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SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE

Executive Summary of the Thesis

On existence of rotund Gâteaux smooth norms which are not midpoint locally uniformly rotund

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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Academic year: 2022-2023

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1. Introduction

Banach spaces are the foundations to most of the development made in functional analysis. It is not surprising that research in the field grew wider as the years passed, among the 100 years that have passed since the inception of the concept, many surprising facts have arisen. In particular, it became apparent how the understanding of the properties of Banach spaces was strongly connected to the geometrical characteristics of their norm functions. One of the first instances of this connection lies in the Milman-Pettis theorem in which we have a connection between convexity and reflexivity. From this starting point many more discoveries were made in this regard creating a vast and rich literature from which we start to perform our research. Our ultimate goal is to address two open questions in the same field found at [7, Section 52.3], the first one being whether it is possible for every infinite-dimensional separable space to be renormed with a rotund and not midpoint locally uniformly rotund norm. The second open question is whether it is possible to renorm all infinite dimensional Banach spaces with separable dual with a norm which is weakly uniformly rotund but not midpoint locally uniformly rotund.

2. State of the art

2.1. Banach spaces

The main protagonist of our research are complete real normed spaces, or as Fréchet called them, Banach spaces. We recall that a norm is a positive function on a real vector space X for which three conditions hold: $\|x\| = 0$ if and only if $x = 0$, it's homogeneous and triangle inequality holds. Two norms are said to be equivalent on the same space if and only if they induce the same norm topology on the space X. From this notion is possible to define a metric space P of all equivalent norm, given an initial Banach space $(X, \|\cdot\|)$. P can be equipped with a distance ρ, generating a metric space (P, ρ) for which all the topological concepts assume some meaning (such as interior, closure, density...). A very important topic that has been extensively studied are basis in Banach spaces. We now wish to introduce the main basis used in our research: Markushevich basis. Given a Banach space X, a fundamental and total biorthogonal system $\{e_n, e_n^*\}$ is called Markushevich basis (M-basis for short), moreover if we have $\overline{\operatorname{span}\, e_n^*} = X^*$ we also say that the M-basis is shrinking. Given an M-basis $\{e_n, e_n^*\}$, we say it is bounded if $\sup \|e_n\|\|e_n^*\| \leq \infty$ for all $n \in \mathbb{N}$. We now wish to state a closing result: every separable Banach space X admits a bounded M-basis. If, moreover, X^* is separable, then the M-basis can be taken to be shrinking and bounded.

2.2. Geometrical properties

This section is devoted to introducing some concepts strengthening the one of convexity, around which our research will revolve. The first condition is called rotundity, for which we ask that for all $x, y \in S_{(X,\|\cdot\|)}$ (notation for the unit sphere), the inequality : $\|x + y\| < 2$ holds. Intuitively, this condition is needed to ensure that the unit sphere doesn't contain any "straight" segment. After this first notion, we consider the one called midpoint local uniform rotundity, introduced by K.Anderson in 1960 as his Phd thesis. This second condition states the following: let $(X, \|\cdot\|)$ be a Banach space and let $x_n, y_n \subseteq S_{(X,\|\cdot\|)}$ be two sequences. We say that $\|\cdot\|$ is MLUR if, whenever $\frac{1}{2}(x_n + y_n)$ converges to some element of the unit sphere, we have $\|x_n - y_n\| \to 0$. This will hold a crucial importance for our research. The last condition that we will see is called weak uniform rotundity. We let $(X, \|\cdot\|)$ be a Banach space and let $x_n, y_n \subseteq S_{(X,\|\cdot\|)}$. The norm $\|\cdot\|$ is said to be weakly uniformly rotund (wUR) if every time we have $\|x_n + y_n\| \to 2$, it then follows $\|x_n - y_n\| \to 0$. If a norm satisfies the MLUR condition, then it is automatically rotund. The same is true for the weak uniform rotundity condition, if it holds then the norm is automatically rotund. No relation can be established between the MLUR condition and the wUR one, there are norms which are MLUR but not wUR and wUR but not MLUR (even in our research).

2.3. Analytical properties

The traditional concept of derivative needs to be extended before it can be applied, for instance, to the norms. This problem arises by the fact that normed spaces can be infinite dimensional and the concept of derivative fails to be extended easily to this new setting. In literature the most important kind of derivatives are: Gâteaux and Fréchet derivatives. Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_2)$ be Banach spaces and $f : U \to Y$ a function, where $U \subseteq X$ is open. We say that f is Gâteaux differentiable at $x \in X$ if for each $h \in X$ the following limit $f'(x)(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$ exists and defines a linear and continuous mapping $f' : X \to Y$. Given a Banach space $(X, \|\cdot\|)$ we say $\|\cdot\|$ is Gâteaux differentiable if the norm satisfies this condition for all $x \in S_{(X,\|\cdot\|)}$. On the other side Fréchet derivative is obtained by imposing the uniform convergence with respect to h. Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_2)$ be Banach spaces and $f : X \to Y$ a function. We say that f is Fréchet differentiable at $x \in X$ if the following limit $f'(x)(h) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t}$ exists, defines a linear and continuous mapping and it is uniform for each $h \in S_{(X,\|\cdot\|)}$. Given a Banach space $(X, \|\cdot\|)$ we say $\|\cdot\|$ is Fréchet differentiable if the norm satisfies this condition for all $x \in S_{(X,\|\cdot\|)}$. All the concept that we have introduced in the previous section are strongly connected through duality to the two smoothness condition that we have defined. For instance, if the dual norm is rotund then the norm on the predual is Gâteaux differentiable while if the dual norm is locally uniformly rotund (a notion weaker than uniform rotundity) then his predual norm is Fréchet differentiable. To close this section we wish to state that every separable Banach space can be renormed with a Gâteaux differentiable norm, while every Banach space with separable dual can be renormed with a Fréchet differentiable norm.

2.4. Tools for renormings

We start by renorming a separable (separable dual) Banach space with a norm $\|\cdot\|$ which is Gâteaux (Fréchet) differentiable. Let us also consider a bounded M-basis $\{e_n, e_n^*\}$ which can be taken to be shrinking whenever X^* is separable. Then we decompose X as the direct sum $Y_1 \oplus Y_2$, where $Y_1 = \text{span}(e_1)$ and $Y_2 = \overline{\text{span}(e_n)_{n\geq 2}}$. We consider a new vector $\in \mathbb{R}^2$: $(\|y_1\|, \|y_2\|)$, where $\|\cdot\|$ is the Gâteaux (Fréchet) renorming. Finally, we define an equivalent norm $|||\cdot|||$ on X by means of a smooth lattice norm of $\mathbb{R}^2$ applied to the vector $(\|y_1\|, \|y_2\|)$. The key property is that the corresponding unit ball of the lattice norm in $\mathbb{R}^2$ presents a flat edge along one of the two directions of the canonical basis of $\mathbb{R}^2$. This last geometrical property will be crucial in order to prove that the final norm, involving $|||\cdot|||$ in its definition, is not MLUR. Lastly, it can be proven that this norm preserves

the smoothness condition of the norm $\|\cdot\|$, provided that also the norm on $\mathbb{R}^2$ is smooth.
We now introduce a new norm $\|\cdot\|_D$, which is a renorming of the initial space X. The unit ball of this norm is the intersection of the initial unit ball and the region delimited by two parallel hyperplanes defined by an element of the dual X^* (it will be used e_1^*, the first element of the M-basis). Thanks to its flat part, like the previous norm, we will manage to find some sequences that will violate MLUR condition. This kind of renorming procedure can be used to approximate arbitrarily well all the norms in the metric space of equivalent renormings, that is, we can suppose that given $\varepsilon \in (0,1)$ then for all $x \in X$ the following holds $\|x\|_0 \leq \|x\|_D \leq (1+\varepsilon)\|x\|_0$, where $\|\cdot\|_0$ is the initial norm.

3. Results

Both initial questions are answered in the affirmative with an explicit proposal of norms that not only match the requirements, but they satisfy stronger conditions. In the first case we add the Gâteaux differentiability of the norm, while in the second case we add the Fréchet differentiability. We also complement this two solutions with other two renorming results that show the density of norms, satisfying the initial requirement, in the space of all renorming.

3.1. Norms

We will now state formally all the original results and we will also state explicitly all the norms that allow us to prove them:

Theorem 3.1. *Every separable Banach space can be renormed with a norm which is Gâteaux differentiable, rotund but not MLUR.*

To show this we propose the following norm that can be generally adapted to every separable Banach space:

$$|x|^2 = |||x|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} |e_m^*(x)|^2.$$

Where norm $|||\cdot|||$ and $\{e_n, e_n^*\}$ are defined as above. Rotundity follows from the mathematical argument at page 147 of [7] applied to the norm, Gâteaux differentiability follows from chain rule of differentiation Theorem 70 [8] and the MLUR condition can be violated by the sequences: $x_n = \sqrt{\frac{4}{5}}(e_1 + e_n)$ and $y_n = \sqrt{\frac{4}{5}}(e_1 - e_n)$.

Theorem 3.2. *Given a separable Banach space X, the metric space (P', ρ) of all rotund and not MLUR equivalent norms on X is a dense subset of the metric space (P, ρ) of all equivalent norms on X.*

Even in this case we propose the following norm that can be generally adapted to every separable Banach space, given $\varepsilon \in (0,1)$:

$$\|x\|_2^2 = \|x\|_D^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m} |e_m^*(x)|^2.$$

Where norm $\|\cdot\|_D$ and $\{e_n, e_n^*\}$ are defined as above. Rotundity follows from the mathematical argument at page 147 of [7] applied to the norm and the MLUR condition can be violated by the sequences: $x_n = (1-\varepsilon)e_1 + \eta e_n$ and $y_n = (1-\varepsilon)e_1 + \eta e_n$ for some value of $\eta \in \mathbb{R}^+$.

Theorem 3.3. *Every Infinite dimensional Banach space with separable dual can be renormed with a norm which is Fréchet, wUR and not MLUR.*

To show this we propose the following norm that can be generally adapted to every Banach space with separable dual:

$$|x|^2 = |||x|||^2 + \sum_{m=1}^{\infty} \frac{1}{4^m} |e_m^*(x)|^2.$$

Where norm $|||\cdot|||$ and $\{e_n, e_n^*\}$ are defined as above. Fréchet differentiability follows from chain rule of differentiation Theorem 69 [8] and the MLUR condition can be violated by the sequences: $x_n = \sqrt{\frac{4}{5}}(e_1 + e_n)$ and $y_n = \sqrt{\frac{4}{5}}(e_1 - e_n)$. It can also be shown that this norm is wUR, but the calculations are too involved.

Theorem 3.4. *Given a Banach space X with separable dual, the metric space (Q, ρ) of all wUR and not MLUR equivalent norms on X is a dense subset of the metric space (P, ρ) of all equivalent norms on X.*

We propose the following norm that can be generally adapted to every Banach space with separable dual, given $\varepsilon \in (0,1)$:

$$\|x\|_2^2 = \|x\|_D^2 + \varepsilon^2 \sum_{m=1}^{\infty} \frac{1}{4^m} |e_m^*(x)|^2.$$

Where norm $\|\cdot\|_D$ and $\{e_n, e_n^*\}$ are defined as above. The wUR condition can be shown by some involved calculations while the MLUR condition can be violated by the sequences: $x_n = (1-\varepsilon)e_1 + \eta e_n$ and $y_n = (1-\varepsilon)e_1 + \eta e_n$ for some value of $\eta \in \mathbb{R}^+$.

4. Conclusions

In conclusion we were able to answer the questions of whether a separable Banach space can be renormed with a norm which is rotund but not MLUR in the affirmative, adding an extra property of the norm: Gâteaux differentiability. We also proved that if the smoothness condition of Gâteaux differentiability is dropped such norms are dense in the space of all renormings. The second question answered in the affirmative is whether a Banach space with separable dual can be renormed with a norm which is weakly uniformly rotund but not MLUR, adding an extra property of the norm: Fréchet differentiability. Even in this case by dropping the smoothness condition it was shown that one could retrieve the density property. Indeed, a continuation of this work lies in answering if both the smoothness and the density conditions can be simultaneously kept:

Problem Given a separable Banach space X, is it true that the set of all equivalent Gâteaux smooth, rotund and not MLUR norms is dense in (P, ρ)?.

Problem Given a Banach space X with separable dual, is it true that the set of all equivalent Fréchet smooth, weakly uniformly rotund and not MLUR norms is dense in (P, ρ)?.

5. Acknowledgements

I would like to express my sincere gratitude to prof. Jacopo Somaglia and prof. Carlo Alberto De Bernardi for the opportunity I was given, the passion that they showed and for their constant and meticulous guidance.

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