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Convergence Analysis for Sponsored Seach Auctions with Price Displaying

Tesi di Laurea Magistrale in
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Abstract

Since the early 2000's online advertising has become one of the main revenue sources for companies, which have been using it to advertise their content and sponsor their goods. One of the main application fields for online advertising is search advertising, where the search engines submit to their users some sponsored contents next to the organic results. The way in which the displayed advertisements are selected is through a suitable auction mechanism. Mechanism design is a well studied sub field of game theory, and the problem of auctioneer's mechanism selection and advertisers' bidding strategy have a solid base in literature. However, very few studies have analyzed auctions in the context of e-commerce, namely *sponsored auction with price display*, where advertisements are associated to a price for the sponsored good and customers' clicks are influenced by the comparison of those selling prices. In this work, we initially formalize the extension of a well known bidding strategy for the generalized second price auction, the balance bidding, to the scenario with price displaying. Our main result consists in the proposal of a new auction mechanism based on GSP that is guaranteed to converge to its equilibrium if advertisers bid according to the extended bidding strategy when prices are fixed. We also study the efficiency of the equilibrium with respect to a parameter of the mechanism called *cut price*, proposing a randomized algorithm that guarantees a lower bound of the equilibria social welfare in expectation. Lastly, we provide some experimental results, in order to empirically analyze the average convergence time and the average efficiency for different auction settings.

Keywords: mechanism design, generalized second price, selling price, convergence, efficiency

Sommario

Sin dai primi anni 2000 la pubblicità online è diventata una delle maggiori fonti di guadagno per le aziende, che la utilizzano per sponsorizzare i loro contenuti e vedere i loro prodotti. Uno dei suoi maggiori campi di applicazione risiede nella pubblicità per motori di ricerca, dove i motori di ricerca mostrano agli utenti contenuti sponsorizzati assieme ai contenuti organici. Il modo in cui i contenuti pubblicitari vengono selezionati consiste nell'applicazione di un meccanismo di asta appropriato. La progettazione di meccanismi di asta è una branca della teoria dei giochi molto studiata, e sia il problema di implementazione di questi meccanismi da parte del venditore che lo studio di strategie di offerta da parte dei partecipanti all'asta sono problemi che trovano una base solida in letteratura. Tuttavia, pochi studi analizzano le aste nel contesto della vendita online, dove gli annunci sono associati a un prezzo di vendita e i click sono influenzati dal confronto degli utenti tra i prezzi stessi. In questo lavoro, formalizziamo inizialmente l'estensione di una nota strategia di offerta per meccanismi GSP, chiamata balance bidding, allo scenario in cui sono presenti i prezzi di vendita. Successivamente, il nostro risultato più significativo consiste nella proposta di un nuovo meccanismo di asta basato su GSP che fornisce garanzie di convergenza quando gli inserzionisti adottano la strategia di offerta descritta e i prezzi di vendita sono fissi. Studiamo inoltre la l'efficienza dell'equilibrio raggiunto, in relazione ad un parametro del meccanismo chiamato *prezzo di taglio*, proponendo un algoritmo randomizzato che garantisce un social welfare minimo atteso all'equilibrio. Per concludere, forniamo alcuni risultati sperimentali per analizzare empiricamente il tempo di convergenza del meccanismo e l'efficienza dell'equilibrio in relazione a diversi parametri dell'asta.

Parole chiave: progettazione di meccanismi, generalized second price, prezzo di vendita, convergenza, efficienza

Contents

1 | Introduction

1.1. Research Area

The revenues of online advertising have steadily increased since the early 2000's, becoming a nearly $190 billion dollars industry in the U.S. in 2021 with a growth of 35.4% with respect to 2020, as reported by the Interacting Advertising Bureau [2]. The main source of Internet advertising revenue comes from the search advertising, which alone contributes for $78.3 billion dollars. In 2021 Alphabet, the parent holding company of Google, announced a revenue of $257 billion dollars, of which $209 billion dollars coming from Google advertising, and $148 billion resulting from just search advertising alone [14]. These results underline how the selling of online advertising constitutes a fundamental component of the business models of many Internet companies, and stands as main revenue factor for search engine companies like Google, Bing and Yahoo!.

The search advertising is a subclass of Internet advertising that consists on the displacing of sponsored contents, called ads, on web pages that show the results of a search query. The way in which ads are selected and displayed is managed by a particular type of auction called *sponsored search auctions*, or *keyword auctions*, where each advertiser bid represent the amount that he is willing to pay to be displayed. When a search engine is queried by an user request, two sets of results are returned: besides the most valuable results selected by the engine (called organic results) a set of ads in the form of links that redirect to the sponsored web page is shown. The ads are different for different searches and are coherent with the context of the query itself. In fact, when an advertiser tries to "buy" a position for his ad he can associate it with a set of keywords, and the sponsored content will compete to appear only in the searches that explicitly contain one of those keywords. For instance, if a travel agency wants to sponsor its website, it can bind his ads to the word "hotel"; consequently, each time a user query will contains the word "hotel" the ad will compete to be displayed. From the point of view of the advertiser this represents a great advantage since it offers the ability to target precisely any possible customer. The way in which ads are selected and displayed is determined by the search engine and the way in which the sponsored search auction is designed, as well as the amount that the advertisers have to pay. In general, the advertisers will submits a bid for their ads stating the maximum amount they are willing to pay for it, and the engine use this information to allocate the ads in order to solve an optimization problem, like maximize its own revenue or the social welfare of the advertisers.

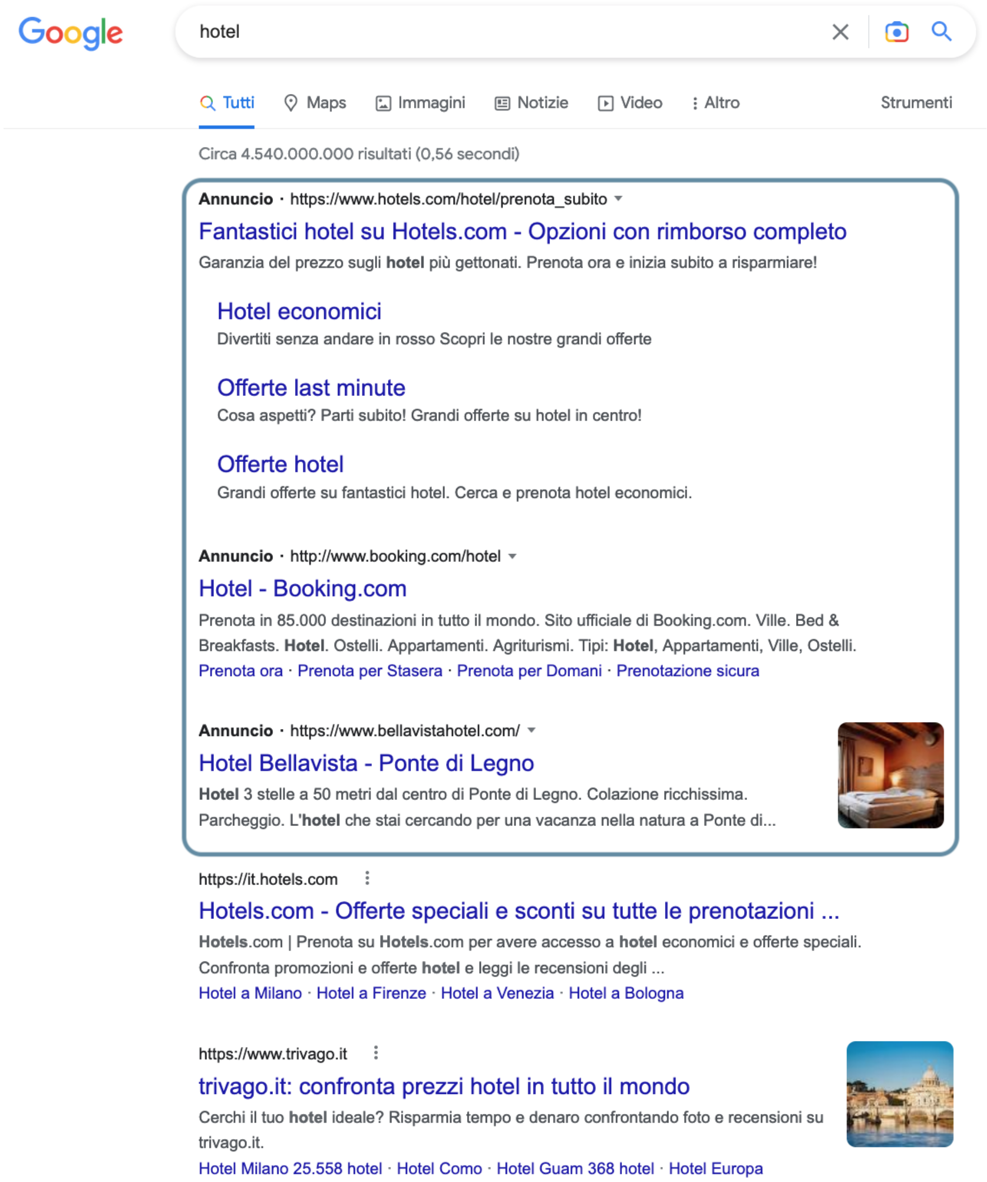

Figure 1.1: Example of search advertising
Google and the Google logo are trademarks of Google LLC.

The engine dedicate a maximum number of positions, called *slots*, to advertising, and typically the slots are disposed sequentially at the top of the page, one after the other (Figure 1.1). The visibility that an ad will receive depends on the slot to which it is assigned, and that is the reason why the advertisers are generally willing to pay more to be allocated to the top slots. In principle the advertisers can declare an untruthful valuation for their ad because that could lead them to a grater utility. The way in which the auctioneer can make them willing to report their true valuation is by adopting a truthful mechanism, like Vickrey-Clarke-Groves (VCG) auction [6, 12, 19], where truth telling is in the best interest of the advertisers. However, despite VCG being used in contextual advertising auctions by many company like Facebook, the most of the search engine do not actually adopt it for search advertising, preferring the Generalized Second Price (GSP) auction. Since GSP is non truthful, the advertisers need to elaborate complex bidding strategies to maximize their utility in the auctions. As showed by Edelman et al. [8], the GSP auction can produce instability and bidding wars among the advertisers. Varian [18] and Edelman et al. [8] independently studied the equilibria of GSP auction, defining a particular class of equilibria respectively called symmetric equilibria and locally envy-free equilibria, showing that these equilibrium always generate a revenue for the auctioneer at least as high as the VCG one. From the point of view of the advertisers Leme and Tardos [15] studied the efficiency of the GSP equilibrium in terms of social welfare.

To face the advertisers' problem of elaborating a strategy in GSP auctions, Cary et al. [3] and Bu et al. [1] independently studied the same bidding strategy, which they named respectively balance bidding and forward looking. This strategy is based on the notion of local envy-free or symmetric equilibrium proposed by Edelman et al. and Varian. The relevance of such strategy consists in its granted convergence, and on the fact that at the equilibrium the allocation and the payment are the same of VCG. Bu et al. extended their work by including the study of the effect of vindictive bidding strategies with respect to forward looking, showing that the proposed bidding strategy is indeed robust against them in the most cases. The effects of vindictive bidders on sponsored auctions has also been studied by Zhou and Lukose [20]. Hashimoto [13] analyzed the VCG equilibrium of the GSP auction when a non strategic bidder participate to the auction, showing it is no longer an equilibrium. Cary et al. [4] extended their previous work by analyzing the case when the click through probability are determined by an endogenous consumer search process. Nisan et al. [17] defined the concept of best response auctions, studying the effects of a best response bidding strategy for different auction settings, while Dütting and Kesselheim [7] showed how the convergence of a best response bidding strategy is not necessary in order to have some good social welfare guarantees.

1.2. Original contributions

The sponsored search auctions described so far are presented in the classical sense, which is by far the most studied and applied scenario. With the exponential growth of e-commerce in the last years, however, a new format of search advertising is becoming more and more popular. The search engine like Google and Yahoo! dedicate some of their advertising spaces to vendors that sponsor their products: in this context, the sponsored ad is not anymore just a link that redirect to the advertiser web page, but includes instead also a picture of the good and the selling price. For example, when searching "smartphone" on a search engine, among the organic results there will be a portion of the web page showing the images of some smartphones, as well as their prices. The keyword auctions are still managed by a bidding mechanism, but the new components add externalities among the ads, since the users now have an immediate way to compare them. An user will more likely click on an ad that sells an item to a low price instead of a similar item for which he would need to pay more. The auction mechanisms need to take this into consideration when allocating the ads to the slots and assign the payment to the advertisers.

Castiglioni et al. [5] studied the efficiency of the equilibrium of a sponsored search auction where the click probability of the ads is influenced by the prices of the allocated ads themselves. However, to the best of our knowledge, no previous work has been conduced to study the convergence of this type of auctions. We studied the same problem under the assumption of fixed prices for the sponsored goods. Even if the selling price constitutes a degree of freedom for the advertiser and can potentially be updated at anytime like the bids, in many practical scenarios the advertisers decide the price a priori and do not update except than in an abrupt way. We show that a sponsored auction implementing the GSP mechanism proposed by Castiglioni et al. [5] is not guaranteed to converge when the advertisers bid according to the selling prices quality-based bidding strategy studied by Cary et al. [3] and Bu et al. [1]. The main result of our work is the proposal of an incentive compatible GSP based mechanism, namely $\mathcal{M}_{ord}^{GSP}$, which is guaranteed to converge to its unique equilibrium when the bidding strategy respects the same assumptions. We will furthermore provide a bound for the efficiency of the equilibrium, in expectation with respect to a parameter of the mechanism itself. Finally, we will provide some experimental results, analyzing the dependence of the convergence time and equilibrium efficiency with respect to the auction settings.

1.3. Structure of the thesis

This thesis is structured as follows:

- Chapter 2 provides an introduction to the basic concepts of mechanism design, later formalizing the model of the sponsored search auction, the VCG mechanism and GSP mechanism. Lastly, the bidding strategies for GSP auctions that will be the base for the consequent work are presented.

- Chapter 3 introduces a formalization of the sponsored search auction with price displaying as well as the already known results.
- Chapter 4 presents our work, which consists in the formalization of a new GSP-based mechanism for sponsored search auction with price displaying and the study of its convergence property and the efficiency of the reached equilibrium.
- Chapter 5 presents the experimental results, empirically analyzing the convergence time and the efficiency of the proposed mechanism in relation with the environment parameters.
- Chapter 6 summarizes the results and analyzes the left open problems.

2 | State of the art

This section will introduce the groundings and the most relevant notions in the literature concerning our work. In Section 2.1 will be introduced the notion of game mechanism, its properties and some relevant class of mechanisms, in Section 2.2 will be described the sponsored search auction and in Section 2.3 the most relevant results regarding the bidding strategies for the sponsored search auctions will be presented.

2.1. Mechanism Design

Mechanism design is a sub-field of economic game theory that attempts to implement desired aggregations of preferences of different participants toward a single joint decision in a strategic setting, assuming that the agents act rationally [16][9]. A rational agent is an agent who makes decisions in order to maximize his own profit, which is measured by some utility function. Such strategic design is adopted since the agent's preferences are usually private, and they may not be willing to disclose them. If the agent's preferences were public knowledge there would be no need for mechanism design, since the same result could be achieved by an optimization problem. The main goal is to implement some social choice function with some desirable features for a non-cooperative like game with incomplete information that aggregates the preferences of the players taking part to the game, assuming that they would act in an intelligent and rational way. This preferences aggregation is an abstraction of many practical settings in social and economical scenarios, like elections, markets, auctions or government policies.

2.1.1. Definition of Mechanism

We will first define the environment that our work will be focused on, which is that of a game with strict incomplete information. The incompleteness of information is referred to the missing knowledge of the agents and the mechanism designer to the players' private preferences.

Definition 2.1. *A game with strict incomplete information for a set of n players defines,* *for each player i:*

- *A set of actions $A_i \in A$.*
- *A set of types $\Theta_i \in \Theta$, where θ_i represents the private information of player i.*
- *An utility function $u_i \colon \theta_i \times A_1 \times ... \times A_n \rightarrow \mathbb{R}$, where $u_i(\theta_i, A_1, ..., A_n)$ is the utility achieved by player i with type θ_i when the action profile of the agents is $\boldsymbol{a} = (a_1, ..., a_n)$.*

It can be observed that each player i must choose his action $a_i \in A_i$ without the knowledge of the other types θ_j, which indirectly affects the utility of i by determining the action of the other players. The behavior of player i in the described setting is defined by a strategy function $s_i \colon \Theta_i \rightarrow A_i$, that specifies which action $a_i \in A_i$ is taken for every possible type θ_i.

When dealing with the mechanism design problem, there are two main issues that the mechanism designer needs to deal with. The first one is the *preference aggregation problem*: for a given type profile $\theta = (\theta_1, ..., \theta_n)$ of the agents, an outcome $x \in X$ needs to be chosen. In order to solve it, the designer has to implement a social choice function.

Definition 2.2 (Social Choice Function)**.** *A Social Choice Function (SCF) is a function $f \colon \Theta_1 \times ... \times \Theta_n \rightarrow X$ that assign to each possible type profile $\theta = (\theta_1, ..., \theta_n)$ a collective outcome $f(\theta_1, ..., \theta_n) \in X$.*

Assuming that the preference aggregation problem has been solved, a second issue appear in the definition of mechanism, the *information revelation problem*: how can the mechanism designer extract the true type θ_i of each agent i? The agents' true type is a private information of the players and they may not be willing to disclose it, since an untruthful revelation could result in a preferred outcome.
The information revelation problem is faced by the construction of a suitable mechanism.

Definition 2.3 (Mechanism)**.** *A mechanism $\mathcal{M} = (A_1, ..., A_n, X, g)$ is a collection of actions sets $(A_1, ..., A_n)$ and an outcome function $g \colon A_1 \times ... \times A_n \rightarrow X$.*

Based on their true type and strategy function, each player i will choose their action $s_i(\theta_i)$ and the mechanism will determine an outcome $x = g(s_1, ..., s_n)$.
We can observe that asking the agents their true type is a particular case of mechanism, in which $A = \Theta$ and $g(\cdot) = f(\cdot)$ is a social choice function.

Definition 2.4 (Direct Revelation Mechanism)**.** *Given a social choice function $f \colon \Theta_1 \times$*

... $\times\, \Theta_n \rightarrow X$, a mechanism $\mathcal{D} = (\Theta_1, ...\Theta_n, X, f)$ is called Direct Revelation Mechanism (DRM) corresponding to $f(\cdot)$. A mechanism that is not a DRM is called Indirect Revelation Mechanism (IRM).

Since the agents are assumed to be intelligent and rational, after knowing about the the mechanism $\mathcal{M} = (A_1, ..., A_n, X, g)$ they are taking part to (whether it is direct or indirect), they will start to analyze their strategic actions in order to come up with the strategy s_i that will maximize their utility. This phenomenon leads to a game among the agents, which is called Bayesian game of incomplete information induced by the game by the mechanism $\mathcal{M}$. The formal definition of the induced Bayesian game is the following:

$$\Gamma^b = (N, (A_i)_{i\in N}, (\Theta_i)_{i\in N}, \Phi(\cdot), (u_i)_{i\in N})$$

where $\Phi(\cdot)$ is the probability density function of the type sets.

Definition 2.5. *A mechanism $\mathcal{M} = ((A_i)_{i\in N}, X, g)$ implements a social choice function $f(\cdot)$ if there is a pure strategy equilibrium $(s_1^*, ..., s_n^*)$ of the Bayesian game induced by the mechanism such that*

$$g(s_1^*(\theta_1), ..., s_n^*(\theta_n)) = f(\theta_1, ..., \theta_n),\ \forall\, (\theta_1, ..., \theta_n) \in \Theta_1 \times ... \times \Theta_n.$$

It is interesting to define and study the mechanism that induces the agents to report their true type:

Definition 2.6. *A social choice function $f(\cdot)$ is said to be incentive compatible (or truthfully rational) if the direct revelation mechanism $\mathcal{D} = (\Theta_1, ..., \Theta_n, X, f)$ has a pure strategy equilibrium $s^* = (s_1^*, ..., s_n^*)$ in which $s_i^*(\theta_i) = \theta_i$, $\forall\, \theta_i \in \Theta_i$, $\forall\, i \in N$.*

That is, a SCF is incentive compatible if truth telling by each agent leads to an equilibrium by the game induced by D.

Definition 2.7 (Dominant strategy incentive compatibility)**.** *A social choice function $f(\cdot)$ is said to be dominant strategy incentive compatible (DSIC) if the direct revelation mechanism that implements $f(\cdot)$ has a dominant strategy equilibrium s^* in which $s_i^*(\theta_i) = \theta_i\, \forall\, \theta_i \in \Theta_i$, $\forall\, i \in N$.*

That is, truth telling by each agent constitutes a dominant strategy equilibrium in the game induced by the mechanism. A necessary and sufficient condition for a SCF f to be dominant strategy incentive compatible is that, for each agent, truth telling is the best action for an agent no matter what other players report:

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i),\ \forall\, i \in N,\ \forall\, \theta_i, \hat{\theta}_i \in \Theta_i,\ \forall\, \theta_{-i} \in \Theta_{-i}$$

It may seems that the more general definition of indirect revelation mechanism could be more versatile than the incentive compatible direct revelation one. This is actually false, since any general mechanism that implements a social choice function in dominant strategy can always be converted into an incentive compatible one.

Definition 2.8 (Revelation Principle). *If an arbitrary mechanism that implements a social choice function f in dominant strategy exists, then an incentive compatible mechanism that implement f exists as well.*

2.1.2. Other desirable properties

It is usually desirable for a mechanism to also ensure some other properties.

Allocative Efficiency

A SCF is allocative efficient if, for any profile of agents' type, the social choice function result in an outcome which is not Pareto dominated. Therefore, it is not possible for an agent to improve his utility without decreasing the utility of at least one other player.

Definition 2.9. *A SCF $f\colon \Theta_1 \times ... \times \Theta_n \to X$ is said to be allocative efficient if, for no agents' type profile $\theta = (\theta_1, ..., \theta_n)$, exists an outcome $x \in X$ such that $u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i)\, \forall\, i$ and $u_j(x, \theta_j) > u_j(f(\theta), \theta_j)$ for some $j \neq i$.*

Individual Rationality

The individual rationality property ensures that no player can receive a negative utility by taking part to the game induced by the mechanism and reporting his true type.
This property is classified in three categories:

- Ex post: when the type set is given, the utility of each player i is weakly grater than the utility given by the status-quo $\bar{U}_i(\theta_i)$.

$$U_i(f(\theta), \theta_i) \geq \bar{U}_i(\theta_i)\, \forall\, i \in N,\ \forall\, \theta \in \Theta$$

- Ex interim: the utility of each player i is weakly greater than the utility given by the status-quo $\bar{U}_i(\theta_i)$, in expectation with respect to the types of the opponents.

$$\mathbb{E}_{\theta_{-i}}[U_i(f(\theta),\theta_i)] \geq \bar{U}_i(\theta_i)\,\forall\, i \in N,\, \forall\,\theta_i \in \Theta_i$$

- Ex ante: the utility of each player i is weakly greater that the expected utility of the status-quo $\bar{U}_i(\theta_i)$, in expectation with respect to the types of all the agents.

$$\mathbb{E}_{\theta}[U_i(f(\theta),\theta_i)] \geq \mathbb{E}_{\theta}[\bar{U}_i(\theta_i)]\,\forall\, i \in N$$

An immediate observation is that

$$\text{Ex post} \implies \text{Ex interim} \implies \text{Ex ante.}$$

Non-dictatorship

If a SCF is dictatorial, it will always choose an outcome that maximizes the utility of an agents known as dictator. A SCF is said to be non-dictatorial if is not dictatorial.

Definition 2.10. *A SCF $f\colon \Theta_1 \times ... \times \Theta_n \to X$ is said to be dictatorial if for every profile of agents' type $\theta = (\theta_1, ..., \theta_n)$, we have $f(\theta_1, ..., \theta_n) \in \{x \in X : u_d(x, \theta_d) \geq u_d(y, \theta_d)\,\forall\, y \in X\}$, where d is a particular agent call dictator.*

The Gibbard-Satterthwaite impossibility theorem [10] states that, in a very large set of problems, the incentive compatible and non-dictatorial property are mutually exclusive:

Theorem 2.1 (Gibbard-Satterthwaite impossibility result)**.** *Let f be an incentive compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.*

2.1.3. Quasi-linear environments

Quasi-linear environment (QLE) is a special class of environments where the Gibbard-Satterthwaite theorem does not hold. In fact, all the social choice functions in a quasi-linear environment are non-dictatorial [9].
In a QLE, the set of outcomes is defined as a set of vectors $X = \{(k, p_1, ..., p_n)\colon k \in K, p_i \in \mathbb{R}\}$, where K is the set of the possible allocations and $p = (p_1, ..., p_n)$ is the vector of payments. The payments generally refer to money transfer in real world scenarios, and they are meant to be positive if the agent receives the money and negative if the agent

pays the money. The utility functions are defined as

$$U_i(x, \theta_i) = U_i((k, p_1, ..., p_n), \theta_i) = v_i(k, \theta_i) + p_i$$

where $v_i \colon K \times \Theta \to \mathbb{R}$ is called i's valuation function.
We can define two important properties of the social choice functions in quasi-linear environments.

Definition 2.11 (Allocative efficiency). *A SCF $f(\theta) = (k(\theta), p_1(\theta), ..., p_n(\theta))$ is allocative efficient if for each $\theta \in \Theta$, $k(\theta)$ satisfies the following condition*

$$k(\theta) \in \arg\max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i).$$

Informally, a SCF in a quasi-linear environment is allocative efficient if for every agents type profile selects the allocation that maximizes the cumulative value of the players, also known as social welfare.

Definition 2.12 (Budget balance). *A SCF $f(\cdot) = (k(\theta), p_1(\theta), ..., p_n(\theta)))$ is weakly budget balance*[1] *if for each $\theta \in \Theta$, $p_1(\theta), ..., p_n(\theta)$ satisfies the following condition*

$$\sum_{i=1}^{n} p_i(\theta) \geq 0$$

Informally, a SCF is weakly budget balance if the sum of the payments of the agents, and thus the revenue of the auctioneer, is non negative.

2.1.4. Vickrey-Clarke-Groves mechanism

Groves mechanisms constitutes an important set of mechanism in quasi-linear environment. The main property related to them is known as Groves' theorem [12], which confirms that there exists social choice functions which are both allocative efficient and incentive compatible in dominant strategy.

Definition 2.13 (Groves mechanism). *A direct revelation mechanism $\mathcal{D} = (\Theta_1, ..., \Theta_n, X, f)$ where $f(\theta) = (k(\theta), p_1(\theta), ..., p_2(\theta))$ is called a Groves mechanism if the following properties are satisfied:*

[1] The weak budget balance property can be empowered to the strict budget balance property, that apply if under the conditions $\sum_{i=1}^{n} p_i(\theta) = 0$.

- $k(\theta) \in \arg\max_{k \in K} \sum_{i=1}^{n} v_i(k, \theta_i)$
- $p_i(\theta) = h_i(\theta_{-i}) - \sum_{j \in N\{i\}} v_j(k(\theta), \theta_j) \, \forall \, i \in N$

where $h_i \colon \Theta_{-i} \to \mathbb{R}$.

The function h is a degree of freedom of the mechanism, and can be implemented to grant some properties like individual rationality or budget balance. The main property of the Groves mechanisms is stated in the following.

Theorem 2.2 (Groves' theorem)**.** *Any social choice function such that the corresponding direct revelation mechanism is a Groves mechanism is incentive compatible in dominant strategies.*

Furthermore, the Green-Laffont theorem [11] proves that, if the types of the players can be any and an allocation efficient social choice function is used, the corresponding direct-revelation mechanism is DSIC if and only if the payments are those defined by Groves.

A special case of Groves mechanisms is called Vickrey-Clarke-Groves mechanisms:

Definition 2.14. *A Vickrey-Clarke-Groves (VCG) mechanism is a Groves mechanism where the function* $h(\theta_{-i})$ *is defined as*

$$h_i(\theta_{-i}) = \max_{k' \in K_{-i}} \sum_{j \in N \setminus \{i\}} v_j(k', \theta_j).$$

Informally, the function h_i is designed to represent the social welfare of the outcome when the player i does not take part to the game.
VCG mechanism are the most important class of truthful mechanisms and their application in sponsored auction is widely studied. An example of how they work is presented in the following for a single item auction, also known as Vickrey auction.

Example 2.1. *Suppose that three agents, namely* a_1*,* a_2 *and* a_3*, are competing in an auction for a single good. The valuation that the agents give to the good is expressed by their type* θ_i*. In particular, suppose that* $\theta_1 > \theta_2 > \theta_3$*. Suppose that the agents offer, for the good that the auctioneer is selling, their true value* θ_i*. Thus agent 1 wins the good and receives a value of* θ_a*, while agent 2 and agent 3 receive a value of* 0*.*
Recall that, in a VCG mechanism, each agent i *is charged with a payment*

$$p_i(\theta) = \max_{k' \in K_{-i}} \sum_{j \in N\{i\}} v_j(k', \theta_j) - \sum_{j \in N\{i\}} v_j(k(\theta), \theta_j)$$

Let $Y_i = \sum_{j \in N\{i\}} v_j(k(\theta), \theta_j)$ be the sum of the values generated by the outcome of all the agents except i, while $X_i = \max_{k' \in K_{-i}} \sum_{j \in N\{i\}} v_j(k', \theta_j)$ is the sum of the values of all the player except i for the outcome that would be selected if i did not take part to the auction. Let's analyse the payments that the agents owe to the auctioneer.

- *Agent 1 won the good. Since the cumulative value generated by the auction is equal to the value of agent 1, it is $Y_1 = 0$. If he had not taken part to the auction, the winner of the good would be agent 2, with a cumulative value of $X_1 = \theta_2$. Thus, the payment agent 1 is charged is $p_1 = \theta_2$.*
- *Agent 2 did not win the good. The cumulative value for the allocation without agent 2's value is $Y_2 = \theta_1 - 0 = \theta_1$. If he did not take part to the auction, the good would have still be won by agent 1, and thus $X_2 = \theta_1$. Thus, the payment agent 2 is charged is $p_2 = 0$.*
- *Agent 3 did not win the good. Similarly to agent 2, the price he is charged of is $p_3 = 0$.*

VCG mechanisms are the most popular known set of Groves mechanisms because of the many properties that belong to them, other than being truthful in dominant strategy.

Theorem 2.3 (Krishna and Perry's theorem)**.** *Among all the auctions that are allocatively efficient and ex-interim individually rational, the VCG mechanisms maximize the expected payments of the players.*

VCG is guaranteed to be weakly budget balance if the environment exhibits the no single-agent effect.

Definition 2.15 (No single-agent effect)**.** *An environment exhibits the no single-agent effect property if for every player i and every profile of types $\theta = (\theta_1, ..., \theta_n)$ there is an allocation $k' \in K_{-i}$ such that*

$$\sum_{j \in N\{i\}} v_j(k', \theta_j) \geq \sum_{j \in N\{i\}} v_j(k(\theta), \theta_j).$$

Theorem 2.4. *If no single-agent effect hold, then all the VCG payments are non-negative and therefore the VCG mechanism is weakly budget balanced.*

Lastly, if choice-set monotonicity and no negative externality properties hold, then VCG is individually rational in ex-post.

Definition 2.16 (Choice-set monotonicity). *An environment exhibits choice-set monotonicity if for every player i holds $K_{-i} \subseteq K$.*

That is, removing an agent weakly decreases the mechanism's set of possible allocations K.

Definition 2.17 (No negative externality). *An environment exhibit the no-negative externality if*

$$v_i(k(\theta_{-i}), \theta_i) \geq 0 \, \forall \, i \in N, \, \forall \, \theta \in \Theta$$

where $k(\theta_{-i}) = \max_{k' \in K_{-i}} \sum_{j \in N\{i\}} v_j(k', \theta_j)$.

That is, every agents has zero or positive valuation for any allocation that can be made without his partecipation.

Theorem 2.5. *If choice-set monotonicity and no negative externality property hold, the VCG mechanism is ex-post individually rational.*

2.2. Sponsored search auctions

Among the different fields in which mechanism theory can apply, one of great interest are sponsored search auctions (SSA), often referred also as keyword auctions. Web search engines uses SSAs to monetize their services by selling advertising spaces next to their algorithmic content: whenever a user submits a query, he receives back a list of advertising results in addition to the search outcome. These sponsored advertises are links that will redirect the user to the advertiser's website. The ads are displaced in a fixed set of positions, called *slots*, on the web page, and the slots at the top of the page generate more clicks from the users. Thus, advertisers generally prefer higher slots instead of lower ones and are willing to pay more to be assigned to them. We will call the maximum amount that an advertiser declare to be willing to pay *bid*. The way in which the search engine selects which ads to show and their positions in the user client is through an auction mechanism.

The advertisers can decide to compete for search queries related to a set of keywords, each one with a different payment, and set a budget for each of them. A new auction is performed every time a search query is submitted, meaning that the advertisers can dynamically change their bids to adapt to the environment.

Generally the advertisers pay the search engine per-click, meaning that any time the sponsored ad is clicked the advertiser is charged a cost. Other payment policy are possible, like per-impression or per-action, but are less adopted for SSAs.

2.2.1. General model

The general model of a sponsored search auction can be described as follows:

- There is a set $N = \{1, ..., n\}$ of players, representing the advertisers, that take part to the auction game with a single advertisement (or ad). The ad referred to the player i will be denoted as a_i.
- The search engine makes available $M = \{1, ..., m\}$ slots that represents the positions in which the selected ads will be displayed on the user client. Each slot is associated with a prominence $\lambda_j \in [0, 1]$, that represent the probability that the user observe the slot j. We suppose that the slots are disposed such that $\lambda_i \geq \lambda_j \, \forall \, i > j$.
- Each player i has a (per-click) value $v_i \in \mathbb{R}$ that the advertiser will collect every time his ad is clicked. The value is (part of) the type θ_i of the player and is independent from the assigned slot.
- When they take part to the auction, each player i reports to the auctioneer a parameter $b_i \in \mathbb{R}$ called bid, that represent the maximum amount that i is willing to pay for a click on his ad.
- Each ad a_i is associated to a quality $q_i \in [0, 1]$ that represents the advertisement click probability given that the ad has been observed by the user. If the ad a_i is allocated in the slot j, we will refer as click-through rate $CTR_i = \lambda_j q_i$ to the probability of being clicked, given the ad quality q_i and the slot prominence λ_j.
- Each advertiser i that joins the mechanism is charged with a pay-per-click $\pi_i \in \mathbb{R}$, which he has to pay to the auctioneer every time his advertise is clicked.

Every time a search query is generated, the search engine runs an auction: the agents report a bid profile $b = (b_1, ..., b_n)$, and the auction mechanism selects an allocation $x \in X$ and a price profile $\pi = (\pi_1, ..., \pi_n)$. Since a new auction is performed for every keyword research among the same agents, the advertisers have the possibility to dynamically modify their bids in order to maximize their utility.

The way in which the auction mechanism is implemented is usually to solve a maximization problem, often referring to social welfare or auctioneer revenue.

2.2.2. VCG

The already discussed VCG mechanism finds one of its application in sponsored auctions and several companies, like Facebook, use this design to implement their advertising

management.
Recalling that VCG mechanism are truthful, in VCG-based SSAs the advertisers can always maximize their utility by bidding their true types.
An example of SSA based on VCG mechanism is provided in the following:

Example 2.2 (VCG based SSA). *A set $N = \{1, 2, 3\}$ of advertisers competes in an auction with $M = 2$ slots whose prominences are $\lambda_1 = 1$, $\lambda_2 = 0.5$. The advertisers values are, respectively, $v_1 = 10$, $v_2 = 7$ and $v_3 = 4$.*
Accordingly to VCG, the ads will be allocated by bid: a_1 will be allocated to the first slot, a_2 to the second one and a_3 will not be allocated. The payment each advertiser is charged are $\pi_1 = 7.5$, $\pi_2 = 2$ and $\pi_3 = 0$.

Even if VCG mechanism can benefit of many desirable properties, most of the companies do not adopt it for to manage sponsored search auctions. The reasons behind that are several, but they can be synthesized in two main aspects:

- Computation complexity: SSA requires to perform a real time computation to determine both the allocation and the payments of the advertisers. VCG requires to compute the harm that each agent cause to the other players, and this can become very expensive as the number of advertisers grows, especially if we consider scenarios in which thousands of requests are generated every second.

- Revenue maximization: VCG does not guarantee that the auctioneer gain the maximum revenue with respect to other mechanisms.

2.2.3. Generalized second price

The most widely used mechanism by search engine to allocate the advertisers are based on Generalize Second Price (GSP). Let assume in our analysis, without loss of generality, that the advertisers are named in such a way that $b_i \geq b_j \, \forall \, i, j \in N$. Like in VCG auctions, in standard GSP auctions the mechanism allocates the slots in decreasing order of payers' bid. However, in GSP-based mechanism the price p_i of the player i allocated to the slot j is equal to the bid of the player $i+1$. So bidder i's total payment p_i is equal to $\lambda_j b_{i+1}$ for $i \in \{1, ..., \min\{N, K\}\}$, and his payoff is equal to $\lambda_j(v_i - b_{i+1})$. If the number of advertisers is not higher than the number of slots $(N \leq M)$ the last bidder's payment p_n is equal to 0. Every non allocated player gets a payment $p_i = 0$.

GSP is similar to VCG, since they share the same allocating rule and in both cases the payment of player i is function of the bids b_{-i} but not of the bid b_i. For the bidder who gets allocated in the last slot the payment is the same under GSP and VCG: 0 if $N \geq M$,

$\lambda_m b_{n+1}$ otherwise. As a particular case, in a single slot auction the outcome under GSP mechanism is the same as under a VCG one.
However, GSP lacks some of the VCG's desirable properties.

GSP properties

The main property that GSP lacks with respect to VCG is having in truth telling a dominant strategy.

Proposition 2.1. *Truth telling is not a dominant strategy under GSP.*

The following example provides an instance in which truth telling is not an optimal strategy.

Example 2.3. *There are three advertisers associated with values $v_1 = 1$, $v_2 = 5$, $v_3 = 10$, who compete for two slots with prominence $\lambda_1 = 1$ and $\lambda_2 = 0.99$. If the players bid truthfully, agent 3 is allocated to the first slot with a payment $\pi_3 = 5$ and an expected utility $u_3 = 1 \cdot (10 - 5) = 5$. However, if he untruthfully declares a bid $b_3 = 4.9$ he will be allocated to the second slot, with a payment $\pi_3 = 1$ and an expected utility $u_3 = 0.99 \cdot (10 - 1) = 8.99 \geq 5$.*

The following result compare the revenue of the auctioneer between GSP and VCG mechanisms.

Proposition 2.2. *If all advertisers were to bid the same amount under the two mechanisms, then each advertiser's payment would be at least as large under GSP as under VCG.*

Being that truth telling is not a dominant strategy for GSP, it is interesting to study the bidding strategies that the agents can adopt and the set of equilibrium that those strategies support. Edelman et al. [8] and Varian [18] studied a subset of the GSP auction equilibrium called *locally envy-free* equilibrium[2].

Definition 2.18. *An equilibrium of the simultaneous-move game induced by GSP is locally envy-free if a player cannot improve his payoff by exchanging bids with the player ranked one position above him. More formally, in a locally envy-free equilibrium, for any $i \leq \min\{N+1, M\}$, it is $\lambda_i v_{f(i)} - \pi^{(i)} \geq \lambda_{i-1} v_i - \pi^{(i-1)}$, where $\pi^{(i)}$ is the price for being allocated to slot i.*

[2]The two studies has been done independently. Varian call the same set of equilibrium *symmetric Nash equilibrium.*

In their analysis, Edelman et al. [8] show that any locally envy-free equilibrium is a stable assignment of the game Γ induced by GSP, and if the number of advertisers is higher than the number of slots, then any stable assignment is an outcome of a locally envy-free equilibrium of Γ.

Futhermore, Edelman et al. [8] studied a special case of local envy-free equilibrium, that results from the strategy profile B^* where, for every advertiser i such that $f(i) \in \{2, ..., \min\{N+1, M\}\}$, the bid b_i is equal to $\pi^{VCG,(j-1)}/\lambda_{j-1}$, where $\pi^{VCG,(j-1)}$ is the payment of the advertiser $j-1$ in the dominant strategy equilibrium of VCG when all the advertisers bid truthfully [3].

Theorem 2.6. *The strategy profile B^* is a locally envy-free equilibrium of the game Γ induced by the GSP auction. In this equilibrium, each advertiser's position and payment are equal to those in the dominant-strategy equilibrium of the game induced by VCG. In any other locally-envy free equilibrium of the game Γ, the total revenue of the seller is at least as high as in B^*.*

Efficiency

Since truth telling is not a dominant strategy under GSP, it is of interest to study the efficiency of the equilibrium of the game induced by the GSP mechanism. The efficiency is evaluated throught two parameters, called *price of stability* and *price of anarchy*.

Definition 2.19 (Social best solution)**.** *The social best solution with respect to an evaluation function g is an outcome $x \in X$ of the game such that the action profile a^{SB} associated to x is*

$$a^{SB} \in \arg\max_{a \in A} \sum_{i \in N} g_i(a)$$

Definition 2.20 (Social best Nash equilibrium)**.** *The social best Nash equilibrium with respect to an evaluation function g is an outcome $x \in X$ of the game such that the action profile a^{SBN} associated to x is*

$$a^{SBN} \in \arg\max_{a \in A : a \text{ is a Nash equilibrium}} \sum_{i \in N} g_i(a)$$

Definition 2.21 (Social worst Nash equilibrium)**.** *The social worst Nash equilibrium with respect to an evaluation function g is an outcome $x \in X$ of the game such that the action*

[3] The bid b_i^* such that $f(i) = 1$ is arbitrarily set to v_i.

profile a^{SWN} *associated to* x *is*

$$a^{SWN} \in \underset{a \in A: a \text{ is a Nash equilibrium}}{\arg\min} \sum_{i \in N} g_i(a)$$

Definition 2.22 (Price of stability)**.** *The price of stability (PoS) of a game with respect of a evaluation function* $g(a)$ *is*

$$PoS = \frac{\sum_{i \in N} g_i(a^{SBN})}{\sum_{i \in N} g_i(a^{SB})}$$

Definition 2.23 (Price of anarchy)**.** *The price of anarchy (PoA) of a game with respect of a evaluation function* $g(a)$ *is*

$$PoS = \frac{\sum_{i \in N} g_i(a^{SWN})}{\sum_{i \in N} g_i(a^{SB})}$$

Leme et Tardos [15] studied the efficiency of the generalized second price auction. Their results in terms of price of anarchy of the social welfare are summarized in Table 2.1 in the case of pure strategy equilibria, mixed strategy equilibria and Bayes-Nash equilibria.

	Pure strategy		Mixed strategy	Bayes-Nash
	2 slots	>2 slots		
PoA	1.25	$\frac{1+\sqrt{5}}{2} \approx 1.628$	4	8

Table 2.1: GSP auction efficiency in terms of social welfare

2.3. Bidding strategies for GSP auctions

Being the GSP mechanism not truthful, the advertisers need to elaborate a bidding strategy. As observed by Edelman et al. [8] this can produce instability in the case of repeated auctions and a bidding war among the players. This is caused by the fact that in search auctions multiple slots are available for the advertisers, and the position of a given sponsored link will affect its click probability and consequently the advertiser utility.

Cary et al. [3] and Bu et al. [1] independently studied whether it is possible for the players to implement a bidding strategy that guarantees some desirable properties. They worked on the assumptions that exactly one randomly chosen advertiser gets to update his bid in each period, and that he knows the current bids of all the others advertisers when he updates his bid. The main result of their study is the definition of a myopic best response strategy called balance bidding (Cary et al. [3])[4]. A myopic best response strategy aims to maximize the agent's utility in the next round of a repeated auction assuming that the others player bids b_{-i} stay fixed, but in the context of GSP sponsored search auctions it has some flexibility. Suppose that the player i's best response is to target slot j^*: then his bid b_i can be in the range $(\pi_i(j^*), \pi_i(j^*-1))$, where $p_i(j)$ is the payment of player i allocated to slot j given b_{-i}.

Definition 2.24 (Balance Bidding). *The Balance Bidding (BB) strategy is the strategy for a player i that, given b_{-i}*

- *targets the slot j^* which maximizes his utility, that is*

$$j^* = \arg\max_j \{\lambda_j(v_i - \pi_i(j))\};$$

- *chooses his bid b'_i for the next round so as to satisfy the following equation:*

$$\lambda_{j^*}(v_i - \pi_i(j^*)) = \lambda_{j^*-1}(v_i - b'_i).$$

If j^* is the first slot, we (arbitrarily) choose $b'_i = \frac{(v_i+\pi_i(1))}{2}$.

The intuition behind the BB strategy is that a player wants to bid high enough to win his preferred slot j^*, but not so high that he would regret it if one of his competitors undercut him. Notice that if an advertiser targets the first slot his bid can be arbitrarily chosen in the range $(\pi_i(1), v_i)$, since it does not affect any agent's cost[5].

Notice that for the agents that can not be assigned a slot for any bid $b_i \leq v_i$ the best bidding response is to bid $b_i = v_i$. This is due to a player wanting to maximize the possibility to be allocated, since GSP is individually rational if agents do not overbid.

Suppose, without loss of generality, that agent i is assigned to the position j. The main property of the BB strategy is illustrated in the following proposition:

Proposition 2.3. *If all players follow the BB strategy in an auction with all distinct λ's and asynchronous bidding, then the system converges to its unique fixed point. At this fixed point the revenue of the auctioneer (and the payment of each player) is equal to that of the VCG equilibrium. The bid b_i of the player i allocated to the slot j at the equilibrium*

[4]Bu et al. [1] refer to the sane bidding strategy as *forward looking.*

[5]Bo et al. [1] studied the same strategy where an agent who target the first slot bid his value.

satisfies the following equations:

$$BB(\boldsymbol{b}_{-i}, v_i) : b_i^* = \begin{cases} 2b_{i+1}^* & \textit{if } j = 1, \\ \frac{\lambda_j}{\lambda_{j-1}} b_{i+1}^* + (1 - \frac{\lambda_j}{\lambda_{j-1}}) v_i & \textit{if } 2 \leq j \leq M, \\ v_i & \textit{if } M < j \leq N. \end{cases}$$

The value of this result is double. On the one hand, it states that if the players adopt the BB strategy in an asynchronous bidding setting the repeated auction will converge to a pure Nash Equilibrium of the induced game Γ. On the other hand, the equilibrium is unique, and in such equilibrium the payments are the same that the players would be charged under the VCG mechanism. This means that it is a locally envy-free equilibrium, and in particular it is the equilibrium related to the strategy B^* described by Edelman et al. [8].

3 | Sponsored search auctions with price displaying

Se far we have discussed sponsored search auctions in a classical sense, where the advertisers sponsor some links with the purpose of redirecting the users towards their website. Whenever an advertiser wants to sell some goods, however, they will associate to their ads a selling price. This is the case, for example, of companies like Amazon and Booking, but the search engine themselves dedicate some of their sponsored ads to e-commerce (Figure 3.1). The web pages display the banners advertising similar goods, as well as their costs. Such setting is similar to that of standard auctions, but it introduces a new externality that affects the users behaviour, as well as the properties of the auction. The users prefer to spend less money, and this prompt them to compare the ads' prices in order to decide which ad to click, thus affecting the click probability for each ad. We will refer to this new class of sponsored search actions as sponsored search action with price displaying (SSAPD).

The problem of SSAPD has already been studied by Castiglioni et al. [5], and their work will constitute the base for the study described in Chapter 4. In Section 3.1 will be formalized the model of SSAPD, in Section 3.2 will be described a direct and an indirect mechanism for SSAPD, while in Section 3.3 will be analyzed the known properties.

3.1. Formal model

The formal model of the SSAPD auctions shares the same structure of the standard sponsored auctions, but introduces some new elements.

- There is a set $N = \{1, ..., n\}$ of agents, representing the advertisers and the sellers, that take part to the auction game to sell a single good with a single advertise. The ad referred to the player i will be denoted as a_i.
- For every i, it is denoted as $c_i \in \mathbb{R}^+$ the production cost and as $p_i \in \mathbb{R}$ the selling price of agent i. We will also denote as $\alpha_i \in [0, 1]$ the probability with which a user

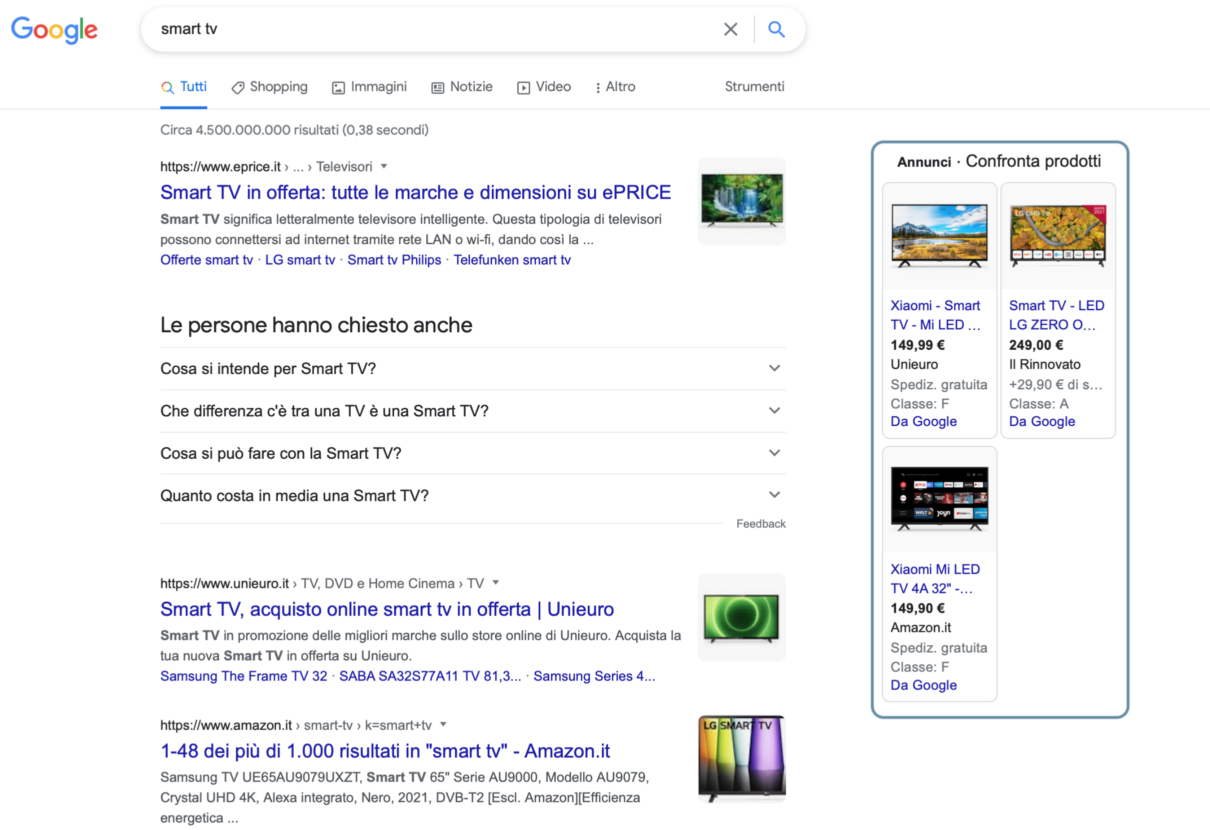

Figure 3.1: Example of Internet advertising with price displaying
Google and the Google logo are trademarks of Google LLC.

that has clicked on the advertise a_i will buy the sponsored item. The pair (α_i, c_i) is a private information of agent i, and will be his type $\theta_i \in \Theta_i$. We will refer to $\mu_i = \alpha_i(p_i - c_i)$ as the gain of player i.

- The search engine makes available $M = \{1, ..., m\}$ slots, which represents the positions in which the ads will be displayed on the user client. Each slot is associated with a prominence $\lambda_j \in [0, 1]$ that represent the probability with which the user observe the slot j. It will be assumed a cascade model where $\lambda_i \geq \lambda_j \,\forall\, i > j$. The function $f : N \to M \cup \{\perp\}$ assigns to each slot at most one ad. If $f(i) = \perp$ the ad is not displayed, and $\lambda_\perp = 0$.

- Each displayed ad a_i is associated to a quality $q_i \in [0, 1]$ that represents the advertisement's click probability given that the ad has been observed by the user. We consider the quality of an ad independent to the advertiser, and so a function of the price profile $\boldsymbol{p} = (p_1, ..., p_n)$ only. It will be also assumed that $q_i : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, 1]$, where $q_i(p_i, p_{min})$ denotes the player i's quality when his price is p_i and $p_{min} = \min_{k:f(k)\neq\perp} p_k$ is the minimum price among the displayed ads. Moreover, given p_{min},

$q_i(p_i, p_{min})$ is non strictly monotonically decreasing in p_i and, given p_i, is non strictly monotonically increasing in p_{min}. This assumption is due to the fact that users will compare the advertisement based on their prices, and their interest in a specific ad will decrease with the price difference with respect to the others.

- Each advertiser i that joins to the mechanism is charged with a per-click payment $\pi_i \in \mathbb{R}$, that he has to pay to the auctioneer every time his advertise is clicked.

We will call bid b_i the declared gain of player i, and we will say that an agent does not overbid if $b_i \leq \mu_i$. We will refer as declared (expected) value to the value $\hat{v}_i(f, \boldsymbol{p}, b_i) = \lambda_{f(i)} q_i(p_i, p_{min}) b_i$ computed with the declared gain b_i in the allocation f, while the (true expected) value is $v(f, \boldsymbol{p}, b_i) = \lambda_{f(i)} q_i(p_i, p_{min}) \alpha_i (p_i - c_i)$. The expectation is referred to both the click probability $\lambda_{f(i)} q_i(p_i, p_{min})$ and the buy rate α_i. The social welfare of the allocation f with respect to the declared gains $\boldsymbol{b}$ is $s\hat{w}(f, \boldsymbol{p}, \boldsymbol{b}) = \sum_{i \in N} \hat{v}_i(f, \boldsymbol{p}, b_i)$, while the true social welfare is $sw(f, \boldsymbol{p}, \boldsymbol{b}) = \sum_{i \in N} v_i(f, \boldsymbol{p}, b_i)$. The auctioneer collects a revenue from the players $rev = \sum_{i \in N} \pi_i$.

The following example illustrate the main components of the described model.

Example 3.1. *Consider an auction where 3 agents compete for 2 slots with prominence $\lambda_1 = 1$ and $\lambda_2 = 0.5$ respectively. The advertisers are selling their goods at the displayed price of $p_1 = 100$, $p_2 = 80$ and $p_3 = 50$. For each agent the buy per click rate is $\alpha_i = 1$ and there are no production costs ($c_i = 0$), thus the gain μ_i of each agent is equal to his selling price p_i. Suppose that the quality of the ads is given by the function*

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \textit{if } p = p_{min} \\ 0.2 & \textit{otherwise} \end{cases}$$

A mechanism M allocate the agent 1 to the first slot and the agent 2 to the second slot, while agent 3 is not allocated.
In this scenario, the minimum displayed price is $p_{min} = \min\{p_1, p_2\} = 80$. In this allocation, the value of the agents is $v_1 = \lambda_1 q_1(p_1, p_{min}) \mu_1 = 1 \cdot 0.2 \cdot 100 = 20$, $v_2 = 40$ and $v_3 = 0$ since ad_3 is not allocated.

3.2. Mechanisms

Castiglioni et al. [5] studied the properties of one direct and two indirect revelation mechanisms for the SSAPD.

3.2.1. Direct revelation mechanism

Consider the following direct revelation mechanism $\mathcal{M}_D^{VCG}$. Each agent i report to the auctioneer his (not necessarily true) type $\theta_i = (\alpha_i', c_i') \in \Theta_i$. The mechanism compute the declared gain as $b_i = \alpha_i'(p_i - c_i')$ for every selling price p_i. Then it computes the assignment f^* and the selling prices p^* that maximizes the social welfare with respect to the declared gains. Lastly the mechanism assigns to each allocated advertiser a cost

$$\pi_i = \max_{f,\boldsymbol{p}} \sum_{i \neq j: f(i) \in M} \hat{v}_j(f, p, b_j) - \hat{v}_j(f^*, p^*, b_j).$$

This VCG based mechanism is truthful, individually rational, weakly budget balanced and it maximizes the social welfare. However, in a real word scenario the advertisers will not let the auctioneer select the selling prices for them as it happen in the caso of direct revelation mechanism, but they will most likely report a selling price to propose to the customers. Thus an indirect revelation mechanism is more suitable for a real word application.

3.2.2. Indirect revelation mechanism

Let's now introduce two indirect revelation mechanisms $\mathcal{M}_I^{VCG}$ and $\mathcal{M}_I^{GSP}$, which are based respectively on VCG and GSP. The two mechanisms share the same structure and allocation function, but differ in the way they assign the payments to the advertisers. Each agent i reports to the auctioneer two parameters (p_i, b_i), where p_i is his selling price that the advertiser wants to be displayed and b_i is the declared gain for a click on an ad with price p_i. Both the mechanisms allocate the advertisers by computing an assignment f^* that maximize the social welfare

$$\hat{sw}(f^*, \boldsymbol{p}, \boldsymbol{b}) = \max_f \hat{sw}(f, \boldsymbol{p}, \boldsymbol{b}).$$

Notice that the allocation maximizing the social welfare may not assign an agent to each slot even in the case that $n \geq M$. For example, consider a two slots auction ($\lambda_1 = \lambda_2 = 1$) for two advertisers whose selling prices and bids are $p_1 = 10$, $b_1 = 8$, $p_2 = 2$ and $b_2 = 1$. Assume that the qualities q_1 and q_2 are determined by the function $q_i(p_i, p_{min}) = 1$ if $p = p_{min}$, 0 otherwise. The allocation that maximizes the social welfare is such $f(1) = 1$ and $f(2) = \perp$.

Then $\mathcal{M}_I^{VCG}$ assigns each advertiser i a per-click VCG payment of

$$\pi_i = \max_f \sum_{j \neq i: f(j) \in M} \hat{v}_j(f, \boldsymbol{p}, b_j) - \hat{v}_j(f^*, \boldsymbol{p}, b_j).$$

Consider now GSP based mechanism. Suppose that the first $k \leq M$ slots are assigned. Notice that, since $\lambda_i < \lambda_j \, \forall \, i < j$, an allocation f^* that assigns an agent to $k \leq M$ slots to maximize the social welfare will allocate the slots $\{1, ..., k\}$. $\mathcal{M}_I^{GSP}$ assigns to each agent i such that $f^*(i) < k$ a payment

$$\pi_i = \lambda_{f^*(i)} q_j(p_j, p_{min}) b_j$$

where j is such that $f^*(j) = f^*(i) + 1$. For the agent such that $f^*(i) = k$ there are two possible payments. If all the unassigned advertisers have a lower selling price than the price p_{min} then $\pi_i = 0$, otherwise

$$\pi_i = \lambda_{f^*(i)} \max_{j: p_j \geq p_{min} \,\&\, f^*(j) = \perp} \{q_j(p_j, p_{min}) b_j\}.$$

The payment is 0 for each advertiser such that $f^*(i) = \perp$.

3.3. Properties

Both $\mathcal{M}_I^{VCG}$ and $\mathcal{M}_I^{GSP}$ are weakly budget balance and individually rational. However, not all the selling prices and bids profiles that constitute an equilibrium are such that the true social welfare is maximized. The following section will aim to summarize the known results studied by Castiglioni et al. [5].

Computational complexity

The problem of computing the allocation that maximize the social welfare is generally hard if externalities are introduced. The main result regarding computational complexity for SSAPD auctions can be summarized as follow:

Proposition 3.1. *For both the direct and indirect revelation mechanism, the problem of allocating the advertisers to the slots to maximize the social welfare can be solved in polynomial time.*

Performance

A reasonable question concerns the measure in which the Nash equilibria of the game induced by the SSAPD auction are inefficient. The known results in terms of price of stability and price of anarchy are reported in the following, for both the VCG and GSP indirect mechanisms. The results will refer to non overbidding agents only, since when players overbid the inefficiency can be arbitrarily large even with a single slot.

The efficiency of a mechanism can be related to both the advertisers social welfare or the auctioneer revenue. Considering the social welfare, both VCG and GSP have a price of anarchy equal to 1 in case of a single slot. When $m \geq 2$ slots are available, the VCG still presents a price of stability equal to one, while the price of anarchy is exactly m. Instead, GSP performs worse than VCG, with a price of stability that is at least 2 even if the agents do not overbid, and a price of anarchy lower bounded by m.
Regarding the revenue of the auctioneer, the price of stability goes to infinity for any number of slots $m \geq 2$ for both VCG and GSP mechanisms. In the case of VCG mechanisms, however, the price of stability is equal to 1 in the case of a single slot. This is not true for GSP based mechanisms[1], for which the price of stability goes to infinity even with $m = 1$. The results are summarized in the Table 3.1.

	1 slot			m $\geq$ 2 slots		
	Social Welfare		Revenue	Social Welfare		Revenue
	PoS	PoA	PoS	PoS	PoA	PoS
$\mathcal{M}_I^{VCG}$	1	1	1	1	m	∞
$\mathcal{M}_I^{GSP}$	1	1	∞	≥ 2	$\geq$ m	∞

Table 3.1: SSAPD lower and upper bounds of PoS and PoA for non overbidding agents.

[1]The result for the PoS of the revenue for GSP mechanism with a single slot is referred to GSP payments that guarantee individual rationality.

4 | Convergence analysis for SSAPD

Every time a user submits a query to a search engine, a new sponsored search auction is performed. The advertisers can report an update of their bids to the auctioneer anytime, reacting to other agents' actions to maximize their utility. However, since the advertisers are in competition, some of them may target the same slot. In Section 2.3 we have seen that it is possible for the agents to adopt a strategy that guarantee convergence in GSP based sponsored search auctions. We intend to investigate whether the same bidding strategy could lead to some convergence results in the case of SSAPD introduced in Chapter 3.

In principle, an advertiser could also submit to the auctioneer an update of the selling price anytime. Even if this is true for the proposed model, in many practical scenarios this change is at most abrupt, since the selling price is selected a priori by the advertiser. Thus, we will consider in our analysis the selling prices submitted to the auctioneer as fixed over time, and we will investigate whether the agents can elaborate a myopic best response bidding strategy that converges to an equilibrium.

In Section 4.1 we will extend the notion of balance bidding strategy studied by Cary et al. [3] and Bu et al. [1] in order to consider the externalities introduced by the price displaying and we will show that it is not guaranteed to converge for the GSP mechanism proposed in Chapter 3. In Section 4.2 we will propose a new indirect auction mechanism based on GSP that is a generalization of $\mathcal{M}_I^{GSP}$. In Section 4.3 we will study the properties of the new mechanism, focusing on convergence and efficiency.

4.1. Balance bidding in SSAPD

We want to investigate whether the BB strategy maintains its convergence property in a SSAPD scenario that implements the mechanism $\mathcal{M}_I^{GSP}$ as described in Section 3.2.2. The displaying of the selling price introduces an externality that affects the way ads are allocated and costs are assigned, so the BB strategy needs to be redefined to adapt to the new environment. We will still consider rational, intelligent and myopic advertisers.

According to the definition proposed by Cary et al. [3], when an agent implement the BB strategy he myopically targets the slot j^* that maximizes his utility and, as a tie breaking rule, selects the maximum bid that grants him not to regret it if he is undercut by a competitor. In the context of SSAPD this means that the agent i selects his bid $b_i' \in (\pi_i(j^*), \pi_i(j^*-1))$ given $\boldsymbol{b}_{-i}$ and $\boldsymbol{p}$ in order to verify:

$$\lambda_{j^*} q_i(p_i, p_{min})(\mu_i - \pi_i(j^*)) = \lambda_{j^*-1} q_i(p_i, p_{min})(\mu_i - \tilde{\pi}_i(j^*-1))$$

where $\pi_i(j^*)$ is the payment assigned to i if he is allocated to the slot j^* and $\tilde{\pi}_i(j^*-1)$ is the worst case payment for advertiser i if he is allocated to the slot j^*-1.

Notice that by imposing $q(p_i, p_{min}) = 1 \, \forall \, p_i, p_{min}$ we obtain the notion of BB described in Secion 2.3, as we reduce the problem to a classic SSA.

Definition 4.1 (Balance Bidding for Price Displaying). *The Balance Bidding for Price Dysplaying (BBPD) strategy is the strategy for player i that, given b_{-i} and $p = (p_1, ..., p_n)$*

- *targets the slot j_i^* which maximizes his utility, that is*

$$j_i^* = \arg\max_s \{\lambda_s q_i(p_i, p_{min})(\mu_i - \pi_i(s))\}$$

- *chooses his bid b_i' for the next round so as to satisfy the following equation:*

$$\lambda_{j^*} q_i(p_i, p_{min})(\mu_i - \pi_i(j^*)) = \lambda_{j^*-1} q_i(p_i, p_{min})(\mu_i - b_i')$$

If j_i^* is the first slot, we (arbitrarily) choose $b' = \mu_i$.

Assuming, without loss of generality, that agents are assigned such that $f(i) < f(i+1) \, \forall \, i \in N$, the definition of balance bidding strategy for price displaying can be summarized in the following bidding strategy, where j^* is the target slot of agent i:

$$b_i'(\boldsymbol{b}_{-i}, \boldsymbol{p}) = \begin{cases} (1-\gamma_{j^*})\mu_i + \gamma_{j^*}\pi_i(j^*) & \text{if } 2 \leq j^* \leq M \\ v_i & \text{if } j^* = 1 \text{ or } j^* > M \end{cases} \tag{4.1}$$

where $\gamma_j = \frac{\lambda_j}{\lambda_{j-1}}$.

Unfortunately, BBPD turns out not to be a feasible strategy for a SSAPD implementing the indirect revelation mechanism based on GSP $\mathcal{M}_I^{GSP}$, in the sense that when an agent target a slot j^* and submit a bid b' accordingly to BBPD, he could be allocated to a slot $j \neq j^*$.

Proposition 4.1. *Balance Bid for Price Displaying is not a feasible strategy for SSAPD that implements the mechanism* ${}^{GSP}_{I}$.

Proof. Consider a SSAPD with $N = \{1, 2, 3\}$ agents and $M = \{1, 2\}$ slots with prominence $\lambda_1 = 1$, $\lambda_2 = 0.5$. For all the agents $\alpha_i = 1$ and $c_i = 0$, while the selling prices are $p_1 = 50$, $p_2 = 51$ and $p_3 = 60$. The quality function is $q(p_i, p_{min}) = 1$ if $p_i = p_{min}$, 0.5 otherwise. Suppose that player 3 want to update his bid, and currently $b_1 = 50$ and $b_2 = 51$. In this scenario, by bidding $b_3 \in (0, 47)$ the allocation selected by the mechanism would be $(1, 2)$, while by bidding $b_3 \in (47, 60)$ it would be $(2, 3)$. Thus, player 3 target the slot $j = 2$. But accordingly to BBPD strategy, his bid would be $b_3 = (1 - \frac{0.5}{1})60 + 0 = 30$, that is not a feasible bid for player 3 to be allocated to the slot $j = 2$. $\square$

We propose an extended version of the BBPD strategy that we will call *Extended Balance Bidding for Price Displaying.*

Definition 4.2 (Extended Balance Bidding for Price Displaying). *The Extended Balance Bidding for Price Displaying (EBBPD) is the strategy for player i that, given $\boldsymbol{b}_{-i}$ and $\boldsymbol{p} = (p_1, ..., p_n)$*

- *targets the slot j_i^* which maximizes his utility, that is*

$$j_i^* = \arg\max_s \{\lambda_s q_i(p_i, p_{min})(\mu_i - \pi_i(s))\}$$

- *chooses his bid b_i' for the next round so as:*

$$b_i' = \max\{BBPD(\boldsymbol{b}_{-i}, \boldsymbol{p}), \bar{b}\}$$

where $\bar{b}$ is such that $\bar{b} = \min b_i : f^*(i) = j^*$.

The rationale behind the EBBPD is the following: when an agent targets the slot that grants him the highest utility, he select his bid such that he will not be affected by being undertaken by another agent. If that bid is not feasible, he bids such to minimize the utility loss of being undertaken.

It can be noticed that EBBPD is a feasible strategy for an agent competing in an auction game implementing $\mathcal{M}_I^{GSP}$. In fact, when agent i targets a slot j, his bid will be by definition at least as high as to guarantee to be assigned to the slot. At the same time he will not bid over $\frac{q_{i-1}(p_{i-1},p_{min})b_{i-1}}{q_i(p_i,p_{min})}$, or he would prefer to be allocated to the slot $j-1$. Unfortunately, a SSAPD based on $\mathcal{M}_I^{GSP}$ is not guaranteed to converge when the advertisers choose their bids accordingly with EBBPD.

Proposition 4.2. *The EBBPD is a feasible bidding strategy for the mechanism $\mathcal{M}_I^{GSP}$. However, when all the agents bid accordingly to EBBPD, the mechanism is not guaranteed to reach an equilibrium.*

An example of non-convergence is showed in Appendix A.

4.2. Auction Mechanism Proposal

We will now introduce a new indirect revelation mechanism based on GSP called $\mathcal{M}_{ord}^{GSP}(p^*)$, where $p^* \in \boldsymbol{p} = (p_1, ..., p_N)$ is a parameter called *cut price*. The mechanism works as follows:

- Each advertiser i reports to the auctioneer his selling price p_i and his declared gain b_i.
- The mechanism discards all the advertisers which have a selling price $p_i < p^*$. Notice that by selecting $p^* = \min_{i \in N} p_i$ no advertiser is discarded. The set of remaining agents is denoted as N', with $|N'| \leq |N|$.
- The ads are sorted by $q_i(p_i, p^*)b_i$ [1]. We will refer to such sorting as σ, and as $\sigma(i)$ to the position of the agent i with respect to the ordering σ.
- Select the set of advertisers $N_\sigma = \{a_i : \sigma(i) \leq \min\{M, N'\}\}$. These are the agents that will be allocated. The minimum selling price displayed will be $p_{min} = \min_{i \in N_\sigma}$.
- Assign the selected ads in N_σ according with an allocation function

  $$f^* = \max_{f : f(i) \neq \perp \, \forall i \in N_\sigma} \sum_i \lambda_{f(i)} q_i(p_i, p_{min}) b_i.$$

  For each agent i in $N' \setminus N_\sigma$ we define $f^*(i) = \perp$. Informally, f^* is the allocation that maximize the social welfare among the allocations that assign all and only the agents in N_σ.
- Suppose that $k = \min\{N', M\}$ slots have been allocated. The per-click payment of each agent i such that $f^*(i) < k$ is $\pi_i = \frac{q_j(p_j,p_{min})}{q_i(p_i,p_{min})} b_j$, where j is such

[1] We extend the definition of the quality of player i from a function of the minimum displayed price $q_i(p_i, p_{min})$ to a function of two prices p_1 and p_2, where $p_1 \geq p_2$. The distinction between the true quality of an ad and its extended meaning should be clear from the context.

$f^*(j) = f^*(i) + 1$. For the agent such that $f^*(i) = k$ the payment is $\pi_i = \max_{j \in N' : f^*(j) = \perp} \{q_j(p_j, p^*) b_j\} / q_i(p_i, p_{min})$.

For simplicity purposes, we will often refer to the quality function $q_i(p_i, p_{min})$ as q_i and to the extended definition of the quality $q_i(p_i, p^*)$ as q_i^*. We will also refer to the social welfare $sw(f, \boldsymbol{p}, \boldsymbol{b})$ as sw_f or sw_A, to represent the social welfare associated respectively to the allocation function f or the allocation A.

A graphical representation of the mechanism working process is showed in Figure 4.1. In the first step, two of the five ads taking part to the auction are discarded since their selling price is lower than the cut price p^*. Then, the remaining advertisers are ordered accordingly to the ordering function σ. Lastly, the top two advertisers accordingly to σ are assigned to the slots by the allocation function f^*.

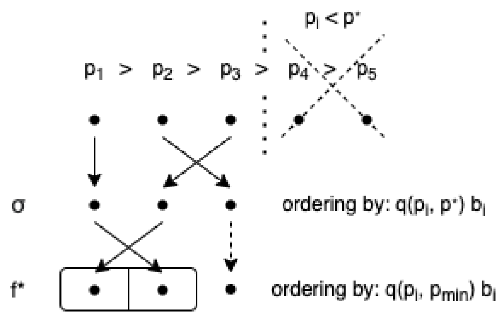

Figure 4.1: Graphical example of $\mathcal{M}_{ord}^{GSP}$ mechanism workflow in a 5 players and 2 slots game.

We also propose a practical example of the mechanism $\mathcal{M}_{ord}^{GSP}$.

Example 4.1. *Consider a set of 4 agents $N = \{1, 2, 3, 4\}$ competing in a SSAPD for $M = 2$ slots, associated to a prominence $\lambda_1 = 1$ and $\lambda_2 = 0.5$. The players are such that:*

- $\alpha_i = 1,\ c_i = 0 \ \ \forall i \in \{1, 2, 3, 4\}$
- $p_1 = b_1 = 10$
- $p_2 = b_2 = 8$
- $p_3 = b_3 = 1$
- $p_4 = b_4 = 0.5$

The quality function is defined as follows:

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \text{if } p_i = p_{min} \\ 0.5 & \text{otherwhise} \end{cases}$$

The cut price is $p^* = 1$. Thus the agent 4 whose selling price is $p_4 = 0.5$ is discarded and will not take part to the allocation process.

The remaining agents are sorted by $q_i(p_i, p^*)b_i$: it results that $ad_1 < ad_2 < ad_3$. Being $M = 2$, the ads in $N_\sigma = \{1, 2\}$ are allocated and $p_{min} = 8$. The agents are assigned such that $f^*(2) = 1$, $f^*(1) = 2$, $f^*(3) = \perp$. The auction payments are $\pi_2 = \frac{q_1(p_1,p_{min})}{q_2(p_2,p_{min})}b_1 = 5$, $\pi_1 = \frac{q_1(p_3,p^*)}{q_2(p_2,p_{min})}b_3 = 1$, $\pi_3 = 0$.

In the following we will state some general observations about the mechanism $\mathcal{M}_{ord}^{GSP}(p^*)$.

Lemma 4.1. *The allocation function f^* implemented by $\mathcal{M}_{ord}^{GSP}$ assigns the ads to the slots such that $q_i(p_i, p_{min})b_i \geq q_j(p_j, p_{min})b_j$, $\forall\, i, j$ such that $f^*(i) \neq \perp, f^*(j) \neq \perp, f^*(i) < f^*(j)$.*

Proof. Since slots are ordered by $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_M$, if $q_i(p_i, p_{min})b_i > q_j(p_j, p_{min})b_j$ for some i, j then a function f such that $f(j) = f^*(i)$, $f(i) = f^*(j)$ and $f(h) = f^*(h)\,\forall\, h \notin \{i, j\}$ would be associated to a social welfare higher than f^*. □

It may be interesting to compare the mechanism $\mathcal{M}_{ord}^{GSP}$ with classical GSP and the indirect revelation mechanism proposed by Castiglioni et al. [5]. As stated in the following, the proposed mechanism is a generalization of both.

Proposition 4.3. *For a SSAPD where the quality is defined as $q_i(p_i, p_{min}) = k\,\forall\, p_i, p_{min} \in$ **R** (i.e. SSAPD $\equiv$ SSA) and $p^* = 0$, $\mathcal{M}_{ord}^{GSP}$ is equivalent to a classic GSP.*

The truthfulness of this proposition can be evinced by noticing that $q_i = q^* = k$, thus ads are allocated in decreasing bid order and advertisers are charged an auction cost equal to the next higher bid.

We are now going to analyze now the relation between $\mathcal{M}_{ord}^{GSP}(p^*)$ and $\mathcal{M}_{I}^{GSP}$, showing that the former is a generalization of the latter. We first introduce the following lemma.

Lemma 4.2. *There is at least one cut price p^* that lead to the allocation that maximize the (declared) social welfare.*

Proof. Given N advertisers, their selling prices $\boldsymbol{p}$ and their bids $\boldsymbol{b}$, consider the allocation A^* that maximize the social welfare. Let p^*_{min} be the minimum selling price displayed in A^*. Suppose that $p^* = p^*_{min}$ and, by absurd, that the selected allocation is $A \neq A^*$. We have two possibles scenarios: either the ad a^* associated with the price p^*_{min} is allocated or not.
If a^* is allocated, then $p_{min} = p^* = p^*_{min}$ and so $f^*(i) = \sigma(i)\,\forall\, i \in N_\sigma$. The social welfare

of the allocation is $s\hat{w}_A = \sum_i \lambda_{f^*(i)} q_i(p_i, p^*_{min}) b_i$. By Lemma 4.1 we can see that the social welfare is maximum and so that $A = A^*$.
If a^* is not allocated, the declared social welfare of the allocation A would be $s\hat{w}_A = \sum_i \lambda_{f^*(i)} q_i(p_i, p_{min}) b_i$, with $p_{min} \geq p^*$, so by definition $q_i(p_i, p_{min}) \geq q_i(p_i, p^*)$ and $s\hat{w}_A \geq s\hat{w}_\sigma$, where $s\hat{w}_\sigma$ is the social welfare when the allocation function is σ. But since a^* is not allocated, it means that $s\hat{w}_\sigma > s\hat{w}_{A^*}$, and so $s\hat{w}_A > s\hat{w}_{A^*}$. □

We can now formalize the relationship between $\mathcal{M}^{GSP}_{ord}$ and $\mathcal{M}^{GSP}_{I}$

Proposition 4.4. *The mechanism $\mathcal{M}^{GSP}_{I}$ is a particular case of the mechanism $\mathcal{M}^{GSP}_{ord}$ where the cut price p^* is chosen to maximize the declared social welfare.*

Proof. We have seen in Lemma 4.2 that we can always select the price p^*_{min} to realize the allocation that maximize the declared social welfare. To be $\mathcal{M}^{GSP}_{I} = \mathcal{M}^{GSP}_{ord}(p^*_{min})$ the mechanisms have to assign the same payments. Notice that, if k ads are allocated, the payment rule is the same for both the advertisers such that $f(i) < k$ and $f(i) = \perp$. In Lemma 4.2 we have shown that, by selecting $p^* = p^*_{min}$, it must occur that $p_{min} = p^*$, and so $p_i \geq p_{min} \, \forall \, i : f^*(i) = \perp$, while all the cut ads have a selling price lower than p_i by definition. Thus $\pi_k = \lambda_k \max_{j \in N' : f^*(j) = \perp} \{q_j(p_j, p^*) b_j\} = \lambda_k \max_{j : p_j \geq p_{min} \,\&\, f^*(j) = \perp} \{q_j(p_j, p_{min}) b_j\}$. □

Individual rationality

One of the main property of the mechanism is individual rationality under the assumption of non overbidding by the advertisers. Non overbidding is a reasonable assumption for rational and intelligent agents since it represents a dominated bidding strategy.

Proposition 4.5. *Overbidding is a weakly dominated strategy for the mechanism $\mathcal{M}^{GSP}_{ord}$.*

Proof. Suppose that the agent i target the slot j, associated with value $v_i = \lambda_j q_i \mu_i$. Being the agents allocated in order of $q(p, p_{min}) b$, decreasing the bid from b_i to μ_i affect the assigned slot or the payments only if there is an agent $k \neq i$ such that $q_i \mu_i < q_k b_k < q_i b_i$. However, in this case the payment for the agent i would be $\pi_i = \lambda_j q_k b_k > \lambda_j q_i \mu_i$, resulting in a negative utility. □

We can now formalize the individual rationality property.

Theorem 4.1 (Individual rationality)**.** *The mechanism $\mathcal{M}^{GSP}_{ord}$ is individually rational in ex-post if the advertisers do not overbid.*

Proof. Suppose that k slots have been allocated. The utility of an advertiser i assigned to a slot $f^*(i) < k$ is $u_i = \lambda_{f^*(i)} q_i(v_i - \pi_i) = \lambda_{f^*(i)}(q_i v_i - q_j b_j)$, where j is the advertiser such that $f^*(j) = f^*(i) + 1$. If i does not overbid, $q_i \mu_i \geq q_i b_i$. By Lemma 4.1 $q_i b_i \geq q_j b_j$, and so $u_i \geq 0$. Consider now the utility of the agent k allocated to the slot k. His utility is $u_k = \lambda_k q_k (v_k - \pi_k) = \lambda_k (q_k v_k - \max_{j: f^*(j) = \perp}\{q_j(p_j, p^*) b_j\})$. Being σ an ordering over $q_i(p_i, p^*) b_i$, $q_k(p_k, p^*) b_k \geq \max_{j: f^*(j)=\perp}\{q_j(p_j, p^*) b_j\}$. Since the quality $q(p, p_{min})$ is a non decreasing function in p_{min}, $q_k(p_k, p_{min}) b_k \geq q_k(p_k, p^*) b_k$, and so $u_k \geq 0$. □

The individual rationality property ensures that an agent will never receive a negative utility by taking part to the game induced by the mechanism $\mathcal{M}^{GSP}_{ord}$. Thus it is reasonable to assume that rational and intelligent advertisers will always prefer to be allocated rather than not, as long as they do not overbid. As a final observation, we can notice that the mechanism is weakly budget balanced, since the auction payments assigned to the advertisers are such that $\pi_i \geq 0 \,\forall\, i \in N$.

4.2.1. Bidding strategies for $\mathcal{M}^{GSP}_{ord}$

We will analyze whether the generalization of the BB strategy for a SSAPD is a feasible bidding strategy for an auction implementing $\mathcal{M}^{GSP}_{ord}$. As we can see, BBPD in not suitable for an advertiser taking part to the auction.

Proposition 4.6. *The BBPD is not a feasible bidding strategy for a SSAPD that implements the mechanism* $\mathcal{M}^{GSP}_{ord}$

The following example provide an instance in which the BBPD is not a feasible bidding strategy.

Example 4.2. *Consider a game with two slots $\lambda_1 = 1$, $\lambda_2 = 0.5$ and three players, such that $p_1 = \mu_1 = b_1 = 100$, $p_2 = \mu_2 = b_2 = 1$ and $p_3 = \mu_3 = 5$, $b_3 = 0$. The quality function is*

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \text{if } p_i = p_{min} \\ 0.25 & \text{otherwhise} \end{cases}$$

When a_3 updates his bid, he can target the slot $j = 2$ by bidding $b_3 \in [4, 5]$, and will not be allocated for any bid in $[0, 4)$. But accordingly to BBPD he would bid $b_3 = (1 - \gamma_2)\mu_3 + \gamma_2 b_2 q_2^*/q_3 = 0.5 \cdot 5 + 0.5 = 3$.

By considering EBBPD instead, for the same reasons of $\mathcal{M}^{GSP}_{I}$ the bidding strategy is feasible for $\mathcal{M}^{GSP}_{ord}$.

Proposition 4.7. *The EBBPD is a feasible bidding strategy for a SSAPD that implements the mechanism* $\mathcal{M}_{ord}^{GSP}$.

We can notice that the two bidding strategies BBPD and EBBPD behave similarly when submitted to the mechanisms $\mathcal{M}_{I}^{GSP}$ and $\mathcal{M}_{ord}^{GSP}$. However, as we will show in Section 4.3, when the agents bid accordingly to EBBPD in a SSAPD implementing $\mathcal{M}_{ord}^{GSP}$ is guaranteed to converges to an equilibrium where the bids are the same as under BBPD.

4.2.2. Cut price

The cut price p^* is a parameter of the mechanism that affects the allocation for a given bid and price profile $\boldsymbol{b}$ and $\boldsymbol{p}$. The rationality behind this parameter relies on the fact that when the advertisers need to be allocated they are selected accordingly to the ordering σ, and the agents with higher selling prices can be penalized with respect to the others. This can negatively affect the social welfare, as showed in the following.

Proposition 4.8. *The social welfare of the equilibrium of the game induced by the mechanism* $\mathcal{M}_{ord}^{GSP}(p^*)$ *can be arbitrarily worst than the optimal equilibrium for some cut price* p^*.

Proof. Consider the following game. Two agents $\{1, 2\}$ compete for a single slot with prominence $\lambda = 1$. The agents have a gain $\mu_1 = p_1 = \bar{p}$ and $\mu_2 = p_2 = \underline{p}$, with $\bar{p} > \underline{p}$. The quality function is defined as

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \text{if } p_i = p_{min} \\ 0 & \text{otherwhise} \end{cases}$$

If $p^* = \underline{p}$ then $q_2(p_2, p^*)b_2 > q_1(p_1, p^*)b_1 \,\forall\, b_1 \neq 0, b_2 \neq 0$, thus agent a_2 is allocated with a corresponding social welfare of $\lambda q_2(p_2, p_2)\mu_2 = \underline{p}$. The optimal allocation is the one that assign to the slot the agent a_1, with a social welfare of $\lambda q_1(p_1, p_1)\mu_1 = \bar{p}$. The ratio among the two social welfare is $\frac{\underline{p}}{\bar{p}}$, which can be arbitrarily small. □

In Section 4.3.2 we will discuss how the cut price can be selected to guarantee some desirable results in terms of social welfare. We will first remark the following lemma.

Lemma 4.3. *If* $p_{min} = p^*$*, then EBBPD* $\equiv$ *BBPD.*

Proof. If $p_{min} = p^*$, then $q_i(p_i, p_min) = q_i(p_i, p^*) \,\forall\, i \in N'$. Thus, if agent i target the slot j, $q_i b_i = (1 - \gamma_j) q_i \mu_i + \gamma_j q_{i+1} b_{i+1} \geq q_{i+1} b_{i+1}$ □

In a repeated auction, the cut price could in principle changes in different instance. We will call a cut price selection strategy as *cut price dynamics*. From Theorem 4.1 we know that the mechanism is individually rational, thus a reasonable assumption is that every rational agent prefers to be allocated rather than not to. Thus, if anytime an agent does not select a bid that grant him a slot, it means that he can not be allocated without overbidding. Based on this assumption, we define a particular cut price dynamics.

Definition 4.3. *Consider a SSAPD based on $\mathcal{M}_{ord}^{GSP}(p^*)$. At any time t, let $B \subseteq N'$ be the set of non discarded agents whose last submitted bid did not grant them a slot in the next iteration of the auction and $K = \{a_i : p_i = p^* \, \forall \, i \in N\}$ the set of agents such that their selling price is equal to the cut price. We call $\mathcal{D}$ the cut price dynamic that selects for the auction at time $t+1$ a cut price $p^{*\prime} = \min\{p_i : p_i > p^* \, \forall \, i \in N\}$ if $K \subseteq B$, or $p^{*\prime} = p^*$ otherwise.*

Informally, the described strategy increases the cut price to the next higher selling price each time all the agents whose selling price is equal to the cut price update their bids without being allocated. Notice that, if the assumption of agents preferring to taking part to the allocation rather than not holds, the dynamic $\mathcal{D}$ is weakly monotonic. No assumption are made on the initial value of p^*.

4.3. Properties of $\mathcal{M}_{ord}^{GSP}$

This section will present our two main results. First, we will analyze the convergence of SSAPD based on $\mathcal{M}_{ord}^{GSP}(p^*)$. Then we will study the efficiency of the equilibrium reached by a repeated auction, proposing a policy for selecting the cut price in order to guarantee some efficiency results.

4.3.1. Convergence

We studied whether the BBPD bidding strategy behave when adopted in a SSAPD that implements the mechanism $\mathcal{M}_{ord}^{GSP}$, proving that it guarantees the convergence of the outcome to an equilibrium in finite time.

Theorem 4.2. *A repeated SSAPD based on the mechanism $\mathcal{M}_{ord}^{GSP}(p^*)$ that implements $\mathcal{D}$ converges to its Nash equilibrium when selling prices are fixed and the agents select their bids accordingly to EBBPD in a random and asynchronous way.*

Proof. The proof is divided in two steps. First we will prove that in a finite time and with probability 1 the set of assigned advertisers N_σ converges to a fixed point, where the unallocated agents bids their value and the allocated ones bid over a threshold. In the second step we will prove that the Nash equilibrium is reached with probability 1.

Lemma 4.4. *There exist $\bar{t}$ for which at every time $t \geq \bar{t}$, for each agent i such that at time $\bar{t}$ $f^*(i) = \perp$, at time t $f^*(i) = \perp$. The agents' bids are such that*

$$b_i = \mu_i \, \forall \, i : f^*(i) = \perp$$

$$q_i(p_i, p^*)b_i \geq \max_{j:f^*(j)=\perp} \{q_j(p_j, p^*)\mu_j\} \, \forall \, i : f^*(i) \neq \perp$$

for every $t \geq \bar{t}$. Moreover, for each $t \geq \bar{t}$ it is $p_{min} = p^*$.

Proof. The number of the allocated advertisers is $\max\{M, N'\}$. If $N' \leq M$ all the agents are assigned to a slot and the proof is trivial. Suppose that $N' > M$. Let k be the advertiser such that $k = \max_{i \in N' : f^*(i) = \perp}\{q_k(p_k, p^*)b_k\}$. For convenience we rename the agents such that the advertiser i is assigned to the position i. By Theorem 4.1 the mechanism is individually rational, so an agent will prefer to be allocated rather than not to. Thus, no agent i who is assigned to a slot will be willing to change his bid under $q_k(p_k, p^*)b_k/q_i(p_i, p^*)$. In the same way, every agent that has not been assigned to a slot will prefer to increase his bid, up to μ_i, in order be allocated. Thus, the vector $(q_{M+1}(p_{M+1}, p^*)b_{M+1}, ..., q_{N'}(p_{N'}, p^*)b_{N'})$ is strictly non decreasing in all its elements. To prove the convergence, we need to show the increments are finite.

We face two possible scenarios: either an agent whose selling price is p^* is currently allocated or he is not. In the former case, let i be the first unallocated agent. In the latter case, let i be the unallocated agent such that $p_i = p^*$. Suppose that, at some time t, the agent i is bidding $b_i < \mu_i$. In the next activation the advertiser will either bid his gain or target a position j, by bidding $b_i' = (1-\gamma_j)\mu_i + \gamma_j \frac{q_{j+1}}{q_i} b_{j+1} = (1-\gamma_j)\mu_i + \gamma_j \frac{q_{j+1}}{q_i^*} b_{j+1}$. Let $\gamma^* = \max_{j>1} \gamma_j$. We can see that

$$\begin{aligned} q_i^* b_i' &\geq (1-\gamma_j) q_i^* \mu_i + \gamma_j q_{j+1} b_{j+1} \\ &\geq (1-\gamma_j) q_i^* \mu_i + \gamma_j q_i^* b_i \\ &= q_i^* b_i + (1-\gamma_j)(q_i \mu_i - q_i^* b_i) \\ &\geq q_i^* b_i + (1-\gamma^*)(q_i^* \mu_i - q_i^* b_i). \end{aligned}$$

Thus, whenever the advertiser i updates his bid, it is

$$q_i^* \mu_i - q_i^* b_i' \leq \gamma^* (q_i^* \mu_i - q_i^* b_i).$$

Be $\delta = \min_{k:q_k^*\mu_k > q_i^*\mu_i}(q_k^*\mu_k - q^*\mu_i)$, if agent i is not in the top M ads by $q_i^*\mu_i$ he will bid his value within $\bar{t}_i$ updates, where $\bar{t}_i$ is such that $q_i b_i {\gamma^*}^{\bar{t}_i} < \delta$.

To conclude the proof, consider the agents such that $p_i = p^*$. If all of them will not be allocated, they will all eventually bid their gain in finite time. Thus, the cut price will be increased by the dynamic $\mathcal{D}$. The monotonicity of $\mathcal{D}$ guarantees the convergence. $\square$

By Lemma 4.4 we know that after a certain time $\bar{t}$ all the non-allocated agents will not be able to win a slot and will bid their true gain $b_i = \mu_i$. Thus the advertisers who are allocated are a fixed set, namely $\bar{N}$, after $\bar{t}$. Notice that being $\bar{N}$ fixed, the minimum displayed price is $p_{min} = \min_{i\in\bar{N}} p_i$, and the quality of the agents are constant $(q_i(p_i, p_{min}) = q_i)$. Moreover, by Lemma 4.4 and Lemma 4.3, $p_{min} = p^*$ and the advertisers will bid accordingly to the BBPD strategy.

To prove that the allocation converges, we first show that it does for a special updating sequence, and then state that this sequences happens with a probability 1 in finite time. The updating rule is called Lowest-First(k, $\boldsymbol{b}$, $\boldsymbol{p}$)[2], where k is the next agent that will activate and update his bid, while $\boldsymbol{b}$, $\boldsymbol{p}$ are the bid and price profiles. The algorithm is summarized by Algorithm 4.1 and works as follow. Rename the agents in $\bar{N}$ such that the $q_1 b_1 > ... > q_{\bar{N}} b_{\bar{N}}$. At each round, the agent i updates his bid accordingly with BBPD by targeting the slot j, then the agents are renamed in order of $q_i b_i$. If $j < i$, the next advertiser updating his bid will be the new agent i, otherwise it will be new agent $j-1$. The update sequence is first called with $i = |\bar{N}|$, and terminate if is called with $i = 0$.

The proof is conduced by induction. If $\bar{N} = 1$, the convergence is trivial, as the agent would bid his value. Consider then the general case when first is called Lowest-First($\bar{N}$, $\boldsymbol{b}$, $\boldsymbol{p}$). Notice that, if Lowest-First(k, $\boldsymbol{b}$, $\boldsymbol{p}$) is called, then the bids $(b_{k+1}, ..., b_{\bar{N}})$ are all consistent with the BBPD strategy. Thus, if Lowest-First(0, $\boldsymbol{b}$, $\boldsymbol{p}$) is called, all the agents are bidding coherently with BBPD and the result is a Nash equilibrium.

When Lowest-First(i, $\boldsymbol{b}$, $\boldsymbol{p}$) is called a consecutive number of times with $i = \bar{N}$, the bidding vector $(b_1, ..., b_{\bar{N}})$ increase each time in at least one coordinate. The active bidder increments his bid by moving closer to $\mu_{\bar{N}}$ with a ratio at least γ^*, so the consecutive calls to Lowest-First($\bar{N}$, $\boldsymbol{b}$, $\boldsymbol{p}$) will eventually terminate.

Finally, notice that since agent i targets a slot j if

$$\lambda_j q_i(p_i, p_{min})(\mu_i - \pi_i(j)) > \lambda_{j-1} q_i(p_i, p_{min})(\mu_i - \pi_i(j-1))$$

[2]The Lowest-First algorithm is based on the Lowest-First algorithm proposed by Bu et al. [1] for the proof of convergence of forward looking strategy in GSP auctions.

it is $q_i b_i = (1-\gamma_j)q_i\mu_i + \gamma_j \pi_i(j) < \pi_i(j-1)$. Thus, a recursive call to Lowest-First($\bar{N}$, $\boldsymbol{b}$, $\boldsymbol{p}$) from a call of Lowest-First(i, $\boldsymbol{b}$, $\boldsymbol{p}$) can happen only if $q_i\mu_i < q_{\bar{N}}\mu_{\bar{N}}$, and so the procedure will terminate.

The lowest-first update sequence happen with a fixed non-zero probability when the bidder is randomly chosen at each round if the run is long enough, and this probability increases with the number of rounds. It follows that the equilibrium will be reached with probability 1 in a finite number of steps.

$\square$

Algorithm 4.1 Lowest-First$(i, \boldsymbol{b} = (b_1, ..., b_N), \boldsymbol{p} = (p_1, ..., p_N))$

---

if i = 0 **then**
    return
end if
Let j be bidder i's favourite slot given $\boldsymbol{b_{-i}}$ and $\boldsymbol{p}$
Agent i updates his bid accordingly with BBPD.
Rename the agents such that $q_1 b_1 < q_2 b_2 < ... < q_N b_N$ (agent i is now indexed by j).
if $j < i$ **then**
    Lowest-First$(i, \boldsymbol{b}, \boldsymbol{p})$
else
    Lowest-First$(h-1, \boldsymbol{b}, \boldsymbol{p})$
end if

We can notice that the equilibrium reached at convergence is unique. In particular, at equilibrium the allocated agents are assigned such that $f^*(i) < f^*(j)\, \forall\, i, j \in N$ such that $f(i), f(j) \neq \perp$ and $q_i\mu_i > q_j\mu_j$.

Lemma 4.5. *In a SSAPD implementing $M^{GSP}_{ord}(p^*)$ that select p^* accordingly to $\mathcal{D}$, if agents select their bids according to EBBPD strategy at the equilibrium the allocated players are ordered by $q_i(p_i, p_{min})\mu_i$.*

Proof. Suppose, by absurd, that two agents i and j are allocated respectively to the slots k and $k+1$, and $q_i\mu_i < q_j\mu_j$. Then, we have that the utility of agent i for being assigned to slot k is $u_i^k = \lambda_k(q_i\mu_i - q_j b_j) = \lambda_k(q_i\mu_i - q_j(\mu_j - \frac{\lambda_{k+1}}{\lambda_k}(\mu_j - \frac{q_{f^*(k+2)} b_{f^*(k+2)}}{q_j}))) = \lambda_{k+1}(q_i\mu_i - q_{f^*(k+2)} b_{f^*(k+2)}) + (\lambda_k - \lambda_{k+1})(q_i\mu_i - q_j\mu_j) < \lambda_{k+1}(q_i\mu_i - q_{f^*(k+2)} b_{f^*(k+2)}) = u_i^{k+1}$. Thus agent i would prefer to be assigned to slot $k+1$ and the allocation is not an equilibrium. $\square$

4.3.2. Efficiency

Proven that the mechanism is guaranteed to converge when agents follow the EBBPD strategy, we question whether the equilibrium is efficient. In Proposition 4.8 we showed that the efficiency with respect of the social welfare for the mechanism $\mathcal{M}_{ord}^{GSP}(p^*)$ can be arbitrarily small if the social welfare is not properly selected. In general, the social welfare of the allocation at convergence can be arbitrarily worse than the optimal one also at convergence. This is trivially true by noticing that the example proposed in Theorem 4.8 represents an equilibrium.

We can formalize the inefficiency of the mechanism given a cut price in terms of price of stability.

Theorem 4.3. *The PoS of the social welfare for the mechanism $\mathcal{M}_{ord}^{GSP}(p^*)$ can be arbitrarily large when the agents select their bids accordingly with the EBBPD strategy.*

Proof. Consider the following game. Two agents $\{1, 2\}$ compete for a single slot with prominence $\lambda_1 = 1$. The agents have a gain $\mu_1 = p_1 = \bar{p}$ and $\mu_2 = p_2 = \underline{p}$, with $\bar{p} > \underline{p}$. The quality function is defined as

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \text{if } p_i = p_{min} \\ \frac{\underline{p}}{\bar{p}} - \epsilon & \text{otherwhise} \end{cases}$$

Suppose that $p^* = \underline{p}$, and thus all the agents will participate to the game. At convergence $b_1 = \mu_1$ and $b_2 = \mu_2$. Thus, being $q_2(p_2, p^*)b_2 = \underline{p} > q_1(p_1, p^*)b_1 = \underline{p} - \epsilon\bar{p}$ ad a_2 will be allocated, with a social welfare of $q_2(\underline{p}, \underline{p})\underline{p} = \underline{p}$. If instead $b_1 = 0$ the ad a_1 is allocated, and the social welfare would be $q_1(\bar{p}, \bar{p})\bar{p} = \bar{p}$. The price of stability is $PoS = \frac{\bar{p}}{\underline{p}}$, that can be arbitrarily large. □

So far we have proven that the social welfare of the equilibrium of the mechanism $\mathcal{M}_{ord}^{GSP}$ can be arbitrarily small for some cut prices and some utility functions. However we show that, for any quality function q, there is at least one cut price that ensures that the mechanism converge to the allocation that maximize the true social welfare.

Theorem 4.4. *In a SSAPD implementing $\mathcal{M}_{ord}^{GSP}(p^*)$ where agents select their bids accordingly with EBBPD, there is at least one cut price p^* such that the mechanism converges to an equilibrium that maximise the true social welfare.*

Proof. Let A^* be the allocation that maximize the true social welfare

$$sw_{A^*} = \max_{f^*} \sum_{i:f^*(i)\neq\perp} \lambda_{f^*(i)} q_i(p_i, p_{min})\mu_i.$$

We select the cut price such that $p^* = \min p_i : A^*(i) \neq \perp$. Notice that no player that is allocated in A^* is discarded by the choice of p^*.

If the agents select their bids accordingly to EBBPD, we know by Theorem 4.2 that the auction will converge to an allocation A. Consider two possible scenarios for A: the ad a^* associated to the advertiser with the minimum displayed price in A^* is either allocated or not.

a^* is allocated: Since A is a Nash equilibrium of the game induced by $\mathcal{M}^{GSP}_{ord}$, we know that $b_j = \mu_j \,\forall\, j : f^*(j) = \perp$. Let $k = \max_{j:f^*(j)=\perp}\{q_j(p_j, p^*)b_j\}$ be the best unassigned agent and $h = \min\{M, N'\}$ be the number of assigned slots. Being $p^* = p_{min}$, $\sigma(i) = f^*(i) \,\forall\, i : f^*(i) \neq \perp$. For each allocated agent i it is $q_i(p_i, p_{min})\mu_i \geq q_i(p_i, p_{min})b_i \geq q_k(p_k, p^*)b_k = q_k(p_k, p^*)\mu_k$, thus the allocated players are the top h according to $q_i\mu_i$. From Lemma 4.5 the agents are assigned in decreasing order of $q_i\mu_i$, and so $A = A^*$ is the allocation that maximize the true social welfare.

a^* is not allocated: Being A an allocation at convergence, the agent a^* is bidding his gain μ_i. Since $p^* \leq p_{min}$ and the quality $q(p_i, p_{min})$ is non decreasing in p_{min}, it is $sw_A \geq sw_\sigma$, where sw_σ is the social welfare associated with the allocation given by the ordering σ. At convergence, all the non-allocated players are bidding their value, while the allocated agents have $q_i(p_i, p_{min})\mu_i \geq q_i(p_i, p_{min})b_i \geq q_i(p_i, p^*)b_i \geq q_k(p_k, p^*)b_k = q_k(p_k, p^*)\mu_k$. Thus, $sw_\sigma > sw_{A^*}$, and so $sw_A > sw_{A^*}$. But since A^* is the allocation that maximize the social welfare, this conclude the proof.

□

By Theorem 4.3 and Theorem 4.4 we can see that by selecting the proper cut price the social welfare of the equilibrium can span from the optimum to one that is arbitrarily bad. We investigate whether there is a way to select the cut price in order to have some guarantees on the resulting true social welfare.

Definition 4.4. *We define as* $\boldsymbol{Part_{\bar{q}}}(N) = \{N_1, N_2, ..., N_k\}$ *a partition over* N *such that*

$$N_i = \{a_j : q(p_j, \min\{p \in N_i\}) \geq \bar{p}\}$$

$$\bar{q} \leq q(p, p).$$

We will refer as $\boldsymbol{P}(Part_{\bar{q}}(N))$ *to the set of selling prices such that*

$$P(Part_{\bar{q}}(N)) = \{p : p = \min_{j \in N_i} p_j \,\forall\, N_i \in Part_{\bar{q}}(N)\}.$$

By means of the partition of the advertisers defined, we can formalize a theoretical bound of the efficiency of the mechanism's equilibrium.

Theorem 4.5. *In a SSAPD implementing* $\mathcal{M}^{GSP}_{ord}(p^*)$ *where the cut price dynamic is* $\mathcal{D}$ *and agents update their bids accordingly to EBBPD, if the the initial cut price is selected from* $P(Part_{\bar{q}}(N))$ *then the social welfare at the equilibrium is in expectation at least* $\frac{\bar{q}}{|Part_{\bar{q}}(N)|}$ *of the optimal social welfare.*

Proof. First we introduce some notation. We refer as $sw_{\mathcal{M}^{GSP}_{ord}(p_i)}$ to the social welfare at equilibrium of the game induced by the mechanism $\mathcal{M}^{GSP}_{ord}$ when the cut price is $p^* = p_i$. We also identify as

$$sw^*(K) = \max_f \sum_{i \in K} \lambda_{f(i)} q_i(p_i, p_{min}(f))\mu_i$$

the maximum true social welfare among the allocation that assign agents from a set K.

Lemma 4.6. *Given a partition of the agents set* $Part_{\bar{q}}(N)$*, when* $N_i \in Part_{\bar{q}}(N)$ *and* $p_i = \min_{i \in N_i} p_i$ *it is*

$$sw_{\mathcal{M}^{GSP}_{ord}(p_i)} \geq sw^*(N_i)\bar{q}.$$

Proof. Let σ^* be the allocation function that orders the agents by their $q_i(p_i, p^*)\mu_i$ and $p_{\mathcal{D}}$ is the cut price at convergence starting from $p^* = p_i$ accordingly to the dynamic $\mathcal{D}$. Since the quality function is non decreasing in p_{min}, it is

$$\begin{aligned} sw_{\mathcal{M}^{GSP}_{ord}(p_i)} &= \sum_{j : p_j \geq p_{\mathcal{D}}} \lambda_{f^*(j)} q_j(p_j, p_{\mathcal{D}})\mu_j \\ &= \sum_{j : p_j \geq p_{\mathcal{D}}} \lambda_{\sigma^*(j)} q_j(p_j, p_{\mathcal{D}})\mu_j \\ &\geq \sum_{j : p_j \geq p_i} \lambda_{\sigma^*(j)} q_j(p_j, p_i)\mu_j. \end{aligned}$$

Consider the allocation $A^*_{N_i}$ that assigns the agents in N_i in order to maximize the true social welfare. By construction the allocation σ^* is the allocation that maximize the true social welfare if the qualities are computed with respect to the cut price. Noticing that

$N_i \subseteq \{a_k : p_k \geq p_i\}$ and that $q_j(p_j, p_i) \geq \bar{q} \,\forall\, j \in N_i$, it is

$$\begin{aligned}
\sum_{j:p_j \geq p_i} \lambda_{\sigma^*}(j) q_j(p_j, p_i)\mu_j &\geq \sum_{j \in N_i} \lambda_{A^*_{N_i}(j)} q_j(p_j, p_i)\mu_j \\
&\geq \sum_{j \in N_i} \lambda_{A^*_{N_i}(j)} \bar{q}\mu_j \\
&\geq \bar{q} \sum_{j \in N_i} \lambda_{A^*_{N_i}(j)} q_j(p_j, p_{min}(A^*_{N_i}))\mu_j \\
&= sw^*(N_i)\bar{q}
\end{aligned}$$

where $p_{min}(A)$ is the minimum price in the allocation A. □

By Lemma 4.6 we know that $sw_{\mathcal{M}^{GSP}_{ord}(p_i)} \geq sw^*(N_i)\bar{q}$, thus

$$\begin{aligned}
\mathop{\mathbb{E}}_{i \sim P(Part_{\bar{q}}(N))}[sw_{\mathcal{M}^{GSP}_{ord}(p_i)}] &\geq \mathop{\mathbb{E}}_{i \sim Part_{\bar{q}}(N)}[sw^*(N_i)]\bar{q} \\
&= \frac{\sum_{i \in Part_{\bar{q}}(N)} sw^*(N_i)\bar{q}}{|Part_{\bar{q}}(N)|} \\
&= \frac{\bar{q}}{|Part_{\bar{q}}(N)|} \sum_{i \in Part_{\bar{q}}(N)} sw^*(N_i)
\end{aligned}$$

To finalize the proof, we show that the sum of the optimal social welfare of the subset in the partition $Part_{\bar{q}}(N)$ is greater than the optimal social welfare of the game, formally $\sum_{N_i \in Part_{\bar{q}}(N)} sw^*(N_i) \geq sw^*(N)$.

Consider the allocation associated with the optimal social welfare A^*_N and a particular allocation A_{N_i} for each subset N_i buildt as follow. Assign the agent j such that $A^*_N(j) = 1$, belonging to N_i to the first slot of A_{N_i}. Then, assign the agent such that $A^*_N(j) = 2$ belonging to N_h (it may be $i = h$) to the most relevant free slot in A_{N_h}. Iterate this procedure for all the agents assigned in A^*_N. Notice that, being the number of available slots the same for A^*_N and each A_{N_i}, then all the agents in A^*_N are eventually allocated in a A_{N_i} in, at worst, the same slot, while the minimum displayed price of each A_{N_i} is at least the minimum selling price in A^*_N. Because of the non decreasing property of the quality in p_{min}, it is

$$\begin{aligned}
\sum_{i \in Part_{\bar{q}}(N)} sw_{A_{N_i}} &= \sum_{i \in Part_{\bar{q}(N)}} \sum_{j \in N_i} \lambda_{f_i(j)} q_j(p_j, p_{min}(A_{N_i}))\mu_j \\
&\geq \max_f \sum_{j \in N} \lambda_{f(j)} q_j(p_j, p_{min}(f))\mu_j \\
&= sw^*(N).
\end{aligned}$$

Finally notice that by definition $sw^*(N_i) \geq sw_{A_{N_i}}$, and so

$$\mathop{\mathbb{E}}_{i \sim P(Part_{\bar{q}}(N))} [sw_{\mathcal{M}_{ord}^{GSP}(p_i)}] \geq \frac{\bar{q}}{|Part_{\bar{q}}(N)|} sw^*(N). \tag{4.2}$$

$\square$

It is possible to extend the result to the following corollary.

Corollary 4.1. *A SSAPD implementing $\mathcal{M}_{ord}^{GSP}(p^*)$ where $p^* \leq \min_{i \in N} p_i$ and agents select their bids accordingly with EBBPD converges to an allocation with an expected social welfare that is at least $q(\bar{p}, \underline{p}) sw^*(N)$, where $\underline{p} = \min_{i \in N} p_i$ and $\bar{p} = \max_{i \in N} p_i$.*

In an auction where the agents' quality function is a generic function $q(\cdot)$, the upper bound of the factor $\frac{\bar{q}}{|Part_{\bar{q}}(N)|}$ can be selected by solving a maximization problem. Unfortunately, in the general case the cardinality of $Part_{\bar{q}}(N)$ can go to infinity with the number of advertisers competing to the auction. Consider, for example, the following quality function:

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \text{if } p_i = p_{min} \\ 0 & \text{otherwhise} \end{cases}$$

It is trivial to notice that, for any $\bar{q} > 0$, $|\,\mathrm{Part}_{\bar{q}}(N)|$ is equal to the number of different selling prices among the agents. We question whether we can give a bound to the cardinality of the partition $\mathrm{Part}_{\bar{q}}(N)$ for a general quality function $q(p, p_{min})$. The following result holds:

Theorem 4.6. *For a general quality function $q(p, p_{min})$ and a given quality threshold $\bar{q}$, it is possible to compute a partition $\mathrm{Part}_{\bar{q}}(N)$ such that*

$$|\,\mathrm{Part}_{\bar{q}}(N)| \leq \left\lceil \frac{(\max_{i \in N} p_i - \min_{i \in N} p_i)}{\Delta p_{\bar{q}}} \right\rceil \tag{4.3}$$

where

$$\Delta p_{\bar{q}} = \min_{i \in N} p_i - \max\{\bar{p}_i : q(\bar{p}_i, p_i) \geq \bar{q} \wedge \bar{p}_i \geq p_i\}.$$

Proof. The proof follow by construction of the parameter $\Delta p_{\bar{q}}$. Consider first the maximization problem. For some price p_i, $\bar{p}_i$ is the maximum price greater than p_i such that, if p_i is the minimum price, then the quality $q(\bar{p}_i, p_i) \geq \bar{q}$. Being the quality function non increasing, it is $q(p, p_i) \geq \bar{q} \, \forall \, p \in [p_i, \bar{p}_i]$ and $q(p, p_i) < \bar{q} \, \forall \, p \in (\bar{p}_i, \infty]$. Then $\Delta p_{\bar{q}}$ is selected by means of a minimization problem as the length of the smallest interval $[p_i, \bar{p}_i]$.

Thus, it is possible to build a partition $\mathrm{Part}_{\bar{q}}(N)$ such that $|N_j| \geq \Delta p_{\bar{q}} \,\forall\, N_j \in \mathrm{Part}_{\bar{q}}(N)$. It follows that

$$|\,\mathrm{Part}_{\bar{q}}(N)| \leq \left\lceil \frac{(\max_{i \in N} p_i - \min_{i \in N} p_i)}{\min_{N_i \in \mathrm{Part}_{\bar{q}}(N)} |N_i|} \right\rceil = \left\lceil \frac{(\max_{i \in N} p_i - \min_{i \in N} p_i)}{\Delta p_{\bar{q}}} \right\rceil.$$

□

The Eq. 4.3 provide an upper bound for the partition of the agent set. However, the computation of parameter $\Delta p_{\bar{q}}$ can be expensive since require to solve both a maximization and a minimization problem. A more relaxed bound can be provided when the quality function is k-Lipschitz continuous, noticing that by definition

$$|q_i(p_i, p_{min_1}) - q_i(p_i, p_{min_2})| \leq k|p_{min_1} - p_{min_2}| \,\forall\, i \in N$$

and thus

$$\frac{q(p,p) - \bar{q}}{k} \leq \Delta p_{\bar{q}}$$

Corollary 4.2. *For a k-Lipschitz quality function $q(p, p_{min})$ and a given quality threshold $\bar{q}$, it is possible to compute a partition $\mathrm{Part}_{\bar{q}}(N)$ such that*

$$|\,\mathrm{Part}_{\bar{q}}(N)| \leq \left\lceil \frac{(\max_{i \in N} p_i - \min_{i \in N} p_i)k}{q(p,p) - \bar{q}} \right\rceil \tag{4.4}$$

In the following we provide an example of the computation of the equilibrium efficiency bound.

Example 4.3. *Consider a SSAPD where an arbitrary set of agents submit ads with selling prices in $[50, 100]$. The agents' quality function is $q_i(p_i, p_{min}) = 1 - \frac{p_i - p_{min}}{50}$, and thus $\frac{1}{50}$-Lipschitz continuous. By Theorem 4.5 and Corollary 4.2, we can select a cut price such that*

$$\begin{aligned}
\mathop{\mathbb{E}}_{i \sim P(Part_{\bar{q}}(N))}[sw_{\mathcal{M}^{GSP}_{ord}(p_i)}] &\geq \frac{\bar{q}}{|Part_{\bar{q}}(N)|} sw^*(N) \\
&\geq \frac{\bar{q}}{\left\lceil \frac{(\max_i p_i - \min_i p_i)k}{\Delta_{\bar{q}}} \right\rceil} sw^*(N) \\
&= \frac{\bar{q}}{\left\lceil \frac{1}{1-\bar{q}} \right\rceil} sw^*(N)
\end{aligned}$$

which is maximized by selecting $\bar{q} = 0.5$*, with an associated expected social welfare*

$$\mathop{\mathbb{E}}_{i \sim P(Part_{\bar{q}}(N))}[sw_{\mathcal{M}^{GSP}_{ord}(p_i)}] \geq \frac{1}{4} sw^*(N)$$

.

5 | Experimental results

In this chapter we will describe how we modeled our simulations in order to validate the mechanism $\mathcal{M}_{ord}^{GSP}$. Then we will provide the experimental results, focusing on how the parameters selection of the simulation affects the convergence time of the auction and on the efficiency of the equilibrium.

5.1. Experimental setting

The difference between a standard SSA and a SSAPD relies on the affection of the ads quality on the other agents, affecting both their click through probability and the auction prices. The shape of the quality function is determined by the preferences of the users over the goods sold by the auction and specifically, in our first approximation, by the difference between the ads prices with respect to the minimum displayed price. However, different goods could be associated with different quality functions. Even if it is reasonable that, given enough time, both the auctioneer and the advertisers can estimate the preferences of the customers, we do not know how it could be computed in a real world scenario. Therefore, we relied on a very reasonable assumption: among the ads displayed, the one with the lowest selling prices p_{min} is associated with the highest social welfare, while the others agents quality goes to zero with the difference of the selling price with respect to p_{min}. In particular, we studied the rectified linear and a sigmoid like quality function.

The rectified linear quality function is formally defined as:

$$ReL(k, q_{max}, q_{min}) : q_i(p_i, p_{min}) = \max\{q_{max} + k(p_i - p_{min}); q_{min}\}.$$

The parameter k represents the gradient of the function: since the quality is non strictly decreasing in the difference $p_i - p_{min}$, it will be $k \leq 0$, where $k = 0$ is a special case when the quality function is constant for all the advertisers. Notice that in that case the SSAPD is equivalent to a classic SSA. The parameter $q_{min} \in [0, 1]$ represents the quality of the advertiser when his price is p_{min}. Parameter $q_{min} \in [0, q_{max}]$, instead, is the minimum quality for each advertiser: since the quality is defined in [0, 1], the linear function has to be bounded in the interval. The shape of $ReL(\cdot)$ for different values of k is showed in Figure 5.1, when $q_{max} = 1$, $q_{min} = 0$ and $p_{min} = 50$.

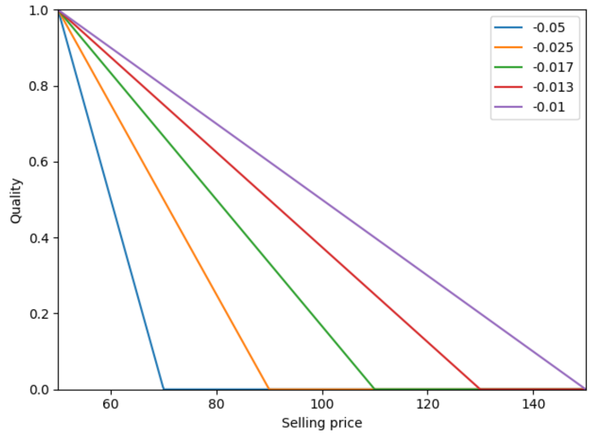

Figure 5.1: $ReL(k, 1, 0)$ quality function for different values of k

The sigmoid like quality function is formally defined as:

$$Sig(q_{max}, h, k, \epsilon) : q_i(p_i, p_{min}) = \frac{(1+h)q_{max}}{h + \left(\frac{(1+h)q_{max}-\epsilon}{\epsilon}\right)^{\frac{p-p_{min}}{(k-1)p_{min}}}}$$

The parameter $q_{max} \in [0, 1]$ represents the quality when $p_i = p_{min}$. Parameters h, k and ϵ shape the slope of the function: h adjust the height of the horizontal asymptote, while for the price $k \cdot p_{min}$ the quality is $h \cdot q_{max}$. In Figure 5.2 is showed the shape of $Sig(\cdot)$ for different values of k, when $p_{min} = 50, q_{max} = 1, h = 20$ and $\epsilon = 0.001$.

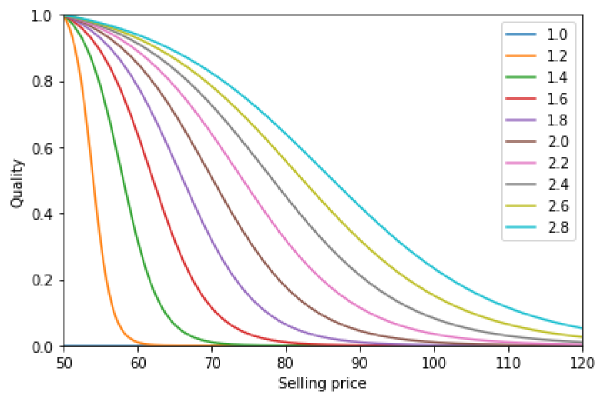

Figure 5.2: $Sig(q_{max}, h, k, \epsilon)$ quality function for different values of k

As well as the quality function, there are other parameters of the auction setting that are interesting to consider since they can affect the convergence time and the efficiency of the equilibrium. First, we will analyze how the number of agents and their selling prices distribution will affect both the convergence time and the equilibrium results. We will also show how the increase of the number of slots result in an increase of the absolute convergence time and a decrease of the efficiency, but a decrease of the average number of updates per advertiser.

5.2. Convergence

In this section we will present some experimental results that illustrate how fast an auction based on $\mathcal{M}_{ord}^{GSP}(p^*)$ implementing the cut price dynamics $\mathcal{D}$ converges to its equilibrium when the advertisers updates their bids accordingly to the EBBPD bidding strategy in an asynchronous and randomized way. At each time t, a random advertiser activates and eventually submits his bid. We will consider, for the analysis of the convergence time, the average number of bids updates per agent (BPA), i.e. the average number of times that an agent activates and submits a bid different from his previous one. We will also provide some experiments that show how the auction settings described in the previous section affect the convergence time.

5.2.1. Experiment 1

In the first experiment, we wanted to see how the number of slots and the number of advertisers affected the average BPA. In particular, we run the experiment in three different settings. For each one, the prominence of the slots follow a geometric series of parameter $\frac{1}{2}$, the buy rate of the agents is drawn from a uniform distribution $\mathcal{U}(0,1)$ and the production costs c are drawn from $\mathcal{U}(0, p_i)$. The initial cut price is always $p^* = 0$, so that all the agents will take part to the auction. All the datasets are the results of at least 800 instances per couple (number of agents, number of slots).

Figure 5.3 shows the results when the agents selling prices are drawn from a uniform distribution $\mathcal{U}(50, 100)$, with a quality function $ReL(-\frac{1}{50}, 1, \epsilon)$, with $\epsilon \to 0$[1].

The first thing that we can notice is that the average number of bids updates is quite small, mainly between 2 and 5 activations per advertiser on average. As expected, we see that the average number of iterations increases with the number of slots. The exception to this trend is showed for 5 and 6 slots when 5 advertisers are competing: being $N \leq M$ the BPA is the same in the two scenarios, but it is interesting to notice that it is also lower than an auction with the same number of advertisers and less slots. Finally, the less predictable result is that the BPA decrease with the number of advertisers with any

[1]The choice of $\epsilon \to 0$ rise from computational reasons where q_i/q_j need to be executed.

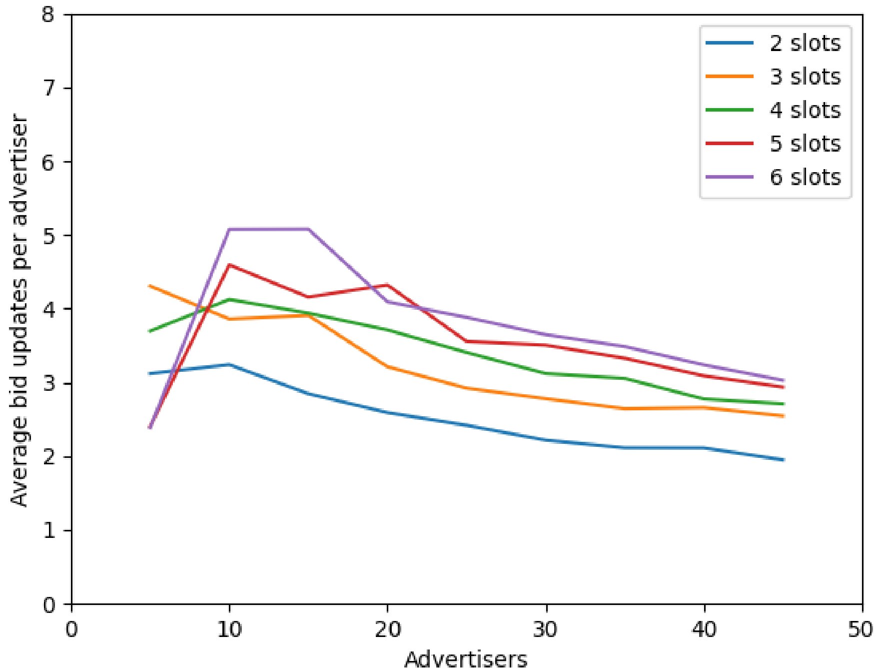

Figure 5.3: Convergence time for $q : ReL(-0.02, 1, \epsilon)$, $p \sim \mathcal{U}(50, 150)$

number of slots.

In Figure 5.4 agents selling prices are drawn from a normal distribution $\mathcal{N}(100, 20)$, with a quality function $ReL(-\frac{1}{50}, 1, \epsilon)$.

As we can see, the results are consistent with the outcomes of the prices drawn form a uniform distribution. The same considerations for the dependencies of the BPA to the number of agents and number of slots can be extended to this setting.

In Figure 5.5 agents selling prices are drawn from a normal distribution $\mathcal{N}(100, 20)$ and the quality function is $Sig(1, 20, 2, 0.001)$. We can notice that the relation of the BPA with the number of advertisers and the number of slots is coherent with the previous results. From the point of view of the performances, instead, the experiments show that the proposed sigmoid quality function converges on average faster than the corresponding scenario with a linear quality.

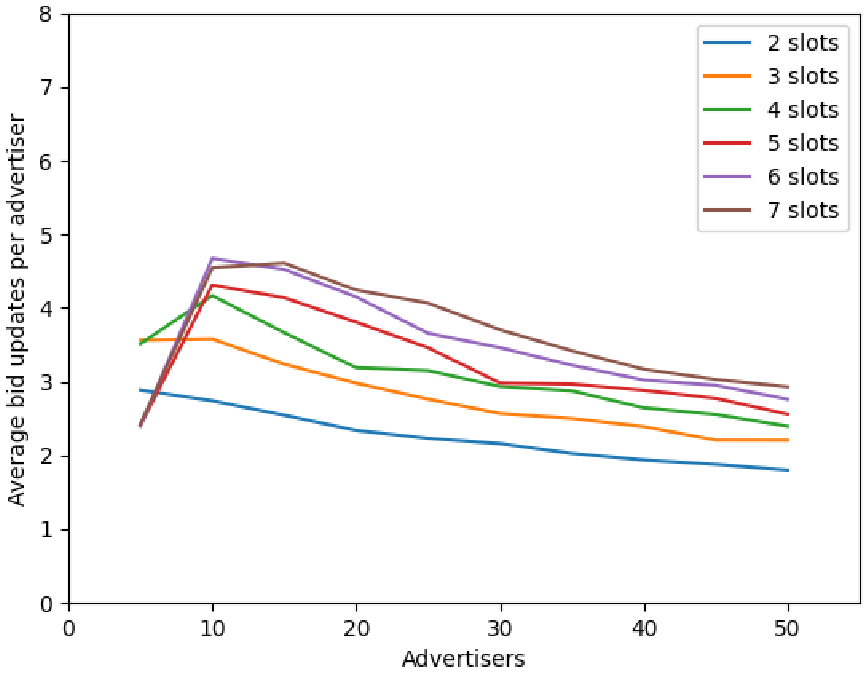

Figure 5.4: Convergence time for $q : ReL(-0.02, 1, \epsilon)$, $p \sim \mathcal{N}(100, 20)$

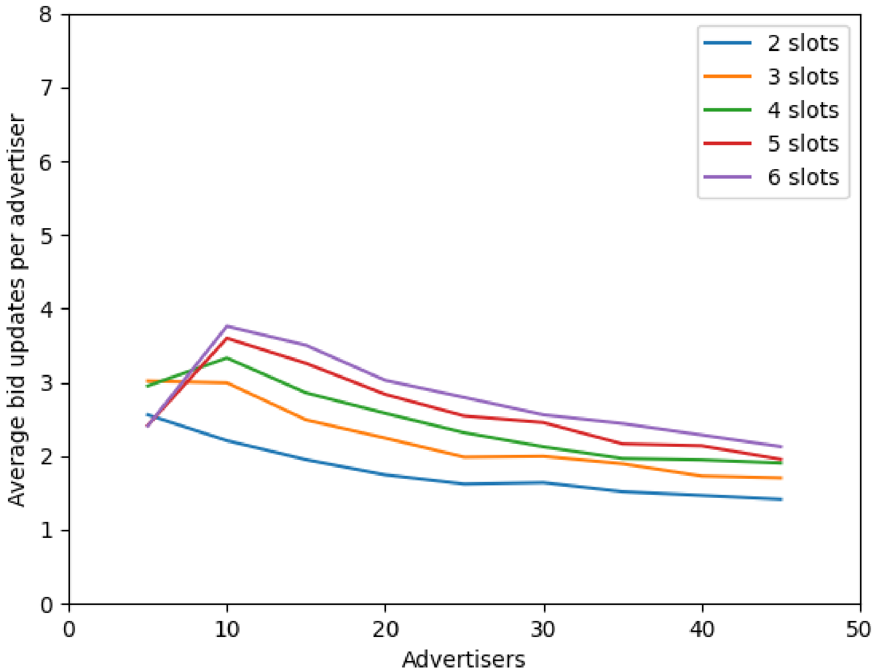

Figure 5.5: Convergence time for $q : Sig(1, 20, 2, 0.001)$, $p \sim \mathcal{N}(100, 20)$

5.3. Efficiency

In this section we will propose some experimental results that measure the efficiency of the equilibrium reached by mechanism $\mathcal{M}_{ord}^{GSP}$ implementing the cut price dynamics $\mathcal{D}$ when agents bids accordingly to the EBBPD bidding strategy. For each experiment, we will estimate the ratio between the true social welfare of the equilibrium and the true social welfare of the optimum equilibrium for the set of advertisers taking part to the auction. We will also compare the experimental results with the theoretical guarantees studied in Chapter 4.

5.3.1. Experiment 2

In this experiment we want to compare how the two proposed quality functions $ReL(\cdot)$ and $Sig(\cdot)$ and the selling prices distributions affects the social welfare of the equilibrium at the increasing of the number of advertisers and the number of slots. For all the settings, the prominence of the slots is a geometric progression of parameter $\frac{1}{2}$ and both the production costs and the buy rates are draw from a uniform distribution. The initial cut price for each auction is selected accordingly to Equation 4.2 and Equation 4.4. The number of samples for each pair of agents number and slots number is at least 700.

Figure 5.6 presents the results for the quality function $ReL(-0.02, 1, \epsilon)$ when selling prices are drawn from a $\mathcal{N}(100, 20)$ distribution. To select the initial cut price, we relied on the theoretical results described by Theorem 4.5 and Corollary 4.2. Being the agents' selling prices drawn from a normal distribution $N(100, 20)$ we can estimate the theoretical guarantee of the social welfare by considering the range of prices in [50, 150], being $P(50 < p < 150) = 0.98759$. The initial cut price is selected as

$$p^* \in \boldsymbol{P}(Part_{\bar{q}}(N))$$

where

$$\bar{q} = \arg\max_{q} \frac{q}{\left\lceil \frac{100 \cdot 0.02}{1-\bar{q}} \right\rceil} = 0.5$$

The theoretical guarantee of the social welfare accordingly to Equation 4.2 and Eq 4.4 is $\frac{1}{8}$ of the optimal allocation. It can be noticed that, by using Equation 4.3 instead of Equation 4.4 the result does not change, due to the fact that the quality function is linear in the difference between p_i and p_{min}.

The main thing we can notice in the experiment is that the empirical efficiency is much higher than the theoretical guarantee, spanning in the range of [0.82, 0.92] for the given

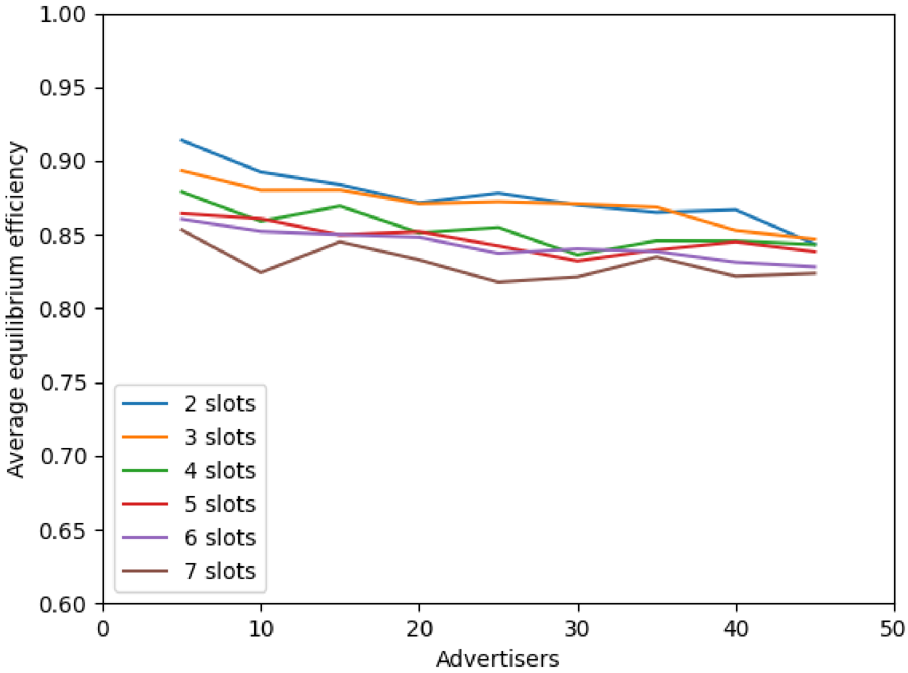

Figure 5.6: Average efficiency for $q : ReL(-0.02, 1, \epsilon)$, $p \sim \mathcal{N}(100, 20)$

number of agents and slots. Moreover, there are two main experimental evidences that we can notice. First of all, the slightly efficiency decrease with the number of agents. This is an expected result, since as the number of advertisers increases the probability of converge to the optimal allocation decreases. Second, the efficiency decrease with the number of slots. An explanation to this phenomenon relies on the fact that when more slots are available, the probability of allocating an ad such that his price is equal to the cut price increase, and thus some better allocation associated to higher selling prices are not reached.

Figure 5.7 represents the experimental results with a quality function $Sig(1, 20, 2, 0.001)$ and $p \sim \mathcal{N}(100, 20)$. We can estimate the theoretical bound of the social welfare by considering the selling prices in the range [50, 150]. The threshold quality for the initial cut price selection accordingly to Equation 4.2 and Equation 4.4 is

$$\bar{q} = \arg\max_{q} \frac{q}{\left\lceil \frac{100 \cdot 0.052249}{1-q} \right\rceil} \approx 0.4667$$

with a theoretical social welfare guarantee of 0.04667 of the optimal allocation. Notice that, by using Equation 4.3 instead of Equation 4.4 to compute partition's cardinality bound, the threshold quality would be

$$\bar{q} = \arg\max_{q} \frac{q}{\left\lceil \frac{100}{12.512522} \right\rceil} \approx 0.64838$$

with a theoretical social welfare guarantee of 0.08105.

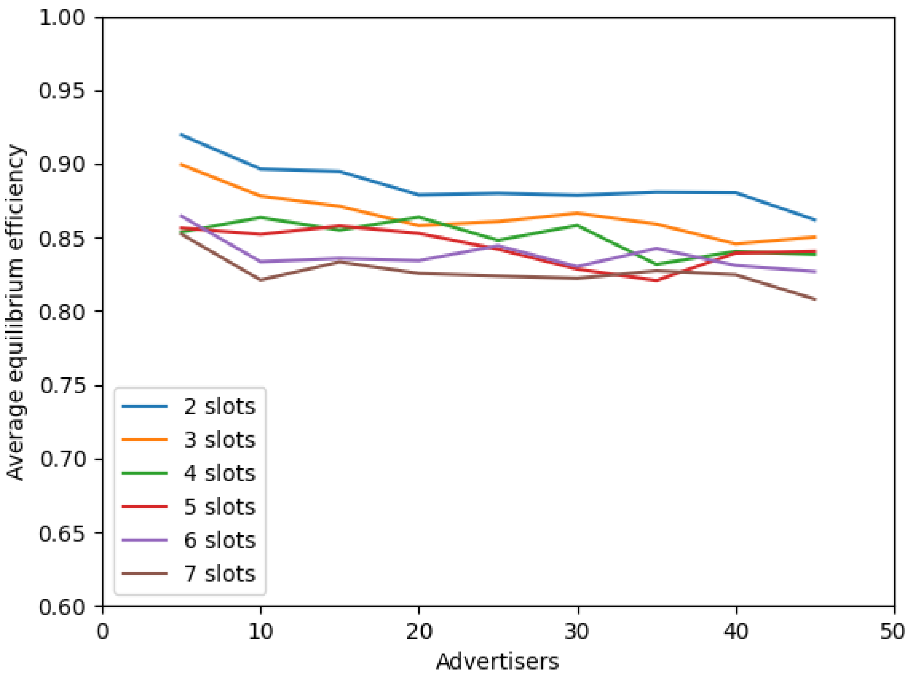

Figure 5.7: Average efficiency for $q : Sig(1, 20, 2, 0.001)$, $p \sim \mathcal{N}(100, 20)$

As we can notice, the results in terms of efficiency are analogous to the one obtained in Figure 5.6: the efficiency is much higher than the theoretical guarantees, slightly decreasing in both the number of slots and the number of advertisers.

Lastly, Figure 5.8 shows the results when the quality function is $Sig(1, 20, 2, 0.001)$ and $p \sim \mathcal{U}(50, 150)$. It is interesting to compare these results to the ones in Figure 5.7: in the very same context, the efficiency of the equilibrium drops from the range [0.8, 0.9] to the range [0.65, 0.75] just by changing the distribution of the selling prices from a gaussian to a uniform. This is due to the fact that, when the selling prices are uniformly distributed, the selling price difference among the agents grows, and thus lower prices allocations (that are more likely to be associated with a lower social welfare) are assigned more frequently.

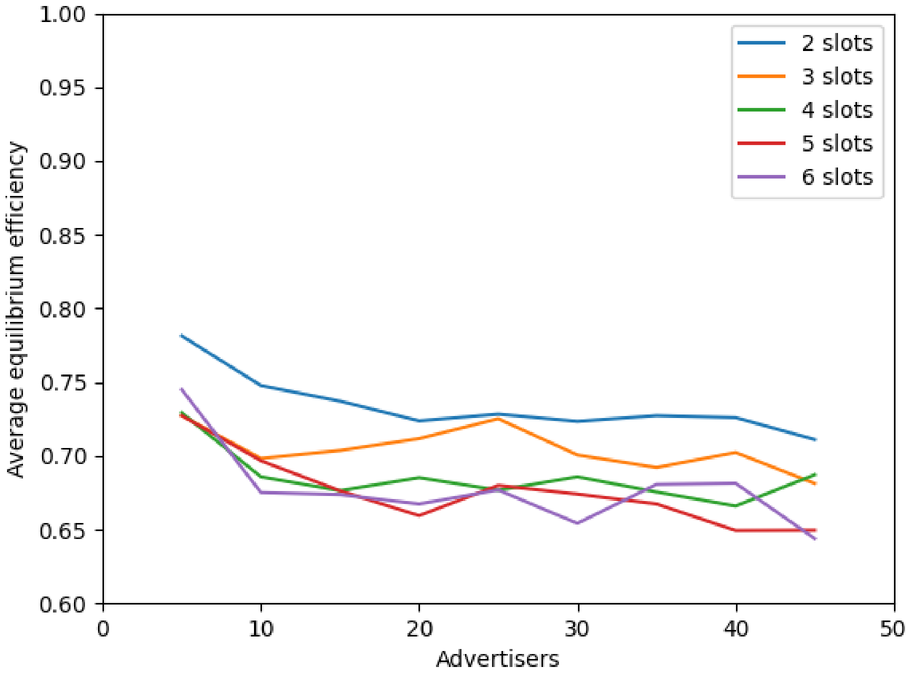

Figure 5.8: Average efficiency for $q : Sig(1, 20, 2, 0.001)$, $p \sim \mathcal{U}(50, 150)$

5.3.2. Experiment 3

In this last experiment we tested how the range of selling prices affects the efficiency of the equilibrium. For this experiment we considered a quality function shaped as a ReL(-0.02, 1, ϵ) and five slots whose prominence follow a geometric series of constant 0.5. The results are showed for selling prices drawn from a uniform distribution of mean 100 and increasing range, while $c_i \sim U(0, p_i)$ and $\alpha \sim U(0, 1)$. Figure 5.9 summarizes the experimental results, with at least 700 instances per point.

It can be immediately noticed how the increase of the price range results in a decrease of the equilibrium efficiency. This confirm the outcome noticed in the Experiment 2, where the empirical results suggested a loss of efficiency of the uniform distribution with respect to the normal distribution in the same range. The result is expected, as when the prices range differences gets higher the advertisers are more distributed on the range, increasing the probability to reach an equilibrium where the optimal minimum price is excluded by the allocation.
This experiment is proposed with the purpuse of providing an empirical result that associate the efficiency to the selling price range. However, it should be noticed that in a real world scenario a conspicuous increase of the the prices slope will likely be associated with

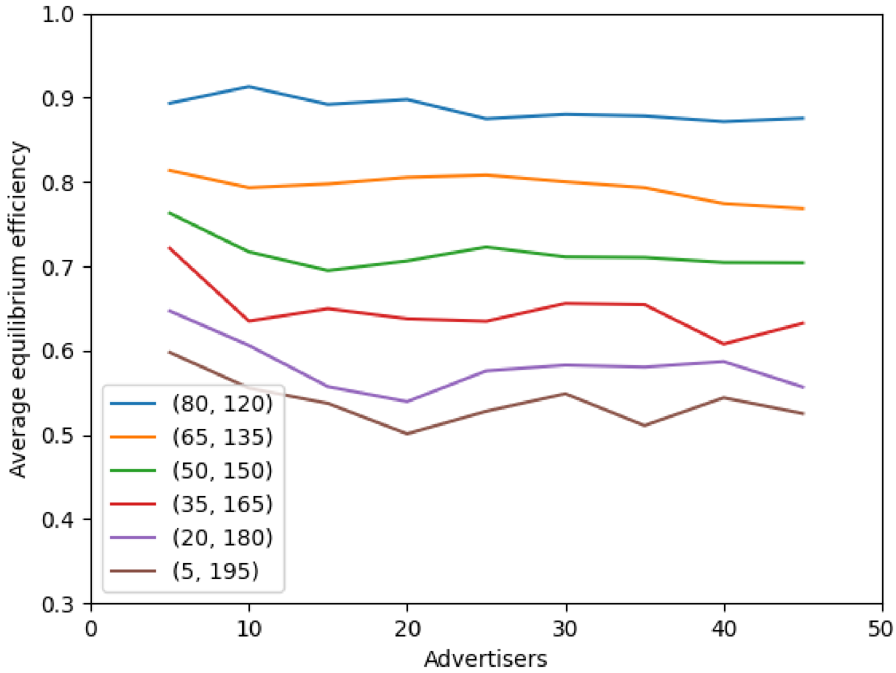

Figure 5.9: Average efficiency for $q : Rel(0.02, 1, \epsilon)$ for different selling price ranges

a change of the quality function shape.

6 | Conclusions and future work

In this final chapter we summarize the results presented in the thesis. Lastly, we conclude by analyzing the left open problems and the possible future works.

6.1. Conclusions

In this work we approached a particular case of sponsored search auctions called sponsored search auctions with price display. This type of auctions differs from the classic one by showing to the users a price for each advertised ad, introducing a new externality among the customers. We modeled this new feature as a click probability of the ads depending on the ad price and the lowest price displayed, and analyzed whether such auction converges when the advertisers bid accordingly to a rational bidding strategy.

Firstly, we formalized the price dependent version of the balance bidding strategy (EBBPD), a strategy studied by Cary et al. [3] and Bu et al. [1] that guarantees the convergence for a search auction based on GSP, and showed that the same convergence result can not be ensured if the allocation is selected by maximizing the declared social welfare.

We proposed a new GSP based allocation mechanism, namely $\mathcal{M}_{ord}^{GSP}(p^*)$, that first selects the ads to allocate by maximizing the social welfare with respect to a minimum price computed among all the advertisers over a threshold called cut price, and then selects the allocation that maximizes the social welfare with respect to the minimum price allocated. We also described a strategy $\mathcal{D}$ to dynamically update the cut price at each iteration of the auction. We showed that this mechanism is individually rational and its social choice function is a generalization of the mechanism that assigns the allocation maximizing the declared social welfare. Moreover, we showed that the mechanism is guaranteed to converge in finite time if when the advertisers bidding strategy is EBBPD and the cut price is updated with the strategy $\mathcal{D}$. Then, we bounded the efficiency of the equilibrium of the mechanism in expectation over the selection of the initial cut price.

Finally, we proposed some experimental results, empirically analyzing the convergence time and the efficiency of the equilibrium in relation the main auction features, such as the number of agents, the number of slots and their prominence, and two different quality functions, showing that the convergence is reached on average within few bids update per advertiser, with an average efficiency much higher than the theoretical bounds.

6.2. Future works

In the study of the mechanism $\mathcal{M}_{ord}^{GSP}$ we worked on the assumption of advertisers submitting to the auctioneer a selling price that does not change over time. The most interesting future research concerns how the mechanism behaves when the advertisers can update at any time both their bids and selling prices, especially in terms of convergence.
Experimental results showed that the efficiency of the equilibrium is much higher than the theoretical guarantees. Thus, future research could study whether it is possible to guarantee a stricter bound of the efficiency and if there exists a better cut price selection policy.
Another open question is how the presence of one or more advertisers that do not bid accordingly to the EBBPD bidding strategy affects the equilibrium property of the auction. In the same way, it is still unknown if non myopic advertisers can elaborate better bidding strategies that take in consideration other agents' reaction, and whether those strategies maintain some desirable properties.

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A | Appendix A

In this appendix we will provide an example of non convergence for the mechanism $\mathcal{M}_I^{GSP}$ when the advertisers select their bids accordingly with the EBBPD bidding strategy described in Chapter 4.

Consider the following settings: three agents, namely A, B and C compete in a three slots auction, such that $\lambda_1 = 1$, $\lambda_2 = 0.8$ and $\lambda_3 = 0.5$. The agents are such that $\alpha_i = 1$, $c_i = 0 \ \forall i \in \{A, B, C\}$, while the selling prices are $p_A = 7$, $p_B = 9$ and $p_C = 15$. The quality function is

$$q_i(p_i, p_{min}) = \begin{cases} 1 & \text{if } p_i = p_{min} \\ 0.5 & \text{otherwhise} \end{cases}$$

Suppose that, at some time t, the agents' bids are $b_A = 7$, $b_B = 3.375$, $b_C = 15$. Ties are broken in lexicographical order. We show that, by bidding accordingly to EBBPD, the auction will never converge to an equilibrium. The following bidding tree validate the proposition. The tree is designed as follows. Node's name represent the allocation assigned by the mechanism $\mathcal{M}_I^{GSP}$ for the corresponding set of bids: the allocated ads are, in order, the ones corresponding to the agents within the parenthesis, while other ads are not assigned to any slot. To each advertiser is associated the corresponding bid, and the advertiser whose name is marked in bold represents the one that has updated his bid in the last iteration. For example, in the state

(**B**:9.0, C:3.0) A:2.75

the agents B and C are respectively assigned to the first and the second slot, and B is the agent that updated his bid in the last iteration.

Whenever a state already in the tree is reached, the exploration on that branch stops: if, for at least one state, the bid profile of a node is the same one of all his child nodes, then the mechanism converges.

```
(C:15) A:7, B:3.375
├── (C:15, A:7, B:7.6001)
│   ├── (A:7, B:7.6001, C:5.625)
│   │   ├── (A:7, C:5.625, B:3.375)
│   │   │   ├── (C:15) A: 7, B:3.375
│   │   │   └── (C:5.625, A:2.75, B:3.375)
│   │   │       ├── (B:9, C:5.625) A:2.75
│   │   │       │   ├── (A:7, B:9, C:5.625)
│   │   │       │   │   ├── (A:7, B:9, C:5.625)
│   │   │       │   │   └── (A:7, C:5.625, B:3.375)
│   │   │       │   └── (B:9, C:3) A:2.75
│   │   │       │       ├── (A:7, B:9, C:3)
│   │   │       │       │   ├── (A:7, B:4.2, C:3)
│   │   │       │       │   │   ├── (A:7, B:4.2, C:3)
│   │   │       │       │   │   └── (C:15) A: 7, B:4.2
│   │   │       │       │   │       ├── (C:15) A:7, B:4.2
│   │   │       │       │   │       └── (C:15, A:7, B:7.6001)
│   │   │       │       │   └── (A:7, B:9, C:5.625)
│   │   │       │       └── (B:9, C:3) A:2.75
│   │   │       └── (C:15) A: 2.75, B:3.375
│   │   │           ├── (B:9, C:15) A:2.75
│   │   │           │   ├── (B:9, C:3) A:2.75
│   │   │           │   └── (C:15, A:6.5626, B:9)
│   │   │           │       ├── (B:9, C:7.7507) A:6.5626
│   │   │           │       │   ├── (A:7, B:9, C:7.7507)
│   │   │           │       │   │   ├── (A:7, B:9, C:5.625)
│   │   │           │       │   │   └── (A:7, C:7.7507, B:3.375)
│   │   │           │       │   │       ├── (C:15) A: 7, B:3.375
│   │   │           │       │   │       └── (C:7.7507, A:3.7896, B:3.375)
│   │   │           │       │   │           ├── (B:9, C:7.7507) A:3.7896
│   │   │           │       │   │           │   ├── (A:7, B:9, C:7.7507)
│   │   │           │       │   │           │   └── (B:9, C:3) A: 3.7896
│   │   │           │       │   │           │       ├── (A:7, B:9, C:3)
│   │   │           │       │   │           │       └── (B:9, C:3) A:3.7896
│   │   │           │       │   │           └── (C:15) A:3.7896, B:3.375
│   │   │           │       │   │               ├── (B:9, C:15) A:3.789
│   │   │           │       │   │               │   ├── (B:9, C:3) A:3.7896
│   │   │           │       │   │               │   └── (C:15, A:6.5626, B:9)
│   │   │           │       │   │               └── (C:15) A:7, B:3.375
│   │   │           │       │   └── (B:9, C:7.7507) A:6.5626
│   │   │           │       └── (C:15, A:6.5626, B:8.9997)
│   │   │           │           ├── (B:8.9997, C:7.7519) A:6.5626
│   │   │           │           │   ├── (A:7, B:8.9997, C:7.7519)
│   │   │           │           │   │   ├── (A:7, B:8.9997, C:5.625)
│   │   │           │           │   │   │   ├── (A:7, B:8.9997, C:5.625)
│   │   │           │           │   │   │   └── (A:7, C:5.625, B:3.375)
│   │   │           │           │   │   └── (A:7, C:7.7519, B:3.375)
│   │   │           │           │   │       ├── (C:15) A:7, B:3.375
│   │   │           │           │   │       └── (C:7.7519, A:3.7903, B:3.375)
│   │   │           │           │   │           ├── (B:9, C:7.7519) A:3.7903
│   │   │           │           │   │           │   ├── (A:7, B:9, C:7.7519)
│   │   │           │           │   │           │   └── (B:9, C:3) A:3.7903
│   │   │           │           │   │           │       ├── (A:7, B:9, C:3)
│   │   │           │           │   │           │       └── (B:9, C:3) A:3.7903
│   │   │           │           │   │           └── (C:15) A: 3.7903, B:3.375
│   │   │           │           │   │               ├── (B:9, C:15) A:3.7903
│   │   │           │           │   │               │   ├── (B:9, C:3) A:3.7903
│   │   │           │           │   │               │   └── (C:15, A:6.5626, B:9)
│   │   │           │           │   │               └── (C:15) A:7, B:3.375
│   │   │           │           │   └── (B:9, C:7.7519) A:6.5626
│   │   │           │           │       ├── (A:7, B:9, C:7.7519)
│   │   │           │           │       │   ├── (A:7, B:9, C:5.625)
│   │   │           │           │       │   └── (A:7, C:7.7519, B:3.375)
│   │   │           │           │       └── (B:9, C:7.7507) A:6.5626
│   │   │           │           └── (C:15, A:6.5626, B:8.9997)
│   │   │           └── (C:15) A:7, B:3.375
│   │   └── (A:7, B:7.6001, C:5.625)
│   └── (C:15, A:7, B:7.6001)
├── (C:15) A:7, B:3.375
└── (C:15) A:7, B:3.375
```

Figure A.1: Example of non convergence of the mechanism $\mathcal{M}_I^{GSP}$ when advertisers bid accordingly to EBBPD

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Variable	Description
N	set of advertisers
a_i	advertiser
p_i	advertiser's selling price
c_i	advertiser's production cost
α_i	advertiser's buy rate
μ_i	advertiser's gain
v_i	advertiser's value
u_i	advertiser's utility
q_i	advertiser's quality
p_{min}	minimum displayed price
M	set of slots
λ_j	slot's prominence
$\mathcal{M}_{ord}^{GSP}$	proposed mechanism
p^*	cut price
f	allocation function
σ	advertisers ordering function
π	auction price
sw	social welfare
$Part$	advertisers partition

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