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Nonlinear Parabolic Differential Equations: global existence and blow-up of solutions

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Abstract

The main topic of this thesis concerns the study of global existence and blow-up of solutions to certain nonlinear parabolic differential equations. The thesis is divided into three parts where three different equations are considered. In Part I, we analyze the Cauchy problem for the porous medium equation with a variable density, which depends on the space variable, and a power-like reaction term: this is a mathematical model of a thermal evolution of a heated plasma. Depending on the rate of decaying at infinity of the density function, by comparison method and suitable sub- and supersolutions, we determine whether the solution exists globally in time or blows up in finite time. In Part II, we consider reaction-diffusion equations posed on complete, noncompact, Riemannian manifolds of infinite volume. Such equations contain power-type nonlinearity and slow diffusion of the porous medium type. For the Cauchy problem related to this equation we prove global existence for positive initial data belonging to suitable L^p spaces, and that solutions corresponding to such data are bounded at all positive times with a quantitative bound on their L^∞ norm. The methods of proof are functional analytic in character, as they depend solely on the validity of the Sobolev and the Poincaré inequalities. In Part III, we are concerned with nonexistence results for a class of quasilinear parabolic differential equations with a potential in bounded domains. In particular, we investigate how the behavior of the potential near the boundary of the domain and the power nonlinearity affect the nonexistence of solutions.

Sommario

L'argomento principale della tesi é lo studio dell'esistenza globale e del blow-up di soluzioni ad alcune equazioni differenziali paraboliche nonlineari. La tesi ÃÍ suddivisa in tre parti in ciascuna delle quali si prende in considerazione una diversa equazione. Nella Parte I, viene analizzato un problema di Cauchy per una equazione dei mezzi porosi con densitá variabile che dipende solo dallo spazio, e un termine di diffusione del tipo potenza: questa equazione rappresenta un modello matematico per l'evoluzione della temperatura del plasma. Utilizzando metodi di sotto- e soprasoluzioni, grazie anche al principio del confronto, si determina quando la soluzione del problema esiste globalmente in tempo e quando invece avviene blow-up in tempo finito. Nella seconda parte, Part II, si studia una classe di equazioni di reazione-diffusione definita su varietá Riemanniane complete, noncompatte e di volume infinito. Queste equazioni contengono nonlinearitá di tipo potenza e una diffusione lenta del tipo mezzi porosi. Per il problema di Cauchy relativo a queste equazioni, si dimostra esistenza globale in tempo delle soluzioni per dati iniziali positivi e che siano appartenenti ad opportuni spazi L^p. Inoltre, per queste soluzioni, si dimostra che esse sono limitate per tutti i tempi e si propone una stima quantitativa sulla loro norma L^∞. I metodi utilizzati per le dimostrazioni sono funzionali e si basano principalmente sulla validitá delle disuguaglianze di Sobolev e Poincaré. Infine, nella Part III, si studia la nonesistenza di soluzioni per una classe di equazioni differenziali paraboliche quasilineari con un potenziale, definite in domini limitati. In particolare, si mostra come il comportamento del potenziale vicino alla frontiera del dominio e la nonlinearitá di tipo potenza influenzano la nonesistenza delle soluzioni.

Contents

Introduction

I.1 The problems

In this thesis we investigate global existence and blow-up of solutions to nonlinear degenerate parabolic partial differential equations on both the Euclidian space and more general complete noncompact Riemannian manifolds. Specifically, we address equations of the following type:

$$\rho(x)\, u_t = \Delta(u^m) + \rho(x)\, u^p \quad \text{in } \mathbb{R}^N \times (0,T), \tag{I.1.1}$$

$$u_t = \Delta(u^m) + u^p \quad \text{in } M \times (0,T), \tag{I.1.2}$$

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = V\, u^q \quad \text{in } \Omega \times (0,T), \tag{I.1.3}$$

where $T > 0$.

In equation (I.1.1), $N \geq 3$ and ρ is a suitable positive function, to which we refer as *weight* from here on; such equation arises in various physical models, see [73] and Section I.2.1. Moreover, we assume that $p > 1$ and $m > 1$. More precisely, we are concerned with global existence and blow-up of solutions to the Cauchy problem associated to equation (I.1.1) when $\rho(x) \sim |x|^{-q}$ as $|x| \to +\infty$ with $q \geq 0$ or $\rho(x) \sim (\log|x|)^{\alpha}\, |x|^{-2}$ as $|x| \to +\infty$ with $|\alpha| > 1$, see Chapters 1, 2, 3.

In equation (I.1.2), M is a complete, noncompact, Riemannian manifold of infinite volume and dimension $N \geq 3$; moreover $p > 1$ and $m > 1$. In Chapters 4, 5, we investigate existence of global in time solutions to the Cauchy problem for equation (I.1.2), depending on the initial datum providing also suitable estimates on the L^{∞}-norm of the solutions for $t > 0$.

Finally, in equation (I.1.3), Ω is an open bounded connected subset of $\mathbb{R}^N$. Moreover, we assume that $p > 1$, $q > \max\{p-1, 1\}$ and $V = V(x,t)$ is a given positive function, to which we refer as *potential* from here on. We study nonexistence of nonnegative, nontrivial global weak solutions to the Cauchy-Dirichlet problem associated to equation (I.1.3), see Chapter 6.

The thesis is organized in three parts which correspond to equations (I.1.1), (I.1.2) and (I.1.3), respectively. Chapters 1, 2 and 3 are contained in Part I. Chapters 4 and 5 are contained in Part II whereas Part III is Chapter 6. In what follows, for each part, we give a brief overview of known results in literature and we outline our main results.

I.2 Part I: The inhomogeneous porous medium equation with reaction on $\mathbb{R}^N$

I.2.1 A survey of the literature

The problem of global solvability of nonlinear evolution problems, such as the Cauchy problem associated to equation (I.1.1), occupies a special place in the theory of nonlinear equations. We say that a problem is globally solvable in time if it admits a bounded solution for any $t \in (0, +\infty)$. On the contrary, we say that the solution to a given problem blows up in finite time when there exists a time $S > 0$ such that

$$\|u(t)\|_\infty \longrightarrow +\infty \quad \text{as } t \to S^- ,$$

If $S = +\infty$ then we say that the blow up occurs in infinite time.

The differential equation in (I.1.1) for $N = 1$, posed in the interval $(-1, 1)$ with homogeneous Dirichlet boundary conditions, has been introduced in [73] as a mathematical model of evolution of plasma temperature, where u is the temperature, $\rho(x)$ is the particle density and $\rho(x)u^p$ represents the volumetric heating of plasma. The interest in thermal waves arises in plasma physics in various laboratory and terrestrial situations where the ambience is at rest but cannot be considered homogeneous. Indeed, in [73, Introduction] a more general source term of the type $A(x)u^p$ has also been considered; however, then the authors assume that $A \equiv 0$; only some remarks for the case $A(x) = \rho(x)$ are made in [73, Section 4].

Equation (I.1.1) is a generalization of the very well-known *Porous Medium Equation* (PME), that is,

$$u_t = \Delta(u^m), \quad \text{in } \Omega \times (0, +\infty) \quad (m > 1)\,, \tag{I.2.4}$$

where Ω is a domain of $\mathbb{R}^N$. The PME is one of the simplest examples of a nonlinear evolution equation of parabolic type. It appears in the description of different natural phenomena such as the flow of a fluid through a porous medium [81, 101], the study of groundwater infiltration [14] or the heat radiation in plasmas [136]. The equation can be posed both in $\Omega \equiv \mathbb{R}^N$ or in bounded subdomains $\Omega \subset \mathbb{R}^N$ and completed with initial and boundary conditions. Observe that, from (I.2.4) we get

$$u_t = \operatorname{div}\left(mu^{m-1}\nabla u\right) \text{ in } \Omega \times (0, +\infty).$$

The diffusivity coefficient appearing in the latter is mu^{m-1} that motivates the finite speed of propagation of solutions to (I.2.4). This means that, if for instance the initial value has compact support, then the solution has compact support for every fixed time. This is the main difference between the solutions to (I.2.4) and the solutions to the heat equation (that is (I.2.4) with $m = 1$). As for the heat equation, also the PME has a family of fundamental solutions whose existence has been shown in [10]. Their explicit form is

$$U(x,t) = t^{-\alpha}\left[C - k|x|^2 t^{-2\beta}\right]_+^{\frac{1}{m-1}}\,, \tag{I.2.5}$$

where $(A)_+ := \max\{A, 0\}$, C is an arbitrary positive constant depending on the mass $M = \int_{\mathbb{R}^N} U(x,t)\,dx$, which is constant in time, and

$$\alpha = \frac{N}{N(m-1)+2}, \quad \beta = \frac{\alpha}{N}, \quad k = \frac{\alpha(m-1)}{2mN}\,.$$

The functions in (I.2.5) are usually referred to as *Barenblatt solutions*. We refer the reader to [5, 103, 128] for a complete overview of the PME equation.

The inhomogeneous version of equation (I.2.4) has also been widely examined in literature. It is given by

$$\rho(x)u_t = \Delta(u^m), \quad \text{in } \Omega \times (0, +\infty) \quad (m > 1)\,, \tag{I.2.6}$$

where Ω is a domain of $\mathbb{R}^N$. The case when $\Omega \equiv \mathbb{R}^N$ and ρ decays at infinity as a negative power of $|x|$, is the most studied one, see e.g. [25, 26, 27, 62, 63, 64, 65, 66, 67, 68, 69, 70, 116, 117, 120]. Moreover, in [71] and [72] the Cauchy problem related to equation (I.2.6) for $N = 1$ is investigated. They explain that equation (I.2.6) models the propagation of a nonlinear thermal wave in an inhomogeneous medium. They suppose that ρ is a positive and smooth function. Let

$$M = \int_{-\infty}^{+\infty} \rho(x)\,dx\,,$$

then the authors investigate the behavior of solutions for both the cases of $M < \infty$ and $M = \infty$ showing remarkable differences between them. Moreover, in [109] for $N \geq 3$, assuming that $\rho(x) \sim |x|^{-q}$, it is proved that equation (I.2.6), for any bounded initial datum u_0, has infinitely many *very weak* solutions if $q > 2$ whereas it admits a unique *very weak* solution if $q \leq 2$. This different behavior of solutions determines $q = 2$ as a threshold value.

Equation (I.2.6) can be further generalized to the following weighted PME

$$\rho_\nu(x)u_t = \operatorname{div}\left[\rho_\mu(x)\nabla(u^m)\right], \quad \text{in } \Omega \times (0, +\infty) \quad (m > 1)\,, \tag{I.2.7}$$

where ρ_ν and $\rho_\mu > 0$ are two weights independent of the time variable. With no claim of generality, we refer the reader e.g. to [49]. Depending on the behavior of ρ_ν and ρ_μ as $|x| \to \infty$, existence and uniqueness and the asymptotic behavior of *energy* solutions for large times have been addressed.

We also recall the well known semilinear heat equation defined as follows

$$u_t = \Delta u + f(u) \quad \text{in } \Omega \times (0, T), \tag{I.2.8}$$

where $T > 0$, Ω is a possibile unbounded domain of $\mathbb{R}^N$ and $f(u)$ is a nonnegative function, thus we are in presence of reaction. The classical choice in equation (I.2.8) is $f(u) = u^p$ for $p > 1$. Such equation models various natural phenomena. Here we have a competition between the diffusion due to the Laplacian and the reaction term, which may drive the solution towards blow-up. In particular, we mention the pioneering work by Fujita [31] where global existence and blow-up of solutions to the Cauchy problem associated to (I.2.8) is investigated when $\Omega \equiv \mathbb{R}^N$. It is shown that

- finite time blow-up occurs for all nontrivial nonnegative initial data, for any
$$1 < p \leq 1 + \frac{2}{N};$$
- global existence of solutions for sufficiently small initial data, for any
$$p > 1 + \frac{2}{N}.$$

The value $p_c = 1 + \frac{2}{N}$ is usually referred to as *the Fujita exponent.* We remark that the critical case $p = p_c$ was left open by Fujita, it was proved later in [58, 77, 133]. In [35] the authors propose a different method to obtain the Fujita exponent in terms of sub and supersolutions and comparison principles. Moreover they apply this method to different reaction-diffusion problems. Observe that, if $p > 1$, due to Kaplan's argument we can say that the solution blows up if it accumulates enough mass, and this is the case of all solutions if $1 < p < p_c$. On the contrary, if $p > p_c$, diffusion does not allow small initial values to grow, and in fact the solutions tend to zero. The fact that when $p > p_c$ there exist small global solutions is easily proved by comparison with a supersolution. Finally we refer the reader to another way of proving Fujita's result introduced in [82, 84].

For more details on equation (I.2.8) for a general nonnegative function $f(u)$ we refer the reader for instance to [24, 29, 30, 31, 32, 58, 74, 83, 114, 121, 135]. Also the weighted version of equation (I.2.8) has been studied in literature, see e.g. [21, 85]. In particular, in [85], they consider the Cauchy problem associated to equation

$$\rho(x)u_t = \Delta u + \rho(x)u^p \quad \text{in } \mathbb{R}^N \times (0, T) \quad (p > 1),$$

where $T > 0$ and $\rho \sim |x|^{-q}$ as $|x| \to \infty$ for $0 \le q < 2$. Is is shown that

- solutions blow-up in finite time, for all nontrivial nonnegative initial data, for any
$$1 < p \le 1 + \frac{2-q}{N-q};$$
- global in time solutions exist for sufficiently small nonnegative initial data, for any
$$p > 1 + \frac{2-q}{N-q}.$$

We also recall the well known nonlinear parabolic equation

$$u_t = \Delta(u^m) + u^p \quad \text{in } \Omega \times (0, +\infty), \qquad (m \ge 1,\ p > 1), \tag{I.2.9}$$

where Ω is a domain of $\mathbb{R}^N$. Equation (I.2.9) is usually referred to as the *Porous Medium Equation with reaction.* The Cauchy problem related to equation (I.2.9) with nonnegative continuous initial datum has been mainly investigated in [99, 119]. In the case of $\Omega \equiv \mathbb{R}^N$, it is shown that the Fujita exponent is $p_c = m + \frac{2}{N}$. More precisely, we have

- finite time blow-up for all nontrivial nonnegative initial data, for any
$$1 < p < m + \frac{2}{N};$$
- global existence in time of solutions for sufficiently small nonnegative initial data, for any
$$p > m + \frac{2}{N};$$

- finite time blow-up for sufficiently large nonnegative initial data, for any

$$p > 1.$$

The results in [119] has been proved by means of comparison principles and suitable sub- and supersolutions of the form

$$u(x,t) = C\zeta(t)\left[1 - \frac{|x|^2}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \mathbb{R}^N \times [0,T),$$

where $\zeta(t)$ and $\eta(t)$ are appropriate auxiliary functions and C and a are positive constants.

The Cauchy problem related to equation (I.1.1) has also been investigated in [86, 87]. More precisely, they consider a class of double-nonlinear operators among which equation (I.1.1) is included. They show that, (see [86, Theorem 1]) if $\rho(x) = |x|^{-q}$ with $q \in (0,2)$, for any $x \in \mathbb{R}^N \setminus \{0\}$,

$$p > m + \frac{2-q}{N-q},$$

the initial datum u_0 is nonnegative and

$$\int_{\mathbb{R}^N} \left\{u_0(x) + [u_0(x)]^{\bar{q}}\right\} \rho(x)dx < \delta,$$

for some $\delta > 0$ small enough and $\bar{q} > \frac{N}{2}(p-m)$, then there exists a global solution of the Cauchy problem associated to (I.1.1). In addition, a smoothing estimate holds. On the other hand, if $\rho(x) = |x|^{-q}$ or $\rho(x) = (1+|x|)^{-q}$ with $q \in [0,2)$, for any initial datum $u_0 \not\equiv 0$ and

$$p < m + \frac{2-q}{N-q},$$

then blow-up prevails, in the sense that there exist $\theta \in (0,1), R > 0, T > 0$ such that

$$\int_{B_R} [u(x,t)]^{\theta} \rho(x)dx \to +\infty \quad \text{as } t \to T^-.$$

Such results have also been generalized to more general initial data, decaying at infinity with a certain rate (see [87]).

I.2.2 Outline of the results

In Chapter 1 (where [92] is reproduced) we address the inhomogeneous porous medium equation with reaction of the form (I.1.1). We assume that the function $\rho : \mathbb{R}^N \to (0, +\infty)$ is such that

$$\begin{aligned} &\text{(i) } \rho \in C(\mathbb{R}^N), \\ &\text{(ii) there exist } k_1, k_2 \in (0,+\infty), \text{ with } k_1 \le k_2, \text{ and } 0 \le q < 2 \text{ such that} \\ &\qquad k_1|x|^q \le \frac{1}{\rho(x)} \le k_2|x|^q \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0)\,. \end{aligned} \tag{I.2.10}$$

Due to hypotheses (I.2.10), we refer to $\rho(x)$ as a *slowly decaying density* at infinity.

We investigate global existence and blow-up of solutions to the Cauchy problem

$$\begin{cases} u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p & \text{in } \mathbb{R}^N \times (0,T) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{I.2.11}$$

where the initial datum $u_0 : \mathbb{R}^N \to [0, +\infty)$ is a compactly supported function and it is such that

$$u_0 \in L^\infty(\mathbb{R}^N).$$

Therefore, the related diffusion operator is $\frac{1}{\rho(x)}\Delta$, and in view of $(i)-(ii)$, the coefficient $\dfrac{1}{\rho(x)}$ can diverge at infinity.

For problem (I.2.11), we prove global existence in time or blow-up in finite time of solutions, depending on the interplay between $p > 1$ and $m > 1$. In particular, suppose that

$$\frac{k_2}{k_1} < m + \frac{(m-1)(N-2)}{2-q},$$

then we introduce the values

$$\bar{p} := \frac{m(N-q) + \frac{2-q}{m-1}(m - \frac{k_2}{k_1})}{N - 2 + \frac{2-q}{m-1}\left(m - \frac{k_2}{k_1}\right)},$$

$$\underline{p} = \frac{m\,(N-q) + \frac{2-q}{m-1}\left(m - \frac{k_1}{k_2}\right)}{N - 2 + \frac{2-q}{m-1}\left(m - \frac{k_1}{k_2}\right)}.$$

It can be easily checked that

$$\underline{p} \le \bar{p}\,.$$

In particular, $\underline{p} = \bar{p}$ whenever $k_1 = k_2$. Then we prove that

- for $p > \bar{p}$, if the initial datum, $u_0 \in L^\infty(\mathbb{R}^N)$, is small enough, then global solutions to problem (I.2.11) exist;
- for any $p > 1$, if the initial datum is sufficiently large, the solutions of problem (I.2.11) blow-up in finite time;
- for $1 < p \le m$, then for any $u_0 \not\equiv 0$, solutions to problem (I.2.11) blow-up in finite time;
- for $m < p \le \underline{p}$, if in addition $q \in [0, \epsilon)$ for $\epsilon > 0$ small enough, then for any $u_0 \not\equiv 0$, solutions to problem (I.2.11) blow-up in finite time.

In the special case, $k_1 = k_2$, the results stated so far can be understood as follows

- for $p > m + \frac{2-q}{N-q}$ and for small enough initial data with $u_0 \in L^\infty(\mathbb{R}^N)$, global solutions to problem (I.2.11) exist;

- for any $p > 1$ and for sufficiently large initial data, solutions of problem (I.2.11) blow-up in finite time;
- for $1 < p \leq m$, then for any $u_0 \not\equiv 0$, solutions to problem (I.2.11) blow-up in finite time;
- for $m < p \leq m + \frac{2-q}{N-q}$, if in addition $q \in [0, \epsilon)$ for $\epsilon > 0$ small enough, then for any $u_0 \not\equiv 0$, solutions to problem (I.2.11) blow-up in finite time.

Our proofs mainly rely on suitable comparison principles and properly constructed sub- and supersolutions. Let us mention that the arguments exploited in [119] cannot be directly used in our case, due to the presence of the coefficient $\rho(x)$. In fact, we construct appropriate sub- and supersolutions, which crucially depend on the behavior at infinity of the inhomogeneity term $\rho(x)$. More precisely, whenever $|x| > 1$, they are of the type

$$w(x,t) = C\zeta(t)\left[1 - \frac{|x|^b}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \left[\mathbb{R}^N \setminus B_1(0)\right] \times [0,T),$$

for suitable functions $\zeta = \zeta(t)$, $\eta = \eta(t)$ and constants $C > 0$, $a > 0$, where $b := 2 - q$. In view of the term $|x|^b$ with $b \in (0, 2]$, we cannot show that such functions are sub- and supersolutions in $B_1(0) \times (0, T)$. Thus we have to extend them in a suitable way in $B_1(0) \times (0, T)$. In order to extend our sub- and supersolutions, we need to impose some extra conditions on $\zeta = \zeta(t)$, $\eta = \eta(t)$, C and a. Thus, it appears a sort of interplay between the behavior of the density $\rho(x)$ in compact sets, say $B_1(0)$, and its behavior for large values of $|x|$.

Finally, let us comment about the proofs of the blow-up result for any nontrivial initial datum. For $1 < p \leq m$, the result follows by a direct application of the previous results. For $m < p < \underline{p}$, the proof is more involved. The corresponding result for the case $\rho \equiv 1$ established in [119] is proved by means of the Barenblatt solutions of the porous medium equation

$$u_t = \Delta(u^m) \quad \text{in } \mathbb{R}^N \times (0, +\infty).$$

In our situation, we do not have self-similar solutions, since our equation in (I.2.11) is not scaling invariant, in view of the presence of the term $\rho(x)$. Indeed, we construct a suitable subsolution z of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) \quad \text{in } \mathbb{R}^N \times (0, +\infty).$$

By means of z, we can show that after a certain time, the solution u of problem (I.2.11) is sufficiently large, then we get finite time blow-up as in the previous situation.

In Chapter 2 (where [93] is reproduced) we study problem (I.2.11) with the following assumptions on $\rho : \mathbb{R}^N \to (0, +\infty)$

$$\begin{aligned} &\text{(i) } \rho \in C(\mathbb{R}^N), \\ &\text{(ii) there exist } k_1, k_2 \in (0, +\infty), \text{ with } k_1 \leq k_2,\ r_0 > 0 \text{ and } q \geq 2 \text{ s.t.} \\ &\qquad k_1(|x| + r_0)^q \leq \frac{1}{\rho(x)} \leq k_2(|x| + r_0)^q \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0). \end{aligned} \tag{I.2.12}$$

Due to hypotheses (I.2.12), we refer to $\rho(x)$ as a *fast decaying density* at infinity. We distinguish between two cases: $q = 2$ and $q > 2$.

First, assume that (I.2.12) holds with $q = 2$, then we prove that

- for any $p > m$, if the initial datum $u_0 \in L^\infty(\mathbb{R}^N)$ is small enough, then there exist global in time solutions to problem (I.2.11), which belong to $L^\infty(\mathbb{R}^N \times (0, +\infty))$;
- for any $p > m$, if the initial datum u_0 is sufficiently large, then solutions to problem (I.2.11) blow-up in finite time.

The proofs mainly relies on suitable comparison principles and properly constructed sub- and supersolutions, which crucially depend on the behavior at infinity of the inhomogeneity term $\rho(x)$. More precisely, they are of the type

$$w(x,t) = C\zeta(t)\left[1 - \frac{\log(|x| + r_0)}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \left[\mathbb{R}^N \setminus B_1(0)\right] \times [0,T),$$

for suitable functions $\zeta = \zeta(t)$, $\eta = \eta(t)$ and constants $C > 0$, $a > 0$. The presence of $\log(|x| + r_0)$ in w is strictly related to the assumption that $q = 2$. Observe that the barriers used in the slowly decaying density case, i.e. $0 \le q < 2$, which are of power type in $|x|$, do not work in the present situation. Furthermore, note that the exponent $\bar{p}$ introduced before for $0 \le q < 2$, when $q = 2$ becomes $\bar{p} = m$.

Now, assume that $q > 2$. We have the following results

- for $1 < p < m$, if $u_0 \in L^\infty(\mathbb{R}^N)$ then there exist global in time solutions to problem (I.2.11). We do not assume that u_0 has compact support, but we need that it fulfills a decay condition as $|x| \to +\infty$. However, u_0 in a compact subset of $\mathbb{R}^N$ can be arbitrarily large. We cannot deduce that the corresponding solution belongs to $L^\infty(\mathbb{R}^N \times (0, +\infty))$, but it is in $L^\infty(\mathbb{R}^N \times (0, \tau))$ for each $\tau > 0$.
- for $p > m \ge 1$, if $u_0 \in L^\infty(\mathbb{R}^N)$ then problem (I.2.11) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$. We need that

  $$0 \le u_0(x) \le CW(x) \quad \text{for all } x \in \mathbb{R}^N,$$

  where $C > 0$ is small enough and $W(x)$ is a suitable function, which vanishes as $|x| \to +\infty$.
- for $p = m > 1$, if $u_0 \in L^\infty(\mathbb{R}^N)$ then problem (I.2.11) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$, provided that $r_0 > 0$ in (I.2.12) is big enough.

Such results are very different with respect to the cases $0 \le q < 2$ and $q = 2$. In fact, we do not have finite-time blow-up, but global existence, for suitable initial data always prevails. The results follow by comparison principles, once we have constructed appropriate supersolutions, that have the form

$$w(x,t) = \zeta(t)W(x) \quad \text{for all } (x,t) \in \mathbb{R}^N \times (0, +\infty),$$

for suitable $\zeta(t)$ and $W(x)$. When $p \ge m$, $\zeta(t) \equiv 1$. Observe that we can also include the linear case $m = 1$, whenever $p > m$.

In Chapter 3 (where [94] is reproduced), we have considered problem (I.2.11) for a different choice of weight $\rho : \mathbb{R}^N \to (0, +\infty)$, $\rho \in C(\mathbb{R}^N)$. In particular, we always make one of the following assumptions:

$$\begin{aligned} &\text{there exist } k \in (0, +\infty) \text{ and } \alpha > 1 \text{ such that} \\ &\frac{1}{\rho(x)} \geq k\,(\log|x|)^{\alpha}\,|x|^2 \quad \text{for all } x \in \mathbb{R}^N \setminus B_e(0)\,; \end{aligned} \tag{I.2.13}$$

or

$$\begin{aligned} &\text{there exist } k_1, k_2 \in (0, +\infty) \text{ with } k_1 \leq k_2 \text{ and } \alpha > 1 \text{ such that} \\ &k_1 \frac{|x|^2}{(\log|x|)^{\alpha}} \leq \frac{1}{\rho(x)} \leq k_2 \frac{|x|^2}{(\log|x|)^{\alpha}} \quad \text{for all } x \in \mathbb{R}^N \setminus B_e(0)\,. \end{aligned} \tag{I.2.14}$$

Global existence and blow-up of solutions are addressed depending on the interplay between $m > 1$, $p > 1$ and α. The method of proofs and the results are similar to those obtained in the case of slowly and fast decaying densities.

In particular, if ρ satisfies (I.2.13), then

- for $1 < p < m$, if the initial datum $u_0 \in L^{\infty}(\mathbb{R}^N)$, then problem (I.2.11) admits a global solution belonging to $L^{\infty}(\mathbb{R}^N \times (0, \tau))$ for every $\tau > 0$;
- for $p > m > 1$, if u_0 satisfies a suitable decaying condition as $|x| \to +\infty$, then problem (I.2.11) admits a global solution in $L^{\infty}(\mathbb{R}^N \times (0, +\infty))$.

On the other hand, if ρ satisfies (I.2.14), then

- for $p > m > 1$, if $u_0 \in L^{\infty}(\mathbb{R}^N)$ is sufficiently large, then the solutions to problem (I.2.11) blow-up in finite time;
- for $p > m > 1$, if $u_0 \in L^{\infty}(\mathbb{R}^N)$ is sufficiently small and compact supported, then, under suitable assumptions on k_1 and k_2, there exist global in time solutions to problem (I.2.11) in $L^{\infty}(\mathbb{R}^N \times (0, +\infty))$.

We construct suitable sub- and supersolutions that are of the form

$$w(x,t) = C\zeta(t)\left[1 - \frac{(\log(|x| + r_0))^q}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \left[\mathbb{R}^N \setminus B_e(0)\right] \times [0, T),$$

for appropriate functions $\zeta = \zeta(t)$, $\eta = \eta(t)$ and constants $C > 0$, $a > 0$, $r_0 > 0$ and $q > 1$.

I.3 Part II: The porous medium equation with reaction on noncompact Riemannian manifolds

I.3.1 A survey of the literature

The problem of global solvability of nonlinear evolution problems has been recently deep investigated also on general Riemannian manifolds. All the problems that we have mentioned so far have a counterpart in the Riemannian setting, i.e. when the Euclidian space $\mathbb{R}^N$ is replaced by a general complete, noncompact Riemannian manifold M. We focus on those results in literature that motivates our investigation on the Cauchy problem associated to equation (I.1.2). We observe that the behavior of solutions is mostly influenced by two competing phenomena.

The first one is the diffusive pattern associated to the *Porous Medium Equation* (PME)

$$u_t = \Delta(u^m), \quad \text{in } M \times (0,T) \quad (m > 1)\,, \tag{I.3.15}$$

where $T > 0$, M is a Riemannian manifold of dimension $N \geq 2$ and Δ is the Laplace-Beltrami operator. Observe that the fact that we assume $m > 1$ puts us in the *slow diffusion case*. On the other hand, if $m < 1$, equation (I.3.15) is called *fast diffusion equation*, on Riemannian manifolds it has been e.g. addressed in [13]. We mention that the investigation on the behavior of solutions to the Cauchy problem associated to equation (I.3.15) when M is the hyperbolic space $\mathbb{H}^N$, has been addressed in [111, 129]. We recall that $\mathbb{H}^N$ is the complete, simply connected manifold of dimension N with sectional curvature everywhere equal to -1. In particular, in [129], the fundamental solution to equation (I.3.15) posed in $\mathbb{H}^N$ has been constructed. It has been shown that the behavior of the fundamental solutions for short and long times is completely different. More precisely, for short times, the fundamental solutions behave like the Barenblatt solutions introduced in (I.2.5). On the other hand, for large times, it has been proved that radial and compactly supported data give rise to solutions that grow logarithmically; in particular the following bound holds

$$\|u(t)\|_{L^\infty(\mathbb{H}^N)} \leq C \left(\frac{\log t}{t}\right)^{\frac{1}{m-1}} \quad \text{for any } t > 2, \tag{I.3.16}$$

where $C > 0$ is a suitable constant. This different behavior depending on the time is a remarkable difference with the Euclidean case and it is due to the gradual influence of the curvature of the hyperbolic space on the form of the fundamental solutions. Moreover, (I.3.16) is in contrast with the well-known power-like growth of the PME in the Euclidean space: the decay rate predicted by (I.3.16) is *faster* than its Euclidean counterpart. Qualitatively speaking, *negative curvature accelerates diffusions*, a fact that is apparent first of all from the behavior of solutions of the classical heat equation. In fact, it can be shown that the standard deviation of a Brownian particle on the hyperbolic space $\mathbb{H}^N$ behaves *linearly* in time, whereas in the Euclidean situation it is proportional to $\sqrt{t}$. Similarly, the heat kernel decays exponentially as $t \to +\infty$ in the hyperbolic space $\mathbb{H}^N$ whereas one has a power-type decay in the Euclidean situation.

Equation (I.3.15) has also been studied in [48, 55] when M is a *Cartan-Hadamard manifold*, namely an N-dimensional complete, simply connected Riemannian manifold with nonpositive sectional curvature. It is further assumed that the sectional curvature is bounded above by a suitable constant $-k < 0$. It is investigated the behavior of solutions to the Cauchy problem related to (I.3.15) when the initial datum u_0 is integrable and bounded on M. It is proved that the smoothing estimate (I.3.16) holds also in this case, i.e.

$$\|u(t)\|_{L^\infty(M)} \leq C \left[\frac{\log(2+t)\|u_0\|_{L^1(M)}^{m-1}}{t}\right]^{\frac{1}{m-1}} \quad \text{for any } t > 0,$$

where $C > 0$ is a suitable constant.

The Cauchy problem associated to equation (I.3.15) has also been investigated on more general Riemannian manifolds, see e.g. [43, 52, 53, 112].

The second driving factor influencing the behavior of solutions to the Cauchy problem associated to equation (I.1.2), is the reaction term u^p, which has the positive sign and, thus, might drive solutions towards blow-up. Let us first recall the well known semilinear heat equation

$$u_t = \Delta u + f(u) \quad \text{in } M \times (0, T), \tag{I.3.17}$$

where $T > 0$, M is a Riemannian manifold of dimension $N \geq 2$, Δ is the Laplace-Beltrami operator and f is a positive function, thus we are in presence of reaction. The choice $f(u) = u^p$, for $p > 1$, has been considered in [9]. It has been shown that the Cauchy problem associated to equation (I.3.17) when $M = \mathbb{H}^N$, always admits a global solution, if the initial datum is sufficiently small. We underline that this behavior of solutions is in contrast with the Euclidian counterpart ($M = \mathbb{R}^N$) where the Fujita phenomenon arises. On the other hand, it is similar to the behavior of solutions to the Cauchy problem posed in bounded domains $\Omega \subset \mathbb{R}^N$ with homogeneous Dirichlet boundary conditions. For other choices of $f(u)$ and more general Riemannian manifolds we refer the reader to e.g. [110, 113, 130, 131].

Let us now recall some results concerning the equation in (I.1.2), i.e.

$$u_t = \Delta(u^m) + u^p \quad \text{in } M \times (0, T),$$

where M is a complete noncompact Riemannian manifold of dimension N, $p > 1$, $m > 1$ and $T > 0$. The Cauchy problem associated to equation (I.1.2) has been studied in [137], under the assumption that the volume of geodesic balls of radius R grows as R^α with $\alpha \geq 2$; this kind of assumption is typically associated to *nonnegative* Ricci curvature.

The situation on *negatively* curved manifolds M is significantly different, and the results in this connection have been shown in [54]. More precisely, in [54], the behavior of solutions to the Cauchy problem associated to equation (I.1.2) when M is a Cartan-Hadamard manifold and the initial datum is nonnegative and compactly supported, has been addressed. Moreover, suitable curvature conditions have been assumed, i.e.

$$\mathrm{Ric}_o(x) \leq -(N-1)h^2 \quad \text{or} \quad \mathrm{Ric}_o(x) \geq -(N-1)k^2, \tag{I.3.18}$$

where $h, k > 0$ and $\mathrm{Ric}_o(x)$ is the Ricci curvature at x in the radial direction $\frac{\partial}{\partial r}$ w.r.t. a given pole o. For $p > m$, a dichotomy phenomenon has been proved. In particular, it has been shown global existence of solutions for small enough initial datum assuming that the upper bound on the Ricci curvature given in (I.3.18) holds. Moreover, a class of sufficiently small data shows propagation properties identical to the ones valid for the unforced porous medium equation (I.3.15). On the other hand, blow-up occurs if the initial datum is large enough and the lower bound on the Ricci curvature given in (I.3.18) holds. For $p \in \left(1, \frac{1+m}{2}\right]$, it is shown that pointwise everywhere blowup in infinite time occurs. Whereas, in the range $p \in \left(\frac{1+m}{2}, m\right]$, they show that, if the solution is global in time, then blowup occurs in infinite time. Thus we can observe that the behavior of solutions is considerably different from the Euclidean setting. In the Riemannian setting, a dichotomy phenomenon between large and small data occurs when $p > m$, whereas in the Euclidean one the same dichotomy occurs when $p > m + \frac{2}{N}$. Finally, a completely new phenomenon appears when $p \in \left(1, \frac{1+m}{2}\right]$, i.e. blow-up of solutions in infinite time.

I.3.2 Outline of the results

In Chapters 4 and 5, we address the porous medium equation with reaction of the form (I.1.2) posed in the Riemannian setting.

We investigate global existence of solutions to the Cauchy problem

$$\begin{cases} u_t = \Delta(u^m) + u^p & \text{in } M \times (0, T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{I.3.19}$$

where M is a complete noncompact Riemannian manifold of infinite volume, of dimension $N \geq 3$, $T > 0$ and $m > 1$, $p > 1$. Moreover, the initial datum u_0 is a nonnegative function.

In particular, in Chapter 4 (where [45] is reproduced), we consider the case when

$$p > m > 1$$

and we assume the validity of the Sobolev or the Poincaré inequalities on M, i.e.

$$\|v\|_{L^{2^*}(M)} \leq \frac{1}{C_s} \|\nabla v\|_{L^2(M)} \quad \text{for any } v \in C_c^\infty(M), \tag{I.3.20}$$

$$\|v\|_{L^2(M)} \leq \frac{1}{C_p} \|\nabla v\|_{L^2(M)} \quad \text{for any } v \in C_c^\infty(M); \tag{I.3.21}$$

where C_p and C_s are numerical constants and $2^* := \frac{2N}{N-2}$. This assumption puts constraints on the Riemannian manifold M. In particular, we recall that it is e.g. well known that the Sobolev inequality always holds on Cartan-Hadamard manifolds, namely complete and simply connected manifolds that have everywhere non-positive sectional curvature. Furthermore, if we assume that sec $\leq -k < 0$ then also the Poincaré inequality holds.

Our results can be summarized as follows.

- For
$$p > m + \frac{2}{N},$$
we assume that the Sobolev inequality in (I.3.20) holds on M. Then we prove that any sufficiently small initial datum
$$u_0 \in L^m(M) \cap L^{(p-m)\frac{N}{2}}(M)$$
gives rise to a global solution $u(t)$ such that $u(t) \in L^\infty(M)$ for all $t > 0$. Moreover, we prove a quantitative bound on the L^∞ of the solution $u(t)$ for any $t > 0$.

- For
$$p > m,$$
we assume that both the Sobolev and Poincaré in (I.3.20), (I.3.21) inequalities hold on M. Then we prove that any sufficiently small initial datum u_0, where u_0 belongs to a suitable Lebesgue space, give rise to a global solution $u(t)$ such that $u(t) \in L^\infty(M)$ for all $t > 0$. Moreover we provide a quantitative bound on the L^∞ norm of the solution $u(t)$ for any $t > 0$.

Observe that, if we only assume that the Sobolev inequality holds, then we need to restrict the range of p asking for $p > m + \frac{2}{N}$. On the other hand, we can relax the assumption on the exponent p, i.e. $p > m$, if we further assume the validity of the Poincaré inequality.

The strategy of the proof of both results mainly relies on the validity of the functional inequalities (I.3.20) and (I.3.21). For this reason, our results can be generalized to different context among which we outline the case of inhomogeneous porous medium equation with reaction in the Euclidean setting, see problem (I.2.11). The problem is naturally posed in the weighted spaces

$$L^q_\rho(\mathbb{R}^N) = \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable }, \; \|v\|_{L^q_\rho} := \left(\int_{\mathbb{R}^N} v^q \rho(x)\, dx \right)^{1/q} < +\infty \right\}.$$

Then we introduce the weighted Sobolev and Poincaré inequalities

$$\|v\|_{L^{2^*}_\rho(\mathbb{R}^N)} \leq \frac{1}{C_s} \|\nabla v\|_{L^2(\mathbb{R}^N)} \quad \text{for any } \; v \in C^\infty_c(\mathbb{R}^N), \tag{I.3.22}$$

$$\|v\|_{L^2_\rho(\mathbb{R}^N)} \leq \frac{1}{C_p} \|\nabla v\|_{L^2(\mathbb{R}^N)} \quad \text{for any } \; v \in C^\infty_c(\mathbb{R}^N), \tag{I.3.23}$$

for suitable positive constants C_s and C_p. The main results of this case can be summarized as follows.

- For
$$p > m + \frac{2}{N},$$
we assume that the Sobolev inequality in (I.3.22) holds on M. Then we prove that any sufficiently small initial datum
$$u_0 \in L^m_\rho(\mathbb{R}^N) \cap L^{(p-m)\frac{N}{2}}_\rho(\mathbb{R}^N)$$
gives rise to a global solution $u(t)$ such that $u(t) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$. Moreover, we prove a quantitative bound on the L^∞ of the solution $u(t)$ for any $t > 0$.

- For
$$p > m,$$
we assume that both the Sobolev and Poincaré inequalities in (I.3.22), (I.3.23) hold on M. Then we prove that any sufficiently small initial datum u_0, where u_0 belongs to a suitable Lebesgue space, give rise to a global solution $u(t)$ such that $u(t) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$. Moreover we provide a quantitative bound on the L^∞ norm of the solution $u(t)$ for any $t > 0$.

In Chapter 5 (where [46] is reproduced), we investigate global existence of solutions to the Cauchy problem in (I.3.19) in the case when

$$1 < p < m.$$

We also assume the validity of the Sobolev and Poincaré inequalities (I.3.20) and (I.3.21) on M. Moreover, we suppose that the initial datum u_0 is a nonnegative function such that

$$u_0 \in L^m(M) .$$

We summarize the main results as follows.

- For any

  $$1 < p < m ,$$

  we suppose that the initial datum u_0 is a nonnegative function such that $u_0 \in L^m(M)$. Then we prove global existence of solutions to problem (I.3.19). Moreover, we show a smoothing effects for solutions, in the sense that L^m data give rise to global solutions $u(t)$ such that $u(t) \in L^\infty(M)$ for all $t > 0$, with a quantitative bound on their L^∞ norm.

- As a consequence, combining this fact with some results proved in [54], we can prove that, on manifolds satisfying e.g. $-c_1 \le \sec \le -c_2$ with $c_1 \ge c_2 > 0$, any solution $u(t)$ to (I.3.19) corresponding to an initial datum $u_0 \in \mathrm{L}^m(M)$ exists globally and, provided u_0 is sufficiently large, it satisfies the property

  $$\lim_{t\to+\infty} u(x,t) = +\infty \quad \forall x \in M,$$

  namely *complete blowup in infinite time* occurs for such solutions to (I.3.19) in the whole range $p \in (1, m)$. We recall that e.g. the above hypothesis on the sectional curvature, sec, includes the particularly important case of the hyperbolic space $\mathbb{H}^N$.

Similarly to the case when $p > m > 1$, our results depend essentially only on the validity of the functional inequalities (I.3.20) and (I.3.21), hence they are generalizable to different contexts. As a particularly significant situation, we single out the case of Euclidean, inhomogeneous porous medium equation with reaction introduced in (I.2.11). Assuming that the weight ρ is such that the weighted Sobolev and Poincaré inequalities in (I.3.22) and (I.3.23) hold, we prove that

- for any

  $$1 < p < m ,$$

  we suppose that the initial datum u_0 is a nonnegative function such that $u_0 \in L^m_\rho(\mathbb{R}^N)$. Then we prove global existence of solutions to problem (I.3.19). Moreover, we show a smoothing effects for solutions, in the sense that L^m_ρ data give rise to global solutions $u(t)$ such that $u(t) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$, with a quantitative bound on their L^∞ norm;

- for $\rho \asymp |x|^{-2}$ as $|x| \to +\infty$, we are able to construct a subsolution of equation (I.2.11) which blows up in infinite time. As a consequence, combining this fact with the previous result of global existence of the solution, we can prove that, any solution $u(t)$ to (I.2.11) corresponding to an initial datum $u_0 \in \mathrm{L}^m_\rho(\mathbb{R}^N)$ exists globally and, provided u_0 is sufficiently large, it satisfies the property

$$\lim_{t\to+\infty} u(x,t) = +\infty \quad \forall x \in \mathbb{R}^N,$$

namely *complete blowup in infinite time* occurs for such solutions to (I.2.11) in the whole range $p \in (1, m)$.

I.4 Part III: Quasilinear parabolic differential inequalities

I.4.1 A survey of the literature

The study of nonexistence of solutions to partial differential equations, such as equation in (I.1.3), has received considerable attention in the literature. Observe that equation in (I.1.3) represents a wide class of nonlinear problems. The approach used to study nonexistence of solutions has been exploited by Mitidieri and Pohozaev in [95, 96] and it is mainly based on the construction of suitable test functions and integral estimates. For a comprehensive description of such approach we refer the reader to [98].

One of the most important and well-studied class of elliptic differential inequalities, due to its ubiquitous presence in many applications, is

$$\Delta u + V(x)u^q \le 0, \tag{I.4.24}$$

both on $\mathbb{R}^N$ and on general Riemannian manifolds M, for $q > 1$. In particular, in many instances it is also required that the solution u of the problem is positive. The Cauchy problem related to inequality (I.4.24) has been investigated by Gidas in [37] and Gidas and Spruck in [38]. In those papers the authors show, among other results, that any nonnegative solution of inequality (I.4.24) is in fact identically null if and only if $q \le \frac{N}{N-2}$, in case $V \equiv 1$ and the dimension of the Euclidean space is $N \ge 3$. Moreover, in [96], the authors show that inequality (I.4.24) on $\mathbb{R}^N$ does not admit any nontrivial nonnegative solution, provided that

$$\liminf_{R\to\infty} R^{-\frac{2q}{q-1}} \int_{B_{\sqrt{2}R}\setminus B_R} V^{-\frac{1}{q-1}}\, dx < \infty$$

We also mention that nonexistence results of nonnegative nontrivial solutions have been much investigated for solutions to elliptic quasilinear inequalities of the form:

$$\frac{1}{a(x)} \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + V(x)u^q \le 0 \quad \text{in } \mathbb{R}^N, \tag{I.4.25}$$

where

$$a > 0,\;\; a \in \mathrm{Lip}_{loc}(\mathbb{R}^N), \qquad V > 0 \;\text{ a.e. on } \mathbb{R}^N,\;\; V \in L^1_{loc}(\mathbb{R}^N),$$

$p > 1$, $q > p - 1$. We refer to [17, 95, 96, 97, 98] for a comprehensive description of results related to problem (I.4.25) and also more general problems on $\mathbb{R}^N$.

Inequality (I.4.25) has also been considered when the Euclidean space is replaced by a complete noncompact Riemannian manifold M. The results in this case have a more recent history, we refer the reader to the inspiring papers of Grigor'yan and Kondratiev [41] and Grigor'yan and Sun [42], whose approach originates from the work of Kurta [80], and the papers by Sun [123, 124].

In particular, in [90], the authors prove nonexistence of solutions to inequality (I.4.25), for any $p \geq 2$, provided that there exists $C_0 > 0$ and $k \in [0, \beta)$, such that, for every $R > 0$ sufficiently large and every small enough $\varepsilon > 0$

$$\int_{B_R \setminus B_{R/2}} V^{-\beta+\varepsilon}\, d\mu \;\leq\; C\, R^{\alpha + C_0\, \varepsilon} (\log R)^k \,,$$

where $d\mu$ is the canonical Riemannian measure on M, B_R is the geodesic ball centered at a point $x_0 \in M$ and

$$\alpha = \frac{pq}{q-p+1}, \quad \beta = \frac{p-1}{q-p+1}\,.$$

Finally, we mention that (I.4.25) posed on an open relatively compact connected domain $\Omega \subset M$ has been studied in [100]. Under the assumptions that

$$a > 0,\;\; a \in \mathrm{Lip}_{loc}(\Omega), \qquad V > 0 \;\text{ a.e. on } \Omega,\;\; V \in L^1_{loc}(\Omega),$$

$p > 1$, $q > p - 1$, the authors investigate the relation between the behavior of the potential V at the boundary of Ω and nonexistence of nonnegative weak solutions.

We now consider the evolutive counterpart of the elliptic inequalities introduced so far, such as the parabolic problem in (I.1.3). Global existence and finite time blow-up of solutions for problem (I.1.3), together with its generalization to a wider class of operators of p-Laplace type or related to the porous medium equation, has been deeply studied in the Euclidean space; without claim of completeness we refer the reader to [33, 34, 35, 97, 98, 105, 104] and references therein. In particular, in [98], the authors consider the Cauchy problem associated to the following inequality

$$u_t - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq u^q \quad \text{in } \mathbb{R}^N \times (0,T),$$

where the initial datum is $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and

$$q > 1\,.$$

They prove nonexistence of nontrivial weak solutions with the assumptions

$$p > \frac{2N}{N+1}, \qquad \max\{1, p-1\} < q \leq p - 1 + \frac{p}{N}\,.$$

We refer the reader to [98] for nonexistence results of more general quasilinear evolution inequalities.

Moreover, problem (I.1.3) has been investigated in the Riemannian setting, see e.g. [9, 89, 110, 56, 137] and references therein.

In [89] problem (I.1.3) is studied when $\Omega = M$ is a complete, N-dimensional, non-compact Riemannian manifold; it is investigated nonexistence of nonnegative nontrivial weak solutions depending on the interplay between the geometry of the underlying manifold, the power nonlinearity and the behavior of the potential at infinity, assuming that $u_0 \in L^1_{loc}(M)$, $u \geq 0$ a.e. in M and $V \in L^1_{loc}(M \times [0, +\infty))$, $V > 0$ a.e. in M.

I.4.2 Outline of the results

In Chapter 6 (where [91] is reproduced), we address a class of quasilinear parabolic differential equations with a potential of the form (I.1.3).

We are concerned with nonexistence of nonnegative weak solutions to the following problem

$$\begin{cases} \partial_t u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq V\, u^q & \text{in } \Omega\times(0,T) \\ u = 0 & \text{on } \partial\Omega\times(0,T) \\ u = u_0 & \text{in } \Omega\times\{0\}\,; \end{cases} \tag{I.4.26}$$

where Ω is an open bounded connected subset of $\mathbb{R}^N$, $T>0$, $p>1$ and $q>\max\{p-1,1\}$.

Under suitable hypotheses on V and q, we obtain nonexistence of global weak solutions. In particular, we assume that

- $p>1$, $q>\max\{p-1,1\}$;
- $V\in L^1_{loc}(\Omega\times[0,+\infty))$, $V>0$ a.e. in $\Omega\times[0,+\infty)$;
- V satisfies some integral conditions which describe its behavior near the boundary $\partial\Omega$,
- $u_0\in L^1_{loc}(\Omega)$, $u_0\geq 0$ a.e. in Ω.

Then we prove that, if u is a nonnegative weak solution of problem (I.4.26) then $u\equiv 0$ a.e. in $\Omega\times[0,+\infty)$.

The proof is mainly based on the choice of a family of suitable test functions, depending on two parameters, that enables us to deduce first some appropriate a priori estimates, then that the unique global solution is $u\equiv 0$. Such test functions are defined by adapting to the present situation those used in [89]; however, some important differences occur, since in [89] an unbounded underlying manifold is considered, whereas now we consider a bounded domain. In some sense, the role of *infinity* of [89] is now played by the boundary $\partial\Omega$. Obviously, this implies that such test functions satisfy different properties.

Moreover, as a special case, we consider the semilinear parabolic problem

$$\begin{cases} \partial_t u - \Delta u = V(x)u^q & \text{in } \Omega\times(0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega\times\{0\}\,. \end{cases} \tag{I.4.27}$$

where Ω is an open bounded connected subset of $\mathbb{R}^N$, $N\geq 3$ and $u_0:\Omega\to[0,+\infty)$, $q>1$ and $T>0$.

We can summarize our results for problem (I.4.27) as follows. As a consequence of our general result, we infer that nonexistence of global solutions for problem (I.4.27) prevails, when

$$V(x)\geq Cd(x)^{-\sigma_1}\quad \text{for all}\ \ x\in\Omega,$$

for some $C>0$ and

$$\sigma_1>q+1,$$

where

$$d(x) := \mathrm{dist}(x, \partial\Omega) \quad \text{for any } \ x \in \overline{\Omega}.$$

Furthermore, we show sharpness of this result for the semilinear problem in the case $\partial\Omega$ is regular enough and $V = V(x)$ is continuous and independent of t. Indeed, under the assumption that

$$0 \leq V(x) \leq Cd(x)^{-\sigma_1} \quad \text{for all } \ x \in \Omega,$$

for some $C > 0$ and

$$0 \leq \sigma_1 < q + 1,$$

we prove the existence of a global classical solution for problem (I.4.27), if the initial datum u_0 is small enough. This existence result is obtained by means of the sub– and supersolutions method. In particular, we construct a supersolution to problem (I.4.27), which actually is a supersolution of the associated stationary equation. Such supersolution is obtained as the fixed point of a suitable contraction map. In order to show that such a fixed point exists, we need to estimate some integrals involving the Green function associated to the Laplace operator $-\Delta$ in Ω and we prove that there exists $C > 0$ such that

$$0 \leq \int_{\Omega} G(x, y) d(y)^{\beta} \, dy \leq C \, d(x) \, , \quad \text{for any } \ \beta > -2 \, .$$

Finally, we study the *slightly supercritical* case

$$V(x, t) \geq d(x)^{-q-1} f(d(x))^{q-1} \quad \text{for all } \ x \in \Omega, \ t \in [0, +\infty)$$

where f is a function satisfying suitable assumptions and such that $\lim_{\varepsilon \to 0^+} f(\varepsilon) = +\infty$, for which we prove nonexistence of nonnegative nontrivial weak solutions in $\Omega \times (0, +\infty))$. The proof of this result require a different argument with respect to the previous nonexistence results, which makes use of linearity of the operator and of the special form of the potential. Then the critical rate of growth $d(x)^{-q-1}$ as x approaches $\partial\Omega$ is indeed sharp for the nonexistence of solutions to problem (I.4.27). Our results do not cover the case of critical rate of growth, i.e.

$$C_1 d(x)^{-q-1} \leq V(x, t) \leq C_2 d(x)^{-q-1}$$

for some $C_1, C_2 > 0$, but we conjecture that also in this case no nonnegative nontrivial solution of problem (I.4.27) exists.

Part I

The inhomogeneous porous medium equation with reaction on $\mathbb{R}^N$

Chapter 1

The slowly decaying density case

1.1 Introduction

We investigate global existence and blow-up of nonnegative solutions to the Cauchy parabolic problem

$$\begin{cases} \rho(x)u_t = \Delta(u^m) + \rho(x)u^p & \text{in } \mathbb{R}^N \times (0,\tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}\,, \end{cases} \tag{1.1.1}$$

where $m > 1, p > 1,\ N \geq 3, \tau > 0$; furthermore, we always assume that

$$\begin{cases} \text{(i) } \rho \in C(\mathbb{R}^N),\ \rho > 0 \text{ in } \mathbb{R}^N\,; \\ \text{(ii) there exist } k_1, k_2 \in (0,+\infty) \text{ with } k_1 \leq k_2 \text{ and } 0 \leq q < 2 \text{ such that} \\ \qquad k_1|x|^q \leq \dfrac{1}{\rho(x)} \leq k_2|x|^q \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0)\,; \\ \text{(iii) } u_0 \in L^\infty(\mathbb{R}^N),\ u_0 \geq 0 \text{ in } \mathbb{R}^N\,. \end{cases} \tag{H}$$

The parabolic equation in problem (1.1.1) is of the *porous medium* type, with a variable density $\rho(x)$ and a reaction term $\rho(x)u^p$. Clearly, such parabolic equation is degenerate, since $m > 1$. Moreover, the differential equation in (1.1.1) is equivalent to

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \qquad \text{in } \mathbb{R}^N \times (0,\tau);$$

therefore, the related diffusion operator is $\frac{1}{\rho(x)}\Delta$, and in view of (H), the coefficient $\dfrac{1}{\rho(x)}$ can positively diverge at infinity. Problem (1.1.1) has been introduced in [73] as a mathematical model of evolution of plasma temperature, where u is the temperature, $\rho(x)$ is the particle density, $\rho(x)u^p$ represents the volumetric heating of plasma. Indeed, in [73, Introduction] a more general source term of the type $A(x)u^p$ has also been considered; however, then the authors assume that $A \equiv 0$; only some remarks for the case $A(x) = \rho(x)$ are made in [73, Section 4], when the problem is set in a slab in one space dimension. Then in [71] and [72] problem (1.1.1) is dealt with in the case without the reaction term $\rho(x)u^p$.

We refer to $\rho(x)$ as a *slowly decaying density* at infinity because, in view of (H),

$$\frac{1}{k_2|x|^q} \leq \rho(x) \leq \frac{1}{k_1|x|^q} \quad \text{for all} \quad |x| > 1\,,$$

with

$$0 \leq q < 2.$$

Global existence and blow-up of solutions for problem (1.1.1) with *fast decaying density* at infinity, i.e. $q \geq 2$, is investigated in [93] . We regard the value $q = 2$ as the threshold one, indeed, the behavior of solutions is very different according to the fact that $q < 2$ or $q = 2$ or $q > 2$. Such important role played by the value $q = 2$ does not surprise. In fact, for problem (1.1.1) without the reaction term u^p, that is

$$\begin{cases} \rho u_t = \Delta(u^m) & \text{in } \mathbb{R}^N \times (0,\tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}\,, \end{cases} \tag{1.1.2}$$

in [109], it is shown that for $q \leq 2$ there exists a unique bounded solution, whereas for $q > 2$, for any $u_0 \in L^\infty(\mathbb{R}^N)$ there exist infinitely many bounded solutions.

Let us briefly recall some results in the literature concerning well-posedness for problems related to (1.1.1). Problem (1.1.1) with $\rho \equiv 1$ and without the reaction term, that is

$$\begin{cases} u_t = \Delta(u^m) & \text{in } \mathbb{R}^N \times (0,\tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{1.1.3}$$

has been the object of detailed investigations. We refer the reader to the book [128] and references therein, for a comprehensive account of the main results. Also problem (1.1.1) with variable density, without reaction term, that is problem (1.1.2), has been widely examined. In particular, depending on the behaviour of $\rho(x)$ as $|x| \to \infty$, existence and uniqueness of solutions and the asymptotic behaviour of solutions for large times have been addressed (see, e.g., [25, 27, 49, 51, 50, 62, 63, 64, 66, 67, 68, 69, 70, 71, 72, 109, 115, 116, 117]).

For problem (1.1.1) with $m = 1$ and $\rho \equiv 1$, global existence and blow-up of solutions have been studied. To be specific, if

$$p \leq 1 + \frac{2}{N},$$

then finite time blow-up occurs, for all nontrivial nonnegative data, whereas, for

$$p > 1 + \frac{2}{N},$$

global existence prevails for sufficiently small initial conditions (see, e.g., [16, 24, 30, 31, 58, 83, 114, 118, 121, 135]). In addition, in [85] (see also [21]), problem (1.1.1) with $m = 1$ has been considered. Let assumption (H) be satisfied, and let

$$b := 2 - q. \tag{1.1.4}$$

Obviously, since $q \in [0,2)$, we have that

$$b \in (0,2]\,.$$

It is shown that if

$$p \leq 1 + \frac{b}{N-2+b},$$

then solutions blow-up in finite time, for all nontrivial nonnegative data, whereas, for

$$p > 1 + \frac{b}{N-2+b},$$

global in time solutions exist, provided that u_0 is small enough.

Now, let us recall some results established in [119] for problem (1.1.1) with $\rho \equiv 1$, $m > 1, p > 1$ (see also [36, 99]). We have:

- ([119, Theorem 1, p. 216]) For any $p > 1$, for all sufficiently large initial data, solutions blow-up in finite time;
- ([119, Theorem 2, p. 217]) if $p \in \left(1, m + \frac{2}{N}\right)$, for *all* initial data, solutions blow-up in finite time;
- ([119, Theorem 3, p. 220]) if $p > m + \frac{2}{N}$, for all sufficiently small initial data, solutions exist globally in time.

Similar results for quasilinear parabolic equations, also involving p-Laplace type operators or double-nonlinear operators, have been stated in [1], [3], [4], [20], [22], [23], [60], [61], [86], [87], [88], [97], [98], [104], [125], [132] (see also [89] for the case of Riemannian manifolds); moreover, in [54] the same problem on Cartan-Hadamard manifolds has been investigated.

Let us observe that the results in [119] illustrated above have been proved by means of comparison principles and suitable sub– and supersolutions of the form

$$w(x,t) = C\zeta(t)\left[1 - \frac{|x|^2}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } \ (x,t) \in \mathbb{R}^N \times [0,T),$$

for appropriate auxiliary functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0$.

In [86, 87] double-nonlinear operators, including in particular problem (1.1.1), are investigated. It is shown that (see [86, Theorem 1]) if $\rho(x) = |x|^{-q}$ with $q \in (0,2)$, for any $x \in \mathbb{R}^N \setminus \{0\}$,

$$p > m + \frac{b}{N-2+b},$$

$u_0 \geq 0$ and

$$\int_{\mathbb{R}^N} \left\{u_0(x) + [u_0(x)]^{\bar{q}}\right\} \rho(x) dx < \delta, \tag{1.1.5}$$

for some $\delta > 0$ small enough and $\bar{q} > \frac{N}{2}(p-m)$, then there exists a global solution of problem (1.1.1). In addition, a smoothing estimate holds. On the other hand, if $\rho(x) = |x|^{-q}$ or $\rho(x) = (1+|x|)^{-q}$ with $q \in [0,2)$, $u_0 \not\equiv 0$ and

$$p < m + \frac{b}{N-2+b},$$

then blow-up prevails, in the sense that there exist $\theta \in (0,1), R>0, T>0$ such that

$$\int_{B_R} [u(x,t)]^{\theta} \rho(x)dx \to +\infty \quad \text{as } t \to T^- .$$

Such results have also been generalized to more general initial data, decaying at infinity with a certain rate (see [87]). We compare the results in [86] with ours below (see Remarks 1.2.3, 1.2.5 and 1.2.8).

1.1.1 Outline of our results

We prove the following results.

- (See Theorem 1.2.1). Suppose that

$$\frac{k_2}{k_1} < m + \frac{(m-1)(N-2)}{b}\,, \tag{1.1.6}$$

and define

$$\overline{p} := \frac{m(N-2+b) + \frac{b}{m-1}(m - \frac{k_2}{k_1})}{N-2+\frac{b}{m-1}\left(m-\frac{k_2}{k_1}\right)}\,. \tag{1.1.7}$$

If u_0 has compact support and is small enough,

$$p > \overline{p},$$

then global solutions exist.

Note that for $k_1 = k_2$,

$$\overline{p} = m + \frac{b}{N+2-b};$$

this is coherent with [86, Theorem 1] (see Remark 1.2.3 below for more details). If in addition $\rho \equiv 1$, and so $b=2$, we have

$$\overline{p} = m + \frac{2}{N}\,.$$

Thus, our results are in accordance with those in [119]. Furthermore, for $m=1$, they are in agreement with the results established in [85], and in [31, 58] when $\rho \equiv 1$.

- (See Theorem 1.2.4). For any $p>1$, if u_0 is sufficiently large, then solutions to problem (1.1.1) blow-up in finite time.

- (see Theorem 1.2.6). If $1<p<m$, then for any $u_0 \not\equiv 0$, solutions to problem (1.1.1) blow-up in finite time. In addition (see Theorem 1.2.7), if

$$m \le p < m + \frac{b}{N-2+b}$$

and $q \in [0,\epsilon)$ for $\epsilon>0$ small enough, then for any $u_0 \not\equiv 0$, solutions to problem (1.1.1) blow-up in finite time.

It remains to be understood if the restriction $q \in [0, \epsilon)$ can be removed.

Actually, we obtain similar results to those described above, also when assumption (H) is fulfilled for general $0 < k_1 < k_2$. In that case, the blow-up result for large initial data can be stated exactly as in the previous case $k_1 = k_2$. Instead, in order to get global existence, the assumption on p changes, since it also depends on the parameters k_1 and k_2. More precisely, Indeed, also our blow-up results for any nontrivial initial datum holds when $0 < k_1 < k_2$. The case $1 < p < m$ is exactly as before. Moreover (see Theorem 1.2.7), if

$$m \leq p < \underline{p},$$

where

$$\underline{p} = \frac{m\,(N-2+b) + \frac{b}{m-1}\left(m - \frac{k_1}{k_2}\right)}{N-2+\frac{b}{m-1}\left(m-\frac{k_1}{k_2}\right)}\,, \tag{1.1.8}$$

then the solution blows-up for any nontrivial initial datum, under the extra hypothesis that $q \in [0, \epsilon)$ for $\epsilon > 0$ small enough. Note that in view of (1.1.6), it can be easily checked that

$$\underline{p} \leq \bar{p}\,.$$

In particular, $\underline{p} = \bar{p}$ whenever $k_1 = k_2$.

The methods used in [21, 31, 58, 85] cannot work in the present situation, since they strongly require $m = 1$. Indeed, our proofs mainly relies on suitable comparison principles (see Propositions 1.3.6, 1.3.7) and properly constructed sub- and supersolutions. Let us mention that the arguments exploited in [119] cannot be directly used in our case, due to the presence of the coefficient $\rho(x)$. In fact, we construct appropriate sub– and supersolutions, which crucially depend on the behavior at infinity of the inhomogeneity term $\rho(x)$. More precisely, whenever $|x| > 1$, they are of the type

$$w(x,t) = C\zeta(t)\left[1 - \frac{|x|^b}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } \ (x,t) \in \left[\mathbb{R}^N \setminus B_1(0)\right] \times [0,T),$$

for suitable functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0$. In view of the term $|x|^b$ with $b \in (0, 2]$, we cannot show that such functions are sub- and supersolutions in $B_1(0) \times (0,T)$. Thus we have to extend them in a suitable way in $B_1(0) \times (0,T)$. This is not only a technical aspect. In fact, in order to extend our sub– and supersolutions, we need to impose some extra conditions on $\zeta = \zeta(t)$, $\eta = \eta(t)$, C and a. Thus, it appears a sort of interplay between the behavior of the density $\rho(x)$ in compact sets, say $B_1(0)$, and its behavior for large values of $|x|$. Finally, let us comment about the proofs of the blow-up result for any nontrivial initial datum. For $1 < p < m$, the result follows by a direct application of Theorem 1.2.4. For $m < p < \underline{p}$, the proof is more involved. The corresponding result for the case $\rho \equiv 1$ established in [119] is proved by means of the Barenblatt solutions of the porous medium equation

$$u_t = \Delta(u^m) \quad \text{in } \mathbb{R}^N \times (0, +\infty)\,.$$

In our situation, we do not have self-similar solutions, since our equation in (1.1.1) is not scaling invariant, in view of the presence of the term $\rho(x)$. Indeed, we construct a suitable subsolution z of equation

$$u_t = \frac{1}{\rho}\Delta(u^m) \quad \text{in } \mathbb{R}^N \times (0, +\infty)\,.$$

By means of z, we can show that after a certain time, the solution u of problem (1.1.1) satisfies the hypotheses required by Theorem 1.2.4. Hence u blows-up in finite time.

Chapter 1 is organized as follows. In Section 1.2 we state our main results, in Section 1.3 we give the precise definitions of solutions, we establish a local in time existence result and some useful comparison principles. In Section 1.4 we prove the global existence theorem. The blow-up results are proved in Section 1.5 for sufficiently big initial data, and in Section 1.6 for any initial datum.

1.2 Statements of the main results

In view of (H)-(i), there exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 \le \rho_2$ such that

$$\rho_1 \le \frac{1}{\rho(x)} \le \rho_2 \quad \text{for all } x \in \overline{B_1(0)}. \tag{1.2.9}$$

As a consequence of hypothesis (H) and (1.2.9), we can assume that

$$k_1 = \rho_1, \quad k_2 = \rho_2\,. \tag{1.2.10}$$

Let $\overline{p}$ be defined by (1.1.7). It is immediate to see that $\overline{p}$ is monotonically increasing with respect to the ratio $\frac{k_2}{k_1}$; furthermore,

$$\overline{p} > m\,.$$

Define

$$\mathfrak{r}(x) := \begin{cases} |x|^b & \text{if} \quad |x| \ge 1, \\ \frac{b|x|^b + 2 - b}{b} & \text{if} \quad |x| < 1\,. \end{cases} \tag{1.2.11}$$

The first result concerns the global existence of solutions to problem (1.1.1) for $p > \overline{p}$.

Theorem 1.2.1. *Let assumptions (H), (1.1.6) and (1.2.10) be satisfied. Suppose that*

$$p > \overline{p},$$

where $\overline{p}$ is given in (1.1.7), and that u_0 is small enough and has compact support. Then problem (1.1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.

More precisely, if $C > 0$ is small enough, $T > 0$ is big enough, $a > 0$ with

$$\omega_0 \le \frac{C^{m-1}}{a} \le \omega_1,$$

for suitable $0 < \omega_0 < \omega_1$,

$$\alpha \in \left(\frac{1}{p-1}, \frac{1}{m-1}\right), \quad \beta = 1 - \alpha(m-1), \tag{1.2.12}$$

$$u_0(x) \leq CT^{-\alpha}\left[1 - \frac{\mathfrak{r}(x)}{a}T^{-\beta}\right]_+^{\frac{1}{m-1}} \quad \text{for any } x \in \mathbb{R}^N\,, \tag{1.2.13}$$

then problem (1.1.1) *admits a global solution* $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$. *Moreover,*

$$u(x,t) \leq C(T+t)^{-\alpha}\left[1 - \frac{\mathfrak{r}(x)}{a}(T+t)^{-\beta}\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \mathbb{R}^N \times [0, +\infty)\,. \tag{1.2.14}$$

The precise choice of the parameters $C > 0, T > 0$ and $a > 0$ in Theorem 1.2.1 is discussed in Remark 1.4.2 below. Observe that if u_0 satisfies (1.2.13), then

$$\|u_0\|_\infty \leq CT^{-\alpha},$$

$$\operatorname{supp} u_0 \subseteq \{x \in \mathbb{R}^N \,:\, \mathfrak{r}(x) \leq aT^\beta\}\,.$$

In view of the choice of C, T, a (see also Remark 1.4.2), $\|u_0\|_\infty$ is small enough, but supp u_0 can be large, since we can select $aT^\beta > r_0$ for any fixed $r_0 > 0$.

Moreover, from (1.2.14) we can infer that

$$\operatorname{supp} u(\cdot, t) \subseteq \{x \in \mathbb{R}^N \,:\, \mathfrak{r}(x) \leq a(T+t)^\beta\} \quad \text{for all } t > 0\,. \tag{1.2.15}$$

Remark 1.2.2. *Note that if* $k_1 = k_2$, *then*

$$\overline{p} = m + \frac{b}{N-2+b}\,.$$

In particular, for $q = 0$, *i.e.* $b = 2$, *we obtain*

$$\overline{p} = m + \frac{2}{N}\,.$$

Hence, Theorem 1.2.1 is coherent with the results in [119].

Remark 1.2.3. *In [86, Theorem 1] a similar global existence result is proved, for* $\rho(x) = |x|^{-q}$ *for any* $x \in \mathbb{R}^N \setminus \{0\}$ *with* $q \in [0, 2)$ *and for suitable* u_0 *not necessarily compactly supported. Clearly, such* ρ *does not satisfy assumption* (H). *Moreover, we can consider a more general behaviour of* $\rho(x)$ *for* $|x|$ *large; this affects the definition of* $\bar{p}$, *and consequently the choice of* p. *The smallness condition in Theorem 1.2.1 is different from that in [86], and it is not possible in general to say which is stronger. Moreover, since we consider* u_0 *with compact support, we can obtain the estimates* (1.2.14) *and* (1.2.15), *which do not have a counterpart in [86]. Finally, in [86] energy methods are used and a smoothing estimate is derived; hence the proof is completely different from our.*

The next result concerns the blow-up of solutions in finite time, for every $p > 1$ and $m > 1$, provided that the initial datum is sufficiently large.

Let

$$\mathfrak{s}(x) := \begin{cases} |x|^b & \text{if} \quad |x| > 1, \\ |x|^2 & \text{if} \quad |x| \leq 1\,. \end{cases}$$

Theorem 1.2.4. *Let assumptions* (H) *and* (1.2.10) *be satisfied. For any* $p>1, m>1$ *and for any* $T>0$*, if the initial datum* u_0 *is large enough, then the solution* u *of problem* (1.1.1) *blows-up in a finite time* $S\in(0,T]$*, in the sense that*

$$\|u(t)\|_\infty \to +\infty \quad \text{as } t\to S^- . \tag{1.2.16}$$

More precisely, we have the following three cases.

(a) Let $p>m$*. If* $C>0, a>0$ *are large enough,* $T>0$*,*

$$u_0(x) \geq CT^{-\frac{1}{p-1}}\left[1-\frac{\mathfrak{s}(x)}{a}T^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}}, \tag{1.2.17}$$

then the solution u *of problem* (1.1.1) *blows-up and satisfies the bound from below*

$$u(x,t) \geq C(T-t)^{-\frac{1}{p-1}}\left[1-\frac{\mathfrak{s}(x)}{a}(T-t)^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t)\in\mathbb{R}^N\times[0,S). \tag{1.2.18}$$

(b) Let $p<m$*. If* $\frac{C^{m-1}}{a}>0$ *and* $a>0$ *are big enough,* $T>0$ *and* (1.2.17) *holds, then the solution* u *of problem* (1.1.1) *blows-up and satisfies the bound from below* (1.2.18)*.*

(c) Let $p=m$*. If* $\frac{C^{m-1}}{a}>0$ *and* $a>0$ *are big enough,* $T>0$ *and* (1.2.17) *holds, then the solution* u *of problem* (1.1.1) *blows-up and satisfies the bound from below* (1.2.18)*.*

Observe that if u_0 satisfies (1.2.17), then

$$\operatorname{supp} u_0 \supseteq \{x\in\mathbb{R}^N \,:\, \mathfrak{s}(x) < aT^{\frac{p-m}{p-1}}\}.$$

In all the cases $(a),(b),(c)$, from (1.2.18) we can infer that

$$\operatorname{supp} u(\cdot,t) \supseteq \{x\in\mathbb{R}^N \,:\, \mathfrak{s}(x) < a(T-t)^{\frac{p-m}{p-1}}\} \quad \text{for all } t\in[0,S). \tag{1.2.19}$$

The precise choice of parameters $C>0$, $T>0$, $a>0$ in Theorem 1.2.4 is discussed in Remark 1.5.2 below.

Remark 1.2.5. *Let us mention that in [86], where some blow-up results are shown for problem* (1.1.1)*, there is not a counterpart of Theorem 1.2.4, since our result concerns* any $p>1$ *and sufficiently large initial data.*

1.2.1 Blow-up for any nontrivial initial datum

In this Subsection we discuss a further result concerning the blow-up of the solution to problem (1.1.1) for any initial datum $u_0\in C(\mathbb{R}^N), u_0\geq 0, u_0\not\equiv 0$.

Let $\underline{p}$ and be defined by (1.1.8) and (1.1.7), respectively. Assume (1.1.6). It is direct to see that

$$\underline{p} \leq \bar{p}. \tag{1.2.20}$$

In particular, $\underline{p}=\bar{p}$, whenever $k_1=k_2$. We distinguish between two cases:

1) $1 < p < m$,

2) $m \leq p < \underline{p}$.

In case 2), we need an extra hypothesis. In fact, we assume that (H) holds with

$$q \in (0, \epsilon), \tag{1.2.21}$$

for some $\epsilon > 0$ to be fixed small enough later. Then, b defined by (1.1.4), satisfies

$$2 - \epsilon \; < \; b \; < \; 2. \tag{1.2.22}$$

Theorem 1.2.6. *Let assumption (H) be satisfied. Suppose that*

$$1 < p < m,$$

and that $u_0 \in C(\mathbb{R}^N), u_0(x) \not\equiv 0$. Then, for any sufficiently large $T > 0$, the solution u of problem (1.1.1) blows-up in a finite time $S \in (0, T]$, in the sense that

$$\|u(t)\|_\infty \to +\infty \quad \text{as } t \to S^-.$$

More precisely, the bound from below (1.2.18) holds, with b, C, a, ζ, η as in Theorem 1.2.4-(b).

Theorem 1.2.7. *Let assumptions (H) and (1.2.21) be satisfied for $\epsilon > 0$ small enough. Let $u_0 \in C^\infty(\mathbb{R}^N)$ and $u_0 \not\equiv 0$. If*

$$m \leq p < \underline{p}, \tag{1.2.23}$$

then there exist sufficiently large $t_1 > 0$ and $T > 0$ such that the solution u of problem (1.1.1) blows-up in a finite time $S \in (0, T + t_1]$, in the sense that

$$\|u(t)\|_\infty \to +\infty \quad \text{as } t \to S^-.$$

More precisely, when $S > t_1$, we have the bound from below

$$u(x,t) \geq C(T+t_1-t)^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a}(T + t_1 - t)^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \mathbb{R}^N \times (t_1, S), \tag{1.2.24}$$

with C, a as in Theorem 1.2.4-(a).

Remark 1.2.8. *As it has been mentioned in the Introduction, in [86, Theorem 3] blow-up of solutions to problem (1.1.1) is shown when $\rho(x) = |x|^{-q}$ or $\rho(x) = (1+|x|)^{-q}$ with $q \in [0, 2)$. However, the results in [86] are different, in fact it is obtained an integral blow-up, that is, for some $R > 0$, $\theta \in (0,1)$, $T > 0$, $\int_{B_R}[u(x,t)^\theta]\rho(x)dx \to +\infty$ as $t \to T^-$. On the other hand, we should mention that the extra hypothesis (1.2.21), that we need in Theorem 1.2.7, in [86] is not used. Furthermore, the methods of proofs in [86] are completely different, since they are based on the choice of a special test function and integration by parts.*

1.3 Preliminaries

In this section we give the precise definitions of solution of all problems we address, then we state a local in time existence result for problem (1.1.1). Moreover, we recall some useful comparison principles.

Throughout the chapter we deal with *very weak* solutions to problem (1.1.1) and to the same problem set in different domains, according to the following definitions.

Definition 1.3.1. *Let $u_0 \in L^\infty(\mathbb{R}^N)$ with $u_0 \geq 0$. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$ is a solution of problem* (1.1.1) *if*

$$\begin{aligned} -\int_{\mathbb{R}^N}\int_0^\tau \rho(x) u \varphi_t \, dt \, dx = & \int_{\mathbb{R}^N} \rho(x) u_0(x) \varphi(x, 0) \, dx \\ & + \int_{\mathbb{R}^N}\int_0^\tau u^m \Delta\varphi \, dt \, dx \\ & + \int_{\mathbb{R}^N}\int_0^\tau \rho(x) u^p \varphi \, dt \, dx \end{aligned} \tag{1.3.25}$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \tau)), \varphi \geq 0$. Moreover, we say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (1.3.25) *with the inequality "$\leq$" ("$\geq$") instead of "$=$" with $\varphi \geq 0$.*

For any $x_0 \in \mathbb{R}^N$ and $R > 0$ we set

$$B_R(x_0) = \{x \in \mathbb{R}^N : \|x - x_0\| < R\}. \tag{1.3.26}$$

When $x_0 = 0$, we write $B_R \equiv B_R(0)$. For every $R > 0$, we consider the auxiliary problem

$$\begin{cases} u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p & \text{in } B_R \times (0, \tau) \\ u = 0 & \text{on } \partial B_R \times (0, \tau) \\ u = u_0 & \text{in } B_R \times \{0\}. \end{cases} \tag{1.3.27}$$

Definition 1.3.2. *Let $u_0 \in L^\infty(B_R)$ with $u_0 \geq 0$. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(B_R \times (0, S))$ for any $S < \tau$ is a solution of problem* (1.3.27) *if*

$$\begin{aligned} -\int_{B_R}\int_0^\tau \rho(x) u \, \varphi_t \, dt \, dx = & \int_{B_R} \rho(x) u_0(x) \varphi(x, 0) \, dx \\ & + \int_{B_R}\int_0^\tau u^m \Delta\varphi \, dt \, dx \\ & + \int_{B_R}\int_0^\tau \rho(x) u^p \varphi \, dt \, dx \end{aligned} \tag{1.3.28}$$

for any $\varphi \in C_c^\infty(\overline{B_R} \times [0, \tau))$ with $\varphi|_{\partial B_R} = 0$ for all $t \in [0, \tau)$. Moreover, we say that a nonnegative function $u \in L^\infty(B_R \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (1.3.28) *with the inequality "$\leq$" ("$\geq$") instead of "$=$", with $\varphi \geq 0$.*

Proposition 1.3.3. *Let hypothesis* (H) *be satisfied. Then there exists a solution* u *to problem* (1.3.27) *with*

$$\tau \geq \tau_R := \frac{1}{(p-1)\|u_0\|_{L^\infty(B_R)}^{p-1}}.$$

Proof. Note that $\underline{u} \equiv 0$ is a subsolution to (1.3.27). Moreover, let $\bar{u}_R(t)$ be the solution of the Cauchy problem

$$\begin{cases} \bar{u}'(t) = \bar{u}^p \\ \bar{u}(0) = \|u_0\|_{L^\infty(B_R)}\,, \end{cases}$$

that is

$$\bar{u}_R(t) = \frac{\|u_0\|_{L^\infty(B_R)}}{\left[1-(p-1)t\|u_0\|_{L^\infty(B_R)}^{p-1}\right]^{\frac{1}{p-1}}} \quad \text{for all } \ t \in [0, \tau_R)\,.$$

Clearly, for every $R > 0$, $\bar{u}_R$ is a supersolution of problem (1.3.27). Due to hypothesis (H),

$$0 < \min_{\bar{B}_R} \frac{1}{\rho} \leq \frac{1}{\rho(x)} \leq \max_{\bar{B}_R} \frac{1}{\rho} \quad \text{for all } x \in \overline{B_R}\,.$$

Hence, by standard results (see, e.g., [128]), problem (1.3.27) admits a nonnegative solution $u_R \in L^\infty(B_R \times (0, S))$ for any $S < \tau$, where $\tau \geq \tau_R$ is the maximal time of existence, i.e.

$$\|u_R(t)\|_\infty \to \infty \quad \text{as } \ t \to \tau_R^-.$$

□

Moreover, the following comparison principle for problem (1.3.27) holds (see [7] for the proof).

Proposition 1.3.4. *Let assumption* (H) *hold. If* u *is a subsolution of problem* (1.3.27) *and* v *is a supersolution of* (1.3.27)*, then*

$$u \leq v \quad a.e. \ in \ \ B_R \times (0, \tau).$$

Proposition 1.3.5. *Let hypothesis* (H) *be satisfied. Then there exists a solution* u *to problem* (1.1.1) *with*

$$\tau \geq \tau_0 := \frac{1}{(p-1)\|u_0\|_\infty^{p-1}}.$$

Moreover, u *is the* minimal solution*, in the sense that for any solution* v *to problem* (1.1.1) *there holds*

$$u \leq v \quad in \ \ \mathbb{R}^N \times (0, \tau)\,.$$

Proof. For every $R>0$ let u_R be the unique solution of problem (1.3.27). It is easily seen that if $0<R_1<R_2$, then

$$u_{R_1} \le u_{R_2} \quad \text{in } \; B_{R_1}\times(0,\tau_0)\,. \tag{1.3.29}$$

In fact, u_{R_2} is a supersolution, while u_{R_1} is a solution of problem (1.3.27) with $R=R_1$. Hence, by Proposition 1.3.4, (1.3.29) follows. Let $\bar u(t)$ be the solution of

$$\begin{cases} \bar u'(t) = \bar u^p \\ \bar u(0) = \|u_0\|_\infty\,, \end{cases}$$

that is

$$\bar u(t) = \frac{\|u_0\|_\infty}{\left[1-(p-1)t\|u_0\|_\infty^{p-1}\right]^{\frac{1}{p-1}}} \quad \text{for all } \; t\in[0,\tau_0)\,.$$

Clearly, for every $R>0$, $\bar u$ is a supersolution of problem (1.3.27). Hence

$$0 \le u_R(x,t) \le \bar u \quad \text{in } \; B_R\times(0,\tau_0)\,. \tag{1.3.30}$$

In view of (1.3.29), the family $\{u_R\}_{R>0}$ is monotone increasing w.r.t. R. Moreover, (1.3.30) implies that the family $\{u_R\}$ is uniformly bounded. Hence $\{u_R\}_{R>0}$ converges point-wise to a function, say $u(x,t)$, as $R\to+\infty$, i.e.

$$\lim_{R\to+\infty} u_R(x,t) = u(x,t) \quad \text{a.e. in } \mathbb{R}^N\times(0,\tau_0)\,.$$

Moreover, by the monotone convergence theorem, passing to the limit as $R\to+\infty$ in (1.3.28) we obtain

$$\begin{aligned} -\int_{\mathbb{R}^N}\int_0^{\tau_0}\rho(x)u\varphi_t\,dt\,dx &= \int_{\mathbb{R}^N}\rho(x)u_0(x)\varphi(x,0)\,dx \\ &+ \int_{\mathbb{R}^N}\int_0^{\tau_0} u^m\Delta\varphi\,dt\,dx \\ &+ \int_{\mathbb{R}^N}\int_0^{\tau_0}\rho(x)u^p\varphi\,dt\,dx \end{aligned}$$

for any $\varphi\in C_c^\infty(\mathbb{R}^N\times[0,\tau_0)), \varphi\ge 0$. Hence u is a solution of problem (1.1.1) $u\in L^\infty(\mathbb{R}^N\times(0,S))$ for any $S<\tau$, where $\tau\ge\tau_0$ is the maximal time of existence, i.e.

$$\|u(t)\|_\infty \to \infty \quad \text{as } \; t\to\tau^-\,.$$

Let us now prove that u is the minimal nonnegative solution to problem (1.1.1). Let v be any other solution to problem (1.1.1). Note that, for every $R>0$, v is a supersolution to problem (1.3.27). Hence, thanks to Proposition 1.3.4,

$$u_R \le v \quad \text{ in } \; B_R\times(0,\tau).$$

Then passing to the limit as $R\to\infty$, we get

$$u\le v \text{ in } \mathbb{R}^N\times(0,\tau)\,.$$

Therefore, u is the minimal nonnegative solution. □

In conclusion, we can state the following two comparison results.

Proposition 1.3.6. *Let hypothesis* (H) *be satisfied. Let* $\bar{u}$ *be a supersolution to problem* (1.1.1)*. Then, if* u *is the minimal solution to problem* (1.1.1) *given by Proposition 1.3.5, then*

$$u \leq \bar{u} \quad a.e.\ in\ \mathbb{R}^N \times (0, \tau)\,. \tag{1.3.31}$$

In particular, if $\bar{u}$ *exists until time* τ*, then also* u *exists at least until time* τ*.*

Proof. Clearly, for any $R > 0$, $\bar{u}$ is a supersolution to problem 1.3.27. Hence, by Proposition 1.3.4,

$$u_R \leq \bar{u} \quad \text{in} \ \ B_R \times (0, \tau)\,.$$

By passing to the limit as $R \to +\infty$, we easily obtain (1.3.31), which trivially ensures that u does exist at least up to τ, by the definition of maximal existence time. □

Proposition 1.3.7. *Let hypothesis* (H) *be satisfied. Let* u *be a solution to problem* (1.1.1) *for some time* $\tau = \tau_1 > 0$ *and* $\underline{u}$ *a subsolution to problem* (1.1.1) *for some time* $\tau = \tau_2 > 0$*. Suppose also that*

$$\operatorname{supp} \underline{u}|_{\mathbb{R}^N \times [0,S]} \ \ is\ compact\ for\ every\ \ S \in (0, \tau_2)\,.$$

Then

$$u \geq \underline{u} \quad\ in\ \ \mathbb{R}^N \times (0, \min\{\tau_1, \tau_2\})\ . \tag{1.3.32}$$

Proof. We fix any $S < \min\{\tau_1, \tau_2\}$. It $R > 0$ is so large that

$$\operatorname{supp} \underline{u}|_{\mathbb{R}^N \times [0,S]} \subseteq B_R \times [0, S],$$

then u and $\underline{u}$ are a supersolution and a subsolution, respectively, to 1.3.27. Hence

$$u \geq \underline{u} \quad \text{in} \ \ B_R \times (0, S)\,.$$

Inequality (1.3.32) then just follows by letting $R \to +\infty$ and using the arbitrariness of S. □

Remark 1.3.8. *Note that by minor modifications in the proof of [109, Theorem 2.3] one could show that problem* (1.1.1) *admits at most one bounded solution.*

In what follows we also consider solutions of equations of the form

$$u_t = \frac{1}{\rho(x)} \Delta(u^m) + u^p \quad \text{in} \ \ \Omega \times (0, \tau), \tag{1.3.33}$$

where $\Omega \subseteq \mathbb{R}^N$. Solutions are meant in the following sense.

Definition 1.3.9. *Let* $\tau > 0$, $p > 1, m > 1$*. We say that a nonnegative function* $u \in L^\infty(\Omega \times (0, S))$ *for any* $S < \tau$ *is a solution of problem* (1.3.27) *if*

$$\begin{aligned} -\int_\Omega \int_0^\tau \rho(x) u\, \varphi_t \, dt\, dx = &\int_\Omega \int_0^\tau u^m \Delta\varphi \, dt\, dx \\ &+ \int_\Omega \int_0^\tau \rho(x) u^p \varphi \, dt\, dx \end{aligned} \tag{1.3.34}$$

for any $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\tau))$ *with* $\varphi|_{\partial\Omega} = 0$ *for all* $t \in [0,\tau)$*. Moreover, we say that a nonnegative function* $u \in L^\infty(\Omega \times (0,S))$ *for any* $S < \tau$ *is a subsolution (supersolution) if it satisfies* (1.3.28) *with the inequality* " $\leq$ " (" $\geq$ ") *instead of* " $=$ ", *with* $\varphi \geq 0$.

Finally, let us recall the following well-known criterion, that will be used in the sequel; we reproduce it for reader's convenience. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Suppose that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$, and that $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$ is of class C^1. Let n be the unit outwards normal to Ω_1 at Σ. Let

$$u = \begin{cases} u_1 & \text{in } \Omega_1 \times [0,T), \\ u_2 & \text{in } \Omega_2 \times [0,T), \end{cases} \tag{1.3.35}$$

where $\partial_t u \in C(\Omega_1 \times (0,T)), u_1^m \in C^2(\Omega_1 \times (0,T)) \cap C^1(\overline{\Omega}_1 \times (0,T)), \partial_t u_2 \in C(\Omega_2 \times (0,T)),)u_2^m \in C^2(\Omega_2 \times (0,T)) \cap C^1(\overline{\Omega}_2 \times (0,T))$.

Lemma 1.3.10. *Let assumption* (H) *be satisfied.*

(i) Suppose that

$$\begin{aligned} \partial_t u_1 &\geq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \textit{for any } (x,t) \in \Omega_1 \times (0,T), \\ \partial_t u_2 &\geq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \textit{for any } (x,t) \in \Omega_2 \times (0,T), \end{aligned} \tag{1.3.36}$$

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \geq \frac{\partial u_2^m}{\partial n} \quad \textit{for any } (x,t) \in \Sigma \times (0,T). \tag{1.3.37}$$

Then u*, defined in* (1.3.35)*, is a supersolution to equation* (1.3.33)*, in the sense of Definition 1.3.9.*

(ii) Suppose that

$$\begin{aligned} \partial_t u_1 &\leq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \textit{for any } (x,t) \in \Omega_1 \times (0,T), \\ \partial_t u_2 &\leq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \textit{for any } (x,t) \in \Omega_2 \times (0,T), \end{aligned} \tag{1.3.38}$$

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \leq \frac{\partial u_2^m}{\partial n} \quad \textit{for any } (x,t) \in \Sigma \times (0,T). \tag{1.3.39}$$

Then u*, defined in* (1.3.35)*, is a subsolution to equation* (1.3.33)*, in the sense of Definition 1.3.9.*

Proof. Take any $\varphi \in C_c^\infty(\overline{\Omega} \times [0,\tau))$ with $\varphi|_{\partial\Omega} = 0$ for all $t \in [0,\tau), \varphi \geq 0$.

(i) We multiply by φ both sides of the two inequalities in (1.3.36), then integrating two times by parts we get

$$\begin{aligned} &-\int_0^\tau \int_{\Omega_1} \rho(u_1\varphi_t + u_1^p\varphi)dxdt \\ &\geq \int_0^\tau u_1^m \Delta\varphi dxdt - \int_0^\tau \int_\Sigma u_1^m \frac{\partial\varphi}{\partial n} d\sigma dt + \int_0^\tau \int_\Sigma \varphi \frac{\partial u_1^m}{\partial n} d\sigma dt\,, \end{aligned}$$

$$
\begin{aligned}
&- \int_0^\tau \int_{\Omega_2} \rho(u_2\varphi_t - u_2^p\varphi) dxdt \\
&\geq \int_0^\tau u_2^m \Delta\varphi dxdt + \int_0^\tau \int_\Sigma u_2^m \frac{\partial\varphi}{\partial n} d\sigma dt - \int_0^\tau \int_\Sigma \varphi \frac{\partial u_2^m}{\partial n} d\sigma dt \,.
\end{aligned}
$$

Summing up the previous two inequalities and using (1.3.37) we obtain

$$
- \int_0^\tau \int_\Omega \rho(u\varphi_t + u^p\varphi)\, dxdt \geq \int_0^\tau u^m \Delta\varphi\, dxdt \,.
$$

Hence the conclusion follows in this case. The statement (ii) can be obtained in the same way. This completes the proof. □

1.4 Global existence: proofs

In what follows we set $r \equiv |x|$. We want to construct a suitable family of supersolutions of equation

$$
u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{1.4.40}
$$

To this purpose, we define, for all $(x,t) \in \left[\mathbb{R}^N \setminus B_1(0)\right] \times [0, +\infty)$,

$$
u(x,t) \equiv u(r(x),t) := C\zeta(t) \left[1 - \frac{r^b}{a}\eta(t)\right]_+^{\frac{1}{m-1}} , \tag{1.4.41}
$$

where $\eta,\ \zeta \in C^1([0,+\infty);[0,+\infty))$ and $C > 0,\ a > 0$.

Now, we compute

$$
u_t - \frac{1}{\rho}\Delta(u^m) - u^p.
$$

To do this, let us set

$$
F(r,t) := 1 - \frac{r^b}{a}\eta(t)
$$

and define

$$
D_1 := \left\{(x,t) \in [\mathbb{R}^N \setminus B_1(0)] \times (0,+\infty) \mid 0 < F(r,t) < 1\right\}.
$$

For any $(x,t) \in D_1$, we have:

$$
\begin{aligned}
u_t &= C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} F^{\frac{1}{m-1}-1} \left(-\frac{r^b}{a}\eta'\right) \\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} \left(1 - \frac{r^b}{a}\eta\right) \frac{\eta'}{\eta} F^{\frac{1}{m-1}-1} - C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}-1} \\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}} - C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}-1};
\end{aligned} \tag{1.4.42}
$$

$$
(u^m)_r = -C^m\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}} \frac{b}{a}\eta r^{b-1}; \tag{1.4.43}
$$

$$\begin{aligned}(u^m)_{rr} = & -C^m\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1}\frac{b^2}{a}\eta r^{b-2}\left(1-\frac{r^b}{a}\eta\right) \\ & + C^m\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1}\frac{b^2}{a}\eta r^{b-2} \\ & - C^m\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}}\frac{b(b-1)}{a}\eta r^{b-2}.\end{aligned} \tag{1.4.44}$$

$$\begin{aligned}\Delta(u^m) &= (u^m)_{rr} + \frac{(N-1)}{r}(u^m)_r \\ &= C^m\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1}\frac{b^2}{a}\eta r^{b-2} \\ &- C^m\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}}\frac{b^2}{a}\eta r^{b-2} \\ &- C^m\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}}\frac{b(b-1)}{a}\eta r^{b-2} \\ &+ \frac{(N-1)}{r}\left(-C^m\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{b}{a}\eta r^{b-1}\right) \\ &= C^m\zeta^m \frac{m}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}-1} r^{b-2} \\ &- C^m(N-2)\zeta^m \frac{m}{m-1}\frac{b}{a}\eta F^{\frac{1}{m-1}} r^{b-2} \\ &- C^m\zeta^m \frac{m^2}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}} r^{b-2}\,.\end{aligned} \tag{1.4.45}$$

We set $\overline{u} \equiv u$,

$$\overline{w}(x,t) \equiv \overline{w}(r(x),t) := \begin{cases}\overline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_1(0)] \times [0,+\infty), \\ \overline{v}(x,t) & \text{in } B_1(0) \times [0,+\infty),\end{cases} \tag{1.4.46}$$

where

$$\overline{v}(x,t) \equiv \overline{v}(r(x),t) := C\zeta(t)\left[1 - \frac{(br^2+2-b)}{2}\frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}}\,. \tag{1.4.47}$$

We also define

$$\begin{aligned} K &:= \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0\,, \\ \bar\sigma(t) &:= \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a}\eta k_1\left(N-2+\frac{bm}{m-1}\right), \\ \bar\delta(t) &:= \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_2, \\ \bar\gamma(t) &:= C^{p-1}\zeta^p(t)\,, \\ \bar\sigma_0(t) &:= \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{m-1}Nb\,k_1\frac{\eta}{a}, \\ \bar\delta_0(t) &:= \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}b^2\,k_2\zeta^m\frac{m}{(m-1)^2}\frac{\eta^2}{a^2}. \end{aligned} \tag{1.4.48}$$

Proposition 1.4.1. *Let $\zeta = \zeta(t)$, $\eta = \eta(t) \in C^1([0,+\infty);[0,+\infty))$. Let $K, \bar\sigma, \bar\delta, \bar\gamma, \bar\sigma_0, \bar\delta_0$ be defined in* (1.4.48). *Assume* (1.1.6), (1.2.10), *and that, for all $t \in (0,+\infty)$,*

$$\eta(t) < a\,, \tag{1.4.49}$$

$$-\frac{\eta'}{\eta^2} \geq \frac{b^2}{a} C^{m-1} \zeta^{m-1}(t) \frac{m}{m-1} k_2, \tag{1.4.50}$$

$$\zeta' + C^{m-1}\zeta^m \frac{b}{a}\frac{m}{m-1}\eta\left[k_1\left(N-2+\frac{bm}{m-1}\right) - \frac{k_2 b}{m-1}\right] - C^{p-1}\zeta^p \geq 0, \tag{1.4.51}$$

$$-\frac{\eta'}{\eta^3} \geq \frac{C^{m-1}}{a^2} k_2 \zeta^{m-1} \frac{m}{m-1}, \tag{1.4.52}$$

$$\zeta' + N\zeta^m \frac{C^{m-1}}{a}\frac{m}{m-1}\eta\, k_1 - N\zeta^m \frac{C^{m-1}}{a^2}\frac{m}{(m-1)^2}\eta^2\, k_2 - C^{p-1}\zeta^p \geq 0. \tag{1.4.53}$$

Then w defined in (1.4.46) *is a supersolution of equation* (1.4.40).

Proof of Proposition 1.4.1. In view of (1.4.42), (1.4.43), (1.4.44) and (1.4.45), for any $(x,t) \in D_1$,

$$\begin{aligned}
&\bar u_t - \frac{1}{\rho}\Delta(\bar u^m) - \bar u^p \\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \\
&-\frac{r^{b-2}}{\rho}\left\{C^m\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}-1} - C^m(N-2)\zeta^m\frac{m}{m-1}\frac{b}{a}\eta F^{\frac{1}{m-1}}\right. \\
&\left.-C^m\zeta^m\frac{m^2}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}}\right\} - C^p\zeta^p F^{\frac{p}{m-1}}\,.
\end{aligned} \tag{1.4.54}$$

Thanks to hypothesis (H), we have

$$\frac{r^{b-2}}{\rho} \geq k_1, \quad -\frac{r^{b-2}}{\rho} \geq -k_2 \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0)\,. \tag{1.4.55}$$

From (1.4.54) and (1.4.55) we get

$$\begin{aligned}
&\bar u_t - \frac{1}{\rho}\Delta(\bar u^m) - \bar u^p \\
&\geq CF^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a}\eta k_1\left(N-2+\frac{bm}{m-1}\right)\right]\right. \\
&\left.-\zeta\frac{1}{m-1}\frac{\eta'}{\eta} - C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_2 - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}.
\end{aligned} \tag{1.4.56}$$

From (1.4.56), taking advantage from $\bar\sigma(t)$, $\bar\delta(t)$ and $\bar\gamma(t)$ defined in (1.4.48), we have

$$\bar u_t - \frac{1}{\rho}\Delta(\bar u^m) - \bar u^p \geq CF^{\frac{1}{m-1}-1}\left[\bar\sigma(t)F - \bar\delta(t) - \bar\gamma(t)F^{\frac{p+m-2}{m-1}}\right]. \tag{1.4.57}$$

For each $t > 0$, set

$$\varphi(F) := \bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}, \quad F \in (0,1)\,.$$

Now our goal is to find suitable C, a, ζ, η such that, for each $t > 0$,

$$\varphi(F) \geq 0 \quad \text{for any } F \in (0,1)\,.$$

We observe that $\varphi(F)$ is concave in the variable F, hence it is sufficient to have $\varphi(F)$ positive in the extrema of the interval of definition $(0, 1)$. This reduces to the system

$$\begin{cases} \varphi(0) \geq 0 \\ \varphi(1) \geq 0\,, \end{cases} \tag{1.4.58}$$

for each $t > 0$. The system is equivalent to

$$\begin{cases} -\bar{\delta}(t) \geq 0 \\ \bar{\sigma}(t) - \bar{\delta}(t) - \bar{\gamma}(t) \geq 0\,, \end{cases}$$

that is

$$\begin{cases} -\frac{\eta'}{\eta^2} \geq \frac{b^2}{a} C^{m-1} \zeta^{m-1} \frac{m}{m-1} k_2 \\ \\ \zeta' + C^{m-1}\zeta^m \frac{b}{a}\frac{m}{m-1}\eta\left[k_1\left(N-2+\frac{bm}{m-1}\right) - \frac{k_2 b}{m-1}\right] - C^{p-1}\zeta^p \geq 0, \end{cases}$$

which is guaranteed by (1.1.6), (1.4.50) and (1.4.51). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in} \quad D_1\,.$$

Since $\bar{u}^m \in C^1([\mathbb{R}^N \setminus B_1(0)] \times (0,T))$, in view of Lemma 1.3.10-(i) (applied with $\Omega_1 = D_1, \Omega_2 = \mathbb{R}^N \setminus [B_1(0) \cup D_1], u_1 = \bar{u}, u_2 = 0, u = \bar{u}$), we can deduce that $\bar{u}$ is a supersolution of equation

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p = 0 \quad \text{in} \quad \left[\mathbb{R}^N \setminus B_1(0)\right] \times (0, +\infty)\,, \tag{1.4.59}$$

in the sense of Definition 1.3.9. Now let v be as in (1.4.47). Set

$$G(r,t) := 1 - \frac{br^2 + 2 - b}{2}\frac{\eta(t)}{a}.$$

Due to (1.4.49),

$$0 < G(r,t) < 1 \quad \text{for all } (x,t) \in B_1(0) \times (0, +\infty)\,.$$

For any $(x,t) \in B_1(0) \times (0, +\infty)$, we have:

$$\bar{v}_t = C\zeta' G^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1}; \tag{1.4.60}$$

$$(\bar{v}^m)_r = -C^m b\zeta^m \frac{m}{m-1} G^{\frac{1}{m-1}} \frac{\eta}{a} r; \tag{1.4.61}$$

$$(\bar{v}^m)_{rr} = C^m \zeta^m \frac{m}{(m-1)^2} G^{\frac{1}{m-1}-1} \frac{\eta^2}{a^2} b^2 r^2 - C^m b\zeta^m \frac{m}{m-1} G^{\frac{1}{m-1}} \frac{\eta}{a}. \tag{1.4.62}$$

Therefore, for all $(x,t) \in B_1(0) \times (0,+\infty)$,

$$\begin{aligned}
&\bar{v}_t - \frac{1}{\rho}\Delta(\bar{v}^m) - \bar{v}^p \\
&= CG^{\frac{1}{m-1}-1}\Big\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + b\frac{N-1}{r}C^{m-1}\zeta^m \frac{m}{m-1}\frac{r}{\rho}\frac{\eta}{a} + \frac{b}{\rho}C^{m-1}\zeta^m\frac{m}{m-1}\frac{\eta}{a}\right] \\
&- \frac{\zeta}{m-1}\frac{\eta'}{\eta} - \frac{r^2}{\rho} b^2 C^{m-1}\frac{m}{(m-1)^2}\zeta^m\frac{\eta^2}{a^2} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\Big\}.
\end{aligned} \tag{1.4.63}$$

Using (1.2.9) and the fact that $r \in (0,1)$, (1.4.63) yields, for all $(x,t) \in B_1(0)\times(0,+\infty)$,

$$\begin{aligned}
&\bar{v}_t - \frac{1}{\rho}\Delta(\bar{v}^m) - \bar{v}^p \\
&\geq CG^{\frac{1}{m-1}-1}\Big\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + Nb\,k_1 C^{m-1}\zeta^m\frac{m}{m-1}\frac{\eta}{a}\right] \\
&- \frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{m-1}b^2\,k_2\frac{m}{(m-1)^2}\frac{\eta^2}{a^2} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\Big\} \\
&= CG^{\frac{1}{m-1}-1}\left[\bar{\sigma}_0(t)G - \bar{\delta}_0(t) - \bar{\gamma}(t)G^{\frac{p+m-2}{m-1}}\right].
\end{aligned} \tag{1.4.64}$$

Hence, due to (1.4.64), we obtain for all $(x,t) \in B_1(0) \times (0,+\infty)$,

$$\bar{v}_t - \frac{1}{\rho}\Delta(\bar{v}^m) - \bar{v}^p \geq CG^{\frac{1}{m-1}-1}\left[\bar{\sigma}_0(t)G - \bar{\delta}_0(t) - \bar{\gamma}(t)G^{\frac{p+m-2}{m-1}}\right]. \tag{1.4.65}$$

For each $t > 0$, set

$$\psi(G) := \bar{\sigma}_0(t)G - \bar{\delta}_0(t) - \bar{\gamma}(t)G^{\frac{p+m-2}{m-1}}, \quad G \in (0,1).$$

Now our goal is to verify that, for each $t > 0$,

$$\psi(G) \geq 0 \quad \text{for any } G \in (0,1).$$

We observe that $\psi(G)$ is concave in the variable G, hence it is sufficient to have $\psi(G)$ positive in the extrema of the interval of definition $(0,1)$. This reduces to the system

$$\begin{cases} \psi(0) \geq 0 \\ \psi(1) \geq 0, \end{cases} \tag{1.4.66}$$

for each $t > 0$. The system is equivalent to

$$\begin{cases} -\bar{\delta}_0(t) \geq 0 \\ \bar{\sigma}_0(t) - \bar{\delta}_0(t) - \bar{\gamma}(t) \geq 0, \end{cases}$$

that is

$$\begin{cases} -\frac{\eta'}{\eta^3} \geq b^2 \frac{C^{m-1}}{a^2} k_2 \zeta^{m-1} \frac{m}{m-1} \\ \zeta' + \frac{C^{m-1}}{a} bN\, k_1 \zeta^m \frac{m}{m-1} \eta - b^2 \frac{C^{m-1}}{a^2} k_2 \zeta^m \frac{m}{(m-1)^2} \eta^2 - C^{p-1}\zeta^p \geq 0, \end{cases}$$

which is guaranteed by (1.1.6), (1.4.52) and (1.4.53). Hence we have proved that

$$\bar{v}_t - \frac{1}{\rho}\Delta(\bar{v}^m) - \bar{v}^p \geq 0 \quad \text{for all } (x,t) \in B_1(0) \times (0,+\infty) \tag{1.4.67}$$

Now, observe that $\bar{w} \in C(\mathbb{R}^N \times [0,+\infty))$; indeed,

$$\bar{u} = \bar{v} = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_1(0) \times (0,+\infty)\,.$$

Moreover, $\bar{w}^m \in C^1(\mathbb{R}^N \times [0,+\infty))$; indeed,

$$(\bar{u}^m)_r = (\bar{v}^m)_r = -C^m\zeta(t)^m \frac{m}{m-1} b\frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_1(0)\times(0,+\infty)\,. \tag{1.4.68}$$

In conclusion, by (1.4.59), (1.4.64), (1.4.68) and Lemma 1.3.10-(i) (applied with $\Omega_1 = \mathbb{R}^N \setminus B_1(0), \Omega_2 = B_1(0), u_1 = \bar{u}, u_2 = \bar{v}, u = \bar{w}$), we can infer that $\bar{w}$ is a supersolution to equation (1.4.40) in the sense of Definition 1.3.9. □

Remark 1.4.2. *Let*

$$p > \overline{p},$$

and assumptions (1.1.6) *and* (1.2.10) *be satisfied. Let* $\omega := \frac{C^{m-1}}{a}$. *In Theorem 1.2.1, the precise hypotheses on parameters* $\alpha, \beta, C > 0, \omega > 0, T > 0$ *are the following:*

condition (1.2.12),

$$\beta - b^2\omega k_2 \frac{m}{m-1} \geq 0\,, \tag{1.4.69}$$

$$-\alpha + b\omega\frac{m}{m-1}\left[k_1\left(N - 2 + \frac{bm}{m-1}\right) - \frac{k_2 b}{m-1}\right] \geq C^{p-1}\,, \tag{1.4.70}$$

$$\beta T^\beta \;\geq\; b^2\frac{\omega}{a}k_2\frac{m}{m-1}, \tag{1.4.71}$$

$$T^\beta > \frac{r_0}{a} \quad (\textit{for } r_0 > 1), \tag{1.4.72}$$

$$-\alpha + b\omega\frac{m}{m-1}\left(k_1 N - b\frac{T^{-\beta}}{(m-1)a}k_2\right) \;\geq\; C^{p-1}\,. \tag{1.4.73}$$

Lemma 1.4.3. *All the conditions in Remark 1.4.2 can be satisfied simoultaneously.*

Proof. We take α satisfying (1.2.12) and

$$\alpha < \min\left\{ \frac{k_1\left(N-2+\frac{bm}{m-1}\right)-\frac{k_2 b}{m-1}}{k_1\left[m\left(N-2+b\right)-(N-2)\right]}, \frac{k_1 N}{bk_2+(m-1)k_1 N}, \frac{1}{m-1} \right\}. \tag{1.4.74}$$

This is possible, since

$$p > \overline{p} > m + \frac{k_2 b}{k_1 N} > m\,.$$

In view of (1.4.74), (1.1.6) and the fact that $\beta = 1-\alpha(m-1)$, we can take $\omega > 0$ so that (1.4.69) holds, the left-hand-side of (1.4.70) is positive, and

$$-\alpha + b\omega\frac{m}{m-1}(k_1 N - \epsilon) > 0\,,$$

for some $\epsilon > 0$. Then, we choose $C > 0$ so small that (1.4.70) holds and

$$-\alpha + b\omega\frac{m}{m-1}(k_1 N - \epsilon) > C^{p-1}; \tag{1.4.75}$$

therefore, also $a > 0$ is properly fixed, in view of the definition of ω. We select $T > 0$ so big that (1.4.71), (1.4.72) are valid and

$$k_1 N - b\frac{T^{-\beta}}{(m-1)a}k_2 \geq \epsilon\,. \tag{1.4.76}$$

From (1.4.76) and (1.4.75) inequality (1.4.73) follows. □

Proof of Theorem 1.2.1. We prove Theorem 1.2.1 by means of Proposition 1.2.1. In view of Lemma 1.4.3, we can assume that all the conditions of Remark 1.4.2 are fulfilled.

Set

$$\zeta(t) = (T+t)^{-\alpha}, \quad \eta(t) = (T+t)^{-\beta}, \quad \text{for all } t > 0\,.$$

Observe that condition (1.4.72) implies (1.4.49). Moreover, consider conditions (1.4.50), (1.4.51) of Proposition 1.4.1 with this choice of $\zeta(t)$ and $\eta(t)$. Therefore we obtain

$$\beta - \frac{b^2}{a}C^{m-1}\frac{m}{m-1}k_2(T+t)^{-\alpha(m-1)-\beta+1} \geq 0 \tag{1.4.77}$$

and

$$\begin{aligned} &-\alpha(T+t)^{-\alpha-1} + \frac{C^{m-1}}{a}\frac{mb}{m-1}\left[k_1\left(N-2+\frac{bm}{m-1}\right)-\frac{k_2 b}{m-1}\right](T+t)^{-\alpha m-\beta} \\ &-C^{p-1}(T+t)^{-\alpha p} \geq 0\,. \end{aligned} \tag{1.4.78}$$

Since, $\beta = 1-\alpha(m-1)$, (1.4.77) and (1.4.78) become

$$C^{m-1}\frac{m}{m-1}\frac{b}{a} \;\leq\; \frac{1-\alpha(m-1)}{k_2 b}\,, \tag{1.4.79}$$

$$\left\{-\alpha+b\frac{C^{m-1}}{a}\frac{m}{m-1}\left[k_1\left(N-2+\frac{bm}{m-1}\right)-\frac{k_2 b}{m-1}\right]\right\}(T+t)^{-\alpha-1} \\ \geq C^{p-1}(T+t)^{-\alpha p}\,. \tag{1.4.80}$$

Due to assumption (1.2.12),

$$\beta>0, \quad -\alpha-1\geq -p\alpha. \tag{1.4.81}$$

Thus (1.4.79) and (1.4.80) follow from (1.6.157), (1.4.69) and (1.4.70).

We now consider conditions (1.4.52) and (1.4.53) of Proposition 1.4.1. Substituting $\zeta(t)$, $\eta(t)$, α and β previously chosen, we get (1.4.71) and

$$\left[-\alpha+b\frac{C^{m-1}}{a}\frac{m}{m-1}\left(k_1N-b\frac{(T+t)^{-\beta}}{(m-1)a}\,k_2\right)\right](T+t)^{-\alpha-1}\;\geq\;C^{p-1}(T+t)^{-p\alpha}\,. \tag{1.4.82}$$

Condition (1.4.82) follows from (1.6.157) and (1.4.73).

Hence, we can choose α,β, $C>0$, $a>0$ and T so that (1.4.79), (1.4.80), (1.4.71) and (1.4.82) hold. Thus the conclusion follows by Propositions 1.4.1 and 1.3.6. □

1.5 Blow-up: proofs

Let

$$\underline{w}(x,t)\equiv\underline{w}(r(x),t):=\begin{cases}\underline{u}(x,t) & \text{in } [\mathbb{R}^N\setminus B_1(0)]\times[0,T),\\ \underline{v}(x,t) & \text{in } B_1(0)\times[0,T),\end{cases} \tag{1.5.83}$$

where $\underline{u}\equiv u$ is defined in (1.4.41) and $\underline{v}$ is defined as follows

$$\underline{v}(x,t)\equiv\underline{v}(r(x),t):=C\zeta(t)\left[1-r^2\frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}}\,. \tag{1.5.84}$$

Observe that for any $(x,t)\in B_1(0)\times(0,T)$, we have:

$$\underline{v}_t=C\zeta' G^{\frac{1}{m-1}}+C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}}-C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1}; \tag{1.5.85}$$

$$(\underline{v}^m)_r=-2C^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{\eta}{a}r;$$

$$(\underline{v}^m)_{rr}=4C^m\zeta^m\frac{m}{(m-1)^2}G^{\frac{1}{m-1}-1}\frac{\eta}{a}-2C^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{\eta}{a} \\ -4C^m\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}G^{\frac{1}{m-1}},$$

$$\Delta(\underline{v}^m)=4C^m\zeta^m\frac{m}{(m-1)^2}G^{\frac{1}{m-1}-1}\frac{\eta}{a}-4C^m\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}G^{\frac{1}{m-1}} \\ -2NC^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{\eta}{a}\,. \tag{1.5.86}$$

Therefore, from (1.5.85) and (1.5.86) we get, for all $(x,t) \in B_1(0) \times (0,T)$,

$$
\begin{aligned}
&\underline{v}_t - \frac{1}{\rho}\Delta(\underline{v}^m) - \underline{v}^p \\
&= CG^{\frac{1}{m-1}-1}\Big\{ G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + 2NC^{m-1}\zeta^m \frac{m}{m-1}\frac{1}{\rho}\frac{\eta}{a} + \frac{4}{\rho}C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}\right] \\
&- \frac{\zeta}{m-1}\frac{\eta'}{\eta} - \frac{4}{\rho}C^{m-1}\frac{m}{(m-1)^2}\frac{\eta}{a} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\Big\}.
\end{aligned} \tag{1.5.87}
$$

We also define

$$
\begin{aligned}
&\underline{\sigma}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a}\eta k_2\left(N-2+\frac{bm}{m-1}\right), \\
&\underline{\delta}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_1, \\
&\underline{\gamma}(t) := C^{p-1}\zeta^p, \\
&\underline{\sigma_0}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + 2C^{m-1}\zeta^m\frac{m}{m-1}\left(N+\frac{2}{m-1}\right)\rho_2\frac{\eta}{a}, \\
&\underline{\delta_0}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + 4\frac{C^{m-1}}{a}\zeta^m\rho_1\frac{m}{(m-1)^2}\eta, \\
&K := \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0.
\end{aligned} \tag{1.5.88}
$$

Proposition 1.5.1. *Let $T \in (0,\infty)$, $\zeta,\ \eta \in C^1([0,T);[0,+\infty))$. Let $\underline{\sigma}, \underline{\delta}, \underline{\gamma}, \underline{\sigma_0}, \underline{\delta_0}, K$ be defined in* (1.5.88). *Assume* (1.2.10) *and that, for all $t \in (0,T)$,*

$$K[\underline{\sigma}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)[\underline{\gamma}(t)]^{\frac{m-1}{p-1}}, \tag{1.5.89}$$

$$(m-1)\underline{\sigma}(t) \le (p+m-2)\underline{\gamma}(t)\,, \tag{1.5.90}$$

$$K[\underline{\sigma_0}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta_0}(t)[\underline{\gamma}(t)]^{\frac{m-1}{p-1}}, \tag{1.5.91}$$

$$(m-1)\underline{\sigma_0}(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{1.5.92}$$

Then w defined in (1.5.83) *is a weak subsolution of equation* (1.4.40).

Proof of Proposition 1.5.1. In view of (1.4.42), (1.4.43), (1.4.44) and (1.4.45) we obtain

$$
\begin{aligned}
&\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \\
&- \frac{r^{b-2}}{\rho}\left\{C^m\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}-1} - C^m\zeta^m\frac{m}{m-1}\frac{b}{a}\eta F^{\frac{1}{m-1}} - C^m\zeta^m\frac{m^2}{(m-1)^2}\frac{b^2}{a}\eta F^{\frac{1}{m-1}}\right\} \\
&- C^p\zeta^p F^{\frac{p}{m-1}} \quad \text{for all } (x,t) \in D_1\,.
\end{aligned} \tag{1.5.93}
$$

In view of hypothesis (H), we can infer that

$$\frac{r^{b-2}}{\rho} \leq k_2, \quad -\frac{r^{b-2}}{\rho} \leq -k_1 \quad \text{for all } x \in \mathbb{R}^N \setminus B_1(0)\,. \tag{1.5.94}$$

From (1.5.93) and (1.5.94) we have

$$\begin{aligned} &\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\ &\leq CF^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a}\eta k_2\left(N-2+\frac{bm}{m-1}\right)\right]\right. \\ &\left.-\zeta\frac{1}{m-1}\frac{\eta'}{\eta} - C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_1 - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}. \end{aligned} \tag{1.5.95}$$

Thanks to (1.5.88), (1.5.95) becomes

$$\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \leq CF^{\frac{1}{m-1}-1}\varphi(F), \tag{1.5.96}$$

where, for each $t \in (0,T)$,

$$\varphi(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{p+m-2}{m-1}}.$$

Our goal is to find suitable C, a, ζ, η such that, for each $t \in (0,T)$,

$$\varphi(F) \leq 0 \quad \text{for any } F \in (0,1)\,.$$

To this aim, we impose that

$$\sup_{F\in(0,1)} \varphi(F) = \max_{F\in(0,1)} \varphi(F) = \varphi(F_0) \leq 0\,,$$

for some $F_0 \in (0,1)$. We have

$$\begin{aligned} \frac{d\varphi}{dF} = 0 &\iff \underline{\sigma}(t) - \frac{p+m-2}{m-1}\underline{\gamma}(t)F^{\frac{p-1}{m-1}} = 0 \\ &\iff F = F_0 = \left[\frac{m-1}{p+m-2}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)}\right]^{\frac{m-1}{p-1}}. \end{aligned}$$

Then

$$\varphi(F_0) = K\,\frac{\underline{\sigma}(t)^{\frac{p+m-2}{p-1}}}{\underline{\gamma}(t)^{\frac{m-1}{p-1}}} - \underline{\delta}(t)\,,$$

where the coefficient K depending on m and p has been defined in (1.5.88). By hypoteses (1.5.89) and (1.5.90), for each $t \in (0,T)$,

$$\varphi(F_0) \leq 0\,, \quad F_0 \leq 1\,. \tag{1.5.97}$$

So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p \leq 0 \quad \text{in } D_1. \tag{1.5.98}$$

Furthermore, since $\underline{u}^m \in C^1([\mathbb{R}^N \setminus B_1(0)] \times (0,T))$, due to Lemma 1.3.10 (applied with $\Omega_1 = D_1, \Omega_2 = \mathbb{R}^N \setminus [B_1(0) \cup D_1], u_1 = \underline{u}, u_2 = 0, u = \underline{u}$), it follows that $\underline{u}$ is a subsolution to equation

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p = 0 \quad \text{in} \quad [\mathbb{R}^N \setminus B_1(0)] \times (0,T),$$

in the sense of Definition 1.3.9.

Let

$$D_2 := \{(x,t) \in B_1(0) \times (0,T) \,:\, 0 < G(r,t) < 1\}\,.$$

Using (1.2.9), (1.5.87) yields, for all $(x,t) \in D_2$,

$$\begin{aligned}
&v_t - \frac{1}{\rho}\Delta(v^m) - v^p \\
&\le CG^{\frac{1}{m-1}-1}\Big\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + 2\left(N + \frac{2}{m-1}\right)k_2 C^{m-1}\zeta^m \frac{m}{m-1}\frac{\eta}{a}\right] \\
&- \frac{\zeta}{m-1}\frac{\eta'}{\eta} - 4C^{m-1}k_1\frac{m}{(m-1)^2}\frac{\eta}{a} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\Big\} \\
&= CG^{\frac{1}{m-1}-1}\left[\underline{\sigma}_0(t)G - \underline{\delta}_0(t) - \underline{\gamma}(t)G^{\frac{p+m-2}{m-1}}\right].
\end{aligned} \tag{1.5.99}$$

Now, by the same arguments used to obtain (1.5.98), in view of (1.5.91) and (1.5.92) we can infer that

$$\underline{v}_t - \frac{1}{\rho}\Delta\underline{v}^m \le \underline{v}^p \quad \text{for any } (x,t) \in D_2\,. \tag{1.5.100}$$

Moreover, since $\underline{v}^m \in C^1(B_1(0) \times (0,T))$, in view of Lemma 1.3.10 (applied with $\Omega_1 = D_2, \Omega_2 = B_1(0) \setminus D_2, u_1 = \underline{v}, u_2 = 0, u = \underline{v}$), we get that $\underline{v}$ is a subsolution to equation

$$\underline{v}_t - \frac{1}{\rho}\Delta\underline{v}^m = \underline{v}^p \quad \text{in } B_1(0) \times (0,T)\,, \tag{1.5.101}$$

in the sense of Definition 1.3.9. Now, observe that $\underline{w} \in C(\mathbb{R}^N \times [0,T))$; indeed,

$$\underline{u} = \underline{v} = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_1(0) \times (0,T)\,.$$

Moreover, since $b \in (0,2]$,

$$(\underline{u}^m)_r \ge (\underline{v}^m)_r = -2C^m\zeta(t)^m\frac{m}{m-1}\frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_1(0) \times (0,T)\,. \tag{1.5.102}$$

In conclusion, in view of (1.5.102) and Lemma 1.3.10 (applied with $\Omega_1 = B_1(0), \Omega_2 = \mathbb{R}^N \setminus B_1(0), u_1 = \underline{v}, u_2 = \underline{u}, u = \underline{w}$), we can infer that $\underline{w}$ is a subsolution to equation (1.4.40), in the sense of Definition 1.3.9. □

Remark 1.5.2. *Let $\omega := \frac{C^{m-1}}{a}$. In Theorem 1.2.4 the precise choice of the parameters $C > 0, a > 0, T > 0$ are as follows.*

(a) Let $p > m$. We require that

$$K\left\{\frac{1}{m-1}+bk_2\omega\frac{m}{m-1}\left(\frac{bm}{m-1}+N-2\right)\right\}^{\frac{p+m-2}{p-1}} \leq \frac{C^{m-1}}{m-1}\left[b^2k_1\omega\frac{m}{m-1}+\frac{p-m}{p-1}\right], \tag{1.5.103}$$

$$1+\omega mbk_2\left(N-2+\frac{bm}{m-1}\right)\leq (p+m-2)\,C^{p-1}, \tag{1.5.104}$$

$$K\left[\frac{1}{m-1}+2\,k_2\omega\frac{m}{m-1}\left(N+\frac{2}{m-1}\right)\right]^{\frac{p+m-2}{p-1}} \leq \frac{C^{m-1}}{m-1}\left[4\,k_1\omega\frac{m}{m-1}+\frac{p-m}{p-1}\right], \tag{1.5.105}$$

$$1+k_2\omega\left(N+\frac{2}{m-1}\right)\leq (p+m-2)C^{p-1}; \tag{1.5.106}$$

(b) Let $p < m$. We require that

$$\omega > \frac{(m-p)(m-1)}{b^2(p-1)mk_1}, \tag{1.5.107}$$

$$a \geq \max\left\{\frac{K\left\{\frac{1}{m-1}+\omega k_2\frac{m}{m-1}b\left(N-2+\frac{bm}{m-1}\right)\right\}^{\frac{p+m-2}{p-1}}}{\omega\frac{1}{m-1}\left[\omega\frac{m}{m-1}k_1b^2-\frac{m-p}{p-1}\right]}, \frac{K\left\{\frac{1}{m-1}+2\omega\,k_2\frac{m}{m-1}\left(N+\frac{2}{m-1}\right)\right\}^{\frac{p+m-2}{p-1}}}{\omega\frac{1}{m-1}\left[4\,k_1\omega\frac{m}{m-1}-\frac{m-p}{p-1}\right]}\right\}, \tag{1.5.108}$$

$$(p+m-2)\,(a\omega)^{\frac{p-1}{m-1}} \geq \max\left\{1+\omega m\,b\,k_2\left(\frac{bm}{m-1}+N-2\right), 1+\omega\,k_2\left(N+\frac{2}{m-1}\right)\right\}. \tag{1.5.109}$$

(c) Let $p = m$. We require that $\omega > 0$,

$$a \geq \max\left\{\frac{K\left\{\frac{1}{m-1}+\omega k_2\frac{m}{m-1}b\left(N-2+\frac{bm}{m-1}\right)\right\}^2}{b^2k_1\omega^2\frac{m}{(m-1)^2}}, \frac{K\left\{\frac{1}{m-1}+2\omega\,k_2\frac{m}{m-1}\left(N+\frac{2}{m-1}\right)\right\}^2}{4\,k_1\omega^2\frac{m}{(m-1)^2}}, \frac{1}{2(m-1)\omega}\left[1+\omega\,m\,b\,k_2\left(\frac{bm}{m-1}+N-2\right)\right], \frac{1}{2(m-1)\omega}\left[1+\omega\,k_2\left(N+\frac{2}{m-1}\right)\right]\right\}. \tag{1.5.110}$$

Lemma 1.5.3. *All the conditions of Remark 1.5.2 can hold simultaneously.*

Proof. (a) We take any $\omega > 0$, then we select $C > 0$ big enough (therefore, $a > 0$ is also fixed, due to the definition of ω) so that (1.5.103)-(1.5.106) hold.

(b) We can take $\omega > 0$ so that (1.5.107) holds, then we take $a > 0$ sufficiently large to guarantee (1.5.108) and (1.5.109) (therefore, $C > 0$ is also fixed).

(c) For any $\omega > 0$, we take $a > 0$ sufficiently large to guarantee (1.5.110) (thus, $C > 0$ is also fixed). □

Proof of Theorem 1.2.4. We now prove Theorem 1.2.4, by means of Proposition 1.5.1. In view of Lemma 1.5.3, we can assume that all the conditions in Remark 1.5.2 are fulfilled. Set

$$\zeta(t) = (T-t)^{-\alpha}, \quad \eta(t) = (T-t)^{\beta}$$

and

$$\alpha = \frac{1}{p-1}, \quad \beta = \frac{m-p}{p-1}.$$

Then

$$\underline{\sigma}(t) = \left[\frac{1}{m-1} + C^{m-1}\frac{m}{m-1}\frac{b}{a}k_2\left(N-2+\frac{bm}{m-1}\right)\right](T-t)^{\frac{-p}{p-1}},$$

$$\underline{\delta}(t) := \left[\frac{m-p}{(m-1)(p-1)} + C^{m-1}\frac{m}{(m-1)^2}\frac{b^2}{a}k_1\right](T-t)^{\frac{-p}{p-1}},$$

$$\underline{\gamma}(t) := C^{p-1}(T-t)^{\frac{-p}{p-1}}.$$

Let $p > m$. Conditions (1.5.103) and (1.5.104) imply (1.5.89) and (1.5.90), whereas (1.5.105) and (1.5.106) imply (1.5.91) and (1.5.92). Hence, by Propositions 1.5.1 and 1.3.7 the thesis follows in this case.

Let $p < m$. Conditions (1.5.108) and (1.5.109) imply (1.5.89) and (1.5.90), whereas conditions (1.5.105) and (1.5.106) imply (1.5.91) and (1.5.92). Hence, by Propositions 1.5.1 and 1.3.7 the thesis follows in this case, too.

Finally, let $p = m$. Condition (1.5.110) implies (1.5.89), (1.5.90), (1.5.91) and (1.5.92). Hence, by Propositions 1.5.1 and 1.3.7 the thesis follows in this case, too. The proof is complete.

□

1.6 Blow-up for any nontrivial initial datum: proofs

Proof of Theorem 1.2.6. Since $u_0 \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, there exist $\varepsilon > 0, r_0 > 0$ and $x_0 \in \mathbb{R}^N$ such that

$$u_0(x) \geq \varepsilon, \quad \text{for all } x \in B_{r_0}(x_0).$$

Without loss of generality, we can assume that $x_0 = 0$. Let $\underline{w}$ be the subsolution of problem (1.1.1) considered in Theorem 1.2.4 (with $a > 0$ and $C > 0$ properly fixed). We can find $T > 0$ sufficiently big in such a way that

$$C\,T^{-\frac{1}{p-1}} \leq \varepsilon, \quad a\,T^{-\frac{m-p}{p-1}} \leq \min\{r_0^b, r_0^2\}. \tag{1.6.111}$$

From inequalities in (1.6.111), we can deduce that

$$\underline{w}(x,0) \le u_0(x) \quad \text{for any } \ x \in \mathbb{R}.$$

Hence, by Theorem 1.2.4 and the comparison principle, the thesis follows. □

Let us explain the strategy of the proof of Theorem 1.2.7. Let u be a solution to problem (1.1.1) and let $\underline{w}$ be the subsolution to problem (1.1.1) given by Theorem 1.2.4. We look for a subsolution z to the equation

$$z_t = \frac{1}{\rho(x)}\Delta(z^m) \quad \text{in } \mathbb{R}^N \times (0,\infty)\,, \tag{1.6.112}$$

such that

$$z(x,0) \le u_0(x) \quad \text{for any } \ x \in \mathbb{R}^N\,, \tag{1.6.113}$$

and

$$z(x,t_1) \ge \underline{w}(x,0) \quad \text{for any } \ x \in \mathbb{R}^N \tag{1.6.114}$$

for $t_1 > 0$ and $T > 0$ large enough. Let $\tau > 0$ be the maximal existence time of u. If $\tau \le t_1$, then nothing has to be proved, and $u(x,t)$ blows-up at a certain time $S \in (0,t_1]$. Suppose that $\tau > t_1$. Since z is also a subsolution to problem (1.1.1), due to (1.6.113) and the comparison principle,

$$z(x,t) \le u(x,t) \quad \text{for any } \ (x,t) \in \mathbb{R}^N \times (0,\tau)\,. \tag{1.6.115}$$

From (1.6.114) and (1.6.115),

$$u(x,t_1) \ge z(x,t_1) \ge \underline{w}(x,0) \quad \text{for any } \ x \in \mathbb{R}^N.$$

Thus $u(x,t+t_1)$ is a supersolution, whereas $\underline{w}(x,t)$ is a subsolution of problem

$$\begin{cases} u_t = \frac{1}{\rho}\Delta(u^m) + u^p & \text{in } \ \mathbb{R}^N \times (0,+\infty) \\ u(x,t_1) = \underline{w}(x,0) & \text{in } \ \mathbb{R}^N \times \{0\}\,. \end{cases}$$

Hence by Theorem 1.2.4, $u(x,t)$ blows-up in a finite time $S \in (t_1, t_1+T)$.

In order to construct a suitable family of subsolutions of equation (1.6.112), let us consider two functions $\eta(t), \zeta(t) \in C^1([0,+\infty);[0,+\infty))$ and two constants $C_1 > 0$, $a_1 > 0$. Define

$$z(x,t) \equiv z(r(x),t) := \begin{cases} \xi(x,t) & \text{in } [\mathbb{R}^N \setminus B_1(0)] \times (0,+\infty) \\ \mu(x,t) & \text{in } B_1(0) \times (0,+\infty)\,, \end{cases} \tag{1.6.116}$$

where

$$\xi(x,t) \equiv \xi(r(x),t) := C_1\zeta(t)\left[1 - \frac{r^b}{a_1}\eta(t)\right]_+^{\frac{1}{m-1}} \tag{1.6.117}$$

and

$$\mu(x,t) \equiv \xi(r(x),t) := C_1\zeta(t)\left[1 - \frac{br^2+2-b}{2a_1}\eta(t)\right]_+^{\frac{1}{m-1}}\,. \tag{1.6.118}$$

Let us set

$$F(r,t) := 1 - \frac{r^b}{a_1}\eta(t)\,, \quad G(r,t) := 1 - \frac{br^2+2-b}{2a_1}\eta(t)$$

and define

$$D_1 := \big\{(x,t) \in [\mathbb{R}^N \setminus B_1(0)] \times (0,+\infty) \,|\, 0 < F(r,t) < 1\big\}\,,$$

$$D_2 := \{(x,t) \in B_1(0) \times (0,+\infty) \,|\, 0 < G(r,t) < 1\}\,.$$

Furthermore, for $\epsilon_0 > 0$ small enough, let

$$\beta_0 = \frac{b\,\dfrac{k_1}{k_2}}{(m-1)\,(N-2)+bm}\,, \tag{1.6.119}$$

$$\alpha_0 := \frac{1-\beta_0}{m-1} = \frac{N-2+\dfrac{b}{m-1}\left(m-\dfrac{k_1}{k_2}\right)}{(m-1)\,(N-2)+bm}\,, \tag{1.6.120}$$

$$\tilde\beta_0 = \frac{2\dfrac{k_1}{k_2}-\epsilon_0}{N(m-1)+2}\,, \tag{1.6.121}$$

$$\tilde\alpha_0 := \frac{1-\tilde\beta_0}{m-1} = \frac{N(m-1)+2-2\dfrac{k_1}{k_2}+\epsilon_0}{(m-1)\,[N(m-1)+2]}\,, \tag{1.6.122}$$

Observe that

$$0 \,<\, \beta_0 \,<\, 1, \qquad 0 \,<\, \tilde\beta_0 \,<\, 1. \tag{1.6.123}$$

Note that, if $\epsilon_0 > 0$ is small enough, then

$$0 \,<\, \beta_0 \,<\, \tilde\beta_0. \tag{1.6.124}$$

Proposition 1.6.1. *Let assumption* (H) *be satisfied. Assume that* (1.2.21) *holds, for* $\epsilon > 0$ *small enough. Let*

$$\bar\beta \in (0, \beta_0)\,, \tag{1.6.125}$$

$$\bar\alpha := \frac{1-\bar\beta}{m-1}\,. \tag{1.6.126}$$

Suppose that

$$1 < p < m + \frac{\bar\beta}{\bar\alpha}\,. \tag{1.6.127}$$

Let $T_1 \in (0,\infty)$*,*

$$\zeta(t) = (T_1+t)^{-\bar\alpha}, \quad \eta(t) = (T_1+t)^{-\bar\beta}\,.$$

Then there exist $\omega_1 := \frac{C_1^{m-1}}{a_1} > 0$*,* $t_1 > 0$ *and* $T > 0$ *such that* z *defined in* (1.6.116) *is a subsolution of equation* (1.6.112) *and satisfies* (1.6.113) *and* (1.6.114)*.*

Proof. We can argue as we have done to obtain (1.5.95), in order to get

$$\begin{aligned} &\xi_t - \frac{1}{\rho}\Delta(\xi^m) \\ &\le C_1 F^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C_1^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a_1}\eta k_2\left(N-2+\frac{bm}{m-1}\right)\right]\right. \\ &\left. -\zeta\frac{1}{m-1}\frac{\eta'}{\eta} - C_1^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a_1}\eta k_1\right\} \quad \text{for all } (x,t)\in D_1 . \end{aligned} \tag{1.6.128}$$

We now define

$$\begin{aligned} \sigma(t) &:= \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C_1^{m-1}\zeta^m\frac{m}{m-1}\frac{b}{a}\eta k_2\left(N-2+\frac{bm}{m-1}\right), \\ \delta(t) &:= \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C_1^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{b^2}{a}\eta k_1. \end{aligned} \tag{1.6.129}$$

Hence, (1.6.128) becomes

$$\xi_t - \frac{1}{\rho}\Delta(\xi^m) \le C_1 F^{\frac{1}{m-1}-1}\bar{\varphi}(F) \quad \text{in } D_1 , \tag{1.6.130}$$

where

$$\bar{\varphi}(F) := \sigma(t)F - \delta(t). \tag{1.6.131}$$

Observe that ξ is a subsolution to equation

$$\xi_t - \frac{1}{\rho}\Delta(\xi^m) = 0 \quad \text{in } D_1 , \tag{1.6.132}$$

whenever, for any $t > 0$

$$\bar{\varphi}(F) \le 0,$$

that is

$$\begin{cases} \sigma(t) > 0 \\ \delta(t) > 0 \\ \sigma(t) - \delta(t) \le 0. \end{cases} \quad \text{for any } t > 0 \tag{1.6.133}$$

By using the very definition of ζ and η, we get

$$\sigma(t) = -\bar{\alpha}(T_1+t)^{-\bar{\alpha}-1} - \frac{\bar{\beta}}{m-1}(T_1+t)^{-\bar{\alpha}-1} + \frac{C_1^{m-1}}{a_1}k_2\frac{m}{m-1}b\left(N-2+\frac{bm}{m-1}\right)(T_1+t)^{-\bar{\alpha}m-\bar{\beta}},$$

$$\delta(t) = -\frac{\bar{\beta}}{m-1}(T_1+t)^{-\bar{\alpha}-1} + \frac{C_1^{m-1}}{a_1}k_1\frac{m}{(m-1)^2}b^2(T_1+t)^{-\bar{\alpha}m-\bar{\beta}} .$$

By (1.6.123), (1.6.125) and (1.6.126),

$$0 < \bar{\beta} < 1, \quad \bar{\alpha} > 0 . \tag{1.6.134}$$

Due to (1.6.126), (1.6.133) becomes

$$\begin{cases} -1 + \dfrac{C_1^{m-1}}{a_1} k_2\, m\, b \left(N - 2 + \dfrac{bm}{m-1}\right) > 0, \\ -\bar{\beta} + \dfrac{C_1^{m-1}}{a_1}\, k_1\, \dfrac{m}{m-1}\, b^2 > 0, \\ \bar{\beta} - 1 + \dfrac{C_1^{m-1}}{a_1}\, b\, m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1}\right) - k_1 \dfrac{b}{m-1}\right] \le 0\,, \end{cases} \tag{1.6.135}$$

which reduces to

$$\frac{C_1^{m-1}}{a_1} \ge \max\left\{\frac{1}{b\, m\, k_2 \left(N - 2 + \dfrac{bm}{m-1}\right)}, \frac{\bar{\beta}(m-1)}{b^2\, m\, k_1}\right\}, \tag{1.6.136}$$

$$\frac{C_1^{m-1}}{a_1} \le \frac{1 - \bar{\beta}}{b\, m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1}\right) - k_1 \dfrac{b}{m-1}\right]}\,. \tag{1.6.137}$$

If (1.6.136) and (1.6.137) are verified, then ξ is a subsolution to equation (1.6.132). We now show that it is possible to find $\omega_1 := \frac{C_1^{m-1}}{a_1}$ such that (1.6.136) (1.6.137) hold. Such ω_1 can be selected, if

$$\frac{1}{b\, m\, k_2 \left(N - 2 + \dfrac{bm}{m-1}\right)} \;<\; \frac{1 - \bar{\beta}}{b\, m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1}\right) - k_1 \dfrac{b}{m-1}\right]}\,, \tag{1.6.138}$$

and

$$\frac{\bar{\beta}(m-1)}{b^2\, m\, k_1} \;<\; \frac{1 - \bar{\beta}}{b\, m \left[k_2 \left(N - 2 + \dfrac{bm}{m-1}\right) - k_1 \dfrac{b}{m-1}\right]}\,. \tag{1.6.139}$$

Conditions (1.6.138) and (1.6.139) are satisfied, if

$$\bar{\beta} < \beta_0\,. \tag{1.6.140}$$

Finally, condition (1.6.140) is guaranteed by hypothesis (1.6.125). Moreover, by Lemma 1.3.10, ξ is a subsolution to equation

$$\xi_t - \frac{1}{\rho(x)} \Delta \xi^m = 0 \quad \text{in } [\mathbb{R}^N \setminus B_1(0)] \times (0, T)\,. \tag{1.6.141}$$

in the sense of Definition 1.3.9. We can argue as we have done to obtain (1.5.99), in order to get

$$\begin{aligned} &\mu_t - \frac{1}{\rho} \Delta(\mu^m) \\ &\le C_1 G^{\frac{1}{m-1}-1} \left\{ G\left[\zeta' + \frac{\zeta}{m-1} \frac{\eta'}{\eta} + b\, k_2 \frac{m}{m-1} \frac{C_1^{m-1}}{a_1} \zeta^m\, \eta \left(N + \frac{2}{m-1}\right)\right] \right. \\ &\left. - \frac{\zeta}{m-1} \frac{\eta'}{\eta} - 2\, k_1\, b\, \frac{C_1^{m-1}}{a_1} \frac{m}{(m-1)^2} \zeta^m\, \eta \;+\; (2-b)\, k_2\, b\, \frac{C_1^{m-1}}{a_1^2} \frac{m}{(m-1)^2}\, \zeta^m\, \eta^2 \right\} \\ &\quad \text{for any } (x,t) \in D_2\,. \end{aligned} \tag{1.6.142}$$

We now define

$$\underline{\sigma}_0(t) := \zeta' + \zeta \frac{1}{m-1}\frac{\eta'}{\eta} + b\,k_2\,\frac{C_1^{m-1}}{a_1}\zeta^m \frac{m}{m-1}\left(N + \frac{2}{m-1}\right)\eta,$$
$$\underline{\delta}_0(t) := -\frac{\zeta}{m-1}\frac{\eta'}{\eta} + 2\,k_1\,b\,\frac{C_1^{m-1}}{a_1}\frac{m}{(m-1)^2}\zeta^m\,\eta \,-\, (2-b)\,k_2\,b\,\frac{C_1^{m-1}}{a_1^2}\frac{m}{(m-1)^2}\,\zeta^m\,\eta^2\,.$$

Hence, (1.6.142) becomes,

$$\mu_t - \frac{1}{\rho}\Delta(\mu^m) \le C_1\,G^{\frac{1}{m-1}-1}\,\phi(G) \;\text{ in }\; D_2\,, \tag{1.6.143}$$

where

$$\phi(G) := \underline{\sigma}_0(t)G - \underline{\delta}_0(t)\,.$$

By arguing as above, we can infer that

$$\mu_t - \frac{1}{\rho}\Delta(\mu^m) \le 0 \quad \text{in }\; D_2\,, \tag{1.6.144}$$

provided that

$$\begin{cases} \sigma_0(t) > 0 \\ \delta_0(t) > 0 \\ \sigma_0(t) - \delta_0(t) \le 0\,. \end{cases} \qquad \text{for any }\; t \in (0, T_1) \tag{1.6.145}$$

By using the very definition of ζ and η, (1.6.145) becomes

$$\begin{aligned}
&-1 + b\,k_2\,\frac{C_1^{m-1}}{a_1}\,m\left(N + \frac{2}{m-1}\right) > 0,\\
&-\bar\beta \,+\, 2\,b\,k_1\,\frac{C_1^{m-1}}{a_1}\frac{m}{m-1} \,-\, (2-b)\,b\,k_2\,\frac{C_1^{m-1}}{a_1^2}\frac{m}{m-1}(T_1+t)^{-\bar\beta} > 0,\\
&\bar\beta \,-\, 1 \,+\, b\,k_2\,m\,\frac{C_1^{m-1}}{a_1}\,N \,+\, \frac{2}{m-1}\left(1 - \frac{k_1}{k_2}\right) \,+\, (2-b)\,k_2\,b\,\frac{C_1^{m-1}}{a_1^2}\frac{m}{m-1}(T_1+t)^{-\bar\beta} \le 0\,,
\end{aligned} \tag{1.6.146}$$

which reduces to

$$\frac{C_1^{m-1}}{a_1} > \max\left\{\frac{1}{b\,m\,k_2\left(N + \dfrac{2}{m-1}\right)}\,,\; \frac{\bar\beta(m-1)}{b\,m\,k_2\left[2\dfrac{k_1}{k_2} - \dfrac{2-b}{a_1}(T_1+t)^{-\bar\beta}\right]}\right\}, \tag{1.6.147}$$

$$\frac{C_1^{m-1}}{a_1} \le \frac{1-\bar\beta}{b\,m\,k_2\left[N + \dfrac{2}{m-1}\left(1 - \dfrac{k_1}{k_2}\right) + \dfrac{2-b}{a_1}\dfrac{(T_1+t)^{-\bar\beta}}{m-1}\right]}\,. \tag{1.6.148}$$

If (1.6.147) and (1.6.148) are verified then μ is a subsolution to equation

$$\mu_t - \frac{1}{\rho}\Delta\mu^m = 0 \quad \text{in }\; D_2\,.$$

In order to find $\omega_1 = \frac{C_1^{m-1}}{a_1}$ satisfying (1.6.147) and (1.6.148), we need

$$\frac{1}{bmk_2\left(N+\dfrac{2}{m-1}\right)} < \frac{1-\bar{\beta}}{bmk_2\left[N+\dfrac{2}{m-1}\left(1-\dfrac{k_1}{k_2}\right)+\dfrac{2-b}{a_1}\dfrac{(T_1+t)^{-\bar{\beta}}}{m-1}\right]}\,, \tag{1.6.149}$$

and

$$\frac{\bar{\beta}(m-1)}{bmk_2\left[2\dfrac{k_1}{k_2}-\dfrac{2-b}{a_1}(T_1+t)^{-\bar{\beta}}\right]} < \frac{1-\bar{\beta}}{bmk_2\left[N+\dfrac{2}{m-1}\left(1-\dfrac{k_1}{k_2}\right)+\dfrac{2-b}{a_1}\dfrac{(T_1+t)^{-\bar{\beta}}}{m-1}\right]}\,. \tag{1.6.150}$$

Now we choose in (1.2.21) $\epsilon = \epsilon(a_1, T_1) > 0$ so that

$$\frac{\epsilon}{a_1}T_1^{-\bar{\beta}} \le \epsilon_0\,, \tag{1.6.151}$$

with ϵ_0 used in (1.6.121) and (1.6.122) to be appropriately fixed. By (1.2.21), (1.2.22) and (1.6.151),

$$\frac{2-b}{a_1}(T_1+t)^{-\bar{\beta}} < \frac{\epsilon}{a_1}T_1^{-\bar{\beta}} \le \epsilon_0.$$

So, conditions (1.6.149) and (1.6.150) are fulfilled, if

$$\frac{1}{b\,m\,k_2\left(N+\dfrac{2}{m-1}\right)} < \frac{1-\bar{\beta}}{b\,m\,k_2\left[N+\dfrac{2}{m-1}\left(1-\dfrac{k_1}{k_2}\right)+\dfrac{\epsilon_0}{m-1}\right]}\,, \tag{1.6.152}$$

and

$$\frac{\bar{\beta}(m-1)}{b\,m\,k_2\left[2\dfrac{k_1}{k_2}-\epsilon\right]} < \frac{1-\bar{\beta}}{b\,m\,k_2\left[N+\dfrac{2}{m-1}\left(1-\dfrac{k_1}{k_2}\right)+\dfrac{\epsilon_0}{m-1}\right]}\,. \tag{1.6.153}$$

Finally, conditions (1.6.152) and (1.6.153) are satisfied, if

$$\bar{\beta} < \tilde{\beta}_0\,, \tag{1.6.154}$$

provided that $\epsilon_0 > 0$ is small enough. Observe that (1.6.154) is guaranteed due to hypothesis (1.6.124) and (1.6.125). Moreover, since $\mu^m \in C^1(B_1(0)\times(0,T_1))$, by Lemma 1.3.10, μ is a subsolution to

$$\mu_t - \frac{1}{\rho}\Delta(\mu^m) = 0 \quad \text{in } B_1(0)\times(0,T_1)\,, \tag{1.6.155}$$

in the sense of Definition 1.3.9. Hence z is a subsolution of equation (1.6.112).

Since $u_0 \not\equiv 0$ and $u_0 \in C(\mathbb{R}^N)$, there exist $r_0 > 0$ and $\varepsilon > 0$ such that

$$u_0(x) > \varepsilon \quad \text{in } B_{r_0}(0).$$

Hence, if

$$\operatorname{supp} z(\cdot,0) \subset B_{r_0}(0), \tag{1.6.156}$$

and

$$z(x,0) \le \varepsilon \quad \text{in } B_{r_0}(0), \tag{1.6.157}$$

then (1.6.113) follows. Moreover, if

$$\operatorname{supp}\ \underline{w}(\cdot,0) \subset \operatorname{supp}\ z(\cdot,t_1)\,, \tag{1.6.158}$$

and

$$\underline{w}(x,0) \le z(x,t_1) \ \text{ for all } x \in \mathbb{R}^N, \tag{1.6.159}$$

then (1.6.114) follows.

We first verify that z satisfies condition (1.6.156) and (1.6.157). If we require that

$$a_1\, T_1^{\bar\beta} \le \frac{r_0^2}{2}\,. \tag{1.6.160}$$

then

$$\operatorname{supp} z(\cdot,0) \cap B_1(0) \subset B_{r_0}(0)\,,$$

and

$$\operatorname{supp} z(\cdot,0) \cap [\mathbb{R}^N \setminus B_1(0)] \subset B_{r_0}(0)\,,$$

therefore (1.6.156) holds. Moreover, if

$$(a_1\,\omega)^{\frac{1}{m-1}} \ \le \varepsilon\, T_1^{\bar\alpha}, \tag{1.6.161}$$

then (1.6.157) holds. Obviously, for any $T_1 > 0$ we can choose $a_1 = a_1(T_1) > 0$ such that (1.6.160) and (1.6.161) are valid. On the other hand,

$$\operatorname{supp} \underline{w}(\cdot,0) \cap B_1(0) \subset \operatorname{supp} z(\cdot,t_1) \cap B_1(0)\,,$$

and if

$$a_1\,(T_1+t_1)^{\bar\beta} \ge a\,T^{\frac{p-m}{p-1}} \tag{1.6.162}$$

then,

$$\operatorname{supp} \underline{w}(\cdot,0) \cap [\mathbb{R}^N \setminus B_1(0)] \subset \operatorname{supp} z(\cdot,t_1) \cap [\mathbb{R}^N \setminus B_1(0)].$$

Hence, (1.6.158) holds. If

$$C_1\,(T_1+t_1)^{-\bar\alpha} \ge C\,T^{-\frac{1}{p-1}}, \tag{1.6.163}$$

then (1.6.159) holds. If we choose the equality in (1.6.163),

$$T_1 + t_1 = \left(\frac{C}{C_1}\right)^{-\frac{1}{\bar\alpha}} T^{\frac{1}{(p-1)\bar\alpha}},$$

then (1.6.162) becomes

$$\left(\frac{C}{C_1}\right)^{-\frac{\bar\beta}{\bar\alpha}} a_1\, T^{\frac{\bar\beta}{\bar\alpha}\frac{1}{(p-1)}} \ge a\,T^{\frac{p-m}{p-1}}\,.$$

The latter holds, if

$$T^{\frac{p-m-\frac{\bar\beta}{\bar\alpha}}{p-1}} \leq \left(\frac{C}{C_1}\right)^{-\frac{\bar\beta}{\bar\alpha}} \frac{a_1}{a}\,. \tag{1.6.164}$$

Condition (1.6.164) is satisfied thanks to (1.6.127), for $T > 0$ sufficiently large. This completes the proof. □

Proof of Theorem 1.2.7. Let $\tau > 0$ be the maximal existence time of u. If $\tau \leq t_1$, then nothing has to be showed, and u blows-up at a certain time $S \in (0, t_1]$. Suppose $\tau > t_1$. Let us consider the subsolution z of equation (1.6.112) as defined in (1.6.116). Since $p < \underline{p}$, we can find $\bar\beta$ (and so $\bar\alpha$) such that (1.6.125), (1.6.126) and (1.6.127) hold. By Proposition 1.6.1, z satisfies (1.6.113) and (1.6.114). Thanks to condition (1.6.113) and the comparison principle, we have (1.6.115). From (1.6.114) and (1.6.115),

$$u(x, t_1) \geq z(x, t_1) \geq \underline{w}(x, 0) \quad \text{for any } \; x \in \mathbb{R}^N.$$

Thus $u(x, t + t_1)$ is a supersolution, whereas $\underline{w}(x, t)$ is a subsolution of problem

$$\begin{cases} u_t = \frac{1}{\rho}\Delta(u^m) + u^p & \text{in } \; \mathbb{R}^N \times (0, +\infty) \\ u = \underline{w} & \text{in } \; \mathbb{R}^N \times \{0\}\,. \end{cases}$$

Hence by Theorem 1.2.4, $u(x, t)$ blows-up in a finite time $S \in (t_1, t_1 + T)$. This completes the proof. □

1.7 Further results: uniqueness

Proposition 1.7.1. *Let assumption* (H) *be satisfied. Then there exist at most one bounded solution* u *to problem* (1.1.1).

1.7.1 Proof of Proposition 1.7.1

We denote by ν the outer normal at any point of the boundary. Let us consider any two solutions to problem (1.3.27), u_1 and u_2. We define

$$a := \frac{u_1^m - u_2^m}{u_1 - u_2} \quad \text{when } u_1 \neq u_2\,, \tag{1.7.165}$$

Observe that $a \in L^\infty(\mathbb{R}^N \times [0, T])$. Let us also define the domains,

$$B_R := \{x \in \mathbb{R}^n : |x| < R\}\,, \qquad Q_{RT} := B_R \times (0, T]\,. \tag{1.7.166}$$

We introduce the approximation of a

$$a_n := \overline{a}\, q_n + \frac{1}{n},$$

where $\overline{a}$ is the extension by 0 of a to $\mathbb{R}^N \times \mathbb{R}$ and q_n is a sequence of mollifiers in $\mathbb{R}^N \times \mathbb{R}$. Then a_n satisfies the following properties

$$\begin{aligned} &\text{(i)}\ \ a_n \in C^\infty(\mathbb{R}^N \times [0,T])\,, \\ &\text{(ii)}\ \ a_n > \frac{1}{n}\,, \\ &\text{(iii)}\ \ a_n \le k\,, \end{aligned} \tag{$H3$}$$

for some $k > 0$. Moreover assume that

$$||a - \overline{a}\, q_n||^2_{L^2(Q_{RT})} \le \frac{1}{n^2} \longrightarrow 0 \text{ as } n \longrightarrow +\infty \tag{1.7.167}$$

We now consider the backward problem

$$\begin{cases} \rho(x)\psi_{nt} + a_n \Delta\psi_n = \rho(x)\lambda\psi_n & 0 < t < T,\ \ x \in B_R \\ \psi_n = 0 & 0 < t < T,\ \ x \in \partial B_R \\ \psi_n(x,T) = \theta(x) & x \in B_R \end{cases} \tag{1.7.168}$$

where ρ satisfies hypotheses $(i) - (ii)$ in (H), $\lambda > 0$ and

$$\theta \in C_0^\infty(B_R), \quad 0 \le \theta \le 1.$$

To prove Proposition 1.7.1 we need the following lemma.

Lemma 1.7.2. *Let assumptions (H)-$(i)$$-$$(ii)$ and $(H3)$ be satisfied. Moreover, consider α, β, $\mu \in \mathbb{R}$ such that*

$$\beta > \frac{N-1}{2}\,, \tag{1.7.169}$$

$$\alpha > 4kN\beta(\beta+1)\,, \tag{1.7.170}$$

and

$$\frac{\mu}{(1+|x|^2)^\beta} > \theta(x) \quad \textit{for all } x \in B_R. \tag{1.7.171}$$

Then the solution ψ_n to problem (1.7.168) has the following properties:

(i)

$$0 \le \psi_n \le \mu\,\frac{e^{(\alpha-\lambda)(T-t)}}{(1+|x|^2)^\beta} \qquad \textit{in } B_R \times [0,T],$$

(ii)

$$\int_0^T \int_{B_R} a_n |\Delta\psi_n|^2\, dx\, dt \le c_1,$$

(iii)

$$\sup_{0<t<T} \int_{B_R} |\nabla\psi_n|^2(t)\, dx \le c_2,$$

for some c_1, c_2 independent of u.

Proof of Lemma 1.7.2. Let us start by proving property (i). Consider the function

$$\phi(x,t) := \frac{e^{(\alpha-\lambda)(T-t)}}{(1+|x|^2)^\beta}. \tag{1.7.172}$$

We compute

$$\Delta\phi = e^{(\alpha-\lambda)(T-t)}\left[-\frac{2\beta N}{(1+|x|^2)^{\beta+1}} + \frac{4\beta(\beta+1)|x|^2}{(1+|x|^2)^{\beta+2}}\right]$$

Observe that, by (H)-(i), $(H3)$ and (1.7.170)

$$\begin{aligned} |a_n\Delta\phi| &\le k|\Delta\phi| \\ &\le k\, e^{(\alpha-\lambda)(T-t)}\left[\frac{2\beta N}{(1+|x|^2)^{\beta+1}} + \frac{4\beta(\beta+1)|x|^2}{(1+|x|^2)^{\beta+2}}\right] \\ &\le k\, e^{(\alpha-\lambda)(T-t)}\left[\frac{2\beta N}{(1+|x|^2)^{\beta+1}} + \frac{4\beta(\beta+1)(1+|x|^2)}{(1+|x|^2)^{\beta+2}}\right] \\ &\le k\,\frac{e^{(\alpha-\lambda)(T-t)}}{(1+|x|^2)^{\beta+1}}\,[2\beta N + 4\beta(\beta+1)] \\ &\le k\,\frac{e^{(\alpha-\lambda)(T-t)}}{(1+|x|^2)^{\beta}}\,[2\beta N + 4\beta(\beta+1)]\,\frac{1}{(1+|x|^2)} \\ &\le 4\,k\,N\beta(\beta+1)\,\phi\,\rho \\ &< \alpha\rho\phi. \end{aligned}$$

where k has been introduced in $(H3)$. Hence we get

$$\rho\,\phi_t + a_n\,\Delta\phi \le \rho\,\phi_t + \rho\,\alpha\,\phi = -\rho\,\alpha\,\phi + \rho\,\lambda\,\phi + \rho\,\alpha\,\phi = \rho\,\lambda\,\phi.$$

Moreover, by (1.7.171),

$$\psi_n(x,T) = \theta(x) \le \frac{\mu}{(1+|x|^2)^\beta} = \mu\,\phi(x,T).$$

Thus, by the maximum principle we have

$$0 \le \psi_n(x,t) \le \mu\,\phi(x,t) \qquad \text{for any } \; 0<t<T,\; x\in B_R.$$

To prove (ii) and (iii), let us multiply the equation in problem (1.7.168) by $\dfrac{\Delta\psi_n}{\rho}$ and integrate in $B_R\times(t,T)$,

$$\begin{aligned} &\int_t^T\int_{B_R}\psi_{nt}\Delta\psi_n\,dx\,dt + \int_t^T\int_{B_R}\frac{a_n}{\rho}|\Delta\psi_n|^2\,dx\,dt \\ &= \int_t^T\int_{B_R}\lambda\psi_n\Delta\psi_n\,dx\,dt\,. \end{aligned}$$

Then by the integration by parts on B_R and in the time interval (t,T), we get,

$$\begin{aligned} &-\frac{1}{2}\int_{B_R}|\nabla\psi_n|^2(x,T)\,dx + \frac{1}{2}\int_{B_R}|\nabla\psi_n|^2(x,t)\,dx + \int_t^T\int_{B_R}\frac{a_n}{\rho}|\Delta\psi_n|^2\,dx\,dt \\ &= -\,\lambda\int_t^T\int_{B_R}|\nabla\psi_n|^2\,dx\,dt \end{aligned} \tag{1.7.173}$$

From the latter we deduce

$$\int_t^T \int_{B_R} \frac{a_n}{\rho} |\Delta\psi_n|^2 \, dx \, dt \le \frac{1}{2} \int_{B_R} |\nabla\theta|^2(x) \, dx - \frac{1}{2} \int_{B_R} |\nabla\psi_n|^2(x,t) \, dx \le c,$$

where c is a positive real constant independent of u. Moreover, since $\rho \in C(\mathbb{R}^N)$, then $\frac{1}{\rho}$ has a finite minimum in $\overline{B}_R$, thus

$$\min_{\overline{B}_R} \frac{1}{\rho} \int_t^T \int_{B_R} a_n |\Delta\psi_n|^2 \, dx \, dt \le \int_t^T \int_{B_R} \frac{a_n}{\rho} |\Delta\psi_n|^2 \, dx \, dt.$$

This ensures property (ii). From equality (1.7.173) we also deduce

$$\int_{B_R} |\nabla\psi_n|^2(x,t) \, dx \le \int_{B_R} |\nabla\theta|^2(x) \, dx \le \hat{c}$$

where $\hat{c}$ is a positive real constant. By the arbitrarily of $t \in (0,T)$ we deduce property (iii). □

We now prove Proposition 1.7.1.

Proof of Proposition 1.7.1. Consider any two solutions u_1 and u_2 to problem (1.3.27). By (1.3.9) and (1.7.166), subtracting u_1 and u_2 we get

$$\begin{aligned} &-\int_0^T \int_{B_R} \rho(u_1 - u_2)\psi_t \, dx \, dt + \int_{B_R} \rho(x)[u_1(x,T) - u_2(x,T)]\psi(x,T) \, dx \\ &= \int_0^T \int_{B_R} (u_1^m - u_2^m)\Delta\psi \, dx \, dt - \int_0^T \int_{\partial B_R} (u_1^m - u_2^m)\nabla\psi \cdot \nu \, d\sigma \, dt \\ &+ \int_0^T \int_{B_R} \rho(u_1^p - u_2^p)\psi \, dx \, dt \end{aligned} \tag{1.7.174}$$

Using the definition of a in (1.7.165), (1.7.174) can be rewritten as

$$\begin{aligned} &\int_{B_R} \rho(x)[u_1(x,T) - u_2(x,T)]\psi(x,T) \, dx \\ &= \int_0^T \int_{B_R} (u_1 - u_2)(\rho\psi_t + a\Delta\psi) \, dx \, dt - \int_0^T \int_{\partial B_R} (u_1^m - u_2^m)\nabla\psi \cdot \nu \, d\sigma \, dt \\ &+ \int_0^T \int_{B_R} \rho(u_1^p - u_2^p)\psi \, dx \, dt \end{aligned} \tag{1.7.175}$$

We aim to prove that

$$\int_{B_R} \rho(x)[u_1(x,T) - u_2(x,T)]\psi(x,T) \, dx \quad \longrightarrow 0 \quad \text{as } R \to +\infty.$$

Let us choose the test function in (1.7.175) equal to the solution ψ_n of problem (1.7.168). Thus, (1.7.175) becomes

$$\begin{aligned}&\int_{B_R}\rho(x)[u_1(x,T)-u_2(x,T)]\theta(x)\,dx\\&=\int_0^T\int_{B_R}\rho\,(u_1-u_2)\,\lambda\,\psi_n\,dx\,dt+\int_0^T\int_{B_R}(u_1-u_2)(a-a_n)\Delta\psi_n\,dx\,dt\\&-\int_0^T\int_{\partial B_R}(u_1^m-u_2^m)\frac{\partial\psi_n}{\partial\nu}\,d\sigma\,dt+\int_0^T\int_{B_R}\rho\,(u_1^p-u_2^p)\psi_n\,dx\,dt\end{aligned}\tag{1.7.176}$$

Let us now define

$$I_1:=\int_0^T\int_{B_R}(u_1-u_2)(a-a_n)\Delta\psi_n\,dx\,dt,\tag{1.7.177}$$

$$I_2:=-\int_0^T\int_{\partial B_R}(u_1^m-u_2^m)\frac{\partial\psi_n}{\partial\nu}\,d\sigma\,dt\tag{1.7.178}$$

and

$$I_3:=\int_0^T\int_{B_R}\rho\,(u_1-u_2)\,\lambda\,\psi_n\,dx\,dt+\int_0^T\int_{B_R}\rho\,(u_1^p-u_2^p)\psi_n\,dx\,dt.\tag{1.7.179}$$

Then we estimate I_1. Thanks to Hölder inequality and since $u_1,u_2\in L^\infty(\mathbb{R}^N\times(0,T))$, we get

$$\begin{aligned}|I_1|&\le\int_0^T\int_{B_R}|u_1-u_2|\left|\frac{a-a_n}{\sqrt{a_n}}\right||\sqrt{a_n}\Delta\psi_n|\,dx\,dt,\\&\le C\left(\int_0^T\int_{B_R}\frac{(a-a_n)^2}{a_n}\,dx\,dt,\right)^{1/2}\left(\int_0^T\int_{B_R}a_n|\Delta\psi_n|^2\,dx\,dt,\right)^{1/2}\end{aligned}\tag{1.7.180}$$

Now,

$$\begin{aligned}\left(\int_0^T\int_{B_R}\frac{(a-a_n)^2}{a_n}\,dx\,dt,\right)^{1/2}&\le\left(\int_0^T\int_{B_R}\frac{\left(a-\overline{a}q_n-\frac{1}{n}\right)^2}{1/n}\,dx\,dt\right)^{1/2}\\&\le\sqrt{n}\left(\int_0^T\int_{B_R}\left[(a-\overline{a}\,q_n)^2+\frac{1}{n^2}\right]\,dx\,dt\right)^{1/2}\\&\le\sqrt{n}\left(||a-\overline{a}\,q_n||^2_{L^2(Q_{RT})}+\frac{1}{n^2}\int_0^T\int_{B_R}\,dx\,dt\right)^{1/2}\\&\le\sqrt{n}\left(\frac{1}{n^2}+\frac{TR^N}{n^2}\right)^{1/2}\\&\le\frac{1}{\sqrt{n}}\left(1+TR^N\right)^{1/2}\le\frac{c(R)}{\sqrt{n}}.\end{aligned}\tag{1.7.181}$$

Moreover, by Lemma 1.7.2,

$$\left(\int_0^T \int_{B_R} a_n |\Delta\psi_n|^2 \, dx \, dt,\right)^{1/2} \leq c_1. \tag{1.7.182}$$

Hence,

$$|I_1| \leq \frac{c_1\, c(R)}{\sqrt{n}} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty. \tag{1.7.183}$$

Let

$$J := \{(x,t) : R-1 \leq |x| \leq R,\, 0 < t < T\}.$$

Let g, which is independent of t, be defined by

$$g(x,t) = g(x) := \frac{d}{|x|^{N-2}} + e \quad \text{for } (x,t) \in J \tag{1.7.184}$$

where d and e satisfy,

$$\begin{aligned} \frac{d}{(R-1)^{N-2}} + e &= \frac{\lambda e^{(\alpha-\lambda)T}}{(1+(R-1)^2)^\beta}, \\ \frac{d}{R^{N-2}} + e &= 0. \end{aligned} \tag{1.7.185}$$

Then g is such that

$$g \geq \psi_n\,, \qquad g(x,t) = 0 \quad \text{on } 0 < t < T,\ |x| = R. \tag{1.7.186}$$

and

$$\frac{\partial}{\partial \nu}\left(g - \psi_n\right)(x,t) \leq 0 \quad \text{for } |x| = R\,,\ 0 < t < T. \tag{1.7.187}$$

Then, since $\dfrac{\partial \psi_n}{\partial \nu} \leq 0$,

$$\sup_{\substack{|x|=R \\ 0<t<T}} \left| \frac{\partial \psi_n}{\partial \nu}(x,t) \right| \leq \sup_{\substack{|x|=R \\ 0<t<T}} \left| \frac{\partial g}{\partial \nu}(x,t) \right| \quad \text{for } |x| = R\,,\ 0 < t < T, \tag{1.7.188}$$

which gives an estimate on the normal derivative of ψ_n. Note that,

$$\Delta g = 0 \quad \text{on } J.$$

Moreover, by (1.7.185),

$$\begin{aligned} &g(R-1) \geq \psi_n(R-1,t) \quad &&\text{for } 0 < t < T, \\ &g(R) = \psi_n(R,t) = 0 &&\text{for } 0 < t < T, \end{aligned}$$

and by (1.7.168),

$$g(x) \geq \psi_n(x,T) \quad \text{for } R-1 < |x| < R.$$

Therefore (1.7.186) holds by maximum principle and so does (1.7.187). It remains to estimate $\frac{\partial g}{\partial \nu}$ over ∂B_R. We have

$$\begin{aligned}\frac{\partial g}{\partial \nu}(x)\bigg|_{|x|=R} &= \frac{(2-N)d}{R^{N-1}} \\ &= \frac{2-N}{R^{N-1}}\frac{\lambda e^{(\alpha-\lambda)T}}{(1+(R-1)^2)^\beta}\left(\frac{1}{(R-1)^{N-2}}-\frac{1}{R^{N-2}}\right)^{-1}.\end{aligned} \tag{1.7.189}$$

Hence,

$$\begin{aligned}\left|\frac{\partial g}{\partial \nu}(x)\right|_{|x|=R} &\le \frac{N-2}{R^{N-1}}\frac{\lambda e^{(\alpha-\lambda)T}}{(1+(R-1)^2)^\beta}\frac{R^{N-2}(R-1)^{N-2}}{R^{N-2}-(R-1)^{N-2}} \\ &\le \frac{c}{R^{N-1}}\frac{1}{(1+(R-1)^2)^\beta}R^{N-2}\frac{R^{2(N-2)}}{1+\left(\frac{R-1}{R}\right)^{N-2}} \\ &\le \frac{c}{R^{N-1}}\frac{1}{(1+(R-1)^2)^\beta}\frac{R^{N-2}}{1+\left(1-\frac{1}{R}\right)^{N-2}} \\ &\le \frac{c}{R^{N-1}}\frac{1}{(1+(R-1)^2)^\beta}\frac{R^{N-2}}{(N-2)R^{-1}} \\ &\le \frac{c}{R^{N-1}}\frac{1}{(1+(R-1)^2)^\beta}R^{N-1} \\ &\le \frac{c}{R^{2\beta}}.\end{aligned} \tag{1.7.190}$$

Combining (1.7.190) together with (1.7.188), we have

$$\sup_{\substack{|x|=R\\0<t<T}}\left|\frac{\partial \psi_n}{\partial \nu}(x,t)\right| \le \frac{c}{R^{2\beta}}. \tag{1.7.191}$$

Going back to (1.7.178),

$$|I_2| \le ||u_1^m - u_2^m||_{L^\infty(\partial B_R\times(0,T))}\frac{c}{R^{2\beta}}\,T\,R^{N-1} \le c\,R^{N-1-2\beta}\,. \tag{1.7.192}$$

Thus in (1.7.176) we get

$$\int_{B_R}\rho(x)[u_1(x,T)-u_2(x,T)]\theta(x)\,dx \le \frac{c_1\,c(R)}{\sqrt{n}} + c\,R^{N-1-2\beta} + |I_3|. \tag{1.7.193}$$

Without lost of generality, we can set

$$\theta(x) = \mathrm{sign}[u_1(x,T)-u_2(x,T)]^+ \quad \text{for } x\in B_R,$$

thus we have

$$\begin{aligned}&\int_{B_R}\rho(x)[u_1(x,T)-u_2(x,T)]^+\,dx \\ &\le \frac{c_1\,c(R)}{\sqrt{n}} + c\,R^{N-1-2\beta} + \int_0^T\int_{B_R}\rho\,[\lambda(u_1-u_2)+(u_1^p-u_2^p)]^+\psi_n\,dx\,dt.\end{aligned} \tag{1.7.194}$$

By Lemma 1.7.2,

$$\psi_n(x,t) \le \frac{e^{(\alpha-\lambda)(T-t)}}{(1+|x|^2)^\beta} \quad \text{for} x \in B_R,\ \ 0<t<T\,.$$

Thus, (1.7.194), letting $n \to +\infty$, reads

$$\begin{aligned} &\int_{B_R} \rho(x)[u_1(x,T)-u_2(x,T)]^+\,dx \\ &\le c\,R^{N-1-2\beta} + \int_0^T \int_{B_R} \rho\, \frac{e^{(\alpha-\lambda)(T-t)}}{(1+|x|^2)^\beta}\,[\lambda(u_1-u_2)+(u_1^p-u_2^p)]^+\,dx\,dt. \end{aligned} \tag{1.7.195}$$

Let

$$s_0 := \max\{\|u_1\|_\infty,\ \|u_2\|_\infty\}.$$

Let L be the Lipschitz constant of the function $s \to s^p$ over $[-s_0.s_0]$. Then choosing

$$\lambda > \max\{L, \alpha\}, \tag{1.7.196}$$

(1.7.195) becomes

$$\begin{aligned} e^{(\lambda-\alpha)T} &\int_{B_R} \rho(x)[u_1(x,T)-u_2(x,T)]^+\,dx \\ &\le e^{(\lambda-\alpha)T}\,c\,R^{N-1-2\beta} + \int_0^T e^{(\lambda-\alpha)t} \int_{B_R} \rho\; 2\,\lambda\,(u_1-u_2)^+\,dx\,dt. \end{aligned} \tag{1.7.197}$$

Let

$$\begin{aligned} h(t) &:= e^{(\lambda-\alpha)t} \int_{B_R} \rho(x)\,(u_1-u_2)^+\,dx, \\ \gamma(t) &:= e^{(\lambda-\alpha)T}\,c\,R^{N-1-2\beta}, \end{aligned}$$

so, (1.7.197) implies

$$h(T) \le \gamma(t) + 2\lambda \int_0^T h(t)\,dt.$$

Thanks to (1.7.196), by Gronwall's Lemma,

$$\int_{B_R} \rho(x)[u_1(x,T)-u_2(x,T)]^+\,dx \le e^{2\lambda T}\,c\,R^{N-1-2\beta} \tag{1.7.198}$$

If we change the role of u_1 and u_2, we obtain symmetrically,

$$\int_{B_R} \rho(x)[u_2(x,T)-u_1(x,T)]^+\,dx \le e^{2\lambda T}\,c\,R^{N-1-2\beta} \tag{1.7.199}$$

By adding (1.7.198) and (1.7.199) we get

$$\int_{B_R} \rho(x)\,|u_1(x,T)-u_2(x,T)|\,dx \le e^{2\lambda T}\,c\,R^{N-1-2\beta} \tag{1.7.200}$$

Finally, by (1.7.169), letting R going to $+\infty$ in (1.7.200), we have

$$\int_{\mathbb{R}^N} \rho(x)\,|u_1(x,T)-u_2(x,T)|\,dx \;\le\; 0.$$

This completes the proof.

□

Chapter 2

The fast decaying density case

2.1 Introduction

We investigate global existence and blow-up of nonnegative solutions to problem

$$\begin{cases} \rho(x)u_t = \Delta(u^m) + \rho(x)u^p & \text{in } \mathbb{R}^N \times (0,\tau) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \times \{0\} \end{cases} \tag{2.1.1}$$

where $N \geq 3, p > 1, m > 1$ and $\tau > 0$. We always assume that

$$\begin{cases} \text{(i) } \rho \in C(\mathbb{R}^N),\ \rho > 0 \text{ in } \mathbb{R}^N, \\ \text{(ii) } u_0 \in L^\infty(\mathbb{R}^N), u_0 \geq 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{H}$$

and that

$$\begin{aligned} &\text{there exist } k_1, k_2 \in (0,+\infty) \text{ with } k_1 \leq k_2, r_0 > 0\,, q \geq 2 \text{ such that} \\ &k_1(|x|+r_0)^q \leq \frac{1}{\rho(x)} \leq k_2(|x|+r_0)^q \quad \text{for all } x \in \mathbb{R}^N\,. \end{aligned} \tag{2.1.2}$$

The parabolic equation in problem (2.1.1) is of the *porous medium* type, with a variable density $\rho(x)$ and a reaction term $\rho(x)u^p$. Clearly, such parabolic equation is degenerate, since $m > 1$. Moreover, the differential equation in (2.1.1) is equivalent to

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \qquad \text{in } \mathbb{R}^N \times (0,\tau);$$

thus the related diffusion operator is $\frac{1}{\rho(x)}\Delta$, and in view of (2.1.2), the coefficient $\dfrac{1}{\rho(x)}$ can positively diverge at infinity. The differential equation in (2.1.1), posed in the interval $(-1,1)$ with homogeneous Dirichlet boundary conditions, has been introduced in [73] as a mathematical model of evolution of plasma temperature, where u is the temperature, $\rho(x)$ is the particle density, $\rho(x)u^p$ represents the volumetric heating of plasma. Indeed, in [73, Introduction] a more general source term of the type $A(x)u^p$ has also been considered; however, then the authors assume that $A \equiv 0$; only some remarks for the case $A(x) = \rho(x)$ are made in [73, Section 4]. Then in [71] and [72] the Cauchy problem (2.1.1) is dealt with in the case without the reaction term $\rho(x)u^p$.

In view of (2.1.2) the density ρ decays at infinity. Indeed,

$$\frac{1}{k_2(|x|+r_0)^q} \le \rho(x) \le \frac{1}{k_1(|x|+r_0)^q} \quad \text{for all} \quad |x|>1\,. \tag{2.1.3}$$

Since we assume (2.1.2), we refer to $\rho(x)$ as a *fast decaying density* at infinity. On the other hand, in [92] it is studied problem (2.1.1) with a *slowly decaying density,* that is (2.1.2) is assumed with $q<2$.

There is a huge literature concerning various problems related to (2.1.1). For instance, problem (2.1.1) with $\rho \equiv 1, m=1$ is studied in [16, 24, 30, 31, 58, 62, 64, 83, 114, 118, 121, 135], problem (2.1.1) without the reaction term u^p is treated in [25, 27, 49, 51, 50, 66, 67, 68, 69, 70, 71, 72, 73, 102, 59, 115, 116, 117]. Moreover, problem (2.1.1) with $m=1$ is addressed in [85] (see also [21]), where ρ satisfies (2.1.3) with $0 \le q < 2$. In particular, let us recall some results established in [119] for problem (2.1.1) with $\rho \equiv 1, m>1, p>1$ (see also [36, 99]). We have:

- ([119, Theorem 1, p. 216]) For any $p>1$, for all sufficiently large initial data, solutions blow-up in finite time;
- ([119, Theorem 2, p. 217]) if $p \in \left(1, m+\frac{2}{N}\right)$, for *all* initial data, solutions blow-up in finite time;
- ([119, Theorem 3, p. 220]) if $p > m+\frac{2}{N}$, for all sufficiently small initial data with compact support, solutions exist globally in time and belong to $L^\infty(\mathbb{R}^N \times (0,+\infty))$.

Similar results for quasilinear parabolic equations, also involving p-Laplace type operators or double-nonlinear operators, have been stated in [1], [3], [4], [20], [22], [23], [60], [61], [86], [87], [88], [97], [98], [104], [125], [132] (see also [89] for the case of Riemannian manifolds); moreover, in [54] the same problem on Cartan-Hadamard manifolds has been investigated. In particular, in [86, Theorem 2] it is shown that if $\rho(x)=(1+|x|)^{-q}$ with $0<q<2$, $p>m$, and u_0 is small enough (in an appropriate sense), then there exists a global solution; moreover, a smoothing estimate is given. Such result will be compared below with one of our results (see Remark 2.2.5).

In [92] the following results for problem (2.1.1) are established, assuming (2.1.2) with $0 \le q < 2$.

- ([92, Theorem 2.1]. If
  $$p > \bar{p},$$
  u_0 has compact support and is small enough, then there exist global in time solutions to problem (2.1.1) which belong to $L^\infty(\mathbb{R}^N \times (0,+\infty))$; here $\bar{p}$ is a certain exponent, which depends on N, m, q, k_1, k_2. In particular, for $k_1 = k_2$ we have
  $$\bar{p} = m + \frac{2-q}{N-q}\,.$$
- ([92, Theorem 2.3]). For any $p>1$, if u_0 is sufficiently large, then solutions to problem (2.1.1) blow-up in finite time.

- ([92, Corollary 2.4, Theorem 2.5]). If $1 < p < \underline{p}$, then for any $u_0 \not\equiv 0$, solutions to problem (2.1.1) blow-up in finite time. Here $\underline{p} \in (m, \bar{p})$ is a certain exponent depending on N, m, q, k_1, k_2. For $k_1 = k_2$, $\underline{p} = \bar{p}$. Observe that for $m < p < \underline{p}$, some extra conditions are needed.

Analogous results, proved by different methods, can be found also in [86, 87], where also more general double-nonlinear operators are treated.

2.1.1 Outline of our results

Let us now describe our main results. We distinguish between two cases: $q = 2$ and $q > 2$. First, assume that (2.1.2) holds with $q = 2$.

- (Theorem 2.2.1). If
$$p > m,$$
u_0 has compact support and is small enough, then there exist global in time solutions to problem (2.1.1), which belong to $L^\infty(\mathbb{R}^N \times (0, +\infty))$;

- (Theorem 2.2.2). For any $p > m$, if u_0 is sufficiently large, then solutions to problem (2.1.1) blow-up in finite time.

The proofs mainly relies on suitable comparison principles and properly constructed sub- and supersolutions, which crucially depend on the behavior at infinity of the inhomogeneity term $\rho(x)$. More precisely, they are of the type

$$w(x,t) = C\zeta(t)\left[1 - \frac{\log(|x| + r_0)}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \left[\mathbb{R}^N \setminus B_1(0)\right] \times [0,T), \tag{2.1.4}$$

for suitable functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0$. The presence of $\log(|x| + r_0)$ in w is strictly related to the assumption that $q = 2$. Observe that the barriers used in [92] for the case $0 \leq q < 2$, which are of power type in $|x|$, do not work in the present situation. Furthermore, note that the exponent $\bar{p}$ introduced in [92] for $0 \leq q < 2$, when $q = 2$ becomes $\bar{p} = m$. Hence Theorem 2.2.1 can be seen as a generalization of [92, Theorem 2.1] to the case $q = 2$.

Now, assume that $q > 2$. We have the following results (see Theorem 2.2.3 and Remark 2.2.4).

- Let $1 < p < m$. Then for suitable $u_0 \in L^\infty(\mathbb{R}^N)$ there exist global in time solutions to problem (2.1.1). We do not assume that u_0 has compact support, but we need that it fulfills a decay condition as $|x| \to +\infty$. However, u_0 in a compact subset of $\mathbb{R}^N$ can be arbitrarily large. We cannot deduce that the corresponding solution belongs to $L^\infty(\mathbb{R}^N \times (0, +\infty))$, but it is in $L^\infty(\mathbb{R}^N \times (0, \tau))$ for each $\tau > 0$.

- Let $p > m \geq 1$. Then for suitable $u_0 \in L^\infty(\mathbb{R}^N)$, problem (2.1.1) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$. We need that
$$0 \leq u_0(x) \leq CW(x) \quad \text{for all } x \in \mathbb{R}^N,$$

where $C > 0$ is small enough and $W(x)$ is a suitable function, which vanishes as $|x| \to +\infty$. We should mention that, as recalled above, a similar result was been obtained in [86, Theorem 2], where also double-non linear operators are treated; see Remark 2.2.5 below.

- Let $p = m > 1$. Then for suitable $u_0 \in L^\infty(\mathbb{R}^N)$, problem (2.1.1) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$, provided that $r_0 > 0$ in (2.1.2) is big enough.

Such results are very different with respect to the cases $0 \le q < 2$ and $q = 2$. In fact, we do not have finite-time blow-up, but global existence prevails, for suitable initial data. The results follow by comparison principles, once we have constructed appropriate supersolutions, that have the form

$$w(x, t) = \zeta(t)W(x) \quad \text{for all} \quad (x, t) \in \mathbb{R}^N \times (0, +\infty),$$

for suitable $\zeta(t)$ and $W(x)$. When $p \ge m$, $\zeta(t) \equiv 1$. Observe that we can also include the linear case $m = 1$, whenever $p > m$. In this respect, our result complement the results in [85], where only the case $q < 2$ is addressed. Finally, let us mention that it remains to be understood whether in the case $1 < p < m$ solutions can blow-up in infinite time or not.

2.2 Statements of the main results

For any $R > 0$, let B_R as in (1.3.26).

For the sake of simplicity, sometimes instead of (2.1.2), we suppose that

$$\begin{aligned} &\text{there exist } k_1, k_2 \in (0, +\infty) \text{ with } k_1 \le k_2\,, q \ge 2\ , R > 0 \text{ such that} \\ &k_1|x|^q \le \frac{1}{\rho(x)} \le k_2|x|^q \quad \text{for all} \ \ x \in \mathbb{R}^N \setminus B_R\,. \end{aligned} \tag{2.2.5}$$

In view of (H)-(i),

$$\begin{aligned} &\text{for any } R > 0 \text{ there exist } \rho_1(R), \rho_2(R) \in (0, +\infty) \text{ with } \rho_1(R) \le \rho_2(R) \\ &\quad \text{such that } \rho_1(R) \le \frac{1}{\rho(x)} \le \rho_2(R) \quad \text{for all} \ \ x \in \overline{B_R}\,. \end{aligned} \tag{2.2.6}$$

Obviously, (2.1.2) is equivalent to (2.2.5) and (2.2.6).

In the sequel we shall refer to q as the order of decaying of $\rho(x)$ as $|x| \to +\infty$.

2.2.1 Order of decaying: $q = 2$

Let $q = 2$. The first result concerns the global existence of solutions to problem (2.1.1) for $p > m$. We assume that

$$r_0 > e, \quad \frac{k_2}{k_1} < (N - 2)(m - 1)\frac{p - m}{p - 1} \log r_0\,. \tag{2.2.7}$$

Such technical request allow us to construct an appropriate supersolution, as it will be apparent in the proof of Proposition 2.3.1 below.

Theorem 2.2.1. *Assume* (H), (2.1.2) *for* $q = 2$ *and* (2.2.7). *Suppose that*

$$p > m\,,$$

and that u_0 *is small enough and has compact support. Then problem* (2.1.1) *admits a global solution* $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.
More precisely, if $C > 0$ *is small enough,* $a > 0$ *is so that*

$$0 < \omega_0 \le \frac{C^{m-1}}{a} \le \omega_1$$

for suitable $0 < \omega_0 < \omega_1$, $T > 0$,

$$u_0(x) \le C T^{-\frac{1}{p-1}} \left[1 - \frac{\log(|x| + r_0)}{a}\, T^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \textit{for a.e. } x \in \mathbb{R}^N\,, \tag{2.2.8}$$

then problem (2.1.1) *admits a global solution* $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$. *Moreover,*

$$u(x,t) \le C(T+t)^{-\frac{1}{p-1}} \left[1 - \frac{\log(|x| + r_0)}{a}\, (T+t)^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \textit{for a.e. } (x,t) \in \mathbb{R}^N \times (0, +\infty)\,. \tag{2.2.9}$$

Observe that if u_0 satisfies (2.2.8), then

$$\operatorname{supp} u_0 \subseteq \{x \in \mathbb{R}^N \,:\, \log(|x| + r_0) \le a T^{\frac{p-m}{p-1}}\}\,.$$

From (2.2.9) we can infer that

$$\operatorname{supp} u(\cdot, t) \subseteq \{x \in \mathbb{R}^N \,:\, \log(|x| + r_0) \le a(T+t)^{\frac{p-m}{p-1}}\} \quad \text{for a.e. } t > 0\,. \tag{2.2.10}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 2.3.2.

The next result concerns the blow-up of solutions in finite time, for every $p > m > 1$, provided that the initial datum is sufficiently large. We assume that hypothesis (2.2.5) holds with the choice

$$q = 2\,, \quad R = e\,. \tag{2.2.11}$$

So we fix, in assumption (2.2.6),

$$\rho_1(R) = \rho_1(e) =: \rho_1\,, \quad \rho_2(R) = \rho_2(e) =: \rho_2\,.$$

Let

$$\mathfrak{s}(x) := \begin{cases} \log(|x|) & \text{if } \; x \in \mathbb{R}^N \setminus B_e, \\ \dfrac{|x|^2 + e^2}{2e^2} & \text{if } \; x \in B_e\,. \end{cases}$$

Theorem 2.2.2. *Let assumption (H), (2.2.5) and (2.2.11) hold. For any*

$$p > m$$

and for any $T > 0$, if the initial datum u_0 is large enough, then the solution u of problem (2.1.1) blows-up in a finite time $S \in (0, T]$, in the sense that

$$\|u(t)\|_\infty \to \infty \text{ as } t \to S^- . \tag{2.2.12}$$

More precisely, if $C > 0$ and $a > 0$ are large enough, $T > 0$,

$$u_0(x) \geq CT^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} T^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}} , \quad \text{for a.e. } x \in \mathbb{R}^N , \tag{2.2.13}$$

then the solution u of problem (2.1.1) blows-up and satisfies the bound from below

$$u(x,t) \geq C(T-t)^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} (T-t)^{\frac{m-p}{p-1}}\right]_+^{\frac{1}{m-1}} , \text{ for a.e. } (x,t) \in \mathbb{R}^N \times (0, S) . \tag{2.2.14}$$

Observe that if u_0 satisfies (2.2.13), then

$$\operatorname{supp} u_0 \supseteq \{x \in \mathbb{R}^N \,:\, \mathfrak{s}(x) < aT^{\frac{p-m}{p-1}}\} .$$

From (2.2.14) we can infer that

$$\operatorname{supp} u(\cdot, t) \supseteq \{x \in \mathbb{R}^N \,:\, \mathfrak{s}(x) < a(T-t)^{\frac{p-m}{p-1}}\} \quad \text{for a.e. } t \in [0, S) . \tag{2.2.15}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 2.4.2.

2.2.2 Order of decaying: $q > 2$

Let $q > 2$. The first result concerns the global existence of solutions to problem (2.1.1) for any $p > 1$ and $m > 1$, $p \neq m$. Let us introduce the parameter $\bar{b} \in \mathbb{R}$ such that

$$0 < \bar{b} < \min\{N-2\,,\ q-2\} . \tag{2.2.16}$$

Moreover, we can find $\bar{c} > 0$ such that

$$(r + r_0)^{-\frac{\bar{b}p}{m}} \leq \bar{c} \quad \text{for any } r \geq 0 , \tag{2.2.17}$$

with $r_0 > 0$ as in hypothesis (2.1.2).

Theorem 2.2.3. *Let assumptions (H), (2.1.2) and (2.2.16) be satisfied with $q > 2$. Suppose that*

$$1 < p < m , \quad \text{or } p > m \geq 1 ,$$

and that u_0 is small enough. Then problem (2.1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, \tau))$ for any $\tau > 0$. More precisely, we have the following cases.

(a) *Let* $1 < p < m$. *If* $C > 0$ *is big enough,* $r_0 > 0$, $T > 1$, $\alpha > 0$,

$$u_0(x) \leq CT^{\alpha}\left(|x| + r_0\right)^{-\frac{\bar{b}}{m}} \quad \text{for a.e. } x \in \mathbb{R}^N\,, \tag{2.2.18}$$

then problem (2.1.1) *admits a global solution* u, *which satisfies the bound from above*

$$u(x,t) \leq C(T+t)^{\alpha}\left(|x| + r_0\right)^{-\frac{\bar{b}}{m}} \text{ for a.e. } (x,t) \in \mathbb{R}^N \times (0,+\infty)\,. \tag{2.2.19}$$

(b) *Let* $p > m \geq 1$. *If* $C > 0$ *is small enough,* $r_0 > 0$ *and* (2.2.18) *holds with* $\alpha = 0$, *then problem* (2.1.1) *admits a global solution* $u \in L^{\infty}(\mathbb{R}^N \times (0,+\infty))$, *which satisfies the bound from above* (2.2.19) *with* $\alpha = 0$.

Remark 2.2.4. *Observe that, in the case when* $p = m$, *if* $C > 0$ *is small enough,* $r_0 > 0$ *big enough to have*

$$\left(\frac{1}{r_0}\right)^{\frac{\bar{b}p}{m}} \leq \bar{b}k_1(N-2-\bar{b})\,,$$

$T > 0$ *and* (2.2.18) *holds with* $\alpha = 0$, *then problem* (2.1.1) *admits a global solution* $u \in L^{\infty}(\mathbb{R}^N \times (0,+\infty))$ *which satisfies the bound from above* (2.2.19) *for* $\alpha = 0$.

Note that in Theorem 2.2.3 we do not require that supp u_0 is compact.

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 2.3.5.

Remark 2.2.5. *The statement in Theorem 2.2.3-(b) is in agreement with [86, Theorem 2], where it is assumed that* $p > m$, $\rho(x) = (1+|x|)^{-q}$ *with* $q > 2$, $\int_{\mathbb{R}^N} \rho(x)u_0(x)dx < +\infty$, $\int_{\mathbb{R}^N} \rho(x)[u_0(x)]^{\bar{q}}dx < \delta$, *for some* $\delta > 0$ *small enough and* $\bar{q} > \frac{N}{2}(p-m)$.

Note that the assumption on u_0 *is of a different type. In particular, in view of* (2.2.18) *and* (2.2.16), *the initial datum* u_0 *considered in Theorem 2.2.3-(b) not necessarily satisfies* $\int_{\mathbb{R}^N} \rho(x)u_0(x)dx < +\infty$.

In [86] the proofs are based on the energy method, so they are completely different with respect to our approach.

2.3 Global existence: proofs

Throughout this Chapter we deal with *very weak* solutions to problem (2.1.1) and to the same problem set in different domains (see Section 1.3).

For every $R > 0$, let B_R as in (1.3.26), then, for $\tau > 0$, we consider the auxiliary problem

$$\begin{cases} u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p & \text{in } B_R \times (0,\tau) \\ u = 0 & \text{on } \partial B_R \times (0,\tau) \\ u = u_0 & \text{in } B_R \times \{0\}\,. \end{cases} \tag{2.3.20}$$

The definition of solution to problem (2.3.20) is given in Definition 1.3.9.

In what follows we set $r \equiv |x|$. We construct a suitable family of supersolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0,+\infty). \tag{2.3.21}$$

2.3.1 Order of decaying: $q = 2$

We assume (H), (2.1.2) with $q = 2$ and (2.2.7). In order to construct a suitable family of supersolutions of (2.3.21), we define, for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$\bar{u}(x,t) := C\zeta(t)\left[1 - \frac{\log(r+r_0)}{a}\eta(t)\right]_+^{\frac{1}{m-1}}, \tag{2.3.22}$$

where $\eta,\, \zeta \in C^1([0,+\infty);[0,+\infty))$ and $C > 0$, $a > 0$, $r_0 > e$.

Now, we compute

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p.$$

To this aim, let us set

$$F(r,t) := 1 - \frac{\log(r+r_0)}{a}\,\eta(t)\,,$$

and define

$$D_1 := \left\{(x,t) \in [\mathbb{R}^N \setminus \{0\}] \times (0,+\infty) \mid 0 < F(r,t) < 1\right\}.$$

For any $(x,t) \in D_1$, we have:

$$\begin{aligned}
\bar{u}_t &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}F^{\frac{1}{m-1}-1}\left(-\frac{\log(r+r_0)}{a}\eta'\right)\\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\left(1 - \frac{\log(r+r_0)}{a}\eta\right)\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}\\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}.
\end{aligned} \tag{2.3.23}$$

$$(\bar{u}^m)_r = -\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)}\eta. \tag{2.3.24}$$

$$\begin{aligned}
(\bar{u}^m)_{rr} &= -\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\left(1 - \frac{\log(r+r_0)}{a}\eta\right)\eta\frac{1}{(r+r_0)^2\log(r+r_0)}\\
&\quad + \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r+r_0)^2\log(r+r_0)} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)^2}\eta\\
&= -\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta\frac{1}{(r+r_0)^2\log(r+r_0)}\\
&\quad + \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r+r_0)^2\log(r+r_0)} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)^2}\eta.
\end{aligned} \tag{2.3.25}$$

$$
\begin{aligned}
\Delta(\bar{u}^m) &= \frac{(N-1)}{r}(\bar{u}^m)_r + (\bar{u}^m)_{rr} \\
&= \frac{(N-1)}{r}\left(-\frac{C^m}{a}\zeta^m \frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)}\eta\right) \\
&\quad - \frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta \frac{1}{(r+r_0)^2\log(r+r_0)} \\
&\quad + \frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r+r_0)^2\log(r+r_0)} \\
&\quad + \frac{C^m}{a}\zeta^m \frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)^2}\eta \\
&\le \frac{N-1}{r+r_0}\left(-\frac{C^m}{a}\zeta^m \frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)}\eta\right) \\
&\quad - \frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta \frac{1}{(r+r_0)^2\log(r+r_0)} \\
&\quad + \frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r+r_0)^2\log(r+r_0)} \\
&\quad + \frac{C^m}{a}\zeta^m \frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)^2}\eta
\end{aligned}
\tag{2.3.26}
$$

We also define

$$
\begin{aligned}
K &:= \left[\left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}}\right] > 0\,, \\
\bar{\sigma}(t) &:= \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta k_1\,(N-2)\,, \\
\bar{\delta}(t) &:= \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}\frac{1}{\log(r_0)}k_2\,, \\
\bar{\gamma}(t) &:= C^{p-1}\zeta^p\,.
\end{aligned}
\tag{2.3.27}
$$

Proposition 2.3.1. *Let $\zeta = \zeta(t)$, $\eta = \eta(t) \in C^1([0,+\infty);[0,+\infty))$. Let K, $\bar{\sigma}$, $\bar{\delta}$, $\bar{\gamma}$ be as defined in (2.3.27). Assume (H), (2.1.2) with $q = 2$, (2.2.7) and that, for all $t \in (0,+\infty)$,*

$$
-\frac{\eta'}{\eta^2} \ge \frac{1}{\log(r_0)}\frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}k_2 \tag{2.3.28}
$$

and

$$
\zeta' + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[(N-2)k_1 - \frac{k_2}{(m-1)\log(r_0)}\right] - C^{p-1}\zeta^p \ge 0\,. \tag{2.3.29}
$$

then $\bar{u}$ defined in (2.3.22) is a supersolution of equation (2.3.21).

Proof. In view of (2.3.23), (2.3.24), (2.3.25) and (2.3.26), for any $(x,t) \in D_1$,

$$\begin{aligned}
&\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \\
&\geq C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \\
&+ \frac{C^m}{a}\zeta^m \frac{m}{m-1}\eta\frac{1}{\rho(r+r_0)^2}F^{\frac{1}{m-1}}\left(\frac{1}{(m-1)\log(r+r_0)} + N - 2\right) \\
&- \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{\log(r+r_0)}\frac{1}{\rho(r+r_0)^2} - C^p\zeta^p F^{\frac{p}{m-1}}.
\end{aligned} \tag{2.3.30}$$

Thanks to hypothesis (H), (2.1.2) and (2.2.7), we have

$$\frac{1}{\log(r+r_0)} \geq 0\,, \quad -\frac{1}{\log(r+r_0)} \geq -\frac{1}{\log(r_0)} \quad \text{for all } \ x \in \mathbb{R}^N\,, \tag{2.3.31}$$

$$\frac{1}{\rho(r+r_0)^2} \geq k_1\,, \quad -\frac{1}{\rho(r+r_0)^2} \geq -k_2 \quad \text{for all } \ x \in \mathbb{R}^N\,. \tag{2.3.32}$$

From (2.3.30), (2.3.31) and (2.3.32) we get,

$$\begin{aligned}
&\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \\
&\geq CF^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta(N-2)k_1\right]\right. \\
&\left. -\zeta\frac{1}{m-1}\frac{\eta'}{\eta} - \frac{C^{m-1}}{a}\zeta^m\frac{m}{(m-1)^2}\frac{1}{\log(r_0)}\eta k_2 - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}
\end{aligned} \tag{2.3.33}$$

From (2.3.33) and (2.3.27), we have

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq CF^{\frac{1}{m-1}-1}\left[\bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}\right]. \tag{2.3.34}$$

For each $t > 0$, set

$$\varphi(F) := \bar{\sigma}(t)F - \bar{\delta}(t) - \bar{\gamma}(t)F^{\frac{p+m-2}{m-1}}, \quad F \in (0,1)\,.$$

Now our goal is to find suitable C, a, ζ, η such that, for each $t > 0$,

$$\varphi(F) \geq 0 \quad \text{for any } \ F \in (0,1)\,.$$

We observe that $\varphi(F)$ is concave in the variable F. Hence it is sufficient to have $\varphi(F)$ positive in the extrema of the interval $(0,1)$. This reduces, for any $t > 0$, to the conditions

$$\begin{aligned}
\varphi(0) \geq 0\,, \\
\varphi(1) \geq 0\,.
\end{aligned} \tag{2.3.35}$$

These are equivalent to

$$-\bar{\delta}(t) \geq 0\,, \quad \bar{\sigma}(t) - \bar{\delta}(t) - \bar{\gamma}(t) \geq 0\,,$$

that is

$$-\frac{\eta'}{\eta^2} \geq \frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}\frac{1}{\log(r_0)}k_2\,,$$
$$\zeta' + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[(N-2)\,k_1 - \frac{k_2}{(m-1)\log(r_0)}\right] - C^{p-1}\zeta^p \geq 0\,,$$

which are guaranteed by (2.2.7), (2.3.28) and (2.3.29). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in } D_1\,.$$

Now observe that

$\bar{u} \in C(\mathbb{R}^N \times [0, +\infty))\,,$

$\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0, +\infty))$, and by the definition of $\bar{u}$,

$\bar{u} \equiv 0$ in $[\mathbb{R}^N \setminus D_1] \times [0, +\infty))$.

Hence, by Lemma 1.3.10 (applied with $\Omega_1 = D_1$, $\Omega_2 = \mathbb{R}^N \setminus D_1$, $u_1 = \bar{u}$, $u_2 = 0$, $u = \bar{u}$), $\bar{u}$ is a supersolution of equation

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p = 0 \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times (0, +\infty)$$

in the sense of Definition 1.3.9. Since $\bar{u}^m_r(0, t) \leq 0$ for any $t > 0$,

$$\Delta[\bar{u}^m(x,t)] \leq \rho(x)[\bar{u}_t(x,t) - \bar{u}^p(x,t)] \quad \text{for all } (x,t) \in (\mathbb{R}^N \setminus \{0\}) \times (0, +\infty),$$

by the same arguments as in the proof of the so-called Kato inequality (see [75, Lemma A]), it can be easily seen that

$$\Delta(\bar{u}^m) \leq \rho(\bar{u}_t - \bar{u}^p) \quad \text{in } \mathfrak{D}'\big(\mathbb{R}^N \times (0, +\infty)\big)$$

(see also [54, proof of Proposition 4.1]. So, $\bar{u}$ is a supersolution of equation (2.3.21) in the sense of Definition 1.3.9. □

Remark 2.3.2. *Let*

$$p > m$$

and assumption (2.2.7) *be satisfied. Let* $\omega := \frac{C^{m-1}}{a}$. *In Theorem 2.2.1 the precise hypotheses on parameters* $C > 0$, $\omega > 0$, $T > 0$ *are the following:*

$$\frac{p-m}{p-1} \geq \omega\frac{m}{m-1}k_2\frac{1}{\log(r_0)}, \tag{2.3.36}$$

$$\omega\frac{m}{m-1}\left[k_1(N-2) - \frac{k_2}{(m-1)\log(r_0)}\right] \geq C^{p-1} + \frac{1}{p-1}\,. \tag{2.3.37}$$

Lemma 2.3.3. *All the conditions in Remark 2.3.2 can be satisfied simultaneously.*

Proof. Since $p > m$ the left-hand-side of (2.3.36) is positive. In view of (2.2.7), we can tale $\omega > 0$ such that (2.3.36) holds and

$$\omega\frac{m}{m-1}\left[k_1(N-2)-\frac{k_2}{(m-1)\log(r_0)}\right] > \frac{1}{p-1}\,.$$

Then we take $C > 0$ so small that (2.3.37) holds (and so $a > 0$ is accordingly fixed). □

Proof of Theorem 2.2.1. We prove Theorem 2.2.1 by means of Proposition 2.3.1. In view of Lemma 2.3.3, we can assume that alla conditions in Remark 2.3.2 are fulfilled. Set

$$\zeta = (T+t)^{-\alpha}\,, \quad \eta = (T+t)^{-\beta}, \quad \text{for all} \quad t > 0\,.$$

Consider conditions (2.3.28), (2.3.29) of Proposition 2.3.1 with this choice of $\zeta(t)$ and $\eta(t)$. Therefore we obtain

$$\beta - \frac{C^{m-1}}{a}\frac{m}{m-1}k_2(T+t)^{-\alpha(m-1)-\beta+1} \geq 0 \tag{2.3.38}$$

and

$$\begin{aligned}-\alpha(T+t)^{-\alpha-1} + \frac{C^{m-1}}{a}\frac{m}{m-1}\left[k_1(N-2)-\frac{k_2}{(m-1)\log(r_0)}\right](T+t)^{-\alpha m-\beta}\\ - C^{p-1}(T+t)^{-\alpha p} \geq 0\,.\end{aligned} \tag{2.3.39}$$

We take

$$\alpha = \frac{1}{p-1}\,, \quad \beta = \frac{p-m}{p-1}\,. \tag{2.3.40}$$

Due to (2.3.40), (2.3.38) and (2.3.39) become

$$\frac{p-m}{p-1} \geq \frac{C^{m-1}}{a}\frac{m}{m-1}\frac{k_2}{\log(r_0)}\,, \tag{2.3.41}$$

$$\frac{C^{m-1}}{a}\frac{m}{m-1}\left[k_1(N-2)-\frac{k_2}{(m-1)\log(r_0)}\right] \geq C^{p-1} + \frac{1}{p-1}\,. \tag{2.3.42}$$

Therefore, (2.3.28) and (2.3.29) follow from assumptions (2.3.36) and (2.3.37). Thus the conclusion follows by Propositions 2.3.1 and 1.3.6. □

2.3.2 Order of decaying: $q > 2$

We assume (H), (2.1.2) and (2.2.16) for $q > 2$ and (2.2.17). In order to construct a suitable family of supersolutions of (2.3.21), we define, for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$\bar{u}(x,t) \equiv \bar{u}(r(x),t) := C\zeta(t)(r+r_0)^{-\frac{\bar{b}}{m}}; \tag{2.3.43}$$

where $\zeta \in C^1([0,+\infty);[0,+\infty))$ and $C > 0$, $r_0 > 0$.

Now, we compute

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p.$$

For any $(x,t) \in \left[\mathbb{R}^N \setminus \{0\}\right] \times (0,+\infty)$, we have:

$$\bar{u}_t = C\,\zeta'\,(r+r_0)^{-\frac{\bar{b}}{m}}\,. \tag{2.3.44}$$

$$(\bar{u}^m)_r = -\,\bar{b}\,C^m\,\zeta^m\,(r+r_0)^{-\bar{b}-1}\,. \tag{2.3.45}$$

$$(\bar{u}^m)_{rr} = \bar{b}\,(\bar{b}+1)\,C^m\,\zeta^m\,(r+r_0)^{-\bar{b}-2}\,. \tag{2.3.46}$$

Proposition 2.3.4. *Let* $\zeta = \zeta(t) \in C^1[0,+\infty);[0,+\infty)), \zeta' \geq 0$*. Assume* (H)*,* (2.1.2) *and* (2.2.16) *for* $q > 2$*,* (2.2.17)*, and that*

$$\bar{b}k_1(N-2-\bar{b})C^m\zeta^m - \bar{c}\,C^p\zeta^p > 0\,. \tag{2.3.47}$$

Then $\bar{u}$ *defined in* (2.3.43) *is a supersolution of equation* (2.3.21)*.*

Proof of Proposition 2.3.4. In view of (2.3.44), (2.3.45), (2.3.46) and the fact that

$$\frac{1}{(r+r_0)^{\bar{b}+1}r} \geq \frac{1}{(r+r_0)^{\bar{b}+2}} \quad \text{for any } x \in \mathbb{R}^N,$$

we get, for any $(x,t) \in (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty)$,

$$\begin{aligned} &\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \\ &\geq C\zeta'(r+r_0)^{-\frac{\bar{b}}{m}} + \frac{1}{\rho}\left\{(N-2-\bar{b})C^m\zeta^m\bar{b}(r+r_0)^{-\bar{b}-2}\right\} - C^p\zeta^p(r+r_0)^{-\frac{\bar{b}p}{m}}. \end{aligned} \tag{2.3.48}$$

Thanks to hypothesis (2.1.2), (2.2.16) and (2.2.17), we have

$$\begin{aligned} &\frac{(r+r_0)^{-\bar{b}-2}}{\rho} \geq k_1(r+r_0)^{-\bar{b}-2+q} = k_1\,, \\ &-(r+r_0)^{-\frac{\bar{b}p}{m}} \geq -\bar{c} \end{aligned} \tag{2.3.49}$$

Since $\zeta' \geq 0$, from (2.3.48) and (2.3.49) we get

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq k_1\bar{b}(N-2-\bar{b})C^m\zeta^m - \bar{c}\,C^p\zeta^p\,. \tag{2.3.50}$$

Hence we get the condition

$$k_1\bar{b}(N-2-\bar{b})C^m\zeta^m - \bar{c}\,C^p\zeta^p \geq 0\,, \tag{2.3.51}$$

which is guaranteed by (2.2.16) and (2.3.47). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in} \quad (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty)\,.$$

Now observe that

$$\begin{aligned} &\bar{u} \in C(\mathbb{R}^N \times [0,+\infty))\,, \\ &\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0,+\infty))\,, \\ &\bar{u}^m_r(0,t) \leq 0\,. \end{aligned}$$

Hence, thanks to a Kato-type inequality we can infer that $\bar{u}$ is a supersolution to equation (2.3.21) in the sense of Definition 1.3.9.

□

Remark 2.3.5. *Let*

$$q > 2$$

and assumption (2.2.16) *be satisfied. In Theorem 2.2.3 the precise hypotheses on parameters* α, $C > 0$, $T > 0$ *are as follows.*

(a) Let $p < m$. *We require that*

$$\alpha > 0, \tag{2.3.52}$$

$$\bar{b}\, k_1(N - 2 - \bar{b})C^m \; - \bar{c}\, C^p \; \geq 0 \tag{2.3.53}$$

(b) Let $p > m$. *We require that*

$$\alpha = 0, \tag{2.3.54}$$

$$\bar{b}\, k_1(N - 2 - \bar{b})C^m - \bar{c}\, C^p \geq 0 \tag{2.3.55}$$

Lemma 2.3.6. *All the conditions in Remark 2.3.5 can hold simultaneously.*

Proof. (a) We observe that, due to (2.2.16),

$$N - 2 - \bar{b} > 0.$$

Therefore, we can select $C > 0$ sufficiently large to guarantee (2.3.53).
(b) We choose $C > 0$ sufficiently small to guarantee (2.3.55). □

Proof of Theorem 2.2.3. We now prove Theorem 2.2.3 in view of Proposition 2.3.4. In view of Lemma 2.3.6 we can assume that all conditions in Remark 2.3.5 are fulfilled. Set

$$\zeta(t) = (T + t)^{\alpha}, \quad \text{for all} \quad t \geq 0\,.$$

Let $p < m$. Inequality (2.3.47) reads

$$\bar{b}\, k_1(N - 2 - \bar{b})C^m(T + t)^{m\alpha} \; - \bar{c}\, C^p(T + t)^{p\alpha} \; \geq 0 \quad \text{for all} \;\; t > 0\,.$$

This follows from (2.3.52) and (2.3.53), for $T > 1$. Hence, by Propositions 2.3.4 and 1.3.5 the thesis follows in this case.

Let $p > m$. Conditions (2.3.54) and (2.3.55) are equivalent to (2.3.47). Hence, by Propositions 2.3.4 and 1.3.5 the thesis follows in this case too. The proof is complete. □

2.4 Blow-up: proofs

In what follows we set $r \equiv |x|$. We construct a suitable family of subsolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{ in } \mathbb{R}^N \times (0, T). \tag{2.4.56}$$

2.4.1 Order of decaying: $q = 2$

Suppose (H), (2.2.5) and (2.2.11). To construct a suitable family of subsolution of (2.4.56), we define, for all $(x,t) \in [\mathbb{R}^N \setminus B_e] \times (0,T)$,

$$\underline{u}(x,t) \equiv \underline{u}(r(x),t) := C\zeta(t)\left[1 - \frac{\log(r)}{a}\eta(t)\right]_+^{\frac{1}{m-1}}, \tag{2.4.57}$$

and

$$w(x,t) \equiv w(r(x),t) := \begin{cases} \underline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_e] \times (0,T), \\ v(x,t) & \text{in } B_e \times (0,T), \end{cases} \tag{2.4.58}$$

where

$$v(x,t) \equiv v(r(x),t) := C\zeta(t)\left[1 - \frac{r^2+e^2}{2e^2}\frac{\eta}{a}\right]_+^{\frac{1}{m-1}}. \tag{2.4.59}$$

Let us set

$$F(r,t) := 1 - \frac{\log(r)}{a}\eta(t),$$

and

$$G(r,t) := 1 - \frac{r^2+e^2}{2e^2}\frac{\eta(t)}{a}.$$

For any $(x,t) \in (\mathbb{R}^N \setminus B_e) \times (0,T)$, we have:

$$\begin{aligned} \underline{u}_t &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}F^{\frac{1}{m-1}-1}\left(-\frac{\log(r)}{a}\eta'\right) = \\ &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\left(1 - \frac{\log(r)}{a}\eta\right)\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} = \\ &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}. \end{aligned} \tag{2.4.60}$$

$$(\underline{u}^m)_r = -\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{r}\eta. \tag{2.4.61}$$

$$\begin{aligned} (\underline{u}^m)_{rr} &= -C^m\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\left(1 - \frac{\log(r)}{a}\eta\right)\eta\frac{1}{r^2\log(r)} \\ &+ \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{r^2\log(r)} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{r^2}\eta = \\ &= -\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta\frac{1}{r^2\log(r)} \\ &+ \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{r^2\log(r)} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{r^2}\eta. \end{aligned} \tag{2.4.62}$$

For any $(x,t) \in B_e \times (0,T)$, we have:

$$\begin{aligned} v_t &= C\zeta' G^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} G^{\frac{1}{m-1}-1}\left(-\frac{r^2+e^2}{2e^2}\frac{\eta'}{a}\right) = \\ &= C\zeta' G^{\frac{1}{m-1}} + C\frac{\zeta}{m-1}\left(1-\frac{r^2+e^2}{2e^2}\frac{\eta}{a}\right)\frac{\eta'}{\eta} G^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1} = \\ &= C\zeta' G^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1}. \end{aligned} \tag{2.4.63}$$

$$(v^m)_r = -\frac{C^m}{a}\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{r}{e^2}\eta\,. \tag{2.4.64}$$

$$(v^m)_{rr} = -C^m\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{1}{e^2}\frac{\eta}{a} + \frac{C^m}{a^2}\zeta^m\frac{m}{(m-1)^2}G^{\frac{1}{m-1}-1}\eta^2\frac{r^2}{e^4}\,. \tag{2.4.65}$$

We also define

$$\begin{aligned} &\underline{\sigma}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta k_2\left(N-2+\frac{1}{m-1}\right), \\ &\underline{\delta}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta}, \\ &\underline{\gamma}(t) := C^{p-1}\zeta^p, \\ &\underline{\sigma}_0(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{N}{e^2}\rho_2\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\,, \\ &K := \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0. \end{aligned} \tag{2.4.66}$$

Proposition 2.4.1. *Let $p > m$. Let $T \in (0,\infty)$, ζ, $\eta \in C^1([0,T);[0,T))$. Let $\underline{\sigma}$, $\underline{\delta}$, $\underline{\gamma}$, $\underline{\sigma}_0$, K be defined in* (2.4.66). *Assume that, for all $t \in (0,T)$,*

$$\underline{\sigma}(t) > 0, \quad K[\underline{\sigma}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}\,, \tag{2.4.67}$$

$$(m-1)\underline{\sigma}(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{2.4.68}$$

$$\underline{\sigma}_0(t) > 0, \quad K[\underline{\sigma}_0](t)^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}, \tag{2.4.69}$$

$$(m-1)\underline{\sigma}_0(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{2.4.70}$$

Then w defined in (2.4.58) *is a subsolution of equation* (2.4.56).

Proof of Proposition 2.4.1. Let $\underline{u}$ be as in (2.4.57) and set

$$D_2 := \left\{(x,t) \in (\mathbb{R}^N \setminus B_e) \times (0,T) \mid 0 < F(r,t) < 1\right\}.$$

In view of (2.4.60), (2.4.61), (2.4.62), we obtain, for all $(x,t) \in D_2$,

$$\begin{aligned} &\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\ &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \\ &+ F^{\frac{1}{m-1}}\frac{C^m}{a}\zeta^m\frac{m}{m-1}\eta\frac{1}{\rho r^2}\left(\frac{1}{(m-1)\log(r)}+N-1\right) - \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{\log(r)}\frac{1}{\rho r^2} \\ &- C^p\zeta^p F^{\frac{p}{m-1}}. \end{aligned}$$

In view of hypotheses (2.2.5) and (2.2.11), we can infer that

$$\frac{1}{\rho r^2} \leq k_2\,, \quad -\frac{1}{\rho r^2} \leq -k_1 \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_e\,. \tag{2.4.71}$$

Moreover,

$$-1 \leq -\frac{1}{\log(r)} \leq 0\,, \quad \frac{1}{\log(r)} \leq 1\,, \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_e\,. \tag{2.4.72}$$

From (2.4.71) and (2.4.72) we have

$$\begin{aligned} &\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\ &\leq CF^{\frac{1}{m-1}-1}\Big\{F\Big[\zeta' + \zeta\,\frac{1}{m-1}\,\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\,\zeta^m\,\frac{m}{m-1}\,\eta\,k_2 \\ &\quad \times\Big(N-2+\frac{1}{m-1}\Big)\Big] - \zeta\frac{1}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\Big\}\,. \end{aligned} \tag{2.4.73}$$

Thanks to (2.4.66) and (2.4.73)

$$\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \leq CF^{\frac{1}{m-1}-1}\varphi(F), \tag{2.4.74}$$

where

$$\varphi(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{p+m-2}{m-1}}\,. \tag{2.4.75}$$

Due to (2.4.74), our goal is to find suitable $C > 0$, $a > 0$, ζ, η such that

$$\varphi(F) \leq 0\,, \quad \text{for all } F \in (0,1)\,.$$

To this aim, we impose that

$$\sup_{F\in(0,1)} \varphi(F) = \max_{F\in(0,1)} \varphi(F) = \varphi(F_0) \leq 0\,,$$

for some $F_0 \in (0,1)$. We have

$$\begin{aligned} \frac{d\varphi}{dF} = 0 &\iff \underline{\sigma}(t) - \frac{p+m-2}{m-1}\underline{\gamma}(t)F^{\frac{p-1}{m-1}} = 0 \\ &\iff F_0 = \left[\frac{m-1}{p+m-2}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)}\right]^{\frac{m-1}{p-1}}\,. \end{aligned}$$

Then,

$$\varphi(F_0) = K\frac{\underline{\sigma}(t)^{\frac{p+m-2}{p-1}}}{\underline{\gamma}(t)^{\frac{m-1}{p-1}}} - \underline{\delta}(t)$$

where the coefficient $K = K(m,p)$ has been defined in (2.4.66). By hypotheses (2.4.67) and (2.4.68)

$$\varphi(F_0) \leq 0\,, \quad 0 < F_0 \leq 1\,. \tag{2.4.76}$$

So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p \leq 0 \quad \text{in } D_2 . \tag{2.4.77}$$

Furthermore, since $\underline{u}^m \in C^1([\mathbb{R}^N \setminus B_e] \times [0,T))$, due to Lemma 1.3.10 (applied with $\Omega_1 = D_2, \Omega_2 = \mathbb{R}^N \setminus [B_e \cup D_2], u_1 = \underline{u}, u_2 = 0, u = \underline{u}$), it follows that $\underline{u}$ is a subsolution to equation

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p = 0 \quad \text{in } [\mathbb{R}^N \setminus B_e] \times (0,T) , \tag{2.4.78}$$

in the sense of Definition 1.3.9.

Let

$$D_3 := \{(x,t) \in B_e \times (0,T) \mid 0 < G < 1\} .$$

In view of (2.4.63), (2.4.64) and (2.4.65), for all $(x,t) \in D_3$,

$$\begin{aligned}
&v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p \\
&= CG^{\frac{1}{m-1}-1}\left\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{1}{\rho}\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\frac{N-1}{e^2}\eta\frac{1}{\rho}\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\frac{1}{e^2}\eta\right]\right. \\
&\left. + -\frac{\zeta}{m-1}\frac{\eta'}{\eta} - \frac{1}{\rho}\frac{C^{m-1}}{a^2}\zeta^m\frac{m}{(m-1)^2}\frac{r^2}{e^4}\eta^2 - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\right\}
\end{aligned} \tag{2.4.79}$$

Using (2.2.6), (2.4.79) yield, for all $(x,t) \in D_3$,

$$\begin{aligned}
v_t - &\frac{1}{\rho}\Delta(v^m) - v^p \\
&\leq CG^{\frac{1}{m-1}-1}\left\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \rho_2\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\frac{N}{e^2}\eta\right]\right. \\
&\left. - \frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\right\} .
\end{aligned} \tag{2.4.80}$$

Thanks to (2.3.27) and (2.4.80),

$$v_t - \frac{1}{\rho}\Delta(v^m) - v^p \leq CG^{\frac{1}{m-1}-1}\psi(G), \tag{2.4.81}$$

where

$$\psi(G) := \underline{\sigma}_0(t)G - \underline{\delta}(t) - \underline{\gamma}(t)G^{\frac{p+m-2}{m-1}} . \tag{2.4.82}$$

Now, by the same arguments used to obtain (2.4.78), in view of (2.4.69) and (2.4.70) we can infer that

$$\psi(G) \leq 0 \quad 0 < G \leq 1 .$$

So far, due to (2.4.81), we have proved that

$$v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p \leq 0 \quad \text{for any } (x,t) \in D_3 . \tag{2.4.83}$$

Moreover, by Lemma 1.3.10 v is a subsolution of equation

$$v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p = 0 \quad \text{in} \quad B_e \times (0,T)\,, \tag{2.4.84}$$

in the sense of Definition 1.3.9. Now, observe that $w \in C(\mathbb{R}^N \times [0,T))$, indeed,

$$\underline{u} = v = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e \times (0,T)\,.$$

Moreover, $w^m \in C^1(\mathbb{R}^N \times [0,T))$, indeed,

$$(\underline{u}^m)_r = (v^m)_r = -C^m\zeta(t)^m \frac{m}{m-1}\frac{1}{e}\frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e \times (0,T)\,.$$

Hence, by Lemma 1.3.10 again, w is a subsolution to equation (2.4.56) in the sense of Definition 1.3.9. □

Remark 2.4.2. *Let*

$$p > m\,,$$

and assumptions (2.2.5) *and* (2.2.11) *be satisfied. Let define* $\omega := \frac{C^{m-1}}{a}$. *In Theorem 2.2.2, the precise hypotheses on parameters* $C > 0$, $a > 0$, $\omega > 0$ *and* $T > 0$ *is the following.*

$$\max\left\{1 + mk_2\frac{C^{m-1}}{a}\left(N - 2 + \frac{1}{m-1}\right); 1 + m\rho_2\frac{C^{m-1}}{a}\frac{N}{e^2}\right\} \le (p+m-2)C^{p-1}\,, \tag{2.4.85}$$

$$\begin{aligned} \frac{K}{(m-1)^{\frac{p+m-2}{p-1}}}\ \max\Bigg\{&\left[1 + mk_2\frac{C^{m-1}}{a}\left(N - 2 + \frac{1}{m-1}\right)\right]^{\frac{p+m-2}{p-1}}; \\ &\left(1 + m\rho_2\frac{C^{m-1}}{a}\frac{N}{e^2}\right)^{\frac{p+m-2}{p-1}}\Bigg\} \le \frac{p-m}{(m-1)(p-1)}C^{m-1}\,. \end{aligned} \tag{2.4.86}$$

Lemma 2.4.3. *All the conditions in Remark 2.4.2 can hold simultaneously.*

Proof. We can take $\omega > 0$ such that

$$\omega_0 \le \omega \le \omega_1$$

for suitable $0 < \omega_0 < \omega_1$ and we can choose $C > 0$ sufficiently large to guarantee (2.4.85) and (2.4.86) (so, $a > 0$ is fixed, too). □

Proof of Theorem 2.2.2. We now prove Theorem 2.2.2, by means of Proposition 2.4.1. In view of Lemma 2.4.3 we can assume that all conditions of Remark 2.4.2 are fulfilled. Set

$$\zeta = (T-t)^{-\alpha}\,, \quad \eta = (T-t)^{-\beta}\,, \quad \text{for all} \quad t > 0\,,$$

and α and β as defined in (2.3.40). Then

$$\begin{aligned}
&\underline{\sigma}(t) := \left[\frac{1}{m-1} + \frac{C^{m-1}}{a}\frac{m}{m-1}k_2\left(\frac{1}{m-1}+N-2\right)\right](T-t)^{-\frac{p}{p-1}}\,,\\
&\underline{\delta}(t) := \frac{p-m}{(m-1)(p-1)}(T-t)^{-\frac{p}{p-1}}\,,\\
&\underline{\gamma}(t) := C^{p-1}(T-t)^{-\frac{p}{p-1}}\,,\\
&\underline{\sigma}_0(t) := \frac{1}{m-1}\left[1+\frac{\rho_2 Nm}{e^2}\frac{C^{m-1}}{a}\right](T-t)^{-\frac{p}{p-1}}\,.
\end{aligned} \tag{2.4.87}$$

Let $p > m$. Condition (2.4.85) implies (2.4.67), (2.4.68), while condition (2.4.86) implies (2.4.69), (2.4.70). Moreover, $w(x,0)$, with w defined in (2.4.58), coincides with the right hand side of (2.2.13). Hence by Propositions 2.4.1 and 1.3.7 the thesis follows.
□

2.5 Further results: non-uniqueness for $q > 2$

In the case when $q > 2$ we can prove a result of non-uniqueness of the solution to problem (2.1.1) in the space $L^\infty(\mathbb{R}^N \times (0,T])$.

Proposition 2.5.1. *Let hypothesis* (H) *be satisfied. Let*

$$q > 2.$$

If there exists a supersolution $V > 0$ *of problem*

$$\begin{aligned}
\frac{1}{\rho}\Delta V &= -1\\
\lim_{|x|\to+\infty} V(x) &= 0\,,
\end{aligned} \tag{2.5.88}$$

then there exist infinitely many solutions u *of problem* (2.1.1) *that belong to* $L^\infty(\mathbb{R}^N \times (0,T])$, *for some* $T > 0$. *In particular, for any* $c > 0$, *there exists a solution* u_c *of problem* (2.1.1) *such that*

$$\lim_{|x|\to+\infty} \frac{1}{T}\int_0^T u_c^m(x,t)\,dt = c$$

To prove Proposition 2.5.1, we introduce the following definitions.

Definition 2.5.2. *Let* $g = \{1, -1\}$. *By a solution to the problem*

$$\Delta V = g\,\rho(x) \quad \text{in } \mathbb{R}^N \tag{2.5.89}$$

we mean any function $V \in C(\mathbb{R}^N)$ *such that*

$$\int_\Omega V\Delta\psi\,dx = \int_{\partial\Omega} V\nabla\psi\cdot\nu\,d\sigma + g\int_\Omega \rho(x)\psi\,dx \tag{2.5.90}$$

for any open bounded set $\Omega \subset \mathbb{R}^N$ *with regular boundary* $\partial\Omega$ *and for any* $\psi \in C^\infty(\overline{\Omega})$, $\psi \geq 0$ *and* $\psi|_{\partial\Omega} = 0$. *Subsolutions (supersolutions) of* (2.5.89) *are defined replacing* " = " *by* " $\geq$ " *(respectively* " $\leq$ "*) in equality* (2.5.90).

Moreover, we shall also consider the following auxiliary problems:

$$\begin{cases} u_t &= \frac{1}{\rho}\Delta(u^m) + u^p \quad \text{in } B_R \times (0,T] \\ u &= \varphi \qquad\qquad\quad\ \ \text{on } \partial B_R \times (0,T] \\ u &= u_0 \qquad\qquad\quad \text{on } B_R \times \{0\}, \end{cases} \tag{2.5.91}$$

and

$$\begin{cases} \frac{1}{\rho}\Delta V &= g \quad \text{in } B_R \\ V &= \chi \quad \text{on } \partial B_R. \end{cases} \tag{2.5.92}$$

Here $B_R := \{x \in \mathbb{R}^N \ : \ |x| < R\}$ and the functions $\varphi \in C(\partial B_R \times (0,T])$ and $\chi \in C(\partial B_R)$ are given functions. Solutions to problems (2.5.91) and (2.5.92) are defined as follows.

Definition 2.5.3. *By a solution to the problem* (2.5.91) *we mean any function* $u \in C(B_R \times (0,T])$ *such that*

$$\begin{aligned} &\int_0^t \int_{B_R} \{\rho\, u\, \psi_t + u^m\, \Delta\psi + \rho\, u^p\, \psi\}\, dx\, d\tau = \int_{B_R} \rho(x)\{u(t)\,\psi(t) - u_0\,\psi(0)\}\, dx \\ &+ \int_0^t \int_{\partial B_R} u^m\, \nabla\psi \cdot \nu\, d\sigma\, d\tau\,, \end{aligned} \tag{2.5.93}$$

for any $t \in (0,T]$ *and any* $\psi \in C^\infty(\overline{B_R} \times (0,T])$, $\psi \geq 0$ *and* $\psi = 0$ *on* $\partial B_R \times (0,T]$. *Subsolutions (supersolutions) of* (2.5.91) *are defined replacing* " $=$ " *by* " $\geq$ " *(respectively* " $\leq$ "*) in equality* (2.5.93).

Definition 2.5.4. *Let* $g = \{1, -1\}$. *By a solution to the problem* (2.5.92) *we mean any function* $V \in C(\overline{\Omega})$ *such that*

$$\int_{B_R} V \Delta\psi\, dx = \int_{\partial B_R} V \nabla\psi \cdot \nu\, d\sigma + g \int_{B_R} \rho(x)\psi\, dx \tag{2.5.94}$$

for any $\psi \in C^\infty(\overline{B_R})$, $\psi \geq 0$ *and* $\psi = 0$ *on* ∂B_R. *Subsolutions (supersolutions) of* (2.5.92) *are defined replacing* " $=$ " *by* " $\geq$ " *(respectively* " $\leq$ "*) in equality* (2.5.97).

We now prove the existence of at least one solutions in $L^\infty(\mathbb{R}^N \times (0,T])$ to problem (2.1.1) when $q > 2$.

We now prove the following Lemma.

Lemma 2.5.5. *Let assumptions* (H) *and* (2.1.2) *follow. Then there exists a supersolution* V *to the problem*

$$\frac{1}{\rho}\Delta V = -1 \quad \text{in } \mathbb{R}^N \tag{2.5.95}$$

such that

$$V(x) \longrightarrow 0 \quad \text{as } |x| \to +\infty.$$

Proof of Lemma 2.5.5. Let us consider

$$V = (r_0 + |x|)^{-b}, \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where

$$\begin{aligned} &0 < b < \min\{N-2\,,\, q-2\}, \\ &r_0 \geq [b\,(N-2-b)]^{q-2-b}\,. \end{aligned} \tag{2.5.96}$$

Then, thanks to (2.1.2) and (2.5.96),

$$\begin{aligned} \frac{1}{\rho}\Delta V &= \frac{1}{\rho}[V_{rr} + \frac{(N-1)}{|x|}V_r] \\ &= \frac{1}{\rho}\left[(b^2+b)(r_0+|x|)^{-b-2} - \frac{(N-1)}{|x|}\,b\,(r_0+|x|)^{-b-1}\right] \\ &\leq k_2\,(r_0+|x|)^q\left[(b^2+b-Nb+b)(r_0+|x|)^{-b-2}\right] \\ &\leq -k_2\,b\,(N-2-b)(r_0+|x|)^{q-2-b} \\ &\leq -1, \qquad \text{in } \mathbb{R}^N \setminus \{0\}. \end{aligned}$$

Hence V is a supersolution to (2.5.95) in $\mathbb{R}^N \setminus \{0\}$, for any $x \in \mathbb{R}^N \setminus \{0\}$. Thanks to a Kato-type inequality, since $V_r \leq 0$, we can easily infer that V is a supersolution of equation (2.5.95) , in the sense of Definition 2.5.4. Moreover,

$$\lim_{|x|\to\infty} V(x) = 0,$$

and

$$V > 0.$$

Finally we can say that

$$0 = \inf_{\mathbb{R}^N} V = \lim_{|x|\to\infty} V,$$

that corresponds to the desired hypotheses of Proposition 2.5.1. □

We can finally prove Proposition 2.5.1.

Proof of Proposition 2.5.1. For any fixed $c > 0$, consider the problem

$$\begin{cases} u_t &= u^p \quad \text{in } (0,T] \\ u(0) &= \omega \end{cases} \tag{2.5.97}$$

such that $\omega \geq \max\{||u_0||_\infty, c^{\frac{1}{m}} 2^{\frac{1}{p-1}}\}$. Then the solution is

$$\overline{u}(t) = \frac{\omega}{[1-(p-1)\,t\,\omega^{p-1}]^{1/(p-1)}} \ (\geq 0).$$

Take any $0 < T \leq \frac{1}{2(p-1)\omega^{p-1}}$. Let $\{u_{R,c}\}$ be the solution to

$$\begin{cases} \rho u_t &= \Delta(u^m) + \rho u^p \quad \text{in } B_R \times (0,T] \\ u &= c^{\frac{1}{m}} \quad \text{on } \partial B_R \times (0,T] \\ u &= u_0 \quad \text{in } B_R \times \{0\}. \end{cases} \tag{2.5.98}$$

Then, thanks to the comparison principle,

$$u_{R,c} \leq \overline{u},$$

and in particular,

$$u_{R,c} \leq \overline{u}(T) =: k \quad \text{ in } B_R \times (0,T].$$

Moreover, $\hat{u} \equiv 0$ is a subsolution to problem (2.5.98). Therefore

$$u_{R,c} \geq 0 \quad \text{ in } B_R \times (0,T].$$

Thus we can say that

$$0 \leq u_{R,c} \leq k \quad \text{ in } B_R \times (0,T].$$

Thanks to compactness argument, we can extract a subsequence $\{u_{R_l,c}\}$ where $R_l \to 0$ as $l \to +\infty$ that converges to u_c, u_c being a solution to problem (2.1.1) in the sense of Definition 1.3.1. Finally, it is still true that

$$0 \leq u \leq k \quad \text{ in } \mathbb{R}^N \times (0,T]. \tag{2.5.99}$$

It remains to show that

$$\lim_{|x|\to\infty} \frac{1}{T} \int_0^T u_c^m(x,T)\, dx = c.$$

Define,

$$v_{R_l,c} := \int_0^T u_{R_l,c}^m(x,t)\, dt \quad x \in B_R. \tag{2.5.100}$$

For any ψ as in Definition 2.5.3, choosing $\psi = \psi(x)$ in equality (2.5.93), we easily obtain

$$\int_{B_{R_l}} v_{R_l} \Delta\psi\, dx = \int_{B_{R_l}} \rho\, [u_{R_l,c}(x,T) - u_0(x)]\, \psi\, dx - \int_{B_{R_l}} \rho \left[\int_0^T u_{R_l,c}^p\, dt\right] \psi\, dx$$
$$+ \int_{\partial B_{R_l}} c\, T\, \nabla\psi \cdot \nu\, d\sigma$$

Moreover, observe that, thanks to (2.5.99)

$$|u_{R_l,c}(x,T)| + |u_0(x)| + \int_0^T \left|u_{R_l,c}^p\right| dt \;\leq\; 2\,k + T\,k^p \;:= M. \tag{2.5.101}$$

Thus,

$$\int_{B_{R_l}} v_{R_l,c} \Delta\psi\, dx \geq \int_{\partial B_{R_l}} v_{R_l,c} \nabla\psi \cdot \nu\, d\sigma - M \int_{B_{R_l}} \rho\, \psi\, dx \tag{2.5.102}$$

Inequality (2.5.102) shows that, for any integer l, the function

$$F_{1,l} := \frac{v_{R_l,c}}{M}, \tag{2.5.103}$$

is a subsolution of problem (2.5.92) for

$$g = -1, \qquad \chi := \frac{c\,T}{M}.$$

Similarly, from Lemma 2.5.5, we have

$$\int_{B_{R_l}} V\Delta\psi\,dx \le \int_{\partial B_{R_l}} V\nabla\psi\cdot\nu\,d\sigma - \int_{B_{R_l}} \rho\psi\,dx \tag{2.5.104}$$

where we have used that $V \geq 0$. We now consider the constant solution $W = c$ of the problem

$$\begin{cases} -\dfrac{1}{\rho}\Delta W = 0 & \text{in } B_R \\ W = c & \text{on } \partial B_R. \end{cases}$$

Then, for any $\psi \in C^\infty(\overline{B_R})$, $\psi \geq 0$ and $\psi = 0$ on ∂B_R, it follows that

$$\int_{B_R} W\Delta\psi\,dx = \int_{\partial B_R} W\nabla\psi\cdot\nu\,d\sigma. \tag{2.5.105}$$

We multiply (2.5.105) by $\dfrac{T}{M}$ and we sum the result together with (2.5.104). Using the definition of W we get,

$$\int_{B_{R_l}} \left(V + \frac{cT}{M}\Delta\psi\right) dx \le \int_{\partial B_{R_l}} \left(V + \frac{cT}{M}\nabla\psi\cdot\nu\right) d\sigma - \int_{B_{R_l}} \rho\psi\,dx. \tag{2.5.106}$$

Defining,

$$F_{2,l} := V + \frac{cT}{M}, \tag{2.5.107}$$

inequality (2.5.106) becomes,

$$\int_{B_{R_l}} F_{2,l}\Delta\psi\,dx \le \int_{\partial B_{R_l}} F_{2,l}\nabla\psi\cdot\nu\,d\sigma - \int_{B_{R_l}} \rho\psi\,dx. \tag{2.5.108}$$

This proves that $F_{2,l}$ is a supersolution to problem (2.5.92). By comparison results, it follows that

$$F_{1,l} \le F_{2,l}.$$

Hence

$$v_{R_l,c} = M\,F_{1,l} \le M\,F_{2,l} = M\left[V + \frac{cT}{M}\right] = M\,V + c\,T \quad \text{in } B_{R_l}.$$

Letting $l \longrightarrow \infty$ we obtain,

$$v_c \le M\,V + c\,T \quad \text{in } \mathbb{R}^N, \tag{2.5.109}$$

where

$$\begin{aligned} &u_c := \lim_{l\to\infty} u_{R_l,c} \quad \text{in } \mathbb{R}^N \times (0,T], \\ &v_c := \int_0^T u_c^m(x,t)\,dt = \lim_{l\to\infty} v_{R_l,c} \quad \text{for } x \in \mathbb{R}^N. \end{aligned} \tag{2.5.110}$$

On the other hand, thanks to (2.5.99),

$$\int_{B_{R_l}} v_{R_l,c}\Delta\psi\,dx \le \int_{\partial B_{R_l}} v_{R_l,c}\nabla\psi\cdot\nu\,d\sigma + M\int_{B_{R_l}} \rho\,\psi\,dx \tag{2.5.111}$$

Inequality (2.5.111) shows that, for any integer l, the function $F_{1,l}$ defined in (2.5.103), is a supersolution of problem (2.5.92) for

$$g = 1, \qquad \chi := \frac{cT}{M}.$$

Similarly, from Lemma 2.5.5, we have

$$\int_{B_{R_l}} (-V\Delta\psi)\, dx \geq \int_{\partial B_{R_l}} (-V\nabla\psi \cdot \nu)\, d\sigma + \int_{B_{R_l}} \rho\psi\, dx, \tag{2.5.112}$$

where we have used that $V \geq 0$. We now consider the constant solution $W = c$ as defined in (2.5.105). Then, we multiply (2.5.105) by $\frac{T}{M}$ and we sum the result together with (2.5.112). Using the definition of W we get,

$$\int_{B_{R_l}} \left(-V + \frac{cT}{M}\Delta\psi\right) dx \geq \int_{\partial B_{R_l}} \left(-V + \frac{cT}{M}\nabla\psi \cdot \nu\right) d\sigma + \int_{\Omega} \rho\psi\, dx. \tag{2.5.113}$$

Defining

$$F_{3,l} := -V + \frac{cT}{M}, \tag{2.5.114}$$

inequality (2.5.113) becomes

$$\int_{B_{R_l}} F_{3,l}\Delta\psi\, dx \geq \int_{\partial B_R} \frac{cT}{M}\nabla\psi \cdot \nu\, d\sigma + \int_{B_{R_l}} \rho\psi\, dx. \tag{2.5.115}$$

This proves that $F_{3,l}$ is a subsolution to problem (2.5.92) with the choice

$$g = 1, \qquad \chi := \frac{cT}{M}.$$

By comparison results, it follows that

$$F_{1,l} \geq F_{3,l},$$

hence

$$v_{R_l,c} = M\,F_{1,l} \geq M\,F_{3,l} = M\left[-V + \frac{cT}{M}\right] = -M\,V + cT \quad \text{in } B_{R_l}.$$

Letting $l \longrightarrow \infty$ we obtain,

$$v_c \geq -M\,V + cT \quad \text{in } \mathbb{R}^N. \tag{2.5.116}$$

Combining (2.5.109) and (2.5.116), thanks to the property of V showed in Lemma 2.5.5, we obtain

$$cT \;=\; \lim_{|x|\to+\infty} (-M\,V + cT) \;\leq\; \lim_{|x|\to+\infty} v_c(x) \;\leq\; \lim_{|x|\to+\infty} (M\,V + cT) \;=\; cT. \tag{2.5.117}$$

Thus

$$\lim_{|x|\to+\infty} v_c(x) \;=\; cT, \tag{2.5.118}$$

Recalling the definition of v_c and u_c in (2.5.110), the thesis follows. □

Chapter 3

The logarithmic decaying density case

3.1 Introduction

We are concerned with global existence and blow-up of nonnegative solutions to the Cauchy parabolic problem

$$\begin{cases} \rho(x)u_t = \Delta(u^m) + \rho(x)u^p & \text{in } \mathbb{R}^N \times (0,\tau) \\ u = u_0 & \text{in } \mathbb{R}^N \times \{0\}\,, \end{cases} \tag{3.1.1}$$

where $m > 1$, $p > 1$, $N \geq 3$, $\tau > 0$. Furthermore, we always assume that

$$\begin{cases} \text{(i) } u_0 \in L^\infty(\mathbb{R}^N),\ u_0 \geq 0 \text{ in } \mathbb{R}^N\,; \\ \text{(ii) } \rho \in C(\mathbb{R}^N),\ \rho > 0 \text{ in } \mathbb{R}^N\,; \end{cases} \tag{3.1.2}$$

the function $\rho = \rho(x)$ is usually referred to as a *variable density.*

The differential equation in problem (3.1.1), posed in $(-1,1)$ with homogeneous Dirichlet boundary conditions, has been introduced in [73] as a mathematical model of a thermal evolution of a heated plasma.

We refer the reader to [92, Introduction], [93, Introduction] for a comprehensive account of the literature concerning various problems related to (3.1.1). Here we limit ourselves to recall only some contribution of that literature. Problem (3.1.1) without the reaction term has been widely examined, e.g., in [25, 27, 49, 51, 50, 66, 67, 68, 69, 70, 71, 72, 108, 106, 107, 109, 115]. Furthermore, global existence and blow-up of solutions of problem (3.1.1) with $m = 1$ and $\rho \equiv 1$ have been studied, e.g., in [31, 58]). If

$$p \leq 1 + \frac{2}{N},$$

then finite time blow-up occurs, for all nontrivial nonnegative data, whereas, for

$$p > 1 + \frac{2}{N},$$

global existence prevails for sufficiently small initial conditions. In addition, in [85] (see also [21]), problem (3.1.1) with $m = 1$ has been considered.

Similar results for quasilinear parabolic equations, also involving p-Laplace type operators or double-nonlinear operators, have been stated in [60], [86], [87], [88], [104], [132] (see also [46] and [89] for the case of Riemannian manifolds); moreover, in [54] the same problem on Cartan-Hadamard manifolds has been investigated.

Global existence and blow-up of solutions for problem (3.1.1) with ρ satisfying

$$\frac{1}{k_1|x|^q} \leq \rho(x) \leq \frac{1}{k_2|x|^q} \quad \text{for all } \ |x| > 1 \tag{3.1.3}$$

have been investigated in [92] for $q \in [0, 2)$, and in [93] for $q \geq 2$. In [92], for $q \in [0, 2)$, the following results have been established.

- ([92, Theorem 2.1]) If $p > \overline{p}$, for a certain $\overline{p} = \overline{p}(k_1, k_2, q, m, N) > m$ and the initial datum is sufficiently small, then solutions exist globally in time. Observe that
$$\overline{p} = m + \frac{2-q}{N-q} \ \text{ when } \ k_1 = k_2.$$
- ([92, Theorem 2.4]) For any $p > 1$, for all sufficiently large initial data, solutions blow-up in finite time.
- ([92, Theorem 2.6]) For $1 < p < m$, for any non trivial initial data, solutions blow-ip in finite time.
- ([92, Theorem 2.7]) If $m < p < \underline{p}$, for a certain $\underline{p} = \underline{p}(k_1, k_2, q, m, N) \leq \overline{p}$, then, for any non trivial initial data, solutions blow-up in finite time, under specific extra assumptions on ρ.

Such results extend those stated in [119] for problem (3.1.1) with $\rho \equiv 1$, $m > 1$, $p > 1$ (see also [36]).

Furthermore, assume that (3.1.3) holds with $q \geq 2$. In [93] the following results have been showed.

- ([93, Theorem 2.1]) If $q = 2$ and $p > m$, then, for sufficiently small initial data, solutions exist globally in time.
- ([93, Theorem 2.2]) If $q = 2$ and $p > m$, then, for sufficiently large initial data, solutions blow-up in finite time.
- ([93, Theorem 2.3]) If $q > 2$, then, for any $p > 1$, for sufficiently small initial data, solutions exist globally in time.

Finally, in [46], (3.1.1) is addressed, when $p < m$. It is assumed that (3.1.2) is satisfied, and that the weighted Poincaré inequality with weight ρ holds. Moreover, in view of the assumption on ρ also the weighted Sobolev inequality is fulfilled. By using such functional inequalities, it is showed that global existence for L^m data occurs, as well as a smoothing effect for the solution, i.e. solutions corresponding to such data are bounded for any positive time. In addition, a quantitative bound on the L^∞ norm of the solution is given.

In what follows, we always consider two types of density functions ρ. To be more specific, we always make one of the following two assumptions:

$$\begin{aligned}&\text{there exist } k \in (0,+\infty) \text{ and } \alpha > 1 \text{ such that}\\ &\frac{1}{\rho(x)} \geq k\,(\log|x|)^\alpha\,|x|^2 \quad \text{for all } x \in \mathbb{R}^N \setminus B_e(0)\,;\end{aligned} \tag{H_1}$$

$$\begin{aligned}&\text{there exist } k_1, k_2 \in (0,+\infty) \text{ with } k_1 \leq k_2 \text{ and } \alpha > 1 \text{ such that}\\ &k_1 \frac{|x|^2}{(\log|x|)^\alpha} \leq \frac{1}{\rho(x)} \leq k_2 \frac{|x|^2}{(\log|x|)^\alpha} \quad \text{for all } x \in \mathbb{R}^N \setminus B_e(0)\,.\end{aligned} \tag{H_2}$$

Assume (H_1). For $1 < p < m$ and for suitable initial data $u_0 \in L^\infty(\mathbb{R}^N)$, we show the existence of global solutions belonging to $L^\infty(\mathbb{R}^N \times (0,\tau))$ for each $\tau > 0$. Indeed, in this case, the global existence follows from the results in [46] for $u_0 \in L^m_\rho(\mathbb{R}^N)$. However, now we consider a different class of initial data u_0. In fact, $u_0 \in L^\infty(\mathbb{R}^N)$ and satisfies a decaying condition as $|x| \to +\infty$; however, u_0 not necessarily belongs to $L^m_\rho(\mathbb{R}^N)$.

On the other hand, for $p > m > 1$, if u_0 satisfies a suitable decaying condition as $|x| \to +\infty$, then problem (3.1.1) admits a solution in $L^\infty(\mathbb{R}^N \times (0,+\infty))$.

Now, assume (H_2). For any $p > m$, if u_0 is sufficiently large, then the solutions to problem (3.1.1) blow-up in finite time. Moreover, if $p > m$, u_0 has compact support and is small enough, then, under suitable assumptions on k_1 and k_2, there exist global in time solutions to problem (3.1.1), which belong to $L^\infty(\mathbb{R}^N \times (0,+\infty))$.

The proofs mainly relies on suitable comparison principles and properly constructed sub- and supersolutions, which crucially depend on the behavior at infinity of the density function $\rho(x)$. More precisely, they are of the type

$$w(x,t) = C\zeta(t)\left[1 - \frac{(\log(|x|+r_0))^q}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \left[\mathbb{R}^N \setminus B_e(0)\right] \times [0,T), \tag{3.1.4}$$

for suitable functions $\zeta = \zeta(t), \eta = \eta(t)$ and constants $C > 0, a > 0, r_0 > 0$ and $q > 1$. Chapter 3 is organized as follows. In Section 3.2 we state our main results. In Section 3.3 we prove Theorem 3.2.1. The blow-up result (that is, Theorem 3.2.2) is proved in Section 3.4. Finally, in Section 3.5 Theorem 3.2.3 is proved .

3.2 Statements of the main results

For any $x_0 \in \mathbb{R}^N$ and $R > 0$ we set

$$B_R(x_0) = \{x \in \mathbb{R}^N : \|x - x_0\| < R\}. \tag{3.2.5}$$

When $x_0 = 0$, we write $B_R \equiv B_R(0)$.

3.2.1 Density ρ satisfying (H_1)

The first result concerns the global existence of solutions to problem (3.1.1) for any $p > 1$ and $m > 1$, $p \neq m$. We introduce the parameter $b \in \mathbb{R}$ such that

$$0 < b < \alpha - 1. \tag{3.2.6}$$

Moreover, since $N \geq 3$, we can choose $\varepsilon > 0$ so that

$$N - 2 - \varepsilon(b + 1) > 0, \tag{3.2.7}$$

and $r_0 > e$ so that

$$\frac{1}{\log(|x| + r_0)} < \varepsilon \quad \text{for any } x \in \mathbb{R}^N. \tag{3.2.8}$$

Finally, we can find $\bar{c} > 0$ such that

$$[\log(|x| + r_0)]^{-\frac{\bar{b}p}{m}} \leq \bar{c} \quad \text{for any } x \in \mathbb{R}^N. \tag{3.2.9}$$

Observe that, thanks to (3.1.2)-(i) and (H_1), we can say that there exists $k_0 > 0$ such that

$$\frac{1}{\rho(x)} \geq k_0 \, [\log(|x| + r_0)]^{\alpha} \, (|x| + r_0)^2 \quad \text{for any } \; x \in \mathbb{R}^N. \tag{3.2.10}$$

Theorem 3.2.1. *Let assumptions* (3.1.2), (H_1), (3.2.6), (3.2.7) *and* (3.2.8) *be satisfied. Suppose that*

$$1 < p < m \,, \quad \textit{or} \quad p > m > 1 \,,$$

and that u_0 is small enough. Then problem (3.1.1) *admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, \tau))$ for any $\tau > 0$. More precisely, we have the following cases.*

(a) Let $1 < p < m$. If $C > 0$ is big enough, $T > 1$, $\beta > 0$,

$$u_0(x) \leq C T^{\beta} \, (\log(|x| + r_0))^{-\frac{b}{m}} \quad \textit{for any } x \in \mathbb{R}^N, \tag{3.2.11}$$

then problem (3.1.1) *admits a global solution u, which satisfies the bound from above*

$$u(x,t) \leq C(T + t)^{\beta} \, (\log(|x| + r_0))^{-\frac{b}{m}} \; \textit{for any } (x,t) \in \mathbb{R}^N \times (0, +\infty). \tag{3.2.12}$$

(b) Let $p > m > 1$. If $C > 0$ is small enough, $T > 0$ and (3.2.11) *holds with $\beta = 0$, then problem* (3.1.1) *admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$, which satisfies the bound from above* (3.2.12) *with $\beta = 0$.*

3.2.2 Density ρ satisfying (H_2)

The next result concerns the blow-up of solutions in finite time, for every $p > m > 1$, provided that the initial datum is sufficiently large. We assume that hypotheses (3.1.2) and (H_2) hold. In view of (3.1.2)-(i), there exist $\rho_1, \rho_2 \in (0, +\infty)$ with $\rho_1 \leq \rho_2$ such that

$$\rho_1 \leq \frac{1}{\rho(x)} \leq \rho_2 \quad \text{for all } \; x \in \overline{B_e(0)}. \tag{3.2.13}$$

Let

$$\underline{b} := \alpha + 1, \tag{3.2.14}$$

and

$$\mathfrak{s}(x) := \begin{cases} (\log|x|)^{\underline{b}} & \text{if} \quad x \in \mathbb{R}^N \setminus B_e, \\ \\ \dfrac{\underline{b}\,|x|^2}{2e^2} + 1 - \dfrac{\underline{b}}{2} & \text{if} \quad x \in B_e \,. \end{cases}$$

Theorem 3.2.2. *Let assumptions* (3.1.2), (H_2), (3.2.13) *and* (3.2.14) *hold. For any*

$$p > m$$

and for any $T > 0$, *if the initial datum* u_0 *is large enough, then the solution* u *of problem* (3.1.1) *blows-up in a finite time* $S \in (0, T]$, *in the sense that*

$$\|u(t)\|_\infty \to \infty \ \text{as} \ t \to S^- \,. \tag{3.2.15}$$

More precisely, if $C > 0$ *and* $a > 0$ *are large enough,* $T > 0$,

$$u_0(x) \ge C T^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} T^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}} \quad \text{for any } x \in \mathbb{R}^N \,, \tag{3.2.16}$$

then the solution u *of problem* (3.1.1) *blows-up and satisfies the bound from below*

$$u(x,t) \ge C(T-t)^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} (T-t)^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \mathbb{R}^N \times (0, S) \,. \tag{3.2.17}$$

Observe that if u_0 satisfies (3.2.16), then

$$\operatorname{supp} u_0 \supseteq \{ x \in \mathbb{R}^N \,:\, \mathfrak{s}(x) < a T^{\frac{p-m}{p-1}} \} \,.$$

From (3.2.17) we can infer that

$$\operatorname{supp} u(\cdot, t) \supseteq \{ x \in \mathbb{R}^N \,:\, \mathfrak{s}(x) < a (T-t)^{\frac{p-m}{p-1}} \} \quad \text{for all } t \in [0, S) \,. \tag{3.2.18}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 3.4.2.

The next result concerns the global existence of solutions to problem (3.1.1) for $p > m$. We assume that ρ satisfies a stronger condition than (H_2). Indeed, we suppose that

$$k_1 \frac{(|x| + r_0)^2}{(\log(|x| + r_0))^\alpha} \le \frac{1}{\rho(x)} \le k_2 \frac{(|x| + r_0)^2}{(\log(|x| + r_0))^\alpha} \quad \text{for all } \ x \in \mathbb{R}^N \,, \tag{3.2.19}$$

where

$$r_0 > e, \quad \frac{k_2}{k_1} < m + (N-3) \left(\frac{m-1}{\bar{b}} \right) , \tag{3.2.20}$$

and

$$\bar{b} := \alpha + 2. \tag{3.2.21}$$

Theorem 3.2.3. *Assume* (3.1.2), (3.2.19), (3.2.20) *and* (3.2.21). *Suppose that*

$$p > m\,,$$

and that u_0 *is small enough and has compact support. Then problem* (3.1.1) *admits a global solution* $u \in L^\infty(\mathbb{R}^N \times (0,+\infty))$.
More precisely, if $C > 0$ *is small enough,* $a > 0$ *is so that*

$$0 < \omega_0 \le \frac{C^{m-1}}{a} \le \omega_1$$

for suitable $0 < \omega_0 < \omega_1$, $T > 0$,

$$u_0(x) \le CT^{-\frac{1}{p-1}} \left[1 - \frac{(\log(|x| + r_0))^{\bar b}}{a} T^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \text{for any } x \in \mathbb{R}^N\,, \tag{3.2.22}$$

then problem (3.1.1) *admits a global solution* $u \in L^\infty(\mathbb{R}^N \times (0,+\infty))$. *Moreover,*

$$u(x,t) \le C(T+t)^{-\frac{1}{p-1}} \left[1 - \frac{(\log(|x| + r_0))^{\bar b}}{a} (T+t)^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \tag{3.2.23}$$

for any $(x,t) \in \mathbb{R}^N \times (0,+\infty)$.

Observe that if u_0 satisfies (3.2.22), then

$$\operatorname{supp} u_0 \subseteq \{x \in \mathbb{R}^N \,:\, (\log(|x| + r_0))^{\bar b} \le aT^{\frac{p-m}{p-1}}\}\,.$$

From (3.2.23) we can infer that

$$\operatorname{supp} u(\cdot,t) \subseteq \{x \in \mathbb{R}^N \,:\, (\log(|x| + r_0))^{\bar b} \le a(T+t)^{\frac{p-m}{p-1}}\} \quad \text{for all } t > 0\,. \tag{3.2.24}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 3.5.2.

3.3 Proof of Theorem 3.2.1

In what follows, we deal with *very weak* solutions to problem (3.1.1) and to the same problem set in different domains (see Section 1.3).

In what follows we set $r \equiv |x|$. We assume (3.1.2), (H_1), (3.2.6) and (3.2.7). We want to construct a suitable family of supersolutions of equation

$$u_t = \frac{1}{\rho(x)} \Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0,+\infty). \tag{3.3.25}$$

In order to do this, we define, for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$\bar u(x,t) \equiv \bar u(r(x),t) := C\zeta(t)\,(\log(r + r_0))^{-\frac{b}{m}}\,; \tag{3.3.26}$$

where $\zeta \in C^1([0,+\infty);[0,+\infty))$, $C>0$ and $r_0>e$ such that (3.2.8) is verified.

Now, we compute

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p .$$

For any $(x,t) \in \left[\mathbb{R}^N \setminus \{0\}\right] \times (0,+\infty)$, we have:

$$\bar{u}_t = C\,\zeta'\,(\log(r+r_0))^{-\frac{b}{m}} \,. \tag{3.3.27}$$

$$(\bar{u}^m)_r = -\,b\,C^m\,\zeta^m\,\frac{(\log(r+r_0))^{-b-1}}{r+r_0} \,. \tag{3.3.28}$$

$$(\bar{u}^m)_{rr} = b\,C^m\,\zeta^m \left\{(b+1)\frac{(\log(r+r_0))^{-b-2}}{(r+r_0)^2} + \frac{(\log(r+r_0))^{-b-1}}{(r+r_0)^2}\right\} . \tag{3.3.29}$$

Proposition 3.3.1. *Let* $\zeta \in C^1([0,+\infty);[0,+\infty)), \zeta' \geq 0$. *Assume* (3.1.2), (H_1), (3.2.6), (3.2.7), (3.2.8), (3.2.9), (3.2.10) *and that*

$$k_0 b(N-2-\varepsilon(b+1))C^m\zeta^m - \bar{c}\,C^p\zeta^p \geq 0. \tag{3.3.30}$$

Then $\bar{u}$ *defined in* (3.3.26) *is a supersolution of equation* (3.3.25).

Proof of Proposition 3.3.1. In view of (3.3.27), (3.3.28), (3.3.29), (3.2.7) and (3.2.8), for any $(x,t) \in (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty)$,

$$\begin{aligned} &\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \\ &\geq C\zeta'\,(\log(r+r_0))^{-\frac{b}{m}} + \frac{1}{\rho}\left\{N-2-\varepsilon(b+1)\right\}C^m\zeta^m b\frac{(\log(r+r_0))^{-b-1}}{(r+r_0)^2} \\ &- C^p\zeta^p\,(\log(r+r_0))^{-\frac{bp}{m}} \,. \end{aligned} \tag{3.3.31}$$

Thanks to hypotheses (3.2.6), (3.2.9) and (3.2.10), we have

$$\begin{aligned} &\frac{1}{\rho}\frac{(\log(r+r_0))^{-\bar{b}-1}}{(r+r_0)^2} \geq k_0\,\frac{(\log(r+r_0))^{\alpha-\bar{b}-1}}{(r+r_0)^2}(r+r_0)^2 \geq k_0\,, \\ &-(\log(r+r_0))^{-\frac{bp}{m}} \geq -\bar{c}\,. \end{aligned} \tag{3.3.32}$$

Since $\zeta' \geq 0$, from (3.3.32) we get

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq k_0\,b(N-2-\varepsilon(b+1))C^m\zeta^m - \bar{c}\,C^p\zeta^p\,. \tag{3.3.33}$$

Hence (3.3.33) is nonnegative if

$$k_0\,b(N-2-\varepsilon(b+1))C^m\zeta^m - \bar{c}\,C^p\zeta^p \geq 0\,, \tag{3.3.34}$$

which is guaranteed by (3.2.7) and (3.3.30). So, we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in} \;\; (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty)\,.$$

Now observe that

$$\bar{u} \in C(\mathbb{R}^N \times [0, +\infty)),$$
$$\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0, +\infty)),$$
$$\bar{u}_r^m(0, t) \leq 0.$$

Hence, thanks to a Kato-type inequality we can infer that $\bar{u}$ is a supersolution to equation (3.3.25) in the sense of Definition 1.3.9.

□

Remark 3.3.2. *Let assumption (H_1) be satisfied. In Theorem 3.2.1 the precise hypotheses on parameters β, $C > 0$, $T > 0$ are as follows.*

(a) Let $p < m$. We require that

$$\beta > 0, \tag{3.3.35}$$

$$k_0\, b(N - 2 - \varepsilon(b + 1))C^m - \bar{c}\, C^p \geq 0. \tag{3.3.36}$$

(b) Let $p > m$. We require that

$$\beta = 0, \tag{3.3.37}$$

$$k_0\, b(N - 2 - \varepsilon(b + 1))C^m - \bar{c}\, C^p \geq 0. \tag{3.3.38}$$

Lemma 3.3.3. *All the conditions in Remark 3.3.2 can hold simultaneously.*

Proof. (a) We observe that, due to (3.2.7),

$$N - 2 - \varepsilon(b + 1) > 0.$$

Therefore, we can select $C > 0$ sufficiently large to guarantee (3.3.36).
(b) We choose $C > 0$ sufficiently small to guarantee (3.3.38). □

Proof of Theorem 3.2.1. We now prove Theorem 3.2.1 in view of Proposition 3.3.1. In view of Lemma 3.3.3 we can assume that all conditions in Remark 3.3.2 are fulfilled. Set

$$\zeta(t) = (T + t)^{\beta}, \quad \text{for all} \quad t \geq 0.$$

Let $p < m$. Inequality (3.3.30) reads

$$k_0\, b(N - 2 - \varepsilon(b + 1))C^m(T + t)^{m\beta} - \bar{c}\, C^p(T + t)^{p\beta} \geq 0 \quad \text{for all} \quad t > 0.$$

This follows from (3.3.35) and (3.3.36), for $T > 1$. Hence, by Propositions 3.3.1 and 1.3.5 the thesis follows in this case.

Let $p > m$. Conditions (3.3.37) and (3.3.38) are equivalent to (3.3.30). Hence, by Propositions 3.3.1 and 1.3.5 the thesis follows in this case too. The proof is complete. □

3.4 Proof of Theorem 3.2.2

We construct a suitable family of subsolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0,T). \tag{3.4.39}$$

We assume (3.1.2) and (H_2). Let

$$\underline{w}(x,t) \equiv \underline{w}(r(x),t) := \begin{cases} \underline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_e(0)] \times [0,T), \\ \underline{v}(x,t) & \text{in } B_e(0) \times [0,T), \end{cases} \tag{3.4.40}$$

where

$$\underline{u}(x,t) \equiv \underline{u}(r(x),t) := C\zeta(t)\left[1 - \frac{(\log r)^{\underline{b}}}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \tag{3.4.41}$$

and

$$\underline{v}(x,t) \equiv \underline{v}(r(x),t) := C\zeta(t)\left[1 - \left(\frac{\underline{b}r^2}{2e^2} + 1 - \frac{\underline{b}}{2}\right)\frac{\eta}{a}\right]_+^{\frac{1}{m-1}}. \tag{3.4.42}$$

Let

$$F(r,t) := 1 - \frac{(\log r)^{\underline{b}}}{a}\eta(t)\,,$$

and

$$G(r,t) := 1 - \left(\frac{\underline{b}r^2}{2e^2} + 1 - \frac{\underline{b}}{2}\right)\frac{\eta}{a}\,.$$

Observe that for any $(x,t) \in [\mathbb{R}^N \setminus B_e(0)] \times (0,T)$, we have:

$$\underline{u}_t = C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}. \tag{3.4.43}$$

$$(\underline{u}^m)_r = -\underline{b}\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{(\log r)^{\underline{b}-1}}{r}\eta; \tag{3.4.44}$$

$$\begin{aligned}
(\underline{u}^m)_{rr} = &-\underline{b}\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}\eta\left\{F^{\frac{1}{m-1}}\left[(\underline{b}-1)\frac{(\log r)^{\underline{b}-2}}{r^2} - \frac{(\log r)^{\underline{b}-1}}{r^2}\right]\right.\\
&+\frac{\underline{b}}{m-1}\frac{(\log r)^{\underline{b}-2}}{r^2}\left(1-(\log r)^{\underline{b}}\,\frac{\eta}{a}\right)F^{\frac{1}{m-1}-1}\\
&\left.-\frac{\underline{b}}{m-1}\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}-1}\right\}\\
=&-\underline{b}^2\frac{C^m}{a}\left(\frac{m}{m-1}\right)^2\zeta^m\eta\frac{(\log r)^{\bar{b}-2}}{r^2}F^{\frac{1}{m-1}}\\
&+\underline{b}\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\frac{(\log r)^{\bar{b}-2}}{r^2}F^{\frac{1}{m-1}}\\
&+\underline{b}\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\frac{(\log r)^{\underline{b}-1}}{r^2}F^{\frac{1}{m-1}}\\
&+\underline{b}^2\frac{C^m}{a}\frac{m}{(m-1)^2}\zeta^m\eta\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}-1}\,.
\end{aligned} \tag{3.4.45}$$

$$\begin{aligned}\Delta(\underline{u}^m) = &\frac{C^m}{a}\zeta^m\eta\,\frac{m}{(m-1)^2}\,\underline{b}^2\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}-1}\\ &-\frac{C^m}{a}\zeta^m\eta\left(\frac{m}{m-1}\right)^2\underline{b}^2\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}}\\ &+\frac{C^m}{a}\zeta^m\eta\,\frac{m}{m-1}\,\underline{b}\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}}\\ &-\frac{C^m}{a}\zeta^m\eta\,\frac{m}{m-1}\,\underline{b}\,\frac{(\log r)^{\underline{b}-1}}{r^2}F^{\frac{1}{m-1}}(N-2)\end{aligned} \tag{3.4.46}$$

Observe that for any $(x,t)\in B_e(0)\times(0,T)$, we have:

$$\underline{v}_t = C\zeta' G^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}G^{\frac{1}{m-1}-1}, \tag{3.4.47}$$

$$(\underline{v}^m)_r = -\frac{C^m}{a}\zeta^m\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{br}{e^2}\eta\,, \tag{3.4.48}$$

$$(\underline{v}^m)_{rr} = -\frac{C^m}{a}\zeta^m\frac{m}{m-1}\frac{b}{e^2}\eta\left[G^{\frac{1}{m-1}} - \frac{r}{m-1}G^{\frac{1}{m-1}-1}\frac{\eta}{a}\frac{br}{e^2}\right]. \tag{3.4.49}$$

$$\begin{aligned}\Delta(\underline{v}^m) &= -\frac{C^m}{a}\zeta^m\frac{m}{m-1}\frac{b}{e^2}\eta G^{\frac{1}{m-1}} + \frac{C^m}{a^2}\zeta^m\frac{m}{(m-1)^2}\frac{\underline{b}^2\,r^2}{e^4}\eta^2 G^{\frac{1}{m-1}-1}\\ &\quad-(N-1)\frac{C^m}{a}\zeta^m\frac{m}{m-1}\frac{b}{e^2}\eta G^{\frac{1}{m-1}}\\ &= \frac{C^m}{a^2}\zeta^m\frac{m}{(m-1)^2}\frac{\underline{b}^2\,r^2}{e^4}\eta^2 G^{\frac{1}{m-1}-1} - N\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}\frac{b}{e^2}\eta G^{\frac{1}{m-1}}\end{aligned} \tag{3.4.50}$$

We also define

$$\begin{aligned}&\underline{\sigma}(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\,k_2\left(\underline{b}\frac{m}{m-1}+N-2\right),\\ &\underline{\delta}(t) := \frac{\zeta}{m-1}\frac{\eta'}{\eta}\\ &\underline{\gamma}(t) := C^{p-1}\zeta^p,\\ &\underline{\sigma}_0(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \rho_2\,N\,\frac{\underline{b}}{e^2}\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\,,\\ &K := \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0.\end{aligned} \tag{3.4.51}$$

Proposition 3.4.1. *Let $T\in(0,\infty)$, $\zeta,\ \eta\in C^1([0,T);[0,+\infty))$. Let $\underline{\sigma},\underline{\delta},\underline{\gamma},\underline{\sigma}_0,K$ be defined in* (3.4.51)*. Assume that, for all $t\in(0,T)$,*

$$\underline{\sigma}(t)>0,\quad K[\underline{\sigma}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}\,, \tag{3.4.52}$$

$$(m-1)\underline{\sigma}(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{3.4.53}$$

$$\underline{\sigma}_0(t)>0,\quad K[\underline{\sigma}_0](t)^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}, \tag{3.4.54}$$

$$(m-1)\underline{\sigma}_0(t) \le (p+m-2)\underline{\gamma}(t)\,. \tag{3.4.55}$$

Then $\underline{w}$ defined in (3.4.40) *is a subsolution of equation* (3.4.39).

Proof of Proposition 3.4.1. In view of (3.4.43), (3.4.44), (3.4.45) and (3.4.46) we obtain

$$
\begin{aligned}
&\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\
&= C\zeta' F^{\frac{1}{m-1}} + C\frac{\zeta}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\frac{\zeta}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} \\
&- \frac{1}{\rho}\left\{\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}\underline{b}^2\,\eta\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}-1} + \frac{C^m}{a}\zeta^m\left(\frac{m}{m-1}\right)^2\underline{b}\,\eta\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}}\right. \\
&\left. -\frac{C^m}{a}\zeta^m\frac{m}{m-1}\underline{b}\,\eta\,\frac{(\log r)^{\underline{b}-2}}{r^2}F^{\frac{1}{m-1}} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}\underline{b}\,\eta\,\frac{(\log r)^{\underline{b}-1}}{r^2}F^{\frac{1}{m-1}}\,(N-2)\right\} \\
&- C^p\zeta^p F^{\frac{p}{m-1}}, \quad \text{for all } (x,t)\in D_1\,.
\end{aligned}
\tag{3.4.56}
$$

In view of (H_2) and (3.2.14), we can infer that

$$
-\frac{1}{\rho}\frac{(\log r)^{\underline{b}-2}}{r^2} \le -\frac{k_1}{\log r} \le -k_1, \quad \text{for all } x\in\mathbb{R}^N\setminus B_e(0)\,, \tag{3.4.57}
$$

$$
\frac{1}{\rho}\frac{(\log r)^{\underline{b}-2}}{r^2} \le \frac{k_2}{\log r} \le k_2, \quad \text{for all } x\in\mathbb{R}^N\setminus B_e(0)\,, \tag{3.4.58}
$$

$$
\frac{1}{\rho}\frac{(\log r)^{\underline{b}-1}}{r^2} \le k_2, \quad \text{for all } x\in\mathbb{R}^N\setminus B_e(0)\,. \tag{3.4.59}
$$

From (3.4.56), (3.4.57), (3.4.58) and (3.4.59) we have

$$
\begin{aligned}
&\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\
&\le CF^{\frac{1}{m-1}-1}\left\{F\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\,\underline{b}\,\eta k_2\left(N-2+\underline{b}\frac{m}{m-1}\right)\right]\right. \\
&\left. -\frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}.
\end{aligned}
\tag{3.4.60}
$$

Thanks to (3.4.51), (3.4.60) becomes

$$
\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \le CF^{\frac{1}{m-1}-1}\varphi(F), \tag{3.4.61}
$$

where, for each $t\in(0,T)$,

$$
\varphi(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{p+m-2}{m-1}}.
$$

Our goal is to find suitable C, a, ζ, η such that, for each $t\in(0,T)$,

$$
\varphi(F)\le 0 \quad \text{for any } F\in(0,1)\,.
$$

To this aim, we impose that

$$
\sup_{F\in(0,1)}\varphi(F) = \max_{F\in(0,1)}\varphi(F) = \varphi(F_0)\le 0\,,
$$

for some $F_0 \in (0,1)$. We have

$$\frac{d\varphi}{dF} = 0 \iff \underline{\sigma}(t) - \frac{p+m-2}{m-1}\underline{\gamma}(t)F^{\frac{p-1}{m-1}} = 0$$
$$\iff F = F_0 = \left[\frac{m-1}{p+m-2}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)}\right]^{\frac{m-1}{p-1}}.$$

Then

$$\varphi(F_0) = K\,\frac{\underline{\sigma}(t)^{\frac{p+m-2}{p-1}}}{\underline{\gamma}(t)^{\frac{m-1}{p-1}}} - \underline{\delta}(t)\,,$$

where the coefficient K depending on m and p has been defined in (3.4.51). By (3.4.52) and (3.4.53), for each $t \in (0,T)$,

$$\varphi(F_0) \le 0\,, \quad F_0 \le 1\,. \tag{3.4.62}$$

So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p \le 0 \quad \text{in } D_1. \tag{3.4.63}$$

Furthermore, since $\underline{u}^m \in C^1([\mathbb{R}^N \setminus B_e(0)] \times (0,T))$, due to Lemma 1.3.10 (applied with $\Omega_1 = D_1, \Omega_2 = \mathbb{R}^N \setminus [B_e(0) \cup D_1], u_1 = \underline{u}, u_2 = 0, u = \underline{u}$), it follows that $\underline{u}$ is a subsolution to equation

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p = 0 \quad \text{in } [\mathbb{R}^N \setminus B_e(0)] \times (0,T),$$

in the sense of Definition 1.3.9.

Let

$$D_2 := \{(x,t) \in B_e(0) \times (0,T) \,:\, 0 < G(r,t) < 1\}\,.$$

Using (3.2.13), (3.4.39) yields, for all $(x,t) \in D_2$,

$$\begin{aligned} &\underline{v}_t - \frac{1}{\rho}\Delta(\underline{v}^m) - \underline{v}^p \\ &\le CG^{\frac{1}{m-1}-1}\Big\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + N\,\rho_2\,\frac{b}{e^2}\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\right] \\ &\quad - \frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\Big\} \\ &= CG^{\frac{1}{m-1}-1}\left[\underline{\sigma}_0(t)G - \underline{\delta}(t) - \underline{\gamma}(t)G^{\frac{p+m-2}{m-1}}\right]. \end{aligned} \tag{3.4.64}$$

Now, by the same arguments used to obtain (3.4.63), in view of (3.4.55) and (3.4.56) we can infer that

$$\underline{v}_t - \frac{1}{\rho}\Delta\underline{v}^m \le \underline{v}^p \quad \text{for any } (x,t) \in D_2\,. \tag{3.4.65}$$

Moreover, since $\underline{v}^m \in C^1(B_e(0) \times (0,T))$, in view of Lemma 1.3.10 (applied with $\Omega_1 = D_2, \Omega_2 = B_e(0) \setminus D_2, u_1 = \underline{v}, u_2 = 0, u = \underline{v}$), we get that $\underline{v}$ is a subsolution to equation

$$\underline{v}_t - \frac{1}{\rho}\Delta\underline{v}^m = \underline{v}^p \quad \text{in } B_e(0) \times (0,T)\,, \tag{3.4.66}$$

in the sense of Definition 1.3.9. Now, observe that $\underline{w} \in C(\mathbb{R}^N \times [0,T))$; indeed,

$$\underline{u} = \underline{v} = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e(0) \times (0,T)\,.$$

Moreover, $\underline{w}^m \in C^1(\mathbb{R}^N \times [0,T))$; indeed,

$$(\underline{u}^m)_r = (\underline{v}^m)_r = -C^m\zeta(t)^m \frac{m}{m-1}\frac{\eta(t)}{a}\frac{\underline{b}}{e}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e(0) \times (0,T)\,. \tag{3.4.67}$$

In conclusion, in view of (3.4.67) and Lemma 1.3.10 (applied with $\Omega_1 = B_e(0), \Omega_2 = \mathbb{R}^N \setminus B_e(0), u_1 = \underline{v}, u_2 = \underline{u}, u = \underline{w}$), we can infer that $\underline{w}$ is a subsolution to equation (3.4.39), in the sense of Definition 1.3.9. □

Remark 3.4.2. *Let*

$$p > m\,,$$

and assumptions (H_2) *and* (3.2.13) *be satisfied. Let define* $\omega := \frac{C^{m-1}}{a}$. *In Theorem 3.2.2, the precise hypotheses on parameters* $C > 0$, $a > 0$, $\omega > 0$ *and* $T > 0$ *are the following.*

$$\max\left\{1 + m\,k_2\,\underline{b}\,\frac{C^{m-1}}{a}\left(N - 2 + \underline{b}\frac{m}{m-1}\right); 1 + m\rho_2\frac{C^{m-1}}{a}\,\underline{b}\,\frac{N}{e^2}\right\} \le (p+m-2)C^{p-1}\,, \tag{3.4.68}$$

$$\begin{aligned}\frac{K}{(m-1)^{\frac{p+m-2}{p-1}}}\max\Bigg\{&\left[1 + m\,k_2\underline{b}\,\frac{C^{m-1}}{a}\left(N - 2 + \underline{b}\frac{m}{m-1}\right)\right]^{\frac{p+m-2}{p-1}};\\ &\left(1 + m\,\rho_2\,\frac{C^{m-1}}{a}\,\underline{b}\,\frac{N}{e^2}\right)^{\frac{p+m-2}{p-1}}\Bigg\} \le \frac{p-m}{(m-1)(p-1)}C^{m-1}\,.\end{aligned} \tag{3.4.69}$$

Lemma 3.4.3. *All the conditions in Remark 3.4.2 can hold simultaneously.*

Proof. We can take $\omega > 0$ such that

$$\omega_0 \le \omega \le \omega_1$$

for suitable $0 < \omega_0 < \omega_1$ and we can choose $C > 0$ sufficiently large to guarantee (3.4.68) and (3.4.69) (so, $a > 0$ is fixed, too). □

Proof of Theorem 3.2.2. We now prove Theorem 3.2.2, by means of Proposition 3.4.1. In view of Lemma 3.4.3 we can assume that all conditions of Remark 3.4.2 are fulfilled. Set

$$\zeta = (T-t)^{-\beta}\,, \quad \eta = (T-t)^{\lambda}\,, \quad \text{for all} \quad t > 0\,,$$

$$\beta = \frac{1}{p-1}\,, \qquad \lambda = \frac{m-p}{p-1}\,.$$

Then

$$\begin{aligned}
&\underline{\sigma}(t) := \left[\frac{1}{m-1} + \frac{C^{m-1}}{a}\frac{m}{m-1}\underline{b}\,k_2\left(\underline{b}\,\frac{m}{m-1}+N-2\right)\right](T-t)^{-\frac{p}{p-1}}\,,\\
&\underline{\delta}(t) := \frac{p-m}{(m-1)(p-1)}(T-t)^{-\frac{p}{p-1}}\,,\\
&\underline{\gamma}(t) := C^{p-1}(T-t)^{-\frac{p}{p-1}}\,,\\
&\underline{\sigma}_0(t) := \frac{1}{m-1}\left[1+\frac{\rho_2\,N\,m\,\underline{b}}{e^2}\,\frac{C^{m-1}}{a}\right](T-t)^{-\frac{p}{p-1}}\,.
\end{aligned} \tag{3.4.70}$$

Let $p > m$. Condition (3.4.68) implies (3.4.53), (3.4.55), while condition (3.4.69) implies (3.4.52), (3.4.54). Hence by Propositions 3.4.1 and 1.3.7 the thesis follows. □

3.5 Proof of Theorem 3.2.3

We assume (3.1.2), (3.2.19) and (3.2.20). In order to construct a suitable family of supersolutions of (3.3.25), we define, for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$\bar{u}(x,t) \equiv \bar{u}(r(x),t) := C\zeta(t)\left[1 - \frac{(\log(r+r_0))^{\bar{b}}}{a}\eta(t)\right]_+^{\frac{1}{m-1}}\,, \tag{3.5.71}$$

where $\eta,\,\zeta \in C^1([0,+\infty);[0,+\infty))$, $C>0$, $a>0$, $r_0>e$ and $\bar{b}$ as in (3.2.21).

Now, we compute

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p.$$

To this aim, set

$$F(r,t) := 1 - \frac{(\log(r+r_0))^{\bar{b}}}{a}\,\eta(t)\,,$$

and

$$D_1 := \left\{(x,t) \in [\mathbb{R}^N \setminus \{0\}] \times (0,+\infty) \mid 0 < F(r,t) < 1\right\}.$$

For any $(x,t) \in D_1$, we have:

$$\begin{aligned}
\bar{u}_t &= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}F^{\frac{1}{m-1}-1}\left(-\frac{(\log(r+r_0))^{\bar{b}}}{a}\eta'\right)\\
&= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}.
\end{aligned} \tag{3.5.72}$$

$$(\bar{u}^m)_r = -\bar{b}\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{(\log(r+r_0))^{\bar{b}-1}}{(r+r_0)}\eta. \tag{3.5.73}$$

$$
\begin{aligned}
(\bar u^m)_{rr} = & -\bar b\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}\eta\left\{F^{\frac{1}{m-1}}\left[(\bar b-1)\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}-\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2}\right]\right.\\
& +\frac{\bar b}{m-1}\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}\left(1-(\log(r+r_0))^{\bar b}\,\frac{\eta}{a}\right)F^{\frac{1}{m-1}-1}\\
& \left.-\frac{\bar b}{m-1}\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}-1}\right\}\\
& = -\bar b\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\left[\bar b\frac{m}{m-1}-1\right]\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\
& +\bar b\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\
& +\bar b^2\frac{C^m}{a}\frac{m}{(m-1)^2}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}-1}\,.
\end{aligned}
\tag{3.5.74}
$$

$$
\begin{aligned}
\Delta(\bar u^m) = & \frac{(N-1)}{r}(\bar u^m)_r+(\bar u^m)_{rr}\\
= & \frac{(N-1)}{r}\left(-\bar b\,\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)}\eta\right)\\
& -\bar b\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\left[\bar b\frac{m}{m-1}-1\right]\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\
& +\bar b\frac{C^m}{a}\frac{m}{m-1}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2}F^{\frac{1}{m-1}}\\
& +\bar b^2\frac{C^m}{a}\frac{m}{(m-1)^2}\zeta^m\eta\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2}F^{\frac{1}{m-1}-1}\,.
\end{aligned}
\tag{3.5.75}
$$

We also define

$$
\begin{aligned}
\bar\sigma(t) &:= \zeta'+\frac{\zeta}{m-1}\frac{\eta'}{\eta}+\bar b\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta k_1\left(\bar b\frac{m}{m-1}+N-3\right),\\
\bar\delta(t) &:= \frac{\zeta}{m-1}\frac{\eta'}{\eta}+\bar b^2\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{(m-1)^2}\eta k_2\,,\\
\bar\gamma(t) &:= C^{p-1}\zeta^p\,.
\end{aligned}
\tag{3.5.76}
$$

Proposition 3.5.1. *Let* $\zeta,\eta\in C^1([0,+\infty);[0,+\infty))$. *Let* $\bar\sigma$, $\bar\delta$, $\bar\gamma$ *be as defined in* (3.5.76). *Assume* (H_2), (3.2.19), (3.2.20), (3.2.21) *and that, for all* $t\in(0,+\infty)$,

$$
-\frac{\eta'}{\eta^2}\geq \bar b^2\,\frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}k_2,
\tag{3.5.77}
$$

and

$$
\zeta'+\bar b\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[\left(\bar b\frac{m}{m-1}+N-3\right)k_1-\frac{\bar b}{(m-1)}k_2\right]-C^{p-1}\zeta^p\geq 0\,.
\tag{3.5.78}
$$

Then $\bar u$ *defined in* (3.5.71) *is a supersolution of equation* (3.3.25).

Proof of Proposition 3.5.1. In view of (3.5.72), (3.5.73), (3.5.74) and (3.5.75), for any $(x,t) \in D_1$,

$$
\begin{aligned}
\bar u_t - &\frac{1}{\rho}\Delta(\bar u^m) - \bar u^p \\
&\geq C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}} - C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}-1} \\
&\quad + \frac{1}{\rho}(N-2)\,\bar b\,\frac{C^m}{a}\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}} \frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2}\eta \\
&\quad + \frac{1}{\rho}\,\bar b\,\frac{C^m}{a}\frac{m}{m-1}\zeta^m \eta \left[\bar b \frac{m}{m-1} - 1\right] \frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2} F^{\frac{1}{m-1}} \\
&\quad - \frac{1}{\rho}\bar b^2 \frac{C^m}{a}\frac{m}{(m-1)^2}\zeta^m \eta \frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2} F^{\frac{1}{m-1}-1} - C^p \zeta^p F^{\frac{p}{m-1}},
\end{aligned}
\tag{3.5.79}
$$

where we have used the inequality

$$\frac{1}{r(r+r_0)} \geq \frac{1}{(r+r_0)^2}\,.$$

Thanks to (3.2.19) and (3.2.21), we have

$$\frac{1}{\rho}\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2} \geq k_1 \quad \text{for all} \quad x \in \mathbb{R}^N\,, \tag{3.5.80}$$

$$-\frac{1}{\rho}\frac{(\log(r+r_0))^{\bar b-2}}{(r+r_0)^2} \geq -k_2 \quad \text{for all} \quad x \in \mathbb{R}^N\,, \tag{3.5.81}$$

$$\frac{1}{\rho}\frac{(\log(r+r_0))^{\bar b-1}}{(r+r_0)^2} \geq k_1 \log(r+r_0) \geq k_1 \quad \text{for all} \quad x \in \mathbb{R}^N\,. \tag{3.5.82}$$

From (3.5.80), (3.5.81) and (3.5.82) we get

$$
\begin{aligned}
\bar u_t - &\frac{1}{\rho}\Delta(\bar u^m) - \bar u^p \\
&\geq C F^{\frac{1}{m-1}-1}\left\{ F\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \bar b\,\frac{C^{m-1}}{a}\zeta^m \frac{m}{m-1}\eta\, k_1 \left(\bar b \frac{m}{m-1} + N - 3\right)\right] \right. \\
&\quad \left. - \frac{\zeta}{m-1}\frac{\eta'}{\eta} - \bar b^2\,\frac{C^{m-1}}{a}\zeta^m \frac{m}{(m-1)^2}\eta\, k_2 - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}} \right\}
\end{aligned}
\tag{3.5.83}
$$

From (3.5.83) and (3.5.76), we have

$$\bar u_t - \frac{1}{\rho}\Delta(\bar u^m) - \bar u^p \geq C F^{\frac{1}{m-1}-1}\left[\bar\sigma(t) F - \bar\delta(t) - \bar\gamma(t) F^{\frac{p+m-2}{m-1}}\right]\,. \tag{3.5.84}$$

For each $t > 0$, set

$$\varphi(F) := \bar\sigma(t) F - \bar\delta(t) - \bar\gamma(t) F^{\frac{p+m-2}{m-1}}, \quad F \in (0,1)\,.$$

Now our goal is to find suitable C, a, ζ, η such that, for each $t > 0$,

$$\varphi(F) \geq 0 \quad \text{for any } \; F \in (0,1)\,.$$

We observe that $\varphi(F)$ is concave in the variable F. Hence it is sufficient to have that $\varphi(F)$ is positive at the extrema of the interval $(0, 1)$. This reduces, for any $t > 0$, to the conditions

$$\begin{aligned} &\varphi(0) \geq 0\,, \\ &\varphi(1) \geq 0\,. \end{aligned} \tag{3.5.85}$$

These are equivalent to

$$-\bar{\delta}(t) \geq 0\,, \quad \bar{\sigma}(t) - \bar{\delta}(t) - \bar{\gamma}(t) \geq 0\,,$$

that is

$$\begin{aligned} &-\frac{\eta'}{\eta^2} \geq \bar{b}^2\,\frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}k_2\,, \\ &\zeta' + \bar{b}\,\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[\left(\bar{b}\frac{m}{m-1} + N - 3\right)k_1 - \frac{\bar{b}}{(m-1)}k_2\right] - C^{p-1}\zeta^p \geq 0\,. \end{aligned}$$

which are guaranteed by (3.2.20), (3.5.77) and (3.5.78). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in } D_1\,.$$

Now observe that

$$\begin{aligned} &\bar{u} \in C(\mathbb{R}^N \times [0, +\infty))\,, \\ &\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0, +\infty))\,, \text{ and by the definition of } \bar{u}\,, \\ &\bar{u} \equiv 0 \text{ in } [\mathbb{R}^N \setminus D_1] \times [0, +\infty))\,. \end{aligned}$$

Hence, by Lemma 1.3.10 (applied with $\Omega_1 = D_1$, $\Omega_2 = \mathbb{R}^N \setminus D_1$, $u_1 = \bar{u}$, $u_2 = 0$, $u = \bar{u}$), $\bar{u}$ is a supersolution of equation

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p = 0 \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times (0, +\infty)$$

in the sense of Definition 1.3.9. Thanks to a Kato-type inequality, since $\bar{u}^m_r(0, t) \leq 0$, we can easily infer that $\bar{u}$ is a supersolution of equation (3.3.25) in the sense of Definition 1.3.9. □

Remark 3.5.2. *Let*

$$p > m$$

and assumption (3.2.20) *be satisfied. Let* $\omega := \frac{C^{m-1}}{a}$. *In Theorem 3.2.3 the precise hypotheses on parameters* $C > 0$, $\omega > 0$, $T > 0$ *are the following:*

$$\frac{p-m}{p-1} \geq \bar{b}^2\,\omega\frac{m}{m-1}k_2, \tag{3.5.86}$$

$$\bar{b}\,\omega\frac{m}{m-1}\left[k_1\left(\bar{b}\frac{m}{m-1} + N - 3\right) - \frac{k_2}{(m-1)}\bar{b}\right] \geq C^{p-1} + \frac{1}{p-1}\,. \tag{3.5.87}$$

Lemma 3.5.3. *All the conditions in Remark 3.5.2 can be satisfied simultaneously.*

Proof. Since $p > m$ the left-hand-side of (3.5.86) is positive. By (3.2.20), we can select $\omega > 0$ so that (3.5.86) holds and

$$\bar{b}\,\omega\frac{m}{m-1}\left[k_1\left(\bar{b}\frac{m}{m-1}+N-3\right)-\frac{k_2}{(m-1)}\,\bar{b}\right]\geq\frac{1}{p-1}\,.$$

Then we take $C > 0$ so small that (3.5.87) holds (and so $a > 0$ is accordingly fixed). □

Proof of Theorem 3.2.3. In view of Lemma 3.5.3, we can assume that all the conditions in Remark 3.5.2 are fulfilled. Set

$$\zeta(t)=(T+t)^{-\frac{1}{p-1}},\quad \text{for all}\quad t\geq 0\,,$$

and

$$\eta(t)=(T+t)^{-\frac{p-m}{p-1}},\quad \text{for all}\quad t\geq 0\,.$$

Let $p > m$. Consider conditions (3.5.77) and (3.5.78) with this choice of ζ and η. They read

$$\frac{p-m}{p-1}\geq\bar{b}^2\frac{C^{m-1}}{a}\frac{m}{m-1}k_2,$$

$$-\frac{1}{p-1}+\bar{b}\,\frac{C^{m-1}}{a}\frac{m}{m-1}\left[\left(\bar{b}\frac{m}{m-1}+N-3\right)k_1-\frac{\bar{b}}{(m-1)}k_2\right]-C^{p-1}\geq 0\,.$$

Therefore, (3.5.77) and (3.5.78) follow from assumptions (3.5.86) and (3.5.87). Hence, by Propositions 3.5.1 and 1.3.5 the thesis follows. □

Part II

The porous medium equation with reaction on Riemannian manifolds

Chapter 4

Global existence and smoothing estimates for $p > m$

4.1 Introduction

We investigate existence of global in time solutions to nonlinear reaction-diffusion problems of the following type:

$$\begin{cases} u_t = \Delta u^m + u^p & \text{in } M \times (0,T) \\ u = u_0 & \text{in } M \times \{0\}, \end{cases} \tag{4.1.1}$$

where M is an $N-$dimensional complete noncompact Riemannian manifold of infinite volume, Δ being the Laplace-Beltrami operator on M and $T \in (0, \infty]$. We shall assume throughout this chapter that

$$N \geq 3, \qquad m > 1, \qquad p > m,$$

so that we are concerned with the case of *degenerate diffusions* of porous medium type (see [128]), and that the initial datum u_0 is nonnegative.

Let $\mathrm{L}^q(M)$ be the space of those measurable functions f such that $|f|^q$ is integrable w.r.t. the Riemannian measure μ. We shall always assume that M supports the Sobolev inequality, namely that:

$$\text{(Sobolev inequality)} \qquad \|v\|_{L^{2^*}(M)} \leq \frac{1}{C_s}\|\nabla v\|_{L^2(M)} \quad \text{for any } v \in C_c^\infty(M), \tag{4.1.2}$$

where C_s is a positive constant and $2^* := \frac{2N}{N-2}$. In one of our main results, we shall also suppose that M supports the Poincaré inequality, namely that:

$$\text{(Poincaré inequality)} \qquad \|v\|_{L^2(M)} \leq \frac{1}{C_p}\|\nabla v\|_{L^2(M)} \quad \text{for any } v \in C_c^\infty(M), \tag{4.1.3}$$

for some $C_p > 0$. Observe that, for instance, (4.1.2) holds if M is a Cartan-Hadamard manifold, i.e. a simply connected Riemannian manifold with nonpositive sectional curvatures, while (4.1.3) is valid when M is a Cartan-Hadamard manifold satisfying the additional condition of having sectional curvatures bounded above by a constant $-c < 0$ (see, e.g., [39, 40]). Therefore, as is well known, in $\mathbb{R}^N$ (4.1.2) holds, but (4.1.3) fails, whereas on the hyperbolic space both (4.1.2) and (4.1.3) are fulfilled.

4.1.1 On some existing results

The behaviour of solutions to (4.1.1) is influenced by competing phenomena. First of all there is a diffusive pattern associated with the so-called *porous medium equation*, namely the equation

$$u_t = \Delta u^m \quad \text{in } M \times (0,T), \tag{4.1.4}$$

where the fact that we keep on assuming $m > 1$ puts us in the *slow diffusion case*. It is known that when $M = \mathbb{R}^n$ and, more generally, e.g. when M is a Cartan-Hadamard manifold, solutions corresponding to compactly supported data have compact support for all time, in contrast with the properties valid for solutions to the heat equation, see [128]. But it is also well-known that, qualitatively speaking, *negative curvature accelerates diffusions*, a fact that is apparent first of all from the behaviour of solutions of the classical heat equation. In fact, it can be shown that the standard deviation of a Brownian particle on the hyperbolic space $\mathbb{H}^n$ behaves *linearly* in time, whereas in the Euclidean situation it is proportional to $\sqrt{t}$. Similarly, the heat kernel decays exponentially as $t \to +\infty$ whereas one has a power-type decay in the Euclidean situation.

In the Riemannian setting the study of (4.1.4) has started recently, see e.g. [43], [47], [48], [52], [53], [55], [111], [129], noting that in some of those papers also the case $m < 1$ in (4.1.4), usually referred to as the *fast diffusion* case, is studied. Nonlinear diffusion gives rise to speedup phenomena as well. In fact, considering again the particularly important example of the hyperbolic space $\mathbb{H}^n$ (cf. [129], [48]), the L^∞ norm of a solution to (4.1.4) satisfies $\|u(t)\|_\infty \asymp \left(\frac{\log t}{t}\right)^{1/(m-1)}$ as $t \to +\infty$, a time decay which is *faster* than the corresponding Euclidean bound. Besides, if the initial datum is compactly supported, the volume $\mathsf{V}(t)$ of the support of the solution $u(t)$ satisfies $\mathsf{V}(t) \asymp t^{1/(m-1)}$ as $t \to +\infty$, while in the Euclidean situation one has $\mathsf{V}(t) \asymp t^{\beta(N,m)}$ with $\beta(N,m) < 1/(m-1)$.

The second driving factor influencing the behaviour of solutions to (4.1.1) is the *reaction term* u^p, which has the positive sign and, thus, might drive solutions towards blow-up. This kind of problems has been widely studied in the Euclidean case $M = \mathbb{R}^N$, especially in the case $m = 1$ (linear diffusion). The literature for this problem is huge and there is no hope to give a comprehensive review here, hence we just mention that blow-up occurs for all nontrivial nonnegative data when $p \leq 1 + 2/N$, while global existence prevails for $p > 1 + 2/N$ (for specific results see e.g. [16], [24], [30], [31], [58], [83], [114], [121], [134], [135]). On the other hand, it is known that when $M = \mathbb{H}^N$ and $m = 1$, for all $p > 1$ and sufficiently small nonnegative data there exists a global in time solution, see [9], [130], [131], [110].

As concerns the slow diffusion case $m > 1$, in the Euclidean setting it is shown in [119] that, when the initial datum is nonnegative, nontrivial and compactly supported, for any $p > 1$, all sufficiently large data give rise to solutions blowing up in finite time. Besides, if $p \in \left(1, m + \frac{2}{N}\right)$, *all* such solutions blow up in finite time. Finally, if $p > m + \frac{2}{N}$, all sufficiently small data give rise to global solutions. For subsequent, very detailed results e.g. about the type of possible blow-up and, in some case, on continuation after blow-up, see [36], [99], [126] and references quoted therein.

For any $x_0 \in M$, $r > 0$ let $B_r(x_0)$ be the geodesic ball centered in x_0 and radius r, let g_{ij} the metric tensor. In [137], problem (4.1.1) is studied when M is a manifold with a pole, $\mu(B_r(x_0)) \leq Cr^\alpha$ for some $\alpha > 2$ and $C > 0$. Under an additional smallness condition on curvature at infinity, if u_0 is sufficiently small and with compact support, then there exists a global solution to problem (4.1.1). Global existence is also proved, for some initial data u_0, under the assumption that M has nonnegative Ricci curvature and $p > \frac{\alpha}{\alpha-2}m$. It should be noticed that such result do not cover cases in which negative curvature either does not tend to zero at infinity, or does so not sufficiently fast, in particular the case of the hyperbolic space cannot be addressed.

The situation on negatively curved manifolds is significantly different, and the first results in this connection have been shown in [54], where only the case of nonnegative, compactly supported data is considered. Among the results of that paper, we mention the case that a *dichotomy phenomenon* holds when $p > m$, in the sense that under appropriate curvature conditions, compatible with the assumptions made in the present work, all sufficiently small data give rise to solutions existing globally in time, whereas sufficiently large data give rise to solutions blowing up in finite time. Results were only partial when $p < m$, since it has been shown that when $p \in \left(1, \frac{1+m}{2}\right]$ and again under suitable curvature conditions, all solutions corresponding to compactly supported initial data exist globally in time, and blow up everywhere pointwise in infinite time. When $p \in \left(\frac{1+m}{2}, m\right)$, precise information on the asymptotic behaviour is not known, since blowup is shown to occur at worse in infinite time, but could in principle occur before.

4.1.2 Qualitative statements of our new results in the Riemannian setting

Our results concerning problem (4.1.1) can be summarized as follows.

- (See Theorem 4.2.2) We prove global existence of solutions to (4.1.1), assuming that the initial datum is sufficiently small, that

$$p > m + \frac{2}{N},$$

and that the Sobolev inequality (4.1.2) holds; moreover, smoothing effects and the fact that suitable L^q norms of solutions decrease in time are obtained. To be specific, any sufficiently small initial datum $u_0 \in L^m(M) \cap L^{(p-m)\frac{N}{2}}(M)$ gives rise to a global solution $u(t)$ such that $u(t) \in L^\infty(M)$ for all $t > 0$ with a quantitative bound on the L^∞ norm of the solution.

- (See Theorem 4.2.5) We show that, if both the Sobolev and the Poincaré inequality (i.e. (4.1.2), (4.1.3)) hold, then for any

$$p > m,$$

for any sufficiently small initial datum u_0, belonging to suitable Lebesgue spaces, there exists a global solution $u(t)$ such that $u(t) \in L^\infty(M)$. Furthermore, a quantitative bound for the L^∞ norm of the solution is satisfied for all $t > 0$.

Note that in Theorem 4.2.2 we only assume the Sobolev inequality and we require that $p > m + \frac{2}{N}$, instead in Theorem 4.2.5 we can relax the assumption on the exponent p, indeed we assume $p > m$, but we need to further require that the Poincaré inequality holds. Moreover, in the two theorems, the hypotheses on the initial data are different.

The main results given in Theorems 4.2.2 and 4.2.5 depend essentially only on the validity of inequalities (4.1.2) and (4.1.3), are functional analytic in character and hence can be generalized to different contexts.

4.1.3 The case of Euclidean, weighted diffusion

As a particularly significant setting, we single out the case of Euclidean, mass-weighted reaction diffusion equations, that has been the object of intense research. In fact we consider the problem

$$\begin{cases} \rho\, u_t = \Delta u^m + \rho\, u^p & \text{in } \mathbb{R}^N \times (0,T) \\ u \;= u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{4.1.5}$$

where $\rho : \mathbb{R}^N \to \mathbb{R}$ is strictly positive, continuous and bounded, and represents a *mass variable density* . The problem is naturally posed in the weighted spaces

$$L^q_\rho(\mathbb{R}^N) = \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable }, \; \|v\|_{L^q_\rho} := \left(\int_{\mathbb{R}^N} v^q \rho(x)\, dx \right)^{1/q} < +\infty \right\}.$$

This kind of problem arises in a physical model provided in [73]. Such choice of ρ ensures that the following analogue of (4.1.2) holds:

$$\|v\|_{L^{2^*}_\rho(\mathbb{R}^N)} \le \frac{1}{C_s} \|\nabla v\|_{L^2(\mathbb{R}^N)} \quad \text{for any } \; v \in C_c^\infty(\mathbb{R}^N) \tag{4.1.6}$$

for a suitable positive constant C_s. In some cases we also assume that the weighted Poincaré inequality is valid, that is

$$\|v\|_{L^2_\rho(\mathbb{R}^N)} \le \frac{1}{C_p} \|\nabla v\|_{L^2(\mathbb{R}^N)} \quad \text{for any } \; v \in C_c^\infty(\mathbb{R}^N), \tag{4.1.7}$$

for some $C_p > 0$. For example, (4.1.7) is fulfilled when $\rho(x) \asymp |x|^{-a}$, as $|x| \to +\infty$, for every $a \ge 2$, whereas, (4.1.6) is valid for every $a > 0$.

Problem (4.1.5) under the assumption $1 < p < m$ has been investigated in [46]. Under the assumption that the Poincaré inequality is valid on M, it is shown that global existence and a smoothing effect for small L^m initial data hold, that is solutions corresponding to such data are bounded for all positive times with a quantitative bound on their L^∞ norm.

In [86, 87] problem (4.1.5) is also investigated, under certain conditions on ρ. It is proved that if $\rho(x) = |x|^{-a}$ with $a \in (0, 2)$,

$$p > m + \frac{2-a}{N-a},$$

and $u_0 \ge 0$ is small enough, then a global solution exists (see [86, Theorem 1]). Note that the homogeneity of the weight $\rho(x) = |x|^{-a}$ is essentially used in the proof, since the Caffarelli-Kohn-Nirenberg estimate is exploited, which requires such a type of weight. In addition, a smoothing estimate holds. On the other hand, any nonnegative solution blows up, in a suitable sense, when $\rho(x) = |x|^{-a}$ or $\rho(x) = (1+|x|)^{-a}$ with $a \in [0, 2)$, $u_0 \not\equiv 0$ and

$$1 < p < m + \frac{2-a}{N-a}.$$

Furthermore, in [87, 88], such results have been extended to more general initial data, decaying at infinity with a certain rate (see [87]). Finally, in [86, Theorem 2], it is shown that if $p > m$, $\rho(x) = (1+|x|)^{-a}$ with $a > 2$, and u_0 is small enough, a global solution exists.

Problem (4.1.5) has also been studied in [92], [93], by means of suitable barriers, supposing that the initial datum is continuous and with compact support. In particular, in [92] the case that $\rho(x) \asymp |x|^{-a}$ for $|x| \to +\infty$ with $a \in (0, 2)$ is addressed. It is proved that for any $p > 1$, if u_0 is large enough, then the solution blows up in finite time. On the other hand, if $p > \bar{p}$, for a certain $\bar{p} > m$ depending on m, p and ρ, and u_0 is small enough, then there exists a global bounded solution. Moreover, in [93] the case that $a \geq 2$ is investigated. For $a = 2$, blowup is shown to occur when u_0 is big enough, whereas global existence holds when u_0 is small enough. For $a > 2$ it is proved that if $p > m$, $u_0 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and goes to 0 at infinity with a suitable rate, then there exists a global bounded solution. Furthermore, for the same initial datum u_0, if $1 < p < m$, then there exists a global solution, which could blow up as $t \to +\infty$.

Our main results concerning problem (4.1.5) can be summarized as follows. Assume that $\rho \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \rho > 0$.

- (See Theorem 4.2.8) We prove that (4.1.5) admits a global solution, provided that

  $$p > m + \frac{2}{N};$$

  moreover, certain smoothing effects for solutions are fulfilled. More precisely, for any sufficiently small initial datum $u_0 \in L^m_\rho(\mathbb{R}^N) \cap L^{(p-m)\frac{N}{2}}_\rho(\mathbb{R}^N)$ there exists a global solution $u(t)$ such that $u(t) \in L^\infty(\mathbb{R}^N)$ for all $t > 0$ and a quantitative bound on the L^∞ norm is verified. Moreover, suitable L^q norms of solutions decrease in time.

- (See Theorem 4.2.9) We show that, if the Poincaré inequality (4.1.7) holds and one assumes the condition

  $$p > m,$$

  then, for any sufficiently small initial datum u_0 belonging to suitable Lebesgue spaces, there exists a global solution $u(t)$ to (4.1.5) such that $u(t) \in L^\infty(\mathbb{R}^N)$, with a quantitative bound on the L^∞ norm.

Let us compare our results with those in [86]. Theorem 4.2.8 deals with a different class of weights ρ with respect to [86, Theorem 1], where $\rho(x) = |x|^{-a}$ for $a \in (0, 2)$, and the homogeneity of ρ is used. As a consequence, also the hypotheses on p and the methods of proofs are different. Furthermore, Theorem 4.2.9 requires the validity of the Poincaré inequality, hence, in particular, it can be applied when $\rho(x) = (1+|x|)^{-a}$ with $a \geq 2$ (see [49]). On the other hand, in Theorem [86, Theorem 2] it is assumed that $\rho(x) = (1+|x|)^{-a}$ for $a > 2$, so, the case $a = 2$ is not included.

4.1.4 Organization of the chapter

In Section 4.2 we state all our main results. In Section 4.3 some auxiliary results concerning elliptic problems are deduced together with a Benilan-Crandal type estimate. In Section 4.4 we introduce a family of approximating problems. Then, for such solutions, we prove that suitable L^q norms of solutions decrease in time, and a smoothing estimate, in the case $p > m + \frac{2}{N}$, supposing that M supports the Sobolev inequality. Under such assumptions, global existence for problem (4.1.1) is shown in Section 4.5. In Section 4.6 we prove that suitable L^q norms of solutions decrease in time, and L^∞ bounds for solutions of the approximating problems, under the assumptions that $p > m$ and that M supports the Poincaré inequality as well. Then, under such hypotheses, existence of global solutions to problem (4.1.1) is proved. Finally, a concise proof of the results concerning problem (4.1.5) is given in Section 4.7 by adapting the previous methods to that situation.

4.2 Statements of main results

We state first our results concerning solutions to problem (4.1.1), then we pass to the ones valid for solutions to problem (4.1.5).

4.2.1 Global existence on Riemannian manifolds

Solutions to (4.1.1) will be meant in the very weak, or distributional, sense, according to the following definition.

Definition 4.2.1. *Let M be a complete noncompact Riemannian manifold of infinite volume. Let $m > 1$, $p > m$ and $u_0 \in L^1_{loc}(M)$, $u_0 \geq 0$. We say that the function u is a solution to problem* (4.1.1) *in the time interval* $[0, T)$ *if*

$$u \in L^p_{loc}(M \times (0, T))$$

and for any $\varphi \in C_c^\infty(M \times [0, T])$ such that $\varphi(x, T) = 0$ for any $x \in M$, u satisfies the equality:

$$-\int_0^T \int_M u\, \varphi_t \, d\mu\, dt = \int_0^T \int_M u^m\, \Delta\varphi \, d\mu\, dt + \int_0^T \int_M u^p\, \varphi \, d\mu\, dt + \int_M u_0(x)\, \varphi(x, 0)\, d\mu.$$

First we consider the case that $p > m + \frac{2}{N}$ and the Sobolev inequality holds on M. In order to state our results we define

$$p_0 := (p - m)\frac{N}{2}. \tag{4.2.8}$$

Observe that $p_0 > 1$ whenever $p > m + \frac{2}{N}$.

Theorem 4.2.2. *Let M be a complete, noncompact manifold of infinite volume such that the Sobolev inequality (4.1.2) holds. Let $m > 1$, $p > m + \frac{2}{N}$ and $u_0 \in L^m(M) \cap L^{p_0}(M)$, $u_0 \geq 0$ where p_0 has been defined in (4.2.8). Let*

$$r > \max\left\{p_0, \frac{N}{2}\right\}, \qquad s = 1 + \frac{2}{N} - \frac{1}{r}.$$

Assume that

$$\|u_0\|_{L^{p_0}(M)} < \varepsilon_0 \tag{4.2.9}$$

with $\varepsilon_0 = \varepsilon_0(p, m, N, r, C_s)$ sufficiently small. Then problem (4.1.1) admits a solution for any $T > 0$, in the sense of Definition 4.2.1. Moreover, for any $\tau > 0$, one has $u \in L^\infty(M \times (\tau, +\infty))$ and there exists a numerical constant $\Gamma > 0$ such that, for all $t > 0$, one has

$$\|u(t)\|_{L^\infty(M)} \leq \Gamma\, t^{-\frac{\gamma}{ms}} \left\{\|u_0\|^{\delta_1}_{L^{p_0}(M)} + \|u_0\|^{\delta_2}_{L^{p_0}(M)}\right\}^{\frac{1}{ms}} \|u_0\|^{\frac{s-1}{s}}_{L^m(M)},$$

where

$$\gamma = \frac{p}{p-1}\left[1 - \frac{N(p-m)}{2\,p\,r}\right], \quad \delta_1 = p\,\frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2\,p\,r}\right], \quad \delta_2 = \frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2\,r}\right].$$

Moreover, let $p_0 \leq q < \infty$ and

$$\|u_0\|_{L^{p_0}(M)} < \hat{\varepsilon}_0 \tag{4.2.10}$$

for $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(p, m, N, r, C_s, q)$ small enough. Then there exists a constant $C = C(m, p, N, \varepsilon_0, C_s, q) > 0$ such that

$$\|u(t)\|_{L^q(M)} \leq C\, t^{-\gamma_q} \|u_0\|^{\delta_q}_{L^{p_0}(M)} \quad \text{for all } \ t > 0\,, \tag{4.2.11}$$

where

$$\gamma_q = \frac{1}{p-1}\left[1 - \frac{N(p-m)}{2q}\right], \quad \delta_q = \frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2q}\right].$$

Finally, for any $1 < q < \infty$, if $u_0 \in L^q(M) \cap L^{p_0}(M) \cap L^m(M)$ and

$$\|u_0\|_{L^{p_0}(M)} < \varepsilon \tag{4.2.12}$$

with $\varepsilon = \varepsilon(p, m, N, r, C_s, q)$ sufficiently small, then

$$\|u(t)\|_{L^q(M)} \leq \|u_0\|_{L^q(M)} \quad \text{for all } \ t > 0\,. \tag{4.2.13}$$

Remark 4.2.3. We notice that the proof of the above theorem will show that one can take an explicit value of ε_0 in (4.2.9). In fact, let $q_0 > 1$ be fixed and $\{q_n\}_{n\in\mathbb{N}}$ be the sequence defined by:

$$q_n = \frac{N}{N-2}(m + q_{n-1} - 1), \qquad \forall n \in \mathbb{N},$$

so that

$$q_n = \left(\frac{N}{N-2}\right)^n q_0 + \frac{N(m-1)}{N-2}\sum_{i=0}^{n-1}\left(\frac{N}{N-2}\right)^i. \tag{4.2.14}$$

Clearly, $\{q_n\}$ is increasing and $q_n \longrightarrow +\infty$ as $n \to +\infty$. Fix $q \in [q_0, +\infty)$ and let $\bar{n}$ be the first index such that $q_{\bar{n}} \geq q$. Define

$$\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(p, m, N, C_s, q, q_0) := \left[\min\left\{\min_{n=0,\dots,\bar{n}} \frac{2m(q_n - 1)}{(m + q_n - 1)^2}C_s^2;\ \frac{2m(p_0 - 1)}{(m + p_0 - 1)^2}C_s^2\right\}\right]^{\frac{1}{p-m}}. \tag{4.2.15}$$

Observe that ε_0 in (4.2.15) depends on the value of q through the sequence $\{q_n\}$. More precisely, $\bar{n}$ is increasing with respect to q, while the quantity $\min_{n=0,\dots,\bar{n}} \frac{2m(q_n-1)}{(m+q_n-1)^2}C_s^2$ decreases w.r.t. q. We then let $q_0 = p_0$, take $q = pr$ and define, for these choice of q_0, q,

$$\varepsilon_0 = \varepsilon_0(p, m, N, C_s, r) = \tilde{\varepsilon}_0(p, m, N, C_s, pr, p_0)\,.$$

Furthermore, in (4.2.10) we can take

$$\hat{\varepsilon}_0 = \hat{\varepsilon}_0(p, m, N, C_s, q) = \tilde{\varepsilon}_0(p, m, N, C_s, q, p_0)\,. \tag{4.2.16}$$

Similarly, one can choose the following explicit value for ε in (4.2.12):

$$\varepsilon = \bar{\varepsilon} \wedge \varepsilon_0, \tag{4.2.17}$$

where

$$\bar{\varepsilon} = \bar{\varepsilon}(p, m, C_s, q) := \left[\min\left\{\frac{2m(q - 1)}{(m + q - 1)^2}C_s^2;\ \frac{2m(p_0 - 1)}{(m + p_0 - 1)^2}C_s^2\right\}\right]^{\frac{1}{p-m}}.$$

Remark 4.2.4. Observe that, for $M = \mathbb{R}^N$, in [119, Theorem 3, pag. 220] it is shown that if $p > m + \frac{2}{N}$ and u_0 has compact support and is small enough, then the solution to problem (4.1.1) globally exists and decays like

$$t^{-\frac{1}{p-1}} \quad \text{as} \quad t \longrightarrow +\infty.$$

Note that under these assumptions, Theorem 4.2.2 can be applied. It implies that the solution to problem (4.1.1) globally exists and decays like

$$t^{-\frac{\gamma}{ms}} \quad \text{as} \quad t \longrightarrow +\infty.$$

It is easily seen that, for any $p \geq m\left(1 + \frac{2}{N}\right)$,

$$\frac{\gamma}{ms} \geq \frac{1}{p-1};$$

instead, for any $m + \frac{2}{N} < p < m\left(1 + \frac{2}{N}\right)$,

$$\frac{\gamma}{ms} < \frac{1}{p-1}.$$

Hence, when $p \geq m\left(1 + \frac{2}{N}\right)$ the decay's rate of the solution $u(t)$, for large times, given by Theorem 4.2.2 is better than that of [119, Theorem 3, pag. 220], while the opposite is true for $m + \frac{2}{N} < p < m\left(1 + \frac{2}{N}\right)$. In both cases, the class of initial data considered in Theorem 4.2.2 is wider.

In the next theorem, we address the case that $p > m$, supposing that both the inequalities (4.1.2) and (4.1.3) hold on M.

Theorem 4.2.5. *Let M be a complete, noncompact manifold of infinite volume such that the Sobolev inequality* (4.1.2) *and the Poincaré inequality* (4.1.3) *hold. Let*

$$m > 1, \quad p > m, \quad r > \frac{N}{2},$$

and $u_0 \in L^\theta(M) \cap L^{pr}(M)$ where $\theta = \min\{m, r\}$, $u_0 \geq 0$. Let

$$s = 1 + \frac{2}{N} - \frac{1}{r}.$$

Assume that

$$\|u_0\|_{L^{p\frac{N}{2}}(M)} < \varepsilon_1 \tag{4.2.18}$$

holds with $\varepsilon_1 = \varepsilon_1(m, p, N, r, C_p, C_s)$ sufficiently small. Then problem (4.1.1) *admits a solution for any $T > 0$, in the sense of Definition 4.2.1. Moreover for any $\tau > 0$ one has $u \in L^\infty(M \times (\tau, +\infty))$ and for all $t > 0$ one has*

$$\|u(t)\|_{L^\infty(M)} \leq \left(\frac{s}{s-1}\right)^{\frac{1}{m}} \|u_0\|_{L^m(M)}^{\frac{s-1}{s}} \left[\|u_0\|_{L^{pr}(M)}^p + \frac{1}{(m-1)t}\|u_0\|_{L^r(M)}\right]^{\frac{1}{ms}}.$$

Moreover, suppose that $u_0 \in L^q(M) \cap L^\theta(M) \cap L^{pr}(M)$ for some for $1 < q < \infty$,

$$\|u_0\|_{L^{p\frac{N}{2}}(M)} < \varepsilon_2, \tag{4.2.19}$$

for some $\varepsilon_2 = \varepsilon_2(p, m, N, r, C_p, C_s, q)$ sufficiently small. Then

$$\|u(t)\|_{L^q(M)} \leq \|u_0\|_{L^q(M)} \quad \textit{for all} \;\; t > 0\,. \tag{4.2.20}$$

Remark 4.2.6. We define, given $q > 1$:

$$\tilde{\varepsilon}_1(q) := \left[\min\left\{\frac{2m(q-1)}{(m+q-1)^2}C;\ \frac{2m\left(p\frac{N}{2}-1\right)}{\left(m+p\frac{N}{2}-1\right)^2}C\right\}\right]^{\frac{p+m+q-1}{p(p+q-1)-m(m+q-1)}} \tag{4.2.21}$$

where $C = C_p^{2m/p}\,\tilde{C}$ and $\tilde{C} = \tilde{C}(C_s, m, p, q) > 0$ is defined in (4.6.91) below, with the choice $\theta := \frac{m(m+q-1)}{p(p+q-1)}$. The proof will show that one can choose $\varepsilon_1 := \min_{i=1,\dots,4} \tilde{\varepsilon}_1(q_i)$ where $q_1 = m$, $q_2 = p$, $q_3 = pr$ and $q_4 = r$.

Similarly, we observe that in (4.2.19) we can choose

$$\varepsilon_2 = \varepsilon_1 \wedge \tilde{\varepsilon}_1(q)\,. \tag{4.2.22}$$

In the next sections we always keep the notation as in Remarks 4.2.3 and 4.2.6.

4.2.2 Weighted, Euclidean reaction-diffusion problems

We consider a *weight* $\rho : \mathbb{R}^N \to \mathbb{R}$ such that

$$\rho \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \rho(x) > 0 \text{ for any } x \in \mathbb{R}^N. \tag{4.2.23}$$

Solutions to problem (4.1.5) are meant according to the following definition.

Definition 4.2.7. *Let $m > 1$, $p > m$ and $u_0 \in L^1_{\rho,loc}(\mathbb{R}^N)$, $u_0 \geq 0$. Let the weight ρ satisfy* (4.2.23)*. We say that the function u is a solution to problem* (4.1.5) *in the interval $[0,T)$ if*

$$u \in L^p_{\rho,loc}(\mathbb{R}^N \times (0,T))$$

and for any $\varphi \in C^\infty_c(\mathbb{R}^N \times [0,T])$ such that $\varphi(x,T) = 0$ for any $x \in \mathbb{R}^N$, u satisfies the equality:

$$-\int_0^T \int_{\mathbb{R}^N} u\,\varphi_t\,\rho(x)\,dx\,dt = \int_0^T \int_{\mathbb{R}^N} u^m\,\Delta\varphi\,dx\,dt + \int_0^T \int_{\mathbb{R}^N} u^p\,\varphi\,\rho(x)\,dx\,dt + \int_{\mathbb{R}^N} u_0(x)\,\varphi(x,0)\,\rho(x)\,dx.$$

First we consider the case that $p > m + \frac{2}{N}$. Recall that since ρ is bounded, the Sobolev inequality (4.1.6) necessarily holds.

Theorem 4.2.8. *Let ρ satisfy* (4.2.23)*. Let $m > 1$, $p > m + \frac{2}{N}$ and $u_0 \in L^m_\rho(\mathbb{R}^N) \cap L^{p_0}_\rho(\mathbb{R}^N)$, $u_0 \geq 0$ with p_0 defined in* (4.2.8)*. Let*

$$r > \max\left\{p_0, \frac{N}{2}\right\}, \qquad s = 1 + \frac{2}{N} - \frac{1}{r}.$$

Assume that

$$\|u_0\|_{L^{p_0}_\rho(\mathbb{R}^N)} < \varepsilon_0$$

with $\varepsilon_0 = \varepsilon_0(p, m, N, r, C_s)$ sufficiently small. Then problem (4.1.5) *admits a solution for any $T > 0$, in the sense of Definition 4.2.7. Moreover, for any $\tau > 0$, one has $u \in L^\infty(\mathbb{R}^N \times (\tau, +\infty))$ and there exist $\Gamma > 0$ such that, for all $t > 0$, one has*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \Gamma\, t^{-\frac{\gamma}{ms}} \left\{ \|u_0\|^{\delta_1}_{L^{p_0}_\rho(\mathbb{R}^N)} + \frac{1}{m-1}\, \|u_0\|^{\delta_2}_{L^{p_0}_\rho(\mathbb{R}^N)} \right\}^{\frac{1}{ms}} \|u_0\|^{\frac{s-1}{s}}_{L^m_\rho(\mathbb{R}^N)},$$

where

$$\gamma = \frac{p}{p-1}\left[1 - \frac{N(p-m)}{2\,p\,r}\right], \; \delta_1 = p\,\frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2\,p\,r}\right], \; \delta_2 = \frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2\,r}\right].$$

Moreover, let $p_0 \leq q < \infty$ and

$$\|u_0\|_{L^{p_0}_\rho(\mathbb{R}^N)} < \hat{\varepsilon}_0$$

for $\hat{\varepsilon}_0 = \hat{\varepsilon}_0(p, m, N, r, C_s, q)$ small enough. Then there exists a constant $C = C(m, p, N, \varepsilon_0, C_s, q) > 0$ such that

$$\|u(t)\|_{L^q_\rho(\mathbb{R}^N)} \leq C\, t^{-\gamma_q} \|u_0\|^{\delta_q}_{L^{p_0}_\rho(\mathbb{R}^N)} \quad \text{for all } \; t > 0\,,$$

where

$$\gamma_q = \frac{1}{p-1}\left[1 - \frac{N(p-m)}{2q}\right], \quad \delta_q = \frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2q}\right].$$

Finally, for any $1 < q < \infty$, *if* $u_0 \in L^q_\rho(\mathbb{R}^N) \cap L^{p_0}_\rho(\mathbb{R}^N) \cap L^m_\rho(\mathbb{R}^N)$ *and*

$$\|u_0\|_{L^{p_0}_\rho(\mathbb{R}^N)} < \varepsilon$$

holds, with $\varepsilon = \varepsilon(p, m, N, r, C_s, q)$ *sufficiently small, then*

$$\|u(t)\|_{L^q_\rho(\mathbb{R}^N)} \le \|u_0\|_{L^q_\rho(\mathbb{R}^N)} \quad \text{for all } \; t > 0\,.$$

A quantitative form of the smallness condition on u_0 in the above theorem can be given exactly as in Remark 4.2.3, see in particular (4.2.15), (4.2.16) and (4.2.17).

In the next theorem, we address the case $p > m$. We suppose that the Poincaré inequality (4.1.7) holds.

Theorem 4.2.9. *Let* ρ *satisfy* (4.2.23) *and assume that the inequality* (4.1.7) *hold. Let*

$$m > 1, \quad p > m, \quad r > \frac{N}{2},$$

and $u_0 \in L^\theta_\rho(\mathbb{R}^N) \cap L^{pr}_\rho(\mathbb{R}^N)$ *where* $\theta = \min\{m, r\}$, $u_0 \ge 0$. *Let*

$$s = 1 + \frac{2}{N} - \frac{1}{r}.$$

Assume that

$$\|u_0\|_{L^{p\frac{N}{2}}_\rho(\mathbb{R}^N)} < \varepsilon_1$$

holds with $\varepsilon_1 = \varepsilon_1(m, p, N, r, C_p, C_s)$ *sufficiently small Then problem* (4.1.5) *admits a solution for any* $T > 0$, *in the sense of Definition 4.2.7. Moreover, for any* $\tau > 0$ *one has* $u \in L^\infty(\mathbb{R}^N \times (\tau, +\infty))$ *and for all* $t > 0$ *one has*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le \left(\frac{s}{s-1}\right)^{\frac{1}{m}} \|u_0\|^{\frac{s-1}{s}}_{L^m_\rho(\mathbb{R}^N)} \left[\|u_0\|^p_{L^{pr}_\rho(\mathbb{R}^N)} + \frac{1}{(m-1)t}\|u_0\|_{L^r_\rho(\mathbb{R}^N)}\right]^{\frac{1}{ms}}.$$

Moreover, suppose that $u_0 \in L^q_\rho(\mathbb{R}^N) \cap L^\theta_\rho(\mathbb{R}^N) \cap L^{pr}_\rho(\mathbb{R}^N)$ *for some for* $1 < q < \infty$,

$$\|u_0\|_{L^{p\frac{N}{2}}_\rho(\mathbb{R}^N)} < \varepsilon_2,$$

for some $\varepsilon_2 = \varepsilon_2(p, m, N, r, C_p, C_s, q)$ *small enough. Then*

$$\|u(t)\|_{L^q_\rho(\mathbb{R}^N)} \le \|u_0\|_{L^q_\rho(\mathbb{R}^N)} \quad \text{for all } \; t > 0\,.$$

A quantitative form of the smallness condition on u_0 in the above Theorem can be given exactly as in Remark 4.2.6, see in particular (4.2.21) and (4.2.22).

4.3 Auxiliary results for elliptic problems

Let $x_0, x \in M$. We denote by $r(x) = \operatorname{dist}(x_0, x)$ the Riemannian distance between x_0 and x. Moreover, we let $B_R(x_0) := \{x \in M, \operatorname{dist}(x_0, x) < R\}$ be the geodesic ball with centre $x_0 \in M$ and radius $R > 0$. If a reference point $x_0 \in M$ is fixed, we shall simply denote by B_R the ball with centre x_0 and radius R. Moreover we denote by μ the Riemannian measure on M.

For any given function v, we define for any $k \in \mathbb{R}^+$

$$T_k(v) := \begin{cases} k & \text{if } v \geq k\,, \\ v & \text{if } |v| < k\,, \\ -k & \text{if } v \leq -k\,. \end{cases} \tag{4.3.24}$$

For every $R > 0$, $k > 0$, consider the problem

$$\begin{cases} u_t = \Delta u^m + T_k(u^p) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, +\infty) \\ u = u_0 & \text{in } B_R \times \{0\}, \end{cases} \tag{4.3.25}$$

where $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Solutions to problem (4.3.25) are meant in the weak sense as follows.

Definition 4.3.1. *Let $m > 1$ and $p > m$. Let $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. We say that a nonnegative function u is a solution to problem* (4.3.25) *if*

$$u \in L^\infty(B_R \times (0, +\infty)), \quad u^m \in L^2\big((0,T); H_0^1(B_R)\big) \qquad \text{for any } T > 0,$$

and for any $T > 0$, $\varphi \in C_c^\infty(B_R \times [0,T])$ such that $\varphi(x,T) = 0$ for every $x \in B_R$, u satisfies the equality:

$$\begin{aligned} -\int_0^T \int_{B_R} u\,\varphi_t \, d\mu\, dt = &-\int_0^T \int_{B_R} \langle \nabla u^m, \nabla \varphi \rangle \, d\mu\, dt + \int_0^T \int_{B_R} T_k(u^p)\, \varphi \, d\mu\, dt \\ &+ \int_{B_R} u_0(x)\, \varphi(x,0)\, d\mu. \end{aligned}$$

We also consider elliptic problems of the type

$$\begin{cases} -\Delta u = f & \text{in } B_R\,, \\ \quad u = 0 & \text{in } \partial B_R\,, \end{cases} \tag{4.3.26}$$

where $f \in L^q(B_R)$ for some $q > 1$.

Definition 4.3.2. *We say that $u \in H_0^1(B_R)$, $u \geq 0$ is a weak subsolution to problem* (4.3.26) *if*

$$\int_{B_R} \langle \nabla u, \nabla \varphi \rangle \, d\mu \leq \int_{B_R} f \varphi \, d\mu,$$

for any $\varphi \in H_0^1(B_R), \varphi \geq 0$.

In the next lemma we recall [46, Lemma 3.6], which will be used later.

Lemma 4.3.3. *Let $v \in L^1(B_R)$. Let $\overline{k} > 0$. Suppose that there exist $C > 0$ and $s > 1$ such that*

$$g(k) \leq C\mu(A_k)^s \quad \text{for any } k \geq \bar{k}.$$

Then $v \in L^\infty(B_R)$ and

$$\|v\|_{L^\infty(B_R)} \leq \frac{s}{s-1} C^{\frac{1}{s}} \|v\|_{L^1(B_R)}^{1-\frac{1}{s}} + \bar{k}.$$

The following proposition contains an estimate in the spirit of the L^∞ one of Stampacchia (see, e.g., [76], [11] and references therein) in the ball B_R; however, some differences are in order. In fact, we aim at obtaining an estimate independent of the radius R (see Remark 4.3.5). Since the volume of M is infinite, the classical estimate of Stampacchia cannot be directly applied.

Proposition 4.3.4. *Let $f \in L^m(B_R)$ where $m > \frac{N}{2}$. Assume that $v \in H_0^1(B_R)$, $v \geq 0$ is a subsolution to problem*

$$\begin{cases} -\Delta v = f & \text{in } B_R\,, \\ \quad v = 0 & \text{on } \partial B_R\,, \end{cases} \tag{4.3.27}$$

in the sense of Definition 4.3.2. Then

$$\|v\|_{L^\infty(B_R)} \leq \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} \|f\|_{L^m(B_R)}^{\frac{1}{s}} \|v\|_{L^1(B_R)}^{\frac{s-1}{s}}, \tag{4.3.28}$$

where

$$s = 1 + \frac{2}{N} - \frac{1}{m}\,. \tag{4.3.29}$$

Remark 4.3.5. If in Proposition 4.3.4 we further assume that there exists a constant $k_0 > 0$ such that

$$\max\left\{\|v\|_{L^1(B_R)}, \|f\|_{L^m(B_R)}\right\} \leq k_0 \quad \text{for all } \; R > 0,$$

then from (4.3.28), we infer that the bound from above on $\|v\|_{L^\infty(B_R)}$ is independent of R. This fact will have a key role in the proof of global existence for problem (4.1.1).

Proof of Proposition 4.3.4. We define

$$G_k(v) := v - T_k(v)$$

where $T_k(v)$ has been defined in (4.3.24) and

$$A_k := \{x \in B_R \,:\, |v(x)| > k\}.$$

Since $G_k(v) \in H_0^1(B_R)$ and $G_k(v) \geq 0$, we can take $G_k(v)$ as test function in problem (4.3.27). Arguing as in the proof of [46, Proposition 3.3] we obtain

$$\int_{B_R} |G_k(v)|\, d\mu \leq \frac{1}{C_s^2} \|f\|_{L^m(B_R)} \mu(A_k)^{\frac{N+2}{N} - \frac{1}{m}}. \tag{4.3.30}$$

By (4.3.29), setting

$$C = \frac{1}{C_s^2}\|f\|_{L^m(B_R)},$$

we rewrite (4.3.30) as

$$\int_{B_R} |G_k(v)|\, d\mu \leq C\mu(A_k)^s.$$

Hence we can apply Lemma 4.3.3 to v and we obtain

$$\|v\|_{L^\infty(B_R)} \leq C^{\frac{1}{s}} \frac{s}{s-1} \|v\|_{L^1(B_R)}^{\frac{s-1}{s}} + \overline{k}.$$

Taking the limit as $\overline{k} \longrightarrow 0$ and we get the thesis.

□

We shall use the following Aronson-Benilan type estimate (see [6]; see also [118, Proposition 2.3]).

Proposition 4.3.6. *Let $m > 1$, $p > m$, $u_0 \in H_0^1(B_R) \cap L^\infty(B_R)$, $u_0 \geq 0$. Let u be the solution to problem* (4.3.25)*. Then, for a.e. $t \in (0, T)$,*

$$-\Delta u^m(\cdot, t) \leq u^p(\cdot, t) + \frac{1}{(m-1)t} u(\cdot, t) \quad \text{in } \mathfrak{D}'(B_R).$$

Proof. The conclusion follows by minor modifications of the proof of [118, Proposition 2.3] (where $p < m$), due to the fact that we have $p > m$. We define

$$z = u_t + \frac{u}{m-1}$$

and the operator

$$Lz = \Delta\left(mu^{m-1}z\right) + mu^{p-1}z\,,$$

where u is the solution to problem (4.3.25). Observe that

$$\begin{aligned} &z(x,0) \geq 0 \quad \text{for} \quad x \in B_R\,, \\ &z(x,t) \geq 0 \quad \text{for} \quad x \in \partial B_R \text{ and } \ t \in (0,T)\,. \end{aligned}$$

Moreover, by direct computation, we get

$$z_t - Lz \geq 0 \quad \text{in } B_R \times (0, T).$$

Thus, arguing as in [118, Proposition 2.3], thanks to the comparison principle, we get, for a.e. $t \in (0, T)$,

$$-\Delta u^m(\cdot, t) \leq T_k[u^p(\cdot, t)] + \frac{1}{(m-1)t} u(\cdot, t) \leq u^p(\cdot, t) + \frac{1}{(m-1)t} u(\cdot, t) \quad \text{in } \mathfrak{D}'(B_R),$$

where we have used that $T_k(u^p) \leq u^p$. □

4.4 L^q and smoothing estimates for $p > m + \frac{2}{N}$

Lemma 4.4.1. *Let $m > 1, p > m + \frac{2}{N}$. Assume that inequality (4.1.2) holds. Suppose that $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Let $1 < q < \infty$, p_0 as in (4.2.8) and assume that*

$$\|u_0\|_{L^{p_0}(B_R)} < \bar{\varepsilon} \tag{4.4.31}$$

with $\bar{\varepsilon} = \bar{\varepsilon}(p, m, q, C_s)$ sufficiently small. Let u be the solution of problem (4.3.25) in the sense of Definition 4.3.1, such that in addition $u \in C([0,T), L^q(B_R))$ for any $q \in (1, +\infty)$, for any $T > 0$. Then

$$\|u(t)\|_{L^q(B_R)} \leq \|u_0\|_{L^q(B_R)} \quad \textit{for all } \ t > 0\,. \tag{4.4.32}$$

Note that the request $u \in C([0,T), L^q(B_R))$ for any $q \in (1, +\infty)$, for any $T > 0$ is not restrictive, since we will construct solutions belonging to that class (see the proof of Theorem 4.2.2 below). This remark also applies to several other intermediate results below.

Proof. Since u_0 is bounded and T_k is a bounded and Lipschitz function, by standard results, there exists a unique solution of problem (4.3.25) in the sense of Definition 4.3.1. We now multiply both sides of the differential equation in problem (4.3.25) by u^{q-1},

$$\int_{B_R} u_t\, u^{q-1}\, d\mu = \int_{B_R} \Delta(u^m)\, u^{q-1}\, d\mu + \int_{B_R} T_k(u^p)\, u^{q-1}\, d\mu\,.$$

Now, formally integrating by parts in B_R. This can be justified by standard tools, by an approximation procedure. We get

$$\frac{1}{q}\frac{d}{dt}\int_{B_R} u^q\, d\mu = -m(q-1)\int_{B_R} u^{m+q-3}\, |\nabla u|^2\, d\mu + \int_{B_R} T_k(u^p)\, u^{q-1}\, d\mu\,. \tag{4.4.33}$$

Observe that, thanks to Sobolev inequality (4.1.2), we have

$$\begin{aligned}\int_{B_R} u^{m+q-3}\, |\nabla u|^2\, d\mu &= \frac{4}{(m+q-1)^2}\int_{B_R} \left|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right|^2 d\mu \\ &\geq \frac{4}{(m+q-1)^2} C_s^2 \left(\int_{B_R} u^{\frac{m+q-1}{2}\frac{2N}{N-2}}\, d\mu\right)^{\frac{N-2}{N}}\,.\end{aligned} \tag{4.4.34}$$

Moreover, the last term in the right hand side of (4.4.33), thanks to Hölder inequality with exponents $\frac{N}{N-2}$ and $\frac{N}{2}$, becomes

$$\begin{aligned}\int_{B_R} T_k(u^p)\, u^{q-1}\, d\mu &\leq \int_{B_R} u^p\, u^{q-1}\, d\mu = \int_{B_R} u^{p-m}\, u^{m+q-1}\, d\mu \\ &\leq \|u(t)\|^{p-m}_{L^{(p-m)\frac{N}{2}}(B_R)} \|u(t)\|^{m+q-1}_{L^{(m+q-1)\frac{N}{N-2}}(B_R)}\,.\end{aligned} \tag{4.4.35}$$

Combining (4.4.34) and (4.4.35) we get

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} \leq -\left[\frac{4\,m(q-1)}{(m+q-1)^2}C_s^2 - \|u(t)\|^{p-m}_{L^{p_0}(B_R)}\right] \|u(t)\|^{m+q-1}_{L^{(m+q-1)\frac{N}{N-2}}(B_R)}\,. \tag{4.4.36}$$

Take any $T > 0$. Observe that, thanks to hypothesis (4.4.31) and the known continuity of the map $t \mapsto u(t)$ in $[0, T]$, there exists $t_0 > 0$ such that

$$\|u(t)\|_{L^{p_0}(B_R)} \le 2\,\bar{\varepsilon} \quad \text{for any} \quad t \in [0, t_0]\,.$$

Hence (4.4.36) becomes, for any $t \in (0, t_0]$,

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} \le -\left[\frac{4\,m(q-1)}{(m+q-1)^2}C_s^2 - 2\,\bar{\varepsilon}^{p-m}\right]\|u(t)\|^{m+q-1}_{L^{(m+q-1)\frac{N}{N-2}}(B_R)} \le 0\,,$$

where the last inequality is obtained thanks to (4.4.31). We have proved that $t \mapsto \|u(t)\|_{L^q(B_R)}$ is decreasing in time for any $t \in (0, t_0]$, i.e.

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_0]\,. \tag{4.4.37}$$

In particular, inequality (4.4.37) follows for the choice $q = p_0$, in view of hypothesis (4.4.31). Hence we have

$$\|u(t)\|_{L^{p_0}(B_R)} \le \|u_0\|_{L^{p_0}(B_R)} < \bar{\varepsilon} \quad \text{for any} \quad t \in (0, t_0]\,.$$

Now, we can repeat the same argument in the time interval $(t_0, t_1]$, where t_1 is chosen, due to the continuity of u, in such a way that

$$\|u(t)\|_{L^{p_0}(B_R)} \le 2\bar{\varepsilon} \quad \text{for any} \quad t \in (t_0, t_1]\,.$$

Thus we get

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_1]\,.$$

Iterating this procedure we obtain that $t \mapsto \|u(t)\|_{L^q(B_R)}$ is decreasing in $[0, T]$. Since $T > 0$ was arbitrary, the thesis follows. □

Using a Moser type iteration procedure we prove the following result:

Proposition 4.4.2. *Let $m > 1$, $p > m + \frac{2}{N}$. Assume that inequality (4.1.2) holds. Suppose that $u_0 \in L^\infty(B_R)$, $u_0 \ge 0$. Let u be the solution of problem (4.3.25) in the sense of Definition 4.3.1, such that in addition $u \in C([0,T), L^q(B_R))$ for any $q \in (1, +\infty)$, for any $T > 0$. Let $1 < q_0 \le q < +\infty$ and assume that*

$$\|u_0\|_{L^{p_0}(B_R)} < \tilde{\varepsilon}_0 \tag{4.4.38}$$

for $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(p, m, N, C_s, q, q_0)$ sufficiently small. Then there exists $C(m, q_0, C_s, \tilde{\varepsilon}_0, N, q) > 0$ such that

$$\|u(t)\|_{L^q(B_R)} \le C\,t^{-\gamma_q}\|u_0\|^{\delta_q}_{L^{q_0}(B_R)} \quad \textit{for all} \quad t > 0\,,$$

where

$$\gamma_q = \left(\frac{1}{q_0} - \frac{1}{q}\right)\frac{N\,q_0}{2\,q_0 + N(m-1)}\,, \quad \delta_q = \frac{q_0}{q}\left(\frac{q + \frac{N}{2}(m-1)}{q_0 + \frac{N}{2}(m-1)}\right)\,. \tag{4.4.39}$$

Proof. Let $\{q_n\}$ be the sequence defined in (4.2.14). We start by proving a smoothing estimate from q_0 to $q_{\bar{n}}$ using a Moser iteration technique (see also [2]).

Let $t > 0$, we define

$$s = \frac{t}{2^{\bar{n}} - 1}, \quad t_n = (2^n - 1)s\,. \tag{4.4.40}$$

Observe that $t_0 = 0, \quad t_{\bar{n}} = t, \quad \{t_n\}$ is an increasing sequence w.r.t. n. Now, for any $1 \le n \le \overline{n}$, we multiply equation (4.3.25) by $u^{q_{n-1}-1}$ and integrate in $B_R \times [t_{n-1}, t_n]$. Thus we get

$$\int_{t_{n-1}}^{t_n} \int_{B_R} u_t\, u^{q_{n-1}-1}\, d\mu\, dt = \int_{t_{n-1}}^{t_n} \int_{B_R} \Delta(u^m)\, u^{q_{n-1}-1}\, d\mu\, dt + \int_{t_{n-1}}^{t_n} \int_{B_R} T_k(u^p)\, u^{q_{n-1}-1}\, d\mu\, dt.$$

Then we integrate by parts in $B_R \times [t_{n-1}, t_n]$. Thanks to Sobolev inequality and hypothesis (4.4.38) we get

$$\begin{aligned} &\frac{1}{q_{n-1}} \left[\|u(\cdot, t_n)\|^{q_{n-1}}_{L^{q_{n-1}}(B_R)} - \|u(\cdot, t_{n-1})\|^{q_{n-1}}_{L^{q_{n-1}}(B_R)} \right] \\ &\le - \left[\frac{4m(q_{n-1} - 1)}{(m + q_{n-1} - 1)^2} C_s^2 - 2\tilde{\varepsilon}_0^{\frac{1}{p-m}} \right] \int_{t_{n-1}}^{t_n} \|u(\tau)\|^{m+q_{n-1}-1}_{L^{(m+q_{n-1}-1)\frac{N}{N-2}}(B_R)}\, d\tau, \end{aligned} \tag{4.4.41}$$

where we have used the fact that $T_k(u^p) \le u^p$. We define q_n as in (4.2.14), so that $(m + q_{n-1} - 1)\dfrac{N}{N-2} = q_n$. Hence, in view of hypothesis (4.4.38) we can apply Lemma 4.4.1 to the integral on the right hand side of (4.4.41), hence we get

$$\begin{aligned} &\frac{1}{q_{n-1}} \left[\|u(\cdot, t_n)\|^{q_{n-1}}_{L^{q_{n-1}}(B_R)} - \|u(\cdot, t_{n-1})\|^{q_{n-1}}_{L^{q_{n-1}}(B_R)} \right] \\ &\le - \left[\frac{4m(q_{n-1} - 1)}{(m + q_{n-1} - 1)^2} C_s^2 - 2\tilde{\varepsilon}_0^{\frac{1}{p-m}} \right] \|u(\cdot, t_n)\|^{m+q_{n-1}-1}_{L^{q_n}(B_R)} |t_n - t_{n-1}|. \end{aligned} \tag{4.4.42}$$

Observe that

$$\begin{aligned} &\|u(\cdot, t_n)\|^{q_{n-1}}_{L^{q_{n-1}}(B_R)} \ge 0, \\ &|t_n - t_{n-1}| = \frac{2^{n-1}\, t}{2^{\bar{n}} - 1}. \end{aligned} \tag{4.4.43}$$

We define

$$d_{n-1} := \left[\frac{4\, m\, (q_{n-1} - 1)}{(m + q_{n-1} - 1)^2} C_s^2 - 2\tilde{\varepsilon}_0^{\frac{1}{p-m}} \right]^{-1} \frac{1}{q_{n-1}}. \tag{4.4.44}$$

By plugging (4.4.43) and (4.4.44) into (4.4.42) we get

$$\|u(\cdot, t_n)\|^{m+q_{n-1}-1}_{L^{q_n}(B_R)} \le \frac{(2^{\bar{n}} - 1) d_n}{2^{n-1}\, t} \|u(\cdot, t_{n-1})\|^{q_{n-1}}_{L^{q_{n-1}}(B_R)}.$$

The latter formula can be rewritten as

$$\|u(\cdot, t_n)\|_{L^{q_n}(B_R)} \le \left(\frac{(2^{\bar{n}} - 1) d_n}{2^{n-1}} \right)^{\frac{1}{m+q_{n-1}-1}} t^{-\frac{1}{m+q_{n-1}-1}} \|u(\cdot, t_{n-1})\|^{\frac{q_{n-1}}{m+q_{n-1}-1}}_{L^{q_{n-1}}(B_R)}.$$

Thanks to the definition of the sequence $\{q_n\}$ in (4.2.14) we write

$$\|u(\cdot,t_n)\|_{L^{q_n}(B_R)} \le \left(\frac{(2^{\bar n}-1)d_{n-1}}{2^{n-1}}\right)^{\frac{N}{(N-2)}\frac{1}{q_n}} t^{-\frac{N}{(N-2)}\frac{1}{q_n}} \|u(\cdot,t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{\frac{q_{n-1}}{q_n}\frac{N}{N-2}}. \quad (4.4.45)$$

Define $\sigma := \frac{N}{N-2}$. Observe that, for any $1 \le n \le \bar n$, we have

$$\begin{aligned}\left(\frac{(2^{\bar n}-1)d_{n-1}}{2^{n-1}}\right)^{\sigma} &= \left[\frac{2^{\bar n}-1}{2^{n-1}}\left(\frac{4\,m(q_{n-1}-1)}{(m+q_{n-1}-1)^2}C_s^2 - 2\varepsilon^{\frac{1}{p-m}}\right)^{-1}\frac{1}{q_{n-1}}\right]^{\sigma} \\ &= \left[\frac{2^{\bar n}-1}{2^{n-1}}\frac{1}{\frac{4\,m\,q_{n-1}(q_{n-1}-1)}{(m+q_{n-1}-1)^2}C_s^2 - 2\tilde\varepsilon_0^{\frac{1}{p-m}}q_{n-1}}\right]^{\sigma},\end{aligned} \quad (4.4.46)$$

where

$$\frac{2^{\bar n}-1}{2^{n-1}} \le 2^{\bar n+1} \qquad \text{for all } 1 \le n \le \bar n. \quad (4.4.47)$$

Consider the function

$$g(x) := \left[\frac{4\,m(x-1)}{(m+x-1)^2}C_s^2 - 2\tilde\varepsilon_0^{\frac{1}{p-m}}\right]x \qquad \text{for } q_0 \le x \le q_{\bar n},\ \ x \in \mathbb{R}.$$

Observe that, thanks to the definition of σ, $g(x) > 0$ for any $q_0 \le x \le q_{\bar n}$. Moreover, g has a minimum in the interval $q_0 \le x \le q_{\bar n}$, call it $\tilde x$. Then we have

$$\frac{1}{g(x)} \le \frac{1}{g(\tilde x)} \qquad \text{for any} \quad q_0 \le x \le q_{\bar n},\ x \in \mathbb{R}. \quad (4.4.48)$$

Thanks to (4.4.46), (4.4.47) and (4.4.48), we can say that there exist a positive constant C, where $C = C(N, C_s, \varepsilon, \bar n, m, q_0)$, such that

$$\left(\frac{(2^{\bar n}-1)d_{n-1}}{2^{n-1}}\right)^{\sigma} \le C\,, \quad \text{for all } 1 \le n \le \bar n. \quad (4.4.49)$$

By using (4.4.49) and (4.4.45) we get, for any $1 \le n \le \bar n$

$$\|u(\cdot,t_n)\|_{L^{q_n}(B_R)} \le C^{\frac{1}{q_n}} t^{-\frac{\sigma}{q_n}} \|u(\cdot,t_{n-1})\|_{L^{q_{n-1}}(B_R)}^{\frac{q_{n-1}\sigma}{q_n}}. \quad (4.4.50)$$

Let us set

$$U_n := \|u(\cdot,t_n)\|_{L^{q_n}(B_R)}.$$

Then (4.4.50) becomes

$$\begin{aligned} U_n &\le C^{\frac{1}{q_n}} t^{-\frac{\sigma}{q_n}} U_{n-1}^{\frac{q_{n-1}\sigma}{q_n}} \\ &\le C^{\frac{1}{q_n}} t^{-\frac{\sigma}{q_n}} \left[C^{\frac{\sigma}{q_n}} t^{-\frac{\sigma^2}{q_n}} U_{k-2}^{\sigma^2\frac{q_{n-2}}{q_n}}\right] \\ &\le \dots \\ &\le C^{\frac{1}{q_n}\sum_{i=0}^{n-1}\sigma^i} t^{-\frac{\sigma}{q_n}\sum_{i=0}^{n-1}\sigma^i} U_0^{\sigma^n\frac{q_0}{q_n}}. \end{aligned}$$

We define

$$\alpha_n := \frac{1}{q_n}\sum_{i=0}^{n-1}\sigma^i, \quad \beta_n := \frac{\sigma}{q_n}\sum_{i=0}^{n-1}\sigma^i = \sigma\,\alpha_n, \quad \delta_n := \sigma^n\frac{q_0}{q_n}. \tag{4.4.51}$$

By substituting n with $\bar{n}$ into (4.4.51) we get

$$\alpha_{\bar{n}} := \frac{N-2}{2}\frac{A}{q_{\bar{n}}}, \quad \beta_{\bar{n}} := \frac{N}{2}\frac{A}{q_{\bar{n}}}, \quad \delta_{\bar{n}} := (A+1)\frac{q_0}{q_{\bar{n}}}, \tag{4.4.52}$$

where $A := \left(\frac{N}{N-2}\right)^{\bar{n}} - 1$. Hence, in view of (4.4.40) and (4.4.52), (4.4.50) with $n = \bar{n}$ yields

$$\|u(\cdot,t)\|_{L^{q_{\bar{n}}}(B_R)} \le C^{\frac{N-2}{2}\frac{A}{q_{\bar{n}}}}\, t^{-\frac{N}{2}\frac{A}{q_{\bar{n}}}}\, \|u_0\|_{L^{q_0}(B_R)}^{q_0\frac{A+1}{q_{\bar{n}}}}. \tag{4.4.53}$$

We have proved a smoothing estimate from q_0 to $q_{\bar{n}}$. Observe that if $q_{\bar{n}} = q$ then the thesis is proved. Now suppose that $q > q_{\bar{n}}$. Observe that $q_0 \le q < q_{\bar{n}}$ and define

$$B := N(m-1)A + 2\,q_0(A+1).$$

From (4.4.53) and Lemma 4.4.1 we get, by interpolation,

$$\begin{aligned}
\|u(\cdot,t)\|_{L^q(B_R)} &\le \|u(\cdot,t)\|_{L^{q_0}(B_R)}^{\theta}\|u(\cdot,t)\|_{L^{q_{\bar{n}}}(B_R)}^{1-\theta} \\
&\le \|u_0(\cdot)\|_{L^{q_0}(B_R)}^{\theta} C\, t^{-\frac{N\,A}{B}(1-\theta)}\, \|u_0\|_{L^{q_0}(B_R)}^{2q_0\frac{A+1}{B}(1-\theta)} \\
&= C\, t^{-\frac{N\,A}{B}(1-\theta)}\, \|u_0\|_{L^{q_0}(B_R)}^{2q_0\frac{A+1}{B}(1-\theta)+\theta},
\end{aligned} \tag{4.4.54}$$

where

$$\theta = \frac{q_0}{q}\left(\frac{q_{\bar{n}} - q}{q_{\bar{n}} - q_0}\right). \tag{4.4.55}$$

Combining (4.4.54), (4.4.39) and (4.4.55) we get the claim, noticing that q was arbitrary in $[q_0, \infty)$. □

Remark 4.4.3. *One can not let $q \to +\infty$ in the above bound. In fact, one can show that $\varepsilon \longrightarrow 0$ as $q \to \infty$. So in such limit the hypothesis on the norm of the initial datum (4.2.9) is satisfied only when $u_0 \equiv 0$.*

Proposition 4.4.4. *Let $m > 1$, $p > m + \frac{2}{N}$, $R > 0$, p_0 be as in (4.2.8), $u_0 \in L^\infty(B_R)$, $u_0 \ge 0$. Let*

$$r > \max\left\{p_0, \frac{N}{2}\right\}, \qquad s = 1 + \frac{2}{N} - \frac{1}{r}. \tag{4.4.56}$$

Suppose that (4.2.9) holds for $\varepsilon_0 = \varepsilon_0(p, m, N, C_s, r)$ sufficiently small. Let u be the solution to problem (4.3.25), such that in addition $u \in C([0,T), L^q(B_R))$ for any $q \in (1, +\infty)$, for any $T > 0$. Let M be such that inequality (4.1.2) holds. Then there exists $\Gamma = \Gamma(p, m, N, r) > 0$ such that, for all $t > 0$,

$$\|u(t)\|_{L^\infty(B_R)} \le \Gamma\, t^{-\frac{\gamma}{ms}} \left\{\|u_0\|_{L^{p_0}(B_R)}^{\delta_1} + \frac{1}{m-1}\|u_0\|_{L^{p_0}(B_R)}^{\delta_2}\right\}^{\frac{1}{ms}} \|u_0\|_{L^m(B_R)}^{\frac{s-1}{s}}, \tag{4.4.57}$$

where

$$\gamma = \frac{p}{p-1}\left[1-\frac{N(p-m)}{2\,p\,r}\right],\ \delta_1 = p\,\frac{p-m}{m-1}\left[1+\frac{N(m-1)}{2\,p\,r}\right],\ \delta_2 = \frac{p-m}{m-1}\left[1+\frac{N(m-1)}{2\,r}\right]. \tag{4.4.58}$$

Remark 4.4.5. If in Proposition 4.4.4, in addition, we assume that for some $k_0 > 0$

$$\max\left\{\|u_0\|_{L^m(B_R)};\ \|u_0\|_{L^{p_0}(B_R)}\right\} \le k_0 \quad \text{for every} \quad R > 0\,,$$

then the bound from above for $\|u(t)\|_{L^\infty(B_R)}$ in (4.4.57) is independent of R.

Proof of Proposition 4.4.4. Let us set $w = u(\cdot, t)$. Observe that $w^m \in H_0^1(B_R)$ and $w \ge 0$. Due to Proposition 4.3.6 we know that

$$-\Delta(w^m) \le \left[w^p + \frac{w}{(m-1)t}\right].$$

Observe that, since $u_0 \in L^\infty(B_R)$ also $w \in L^\infty(B_R)$. Due to (4.4.56), we can apply Proposition 4.3.4. So, we have that

$$\begin{aligned}\|w\|^m_{L^\infty(B_R)} &\le \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} \left\|w^p + \frac{w}{(m-1)t}\right\|^{\frac{1}{s}}_{L^r(B_R)} \|w^m\|^{\frac{s-1}{s}}_{L^1(B_R)} \\ &\le \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} \left\{\|w^p\|_{L^r(B_R)} + \frac{1}{(m-1)t}\,\|w\|_{L^r(B_R)}\right\}^{\frac{1}{s}} \|w\|^{m\frac{s-1}{s}}_{L^m(B_R)}\end{aligned} \tag{4.4.59}$$

where s has been defined in (4.3.29). Thanks to (4.2.9), with an appropriate choice of ε_0, and (4.4.56) we can apply Proposition 4.4.2 with

$$q = pr, \quad q_0 = p_0, \quad \gamma_{pr} = \frac{1}{p-1}\left[1-\frac{N(p-m)}{2pr}\right]$$

and $\delta_{pr} = \delta_1/p$, δ_1 defined in (4.4.58). Hence we obtain

$$\|w^p\|_{L^r(B_R)} = \|w\|^p_{L^{pr}(B_R)} \le \left[C\,t^{-\gamma_{pr}}\|u_0\|^{\delta_1/p}_{L^{p_0}(B_R)}\right]^p, \tag{4.4.60}$$

where $C > 0$ is defined in Proposition 4.4.2. Similarly, by (4.2.9), with an appropriate choice of ε_0, and (4.4.56), we can apply Proposition 4.4.2 with

$$q = r, \quad q_0 = p_0, \quad \gamma_r = \frac{1}{p-1}\left[1-\frac{N(p-m)}{2r}\right]$$

and $\delta_r = \delta_2$ as defined in (4.4.58). Hence we obtain

$$\|w\|_{L^r(B_R)} \le Ct^{-\gamma_r}\|u_0\|^{\delta_2}_{L^{p_0}(B_R)}, \tag{4.4.61}$$

where $C > 0$ is defined in Proposition 4.4.2. Plugging (4.4.60) and (4.4.61) into (4.4.59) we obtain

$$\begin{aligned}\|w\|^m_{L^\infty(B_R)} &\le \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} \left\{\|w^p\|_{L^r(B_R)} + \frac{1}{(m-1)t}\,\|w\|_{L^r(B_R)}\right\}^{\frac{1}{s}} \|w\|^{m\frac{s-1}{s}}_{L^m(B_R)} \\ &\le \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} \left\{C^p\,t^{-p\,\gamma_{pr}}\|u_0\|^{\delta_1}_{L^{p_0}(B_R)} + \frac{1}{(m-1)t}\,C\,t^{-\gamma_r}\|u_0\|^{\delta_2}_{L^{p_0}(B_R)}\right\}^{\frac{1}{s}} \|w\|^{m\frac{s-1}{s}}_{L^m(B_R)}.\end{aligned}$$

Observe that $-p\gamma_{pr} = -\gamma_r - 1 = \gamma$, where γ has been defined in (4.4.58). Hence we obtain

$$\|w\|^m_{L^\infty(B_R)} \le \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} t^{-\frac{\gamma}{s}} \left\{C^p \|u_0\|^{\delta_1}_{L^{p_0}(B_R)} + \frac{1}{m-1} C \|u_0\|^{\delta_2}_{L^{p_0}(B_R)}\right\}^{\frac{1}{s}} \|w\|^{m\frac{s-1}{s}}_{L^m(B_R)}.$$

Moreover, since $u_0 \in L^\infty(B_R)$, we can apply Lemma 4.4.1 to w with $q = m$. Thus from (4.4.32) with $q = m$ we get

$$\|w\|^m_{L^\infty(B_R)} \le \frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} t^{-\frac{\gamma}{s}} \left\{C^p \|u_0\|^{\delta_1}_{L^{p_0}(B_R)} + \frac{1}{m-1} C \|u_0\|^{\delta_2}_{L^{p_0}(B_R)}\right\}^{\frac{1}{s}} \|u_0\|^{m\frac{s-1}{s}}_{L^m(B_R)}.$$

Finally define

$$\Gamma := \left[\frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}} \max\left\{C^{\frac{p}{s}}\,;\ C^{\frac{1}{s}}\right\}\right]^{\frac{1}{m}}.$$

Hence we obtain

$$\|w\|_{L^\infty(B_R)} \le \Gamma\, t^{-\frac{\gamma}{ms}} \left\{\|u_0\|^{\delta_1}_{L^{p_0}(B_R)} + \frac{1}{m-1} \|u_0\|^{\delta_2}_{L^{p_0}(B_R)}\right\}^{\frac{1}{ms}} \|u_0\|^{\frac{s-1}{s}}_{L^m(B_R)}.$$

□

4.5 Proof of Theorem 4.2.2

Proof of Theorem 4.2.2. Let $\{u_{0,h}\}_{h\ge 0}$ be a sequence of functions such that

(a) $u_{0,h} \in L^\infty(M) \cap C^\infty_c(M)$ for all $h \ge 0$,
(b) $u_{0,h} \ge 0$ for all $h \ge 0$,
(c) $u_{0,h_1} \le u_{0,h_2}$ for any $h_1 < h_2$,
(d) $u_{0,h} \longrightarrow u_0$ in $L^m(M) \cap L^{p_0}(M)$ as $h \to +\infty$,

where p_0 has been defined in (4.2.8). Observe that, due to assumptions (c) and (d), $u_{0,h}$ satisfies (4.2.9). For any $R > 0$, $k > 0$, $h > 0$, consider the problem

$$\begin{cases} u_t = \Delta u^m + T_k(u^p) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, \infty) \\ u = u_{0,h} & \text{in } B_R \times \{0\}\,. \end{cases} \tag{4.5.62}$$

From standard results it follows that problem (4.5.62) has a solution $u^R_{h,k}$ in the sense of Definition 4.3.1; moreover, $u^R_{h,k} \in C\big([0,T]; L^q(B_R)\big)$ for any $q > 1$. Hence, by Lemma 4.4.1, in Proposition 4.4.2 and in Proposition 4.4.4, we have for any $t \in (0, +\infty)$,

$$\|u^R_{h,k}(t)\|_{L^m(B_R)} \le \|u_{0,h}\|_{L^m(B_R)}; \tag{4.5.63}$$

$$\|u^R_{h,k}(t)\|_{L^p(B_R)} \le C\, t^{-\gamma_p} \|u_{0,h}\|^{\delta_p}_{L^{p_0}(B_R)}\,; \tag{4.5.64}$$

where

$$\gamma_p = \frac{1}{p-1}\left[1 - \frac{N(p-m)}{2p}\right], \quad \delta_p = \frac{p-m}{p-1}\left[1 + \frac{N(m-1)}{2p}\right],$$

$$\|u^R_{h,k}\|_{L^\infty(B_R)} \le \Gamma\, t^{-\frac{\gamma}{ms}} \left\{\|u_{0,h}\|^{\delta_1}_{L^{p_0}(B_R)} + \frac{1}{m-1}\|u_{0,h}\|^{\delta_2}_{L^{p_0}(B_R)}\right\}^{\frac{1}{ms}} \|u_{0,h}\|^{\frac{s-1}{s}}_{L^m(B_R)}, \tag{4.5.65}$$

with s as in (4.4.56) and γ, δ_1, δ_2 as in (4.4.58). In addition, for any $\tau \in (0,T), \zeta \in C^1_c((\tau,T)), \zeta \ge 0$, $\max_{[\tau,T]} \zeta' > 0$,

$$\begin{aligned}\int_\tau^T \zeta(t)\left[\left((u^R_{h,k})^{\frac{m+1}{2}}\right)_t\right]^2 d\mu dt &\le \max_{[\tau,T]} \zeta' \bar{C} \int_{B_R} (u^R_{h,k})^{m+1}(x,\tau)d\mu \\ &\quad + \bar{C} \max_{[\tau,T]} \zeta \int_{B_R} F\big(u^R_{h,k}(x,T)\big)d\mu \\ &\le \max_{[\tau,T]} \zeta'(t)\bar{C}\|u^R_{h,k}(\tau)\|_{L^\infty(B_R)}\|u^R_{h,k}(\tau)\|^m_{L^m(B_R)} \\ &\quad + \frac{\bar{C}}{m+p}\|u^R_{h,k}(T)\|^p_{L^\infty(B_R)}\|u^R_{h,k}(T)\|^m_{L^m(B_R)}\end{aligned} \tag{4.5.66}$$

where

$$F(u) = \int_0^u s^{m-1+p}\, ds\,,$$

and $\bar{C} > 0$ is a constant only depending on m. Inequality (4.5.66) is formally obtained by multiplying the differential inequality in problem (4.3.25) by $\zeta(t)[(u^m)_t]$, and integrating by parts; indeed, a standard approximation procedure is needed (see [49, Lemma 3.3] and [7, Theorem 13]).

Moreover, as a consequence of Definition 4.3.1, for any $\varphi \in C^\infty_c(B_R \times [0,T])$ such that $\varphi(x,T) = 0$ for any $x \in B_R$, $u^R_{h,k}$ satisfies

$$\begin{aligned}-\int_0^T \int_{B_R} u^R_{h,k}\, \varphi_t\, d\mu\, dt = &\int_0^T \int_{B_R} (u^R_{h,k})^m\, \Delta\varphi\, d\mu\, dt + \int_0^T \int_{B_R} T_k[(u^R_{h,k})^p]\, \varphi\, d\mu\, dt \\ &+ \int_{B_R} u_{0,h}(x)\, \varphi(x,0)\, d\mu,\end{aligned} \tag{4.5.67}$$

where all the integrals are finite. Now, observe that, for any $h > 0$ and $R > 0$ the sequence of solutions $\{u^R_{h,k}\}_{k\ge 0}$ is monotone increasing in k hence it has a pointwise limit for $k \to \infty$. Let u^R_h be such limit so that we have

$$u^R_{h,k} \longrightarrow u^R_h \quad \text{as } k \to \infty \text{ pointwise.}$$

In view of (4.5.63), (4.5.64) and (4.5.65), the right hand side of (4.5.66) is independent of k. So, $(u^R_h)^{\frac{m+1}{2}} \in H^1((\tau,T); L^2(B_R))$. Therefore, $(u^R_h)^{\frac{m+1}{2}} \in C\big([\tau,T]; L^2(B_R)\big)$. We can now pass to the limit as $k \to +\infty$ in inequalities (4.5.63), (4.5.64) and (4.5.65) arguing as follows. From inequality (4.5.63) and (4.5.64), thanks to the Fatou's Lemma, one has for all $t > 0$

$$\|u^R_h(t)\|_{L^m(B_R)} \le \|u_{0,h}\|_{L^m(B_R)}. \tag{4.5.68}$$

$$\|u_h^R(t)\|_{L^p(B_R)} \le C\, t^{-\gamma_p} \|u_{0,h}\|_{L^{p_0}(B_R)}^{\delta_p}\,; \tag{4.5.69}$$

On the other hand, from (4.5.65), since $u_{h,k}^R \longrightarrow u_h^R$ as $k \to \infty$ pointwise and the right hand side of (4.5.65) is independent of k, one has for all $t > 0$

$$\|u_h^R\|_{L^\infty(B_R)} \le \Gamma\, t^{-\frac{\gamma}{ms}} \left\{ \|u_{0,h}\|_{L^{p_0}(B_R)}^{\delta_1} + \frac{1}{m-1} \|u_{0,h}\|_{L^{p_0}(B_R)}^{\delta_2} \right\}^{\frac{1}{ms}} \|u_{0,h}\|_{L^m(B_R)}^{\frac{s-1}{s}}, \tag{4.5.70}$$

with s as in (4.4.56) and γ, δ_1, δ_2 as in (4.4.58). Note that (4.5.68), (4.5.69) and (4.5.70) hold *for all* $t > 0$, in view of the continuity property of u deduced above. Moreover, thanks to Beppo Levi's monotone convergence theorem, it is possible to compute the limit as $k \to +\infty$ in the integrals of equality (4.5.67) and hence obtain that, for any $\varphi \in C_c^\infty(B_R \times (0,T))$ such that $\varphi(x,T) = 0$ for any $x \in B_R$, the function u_h^R satisfies

$$\begin{aligned} -\int_0^T \int_{B_R} u_h^R \, \varphi_t \, d\mu \, dt = \int_0^T \int_{B_R} \left(u_h^R\right)^m \Delta\varphi \, d\mu \, dt + \int_0^T \int_{B_R} \left(u_h^R\right)^p \varphi \, d\mu \, dt \\ + \int_{B_R} u_{0,h}(x)\, \varphi(x,0)\, d\mu. \end{aligned} \tag{4.5.71}$$

Observe that all the integrals in (4.5.71) are finite, hence u_h^R is a solution to problem (4.5.62), where we replace $T_k(u^p)$ with u^p itself, in the sense of Definition 4.3.1. Indeed we have, due to (4.5.68), $u_h^R \in L^m(B_R \times (0,T))$ hence $u_h^R \in L^1(B_R \times (0,T))$. Moreover, due to (4.5.69), $u_h^R \in L^p(B_R \times (0,T))$ indeed we can write

$$\begin{aligned} \int_0^T \int_{B_R} \left(u_h^R\right)^p d\mu \, dt &= \int_0^T \|u_h^R\|_{L^p(B_R)}^p \, dt \\ &\le \int_0^T \left(C\, t^{-\gamma_p} \|u_{0,h}\|_{L^{p_0}(B_R)}^{\delta_p} \right)^p dt \\ &= C^p \, \|u_{0,h}\|_{L^{p_0}(B_R)}^{p\delta_p} \int_0^T t^{-p\gamma_p} \, dt. \end{aligned} \tag{4.5.72}$$

Now observe that the integral in (4.5.72) is finite if and only if $p\,\gamma_p < 1$. The latter reads $p > m + \frac{2}{N}$, which is guaranteed by the hypotheses of Theorem 4.2.2.

Let us now observe that, for any $h > 0$, the sequence of solutions $\{u_h^R\}_{R>0}$ is monotone increasing in R, hence it has a pointwise limit as $R \to +\infty$. We call its limit function u_h so that

$$u_h^R \longrightarrow u_h \quad \text{as } R \to +\infty \text{ pointwise.}$$

In view of (4.5.63), (4.5.64), (4.5.65), (4.5.68), (4.5.69), (4.5.70), the right hand side of (4.5.66) is independent of k and R. So, $(u_h)^{\frac{m+1}{2}} \in H^1((\tau,T); L^2(M))$. Therefore, $(u_h)^{\frac{m+1}{2}} \in C\big([\tau,T]; L^2(M)\big)$. Since $u_0 \in L^m(M) \cap L^{p_0}(M)$, there exists $k_0 > 0$ and $k_1 > 0$ such that

$$\begin{aligned} &\|u_{0h}\|_{L^m(B_R)} \le k_0 \qquad \forall\, h > 0, \;\; \forall\, R > 0\,, \\ &\|u_{0h}\|_{L^{p_0}(B_R)} \le k_1 \quad \forall\, h > 0, \;\; \forall\, R > 0\,. \end{aligned} \tag{4.5.73}$$

Note that, in view of (4.5.73), the norms in (4.5.68), (4.5.69) and (4.5.70) do not depend on R (see Lemma 4.4.1, Proposition 4.4.2, Proposition 4.4.4 and Remark 4.4.5). Therefore, we pass to the limit as $R \to +\infty$ in (4.5.68), (4.5.69) and (4.5.70). By Fatou's

Lemma,

$$\|u_h(t)\|_{L^m(M)} \le \|u_{0,h}\|_{L^m(M)}, \tag{4.5.74}$$

$$\|u_h(t)\|_{L^p(M)} \le C\, t^{-\gamma_p} \|u_{0,h}\|^{\delta_p}_{L^{p_0}(M)}\,, \tag{4.5.75}$$

furthermore, since $u_h^R \longrightarrow u_h$ as $R \to +\infty$ pointwise,

$$\|u_h\|_{L^\infty(M)} \le \Gamma\, t^{-\frac{\gamma}{ms}} \left\{ \|u_{0,h}\|^{\delta_1}_{L^{p_0}(M)} + \frac{1}{m-1} \|u_{0,h}\|^{\delta_2}_{L^{p_0}(M)} \right\}^{\frac{1}{ms}} \|u_{0,h}\|^{\frac{s-1}{s}}_{L^m(M)}, \tag{4.5.76}$$

with s as in (4.4.56) and γ, δ_1, δ_2 as in (4.4.58). Note that (4.5.74), (4.5.75) and (4.5.76) hold *for all* $t > 0$, in view of the continuity property of u_h^R deduced above.

Moreover, again by monotone convergence, it is possible to compute the limit as $R \to +\infty$ in the integrals of equality (4.5.71) and hence obtain that, for any $\varphi \in C_c^\infty(M \times (0,T))$ such that $\varphi(x,T) = 0$ for any $x \in M$, the function u_h satisfies,

$$\begin{aligned} -\int_0^T \int_M u_h\, \varphi_t \, d\mu\, dt = \int_0^T \int_M (u_h)^m\, \Delta\varphi \, d\mu\, dt + \int_0^T \int_M (u_h)^p \, \varphi \, d\mu\, dt \\ + \int_M u_{0,h}(x)\, \varphi(x,0)\, d\mu. \end{aligned} \tag{4.5.77}$$

Observe that, arguing as above, due to inequalities (4.5.74) and (4.5.75), all the integrals in (4.5.77) are well posed hence u_h is a solution to problem (4.1.1), where we replace u_0 with $u_{0,h}$, in the sense of Definition 4.2.1. Finally, let us observe that $\{u_{0,h}\}_{h\ge 0}$ has been chosen in such a way that

$$u_{0,h} \longrightarrow u_0 \ \text{ in } L^m(M) \cap L^{p_0}(M).$$

Observe also that $\{u_h\}_{h\ge 0}$ is a monotone increasing function in h hence it has a limit as $h \to +\infty$. We call u the limit function. In view (4.5.63), (4.5.64), (4.5.65), (4.5.68), (4.5.69), (4.5.70), (4.5.74), (4.5.75) and (4.5.76) the right hand side of (4.5.66) is independent of k, R and h. So, $u^{\frac{m+1}{2}} \in H^1((\tau,T); L^2(M))$. Therefore, $u^{\frac{m+1}{2}} \in C\big([\tau,T]; L^2(M)\big)$. Hence, we can pass to the limit as $h \to +\infty$ in (4.5.74), (4.5.75) and (4.5.76) and similarly to what we have seen above, we get

$$\|u(t)\|_{L^m(M)} \le \|u_0\|_{L^m(M)}, \tag{4.5.78}$$

$$\|u(t)\|_{L^p(M)} \le C\, t^{-\gamma_p} \|u_0\|^{\delta_p}_{L^{p_0}(M)}\,, \tag{4.5.79}$$

and

$$\|u\|_{L^\infty(M)} \le \Gamma\, t^{-\frac{\gamma}{ms}} \left\{ \|u_0\|^{\delta_1}_{L^{p_0}(M)} + \frac{1}{m-1} \|u_0\|^{\delta_2}_{L^{p_0}(M)} \right\}^{\frac{1}{ms}} \|u_0\|^{\frac{s-1}{s}}_{L^m(M)}, \tag{4.5.80}$$

with s as in (4.4.56) and γ, δ_1, δ_2 as in (4.4.58). Note that both (4.5.78), (4.5.79) and (4.5.80) hold *for all* $t > 0$, in view of the continuity property of u deduced above.

Moreover, again by monotone convergence, it is possible to compute the limit as $h \to +\infty$ in the integrals of equality (4.5.77) and hence obtain that, for any $\varphi \in C_c^\infty(M \times (0,T))$ such that $\varphi(x,T) = 0$ for any $x \in M$, the function u satisfies,

$$\begin{aligned} -\int_0^T \int_M u\,\varphi_t\,d\mu\,dt = \int_0^T \int_M u^m\,\Delta\varphi\,d\mu\,dt + \int_0^T \int_M u^p\,\varphi\,d\mu\,dt \\ + \int_M u_0(x)\,\varphi(x,0)\,d\mu. \end{aligned} \tag{4.5.81}$$

Observe that, due to inequalities (4.5.78) and (4.5.79), all the integrals in (4.5.81) are finite, hence u is a solution to problem (4.1.1) in the sense of Definition 4.2.1.

Finally, let us discuss (4.2.13) and (4.2.11). Let $p_0 \le q < \infty$, and observe that, thanks to hypotheses (c) and (d), u_{0h} satisfies hypothesis (4.2.10) for such q and $q_0 = p_0$ as u_0, then we have

$$\|u_{h,k}^R(t)\|_{L^q(B_R)} \le C\,t^{-\gamma_q}\|u_{0,h}\|_{L^{p_0}(B_R)}^{\delta_q}. \tag{4.5.82}$$

Hence, due to (4.5.82), letting $k \to +\infty$, $R \to +\infty$, $h \to +\infty$, by Fatou's Lemma we deduce (4.2.11).

Now let $1 < q < \infty$. If $u_0 \in L^q(M) \cap L^m(M) \cap L^{p_0}(M)$, we choose the sequence u_{0h} in such a way that it further satisfies

$$u_{0,h} \longrightarrow u_0 \quad \text{in } \ L^q(M) \quad \text{ as } \ h \to +\infty\,,$$

and observe that u_{0h} satisfies also (4.2.12) for such q. Then we have that

$$\|u_{h,k}^R(t)\|_{L^q(B_R)} \le \|u_{0,h}\|_{L^q(B_R)}. \tag{4.5.83}$$

Hence, due to (4.5.83), letting $k \to +\infty$, $R \to +\infty$, $h \to +\infty$, by Fatou's Lemma we deduce (4.2.13). $\square$

4.6 Estimates for $p > m$

Lemma 4.6.1. *Let $m > 1, p > m$. Assume that inequalities* (4.1.3) *and* (4.1.2) *hold. Suppose that $u_0 \in L^\infty(B_R)$, $u_0 \ge 0$. Let $1 < q < \infty$ and assume that*

$$\|u_0\|_{L^{p\frac{N}{2}}(B_R)} < \tilde{\varepsilon}_1 \tag{4.6.84}$$

for a suitable $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(p,m,N,C_p,C_s,q)$ sufficiently small. Let u be the solution of problem (4.3.25) *in the sense of Definition 4.3.1, such that in addition $u \in C([0,T);L^q(B_R))$. Then*

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for all } \ t > 0\,. \tag{4.6.85}$$

Proof. Since u_0 is bounded and T_k is a bounded and Lipschitz function, by standard results, there exists a unique solution of problem (4.3.25) in the sense of Definition

4.3.1. We now multiply both sides of the differential equation in problem (4.3.25) by u^{q-1}, therefore

$$\int_{B_R} u_t\, u^{q-1}\, d\mu = \int_{B_R} \Delta(u^m)\, u^{q-1}\, d\mu + \int_{B_R} T_k(u^p)\, u^{q-1}\, d\mu\,.$$

We integrate by parts. This can be justified by standard tools, by an approximation procedure. Using the fact that $T(u^p) \le u^p$, we can write

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_{B_R} u^q\, d\mu &\le -m(q-1)\int_{B_R} u^{m+q-3}\,|\nabla u|^2\, d\mu + \int_{B_R} u^p\, u^{q-1}\, d\mu \\ &\le -\frac{4m(q-1)}{(m+q-1)^2}\int_{B_R}\left|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right|^2 d\mu + \int_{B_R} u^{p+q-1}\, d\mu.\end{aligned} \tag{4.6.86}$$

Now we take $c_1 > 0$, $c_2 > 0$ such that $c_1 + c_2 = 1$. Thus

$$\int_{B_R}\left|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right|^2 d\mu = c_1\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^2_{L^2(B_R)} + c_2\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^2_{L^2(B_R)}. \tag{4.6.87}$$

Take any $\alpha \in (0,1)$. Thanks to (4.1.3), (4.6.87) becomes

$$\begin{aligned}\int_{B_R}\left|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right|^2 d\mu &\ge c_1\, C_p^2\, \|u\|^{m+q-1}_{L^{m+q-1}(B_R)} + c_2\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^2_{L^2(B_R)} \\ &\ge c_1\, C_p^2\, \|u\|^{m+q-1}_{L^{m+q-1}(B_R)} + c_2\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^{2+2\alpha-2\alpha}_{L^2(B_R)} \\ &\ge c_1 C_p^2\, \|u\|^{m+q-1}_{L^{m+q-1}(B_R)} + c_2 C_p^{2\alpha}\, \|u\|^{\alpha(m+q-1)}_{L^{m+q-1}(B_R)}\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^{2-2\alpha}_{L^2(B_R)}.\end{aligned} \tag{4.6.88}$$

Moreover, using the interpolation inequality, Hölder inequality and (4.1.2), we have

$$\begin{aligned}\int_{B_R} u^{p+q-1}\, d\mu, &= \|u\|^{p+q-1}_{L^{p+q-1}} \\ &\le \|u\|^{\theta(p+q-1)}_{L^{m+q-1}(B_R)}\, \|u\|^{(1-\theta)(p+q-1)}_{L^{p+m+q-1}(B_R)} \\ &\le \|u\|^{\theta(p+q-1)}_{L^{m+q-1}(B_R)}\left[\|u\|^{(1-\theta)\frac{p}{p+m+q-1}}_{L^{p\frac{N}{2}}(B_R)}\, \|u\|^{(1-\theta)\frac{m+q-1}{p+m+q-1}}_{L^{(m+q-1)\frac{N}{N-2}}(B_R)}\right]^{p+q-1} \\ &\le \|u\|^{\theta(p+q-1)}_{L^{m+q-1}(B_R)}\, \|u\|^{(1-\theta)\frac{p(p+q-1)}{p+m+q-1}}_{L^{p\frac{N}{2}}(B_R)}\left(\frac{1}{C_s}\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|_{L^2(B_R)}\right)^{2(1-\theta)\frac{p+q-1}{p+m+q-1}}\end{aligned} \tag{4.6.89}$$

where $\theta := \frac{m(m+q-1)}{p(p+q-1)}$. By plugging (4.6.88) and (4.6.89) into (4.6.86) we obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} &\le -\frac{4m(q-1)}{(m+q-1)^2}\, c_1\, C_p^2\, \|u(t)\|^{m+q-1}_{L^{m+q-1}(B_R)} \\ &\quad - \frac{4m(q-1)}{(m+q-1)^2}\, c_2\, C_p^{2\alpha}\, \|u(t)\|^{\alpha(m+q-1)}_{L^{m+q-1}(B_R)}\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^{2-2\alpha}_{L^2(B_R)} \\ &\quad + \tilde{C}\|u(t)\|^{\theta(p+q-1)}_{L^{m+q-1}(B_R)}\, \|u(t)\|^{(1-\theta)\frac{p(p+q-1)}{p+m+q-1}}_{L^{p\frac{N}{2}}(B_R)}\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^{2(1-\theta)\frac{p+q-1}{p+m+q-1}}_{L^2(B_R)}\end{aligned} \tag{4.6.90}$$

where

$$\tilde{C} = \left(\frac{1}{C_s}\right)^{2(1-\theta)\frac{p+q-1}{p+m+q-1}}. \tag{4.6.91}$$

Let us now fix $\alpha \in (0,1)$ such that

$$2 - 2\alpha = 2(1-\theta)\left(\frac{p+q-1}{p+m+q-1}\right).$$

Hence we have

$$\alpha = \frac{m}{p}. \tag{4.6.92}$$

By substituting (4.6.92) into (4.6.90) we obtain

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} \le & -\frac{4m(q-1)}{(m+q-1)^2}\,c_1\,C_p^2\,\|u(t)\|^{m+q-1}_{L^{m+q-1}(B_R)} \\ & -\frac{1}{\tilde{C}}\left\{\frac{4m(q-1)C}{(m+q-1)^2} - \|u(t)\|^{\frac{p(p+q-1)-m(m+q-1)}{p+m+q-1}}_{L^{p\frac{N}{2}}(B_R)}\right\} \\ & \times \|u(t)\|^{\alpha(m+q-1)}_{L^{m+q-1}(B_R)}\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^{2-2\alpha}_{L^2(B_R)},\end{aligned} \tag{4.6.93}$$

where C has been defined in Remark 4.2.6. Observe that, thanks to hypothesis (4.6.84) and the continuity of the solution $u(t)$, there exists $t_0 > 0$ such that

$$\|u(t)\|_{L^{p\frac{N}{2}}(B_R)} \le 2\,\tilde{\varepsilon}_1 \quad \text{for any} \quad t \in (0, t_0]\,.$$

Hence (4.6.93) becomes, for any $t \in (0, t_0]$

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} \le & -\frac{4m(q-1)}{(m+q-1)^2}\,c_1\,C_p^2\,\|u(t)\|^{m+q-1}_{L^{m+q-1}(B_R)} \\ & -\frac{1}{\tilde{C}}\left\{\frac{4m(q-1)C}{(m+q-1)^2} - 2\tilde{\varepsilon}_1^{\frac{p(p+q-1)-m(m+q-1)}{p+m+q-1}}\right\}\|u(t)\|^{\alpha(m+q-1)}_{L^{m+q-1}(B_R)}\left\|\nabla\left(u^{\frac{m+q-1}{2}}\right)\right\|^{2-2\alpha}_{L^2(B_R)} \\ & \le 0\,,\end{aligned}$$

provided $\tilde{\varepsilon}_1$ is small enough. Hence we have proved that $\|u(t)\|_{L^q(B_R)}$ is decreasing in time for any $t \in (0, t_0]$, i.e.

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_0]\,. \tag{4.6.94}$$

In particular, inequality (4.6.94) holds $q = p\frac{N}{2}$. Hence we have

$$\|u(t)\|_{L^{p\frac{N}{2}}(B_R)} \le \|u_0\|_{L^{p\frac{N}{2}}(B_R)} < \tilde{\varepsilon}_1 \quad \text{for any} \quad t \in (0, t_0]\,.$$

Now, we can repeat the same argument in the time interval $(t_0, t_1]$ where t_1 is chosen, thanks to the continuity of $u(t)$, in such a way that

$$\|u(t)\| \le 2\,\tilde{\varepsilon}_1 \quad \text{for any} \quad t \in (t_0, t_1]\,.$$

Thus we get

$$\|u(t)\|_{L^q(B_R)} \le \|u_0\|_{L^q(B_R)} \quad \text{for any} \quad t \in (0, t_1]\,.$$

Iterating this procedure we obtain the thesis.

□

Proposition 4.6.2. *Let $m > 1$, $p > m$, $R > 0$, $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Let*

$$r > \frac{N}{2}, \qquad s = 1 + \frac{2}{N} - \frac{1}{r}. \tag{4.6.95}$$

Suppose that (4.2.18) holds for $\varepsilon_1 = \varepsilon_1(p, m, N, r, C_s, C_p)$ sufficiently small. Let u be the solution to problem (4.3.25), such that in addition $u \in C([0,T); L^q(B_R))$ for any $1 < q < +\infty$ and $T > 0$. Let M support the Sobolev and Poincaré inequalities (4.1.2) and (4.1.3). Then there exists $\Gamma = \Gamma(N, m, l, C_s) > 0$ independent of T such that, for all $t > 0$,

$$\|u(t)\|_{L^\infty(B_R)} \leq \Gamma \, \|u_0\|_{L^m(B_R)}^{\frac{s-1}{s}} \left[\|u_0\|_{L^{pr}(B_R)}^p + \frac{1}{(m-1)t} \|u_0\|_{L^r(B_R)} \right]^{\frac{1}{ms}}. \tag{4.6.96}$$

Remark 4.6.3. If in Proposition 4.6.2, in addition, we assume that for some $k_0 > 0$

$$\max\left\{ \|u_0\|_{L^m(B_R)};\ \|u_0\|_{L^{pr}(B_R)};\ \|u_0\|_{L^r(B_R)} \right\} \leq k_0 \quad \text{for every} \quad R > 0\,,$$

then the bound from above for $\|u(t)\|_{L^\infty(B_R)}$ in (4.6.96) is independent of R.

Proof of Proposition 4.6.2. Let us set $w = u(\cdot, t)$. Observe that $w^m \in H_0^1(B_R)$ and $w \geq 0$. Due to Proposition 4.3.6 we know that

$$-\Delta(w^m) \leq \left[w^p + \frac{w}{(m-1)t} \right].$$

Observe that, since $u_0 \in L^\infty(B_R)$ also $w \in L^\infty(B_R)$. Due to (4.6.95), we can apply Proposition 4.3.4, so we have that

$$\|w\|_{L^\infty(B_R)}^m \leq \frac{s}{s-1} \left(\frac{1}{C_s} \right)^{\frac{2}{s}} \left\| w^p + \frac{w}{(m-1)t} \right\|_{L^r(B_R)}^{\frac{1}{s}} \|w^m\|_{L^1(B_R)}^{\frac{s-1}{s}}.$$

Therefore

$$\|w\|_{L^\infty(B_R)}^m \leq \frac{s}{s-1} \left(\frac{1}{C_s} \right)^{\frac{2}{s}} \left\{ \|w^p\|_{L^r(B_R)} + \frac{1}{(m-1)t} \|w\|_{L^r(B_R)} \right\}^{\frac{1}{s}} \|w\|_{L^m(B_R)}^{m\frac{s-1}{s}}, \tag{4.6.97}$$

where s has been defined in (4.6.95). In view of (4.2.18) with a suitable ε_1, since $u_0 \in L^\infty(B_R)$, we can apply Lemma 4.6.1. Hence we obtain

$$\|w^p\|_{L^r(B_R)} = \|w\|_{L^{pr}(B_R)}^p \leq \|u_0\|_{L^{pr}(B_R)}^p. \tag{4.6.98}$$

Similarly, again for an appropriate ε_1 in (4.2.18), since $u_0 \in L^\infty(B_R)$, we can apply Lemma 4.6.1 and obtain

$$\|w\|_{L^r(B_R)} \leq \|u_0\|_{L^r(B_R)}. \tag{4.6.99}$$

Plugging (4.6.98) and (4.6.99) into (4.6.97) we obtain

$$\begin{aligned} \|w\|_{L^\infty(B_R)}^m &\leq \frac{s}{s-1} \left(\frac{1}{C_s} \right)^{\frac{2}{s}} \left\{ \|w\|_{L^{pr}(B_R)}^p + \frac{1}{(m-1)t} \|w\|_{L^r(B_R)} \right\}^{\frac{1}{s}} \|w\|_{L^m(B_R)}^{m\frac{s-1}{s}} \\ &\leq \frac{s}{s-1} \left(\frac{1}{C_s} \right)^{\frac{2}{s}} \left\{ \|u_0\|_{L^{pr}(B_R)}^p + \frac{1}{(m-1)t} \|u_0\|_{L^r(B_R)} \right\}^{\frac{1}{s}} \|w\|_{L^m(B_R)}^{m\frac{s-1}{s}}. \end{aligned}$$

Moreover, since $u_0 \in L^\infty(B_R)$, we can apply Lemma 4.6.1 to w with $q = m$. Thus from (4.6.85) with $q = m$ we get

$$\|w\|_{L^\infty(B_R)} \le \left[\frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}}\right]^{\frac{1}{m}} \|u_0\|_{L^m(B_R)}^{\frac{s-1}{s}} \left[\|u_0\|_{L^{pr}(B_R)}^p + \frac{1}{(m-1)t}\|u_0\|_{L^r(B_R)}\right]^{\frac{1}{ms}}. \tag{4.6.100}$$

We define

$$\Gamma := \left[\frac{s}{s-1}\left(\frac{1}{C_s}\right)^{\frac{2}{s}}\right]^{\frac{1}{m}}. \tag{4.6.101}$$

Then from (4.6.100) we get

$$\|w\|_{L^\infty(B_R)} \le \Gamma \|u_0\|_{L^m(B_R)}^{\frac{s-1}{s}} \left[\|u_0\|_{L^{pr}(B_R)}^p + \frac{1}{(m-1)t}\|u_0\|_{L^r(B_R)}\right]^{\frac{1}{ms}}.$$

□

Proof of Theorem 4.2.5. The proof of Theorem 4.2.5 follows the same line of arguments of that of Theorem 4.2.2, with minor differences. Let $\{u_{0,h}\}_{h\ge 0}$ be a family of functions such that

(a) $u_{0,h} \in L^\infty(M) \cap C_c^\infty(M)$ for all $h \ge 0$,

(b) $u_{0,h} \ge 0$ for all $h \ge 0$,

(c) $u_{0,h_1} \le u_{0,h_2}$ for any $h_1 < h_2$,

(d) $u_{0,h} \longrightarrow u_0$ in $L^\theta(M) \cap L^{pr}(M)$ where $\theta := \min\{m, r\}$ as $h \to +\infty$,

Observe that, due to assumptions (c) and (d), $u_{0,h}$ satisfies (4.2.18) for an appropriate ε_1 sufficiently small. Moreover, thanks by interpolation, since $m < p < pr$, we have

$$u_{0,h} \longrightarrow u_0 \text{ in } L^p(M) \quad \text{as } h \to +\infty.$$

For any $R > 0$, $k > 0$, $h > 0$, consider the problem

$$\begin{cases} u_t = \Delta u^m + T_k(u^p) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, \infty) \\ u = u_{0,h} & \text{in } B_R \times \{0\}. \end{cases} \tag{4.6.102}$$

From standard results it follows that problem (4.6.102) has a solution $u_{h,k}^R$ in the sense of Definition 4.3.1; moreover, $u_{h,k}^R \in C\big([0,T]; L^q(B_R)\big)$ for any $q > 1$. Hence, it satisfies the inequalities in Lemma 4.6.1 and in Proposition 4.6.2, i.e., for any $t \in (0, +\infty)$,

$$\|u_{h,k}^R(t)\|_{L^m(B_R)} \le \|u_{0,h}\|_{L^m(B_R)};$$

$$\|u_{h,k}^R(t)\|_{L^p(B_R)} \le \|u_{0,h}\|_{L^p(B_R)};$$

$$\|u_{h,k}^R\|_{L^\infty(B_R)} \le \Gamma \|u_{0,h}\|_{L^m(B_R)}^{\frac{s-1}{s}} \left[\|u_{0,h}\|_{L^{pr}(B_R)}^p + \frac{1}{(m-1)t}\|u_{0,h}\|_{L^r(B_R)}\right]^{\frac{1}{ms}},$$

with r and s as in (4.6.95) and Γ as in (4.6.101). Arguing as in the proof of Theorem (4.2.13), we can pass to the limit as $k \to +\infty, R \to +\infty, h \to \infty$ obtaining a function u, which satisfies

$$\|u(t)\|_{L^m(M)} \leq \|u_0\|_{L^m(M)}, \tag{4.6.103}$$

$$\|u(t)\|_{L^p(M)} \leq \|u_0\|_{L^p(M)}, \tag{4.6.104}$$

and

$$\|u\|_{L^\infty(M)} \leq \Gamma \, \|u_0\|_{L^m(M)}^{\frac{s-1}{s}} \left[\|u_0\|_{L^{pr}(M)}^p + \frac{1}{(m-1)t} \|u_0\|_{L^r(M)} \right]^{\frac{1}{ms}}, \tag{4.6.105}$$

with r and s as in (4.6.95) and Γ as in (4.6.101). Moreover, for any $\varphi \in C_c^\infty(M \times (0,T))$ such that $\varphi(x,T) = 0$ for any $x \in M$, the function u satisfies

$$\begin{aligned} -\int_0^T \int_M u \, \varphi_t \, d\mu \, dt = \int_0^T \int_M u^m \, \Delta\varphi \, d\mu \, dt + \int_0^T \int_M u^p \, \varphi \, d\mu \, dt \\ + \int_M u_0(x) \, \varphi(x,0) \, d\mu. \end{aligned} \tag{4.6.106}$$

Observe that, due to inequalities (4.6.103), (4.6.104) and (4.6.105), all the integrals in (4.6.106) are finite, hence u is a solution to problem (4.1.1) in the sense of Definition 4.2.1. Finally, using hypothesis (4.2.19), inequality (4.2.20) can be derived exactly as (4.2.13). □

4.7 Proofs of Theorems 4.2.8 and 4.2.9

We use the following Aronson-Benilan type estimate (see [6]; see also [118, Proposition 2.3]); it can be shown exactly as Proposition 4.3.6.

Proposition 4.7.1. *Let $m > 1$, $p > m$, $u_0 \in H_0^1(B_R) \cap L^\infty(B_R)$, $u_0 \geq 0$. Let u be the solution to problem* (4.7.107)*. Then, for a.e. $t \in (0,T)$,*

$$-\Delta u^m(\cdot,t) \leq \rho u^p(\cdot,t) + \frac{\rho}{(m-1)t} u(\cdot,t) \quad in \ \mathfrak{D}'(B_R).$$

For any $R > 0$, consider the following approximate problem

$$\begin{cases} \rho(x) u_t = \Delta u^m + \rho(x) u^p & \text{in } B_R \times (0,T) \\ u = 0 & \text{in } \partial B_R \times (0,T) \\ u = u_0 & \text{in } B_R \times \{0\}, \end{cases} \tag{4.7.107}$$

where B_R denotes the Euclidean ball with radius R and centre in the origin O.

We exploit the following estimate, which can be proved as that in Lemma 4.4.1.

Lemma 4.7.2. *Let*

$$m > 1, \qquad p > m + \frac{2}{N}.$$

Suppose that inequality (4.1.6) *holds. Suppose that* $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. *Let* $1 < q < \infty$, p_0 *be as in* (4.2.8) *and assume that*

$$\|u_0\|_{L^{p_0}_\rho(B_R)} < \bar{\varepsilon},$$

for $\bar{\varepsilon} = \bar{\varepsilon}(p, m, C_s, q)$ *small enough. Let* u *be the solution of problem* (4.7.107), *such that in addition* $u \in C([0,T), L^q_\rho(B_R))$ *for any* $q \in (1, +\infty)$, *for any* $T > 0$. *Then*

$$\|u(t)\|_{L^q_\rho(B_R)} \leq \|u_0\|_{L^q_\rho(B_R)} \quad \text{for all } \; t > 0\,.$$

The following smoothing estimate is also used; the proof is the same as that of Proposition 4.4.2.

Proposition 4.7.3. *Let*

$$m > 1, \qquad p > m + \frac{2}{N},$$

Assume (4.2.23) *and* (4.1.6). *Suppose that* $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. *Let* u *be the solution of problem* (4.7.107), *such that in addition* $u \in C([0,T), L^q_\rho(B_R))$ *for any* $q \in (1, +\infty)$, *for any* $T > 0$. *Assume that* (4.2.9) *holds for* $\varepsilon_0 = \varepsilon_0(p, m, N, r, C_s)$ *sufficiently small. There exists* $C(m, q_0, C_s, \varepsilon, N, q) > 0$ *such that*

$$\|u(t)\|_{L^q_\rho(B_R)} \leq C\, t^{-\gamma_q} \|u_0\|^{\delta_q}_{L^{q_0}_\rho(B_R)} \quad \text{for all } \; t > 0\,,$$

where

$$\gamma_q = \left(\frac{1}{q_0} - \frac{1}{q}\right) \frac{N\, q_0}{2\, q_0 + N(m-1)}\,; \quad \delta_q = \frac{q_0}{q} \left(\frac{q + \frac{N}{2}(m-1)}{q_0 + \frac{N}{2}(m-1)}\right)\,.$$

Proof of Theorem 4.2.8. The conclusion follows by repeating the same arguments as in the proof of Theorem 4.2.2. We use Lemma 4.7.2 instead of Lemma 4.4.1, Proposition 4.7.3 instead of 4.4.2 and Proposition 4.7.1 instead of Proposition 4.3.6. □

4.7.1 Proof of Theorem 4.2.9

We consider problem (4.7.107). We use the following estimate, which can be proved as that in Lemma 4.6.1.

Lemma 4.7.4. *Let*

$$m > 1, \qquad p > m.$$

Assume that (4.1.6) *and* (4.1.7) *hold. Suppose that* $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. *Let* $1 < q < \infty$ *and assume that and assume that*

$$\|u_0\|_{L^{p\frac{N}{2}}(B_R)} < \tilde{\varepsilon}_1$$

for a suitable $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(p, m, N, C_p, C_s, q)$ *sufficiently small. Let* u *be the solution of problem* (4.7.107), *such that in addition* $u \in C([0,T), L^q(B_R))$ *for any* $q \in (1, +\infty)$, *for any* $T > 0$. *Then*

$$\|u(t)\|_{L^q(B_R)} \leq \|u_0\|_{L^q(B_R)} \quad \text{for all } \; t > 0\,.$$

Proof of Theorem 4.2.9. The conclusion follows arguing step by step as in the proof of Theorem 4.2.5. We use Lemma 4.7.4 instead of Lemma 4.6.1 and Proposition 4.7.1 instead of Proposition 4.3.6. □

Chapter 5

Global existence and smoothing estimates for $p < m$

5.1 Introduction

Let M be a complete noncompact Riemannian manifold of infinite volume, whose dimension N will be required throughout the chapter to satisfy the bound $N \geq 3$. Let us consider the following Cauchy problem, for any $T > 0$

$$\begin{cases} u_t = \Delta u^m + u^p & \text{in } M \times (0,T) \\ u = u_0 & \text{in } M \times \{0\} \end{cases} \tag{5.1.1}$$

where Δ is the Laplace-Beltrami operator. We shall assume throughout this chapter that $1 < p < m$ and that the initial datum u_0 is nonnegative. We let $\mathrm{L}^q(M)$ be as usual the space of those measurable functions f such that $|f|^q$ is integrable w.r.t. the Riemannian measure μ and make the following basic assumptions on M, which amount to assuming the validity of both the Poincaré and the Sobolev inequalities on M:

$$\text{(Poincaré inequality)} \qquad \|v\|_{L^2(M)} \leq \frac{1}{C_p}\|\nabla v\|_{L^2(M)} \quad \text{for any } v \in C_c^\infty(M); \tag{5.1.2}$$

$$\text{(Sobolev inequality)} \qquad \|v\|_{L^{2^*}(M)} \leq \frac{1}{C_s}\|\nabla v\|_{L^2(M)} \quad \text{for any } v \in C_c^\infty(M), \tag{5.1.3}$$

where C_p and C_s are numerical constants and $2^* := \frac{2N}{N-2}$. The validity of (5.1.2), (5.1.3) puts constraints on M, and we comment that it is e.g. well known that, on *Cartan-Hadamard manifolds*, namely complete and simply connected manifolds that have everywhere non-positive sectional curvature, (5.1.3) always holds. Furthermore, when M is Cartan-Hadamard and, besides, $\sec \leq -c < 0$ everywhere, sec indicating sectional curvature, it is known that (5.1.2) holds as well, see e.g. [39, 40]. Thus, both (5.1.2), (5.1.3) hold when M is Cartan-Hadamard and $\sec \leq -c < 0$ everywhere, a case that strongly departs from the Euclidean situation but covers a wide class of manifolds, including e.g. the fundamental example of the hyperbolic space $\mathbb{H}^n$, namely that Cartan-Hadamard manifold whose sectional curvatures equal -1 everywhere (or the similar case in which $\sec = -k$ everywhere, for a given $k > 0$).

In Chapter 4, (where [46] is reproduced), problem (5.1.1) has been studied when $p > m$. We refer the reader Section 4.1 for a comprehensive account of the literature.

5.1.1 Qualitative statements of main results in the manifold setting

We extend here the results of [54] in two substantial aspects. In fact, we summarize our main results as follows.

- The methods of [54] rely heavily on explicit *barrier arguments*, that by their very same nature are applicable to compactly supported data only and, in addition, require explicit curvature bounds in order to be applicable. We prove here global existence for L^m data and prove *smoothing effects* for solutions to (5.1.1), where by smoothing effect we mean the fact that L^m data give rise to global solutions $u(t)$ such that $u(t) \in \mathrm{L}^\infty$ for all $t > 0$, with quantitative bounds on their L^∞ norm. This will be a consequence *only* of the validity of Sobolev and Poincaré inequalities (5.1.3), (5.1.2), see Theorem 5.2.2.

- As a consequence, combining this fact with some results proved in [54], we can prove that, on manifolds satisfying e.g. $-c_1 \le \sec \le -c_2$ with $c_1 \ge c_2 > 0$, thus encompassing the particularly important case of the hyperbolic space $\mathbb{H}^n$ (somewhat weaker lower curvature bounds can be assumed), any solution $u(t)$ to (5.1.1) corresponding to an initial datum $u_0 \in \mathrm{L}^m$ exists globally and, provided u_0 is sufficiently large, it satisfies the property

  $$\lim_{t\to+\infty} u(x,t) = +\infty \quad \forall x \in M,$$

  namely *complete blowup in infinite time* occurs for such solutions to (5.1.1) in the whole range $p \in (1, m)$, see Theorem 5.2.3.

Our results can also be seen as an extension of some of the results proved in [118]. However, the proof of the smoothing estimate given in [118, Theorem 1.3] is crucially based on the assumption that the measure of the domain where the problem is posed is finite. This is not true in our setting. So, even if we use some general idea introduced in [118], our proofs and results are in general quite different from those in [118].

For detailed reference to smoothing effect for linear evolution equations see [19], whereas we refer to [127] for a general treatment of smoothing effects for nonlinear diffusions, and to [12, 49, 48] for connections with functional inequalities in the nonlinear setting.

We mention phenomena similar to the ones discussed in the present chapter occur in qualitatively related but different settings. For example, we mention that solutions to the heat equation with Dirichlet boundary conditions in a twisted tube (namely a straight tube in $\mathbb{R}^3$ whose cross-section is twisted in a given compact region) give rise to smoothing estimates that are *stronger* for large times than the ones corresponding to the untwisted situation, i.e. the geometry improves the smoothing effects, see [79, 78, 44].

5.1.2 Qualitative statements of main results for Euclidean, weighted reaction-diffusion equations

The main result given in Theorem 5.2.2 depend essentially only on the validity of inequalities (5.1.2) and (5.1.3), and as such is almost immediately generalizable to different contexts. As a particularly significant situation, we single out the case of Euclidean, mass-weighted reaction diffusion equations. In fact we consider the problem

$$\begin{cases} \rho\, u_t = \Delta u^m + \rho\, u^p & \text{in } \mathbb{R}^N \times (0,T) \\ u \ = u_0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{5.1.4}$$

in the Euclidean setting, where $\rho : \mathbb{R}^N \to \mathbb{R}$ is strictly positive, continuous and bounded, and represents a *mass density* . The problem is naturally posed in the weighted spaces

$$L^q_\rho(\mathbb{R}^N) = \left\{ v : \mathbb{R}^N \to \mathbb{R} \text{ measurable },\ \|v\|_{L^q_\rho} := \left(\int_{\mathbb{R}^N} v^q \rho(x)\, dx \right)^{1/q} < +\infty \right\},$$

This kind of models originates in a physical model provided in [73]. There are choices of ρ ensuring that the following analogues of (5.1.2) and (5.1.3) hold:

$$\|v\|_{L^2_\rho(\mathbb{R}^N)} \le \frac{1}{C_p} \|\nabla v\|_{L^2(\mathbb{R}^N)} \quad \text{for any } \ v \in C_c^\infty(\mathbb{R}^N) \tag{5.1.5}$$

and

$$\|v\|_{L^{2^*}_\rho(\mathbb{R}^N)} \le \frac{1}{C_s} \|\nabla v\|_{L^2(\mathbb{R}^N)} \quad \text{for any } \ v \in C_c^\infty(\mathbb{R}^N) \tag{5.1.6}$$

for suitable positive constants. In fact, in order to make a relevant example, if $\rho(x) \asymp |x|^{-a}$ for a suitable $a > 0$, it can be shown that (5.1.5) holds if $a \ge 2$ (see e.g. [49] and references therein), whereas also (5.1.6) is obviously true for any $a > 0$ because of the validity of the usual, unweighted Sobolev inequality and of the assumptions on ρ. Of course more general cases having a similar nature but where the analogue of (5.1.6) is not a priori trivial, could be considered, but we focus on that example since it is widely studied in the literature and because of its physical significance.

In [86, 87] a large class of nonlinear reaction-diffusion equations, including in particular problem (5.1.4) under certain conditions on ρ, is investigated. It is proved that a global solution exists, (see [86, Theorem 1]) provided that $\rho(x) = |x|^{-a}$ with $a \in (0,2)$,

$$p > m + \frac{2-a}{N-a},$$

and $u_0 \ge 0$ is small enough. In addition, a smoothing estimate holds. On the other hand, if $\rho(x) = |x|^{-a}$ or $\rho(x) = (1+|x|)^{-a}$ with $a \in [0,2)$, $u_0 \not\equiv 0$ and

$$1 < p < m + \frac{2-a}{N-a},$$

then any nonnegative solution blows up in a suitable sense. Such results have also been generalized to more general initial data, decaying at infinity with a certain rate (see

[87]). Finally, in [86, Theorem 2], it is shown that if $p > m$, $\rho(x) = (1 + |x|)^{-a}$ with $a > 2$, and u_0 is small enough, a global solution exists.

Problem (5.1.4) has also been studied in [92], [93], by constructing and using suitable barriers, initial data being continuous and compactly supported. In particular, in [92] the case that $\rho(x) \asymp |x|^{-a}$ for $|x| \to +\infty$ with $a \in (0, 2)$ is addressed. It is proved that for any $p > 1$, if u_0 is large enough, then blowup occurs. On the other hand, if $p > \bar{p}$, for a certain $\bar{p} > m$ depending on m, p and ρ, and u_0 is small enough, then global existence of bounded solutions prevails. Moreover, in [93] the case that $a \geq 2$ is investigated. For $a = 2$, blowup is shown to occur when u_0 is big enough, whereas global existence holds when u_0 is small enough. For $a > 2$ it is proved that if $p > m$, $u_0 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ and goes to 0 at infinity with a suitable rate, then there exists a global bounded solution. Furthermore, for the same initial datum u_0, if $1 < p < m$, then there exists a global solution, which could blow up as $t \to +\infty$.

Our main results in this setting can be summarized as follows.

- We prove in Theorem 5.2.5 global existence and smoothing effects for solutions to (5.1.4), assuming that the weight $\rho : \mathbb{R}^N \to \mathbb{R}$ is strictly positive, smooth and bounded, so that (5.1.6) necessarily holds, and assuming the validity of (5.1.5). In particular, L^m data give rise to global solutions $u(t)$ such that $u(t) \in \mathrm{L}^\infty$ for all $t > 0$, with quantitative bounds on their L^∞ norm. By constructing a specific, delicate example, we show in Proposition 5.6.6 that the bound on the L^∞ norm (which involves a quantity diverging as $t \to +\infty$) is qualitatively sharp, in the sense that there are examples of weights for which our running assumption holds and for which blow-up of solutions in infinite time holds pointwise everywhere (we refer to this property by saying that *complete blowup in infinite time* occurs). We also prove, by similar methods which follow the lines of [118], different smoothing effects which are stronger for large times, when ρ is in addition assumed to be integrable, see Theorem 5.2.6.

Let us mention that the results in [93] for $1 < p < m$ are improved here in various directions. In fact, now we consider a larger class of initial data u_0, since we do not require that they are locally bounded; moreover, in [93] no smoothing estimates are addressed. Furthermore, the fact that for integrable weights ρ we have global existence of bounded solutions does not have a counterpart in [93], nor has the blowup results in infinite time.

5.1.3 On some open problems

As stated above, the present chapter settles the problem of global existence of solutions to problem (5.1.1) on manifolds M supporting both the Sobolev and the Poincaré inequalities, in the case $1 < p < m$ and for data belonging to $L^m(M)$. It is also shown that solutions corresponding to such data are bounded for all $t > 0$, with quantitative bounds on the $L^\infty(M)$ norm of solutions for all $t > 0$. We also settle the long-time behaviour of solutions to problem (5.1.1) on manifolds M whose curvature is pinched between two strictly negative constants, where $1 < p < m$ and data belong to $L^m(M)$, showing that they blowup pointwise in infinite time. The following questions are however open for further investigation:

- Does similar results hold for data in Lebesgue spaces $L^q(M)$ with $q \neq m$? The present method of proof does not extend to such data.
- Do *all* initial data in $L^q(M)$ blow up in infinite time, or the long-time asymptotic of small data is different?

5.1.4 Organizazion of the chapter

In Section 5.2 we collect the relevant definitions and state our main results, both in the setting of Riemannian manifolds and in the Euclidean, weighted case. In Section 5.3 we prove some crucial results for an auxiliary elliptic problem, that will then be used in Section 5.4 to show bounds on the L^p norms of solutions to certain evolution problems posed on geodesic balls. In Section 5.5 we conclude the proof of our main results for the case of reaction-diffusion problems on manifolds. In Section 5.6 we briefly comment on the adaptation to be done to deal with the weighted Euclidean case, and prove the additional results valid in the case of an integrable weight. We also discuss there a delicate example showing that complete blowup in infinite time may occur under the running assumptions.

5.2 Preliminaries and statement of main results

We first define the concept of solution to (5.1.1) that we shall use hereafter. It will be meant in the very weak, or distributional, sense.

Definition 5.2.1. *Let M be a complete noncompact Riemannian manifold of infinite volume. Let $1 < p < m$ and $u_0 \in L^m(M)$, $u_0 \geq 0$. We say that the function u is a solution to problem* (5.1.1) *in the time interval* $[0, T)$ *if*

$$u \in L^m(M \times (0,T)),$$

and for any $\varphi \in C_c^\infty(M \times [0,T])$ such that $\varphi(x,T) = 0$ for any $x \in M$, u satisfies the equality:

$$\begin{aligned} -\int_0^T \int_M u\, \varphi_t \, d\mu\, dt = & \int_0^T \int_M u^m \, \Delta\varphi \, d\mu\, dt + \int_0^T \int_M u^p \, \varphi \, d\mu\, dt \\ & + \int_M u_0(x)\, \varphi(x,0)\, d\mu. \end{aligned}$$

Theorem 5.2.2. *Let M be a complete, noncompact manifold of infinite volume such that the Poincaré and Sobolev inequalities* (5.1.2) *and* (5.1.3) *hold on M. Let $1 < p < m$ and $u_0 \in L^m(M)$, $u_0 \geq 0$. Then problem* (5.1.1) *admits a solution for any $T > 0$, in the sense of Definition 5.2.1. Moreover for any $T > \tau > 0$ one has $u \in L^\infty(M \times (\tau, T))$ and there exist numerical constants $c_1, c_2 > 0$, independent of T, such that, for all $t > 0$ one has*

$$\|u(t)\|_{L^\infty(M)} \leq c_1 e^{c_2 t} \left\{ \|u_0\|_{L^m(M)}^{\frac{2m}{2m+N(m-p)}} + \frac{\|u_0\|_{L^m(M)}^{\frac{2m}{2m+N(m-1)}}}{t^{\frac{N}{2m+N(m-1)}}} \right\}. \tag{5.2.7}$$

Besides, if $q > 1$ and $u_0 \in L^q(M) \cap L^m(M)$, then there exists $C(q) > 0$ such that

$$\|u(t)\|_{L^q(M)} \le e^{C(q)t}\|u_0\|_{L^q(M)} \quad \text{for all } \; t > 0\,. \tag{5.2.8}$$

One may wonder whether the upper bound in (5.2.7) is qualitatively sharp, since its r.h.s. involves a function of time that tends to $+\infty$ as $t \to +\infty$. This is indeed the case, since there is a wide class of situations covered by Theorem 5.2.2 in which classes of solutions do indeed satisfy $\|u(t)\|_\infty \to +\infty$ as $t \to +\infty$ and show even the much stronger property of *blowing up pointwise everywhere in infinite time*. In fact, as a direct consequence of Theorem 5.2.2, of known geometrical conditions for the validity of (5.1.2) and (5.1.3), and of some results given in [54], we can prove the following result. We stress that this property has no Euclidean analogue for the corresponding reaction-diffusion problem.

Theorem 5.2.3. *Let M be a Cartan-Hadamard manifold and let* sec *denote sectional curvature,* Ric_o *denote the Ricci tensor in the radial direction with respect to a given pole $o \in M$. Assume that the following curvature bounds hold everywhere on M, for suitable $k_1 \ge k_2 > 0$:*

$$\mathrm{Ric}_o(x) \ge -k_1; \quad \sec \le -k_2\,.$$

Then the results of Theorem 5.2.2 hold. Besides, consider any nonnegative solution u to (5.1.1) *corresponding to an initial datum $u_0 \in L^m(M)$ which is sufficiently large in the sense that $u_0 \ge v_0$ for a suitable nonnegative and sufficiently large function $v_0 \in C^0_c(M)$. Then u satisfies*

$$\lim_{t\to+\infty} u(x,t) = +\infty \quad \forall x \in M.$$

Observe that, as it will appear from the proof, for the function v_0 in Theorem 5.2.3 we require that $v_0 > 0$ in a geodesic ball B_R with $R > 0$ and $m := \inf_{B_R} v_0$ both sufficiently large.

5.2.1 Weighted reaction-diffusion equations in the Euclidean space

As mentioned in the Introduction, the methods used in proving Theorem 5.2.2 are general enough, being based on functional inequalities only, to be easily generalized to different contexts. We single out here the one in which reaction-diffusion equations are considered in the Euclidean setting, but in which diffusion takes place in a medium having a nonhomogeneous density, see e.g. [73], [86], [87], [88] and references quoted therein.

We consider a *weight* $\rho : \mathbb{R}^N \to \mathbb{R}$ such that

$$\rho \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \quad \rho(x) > 0 \text{ for any } x \in \mathbb{R}^N, \tag{5.2.9}$$

and the associated weighted Lebesgue spaces

$$L^q_\rho(\mathbb{R}^N) = \{v : \mathbb{R}^N \to \mathbb{R} \text{ measurable} \mid \|v\|_{L^q_\rho} < +\infty\},$$

where $\|v\|_{L^q_\rho} := \int_{\mathbb{R}^N} \rho(x)\,|v(x)|^q\,dx$. Moreover, we assume that ρ is such that the weighted Poincaré inequality (5.1.5) holds. By construction and by the assumptions in (5.2.9) it follows that the weighted Sobolev inequality (5.1.6) also holds, as a consequence of the usual Sobolev inequality in $\mathbb{R}^N$ and of (5.2.9).

Moreover, we let $u_0 : \mathbb{R}^N \to \mathbb{R}$ be such that

$$u_0 \in L^m_\rho(\mathbb{R}^N), \quad u_0(x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^N$$

and consider, for any $T > 0$ and for any $1 < p < m$, problem (5.1.4).

The definition of solution we use will be again the very weak one, adapted to the present case.

Definition 5.2.4. *Let $1 < p < m$ and $u_0 \in L^m_\rho(\mathbb{R}^N)$, $u_0 \geq 0$. Let the weight ρ satisfy* (5.2.9). *We say that the function u is a solution to problem* (5.1.4) *in the interval $[0,T)$ if*

$$u \in L^m_\rho(\mathbb{R}^N \times (0,T))$$

and for any $\varphi \in C^\infty_c(\mathbb{R}^N \times [0,T])$ such that $\varphi(x,T) = 0$ for any $x \in \mathbb{R}^N$, u satisfies the equality:

$$\begin{aligned} -\int_0^T \int_{\mathbb{R}^N} u\, \varphi_t\, \rho(x)\, dx\, dt = \int_0^T \int_{\mathbb{R}^N} u^m\, \Delta\varphi\, dx\, dt + \int_0^T \int_{\mathbb{R}^N} u^p\, \varphi\, \rho(x)\, dx\, dt \\ + \int_{\mathbb{R}^N} u_0(x)\, \varphi(x,0)\, \rho(x)\, dx. \end{aligned} \tag{5.2.10}$$

Theorem 5.2.5. *Let ρ satisfy* (5.2.9) *and assume that the weighted Poincaré inequality* (5.1.5) *holds. Let $1 < p < m$ and $u_0 \in L^m_\rho(\mathbb{R}^N)$, $u_0 \geq 0$. Then problem* (5.1.4) *admits a solution for any $T > 0$, in the sense of Definition 5.2.4. Moreover for any $T > \tau > 0$ one has $u \in L^\infty(\mathbb{R}^N \times (\tau, T))$ and there exist numerical constants c_1, $c_2 > 0$, independent of T, such that, for all $t > 0$ one has*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq c_1 e^{c_2 t} \left\{ \|u_0\|_{L^m_\rho(\mathbb{R}^N)}^{\frac{2m}{2m+N(m-p)}} + \frac{\|u_0\|_{L^m_\rho(\mathbb{R}^N)}^{\frac{2m}{2m+N(m-1)}}}{t^{\frac{N}{2m+N(m-1)}}} \right\}. \tag{5.2.11}$$

Besides, if $q > 1$ and $u_0 \in L^q_\rho(\mathbb{R}^N) \cap L^m_\rho(\mathbb{R}^N)$, then there exists $C(q) > 0$ such that

$$\|u(t)\|_{L^q_\rho(\mathbb{R}^N)} \leq e^{C(q)t} \|u_0\|_{L^q_\rho(\mathbb{R}^N)} \quad \textit{for all } \ t > 0\,.$$

Finally, there are examples of weights satisfying the assumptions of the present Theorem and such that sufficiently large initial data u_0 give rise to solutions $u(x,t)$ blowing up pointwise everywhere in infinite time, i.e. such that $\lim_{t\to+\infty} u(x,t) = +\infty$ for all $x \in \mathbb{R}^N$, so that in particular $\|u(t)\|_\infty \to +\infty$ as $t \to +\infty$ and hence the upper bound in (5.2.11) *is qualitatively sharp. One can take e.g. $\rho \asymp |x|^{-2}$ as $|x| \to +\infty$ for this to hold.*

In the case of *integrable weights* one can adapt the methods of [118] to prove a stronger result.

Theorem 5.2.6. *Let ρ satisfy* (5.2.9) *and $\rho \in L^1(\mathbb{R}^N)$. Let $1 < p < m$ and $u_0 \in L^1_\rho(\mathbb{R}^N)$, $u_0 \geq 0$. Then problem* (5.1.4) *admits a solution for any $T > 0$, in the sense of Definition 5.2.4. Moreover for any $T > \tau > 0$ one has $u \in L^\infty(\mathbb{R}^N \times (\tau, T))$ and there exists $C = C(m, p, N, \|\rho\|_{L^1(\mathbb{R}^N)}) > 0$, independent of the initial datum u_0, such that, for all $t > 0$, one has*

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C \left\{ 1 + \left[\frac{1}{(m-1)t} \right]^{\frac{1}{m-1}} \right\}. \tag{5.2.12}$$

Remark 5.2.7.

- The bound (5.2.12) cannot be replaced by a similar one in which the r.h.s. is replaced by $\frac{C}{(m-1)t}$, that would entail $\|u(t)\|_\infty \to 0$ as $t \to +\infty$, as customary e.g. in the case of solutions to the Porous Medium Equation posed in bounded, Euclidean domains (see [128]). In fact, it is possible that *stationary, bounded solutions* to (5.1.4) exist, provided a positive bounded solution U to the equation
$$-\Delta U = \rho U^a \tag{5.2.13}$$
exists, where $a = p/m < 1$. If this fact holds, $V := U^{\frac{1}{m}}$ is a stationary, bounded, positive solution to the differential equation in (5.1.4), whose L^∞ norm is of course constant in time. In turn, a celebrated results of [15] entails that positive, bounded solutions to (5.2.13) exist if e.g. $\rho \asymp |x|^{-2-\epsilon}$ for some $\epsilon > 0$ as $|x| \to +\infty$ (in fact, a full characterization of the weights for which this holds is given in [15]), a condition which is of course compatible with the assumptions of Theorem 5.2.6.
- Of course, the bound (5.2.11), which gives stronger information when $t \to 0$, continues to hold under the assumptions of Theorem 5.2.6.

5.3 Auxiliary results for elliptic problems

Let $x_0, x \in M$ be given. We denote by $r(x) = \operatorname{dist}(x_0, x)$ the Riemannian distance between x_0 and x. Moreover, we let

$$B_R(x_0) := \{x \in M, \operatorname{dist}(x_0, x) < R\}$$

be the geodesics ball with center $x_0 \in M$ and radius $R > 0$. Let $x_0 \in M$ any fixed reference point. We set $B_R \equiv B_R(x_0)$. As mentioned above, we denote by μ the Riemannian measure on M.

For any given function v, we define for any $k \in \mathbb{R}^+$

$$T_k(v) := \begin{cases} k & \text{if } v \geq k \\ v & \text{if } |v| < k \\ -k & \text{if } v \leq -k \end{cases}.$$

For every $R > 0, k > 0$, consider the problem

$$\begin{cases} u_t = \Delta u^m + T_k(u^p) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, +\infty) \\ u = u_0 & \text{in } B_R \times \{0\}, \end{cases} \tag{5.3.14}$$

where $u_0 \in L^\infty(B_R), u_0 \geq 0$. Solutions to problem (5.3.14) are meant in the weak sense as follows.

Definition 5.3.1. *Let $p < m$. Let $u_0 \in L^\infty(B_R), u_0 \geq 0$. We say that a nonnegative function u is a solution to problem* (5.3.14) *if*

$$u \in L^\infty(B_R \times (0, +\infty)), u^m \in L^2\big((0,T); H_0^1(B_R)\big) \quad \textit{for any } T > 0,$$

and for any $T > 0, \varphi \in C_c^\infty(B_R \times [0,T])$ such that $\varphi(x,T) = 0$ for every $x \in B_R$, u satisfies the equality:

$$\begin{aligned} -\int_0^T \int_{B_R} u\, \varphi_t \, d\mu \, dt = &-\int_0^T \int_{B_R} \langle \nabla u^m, \nabla \varphi \rangle \, d\mu \, dt + \int_0^T \int_{B_R} T_k(u^p)\, \varphi \, d\mu \, dt \\ &+ \int_{B_R} u_0(x)\, \varphi(x,0)\, d\mu. \end{aligned}$$

We also consider elliptic problems of the type

$$\begin{cases} -\Delta u &= f \quad \text{in } B_R \\ \quad u &= 0 \quad \text{in } \partial B_R \,, \end{cases} \tag{5.3.15}$$

with $f \in L^q(B_R)$ for some $q > 1$.

Definition 5.3.2. *We say that $u \in H_0^1(B_R), u \geq 0$ is a weak subsolution to problem* (5.3.15) *if*

$$\int_{B_R} \langle \nabla u, \nabla \varphi \rangle \, d\mu \leq \int_{B_R} f \varphi \, d\mu,$$

for any $\varphi \in H_0^1(B_R), \varphi \geq 0$.

The following proposition contains an estimate in the spirit of the celebrated L^∞ estimate of Stampacchia (see, e.g., [76], [11] and references therein). However, the obtained bound and the proof are different. This is due to the fact that we need an estimate independent of the measure of B_R, in order to let $R \to +\infty$ when we apply such estimate in the proof of global existence for problem (5.1.1) (see Remark 5.3.4 below). Indeed recall that, obviously, since M has infinite measure, $\mu(B_R) \to +\infty$ as $R \to +\infty$.

Proposition 5.3.3. *Let $f_1 \in L^{m_1}(B_R)$ and $f_2 \in L^{m_2}(B_R)$ where $m_1 > \frac{N}{2}$, $m_2 > \frac{N}{2}$. Assume that $v \in H_0^1(B_R)$, $v \geq 0$ is a subsolution to problem*

$$\begin{cases} -\Delta v = (f_1 + f_2) & \textit{in } B_R \\ v = 0 & \textit{on } \partial B_R \end{cases} . \tag{5.3.16}$$

in the sense of Definition 5.3.2. Let $\bar{k} > 0$. Then

$$\|v\|_{L^\infty(B_R)} \leq \left\{ C_1 \|f_1\|_{L^{m_1}(B_R)} + C_2 \|f_2\|_{L^{m_2}(B_R)} \right\}^{\frac{1}{s}} \|v\|_{L^1(B_R)}^{\frac{s-1}{s}} + \bar{k}, \tag{5.3.17}$$

where

$$s = 1 + \frac{2}{N} - \frac{1}{l}\,, \tag{5.3.18}$$

$$\frac{N}{2} < l < \min\{m_1\,, m_2\}, \tag{5.3.19}$$

$$\overline{C}_1 = \left(\frac{s}{s-1}\right)^s \frac{1}{C_s^2}\left(\frac{2}{\overline{k}}\right)^{\frac{1}{l}-\frac{1}{m_1}}\,, \quad \overline{C}_2 = \left(\frac{s}{s-1}\right)^s \frac{1}{C_s^2}\left(\frac{2}{\overline{k}}\right)^{\frac{1}{l}-\frac{1}{m_2}}\,, \tag{5.3.20}$$

and

$$C_1 = \overline{C}_1\,\|v\|_{L^1(B_R)}^{\frac{1}{l}-\frac{1}{m_1}}, \quad C_2 = \overline{C}_2\,\|v\|_{L^1(B_R)}^{\frac{1}{l}-\frac{1}{m_2}}\,. \tag{5.3.21}$$

Observe that Proposition 5.3.3 generalizes Proposition 4.3.4.

Remark 5.3.4. If in Proposition 5.3.3 we further assume that there exists a constant $k_0 > 0$ such that

$$\max(\|v\|_{L^1(B_R)}, \|f_1\|_{L^{m_1}(B_R)}, \|f_2\|_{L^{m_2}(B_R)}) \le k_0 \quad \text{for all} \;\; R > 0,$$

then from (5.3.17), we infer that the bound from above on $\|v\|_{L^\infty(B_R)}$ is independent of R. This fact will have a key role in the proof of global existence for problem (5.1.1).

5.3.1 Proof of Proposition 5.3.3

Let us first define

$$G_k(v) := v - T_k(v)\;, \tag{5.3.22}$$

$$g(k) := \int_{B_R} |G_k(v)|\,d\mu.$$

For any $R > 0$, for $v \in L^1(B_R)$, we set

$$A_k := \{x \in B_R : \,|v(x)| > k\}. \tag{5.3.23}$$

We first state two technical Lemmas.

Lemma 5.3.5. *Let $v \in L^1(B_R)$. Then $g(k)$ is differentiable almost everywhere in $(0, +\infty)$ and*

$$g'(k) = -\mu(A_k).$$

We omit the proof since it is identical to the one given in [11].

Lemma 5.3.6. *Let $v \in L^1(B_R)$. Let $\overline{k} > 0$. Suppose that there exist $C > 0$ and $s > 1$ such that*

$$g(k) \le C\mu(A_k)^s \quad \text{for any } k \ge \bar{k}. \tag{5.3.24}$$

Then $v \in L^\infty(B_R)$ and

$$\|v\|_{L^\infty(B_R)} \le C^{\frac{1}{s}}\,\frac{s}{s-1}\|v\|_{L^1(B_R)}^{1-\frac{1}{s}} + \bar{k}. \tag{5.3.25}$$

Remark 5.3.7. Observe that if C in (5.3.24) does not depend on R and, for some $k_0 > 0$,

$$\|v\|_{L^1(B_R)} \leq k_0 \quad \text{for all} \;\; R > 0,$$

then, in view of the estimate (5.3.25), the bound on $\|v\|_{L^\infty(B_R)}$ is independent of R.

Proof of Lemma 5.3.6. Thanks to Lemma 5.3.5 together with hypotheses (5.3.24) we have that

$$g'(k) = -\mu(A_k) \leq -\left[C^{-1}\, g(k)\right]^{\frac{1}{s}},$$

hence

$$g(k) \leq C\,[-g'(k)]^s.$$

Integrating between $\bar{k}$ and k we get

$$\int_{\bar{k}}^{k} \left(-\frac{1}{C^{\frac{1}{s}}}\right) d\tau \geq \int_{\bar{k}}^{k} g'(\tau)\, g(\tau)^{-\frac{1}{s}}\, dg, \tag{5.3.26}$$

that is:

$$-C^{-\frac{1}{s}}(k-\bar{k}) \geq \frac{s}{s-1}\left[g(k)^{1-\frac{1}{s}} - g(\bar{k})^{1-\frac{1}{s}}\right].$$

Using the definition of g, this can be rewritten as

$$\begin{aligned} g(k)^{1-\frac{1}{s}} &\leq g\left(\bar{k}\right)^{1-\frac{1}{s}} - \frac{s-1}{s}\, C^{-\frac{1}{s}}(k-\bar{k}) \\ &\leq \|v\|_{L^1(B_R)}^{1-\frac{1}{s}} - \frac{s-1}{s}\, C^{-\frac{1}{s}}(k-\bar{k}) \quad \text{for any } k > \bar{k}. \end{aligned}$$

Choose

$$k = k_0 = C^{\frac{1}{s}} \|v\|_{L^1(B_R)}^{1-\frac{1}{s}} \frac{s}{s-1} + \bar{k},$$

and substitute it in the last inequality. Then $g(k_0) \leq 0$. Due to the definition of g this is equivalent to

$$\int_{B_R} |G_{k_0}(v)|\, d\mu = 0 \iff |G_{k_0}(v)| = 0 \iff |v| \leq k_0.$$

As a consequence we have

$$\|v\|_{L^\infty(B_R)} \leq k_0 = \frac{s}{s-1} C^{\frac{1}{s}} \|v\|_{L^1(B_R)}^{1-\frac{1}{s}} + \bar{k}.$$

□

Proof of Proposition 5.3.3. Take $G_k(v)$ as in (5.3.22) and A_k as in (5.3.23). From now one we write, with a slight abuse of notation,

$$\|f\|_{L^q(B_R)} = \|f\|_{L^q}\,.$$

Since $G_k(v) \in H_0^1(B_R)$ and $G_k(v) \geq 0$, we can take $G_k(v)$ as test function in problem (5.3.16). Then, by means of (5.1.3), we get

$$\begin{aligned} \int_{B_R} \nabla u \cdot \nabla G_k(v)\, d\mu &\geq \int_{A_k} |\nabla v|^2\, d\mu \\ &\geq \int_{B_R} |\nabla G_k(v)|^2\, d\mu \\ &\geq C_s^2 \left(\int_{B_R} |G_k(v)|^{2^*}\, d\mu \right)^{\frac{2}{2^*}} . \end{aligned} \tag{5.3.27}$$

If we now integrate on the right hand side of (5.3.16), thanks to Hölder inequality, we get

$$\begin{aligned} \int_{B_R} (f_1 + f_2)\, G_k(v)\, d\mu &= \int_{A_k} f_1\, G_k(v)\, d\mu + \int_{A_k} f_2\, G_k(v)\, d\mu \\ &\leq \left(\int_{A_k} |G_k(v)|^{2^*}\, d\mu \right)^{\frac{1}{2^*}} \left[\left(\int_{A_k} |f_1|^{\frac{2N}{N+2}}\, d\mu \right)^{\frac{N+2}{2N}} + \left(\int_{A_k} |f_2|^{\frac{2N}{N+2}}\, d\mu \right)^{\frac{N+2}{2N}} \right] \\ &\leq \left(\int_{B_R} |G_k(v)|^{2^*}\, d\mu \right)^{\frac{1}{2^*}} \left[\|f_1\|_{L^{m_1}} \mu(A_k)^{\frac{N+2}{2N}\left(1 - \frac{2N}{m_1(N+2)}\right)} \right. \\ &\left. + \|f_2\|_{L^{m_2}} \mu(A_k)^{\frac{N+2}{2N}\left(1 - \frac{2N}{m_2(N+2)}\right)} \right] . \end{aligned} \tag{5.3.28}$$

Combining (5.3.27) and (5.3.28) we have

$$\begin{aligned} C_s^2 \left(\int_{B_R} |G_k(v)|^{2^*}\, d\mu \right)^{\frac{1}{2^*}} &\leq \left[\|f_1\|_{L^{m_1}} \mu(A_k)^{\frac{N+2}{2N}\left(1 - \frac{2N}{m_1(N+2)}\right)} \right. \\ &\left. + \|f_2\|_{L^{m_2}} \mu(A_k)^{\frac{N+2}{2N}\left(1 - \frac{2N}{m_2(N+2)}\right)} \right] . \end{aligned} \tag{5.3.29}$$

Observe that

$$\int_{B_R} |G_k(v)|\, d\mu \leq \left(\int_{B_R} |G_k(v)|^{2^*}\, d\mu \right)^{\frac{1}{2^*}} \mu(A_k)^{\frac{N+2}{2N}} . \tag{5.3.30}$$

We substitute (5.3.30) in (5.3.29) and we obtain

$$\int_{B_R} |G_k(v)|\, d\mu \leq \frac{1}{C_s^2} \left[\|f_1\|_{L^{m_1}} \mu(A_k)^{1 + \frac{2}{N} - \frac{1}{m_1}} + \|f_2\|_{L^{m_2}} \mu(A_k)^{1 + \frac{2}{N} - \frac{1}{m_2}} \right] .$$

Using the definition of l in (5.3.19), for any $k \geq \overline{k}$, we can write

$$\int_{B_R} |G_k(v)|\, d\mu \leq \frac{1}{C_s^2}\, \mu(A_k)^{1 + \frac{2}{N} - \frac{1}{l}} \left[\|f_1\|_{L^{m_1}} \mu(A_{\overline{k}})^{\frac{1}{l} - \frac{1}{m_1}} + \|f_2\|_{L^{m_2}} \mu(A_{\overline{k}})^{\frac{1}{l} - \frac{1}{m_2}} \right] \tag{5.3.31}$$

Set

$$C = \frac{1}{C_s^2} \left[\|f_1\|_{L^{m_1}} \left(\frac{2}{\overline{k}} \|v\|_{L^1(B_R)} \right)^{\frac{1}{l} - \frac{1}{m_1}} + \|f_2\|_{L^{m_2}} \left(\frac{2}{\overline{k}} \|v\|_{L^1(B_R)} \right)^{\frac{1}{l} - \frac{1}{m_2}} \right] .$$

Hence, by means of Chebychev inequality, (5.3.31) reads, for any $k \geq \bar{k}$,

$$\int_{B_R} |G_k(v)|\, d\mu \leq C\, \mu(A_k)^s\,, \tag{5.3.32}$$

where s has been defined in (5.3.18). Now, (5.3.32) corresponds to the hypotheses of Lemma 5.3.6, hence the thesis of such lemma follows and we have

$$\|v\|_{L^\infty} \leq \frac{s}{s-1} C^{\frac{1}{s}}\, \|v\|_{L^1}^{1-\frac{1}{s}} + \bar{k}\,.$$

Then the thesis follows thanks to (5.3.21). □

5.4 L^q and smoothing estimates

Lemma 5.4.1. *Let $1 < p < m$. Let M be such that inequality* (5.1.2) *holds. Suppose that $u_0 \in L^\infty(B_R)$, $u_0 \geq 0$. Let u be the solution of problem* (5.3.14)*. Then, for any $1 < q < +\infty$, for some constant $C = C(q) > 0$, one has*

$$\|u(t)\|_{L^q(B_R)} \leq e^{C(q)t} \|u_0\|_{L^q(B_R)} \quad \textit{for all}\ \ t > 0\,. \tag{5.4.33}$$

Proof. Let $x \in \mathbb{R}$, $x \geq 0$, $1 < p < m$, $\varepsilon > 0$. Then, for any $1 < q < +\infty$, due to Young's inequality, it follows that

$$\begin{aligned} x^{p+q-1} &= x^{(m+q-1)(\frac{p-1}{m-1})} x^{q(\frac{m-p}{m-1})} \\ &\leq \varepsilon x^{(m+q-1)(\frac{p-1}{m-1})(\frac{m-1}{p-1})} + \left(\frac{1}{\varepsilon}\frac{p-1}{m-1}\right)^{\frac{p-1}{m-p}} x^{q(\frac{m-p}{m-1})(\frac{m-1}{m-p})} \\ &= \varepsilon x^{m+q-1} + \left(\frac{1}{\varepsilon}\frac{p-1}{m-1}\right)^{\frac{p-1}{m-p}} x^q. \end{aligned} \tag{5.4.34}$$

Since u_0 is bounded and $T_k(u^p)$ is a bounded and Lipschitz function, by standard results, there exists a unique solution of problem (5.3.14) in the sense of Definition 5.3.1; moreover, $u \in C\big([0,T]; L^q(B_R)\big)$. We now multiply both sides of the differential equation in problem (5.3.14) by u^{q-1} and integrate by parts. This can be justified by standard tools, by an approximation procedure. Using the fact that

$$T_k(u^p) \leq u^p,$$

thanks to the Poincaré inequality, we obtain for all $t > 0$

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|_{L^q(B_R)}^q \leq -\frac{4m(q-1)}{(m+q-1)^2} C_p^2 \|u(t)\|_{L^{m+q-1}(B_R)}^{m+q-1} + \|u(t)\|_{L^{p+q-1}(B_R)}^{p+q-1}.$$

Now, using inequality (5.4.34), we obtain

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|_{L^q(B_R)}^q \leq -\frac{4m(q-1)}{(m+q-1)^2} C_p^2 \|u(t)\|_{L^{m+q-1}(B_R)}^{m+q-1} + \varepsilon \|u(t)\|_{L^{m+q-1}(B_R)}^{m+q-1} + C(\varepsilon)\|u(t)\|_{L^q(B_R)}^q,$$

where $C(\varepsilon) = \left(\frac{1}{\varepsilon}\frac{p-1}{m-1}\right)^{\frac{p-1}{m-p}}$. Thus, for every $\varepsilon > 0$ so small that

$$0 < \varepsilon < \frac{4m(q-1)}{(m+q-1)^2} C_p^2,$$

we have

$$\frac{1}{q}\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} \le C(\varepsilon)\|u(t)\|^q_{L^q(B_R)}.$$

Hence, we can find $C = C(q) > 0$ such that

$$\frac{d}{dt}\|u(t)\|^q_{L^q(B_R)} \le C(q)\|u(t)\|^q_{L^q(B_R)} \quad \text{for all } \ t > 0\,.$$

If we set $y(t) := \|u(t)\|^q_{L^q(B_R)}$, the previous inequality reads

$$y'(t) \le C(q)y(t) \quad \text{for all } \ t \in (0,T)\,.$$

Thus the thesis follows. □

Note that for the constant $C(q)$ in Lemma 5.4.1 does not depend on R and $k > 0$; moreover, we have that

$$C(q) \to +\infty \quad \text{as } \ q \to +\infty\,.$$

We shall use the following Aronson-Benilan type estimate (see [6]; see also [118, Proposition 2.3]).

Proposition 5.4.2. *Let $1 < p < m$, $u_0 \in H^1_0(B_R) \cap L^\infty(B_R)$, $u_0 \ge 0$. Let u be the solution to problem* (5.3.14)*. Then, for a.e. $t \in (0,T)$,*

$$-\Delta u^m(\cdot,t) \le u^p(\cdot,t) + \frac{1}{(m-1)t}u(\cdot,t) \quad \textit{in } \mathfrak{D}'(B_R).$$

Proof. By arguing as in [6], [118, Proposition 2.3] we get

$$-\Delta u^m(\cdot,t) \le T_k[u^p(\cdot,t)] + \frac{1}{(m-1)t}u(\cdot,t) \le u^p(\cdot,t) + \frac{1}{(m-1)t}u(\cdot,t) \quad \text{in } \mathfrak{D}'(B_R),$$

since $T_k(u^p) \le u^p$. □

Proposition 5.4.3. *Let $1 < p < m$, $R > 0, u_0 \in L^\infty(B_R)$, $u_0 \ge 0$. Let u be the solution to problem* (5.3.14)*. Let M be such that inequality* (5.1.3) *holds. Then there exists $\Gamma = \Gamma(p,m,N,C_s) > 0$ such that, for all $t > 0$,*

$$\begin{aligned}\|u(t)\|_{L^\infty(B_R)} \le \Gamma \Big\{ &\left[e^{Ct}\|u_0\|_{L^m(B_R)}\right]^{\frac{2m}{2m+N(m-p)}} \\ &+ \left[e^{Ct}\|u_0\|_{L^m(B_R)}\right]^{\frac{2m}{2m+N(m-1)}} \left[\frac{1}{(m-1)t}\right]^{\frac{N}{2m+N(m-1)}} \Big\}\,;\end{aligned} \tag{5.4.35}$$

here the constant $C = C(m) > 0$ is the one given in Lemma 5.4.1.

Remark 5.4.4. If in Proposition 5.4.3, in addition, we assume that for some $k_0 > 0$

$$\|u_0\|_{L^m(B_R)} \le k_0 \quad \text{for every} \quad R > 0\,,$$

then the bound from above for $\|u(t)\|_{L^\infty(B_R)}$ in (5.4.35) is independent of R.

Proof of Proposition 5.4.3. Let us set $w = u(\cdot, t)$. Observe that $w^m \in H_0^1(B_R)$ and $w \ge 0$. Due to Proposition 5.4.2 we know that

$$-\Delta(w^m) \le \left[w^p + \frac{w}{(m-1)t}\right]. \tag{5.4.36}$$

Observe that, since $u_0 \in L^\infty(B_R)$ also $w \in L^\infty(B_R)$. Let $q \ge 1$ and

$$r_1 > \max\left\{\frac{q}{p}, \frac{N}{2}\right\}, \quad r_2 > \max\left\{q, \frac{N}{2}\right\}.$$

We can apply Proposition 5.3.3 with

$$r_1 = m_1, \quad r_2 = m_2, \quad \frac{N}{2} < l < \min\{m_1\,, m_2\}\,.$$

So, we have that

$$\|w\|^m_{L^\infty(B_R)} \le \left\{C_1(r_1)\|w^p\|_{L^{r_1}(B_R)} + \gamma C_2(r_2)\|w\|_{L^{r_2}(B_R)}\right\}^{\frac{1}{s}} \|w\|^{m\frac{s-1}{s}}_{L^m(B_R)} + \bar{k}\,, \tag{5.4.37}$$

where $s = 1+2/N-1/l$ and $\gamma = 1/[(m-1)t]$. Thanks to Hölder inequality and Young's inequality with exponents

$$\alpha_1 = \frac{sm}{p - \frac{q}{r_1}} > 1, \qquad \beta_1 = \frac{sm}{sm - \left(p - \frac{q}{r_1}\right)} > 1.$$

we obtain, for any $\varepsilon_1 > 0$

$$\begin{aligned}
\|w^p\|_{L^{r_1}(B_R)} &= \left\|w^{p-q/r_1+q/r_1}\right\|_{L^{r_1}(B_R)} = \left[\int_{B_R} w^{r_1(p-q/r_1)} w^q d\mu\right]^{\frac{1}{r_1}} \\
&\le \left[\|w^{r_1(p-q/r_1)}\|_{L^\infty(B_R)}\|w^q\|_{L^1(B_R)}\right]^{\frac{1}{r_1}} \\
&= \|w\|^{p-q/r_1}_{L^\infty(B_R)} \left(\int_{B_R} w^q\, d\mu\right)^{\frac{1}{r_1}} = \|w\|^{p-q/r_1}_{L^\infty(B_R)} \|w\|^{q/r_1}_{L^q(B_R)} \\
&\le \frac{\varepsilon_1^{\alpha_1}}{\alpha_1} \|w\|^{\frac{sm}{p-q/r_1}(p-q/r_1)}_{L^\infty(B_R)} + \frac{\alpha_1 - 1}{\alpha_1} \varepsilon_1^{-\frac{\alpha_1}{\alpha_1-1}} \|w\|^{\frac{\beta_1 q}{r_1}}_{L^q(B_R)}\,.
\end{aligned} \tag{5.4.38}$$

We set

$$\delta_1 := \frac{\varepsilon_1^{\alpha_1}}{\alpha_1}, \quad \eta(x) = \frac{x-1}{x^{\frac{x}{x-1}}}.$$

Thus from (5.4.38) we obtain

$$\|w^p\|_{L^{r_1}(B_R)} \le \delta_1 \|w\|^{sm}_{L^\infty(B_R)} + \frac{\eta(\alpha_1)}{\delta_1^{\frac{1}{\alpha_1-1}}} \|w\|^{\frac{smq}{r_1}\frac{1}{sm-(p-q/r_1)}}_{L^q(B_R)}\,. \tag{5.4.39}$$

Similarly, again thanks to Hölder inequality and Young's inequality with exponents

$$\alpha_2 = \frac{sm}{1-\frac{q}{r_2}} > 1, \qquad \beta_2 = \frac{sm}{sm - \left(1-\frac{q}{r_2}\right)} > 1.$$

we obtain, for any $\varepsilon_2 > 0$

$$\begin{aligned}\|w\|_{L^{r_2}(B_R)} &\le \left\|w^{1-q/r_2+q/r_2}\right\|_{L^{r_2}(B_R)} \le \|w\|^{1-q/r_2}_{L^\infty(B_R)} \|w\|^{q/r_2}_{L^q(B_R)} \\ &\le \frac{\varepsilon_2^{\alpha_2}}{\alpha_2} \|w\|^{\frac{sm}{1-q/r_2}\left(1-\frac{q}{r_2}\right)}_{L^\infty(B_R)} + \frac{\alpha_2-1}{\alpha_2}\varepsilon_2^{-\frac{\alpha_2}{\alpha_2-1}} \|w\|^{\frac{\beta_2 q}{r_2}}_{L^q(B_R)}.\end{aligned}$$

We set $\delta_2 := \frac{\varepsilon_2^{\alpha_2}}{\alpha_2}$ and thus we obtain

$$\|w\|_{L^{r_2}(B_R)} \le \delta_2 \|w\|^{sm}_{L^\infty(B_R)} + \frac{\eta(\alpha_2)}{\delta_2^{\frac{1}{\alpha_2-1}}} \|w\|^{\frac{smq}{r_2}\frac{1}{sm-(1-q/r_2)}}_{L^q(B_R)}. \tag{5.4.40}$$

Plugging (5.4.39) and (5.4.40) into (5.4.37) we obtain

$$\begin{aligned}\|w\|^{ms}_{L^\infty(B_R)} &\le 2^{s-1}\left\{\left[C_1\|w^p\|_{L^{r_1}(B_R)} + \gamma C_2\|w\|_{L^{r_2}(B_R)}\right]\|w\|^{m(s-1)}_{L^m(B_R)} + \bar{k}^s\right\} \\ &\le 2^{s-1}\left\{C_1\left[\delta_1\|w\|^{sm}_{L^\infty(B_R)} + \frac{\eta(\alpha_1)}{\delta_1^{\frac{1}{\alpha_1-1}}}\|w\|^{\frac{smq}{r_1}\frac{1}{sm-(p-q/r_1)}}_{L^q(B_R)}\right]\right. \\ &\left. +\gamma C_2\left[\delta_2\|w\|^{sm}_{L^\infty(B_R)} + \frac{\eta(\alpha_2)}{\delta_2^{\frac{1}{\alpha_2-1}}}\|w\|^{\frac{smq}{r_2}\frac{1}{sm-(1-q/r_2)}}_{L^q(B_R)}\right]\right\}\|w\|^{m(s-1)}_{L^m(B_R)} + 2^{s-1}\bar{k}^s.\end{aligned}$$

Without loss of generality we can assume that $\|w\|^m_{L^m(B_R)} \neq 0$. Choosing $\varepsilon_1, \varepsilon_2$ such that

$$\delta_1 = \frac{1}{4C_1\|w\|^{m(s-1)}_{L^m(B_R)}2^{s-1}} \qquad \delta_2 = \frac{1}{4\gamma\, C_2\|w\|^{m(s-1)}_{L^m(B_R)}2^{s-1}}$$

we thus have

$$\begin{aligned}\frac{1}{2}\|w\|^{sm}_{L^\infty(B_R)} &\le 4^{\frac{1}{\alpha_1-1}}\eta(\alpha_1)\left(2^{s-1}C_1\|w\|^{m(s-1)}_{L^m(B_R)}\right)^{\frac{\alpha_1}{\alpha_1-1}}\|w\|^{\frac{smq}{r_1}\frac{1}{sm-(p-q/r_1)}}_{L^q(B_R)} \\ &+ 4^{\frac{1}{\alpha_2-1}}\eta(\alpha_2)\left(2^{s-1}\gamma C_2\|w\|^{m(s-1)}_{L^m(B_R)}\right)^{\frac{\alpha_2}{\alpha_2-1}}\|w\|^{\frac{smq}{r_2}\frac{1}{sm-(1-q/r_2)}}_{L^q(B_R)} \\ &+ 2^{s-1}\bar{k}^s.\end{aligned}$$

This reduces to

$$\begin{aligned}\|w\|_{L^\infty(B_R)} &\le (2)^{\frac{1}{sm}}\, 4^{\frac{1}{sm(\alpha_1-1)}}\eta(\alpha_1)^{\frac{1}{sm}}\left(2^{\frac{s-1}{sm}}C_1^{\frac{1}{sm}}\|w\|^{\frac{s-1}{s}}_{L^m(B_R)}\right)^{\frac{\alpha_1}{\alpha_1-1}}\|w\|^{\frac{q}{r_1}\frac{1}{sm-(p-q/r_1)}}_{L^q(B_R)} \\ &+ (2)^{\frac{1}{sm}}\, 4^{\frac{1}{sm(\alpha_2-1)}}\eta(\alpha_2)^{\frac{1}{sm}}\left(2^{\frac{s-1}{sm}}\gamma^{\frac{1}{sm}}C_2^{\frac{1}{sm}}\|w\|^{\frac{s-1}{s}}_{L^m(B_R)}\right)^{\frac{\alpha_2}{\alpha_2-1}}\|w\|^{\frac{q}{r_2}\frac{1}{sm-(1-q/r_2)}}_{L^q(B_R)} \\ &+ (2)^{\frac{1}{sm}}\left(2^{\frac{s-1}{s}}\bar{k}\right)^{\frac{1}{m}}.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}\|w\|_{L^\infty(B_R)} \le & \left[\eta(\alpha_1)\left(2^{\alpha_1 s+1}C_1^{\alpha_1}\right)^{\frac{1}{\alpha_1-1}}\right]^{\frac{1}{sm}} \|w\|_{L^m(B_R)}^{\frac{\alpha_1}{\alpha_1-1}\frac{s-1}{s}} \|w\|_{L^q(B_R)}^{\frac{q}{r_1}\frac{1}{sm-(p-q/r_1)}} \\ & + \left[\eta(\alpha_2)\left(2^{\alpha_2 s+1}\gamma^{\alpha_2}C_2^{\alpha_2}\right)^{\frac{1}{\alpha_2-1}}\right]^{\frac{1}{sm}} \|w\|_{L^m(B_R)}^{\frac{\alpha_2}{\alpha_2-1}\frac{s-1}{s}} \|w\|_{L^q(B_R)}^{\frac{q}{r_2}\frac{1}{sm-(1-q/r_2)}} \\ & + \left(2\bar{k}\right)^{\frac{1}{m}}.\end{aligned} \tag{5.4.41}$$

Now we use the definitions of C_1, $C_2, \overline{C}_1, \overline{C}_2$ introduced in (5.3.21) and (5.3.20), obtaining

$$\begin{aligned}\|w\|_{L^\infty(B_R)} \le & \left[\eta(\alpha_1)\left(2^{\alpha_1 s+1}\,\overline{C}_1^{\alpha_1}\right)^{\frac{1}{\alpha_1-1}}\right]^{\frac{1}{sm}} \|w\|_{L^m(B_R)}^{\frac{\alpha_1}{\alpha_1-1}\frac{1}{s}\left(s-1+\frac{1}{l}-\frac{1}{r_1}\right)} \|w\|_{L^q(B_R)}^{\frac{q}{r_1}\frac{1}{sm-(p-q/r_1)}} \\ & + \left[\eta(\alpha_2)\left(2^{\alpha_2 s+1}\gamma^{\alpha_2}\overline{C}_2^{\alpha_2}\right)^{\frac{1}{\alpha_2-1}}\right]^{\frac{1}{sm}} \|w\|_{L^m(B_R)}^{\frac{\alpha_2}{\alpha_2-1}\frac{1}{s}\left(s-1+\frac{1}{l}-\frac{1}{r_2}\right)} \|w\|_{L^q(B_R)}^{\frac{q}{r_2}\frac{1}{sm-(1-q/r_2)}} \\ & + \left(2\bar{k}\right)^{\frac{1}{m}}.\end{aligned}$$

By taking limits as $r_1 \longrightarrow +\infty$ and $r_2 \longrightarrow +\infty$ we have

$$\begin{aligned}&\frac{\alpha_1}{\alpha_1-1} \longrightarrow \frac{ms}{ms-p};\\ &\frac{\alpha_2}{\alpha_2-1} \longrightarrow \frac{ms}{ms-1};\\ &\eta(\alpha_1) \longrightarrow \left[\frac{p}{ms}\right]^{\frac{p}{ms-p}}\left\{1-\frac{p}{ms}\right\};\\ &\eta(\alpha_2) \longrightarrow \left[\frac{1}{ms}\right]^{\frac{1}{ms-1}}\left\{1-\frac{1}{ms}\right\};\\ &\overline{C}_1 \longrightarrow \left(\frac{s}{s-1}\right)^s \frac{1}{C_s^2}\left(\frac{2}{\bar{\bar{k}}}\right)^{\frac{1}{l}};\\ &\overline{C}_2 \longrightarrow \left(\frac{s}{s-1}\right)^s \frac{1}{C_s^2}\left(\frac{2}{\bar{\bar{k}}}\right)^{\frac{1}{l}}.\end{aligned}$$

Moreover we define

$$\begin{aligned}&\tilde{\Gamma}_1 := \left[\left(\frac{p}{ms}\right)^{\frac{p}{ms-p}}\left(\frac{ms-p}{ms}\right)\right]^{\frac{1}{ms}}\left[2^{ms^2-p}\left(\frac{s}{s-1}\right)^s\frac{1}{C_s^2}\left(\frac{2}{\bar{\bar{k}}}\right)^{\frac{1}{l}}\right]^{\frac{1}{ms-p}},\\ &\tilde{\Gamma}_2 := \left[\left(\frac{1}{ms}\right)^{\frac{1}{ms-1}}\left(\frac{ms-1}{ms}\right)\right]^{\frac{1}{ms}}\left[2^{ms^2-1}\left(\frac{s}{s-1}\right)^s\frac{1}{C_s^2}\left(\frac{2}{\bar{\bar{k}}}\right)^{\frac{1}{l}}\right]^{\frac{1}{ms-1}},\\ &\tilde{\Gamma} := \max\{\tilde{\Gamma}_1\ ,\tilde{\Gamma}_2\}.\end{aligned}$$

Hence by (5.4.41) we get

$$\|w\|_{L^\infty(B_R)} \le \tilde{\Gamma}\left[\|w\|_{L^m(B_R)}^{\frac{m}{ms-p}\left(s-1+\frac{1}{l}\right)} + \|w\|_{L^m(B_R)}^{\frac{m}{ms-1}\left(s-1+\frac{1}{l}\right)}\gamma^{\frac{1}{ms-1}}\right] + \left(2\bar{k}\right)^{\frac{1}{m}}. \tag{5.4.42}$$

Letting $l \to +\infty$ in (5.4.42), we can infer that

$$\|w\|_{L^\infty(B_R)} \le \Gamma \left[\|w\|_{L^m(B_R)}^{\frac{2m}{2m+N(m-p)}} + \|w\|_{L^m(B_R)}^{\frac{2m}{2m+N(m-1)}} \gamma^{\frac{N}{2m+N(m-1)}} \right] + \left(2\bar{k}\right)^{\frac{1}{m}}, \tag{5.4.43}$$

where

$$\Gamma_1 = \left(1 - \frac{pN}{m(N+2)}\right)^{\frac{N}{m(N+2)}} 2 \left[\left(\frac{pN}{m(N+2)}\right)^{\frac{pN}{m(N+2)}} 2^{2m\left(1+\frac{2}{N}\right)} \left(\frac{N+2}{N}\right)^{\frac{N+2}{N}} \frac{1}{C_s^2} \right]^{\frac{N}{2m+N(m-p)}},$$

$$\Gamma_2 = \left(1 - \frac{N}{m(N+2)}\right)^{\frac{N}{m(N+2)}} 2 \left[\left(\frac{N}{m(N+2)}\right)^{\frac{N}{m(N+2)}} 2^{2m\left(1+\frac{2}{N}\right)} \left(\frac{N+2}{N}\right)^{\frac{N+2}{N}} \frac{1}{C_s^2} \right]^{\frac{N}{2m+N(m-1)}};$$

$\Gamma = \max\{\Gamma_1 ; \Gamma_2\}$.

Letting $\bar{k} \to 0$ in (5.4.43) we obtain

$$\|w\|_{L^\infty(B_R)} \le \Gamma \left[\|w\|_{L^m(B_R)}^{\frac{2m}{2m+N(m-p)}} + \|w\|_{L^m(B_R)}^{\frac{2m}{2m+N(m-1)}} \gamma^{\frac{N}{2m+N(m-1)}} \right]. \tag{5.4.44}$$

Finally, since $u_0 \in L^\infty(B_R)$, we can apply Lemma 5.4.1 to w with $q = m$. Thus from (5.4.33) with $q = m$ and (5.4.44), the thesis follows. □

5.5 Proof of Theorems 5.2.2, 5.2.3

Proof of Theorem 5.2.2. Let $\{u_{0,h}\}_{h\ge 0}$ be a sequence of functions such that

$$\begin{aligned} & u_{0,h} \in L^\infty(M) \cap C_c^\infty(M) \ \text{ for all } h \ge 0, \\ & u_{0,h} \ge 0 \ \text{ for all } h \ge 0, \\ & u_{0,h_1} \le u_{0,h_2} \ \text{ for any } h_1 < h_2, \\ & u_{0,h} \longrightarrow u_0 \ \text{ in } L^m(M) \quad \text{as } h \to +\infty. \end{aligned}$$

For any $R > 0, k > 0, h > 0$, consider the problem

$$\begin{cases} u_t = \Delta u^m + T_k(u^p) & \text{in } B_R \times (0, +\infty) \\ u = 0 & \text{in } \partial B_R \times (0, \infty) \\ u = u_{0,h} & \text{in } B_R \times \{0\}. \end{cases} \tag{5.5.45}$$

From standard results it follows that problem (5.5.45) has a solution $u_{h,k}^R$ in the sense of Definition 5.3.1; moreover, $u_{h,k}^R \in C\big([0,T]; L^q(B_R)\big)$ for any $q > 1$. Hence, it satisfies the inequalities in Lemma 5.4.1 and in Proposition 5.4.3, i.e., for any $t \in (0, +\infty)$,

$$\|u_{h,k}^R(t)\|_{L^m(B_R)} \le e^{Ct} \|u_{0,h}\|_{L^m(B_R)}; \tag{5.5.46}$$

$$\begin{aligned} \|u_{h,k}^R(t)\|_{L^\infty(B_R)} \le \Gamma \Bigg\{ & \left[e^{Ct}\|u_{0,h}\|_{L^m(B_R)}\right]^{\frac{2m}{2m+N(m-p)}} \\ & + \left[e^{Ct}\|u_{0,h}\|_{L^m(B_R)}\right]^{\frac{2m}{2m+N(m-1)}} \left[\frac{1}{(m-1)t}\right]^{\frac{N}{2m+N(m-1)}} \Bigg\}. \end{aligned} \tag{5.5.47}$$

In addition, for any $\tau \in (0,T), \zeta \in C_c^1((\tau,T)), \zeta \geq 0$, $\max_{[\tau,T]} \zeta' > 0$,

$$\begin{aligned}\int_\tau^T \zeta(t)\left[\left((u_{h,k}^R)^{\frac{m+1}{2}}\right)_t\right]^2 d\mu dt &\leq \max_{[\tau,T]} \zeta' \bar{C} \int_{B_R} (u_{h,k}^R)^{m+1}(x,\tau) d\mu \\ &+ \bar{C} \max_{[\tau,T]} \zeta \int_{B_R} F\big(u_{h,k}^R(x,T)\big) d\mu \\ &\leq \max_{[\tau,T]} \zeta'(t) \bar{C} \|u_{h,k}^R(\tau)\|_{L^\infty(B_R)} \|u_{h,k}^R(\tau)\|_{L^m(B_R)}^m \\ &+ \frac{\bar{C}}{m+p} \|u_{h,k}^R(T)\|_{L^\infty(B_R)}^p \|u_{h,k}^R(T)\|_{L^m(B_R)}^m\end{aligned} \tag{5.5.48}$$

where

$$F(u) = \int_0^u s^{m-1+p}\, ds\,,$$

and $\bar{C} > 0$ is a constant only depending on m. Inequality (5.5.48) is formally obtained by multiplying the differential inequality in problem (5.3.14) by $\zeta(t)[(u^m)_t]$, and integrating by parts; indeed, a standard approximation procedure is needed (see [49, Lemma 3.3] and [7, Theorem 13]).

Moreover, as a consequence of Definition 5.3.1, for any $\varphi \in C_c^\infty(B_R \times [0,T])$ such that $\varphi(x,T) = 0$ for any $x \in B_R$, $u_{h,k}^R$ satisfies

$$\begin{aligned}-\int_0^T \int_{B_R} u_{h,k}^R\, \varphi_t\, d\mu\, dt = \int_0^T \int_{B_R} (u_{h,k}^R)^m\, \Delta\varphi\, d\mu\, dt + \int_0^T \int_{B_R} T_k[(u_{h,k}^R)^p]\, \varphi\, d\mu\, dt \\ + \int_{B_R} u_{0,h}(x)\, \varphi(x,0)\, d\mu.\end{aligned} \tag{5.5.49}$$

Observe that all the integrals in (5.5.49) are finite. Indeed, due to (5.5.46), $u_{h,k}^R \in L^m(B_R \times (0,T))$ hence, since $p < m$, $u_{h,k}^R \in L^p(B_R \times (0,T))$ and $u_{h,k}^R \in L^1(B_R \times (0,T))$. Moreover, observe that, for any $h > 0$ and $R > 0$ the sequence of solutions $\{u_{h,k}^R\}_{k\geq 0}$ is monotone increasing in k hence it has a pointwise limit for $k \to \infty$. Let u_h^R be such limit so that we have

$$u_{h,k}^R \longrightarrow u_h^R \quad \text{as } k \to \infty \text{ pointwise.}$$

In view of (5.5.46), (5.5.47), the right hand side of (5.5.48) is independent of k. So, $(u_h^R)^{\frac{m+1}{2}} \in H^1((\tau,T);L^2(B_R))$. Therefore, $(u_h^R)^{\frac{m+1}{2}} \in C\big([\tau,T];L^2(B_R)\big)$. We can now pass to the limit as $k \to +\infty$ in inequalities (5.5.46) and (5.5.47) arguing as follows. From inequality (5.5.46), thanks to the Fatou's Lemma, one has for all $t > 0$

$$\|u_h^R(t)\|_{L^m(B_R)} \leq e^{Ct} \|u_{0,h}\|_{L^m(B_R)}. \tag{5.5.50}$$

On the other hand, from (5.5.47), since $u_{h,k}^R \longrightarrow u_h^R$ as $k \to \infty$ pointwise and the right hand side of (5.5.47) is independent of k, one has for all $t > 0$

$$\begin{aligned}\|u_h^R(t)\|_{L^\infty(B_R)} \leq\leq \Gamma \Bigg\{ \left[e^{Ct}\|u_{0,h}\|_{L^m(B_R)}\right]^{\frac{2m}{2m+N(m-p)}} \\ + \left[e^{Ct}\|u_{0,h}\|_{L^m(B_R)}\right]^{\frac{2m}{2m+N(m-1)}} \left[\frac{1}{(m-1)t}\right]^{\frac{N}{2m+N(m-1)}} \Bigg\}.\end{aligned} \tag{5.5.51}$$

Note that both (5.5.50) and (5.5.51) hold *for all* $t > 0$, in view of the continuity property of u deduced above. Moreover, thanks to Beppo Levi's monotone convergence Theorem, it is possible to compute the limit as $k \to +\infty$ in the integrals of equality (5.5.49) and hence obtain that, for any $\varphi \in C_c^\infty(B_R \times (0,T))$ such that $\varphi(x,T) = 0$ for any $x \in B_R$, the function u_h^R satisfies

$$\begin{aligned} -\int_0^T \int_{B_R} u_h^R \varphi_t \, d\mu \, dt = \int_0^T \int_{B_R} (u_h^R)^m \, \Delta\varphi \, d\mu \, dt + \int_0^T \int_{B_R} (u_h^R)^p \, \varphi \, d\mu \, dt \\ + \int_{B_R} u_{0,h}(x) \, \varphi(x,0) \, d\mu. \end{aligned} \tag{5.5.52}$$

Observe that, due to inequality (5.5.50), all the integrals in (5.5.52) are finite, hence u_h^R is a solution to problem (5.5.45), where we replace $T_k(u^p)$ with u^p itself, in the sense of Definition 5.3.1.

Let us now observe that, for any $h > 0$, the sequence of solutions $\{u_h^R\}_{R>0}$ is monotone increasing in R, hence it has a pointwise limit as $R \to +\infty$. We call its limit function u_h so that

$$u_h^R \longrightarrow u_h \quad \text{as } R \to +\infty \text{ pointwise.}$$

In view of (5.5.46), (5.5.47), (5.5.50), (5.5.51), the right hand side of (5.5.48) is independent of k and R. So, $(u_h)^{\frac{m+1}{2}} \in H^1((\tau,T); L^2(M))$. Therefore, $(u_h)^{\frac{m+1}{2}} \in C\big([\tau,T]; L^2(M)\big)$. Since $u_0 \in L^m(M)$, there exists $k_0 > 0$ such that

$$\|u_{0h}\|_{L^m(B_R)} \le k_0 \quad \forall\, h > 0, R > 0\,. \tag{5.5.53}$$

Note that, in view of (5.5.53), the norms in (5.5.50) and (5.5.51) do not depend on R (see Proposition 5.4.3, Lemma 5.4.1 and Remark 5.4.4). Therefore, we pass to the limit as $R \to +\infty$ in (5.5.50) and (5.5.51). By Fatou's Lemma,

$$\|u_h(t)\|_{L^m(M)} \le e^{Ct} \|u_{0,h}\|_{L^m(M)}; \tag{5.5.54}$$

furthermore, since $u_h^R \longrightarrow u_h$ as $R \to +\infty$ pointwise,

$$\begin{aligned} \|u_h(t)\|_{L^\infty(M)} \le \Gamma \Bigg\{ \left[e^{Ct} \|u_{0,h}\|_{L^m(M)}\right]^{\frac{2m}{2m+N(m-p)}} \\ + \left[e^{Ct} \|u_{0,h}\|_{L^m(M)}\right]^{\frac{2m}{2m+N(m-1)}} \left[\frac{1}{(m-1)t}\right]^{\frac{N}{2m+N(m-1)}} \Bigg\}. \end{aligned} \tag{5.5.55}$$

Note that both (5.5.54) and (5.5.55) hold *for all* $t > 0$, in view of the continuity property of u_h^R deduced above.

Moreover, again by monotone convergence, it is possible to compute the limit as $R \to +\infty$ in the integrals of equality (5.5.52) and hence obtain that, for any $\varphi \in C_c^\infty(M \times (0,T))$ such that $\varphi(x,T) = 0$ for any $x \in M$, the function u_h satisfies,

$$\begin{aligned} -\int_0^T \int_M u_h \, \varphi_t \, d\mu \, dt = \int_0^T \int_M (u_h)^m \, \Delta\varphi \, d\mu \, dt + \int_0^T \int_M (u_h)^p \, \varphi \, d\mu \, dt \\ + \int_M u_{0,h}(x) \, \varphi(x,0) \, d\mu. \end{aligned} \tag{5.5.56}$$

Observe that, due to inequality (5.5.54), all the integrals in (5.5.56) are well posed hence u_h is a solution to problem (5.1.1), where we replace u_0 with $u_{0,h}$, in the sense of Definition 5.2.1. Finally, let us observe that $\{u_{0,h}\}_{h\geq 0}$ has been chosen in such a way that

$$u_{0,h} \longrightarrow u_0 \ \text{ in } L^m(M)$$

Observe also that $\{u_h\}_{h\geq 0}$ is a monotone increasing function in h hence it has a limit as $h \to +\infty$. We call u the limit function. In view (5.5.46), (5.5.47), (5.5.50), (5.5.51), (5.5.54), (5.5.55), the right hand side of (5.5.48) is independent of k, R and h. So, $u^{\frac{m+1}{2}} \in H^1((\tau, T); L^2(M))$. Therefore, $u^{\frac{m+1}{2}} \in C\big([\tau, T]; L^2(M)\big)$. Hence, we can pass to the limit as $h \to +\infty$ in (5.5.54) and (5.5.55) and similarly to what we have seen above, we get

$$\|u(t)\|_{L^m(M)} \leq e^{Ct}\|u_0\|_{L^m(M)}, \tag{5.5.57}$$

and

$$\begin{aligned} \|u(t)\|_{L^\infty(M)} \leq \Gamma \Bigg\{ & \left[e^{Ct}\|u_0\|_{L^m(M)}\right]^{\frac{2m}{2m+N(m-p)}} \\ & + \left[e^{Ct}\|u_0\|_{L^m(M)}\right]^{\frac{2m}{2m+N(m-1)}} \left[\frac{1}{(m-1)t}\right]^{\frac{N}{2m+N(m-1)}} \Bigg\} . \end{aligned} \tag{5.5.58}$$

Note that both (5.5.57) and (5.5.58) hold *for all* $t > 0$, in view of the continuity property of u deduced above.

Moreover, again by monotone convergence, it is possible to compute the limit as $h \to +\infty$ in the integrals of equality (5.5.56) and hence obtain that, for any $\varphi \in C_c^\infty(M \times (0, T))$ such that $\varphi(x, T) = 0$ for any $x \in M$, the function u satisfies,

$$\begin{aligned} -\int_0^T \int_M u\, \varphi_t \, d\mu\, dt = & \int_0^T \int_M u^m \, \Delta\varphi \, d\mu\, dt + \int_0^T \int_M u^p \, \varphi \, d\mu\, dt \\ & + \int_M u_0(x)\, \varphi(x, 0)\, d\mu. \end{aligned} \tag{5.5.59}$$

Observe that, due to inequality (5.5.57), all the integrals in (5.5.59) are finite, hence u is a solution to problem (5.1.1) in the sense of Definition 5.2.1.

Finally, let us discuss (5.2.8). Let $q > 1$. If $u_0 \in L^q(M) \cap L^m(M)$, we choose the sequence u_{0h} so that it further satisfies

$$u_{0h} \to u_0 \quad \text{in } \ L^q(M) \quad \text{ as } \ h \to +\infty\,.$$

We have that

$$\|u_{h,k}^R(t)\|_{L^q(B_R)} \ \leq \ e^{Ct}\|u_{0,h}\|_{L^q(B_R)}. \tag{5.5.60}$$

Hence, due to (5.5.60), letting $k \to +\infty, R \to +\infty, h \to +\infty$, by Fatou's Lemma we deduce (5.2.8). □

Proof of Theorem 5.2.3. We note in first place that the geometrical assumptions on M, in particular the upper curvature bound $\sec \leq -k_2 < 0$, ensure that inequalities (5.1.2) and (5.1.3) both hold on M, see e.g. [39, 40]. Hence, all the result of Theorem 5.2.2 hold, in particular solutions corresponding to data $u_0 \in \mathrm{L}^m(M)$ exist globally in time.

Besides, it has been shown in [54] that if u_0 is a continuous, nonnegative, nontrivial datum, which is sufficiently large in the sense given in the statement, under the lower curvature bound being assumed here the corresponding solution u satisfies the bound

$$u(x,t) \geq C\zeta(t)\left[1 - \frac{r}{a}\eta(t)\right]_+^{\frac{1}{m-1}} \qquad \forall t \in (0,S),\ \forall x \in M,$$

possibly up to a finite time explosion time S, which has however been proved in the present chapter not to exist. Here, the functions η, ζ are given by:

$$\zeta(t) := (\tau + t)^{\alpha}\,, \quad \eta(t) := (\tau + t)^{-\beta} \quad \text{for every } t \in [0, \infty)\,,$$

where $C, \tau, R_0, \inf_{B_{R_0}} u_0$ must be large enough and one can take $0 < \alpha < \frac{1}{m-1}\,, \beta = \frac{\alpha(m-1)+1}{2}$. Clearly, u then satisfies $\lim_{t\to+\infty} u(x,t) = +\infty$ for all $x \in M$, and hence u enjoys the same property by comparison. □

5.6 Proof of Theorems 5.2.5, 5.2.6

For any $R > 0$ we consider the following approximate problem

$$\begin{cases} \rho(x)u_t = \Delta u^m + \rho(x)u^p & \text{in } B_R \times (0,T) \\ u = 0 & \text{in } \partial B_R \times (0,T) \\ u = u_0 & \text{in } B_R \times \{0\}\,, \end{cases} \tag{5.6.61}$$

here B_R denotes the Euclidean ball with radius R and centre in O.

We shall use the following Aronson-Benilan type estimate (see [6]; see also [118, Proposition 2.3]).

Proposition 5.6.1. *Let $1 < p < m$, $u_0 \in H_0^1(B_R) \cap L^\infty(B_R)$, $u_0 \geq 0$. Let u be the solution to problem* (5.6.61)*. Then, for a.e. $t \in (0,T)$,*

$$-\Delta u^m(\cdot,t) \leq \rho u^p(\cdot,t) + \frac{\rho}{(m-1)t}u(\cdot,t) \quad \text{in } \mathfrak{D}'(B_R).$$

Proof of Theorem 5.2.5. The conclusion follows using step by step the same arguments given in the proof of Theorem 5.2.2, since the necessary functional inequalities are being assumed. We use Proposition 5.6.1 instead of 5.4.2. The last statement of the Theorem will be proved later on in Section 5.6.1 □

In order to prove Theorem 5.2.6 we adapt the strategy of [118] to the present case, so we shall be concise and limit ourselves to identifying the main steps and differences. Define

$$d\mu := \rho(x)dx\,.$$

For any $R > 0, k > 0$, for any $v \in L^1_\rho(B_R)$, we set

$$A_k := \{x \in B_R : |v(x)| > k\}$$

and

$$g(k) := \int_{B_R} |G_k(v)|\rho(x)\,dx\,,$$

where $G_k(v)$ has been defined in (5.3.22).

Lemma 5.6.2. *Let $v \in L^1_\rho(B_R)$. Suppose that there exist $C > 0$ and $s > 1$ such that*

$$g(k) \le C\mu(A_k)^s \quad \textit{for any } k \in \mathbb{R}^+.$$

Then $v \in L^\infty(B_R)$ and

$$\|v\|_{L^\infty(B_R)} \le C\left(\frac{s}{s-1}\right)^s \|\rho\|^{s-1}_{L^1(\mathbb{R}^N)}.$$

Proof. Arguing as in the proof of Lemma 5.3.6, we integrate inequality (5.3.26) between 0 and k and using the definition of g, we obtain

$$g(k)^{1-\frac{1}{s}} \le \|v\|^{1-\frac{1}{s}}_{L^1_\rho(B_R)} - \frac{s-1}{s}\,C^{-\frac{1}{s}}k \quad \text{for any } k \in \mathbb{R}^+\,.$$

Choose

$$k = k_0 = C^{\frac{1}{s}}\|v\|^{1-\frac{1}{s}}_{L^1_\rho(B_R)}\frac{s}{s-1},$$

and substitute it in the last inequality. Then we have

$$\begin{aligned} g(k_0) \le 0 \iff \int_{B_R} |G_{k_0}(v)|\,d\mu = 0 \iff |G_{k_0}(v)| = 0 \\ \iff |v| \le k_0 \iff |v| \le C^{\frac{1}{s}}\|v\|^{1-\frac{1}{s}}_{L^1_\rho(B_R)}\frac{s}{s-1}. \end{aligned}$$

Thanks to the assumption that $\rho \in L^1(\mathbb{R}^N)$, we can apply the weighted Hölder inequality to get

$$\|v\|_{L^\infty(B_R)} \le \frac{s}{s-1}C^{\frac{1}{s}}\|v\|^{1-\frac{1}{s}}_{L^\infty(B_R)}\|\rho\|^{1-\frac{1}{s}}.$$

Rearranging the terms in the previous inequality we obtain the thesis. □

Lemma 5.6.3. *Let ρ satisfy (5.2.9) and $\rho \in L^1(\mathbb{R}^N)$. Let $f_1 \in L^{m_1}_\rho(B_R)$ and $f_2 \in L^{m_2}_\rho(B_R)$ where*

$$m_1 > \frac{N}{2}, \qquad m_2 > \frac{N}{2}\,.$$

Assume that $v \in H^1_0(B_R)$, $v \ge 0$ is a subsolution to problem

$$\begin{cases} -\Delta v = \rho(f_1 + f_2) & \textit{in } B_R \\ v = 0 & \textit{on } \partial B_R \end{cases}.$$

Then

$$\|v\|_{L^\infty(B_R)} \le C_1\|f_1\|_{L^{m_1}_\rho(B_R)} + C_2\|f_2\|_{L^{m_2}_\rho(B_R)}, \tag{5.6.62}$$

where

$$C_1 = \frac{1}{C_s^2}\left(\frac{s}{s-1}\right)^s \|\rho\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N}-\frac{1}{m_1}}, \qquad C_2 = \frac{1}{C_s^2}\left(\frac{s}{s-1}\right)^s \|\rho\|_{L^1(\mathbb{R}^N)}^{\frac{2}{N}-\frac{1}{m_2}}, \tag{5.6.63}$$

with s given by (5.3.18) .

Remark 5.6.4. If in Lemma 5.6.3 we further assume that there exists a constant $k_0 > 0$ such that

$$\|f_1\|_{L^{m_1}_\rho(B_R)} \le k_0, \quad \|f_2\|_{L^{m_2}_\rho(B_R)} \le k_0 \quad \text{for all } \ R > 0,$$

then from (5.6.62), we infer that the bound from above on $\|v\|_{L^\infty(B_R)}$ is independent of R. This fact will have a key role in the proof of global existence for problem (5.1.4).

Proof of Lemma 5.6.3. By arguing as in the proof of Proposition 5.3.3, we get

$$\int_{B_R} |G_k(v)|\, d\mu \le \frac{1}{C_s^2}\left[\|f_1\|_{L^{m_1}_\rho}\mu(A_k)^{1+\frac{2}{N}-\frac{1}{m_1}} + \|f_2\|_{L^{m_2}_\rho}\mu(A_k)^{1+\frac{2}{N}-\frac{1}{m_2}}\right].$$

Thus

$$\int_{B_R} |G_k(v)|\, d\mu \le \frac{1}{C_s^2}\,\mu(A_k)^{1+\frac{2}{N}-\frac{1}{l}}\left[\|f_1\|_{L^{m_1}_\rho}\|\rho\|_{L^1(\mathbb{R}^N)}^{\frac{1}{l}-\frac{1}{m_1}} + \|f_2\|_{L^{m_2}_\rho}\|\rho\|_{L^1(\mathbb{R}^N)}^{\frac{1}{l}-\frac{1}{m_2}}\right].$$

Now, defining

$$\bar{C} = \frac{1}{C_s^2}\left[\|f_1\|_{L^{m_1}(B_R)}\|\rho\|_{L^1(\mathbb{R}^N)}^{\frac{1}{l}-\frac{1}{m_1}} + \|f_2\|_{L^{m_2}(B_R)}\|\rho\|_{L^1(\mathbb{R}^N)}^{\frac{1}{l}-\frac{1}{m_2}}\right],$$

the last inequality is equivalent to

$$\int_{B_R} |G_k(v)|\, d\mu \le \bar{C}\,\mu(A_k)^s\,, \quad \text{for any } k \in \mathbb{R}^+\,,$$

where s has been defined in (5.3.18). Hence, it is possible to apply Lemma 5.6.2. By using the definitions of C_1 and C_2 in (5.6.63), we thus have

$$\|v\|_{L^\infty(B_R)} \le C_1\,\|f_1\|_{L^{m_1}_\rho(B_R)} + C_2\,\|f_2\|_{L^{m_2}_\rho(B_R)}\,.$$

□

Proposition 5.6.5. *Let $1 < p < m$, $R > 0, u_0 \in L^\infty(B_R)$, $u_0 \ge 0$. Let u be the solution to problem* (5.6.61)*. Let inequality* (5.1.6) *hold. Then there exists $C = C(p, m, N, C_s, \|\rho\|_{L^1(\mathbb{R}^N)}) > 0$ such that, for all $t > 0$,*

$$\|u(t)\|_{L^\infty(B_R)} \le C\left[1 + \left(\frac{1}{(m-1)t}\right)^{\frac{1}{m-1}}\right].$$

Proof. We proceed as in the proof of Proposition 5.4.3, up to inequality (5.4.40). Thanks to the fact that $\rho \in L^1(\mathbb{R}^N)$, we can apply to (5.4.36) the thesis of Lemma 5.6.3. Thus we obtain

$$\|w\|^m_{L^\infty(B_R)} \le C_1 \|w^p\|_{L^{r_1}_\rho(B_R)} + \gamma C_2 \|w\|_{L^{r_2}_\rho(B_R)}. \tag{5.6.64}$$

Now the constants are

$$\alpha_1 = \frac{m}{p - \frac{q}{r_1}};$$
$$\alpha_2 = \frac{m}{1 - \frac{q}{r_2}};$$
$$\varepsilon_1 \text{ such that } \delta_1 = \frac{1}{4C_1};$$
$$\varepsilon_2 \text{ such that } \delta_2 = \frac{1}{4\gamma C_2}.$$

Plugging (5.4.39) and (5.4.40) into (5.6.64) we obtain

$$\begin{aligned}\|w\|^m_{L^\infty(B_R)} &\le C_1 \|w^p\|_{L^{r_1}_\rho(B_R)} + \gamma C_2 \|w\|_{L^{r_2}_\rho(B_R)} \\ &\le C_1 \left[\delta_1 \|w\|^m_{L^\infty(B_R)} + \frac{\eta(\alpha_1)}{\delta_1^{\frac{1}{\alpha_1 - 1}}} \|w\|^{\frac{mq}{r_1}\frac{1}{m-p+q/r_1}}_{L^q_\rho(B_R)} \right] \\ &+ \gamma C_2 \left[\delta_2 \|w\|^m_{L^\infty(B_R)} + \frac{\eta(\alpha_2)}{\delta_2^{\frac{1}{\alpha_2 - 1}}} \|w\|^{\frac{mq}{r_2}\frac{1}{m-1+q/r_2}}_{L^q_\rho(B_R)} \right].\end{aligned} \tag{5.6.65}$$

Inequality (5.6.65) can be rewritten as

$$\begin{aligned}\|w\|_{L^\infty(B_R)} &\le \left[2\eta(\alpha_1) \left(4C_1^{\alpha_1}\right)^{\frac{1}{\alpha_1 - 1}} \right]^{\frac{1}{m}} \|w\|^{\frac{q}{r_1}\frac{1}{m-p+q/r_1}}_{L^q_\rho(B_R)} \\ &+ \left[2\eta(\alpha_2) \left(4\gamma^{\alpha_2} C_2^{\alpha_2}\right)^{\frac{1}{\alpha_2 - 1}} \right]^{\frac{1}{m}} \|w\|^{\frac{q}{r_2}\frac{1}{m-1+q/r_2}}_{L^q_\rho(B_R)}.\end{aligned}$$

Computing the limits as $r_1 \longrightarrow \infty$ and $r_2 \longrightarrow \infty$ we have

$$\eta(\alpha_1) \longrightarrow \left[\frac{p}{m}\right]^{\frac{p}{m-p}} \left\{1 - \frac{p}{m}\right\};$$
$$\eta(\alpha_2) \longrightarrow \left[\frac{1}{m}\right]^{\frac{1}{m-1}} \left\{1 - \frac{1}{m}\right\};$$
$$\|w\|^{\frac{q}{r_1}\frac{1}{(m-p+q/r_1)}}_{L^q_\rho(B_R)} \longrightarrow 1;$$
$$\|w\|^{\frac{q}{r_2}\frac{1}{(m-1+q/r_2)}}_{L^q_\rho(B_R)} \longrightarrow 1.$$

Moreover we define

$$\Gamma_1 := \left[2\left(\frac{p}{m}\right)^{\frac{p}{m-p}} \left(1 - \frac{p}{m}\right) \right]^{\frac{1}{m}} 4^{\frac{mp}{m-p}} C_1^{\frac{mp}{m-p}};$$
$$\Gamma_2 := \left[2\left(\frac{1}{m}\right)^{\frac{1}{m-1}} \left(1 - \frac{1}{m}\right) \right]^{\frac{1}{m}} 4^{\frac{m}{m-1}} C_1^{\frac{m}{m-1}};$$
$$C := \max\{\Gamma_1\ , \Gamma_2\}$$

and notice that, by the above construction, the thesis follows with this choice of C. □

Proof of Theorem 5.2.6. The conclusion follows by the same arguments as in the proof of Theorem 5.2.2. However, some minor differences are in order. We replace Proposition 5.4.3 by Proposition 5.6.5. Moreover, since $u_0 \in L^1_\rho(\mathbb{R}^N)$, the family of functions $\{u_{0h}\}$ is as follows:

$$\begin{aligned} &u_{0,h} \in L^\infty(\mathbb{R}^N) \cap C_c^\infty(\mathbb{R}^N) \ \text{ for all } h \geq 0, \\ &u_{0,h} \geq 0 \ \text{ for all } h \geq 0, \\ &u_{0,h_1} \leq u_{0,h_2} \ \text{ for any } h_1 < h_2, \\ &u_{0,h} \longrightarrow u_0 \ \text{ in } L^1_\rho(\mathbb{R}^N) \quad \text{ as } h \to +\infty . \end{aligned}$$

Furthermore, instead of (5.5.46), (5.5.50), (5.5.54), (5.5.57), we use the following. By standard arguments (see, e.g. proof of [118, Proposition 2.5-(i)]) we have that

$$\|u^R_{h,k}(t)\|_{L^1_\rho(B_R)} \leq C\|u_{0h}\|_{L^1_\rho(B_R)} \quad \text{ for all } t > 0 ,$$

for some positive constant $C = C(p, m, N, \|\rho\|_{L^1(\mathbb{R}^N)})$, and, for any $\varepsilon \in (0, m-p)$,

$$\int_0^1 \int_{B_R} (u^R_{h,k})^{p+\varepsilon} \rho(x) dx dt \leq \tilde{C} ,$$

for some positive constant $\tilde{C} = \tilde{C}(p, m, N, \|\rho\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^1_\rho(\mathbb{R}^N)})$. Hence, after having passed to the limit as $k \to +\infty, R \to +\infty, h \to +\infty$, for any $T > 0, \varphi \in C_c^\infty(\mathbb{R}^N \times (0, T))$ such that $\varphi(x, T) = 0$ for every $x \in \mathbb{R}^N$, we have that

$$\int_0^T \int_{\mathbb{R}^N} u^{p+\varepsilon} \rho(x) \varphi \, dx dt \leq C .$$

Therefore, (5.2.10) holds.

□

5.6.1 End of proof of Theorem 5.2.5: an example of complete blowup in infinite time

We recall that we are assuming $m > 1$ and $1 < p < m$. Let us set $r := |x|$. We now construct a subsolution to equation

$$\rho \, u_t = \Delta u^m + \rho \, u^p \quad \text{in } \mathbb{R}^N \times (0, T) , \tag{5.6.66}$$

under the hypothesis that there exist k_1 and k_2 with $k_2 \geq k_1 > 0$ such that

$$k_1 r^2 \leq \frac{1}{\rho(x)} \leq k_2 r^2 \quad \text{for any } x \in \mathbb{R}^N \setminus B_e. \tag{5.6.67}$$

Moreover, due to the running assumptions on the weight there exist positive constants ρ_1, ρ_2 such that

$$\rho_1 \leq \frac{1}{\rho(x)} \leq \rho_2 \quad \text{for any } x \in B_e . \tag{5.6.68}$$

Let

$$\mathfrak{s}(x) := \begin{cases} \log(|x|) & \text{if} \quad x \in \mathbb{R}^N \setminus B_e, \\ \\ \dfrac{|x|^2 + e^2}{2e^2} & \text{if} \quad x \in B_e\,. \end{cases}$$

The requested statements will follow from the following result.

Proposition 5.6.6. *Let assumption* (5.2.9), (5.6.67) *and* (5.6.68) *be satisfied, and* $1 < p < m$. *If the initial datum* u_0 *is smooth, compactly supported and large enough, then problem* (5.1.4) *has a solution* $u(t) \in L^\infty(\mathbb{R}^N)$ *for any* $t \in (0, \infty)$ *that blows up in infinite time, in the sense that*

$$\lim_{t\to+\infty} u(x,t) = +\infty \quad \forall x \in \mathbb{R}^N. \tag{5.6.69}$$

More precisely, if $C > 0$, $a > 0$, $\alpha > 0$, $\beta > 0$, $T > 0$ *verify*

$$0 < T^{-\beta} < \frac{a}{2}. \tag{5.6.70}$$

$$0 < \alpha < \frac{1}{m-1}\,, \qquad \beta = \frac{\alpha(m-1)+1}{2}\,, \tag{5.6.71}$$

and

$$u_0(x) \ge CT^\alpha \left[1 - \frac{\mathfrak{s}(x)}{a}\, T^{-\beta}\right]_+^{\frac{1}{m-1}}\,, \quad \textit{for any } x \in \mathbb{R}^N\,,$$

then the solution u *of problem* (5.1.4) *satisfies* (5.6.69) *and the bound from below*

$$u(x,t) \ge C(T+t)^\alpha \left[1 - \frac{\mathfrak{s}(x)}{a}\,(T+t)^{-\beta}\right]_+^{\frac{1}{m-1}}\,, \ \textit{for any } (x,t) \in \mathbb{R}^N \times (0,+\infty)\,.$$

Proof. We construct a suitable subsolution of (5.6.66). Define, for all $(x,t) \in \mathbb{R}^N$,

$$w(x,t) \equiv w(r(x),t) := \begin{cases} u(x,t) & \text{in } [\mathbb{R}^N \setminus B_e] \times (0,T), \\ v(x,t) & \text{in } B_e \times (0,T), \end{cases}$$

where

$$u(x,t) \equiv u(r(x),t) := C(T+t)^\alpha \left[1 - \frac{\log(r)}{a}(T+t)^{-\beta}\right]_+^{\frac{1}{m-1}}\,,$$

and

$$v(x,t) \equiv v(r(x),t) := C(T+t)^\alpha \left[1 - \frac{r^2+e^2}{2e^2}\,\frac{(T+t)^{-\beta}}{a}\right]_+^{\frac{1}{m-1}}\,.$$

Moreover, let

$$F(r,t) := 1 - \frac{\log(r)}{a}(T+t)^{-\beta}\,,$$

$$G(r,t) := 1 - \frac{r^2+e^2}{2e^2}\,\frac{(T+t)^{-\beta}}{a}\,.$$

and define

$$D_1 := \left\{(x,t) \in (\mathbb{R}^N \setminus B_e) \times (0,T) \mid 0 < F(r,t) < 1\right\}.$$

For any $(x,t) \in D_1$, we have:

$$u_t = C\alpha(T+t)^{\alpha-1}F^{\frac{1}{m-1}} - C\beta(T+t)^{\alpha-1}\frac{1}{m-1}F^{\frac{1}{m-1}} + C\beta(T+t)^{\alpha-1}\frac{1}{m-1}F^{\frac{1}{m-1}-1}.$$

$$(u^m)_r = -\frac{C^m}{a}(T+t)^{m\alpha}\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{r}(T+t)^{-\beta}.$$

$$\begin{aligned}(u^m)_{rr} = {} & \frac{C^m}{a}(T+t)^{m\alpha}\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{(T+t)^{-\beta}}{r^2} \\ & + \frac{C^m}{a^2}(T+t)^{m\alpha}\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{(T+t)^{-2\beta}}{r^2}.\end{aligned}$$

Due to (5.6.70)

$$0 < G(r,t) < 1 \quad \text{for all } (x,t) \in B_e \times (0,+\infty).$$

For any $(x,t) \in B_e \times (0,T)$, we have:

$$v_t = C\alpha(T+t)^{\alpha-1}G^{\frac{1}{m-1}} - C\beta(T+t)^{\alpha-1}\frac{1}{m-1}G^{\frac{1}{m-1}} + C\beta(T+t)^{\alpha-1}\frac{1}{m-1}G^{\frac{1}{m-1}-1}.$$

$$(v^m)_r = -\frac{C^m}{a}(T+t)^{m\alpha}\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{r}{e^2}(T+t)^{-\beta}.$$

$$(v^m)_{rr} = -\frac{C^m}{a}(T+t)^{m\alpha}\frac{m}{m-1}G^{\frac{1}{m-1}}\frac{(T+t)^{-\beta}}{e^2} + \frac{C^m}{a^2}(T+t)^{m\alpha}\frac{m}{(m-1)^2}G^{\frac{1}{m-1}-1}(T+t)^{-2\beta}\frac{r^2}{e^4}.$$

For every $(x,t) \in D_1$, by the previous computations we have

$$\begin{aligned} u_t - & \frac{1}{\rho}\Delta u^m - u^p \\ = {} & C\alpha(T+t)^{\alpha-1}F^{\frac{1}{m-1}} - C\beta(T+t)^{\alpha-1}\frac{1}{m-1}F^{\frac{1}{m-1}} + C\beta(T+t)^{\alpha-1}\frac{1}{m-1}F^{\frac{1}{m-1}-1} \\ & + \frac{1}{\rho}\left\{-\frac{C^m}{a}(T+t)^{m\alpha-\beta}\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{r^2} - \frac{C^m}{a^2}(T+t)^{m\alpha-2\beta}\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{1}{r^2}\right. \\ & \left. +(N-1)\frac{C^m}{a}(T+t)^{m\alpha-\beta}\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{r^2}\right\} - C^p(T+t)^{p\alpha}F^{\frac{p}{m-1}}. \end{aligned} \tag{5.6.72}$$

Thanks to (5.6.67), (5.6.72) becomes, for every $(x,t) \in D_1$

$$\begin{aligned} u_t - & \frac{1}{\rho}\Delta u^m - u^p \\ \le {} & CF^{\frac{1}{m-1}-1}\left\{F\left[\alpha(T+t)^{\alpha-1} - \frac{\beta}{m-1}(T+t)^{\alpha-1} + (N-2)k_2\frac{C^{m-1}}{a}\frac{m}{m-1}(T+t)^{m\alpha-\beta}\right]\right. \\ & \left. + \frac{\beta}{m-1}(T+t)^{\alpha-1} - \frac{C^{m-1}}{a^2}\frac{m}{(m-1)^2}k_1(T+t)^{m\alpha-2\beta} - C^{p-1}(T+t)^{p\alpha}F^{\frac{p+m-2}{m-1}}\right\} \\ \le {} & CF^{\frac{1}{m-1}-1}\left\{\sigma(t)F - \delta(t) - \gamma(t)F^{\frac{p+m-2}{m-1}}\right\} \end{aligned}$$

where

$$\varphi(F) := \sigma(t)F - \delta(t) - \gamma(t)F^{\frac{p+m-2}{m-1}},$$

with

$$\begin{aligned}
\sigma(t) &= \left[\alpha - \frac{\beta}{m-1}\right](T+t)^{\alpha-1} + \frac{C^{m-1}}{a}\frac{m}{m-1}k_2\,(N-2)\,(T+t)^{m\alpha-\beta}\,,\\
\delta(t) &= -\frac{\beta}{m-1}(T+t)^{\alpha-1} + \frac{C^{m-1}}{a^2}\frac{m}{(m-1)^2}k_1(T+t)^{m\alpha-2\beta}\,,\\
\gamma(t) &= C^{p-1}(T+t)^{p\alpha}\,,
\end{aligned}$$

Our goal is to find suitable $C>0$, $a>0$, such that

$$\varphi(F) \le 0\,, \quad \text{for all } F\in(0,1)\,.$$

To this aim, we impose that

$$\sup_{F\in(0,1)}\varphi(F) = \max_{F\in(0,1)}\varphi(F) = \varphi(F_0) \le 0\,,$$

for some $F_0\in(0,1)$. We have

$$\begin{aligned}
\frac{d\varphi}{dF} = 0 &\iff \sigma(t) - \frac{p+m-2}{m-1}\gamma(t)F^{\frac{p-1}{m-1}} = 0\\
&\iff F_0 = \left[\frac{m-1}{p+m-2}\frac{\sigma(t)}{\gamma(t)}\right]^{\frac{m-1}{p-1}}\,.
\end{aligned}$$

Then

$$\varphi(F_0) = K\frac{\sigma(t)^{\frac{p+m-2}{p-1}}}{\gamma(t)^{\frac{m-1}{p-1}}} - \delta(t)$$

where $K = \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0$. The two conditions we must verify are

$$K[\sigma(t)]^{\frac{p+m-2}{p-1}} \le \delta(t)\gamma(t)^{\frac{m-1}{p-1}}\,,\quad (m-1)\sigma(t) \le (p+m-2)\gamma(t)\,. \tag{5.6.73}$$

Observe that, thanks to the choice in (5.6.71) and by choosing

$$\frac{C^{m-1}}{a} \ge 2\beta\,\frac{(m-1)}{m}\frac{1}{k_1},$$

we have

$$\begin{aligned}
\sigma(t) &\le \frac{C^{m-1}}{a}\frac{m}{m-1}k_2\,(N-2)\,(T+t)^{m\alpha-\beta}\,,\\
\delta(t) &\ge \frac{C^{m-1}}{2a^2}\frac{m}{(m-1)^2}k_1(T+t)^{m\alpha-2\beta}
\end{aligned}$$

and conditions in (5.6.73) follow. So far, we have proved that

$$u_t - \frac{1}{\rho(x)}\Delta(u^m) - u^p \le 0 \quad \text{ in } D_1\,.$$

Furthermore, since $u^m \in C^1([\mathbb{R}^N \setminus B_e] \times [0,T))$ it follows that u is a subsolution to equation (5.6.66) in $[\mathbb{R}^N \setminus B_e] \times (0,T)$. Now, we consider equation (5.6.66) in $B_e \times (0,T)$. We observe that, due to condition (5.6.70),

$$\frac{1}{2} < G < 1 \text{ for all } (x,t) \in B_e \times (0,T). \tag{5.6.74}$$

Similarly to the previous computation we obtain, for all $(x,t) \in B_e \times (0,T)$:

$$v_t - \frac{1}{\rho}\Delta v^m - v^p \le C G^{\frac{1}{m-1}-1}\psi(G)\,,$$

where

$$\psi(G) := \sigma_0 G - \delta_0 - \gamma G^{\frac{p+m-2}{m-1}}\,,$$

with

$$\begin{aligned}
\sigma_0(t) &= \left[\alpha - \frac{\beta}{m-1}\right](T+t)^{\alpha-1} + \rho_2 \frac{N}{e^2}\frac{m}{m-1}\frac{C^{m-1}}{a}(T+t)^{m\alpha-\beta}\,,\\
\delta_0(t) &= -\frac{\beta}{m-1}(T+t)^{\alpha-1}\\
\gamma(t) &= C^{p-1}(T+t)^{p\alpha}\,.
\end{aligned}$$

Due to (5.6.74), v is a subsolution of (5.6.66) for every $(x,t) \in B_e \times (0,T)$, if

$$2^{\frac{p+m-2}{m-1}}(\sigma_0 - \delta_0) \le \gamma\,.$$

This last inequality is always verified thanks to (5.6.71). Hence we have proved that

$$v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p \le 0 \quad \text{in } \; B_e \times (0,T)\,,$$

Moreover, $w^m \in C^1(\mathbb{R}^N \times [0,T))$, indeed,

$$(u^m)_r = (v^m)_r = -C^m \zeta(t)^m \frac{m}{m-1}\frac{1}{e}\frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e \times (0,T)\,.$$

Hence, w is a subsolution to equation (5.6.66) in $\mathbb{R}^N \times (0,T)$.

□

Part III

Quasilinear parabolic differential inequalities

Chapter 6

Nonexistence of solutions for a class of quasilinear parabolic inequalities

6.1 Introduction

In this chapter we investigate nonexistence of nonnegative, nontrivial global weak solutions to quasilinear parabolic inequalities of the following type:

$$\begin{cases} \partial_t u - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \geq V\, u^q & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u = u_0 & \text{in } \Omega \times \{0\}\,; \end{cases} \tag{6.1.1}$$

where Ω is an open bounded connected subset of $\mathbb{R}^N$, $N \geq 3$, $p > 1$ and $q > \max\{p-1, 1\}$. Furthermore, we assume that $V > 0$ a.e. in $\Omega \times (0,T)$ and the initial condition $u_0 \geq 0$ a.e. in Ω.

Global existence and finite time blow-up of solutions for problem (6.1.1) has been deeply studied when $\Omega = \mathbb{R}^N$, see e.g. [33, 34, 35, 97, 98, 105, 104] and references therein. In particular, in [98], nonexistence of nontrivial weak solutions is proved for problem (6.1.1) when $\Omega = \mathbb{R}^N$, $V \equiv 1$ and

$$p > \frac{2N}{N+1}, \qquad q \leq p - 1 + \frac{p}{N}\,.$$

Moreover, problem (6.1.1) has been investigated also in the Riemannian setting, see e.g. [9, 89, 110, 56, 137] and references therein. In [89] problem (6.1.1) is studied when $\Omega = M$ is a complete, N-dimensional, noncompact Riemannian manifold; it is investigated nonexistence of nonnegative nontrivial weak solutions depending on the interplay between the geometry of the underlying manifold, the power nonlinearity and the behavior of the potential at infinity, assuming that $u_0 \in L^1_{loc}(M)$, $u \geq 0$ a.e. in M and $V \in L^1_{loc}(M \times [0, +\infty))$, $V > 0$ a.e. in M.

Furthermore, we mention that nonexistence results of nonnegative nontrivial solutions have been also much investigated for solutions to elliptic quasilinear equation of the form

$$\frac{1}{a(x)}\operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) + V(x)u^q \leq 0 \quad \text{in } M\,, \tag{6.1.2}$$

where

$$a > 0,\ \ a \in \mathrm{Lip}_{loc}(M), \qquad V > 0 \ \text{ a.e. on } M,\ \ V \in L^1_{loc}(M),$$

$p > 1$, $q > p-1$ and M can be either the Euclidean space $\mathbb{R}^N$ or a general Riemannian manifold.

We refer to [17, 95, 96, 97, 98] for a comprehensive description of results related to problem (6.1.2) and also more general problems on $\mathbb{R}^N$. Problem (6.1.2) when M is a complete noncompact Riemannian manifold has been considered e.g. in [41, 42, 90, 123, 124]. In particular, in [90] it is showed how the geometry of the underlying manifold M and the behavior of the potential V at infinity affect the nonexistence of nonnegative nontrivial weak solutions for inequality (6.1.2). Finally, we mention that (6.1.2) posed on an open relatively compact connected domain $\Omega \subset M$ has been studied in [100]. Under the assumptions that

$$a > 0,\ \ a \in \mathrm{Lip}_{loc}(\Omega), \qquad V > 0 \ \text{ a.e. on } \Omega,\ \ V \in L^1_{loc}(\Omega),$$

$p > 1$, $q > p-1$, the authors investigate the relation between the behavior of the potential V at the boundary of Ω and nonexistence of nonnegative weak solutions.

In the present work, we are concerned with nonnegative weak solutions to problem (6.1.1). Under suitable weighted volume growth assumptions involving V and q, we obtain nonexistence of global weak solutions (see Theorems 6.2.1, 6.2.2). The proofs are mainly based on the choice of a family of suitable test functions, depending on two parameters, that enables us to deduce first some appropriate a priori estimates, then that the unique global solution is $u \equiv 0$. Such test functions are defined by adapting to the present situation those used in [89]; however, some important differences occur, since in [89] an unbounded underlying manifold is considered, whereas now we consider a bounded domain. In some sense, the role of *infinity* of [89] is now played by the boundary $\partial\Omega$. Obviously, this implies that such test functions satisfy different properties. To the best of our knowledge, the definition and the use of such test functions are new.

As a special case, we consider in particular the semilinear parabolic problem

$$\begin{cases} \partial_t u - \Delta u = V u^q & \text{in } \Omega \times (0, +\infty) \\ u = 0 & \text{on } \partial\Omega \times (0, +\infty) \\ u = u_0 & \text{in } \Omega \times \{0\}\,, \end{cases} \tag{6.1.3}$$

where $q > 1$, $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, with $V \geq 0$, i.e. problem (6.1.1) with $p = 2$.

As a consequence of our general results, we infer that nonexistence of global solutions for problems (6.1.1) and (6.1.3) prevails, when

$$V(x,t) \geq C d(x)^{-\sigma_1} \quad \text{for a.e. } \ x \in \Omega,\, t \in [0, +\infty)$$

for some $C > 0$ and

$$\sigma_1 > q + 1,$$

where

$$d(x) := \text{dist}(x, \partial\Omega) \quad \text{for any } \; x \in \overline{\Omega}. \tag{6.1.4}$$

Furthermore, we show the sharpness of this result for the semilinear problem (6.1.3) in case $\partial\Omega$ is regular enough and $V = V(x)$ is continuous and independent of t. Indeed, under the assumption that

$$0 \leq V(x) \leq Cd(x)^{-\sigma_1} \quad \text{for all } \; x \in \Omega$$

for some $C > 0$ and

$$0 \leq \sigma_1 < q + 1,$$

we prove the existence of a global classical solution for problem (6.1.3) (see Theorem 6.2.5), if the initial datum u_0 is small enough. This existence result is obtained by means of the sub– and supersolution's method. In particular, we construct a supersolution to problem (6.1.3), which actually is a supersolution of the associated stationary equation. Such supersolution is obtained as the fixed point of a suitable contraction map. In order to show that such a fixed point exists, we need to estimate some integrals involving the Green function associated to the Laplace operator $-\Delta$ in Ω (see Lemmas 6.6.1-6.6.2). Finally, we study the *slightly supercritical* case

$$V(x,t) \geq d(x)^{-q-1} f(d(x))^{q-1} \quad \text{for a.e.} \;\; x \in \Omega,\, t \in [0, +\infty),$$

where f is a function satisfying suitable assumptions and such that $\lim_{\varepsilon\to 0^+} f(\varepsilon) = +\infty$, for which we prove nonexistence of nonnegative nontrivial weak solutions in $\Omega \times (0, +\infty)$ (see Theorem 6.2.6). The proof of this result require a different argument with respect to the previous nonexistence results, which makes use of linearity of the operator and of the special form of the potential. Then the critical rate of growth $d(x)^{-q-1}$ as x approaches $\partial\Omega$ is indeed sharp for the nonexistence of solutions to problem (6.1.3). Our results do not cover the case of *critical* rate of growth, i.e.

$$C_1 d(x)^{-q-1} \leq V(x,t) \leq C_2 d(x)^{-q-1}$$

for some C_1, $C_2 > 0$, but we conjecture that also in this case no nonnegative nontrivial supersolution of problem (6.1.3) exists.

Chapter 6 is organized as follows. In Section 6.2 we describe our main results and some consequences for problem (6.1.1) (see Theorem 6.2.1, 6.2.2 and Corollaries 6.2.3, 6.2.4);). in particular, in Subsection 6.2.1 we give the statements of our results for the semilinear problem (6.1.3) (see Theorem 6.2.5, 6.2.6 and Corollary 6.2.7). The definition of weak solutions and some preliminaries inequalities are stated in Section 6.3. Finally we prove the results obtained for problem (6.1.1) in Sections 6.4 and 6.5, while the proofs of the results concerning the semilinear problem (6.1.3) are shown in Sections 6.6 and 6.7.

6.2 Statements of the main results

We now introduce the following two hypotheses HP1 and HP2 under which we will prove nonexistence for problem (6.1.1). Let $\theta_1 \geq 1$, $\theta_2 \geq 1$; for each $\delta > 0$, we define

$$S := \Omega \times [0, +\infty) \quad \text{and} \quad E_\delta := \left\{(x,t) \in S \, : \, d(x)^{-\theta_2} + t^{\theta_1} \leq \delta^{-\theta_2}\right\}, \tag{6.2.5}$$

Observe that $E_{\delta_1} \subset E_{\delta_2}$ for every $\delta_1 > \delta_2 > 0$ and that $\bigcup_{\delta>0} E_\delta = S$. Moreover let

$$\begin{aligned} \bar{s_1} &:= \frac{q}{q-1}\theta_2\,, \quad \bar{s_2} := \frac{1}{q-1}\,, \\ \bar{s_3} &:= \frac{pq}{q-p+1}\theta_2\,, \quad \bar{s_4} := \frac{p-1}{q-1}\,, \end{aligned} \tag{6.2.6}$$

HP1 Assume that there exist constants $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 > 0$, $C > 0$, $\delta_0 \in (0,1)$ and $\varepsilon_0 > 0$ such that

(i) for any $\delta \in (0,\delta_0)$ and for any $\varepsilon \in (0,\varepsilon_0)$

$$\int_{E_{\left(\frac{1}{2}\right)^{1/\theta_2}\delta} \setminus E_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon}\,dxdt \,\leq\, C\delta^{-\bar{s_1}-C_0\varepsilon}\,|\log(\delta)|^{s_2} \tag{6.2.7}$$

for some $0 < s_2 < \bar{s_2}$;

(ii) for any $\delta \in (0,\delta_0)$ and for any $\varepsilon \in (0,\varepsilon_0)$

$$\int_{E_{\left(\frac{1}{2}\right)^{1/\theta_2}\delta} \setminus E_\delta} d(x)^{-(\theta_2+1)\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\,dxdt \,\leq\, C\delta^{-\bar{s_3}-C_0\varepsilon}\,|\log(\delta)|^{s_4} \tag{6.2.8}$$

for some $0 < s_4 < \bar{s_4}$.

HP2 Assume that there exist constants $\theta_1 \geq 1$, $\theta_2 \geq 1$, $C_0 > 0$, $C > 0$, $\delta_0 \in (0,1)$ and $\varepsilon_0 > 0$ such that

(i) for any $\delta \in (0,\delta_0)$ and for any $\varepsilon \in (0,\varepsilon_0)$

$$\int_{E_{\left(\frac{1}{2}\right)^{1/\theta_2}\delta} \setminus E_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} V^{-\frac{1}{q-1}+\varepsilon}\,dxdt \,\leq\, C\delta^{-\bar{s_1}-C_0\varepsilon}\,|\log(\delta)|^{\bar{s_2}}\,; \tag{6.2.9}$$

$$\int_{E_{\left(\frac{1}{2}\right)^{1/\theta_2}\delta} \setminus E_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} V^{-\frac{1}{q-1}-\varepsilon}\,dxdt \,\leq\, C\delta^{-\bar{s_1}-C_0\varepsilon}\,|\log(\delta)|^{\bar{s_2}}\,; \tag{6.2.10}$$

(ii) for any $\delta \in (0,\delta_0)$ and for any $\varepsilon \in (0,\varepsilon_0)$

$$\int_{E_{\left(\frac{1}{2}\right)^{1/\theta_2}\delta} \setminus E_\delta} d(x)^{-(\theta_2+1)\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\,dxdt \,\leq\, C\delta^{-\bar{s_3}-C_0\varepsilon}\,|\log(\delta)|^{\bar{s_4}}\,; \tag{6.2.11}$$

$$\int_{E_{\left(\frac{1}{2}\right)^{1/\theta_2}\delta} \setminus E_\delta} d(x)^{-(\theta_2+1)\left(\frac{q}{q-p+1}+\varepsilon\right)} V^{-\frac{p-1}{q-p+1}-\varepsilon}\,dxdt \,\leq\, C\delta^{-\bar{s_3}-C_0\varepsilon}\,|\log(\delta)|^{\bar{s_4}}\,. \tag{6.2.12}$$

We can now state our main results.

Theorem 6.2.1. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. Assume that condition HP1 holds. If u is a nonnegative weak solution of problem* (6.1.1), *then $u = 0$ a.e. in S.*

Theorem 6.2.2. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. Assume that condition HP2 holds. If u is a nonnegative weak solution of problem* (6.1.1), *then $u = 0$ a.e. in S.*

As a consequence of Theorem 6.2.1 we introduce the following Corollary 6.2.3. Let $d(x)$ and S be defined as in (6.1.4) and (6.2.5) respectively. Moreover we introduce the functions $h : \Omega \to \mathbb{R}$ and $f : (0, +\infty) \to \mathbb{R}$ such that

$$h(x) \geq C\, d(x)^{-\sigma_1} \, |\log(d(x))|^{-\delta_1} \qquad \text{for a.e. } x \in \Omega, \tag{6.2.13}$$

$$0 < f(t) \leq C\,(1+t)^{\alpha} \qquad \text{for a.e. } t \in (0, +\infty), \tag{6.2.14}$$

where $\sigma_1, \delta_1, \alpha \geq 0$, $C > 0$. We can now state the following

Corollary 6.2.3. *Let $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. Suppose that $V \in L^1_{loc}(\Omega \times [0, +\infty)$ satisfies*

$$V(x,t) \geq h(x) f(t) \quad \textit{for a.e. } (x,t) \in S, \tag{6.2.15}$$

where h and f satisfy (6.2.13) *and* (6.2.14) *respectively. Moreover suppose that*

$$\begin{aligned} &\int_0^T f(t)^{-\frac{1}{q-1}}\, dt \leq C T^{\sigma_2} \,(\log T)^{\delta_2} \\ &\int_0^T f(t)^{-\frac{p-1}{q-p+1}}\, dt \leq C T^{\sigma_4}, \end{aligned} \tag{6.2.16}$$

for $T > 0$, σ_2, σ_4, δ_2, $\delta_4 \geq 0$ and $C > 0$. Finally assume that

(i) $\sigma_1 > q + 1$;

(ii) $0 \leq \sigma_2 \leq \frac{q}{q-1}$;

(iii) $\delta_1 < 1$ *and* $\delta_2 < \frac{1-\delta_1}{q-1}$.

Now, if u is a nonnegative weak solution of problem (6.1.1), *then $u = 0$ a.e. in S.*

As an immediate consequence of Corollary 6.2.3, choosing $f(t) \equiv 1$, $\sigma_2 = \sigma_4 = 1$ and $\delta_1 = \delta_2 = 0$, we obtain the following

Corollary 6.2.4. *Let $p > 1$, $q > \max\{p-1, 1\}$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. Suppose that $V \in L^1_{loc}(\Omega \times [0, +\infty))$ satisfies*

$$V(x,t) \geq C d(x)^{-\sigma_1} \quad \textit{for a.e. } (x,t) \in S, \tag{6.2.17}$$

with $\sigma_1 > q + 1$. If u is a nonnegative weak solution of problem (6.1.1), *then $u = 0$ a.e. in S.*

6.2.1 Further result for semilinear problems

We prove, for the semilinear problem (6.1.3), an existence result when $V = V(x)$ is continuous and independent of t and

$$0 \leq V(x) \leq Cd(x)^{-\sigma_1}, \quad x \in \Omega,$$

with

$$0 \leq \sigma_1 < q + 1$$

(see Theorem 6.2.5). Then we show a nonexistence result that yield that all nonnegative solutions of (6.1.3) are trivial if V blows up at the boundary $\partial\Omega$ faster than $d(x)^{-q-1}$ (see Theorem 6.2.6 and Corollary 6.2.7 for precise statements).

Theorem 6.2.5. *Suppose that $\partial\Omega$ is of class C^3 and let $u_0 \in C(\Omega)$, $u_0 \geq 0$ in Ω, be such that there exists $\varepsilon > 0$ such that*

$$0 \leq u_0 \leq \varepsilon\, d(x) \quad \textit{for any } x \in \overline{\Omega}. \tag{6.2.18}$$

Moreover let $V \in C(\Omega)$, $V > 0$ in Ω and assume that for some $C > 0$

$$V = V(x) \leq Cd(x)^{-\sigma_1} \quad \textit{for any } x \in \overline{\Omega}. \tag{6.2.19}$$

with

$$0 < \sigma_1 < q + 1. \tag{6.2.20}$$

Then problem (6.1.3) admits a classical solution u in $(\Omega \times (0, +\infty))$ if $\varepsilon > 0$ is small enough.

For any $\varepsilon > 0$ sufficiently small, set

$$\Omega_\varepsilon = \{x \in \Omega \,|\, d(x) \geq \varepsilon\}. \tag{6.2.21}$$

Theorem 6.2.6. *Let $V \in L^1_{loc}(\Omega \times [0, \infty))$, $V > 0$ a.e., and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. Assume that there exists a nonincreasing function $f : (0, \varepsilon_0) \to [1, \infty)$ such that $\lim_{\varepsilon\to 0^+} f(\varepsilon) = +\infty$ and such that, for some $C > 0$, for every $\varepsilon > 0$ small enough*

$$\begin{aligned} &\int_0^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} V^{-\frac{1}{q-1}}\, dxdt \,\leq\, C\,\varepsilon^{\frac{2q}{q-1}}\,, \\ &\int_{\frac{1}{2}f(\varepsilon)}^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}}} V^{-\frac{1}{q-1}}\, dxdt \,\leq\, C\, f(\varepsilon)^{\frac{q}{q-1}}\,. \end{aligned} \tag{6.2.22}$$

If u is a nonnegative weak supersolution of problem (6.1.3), see Definition 6.3.2, then $u = 0$ a.e. in $\Omega \times (0, +\infty)$.

As a consequence of Theorem 6.2.6 we have the following

Corollary 6.2.7. *Suppose that $u_0 \in L^1_{loc}(\Omega)$ with $u_0 \geq 0$ a.e. in Ω. Assume that V satisfies for some $C > 0$*

$$V(x,t) \geq Cd(x)^{-q-1} f(d(x))^{q-1} \quad \textit{for a.e. } x \in \Omega,\, t \in [0, +\infty), \tag{6.2.23}$$

where $f : (0, \mathrm{diam}(\Omega)] \to [1, +\infty)$ is nonincreasing in a right-neighborhood of 0 and such that $\lim_{\varepsilon\to 0^+} f(\varepsilon) = +\infty$. If u is a nonnegative weak supersolution of problem (6.1.3), see Definition 6.3.2, then $u = 0$ a.e. in $\Omega \times (0, +\infty)$.

Remark 6.2.8. *We note that an example of function f satisfying the assumptions of Corollary 6.2.7 is*

$$f(r) = \left[\overbrace{\log \circ \log \circ \ldots \circ \log}^{m \text{ times}} \left(K + \frac{1}{r} \right) \right]^{\beta}, \quad r > 0,$$

for any $\beta > 0$, $m \in \mathbb{N}$ and for $K > 0$ sufficiently large.

Remark 6.2.9. *We note that our results do not cover the case of a potential V having critical growth, i.e.*

$$C_1 d(x)^{-q-1} \leq V(x,t) \leq C_2 d(x)^{-q-1},$$

even if we conjecture that Corollary 6.2.4 holds also when $\sigma_1 = q+1$.

Remark 6.2.10. *From Remark 2 in [100] we see that the stationary problem*

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + V u^q \leq 0 \qquad \text{in } \Omega \tag{6.2.24}$$

does not admit any nontrivial nonnegative solution if

$$V(x) \geq C d(x)^{-q-1} |\log d(x)|^{-1}$$

for some $C > 0$. On the other hand, the function φ satisfying (6.6.136)*, which we construct in the proof of Theorem 6.2.5 using a fixed point argument (for small values of the parameter $\lambda > 0$), is a nonnegative nontrivial solution of problem* (6.2.24) *with $p = 2$ in the case when*

$$V \leq C d(x)^{-\sigma_1},$$

with $0 \leq \sigma_1 < q+1$. Thus we see that the exponent $q+1$ plays a special role both for the elliptic and the parabolic problems.

For the sake of completeness, we also observe that in [100] an example was constructed in a unit ball, showing that problem (6.2.24) *for $p = 2$ may admit a nontrivial nonnegative solution if*

$$V(x) = C d(x)^{-q-1} |\log d(x)|^{-1-\varepsilon}$$

for some $\varepsilon > 0$.

6.3 Preliminaries

Let us first give the precise definition of solution to problem (6.1.1).

Definition 6.3.1. *Let $p > 1$, $q > \max\{p-1, 1\}$, $V \in L^1_{loc}(\Omega \times [0,+\infty))$, $V > 0$ a.e. in $\Omega \times (0,+\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. We say that $u \in W^{1,p}_{loc}(\Omega \times [0,+\infty)) \cap L^q_{loc}(\Omega \times [0,+\infty), V dxdt)$ is a weak solution of problem* (6.1.1) *if $u \geq 0$ a.e. in $\Omega \times (0,+\infty)$ and for every $\varphi \in W^{1,p}(\Omega \times [0,+\infty))$, $\varphi \geq 0$ a.e. in $\Omega \times [0,+\infty)$ and with compact support, one has*

$$\begin{aligned} \int_0^\infty \int_\Omega V u^q \varphi \, dxdt \leq & \int_0^\infty \int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla \varphi \rangle \, dxdt \\ & - \int_0^\infty \int_\Omega u \, \partial_t \varphi \, dxdt - \int_\Omega u_0 \, \varphi(x,0) \, dx. \end{aligned} \tag{6.3.25}$$

Definition 6.3.2. *Let $p, q > 1$, $V \in L^1_{loc}(\Omega \times [0, +\infty))$, $V > 0$ a.e. in $\Omega \times (0, +\infty)$ and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. in Ω. We say that $u \in W^{1,2}_{loc}(\Omega \times [0, +\infty)) \cap L^q_{loc}(\Omega \times [0, +\infty), V dxdt)$ is a weak solution of problem* (6.1.3) *if $u \geq 0$ a.e. in $\Omega \times (0, +\infty)$ and for every $\varphi \in \mathrm{Lip}(\Omega \times [0, \infty))$, $\varphi \geq 0$ in $\Omega \times [0, +\infty)$ and with compact support in $\Omega \times [0, \infty)$, one has*

$$\int_0^\infty \int_\Omega V u^q \varphi \, dxdt = \int_0^\infty \int_\Omega \langle \nabla u, \nabla \varphi \rangle \, dxdt - \int_0^\infty \int_\Omega u \, \partial_t \varphi \, dxdt - \int_\Omega u_0 \, \varphi(x, 0) \, dx. \tag{6.3.26}$$

We say that u is a supersolution to problem (6.1.3) *if it satisfies Definition 6.3.1 with $p = 2$.*

We now state some Lemmas that will be used in the proofs of Theorems 6.2.1 and 6.2.2. We omit here the proofs of these Lemmas that can be find in [89].

Lemma 6.3.3. *Let $s \geq \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$ be fixed. Then there exists a constant $C > 0$ such that for every $\alpha \in \frac{1}{2}\left(-\min\{1, p-1\}, 0\right)$, for every nonnegative weak solution u of problem* (6.1.1) *and for every $\varphi \in Lip\,(\Omega \times [0, +\infty))$ with compact support, $0 \leq \varphi \leq 1$ one has*

$$\begin{aligned} &\frac{1}{2} \int_0^\infty \int_\Omega V \, u^{q+\alpha} \, \varphi^s \, dx \, dt + \frac{3}{4}|\alpha| \int_0^\infty \int_\Omega |\nabla u|^p u^{\alpha-1} \, \varphi^s \, dxdt \\ &\qquad \leq C \left\{ |\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}} \, dxdt \right. \\ &\qquad\qquad \left. + \int_0^\infty \int_\Omega |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}} \, dx \, dt \right\}. \end{aligned} \tag{6.3.27}$$

Lemma 6.3.4. *Let $s \geq \max\left\{1, \frac{q+1}{q-1}, \frac{2pq}{q-p+1}\right\}$ be fixed. Then there exists a constant $C > 0$ such that for every $\alpha \in \frac{1}{2}\left(-\min\left\{1, p-1, q-1, \frac{q-p+1}{p-1}\right\}, 0\right)$, for every nonnegative weak solution u of problem* (6.1.1) *and for every $\varphi \in Lip\,(S)$ with compact support and $0 \leq \varphi \leq 1$ one has*

$$\begin{aligned} &\int_0^\infty \int_\Omega V \, u^q \, \varphi^s \, dx \, dt \\ &\quad \leq C \left[|\alpha|^{-1} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} \, dxdt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} \, |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} \, dxdt \right) \right]^{\frac{p-1}{p}} \\ &\quad \times \left(\int\int_{S \setminus K} V \, u^{q+\alpha} \varphi^s \, dx \, dt \right)^{\frac{1}{q+\alpha}} \left(\int\int_{S \setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} \, |\nabla \varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}} \, dxdt \right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\ &\quad + C \left(\int\int_{S \setminus K} V \, u^{q+\alpha} \, \varphi^s \, dxdt \right)^{\frac{1}{q+\alpha}} \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q+\alpha-1}} \, |\partial_t \varphi|^{\frac{q+\alpha}{q+\alpha-1}} \, dxdt \right)^{\frac{q+\alpha-1}{q+\alpha}}, \end{aligned} \tag{6.3.28}$$

where $K := \{(x, t) \in S \, : \, \varphi(x, t) = 1\}$ and S has been defined in (6.2.5).

Corollary 6.3.5. *Under the hypotheses of Lemma 6.3.4 one has*

$$\begin{aligned}
&\int_0^\infty \int_\Omega V\, u^q\, \varphi^s\, dx\, dt \\
&\quad \le C\left[|\alpha|^{-1}\left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}}\, |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}\, dxdt\right)\right]^{\frac{p-1}{p}} \\
&\quad \times \left(\iint_{S\setminus K} V\, u^q \varphi^s\, dx\, dt\right)^{\frac{(1-\alpha)(p-1)}{pq}} \left(\iint_{S\setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}\, |\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\, dxdt\right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&\quad + C\left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}}\, |\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}}\, |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}\, dxdt\right)^{\frac{1}{q+\alpha}} \\
&\quad \times \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q+\alpha-1}}\, |\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\, dxdt\right)^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned} \tag{6.3.29}$$

Lemma 6.3.6. *Let* $s \ge \max\left\{1, \frac{q+1}{q-1}, \frac{2pq}{q-p+1}\right\}$ *be fixed. Then there exists a constant* $C > 0$ *such that for every* $\alpha \in \frac{1}{2}\left(-\min\left\{1, p-1, q-1, \frac{q-p+1}{p-1}\right\}, 0\right)$*, for every nonnegative weak solution* u *of problem* (6.1.1) *and for every* $\varphi \in Lip\,(S)$ *with compact support and* $0 \le \varphi \le 1$ *one has*

$$\begin{aligned}
&\int_0^\infty \int_\Omega V\, u^q\, \varphi^s\, dx\, dt \\
&\quad \le C\left[|\alpha|^{-1}\left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla\varphi|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt + \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}}\, |\partial_t\varphi|^{\frac{q+\alpha}{q-1}}\, dxdt\right)\right]^{\frac{p-1}{p}} \\
&\quad \times \left(\iint_{S\setminus K} V\, u^{q+\alpha} \varphi^s\, dxdt\right)^{\frac{1}{q+\alpha}} \left(\iint_{S\setminus K} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}}\, |\nabla\varphi|^{\frac{pq}{q-(1-\alpha)(p-1)}}\, dxdt\right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&\quad + C\left(\iint_{S\setminus K} V\, u^q\, \varphi^s\, dx\, dt\right)^{\frac{1}{q}} \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q-1}}\, |\partial_t\varphi|^{\frac{q}{q-1}}\, dxdt\right)^{\frac{q-1}{q}},
\end{aligned} \tag{6.3.30}$$

where $K := \{(x,t) \in S \,:\, \varphi(x,t) = 1\}$ *and* S *has been defined in* (6.2.5).

6.4 Proof of Theorem 6.2.1 and Corollary 6.2.3

Proof of Theorem 6.2.1. For any $\delta > 0$ sufficiently small, let $\alpha := \frac{1}{\log\delta}$. Observe that $\alpha < 0$ and $\alpha \to 0^-$ for $\delta \to 0$. We define for any $(x,t) \in S$

$$\varphi(x,t) := \begin{cases} 1 & \text{in } E_\delta \\ \left[\dfrac{d(x)^{-\theta_2} + t^{\theta_1}}{\delta^{-\theta_2}}\right]^{C_1\alpha} & \text{in } (E_\delta)^C \end{cases}. \tag{6.4.31}$$

where

$$C_1 > \frac{2(C_0 + \theta_2 + 1)}{\theta_2 q} \tag{6.4.32}$$

$\theta_1, \theta_2 \geq 1$ as in HP1 and E_δ has been defined in (6.2.5). Moreover, for any $n \in \mathbb{N}$ we define

$$\eta_n(x,t) := \begin{cases} 1 & \text{in } E_{\frac{\delta}{n}} \\ 2 - \left(\frac{\delta}{n}\right)^{\theta_2} \left[d(x)^{-\theta_2} + t^{\theta_1}\right] & \text{in } E_{\left(\frac{1}{2}\right)^{1/\theta_2}\frac{\delta}{n}} \setminus E_{\frac{\delta}{n}} \\ 0 & \text{in } E^C_{\left(\frac{1}{2}\right)^{1/\theta_2}\frac{\delta}{n}} \end{cases}. \tag{6.4.33}$$

Let

$$\varphi_n(x,t) := \eta_n(x,t)\,\varphi(x,t). \tag{6.4.34}$$

Observe that for any $(x,t) \in S$, $\varphi_n \in Lip(S)$ and $0 \leq \varphi \leq 1$. Moreover, for any $a \geq 1$ we have

$$|\partial_t \varphi_n|^a = |\eta_n \partial_t \varphi \,+\, \varphi \partial_t \eta_n|^a \leq 2^{a-1}\left(|\partial_t \varphi|^a + \varphi^a |\partial_t \eta_n|^a\right). \tag{6.4.35}$$

$$|\nabla \varphi_n|^a = |\eta_n \nabla \varphi \,+\, \varphi \nabla \eta_n|^a \leq 2^{a-1}\left(|\nabla \varphi|^a + \varphi^a |\nabla \eta_n|^a\right). \tag{6.4.36}$$

Let $s \geq \max\left\{1, \frac{q}{q-1}, \frac{pq}{q-p+1}\right\}$, we apply Lemma 6.3.3 with φ replaced by the family of functions φ_n. Then, for some positive constant C, for every $n \in \mathbb{N}$ and $|\alpha| > 0$ we have

$$\begin{aligned} &\int_0^\infty \int_\Omega V\, u^{q+\alpha}\, \varphi_n^s \, dx\, dt \\ &\quad \leq C\left\{|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}}\, dxdt + \int_0^\infty \int_\Omega |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, dxdt\right\} \\ &\quad \leq C|\alpha|^{-\frac{(p-1)q}{q-p+1}}\left[\int_0^\infty \int_\Omega |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}}\, dxdt + \int_0^\infty \int_\Omega \varphi^{\frac{p(q+\alpha)}{q-p+1}} |\nabla \eta_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}}\, dx\, dt\right] \\ &\quad + C\left[\int_0^\infty \int_\Omega |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, dxdt + \int_0^\infty \int_\Omega \varphi^{\frac{q+\alpha}{q-1}} |\partial_t \eta_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, dxdt\right]. \end{aligned}$$

Let us define

$$\tilde{E}_{\delta,n} := E_{\left(\frac{1}{2}\right)^{1/\theta_2}\frac{\delta}{n}} \setminus E_{\frac{\delta}{n}}, \tag{6.4.37}$$

and

$$I_1 := \int_0^\infty \int_\Omega |\nabla \varphi|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}}\, dxdt, \tag{6.4.38}$$

$$I_2 := \int\int_{\tilde{E}_{\delta,n}} \varphi^{\frac{p(q+\alpha)}{q-p+1}} |\nabla \eta_n|^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}}\, dx\, dt, \tag{6.4.39}$$

$$I_3 := \int_0^\infty \int_\Omega |\partial_t \varphi|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, dxdt, \tag{6.4.40}$$

$$I_4 := \int\int_{\tilde{E}_{\delta,n}} \varphi^{\frac{q+\alpha}{q-1}} |\partial_t \eta_n|^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, dxdt. \tag{6.4.41}$$

Then the latter inequality can be read, for a positive constant C and for every $n \in \mathbb{N}$, as

$$\int_0^\infty \int_\Omega V\, u^{q+\alpha}\, \varphi_n^s\, dxdt \leq C|\alpha|^{-\frac{(p-1)q}{q-p+1}}\left[I_1 + I_2\right] + C\left[I_3 + I_4\right]. \tag{6.4.42}$$

In view of (6.4.31) and (6.4.33), for $|\alpha| > 0$, $C > 0$ and for every $n \in \mathbb{N}$, we have

$$\begin{aligned} I_2 &\le \int\int_{\tilde{E}_{\delta,n}} C\, n^{C_1\alpha\theta_2 \frac{p(q+\alpha)}{q-p+1}} \left(\frac{\delta}{n}\right)^{\theta_2 \frac{p(q+\alpha)}{q-p+1}} \Big[d(x)^{-\theta_2-1}|\nabla d(x)|\Big]^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}}\, dx\, dt \\ &\le C\, n^{\theta_2 \frac{p(q+\alpha)}{q-p+1}(C_1\alpha-1)} \delta^{\theta_2 \frac{p(q+\alpha)}{q-p+1}} \int\int_{\tilde{E}_{\delta,n}} d(x)^{-(\theta_2+1)\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha+1}{q-p+1}}\, dx\, dt. \end{aligned} \tag{6.4.43}$$

Due to assumption HP1(ii) with $\varepsilon = -\frac{\alpha}{q-p+1} > 0$, (6.4.43) reduces to

$$I_2 \le C\, n^{\theta_2 \frac{p(q+\alpha)}{q-p+1}(C_1\alpha-1)} \delta^{\theta_2 \frac{p(q+\alpha)}{q-p+1}} \left(\frac{\delta}{n}\right)^{-\frac{pq}{q-p+1}-C_0\varepsilon} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_4}, \tag{6.4.44}$$

with s_4 as in HP1. Now observe that, due (6.4.32), we have

$$\frac{|\alpha|}{q-p+1}\,(-\theta_2\, p + C_1\, p\, \theta_2(q+\alpha) - C_0) \ge \frac{|\alpha|}{q-p+1}.$$

Moreover, there exist $\bar{C} > 0$ such that

$$\delta^{\frac{\alpha}{q-p+1}[\theta_2 p + C_0]} = \mathrm{e}^{\frac{\alpha}{q-p+1}[\theta_2 p + C_0]\log(\delta)} = \mathrm{e}^{\frac{\theta_2 p + C_0}{q-p+1}} \le \bar{C}.$$

Then from (6.4.44) we deduce, for some $C > 0$

$$I_2 \le C\, n^{-\frac{|\alpha|}{q-p+1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_4}. \tag{6.4.45}$$

Similarly, in view of (6.4.31) and (6.4.33), for $|\alpha| > 0$, $C > 0$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned} I_4 &\le C \int\int_{\tilde{E}_{\delta,n}} n^{\theta_2 C_1 \alpha\left(\frac{q+\alpha}{q-1}\right)} \left(\frac{\delta}{n}\right)^{\theta_2\left(\frac{q+\alpha}{q-1}\right)} \left(\theta_1 t^{\theta_1-1}\right)^{\frac{q+\alpha}{q-1}} V^{-\frac{\alpha+1}{q-1}}\, dxdt \\ &\le C n^{\theta_2\left(\frac{q+\alpha}{q-1}\right)(C_1\alpha-1)} \delta^{\theta_2\left(\frac{q+\alpha}{q-1}\right)} \int\int_{\tilde{E}_{\delta,n}} t^{(\theta_1-1)\left(\frac{q+\alpha}{q-1}\right)} V^{-\frac{\alpha+1}{q-1}}\, dxdt. \end{aligned} \tag{6.4.46}$$

Due to assumption HP1(i) with $\varepsilon = -\frac{\alpha}{q-1} > 0$, (6.4.46) reduces to

$$\begin{aligned} I_4 &\le C\, n^{\theta_2\left(\frac{q+\alpha}{q-1}\right)(C_1\alpha-1)} \delta^{\theta_2\left(\frac{q+\alpha}{q-1}\right)} \left(\frac{\delta}{n}\right)^{-\frac{q}{q-1}\theta_2 - C_0\varepsilon} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_2} \\ &\le C\, n^{\frac{1}{q-1}[C_1\alpha\theta_2(q+\alpha) - \alpha\theta_2 + C_0|\alpha|]}\, \delta^{\frac{1}{q-1}[\alpha\theta_2 + C_0\alpha]}, \end{aligned} \tag{6.4.47}$$

with s_2 as in HP1. We now observe that, due to (6.4.32), we can write

$$n^{-\frac{|\alpha|}{q-1}[C_1\theta_2(q+\alpha) - \theta_2 - C_0]} \le n^{-\frac{|\alpha|}{q-1}}. \tag{6.4.48}$$

Moreover, observe that there exist $\bar{C} > 0$ such that

$$\delta^{\frac{\alpha}{q-1}(\theta_2 + C_0)} = \mathrm{e}^{\frac{\alpha}{q-1}(\theta_2+C_0)\log(\delta)} = \mathrm{e}^{\frac{\theta_2 + C_0}{q-1}} \le \bar{C}. \tag{6.4.49}$$

By plugging (6.4.48) and (6.4.49) into (6.4.47) we get

$$I_4 \leq C\, n^{-\frac{|\alpha|}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_2}. \tag{6.4.50}$$

Let us now consider integral I_1 defined in (6.4.38). By using the definition of φ in (6.4.31) we can write

$$\begin{aligned} I_1 &\leq \int\int_{E_\delta^C} \left[C_1|\alpha|\theta_2 \left(\frac{d(x)^{-\theta_2}+t^{\theta_1}}{\delta^{-\theta_2}}\right)^{C_1\alpha-1} \frac{d(x)^{-\theta_2-1}}{\delta^{-\theta_2}}\right]^{\frac{p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}}\, dxdt \\ &\leq C\int\int_{E_\delta^C} |\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \left[d(x)^{-\theta_2}+t^{\theta_1}\right]^{\frac{(C_1\alpha-1)p(q+\alpha)}{q-p+1}} d(x)^{-\frac{(\theta_2+1)p(q+\alpha)}{q-p+1}} \delta^{\frac{\theta_2 C_1\alpha p(q+\alpha)}{q-p+1}} V^{-\frac{p+\alpha-1}{q-p+1}}\, dxdt. \end{aligned} \tag{6.4.51}$$

Similarly to (6.4.49), we can say that there exist $\bar{C}>0$ such that

$$\delta^{\frac{\theta_2 C_1\alpha p(q+\alpha)}{q-p+1}} \leq \bar{C},$$

hence (6.4.51), for some constant $C>0$, reduces to

$$I_1 \leq C|\alpha|^{\frac{p(q+\alpha)}{q-p+1}} \int\int_{E_\delta^C} V^{-\frac{p+\alpha-1}{q-p+1}} d(x)^{-\frac{(\theta_2+1)p(q+\alpha)}{q-p+1}} \left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]^{-\frac{\theta_2(C_1\alpha-1)p(q+\alpha)}{q-p+1}} dxdt. \tag{6.4.52}$$

Claim: *Let $f:(0,+\infty)\to[0,+\infty)$ be a non decreasing function and if $HP1(ii)$ holds then, for any $0<\varepsilon<\varepsilon_0$ and for any $0<\delta<\delta_0$ small enough, we can write*

$$\begin{aligned} \int\int_{(E_\delta)^C} f\left(\left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]\right) d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\, dxdt \\ \leq C\int_0^{2^{\frac{1}{\theta_2}}\delta} f(z) z^{-\frac{pq}{q-p+1}\theta_2 - C_0\varepsilon - 1} |\log z|^{s_4}\, dz, \end{aligned} \tag{6.4.53}$$

for some constant $C>0$. To show the claim, we first observe that

$$f(x) \leq f\left(\frac{\delta}{2^{\frac{n}{\theta_2}}}\right) \qquad \text{for all } x\in E_{\frac{\delta}{2}\frac{n+1}{\theta_2}} \setminus E_{\frac{\delta}{2}\frac{n}{\theta_2}}.$$

Hence, due to HP1(ii), we can write

$$
\begin{aligned}
&\int\int_{(E_\delta)^C} f\left(\left[d(x)^{-\theta_2}+t^{\theta_1}\right]^{-\frac{1}{\theta_2}}\right) d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\,dxdt\\
&\quad=\sum_{n=0}^{+\infty}\int\int_{\left(E_{\frac{\delta}{2}\frac{n+1}{\theta_2}}\setminus E_{\frac{\delta}{2}\frac{n}{\theta_2}}\right)} f\left(\left[d(x)^{-\theta_2}+t^{\theta_1}\right]^{-\frac{1}{\theta_2}}\right) d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\,dxdt\\
&\quad\le\sum_{n=0}^{+\infty} f\left(\left[\frac{\delta}{2^{\frac{n}{\theta_2}}}\right]^{-\theta_2\left(-\frac{1}{\theta_2}\right)}\right)\int\int_{\left(E_{\frac{\delta}{2}\frac{n+1}{\theta_2}}\setminus E_{\frac{\delta}{2}\frac{n}{\theta_2}}\right)} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} V^{-\frac{p-1}{q-p+1}+\varepsilon}\,dxdt\\
&\quad\le C\sum_{n=0}^{+\infty} f\left(\frac{\delta}{2^{\frac{n}{\theta_2}}}\right)\left(\frac{\delta}{2^{\frac{n}{\theta_2}}}\right)^{-\frac{pq}{q-p+1}\theta_2-C_0\varepsilon}\left|\log\left(\frac{\delta}{2^{\frac{n}{\theta_2}}}\right)\right|^{s_4}\\
&\quad\le C\sum_{n=0}^{+\infty}\int_{\frac{\delta}{2^{n/\theta_2}}}^{\frac{\delta}{2^{(n-1)/\theta_2}}} f(z)z^{-\frac{pq}{q-p+1}\theta_2-C_0\varepsilon-1}|\log z|^{s_4}\,dz\\
&\quad= C\int_0^{2^{1/\theta_2}\delta} f(z)z^{-\frac{pq}{q-p+1}\theta_2-C_0\varepsilon-1}|\log z|^{s_4}\,dz.
\end{aligned}
$$

We now apply (6.4.53) with $\varepsilon=\frac{|\alpha|}{q-p+1}>0$ to inequality (6.4.52). We get

$$
I_1\le C|\alpha|^{\frac{p(q+\alpha)}{q-p+1}}\int_0^{2^{1/\theta_2}\delta} z^{-\theta_2\frac{(C_1\alpha-1)p(q+\alpha)}{q-p+1}-\frac{pq}{q-p+1}\theta_2+\frac{C_0\alpha}{q-p+1}-1}|\log z|^{s_4}\,dz. \tag{6.4.54}
$$

We define

$$
b:=\frac{1}{q-p+1}\left(-\theta_2C_1\alpha p(q+\alpha)+\theta_2p\alpha+C_0\alpha\right), \tag{6.4.55}
$$

and due to (6.4.32), we observe that

$$
b\ge\frac{|\alpha|}{q-p+1}\ge 0.
$$

By plugging (6.4.55) into inequality (6.4.54) we can write

$$
I_1\le C\,|\alpha|^{\frac{p(q+\alpha)}{q-p+1}}\int_0^{2^{1/\theta_2}\delta} z^{b-1}|\log z|^{s_4}\,dz. \tag{6.4.56}
$$

Let us now perform a change of variable, we define

$$
y:=b\,|\log z|,
$$

hence (6.4.56) reduces to

$$
\begin{aligned}
I_1&\le C\,|\alpha|^{\frac{p(q+\alpha)}{q-p+1}}b^{-s_4-1}\int_{-\infty}^0 e^y|y|^{s_4}\,dy\\
&\le C\,|\alpha|^{\frac{p(q+\alpha)}{q-p+1}}\left(\frac{|\alpha|}{q-p+1}\right)^{-s_4-1}\\
&\le C\,|\alpha|^{\frac{pq}{q-p+1}-s_4-1}.
\end{aligned} \tag{6.4.57}
$$

with s_4 as in HP1(ii).

Finally, let us consider I_3 defined in (6.4.40). Due to the definition of φ in (6.4.31) we get

$$\begin{aligned} I_3 &\leq \int\int_{E_\delta^C}\left[C_1|\alpha|\theta_2\left(\frac{d(x)^{-\theta_2}+t^{\theta_1}}{\delta^{-\theta_2}}\right)^{C_1\alpha-1}\frac{t^{\theta_1-1}}{\delta^{-\theta_2}}\right]^{\frac{q+\alpha}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,dxdt \\ &\leq C\int\int_{E_\delta^C}|\alpha|^{\frac{q+\alpha}{q-1}}\left[d(x)^{-\theta_2}+t^{\theta_1}\right]^{\frac{(C_1\alpha-1)(q+\alpha)}{q-1}}t^{\frac{(\theta_1-1)(q+\alpha)}{q-1}}\delta^{\frac{\theta_2C_1\alpha(q+\alpha)}{q-1}}V^{-\frac{\alpha+1}{q-1}}\,dxdt. \end{aligned} \tag{6.4.58}$$

Arguing as in (6.4.49), we can say that there exist $\bar{C}>0$ such that

$$\delta^{\frac{\theta_2C_1\alpha(q+\alpha)}{q-1}}\leq\bar{C}\,.$$

Hence (6.4.58), for some constant $C>0$, reduces to

$$I_3\leq C|\alpha|^{\frac{q+\alpha}{q-1}}\int\int_{E_\delta^C}V^{-\frac{\alpha+1}{q-1}}t^{\frac{(\theta_1-1)(q+\alpha)}{q-1}}\left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]^{-\theta_2\frac{(C_1\alpha-1)(q+\alpha)}{q-1}}\,dxdt. \tag{6.4.59}$$

Let us now show the following

Claim: *Let $f:(0,+\infty)\to[0,+\infty)$ be a non decreasing function and if HP1(i) holds then, for any $0<\varepsilon<\varepsilon_0$ and for any $0<\delta<\delta_0$ small enough, we can write*

$$\begin{aligned} &\int\int_{E_\delta^C}f\left(\left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]\right)t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}V^{-\frac{1}{q-1}+\varepsilon}\,dxdt \\ &\qquad\leq C\int_0^{2^{\frac{1}{\theta_2}}\delta}f(z)z^{-\frac{q}{q-1}\theta_2-C_0\varepsilon-1}|\log z|^{s_2}\,dz, \end{aligned} \tag{6.4.60}$$

for some constant $C>0$.

Inequality (6.4.60) can be proven similarly to (6.4.53) where one uses HP1(i) instead of HP1(ii). We now apply (6.4.60) with $\varepsilon=\frac{|\alpha|}{q-1}>0$ to inequality (6.4.59). We get

$$I_3\leq C|\alpha|^{\frac{q+\alpha}{q-1}}\int_0^{2^{1/\theta_2}\delta}z^{-\theta_2(C_1\alpha-1)\frac{q+\alpha}{q-1}-\frac{q}{q-1}\theta_2+\frac{C_0\alpha}{q-1}-1}|\log z|^{s_2}\,dz. \tag{6.4.61}$$

We define

$$\beta:=\frac{1}{q-1}\left(-\theta_2C_1\alpha(q+\alpha)+\theta_2\alpha+C_0\alpha\right), \tag{6.4.62}$$

and due to (6.4.32), we can say that

$$\beta\geq\frac{|\alpha|}{q-1}>0.$$

By plugging (6.4.62) into inequality (6.4.61), we get

$$\begin{aligned} I_3 &\leq C|\alpha|^{\frac{q+\alpha}{q-1}}\int_{-\infty}^0 e^y\left|\frac{y}{\beta}\right|^{s_2}\frac{1}{\beta}\,dy \\ &\leq C\,|\alpha|^{\frac{q+\alpha}{q-1}}\,\beta^{-s_2-1} \\ &\leq C\,|\alpha|^{\frac{1}{q-1}-s_2}. \end{aligned} \tag{6.4.63}$$

with s_2 as in HP1(i).

For any $n \in \mathbb{N}$ and $\delta > 0$ small enough, due to inequalities (6.4.45), (6.4.50), (6.4.57) and (6.4.63), inequality (6.4.42) reduces to

$$\begin{aligned}\int_0^\infty \int_\Omega V\, u^{q+\alpha}\, \varphi_n^s \, dx dt \leq C|\alpha|^{-\frac{(p-1)q}{q-p+1}} \left[|\alpha|^{\frac{pq}{q-p+1}-s_4-1} + n^{-\frac{|\alpha|}{q-p+1}} \left| \log\left(\frac{\delta}{n}\right)\right|^{s_4} \right] \\ + C\left[|\alpha|^{\frac{1}{q-1}-s_2} + n^{-\frac{|\alpha|}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{s_2}\right],\end{aligned} \tag{6.4.64}$$

where $C > 0$ does not depend on δ and n. By taking the limit in (6.4.64) as $n \to \infty$ for a fixed $\delta > 0$ small enough, we get

$$\begin{aligned}0 \leq \int\int_{E_\delta} V\, u^{q+\alpha}\, dx dt \leq \int_0^\infty \int_\Omega V\, u^{q+\alpha}\, \varphi_n^s \, dx dt \\ \leq C\left[|\alpha|^{\frac{p-1}{q-p+1}-s_4} + |\alpha|^{\frac{1}{q-1}-s_2}\right].\end{aligned} \tag{6.4.65}$$

Observe that, due to the definitions of s_2 in HP1(i) and s_4 in HP2(ii)

$$\frac{1}{q-1} - s_2 > 0\,, \quad \frac{p-1}{q-p+1} - s_4 > 0\,.$$

Hence we can take the limit in (6.4.65) as $\delta \to 0$ (and thus $\alpha \to 0^-$) obtaining, by Fatou's Lemma

$$\int_0^\infty \int_\Omega V\, u^q \, dx dt = 0,$$

which concludes the proof. □

As a consequence of Theorem 6.2.1 we prove Corollary 6.2.3.

Proof of Corollary 6.2.3. We show that under the assumptions of Corollary 6.2.3, hypothesis HP1 is satisfied. Let us define

$$\overline{E}_\delta := E_{\left(\frac{1}{2}\right)^{\frac{1}{\theta_2}}\delta} \setminus E_\delta$$

and observe that

$$\overline{E}_\delta \subset \left\{ d(x) \geq 2^{-\frac{1}{\theta_2}}\delta \right\} \times \left[0,\, 2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}\right] =: \Omega_\delta \times \left[0,\, 2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}\right],$$

where $d(x)$ has been define in (6.1.4). Observe that

$$
\begin{aligned}
&\int\int_{\overline{E}_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}\, V^{-\frac{1}{q-1}+\varepsilon}\, dxdt \\
&\quad \le \int\int_{\overline{E}_\delta} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)}\, [f(t)h(x)]^{-\frac{1}{q-1}+\varepsilon}\, dxdt \\
&\quad \le C \int_{\Omega_\delta} h(x)^{-\frac{1}{q-1}+\varepsilon}\, dx \int_0^{2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} f(t)^{-\frac{1}{q-1}+\varepsilon} dt \\
&\quad \le C \int_{\Omega_\delta} \left[d(x)^{-\sigma_1} |\log(d(x))|^{-\delta_1}\right]^{-\frac{1}{q-1}+\varepsilon} dx \\
&\quad \times \int_0^{2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}} f(t)^{-\frac{1}{q-1}} (1+t)^{\alpha\varepsilon} t^{(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)} dt \\
&\quad \le C \int_{\Omega_\delta} d(x)^{\frac{\sigma_1}{q-1}-\varepsilon\sigma_1}\, |\log(d(x))|^{\frac{\delta_1}{q-1}-\varepsilon\delta_1}\, dx \\
&\quad \times \left[\delta^{-\frac{\theta_2}{\theta_1}\left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)+\alpha\varepsilon\right]} \int_0^{2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}} f(t)^{-\frac{1}{q-1}} dt\right] \\
&\quad \le C\, |\log(\delta)|^{\frac{\delta_1}{q-1}-\varepsilon\delta_1} \left[\delta^{-\frac{\theta_2}{\theta_1}\left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)+\alpha\varepsilon\right]}\right] \delta^{-\frac{\theta_2}{\theta_1}\sigma_2}\, |\log(\delta)|^{\delta_2} \\
&\quad \le C\delta^{-\frac{\theta_2}{\theta_1}\left[(\theta_1-1)\left(\frac{q}{q-1}-\varepsilon\right)+\alpha\varepsilon+\sigma_2\right]}\, |\log(\delta)|^{\frac{\delta_1}{q-1}-\varepsilon\delta_1+\delta_2},
\end{aligned}
\tag{6.4.66}
$$

for $\theta_1, \theta_2 \ge 1$ as in HP1. For $\varepsilon > 0$ small enough, condition (6.2.7) of HP1 is satisfied because

$$
\frac{\theta_2}{\theta_1}\left[\frac{q}{q-1} - \sigma_2\right] \ge 0 \quad \text{and} \quad \delta_2 + \frac{\delta_1}{q-1} < \bar{s}_2\,. \tag{6.4.67}
$$

On the other hand, for $\varepsilon > 0$ sufficiently small

$$
\begin{aligned}
&\int\int_{\overline{E}_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)}\, V^{-\frac{p-1}{q-p+1}+\varepsilon}\, dxdt \\
&\quad \leq \int\int_{\overline{E}_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)}\, [f(t)h(x)]^{-\frac{p-1}{q-p+1}+\varepsilon}\, dxdt \\
&\quad \leq \int_{\Omega_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} h(x)^{-\frac{p-1}{q-p+1}+\varepsilon} dx \int_0^{2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}} f(t)^{-\frac{p-1}{q-p+1}+\varepsilon} dt \\
&\quad \leq C\int_{\Omega_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)} \left[d(x)^{\sigma_1}|\log(d(x))|^{\delta_1}\right]^{\frac{p-1}{q-p+1}-\varepsilon} dx \\
&\quad \times \left[\delta^{-\frac{\theta_2}{\theta_1}\alpha\varepsilon} \int_0^{2^{\frac{1}{\theta_1}}\delta^{-\frac{\theta_2}{\theta_1}}} f(t)^{-\frac{p-1}{q-p+1}} dt\right] \\
&\quad \leq C\int_{\Omega_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\sigma_1\frac{p-1}{q-p+1}-\varepsilon\sigma_1} |\log(d(x))|^{\delta_1\frac{p-1}{q-p+1}-\varepsilon\delta_1} dx \\
&\quad \times \left[\delta^{-\frac{\theta_2}{\theta_1}\alpha\varepsilon}\delta^{-\frac{\theta_2}{\theta_1}\sigma_4}|\log(\delta)|^{\delta_4}\right] \\
&\quad \leq C\delta^{-\frac{\theta_2}{\theta_1}(\alpha\varepsilon+\sigma_4)}|\log(\delta)|^{\delta_4+\delta_1\left(\frac{p-1}{q-p+1}-\varepsilon\right)} \\
&\quad \times \int_{\Omega_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\sigma_1\frac{p-1}{q-p+1}-\varepsilon\sigma_1} dx
\end{aligned}
\tag{6.4.68}
$$

We define

$$\beta := -(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)+\sigma_1\frac{p-1}{q-p+1}-\varepsilon\sigma_1$$

and we observe that $\beta < -1$ for θ_2 sufficiently big. Therefore, due to the boundedness of Ω_δ, inequality (6.4.68) reduces to

$$
\begin{aligned}
&\int\int_{\overline{E}_\delta} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}-\varepsilon\right)}\, V^{-\frac{p-1}{q-p+1}+\varepsilon}\, dxdt \\
&\qquad \leq C\delta^{-\frac{\theta_2}{\theta_1}(\alpha\varepsilon+\sigma_4)+\beta+1}|\log(\delta)|^{\delta_4+\delta_1\left(\frac{p-1}{q-p+1}-\varepsilon\right)}
\end{aligned}
\tag{6.4.69}
$$

For $\varepsilon > 0$ small enough and for $\theta_2/\theta_1 > 0$ small enough, condition (6.2.8) is satisfied because the hypotheses of the Corollary 6.2.3 guarantee that

$$\sigma_1 - \frac{\theta_2}{\theta_1}\sigma_4\frac{q-p+1}{p-1} \geq q+1 \quad \text{and} \quad \delta_4+\delta_1\left(\frac{p-1}{q-p+1}-\varepsilon\right) < \bar{s}_4.$$

Thus HP1 holds and we can apply Theorem 6.2.1 and obtain the thesis. □

6.5 Proof of Theorem 6.2.2

Proof of Theorem 6.2.2. Let us recall the family of functions φ_n defined in (6.4.34). We claim that $u^q \in L^1(\Omega\times(0,+\infty), V d\mu dt)$. To prove this, we start by showing that for some constants $A > 0$, $B > 0$, $s \geq 1$, for every $\delta > 0$ small enough and every $n \in \mathbb{N}$ we have

$$\int_0^\infty \int_\Omega \varphi_n^s u^q V\, dxdt \leq A \left(\int_0^\infty \int_\Omega \varphi_n^s u^q V\, dxdt \right)^{\frac{p-1}{pq}} + B. \tag{6.5.70}$$

In order to prove (6.5.70) we apply Corollario 6.3.5 with φ replaced by the family of functions φ_n. Let

$$C_1 > \max \left\{ \frac{2(1 + C_0 + \theta_2 p)}{p\theta_2 q}, \frac{2(C_0 + 1)}{\theta_2 (q-1) q}, \frac{C_0 + 1}{\theta_2 q} \right\}, \tag{6.5.71}$$

with $C_0 > 0$ and $\theta_2 \geq 1$ as in HP2. Then for any fixed $s \geq \max\left\{1, \frac{q+1}{q-1}, \frac{2pq}{q-p+1}\right\}$, $\delta > 0$ sufficiently small, $\alpha = \frac{1}{\log \delta} < 0$ and for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_0^\infty \int_\Omega V\, u^q\, \varphi^s\, dx\, dt \\
&\leq C \Bigg[|\alpha|^{-1} \Bigg(|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt \\
&+ \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}}\, dxdt \Bigg) \Bigg]^{\frac{p-1}{p}} \times \left(\int \int_{E_\delta^C} V\, u^q \varphi_n^s\, dx\, dt \right)^{\frac{(1-\alpha)(p-1)}{pq}} \\
&\times \left(\int \int_{E_\delta^C} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}}\, dxdt \right)^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&+ C \Bigg[|\alpha|^{-\frac{(p-1)q}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt \\
&+ \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}}\, dxdt \Bigg]^{\frac{1}{q+\alpha}} \\
&\times \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q+\alpha-1}}\, dxdt \right)^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned} \tag{6.5.72}$$

where E_δ has been defined in (6.2.5). We also define

$$J_1 := \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla \varphi_n|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt; \tag{6.5.73}$$

$$J_2 := \int_0^\infty \int_\Omega V^{-\frac{\alpha+1}{q-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q-1}}\, dxdt; \tag{6.5.74}$$

$$J_3 := \int \int_{E_\delta^C} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla \varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}}\, dxdt; \tag{6.5.75}$$

$$J_4 := \int \int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\partial_t \varphi_n|^{\frac{q+\alpha}{q+\alpha-1}}\, dxdt. \tag{6.5.76}$$

By using (6.5.73), (6.5.74), (6.5.75) and (6.5.76), inequality (6.5.72) reads

$$\begin{aligned}
&\int_0^\infty \int_\Omega V\, u^q\, \varphi^s\, dx\, dt \\
&\quad \le C\left[|\alpha|^{-1-\frac{(p-1)q}{q-p+1}} J_1\right]^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V\, u^q \varphi_n^s\, dx\, dt\right)^{\frac{(1-\alpha)(p-1)}{pq}} J_3^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&\quad + C\left[|\alpha|^{-1} J_2\right]^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V\, u^q \varphi_n^s\, dx\, dt\right)^{\frac{(1-\alpha)(p-1)}{pq}} J_3^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&\quad + C\left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 + J_2\right]^{\frac{1}{q+\alpha}} J_4^{\frac{q+\alpha-1}{q+\alpha}} \\
&\quad \le C\left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1\right]^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V\, u^q \varphi_n^s\, dx\, dt\right)^{\frac{(1-\alpha)(p-1)}{pq}} \\
&\quad \times \left[|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3\right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&\quad + C J_2^{\frac{p-1}{p}} \left(\iint_{E_\delta^C} V\, u^q \varphi_n^s\, dx\, dt\right)^{\frac{(1-\alpha)(p-1)}{pq}} \left[|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3\right]^{\frac{q-(1-\alpha)(p-1)}{pq}} \\
&\quad + C\left[|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1 + J_2\right]^{\frac{1}{q+\alpha}} J_4^{\frac{q+\alpha-1}{q+\alpha}}.
\end{aligned} \tag{6.5.77}$$

Let us prove that, for $\delta > 0$ sufficiently small and $|\alpha| = -\frac{1}{\log \delta} > 0$ sufficiently small

$$\limsup_{n\to\infty} \left(|\alpha|^{-\frac{(p-1)q}{q-p+1}} J_1\right) \le C, \tag{6.5.78}$$

$$\limsup_{n\to\infty}, \left(|\alpha|^{-\frac{(p-1)q}{q-(1-\alpha)(p-1)}} J_3\right) \le C, \tag{6.5.79}$$

$$\limsup_{n\to\infty}\ J_2 \le C, \tag{6.5.80}$$

$$\limsup_{n\to\infty}\ J_4 \le C, \tag{6.5.81}$$

for some $C > 0$ independent of α.

We start by proving (6.5.78). Observe that

$$J_1 \le\ C(I_1 + I_2), \tag{6.5.82}$$

with I_1 and I_2 defined in (6.4.38) and (6.4.39), respectively. Similarly to proof of Theorem 6.2.1, in view of (6.4.31) and (6.4.33) we obtain inequality (6.4.44). Then due to condition (6.2.11) in HP2(ii) with $\varepsilon = -\frac{\alpha}{q-p+1} > 0$ we have, for every $n \in \mathbb{N}$

$$I_2 \le C\, n^{\theta_2 \frac{p(q+\alpha)}{q-p+1}(C_1\alpha - 1)} \delta^{\theta_2 \frac{p(q+\alpha)}{q-p+1}} \left(\frac{\delta}{n}\right)^{-\frac{pq}{q-p+1}\theta_2 - C_0\varepsilon} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_4}, \tag{6.5.83}$$

with $\bar{s}_4$ as in (6.2.6). Now observe that due to (6.5.71)

$$\frac{|\alpha|}{q-p+1}\left(-\theta_2 p + C_1 p\theta_2(q+\alpha) - C_0\right) \geq \frac{|\alpha|}{q-p+1}\,.$$

Moreover, there exists $\bar{C}>0$ such that

$$\delta^{\frac{1}{q-p+1}(\theta_2 pq - \theta_2 pq + \theta_2 p\alpha + C_0\alpha)} = e^{\frac{\alpha}{q-p+1}(\theta_2 p + C_0)\log(\delta)} = e^{\frac{\theta_2 p + C_0}{q-p+1}} \leq \bar{C}.$$

Then from (6.5.83) we deduce, for some positive constant C

$$I_2 \leq C\, n^{-\frac{|\alpha|}{q-p+1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_4}. \tag{6.5.84}$$

On the other hand, arguing as in the proof of Theorem 6.2.1, we deduce inequality (6.4.57). Therefore

$$I_1 \leq C\,|\alpha|^{\frac{pq}{q-p+1} - s_4 - 1} \leq C\,|\alpha|^{\frac{pq}{q-p+1} - \bar{s}_4 - 1} \leq C\,|\alpha|^{\frac{q(p-1)}{q-p+1}}. \tag{6.5.85}$$

Combining (6.5.82), (6.5.84) and (6.5.85), for some $C>0$ and for every $n\in\mathbb{N}$, we have

$$|\alpha|^{-\frac{q(p-1)}{q-p+1}} J_1 \leq C\left(1 + |\alpha|^{-\frac{q(p-1)}{q-p+1}} n^{-\frac{|\alpha|}{q-p+1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_4}\right). \tag{6.5.86}$$

We can compute the limit as $n\to\infty$ on both sides of (6.5.86), thus we obtain (6.5.78).

Now observe that

$$J_2 \leq C(I_3 + I_4), \tag{6.5.87}$$

with I_3 and I_4 defined in (6.4.40) and (6.4.41), respectively. Then arguing as in the proof of Theorem 6.2.1, due to condition (6.2.9) in HP2(i) with $\varepsilon = -\frac{\alpha}{q-1} > 0$ we deduce, for some positive constant C

$$I_4 \leq C\, n^{-\frac{|\alpha|}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2}, \tag{6.5.88}$$

where $\bar{s}_2$ has been defined in (6.2.6). Moreover, from inequality (6.4.65) and (6.2.6) we deduce, for some constant $C>0$

$$I_3 \leq C. \tag{6.5.89}$$

Combining (6.5.87), (6.5.88) and (6.5.89), for some $C>0$ and for every $n\in\mathbb{N}$, we have

$$J_2 \leq C\left(1 + n^{-\frac{|\alpha|}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2}\right).$$

Letting $n\to\infty$ we obtain (6.5.79).

We now proceed to estimate J_4. Observe that

$$J_4 \leq C\,(I_5 + I_6)\,, \tag{6.5.90}$$

where

$$I_5 := \int\int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\partial_t\varphi|^{\frac{q+\alpha}{q+\alpha-1}}\,dxdt; \quad I_6 := \int\int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\partial_t\eta_n|^{\frac{q+\alpha}{q+\alpha-1}}\,dxdt.$$

Due to (6.4.31) we have

$$
\begin{aligned}
I_5 &\le C \int\int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} |\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \left[\frac{d(x)^{-\theta_2}+t^{\theta_1}}{\delta^{-\theta_2}}\right]^{\frac{(C_1\alpha-1)(q+\alpha)}{q+\alpha-1}} \left(\frac{t^{\theta_1-1}}{\delta^{-\theta_2}}\right)^{\frac{q+\alpha}{q+\alpha-1}} dxdt \\
&\le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int\int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} \left[d(x)^{-\theta_2}+t^{\theta_1}\right]^{\frac{(C_1\alpha-1)(q+\alpha)}{q+\alpha-1}} \delta^{\frac{\theta_2 C_1\alpha(q+\alpha)}{q+\alpha-1}} t^{(\theta_1-1)\left(\frac{(q+\alpha)}{q+\alpha-1}\right)} dxdt \\
&\le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int\int_{E_\delta^C} V^{-\frac{1}{q+\alpha-1}} \left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]^{-\theta_2(C_1\alpha-1)\left(\frac{q}{q-1}-\frac{\alpha}{(q+\alpha-1)(q-1)}\right)} \\
&\quad \times t^{(\theta_1-1)\left(\frac{q}{q-1}-\frac{\alpha}{(q+\alpha-1)(q-1)}\right)} dxdt,
\end{aligned}
\tag{6.5.91}
$$

where we have used that there exists a positive constant $\bar{C}$ such that

$$
\delta^{\theta_2 C_1\alpha\left(\frac{q+\alpha}{q+\alpha-1}\right)} = e^{\theta_2 C_1\alpha\left(\frac{q+\alpha}{q+\alpha-1}\right)\log\delta} = e^{\theta_2 C_1\left(\frac{q+\alpha}{q+\alpha-1}\right)} \le \bar{C}\,.
$$

Claim: *Let $f:(0,+\infty)\to[0,+\infty)$ be a non decreasing function and if HP2(i) holds then, for any $0<\varepsilon<\varepsilon_0$ and for any $0<\delta<\delta_0$ small enough, we can write*

$$
\begin{aligned}
&\int\int_{E_\delta^C} f\left(\left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]\right) t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} V^{-\frac{1}{q-1}-\varepsilon}\, dxdt \\
&\qquad\qquad \le C\int_0^{2^{\frac{1}{\theta_2}}\delta} f(z) z^{-\bar{s}_1-C_0\varepsilon-1} |\log z|^{\bar{s}_2}\, dz,
\end{aligned}
\tag{6.5.92}
$$

for some constant $C>0$ with $\bar{s}_1$ and $\bar{s}_2$ as in (6.2.6). Inequality (6.5.92) can be proven similarly to (6.4.53) and (6.4.60) where one uses the condition (6.2.10) in HP2(i) instead of HP1. By using the latter claim with $\varepsilon=\frac{|\alpha|}{(q+\alpha-1)(q-1)}>0$ we obtain

$$
I_5 \le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int_0^{2^{\frac{1}{\theta_2}}\delta} z^{-\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right)-\bar{s}_1-C_0\varepsilon-1} |\log z|^{\bar{s}_2}\, dz.
$$

Then observe that, due to (6.5.71)

$$
-\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right) - \bar{s}_1 - C_0\varepsilon \ge \frac{|\alpha|}{(q-1)^2} =: b
$$

Now we define

$$
y := b\log z,
$$

then there exists $\bar{C}>0$ such that

$$
\begin{aligned}
I_5 &\le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \int_{-\infty}^0 e^y \left|\frac{y}{b}\right|^{\bar{s}_2} \frac{1}{b}\, dy \\
&\le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} b^{-\bar{s}_2-1} \int_{-\infty}^0 e^y |y|^{\bar{s}_2}\, dy \\
&\le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}} \left(\frac{|\alpha|}{(q-1)^2}\right)^{-\bar{s}_2-1} \\
&\le C\,|\alpha|^{\frac{q+\alpha}{q+\alpha-1}-\frac{1}{q-1}-1} \le \bar{C}\,.
\end{aligned}
\tag{6.5.93}
$$

On the other hand, due to (6.4.31) and condition (6.2.10) in HP2(i) with $\varepsilon = \frac{|\alpha|}{(q+\alpha-1)(q-1)}$, by using the definition of $\tilde{E}_{\delta/n}$ in (6.4.37), for every $n \in \mathbb{N}$ we have

$$\begin{aligned} I_6 &\leq C \int\int_{\tilde{E}_{\delta/n}} V^{-\frac{1}{q+\alpha-1}} \left[\frac{d(x)}{n}^{\theta_2} t^{\theta_1-1}\right]^{\frac{q+\alpha}{q+\alpha-1}} n^{\theta_2\alpha C_1\left(\frac{q+\alpha}{q+\alpha-1}\right)} \, dxdt \\ &\leq C\, n^{\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right)} \delta^{\theta_2\left(\frac{q+\alpha}{q+\alpha-1}\right)} \int\int_{\tilde{E}_{\delta/n}} V^{-\frac{1}{q-1}-\varepsilon} t^{(\theta_1-1)\left(\frac{q}{q-1}+\varepsilon\right)} \, dxdt \\ &\leq C\, n^{\theta_2(C_1\alpha-1)\left(\frac{q+\alpha}{q+\alpha-1}\right)} \delta^{\theta_2\left(\frac{q+\alpha}{q+\alpha-1}\right)} \left(\frac{\delta}{n}\right)^{-\bar{s}_1-C_0\varepsilon} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2} \\ &\leq C\, n^{-\frac{|\alpha|}{q+\alpha-1}\left[\theta_2 C_1(q+\alpha)\frac{\theta_2}{q-1} - \frac{C_0}{q-1}\right]} \delta^{\frac{|\alpha|}{(q+\alpha-1)(q-1)}[\theta_2 - C_0]} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2} \end{aligned} \tag{6.5.94}$$

Now observe that there exists a positive constant $\bar{C}$ such that

$$\delta^{\frac{|\alpha|}{(q+\alpha-1)(q-1)}[\theta_2 - C_0]} = e^{\frac{|\alpha|}{(q+\alpha-1)(q-1)}[\theta_2 - C_0]\log\delta} = e^{\frac{C_0-\theta_2}{(q+\alpha-1)(q-1)}} \leq \bar{C}\,, \tag{6.5.95}$$

and due to (6.5.71)

$$-\frac{|\alpha|}{q+\alpha-1}\left[\theta_2 C_1(q+\alpha)\frac{\theta_2}{q-1} - \frac{C_0}{q-1}\right] \leq -\frac{|\alpha|}{(q-1)^2}. \tag{6.5.96}$$

Combining (6.5.95) and (6.5.96) with (6.5.94) we obtain

$$I_6 \,\leq\, C\, n^{-\frac{|\alpha|}{(q-1)^2}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2}. \tag{6.5.97}$$

We now substitute (6.5.93) and (6.5.97) into inequality (6.5.90) thus we have, for some $C > 0$ and for every $n \in \mathbb{N}$

$$J_4 \leq C\left[1 + n^{-\frac{|\alpha|}{(q-1)^2}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2}\right].$$

Letting $n \to \infty$ we get (6.5.81).

In order to estimate integral J_3 defined in (6.5.75), we define, for sufficiently small $|\alpha| > 0$, the positive constant λ

$$\lambda := \frac{|\alpha| q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]}. \tag{6.5.98}$$

Observe that, for sufficiently small $|\alpha| > 0$

$$\frac{|\alpha| q(p-1)}{(q-p+1)^2} \,<\, \lambda \,<\, \frac{2|\alpha| q(p-1)}{(q-p+1)^2}, \tag{6.5.99}$$

and

$$\frac{pq}{q-(1-\alpha)(p-1)} = \frac{\bar{s}_3}{\theta_2} + \lambda\, p, \tag{6.5.100}$$

where $\bar{s}_3$ has been defined in (6.2.6) and $\theta_2 \geq 1$ as in HP2. Thus by the definition of φ_n in (6.4.34) and by (6.5.98), for sufficiently small $|\alpha| > 0$ and for every $n \in \mathbb{N}$ we have

$$\begin{aligned} J_3 &\leq C \int\int_{E_\delta^C} V^{-\lambda-\bar{s}_4} |\nabla\varphi|^{\frac{\bar{s}_3}{\theta_2}+\lambda p}\, dxdt + C \int\int_{\tilde{E}_{\delta/n}} V^{-\lambda-\bar{s}_4} \left(\varphi|\nabla\eta_n|\right)^{\frac{\bar{s}_3}{\theta_2}+\lambda p}\, dxdt \\ &=: I_7 + I_8\,, \end{aligned} \tag{6.5.101}$$

where $\tilde{E}_{\delta/n}$ has been defined in (6.4.37). Due to the very definition of φ and η_n in (6.4.31) and (6.4.33) respectively, and by (6.5.100) we get

$$\begin{aligned} I_8 &\leq C \int\int_{\tilde{E}_{\delta/n}} V^{-\lambda-\bar{s}_4} n^{C_1\alpha\theta_2\left(\frac{\bar{s}_3}{\theta_2}+\lambda p\right)} \left(\frac{\delta}{n}\right)^{\theta_2\left(\frac{\bar{s}_3}{\theta_2}+\lambda p\right)} d(x)^{-(\theta_2+1)\left(\frac{\bar{s}_3}{\theta_2}+\lambda p\right)}\, dxdt \\ &\leq C\, n^{(C_1\alpha-1)(\bar{s}_3+\lambda p\theta_2)} \delta^{\bar{s}_3+\lambda p\theta_2} \int\int_{\tilde{E}_{\delta/n}} V^{-\lambda-\bar{s}_4} d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}+\lambda\right)}\, dxdt \end{aligned}$$

Now we use condition (6.2.12) in HP2(ii) with $\varepsilon = \lambda$ and we obtain, for every $n \in \mathbb{N}$ and for sufficiently small $\delta > 0$

$$\begin{aligned} I_8 &\leq C\, n^{(C_1\alpha-1)p\theta_2\left(\frac{q}{q-p+1}+\lambda\right)} \delta^{p\,\theta_2\left(\frac{q}{q-p+1}+\lambda\right)} \left(\frac{\delta}{n}\right)^{-\frac{pq}{q-p+1}\theta_2 - C_0\lambda} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_4} \\ &\leq C\, n^{C_1\alpha p\theta_2\left(\frac{q}{q-p+1}+\lambda\right)-\lambda\, p\theta_2+C_0\lambda} \delta^{p\,\theta_2\lambda - C_0\lambda} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_4}. \end{aligned}$$

Due to the definition of λ in (6.5.98), inequality (6.5.99) and the definition of C_1 in (6.5.71), for sufficiently small $|\alpha| > 0$ we write

$$\begin{aligned} &C_1\alpha\, p\,\theta_2\left(\frac{q}{q-p+1}+\lambda\right) - \lambda\, p\theta_2 + C_0\lambda \\ &= (C_1\alpha-1)\,p\,\theta_2 \frac{|\alpha| q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} + C_1\alpha\frac{p\,q\,\theta_2}{q-p+1} + \frac{C_0|\alpha|q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} \\ &\leq (C_1\alpha-1)\,p\,\theta_2\frac{|\alpha|q(p-1)}{(q-p+1)^2} + C_1\alpha\frac{p\,q\,\theta_2}{q-p+1} + \frac{C_0|\alpha|q(p-1)}{(q-p+1)^2} \\ &\leq C_1\alpha\,\theta_2\left[\frac{|\alpha|q\,p^2}{(q-p+1)^2} - \frac{|\alpha|q\,p}{(q-p+1)^2} + \frac{q\,p}{q-p+1}\right] - \frac{|\alpha|q(p-1)}{(q-p+1)^2}[p\theta_2 - C_0] \\ &\leq -\frac{|\alpha|}{(q-p+1)^2}\left[C_1\theta_2\, p\, q(q+(p-1)(|\alpha|-1)) + (p\,\theta_2 - C_0)q(p-1)\right] \\ &\leq -\frac{|\alpha|q}{(q-p+1)^2}[C_1\theta_2\, p\, q - C_0(p-1)] \\ &\leq -\frac{|\alpha|q\,p}{(q-p+1)^2}[C_1\theta_2\, q - C_0] \\ &\leq -\frac{|\alpha|q\,p}{(q-p+1)^2}\,. \end{aligned}$$

Moreover, since $\alpha = \frac{1}{\log\delta} < 0$, there exists $\bar{C}$ such that

$$\delta^{\lambda(p\theta_2-C_0)} = e^{\lambda(p\theta_2-C_0)\log\delta} = e^{-\lambda(p\theta_2-C_0)|\log\delta|} < e^{\frac{-|\alpha|\,q\,(p-1)}{(q-p+1)^2}(p\,\theta_2-C_0)|\log\delta|} \leq \bar{C}$$

Therefore we obtain the following bound on I_8

$$I_8 \le C\, n^{-\frac{|\alpha|\, p\, q}{(q-p+1)^2}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_4}. \tag{6.5.102}$$

On the other hand, by using the definition of φ in (6.4.31) we can write

$$I_7 \le C\,|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int\int_{E_\delta^C} V^{-\lambda-\bar{s}_4} \left[\left(\frac{d(x)^{-\theta_2}+t^{\theta_1}}{\delta^{-\theta_2}}\right)^{C_1\alpha-1} \delta^{\theta_2} d(x)^{-(\theta_2-1)}\right]^{\frac{\bar{s}_3}{\theta_2}+\lambda p} dxdt\,,$$

and we observe that there exists $\bar{C}>0$ such that

$$\delta^{C_1\alpha\,\theta_2\left(\frac{\bar{s}_3}{\theta_2}+\lambda\, p\right)} = \delta^{C_1\alpha\,\theta_2\left(\frac{p\,q}{q-(1-\alpha)(p-1)}\right)} < \delta^{C_1\alpha\,\theta_2\left(\frac{p\,q}{q-p+1}\right)} = e^{C_1\alpha\,\theta_2\left(\frac{pq}{q-p+1}\right)\log\delta} \le \bar{C}\,.$$

Therefore we get

$$I_7 \le C\,|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int\int_{E_\delta^C} V^{-\lambda-\bar{s}_4} \left[d(x)^{-\theta_2}+t^{\theta_1}\right]^{(C_1\alpha-1)\frac{\bar{s}_3}{\theta_2}+\lambda\, p} d(x)^{-(\theta_2-1)\left(\frac{\bar{s}_3}{\theta_2}+\lambda\, p\right)}\, dxdt\,,$$

We now state the following

Claim: *Let $f:(0,+\infty)\to[0,+\infty)$ be a non decreasing function and suppose that HP2(ii) holds. Then, for any $0<\varepsilon<\varepsilon_0$ and for any $0<\delta<\delta_0$ small enough, we can write*

$$\begin{aligned}\int\int_{E_\delta^C} f\left(\left[\left(d(x)^{-\theta_2}+t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]\right) d(x)^{-(\theta_2+1)p\left(\frac{q}{q-p+1}+\varepsilon\right)} V^{-\frac{p-1}{q-p+1}-\varepsilon}\, dxdt \\ \le C\int_0^{2^{\frac{1}{\theta_2}}\delta} f(z)\, z^{-\bar{s}_3-C_0\varepsilon-1}|\log z|^{\bar{s}_4}\, dz,\end{aligned} \tag{6.5.103}$$

for some constant $C>0$ with $\bar{s}_3$ and $\bar{s}_4$ as in (6.2.6). Inequality (6.5.103) can be proven similarly to (6.4.53) and (6.4.60) where one uses the condition (6.2.12) in HP2(ii) instead of HP1. By using the latter claim with $\varepsilon=\lambda$ we get

$$I_7 \le C\,|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int_0^{2^{\frac{1}{\theta_2}}\delta} z^{-\theta_2(C_1\alpha-1)\left(\frac{\bar{s}_3}{\theta_2}+\lambda\, p\right)-\bar{s}_3-C_0\,\lambda-1}|\log z|^{\bar{s}_4}\, dz \tag{6.5.104}$$

Observe that, since $\alpha<0$ and due to (6.5.71)

$$\begin{aligned}&-\theta_2(C_1\alpha-1)\left(\frac{\bar{s}_3}{\theta_2}+\lambda\, p\right)-\bar{s}_3-C_0\,\lambda \\ &\quad= -\theta_2 C_1\alpha\frac{p\,q}{q-(1-\alpha)(p-1)} + p\,\theta_2\frac{|\alpha|\,q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} - C_0\frac{|\alpha|\,q(p-1)}{(q-p+1)[q-(1-\alpha)(p-1)]} \\ &\quad\ge |\alpha|\theta_2\, C_1\frac{p\,q}{(q-p+1)^2} + p\,\theta_2\frac{|\alpha|\,q(p-1)}{(q-p+1)^2} - C_0\frac{2\,|\alpha|\,q(p-1)}{(q-p+1)^2} \\ &\quad\ge \frac{|\alpha|\,q(p-1)}{(q-p+1)^2}\{\theta_2\, C_1-2C_0\} \\ &\quad\ge \frac{|\alpha|\,q(p-1)}{(q-p+1)^2} =: a\,.\end{aligned}$$

We now set $y := a\log z$ then, by using the definition of $\bar s_4$ in (6.2.6), (6.5.104) becomes

$$I_7 \le C\,|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}} \int_{-\infty}^{0} e^{y}\,|y|^{\bar s_4}\,dy \;\le\; C\,|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}-\frac{q}{q-p+1}}. \tag{6.5.105}$$

Combining together (6.5.101), (6.5.102) and (6.5.105), for any $\delta > 0$ small enough and for every $n \in \mathbb{N}$ we have

$$|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} J_3 \;\le\; C\,|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} \left[|\alpha|^{\frac{pq}{q-(1-\alpha)(p-1)}-\frac{q}{q-p+1}} + n^{-\frac{|\alpha|\,p\,q}{(q-p+1)^2}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar s_4} \right].$$

Then letting $n \to \infty$, for every $\delta > 0$ small enough we obtain (6.5.79). Now using (6.5.78), (6.5.79), (6.5.80) and (6.5.81) in (6.5.77), for any $\delta > 0$ sufficiently small and for every $n \in \mathbb{N}$ we get

$$\begin{aligned}
\int_0^\infty \int_\Omega \varphi_n^s\, u^q\, V\, d\mu dt \;&\le\; C' \left(\int\int_{E_\delta^C} \varphi_n^s\, u^q\, V\, dxdt \right)^{\frac{(1-\alpha)(p-1)}{p\,q}} + C'' \\
&\le\; C' \left(\int_0^\infty \int_\Omega \varphi_n^s\, u^q\, V\, dxdt \right)^{\frac{(1-\alpha)(p-1)}{p\,q}} + C'' \\
&\le\; C' \left(1 + \int_0^\infty \int_\Omega \varphi_n^s\, u^q\, V\, dxdt \right)^{\frac{p-1}{p\,q}} + C'' \\
&\le\; A \left(\int_0^\infty \int_\Omega \varphi_n^s\, u^q\, V\, dxdt \right)^{\frac{p-1}{p\,q}} + B\,,
\end{aligned}$$

where A and B are positive constants and they are independent of n, δ and $\frac{p-1}{p\,q} \in (0,1)$. This easily implies that there exists $C > 0$ such that, for sufficiently small $\delta > 0$ and for every $n \in \mathbb{N}$

$$\int_0^\infty \int_\Omega \varphi_n^s\, u^q\, V\, dxdt \;\le C\,. \tag{6.5.106}$$

By using the definition of φ_n in (6.4.31) we observe that

$$\begin{aligned}
&\varphi_n = 1 \quad \text{in } E_\delta \\
&\varphi_n \ge 0 \quad \text{in } \Omega \times [0, +\infty)
\end{aligned}$$

hence

$$\int_0^\infty u^q\, V\, dxdt \le \int_0^\infty \int_\Omega \varphi_n^s\, u^q\, V\, dxdt \;\le C\,.$$

Then letting $\delta \to 0$ we obtain that

$$u^q \in L^1\left(\Omega \times (0,\infty);\, V\, dxdt\right) \tag{6.5.107}$$

Now, we want to show that

$$\int_0^\infty u^q\, V\, dxdt = 0.$$

In order to do this, we use Lemma 6.3.6 where φ is replaced by φ_n

$$\begin{aligned}
\int\int_{E_\delta} u^q\, V\, dxdt &\le \int_0^\infty \int_\Omega \varphi_n^s u^q\, V\, dxdt \\
&\le C\,\left[|\alpha|^{-1-\frac{q(p-1)}{q-p+1}} \int_0^\infty \int_\Omega V^{-\frac{p+\alpha-1}{q-p+1}} |\nabla\varphi_n|^{\frac{p(q+\alpha)}{q-p+1}}\, dxdt\right.\\
&+ \left.|\alpha|^{-1} \int_0^\infty \int_\Omega V^{-\frac{\alpha-1}{q-1}} |\partial_t\varphi_n|^{\frac{q+\alpha}{q-1}}\, dxdt\right]^{\frac{p-1}{p}} \left(\int\int_{E_\delta^C} \varphi_n^s u^q V\, dxdt\right)^{\frac{(1-\alpha)(p-1)}{p\,q}}\\
&\times \left[\int\int_{E_\delta^C} V^{-\frac{(1-\alpha)(p-1)}{q-(1-\alpha)(p-1)}} |\nabla\varphi_n|^{\frac{pq}{q-(1-\alpha)(p-1)}}\, dxdt\right]^{\frac{q-(1-\alpha)(p-1)}{p\,q}}\\
&+ C\,\left[\int\int_{E_\delta^C} \varphi_n^s u^q\, V\, dxdt\right]^{\frac{1}{q}} \left[\int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t\varphi_n|^{\frac{q}{q-1}}\, dxdt\right]^{\frac{q-1}{q}}\\
&\le C\,\left[|\alpha|^{-\frac{q(p-1)}{q-p+1}} J_1\right]^{\frac{p-1}{p}} \left(\int\int_{E_\delta^C} \varphi_n^s u^q V\, dxdt\right)^{\frac{(1-\alpha)(p-1)}{p\,q}} \left[|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} J_3\right]^{\frac{q-(1-\alpha)(p-1)}{p\,q}}\\
&+ C\, J_2^{\frac{p-1}{p}} \left(\int\int_{E_\delta^C} \varphi_n^s u^q V\, dxdt\right)^{\frac{(1-\alpha)(p-1)}{p\,q}} \left[|\alpha|^{-\frac{q(p-1)}{q-(1-\alpha)(p-1)}} J_3\right]^{\frac{q-(1-\alpha)(p-1)}{p\,q}}\\
&+ C\,\left(\int\int_{E_\delta^C} \varphi_n^s u^q V\, dxdt\right)^{\frac{1}{q}} J_5^{\frac{q-1}{q}}\,,
\end{aligned} \tag{6.5.108}$$

where J_1, J_2, J_3 have been defined in (6.5.72), (6.5.73), (6.5.74) and

$$J_5 := \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t\varphi_n|^{\frac{q}{q-1}}\, dxdt\,.$$

Due to the definition of φ_n in (6.4.34) we have

$$\begin{aligned}
J_5 &\le C \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t\varphi|^{\frac{q}{q-1}}\, dxdt + \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} |\partial_t\eta_n|^{\frac{q}{q-1}}\, dxdt \\
&:= I_9 + I_{10}\,.
\end{aligned} \tag{6.5.109}$$

By (6.4.31) we have

$$I_9 \le C\,|\alpha|^{\frac{q}{q-1}} \int\int_{E_\delta^C} V^{-\frac{1}{q-1}} \left[\left(d(x)^{-\theta_2} + t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]^{-\theta_2(C_1\alpha-1)\frac{q}{q-1}} t^{(\theta_1-1)\frac{q}{q-1}}\, dxdt \tag{6.5.110}$$

We now state the following

Claim: *Let $f:(0,+\infty)\to[0,+\infty)$ be a non decreasing function and suppose that*

HP2(i) holds. Then, for any $0 < \delta < \delta_0$ small enough, we can write

$$\int\int_{E_\delta^C} f\left(\left[\left(d(x)^{-\theta_2} + t^{\theta_1}\right)^{-\frac{1}{\theta_2}}\right]\right) t^{(\theta_1-1)\left(\frac{q}{q-1}\right)} V^{-\frac{1}{q-1}}\, dxdt \le C \int_0^{2^{\frac{1}{\theta_2}}\delta} f(z) z^{-\bar{s}_1-1} |\log z|^{\bar{s}_2}\, dz, \tag{6.5.111}$$

for some constant $C > 0$ with $\bar{s}_1$ and $\bar{s}_2$ as in (6.2.6). Inequality (6.5.111) can be proven similarly to (6.4.53) and (6.4.60) where one uses the condition HP2(i) with $\varepsilon = 0$ instead of HP1. We now use the latter claim to (6.5.110), thus we have

$$\begin{aligned} I_9 &\le C\,|\alpha|^{\frac{q}{q-1}} \int_0^{2^{\frac{1}{\theta_2}}\delta} z^{-\theta_2(C_1\alpha-1)\frac{q}{q-1} - \frac{q}{q-1}\theta_2 - 1} |\log z|^{\bar{s}_2}\, dz \\ &\le C\,|\alpha|^{\frac{q}{q-1}} \int_0^{2^{\frac{1}{\theta_2}}\delta} z^{-\theta_2 C_1\alpha\frac{q}{q-1} - 1} |\log z|^{\bar{s}_2}\, dz \\ &\le C\,|\alpha|^{\frac{q}{q-1}} \int_{-\infty}^0 e^{\frac{1}{\gamma}} \left|\frac{y}{\gamma}\right|^{\bar{s}_2} \frac{1}{\gamma}\, dy \\ &\le C\,|\alpha|^{\frac{q}{q-1} - \bar{s}_2 - 1} \\ &\le C \end{aligned} \tag{6.5.112}$$

where

$$\gamma := |\alpha|\,\theta_2\,C_1\frac{q}{q-1} \qquad \text{and} \qquad y := \gamma\,\log z\,.$$

On the other hand, by (6.4.33) we have

$$\begin{aligned} I_{10} &\le C \int\int_{\tilde{E}_{\frac{\delta}{n}}} V^{-\frac{1}{q-1}} \left[n^{\theta_2 C_1\alpha} \left(\frac{\delta}{n}\right)^{\theta_2} t^{\theta_1 - 1} \right]^{\frac{q}{q-1}} dxdt \\ &\le C\, n^{\theta_2 (C_1\alpha - 1)\frac{q}{q-1}}\, \delta^{\theta_2\frac{q}{q-1}} \int\int_{\tilde{E}_{\frac{\delta}{n}}} V^{-\frac{1}{q-1}} t^{(\theta_1-1)} dxdt \end{aligned}$$

Then, due to HP2(ii) with $\varepsilon = 0$ we have

$$\begin{aligned} I_{10} &\le C\, n^{\theta_2 (C_1\alpha - 1)\frac{q}{q-1} + \frac{q}{q-1}\theta_2}\, \delta^{\theta_2\frac{q}{q-1} - \frac{q}{q-1}\theta_2} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2} \\ &\le n^{-|\alpha|\theta_2 C_1\frac{q}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2} \end{aligned} \tag{6.5.113}$$

Now, combining (6.5.109), (6.5.112) and (6.5.113) we get

$$J_5 \le C \left[1 + n^{-|\alpha|\theta_2 C_1\frac{q}{q-1}} \left|\log\left(\frac{\delta}{n}\right)\right|^{\bar{s}_2} \right]$$

By letting $n \to \infty$ we obtain

$$J_5 \le C \tag{6.5.114}$$

Finally we substitute inequality (6.5.72), (6.5.73), (6.5.74) and (6.5.114) into (6.5.108) thus we have

$$\int\int_{E_\delta} u^q\,V\,dxdt \le C\left[\left(\int\int_{E_\delta^C} \varphi_n^s\,u^q\,V\,dxdt\right)^{\frac{(1-\alpha)(p-1)}{pq}} + \left(\int\int_{E_\delta^C} \varphi_n^s\,u^q\,V\,dxdt\right)^{\frac{1}{q}}\right].$$

Passing to the lim sup as $n \to \infty$, we obtain for some constant $C > 0$

$$\int\int_{E_\delta} u^q\,V\,dxdt \le C\left[\left(\int\int_{E_\delta^C} u^q\,V\,dxdt\right)^{\frac{(1-\alpha)(p-1)}{p}} + \left(\int\int_{E_\delta^C} u^q\,V\,dxdt\right)^{\frac{1}{q}}\right]. \tag{6.5.115}$$

Now we can pass to the limit in (6.5.115) as $\delta \to 0$, and thus as $\alpha \to 0^-$, and conclude by using Fatou's Lemma and (6.5.107) that

$$\int_0^\infty \int_\Omega u^q\,V\,dxdt = 0.$$

Thus $u = 0$ a.e. in $\Omega \times [0, \infty)$.

□

6.6 Proof of Theorem 6.2.5

Throughout this section we always assume that $\partial\Omega$ is of class C^3. We now introduce two Lemmas that will be used in the proof of Theorem 6.2.5. Let us first observe that, under the assumptions of Theorem 6.2.5, the Green function $G(x, y)$ associated to the laplacian operator $-\Delta$ satisfies the following bound

$$G(x, y) \le C \min\left\{1, \frac{d(x)d(y)}{|x-y|^2}\right\} |x-y|^{2-N}, \tag{6.6.116}$$

for some $C > 0$ and $d(x)$ as in (6.1.4). See [57], [137]; see also [18], [28].

Lemma 6.6.1. *Suppose that* (6.6.116) *holds and define*

$$\psi(x) := \int_\Omega G(x, y)\, d(y)^\beta\, dy, \tag{6.6.117}$$

for $-1 < \beta \le 0$. *Then, there exist* $c = c(\beta) > 0$ *such that*

$$0 \le \psi(x) \le c\, d(x) \quad \text{for every } x \in \Omega, \tag{6.6.118}$$

Proof. Let us fix $x \in \Omega$ such that $d(x) > 0$. Then, for any $y \in \Omega$ either

$$d(y) \ge 2|x-y|, \tag{6.6.119}$$

or

$$d(y) \le 2|x-y|. \tag{6.6.120}$$

Therefore we write

$$\psi(x) = \int_{\{d(y)\geq 2|x-y|\}} G(x,y)d(y)^{\beta}\,dy + \int_{\{d(y)\leq 2|x-y|\}} G(x,y)d(y)^{\beta}\,dy$$

Moreover observe that, for any $z \in \partial\Omega$

$$|y-z| \leq |x-z| + |y-x|.$$

If we fix $z \in \partial\Omega$ such that $d(x) = |x-z|$ then the latter can be rewrite as

$$|y-z| \leq d(x) + |y-x|. \tag{6.6.121}$$

Combining (6.6.119) and (6.6.121), it follows that

$$2|x-y| \leq d(y) \leq |y-z| \leq d(x) + |y-x| \Longrightarrow |x-y| \leq d(x). \tag{6.6.122}$$

Due to (6.6.116), (6.6.119) and (6.6.122)

$$\begin{aligned}
0 &\leq \int_{\{d(y)\geq 2|x-y|\}} G(x,y)d(y)^{\beta}\,dy \\
&\leq c\int_{\{d(y)\geq 2|x-y|\}} \frac{d(y)^{\beta}}{|x-y|^{N-2}}\,dy \\
&\leq c\int_{\{d(y)\geq 2|x-y|\}} \frac{d(x)d(y)^{\beta}}{|x-y|^{N-1}}\,dy \\
&\leq c\int_{\{d(y)\geq 2|x-y|\}} \frac{d(x)}{|x-y|^{N-1-\beta}}\,dy.
\end{aligned}$$

Now, since $-1 < \beta \leq 0$

$$c\int_{\{d(y)\geq 2|x-y|\}} \frac{d(x)}{|x-y|^{N-1-\beta}}\,dy \;\leq\; c\,d(x)\int_{B_R(x)} \frac{1}{|x-y|^{N-1-\beta}}\,dy \leq c\,d(x), \tag{6.6.123}$$

where $R := \operatorname{diam}(\Omega) = \sup\{|x-y| \,:\, x,y \in \Omega\}$. Similarly, due to (6.6.116), (6.6.120) and (6.6.122)

$$\begin{aligned}
0 &\leq \int_{\{d(y)\leq 2|x-y|\}} G(x,y)d(y)^{\beta}\,dy \\
&\leq c\int_{\{d(y)\leq 2|x-y|\}} \frac{d(x)d(y)^{1+\beta}}{|x-y|^{N}}\,dy \\
&\leq c\int_{\{d(y)\leq 2|x-y|\}} \frac{d(x)}{|x-y|^{N-(1+\beta)}}\,dy \\
&\leq c\,d(x)\int_{B_R(x)} \frac{1}{|x-y|^{N-(1+\beta)}}\,dy \\
&\leq c\,d(x)
\end{aligned} \tag{6.6.124}$$

Finally, due to (6.6.123) and (6.6.124), for any $x \in \Omega$, there exists $c = c(\beta)$ such that

$$0 \leq \psi(x) \leq c\,d(x).$$

□

Lemma 6.6.2. *Suppose that* (6.6.116) *holds. Let us recall the definition of* ψ *in* (6.6.117) *and suppose that*

$$-2 < \beta \leq -1. \tag{6.6.125}$$

Then, there exist $M > 0$ *such that*

$$0 \leq \psi(x) \leq M \quad \textit{for any } x \in \Omega, \tag{6.6.126}$$

Proof. By Lemma 6.6.1 we only need to consider the case $-2 < \beta \leq -1$. For every $\varepsilon > 0$ small enough, let Ω_ε be defined as in (6.2.21). Moreover let $G_\varepsilon(x, y)$ be the Green function associated to the operator $-\Delta$ for $x, y \in \Omega_\varepsilon$. For every $\varepsilon > 0$, let

$$u_\varepsilon(x) := \int_{\Omega_\varepsilon} G_\varepsilon(x, y)\, d(y)^\beta\, dy. \tag{6.6.127}$$

Observe that, for every $\varepsilon > 0$, $u_\varepsilon \in C^\infty(\Omega_\varepsilon) \cap C^0(\overline{\Omega}_\varepsilon)$, $u_\varepsilon > 0$ in Ω_ε and it solves the following problem

$$\begin{cases} -\Delta u_\varepsilon(x) = d(x)^\beta & \text{in } \Omega_\varepsilon \\ u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases}.$$

Moreover, due to assumption (6.6.125), see [106], there exists $v : \bar{\Omega} \to \mathbb{R}$, $v \in C^0(\bar{\Omega})$, $v > 0$ in Ω such that v is a solution to problem

$$\begin{cases} -\Delta v(x) = d(x)^\beta & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

Observe that, due to the maximum principle, it follows that

$$0 < u_\varepsilon < v \quad \text{in } \Omega_\varepsilon \quad \text{for any } \varepsilon > 0. \tag{6.6.128}$$

Moreover, for $0 < \varepsilon_1 < \varepsilon_2$ one has

$$u_{\varepsilon_2}(x) \leq u_{\varepsilon_1(x)} \quad \text{for any } x \in \Omega_{\varepsilon_2} \tag{6.6.129}$$

Hence, the sequence of functions $\{u_\varepsilon\}_{\varepsilon>0}$, due to (6.6.128) and (6.6.129), admits a finite limit for $\varepsilon \to 0$, in particular we write

$$\lim_{\varepsilon \to 0} u_\varepsilon(x) = w(x) \quad \text{for any } x \in \Omega, \tag{6.6.130}$$

and $0 < w(x) \leq v(x)$ for any $x \in \Omega$. Now observe that

$$G_\varepsilon(x, y) \to G(x, y) \quad \text{as } \varepsilon \to 0 \text{ for any } x, y \in \Omega\,.$$

It follows, by the Monotone Convergence Theorem that for any $\varepsilon > 0$ one has

$$u_\varepsilon(x) = \int_\Omega G_\varepsilon(x, y) d(y)^\beta\, dy \longrightarrow \int_\Omega G(x, y) d(y)^\beta\, dy \quad \text{as } \varepsilon \to 0. \tag{6.6.131}$$

Hence, due to (6.6.130) and (6.6.131), for any $x \in \Omega$ we can write

$$w(x) = \int_\Omega G(x, y) d(y)^\beta\, dy, \quad \text{and} \quad 0 \leq \int_\Omega G(x, y) d(y)^\beta \leq v(x)\,.$$

Finally, since v is continuous in a closed and bounded domain, there exists $M > 0$ such that

$$v(x) \leq M, \quad \text{for any } \ x \in \bar{\Omega},$$

and

$$0 \leq \int_{\Omega} G(x,y) d(y)^{\beta} \, dy \leq M.$$

□

We are now ready to prove Theorem 6.2.5.

Proof of Theorem 6.2.5. We want to construct a subsolution and a supersolution to problem (6.1.3). Let $\underline{u}$ be the subsolution and $\overline{u}$ be the supersolution. We firstly set

$$\underline{u} \equiv 0.$$

On the other hand, in order to construct $\overline{u}$, let us define, for any $\lambda > 0$

$$S_{\lambda} = \{ v \in C^0(\bar{\Omega}) \, : \, 0 \leq v(x) \leq \lambda \, d(x), \ \forall x \in \Omega \}. \tag{6.6.132}$$

with $d(x)$ as in (6.1.4). Moreover we define the map $T : S_{\lambda} \to S_{\lambda}$ such that

$$Tv(x) = \lambda^q \int_{\Omega} G(x,y) \, dy + \int_{\Omega} G(x,y) V(y) v(y)^q \, dy. \tag{6.6.133}$$

We prove that T is well defined and that it is a contraction map. Observe that due to Lemma 6.6.1 with $\beta = 0$ one has, for some $c_1 > 0$

$$0 \leq \lambda^q \int_{\Omega} G(x,y) \, dy \leq c_1 \, \lambda^q d(x), \quad \text{for every } x \in \Omega. \tag{6.6.134}$$

Similarly, due to (6.2.19), Lemma 6.6.1 with $\beta = -\sigma_1 + q$ and (6.2.20), for some $c_2 > 0$

$$0 \leq \int_{\Omega} G(x,y) V(y) v(y)^q \, dy \leq c_2 \, \lambda^q \int_{\Omega} G(x,y) d(y)^{-\sigma_1 + q} \, dy \leq c_2 \, \lambda^q \, d(x). \tag{6.6.135}$$

By using (6.6.134) and (6.6.135), inequality (6.6.133), for some $C > 0$ and $\lambda > 0$ small enough, reduces to

$$0 \leq Tv(x) \leq C \lambda^q \, d(x) \leq \ \lambda \, d(x) \quad \text{for any } \ x \in \Omega.$$

Hence, for a sufficiently small $\lambda > 0$, the function $Tv : \bar{\Omega} \to \mathbb{R}$ is continuous and thus the map $T : S_{\lambda} \to S_{\lambda}$ is well defined. Let us now show that T is a contraction map. for $\lambda > 0$ small enough. Fix $w, v \in S_{\lambda}$, then for any $x \in \Omega$

$$\begin{aligned} |Tw(x) - Tv(x)| &\leq \int_{\Omega} G(x,y) V(y) |w^q(y) - v^q(y)| \, dy \\ &\leq \int_{\Omega} G(x,y) V(y) q \xi(y)^{q-1} |w(y) - v(y)| \, dy, \end{aligned}$$

for some $\xi(y)$ between $w(y)$ and $v(y)$. Then $0 \le \xi(y) \le \lambda d(y)$ and hence, due to Lemma 6.6.2 with $\beta = -\sigma_1 + q - 1$ and (6.2.20),

$$\begin{aligned}|Tw(x) - Tv(x)| &\le C\left(\int_\Omega G(x,y)d(y)^{-\sigma_1+q-1}\,dy\right)\lambda^{q-1}\|w-v\|_{L^\infty(\Omega)}\\ &\le C\,M\,\lambda^{q-1}\|w-v\|_{L^\infty(\Omega)}.\end{aligned}$$

Thus we have, for $\lambda > 0$ small enough,

$$\|Tw - Tv\|_{L^\infty(\Omega)} \le \frac{1}{2}\|w-v\|_{L^\infty(\Omega)},$$

hence T is a contraction map. Therefore, there exists $\varphi \in S_\lambda$ such that $\varphi = T\varphi$. In particular, we have

(i) $0 \le \varphi(x) \le \lambda\, d(x)$ for any $x \in \bar\Omega$;

(ii) φ is a solution of

$$\begin{cases}-\Delta\varphi = \lambda^q + V\varphi^q & \text{in } \Omega\\ \varphi = 0 & \text{on } \partial\Omega\end{cases} \tag{6.6.136}$$

(iii) $\varphi > 0$ in Ω.

We now set $\overline{u}(x,t) = \varphi(x)$ and show that $\overline{u}$ is a supersolution to problem (6.1.3). Observe that

(i) $\partial_t\overline{u} - \Delta\overline{u} = -\Delta\varphi = \lambda^q + V\,\varphi^q \ge V\,\overline{u}^q$ in $\Omega \times (0,+\infty)$;

(ii) $\overline{u}(x,t) = \varphi(x) = 0$ for any $x \in \partial\Omega$, for any $t \in (0,+\infty)$;

(iii) $\overline{u} \ge 0$ and $\overline{u} \not\equiv 0$;

(iv) $0 \le u_0(x) \le \overline{u}(x,0)$ for any $x \in \Omega$, if ε is small enough; indeed we can apply the Hopf's Lemma and if n denotes the inward normal unit vector to $\partial\Omega$ deduce that

$$\frac{\partial\varphi}{\partial n} > 0, \quad \text{for any } x \in \partial\Omega.$$

Then, due to the compactness of $\bar\Omega$ and the continuity of φ in Ω we observe that there exists $\alpha > 0$ such that

$$\varphi \ge \alpha\, d(x) \quad \text{for any } x \in \bar\Omega.$$

Now, if $\varepsilon > 0$ in (6.2.18) is sufficiently small, we have that

$$0 \le u_0(x) \le \varepsilon\, d(x) \le \alpha\, d(x) \le \varphi(x) = \overline{u}(x,0) \quad \text{for any } x \in \bar\Omega.$$

Thus $\overline{u} : \bar\Omega \times [0,+\infty) \to \mathbb{R}$ is a supersolution to problem (6.1.3), such that $\overline{u} \ge \underline{u}$ in $\bar\Omega \times [0,+\infty)$. Finally, we conclude that there exists a solution $u : \Omega \times [0,+\infty) \to \mathbb{R}$ of problem (6.1.3) such that

$$0 \le u(x) \le \overline{u}(x) \quad \text{for any } x \in \bar\Omega.$$

□

6.7 Proof of Theorem 6.2.6 and of Corollary 6.2.7

We introduce some auxiliary Lemmas that are needed in the proof of Theorem 6.2.6.

Lemma 6.7.1. *Let $u_0 \in C(\Omega)$, $u_0 \geq 0$ in Ω. Moreover let $V \in C(\Omega \times [0,+\infty))$, $V > 0$ in $\Omega \times (0,+\infty)$ and assume that $u \geq 0$ is a classical solution of problem* (6.1.3) *with initial datum u_0. Let $\alpha > \frac{2q}{q-1}$ and $\psi \in C^{2,1}_{x,t}(\Omega \times [0,+\infty))$, $\psi \geq 0$ a.e. in $\Omega \times [0,+\infty)$ with compact support in $\Omega \times [0,+\infty)$ then*

$$\int_0^\infty \int_\Omega u^q V \psi^\alpha \,dxdt \leq 2^{\frac{1}{q-1}} \left\{ \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} \left| \alpha(\alpha-1)|\nabla\psi|^2 + \alpha\psi\Delta\psi \right|^{\frac{q}{q-1}} dxdt \right.$$
$$\left. + \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha\psi\psi_t|^{\frac{q}{q-1}} dxdt \right\}. \tag{6.7.137}$$

Proof. Using Definition 6.3.1 and Young inequality with coefficients q and $\frac{q}{q-1}$ we write

$$\begin{aligned}
\int_0^\infty \int_\Omega u^q V \psi^\alpha \,dxdt &\leq \int_0^\infty \int_\Omega |u| \, |(\psi^\alpha)_t - \Delta(\psi^\alpha)| \,dxdt - \int_\Omega u_0(x)\psi^\alpha(x,0)\,dx \\
&\leq \int_0^\infty \int_\Omega |u| V^{\frac{1}{q}} \psi^{\frac{\alpha}{q}} \psi^{-\frac{\alpha}{q}} V^{-\frac{1}{q}} |(\psi^\alpha)_t - \Delta(\psi^\alpha)| \,dxdt - \int_\Omega u_0(x)\psi^\alpha(x,0)\,dx \\
&\leq \frac{1}{q} \int_0^\infty \int_\Omega |u|^q V \psi^\alpha \,dxdt + \frac{q-1}{q} \int_0^\infty \int_\Omega (V\psi^\alpha)^{-\frac{1}{q-1}} |(\psi^\alpha)_t - \Delta(\psi^\alpha)|^{\frac{q}{q-1}} dxdt
\end{aligned}$$

Reordering the terms we get

$$\begin{aligned}
\int_0^\infty \int_\Omega u^q V \psi^\alpha \,dxdt &\leq \int_0^\infty \int_\Omega (V\psi^\alpha)^{-\frac{1}{q-1}} \left| \alpha\psi^{\alpha-1}\psi_t - \alpha(\alpha-1)\psi^{\alpha-2}|\nabla\psi|^2 - \alpha\psi^{\alpha-1}\Delta\psi \right|^{\frac{q}{q-1}} dxdt \\
&\leq \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \psi^{-\frac{\alpha}{q-1} + \frac{q(\alpha-2)}{q-1}} \left| \alpha\psi\psi_t - \alpha(\alpha-1)|\nabla\psi|^2 - \alpha\psi\Delta\psi \right|^{\frac{q}{q-1}} dxdt \\
&\leq 2^{\frac{1}{q-1}} \left\{ \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} \left| \alpha(\alpha-1)|\nabla\psi|^2 + \alpha\psi\Delta\psi \right|^{\frac{q}{q-1}} dxdt \right. \\
&\left. + \int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \psi^{\alpha + \frac{2q}{q-1}} |\alpha\psi\psi_t|^{\frac{q}{q-1}} dxdt \right\}
\end{aligned}$$

This proves the thesis. □

Lemma 6.7.2. *Let the assumptions of Lemma 6.7.1 hold. Moreover let $K \subset (\Omega \times [0,+\infty))$ be a compact set and let ψ such that $\psi \equiv 1$ in K. Let $S_k := (\Omega \times [0,+\infty)) \setminus K$ then*

$$\int_0^\infty \int_\Omega u^q V \psi^\alpha \,dxdt \leq 2^{\frac{1}{q}} \left(\int\int_{S_k} |u|^q V \psi^\alpha \,dxdt \right)^{\frac{1}{q}}$$
$$\times \left\{ \left[\int\int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} \left| \alpha(\alpha-1)|\nabla\psi|^2 + \alpha\psi\Delta\psi \right|^{\frac{q}{q-1}} dxdt \right]^{\frac{q-1}{q}} \right.$$
$$\left. + \left[\int\int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha\psi\psi_t|^{\frac{q}{q-1}} dxdt \right]^{\frac{q-1}{q}} \right\}. \tag{6.7.138}$$

Proof. Similarly to the proof of Lemma 6.7.1, using the definition of weak solution of problem (6.1.3) and Hölder inequality with coefficients q and $\frac{q}{q-1}$ we get

$$\begin{aligned}
\int_0^\infty \int_\Omega u^q V \psi^\alpha \,dxdt &\leq \left(\int_0^\infty \int_\Omega |u|^q V \psi^\alpha \,dxdt\right)^{\frac{1}{q}} \left(\int_0^\infty \int_\Omega V^{-\frac{1}{q-1}} \psi^{-\frac{\alpha}{q-1}} \left|(\psi^\alpha)_t - \Delta(\psi^\alpha)\right|^{\frac{q}{q-1}} \,dxdt\right)^{\frac{q-1}{q}} \\
&\leq \left(\int\int_{S_k} |u|^q V \psi^\alpha \,dxdt\right)^{\frac{1}{q}} \\
&\times \left(\int\int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha-\frac{2q}{q-1}} \left|\alpha\psi\psi_t - \alpha(\alpha-1)|\nabla\psi|^2 - \alpha\psi\Delta\psi\right|^{\frac{q}{q-1}} \,dxdt\right)^{\frac{q-1}{q}} \\
&\leq 2^{\frac{1}{q}} \left(\int\int_{S_k} |u|^q V \psi^\alpha \,dxdt\right)^{\frac{1}{q}} \\
&\times \left\{\left[\int\int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha-\frac{2q}{q-1}} \left|\alpha(\alpha-1)|\nabla\psi|^2 + \alpha\psi\Delta\psi\right|^{\frac{q}{q-1}} \,dxdt\right]^{\frac{q-1}{q}} \right. \\
&\left. + \left[\int\int_{S_k} V^{-\frac{1}{q-1}} \psi^{\alpha-\frac{2q}{q-1}} |\alpha\psi\psi_t|^{\frac{q}{q-1}} \,dxdt\right]^{\frac{q-1}{q}} \right\}
\end{aligned}$$

This proves the thesis. □

We now need to introduce the so called *Whitney distance* $\delta : \Omega \to \mathbb{R}^+$ which is a function in $C^\infty(\Omega)$, regardless of the regularity of $\partial\Omega$, such that for all $x \in \Omega$

$$\begin{aligned}
&c^{-1}\, d(x) \leq \delta(x) \leq c\; d(x)\,, \\
&|\nabla\delta(x)| \leq c\,, \qquad (6.7.139)\\
&|\Delta\delta(x)| \leq c\,\delta^{-1}(x)\,,
\end{aligned}$$

where $d(x)$ has been defined in (6.1.4) and $c > 0$ is a constant independent of x. These properties of the Whitney distance may be found in [8, 122].

Lemma 6.7.3. *Let $V \in L^1_{loc}(\Omega \times [0,\infty))$, $V > 0$ a.e., and $u_0 \in L^1_{loc}(\Omega)$, $u_0 \geq 0$ a.e. Assume that there exists a nonincreasing function $f : (0, \varepsilon_0) \to [1,\infty)$ such that $\lim_{\varepsilon\to 0^+} f(\varepsilon) = +\infty$ and such that for every $\varepsilon > 0$ small enough conditions* (6.2.22) *hold. Let $u \geq 0$ be a weak solution of problem* (6.1.3)*, then*

$$\int_0^{+\infty} \int_\Omega u^q V \,dxdt < +\infty \qquad (6.7.140)$$

Proof. For every $\varepsilon > 0$ small enough, we consider a smooth function $g_\varepsilon : [0,\infty) \to \mathbb{R}$ such that $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon \equiv 1$ in $[\varepsilon, +\infty)$, $\operatorname{supp} g_\varepsilon \subset [\frac{\varepsilon}{2}, +\infty)$, $0 \leq g'_\varepsilon \leq \frac{C}{\varepsilon}$ and $|g''_\varepsilon| \leq \frac{C}{\varepsilon^2}$ for some constant $C > 0$. We also introduce η a smooth function such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $[0, \frac{1}{2} f(\varepsilon)]$, $\operatorname{supp} \eta \subset [0, f(\varepsilon))$ and $-\frac{C}{f(\varepsilon)} \leq \eta' \leq 0$. Now let

$$\psi_\varepsilon(x,t) := \phi_\varepsilon(x)\,\eta(t), \qquad (6.7.141)$$

where

$$\phi_\varepsilon(x) := g_\varepsilon(d(x)) = \begin{cases} 1 & \delta(x) > 2\varepsilon \\ 0 & \delta(x) < \varepsilon \end{cases}, \qquad (6.7.142)$$

and δ is the Whitney distance defined in (6.7.139). Observe that, due to (6.7.141), (6.7.142) and (6.7.139) for every $x \in \Omega$ we have

$$\begin{aligned} |\nabla\psi_\varepsilon| &= |g'_\varepsilon(\delta(x))\nabla\delta(x)| \le \frac{C}{\varepsilon}, \\ |\Delta\psi_\varepsilon| &= |g''_\varepsilon(\delta(x))|\nabla\delta(x)|^2 + g'_\varepsilon(\delta(x))\Delta\delta(x)| \le \frac{C}{\varepsilon^2}, \end{aligned} \tag{6.7.143}$$

for some constant $C > 0$. Hence for every $x \in \Omega$, $t \in [0, T)$ we have

$$|(\psi_\varepsilon)_t| \le \frac{C}{f(\varepsilon)}, \qquad \left|\alpha(\alpha-1)|\nabla\psi_\varepsilon|^2 + \alpha\psi_\varepsilon\Delta\psi_\varepsilon\right|^{\frac{q}{q-1}} \le \frac{C}{\varepsilon^{\frac{2q}{q-1}}}. \tag{6.7.144}$$

Let $\tilde\Omega_\varepsilon = \{x \in \Omega \,|\, \delta(x) \ge \varepsilon\}$ and note that by (6.7.139) for every $r > 0$ we have

$$\tilde\Omega_r \subset \Omega_{\frac{r}{c_0}}, \qquad \Omega_r \subset \tilde\Omega_{\frac{r}{c_0}}.$$

We now observe, applying Lemma 6.7.1 with the test function ψ_ε defined in (6.7.141), that

$$\begin{aligned} \int_0^T \int_{\Omega_\varepsilon} u^q\, V\, dxdt &\le \int_0^\infty u^q\, \psi_\varepsilon^\alpha V\, dxdt \\ &\le C\left\{\int_0^\infty V^{-\frac{1}{q-1}}\, \psi_\varepsilon^{\alpha-\frac{2q}{q-1}} \left|\alpha(\alpha-1)|\nabla\psi_\varepsilon|^2 + \alpha\psi_\varepsilon\Delta\psi_\varepsilon\right|^{\frac{q}{q-1}} dxdt \right. \\ &\quad \left. + \int_0^\infty V^{-\frac{1}{q-1}}\, \psi_\varepsilon^{\alpha-\frac{2q}{q-1}} |\alpha\psi_\varepsilon(\psi_\varepsilon)_t|^{\frac{q}{q-1}}\, dxdt \right\} \\ &=: I_1 + I_2\,. \end{aligned} \tag{6.7.145}$$

Now, due to the definition of ψ_ε in (6.7.141) and by (6.2.22) and (6.7.144), for every small enough $\varepsilon > 0$ we have

$$\begin{aligned} I_1 &\le \int_0^{2T} \int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} V^{-\frac{1}{q-1}} \left[\left|\alpha(\alpha-1)|\nabla\psi_\varepsilon|^2 + \alpha\psi_\varepsilon\Delta\psi_\varepsilon\right|\right]^{\frac{q}{q-1}} dxdt \\ &\le \frac{C}{\varepsilon^{\frac{2q}{q-1}}} \int_0^{2T} \int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} V^{-\frac{1}{q-1}}\, dxdt \\ &\le C\,. \end{aligned} \tag{6.7.146}$$

where we set $N = [2\log_2 c_0] + 1$. Similarly, due to (6.7.141) and by (6.2.22) and (6.7.144) we can observe that

$$\begin{aligned} I_2 &\le \int_T^{2T} \int_{\Omega_\varepsilon} V^{-\frac{1}{q-1}} \left[|\alpha\psi_\varepsilon(\psi_\varepsilon)_t|\right]^{\frac{q}{q-1}} dxdt \\ &\le \frac{C}{T^{\frac{q}{q-1}}} \int_T^{2T} \int_{\Omega_\varepsilon} V^{-\frac{1}{q-1}}\, dxdt \\ &\le C\,. \end{aligned} \tag{6.7.147}$$

By substituting (6.7.146) and (6.7.147) into (6.7.145) and letting $\varepsilon \to 0$ we obtain the thesis. □

We are now ready to prove Theorem 6.2.6.

Proof of Theorem 6.2.6. For small enough $\varepsilon > 0$ consider the test function ψ_ε defined in (6.7.141). Define

$$K_\varepsilon := \Omega_{2\varepsilon} \times [0, T]\,; \tag{6.7.148}$$

and

$$S_{K_\varepsilon} := (\Omega \times [0, +\infty)) \setminus K_\varepsilon\,. \tag{6.7.149}$$

Observe that $\psi_\varepsilon \equiv 1$ on K_ε, hence we can apply Lemma 6.7.2 with the test function ψ_ε, thus we have

$$\begin{aligned}
\int\!\!\int_{K_\varepsilon} u^q\, V\, dxdt \;&\le \int_0^\infty \int_\Omega u^q\, \psi_\varepsilon^\alpha V\, dxdt \\
&\le C\left(\int\!\!\int_{S_{K_\varepsilon}} |u|^q V \psi^\alpha\, dxdt\right)^{\frac{1}{q}} \\
&\times \left\{ \left[\int\!\!\int_{S_{K_\varepsilon}} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} \left|\alpha(\alpha-1)|\nabla\psi|^2 + \alpha\psi\Delta\psi\right|^{\frac{q}{q-1}}\, dxdt\right]^{\frac{q-1}{q}} \right. \\
&+ \left. \left[\int\!\!\int_{S_{K_\varepsilon}} V^{-\frac{1}{q-1}} \psi^{\alpha - \frac{2q}{q-1}} |\alpha\psi\psi_t|^{\frac{q}{q-1}}\, dxdt\right]^{\frac{q-1}{q}} \right\} \\
&=: \; I_1 + \; I_2\,.
\end{aligned} \tag{6.7.150}$$

We can also use Lemma 6.7.3 hence we say that there exists $C > 0$ such that

$$I_1 \le C\,, \qquad I_2 \le C\,.$$

Thus we have

$$\int\!\!\int_{K_\varepsilon} u^q\, V\, dxdt \;\le\; C\left(\int\!\!\int_{S_{K_\varepsilon}} |u|^q V \psi^\alpha\, dxdt\right)^{\frac{1}{q}}\,.$$

Letting $\varepsilon \to 0$ we obtain

$$\int_0^T \int_\Omega u^q\, V\, dxdt \;= 0\,, \tag{6.7.151}$$

which proves the thesis.

□

Proof of Corollary 6.2.7. By (6.2.23) and the assumptions on f, for $\varepsilon > 0$ small enough we have

$$\begin{aligned}
\int_0^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} V^{-\frac{1}{q-1}}\, dxdt &\le C f(\varepsilon) \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} d(x)^{\frac{q+1}{q-1}} f(d(x))^{-1}\, dx \\
&\le\; C\varepsilon^{\frac{q+1}{q-1}} \int_{\Omega_{\frac{\varepsilon}{2}} \setminus \Omega_\varepsilon} dx \;\le\; C\varepsilon^{\frac{2q}{q-1}}
\end{aligned}$$

and

$$\int_{\frac{1}{2}f(\varepsilon)}^{f(\varepsilon)} \int_{\Omega_{\frac{\varepsilon}{2}}} V^{-\frac{1}{q-1}} \, dxdt \leq \, Cf(\varepsilon) \int_{\Omega_{\frac{\varepsilon}{2}}} d(x)^{\frac{q+1}{q-1}} f(d(x))^{-1} \, dx$$
$$\leq \, Cf(\varepsilon) \, \leq \, Cf(\varepsilon)^{\frac{q}{q-1}}.$$

Thus conditions (6.2.22) are satisfied and by Theorem 6.2.6 $u \equiv 0$ a.e. in $\Omega \times [0, \infty)$. □

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