g||^2 + ||\Delta\theta_g||^2 + ||\partial_t\theta_g||^2)\\ &\le \hat{K}(||\partial_t\Theta^n_{k,\lambda}||^2 + ||\nabla\Theta^n_{k,\lambda}||^2 + ||\theta_g||^2_{H^2(\Omega)} + ||\Delta\theta_g||^2 + ||\partial_t\theta_g||^2)\\ &\le \tilde{K}(||\partial_t\Theta^n_{k,\lambda}||^2 + ||\nabla\Theta^n_{k,\lambda}||^2 + ||g(t)||^2_{H^{3/2}(\partial\Omega)} + ||\partial_t g(t)||^2_{H^{1/2}(\partial\Omega)}). \quad (3.152)\end{aligned}$$

Coming back to the equation for temperature (3.148), since, by Young's inequality and the same estimates as before, $||\Delta\theta_g|| \le ||\theta_g||_{H^2(\Omega)} \le C||g(t)||_{H^{3/2}(\partial\Omega)}$, we obtain:

$$\begin{aligned}\frac{k}{2}\frac{d}{dt}||\nabla\Theta^n_{k,\lambda}||^2 + ||\partial_t\Theta^n_{k,\lambda}||^2 \le& ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\Theta^n_{k,\lambda}||\ ||\partial_t\Theta^n_{k,\lambda}|| + ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g||\ ||\partial_t\Theta^n_{k,\lambda}||\\ &+ ||\partial_t\theta_g||\ ||\partial_t\Theta^n_{k,\lambda}|| + \kappa||\Delta\theta_g||\ ||\partial_t\Theta^n_{k,\lambda}||\\ &\le \frac{1}{2}||\partial_t\Theta^n_{k,\lambda}||^2 + C(||\mathbf{u}^n_{k,\lambda}\cdot\nabla\theta_g||^2\\ &+ ||\mathbf{u}^n_{k,\lambda}\cdot\nabla\Theta^n_{k,\lambda}||^2 + ||\partial_t\theta_g||^2 + ||\Delta\theta_g||^2)\\ &\le \frac{1}{2}||\partial_t\Theta^n_{k,\lambda}||^2 + C(||\mathbf{u}^n_{k,\lambda}\cdot\nabla\Theta^n_{k,\lambda}||^2)\\ &+ \bar{K}(||g(t)||^2_{H^{3/2}(\partial\Omega)} + ||\partial_t g(t)||^2_{H^{1/2}(\partial\Omega)}). \quad (3.153)\end{aligned}$$

Considering $\delta = \dfrac{1}{4C\tilde{K}}$ in (3.151), substituting in (3.150) and then in (3.153) we obtain

$$\begin{aligned}\frac{\kappa}{2}\frac{d}{dt}||\nabla\Theta^n_{k,\lambda}||^2 + ||\partial_t\Theta^n_{k,\lambda}||^2 &\le \frac{1}{2}||\partial_t\Theta^n_{k,\lambda}||^2 + \frac{1}{4\tilde{K}}||\Theta^n_{k,\lambda}||^2_{H^2(\Omega)}\\ &+ \bar{\bar{K}}(||\nabla\Theta^n_{k,\lambda}||^2 + ||g(t)||^2_{H^{3/2}(\partial\Omega)} + ||\partial_t g(t)||^2_{H^{1/2}(\partial\Omega)})\\ &\le \frac{1}{2}||\partial_t\Theta^n_{k,\lambda}||^2 + \frac{1}{4}||\partial_t\Theta^n_{k,\lambda}||^2\\ &+ K'(||\nabla\Theta^n_{k,\lambda}||^2 + ||g(t)||^2_{H^{3/2}(\partial\Omega)} + ||\partial_t g(t)||^2_{H^{1/2}(\partial\Omega)}).\end{aligned}$$

Rearranging the time derivative, we are left with

$$\frac{\kappa}{2}\frac{d}{dt}||\nabla\Theta^n_{k,\lambda}||^2 + \frac{1}{4}||\partial_t\Theta^n_{k,\lambda}||^2 \le K'||\nabla\Theta^n_{k,\lambda}||^2 + K'(||g(t)||^2_{H^{3/2}(\partial\Omega)} + ||\partial_t g(t)||^2_{H^{1/2}(\partial\Omega)}).$$

Since, by assumption on g, the second term in the right-hand side belongs to $L^1(0,T)$, we can apply Gronwall's inequality (the term in the time derivative is absolutely continuous, indeed) with $a(t) = \dfrac{2K'}{\kappa}$ and $b(t) = \dfrac{2K'}{\kappa}(||g(t)||^2_{H^{3/2}(\partial\Omega)} + ||\partial_t g(t)||^2_{H^{1/2}(\partial\Omega)})$, to get, by the orthogonality of the projector on V_θ:

$$||\nabla\Theta^n_{k,\lambda}(t)||^2 \leq e^{\frac{2K'}{\kappa}T}[||\nabla\Theta_0||^2 + \frac{2K'}{\kappa}(||g||^2_{L^2(0,T;H^{3/2}(\partial\Omega))} + ||\partial_t g||^2_{L^2(0,T;H^{1/2}(\partial\Omega))}] \leq K'', \tag{3.154}$$

with K'' independent of k, n, λ. Integrating in time in $(0, T)$ the previous equation we also obtain that

$$\int_0^T ||\partial_t\Theta^n_{k,\lambda}||^2 \leq K,$$

with K independent of k, n, λ. Then we also deduce from (3.152) that $\Theta^n_{k,\lambda} \in L^2(0, T; H^2(\Omega))$ and that it is uniformly bounded also in this norm.

To sum up, differently from the less regular case, we also have these estimates: $\Theta^n_{k,\lambda}$ is uniformly bounded in $L^\infty(0, T; V_\theta) \cap H^1(0, T; H) \cap L^2(0, T; V^2_\theta)$.

Then we can pass to the limit and conclude the proof of the existence of a strong solution, because we have reached the needed regularity and the equation for temperature also holds almost everywhere in $\Omega \times (0, T)$. Being the weak limit in the right convergence, also the limit $\Theta \in L^\infty(0, T; V_\theta) \cap H^1(0, T; H) \cap L^2(0, T; V^2_\theta)$. Since $\theta = \Theta + \theta_g$, the same regularity holds for θ, because by construction $\theta_g \in L^2(0, T; H^2(\Omega))$ and $\partial_t\theta_g \in L^2(0, T; V)$: we have $\theta \in L^\infty(0, T; V) \cap H^1(0, T; H) \cap L^2(0, T; H^2(\Omega))$. Thus the solution $(\mathbf{u}, \varphi, \mu, \theta)$ has the sufficient regularity to be a strong solution. □

Chapter 4

Stability estimates and uniqueness

In this chapter we deal with the proofs of the continuous dependence estimates presented in Chapter 2. These estimates are important to guarantee the stability of the solutions, and thus the well-posedness of the problem, since from these estimates the uniqueness of the solution can be derived as an immediate corollary.

The first estimate proved is the one leading to weak-strong uniqueness, and it gives the possibility of controlling the norms of $\mathbf{u}, \varphi,$ and θ in the respective dual norms, when a weak solution in the sense of Definition 1.1 and a strong solution (Definition 2.1.6) are given.

The second estimate can be achieved when dealing with quasi-strong solutions (see Definition 1.2). It controls the L^2 norms of $\mathbf{u}$ and φ, whereas it controls only the dual norm of θ.

The third estimate is a higher order estimate and can be achieved when dealing with strong solutions (see Definition 1.3). This estimate controls the L^2 norms of $\mathbf{u}$ and φ and also the L^2 norm of θ.

4.1 Proof of Theorem 2.2.1

We now prove the stability estimate (2.1) from which the weak-strong uniqueness is a direct consequence. Let us consider the weak form also for the strong solution and perform the difference between them.

We define $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\varphi = \varphi_1 - \varphi_2$, $\theta = \theta_1 - \theta_2$ and $\mu = -\alpha\Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2)$. Then, having $\theta_i = g(t)$ on $\partial\Omega \times (0,T)$, we get

$$< \partial_t \mathbf{u}, \mathbf{w} > + b(\mathbf{u}_1, \mathbf{u}, \mathbf{w}) + b(\mathbf{u}, \mathbf{u}_2, \mathbf{w}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{w}) \tag{4.1}$$

$$= \alpha(\nabla\varphi_1 \otimes \nabla\varphi, \nabla \mathbf{w}) + \alpha(\nabla\varphi \otimes \nabla\varphi_2, \nabla \mathbf{w}) + (\theta, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_\sigma \tag{4.2}$$

$$< \partial_t\varphi, v > +(\nabla\mu, \nabla v) + (\mathbf{u}_1 \cdot \nabla\varphi, v) + (\mathbf{u} \cdot \nabla\varphi_2, v) = 0 \quad \forall v \in V \tag{4.3}$$

$$< \partial_t\theta, \xi > +\kappa(\nabla\theta, \nabla\xi) - (\mathbf{u}_1 \cdot \theta, \nabla\xi) - (\mathbf{u} \cdot \theta_2, \nabla\xi) = 0 \quad \forall\xi \in V_\theta \tag{4.4}$$

where in the last term we exploited the fact that $\mathbf{u} = 0$ on $\partial\Omega$ and div $\mathbf{u} = 0$ in Ω to say $(\mathbf{u}_1 \cdot \nabla\theta, \xi) = -(\mathbf{u}_1 \cdot \theta, \nabla\xi)$ and $(\mathbf{u} \cdot \nabla\theta_2, \xi) = -(\mathbf{u} \cdot \theta_2, \nabla\xi)$.
First of all, since

$$\mu\nabla\varphi = (-\alpha\Delta\varphi + \Psi'(\varphi))\nabla\varphi,$$

we have

$$\mu\nabla\varphi = \nabla\left(\frac{\alpha}{2}|\nabla\varphi|^2 + \Psi(\varphi)\right) - \alpha \operatorname{div}(\nabla\varphi \otimes \nabla\varphi);$$

therefore the aforementioned weak formulation is equivalent to the one used up to now.

We then recall that, due to the previous regularity properties, we have for $i = 1, 2$,

$$||\mathbf{u}_i(t)|| \le C_0, \quad ||\varphi_i(t)||_V \le C_0, \quad ||\varphi_i(t)||_{L^\infty(\Omega)} \le 1 \quad ||\theta_2(t)||_1 \le C_0. \tag{4.5}$$

for almost any $t \in (0,T)$, for some constant C_0 depending on the initial data energy. We start with the equation (4.3). Remembering the zero divergence of the velocity and its no-slip boundary conditions, we rewrite it by integration by parts as

$$< \partial_t\varphi, v > +(\nabla\mu, \nabla v) - (\mathbf{u}_1\varphi, \nabla v) - (\mathbf{u}\varphi_2, \nabla v) = 0 \quad \forall v \in V. \tag{4.6}$$

We have also, as already noticed many times, that (it is enough testing (4.3) against $v = 1$, remembering that all the integrals apart from the first one vanish due to boundary conditions and zero divergence of velocity) $\bar{\varphi}(t) = \bar{\varphi}(0) = 0$ (since by assumption $\bar{\varphi}_{01} = \bar{\varphi}_{02}$) for all $t \in [0,T]$: then $\varphi \in V_0$ (see (B.16)), and then also $\varphi \in V_0'$. With this remark in mind, we can test the equation with $v = \bar{A}_0^{-1}\varphi$, which is now well defined, obtaining (remembering that, exactly as in (B.14), we have $(\nabla\mu, \nabla\bar{A}_0^{-1}\varphi) = (\mu, \varphi)$):

$$\frac{1}{2}\frac{d}{dt}||\varphi||_*^2 + (\mu, \varphi) = \mathcal{I}_1 + \mathcal{I}_2 \tag{4.7}$$

with

$$\mathcal{I}_1 = (\mathbf{u}_1\varphi, \nabla\bar{A}_0^{-1}\varphi)$$

$$\mathcal{I}_2 = (\mathbf{u}\varphi_2, \nabla\bar{A}_0^{-1}\varphi).$$

Now by Lagrange theorem, from the properties of regularity of Ψ, we have, for some $\xi = \xi(s_1, s_2) \in (-1, 1)$, from property (1.2), that

$$\Psi'(s_2) - \Psi'(s_1) = \Psi''(\xi)(s_2 - s_1) \geq -\tilde{\alpha}(s_2 - s_1) \quad \forall s_1, s_2 \in (-1, 1)$$

which means, since $\varphi_1, \varphi_2 \in (-1, 1)$ almost everywhere in $\Omega \times (0, T)$, that, always almost everywhere,

$$\Psi'(\varphi_1) - \Psi'(\varphi_2) \geq -\tilde{\alpha}(\varphi_1 - \varphi_2) = -\tilde{\alpha}\varphi.$$

We then are able to say that, by integrating by parts the first term, due to homogeneous Neumann boundary conditions,

$$(\mu, \varphi) = -\alpha(\Delta\varphi, \varphi) + (\Psi'(\varphi_1) - \Psi'(\varphi_2), \varphi) \geq \alpha||\nabla\varphi||^2 - \tilde{\alpha}||\varphi||^2. \tag{4.8}$$

From definition of $\bar{A}_0$ we also get, from Cauchy-Schwartz's and Young's inequalities:

$$\tilde{\alpha}||\varphi||^2 = \tilde{\alpha}(\nabla\varphi, \nabla\bar{A}_0^{-1}\varphi) \leq \frac{\alpha}{2}||\nabla\varphi||^2 + C||\varphi||_*^2. \tag{4.9}$$

We can now write

$$\frac{1}{2}\frac{d}{dt}||\varphi||_*^2 + \alpha||\nabla\varphi||^2 - \tilde{\alpha}||\varphi||^2 \leq \frac{1}{2}\frac{d}{dt}||\varphi||_*^2 + (\mu, \varphi) = \mathcal{I}_1 + \mathcal{I}_2$$

implying

$$\frac{1}{2}\frac{d}{dt}||\varphi||_*^2 + \frac{\alpha}{2}||\nabla\varphi||^2 \leq C||\varphi||_*^2 + \mathcal{I}_1 + \mathcal{I}_2.$$

We now estimate the last two terms in the inequality, by Cauchy-Schwartz's and Young's inequalities and the Sobolev embeddings $V \hookrightarrow L^6(\Omega)$ and $\mathbf{V}_\sigma \hookrightarrow [L^3(\Omega)]^2$ and by Poincaré's inequality:

$$\mathcal{I}_1 \leq ||\varphi||_{L^6(\Omega)}||\mathbf{u}_1||_{L^3(\Omega)}||\varphi||_* \leq \frac{\alpha}{8}||\nabla\varphi||^2 + C_1||\nabla\mathbf{u}_1||^2||\varphi||_*^2.$$

Whereas for the other term, by Cauchy-Schwartz's and Young's inequalities, we obtain, applying estimates (4.5):

$$\mathcal{I}_2 \leq ||\varphi_2||_{L^\infty(\Omega)}||\mathbf{u}||\ ||\varphi||_* \leq \frac{\nu}{8}||\mathbf{u}||^2 + C_2||\varphi||_*^2.$$

We now pass to estimate the equation (4.2): testing it against $\mathbf{v} = \mathbf{A}^{-1}\mathbf{u}$ (where $\mathbf{A}$ is the Stokes operator defined in Appendix B.2) remembering that $(\nabla\mathbf{u}, \nabla\mathbf{A}^{-1}\mathbf{u}) = ||\mathbf{u}||^2$, we get

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}||_\flat^2 + \nu||\mathbf{u}||^2 = \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5, \tag{4.10}$$

with

$$\mathcal{I}_3 = -b(\mathbf{u}_1, \mathbf{u}, \mathbf{A}^{-1}\mathbf{u}) - b(\mathbf{u}, \mathbf{u}_2, \mathbf{A}^{-1}\mathbf{u})$$

$$\mathcal{I}_4 = \alpha(\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{A}^{-1}\mathbf{u}) + \alpha(\nabla\varphi \otimes \nabla\varphi_2, \nabla\mathbf{A}^{-1}\mathbf{u})$$

$$\mathcal{I}_5 = (\theta, \mathbf{e}_2 \cdot \mathbf{A}^{-1}\mathbf{u}).$$

By Holder's and Young's inequalities and by Lemma A.1.4 and (B.10), since $[\nabla\mathbf{A}^{-1}\mathbf{u}]_{ij} \in H^1(\Omega)$ for each component and by (4.5):

$$\begin{aligned}
\mathcal{I}_3 &\leq (||\mathbf{u}_1||_{L^4(\Omega)} + ||\mathbf{u}_2||_{L^4(\Omega)})\ ||\mathbf{u}||\ ||\ \nabla\mathbf{A}^{-1}\mathbf{u}||_{[L^4(\Omega)]^2} \\
&\leq C(||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2} + ||\mathbf{u}_2||^{1/2}||\nabla\mathbf{u}_2||^{1/2})\ ||\mathbf{u}||\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||^{1/2}||\nabla\mathbf{A}^{-1}\mathbf{u}||_1^{1/2} \\
&\leq \bar{C}(||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2} + ||\mathbf{u}_2||^{1/2}||\nabla\mathbf{u}_2||^{1/2})\ ||\mathbf{u}||\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||^{1/2}||\mathbf{A}^{-1}\mathbf{u}||_{H^2(\Omega)}^{1/2} \\
&\leq \tilde{C}(||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2} + ||\mathbf{u}_2||^{1/2}||\nabla\mathbf{u}_2||^{1/2})\ ||\mathbf{u}||\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||^{1/2}||\mathbf{A}^{-1}\mathbf{u}||_{\mathbf{W}_\sigma}^{1/2} \\
&= \tilde{C}(||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2} + ||\mathbf{u}_2||^{1/2}||\nabla\mathbf{u}_2||^{1/2})\ ||\mathbf{u}||^{\frac{3}{2}}\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||^{1/2} \\
&\leq \frac{\nu}{8}||\mathbf{u}||^2 + K||\mathbf{u}||_\flat^2(||\nabla\mathbf{u}_1||^{1/2} + ||\nabla\mathbf{u}_2||^{1/2})^4 \leq \frac{\nu}{8}||\mathbf{u}||^2 + \bar{K}||\mathbf{u}||_\flat^2(||\nabla\mathbf{u}_1||^2 + ||\nabla\mathbf{u}_2||^2).
\end{aligned}$$

About $\mathcal{I}_4$, acting in a similar way, we have by Holder's and Young's inequalities and by Lemma A.1.4, estimates (4.5) and Sobolev embedding $V \hookrightarrow L^4(\Omega)$:

$$\begin{aligned}
\mathcal{I}_4 &\le (||\nabla\varphi_1||_{L^4(\Omega)} + ||\nabla\varphi_1||_{L^4(\Omega)})\ ||\nabla\varphi||\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||_{[L^4(\Omega)]^2} \\
&\le C(||\varphi_1||_{H^2(\Omega)} + ||\varphi_2||_{H^2(\Omega)})\ ||\nabla\varphi||\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||^{1/2}||\nabla\mathbf{A}^{-1}\mathbf{u}||_1^{1/2} \\
&\le \bar{C}(||\varphi_1||_{H^2(\Omega)} + ||\varphi_2||_{H^2(\Omega)})\ ||\nabla\varphi||\ ||\nabla\mathbf{A}^{-1}\mathbf{u}||^{1/2}||\mathbf{u}||^{1/2} \\
&\le \frac{\alpha}{4}||\nabla\varphi||^2 + \hat{C}(||\varphi_1||^2_{H^2(\Omega)} + ||\varphi_2||^2_{H^2(\Omega)})||\nabla\mathbf{A}^{-1}\mathbf{u}||\ ||\mathbf{u}|| \\
&\le \frac{\alpha}{4}||\nabla\varphi||^2 + \frac{\nu}{8}||\mathbf{u}||^2 + K(||\varphi_1||^4_{H^2(\Omega)} + ||\varphi_2||^4_{H^2(\Omega)})||\mathbf{u}||^2_\flat.
\end{aligned}$$

In conclusion, we are left with the last term, estimated again by means of Cauchy-Schwartz's, Young's and Poincaré's inequalities:

$$\mathcal{I}_5 \le \frac{\kappa}{6}||\theta||^2 + C||\nabla\mathbf{A}^{-1}\mathbf{u}||^2 = \frac{\kappa}{6}||\theta||^2 + C||\mathbf{u}||^2_\flat. \tag{4.11}$$

In conclusion, we tackle equation (4.4): substituting $\xi = A_0^{-1}\theta$, where operator A_0 is defined in Appendix B.3, we get:

$$\frac{1}{2}\frac{d}{dt}||\theta||^2_* + \kappa||\theta||^2 = \mathcal{I}_6 + \mathcal{I}_7 \tag{4.12}$$

where

$$\begin{aligned}
\mathcal{I}_6 &= (\mathbf{u}_1\cdot\theta, \nabla A_0^{-1}\theta) \\
\mathcal{I}_7 &= (\mathbf{u}\cdot\theta_2, \nabla A_0^{-1}\theta).
\end{aligned}$$

Then, for the last two terms we get, by Holder's and Young's inequalities and by Lemma A.1.4 and (B.15), since $\nabla A_0^{-1}\theta \in [H^1(\Omega)]^2$ and by (4.5):

$$\begin{aligned}
\mathcal{I}_6 &\le ||\theta||\ ||\mathbf{u}_1||_{L^4(\Omega)}||\nabla A_0^{-1}\theta||_{L^4(\Omega)} \\
&\le C||\theta||\ ||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2}||\nabla A_0^{-1}\theta||^{1/2}||\nabla A_0^{-1}\theta||_1^{1/2} \\
&\le \bar{C}||\theta||\ ||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2}||\nabla A_0^{-1}\theta||^{1/2}||A_0^{-1}\theta||^{1/2}_{H^2(\Omega)} \\
&\le K||\theta||^{3/2}\ ||\nabla\mathbf{u}_1||^{1/2}\ ||\nabla A_0^{-1}\theta||^{1/2} \le \frac{\kappa}{6}||\theta||^2 + \tilde{C}||\nabla A_0^{-1}\theta||^2\ ||\nabla\mathbf{u}_1||^2 \\
&= \frac{\kappa}{6}||\theta||^2 + \tilde{C}||\theta||^2_{**}\ ||\nabla\mathbf{u}_1||^2
\end{aligned}$$

and in conclusion by similar arguments, remembering Sobolev embedding $\theta_2 \in V \hookrightarrow L^4(\Omega)$ and $\mathbf{u} \in \mathbf{V}_\sigma \hookrightarrow [L^4(\Omega)]^2$ and remembering (4.5), where we exploited the more regularity of the strong solution (it is exactly at this point that we need more regularity for θ_2), we are left with

$$\begin{aligned}
\mathcal{I}_7 &\leq ||\theta_2||_{L^4(\Omega)}||\mathbf{u}||\ ||\nabla A_0^{-1}\theta||_{[L^4(\Omega)]^2} \\
&\leq K||\theta_2||_1||\mathbf{u}||\ ||\nabla A_0^{-1}\theta||^{1/2}||\nabla A_0^{-1}\theta||_1^{1/2} \\
&\leq K||\theta_2||_1||\mathbf{u}||\ ||\nabla A_0^{-1}\theta||^{1/2}||\nabla A_0^{-1}\theta||_{H^2(\Omega)}^{1/2} \\
&\leq K||\theta_2||_1||\mathbf{u}||\ ||\nabla A_0^{-1}\theta||^{1/2}\ ||\theta||^{1/2} \\
&\leq \frac{\nu}{8}||\mathbf{u}||^2 + \frac{\kappa}{6}||\theta||^2 + C||\theta_2||_1^4\ ||\nabla A_0^{-1}\theta||^2 \\
&\leq \frac{\nu}{8}||\mathbf{u}||^2 + \frac{\kappa}{6}||\theta||^2 + \bar{C}||\theta||_{**}^2,
\end{aligned}$$

where in the last steps we have used the Young's inequality for three terms (with exponents 2, 4, 4).

We define

$$\mathcal{H}(t) = \frac{1}{2}||\mathbf{u}||_\flat^2 + \frac{1}{2}||\varphi||_*^2 + \frac{1}{2}||\theta||_{**}^2 \tag{4.13}$$

and

$$\mathcal{R}(t) = \nu||\mathbf{u}||^2 + \kappa||\theta||^2 + \frac{\alpha}{2}||\nabla\varphi||^2 \tag{4.14}$$

and then sum up all the three inequalities, obtaining

$$\frac{d}{dt}\mathcal{H}(t) + \nu||\mathbf{u}||^2 + \kappa||\theta||^2 + \frac{\alpha}{2}||\nabla\varphi||^2 \leq C||\varphi||_*^2 + \sum_{j=1}^{7}\mathcal{I}_j$$

and substituting all the estimates we get

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}(t) + \mathcal{R}(t) &\leq C||\varphi||_*^2 + \frac{\alpha}{8}||\nabla\varphi||^2 + C_1||\nabla\mathbf{u}_1||^2||\varphi||_*^2 \\
&+ \frac{\nu}{8}||\mathbf{u}||^2 + C_2||\varphi||_*^2 \\
&+ \frac{\nu}{8}||\mathbf{u}||^2 + \bar{K}||\mathbf{u}||_\flat^2(||\nabla\mathbf{u}_1||^2 + ||\nabla\mathbf{u}_2||^2) \\
&+ \frac{\alpha}{4}||\nabla\varphi||^2 + \frac{\nu}{8}||\mathbf{u}||^2 + K(||\varphi_1||_{H^2(\Omega)}^4 + ||\varphi_2||_{H^2(\Omega)}^4)||\mathbf{u}||_\flat^2 \\
&+ \frac{\kappa}{6}||\theta||^2 + C||\mathbf{u}||_\flat^2 \\
&+ \frac{\kappa}{6}||\theta||^2 + \tilde{C}||\theta||_{**}^2\ ||\nabla\mathbf{u}_1||^2 \\
&+ \frac{\nu}{8}||\mathbf{u}||^2 + \frac{\kappa}{6}||\theta||^2 + \bar{C}||\theta||_{**}^2.
\end{aligned}$$

Thus, we can define

$$\mathcal{D}(t) = 1 + ||\nabla \mathbf{u}_1||^2 + ||\nabla \mathbf{u}_2||^2 + ||\varphi_1||^4_{H^2(\Omega)} + ||\varphi_2||^4_{H^2(\Omega)} \tag{4.15}$$

and manipulating the constants we are led to

$$\frac{d}{dt}\mathcal{H}(t) + \frac{\nu}{2}||\mathbf{u}||^2 + \frac{\kappa}{2}||\theta||^2 + \frac{\alpha}{8}||\nabla\varphi||^2 \leq K_0 \mathcal{D}(t)\mathcal{H}(t). \tag{4.16}$$

We now observe that $\mathcal{D} \in L^1(0,T)$, since $\mathbf{u}_1,\ \mathbf{u}_2 \in L^2(0,T;\mathbf{V}_\sigma)$ and $\varphi_1,\ \varphi_2 \in L^4(0,T;H^2(\Omega))$ and $T < +\infty$. Then

$$\int_0^T \mathcal{D}(t) \leq \bar{C}(T).$$

We can now apply Gronwall's Lemma A.1.7 (indeed, we know by Lemma by (B.9), (B.13) and (B.19) that $\mathcal{H}$ is absolutely continuous and $a, b \in L^1(0,T)$) to (4.16), obtaining

$$\mathcal{H}(t) \leq K_0 e^{\bar{C}} \mathcal{H}(0)$$

and this implies the desired estimates, since the norms considered are equivalent to the classical dual norms in the right spaces (see Appendix B.3). □

4.2 Proof of Theorem 2.2.3

To prove the stability estimate stated in Theorem 2.2.3, we define $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\varphi = \varphi_1 - \varphi_2$ and $\theta = \theta_1 - \theta_2$, where $(\mathbf{u}_1, \varphi_1, \theta_1)$ and $(\mathbf{u}_2, \varphi_2, \theta_2)$ are two solutions departing from $(\mathbf{u}_{0,1}, \varphi_{0,1}, \theta_{0,1})$ and $(\mathbf{u}_{0,2}, \varphi_{0,2}, \theta_{0,2})$, respectively and satisfy $\mathbf{u}_{0i} \in \mathbf{V}_\sigma$, $\varphi_{0i} \in V_2$ such that $||\varphi_{0,i}||_\infty < 1$, $|\bar{\varphi}_{0,i}| < 1$, $\mu_{0,i} = -\alpha\Delta\varphi_{0,i} + \Psi'(\varphi_{0,i}) \in \Psi_1$ and $\partial_{\mathbf{n}}\varphi_{0,i} = 0$ on $\partial\Omega$, and $\theta_{0,i} \in H$ and θ_i such that $\theta_i = g(t)$ on $\partial\Omega \times (0,T)$. We recall that this means that $\theta \in V_\theta = H^1_0(\Omega)$ for almost any $t \in (0,T)$.

We define $\mu = -\alpha\Delta\varphi + \Psi'(\varphi_1) - \Psi'(\varphi_2)$ and we get

$$\begin{aligned} &< \partial_t \mathbf{u}, \mathbf{w} > + b(\mathbf{u}_1, \mathbf{u}, \mathbf{w}) + b(\mathbf{u}, \mathbf{u}_2, \mathbf{w}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{w}) \\ &\qquad = \alpha(\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{w}) + \alpha(\nabla\varphi \otimes \nabla\varphi_2, \nabla\mathbf{w}) + (\theta, \mathbf{e}_2 \cdot \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}_\sigma \end{aligned}$$

$$< \partial_t\varphi, v > + (\nabla\mu, \nabla v) + (\mathbf{u}_1 \cdot \nabla\varphi, v) + (\mathbf{u} \cdot \nabla\varphi_2, v) = 0 \quad \forall v \in V \tag{4.17}$$

$$< \partial_t \theta, \xi > + \kappa(\nabla\theta, \nabla\xi) - (\mathbf{u}_1 \cdot \theta, \nabla\xi) - (\mathbf{u} \cdot \theta_2, \nabla\xi) = 0 \quad \forall \xi \in V_\theta \tag{4.18}$$

where in the last term we exploited the fact that $\mathbf{u} = 0$ on $\partial\Omega$ and $\operatorname{div}(u) = 0$ in Ω to say $(\mathbf{u}_1 \cdot \nabla\theta, \xi) = -(\mathbf{u}_1 \cdot \theta, \nabla\xi)$ and $(\mathbf{u} \cdot \nabla\theta_2, \xi) = -(\mathbf{u} \cdot \theta_2, \nabla\xi)$. Then we can test with $\mathbf{w} = \mathbf{u}$, $v = \varphi$ and $\xi = A_0^{-1}\theta$ (the operator A_0 is defined in Appendix B.3). Using the property (B.13) and the property that $(\nabla\theta, \nabla A_0^{-1}\theta) = ||\theta||^2$ (see (B.14)) and summing up the three resulting equalities we find

$$\frac{d}{dt}\mathcal{H}(t) + \nu||\nabla\mathbf{u}||^2 + \kappa||\theta||^2 + (\nabla\mu.\nabla\varphi) = \sum_{j=1}^{6} \mathcal{I}_j,$$

having set

$$\mathcal{H}(t) = \frac{1}{2}||\mathbf{u}||^2 + \frac{1}{2}||\varphi||^2 + \frac{1}{2}||\theta||_{**}^2.$$

Since $b(\mathbf{u}_1, \mathbf{u}, \mathbf{u}) = 0$:

$$\mathcal{I}_1 = -b(\mathbf{u}, \mathbf{u}_2, \mathbf{u})$$

$$\mathcal{I}_2 = \alpha(\nabla\varphi_1 \otimes \nabla\varphi, \nabla\mathbf{u}) + \alpha(\nabla\varphi \otimes \nabla\varphi_2, \nabla\mathbf{u})$$

$$\mathcal{I}_3 = -(\mathbf{u}_1 \cdot \nabla\varphi, v) - (\mathbf{u} \cdot \nabla\varphi_2, v) = (\mathbf{u}_1\varphi, \nabla\varphi) + (\mathbf{u}\varphi_2, \nabla\varphi)$$

$$\mathcal{I}_4 = (\theta, \mathbf{e}_2 \cdot \mathbf{u})$$

$$\mathcal{I}_5 = (\mathbf{u}_1 \cdot \theta, \nabla A_0^{-1}\theta)$$

$$\mathcal{I}_6 = (\mathbf{u} \cdot \theta_2, \nabla A_0^{-1}\theta).$$

In light of the regularity of the solutions, and since we have proven in Theorem 2.1.3 that $F''(\varphi_i) = \Psi''(\varphi_i) + \alpha_0 \in L^\infty(0, T; L^3(\Omega))$, there exists a positive constant C_0 such that

$$\begin{aligned} &||\mathbf{u}_i||_{L^\infty(0,T;\mathbf{V}_\sigma)} + ||\mathbf{u}_i||_{L^\infty(0,T;[L^3(\Omega)]^2)} + ||\varphi_i||_{L^\infty(0,T;V)} \\ &\qquad + ||\varphi_i||_{L^\infty(0,T;W^{2,3}(\Omega))} + ||\Psi''(\varphi_i)||_{L^\infty(0,T;L^3(\Omega))} \leq C_0 \quad i = 1, 2 \end{aligned} \tag{4.19}$$

where C_0 depends on T, on the initial energy and on the norms of the initial data (indeed, all the estimates obtained in the Galerkin setting are still valid also for the solutions). Due to homogeneous Neumann boundary conditions, integrating by parts the term $||\nabla\varphi||$, from (A.4) we obtain the inequality (already used in [61]):

$$||\varphi||_1^2 \leq ||\Delta\varphi||||\varphi|| + ||\varphi||^2, \tag{4.20}$$

where $||\cdot||$ represents the L^2 norm.

Integrating by parts and using the embedding $V \hookrightarrow L^6(\Omega)$, together with (4.19 we observe that

$$(\nabla\mu, \nabla\varphi) = (-\mu, \Delta\varphi) = \alpha||\nabla\varphi||^2 - (\Psi'(\varphi_1), \Delta\varphi) + (\Psi'(\varphi_2), \Delta\varphi).$$

Since F'' is convex in $(-1, 1)$ and $||\varphi_i||_\infty < 1$, also $\Psi'' = F'' - \alpha_0$ is convex, by 4.19 and by Holder's inequality and the aforementioned Sobolev embedding, we deduce

$$\begin{aligned}
(\Psi'(\varphi_1) - \Psi'(\varphi_2), \Delta\varphi) &= ((\varphi_1 - \varphi_2)\int_0^1 \Psi''(\varphi_1 s + (1-s)\varphi_2)ds, \Delta\varphi) \\
&\le (\varphi \int_0^1 \{s\Psi''(\varphi_1) + (1-s)\Psi''(\varphi_2)\}ds, \Delta\varphi) \\
&= \frac{1}{2}(\Psi''(\varphi_1) - \Psi''(\varphi_2), \varphi\Delta\varphi) \\
&\le ||\Psi''(\varphi_1) - \Psi''(\varphi_2)||_{L^3(\Omega)}||\varphi||_{L^6(\Omega)}||\Delta\varphi|| \\
&\le (||\Psi''(\varphi_1)||_{L^3(\Omega)} + ||\Psi''(\varphi_2)||_{L^3(\Omega)})||\varphi||_{L^6(\Omega)}||\Delta\varphi|| \\
&\le 2C_0||\varphi||_{L^6(\Omega)}||\Delta\varphi|| \\
&\le C||\varphi||_1||\Delta\varphi||
\end{aligned}$$

now, since obviously $a \ge -|a|$, we obtain, using (4.20) and Young's inequality twice:

$$\begin{aligned}
(\nabla\mu, \nabla\varphi) &\ge \alpha||\Delta\varphi||^2 - C||\varphi||_1||\Delta\varphi|| \\
&\ge \alpha||\Delta\varphi||^2 - \frac{\alpha}{4}||\Delta\varphi||^2 - \bar{C}||\varphi||_1^2 \\
&\ge \alpha||\Delta\varphi||^2 - \frac{\alpha}{4}||\Delta\varphi||^2 - ||\varphi||^2 - \frac{\alpha}{4}||\Delta\varphi||^2 - C||\varphi||^2 \ge \frac{\alpha}{2}||\Delta\varphi||^2 - \hat{C}||\varphi||^2.
\end{aligned}$$

We now need to estimate the terms in the right-hand side: by Holder's inequality an then Young's inequality, together with the Sobolev embedding $\mathbf{V}_\sigma \hookrightarrow [L^6(\Omega)]^2$ and from (4.19) :

$$\mathcal{I}_1 \le ||\mathbf{u}||\;||\nabla\mathbf{u}_2||_{[L^3(\Omega)]^2}||\mathbf{u}||_{[L^6(\Omega)]^2} \le ||\mathbf{u}||\;||\nabla\mathbf{u}_2||_{[L^3(\Omega)]^2}||\nabla\mathbf{u}|| \le \frac{\nu}{4}||\nabla\mathbf{u}||^2 + C||\mathbf{u}||^2||\nabla\mathbf{u}_2||^2_{[L^3(\Omega)]^2}.$$

Then, by (4.19), the embedding $W^{2,3}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, valid in dimension two (see, e.g.,

[20]), Holder's and Young's inequalities and in the last step by (4.20) we get:

$$\begin{aligned}\mathcal{I}_2 \leq (||\nabla\varphi_1||_\infty + ||\nabla\varphi_2||_\infty)||\nabla\varphi||\ ||\nabla\mathbf{u}|| \leq K||\nabla\varphi||\ ||\nabla\mathbf{u}|| \\ \leq \frac{\nu}{4}||\nabla\mathbf{u}||^2 + \bar{C}||\nabla\varphi||^2 \\ \leq \frac{\nu}{4}||\nabla\mathbf{u}||^2 + \frac{\alpha}{8}||\Delta\varphi||^2 + \tilde{C}||\varphi||^2.\end{aligned}$$

Then, again by Sobolev embeddings, Holder's and Young's inequality and (4.20) we get, since for (4.19) $||\mathbf{u}_1||_{[L^3(\Omega)]^2} \leq C_0$,

$$\mathcal{I}_3 \leq ||\varphi||_{L^6(\Omega)}||\mathbf{u}_1||_{[L^3(\Omega)]^2}||\nabla\varphi|| + ||\varphi_2||_\infty||\mathbf{u}||\ ||\nabla\varphi|| \leq \frac{\alpha}{8}||\Delta\varphi||^2 + \bar{K}(||\varphi||^2 + ||\varphi_2||_\infty^2||\mathbf{u}||^2).$$

For the next term, by Holder's and Young's inequalities we get

$$\mathcal{I}_4 \leq \frac{\kappa}{4}||\theta||^2 + C||\mathbf{u}||^2.$$

Then, for the last two terms we get, by Holder's and Young's inequalities and by Lemma A.1.4 and (B.15), since $\nabla A_0^{-1}\theta \in [H^1(\Omega)]^2$ and by (4.19):

$$\begin{aligned}\mathcal{I}_5 &\leq ||\theta||\ ||\mathbf{u}_1||_{L^4(\Omega)}||\nabla A_0^{-1}||_{L^4(\Omega)} \\ &\leq C||\theta||\ ||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2}||\nabla A_0^{-1}\theta||^{1/2}||\nabla A_0^{-1}\theta||_1^{1/2} \\ &\leq \bar{C}||\theta||\ ||\mathbf{u}_1||^{1/2}||\nabla\mathbf{u}_1||^{1/2}||\nabla A_0^{-1}\theta||^{1/2}||A_0^{-1}\theta||_{H^2(\Omega)}^{1/2} \\ &\leq K||\theta||^{3/2}||\nabla A_0^{-1}\theta||^{1/2} \leq \frac{\kappa}{4}||\theta||^2 + \tilde{C}||\nabla A_0^{-1}\theta||^2 = \frac{\kappa}{4}||\theta||^2 + \tilde{C}||\theta||_{**}^2\end{aligned}$$

and in conclusion by similar arguments, remembering Sobolev embedding $\theta_2 \in V \hookrightarrow L^4(\Omega)$ and $\mathbf{u} \in \mathbf{V}_\sigma \hookrightarrow [L^4(\Omega)]^2$

$$\mathcal{I}_6 \leq ||\theta_2||_{L^4(\Omega)}||\mathbf{u}||_{[L^4(\Omega)]^2}||\nabla A_0^{-1}\theta|| \leq \frac{\nu}{4}||\nabla\mathbf{u}||^2 + C||\theta_2||_1^2||\nabla A_0^{-1}\theta||^2 = \frac{\nu}{4}||\nabla\mathbf{u}||^2 + C||\theta_2||_1^2||\theta||_{**}^2.$$

Then, to sum up we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H}(t) + \nu||\nabla\mathbf{u}||^2 + \kappa||\theta||^2 + \frac{\alpha}{2}||\Delta\varphi||^2 - \hat{C}||\varphi||^2 \leq\ & \frac{\nu}{4}||\nabla\mathbf{u}||^2 + C||\mathbf{u}||^2||\nabla\mathbf{u}_2||^2_{[L^3(\Omega)]^2} \\
& + \frac{\nu}{4}||\nabla\mathbf{u}||^2 + \frac{\alpha}{8}||\Delta\varphi||^2 + \tilde{C}||\varphi||^2 \\
& + \frac{\alpha}{8}||\Delta\varphi||^2 + \bar{K}(||\varphi||^2 + ||\varphi_2||^2_\infty||\mathbf{u}||^2) \\
& + \frac{\kappa}{4}||\theta||^2 + C||\mathbf{u}||^2 \\
& + \frac{\kappa}{4}||\theta||^2 + \tilde{C}||\theta||^2_{**} \\
& + \frac{\nu}{4}||\nabla\mathbf{u}||^2 + C||\theta_2||^2_1||\theta||^2_{**}
\end{aligned}
$$

we then obtain, by rearranging the terms, and considering less constants and setting

$$
\mathcal{Q}(t) = 1 + ||\nabla\mathbf{u}_2||^2_{[L^3(\Omega)]^2} + ||\varphi_2||^2_\infty + ||\theta_2||^2_1
$$

$$
\frac{d}{dt}\mathcal{H}(t) + \frac{\nu}{4}||\nabla\mathbf{u}||^2 + \frac{\kappa}{2}||\theta||^2 + \frac{\alpha}{4}||\Delta\varphi||^2 \leq \bar{K}\mathcal{Q}(t)\mathcal{H}(t). \tag{4.21}
$$

We now observe that $\mathcal{Q} \in L^1(0,T)$, because by Sobolev embedding we get by means of (B.10) since $\mathbf{u}_2 \in \mathbf{W}_\sigma$ (see Appendix B.2 for the definition of this space), $||\nabla\mathbf{u}_2||^2_{[L^3(\Omega)]^2} \leq C||\nabla\mathbf{u}_2||^2_{\mathbf{H}^1} \leq K||\mathbf{A}\mathbf{u}_2||^2$ and $\mathbf{u}_2 \in L^2(0,T;\mathbf{W}_\sigma)$, but also $||\varphi_2||_\infty \leq ||\varphi_2||_{H^2(\Omega)}$ and $\varphi_2 \in L^4(0,T;V_2) \hookrightarrow L^2(0,T;V_2)$ and in conclusion $\theta_2 \in L^2(0,T;V)$. Then

$$
\int_0^T \mathcal{Q}(t) \leq \bar{C}
$$

We can now apply Gronwall's Lemma A.1.7 (indeed, we know by Lemma A.2.2 and by (B.13) that $\mathcal{H}$ is absolutely continuous and $a, b \in L^1(0,T)$) to (4.21), obtaining

$$
\mathcal{H}(t) \leq \bar{K}e^{\bar{C}}\mathcal{H}(0)
$$

and this implies the estimates (2.2). From these estimates, clearly we immediately deduce the uniqueness of the solution. The constants appearing in the stability estimates only depend on T, the initial energy and on the norms of the initial data. □

4.3 Proof of Theorem 2.2.5

Here we show a stability estimate for strong solutions. With respect to the previous one, the difference is that a higher order norm of the temperature can be controlled.

The proof is similar to the one in the previous section, apart from the following higher order estimates.

First of all, we test the equation for the temperature against $\xi = \theta \in V_\theta$. In this way, we get

$$\frac{d}{dt}||\theta||^2 + \kappa||\nabla\theta||^2 - (\mathbf{u}\theta_2, \nabla\theta) = 0. \tag{4.22}$$

Then we have only to estimate the following, since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ ([4]), by Young's inequality:

$$\begin{aligned}(\mathbf{u}\theta_2, \nabla\theta) &\le ||\theta_2||_\infty \; ||\mathbf{u}|| \; ||\nabla\theta|| \\ &\le C||\theta_2||_{H^2(\Omega)}||\mathbf{u}|| \; ||\nabla\theta|| \\ &\le \frac{\kappa}{4}||\nabla\theta||^2 + \bar{C}||\mathbf{u}||^2||\theta_2||^2_{H^2(\Omega)}.\end{aligned}$$

Considering the other equations, the only change is in the term called $\mathcal{I}_4 = (\theta, \mathbf{e}_2 \cdot \mathbf{u})$ with the numbering of the proof of Theorem 2.2.3: since $\theta \in V_\theta$, we can apply Poincaré's inequality to get

$$\mathcal{I}_4 \le ||\theta|| \; ||\mathbf{u}|| \le C_0||\nabla\theta|| \; ||\mathbf{u}|| \le \frac{\kappa}{4}||\nabla\theta||^2 + K||\mathbf{u}||^2.$$

If now we sum up all the terms and consider the other ones estimated in the previous proof, we obtain, setting

$$\mathcal{H}_2(t) = \frac{1}{2}||\mathbf{u}||^2 + \frac{1}{2}||\varphi||^2 + \frac{1}{2}||\theta||^2,$$

that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_2(t) + \nu||\nabla\mathbf{u}||^2 + \kappa||\nabla\theta||^2 + \frac{\alpha}{2}||\Delta\varphi||^2 - \hat{C}||\varphi||^2 \le{}& \frac{\nu}{4}||\nabla\mathbf{u}||^2 + C||\mathbf{u}||^2||\nabla\mathbf{u}_2||^2_{[L^3(\Omega)]^2} \\ &+ \frac{\nu}{4}||\nabla\mathbf{u}||^2 + \frac{\alpha}{8}||\Delta\varphi||^2 + \tilde{C}||\varphi||^2 \\ &+ \frac{\alpha}{8}||\Delta\varphi||^2 + \bar{K}(||\varphi||^2 + ||\varphi_2||^2_\infty||\mathbf{u}||^2) \\ &+ \frac{\kappa}{4}||\nabla\theta||^2 + \bar{C}||\mathbf{u}||^2||\theta_2||^2_{H^2(\Omega)} \\ &+ \frac{\kappa}{4}||\nabla\theta||^2 + K||\mathbf{u}||^2.\end{aligned}$$

We then obtain, by rearranging the terms, and considering less constants and setting

$$\mathcal{R}(t) = 1 + ||\nabla\mathbf{u}_2||^2_{[L^3(\Omega)]^2} + ||\varphi_2||^2_\infty + ||\theta_2||^2_{H^2(\Omega)},$$

that

$$\frac{d}{dt}\mathcal{H}_2(t) + \frac{\nu}{2}||\nabla \mathbf{u}||^2 + \frac{\kappa}{2}||\nabla\theta||^2 + \frac{\alpha}{4}||\Delta\varphi||^2 \leq \bar{K}\mathcal{R}(t)\mathcal{H}_2(t). \tag{4.23}$$

We now observe that $\mathcal{R} \in L^1(0,T)$, because by Sobolev embedding we get by means of (B.10) since $\mathbf{u}_2 \in \mathbf{W}_\sigma$, $||\nabla \mathbf{u}_2||^2_{[L^3(\Omega)]^2} \leq C||\nabla \mathbf{u}_2||^2_{\mathbf{H}^1} \leq K||\mathbf{A}\mathbf{u}_2||^2$ and $\mathbf{u}_2 \in L^2(0,T;\mathbf{W}_\sigma)$, but also, by Sobolev embedding, $||\varphi_2||_\infty \leq C||\varphi_2||_{H^2(\Omega)}$ and $\varphi_2 \in L^4(0,T;V_2) \hookrightarrow L^2(0,T;V_2)$ and in conclusion $\theta_2 \in L^2(0,T;H^2(\Omega))$, whose regularity is the reason why we need a strong solution for this estimate. Then

$$\int_0^T \mathcal{R}(t) \leq \bar{C}.$$

We can now apply Gronwall's Lemma A.1.7 (indeed, we know by Lemma A.2.2 that $\mathcal{H}_2$ is absolutely continuous and $a, b \in L^1(0,T)$) to (4.21), obtaining

$$\mathcal{H}_2(t) \leq \bar{K}e^{\bar{C}}\mathcal{H}_2(0)$$

and this implies the estimates (2.2). All the estimates obtained in the Galerkin setting are still valid also for the solutions, then the constants appearing in the stability estimates only depend on T, the initial energy and on the norms of the initial data. □

Chapter 5

Numerical approximation

5.1 Discretization and numerical scheme

We now discuss a numerical approximation of the Cahn-Hilliard-Boussinesq system. As we have seen in the Introduction, there are not present any numerical schemes for this kind of systems, even though there are studies for similar schemes such as the CH equation (e.g., [64]) and the NSCH system (see, e.g., [95]). We start from the space discretization by means of finite elements, then passing to time discretization. We analyze the properties of the resulting numerical scheme, in terms of stability. This means that, in case of homogeneous Dirichlet boundary conditions for the temperature θ, we obtain that the total energy, defined in (19), does not increase in time. Moreover, we show that the scheme preserves the total mass of the system, which is a fundamental property of the CHB system with homogeneous Neumann boundary conditions for φ and the chemical potential μ.

In conclusion, the development of an adaptive time step gives the possibility of reducing the number of time steps in the simulations since, for small times, the time step needed for the solution of the CH equation is very small, whereas it could increase in the next time steps, since the characteristic time of the NS equation is larger than the former. We linearize the numerical scheme, by means of the Newton's method and then, in Chapter 6, we perform some numerical tests.

5.1.1 Semidiscrete formulation

For the discretization of the equations we followed the discretization of the CH equation proposed in [64]. The semidiscrete formulation reads as follows, by a Galerkin method (see e.g. [89] for a referenece). We will consider ν and κ as positive constants.

Let $\mathbf{V}_\sigma^h \subset [H_0^1(\Omega)]^2$, $Q_h \subset L^2(\Omega)$, $V_h \subset H^1(\Omega)$ and $Y_h \subset H_0^1(\Omega)$ be finite dimensional spaces.

For every $t \geq 0$, find $(\mathbf{u}_h, p_h, \varphi_h, \mu_h, \Theta_h) \in \mathbf{V}_\sigma^h \times Q_h \times V_h \times V_h \times Y_h$ (or $\Theta_h \in \tilde{Y}_h \subset H^1(\Omega)$, $\tilde{Y}_h$ finite dimensional space, for the nonhomogeneous Dirichlet boundary conditions) such that

$$\begin{aligned}
< \partial_t \mathbf{u}_h, \mathbf{w}_h > + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{w}) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{w}_h) - (p_h, \mathrm{div}\mathbf{w}_h) = \\
- (\varphi_h \nabla \mu_h, \mathbf{w}_h) + (\Theta_h, \mathbf{e}_2 \cdot \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{V}_\sigma^h
\end{aligned}$$

$$(\mathrm{div}\ \mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h$$

$$< \partial_t \varphi_h, v_h > + (\nabla \mu_h, \nabla v_h) + (\mathbf{u}_h \cdot \nabla \varphi_h, v_h) = 0 \quad \forall v_h \in V_h$$

$$< \mu_h, q_h >= \alpha < \nabla \varphi_h, \nabla q_h > + < \Psi'(\varphi_h), q_h > \quad \forall q_h \in V_h$$

$$< \partial_t \Theta_h, \xi_h > + \kappa(\nabla \Theta_h, \nabla \xi_h) + (\mathbf{u}_h \cdot \nabla \Theta_h, \xi_h) = 0 \quad \forall \xi_h \in Y_h.$$

Remark 5.1.1. We used the same spaces for the approximate components of the solution φ_h and μ_h.

Remark 5.1.2. The approximate components of the solution can be expressed by means of the bases of the finite dimensional spaces adopted:

$$\mathbf{u}_h(t) = \sum_{i=1}^{N_u} \zeta_i(t) \mathbf{w}_i \in \mathbf{V}_\sigma^h \quad p_h(t) = \sum_{i=1}^{N_p} \rho_i(t) \phi_i \in Q_h \quad \varphi_h(t) = \sum_{i=1}^{N_\varphi} \beta_i(t) \chi_i \in V_h \tag{5.1}$$

$$\mu_h(t) = \sum_{i=1}^{N_\varphi} \gamma_i(t) \chi_i \in V_h \quad \Theta_h(t) = \sum_{i=1}^{N_\Theta} \delta_i(t) v_i \in \mathbf{Y}_h, \tag{5.2}$$

where $\mathbf{V}_\sigma^h := Span(\mathbf{w}_1, \ldots, \mathbf{w}_{N_p})$, $Q_h := Span(\phi_1, \ldots, \phi_{N_p})$, $V_h := Span(\chi_1, \ldots, \chi_{N_\varphi})$ and $Y_h := Span(v_1, \ldots, v_{N_\Theta})$, and $\{\zeta_i\}_i$, $\{\rho_i\}_i$, $\{\beta_i\}_i$, $\{\gamma_i\}_i$ and $\{\delta_i\}_i$ are the coordinates of the variables with respect to the corresponding basis of the finite dimensional space they belong to.

As a particular choice of Galerkin method, we choose the Finite Elements Method (FEM) and we now define the finite dimensional spaces.

Namely, in the sequel we will adopt the following choices for the finite element approximations. We define $\mathcal{T}_h$ as a finite regular triangulation, which is a covering of the domain Ω_h, a polygonal approximation of the domain $\Omega \in \mathbb{R}^2$ (if it is not polygonal itself, otherwise $\Omega_h \equiv \Omega$). Namely, we define

$$\Omega_h = int\left(\bigcup_{K \in \mathcal{T}_h K}\right),$$

where K is each triangle of the triangulation, and, given a set A, $int(A)$ is the interior of A (see, for instance, [89] and references therein, for a more detailed description of FEM).

We then introduce $(\mathbf{V}_\sigma^h, Q_h) = (\mathbb{P}_{1b}$ finite elements, $\mathbb{P}_1$ finite elements), where

$$\mathbb{P}_1 \text{ finite elements} = \{v_h \in C^0(\overline{\Omega}) : v_h|_K \in \mathbb{P}_1 \quad \forall K \in \mathcal{T}_h\},$$

representing the space of the globally continuous functions that are polynomials of degree 1 on each triangle K of the triangulation $\mathcal{T}_h$.

Moreover $\mathbb{P}_{1b}$ finite elements is the classical $\mathbb{P}_1$ bubble space, where the linear velocity space is enriched by additional degrees of freedom (the *bubbles*) which are zero at each element boundary and is either cubic or piecewise linear inside the element (see, e.g., [89] for a reference).

Remark 5.1.3. We notice that the choice of the previous two spaces is inf-sup stable (see [13]).

Then we will consider $V_h = \mathbb{P}_1$ finite elements and $Y_h = \mathbb{P}_1$ finite elements, in order to reduce the computational costs.

5.1.2 Fully discrete scheme

Following [64] for the CH equation, we then approximated all the other time derivatives by means of backward Euler approximation (see [37] for a reference).

For the treatment of the nonlinearities we use a semi-implicit scheme only for the convective term in the velocity equation and for the term whose indexes are highlighted with a $*$ in the following (they should all be $n+1$ if we followed a fully implicit scheme).

Moreover we introduced the trilinear forms $c_1(\mathbf{u},\mathbf{v},\mathbf{w}) = b(\mathbf{u},\mathbf{v},\mathbf{w}) + \frac{1}{2}(\mathbf{u}\ \mathrm{div}\mathbf{w},\mathbf{v})$, with $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}\in [H^1_0(\Omega)]^2$ and $c_2(\mathbf{z},\theta,\xi) = -(\mathbf{z}\cdot\nabla\xi,\theta) - \frac{1}{2}(\mathrm{div}\mathbf{z}\ \theta,\xi)$, with $\mathbf{z}\in [H^1_0(\Omega)]^2$, $\theta,\ \xi \in H^1(\Omega)$, in order to obtain antisymmetric trilinear forms, following the ideas, e.g., in [81] and [97], which are used to guarantee energy stability (see Theorem 5.1.4). Clearly, if $\mathbf{w},\mathbf{z} \in \mathbf{V}_\sigma$, the forms coincide with the original ones, because the divergence of these functions is zero. The scheme thus reads: for $\Delta t_n > 0$ and for all n such that $t_n \leq T$, with $T > 0$ fixed value, find $(\mathbf{u}^h_{n+1}, p^h_{n+1}, \varphi^h_{n+1}, \mu^h_{n+1}, \Theta^h_{n+1}) \in \mathbf{V}^h_\sigma \times Q_h \times V_h \times V_h \times Y_h$ such that

$$\begin{aligned}&\frac{1}{\Delta t_n}(\mathbf{u}^h_{n+1} - \mathbf{u}^h_n, \mathbf{w}_h) + b(\mathbf{u}^h_{*,n}, \mathbf{u}^h_{n+1}, \mathbf{w}_h)\\ &+ \frac{1}{2}(\mathbf{u}^h_{n+1}\mathrm{div}\ \mathbf{u}^h_{*,n}, \mathbf{w}_h) + \nu(\nabla\mathbf{u}^h_{n+1}, \nabla\mathbf{w}_h) - (p^h_{n+1}, \mathrm{div}\mathbf{w}_h)\\ &= -(\varphi^h_{*,n}\nabla\mu^h_{n+1}, \mathbf{w}_h) + (\Theta^h_{n+1}, \mathbf{e}_2\cdot\mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{V}^h_\sigma\end{aligned} \tag{5.3}$$

$$(\mathrm{div}\ \mathbf{u}^h_{n+1}, q_h) = 0 \qquad \forall q_h \in Q_h \tag{5.4}$$

$$\begin{aligned}&\frac{1}{\Delta t_n}(\varphi^h_{n+1} - \varphi^h_n, v_h)\\ &+ (\nabla\mu^h_{n+1}, \nabla v_h) - (\mathbf{u}^h_{n+1}\varphi^h_{*,n}, \nabla v_h) = 0 \quad \forall v_h \in V_h\end{aligned} \tag{5.5}$$

$$\begin{aligned}(\mu^h_{n+1}, \phi) &= \left(\phi, \frac{1}{2}\left(\Psi'(\varphi^h_n) + \Psi'(\varphi^h_{n+1})\right) - \frac{(\varphi^h_{n+1} - \varphi^h_n)^2}{12}\Psi'''(\varphi^h_n)\right)\\ &+ \alpha(\nabla\phi, \nabla\varphi^h_{n+\beta}) \qquad \forall\phi \in V_h\end{aligned} \tag{5.6}$$

$$\begin{aligned}&\frac{1}{\Delta t_n}(\Theta^h_{n+1} - \Theta^h_n, \xi_h) + \kappa(\nabla\Theta^h_{n+1}, \nabla\xi_h)\\ &- (\Theta^h_{n+1}, \mathbf{u}^h_{n+1}\cdot\nabla\xi_h) - \frac{1}{2}(\mathrm{div}\ \mathbf{u}^h_{n+1}\Theta^h_{n+1}, \xi_h) = 0\ \forall\xi_h \in Y_h\end{aligned} \tag{5.7}$$

where we chose $\varphi^h_{*,n} = \varphi^h_n$, $\mathbf{u}^h_{*,n} = \mathbf{u}^h_n$ and

$$\beta = \frac{1}{2} + \eta,$$

with η a real-valued parameter to be chosen and

$$\varphi_{n+\beta}^h = \varphi_n^h + \beta(\varphi_{n+1}^h - \varphi_n^h).$$

For the case of homogeneous Dirichlet boundary conditions for temperature, we have $\Theta_{n+1}^h \in H_0^1(\Omega)$, whereas for a suitable nonhomogeneous datum g at the boundary (see Chapter 2 for a detailed description of the necessary regularity of the boundary datum) we ask for $\Theta_{n+1}^h \in H^1(\Omega)$ such that $\Theta_{n+1}^h = g_h(t_{n+1})$ on $\partial\Omega_h$ for every time step n, where g_h is a suitable projection or interpolation of g on the finite dimensional space of the traces of functions in $\tilde{Y}_h$ on $\partial\Omega_h$.

For the homogeneous case we propose the following theorem, having fixed $T > 0$ and $N \in \mathbb{N}$ the maximum value such that the last time step $t_N \leq T$. We define Δt_n as the time step at each time t_n, in order to highlight its dependence on each step n.

The numerical scheme adopted is the following: for $\Delta t_n > 0$ and for all n such that $t_n \leq T$, with $T > 0$ fixed value, find $(\mathbf{u}_{n+1}^h, p_{n+1}^h, \varphi_{n+1}^h, \mu_{n+1}^h, \Theta_{n+1}^h) \in \mathbf{V}_\sigma^h \times Q_h \times V_h \times V_h \times Y_h$ such that

$$\begin{aligned} &\frac{1}{\Delta t_n}(\mathbf{u}_{n+1}^h - \mathbf{u}_n^h, \mathbf{w}_h) + b(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h, \mathbf{w}_h) \\ &+ \frac{1}{2}(\mathbf{u}_{n+1}^h \text{div } \mathbf{u}_n^h, \mathbf{w}_h) + \nu(\nabla \mathbf{u}_{n+1}^h, \nabla \mathbf{w}_h) - (p_{n+1}^h, \text{div}\mathbf{w}_h) \\ &= -(\varphi_n^h \nabla \mu_{n+1}^h, \mathbf{w}_h) + (\Theta_{n+1}^h, \mathbf{e}_2 \cdot \mathbf{w}_h) \quad \forall \mathbf{w}_h \in \mathbf{V}_\sigma^h \end{aligned} \tag{5.8}$$

$$(\text{div } \mathbf{u}_{n+1}^h, q_h) = 0 \qquad \forall q_h \in Q_h \tag{5.9}$$

$$\begin{aligned} &\frac{1}{\Delta t_n}(\varphi_{n+1}^h - \varphi_n^h, v_h) \\ &+ (\nabla \mu_{n+1}^h, \nabla v_h) - (\mathbf{u}_{n+1}^h \varphi_n^h, \nabla v_h) = 0 \quad \forall v_h \in V_h \end{aligned} \tag{5.10}$$

$$\begin{aligned} (\mu_{n+1}^h, \phi) = &\left(\phi, \frac{1}{2}\left(\Psi'(\varphi_n^h) + \Psi'(\varphi_{n+1}^h)\right) - \frac{(\varphi_{n+1}^h - \varphi_n^h)^2}{12}\Psi'''(\varphi_n^h)\right) \\ &+ \alpha(\nabla\phi, \nabla\varphi_{n+\beta}^h) \quad \forall \phi \in V_h \end{aligned} \tag{5.11}$$

$$\begin{aligned} &\frac{1}{\Delta t_n}(\Theta_{n+1}^h - \Theta_n^h, \xi_h) + \kappa(\nabla \Theta_{n+1}^h, \nabla \xi_h) \\ &- (\Theta_{n+1}^h, \mathbf{u}_{n+1}^h \cdot \nabla \xi_h) - \frac{1}{2}(\text{div } \mathbf{u}_{n+1}^h \Theta_{n+1}^h, \xi_h) = 0 \ \forall \xi_h \in Y_h \end{aligned} \tag{5.12}$$

where

$$\beta = \frac{1}{2} + \eta,$$

with η a real-valued parameter to be chosen and

$$\varphi^h_{n+\beta} = \varphi^h_n + \beta(\varphi^h_{n+1} - \varphi^h_n).$$

Theorem 5.1.4. *Consider the scheme (5.8)-(5.12): in case of homogeneous Dirichlet conditions on the temperature and velocity fields, defining*

$$E_n = \frac{1}{2}||\boldsymbol{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega \Psi(\varphi_n)dx \tag{5.13}$$

enjoys the following properties, $\forall n = 0, \dots, N-1$*:*

- *Mass conservation:*

$$\int_\Omega \varphi^h_{n+1} dx = \int_\Omega \varphi^h_0 \, dx$$

- *Nonlinear stability condition (for* Δt_n *sufficiently small),with* C_0 *ad* C_1 *the Poincaré's constants for the homogeneous Dirichlet condition for temperature and velocity respectively:*

$$E_{n+1} \leq (1 + \frac{\Delta t_n C_1^2}{4\nu})E_n$$

In particular, if $\kappa\nu \geq \dfrac{C_1^2 C_0^2}{4}$*, we have, independently of* Δt_n*,*

$$E_{n+1} \leq E_n \quad \forall n = 0, \dots, N-1 \tag{5.14}$$

Proof. The mass conservation property is immediate, substituting $v_h = 1$ as a test function for (5.10).

In [64] the following quadrature formula is introduced and proven:

Let $f: \ [a,b] \to \mathbb{R}$ be a sufficiently smooth function. Then

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a) + f(b)) - \frac{(b-a)^3}{12} f''(a) - \frac{(b-a)^4}{24} f'''(\chi) \quad \chi \in (a,b). \tag{5.15}$$

This formula in [64] is then used to show the energy stability of a numerical scheme for the approximation of CH equation, whereas here we exploit it for the numerical scheme approximating the CHB system.

As done in [64], we apply the quadrature formula (5.15) to the right-hand side of the identity $\Phi = \Psi'$ (such that $\Phi''' = \Psi^{IV} \geq 0$, as it is for the case of the physically relevant logarithmic potential (2) defined in the Introduction) in integral form, defining $[[a_n^h]] = a_{n+1}^h - a_n^h$:

$$[[\Psi(\varphi_n^h)]] = \int_{\varphi_n^h}^{\varphi_{n+1}^h} \Psi'(t)dt = \int_{\varphi_n^h}^{\varphi_{n+1}^h} \Phi(t)dt. \tag{5.16}$$

We obtain, for the right-hand side:

$$\int_{\varphi_n^h}^{\varphi_{n+1}^h} \Phi(t)dt = \frac{[[\varphi_n^h]]}{2}(\Phi(\varphi_n^h) + \Phi(\varphi_{n+1}^h)) - \frac{[[\varphi_n^h]]^3}{12}\Phi''(\varphi_n^h) - \frac{[[\varphi_n^h]]^4}{24}\Phi'''(\bar{\xi}), \tag{5.17}$$

where $\bar{\xi} = \xi\varphi_{n+1}^h + (1-\xi)\varphi_n^h = \varphi_{n+\xi}^h$, with $\xi \in (0,1)$. After some algebraic manipulations, from (5.16) we can write, dividing by $[[\varphi_n^h]]$:

$$\frac{[[\Psi(\varphi_n^h)]]}{[[\varphi_n^h]]} + \frac{[[\varphi_n^h]]^3}{24}\Phi'''(\varphi_{n+\xi}^h) = \frac{1}{2}(\Phi(\varphi_n^h) + \Phi(\varphi_{n+1}^h)) - \frac{[[\varphi_n^h]]^2}{12}\Phi''(\varphi_n^h). \tag{5.18}$$

Now we follow a similar proof as in [64], but in a more general context than the only CH equation, testing equation (5.10) against $v = \mu_{n+1}^h$ and (5.11) against $\phi = \dfrac{[[\varphi_n^h]]}{\Delta t_n}$. We obtain

$$\begin{aligned}&(\mathbf{u}_{n+1}^h\varphi_n^h, \nabla\mu_{n+1}^h) - (\nabla\mu_{n+1}^h, \nabla\mu_{n+1}^h)\\ &- \left(\frac{[[\varphi_n^h]]}{\Delta t_n}, \frac{[[\Psi(\varphi_n^h)]]}{[[\varphi_n^h]]} + \frac{[[\varphi_n^h]]^3}{24}\Phi'''(\varphi_{n+\xi}^h)\right) - \alpha\left(\nabla\frac{[[\varphi_n^h]]}{\Delta t_n}, \nabla\varphi_{n+\beta}^h\right) = 0\end{aligned} \tag{5.19}$$

Using the relation $\varphi_{n+1/2}^h = \dfrac{1}{2}(\varphi_n^h + \varphi_{n+1}^h)$, we obtain

$$\varphi_{n+\beta}^h = \varphi_{n+1/2}^h + \eta[[\varphi_n^h]].$$

Therefore it follows from (5.19) that

$$\begin{aligned}&(\mathbf{u}_{n+1}^h\varphi_n^h, \nabla\mu_{n+1}^h) - (\nabla\mu_{n+1}^h, \nabla\mu_{n+1}^h) - \left(\frac{[[\varphi_n^h]]}{\Delta t_n}, \frac{[[\Psi(\varphi_n^h)]]}{[[\varphi_n^h]]} + \frac{[[\varphi_n^h]]^3}{24}\Phi'''(\varphi_{n+\xi}^h)\right)\\ &- \alpha\left(\nabla\frac{[[\varphi_n^h]]}{\Delta t_n}, \nabla\varphi_{n+1/2}^h\right) - \alpha\left(\nabla[[\varphi_n^h]], \frac{\eta}{\Delta t_n}\nabla[[\varphi_n^h]]\right) = 0\end{aligned}$$

and, making use of the identity,

$$\alpha\left(\nabla[[\varphi_n^h]], \nabla\varphi_{n+1/2}^h\right) = \frac{\alpha}{2}\int_\Omega [[\ |\nabla\varphi_n^h|^2\ ]]dx$$

we get

$$\frac{1}{\Delta t_n}\left\{\frac{\alpha}{2}\int_\Omega [[\ |\nabla \varphi_n^h|^2\ ]]dx + \int_\Omega [[\Psi(\varphi_n^h)]]dx\right\} - (\mathbf{u}_{n+1}^h \varphi_n^h, \nabla \mu_{n+1}^h)$$
$$= -(\nabla \mu_{n+1}^h, \nabla \mu_{n+1}^h) - \left(\frac{[[\varphi_n^h]]^4}{\Delta t_n}, \frac{1}{24}\Phi'''(\varphi_{n+\xi}^h)\right) - \alpha\left(\nabla[[\varphi_n^h]], \frac{\eta}{\Delta t_n}\nabla[[\varphi_n^h]]\right). \qquad (5.20)$$

Since $\Phi''' \geq 0$ in the case of logarithmic potential, all the terms in the right-hand side of (5.20) are negative, thus

$$\frac{1}{\Delta t_n}\left\{\frac{\alpha}{2}\int_\Omega [[\ |\nabla \varphi_n^h|^2\ ]]dx + \int_\Omega [[\Psi(\varphi_n^h)]]dx\right\} - (\mathbf{u}_{n+1}^h \varphi_n^h, \nabla \mu_{n+1}^h) \leq 0$$

for any $n \geq 0$.

Now, we test the equation for the velocity against $\mathbf{w}_h = \mathbf{u}_{n+1}^h$, the continuity equation against $q_h = p_{n+1}^h$ and the equation for the temperature against $\xi_h = \Theta_{n+1}^h$. Since

$$-(\mathbf{u}_{n+1}^h \cdot \nabla \Theta_{n+1}^h, \Theta_{n+1}^h) - \frac{1}{2}(\text{div } \mathbf{u}_{n+1}^h \Theta_{n+1}^h, \Theta_{n+1}^h) = 0$$

and

$$b(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h, \mathbf{u}_{n+1}^h) + \frac{1}{2}(\mathbf{u}_{n+1}^h \text{div } \mathbf{u}_n^h, \mathbf{u}_{n+1}^h) = 0,$$

due to homogeneous Dirichlet boundary conditions for velocity and temperature, we obtain, summing up all the equations, simplifying the term $\Delta t_n(\varphi_n^h \nabla \mu_{n+1}^h, \mathbf{u}_{n+1}^h)$, which cancels out in the summation:

$$\begin{aligned}
&([[\mathbf{u}_n^h]], \mathbf{u}_{n+1}^h) + \nu \Delta t_n ||\nabla \mathbf{u}_{n+1}^h||^2 \\
&+ \frac{\alpha}{2}[[\ ||\nabla \varphi_n^h||^2\ ]] + \int_\Omega [[\Psi(\varphi_n^h)]]dx \\
&([[\Theta_n^h]], \Theta_{n+1}^h) + \kappa \Delta t_n ||\nabla \Theta_{n+1}^h||^2 \\
&\leq \Delta t_n(\Theta_{n+1}^h, \mathbf{e}_2 \cdot \mathbf{u}_{n+1}^h).
\end{aligned} \qquad (5.21)$$

To reach the first stability result, we can apply Cauchy-Schwartz's and Young's inequalities (also to obtain $(\mathbf{u}_n^h, \mathbf{u}_{n+1}^h) \leq \frac{1}{2}||\mathbf{u}_n^h||^2 + \frac{1}{2}||\mathbf{u}_{n+1}^h||^2$ and $(\Theta_n^h, \Theta_{n+1}^h) \leq \frac{1}{2}||\Theta_n^h||^2 + \frac{1}{2}||\Theta_{n+1}^h||^2$) to get for the right-hand side:

$$\Delta t_n(\Theta_{n+1}^h, \mathbf{e}_2 \cdot \mathbf{u}_{n+1}^h) \leq \Delta t_n ||\Theta_{n+1}^h||\ ||\mathbf{u}_{n+1}^h|| \leq \frac{\Delta t_n C_1^2}{4\nu}||\Theta_{n+1}^h||^2 + \nu \Delta t_n ||\nabla \mathbf{u}_{n+1}^h||^2 \quad (5.22)$$

Analyzing the equation for the temperature (5.12), testing it against Θ_{n+1}^h, we easily get that $||\Theta_{n+1}^h|| \leq ||\Theta_n^h||$. Thus substituting it in inequality (5.21) and rearranging the terms, we get

$$E_{n+1} \leq (1 + \frac{\Delta t_n C_1^2}{4\nu}) E_n$$

Applying Poincaré's inequality also to the temperature in (5.22), we obtain

$$\Delta t_n(\Theta_{n+1}^h, \mathbf{e}_2 \cdot \mathbf{u}_{n+1}^h) \leq \frac{\Delta t_n C_0^2 C_1^2}{4\nu} ||\nabla \Theta_{n+1}^h||^2 + \nu \Delta t_n ||\nabla \mathbf{u}_{n+1}^h||^2$$

and we observe that if $\kappa \geq \dfrac{C_1^2 C_0^2}{4\nu}$ or, in other words, if

$$\kappa\nu \geq \frac{C_1^2 C_0^2}{4} \tag{5.23}$$

we have

$$E_{n+1} \leq E_n \quad \forall n = 0, \dots, N,$$

because we exploit the dissipative term related to thermal conductivity in the left-hand side to compensate the remaining extra term in (5.22), and this concludes the proof. □

Remark 5.1.5. The condition (5.23) on the physical parameters, which depends only on the domain Ω, has a physical interpretation: since in the NS equation a new forcing term due to temperature, $\theta \mathbf{e}_2$, which is the gravitational force, is present, either the viscosity ν in the NS equation or the thermal conductivity κ have to be sufficiently large to be able to dissipate in time a sufficient amount of the total energy, in order to prevent it from non physical increase. To quantify this condition on the parameters, we consider the rectangle $(0, 2) \times (0, 1)$, which is used in the simulations of Chapter 6: a computation by means of the MATLAB command `pdeeig` shows that the best Poincaré's constants for the mesh chosen are $C_0 = C_1 \approx 0.28$ (indeed the first eigenvalue of the Dirichlet laplacian is $\lambda_1 \approx 12.34$ and $C_0 = C_1 = \lambda_1^{-1/2}$). Therefore we obtain, thanks to Theorem 5.1.4, that the sufficient condition is $\kappa\nu \geq \dfrac{C_1^2 C_0^2}{4} \approx 0.0015$.

We can now pass to analyze the linearization of the above numerical scheme, in order to solve it, by means of the FreeFem++ software (see [69] for a reference).

5.1.3 Linearization

We first linearize the two coupled problems, i.e. the equations for temperature, the NS equations and the CH equation, by decoupling them following a fixed point iteration scheme. We fix a maximum number of fixed point iterations for each time step, indicated by $S_m > 0$.

Again we define Δt_n the time step at time t_n which can vary at each iteration. The method thus becomes:

For $\Delta t_n > 0$ and for all n such that $t_n \leq T$, with $T > 0$ fixed value, solve the following iterative scheme:

- Set the initial condition for $n = 0$:

$$(\mathbf{u}_0^h, p_0^h, \varphi_0^h, \mu_0^h, \Theta_0^h) = (\mathbf{u}_n^h, p_n^h, \varphi_n^h, \mu_n^h, \Theta_n^h). \tag{5.24}$$

  and set the initial time step Δt_0.

- For every n such that $t_n \leq T$:

  1. Set $s = 0$ and initialize

$$(\mathbf{u}_{n+1,0}^h, p_{n+1,0}^h, \varphi_{n+1,0}^h, \mu_{n+1,0}^h, \Theta_{n+1,0}^h) = (\mathbf{u}_n^h, p_n^h, \varphi_n^h, \mu_n^h, \Theta_n^h)$$

  2. While $0 \leq s < S_m$ do:

     (a) Compute $\Theta_{n+1,s+1}^h$ such that

$$\begin{aligned}&\frac{1}{\Delta t_n}(\Theta_{n+1,s+1}^h - \Theta_n^h, \xi_h) + \kappa(\nabla\Theta_{n+1,s+1}^h, \nabla\xi_h) - (\Theta_{n+1,s+1}^h, \mathbf{u}_{n+1,s}^h \cdot \nabla\xi_h)\\ &- \frac{1}{2}(\text{div } \mathbf{u}_{n+1,s}^h \Theta_{n+1,s+1}^h, \xi_h) = 0 \qquad \forall \xi_h \in Y_h.\end{aligned} \tag{5.25}$$

     (b) Compute $(\mathbf{u}_{n+1,s+1}^h, p_{n+1,s+1}^h)$ such that

$$\begin{aligned}&\frac{1}{\Delta t_n}(\mathbf{u}_{n+1,s+1}^h - \mathbf{u}_n^h, \mathbf{w}_h) + b(\mathbf{u}_n^h, \mathbf{u}_{n+1,s+1}^h, \mathbf{w}_h)\\ &+ \frac{1}{2}(\mathbf{u}_{n+1,s+1}^h \text{div } \mathbf{u}_n^h, \mathbf{w}_h) + \nu(\nabla\mathbf{u}_{n+1,s+1}^h, \nabla\mathbf{w}_h)\\ &- (p_{n+1,s+1}^h, \text{div}\mathbf{w}_h) = -(\varphi_n^h \nabla\mu_{n+1,s}^h, \mathbf{w}_h) + (\Theta_{n+1,s+1}^h, \mathbf{e}_2 \cdot \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in \mathbf{V}_\sigma^h\end{aligned} \tag{5.26}$$

     and

$$(\text{div } \mathbf{u}_{n+1,s+1}^h, q_h) = 0 \qquad \forall q_h \in Q_h. \tag{5.27}$$

(c) Compute $(\mu^h_{n+1,s+1}, \varphi^h_{n+1,s+1})$ by means of Newton's method, such that

$$\frac{1}{\Delta t_n}(\varphi^h_{n+1,s+1} - \varphi^h_n, v_h) + (\nabla \mu^h_{n+1,s+1}, \nabla v_h) - (\mathbf{u}^h_{n+1,s+1}\varphi^h_n, \nabla v_h) = 0 \quad \forall v_h \in V_h \tag{5.28}$$

and

$$(\mu^h_{n+1,s+1}, \phi) = \left(\phi, \frac{1}{2}\left(\Psi'(\varphi^h_n) + \Psi'(\varphi^h_{n+1,s+1})\right) - \frac{(\varphi^h_{n+1,s+1} - \varphi^h_n)^2}{12}\Psi'''(\varphi^h_n)\right) + \alpha(\nabla\phi, \nabla\varphi^h_{n+\beta,s+1}) \quad \forall \phi \in V_h \tag{5.29}$$

3. Set $s = s + 1$

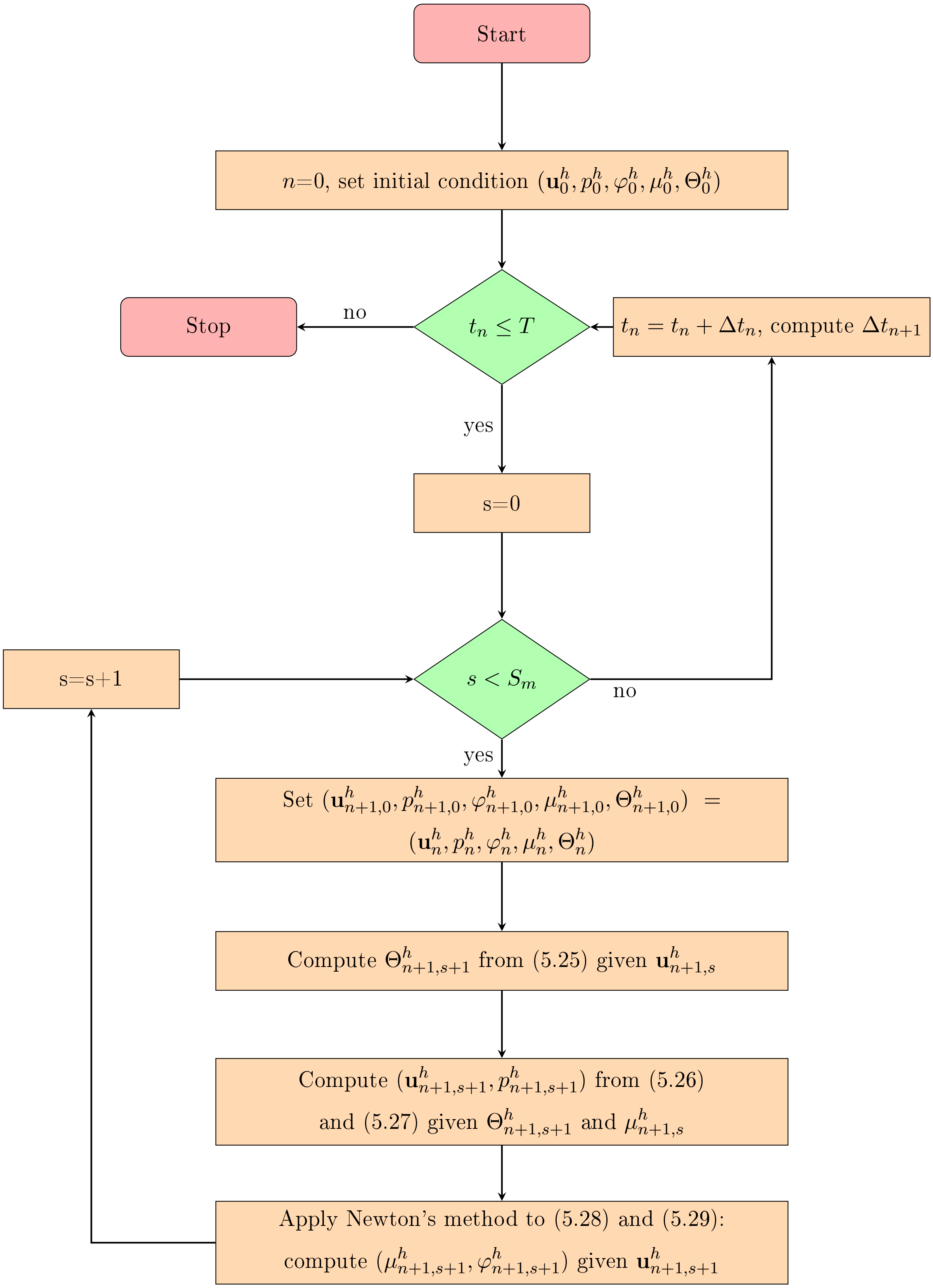
- Set $(\mathbf{u}^h_{n+1}, p^h_{n+1}, \varphi^h_{n+1}, \mu^h_{n+1}, \Theta^h_{n+1}) = (\mathbf{u}^h_{S_m-1}, p^h_{S_m-1}, \varphi^h_{S_m-1}, \mu^h_{S_m-1}, \Theta^h_{S_m-1})$.
- Set $n = n + 1$ and compute Δt_{n+1}.

Remark 5.1.6. Notice that, in step 2.(b), the right-hand side is completely known from the previous step and can be computed. The same goes for step 2.(c), which is decoupled from the other equations, since $\mathbf{u}^h_{n+1,s+1}$ has already been computed in the previous steps. We can visualize the numerical scheme in the flowchart in Figure 5.1.

Remark 5.1.7. In the simulations of Chapter 6, we will sistematically use $S_m = 1$, since we found it enough to get the desired properties of the solution: this means that it is equivalent to solve the linearized problem without fixed point iterations. For further investigations, the fixed point iterations could be implemented, for example with the following stopping criterion based on the increment, instead of the maximum number of iterations: fixed a tolerance *tol*, stop the iterations when

$$||\mathbf{u}^h_{n+1,s+1} - \mathbf{u}^h_{n+1,s}|| + ||\varphi^h_{n+1,s+1} - \varphi^h_{n+1,s}|| + ||\Theta^h_{n+1,s+1} - \Theta^h_{n+1,s}|| \leq \ tol.$$

Figure 5.1: Flowchart of the numerical scheme adopted.



We can now consider the bases for the finite element spaces previously defined: since we have chosen the same spaces for p, φ, μ and θ, we have the same number of basis functions, denoted by M; therefore we have $\mathbf{V}_\sigma^h = Span(\mathbf{w}_1, \dots, \mathbf{w}_{N_u})$, $Q_h = Span(\phi_1, \dots, \phi_M)$, $V_h = Span(\chi_1, \dots, \chi_M)$ and $Y_h = Span(v_1, \dots, v_M)$ and test the previous equations against each element of the bases.

We obtain

$$\begin{aligned}
&\mathbf{u}_h(t_n) \approx \mathbf{u}_n^h = \sum_{i=1}^{N_u} U_{n,i}\mathbf{w}_i \in \mathbf{V}_\sigma^h \qquad p_h(t_n) \approx p_n^h = \sum_{i=1}^{M} P_{n,i}\phi_i \in Q^h \\
&\varphi_h(t_n) \approx \varphi_n^h = \sum_{i=1}^{M} \bar{\varphi}_{n,i}\chi_i \in V_h \\
&\mu_h(t_n) \approx \mu_n^h = \sum_{i=1}^{M} \bar{\mu}_{n,i}\chi_i \in V_h \qquad \Theta_h(t_n) \approx \Theta_n^h = \sum_{i=1}^{M} \bar{\Theta}_{n,i}v_i \in \mathbf{Y}_h.
\end{aligned} \tag{5.30}$$

and we define $\mathbf{U}_n$, $\mathbf{P}_n$, $\bar{\varphi}_n$, $\bar{\mu}_n$ and $\bar{\Theta}_n$ as the vectors of coordinates with respect to the corresponding bases. Therefore we obtain:

For $\Delta t_n > 0$ and for all n such that $t_n \leq T$, with $T > 0$ fixed value, solve the following iterative scheme:

- Set the initial condition for $n = 0$:

$$(\mathbf{U}_0, \mathbf{P}_0, \bar{\varphi}_0, \bar{\mu}_0, \bar{\Theta}_0) = (\mathbf{U}_n, \mathbf{P}_n, \bar{\varphi}_n, \bar{\mu}_n, \bar{\Theta}_n). \tag{5.31}$$

  and set the initial time step Δt_0.

- For every n such that $t_n \leq T$:

  1. Set $s = 0$ and initialize

  $$(\mathbf{U}_{n+1,0}, \mathbf{P}_{n+1,0}, \bar{\varphi}_{n+1,0}, \bar{\mu}_{n+1,0}, \bar{\Theta}_{n+1,0}) = (\mathbf{U}_n, \mathbf{P}_n, \bar{\varphi}_n, \bar{\mu}_n, \bar{\Theta}_n)$$

  2. While $0 \leq s < S_m$ do:

(a) Compute $\bar{\Theta}_{n+1,s+1}$ such that $\forall i = 1, \dots, M$

$$\frac{1}{\Delta t_n} \sum_{j=1}^{M} \{\bar{\Theta}_{n+1,s+1,j} - \bar{\Theta}_{n,j}\}(v_j, v_i) + \kappa \sum_{j=1}^{M} \bar{\Theta}_{n+1,s+1,j}(\nabla v_j, \nabla v_i)$$
$$- \sum_{j=1}^{M} \bar{\Theta}_{n+1,s+1,j}(v_j, \mathbf{u}^h_{n+1,s} \cdot \nabla v_i) - \frac{1}{2} \sum_{j=1}^{M} \bar{\Theta}_{n+1,s+1,j}(\text{div } \mathbf{u}^h_{n+1,s} v_j, v_i) = 0 \tag{5.32}$$

(b) Compute $(\mathbf{U}_{n+1,s+1}, \mathbf{P}_{n+1,s+1})$ such that $\forall i = 1, \dots, N_u$

$$\frac{1}{\Delta t_n} \sum_{j=1}^{N_u} \{U_{n+1,s+1,j} - U_{n,j}\}(\mathbf{w}_j, \mathbf{w}_i) + \sum_{j=1}^{N_u} U_{n+1,s+1,j} b(\mathbf{u}^h_n, \mathbf{w}_j, \mathbf{w}_i) \tag{5.33}$$
$$+ \frac{1}{2} \sum_{j=1}^{N_u} U_{n+1,s+1,j}(\mathbf{w}_j \text{div } \mathbf{u}^h_n, \mathbf{w}_i) + \nu \sum_{j=1}^{N_u} U_{n+1,s+1,j}(\nabla \mathbf{w}_j, \nabla \mathbf{w}_i)$$
$$- \sum_{j=1}^{N_p} P_{n+1,s+1,j}(\phi_j, \text{div} \mathbf{w}_i) = -(\varphi^h_n \nabla \mu^h_{n+1,s}, \mathbf{w}_i) + (\Theta^h_{n+1,s+1}, \mathbf{e}_2 \cdot \mathbf{w}_i)$$

and

$$\sum_{j=1}^{N_u} U_{n+1,s+1,j}(\text{div} \mathbf{w}_j, \phi_i) = 0 \qquad \forall i = 1, \dots, M. \tag{5.34}$$

(c) Compute $(\bar{\mu}^h_{n+1,s+1}, \bar{\varphi}^h_{n+1,s+1})$ by means of Newton's method, such that they solve (5.28) and (5.29).

3. Set $s = s + 1$

- Set $(\mathbf{U}^{n+1}, \mathbf{P}^{n+1}, \bar{\varphi}^{n+1}, \bar{\mu}^{n+1}, \bar{\Theta}^{n+1}) = (\mathbf{U}_{S_m-1}, \mathbf{P}_{S_m-1}, \bar{\varphi}_{S_m-1}, \bar{\mu}_{S_m-1}, \bar{\Theta}_{S_m-1})$.
- Set $n = n + 1$ and compute Δt_{n+1}.

We are left to tackle the last nonlinear step 2.(c), whose expanded formulation reads: $\forall i = 1, \dots, M$

$$\frac{1}{\Delta t_n} \sum_{j=1}^{M} \{\bar{\varphi}_{n+1,s+1,j} - \bar{\varphi}_{n,j}\}(\chi_j, \chi_i) + \sum_{j=1}^{M} \bar{\mu}_{n+1,s+1,j}(\nabla \chi_j, \nabla \chi_i) - (\mathbf{u}^h_{n+1,s+1} \varphi^h_n, \nabla \chi_i) = 0 \tag{5.35}$$

and

$$\sum_{j=1}^{M} \bar{\mu}_{s+1,j}(\chi_j, \chi_i) \tag{5.36}$$

$$= (\chi_i, \frac{1}{2}(\Psi'(\varphi_n^h) + \Psi'(\varphi_{n+1,s+1}^h)) - \frac{(\varphi_{n+1,s+1}^h - \varphi_n^h)^2}{12}\Psi'''(\varphi_n^h)) + \alpha \sum_{j=1}^{M} \bar{\varphi}_{n+\beta,j}(\nabla\chi_j, \nabla\chi_i)$$

where $\bar{\varphi}_{n+\beta,j} = \bar{\varphi}_{n,j} + \beta(\bar{\varphi}_{n+1,s+1,j} - \bar{\varphi}_{n,j}) \ \ \forall j = 1, \dots, M$.

We apply Newton's method (see, e.g., [90]) to linearize the equations: in order to use it, we rewrite the equations in an algebraic form, introducing the following vectors and matrices:

$$\mathbf{A}_{ij} = \frac{1}{\Delta t_n}(\chi_j, \chi_i), \quad i, j = 1, \dots, M \tag{5.37}$$

$$\mathbf{K}_{ij} = (\nabla\chi_j, \nabla\chi_i), \quad i, j = 1, \dots, M \tag{5.38}$$

$$\mathbf{g}_i = \{\mathbf{A}\bar{\varphi}_n\}_i + (\mathbf{u}_{n+1,s+1}^h \varphi_n^h, \nabla\chi_i), \quad i = 1, \dots, M \tag{5.39}$$

$$\mathbf{D}_{ij} = (\chi_j, \chi_i), \quad i, j = 1, \dots, M \tag{5.40}$$

$$\mathbf{h}_i(\bar{\varphi}_{n+1,s+1}) = \left(\chi_i, \frac{1}{2}\left(\Psi'(\varphi_n^h) + \Psi'(\varphi_{n+1,s+1}^h)\right) - \frac{(\varphi_{n+1,s+1}^h - \varphi_n^h)^2}{12}\Psi'''(\varphi_n^h)\right), \ i = 1, \dots, M. \tag{5.41}$$

In this way, the system of equations becomes

$$\Phi(\begin{bmatrix} \bar{\varphi}_{n+1,s+1} \\ \bar{\mu}_{n+1,s+1} \end{bmatrix}) = \begin{bmatrix} \mathbf{A} & \mathbf{K} \\ -\alpha\beta\mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \bar{\varphi}_{n+1,s+1} \\ \bar{\mu}_{n+1,s+1} \end{bmatrix} - \begin{bmatrix} \mathbf{g} \\ \mathbf{h}(\bar{\varphi}_{n+1,s+1}) + \alpha(1-\beta)\mathbf{K}\bar{\varphi}_n \end{bmatrix} = \mathbf{0} \tag{5.42}$$

We clearly see in this way that to apply Newton's method we need to linearize the term $-\mathbf{h}$. In particular, the corresponding Jacobian reads:

$$\mathbf{J}_{ij}^h(\mathbf{q}) = -\frac{\partial \mathbf{h}_i}{\partial \bar{\varphi}_{s+1,j}} = -\frac{1}{2}\Psi''(\mathbf{q})(\chi_i, \chi_j) + \frac{(\mathbf{q} - \varphi_n^h)}{6}\Psi'''(\varphi_n^h)(\chi_i, \chi_j), \quad i, j = 1, \dots, M \tag{5.43}$$

and we can conclude with another iterative loop indexed by $r \geq 0$:

$$\begin{bmatrix} \mathbf{A} & \mathbf{K} \\ \mathbf{J}^h(\bar{\varphi}_{n+1,s+1,r}) - \alpha\beta\mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \delta\bar{\varphi} \\ \delta\bar{\mu} \end{bmatrix} = -\Phi(\begin{bmatrix} \bar{\varphi}_{n+1,s+1,r} \\ \bar{\mu}_{n+1,s+1,r} \end{bmatrix}) \tag{5.44}$$

$$\begin{bmatrix} \bar{\varphi}_{n+1,s+1,r+1} \\ \bar{\mu}_{n+1,s+1,r+1} \end{bmatrix} = \begin{bmatrix} \bar{\varphi}_{n+1,s+1,r} \\ \bar{\mu}_{n+1,s+1,r} \end{bmatrix} + \begin{bmatrix} \delta\bar{\varphi} \\ \delta\bar{\mu} \end{bmatrix} \tag{5.45}$$

As a stopping criterion we decided to use the one based on the residual: stop the iterations when

$$||\Phi(\begin{bmatrix}\bar{\varphi}_{n+1,s+1,r+1}\\ \bar{\mu}_{n+1,s+1,r+1}\end{bmatrix})||_2 \le tol,$$

with $tol = 10^{-4}$, where $||\mathbf{q}||_2^2 = \sum_{i=1}^{N_\varphi} q_i^2$ is the classical Euclidean 2-norm.

For the solution of the linear system we used LinearGMRES as a solver, with a fixed tolerance of 10^{-6}, which is 100 times smaller than the tolerance adopted as stopping criterion for the Newton's method.

In order to reduce the computational effort, we used a suitable block diagonal preconditioner, namely a Jacobi block preconditioner, for the iterative solver GMRES: in particular we made use of the following one

$$\mathbf{P} = \begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{bmatrix} \tag{5.46}$$

where $\mathbf{A}$ and $\mathbf{D}$ are the matrices introduced in (5.37) and (5.40), respectively, used in the discretization of the CH equation. In this way we reduced the GMRES iterations for the solution of the linear system to about 4 or 5 and the computational time is sensibly reduced, whereas the number of Newton's steps needed is about 1 or 2 for each step s.

We can thus restate the final algorithm to be solved in this way:

For $\Delta t_n > 0$ and for all n such that $t_n \le T$, with $T > 0$ fixed value, solve the following iterative scheme:

- Set the initial condition for $n = 0$:

  $$(\mathbf{U}_0, \mathbf{P}_0, \bar{\varphi}_0, \bar{\mu}_0, \bar{\Theta}_0) = (\mathbf{U}_n, \mathbf{P}_n, \bar{\varphi}_n, \bar{\mu}_n, \bar{\Theta}_n). \tag{5.47}$$

  and set the initial time step Δt_0.

- For every n such that $t_n \le T$:

  1. Set $s = 0$ and initialize

  $$(\mathbf{U}_{n+1,0}, \mathbf{P}_{n+1,0}, \bar{\varphi}_{n+1,0}, \bar{\mu}_{n+1,0}, \bar{\Theta}_{n+1,0}) = (\mathbf{U}_n, \mathbf{P}_n, \bar{\varphi}_n, \bar{\mu}_n, \bar{\Theta}_n)$$

2. While $0 \leq s < S_m$ do:

   (a) Compute $\bar{\Theta}_{n+1,s+1}$ such that $\forall i = 1, \ldots, M$

$$\frac{1}{\Delta t_n} \sum_{j=1}^{M} \{\bar{\Theta}_{n+1,s+1,j} - \bar{\Theta}_{n,j}\}(v_j, v_i) + \kappa \sum_{j=1}^{M} \bar{\Theta}_{n+1,s+1,j}(\nabla v_j, \nabla v_i)$$
$$- \sum_{j=1}^{M} \bar{\Theta}_{n+1,s+1,j}(v_j, \mathbf{u}_{n+1,s}^h \cdot \nabla v_i) - \frac{1}{2} \sum_{j=1}^{M} \bar{\Theta}_{n+1,s+1,j}(\text{div } \mathbf{u}_{n+1,s}^h v_j, v_i) = 0 \tag{5.48}$$

   (b) Compute $(\mathbf{U}_{n+1,s+1}, \mathbf{P}_{n+1,s+1})$ such that $\forall i = 1, \ldots, N_u$

$$\frac{1}{\Delta t_n} \sum_{j=1}^{N_u} \{U_{n+1,s+1,j} - U_{n,j}\}(\mathbf{w}_j, \mathbf{w}_i) + \sum_{j=1}^{N_u} U_{n+1,s+1,j} b(\mathbf{u}_n^h, \mathbf{w}_j, \mathbf{w}_i) \tag{5.49}$$
$$+ \frac{1}{2} \sum_{j=1}^{N_u} U_{n+1,s+1,j}(\mathbf{w}_j \text{div } \mathbf{u}_n^h, \mathbf{w}_i) + \nu \sum_{j=1}^{N_u} U_{n+1,s+1,j}(\nabla \mathbf{w}_j, \nabla \mathbf{w}_i)$$
$$- \sum_{j=1}^{N_p} P_{n+1,s+1,j}(\phi_j, \text{div}\mathbf{w}_i) = -(\varphi_n^h \nabla \mu_{n+1,s}^h, \mathbf{w}_i) + (\Theta_{n+1,s+1}^h, \mathbf{e}_2 \cdot \mathbf{w}_i)$$

   and

$$\sum_{j=1}^{N_u} U_{n+1,s+1,j}(\text{div}\mathbf{w}_j, \phi_i) = 0 \qquad \forall i = 1, \ldots, M. \tag{5.50}$$

   (c) Set $r = 0$ and

$$\begin{bmatrix} \bar{\varphi}_{n+1,s+1,0} \\ \bar{\mu}_{n+1,s+1,0} \end{bmatrix} = \begin{bmatrix} \bar{\varphi}_{n+1,s} \\ \bar{\mu}_{n+1,s} \end{bmatrix}$$

   While

$$||\Phi(\begin{bmatrix} \bar{\varphi}_{n+1,s+1,r} \\ \bar{\mu}_{n+1,s+1,r} \end{bmatrix})||_2 > tol$$

   repeat iteratively to find $(\bar{\mu}_{n+1,s+1}, \bar{\varphi}_{n+1,s+1})$:

      i.

$$\begin{bmatrix} \mathbf{A} & \mathbf{K} \\ \mathbf{J}^h(\bar{\varphi}_{n+1,s+1,r}) - \alpha\beta\mathbf{K} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \delta\bar{\varphi} \\ \delta\bar{\mu} \end{bmatrix} = -\Phi(\begin{bmatrix} \bar{\varphi}_{n+1,s+1,r} \\ \bar{\mu}_{n+1,s+1,r} \end{bmatrix} \tag{5.51}$$

      ii.

$$\begin{bmatrix} \bar{\varphi}_{n+1,s+1,r+1} \\ \bar{\mu}_{n+1,s+1,r+1} \end{bmatrix} = \begin{bmatrix} \bar{\varphi}_{n+1,s+1,r} \\ \bar{\mu}_{n+1,s+1,r} \end{bmatrix} + \begin{bmatrix} \delta\bar{\varphi} \\ \delta\bar{\mu} \end{bmatrix} \tag{5.52}$$

iii. $r = r + 1$

3. Set $s = s + 1$

- Set $(\mathbf{U}^{n+1}, \mathbf{P}^{n+1}, \bar{\varphi}^{n+1}, \bar{\mu}^{n+1}, \bar{\Theta}^{n+1}) = (\mathbf{U}_{S_m-1}, \mathbf{P}_{S_m-1}, \bar{\varphi}_{S_m-1}, \bar{\mu}_{S_m-1}, \bar{\Theta}_{S_m-1})$.

- Set $n = n + 1$ and compute Δt_{n+1}.

5.1.4 Time step adaptivity

As we have already noticed, in order to reduce the number of time steps in the simulations, since at the very beginning of them the time step needed for the solution of the CH equation is very small, whereas it could increase a bit in the next steps, we used an adaptive time step in the following iterative procedure, in order to compute Δt_{n+1}, given $\mathbf{u}_n^h$, φ_n^h, μ_n^h:

- Set

$$\Delta t_{1,(0)} = \Delta t_0$$

- For every $n \geq 0$ do:

For $l \geq 0$:

1. Compute $\varphi_{BE,(l)}$, solution of equation (5.5) with a Backward Euler scheme (thus a linear equation):

$$\frac{1}{\Delta t_{n+1,(l)}}(\varphi_{BE,(l)} - \varphi_n^h, v_h) + (\nabla \mu_n^h, \nabla v_h) - (\mathbf{u}_n^h \varphi_n^h, \nabla v_h) = 0 \quad \forall v_h \in V_h.$$

2. Compute $\varphi_{n+1,(l)}^h$ (and all the other variables) with the algorithm proposed in Section 5.1.3 and using $\Delta t_{n+1,(l)}$ as time step.
3. Calculate $e_{n+1,(l)} = \dfrac{||\varphi_{BE,(l)} - \varphi_{n+1,(l)}^h||_2}{||\varphi_{n+1,(l)}^h||_2}$.
4. Update the time step

$$\Delta t_{n+1,(l+1)} = \rho \sqrt{\frac{TOL}{e_{n+1,(l)}}} \Delta t_{n+1,(l)}$$

5. If $e_{n+1} \geq TOL$ then $l = l+1$ and return to step 1., otherwise set $\varphi^h_{n+1} = \varphi^h_{n+1,(l)}$ (and in the same way all the other variables), $\Delta t_{n+1} = \Delta t_{n+1,(l)}$ and $\Delta t_{n+2,0} = \Delta t_{n+1,(l+1)}$

- $n = n + 1$.

This kind of update of the time step is frequently used in adaptive time-stepping algorithms (see, e.g. [64] and [15]): we chose the safety coefficient $\rho = 0.9$ and $TOL = 10^{-3}$, as suggested in [80]. Moreover we set $\Delta t_0 = 6 \times 10^{-12}$. In this way, in the simulations that we present in Chapter 6, we reach a range of time steps from about 10^{-12} to 10^{-5}.

Chapter 6

Numerical tests

In this Chapter we present five simulations performed implementing the numerical scheme presented in Chapter 5, by means of FreeFem++. We verify the mass conservation of property and the stability of the scheme. We observe the phase separation phenomenon and, in particular, the effects of different initial conditions for the velocity and temperature fields on the concentration field.

6.1 Choice of parameters

In the numerical tests performed we decided to consider the concentration field in $[0, 1]$, as proposed in [64] (the theory obtained for the concentration field in $[-1, 1]$ is exactly the same as in this case, up to a rescaling), with the logarithmic potential defined as

$$\Psi(\varphi) = 2(\varphi \log(\varphi) + (1-\varphi) \log(1-\varphi)) + 2\alpha_0\varphi(1-\varphi)$$

with $\alpha_0 = 2.4$, in order to obtain the double well potential in Figure 6.1: it is non-convex, with two local minima to which the concentration is driven, at about 0.16 and 0.84. The derivatives necessaries for the computations of the Newton's method are then the following

$$\Psi'(\varphi) = 2(\log(\varphi) - \log(1-\varphi)) + 2\alpha_0(1-2\varphi)$$

$$\Psi''(\varphi) = 2(\frac{1}{\varphi} - \frac{1}{\varphi-1}) - 4\alpha_0$$

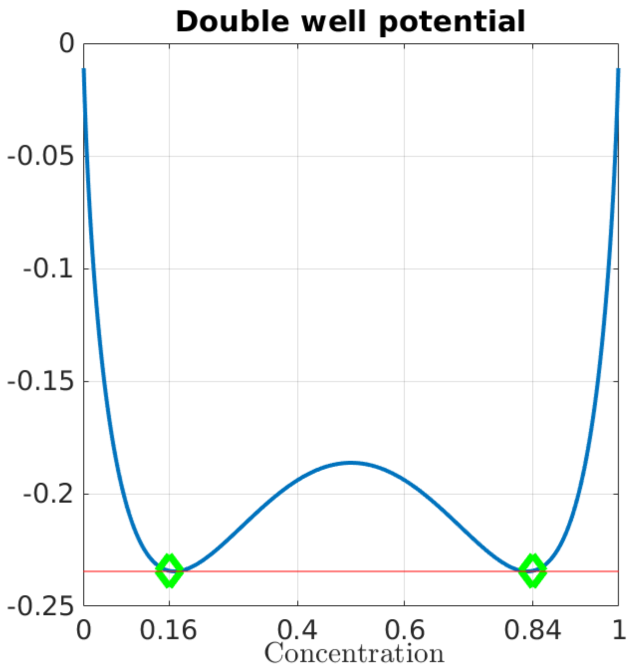


Figure 6.1: Double well potential: we can see the two local minima

$$\Psi'''(\varphi) = 2\frac{2\varphi - 1}{\varphi^2(1-\varphi)^2}$$
$$\Psi^{IV}(\varphi) = 4\frac{3\varphi^2 - 3\varphi + 1}{\varphi^3(1-\varphi)^3}$$

Remark 6.1.1. We can observe that $\Psi^{IV}(\varphi) \geq 0$ for $\varphi \in (0,1)$, thus Theorem 5.1.4 is valid.

The domain Ω chosen is the open rectangle $(0,2) \times (0,1) \subset \mathbb{R}^2$. We then choose the computational mesh, from which we will derive the value of α, the other parameter in the CH equation: we use a structured mesh with 26'080 triangular cells and 13'041 nodes, corresponding to a mean diameter of each cell of about $h \approx 0.009$, as we can see in Figure 6.2.

We then chose the parameter α to be $\alpha = 8 \times 10^{-5} \approx h^2$. The other parameters to be chosen are $\beta = 0.6$, the kinematic viscosity ν for the velocity and the thermal diffusivity κ for the temperature. These two coefficients will be chosen in different ways, but respecting the limitation shown in Remark 5.1.5.

Remark 6.1.2. Clearly, being in a continuous finite elements setting and using only regular

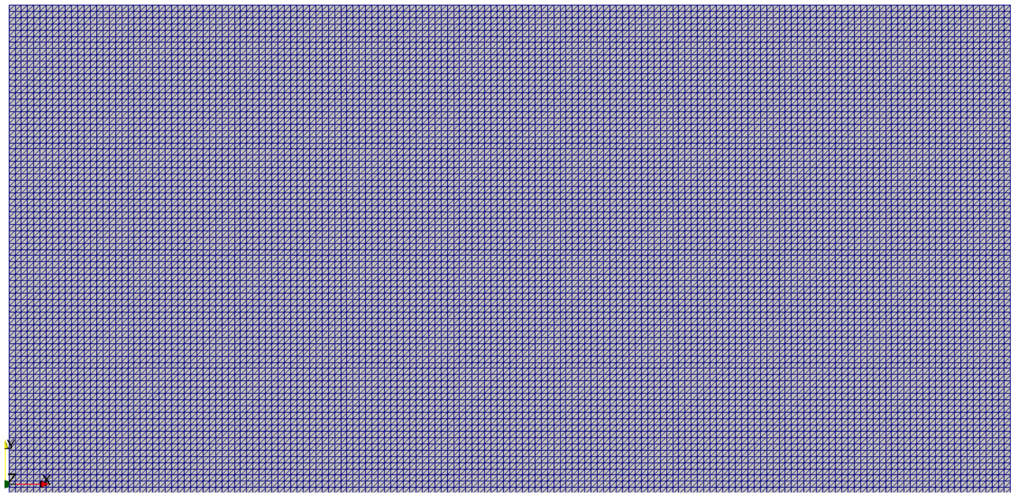


Figure 6.2: Computational mesh

functions, the hypotheses of Theorem 2.1.6, at a continuous level in infinite dimensional spaces, are fulfilled, thus we expect the existence of a solution.

6.2 Initial conditions

For what concerns the initial conditions to be set, we now focus the attention on the concentration field, since for temperature and velocity the initial conditions will vary between the different numerical tests. We decided to consider two different possible initializations:

- Symmetric initialization with zero mean perturbation of a uniform field $\bar{\varphi} \equiv 0.63$:

  $$\varphi_0(x, y) = 0.63 + 0.1\sin(16\pi x)\cos(12\pi y)$$

  which leads to the initial condition in Figure 6.3, with a total initial mass of $m = \int_\Omega \varphi_0 dx = 1.26000$.

- Nonsymmetric initialization with nonzero mean perturbation of a uniform field $\bar{\varphi} \equiv 0.63$: we used the FreeFem++ commands

  $$\varphi_0(x, y) = 0.63 + 0.05(-0.5 + \texttt{randreal1()})$$

  We recall that, since we have set the initial seed for the command `randreal1`, this initial condition is actually deterministic and can be reproduced identically at any

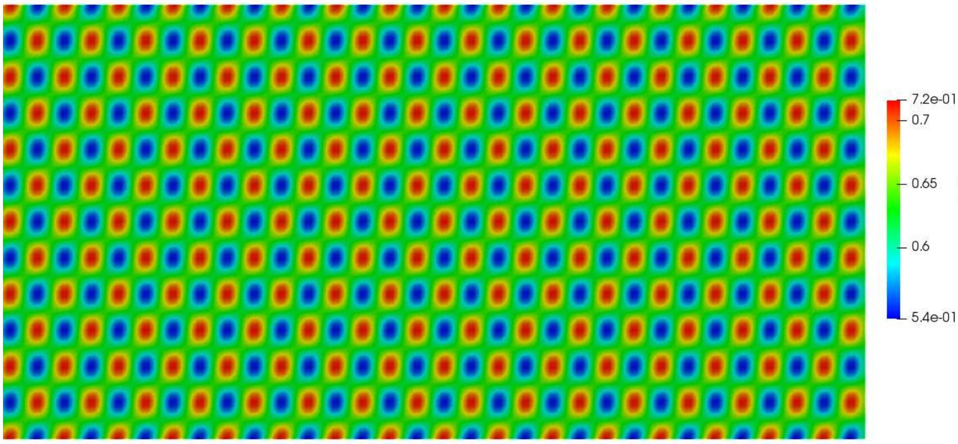


Figure 6.3: Symmetric initialization with zero mean perturbation of $\bar{\varphi}$

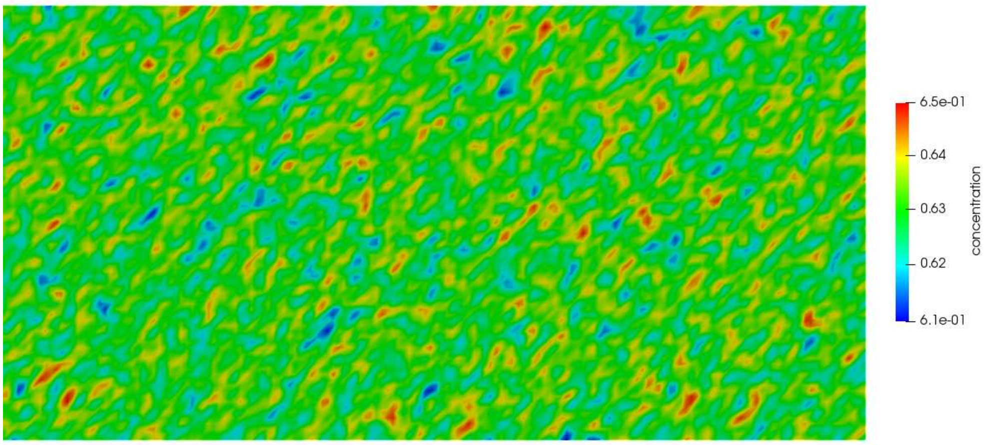


Figure 6.4: Nonsymmetric initialization with nonzero mean perturbation of $\bar{\varphi}$

time. This condition is represented in Figure 6.4 and the total initial mass is $m = \int_\Omega \varphi_0 dx = 1.26036$.

The initial condition on μ_0 is computed from the initial condition on φ_0, by means of the definition:

$$\mu_0 = -\alpha \Delta \varphi_0 + \Psi'(\varphi_0)$$

We can now pass to analyze the numerical tests.

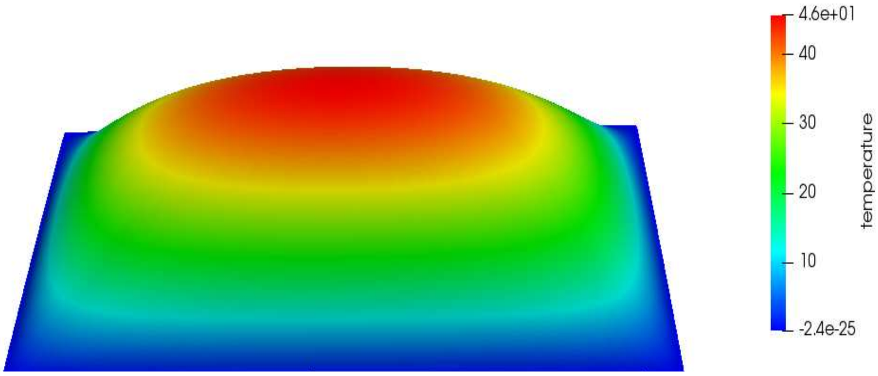


Figure 6.5: Test 1. Initial temperature field in 3D representation.

6.3 Numerical test 1

First, we fixed the remaining parameters: $\nu = 0.01$ and $\kappa = 5$: in this way the sufficient condition is satisfied, since $\kappa\nu = 0.05 > 0.0015$. As initial condition for φ we chose the one in Figure 6.3, the symmetric one with zero mean perturbation of $\bar{\varphi}$, whereas for the temperature we initialized the field solving the equation:

$$\kappa(\nabla\Theta_0, \nabla\xi) = \int_\Omega 2000\xi dx \quad \forall\xi \in H_0^1(\Omega)$$

With homogeneous Dirichlet boundary conditions for Θ_0. This initial condition simulates a sudden injection of a source of heat in the system, which acts as a forcing term in the equation. The resulting symmetric initial condition is shown in Figure 6.5.

We then initialize the last field, i.e. the velocity field, by solving a Stokes equation for $(\mathbf{u}_0, p_0) \in [H_0^1(\Omega)]^2 \times L^2(\Omega)$ keeping into account φ_0, μ_0 and Θ_0:

$$\begin{cases} \nu(\nabla\mathbf{u}_0, \nabla\mathbf{w}) - (p_0, \mathrm{div}(\mathbf{w})) = (-\varphi_0\nabla(\mu_0), \mathbf{w}) + (\Theta_0, \mathbf{w}_y) \quad \forall\mathbf{w} \in [H_0^1(\Omega)]^2 \\ (\mathrm{div}(\mathbf{u}_0), q) = 0 \quad \forall q \in L^2(\Omega) \end{cases}$$

always with no slip boundary conditions, obtaining the velocity field in Figure 6.6.

For this test we reached $T \approx 0.024$. The resulting concentration field in time is represented in Figures 6.7-6.10, where we can see the spinodal decomposition at the very beginning and then the phase separation leads to the formation of larger and larger *bubbles*.

The advective effect of the velocity field is clearly visible: we can see that the motions in the concentration field are regulated by the advection of the velocity field. To see this

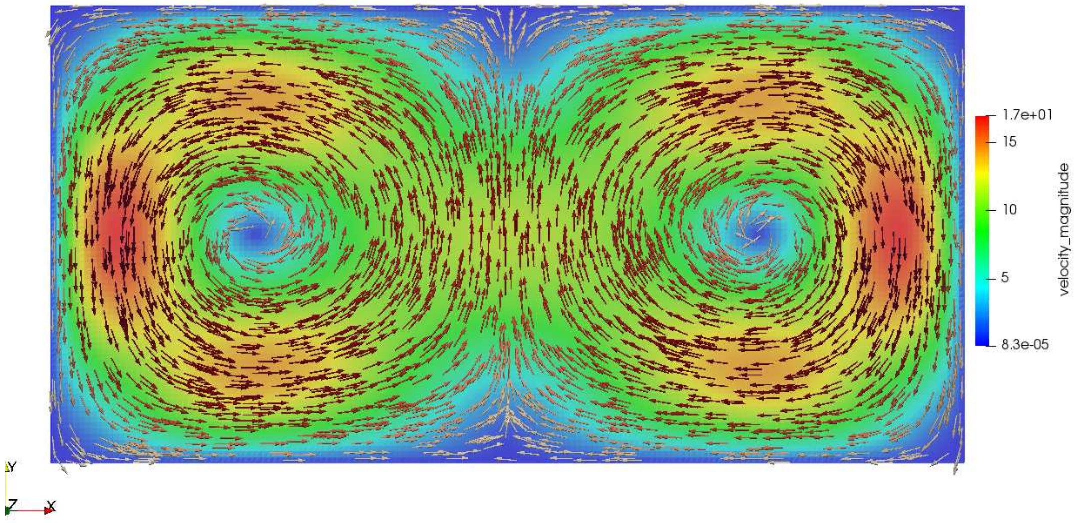


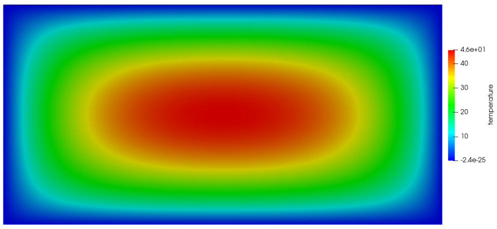
Figure 6.6: Test 1. Initial velocity field.

effect, we can observe Figure 6.11, in which two different time steps are superimposed (t=0.018 and t=0.0223, the former in the background, the latter in the foreground) together with streamlines for the velocity field: we can see that the droplets follow the streamlines.

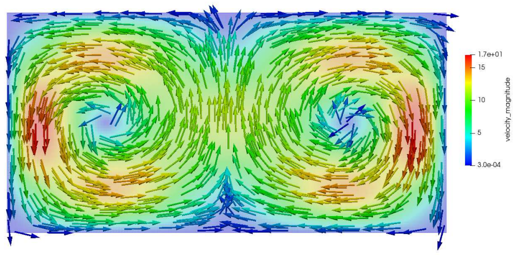
As it is physically reasonable, the concentration is driven to the two local minima of the double well potential previously shown.

For what concerns the velocity field, in Figure 6.6 we can see the evolution of streamlines in time: it is much slower than the evolution of the concentration, having longer characteristic time, nevertheless we can see a difference in the shape of the vortices. Indeed we clearly see the formation of smaller vortices at the corners and in the middle of the domain.

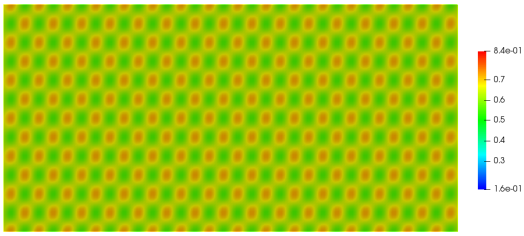
In conclusion, in Figures 6.7-6.10, we see the temperature field in time: the effect of the temperature is to generate the convective cells (i.e, the two vortices in the velocity field) and as a feedback the temperature tends to become stratified in time, due to the motion of the hot fluid from bottom to top. Moreover, we can see that the range of values decreases in time, because most of the energy has been dissipated in time.



(a) Temperature field

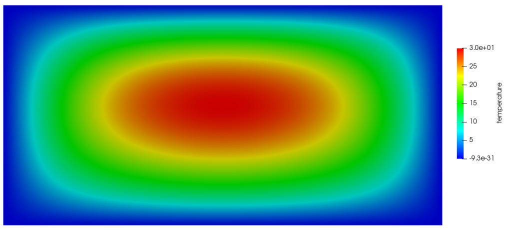


(b) Velocity field

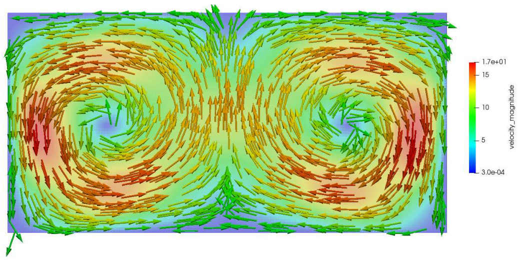


(c) Concentration field

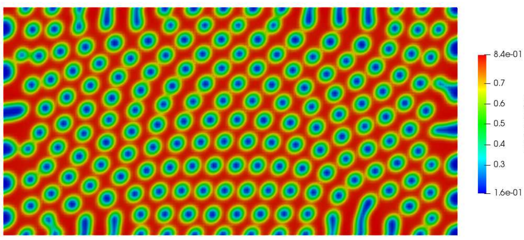
Figure 6.7: Test 1. Time $t = 6 \times 10^{-12}$.



(a) Temperature field

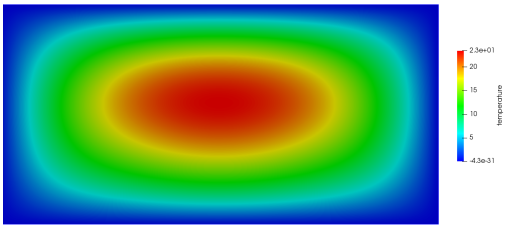


(b) Velocity field

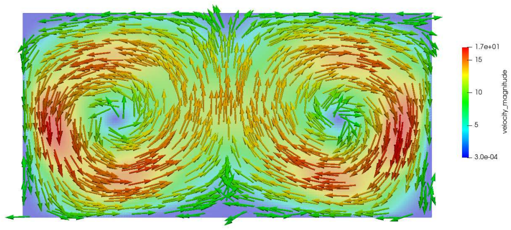


(c) Concentration field

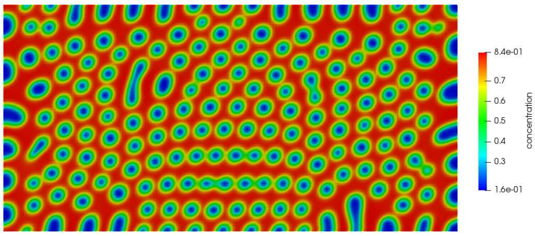
Figure 6.8: Test 1. Time $t = 0.0081$.



(a) Temperature field

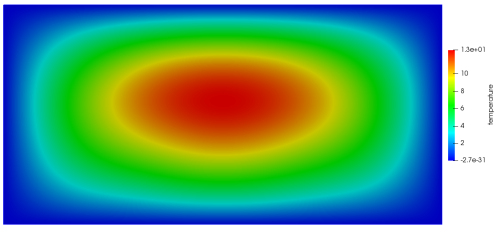


(b) Velocity field

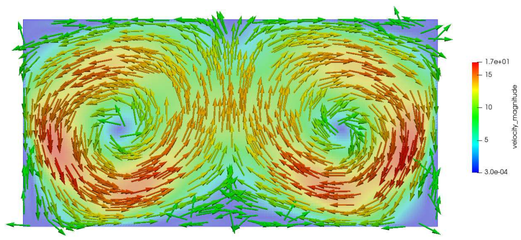


(c) Concentration field

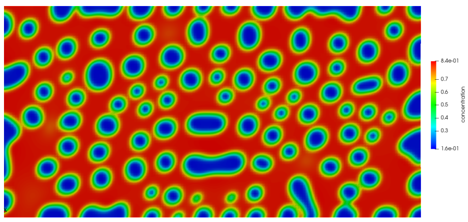
Figure 6.9: Test 1. Time $t = 0.012$.



(a) Temperature field



(b) Velocity field



(c) Concentration field

Figure 6.10: Test 1. Time $t = 0.0223$.

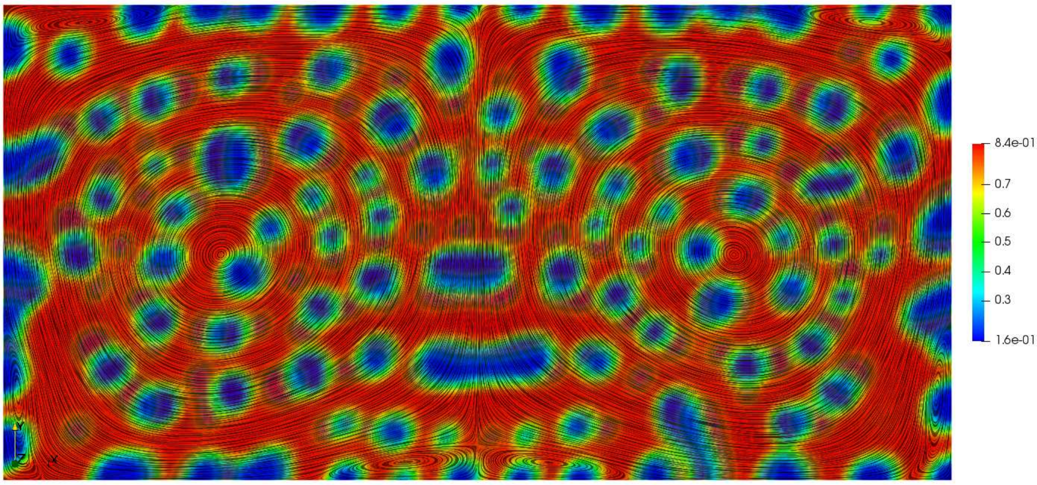
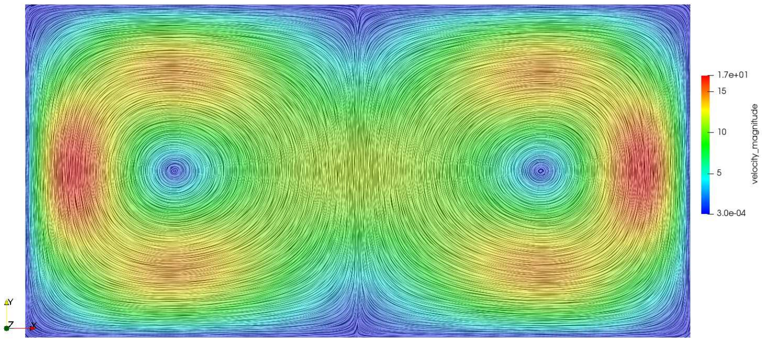


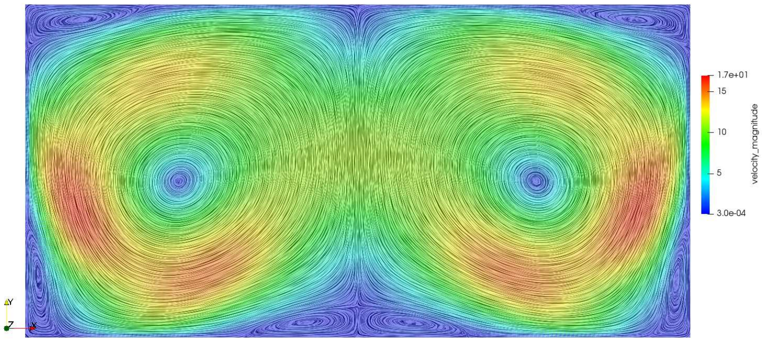
Figure 6.11: Test 1. Superimposition of time steps (in the background the old time step $t = 0.018$, in the foreground time $t = 0.223$): we see that the droplets follow the streamlines.

We can now analyze the properties highlighted in Theorem 5.1.4: first of all, in Figure 6.13, we see the plot of the total mass in time: it is always constant (up to the fifth significant digit) $m = 1.26000$.

If we now compute the total energy for each time step $E_n = \frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega \Psi(\varphi_n)dx$, we obtain, as expected from Theorem 5.1.4, that the energy is nondecreasing, and in particular it is decreasing, as we can see in Figure 6.14. Indeed, the sufficient condition on ν and κ is respected. Moreover, in Figure 6.15, we computed the derivative of the total energy by means of backward finite differences, obtaining that the derivative is always negative, confirming the decrease of the total energy. In conclusion, the time step ranges from 6×10^{-12} at the beginning to 3.0224×10^{-5}, as we can see in Figure 6.16: actually, we can consider it constant, since the oscillations are around approximately the same mean value.



(a) Time t=6×10^{-12}



(b) Time t=0.023

Figure 6.12: Test 1. Streamlines in time, a comparison: we clearly see the formation of smaller vortices at the corners and in the middle of the domain.

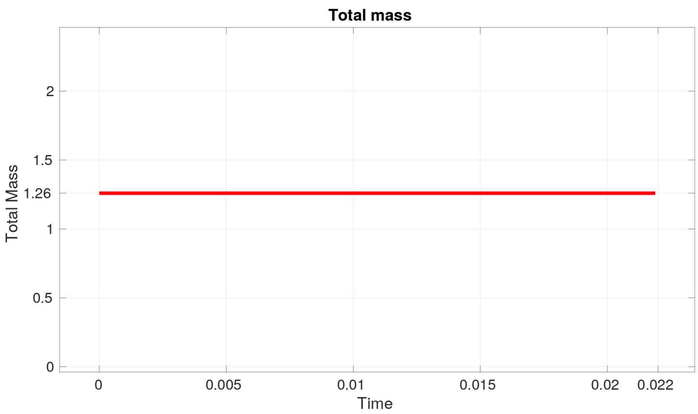


Figure 6.13: Test 1. Total mass of the system: we see that it is constant in time, as expected.

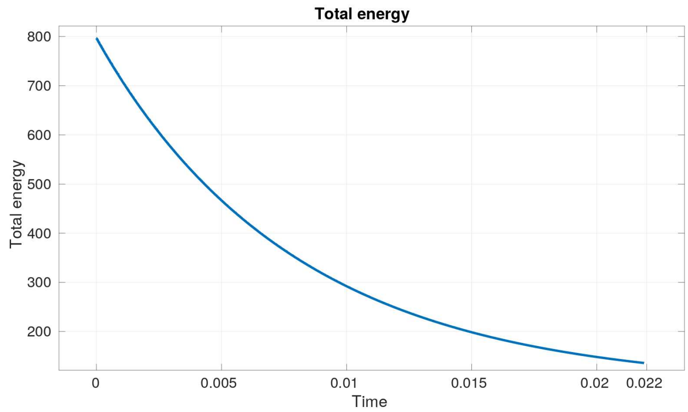


Figure 6.14: Test 1. Total energy E_n: it is always decreasing as expected.

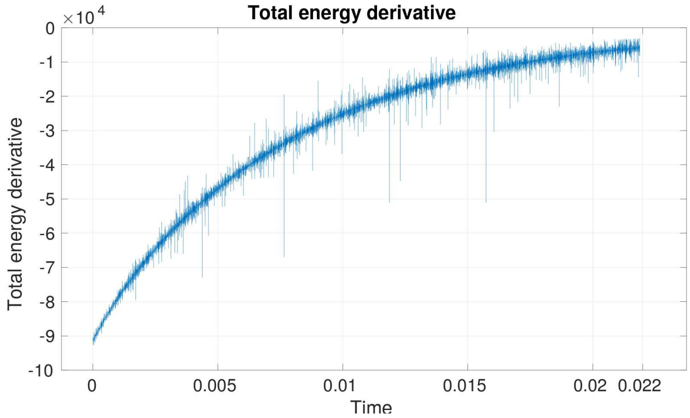


Figure 6.15: Test 1. Derivative of the total energy E_n: it is always negative as expected.

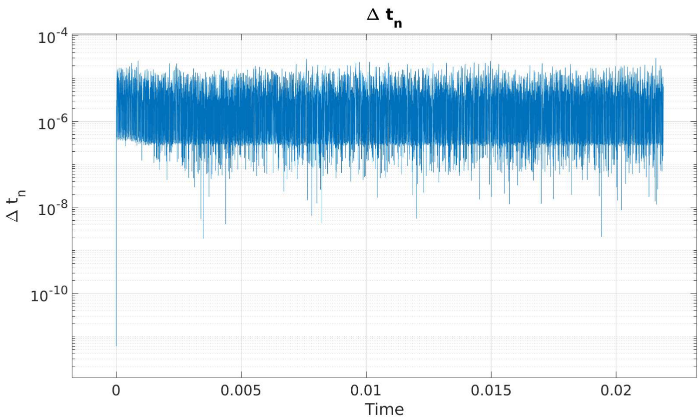


Figure 6.16: Test 1. Semi-log plot of the adaptive time step in time.

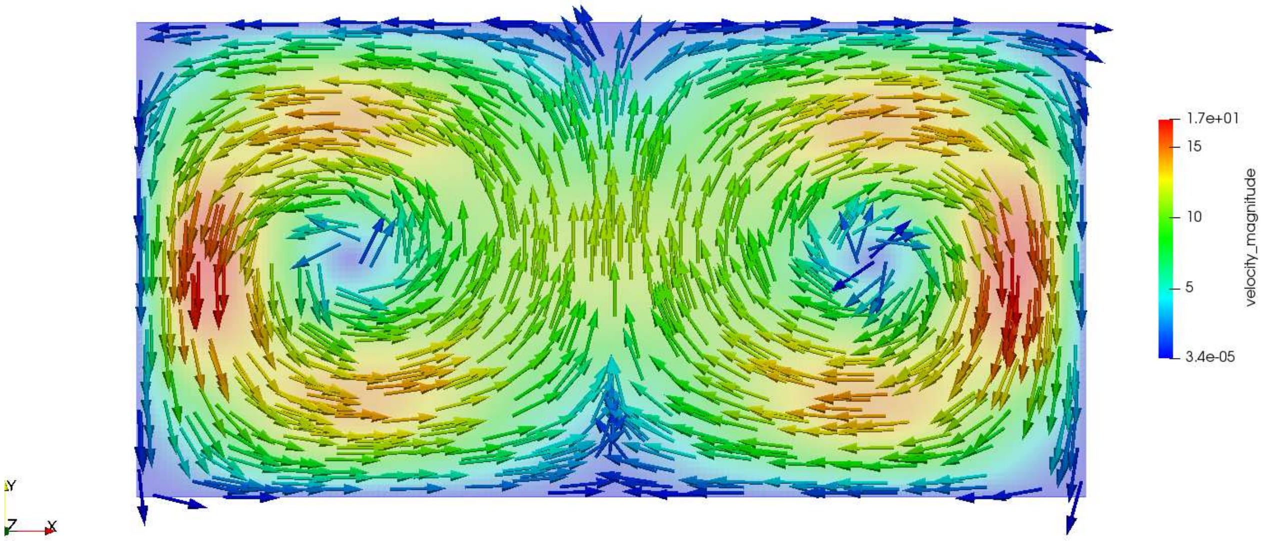


Figure 6.17: Test 2. Initial velocity field.

6.4 Numerical test 2

We fixed the parameters: $\nu = 0.01$ and $\kappa = 5$: in this way the sufficient condition is satisfied, since $\kappa\nu = 0.05 > 0.0015$. As initial condition for φ we chose the one in Figure 6.4, the nonsymmetric one with nonzero mean perturbation of $\bar{\varphi}$, whereas for the temperature we initialized the field solving the equation:

$$\kappa(\nabla\Theta_0, \nabla\xi) = \int_\Omega 2000\xi dx \quad \forall\xi \in H_0^1(\Omega),$$

exactly as in test 1, with homogeneous Dirichlet boundary conditions for Θ_0. The resulting symmetric initial condition is already shown in Figure 6.5.

Also the velocity is initialized exactly as in the previous test, obtaining the velocity field in Figure 6.17.

For this test we reached $T \approx 0.03$. The resulting concentration field in time is represented in Figures 6.18-6.21, where we can see the spinodal decomposition at the very beginning and then the phase separation leads to the formation of larger and larger *bubbles*. The advective effect of the velocity field is clearly visible also in this nonsymmetric case: we can see that the motions in the concentration field are regulated by the advection of the velocity field. To see this effect, we can observe, as done in the previous simulation, Figure 6.22, in which two different time steps are superimposed (t=0.021 and t=0.03, the former in the background, the latter in the foreground) together with the streamlines: we can see that the droplets follow the streamlines.

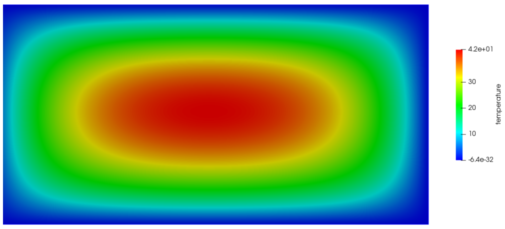
As it is physically reasonable, the concentration is driven to the two local minima of the double well potential previously shown.

For what concerns the velocity field, in Figure 6.23 we clearly see the formation of smaller vortices at the corners and in the middle of the domain.

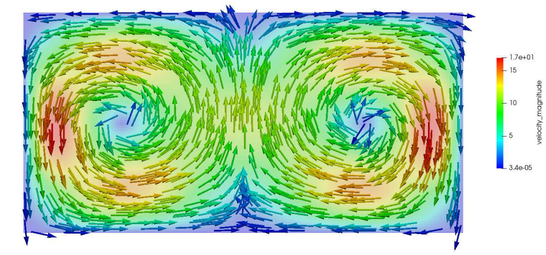
In conclusion, in Figures 6.18-6.21, we see the temperature field vs. time: we can see as in the previous test that the range of values decreases in time, because most of the energy has been dissipated.

We can now analyze the properties highlighted in Theorem 5.1.4: first of all, in Figure 6.24, we see the plot of the total mass in time: it is always constant (up to the fifth significant digit) $m = 1.26036$.

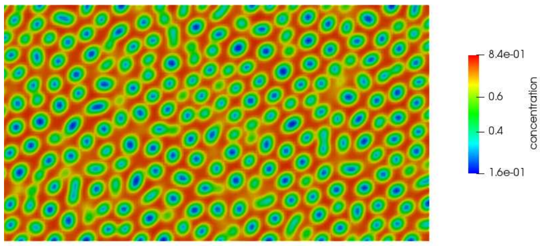
If we now compute the total energy for each time step $E_n = \frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega \Psi(\varphi_n)dx$, we obtain, as expected from Theorem 5.1.4 that the energy is nondecreasing, and in particular it is decreasing, as we can see in Figure 6.25. Indeed, the sufficient condition on ν and κ is respected. Moreover, in Figure 6.26, we computed the derivative of the total energy by means of backward finite differences, obtaining that the derivative is always negative, confirming the decrease of the total energy. In conclusion, the time step ranges from 6×10^{-12} at the beginning to 6.77×10^{-5}, as we can see in Figure 6.27: actually, we can consider it constant, since the oscillations are around approximately the same mean value.



(a) Temperature field

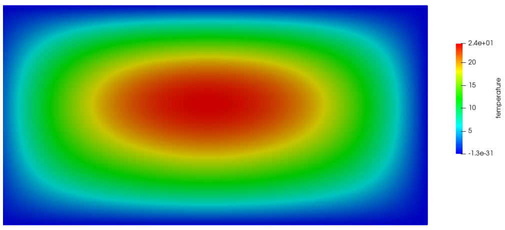


(b) Velocity field

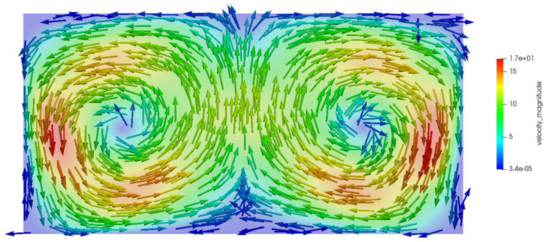


(c) Concentration field

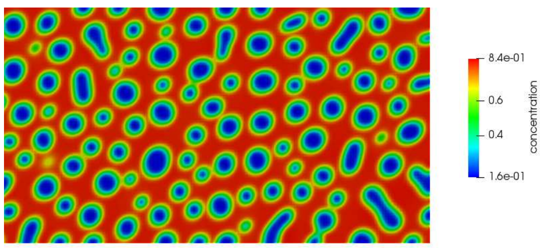
Figure 6.18: Test 2. Time $t = 0.0015$.



(a) Temperature field

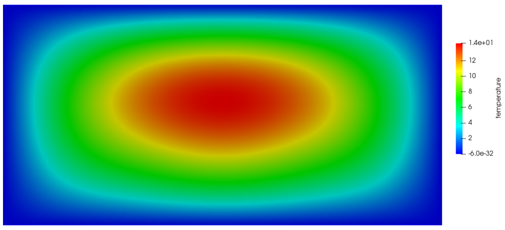


(b) Velocity field

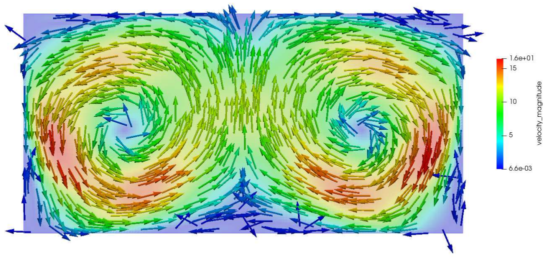


(c) Concentration field

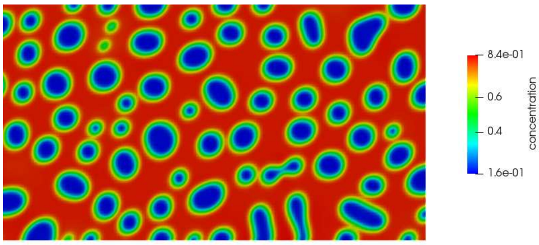
Figure 6.19: Test 2. Time $t = 0.012$.



(a) Temperature field

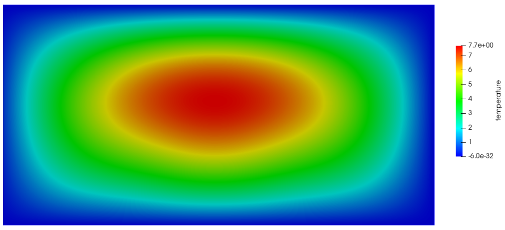


(b) Velocity field

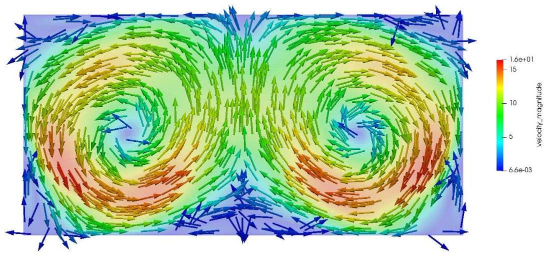


(c) Concentration field

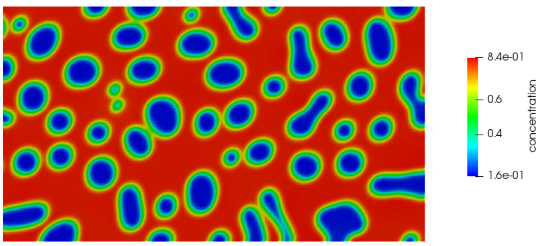
Figure 6.20: Test 2. Time $t = 0.021$.



(a) Temperature field



(b) Velocity field



(c) Concentration field

Figure 6.21: Test 2. Time $t = 0.03$.

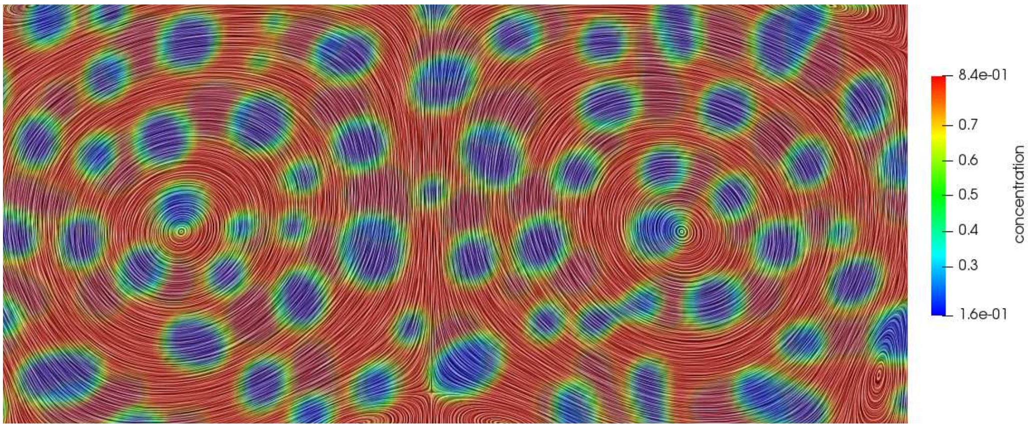
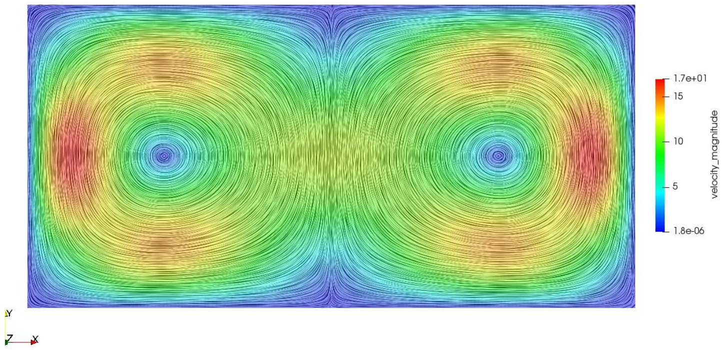
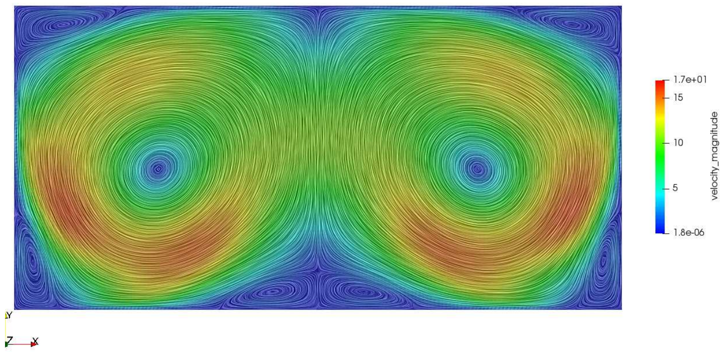


Figure 6.22: Test 2. Superimposition of time steps (in the background the old time step at time $t = 0.021$, in the foreground time $t = 0.03$): we see that the droplets follow the streamlines.



(a) Time t=6×10^{-12}



(b) Time t=0.03

Figure 6.23: Test 2. Streamlines vs. time, a comparison: we clearly see the formation of smaller vortices at the corners and in the middle of the domain.

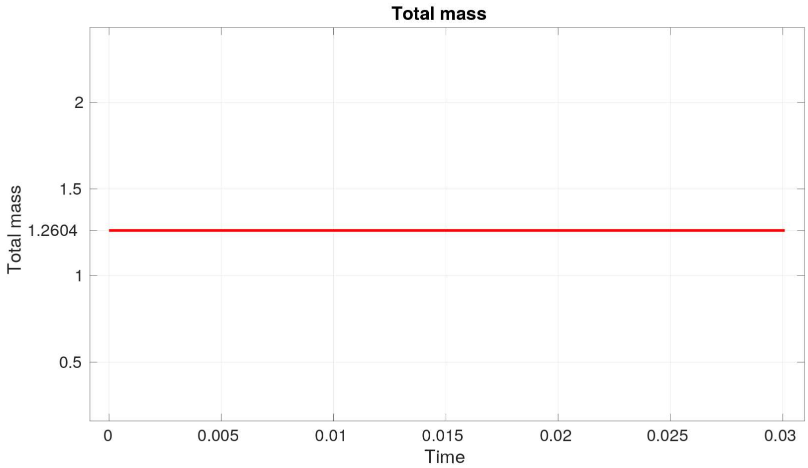


Figure 6.24: Test 2. Total mass of the system: we see that it is constant in time, as expected.

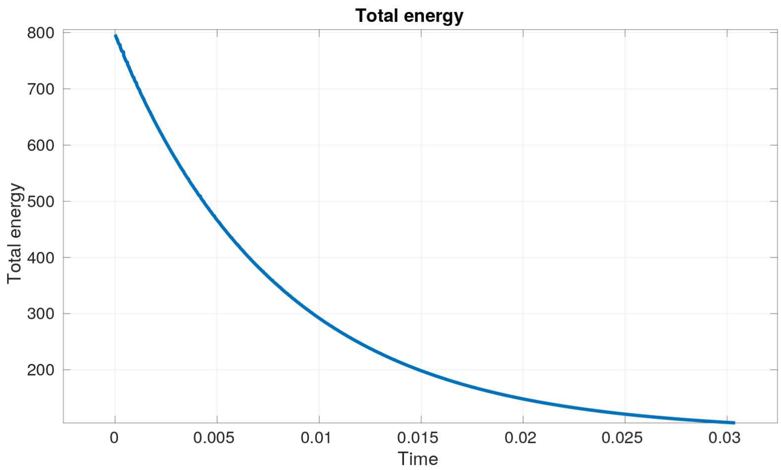


Figure 6.25: Test 2. Total energy E_n: it is always decreasing as expected.

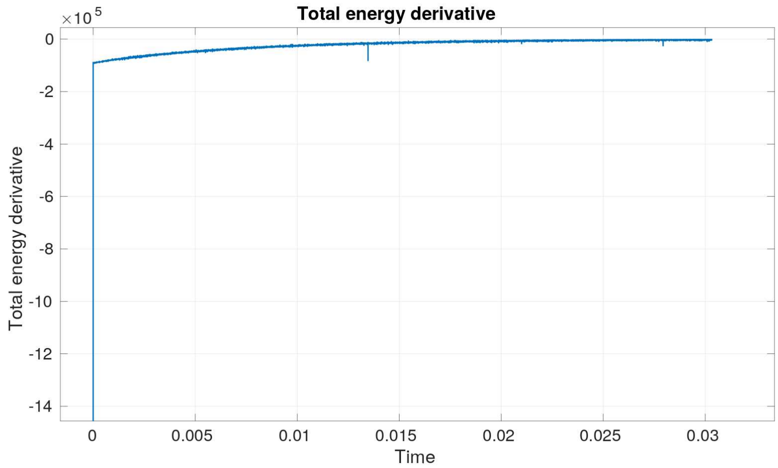


Figure 6.26: Test 2. Derivative of the total energy E_n: it is always negative as expected.

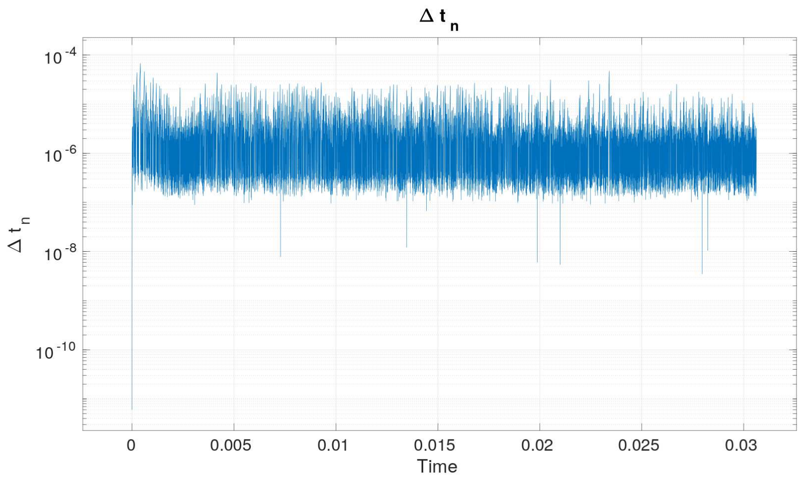


Figure 6.27: Test 2. Semi-log plot of the adaptive time step in time.

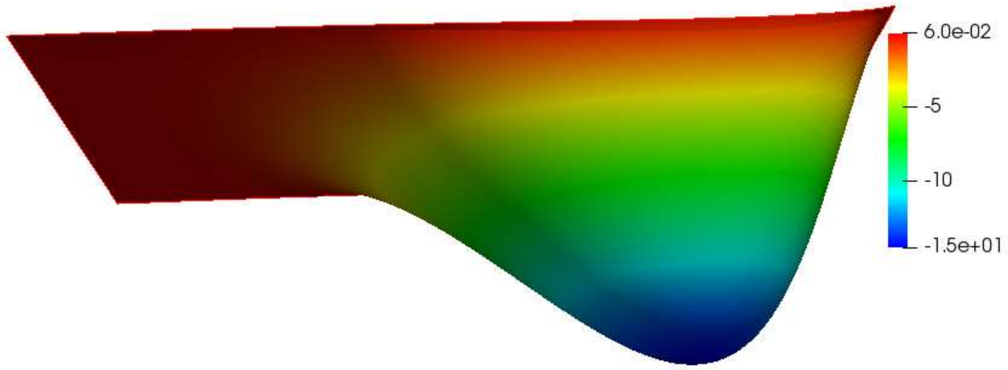


Figure 6.28: Test 3. Initial temperature field in 3D representation.

6.5 Numerical test 3

We fixed the parameters: $\nu = 0.05$ and $\kappa = 10$: in this way the sufficient condition is satisfied, since $\kappa\nu = 0.5 > 0.0015$. As initial condition for φ we chose the one in Figure 6.4, the nonsymmetric initialization with nonzero mean perturbation of $\bar{\varphi}$, whereas for the temperature we initialized the field solving the equation:

$$\kappa(\nabla\Theta_0, \nabla\xi) = \int_\Omega 1000\sin(x)(0.5 - x^2)\xi dx \quad \forall\xi \in H_0^1(\Omega)$$

With homogeneous Dirichlet boundary conditions for Θ_0. This initial condition simulates a sudden injection of a source of heat in the system, which acts as a forcing term in the equation. The resulting nonsymmetric initial condition is shown in Figure 6.28.

We then initialize the last field, i.e. the velocity field, by solving a Stokes equation for $(\mathbf{u}_0, p_0) \in [H_0^1(\Omega)]^2 \times L^2(\Omega)$ keeping into account φ_0, μ_0 and Θ_0:

$$\begin{cases} \nu(\nabla\mathbf{u}_0, \nabla\mathbf{w}) - (p_0, \mathrm{div}(\mathbf{w})) = (-\varphi_0\nabla(\mu_0), \mathbf{w}) + (\Theta_0, \mathbf{w}_y) \quad \forall\mathbf{w} \in [H_0^1(\Omega)]^2 \\ (\mathrm{div}(\mathbf{u}_0), q) = 0 \quad \forall q \in L^2(\Omega) \end{cases}$$

always with no slip boundary conditions, obtaining the velocity field in Figure 6.29.

For this test we reached $T \approx 0.05$, a larger value than before, since the effect of velocity field is less relevant and thus the time step can be a bit larger (in this test it ranges from 6×10^{-12} to 4.9×10^{-5} as we can see in Figure 6.38). The resulting concentration field in

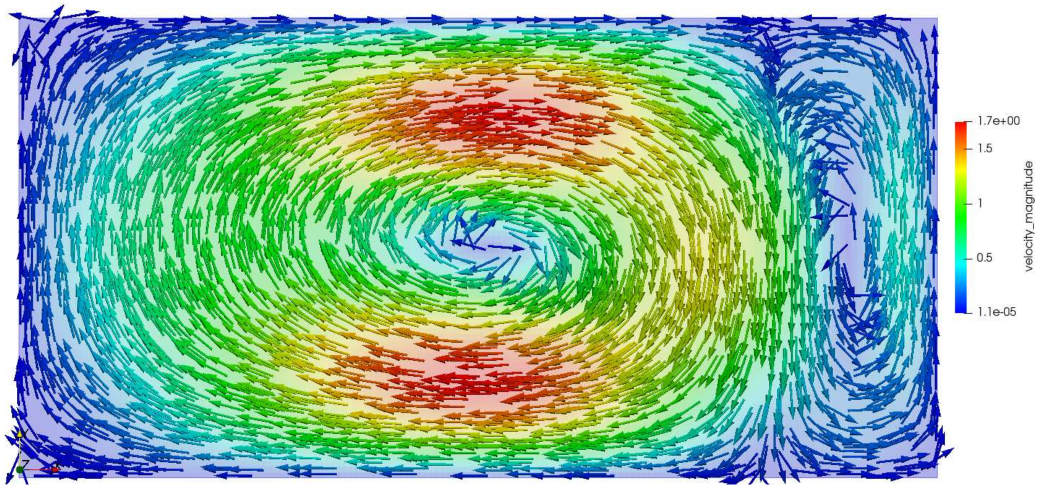


Figure 6.29: Test 3. Initial velocity field.

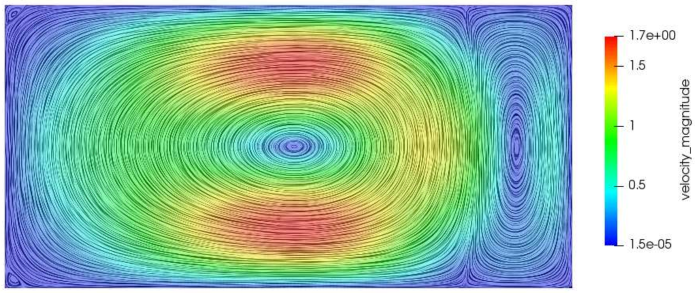
time is represented in Figures 6.31-6.34, where we can see the spinodal decomposition at the very beginning. The advective effect of the velocity field is not clearly visible: indeed the velocity field is not so strong and thus the advection is limited. As it is physically reasonable, the concentration is driven to the two local minima of the double well potential previously shown.

For what concerns the velocity field, in Figure 6.30, we can see the evolution in time: it is much slower than the evolution of the concentration, we can only notice the formation of smaller secondary vortices in the corners of the domain.

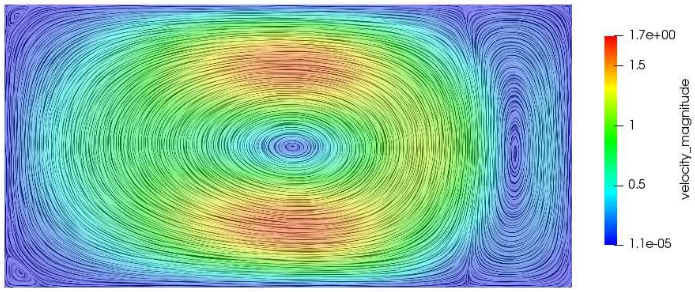
In conclusion, in Figures 6.31-6.34 we see the temperature field vs. time: the high value of the thermal diffusivity κ quickly dissipates most of the energy and it also makes the field symmetric, spreading it into the whole domain.

We can now analyze the properties highlighted in Theorem 5.1.4, as done in the previous tests: in Figure 6.35 we see the plot of the total mass in time: it is always constant (up to the fifth significant digit) $m = 1.26036$.

If we now compute the total energy for each time step $E_n = \frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega \Psi(\varphi_n)dx$, we obtain, as expected from Theorem 5.1.4 that the energy is nondecreasing, and in particular it is decreasing, as we can see in Figure 6.36. Indeed, the sufficient condition on ν and κ is respected. In Figure 6.37 we computed the derivative of the



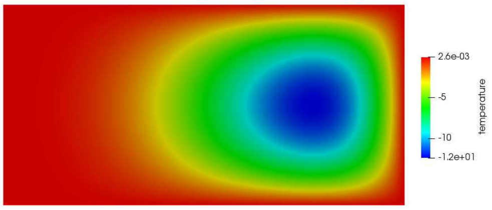
(a) Time t=6×10^{-12}



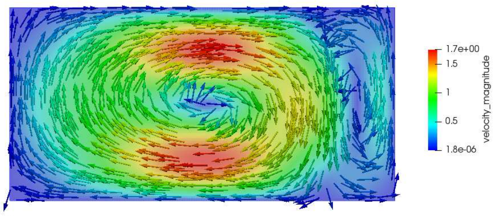
(b) Time $t = 0.053$

Figure 6.30: Test 3. Velocity field vs. time with streamlines.

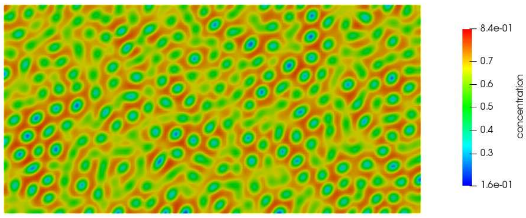
total energy by means of backward finite differences, obtaining that the derivative is always negative, confirming the decrease of the total energy. We can notice that the decrease in the energy is more stressed in this test, since both the coefficients κ and ν, responsible for the dissipation, are larger.



(a) Temperature field

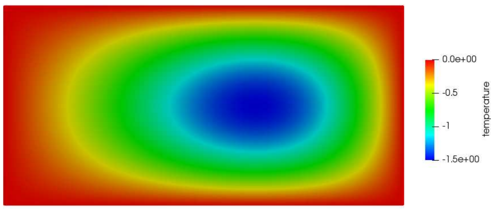


(b) Velocity field

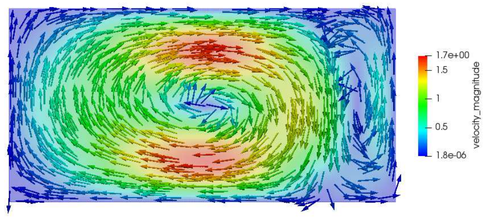


(c) Concentration field

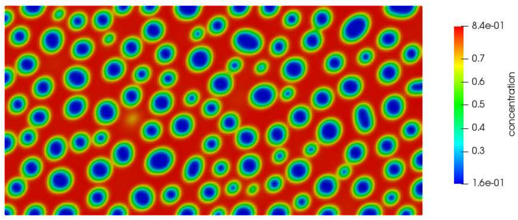
Figure 6.31: Test 3. Time $t = 0.0011$.



(a) Temperature field

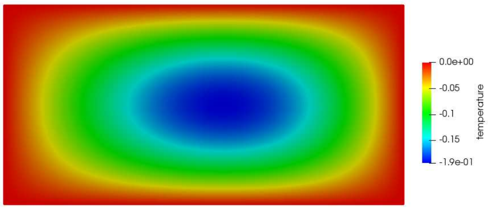


(b) Velocity field

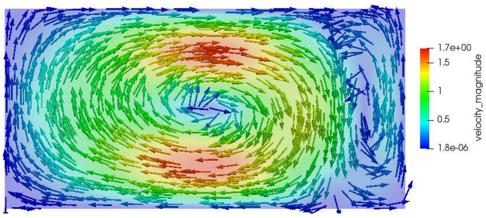


(c) Concentration field

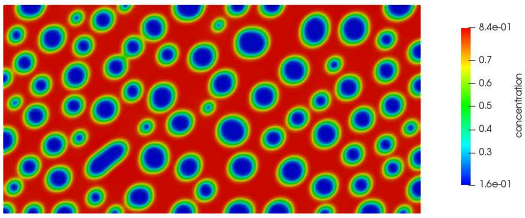
Figure 6.32: Test 3. Time $t = 0.015$.



(a) Temperature field

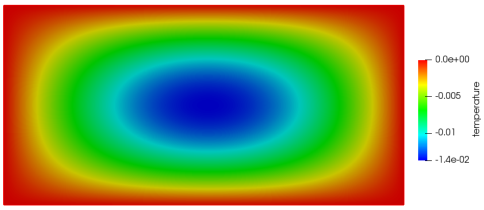


(b) Velocity field

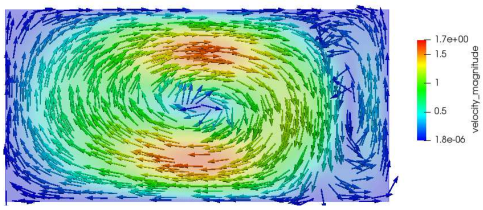


(c) Concentration field

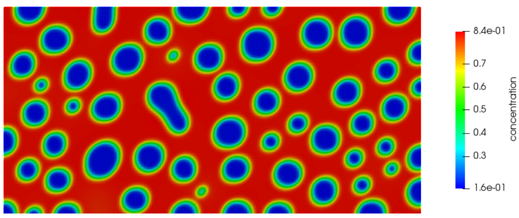
Figure 6.33: Test 3. Time $t = 0.031$.



(a) Temperature field



(b) Velocity field



(c) Concentration field

Figure 6.34: Test 3. Time $t = 0.053$.

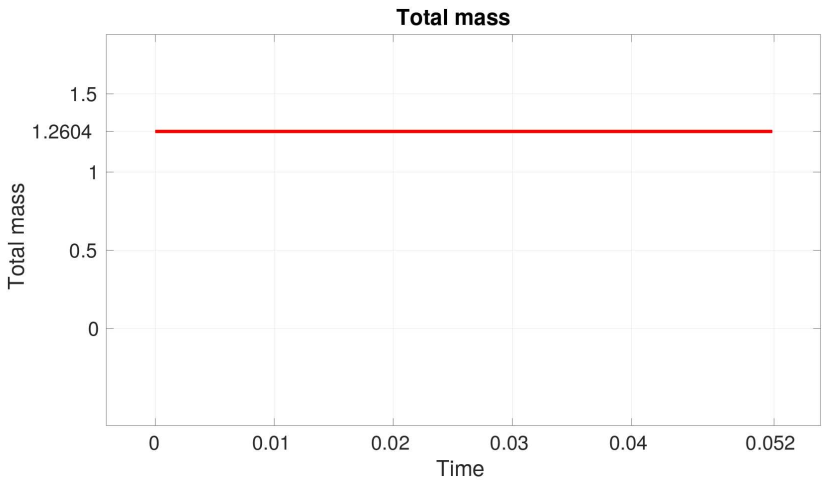


Figure 6.35: Test 3. Total mass of the system: we see that it is constant in time, as expected.

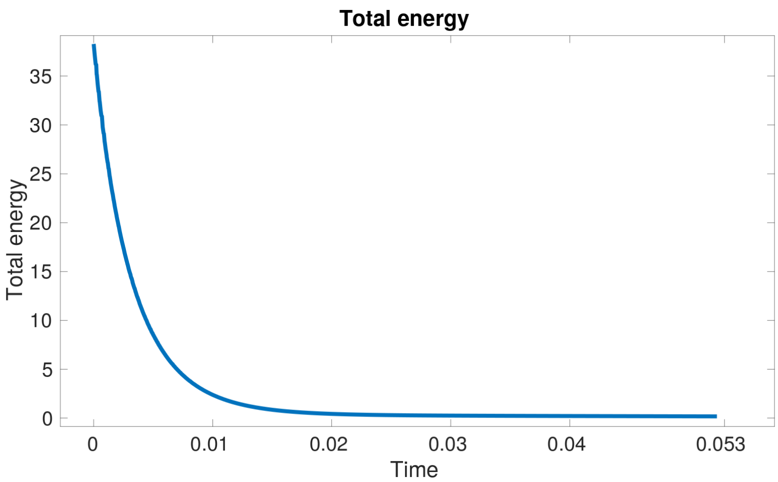


Figure 6.36: Test 3. Total energy E_n: it is always decreasing as expected.

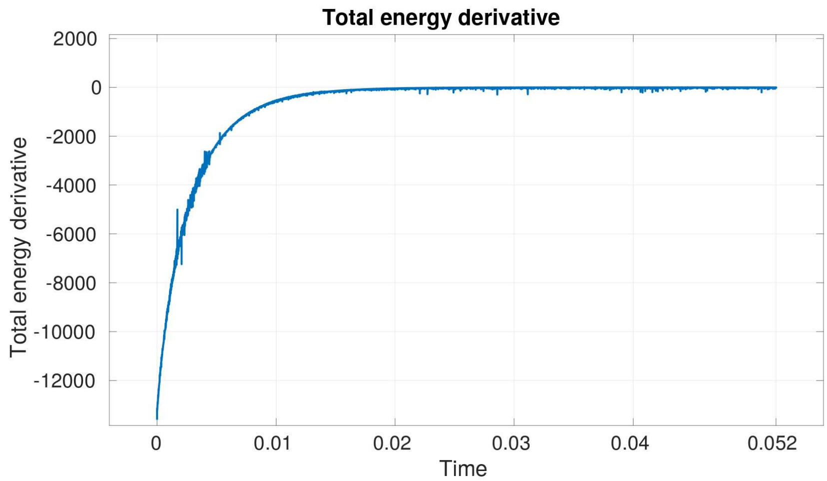


Figure 6.37: Test 3. Derivative of the total energy E_n: it is always negative as expected.

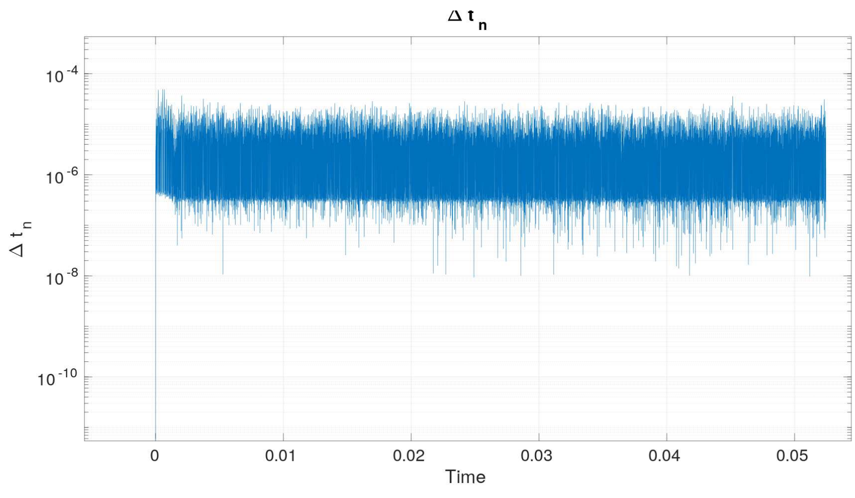


Figure 6.38: Test 3. Semi-log plot of the adaptive time step in time.

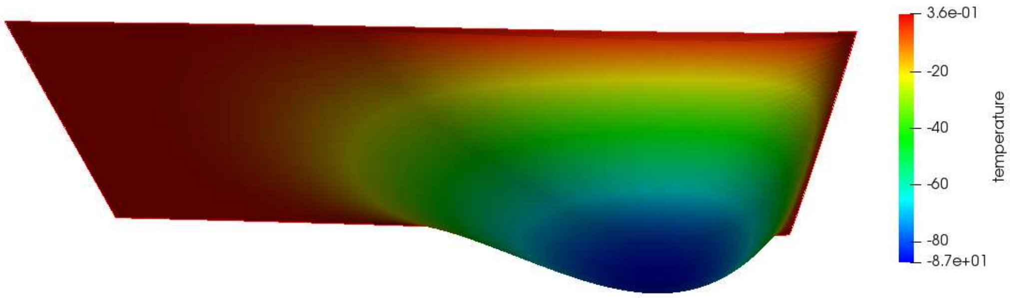


Figure 6.39: Test 4. Initial temperature field in 3D representation.

6.6 Numerical test 4

We fixed the parameters: $\nu = 0.02$ and $\kappa = 20$: in this way the sufficient condition is satisfied, since $\kappa\nu = 0.4 > 0.0015$. We recall that in this case we have a nonhomogenoeus boundary condition for the temperature, but the dissipation, as we will notice, is enough to prevent the energy from increasing also in this case. As initial condition for φ we chose the one in Figure 6.3, the symmetric initialization with zero mean perturbation of $\bar{\varphi}$, whereas for the temperature we initialized the field solving the equation:

$$\kappa(\nabla\theta_0, \nabla\xi) = \int_\Omega 12 \times 10^4 \sin(x)(0.5 - x^2)\xi dx \quad \forall \xi \in H_0^1(\Omega)$$

with non homogeneous Dirichlet boundary conditions g for θ and θ_0:

$$g = 5 \times 10^{-4} x^2 y^3 \text{ on } \partial\Omega \times (0, T) \tag{6.1}$$

The initial condition simulates a sudden injection of a source of heat in the system, which acts as a forcing term in the equation. Moreover, the presence of a boundary datum for the temperature means that there is a continuous injection of energy also during time. The resulting nonsymmetric initial condition is shown in Figure 6.39.

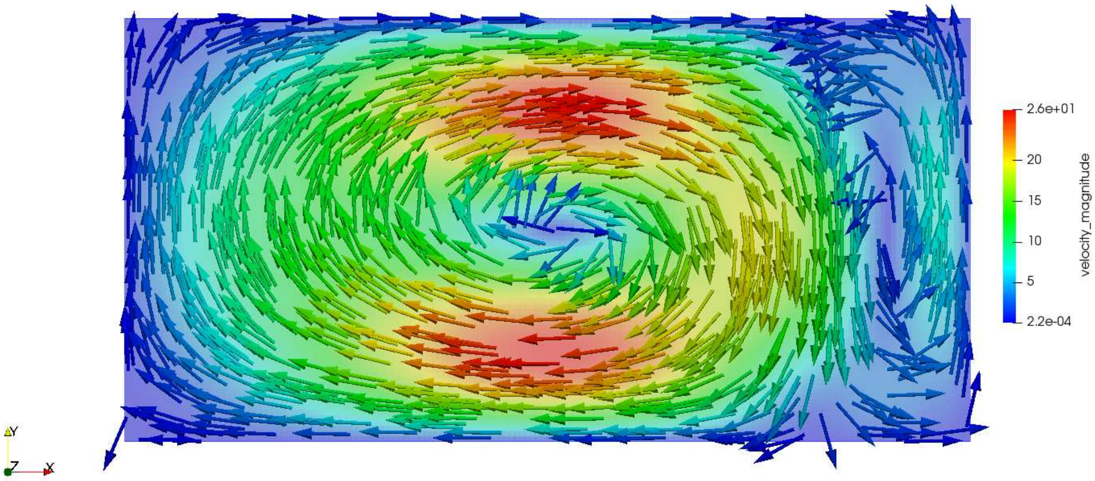


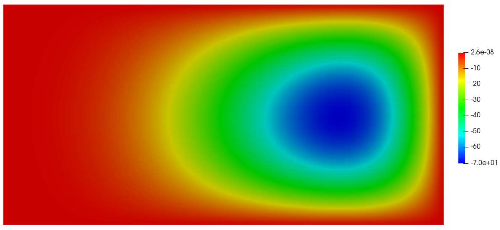
Figure 6.40: Test 4. Initial velocity field.

We then initialize the last field, i.e. the velocity field, by solving a Stokes equation for $(\mathbf{u}_0, p_0) \in [H_0^1(\Omega)]^2 \times L^2(\Omega)$ keeping into account φ_0, μ_0 and θ_0:

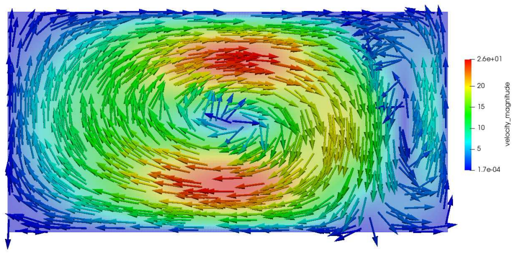
$$\begin{cases} \nu(\nabla \mathbf{u}_0, \nabla \mathbf{w}) - (p_0, \operatorname{div}(\mathbf{w})) = (-\varphi_0 \nabla(\mu_0), \mathbf{w}) + (\theta_0, \mathbf{w}_y) \quad \forall \mathbf{w} \in [H_0^1(\Omega)]^2 \\ (\operatorname{div}(\mathbf{u}_0), q) = 0 \quad \forall q \in L^2(\Omega) \end{cases}$$

always with no slip boundary conditions, obtaining the velocity field in Figure 6.40.

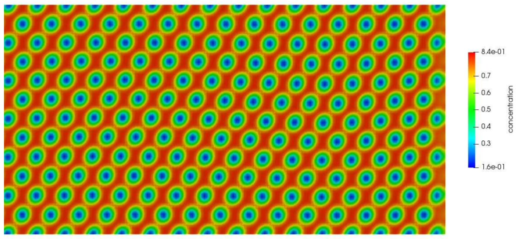
For this test we reached $T \approx 0.02$ and the time step can be a bit larger (in this test it ranges from 6×10^{-12} to 4.99×10^{-5}: actually, we can consider it constant, since the oscillations are around approximately the same mean value).). The resulting concentration field in time is represented in Figures 6.41-6.44, where we can see the spinodal decomposition at the very beginning. The advective effect of the velocity field is visible, since the bubbles move during time. The concentration is driven to the two local minima of the double well potential previously shown.



(a) Temperature field

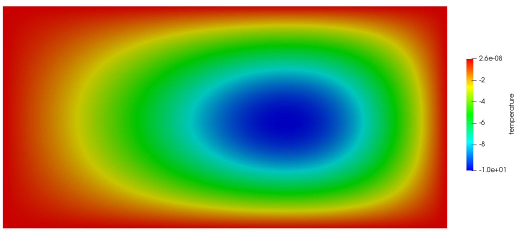


(b) Velocity field

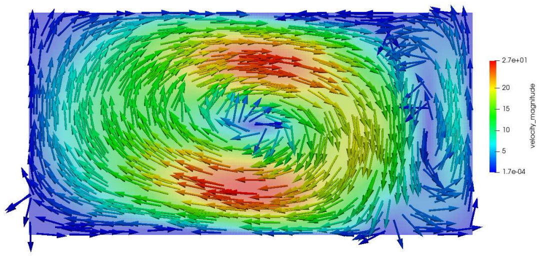


(c) Concentration field

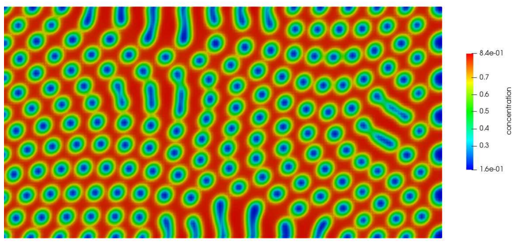
Figure 6.41: Test 4. Time $t = 0.0007$.



(a) Temperature field

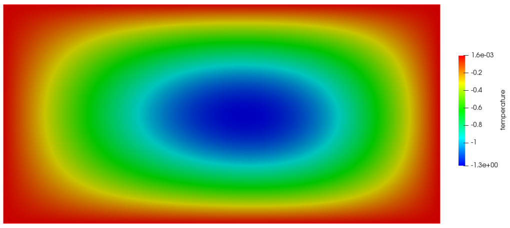


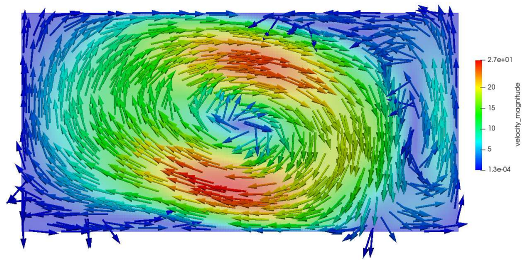
(b) Velocity field



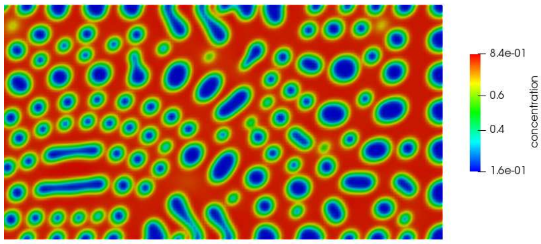
(c) Concentration field

Figure 6.42: Test 4. Time $t = 0.007$.



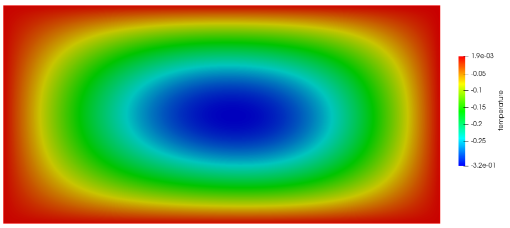


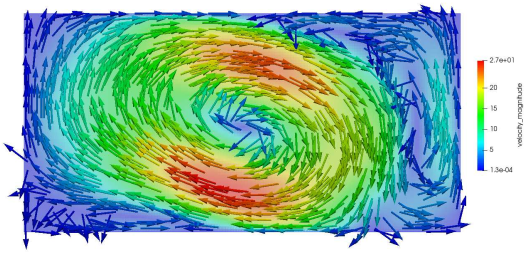
(a) Temperature field (b) Velocity field



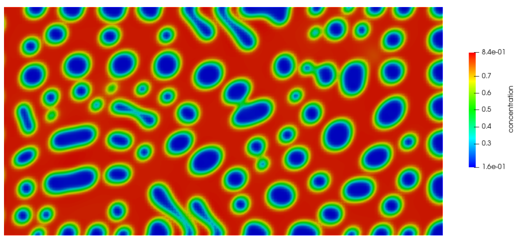
(c) Concentration field

Figure 6.43: Test 4. Time $t = 0.015$.



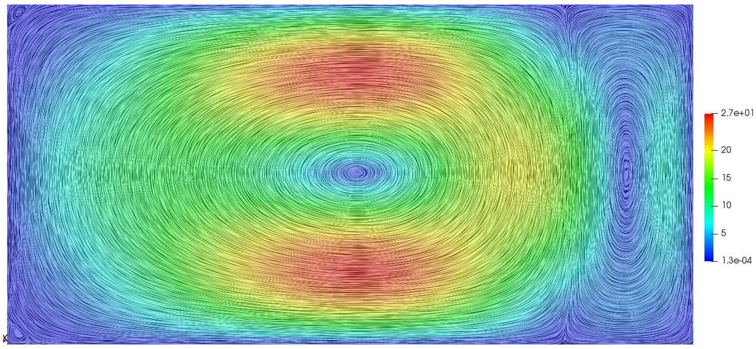


(a) Temperature field (b) Velocity field

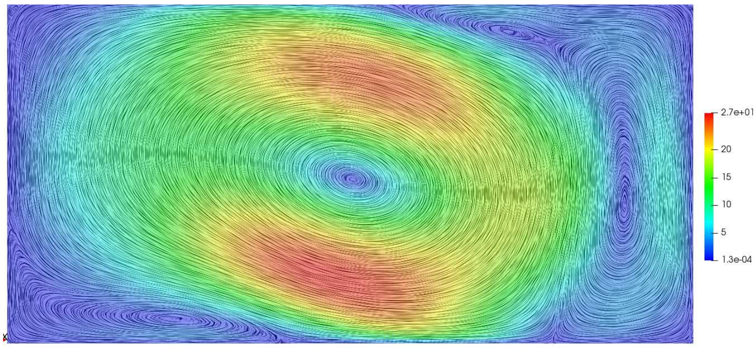


(c) Concentration field

Figure 6.44: Test 4. Time $t = 0.022$.



(a) Time t=6×10^{-12}



(b) Time $t = 0.022$

Figure 6.45: Test 4. Velocity field vs time with streamlines: we see the formation of a secondary vortex under the principal one.

For what concerns the velocity field, in Figure 6.45, we can see the evolution in time: the principal vortex changes its shape and other secondary vortices appear, differently from test 3, since the velocity magnitude is larger.

In Figures 6.31-6.34 we see the temperature field vs. time: the very high value of the thermal diffusivity κ quickly dissipates most of the energy and it also makes the field symmetric.

Moreover, in Figure 6.46 we see the plot of the total mass in time: it is always constant (up to the fifth significant digit) $m = 1.26000$.

If we now compute the total energy for each time step $E_n = \frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega \Psi(\varphi_n)dx$, we obtain that the energy is nondecreasing, and, in particular, it is decreasing, as we can see in Figure 6.47. In Figure 6.48 we computed the derivative of the

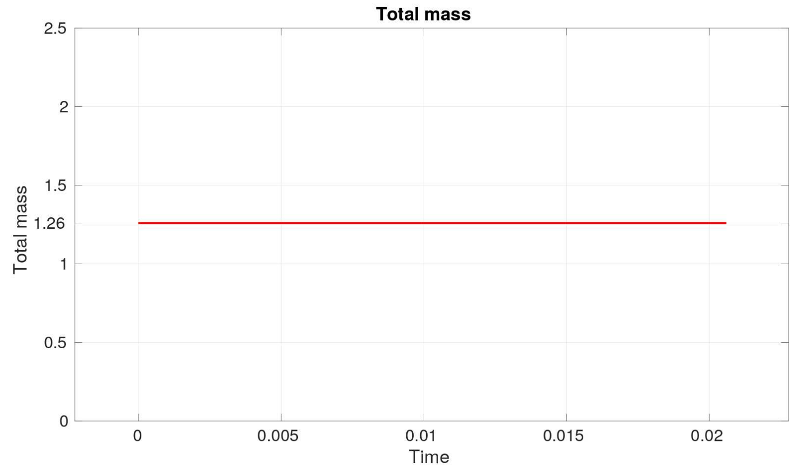


Figure 6.46: Test 4. Total mass of the system: we see that it is constant in time, as expected.

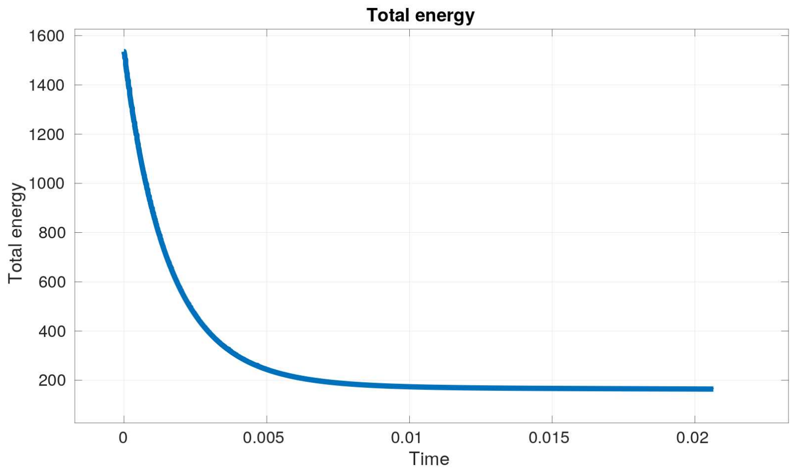


Figure 6.47: Test 4. Total energy E_n: it is always decreasing as expected.

total energy by means of backward finite differences, obtaining that the derivative is always negative, confirming the decrease of the total energy. The decrease of the energy is still observable, even though we have imposed nonhomogeneous Dirichlet boundary conditions for the temperature. This means that the dissipation is still enough to compensate the extra energy injected in the system from the boundary of the domain.

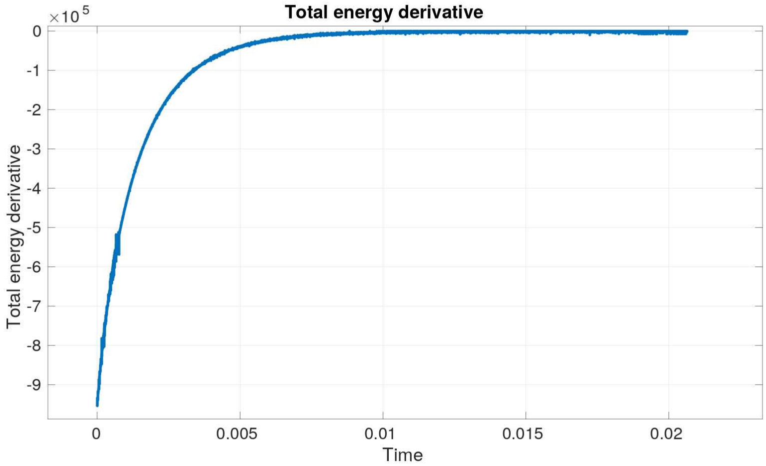


Figure 6.48: Test 4. Derivative of the total energy E_n: it is always negative as expected, and it tends to zero as time increases.

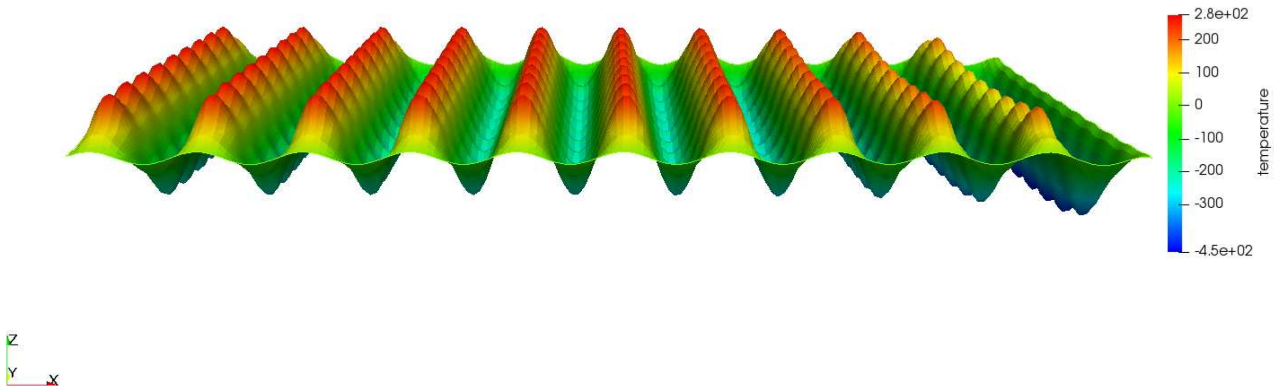


Figure 6.49: Test 5. Initial temperature field in 3D representation.

6.7 Numerical test 5

We fixed the parameters: $\nu = 0.01$ and $\kappa = 1$: in this way the sufficient condition is satisfied, since $\kappa\nu = 0.01 > 0.0015$. As initial condition for φ we chose the one in Figure 6.4, the nonsymmetric initialization with nonzero mean perturbation of $\bar{\varphi}$, whereas for the temperature we initialized the field solving the following equation, which is completely different from all the previous cases:

$$\kappa(\nabla\Theta_0, \nabla\xi) = \int_\Omega 5 \times 10^5 \sin(32x)\cos^2(32y)\xi dx \quad \forall\xi \in H_0^1(\Omega),$$

With homogeneous Dirichlet boundary conditions. The resulting nonsymmetric initial condition is shown in Figure 6.49: we can observe that the range of values is much larger than the previous tests. This will lead to much larger velocity in magnitude.

We then initialize the velocity field, by solving a Stokes equation for $(\mathbf{u}_0, p_0) \in [H_0^1(\Omega)]^2 \times L^2(\Omega)$ keeping into account φ_0, μ_0 and Θ_0:

$$\begin{cases} \nu(\nabla\mathbf{u}_0, \nabla\mathbf{w}) - (p_0, \mathrm{div}(\mathbf{w})) = (-\varphi_0\nabla(\mu_0), \mathbf{w}) + (\Theta_0, \mathbf{w}_y) \quad \forall\mathbf{w} \in [H_0^1(\Omega)]^2 \\ (\mathrm{div}(\mathbf{u}_0), q) = 0 \quad \forall q \in L^2(\Omega) \end{cases}$$

always with no slip boundary conditions, obtaining the velocity field in Figure 6.50: we notice that the velocity field magnitude is much larger than the previous cases, and also the

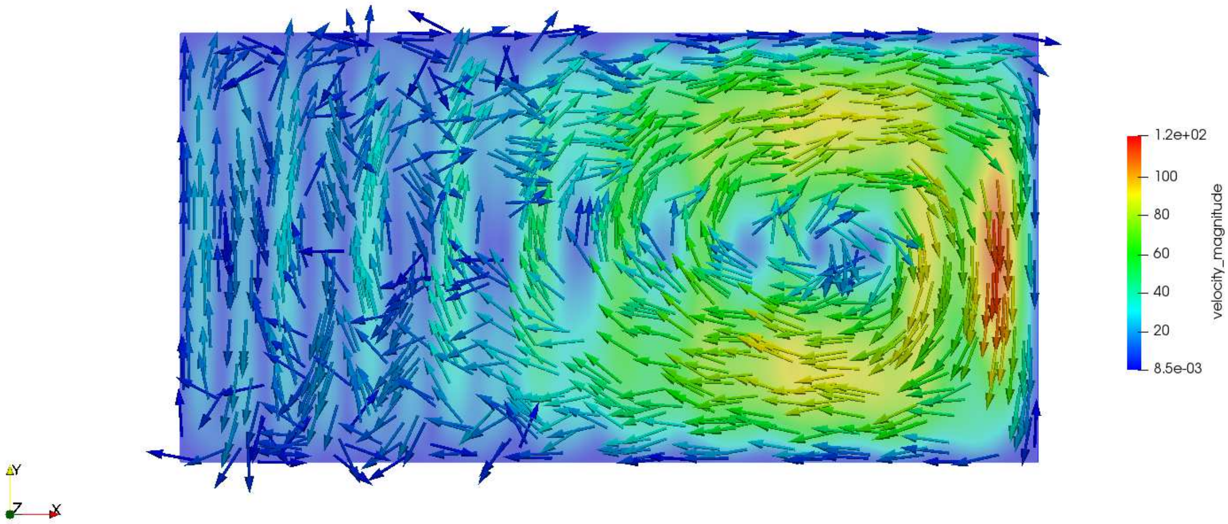
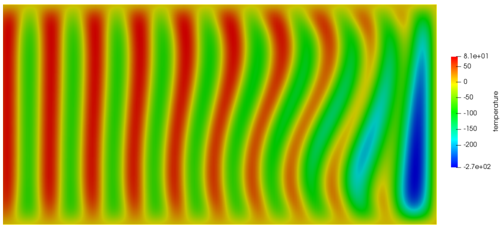


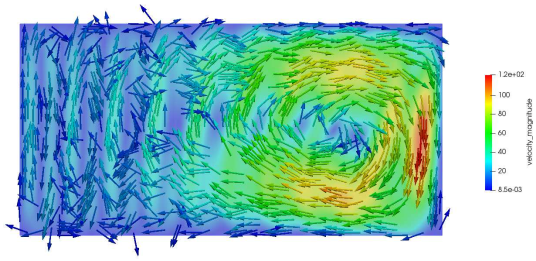
Figure 6.50: Test 5. Initial velocity field.

field is completely different, since it strongly depends on the temperature field Θ_0 already initialized.

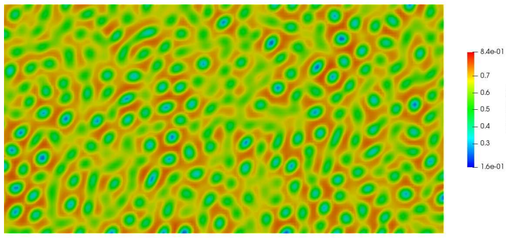
For this test we reached $T \approx 0.017$ and the time step ranges from 6×10^{-12} to 5.4×10^{-5}: as a matter of fact, we can consider it constant, since the oscillations are around approximately the same mean value. The concentration field in time is represented in Figures 6.51-6.54, where we can see the spinodal decomposition at the very beginning. The advective effect of the velocity field is strong, since the bubbles move during time and their shape is sensibly distorted, since it is very elongated. The concentration is driven to the two local minima of the double well potential.



(a) Temperature field

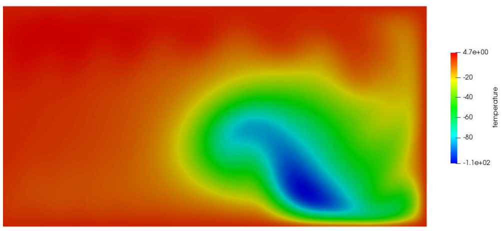


(b) Velocity field

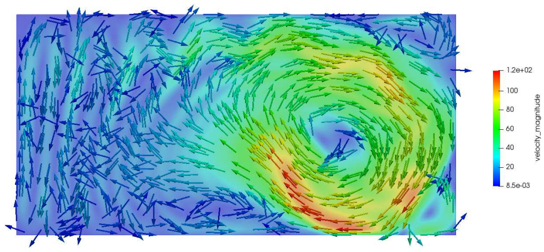


(c) Concentration field

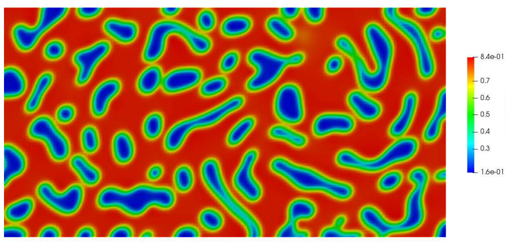
Figure 6.51: Test 5. Time $t = 0.0011$.



(a) Temperature field

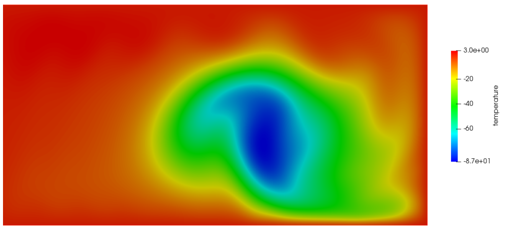


(b) Velocity field

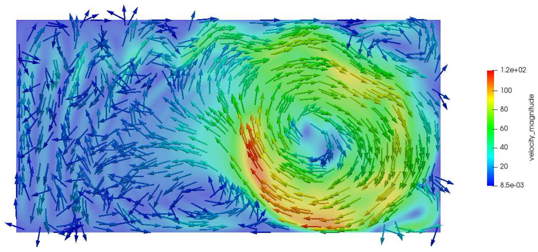


(c) Concentration field

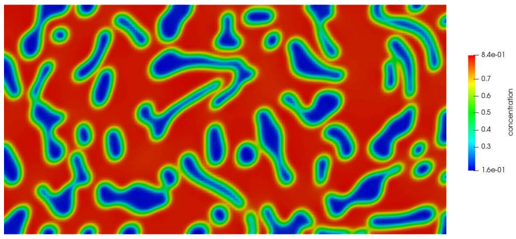
Figure 6.52: Test 5. Time $t = 0.0088$.



(a) Temperature field

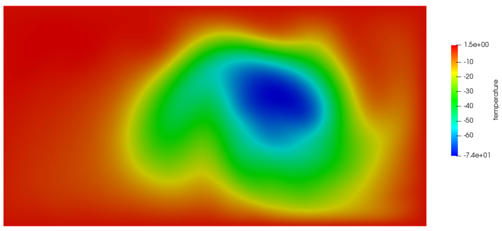


(b) Velocity field

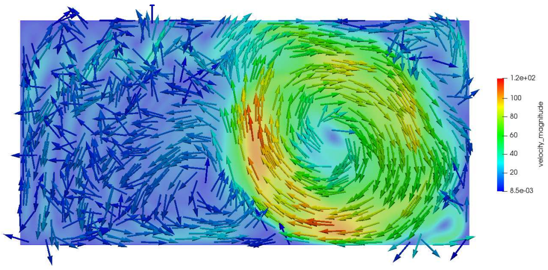


(c) Concentration field

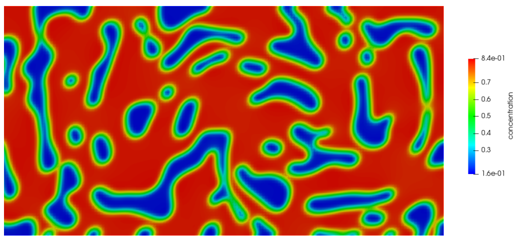
Figure 6.53: Test 5. Time $t = 0.012$.



(a) Temperature field

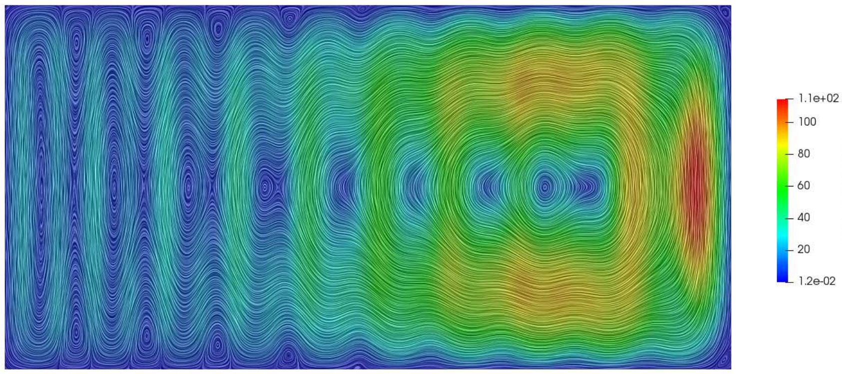


(b) Velocity field

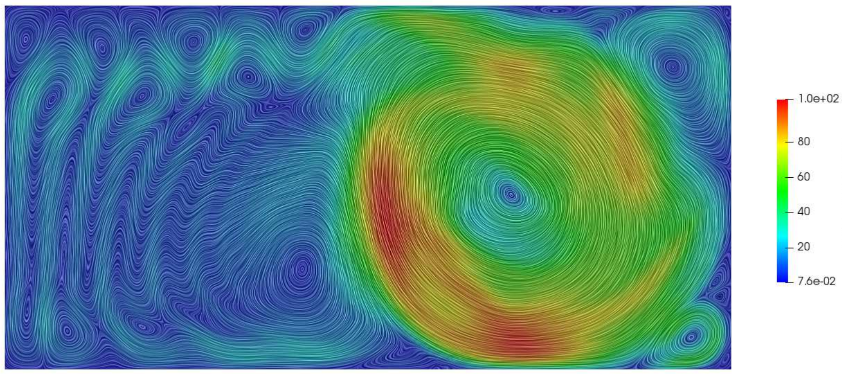


(c) Concentration field

Figure 6.54: Test 5. Time $t = 0.017$.



(a) Time t=6×10^{-12}

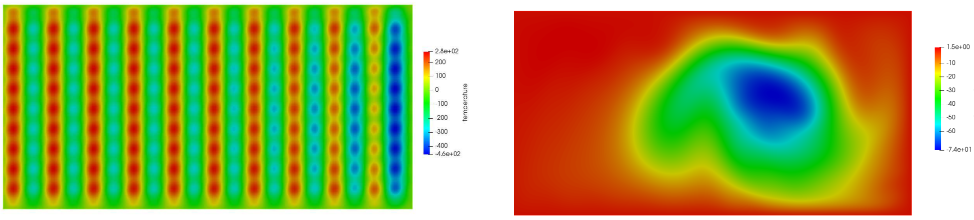


(b) Time t=0.017

Figure 6.55: Test 5. Velocity field vs. time with streamlines: we see the formation of new vortices and the change of shape of the principal one.

For what concerns the velocity field, in Figure 6.55, we can see the evolution in time: the principal vortex completely changes its shape and other secondary vortices appear. Moreover the range of values of velocity magnitude is reduced.

In Figures 6.31-6.34, and, more in detail, in Figure 6.56, we see the temperature field vs. time: the distribution in space completely changes due to two different phenomena. Firstly, the dissipation effect progressively reduces the range of values and tends to homogenize the field, by consuming energy. Secondly, the strong advective effect makes the area with lowest value move quickly across the domain.



(a) Time t=6×10^{-12} (b) Time t=0.017

Figure 6.56: Test 5. Temperature field vs. time: the space distribution completely changes due to diffusion and advection.

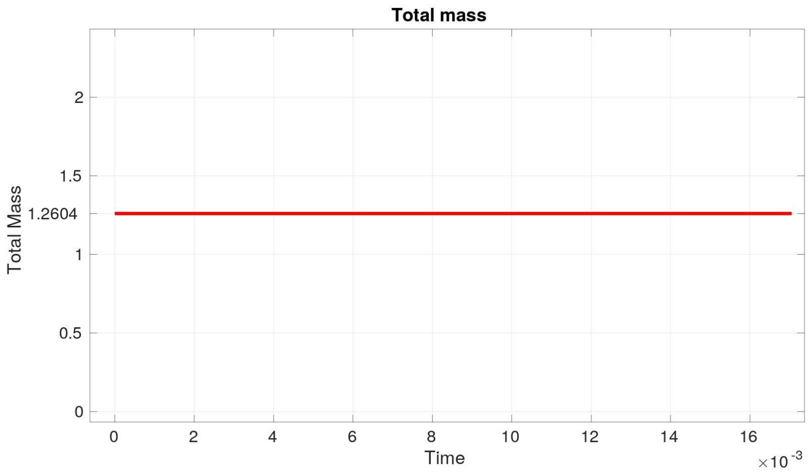


Figure 6.57: Test 5. Total mass of the system: we see that it is constant in time, as expected.

Moreover, in Figure 6.57 we see the plot of the total mass in time: it is always constant (up to the fifth significant digit) $m = 1.26036$.

If we now compute the total energy for each time step $E_n = \frac{1}{2}||\mathbf{u}_n||^2 + \frac{1}{2}||\Theta_n||^2 + \frac{\alpha}{2}||\nabla\varphi_n||^2 + \int_\Omega \Psi(\varphi_n)dx$, we obtain that the energy is nondecreasing, and, in particular, it is decreasing, as we can see in Figure 6.58. We notice that in this test the energy is much larger than the one in the previous tests. In Figure 6.59 we computed the derivative of the total energy by means of backward finite differences, obtaining that the derivative is always negative. This fact confirms the decrease of the total energy, as we expected from Theorem 5.1.4.

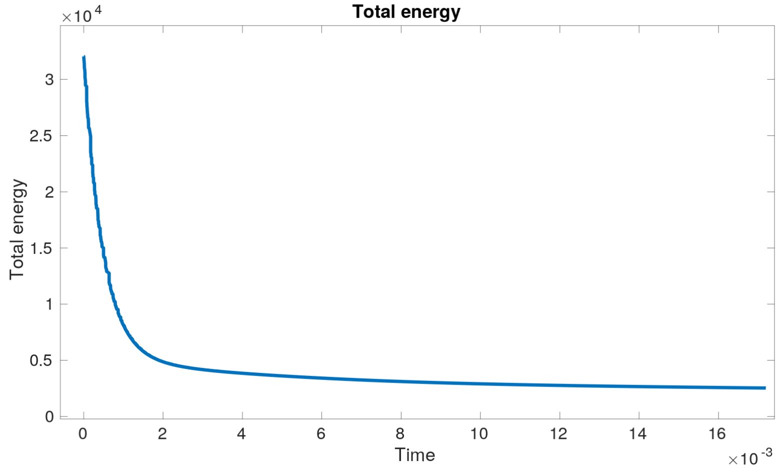


Figure 6.58: Test 5. Total energy E_n: it is always decreasing as expected.

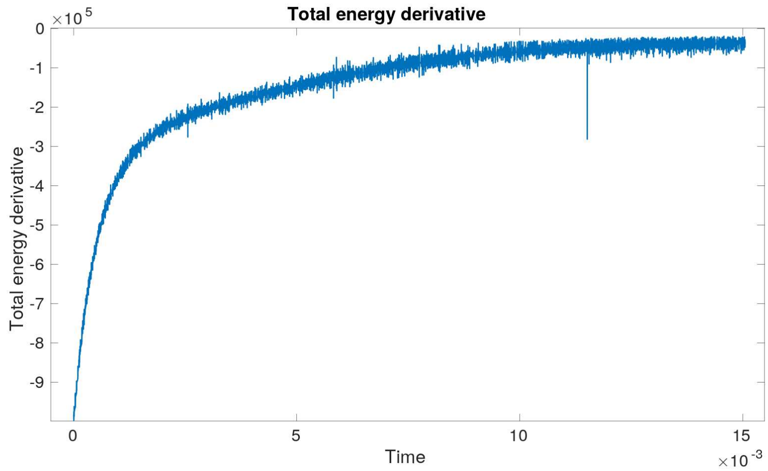


Figure 6.59: Test 5. Derivative of the total energy E_n: it is always negative as expected, and it tends to zero as time increases.

Conclusions and future issues

In this thesis we have investigated the 2D Cahn-Hilliard-Boussinesq system, characterized by a logarithmic potential. More precisely, we have proven the existence of a global weak solution as well as the existence of more regular solutions (quasi-strong and strong). Also, we have obtained some stability estimates. These estimates yield, in particular, the weak-strong uniqueness and the uniqueness of quasi-strong solutions.

However, from the theoretical viewpoint, there are still several issues which deserve to be analyzed.

First of all, having already found a result of existence of strong solutions, on account of the dissipativity properties of the system one could prove the regularization of weak solutions in finite time, that is, any weak solution gets a strong solution instantaneously. Moreover, following what was done in [61] for the case of NSCH, the regularity of the solutions to the CHB system when the viscosity is not constant, but depends on both the concentration and the temperature, could be studied. We recall that here we have proven only the existence of weak solutions for a non-constant viscosity.

Regularity results should allow us to establish the so-called phase separation property from the pure phases. This means that for every $\tau > 0$ there exists $\delta = \delta(\tau) > 0$ such that

$$\sup_{t \geq \tau} ||\varphi(t)||_{L^\infty(\Omega)} \leq 1 - \delta.$$

This property has already been shown for the NSCH system with logarithmic potential (see [61]).

On account of the dissipative nature of the system, a further aspect that could be investigated is the longtime behavior of a given solution, the goal being to establish the convergence to a single stationary state (see, e.g., [3] and [54] for the NSCH system). The longtime behavior can also be studied from a global viewpoint within the theory of dissipative dynamical systems, proving, for instance, the existence of global and exponential attractors (see, e.g., [52] and [55] for the NSCH system).

Other challenging theoretical issues are the study of the inviscid CHB system, which has been studied, e.g., in [102] for a regular potential, but not in case of the logarithmic potential. Moreover, it could be interesting to see whether the solutions of such a system could be regarded as the limit of the solutions of the viscous CHB system as $\nu \to 0$.

Furthermore, it would be particularly meaningful to analyze the system with vanishing thermal conductivity κ and to establish if the solutions of such a system could be obtained from the CHB system when $\kappa \to 0$. We stress again that the system with $\kappa = 0$ is the compressible NSCH system (17), thus it could be a way of finding results about existence and regularity of the solutions for this completely different system of equations, starting from the analysis of the CHB system. Note that, in this context, the equation for the temperature becomes a pure transport equation, namely, the continuity equation for the fluid density.

Aiming at the numerical solution of the CHB system, we have introduced a discretization of the equations in space, by means of finite elements, and in time. We have proved that the proposed numerical scheme preserves the mass of the system and it is stable, in the sense that the total energy does not increase in time, which is fundamental from the physical viewpoint. By means of the simulations that we performed, we have verified that these properties are effectively respected: namely, the total energy decreases in time and the mass is conserved. We have also used an adaptive timestep, but in the simulations it has not presented noticeable variations in magnitude. Thus a possible direction of improvement could be finding a better time step adaptivity algorithm, in order to effectively exploit the different characteristic times of the equations and investigate numerically the long-time behavior of the solutions, in particular the possibility of having stationary solutions.

Moreover, in order to better capture the interface phenomena, some kind of mesh adaptivity could be introduced, together with different basis functions for the Galerkin approximations: Isogeometric Analysis could be a valid alternative, as shown, for instance, in [15] and [64].

In conclusion, it could be interesting to study the numerical simulations of some of the above mentioned systems, namely the vanishing viscosity case and the vanishing thermal conductivity case, in order to obtain a sort of verification of the analytical results.

Appendices

Appendix A

Basic tools of Functional Analysis

A.1 Basic tools used in the proofs

In this Appendix we state some results that were often used throughout proofs. In the following Ω is a bounded domain in $\mathbb{R}^2$ with a sufficiently smooth boundary.

We first recall the well known Poincaré's inequality (see, e.g., [20], Corollary 9.19).

Lemma A.1.1. *For any $\varphi \in V$ it holds*

$$||\varphi - \bar{\varphi}||_V \leq C_0||\nabla\varphi||. \tag{A.1}$$

Then we state another inequality, consequence of the Poincaré's inequality and the elliptic regularity (see, e.g., [98], Chap.2).

Lemma A.1.2. *For any $\varphi \in V_2$ it holds*

$$||\varphi - \bar{\varphi}||_{H_2(\Omega)} \leq C||\Delta\varphi||. \tag{A.2}$$

From [20], Chap.8, Sec.6, we have the following inequalities, known as Gagliardo-Nirenberg interpolation inequalities for $\Omega \subset \mathbb{R}^2$ regular bounded open set:

Theorem A.1.3. *Let $1 \leq q \leq p \leq \infty$. Then*

$$||u||_{L^p(\Omega)} \leq C||u||^{1-a}_{L^q(\Omega)}||u||^a_{H^1(\Omega)} \quad \forall u \in H^1(\Omega), \text{ where } a = 1 - (q/p). \tag{A.3}$$

A similar classical inequality of Ladyzhenskaya type (see, e.g., [78], Theorem 2.2) which will be useful is:

Lemma A.1.4. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Then*

$$||f||^2_{L^4(\Omega)} \leq C\ (||f||\ ||\nabla f|| + ||f||^2) \qquad \forall f \in H^1(\Omega) \tag{A.4}$$

for some constant $C = C(\Omega)$, implying that

$$||f||_{L^4(\Omega)} \leq K\ ||f||^{1/2}||f||^{1/2}_V \qquad \forall f \in H^1(\Omega) \tag{A.5}$$

for some constant K=$K(\Omega)$.

We then define, for any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in H^1(\Omega)$:

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i\ dx$$

from which we obtain the following inequality (from Gagliardo-Nirenberg interpolation inequality, see Theorem A.1.3, and from Poincaré's inequality) for a bi-dimensional domain, as exploited also in [32]:

Lemma A.1.5. *For any $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \boldsymbol{V}_\sigma$ it holds*

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq C||\nabla \boldsymbol{u}||^{\frac{1}{2}}||\boldsymbol{u}||^{\frac{1}{2}}||\nabla \boldsymbol{v}||\ ||\boldsymbol{w}||^{\frac{1}{2}}||\nabla \boldsymbol{w}||^{\frac{1}{2}} \tag{A.6}$$

and by the antysimmetry of the trilinear form $b(\cdot, \cdot, \cdot)$ it also holds, for any $\boldsymbol{u}$, $\boldsymbol{v} \in H^1(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{V}_\sigma$, that

$$|b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})| \leq ||\boldsymbol{u}||^{\frac{1}{2}}||\boldsymbol{u}||_1^{\frac{1}{2}}||\boldsymbol{v}||^{\frac{1}{2}}||\boldsymbol{v}||_1^{\frac{1}{2}}||\boldsymbol{w}||_1. \tag{A.7}$$

We will also make use of the following estimate ([2], Sec.2), which comes from the Agmon's inequality (see, e.g., [5], Lemma 13.2):

Lemma A.1.6. *For every $u \in H^2(\Omega)$, $\Omega \in \mathbb{R}^2$ it holds*

$$||\nabla u||_{L^4(\Omega)} \leq ||u||^{\frac{1}{4}}||u||^{\frac{3}{4}}_{H^2(\Omega)}.$$

We also recall the following version of the well known Gronwall's Lemma (see, e.g., [65], Sec.2, or [93], Chap.10, Lemma 10.8):

Lemma A.1.7. *Let $I = [t_0, t_1]$. Suppose $a : I \to \mathbb{R}$ and $b : I \to \mathbb{R}$ are continuous (or $a, b \in L^1(0,T)$), and suppose $u : I \to \mathbb{R}$ is in $C^1(I)$ (or even in $C(I)$) and satisfies (it is enough in weak sense)*

$$u'(t) \leq a(t)u(t) + b(t) \text{ for } t \in I, \text{ and } u(t_0) = u_0.$$

Then

$$u(t) \leq u_0 e^{\int_{t_0}^t a} + \int_{t_0}^t e^{\int_s^t a} b(s) ds. \tag{A.8}$$

Another important theorem is the following weak form of Lebesgue theorem (see [82], Lemma 1.3):

Theorem A.1.8. *Let $\{f_n\}$ be a sequence in $L^2(\Omega \times (0,T))$ such that it is uniformly bounded, $\sup_{n\in\mathbb{N}} ||f_n|| = M < +\infty$, and $f_n \to f$ almost everywhere in $\Omega \times (0,T)$. Then $f_n \rightharpoonup f$ in $L^2(\Omega \times (0,T))$.*

A.2 Embedding theorems

Here we recall the Aubin-Lions Lemma (see [82], Lemma 1.2 or [83], Chap.1):

Lemma A.2.1. *Let $X \hookrightarrow Y \hookrightarrow Z$ three Hilbert spaces, and suppose that the embedding of X into Y is compact.*

1. *For any $p_1,\ p_2 \in (1, , +\infty)$ the embedding*

$$\left\{ f \in L^{p_1}(0,T;X),\ \frac{df}{dt} \in L^{p_2}(0,T;Z) \right\} \hookrightarrow L^{p_1}(0,T;Y) \tag{A.9}$$

   is compact.

2. *For every $p > 1$ the embedding*

$$\left\{ f \in L^{\infty}(0,T;X),\ \frac{df}{dt} \in L^{p}(0,T;Z) \right\} \hookrightarrow C([0,T];Y) \tag{A.10}$$

   is compact.

We have another important Lemma (see, e.g., [83], Vol.I, Chap.1) for the case of a Hilbert triplet:

Lemma A.2.2. *Let (V,H,V') a Hilbert triplet, with V and H separable spaces. Then*

$$H^1(0,T;V,V') = \left\{ f \in L^2(0,T;V),\ \frac{df}{dt} \in L^2(0,T;V') \right\} \hookrightarrow C([0,T];H).$$

Moreover it holds the following integration by parts rule:
for every $u, v \in H^1(0,T;V,V')$, *for every* $s, t \in [0,T]$*:*

$$\int_s^t \{ < \dot{u}(r), v(r) > + < u(r), \dot{v}(r) > \} \, dr = (u(t), v(t)) - (u(s), v(s)).$$

Appendix B

Further estimates and lemmas

Here we report some well-posedness and regularity results about some stationary problems related to our system.

B.1 A Neumann problem with logarithmic nonlinearity

We start from the Neumann problem: F is the same logarithmic potential as the potential defined in (2):

$$F(s) = \frac{\bar{\alpha}}{2}((1+s)\ln(1+s) + (1-s)\ln(1-s)) \quad \forall s \in (-1,1) \tag{B.1}$$

$$\begin{cases} -\Delta u + F'(u) = f \text{ in } \Omega \\ \partial_{\mathbf{n}} u = 0 \qquad\qquad \text{on } \Omega \end{cases} \tag{B.2}$$

under the assumptions of the potential made in the previous sections, we have the following lemmas (see [61], Lemma A.1):

Lemma B.1.1. *Let $\partial\Omega$ be a bounded domain in $\mathbb{R}^2$, with smooth boundary. Assume that $f \in L^2(\Omega)$. Then there exists a unique solution u to problem (B.2) such that $u \in H^2(\Omega)$, $F'(u) \in L^2(\Omega)$ and satisfies $-\Delta u + F'(u) = f$ for almost every $x \in \Omega$ and $\partial_n u = 0$ for almost every $x \in \partial\Omega$. Moreover we have*

$$||u||_{H^2(\Omega)} + ||F'(u)|| \leq C(1 + ||f||), \tag{B.3}$$

where $||\cdot||$ *is the* L^2 *norm.*

Let us assume that the sequence $f_k \subset L^2(\Omega)$ *and* $f \in L^2(\Omega)$. *We consider the solutions* u_k *and* u *to the problem (B.2) corresponding to* f_k *and* f, *respectively. Then,* $f_k \to f$ *in* $L^2(\Omega)$, *as* $k \to \infty$, *implies*

$$||u_k - u||_1 \to 0 \quad as \quad k \to 0. \tag{B.4}$$

We then report other elliptic estimates, already stated and proven, e.g, in [34], from Lemma A.1 to Lemma A.6 or in [2], Lemma 2.

Theorem B.1.2. *Let* Ω *be a bounded domain in* $\mathbb{R}^2$ *with smooth boundary. Assume that* u *is the solution to problem (B.2). We have the following:*

- *Let* $f \in L^p(\Omega)$, *where* $2 \leq p \leq \infty$. *Then we have*

$$||F'(u)||_{L^p(\Omega)} \leq ||f||_{L^p(\Omega)}.$$

- *Let* $f \in H^1(\Omega)$. *Then we have*

$$||\Delta u|| \leq ||\nabla u||^{\frac{1}{2}} ||\nabla f||^{\frac{1}{2}}. \tag{B.5}$$

- *Let* $f \in H^1(\Omega)$. *Assume that* F *satisfies*

$$F''(s) \leq e^{C|F'(s)|+C} \quad \forall s \in (-1,1)$$

for some positive constant C. *Then for any* $p \geq 1$, *there exists a positive constant* $C = C(p)$ *such that*

$$||F''(u)||_{L^p(\Omega)} \leq C(1 + e^{C||f||_1^2}). \tag{B.6}$$

In addition, there exists a positive constant C=C(p) such that

$$||u||_{W^{2,p}(\Omega)} + ||F'(u)||_{L^p(\Omega)} \leq C(1 + ||f||_1) \tag{B.7}$$

for any $p \geq 2$.

B.2 The homogeneous Dirichlet problem for the Stokes equation

Considering now the Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = f \text{ in } \Omega \\ \text{div } \mathbf{u} = 0 \qquad \text{in } \Omega \\ \mathbf{u} = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{B.8}$$

First we introduce the Stokes operator as the map $\mathbf{A} : \mathbf{V}_\sigma \to \mathbf{V}'_\sigma$ such that

$$< \mathbf{A}\mathbf{u}, \mathbf{v} > = < \nabla \mathbf{u}, \nabla \mathbf{v} > \quad \forall\ \mathbf{u}, \mathbf{v} \in \mathbf{V}_\sigma,$$

namely, $\mathbf{A}$ is the canonical isomorphism from $\mathbf{V}_\sigma$ onto $\mathbf{V}_\sigma$.

We can denote by $\mathbf{A}^{-1} : \mathbf{V}'_\sigma \to \mathbf{V}_\sigma$ the inverse map of the Stokes operator. That is, given $\mathbf{f} \in \mathbf{V}_\sigma$, there exists a unique $\mathbf{u} = \mathbf{A}^{-1}\mathbf{f} \in \mathbf{V}_\sigma$ such that

$$(\nabla \mathbf{A}^{-1}\mathbf{f}, \nabla \mathbf{v}) = < \mathbf{f}, \mathbf{v} > \quad \forall\ \mathbf{v} \in \mathbf{V}_\sigma.$$

It follows that $||\mathbf{f}||_\flat := ||\nabla \mathbf{A}^{-1}\mathbf{f}|| = < \mathbf{f}, \mathbf{A}^{-1}\mathbf{f} >^{\frac{1}{2}}$ is an equivalent norm on $\mathbf{V}'_\sigma$. In addition, for any $\mathbf{f} \in H^1(0, T; \mathbf{V}'_\sigma)$ we have the chain rule (see [97], Chap.3, Lemma 1.1)

$$< \mathbf{f}_t(t), \mathbf{A}^{-1}\mathbf{f}(t) > = \frac{1}{2}\frac{d}{dt}||\mathbf{f}(t)||_\flat^2 \ \text{ a.e. } t \in (0, T). \tag{B.9}$$

After the already mentioned De Rham's theorem, which was already exploited to retrieve the existence of pressure, up to a constant, in $L^2(0, T, L^2(\Omega))$ (see Remark 1.2.3 on Definition 1.1), we have more regular properties: we have that $\mathbf{A}$ is also a positive, unbounded, self-adjoint operator in $\mathbf{H}_\sigma$, with compact inverse and domain $D(\mathbf{A}) = \{\mathbf{u} \in \mathbf{V}_\sigma : \mathbf{A}\mathbf{u} \in \mathbf{H}_\sigma\} = [\mathbf{H}^2(\Omega)]^2 \cap \mathbf{V}_\sigma := \mathbf{W}_\sigma$. As in [97], Chap.2, Prop.2.2, we have, due to regularity results, defining $(\mathbf{u}, \mathbf{v}) = (\mathbf{A}\mathbf{u}, \mathbf{A}\mathbf{v})$ and $||\mathbf{v}||^2_{\mathbf{W}_\sigma} = (\mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{v})$:

$$\exists C > 0 \ \ s.t. \ \ ||\mathbf{u}||_{H^2(\Omega)} \le C||\mathbf{u}||_{\mathbf{W}_\sigma} \ \ \forall \mathbf{u} \in \mathbf{W}_\sigma. \tag{B.10}$$

B.3 The homogenous Dirichlet problem for the Laplace equation

We need to introduce the classical Riesz isomorphism A_0 from V_θ to V'_θ:

$$< A_0 u, v >= (\nabla u, \nabla v) \ \ \forall \ u, v \in V_\theta. \tag{B.11}$$

Then, denoting by A_0^{-1} its inverse map, we have that $f \in V_\theta$, $A_0^{-1} f$ is the unique $u \in V_\theta$ such that $< A_0 u, v >=< f, v >$ for all $v \in V_\theta$. On account of the above definitions, we observe that that

$$< f, A_0^{-1} g >= (\nabla(A_0^{-1} f), \nabla(A_0^{-1} g)) \ \ \forall f, g \in V'_\theta. \tag{B.12}$$

Owing to (B.12) it can be proved that $||f||_{**} := ||\nabla A_0^{-1} f|| =< f, A_0^{-1} f >^{\frac{1}{2}}$ is a norm on V'_θ equivalent to the usual and natural one. In addition, for any $u \in H^1(0, T; V'_\theta)$ we have the chain rule (see [97], Chap.3, Lemma 1.1):

$$< u_t(t), A_0^{-1} u(t) >= \frac{1}{2} \frac{d}{dt} ||u(t)||^2_{**} \ \ \text{a.e. } t \in (0, T). \tag{B.13}$$

We also have that, since $L^2(\Omega) \hookrightarrow V'_\theta$:

$$(\nabla u, \nabla A_0^{-1} u) =< A_0(A_0^{-1} u), u >=< u, u >= ||u||^2. \tag{B.14}$$

Furthermore, as can be found in [98], Chap.II, Secs.2.1-2.2, we obtain, due to regularity theorems, that A_0 is an isomorphism also from $D(A_0) = H^2(\Omega) \cap V_\theta$ to $H = L^2(\Omega)$. We then have that

$$||A_0^{-1} f||_{H^2(\Omega)} \leq C||f|| \ \ \forall f \in H. \tag{B.15}$$

B.4 The homogenous Neumann problem for the Laplace equation

In conclusion, we have to make similar considerations for the Neumann problem as done, e.g., in the Appendix of [61]): for any $\lambda \geq 0$ we consider the system

$$\begin{cases} -\Delta u + \lambda u = f & \text{in } \Omega \\ \partial_{\mathbf{n}} u = 0 & \text{on } \Omega. \end{cases}$$

We introduce the operator $B_\lambda \in \mathcal{L}(V, V')$ defined by

$$< B_\lambda u, v >= (\nabla u \cdot \nabla v + \lambda uv) \quad \forall u, v \in V.$$

We consider the space

$$V_0 = \{v \in V : \bar{v} = 0\} \tag{B.16}$$

and its dual V_0'. The restriction $\bar{A}_0$ of B_0 to V_0 being an isomorphism from V_0 onto V_0', we denote $\bar{A_0}^{-1} : V_0' \to V_0$ its inverse map. It is well known that for all $f \in V_0'$, $\bar{A_0}^{-1} f$ is the unique $u \in V_0$ such that $< \bar{A}_0 u, v >=< f, v >$ for all $v \in V$. On account of the above definitions, we observe that

$$< f, \bar{A_0}^{-1} g >= (\nabla \bar{A_0}^{-1} f, \nabla \bar{A_0}^{-1} g) \quad \forall f, g \in V_0' \tag{B.17}$$

And owing to (B.17) it is straightforward to prove that

$$||f||_* := ||\nabla \bar{A_0}^{-1} f|| =< f, \bar{A_0}^{-1} f >^{\frac{1}{2}} \tag{B.18}$$

is a norm on V_0' equivalent to the natural one. In addition, for any $u \in H^1(0, T; V_0')$ we have the chain rule (see [97], Chap.3, Lemma 1.1)

$$< u_t(t), \bar{A_0}^{-1} u(t) >= \frac{1}{2} \frac{d}{dt} ||u(t)||_*^2 \quad \text{a.e. } t \in (0, T). \tag{B.19}$$

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