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Relationships between Interlocking Directorates and governance mechanism: a multiplex network analysis

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Abstract

We employ multiplex network analysis to evaluate the relationship between Interlocking Directorates and governance mechanism. The multiplex network consists of three layers: the first layer is built using direct interlocking, the second using indirect interlocking and the third using a linear combination of some characteristics of the board of directors for each firm. We consider two different multiplex networks; they differ for the weights assigned in the linear combination of the third layer. In any layer of the network there are 201 nodes representing the Italian listed companies of the FTSE Italia All-share index for which we found the data. We observe that: interlocking and governance layer do not have a remarkable edge overlap unless we consider particular linear combinations on the third layer; the weights of the linear combination influence the multiplex networks and, in particular, the governance layers; the sectors' partition for the network is not remarkable. From the results we can understand that interlocking and governance mechanism do not follow the same pattern to connect the nodes but they are, in some way, complementary, i.e. they connect nodes which are not linked by the other phenomenon.

Keywords: Interlocking, Corporate governance, Complex networks, Multiplex networks

Contents

Introduction

Interlocking has been a phenomenon always present in the Italian economy. For example, in [27] are investigated the interlocks between banks and Italian joint-stock companies in the period $1913 - 1936$, a critical period of Italian industrialisation. So, interlocking has already been studied and, despite there have been limitations to it in the last decade, it still exercises an important role. Particularly, it is present in the Italian listed companies and this phenomenon helps agencies to be in contact one with the other and to exchange information using the directors who are in common between them.

The corporate governance is a system of rules, practices and processes by which a firm is directed and controlled. It is one of the pillar of the ESG. ESG stands for Environmental, Social and Governance and it is an indicator through which it is measured how the firms' activities follow the three pillars previously mentioned. In the last years the ESG performances become very important for every listed firms. An example of the growing importance of ESG indicator is exposed in [32]. This study shows as institutional investors prioritise engagement rather than exiting the market and the engagement is primarily focused on governance issue, secondly on environmental considerations and thirdly on occupational, health and safety. In this environment, governance has a great importance in the functioning of the firms and it is worth it to study how it spreads in the Italian listed companies.

We want to unify these two aspects in a network to understand if a very important trait of the Italian economy, interlocking, affects and is affected by the corporate governance of the firms. Some studies about this field has already been done, as explained in Chapter 1, but it is never been used an instrument as the multiplex network. So, we will perform an analysis using this instrument which is growing more and more interest in the field of the complex networks. Indeed, as it will be presented in Chapter 1, the multiplex network allows to consider on different layers different properties of the nodes and see how the nodes connect one to the other. Moreover, analyzing these layers all together it is possible to find out the properties of the network and to understand how the different layers interact one with the other.

The thesis is structured as follows: in Introduction there is a brief introduction of the arguments debated in the thesis; in Chapter 1 we review the literature about interlocking and governance giving the definitions and presenting the relevant results already studied and we briefly define the networks and the multilayer networks; in Chapter 2 we introduce the mathematical concepts used to perform the analysis; in Chapter 3 we present the data and we give some descriptive statistics about them; in Chapter 4 we build the layers and we describe their properties; in Chapter 5 we build the multiplex networks and we describe their properties; in Chapter 6 we make a discussion on the obtained results and we take the conclusions; in Chapter 7 we briefly travel through again the thesis and we provide the results we find.

1 | Literature review

We want to analyse how interlocking and good governance mechanism are related in Italian listed companies. To do this, we will use a multiplex network. So, firstly, we briefly introduce the concept of network and multilayer network.

A network is represented by a graph $G = (V, E)$ where V is the set of vertices or nodes and E is the set of edges or links. In the thesis, the set V will be represented by all the Italian listed companies. The network can be: undirected, i.e. the link between two nodes does not have a direction and so if there is an edge between node i and node j then there is the edge between node j and node i (for example, social networks where each friend of someone is that someone's friend); directed, i.e. the link has a direction (for example, food webs consist of directed links from predator to preys); weighted, i.e. there is a weight on the link (for example, traffic networks where weight represents the number of commuters between two towns); unweighted, i.e. there is no weight on the link (for example, Zachary network which illustrates friendship between members of an university karate club).

As defined in [10], the multilayer network, of which multiplex is a particular case, is a network made up by multiple layers each one representing a network with a given operation mode, social circle or temporal instance. The multilayer networks are useful because many real-world systems do not operate in isolation, on the contrary they are interconnected and what happens at a single level of interaction affects the structure and function at another interconnected level. In a multiplex network, each type of interaction between the nodes is described by a single layer network and the different layers of the network describe the different modes of interaction. In our network there will be three layers: the first two layers represent Interlocking Directorates and the other one represents a network built on a governance index.

The Interlocking Directorates (ID) is defined as the situation where a person affiliated with one organization sits on the board of directors of another organization [24].

There are both advantages and disadvantages in ID. ID can strengthen strategic alliances, improve corporate performance and obtain external resources more easily [29]; moreover,

as explained in [8], ID has advantages in reducing the environmental uncertainty and encourage and manage organizational learning through a valuable combination of resources leading to better problem-solving capacities. Under this view ID actively contributes to a firm's value creation process. However, there are also some disadvantages in ID. Some of these disadvantages are highlighted in [8] and they are: lower monitoring, creation of conflict of interests for companies sharing directors, lower competition pressure between linked firms and the conclusion of agreements between connected firms at the detriment of customers.
ID has different qualities based on laws and market characteristics of the considered country. In [8], it is analysed the ID network in three different countries: France, Germany and United Kingdom. There are some differences in the laws of these countries. Laws and their enforcement concerning investors' protection are strong in common law countries (United Kingdom), intermediary in German civil law countries and weakest in French civil law countries, in which ownership concentration substitutes laws as governance mechanism. Due to these differences, the French market has a bigger ID network while the British market has a significantly smaller ID network. When shareholder's rights are weak, as in the civil law countries, the development of networks seems to be a way for directors to extract value from the company. From a market point of view, France and Germany are examples of coordinated market economies where the economic coordination depends on nonmarket relationships, such as network monitoring or collaborative relationship, while United Kingdom is an example of liberal market economy where the economic coordination relies on competitive market arrangements. Also for this reason, France and Germany have developed more networks than companies in United Kingdom, indeed they traditionally coordinate through relationships.
Italy is a civil law country, as France, and, indeed, in the Italian case interlocking is not a new phenomenon. Through ID, and other phenomena such as cross-ownership and circular ownership, the Italian system has been characterized by pyramidal groups headed by a small number of families that permanently control the Italian firms. Moreover, between the most important firms there are stable links, despite a remarkable turnover among directors [5]. This means that the firms are interested in maintaining the links during time. As highlighted in [14], these links are created by a small number of directors who sit on different boards and by the intra-group interlocks, i.e. the presence of the same directors in two or more boards which belong to the same business group. Usually, these directors are members of at least one blue chip company and often are also shareholders in the same company [13].
In the last decade there has been a decrease in ID due to different factors: 2008 financial crisis, legislation decrees as the Interlocking Ban (2011) and the spread of responsibility

due to intra-group interlocks. But the number of ID which decrease is related to peripheral units, making the heart of the network even more connected, dense and stable [14].

The second topic we want to analyse is the corporate governance. The corporate governance is the system of rules, practices and processes by which a firm is directed and controlled [20]. The scope of corporate governance is to balance the interests of a company's many stakeholders, such as shareholders; more in details: the stakeholders are parties which has an interest in the company and can either affect or be affected by the business. Balancing the interests of stakeholders is the aim of both financial and non-financial institutions, but: "for financial institutions, the scope of corporate governance goes beyond the shareholders to include debtholders, insurance policy holders and other creditors" [19]. Summing up, [28] defines corporate governance "as the ways in which suppliers of finance to corporations assure themselves of getting a return on their investment".
Corporate governance plays an important role in the company. Indeed, good corporate governance mechanism: could be regarded as key factor in safeguarding the interests of the shareholders of companies operating in capital markets [30]; improves corporate accountability, builds a corporate reputation, provides valuable investment decision-making information [28]; has an important role in the probability of accounting frauds (firms with a weak governance structure are more prone to accounting frauds) [6]; affects dividend policy [2].
A central role in the corporate governance of companies is exercised by the board of directors. Boards fulfil the following roles: supervisory role, i.e. monitor and evaluate management; managerial role, i.e. make managerial decisions such as which projects to undertake and which employees to hire; advisory role, i.e. offer valuable advice [15]. The board has to control the management ensuring managers' goal-congruent behaviour and reporting to the shareholders on its stewardship. Additionally the board sets the strategic aims and provides the leadership to put them into effects [30] and is also responsible for the accountability and transparency of an organization by data disclosure [28].
There are several qualities which characterize a board and which influence the corporate governance of a company and its financial performance. The characteristics of our interest are: board size, board independence, 'busy-ness' of corporate directors, gender diversity, number of minority directors. More in details: board size is the number of directors who compose the boards; board independence is the percentage of independent directors in the board; 'busy-ness' of corporate directors is the number of other tasks that each director has to do besides being in the board; gender diversity can be represented as the percent-

age of women and/or the percentage of men in the board; minority director is a member of the board elected by the minority shareholders. All these qualities of the board are related to the financial performance of the company positively and/or negatively.
As studied in [1], board independence is related with the probability of bankruptcy. In particular, the higher the number of independent members in the board, the higher the probability to not have bankruptcy; almost the same conclusion can be reached with board size: the bigger the board, the lower the probability of bankruptcy. Other works show that these qualities have negative or null impact on the financial performances, such as in [3, 23]. But, even if the board of directors' qualities have a negative impact on firm performances, as in the last two works, there are other corporate governance provisions, such as ownership structure, which have a positive impact on the performances. This shows that is always important to have a good governance structure.
The impact of the boards on corporate performance can be different due to the considered paper, indeed there is lack of homogeneity in the results. This may happen because each study considers a different country with its own market characteristics. Since this thesis focuses on the Italian listed companies, we approach to this framework by looking at studies which refer to more advanced countries. In [28], it is proved that board qualities positively moderate the relationship between Corporate Social Responsibility practices and firm performances. In particular, a larger board is considered positive because it increases the possibility to have different area of expertise among directors. Also board independence positively moderate the link between Corporate Social Responsibility practices and firm performances. Moreover, this study shows that, with a continuous oversight from independent directors, the company would likely favour acquiring a better public image with managing its financial activities. In [16], it is shown that in Italy the gender diversity has a positive impact on firm performance. In particular, it is proved that gender diversity, measured as the percentage of women in the board, has a positive impact on firm performance while the presence of one or some women per se has an insignificant effect on financial performance. So, Italian companies must concentrate on having a correct mixture between men and women in the board. Finally, in [12], another board characteristic as the 'busy-ness' of directors has a positive impact on the performances of Italian companies.
We can notice that in all these papers it is never used a complex network but are used other techniques: in [28] is used linear regression for panel data, in [12, 16] is used a panel data analysis, in [30] is used the multiple regression analysis and so on. However, there are some works in which it is used a complex network approach but the considered network is a board of director network (for example in [18, 25]) and it is not a network built on a governance index.

In literature, there are already some works which investigate the relation between corporate governance mechanism and Interlocking Directorates.
In [31], the author examines how the board interlock network and institutional factors are associated with board governance policy adoption in non-profit organizations. To investigate the policies adopted by the considered organizations, a linear regression model is used. Results show that the presence of board interlocks and central network positions conferred by influential status within network (i.e. having board linkages connected to well-connected non-profit organizations) explain variations in policy adoption. In particular, these two qualities positively influence the adoption of good corporate governance mechanism. Moreover, the findings on network position are interesting because it suggests that not every non-profit organization is connected in the same way within the board interlock and not every organization has the same amount of influence in governance policy adoption.
In [22] another aspect of the relation between the adoption of new corporate governance mechanism and board interlocks is analysed. Indeed, it is shown how board interlocks not only have the potential to significantly speed up reform processes but also have, at the same time, considerable potential to slow down reform processes. In particular, this role is exercised by sent interlocks, i.e. interlocks created when a focal firm's executive serves on the board of another firm. Instead received interlocks, i.e. interlocks formed when an executive of another firm serves on the focal firm's board, have no role in this process. All the analysis is performed using a competing-risk discrete-time event history analysis specified to test the effects of covariates on the likelihood that a firm adopts an institutionally contested corporate governance practice.

Since we will use a multiplex network to study the relation between ID and corporate governance mechanism, we can give some definitions and introduce some concepts which will be useful in the following.
Networks are usually represented by an adjacency matrix A where: $A_{ij} = 1$ if there is a link between node i and node j, $A_{ij} = 0$ otherwise.
The degree of a node is the number of edges incidental to it.
The distance between two nodes is the length of the shortest path connecting the nodes.
A subgraph $H = (V', E')$ is a graph such that $V' \in V$ and $E' \in E$.
The centrality of a node is a measure of its importance in the network; there are several definitions of centrality and each one measures the importance of the node in a different way. In the following we recall some of these definitions. Degree centrality: more the

neighbours of the node are, more the node is important (the neighbours are the nodes which are connected through a link with the considered node); betweenness centrality: more the number of shortest paths which pass through a node is, more this node is important; closeness centrality: more a node is close to the others more it is important; eigenvector centrality: more a node relates with many and important nodes more the node is important. These are some of the most important notions of centrality and they are used, for example, in [18]. In this paper all the measures of centrality introduced before are considered and it is possible to see how these measures differ one from the other. However, even if they measure the importance of the node in a different way, it is shown that all the measures of board network are positively related to firms' ESG performances and that having a richer network can facilitate the firms' ability to manage and meet the needs of multiple stakeholders. More in details: ESG performances are Environmental, Social and Governance performances of the considered firm; moreover, in this framework a network becomes richer when the measures of centrality increase. The notion of centrality is very similar also for the multiplex networks.
There is another quantity that is related to the multiplex network and we will use in our study: edge overlap, i.e. total number of pair of nodes connected at the same time by a link in all the layers. An example of how edge overlap can be used is represented in [4]. In this study, the focus is on the multiple relations among Indonesian terrorists belonging to the so-called Noordin Top terrorist network. There are information about Trust, Operational, Communication and Business relations among a group of terrorists and so it is built a multiplex network with four layers. Using edge overlap, it is shown that the stronger the trust connection between two terrorists the higher the probability for them to operate together, communicate or have a common business.
Moreover, in [17] it is shown an example of how we can use the multilayer network and how important it could be to describe different types of connections between nodes. This paper shows the relationship between firms' characteristics and board of director network. The firms' characteristics are: RoA (Return on Assets) which indicates how profitable a company is in relation to its total assets; Debt-to-Equity ratio which represents leverage risk; volatility which represents market risk; beta which represents management quality; Tobin's Q ratio and P/E (Price-to-Earnings) ratio which represent the market outlook of the firm. The board of director network is a network whose nodes are the directors of all the considered companies. The firms' board member connections are represented by a multiplex network with two layers: the first layer represents the direct connections, i.e. two board members are connected if they belong to the same board; the second layer represents the indirect connections, i.e. two board members connect themselves through other people they have associations with. Moreover, each layer has four sub-layers which

express a different type of connection. Using this complex network, it is proved that the nature of board connections changes the firms' characteristics significantly, and therefore, different types of board member associations cannot be treated equally while analysing the board members' network effects on firms' characteristics.

In our study we want to use a multiplex network to relate interlocking with good corporate governance mechanism. In particular, we want to see if the presence of the same director(s) between two companies improves the corporate governance mechanism of the firms increasing the overall performance of the companies. This is interesting to see nowadays because the decrease of ID have created a more dense and stable ID network and we want to see if the companies in the dense and stable core help each other to have a better corporate governance structure than the firms which are in the network's periphery. Indeed, having a good corporate governance structure helps the firm to increase its overall performance and to obtain external resources more easily.
We perform this analysis in the Italian market and we will create a network where: the nodes are all the Italian listed companies, the first two layers are built using ID and the third layer is built using a governance index. This is a novelty introduced by our study, indeed we will not consider a board of director network but we will use a governance index. Moreover, to understand if the interlocking is positively related to corporate governance mechanism we will use some network measures introduced before, such as centrality and edge overlap.

2 | Methodology

In this chapter we introduce some basic definitions about complex networks and multi-layer networks. In particular, we introduce all the properties and formulas we will use in the following to analyze the layers and the multiplex networks. Some definitions have been already given in Chapter 1 and so they are not brought back again.

2.1. Basic definitions of complex networks

A network is represented by a graph with N nodes (or vertices) and L links (or edges). Nodes represent individuals, objects, subsystems; links represent the interaction between the nodes. The network can be: directed, i.e. links have a direction; undirected, i.e. links do not have a direction and if there is the link between node i and node j then there is the link between node j and node i; weighted, i.e. there are weights on the links; unweighted, i.e. there are no weights on the links. In Figure 2.1, it is possible to see an example of every type of graph.

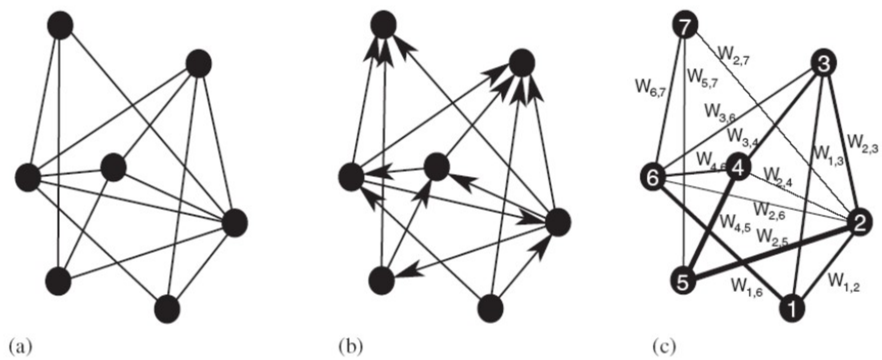

Figure 2.1: Examples of network's graphs

In particular: the directed network is represented in *(b)* since there are arrows indicating the direction of every link; the undirected network is represented in *(a)* and *(c)* because there is no arrow indicating the direction of the links; the weighted network is represented in *(c)* in fact there are weights on the links; the unweighted network is represented in *(a)* and *(b)* indeed there is no weight on the links. In the construction of our layers, we obtain a weighted undirected network.

Now, we introduce some quantities which will be used to analyze the properties of each layer. These definitions are taken from [26].
The first notion we introduce is the degree of a node. The degree of a node is the number of edges incidental to it and it is equal to:

$$k_i = \sum_j a_{ij} \tag{2.1}$$

where a_{ij} is an element of the adjacency matrix A and it is equal to 1 if there exists the link between node i and node j otherwise it is equal to 0. We can use this definition because we are considering an undirected network. Using (2.1) we can compute the average degree ($\langle k \rangle$) for each layer:

$$\langle k \rangle = \frac{1}{N} \cdot \sum_{i=1}^{N} k_i \tag{2.2}$$

where N is the number of nodes in the network.
Moreover, from the notion of degree of a node, we can find the isolated nodes of each layer. Indeed, the isolated nodes are the nodes with degree equal to 0.
Since we are considering a weighted network we can also consider the strength of a node. In an undirected weighted network the strength of a node i is equal to:

$$s_i = \sum_j w_{ij} \tag{2.3}$$

where w_{ij} is the weight of the link between node i and node j. Using (2.3) we can compute the average weighted degree ($\langle s \rangle$) for each layer:

$$\langle s \rangle = \frac{1}{N} \cdot \sum_{i=1}^{N} s_i \tag{2.4}$$

where N is the number of nodes in the network.
The degree distribution is the number of nodes having a given degree. We compare the degree distribution of the layers with the degree distribution of a scale-free network. A scale-free network is a network which contains few largely connected nodes and many

scarcely connected nodes. This network is strongly heterogeneous and its nodes have large fluctuations around the average degree.
Regarding the nodes, we can also describe centrality. Centrality is a measure of the importance of every node in the network. There exist different types of centrality that we already mentioned in Chapter 1. We introduce: degree centrality, betweenness centrality, closeness centrality and eigenvector centrality. These quantities are respectively computed by using the following formulas:

$$D_i = k_i, \tag{2.5}$$

$$b_i = \sum_{j,k} \frac{n_{jk}(i)}{n_{jk}}, \tag{2.6}$$

$$c_i = \frac{N-1}{\sum_j d_{ij}}, \tag{2.7}$$

$$\gamma_i = \alpha \cdot \sum_j a_{ij} \cdot \gamma_j. \tag{2.8}$$

where: n_{jk} is the number of shortest paths connecting j to k and $n_{jk}(i)$ is the number of shortest paths connecting j to k passing through i; d_{ij} is the distance between node i and node j; a_{ij} is the element of the adjacency matrix, γ_j is the eigenvector centrality of node j and α is a coefficient that allows us to write (2.8) as: $A\gamma = \lambda\gamma$ where $\lambda = \frac{1}{\alpha}$.
In equation (2.6) and (2.7), we have introduced two other interesting properties of networks: shortest paths and distances. These two notions are somehow related. In an unweighted network the distance between two nodes is the length of the shortest path connecting them. However, we are considering a weighted network where the bigger the weights the closer the nodes. So, we can define the distance between two nodes as the cost of the lighter path connecting them, where the cost of each link is $\frac{1}{Weight}$. Moreover, if two nodes are not connected (i.e. there is no path which connects node i to node j), then $d_{ij} = Inf$.
We can introduce another quantity: graph density. The graph density measures the density of the links in the network and it is measured using the following equation:

$$Density = \frac{\sum_{i,j} a_{ij}}{N(N-1)} \tag{2.9}$$

where: a_{ij} is the element of the adjacency matrix and $N = 201$ is the number of nodes in the network.
Another property we can consider is the network's giant component. It is a connected component of the network that contains a significant proportion of the entire nodes of

the network. Usually, it contains a big fraction of nodes. We can also consider the other components of the network; they are always connected components of the network but they are not as big as the giant component. Notice that each isolated node is itself a component.
Finally, we want to see if the partition in sectors is significant in our layers. One way to do this is to compute the modularity Q. The modularity quantifies to what extent the intra-/inter-community link densities are anomalous in comparison to the "null model". Modularity is defined as:

$$Q = \frac{1}{2L} \sum_{C_h} \sum_{i,j \in C_h} \left[a_{ij} - \frac{k_i k_j}{2L} \right] \tag{2.10}$$

where: L is the number of links in the network; C_h is the h-community in the considered partition; a_{ij} is the element of the adjacency matrix A; k_i and k_j are respectively the degrees of node i and node j. This quantity is contained in $\left[-\frac{1}{2}, 1\right]$. When Q is equal to 1, there is a perfect partition because there are only intra-community links and there are not inter-community links. So, more Q is near 1, more the partition is worthy of attention.

2.2. Basic definitions of multiplex networks

The multilayer network, of which multiplex is a particular case, is a network made up by multiple layers each one representing a network with a given operation mode, social circle or temporal instance. In a multilayer network, each type of interaction between the nodes is described by a single layer network and the different layers of the network describe the different modes of interaction [10]. Multiplex is a particular case of multilayer network where all the layers are composed by the same nodes. Indeed, in our case there are the same 201 companies in all the layers of the networks.
In multiplex network, there are two types of links: intra-layer links, i.e. links between nodes of the same layer; inter-layer links, i.e. links between nodes of different layers. All these links must be considered in the analysis of the network. However, in our networks there are not inter-layer links because the layers describe properties which do not create this type of edges. So, we have only intra-layer links. Keeping in mind this last consideration, we describe the properties we want to analyze in the networks.

We start describing the centrality of the nodes in the multiplex network. We want to provide an extension of the notions of centrality we already introduced in the previous

section: degree centrality, betweenness centrality, closeness centrality and eigenvector centrality.
First, we consider degree centrality. The degree of node i in a multiplex network is a vector:

$$k_i = \left(k_i^{[1]}, ..., k_i^{[M]}\right) \tag{2.11}$$

where: M is the number of layers in the multiplex network; $k_i^{[\alpha]}$ is the degree of node i in layer α computed using equation (2.1). In our case, $M = 3$ because both the networks have 3 layers. This is a natural extension of the degree centrality, but we want to rank the nodes using this centrality and so we need to find an order in $\mathbb{R}^M$ that allows us to rank the nodes. So, once we find all the vectors k_i using (2.11), we define the overlapping degree of node i as:

$$o_i = \sum_{\alpha=1}^{M} k_i^{[\alpha]} \tag{2.12}$$

where: M is the number of layers in the network. We use this overlapping degree which corresponds to $o_i = \|k_i\|_1$. Notice that there are other aggregation measures that can be used, as the linear combination.
Now, we consider betweenness centrality. We want to extend (2.6) to a multiplex network. This measure of centrality is based on the concept of shortest path which is a metric structure of the network. To generalize the betweenness centrality we have to generalize the metric structure of the network. First of all, we notice that the length of a path in a multiplex network depends on the links we are considering. In our situation, the length of the walk depends only on the intra-layer links. Before giving the notion of shortest path in a multiplex network, we define what is a walk on a multiplex network. A walk is an alternating sequence of nodes and edges:

$$\left\{x_1^{[\alpha_1]}, l_1, x_2^{[\alpha_2]}, l_2, ..., l_{q-1}, x_q^{[\alpha_q]}\right\} \tag{2.13}$$

where: $x_1^{[\alpha_1]}$ is the starting node which is in layer α_1 and $x_q^{[\alpha_q]}$ is the ending node which is in layer α_q; l_1 is the link which connects $x_1^{[\alpha_1]}$ with $x_2^{[\alpha_2]}$ and, since there are only intra-layer links, is a link which is on layer α_2. Now, we can compute the shortest path through which two nodes are connected using the notion introduced in (2.13). To perform this computation in *Matlab*, we create a new network where the nodes are the 201 companies and the links are the ones of the layers. When there are two nodes which have a link in

more than one layer, we select only the link with the biggest weight, because the bigger the weight the closer the nodes. After having created this new network, we apply (2.6) to compute the betweenness centrality of the multiplex.
We can reason in a very similar way to extend the concept of closeness centrality. Indeed, closeness centrality is based on distance which is a metric structure of the network as the shortest path. So, we can proceed as before and we can compute the closeness centrality using (2.7) on the same network created to compute the betweenness centrality.
Finally, there is the eigenvector centrality. To generalize this measure of centrality we use an approach similar to the one used for degree centrality. Indeed, the simplest way to calculate eigenvector centrality in a multiplex network is to consider the eigenvector centrality separately in each layer. In this way, we have that the eigenvector centrality of node i is the following vector:

$$\gamma_i = \left(\gamma_i^{[1]}, ..., \gamma_i^{[M]}\right) \tag{2.14}$$

where: M is the number of layers and $\gamma_i^{[\alpha]}$ is the eigenvector centrality of node i in layer α computed using equation (2.8). Then, to obtain a numeric centrality measure, we use an aggregation function. As done for degree centrality we use the sum and we obtain:

$$\gamma_i = \sum_{\alpha=1}^{M} \gamma_i^{[\alpha]} \tag{2.15}$$

where: M is the number of layers in the network and $\gamma_i^{[\alpha]}$ is the eigenvector centrality of node i in layer α. Notice that there are many other aggregation functions that can be used.

Another quantity is the edge overlap. We want to understand if the number of overlapped links is not negligible with respect to the number of links in the layers.
We can define two types of edge overlap: the total overlap and the local overlap. The total overlap $O^{\alpha\beta}$ between two layers α and β is defined as the number of links that are in common between layer α and layer β:

$$O^{\alpha\beta} = \sum_{i<j} a_{ij}^{\alpha} \cdot a_{ij}^{\beta} \tag{2.16}$$

where: a_{ij}^{α} is an element of the adjacency matrix A^{α} and it is equal to 1 if there exists the link between node i and node j in layer α, it is 0 otherwise; a_{ij}^{β} is equivalent to the previous element but is on layer β instead of layer α. Moreover, we consider $i < j$ because

we have undirected networks and so we do not want to compute a link twice.
Sometimes it can be useful to compute also the local overlap $o_i^{\alpha\beta}$ which is the edge overlap of node i between layer α and layer β. This quantity is defined as the number of neighbours of node i that are neighbours in both layers α and β:

$$o_i^{\alpha\beta} = \sum_{j=1}^{N} a_{ij}^{\alpha} \cdot a_{ij}^{\beta} \tag{2.17}$$

where: N is the number of nodes in the network; a_{ij}^{α} is an element of the adjacency matrix A^{α} and it is equal to 1 if there is a link between node i and node j in layer α and 0 otherwise; a_{ij}^{β} is the same of the previous element but on layer β instead of layer α.
Both the measures of centrality for multiplex networks and edge overlap have been described using [7].

Another important concept is that of the multiplex participation coefficient [11]. It is a value computed for every node of the multiplex by using the following equation:

$$P_i = \frac{M}{M-1}\left[1 - \sum_{\alpha=1}^{M}\left(\frac{k_i^{[\alpha]}}{o_i}\right)^2\right] \tag{2.18}$$

where: M is the number of layers in the multiplex network; $k_i^{[\alpha]}$ is the degree of node i in layer α; o_i is the overlapping degree of node i computed using equation (2.12).
This value belongs to the interval $[0, 1]$ and it is computed for any node i of the multiplex. If $P_i = 1$, then all the links incident in node i are equally distributed across the layers; if $P_i = 0$, then node i is active only in one layer. Obviously, if there is an isolated node i in the multiplex network, P_i cannot be computed because $o_i = 0$.
Moreover, we can use P_i and o_i together to classify the nodes in: multiplex hubs (high P_i and o_i), focused hubs (high o_i and low P_i); multiplex leaves (low o_i and high P_i); focused leaves (low P_i and o_i).

Finally, we want to make a representation of the multiplex networks. To perform this representation, we use the projection network [7]. The projection network of the multiplex network W is the graph $proj(W) = (X_W, E_W)$ where:

$$X_W = \bigcup_{\alpha=1}^{M} X_{\alpha} \tag{2.19}$$

$$E_W = \left(\bigcup_{\alpha=1}^{M} E_\alpha \right) \bigcup \left(\bigcup_{\alpha,\beta=1,\alpha\neq\beta}^{M} E_{\alpha\beta} \right) \tag{2.20}$$

In particular, X_W is equal to any X_α because we are considering a multiplex network while E_W is equal to the sum of all the links that connect two nodes. In our case, the second element of (2.20) is always null because there are no inter-layer links.
In this way we obtain a new graph on which we can perform the same analysis we did for the single layers. Moreover, we can use the same formulas introduced before for the single layers (formulas (2.1) - (2.10)). However, we do not consider the equations (2.5), (2.6), (2.7) and (2.8) because we have already generalized the notions of centrality to the multiplex networks. We also compute the contribution that each layer gives to create the projection network: it is a percentage which is computed in the following way:

$$contribution(\alpha) = \frac{\sum_{i=1}^{N} s_i^{[\alpha]}}{\sum_{\beta=1}^{M} \sum_{i=1}^{N} s_i^{[\beta]}} \tag{2.21}$$

where: α is the considered layer; $s_i^{[\alpha]}$ is the strength of node i in layer α computed using (2.3); $s_i^{[\beta]}$ is the strength of node i in layer β; N is the number of nodes in the dataset; M is the number of layers of the multiplex network.

3 | Data

The dataset is composed by all the Italian companies listed in the Italian market stock exchange, Borsa Italiana. We got the firms in our dataset from [21]. Here there are all the companies which compose FTSE Italia All-Share index. This index is composed by all the companies which make up other three indexes: FTSE MIB, FTSE Italia Mid Cap and FTSE Italia Small Cap. There are 222 companies and all the data are taken as of end of 2020.

We collect data which refer to board of directors and its members for every company. To retrieve these data we search online two documents which each Italian company has to publish: "Relazione sul governo societario e gli assetti proprietari al 31 dicembre 2020" and "Relazione sulla politica in materia di remunerazione e sui compensi corrisposti 2021". In the first document, we recover all the data about board of directors and its composition; this document refers to 2020 and gives all the information till the end of the year. From this document we obtain the following quantities: number of directors and independent directors in the board, number of meetings of board of directors and of board of statutory auditors and of independent directors only, the duration of the meetings, the presence of pre-meeting advice about the arguments that will be discussed, board evaluation, independence criteria through which a director is defined independent, the presence and composition of some committees such as nomination committee, remuneration committee and ESG committee. In the second document, we recover all the data about members, in particular: name and role, age, birth's date, gender, remuneration and also if the member is an executive, not executive or independent member. All these information have been taken from the document which speaks about 2021 because there is a section which reports all the remuneration of the previous year in which we are interested in.

Finally, we have also data about the ownership of each company which have been taken from the official site of CONSOB.

There are some companies for which we are not able to find the two documents previously mentioned, so we decide not to consider these firms in our dataset. This is the reason for which our dataset is composed of 201 companies out of 222.

3.1. Descriptive statistics

Now we concentrate on the data used to create the multiplex network and we give some descriptive statistics of these data.

First we consider the interlocking phenomenon. In our sample of 201 companies, there are 1720 directors who occupy 1971 positions. So each director occupies on average 1,15 positions. Since the total number of directors is less than the available positions, there are some directors who sit on more than one board and so they create the so called interlocking network.

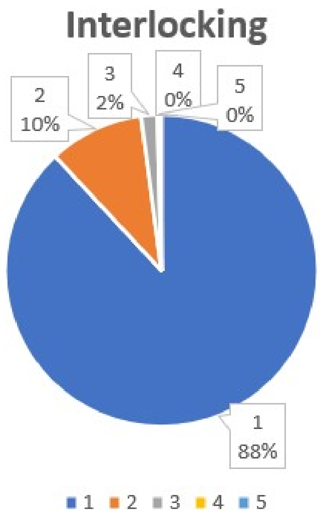

Figure 3.1: Percentage of directors who sits on a given number of boards

In Figure 3.1, it can be seen that the majority of directors (1513 directors, about $87,97\%$) sits on the board of only one company. Moreover, a relevant number of directors (171, about $9,94\%$) sits on the board of two companies. The maximum number of boards on which a director sits is 5, however there are only 2 directors who sit on 5 boards.
Moreover, the firms are divided in different sectors due to their activities. This could be interesting because we can analyze if the interlocking takes place in companies belonging to the same sector. In the dataset there are 11 sectors: Utenze (Utilities), Beni voluttuari (Luxury goods), Immobiliare (Real estate), Industria (Industry), Assistenza sanitaria (Healthcare), Finanza (Finance), Materie prime (Raw materials), Tecnologia (Technology), Prima necessità (Commodities), Energia (Energy) and Telecomunicazioni (Telecommunications). In Figure 3.2, it is shown how the firms are divided in these sectors.

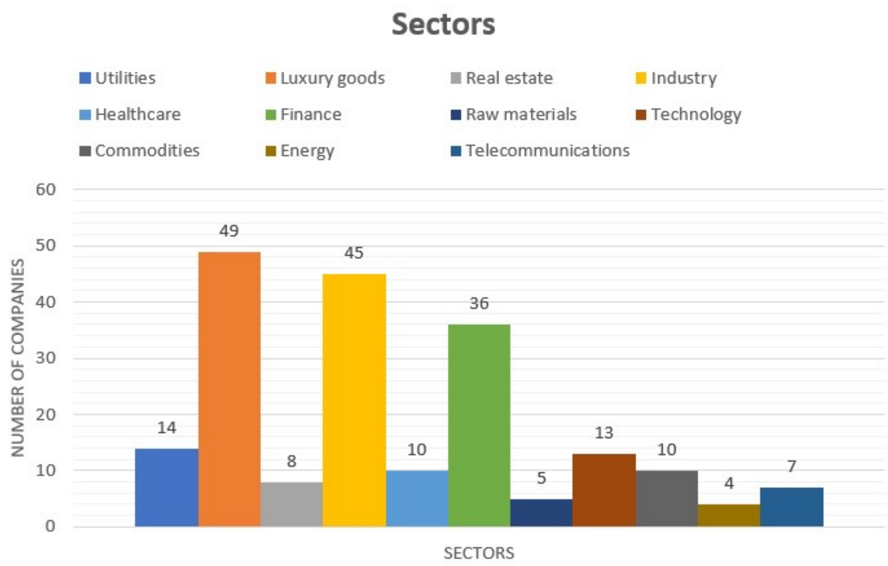

Figure 3.2: Number of companies in each sector

Boards	Utilities	Luxury goods	Real estate	Industry
1	138	433	63	415
2	3	24	4	8
3	0	0	0	1

Boards	Healthcare	Finance	Raw materials	Technology
1	90	393	42	100
2	1	9	0	3
3	0	0	0	0

Boards	Commodities	Energy	Telecommunications
1	85	38	65
2	1	0	0
3	0	0	0

Table 3.1: Interlocking phenomenon per sector

In Figure 3.2, it is pointed out that there are three sectors, Luxury goods, Industry and Finance, which contain the majority of companies (130 out of 201, around 64, 68%), while the other companies are distributed among the other sectors. In Table 3.1, it is illustrated how many positions a director occupies in the board of directors of companies which belong to the same sector.
In Luxury goods, Industry and Finance there is the biggest quantity of directors who sit on more than one board of companies of the same sector. This means that the majority of intra-sector interlocks happen in these sectors; however, this is also due to the fact that the majority of companies belongs to these sectors. Moreover, there is only one director who occupies 3 positions in the same sector and it is in Industry, while there is no director who occupies 4 or 5 positions in the same sector. This means that the few directors who occupy 4 and 5 positions are on board of companies which belong to different sectors. Moreover, the three smallest sectors, Raw materials, Energy and Telecommunications, are the unique sectors in which no director sits on more than one board.

In the following we consider all the quantities that will be used to create the governance index. These quantities are: board size, board independence, 'busy-ness' of corporate directors, gender diversity, number of minority directors. We describe all these quantities both in the general market and in each sector.

First, we concentrate on board size. Board size is the number of directors who compose the board and from Figure 3.3 we can see how many companies have a given number of directors.
There is only one company which has 2 directors: Lazio SPA. This is a particular company because it adopts a two-tier system. The two-tier board of directors is a system in which a company is governed by two distinct boards of directors, a management board and a supervisory board; management board is accountable to supervisory board and makes decisions related to operational and tactical direction of company while the supervisory board makes decisions about long-term strategic direction of business [9]. All the other companies have a one-tier system, i.e. there is a single body of directors that makes strategic decisions of a company, and they have all more than 2 members.
The maximum board size is 19 and it is achieved by 2 companies, Intesa San Paolo SPA and Unipol Gruppo Finanziario SPA. The mean is 9, 80 directors for each company.

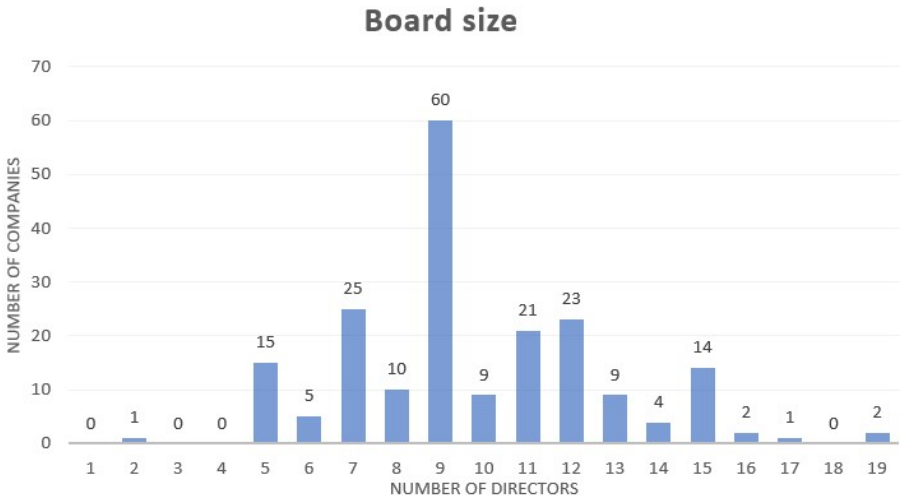

Figure 3.3: Number of companies with a given number of directors

Considering each sector, we obtain the following results:

Statistics	Utilities	Luxury goods	Real estate	Industry
Mean	10,29	9,84	8,88	9,60
Max	15	16	12	15
Min	5	2	5	5

Statistics	Healthcare	Finance	Raw materials	Technology
Mean	9,20	11,42	8,40	8,15
Max	15	19	11	11
Min	6	5	6	5

Statistics	Commodities	Energy	Telecommunications
Mean	8,60	9,50	9,29
Max	11	12	15
Min	5	8	5

Table 3.2: Board size per sector

From Table 3.2, it can be deduced that Finance is the sector which, in mean, has the biggest boards while Technology has the smallest boards. The two firms which have a board size of 19 directors are both in Finance, while Lazio SPA, which is the unique company with 2 directors, is in Luxury goods. Moreover, we notice that the three biggest sectors (Luxury goods, Industry and Finance) have three of the four biggest boards.

The second quantity used to create the governance index is board independence. Board independence is the percentage of independent directors in the board. The criteria through which a director is independent are defined in: "Relazione sul governo societario e gli assetti proprietari al 31 dicembre 2020"; in this document, it is expressed if the criteria used are the ones introduced by laws or if the company has introduced more restrictive criteria to define independent directors.
The highest percentage of independent directors is 90, 91% and it is reached by Finecobank, while the lowest percentage of independent directors is 0, 00% and it is reached by Banca Popolare di Sondrio, Caltagirone Editore, Caltagirone SPA and Sol SPA. Moreover, there is one company, Lazio SPA, for which we do not know the number of independent directors because it adopts a two tier-system.
The mean value of board independence is 48, 19%. It is possible to see the general situation in Figure 3.4.

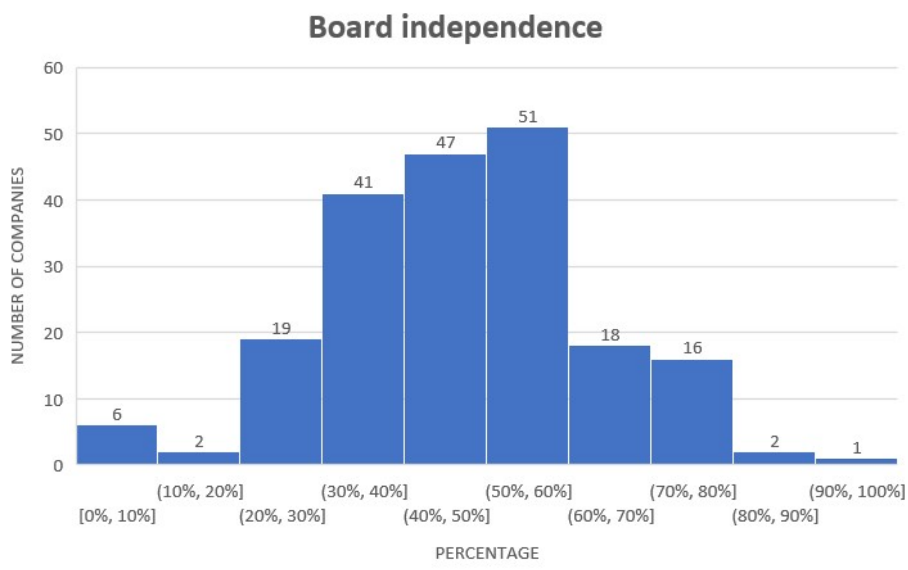

Figure 3.4: Number of companies with a given percentage of independent directors

The majority of companies has a percentage of independent directors contained in the interval [30%, 60%]. They are 139 companies out of 201, about 69, 15%.

The situation sector by sector is represented in Table 3.3.

Statistics	Utilities	Luxury goods	Real estate	Industry
Mean	58,36%	45,08%	48,24%	47,08%
Max	86,67%	72,73%	77,78%	77,78%
Min	33,33%	0,00%	22,22%	0,00%

Statistics	Healthcare	Finance	Raw materials	Technology
Mean	37,31%	56,51%	29,21%	45,89%
Max	66,67%	90,91%	57,14%	66,67%
Min	25,00%	0,00%	0,00%	25,00%

Statistics	Commodities	Energy	Telecommunications
Mean	40,77%	48,61%	50,76%
Max	55,56%	55,56%	80,00%
Min	20,00%	33,33%	38,46%

Table 3.3: Board independence per sector

There are only three sectors, Utilities, Finance and Telecommunications, which have boards with board independence bigger than 50% while all the others are below this value. However, there is at least one company per sector which has board independence bigger than 50%, indeed the maximum is bigger than this value for each sector. Finally, there are also four different sectors which have companies with no independent director in their board: Luxury goods, Industry, Finance and Raw materials. Notice that the first three sectors are the market's biggest sector.

The third quantity used to create the governance index is the 'busy-ness' of corporate directors. This quantity is represented by the number of other tasks that each director has to do besides being in the board. We consider as other task every type of work that each director has to do, inside or outside the firm, which is not related with the board of directors' activities. To compare this value between different companies, we consider the mean value of other tasks that each director of the considered company has to do. If the number is bigger than 1 then each director has to do at least one other task, inside or outside the firm, and so have less time to dedicate to the considered company. In particular, the higher the number the fewer the time that the board of directors dedicates to the company. When this information is not available, we decide to put this value equal to 0. The situation is represented in Figure 3.5.

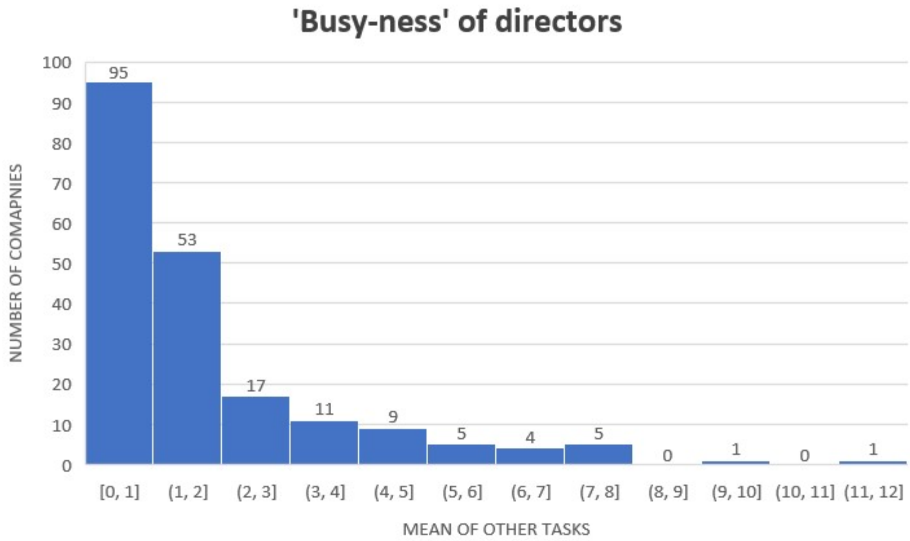

Figure 3.5: Number of companies with a given mean value of other tasks

The maximum value of other tasks per director is $11,92$ and it is reached by Azimut Holding SPA; here the directors have few time to dedicate to the management of the firm. However, there are 24 companies in which this value is equal to 0; in these companies (as Alkemy SPA, Banca Popolare di Sondrio, Titanmet, Valsoia SPA) the directors are fully concentrated on the management of the company and dedicate a lot of time to it. The mean value of 'busy-ness' is equal to $1,76$ other tasks per director of each company.
The situation sector by sector is represented in Table 3.4.
There are only two sectors in which each director has more than 2 other tasks to do: Real estate and Technology. The directors of these sectors have fewer time to dedicate to the firm than the directors of the other sectors. Moreover, there are only two sectors in which each director has less than 1 task to do: Energy and Telecommunications. These directors are the ones who can dedicate the biggest amount of time to the management of the company. However, for almost each sector there is at least a company in which the directors have no other task to do and are fully dedicated to the management of the firm. Indeed, the minimum value is $0,00$ for almost any sector.

Statistics	Utilities	Luxury goods	Real estate	Industry
Mean	1,10	1,91	2,12	1,94
Max	4,44	9,36	6,89	7,86
Min	0,10	0,00	0,00	0,00

Statistics	Healthcare	Finance	Raw materials	Technology
Mean	1,71	1,75	1,66	2,37
Max	5,53	11,92	4,33	7,75
Min	0,00	0,00	0,00	0,00

Statistics	Commodities	Energy	Telecommunications
Mean	1,55	0,45	0,55
Max	7,29	1,00	1,44
Min	0,00	0,08	0,00

Table 3.4: 'Busy-ness' of directors per sector

The fourth quantity used to create the governance index is gender diversity. We express this quantity as the percentage of women in the board. By Italian laws, the percentage of the least represented sex in the board must be of at least $40,00\%$ but there are many companies which do not satisfy this requirement, as it can be seen from Figure 3.6.
Moreover, the majority of companies, 171 companies out of 201, about $85,07\%$, has a percentage contained in $[30\%, 50\%]$. However, a big quantity of companies does not satisfy the requirement of $40,00\%$ and there are also some companies, as D'Amico and Lazio SPA, which have no women in the board. Remember that the situation of Lazio SPA is particular because it is the unique society of the dataset which has a two-tier system.
The mean value of women's percentage is $38,65\%$. This value is below the threshold imposed by laws, this highlights how Italian firms have to work to increase the number of women in the board.

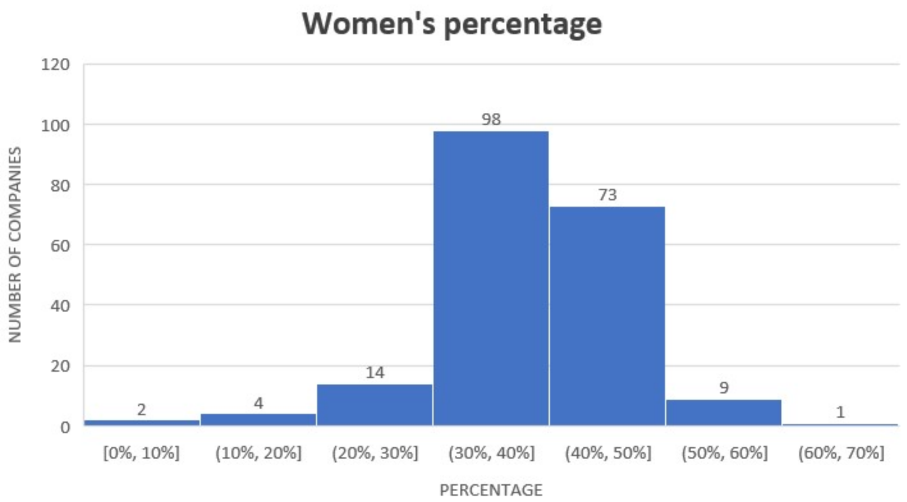

Figure 3.6: Number of companies with a given percentage of women directors

In Table 3.5, it is represented the situation sector by sector:

Statistics	Utilities	Luxury goods	Real estate	Industry
Mean	41,72%	38,19%	41,04%	37,91%
Max	60,00%	55,56%	45,45%	58,33%
Min	33,33%	0,00%	33,33%	0,00%

Statistics	Healthcare	Finance	Raw materials	Technology
Mean	33,78%	39,85%	39,48%	36,19%
Max	54,55%	63,64%	54,55%	45,45%
Min	14,29%	22,22%	33,33%	20,00%

Statistics	Commodities	Energy	Telecommunications
Mean	38,71%	37,15%	43,42%
Max	44,44%	44,44%	57,14%
Min	25,00%	33,33%	33,33%

Table 3.5: Women's percentage per sector

Except for Utilities, Real estate and Telecommunications, all the other sectors have a mean value of women's percentage below the value indicated by Italian laws. Moreover, every sector has at least one company which has a value below the threshold indicated by Italian laws. So all the sectors have to work and increase the presence of women in the boards. However, in each sector there are companies which satisfy the requirement. Notice that in Finance sector there is a company, Finecobank, which has a women's percentage of 63, 64%; this firm does not satisfy the requirement indicated by Italian laws because it has a men's percentage less than 40, 00%. However, this is the unique company to have this problem. So we can conclude that in the Italian market we have to work a lot to increase the quantity of women in the board.

Finally, the last quantity used to create the governance index is minority directors. This quantity is represented by the percentage of minority directors in the board. This is an important quantity because minority directors are members linked to minority shareholders and so they represent their interests in the company. So, the higher the percentage of minority directors the better the minority shareholders' interests are represented.

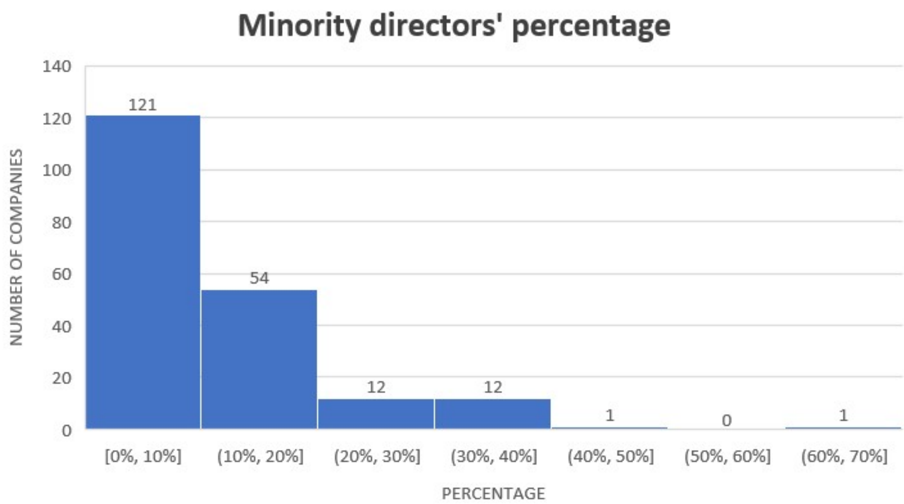

Figure 3.7: Number of companies with a given percentage of minority directors

From Figure 3.7, it is possible to see how more than 50, 00% of the companies have a percentage of minority directors included in [0%, 10%]. In particular, between the 121 companies included in this interval, there are 91 companies which have no minority director (about 45, 27% of the total); this means that in many companies there is no director which represents minority shareholders' interests.
There is only one company which has a percentage of minority directors bigger than 50%;

this firm is ENEL and has $66,67\%$ of minority directors. The mean value of minority directors in the board is $9,06\%$.
In Table 3.6, it is illustrated the situation for each sector.

Statistics	Utilities	Luxury goods	Real estate	Industry
Mean	20,08%	6,97%	14,39%	8,53%
Max	66,67%	27,27%	36,36%	33,33%
Min	0,00%	0,00%	0,00%	0,00%

Statistics	Healthcare	Finance	Raw materials	Technology
Mean	3,86%	11,54%	4,04%	3,53%
Max	12,50%	40,00%	11,11%	22,22%
Min	0,00%	0,00%	0,00%	0,00%

Statistics	Commodities	Energy	Telecommunications
Mean	5,87%	18,75%	6,47%
Max	33,33%	33,33%	23,08%
Min	0,00%	0,00%	0,00%

Table 3.6: Minority directors' percentage per sector

The sectors in which the minority shareholders' interests are better represented are: Utilities, Energy, Real estate and Finance. All these sectors have a mean percentage bigger than $10,00\%$, while all the other sectors are below this value. Utilities has the biggest mean percentage; this is due to the fact that Enel, which is the firm with the biggest minority directors' percentage, belongs to this sector and so increases the mean. Finally, we notice that all the sectors have at least a company in which the minority shareholders' interests are not represented.

4 | Layers' analysis

In this chapter we construct and analyze all the layers of the multiplex network.
Note that all the properties are described using *Matlab* and *Gephi*. Moreover, there is a graphic representation of all the layers in Appendix A.

4.1. Layer 1: direct interlocking

First of all, we construct the interlocking layers. We recall that the interlocking is defined as the situation where a person affiliated with one organization sits on the board of directors of another organization [24].
We construct two different layers of interlocking: the first layer considers the phenomenon of direct interlocking while the second layer is built using the indirect interlocking. We recall that all the layers are constituted by the same nodes and these nodes represent the Italian listed companies for which we find the data.

In the first layer we have a weighted undirected network. There is a link between two nodes if there is at least one director who sits on both the board of directors of the considered companies. It is possible to realize immediately that the network is undirected; indeed, if there is a link between node i and node j then there is at least one director in common between the two companies and so there is also a link between node j and node i. Moreover, there is a weight on the links; the weight represents the number of directors that are in common between the two companies and so it is an integer. However, we want to have all the weights of the network contained in $[0, 1]$ because in this way it is easier to compare all the layers that we will introduce in the next sections. To do this, we normalize all the weights by dividing every weight for the maximum weight. The maximum weight is 9 and it is between Class Editori SPA and Compagnia Immobiliare Azionaria; this means that there are 9 directors of the board of Class Editori SPA who belong to the board of Compagnia Immobiliare Azionaria (and viceversa).

Now, we can describe some properties of the network on the first layer.
We start computing the degree distribution. We compute the degree of each node using (2.1) and then we are able to construct the degree distribution of the layer. The degree distribution is represented in Figure 4.1. From Figure 4.1, we can notice that there are 38 nodes which are isolated, i.e. they have degree 0 because they are not connected with any other node. The maximum degree is 13 and it is reached by OVS SPA. OVS SPA is the most connected company in the network. The majority of the companies has a degree smaller or equal than 2, indeed there are 36 companies with degree 1 and 38 companies with degree 2. So, there are 112 companies (about $55,72\%$) which have degree smaller or equal than 2.

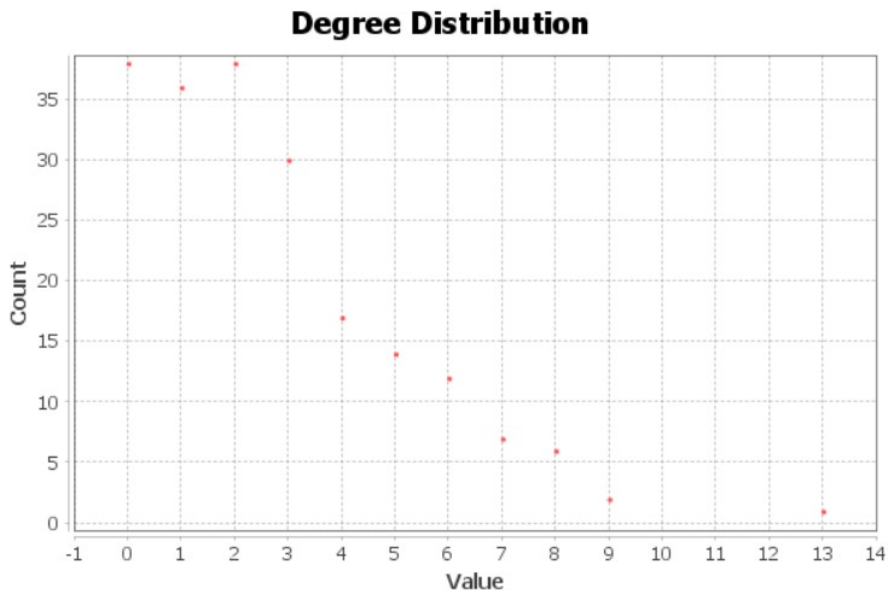

Figure 4.1: Degree distribution of Layer 1

As confirmed by Figure 4.1, there are many nodes which are scarcely connected while there are few largely connected nodes. So, even if the range of degrees is small, we can say that the degree distribution can be compared to that of a scale-free network.
The average degree computed using (2.2) is equal to $2,69$.
Since the layer has a weighted network we can compute the weight of each node using (2.3) and we obtain the average weighted degree using (2.4). This quantity is equal to $0,34$.
Then we can compute the centrality of each node. As previously said, we have different types of centrality which give different notions of importance of the nodes in the network. For this first layer, the nodes with the lowest importance are the 38 isolated nodes. However, it could be interesting to see which are the most important nodes in the network

and how these nodes change by considering different notions of centrality. So, using (2.5), (2.6), (2.7) and (2.8), we compute the centrality of each node. In Table 4.1 we report the ten most important nodes due to the considered notion of centrality.

Degree	Betweenness
OVS SPA	OVS SPA
ATLANTIA SPA	ATLANTIA SPA
ERG SPA	ERG SPA
CERVED GROUP SPA	CERVED GROUP SPA
EUROTECH	BPER
INTERPUMP GROUP SPA	TELECOM ITALIA
NEXI SPA	EUROTECH
TAMBURI INVESTMENT SPA	DIASORIN
TELECOM ITALIA	NEXI SPA
AMPLIFON	INTERPUMP GROUP SPA

Closeness	Eigenvector
ERG SPA	CLASS EDITORI
CERVED GROUP SPA	COMPAGNIA IMMOBILIARE AZIONARIA
OVS SPA	TOD'S
ATLANTIA SPA	INTEK GROUP
TAMBURI INVESTMENT SPA	RCS MEDIAGROUP SPA
DIASORIN	SERVIZI ITALIA SPA
EUROTECH	MONCLER SPA
AMPLIFON	PININFARINA SPA
NEXI SPA	ASSICURAZIONI GENERALI
TELECOM ITALIA	IMMSI SPA

Table 4.1: Ten most important companies of Layer 1 for each centrality

From Table 4.1 we can immediately notice that there are nodes which are in the first places of importance for some measures of centrality and they are not in the ten most important nodes of other measures of centrality, for example OVS SPA, Atlantia SPA, Erg

SPA and Cerved Group SPA are in the first ranks of degree, betweenness and closeness centrality but they are not in eigenvector centrality. However, we can see that the first three notions of centrality give similar results, while eigenvector centrality gives different results. In particular, it is interesting to notice that all the companies in the top ten of eigenvector centrality do not appear in the other measures of centrality; this means that these companies are related with the most important nodes in the network and, maybe, are related one with the others.
Another quantity we can compute is the density of links in the network. Using (2.9), we obtain a value equal to 0,013. This means that the number of links in the network is very small with respect to the possible links in the network.
Moreover, in this network there is a giant component which contains 152 nodes and 263 links. So, this component contains the $75,62\%$ of nodes and $97,41\%$ of links of the network. In this layer there are 44 components: 38 components are constituted by only 1 node and they are the isolated nodes, 1 component is the giant component and there are other 5 components which are very small.
Finally, partitioning the nodes in the sectors which compose the market and using equation (2.10), we can compute the modularity coefficient related to this partition. We obtain a very low value of $0,0042$. This is due to the fact that there are some sectors, Raw materials, Energy and Telecommunications, which do not have direct interlocking inside. Moreover, this value underlines that the partition in sectors is not a good partition of the network because there are more inter-community links than intra-community links.

4.2. Layer 2: indirect interlocking

The second layer is characterized by an interlocking layer which is built using the indirect interlocking. The indirect interlocking is a phenomenon different from the direct interlocking. In particular, there is indirect interlocking between two companies when there are two directors, one on the board of directors of one company and one on the board of the other company, who sit together on the board of directors of a third company. Here there is indirect interlocking because there is no director who sits on both the board of directors but there are two, or more, directors of the two companies who sit on the board of a third company. So, the directors establish a relationship and so we can say that the two companies are indirectly linked. Notice that a director can have both direct and indirect interlocking.
Also in this situation, we obtain a weighted undirected network. By the description above, we can realize that the network is undirected because if company i is linked to company

j by indirect interlocking, then the viceversa holds. Moreover, the network is weighted and the weight is an integer number equal to the number of directors who have indirect interlock. As in the previous case, we want to have all the weights included in $[0, 1]$ and so we normalize by dividing every weight by the maximum weight. The maximum weight is 9 and it is between Aedes Siiq SPA and Banca Profilo and between Banca Generali SPA and Caltagirone Editore: this means that there are 9 directors who have indirect interlocking between these two couples of companies.

Now, we describe some properties of the network on the second layer.
We start computing the degree distribution. We compute the degree of each node using (2.1) and then we are able to construct the degree distribution of the layer. The degree distribution is represented in Figure 4.2. We notice that there are 46 isolated nodes and the maximum degree is bigger than the one in Layer 1 and it is reached by OVS SPA. So, we can notice that the same company has the maximum degree in the first two layers. However, the degree distribution is not similar to the one of a scale-free network; indeed, the degree distribution decreases too slowly towards 0 and so we cannot say that there are few largely connected nodes in the network.

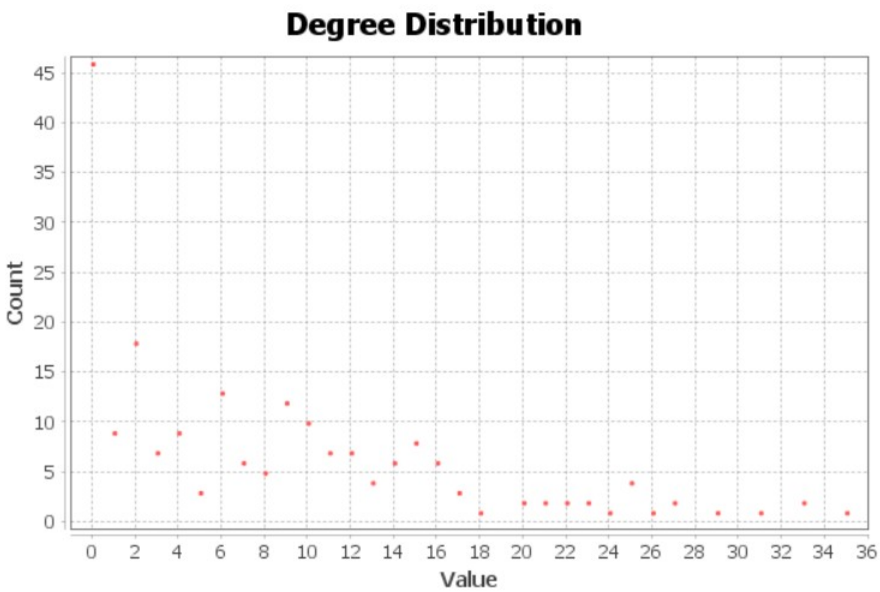

Figure 4.2: Degree distribution of Layer 2

The average degree is computed using (2.2) and it is equal to $7,98$.
Since we have a weighted network, we can compute the strength of a node using (2.3) and then, using (2.4), we obtain the average weighted degree which is equal to $1,10$.
Then we can compute the centrality of each node. To perform this computation we use

the formulas (2.5), (2.6), (2.7) and (2.8). We know that the least important nodes are the 46 isolated nodes but we want to know which are the most important nodes. So, as done for Layer 1, in Table 4.2 we represent the ten most important nodes due to the different notions of centrality.

Degree	Betweenness
OVS SPA	UNIPOL GRUPPO FINANZIARIO
CERVED GROUP SPA	ATLANTIA SPA
ERG SPA	CERVED GROUP SPA
ATLANTIA SPA	ERG SPA
AMPLIFON	OVS SPA
EUROTECH	BPER
TAMBURI INVESTMENT SPA	ACEA SPA
MONCLER SPA	TOD'S
DIASORIN	RCS MEDIAGROUP SPA
ELICA	CALTAGIRONE SPA

Closeness	Eigenvector
UNIPOL GRUPPO FINANZIARIO	TOD'S
CERVED GROUP SPA	INTEK GROUP
ATLANTIA SPA	CLASS EDITORI
RCS MEDIAGROUP SPA	COMPAGNIA IMMOBILIARE AZIONARIA
ERG SPA	RCS MEDIAGROUP SPA
SABAF SPA	UNIPOL GRUPPO FINANZIARIO
TOD'S	TAMBURI INVESTMENT SPA
UNIPOLSAI SPA	MONCLER SPA
BORGOSESIA	OVS SPA
BUZZI UNICEM SPA	CERVED GROUP SPA

Table 4.2: Ten most important companies of Layer 2 for each centrality

In Table 4.2 we can see how the ranks change due to the considered centrality. In particular, OVS SPA is the most important company with respect to degree centrality but loses positions in the other measures of centrality because the weights have a very important role. The role of the weights is confirmed also because there are companies, as Unipol Gruppo Finanziario SPA and Tod's, which are not in the top ten of degree centrality but they are in the top ten of all the other measures of centrality. There are also some companies that appear only in one rank, as Borgosesia and Buzzi Unicem SPA which appear only in closeness centrality. In this second layer there are important nodes which do not appear in the first layer, as Moncler SPA, Unipol Gruppo Finanziario SPA and RCS Mediagroup SPA. Moreover, these companies have an important role in this layer; indeed, Unipol Gruppo Finanziario SPA is in the first place in betweenness and closeness centrality and sixth in eigenvector centrality. In particular, when we consider betweenness and closeness centrality, the most important nodes in Layer 1 are not the same of Layer 2; in degree and eigenvector centrality the differences are less evident.

Another quantity we can compute is density. The links are more than the fisrt layer and so the density is higher than before. Using (2.9) we have that the density is $0,04$. Even if the value is bigger than before, the number of links in the network is very small with respect to the possible links.

Moreover, in the network there is a giant component which is composed by 152 nodes and 799 links. In particular, in the giant component is contained the $75,62\%$ of all the nodes and the $99,63\%$ of all the links. Here the giant component has the same size of the giant component in first layer. In the second layer there are more components; indeed, there are 48 components: 46 are isolated nodes, the giant component and 1 remaining component which is very small.

Finally, using (2.10), we can compute the modularity of this layer. As for Layer 1, the value of the modularity is very small, even negative, and this means that the sectors' partition is not a good partition for this layer. Indeed, there are many sectors, Real estate, Healthcare, Finance, Raw materials, Technology, Commodities, Energy and Telecommunications, which have not indirect interlock inside. So, there are more inter-community indirect interlocks than intra-community indirect interlocks.

4.3. Layer 3: governance layer

Finally, we construct the governance layer. The governance layer is built using the governance index which is made by the following quantities: board size, board independence, 'busy-ness' of corporate directors, gender diversity and minority directors.

We construct two different third layers and so we will have two multiplex networks to study.

We start by considering the first network.
We consider a similarity index between every couple of nodes. For every couple of nodes we want a value included in $[0, 1]$, where the value is equal to 1 if the nodes have the same characteristics, while it is 0 if the nodes are completely different. To obtain this value we use the following formula:

$$Similarity\_Index(i, j) = 1 - Distance(i, j) \tag{4.1}$$

where

$$Distance(i, j) = \frac{1}{5} \cdot Size + \frac{1}{5} \cdot Independence + \frac{1}{5} \cdot Busyness + \frac{1}{5} \cdot Women + \frac{1}{5} \cdot Minority$$

In particular: $Size$ is the normalized absolute difference between the number of directors in the board; $Independence$ is the normalized absolute difference between the percentage of independent directors in the board; $Busyness$ is the normalized absolute difference between the mean task of directors in the board; $Women$ is the normalized absolute difference between the percentage of women directors in the board; $Minority$ is the normalized absolute difference between the percentage of minority directors in the board.
In this way we obtain a completely connected network because every node has a similarity index with respect to any other node in the network. Moreover, this is a symmetric matrix because the normalized distance is a symmetric function. However, we do not want a connected network because we want to compare this layer with the first two layers which are not completely connected networks. Moreover, we want to consider only the relevant relations which are the ones with the biggest weights. So, we need to find a threshold in such a way that allows us to achieve our targets. After having identified the threshold, we normalize the remaining values to obtain values contained in $[0, 1]$.
We choose a value for the threshold equal to $0,945$. The value is pretty high but we need to pass from a network where all the nodes have degree 200 to a network where the degree distribution is significantly smaller. The degree distribution we obtain by imposing this threshold is represented in Figure 4.3. We choose this value because we want to obtain a degree distribution which can be compared to the one of the first layer, represented in Figure 4.1, and so to the scale-free network. Indeed, in Figure 4.3 we can see that there are many nodes with low degree (102 nodes have degree smaller or equal than 3, around $50,75\%$) and there are few nodes which have a big degree.

So, considering the threshold imposed before, we can analyze the properties of this third layer.
We have already shown the degree distribution, so applying (2.2) we obtain the average

degree which is equal to 5, 43. Moreover, since we have a weighted network, applying (2.3) and (2.4), we obtain the average weighted degree which is equal to 1, 43. Both the values are bigger than the one obtained in the first layer, in particular the average weighted degree is substantially high. This shows that there could be some differences in this network with respect to the first two networks obtained before.

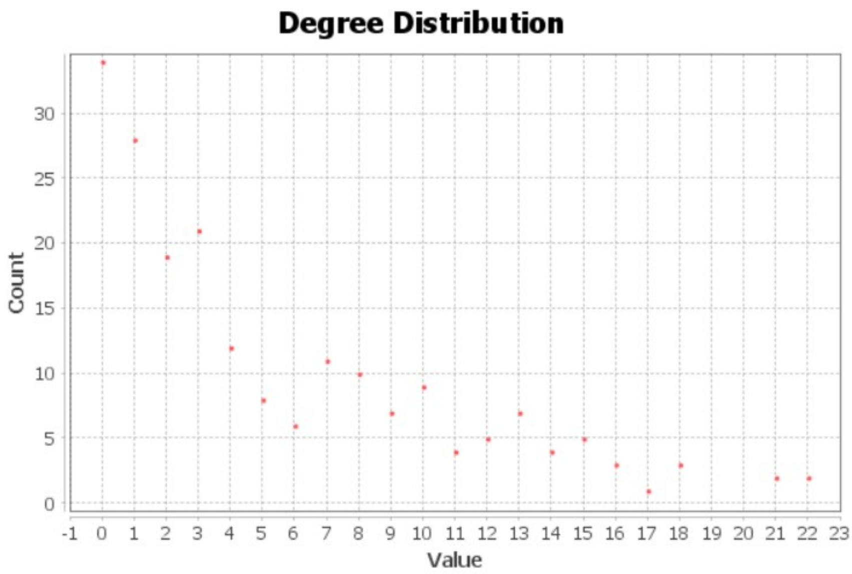

Figure 4.3: Degree distribution of the governance layer associated with first distance

Now, we compute the centrality of each node. To perform this computation we use the equations: (2.5), (2.6), (2.7) and (2.8). As done before, we do not consider the least important nodes because there are 34 isolated nodes and so we consider only the ten most important nodes for each measure of centrality. We report the ranks in Table 4.3. First of all, we notice that the companies in the ranks are different from the ones of the previous layers and this is relevant because it means that there are different companies which occupy the top ten of interlocking layers and governance layer. There is only one company, OVS SPA, which appears in all the layers; indeed, it is in closeness centrality of this layer and also comes to view in degree centrality, betweenness centrality and closeness centrality of Layer 1 and in degree centrality, betweenness centrality and eigenvector centrality of Layer 2. Moreover, we can make the same considerations we already did for the other layers. There are some companies which are in top ten for some measures of centrality and they are not for other measures, such as Giglio Group SPA, Netweek SPA and Ross SPA which are present only in eigenvector centrality; this means that all these companies are related with the most important nodes in the network. Another interesting notation is that the first four nodes of degree centrality are the same of eigenvector centrality and the first two vertices are in the same position. This means that Fullsix and Piquadro SPA, which are the most connected companies of the network, are also related with the most important nodes in the network.

Degree	Betweenness
FULLSIX	LA DORIA SPA
PIQUADRO SPA	JUVENTUS FOOTBALL CLUB
AMBIENTHESIS	ILLIMITY BANK
SOFTLAB	BREMBO
EXPRIVIA	FALCK RENEWABLES
GAS PLUS	PRYSMIAN SPA
JUVENTUS FOOTBALL CLUB	AEFFE SPA
LA DORIA SPA	DE LONGHI
GEFRAN SPA	TAS
NEODERTECH SPA	BANCA FARMAFACTORING SPA

Closeness	Eigenvector
LA DORIA SPA	FULLSIX
AEFFE SPA	PIQUADRO SPA
GEFRAN SPA	SOFTLAB
TAS	AMBIENTHESIS
BANCA FARMAFACTORING SPA	GEQUITY
JUVENTUS FOOTBALL CLUB	GIGLIO GROUP SPA
ILLIMITY BANK	NETWEEK SPA
OVS SPA	ROSS SPA
BANCA GENERALI SPA	LA DORIA SPA
FULLSIX	GEFRAN SPA

Table 4.3: Ten most important companies of Layer 3 - Network 1 for each centrality

Using (2.9), we compute the graph density of the layer which is equal to $0,027$. This is a low value and it is very similar to the value in the first layer, so we can make the same considerations.

In this layer, we find a giant component which is smaller than the ones find in the interlocking layers. It is composed by 141 nodes and 521 edges which correspond respectively to the $70,15\%$ of all the nodes and to the $95,25\%$ of the total links. Despite the giant component is smaller than the first two layers, it is sufficiently big to not have other comparable components in the network. In this layer there are 42 components: 34 components are isolated nodes, 1 is the giant component and 7 are smaller components.
Finally, using (2.10), we compute the modularity of the network and we obtain a negative number. As for interlocking layers, this means that the partition in sectors of the network is not a good partition. Indeed there are some sectors, Utilities, Real estate, Healthcare, Raw materials, Technology, Commodities, Energy and Telecommunications, which have no link inside. Notice that the modularity is bigger than the second layer. In conclusion, there are more inter-community links than intra-community links.

Now, we consider the second network.
This network is made by the first two interlocking layers and a third layer built on the governance index but different from the one just described. To consider a different third layer, we use a linear combination where the weights of the variables are no more equal. In particular, we decide to assign the maximum weight to the variable with the biggest variance and then we proceed in a decreasing order assigning the second maximum weight to the variable with the second biggest variance and so on. Before computing the variance, we decide to normalize the variables by dividing every element of the variable with the maximum value; in this way we obtain comparable results. We decide to use this approach to see what changes in the network. In particular, we want to see if the distance, and so the similarity index, is influenced by the variable with the biggest variance or if the other variables rebalance the situation and there is a result similar to the one obtained in the first network. To construct the new similarity index, we always use equation (4.1) where:

$$Distance(i,j) = \frac{2}{5} \cdot Independence + \frac{3}{10} \cdot Busyness + \frac{3}{20} \cdot Minority + \frac{1}{10} \cdot Size + \frac{1}{20} \cdot Women$$

where: $Independence$ is the normalized absolute difference between the percentage of independent directors in the board; $Busyness$ is the normalized absolute difference between the mean task of directors in the board; $Minority$ is the normalized absolute difference between the percentage of minority directors in the board; $Size$ is the normalized absolute difference between the number of directors in the board; $Women$ is the normalized absolute difference between the percentage of women in the board.
After having computed the similarity index for every couple of nodes, we can make the same considerations done before and then proceed to the choice of the threshold. We

choose a value for the threshold equal to $0,945$. Notice that we choose a value equal to the one before even if we are considering a different similarity index. We make this choice because this value allows us to obtain a degree distribution which can be compared to the ones of the interlocking layers and it allows us to consider only the relevant edges. The degree distribution is represented in Figure 4.4. The degree distribution obtained with this second value is different with respect to the previous one (Figure 4.3), indeed the maximum degree is higher and there is no similarity with the degree distribution of a scale-free network. Indeed, there are 81 nodes with degree lower or equal than 3 (they correspond to the $40,30\%$ of all the nodes) and the number of nodes with high degrees does not decrease so quickly as in scale-free networks.
The degree distribution is built using (2.1) to compute the degree of the nodes. Applying (2.2), we obtain the average degree of the network which is equal to $7,88$. This value is bigger than before. Moreover, reasoning in the same way and applying (2.3) and (2.4), we obtain the average weighted degree which is equal to $2,23$.

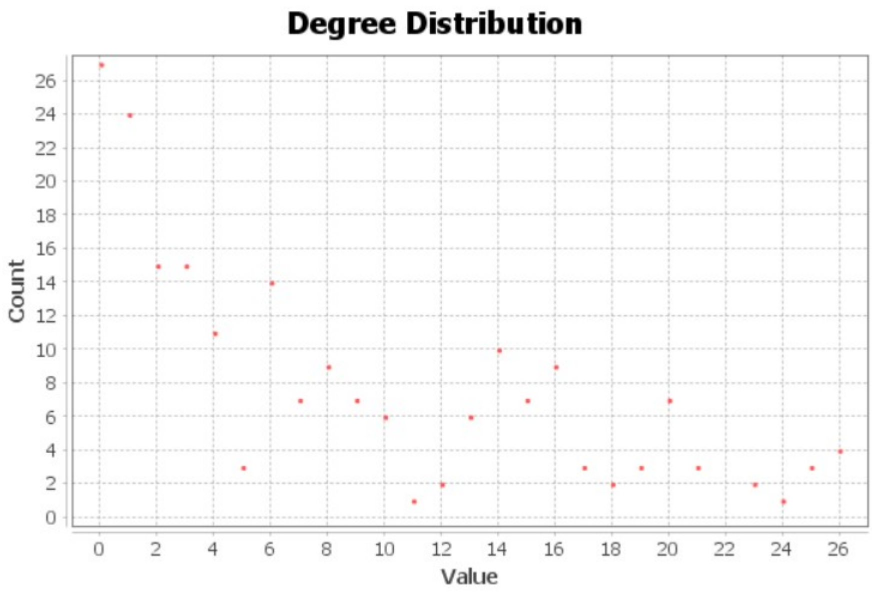

Figure 4.4: Degree distribution of the governance layer associated with second distance

Now, we can analyze the centrality. We use formulas (2.5), (2.6), (2.7) and (2.8) to compute the different measures of centrality. Then, in Table 4.4, we show the ten most important nodes for each notion of centrality. As before we study only the most important nodes because the least important are the 27 isolated nodes. We can immediately notice that the first four nodes of degree centrality and of eigenvector centrality are the same of the previous governance layer. Moreover, the first four nodes of eigenvector centrality are in the same order as before. Also in closeness centrality there are some similarities with the previous layer and the company in the first position is always the same. The things are a little bit different in betweenness centrality. In this layer, OVS SPA does not appear anymore but there is Erg SPA which occupies one of the first positions in betweenness centrality. This is the unique node which is also in the first two layers. However, in every notion of centrality there is at least one node which does not appear in any other top ten of the previous layers.

Degree	Betweenness
AMBIENTHESIS	TAS
FULLSIX	EXPRIVIA
PIQUADRO SPA	DE LONGHI
SOFTLAB	ERG SPA
JUVENTUS FOOTBALL CLUB	LA DORIA SPA
LA DORIA SPA	MONRIF SPA
PHARMANUTRA SPA	LUVE SPA
DIGITAL BROS	GAS PLUS
ALKEMY SPA	IL SOLE 24 ORE
GEFRAN SPA	FALCK RENEWABLES

Closeness	Eigenvector
LA DORIA SPA	FULLSIX
EXPRIVIA	PIQUADRO SPA
GEFRAN SPA	SOFTLAB
JUVENTUS FOOTBALL CLUB	AMBIENTHESIS
GAS PLUS	GEFRAN SPA
TAS	LA DORIA SPA
FULLSIX	PHARMANUTRA SPA
PIQUADRO SPA	TISCALI
BANCA FINNAT EURAMERICA	JUVENTUS FOOTBALL CLUB
BF SPA	GEQUITY

Table 4.4: Ten most important companies of Layer 3 - Network 2 for each centrality

In this layer, the graph density, computed using (2.9), is equal to $0,039$. This value is comparable with the one obtained in Layer 2.
Moreover, in this layer there is a giant component composed by 141 nodes and 757 edges, which correspond to the $70,15\%$ of all the nodes and to the $95,58\%$ of the total links. Notice that the giant component is constituted by the same number of nodes of the giant component of Layer 3 associated to the first network. In this layer there are 35 components: 27 components are the isolated nodes, 1 is the giant component and the remaining 7 components are not comparable with the giant component.
Finally, using (2.10), we compute the modularity of the network considering the sectors as partition. Also here we obtain a negative number and this implies that this is not a good partition. The sectors which have no link inside are: Real estate, Raw materials and Energy. In conclusion, there are more inter-community links than intra-community links.

4.4. Sectors

Now, we want to create the layers for each sector. As previously said, in the Italian market there are 11 sectors and for each sector we want to build the three layers. For each layer, we report data about: Average degree, Maximum degree, number of links between the nodes of the sector (L) and Density of the network. To compute Average degree we use (2.2) and to compute density we use (2.9). Then, we compare these data between the different sectors and also with the layers composed by all the nodes.

4.4.1. Layer 1

We start with the first layer, direct interlocking. In Table 4.5, are collected all the properties for each sector.
First of all, we concentrate on the maximum degrees reported in Table 4.5. The highest maximum degree is 3 and it is reached by Luxury goods and Industry. This is a very small value with respect to the one obtained considering all the nodes (the value in the whole layer is 13). Moreover, OVS SPA, which is the node with the maximum degree, in its sector has a degree equal to 2. So, this means that OVS SPA, and all the other nodes, have many links with nodes which do not belong to the same sector.
This last consideration can be verified also considering the number of links which are inside the sectors. Summing all these values, there are only 45 links which are inside the sectors, they correspond to the $16,67\%$ of all the links in the network. So, this confirms the fact that the nodes connect themselves with many nodes which are outside their sec-

tor.
Regarding the density of the links, there are both sectors which have a density higher than the whole layer, such as Utilities, Luxury goods and Real estate, and sectors which have density smaller than the whole layer, such as Finance and Industry. The sectors in which the density is bigger than the whole network are the sectors in which the direct interlocking phenomenon is more widespread and so the agencies of these sectors communicate and influence one with the other using direct interlocking more than the other companies.

Properties	Utilities	Luxury goods	Real estate	Industry
Avg. Degree	0,4286	0,7755	0,5000	0,4889
Maximum Degree	2	3	1	3
L	3	19	2	11
Density	0,0330	0,0162	0,0714	0,0111

Properties	Healthcare	Finance	Raw materials	Technology
Avg. Degree	0,2000	0,2778	0	0,4615
Maximum Degree	1	1	0	1
L	1	5	0	3
Density	0,0222	0,0079	0	0,0385

Properties	Commodities	Energy	Telecommunications
Avg. Degree	0,2000	0	0
Maximum Degree	1	0	0
L	1	0	0
Density	0,0222	0	0

Table 4.5: Layer 1 properties per sector

We can make one consideration about Raw Materials, Energy and Telecommunications. These sectors have all the values equal to 0. This means that there are no links in these sectors. All the nodes are isolated and so their degree centrality, betweenness centrality

and closeness centrality is equal to 0. Moreover, all the nodes have the same eigenvector centrality because no node is more important than the other. This means that inside the sectors there are no relations between the companies and so they are not interested in communicating and influencing one with the other using the direct interlocking phenomenon.
Finally, we can make one last consideration about centrality in each sector. Recalling Table 4.1 and considering some of the most important nodes, there are some companies which decrease their centrality. Indeed, there are some nodes, as OVS SPA, that in their sector have not the same centrality they have in the whole network. Indeed, OVS SPA, which belongs to Luxury goods, is overcome in degree centrality, betweenness centrality and closeness centrality by RCS Mediagroup SPA. This company does not appear in the top ten centrality of the whole layer. Something very similar happens also to Atlantia SPA. So, it is possible to notice how the things change when we consider only a sector and how are important the links between nodes of different sectors.

4.4.2. Layer 2

Now, we concentrate on the results of the second layer, indirect interlocking. All the results are reported in Table 4.6. To have indirect interlocking in the sectors, two companies must have at least one member of their board sitting on the board of a third company together with the director of the other company; this third company must be in the same sector of the other two companies. So, in the partition per sector, there are not all the links between companies of the same sector where the third company does not belong to the same sector of the other two companies.
From Table 4.6, we immediately notice that there are many sectors which have not links between their nodes. Indeed: Real Estate, Healthcare, Finance, Raw Materials, Technology, Commodities, Energy and Telecommunications have no links. This immediately highlights how important are the links between nodes of different sectors in Layer 2 and how many are the third company which do not belong to the same sector of the other two companies. Indeed, in the whole network there are 146 links between companies of the same sector but, considering the partition in sectors, the total number of links is 18. This means that there are 128 companies of the same sector which have, as third company, a node which belongs to a different sector.
Moreover, in the few sectors which have links, Utilities, Luxury goods and Industry, the maximum degree and the density of the sectors are very low with respect to the maximum degree and to the density of the whole network. As previously said, the number of links

inside the sectors is 18, which corresponds to the $2,24\%$ of all the links in the network. This is a negligible portion of links.
All these considerations are supported from the fact that the modularity of Layer 2 is negative and this highlights how the partition in sectors is not a good partition to describe the network. Finally, we can also notice that the number of total links inside the sectors in Layer 2 is smaller than the number of total links inside the sectors in Layer 1 while the number of total links in Layer 2 is bigger than the number of total links in Layer 1.

Properties	Utilities	Luxury goods	Real estate	Industry
Avg. Degree	0,1429	0,5306	0	0,1778
Maximum Degree	1	3	0	2
L	1	13	0	4
Density	0,0110	0,0111	0	0,0040

Properties	Healtcare	Finance	Raw materials	Technology
Avg. Degree	0	0	0	0
Maximum Degree	0	0	0	0
L	0	0	0	0
Density	0	0	0	0

Properties	Commodities	Energy	Telecommunications
Avg. Degree	0	0	0
Maximum Degree	0	0	0
L	0	0	0
Density	0	0	0

Table 4.6: Layer 2 properties per sector

4.4.3. Layer 3

Finally, we concentrate on the governance layers. First of all, we consider the first network and we report the data in Table 4.7.

Properties	Utilities	Luxury goods	Real estate	Industry
Avg. Degree	0	0,4898	0	0,1778
Maximum Degree	0	3	0	2
L	0	12	0	4
Density	0	0,0102	0	0,0040

Properties	Healthcare	Finance	Raw materials	Technology
Avg. Degree	0	0,5000	0	0
Maximum Degree	0	3	0	0
L	0	9	0	0
Density	0	0,0143	0	0

Properties	Commodities	Energy	Telecommunications
Avg. Degree	0	0	0
Maximum Degree	0	0	0
L	0	0	0
Density	0	0	0

Table 4.7: Layer 3 - Network 1 properties per sector

The situation does not improve with respect to the previous layers. Indeed there are 8 sectors which have no link inside. Moreover, in the remaining 3 sectors, the maximum degree of the nodes is very small with respect to the maximum degree of the whole network. This means that the majority of the neighbours of any node are companies which do not belong to the same sector. Indeed, Fullsix, which is the most connected vertex in the network, has degree 0; this means that all the links that Fullsix has are with companies which do not belong to its sector. Instead, Piquadro SPA, which is the second most

connected node in the layer, has degree 3 in its sector (Luxury goods) but this value is still very low with respect to the degree of the vertex in the layer.
Moreover, the number of links inside the sectors is equal to 25, which corresponds to the 4, 57% of all the links. This value is bigger than the one in Layer 2 but it is smaller than the one in Layer 1. However, it confirms that the great majority of links (about 95, 43%) are between companies which belong to different sectors.
Finally, the discussion we can make about centrality is very similar to the one we made in Layer 1. The only thing which change are the companies to consider (there is no more OVS SPA but there is Piquadro SPA).

Finally we concentrate on Layer 3 of the second network. We report the data in Table 4.8.

Properties	Utilities	Luxury goods	Real estate	Industry
Avg. Degree	0,2857	0,8571	0	0,4000
Maximum Degree	1	5	0	2
L	2	21	0	9
Density	0,0220	0,0179	0	0,0091

Properties	Healthcare	Finance	Raw materials	Technology
Avg. Degree	0,2000	0,8333	0	0,3077
Maximum Degree	1	4	0	1
L	1	15	0	2
Density	0,0222	0,0238	0	0,0256

Properties	Commodities	Energy	Telecommunications
Avg. Degree	0,2000	0	0,2857
Maximum Degree	1	0	1
L	1	0	1
Density	0,0222	0	0,0476

Table 4.8: Layer 3 - Network 2 properties per sector

The situation is better than the previous Layer 3. Indeed, there are only 3 sectors which have no link inside. Particularly interesting is the Telecommunications sector: it is very small (it is constituted by 7 nodes) and so, even if it has only 1 link, its density is bigger than the density of the whole network. Instead, there are other sectors, as Luxury goods, which are big and so, even if they have many links, they have a density lower than the one of the whole network.
In this layer, there is a maximum degree inside the sectors equal to 5. This value is reached by Piquadro SPA and Ross SPA which are both in Luxury goods. Regarding Piquadro SPA, we can say that it has many more neighbours that belong to different sectors than neighbours that belong to its sector. Finally, we can also notice as vertices which do not appear in the top ten of degree centrality of the whole network, as Ross SPA, can increase their centrality inside their sector. Indeed, Ross SPA has the same degree of Piquadro SPA and it has a comparable importance in their sector.
Finally, the number of links which are inside the sectors is 53, which corresponds to the $6,69\%$. This value is bigger than the previous Layer 3, but it is still very little. So, also for this layer we can conclude that the number of links which connects nodes of different sectors is sensibly higher than the number of edges which connects nodes inside the same sector.
Finally, the considerations we can make about the centrality of the nodes in their sector are very similar to the ones made for the previous layers.

5 | Multiplex analysis

In this chapter we describe the properties of the multiplex networks. We have two different multiplex networks which have both three layers: the first two layers are the same in both the networks and they are built using direct and indirect interlocking, the third layer is built using a governance index and it is different in the two networks.
Note that all the properties are computed using *Matlab* and *Gephi*. Moreover, there is a graphic representation of the projection networks of the multiplex in Appendix B.

5.1. First multiplex

The first multiplex is composed by the direct interlocking layer, the indirect interlocking layer and the first governance layer with threshold equal to $0,945$. In this network, there are only intra-layer links and there are not inter-layer links. Now, we start describing the properties of this network.

First of all, we consider centrality. There are different types of centrality and we use different formulas to compute them. In this multiplex there are 6 nodes which are isolated: Conafi, D'Amico, Gpi SPA, Italian Exhibition Group, Landi Renzo SPA and Lazio SPA. These companies are isolated in all the layers of the network and so they are the least central nodes in the multiplex. We want also to see which are the ten most central nodes for each measure of centrality and to compare the results with each single layer. In Table 5.1, we report the ten most central nodes for each notion of centrality.
To compute degree centrality, we use formula (2.12). We sum all the values of the three layers and then we rank the nodes. First of all, we check how many nodes in the top ten of degree centrality of Network 1 are also in the top ten of degree centrality of Layer 1, Layer 2 and Layer 3. There are: 7 nodes in the top ten of Layer 1, 7 nodes in the top ten of Layer 2 and 0 nodes in the top ten of Layer 3; among the 7 nodes of Layer 1 and Layer 2, 6 coincide. The 6 nodes in common between the layers, occupy the first 7 positions in the ranking and the first five nodes are all nodes which appear in the rankings of the

interlocking layers. So, the interlocking layers heavily influence the degree centrality of the multiplex. However, there are 2 nodes, Brembo and Buzzi Unicem SPA, which are new entries (they do not appear in any ranking), they take advantage of the multiplex structure to obtain importance in the network. Finally, notice that OVS SPA is the node with the highest degree in the multiplex network thanks to the high degree it has in the first two layers and Tamburi Investment Partners SPA earns some positions with respect to the position it has in the single layers.

Now, we concentrate on betweenness centrality. To compute this quantity, we build the network as explained in Section 2.2 and then we apply (2.6) on this network. The situation is pretty different with respect to degree centrality. There are: 1 node which appears in the top ten of Layer 1, 2 nodes which appear in the top ten of Layer 2 (1 of these nodes, OVS SPA, is the same of the first layer) and 4 nodes which come to view in the top ten of Layer 3. The first position is occupied by La Doria SPA which is the company in the first place of Layer 3. The first three companies of betweenness centrality of Layer 3 appear also in Table 5.1 even if they are not in the first positions; while, about interlocking layers, the unique company which maintains a strong betweenness centrality is OVS SPA. So, the third layer influences this notion of centrality more than the first two layers. Moreover, there are 4 vertices which are new entries in the ranking: Tamburi Investment Partners SPA, Banca Generali SPA, Aedes Siiq SPA and Compagnia Immobiliare Azionaria; particularly, the first three companies use the structure of the multiplex to gain a very central role in betweenness centrality and to overcome all the agencies which appear in the single layers except for La Doria SPA and OVS SPA.

Then, we concentrate on closeness centrality. To compute this quantity, we create the new network as explained in Section 2.2 and then we apply equation (2.7) on this network. In this notion of centrality, the situation is somehow similar to betwenneess centrality. There are: 2 nodes which appear in the top ten of Layer 1, 0 nodes which appear in the top ten of Layer 2 and 5 nodes which come to view in the top ten of Layer 3; among the 5 companies of Layer 3, one is in common with Layer 1 and it is OVS SPA. Due to the importance it has in the first and in the third layer, OVS SPA is the node with the highest closeness centrality in the network. Layer 3 influences the rankings in the network even if the nodes do not appear in the same order in the network. There are 4 companies which use the structure of the multiplex network to earn closeness centrality: Aedes Siiq SPA, Aeffe SPA, Datalogic and Restart SPA. Notice that Aedes Siiq SPA is one of the companies which is taking more advantage from the structure of the multiplex, indeed it earns positions also in betweenness centrality. However, in the multiplex network, all the nodes are close to the other except for the 6 isolated nodes; so, the presence of new agencies in the top ten is facilitated by the proximity of all the nodes.

Degree	Betweenness
OVS SPA	LA DORIA SPA
ERG SPA	OVS SPA
TAMBURI INVESTMENT SPA	TAMBURI INVESTMENT SPA
ATLANTIA SPA	BANCA GENERALI SPA
CERVED GROUP SPA	AEDES SIIQ SPA
INTERPUMP GROUP SPA	JUVENTUS FOOTBALL CLUB
AMPLIFON	BANCA FARMAFACTORING SPA
BREMBO	UNIPOL GRUPPO FINANZIARIO
BUZZI UNICEM SPA	ILLIMITY BANK
ELICA	COMPAGNIA IMMOBILIARE AZIONARIA

Closeness	Eigenvector
OVS SPA	COMPAGNIA IMMOBILIARE AZIONARIA
BANCA GENERALI SPA	CLASS EDITORI
BANCA FARMAFACTORING SPA	INTEK GROUP
AEDES SIIQ SPA	TOD'S
TAMBURI INVESTMENT SPA	PIQUADRO SPA
AEFFE SPA	FULLSIX
LA DORIA SPA	AMBIENTHESIS
DATALOGIC	SOFTLAB
ILLIMITY BANK	NETWEEK SPA
RESTART SPA	ROSS SPA

Table 5.1: Ten most important companies of network 1 for each centrality

Finally, we concentrate on eigenvector centrality. To compute eigenvector centrality, we use equation (2.15). There are: 4 nodes which appear in Table 4.1, 4 nodes which appear in Table 4.2 (these nodes are the same of Layer 1) and 6 nodes which come to view in Table 4.3. All the 10 vertices appear in one of the rankings of the single layers. Another interesting notation is that the first four positions are occupied by the nodes which are in Layer 1 and in Layer 2 and the remaining positions are occupied by the nodes which are

in Layer 3. However, they do not maintain the same order they have in Table 4.1, Table 4.2 and Table 4.3. We cannot say that there is one layer which influences the ranking of the network more than the other; however, the companies in the top ten of interlocking layers occupy the first four positions maintaining all the importance they have in the first two layers while the companies in the governance layer lose some of their importance but, in its entirely, they continue to keep great importance for the multiplex network. None of the nodes of eigenvector centrality is in degree centrality; this means that these nodes have not the highest degree of the network but are connected with nodes which have great importance in the network, indeed they are connected one with the other. This is interesting because they earn their relevance taking advantage one of the other; particularly, the first four nodes are not related with the last six vertices just as the last six vertices are not related with the first four nodes.
Finally, there is no company which appears in all the measures of centrality represented in Table 5.1. This highlights the different characteristics and organization of the companies. The companies do not behave in the same way: there are some companies that create a lot of links, while there are companies that have less links than the previous firms but with more relevant agencies. This is also the reason for which in each centrality there is at least a node which does not appear in the other notions of centrality: Buzzi Unicem SPA in degree centrality, Unipol Gruppo Finanziario SPA in betweenness centrality, Restart SPA in closeness centrality and Tod's in eigenvector centrality.

Now, we concentrate on edge overlap. We study both local and global edge overlap for each possible couple of layers and for all the layers together using respectively equation (2.17) and (2.16).
We start considering Layer 1 and Layer 2. There are 35 links in common between these two layers. They correspond to the $12,96\%$ of links of Layer 1 and to the $4,36\%$ of links of Layer 2. In particular, there are two companies with 6 links which overlap: OVS SPA and Tamburi Investment Partners SPA. OVS SPA has a degree of 13 in Layer 1 and so the $46,15\%$ of links overlap in Layer 1, while it has a degree of 35 in Layer 2 and so the $17,14\%$ of links overlap in Layer 2. Instead, Tamburi Investment Partners SPA has a degree of 8 in Layer 1 and so the 75% of links overlap in Layer 1, while it has a degree of 27 in Layer 2 and so the $22,22\%$ of links overlap in Layer 2.
We proceed by considering Layer 1 and Layer 3. There are 13 links which overlap. They correspond to the $4,81\%$ of links of Layer 1 and to the $2,38\%$ of links of Layer 3. The number of links which overlap is smaller than the overlapping links between Layer 1 and Layer 2. Indeed, there are many links of Layer 1 that there are not in Layer 3, and

viceversa. The two layers describe different aspects of the company and the links are, in a certain way, complementary, i.e. one index connects nodes which are not connected by the other index. To understand if this relation is only between Layer 1 and Layer 3 or affects all the interlocking layers, we study the relation between Layer 2 and Layer 3.
Between Layer 2 and Layer 3 there are 29 links in common. They correspond to the $3,62\%$ of all the links of Layer 2 and to the $5,30\%$ of all the links of Layer 3. As in the previous case, the percentages are very low and so we can say that indirect interlocking and the governance index are somehow complementary, i.e. one index connects nodes which are not connected by the other index. So, it is possible to say that the interlocking network and the governance network have not a consistent edge overlap. This can be verified by considering the overlap between all the three layers.
If we consider all the layers, there are only 5 links which overlap. They are: Buzzi Unicem SPA - Erg SPA; Caltagirone Editore - Caltagirone SPA; Interpump Group SPA - OVS SPA; Interpump Group SPA - Tamburi Investment Partners SPA; OVS SPA - Tamburi Investment Partners SPA. Among the 35 overlapping links between Layer 1 and Layer 2, only the $14,29\%$ overlap with Layer 3. This means that the interlocking network, constituted by the first two layers, and the governance network connect the nodes in different ways and so the companies which are not linked when we consider the interlocking phenomenon may be linked when we consider the governance index, and viceversa.

Another property on which we can concentrate is the multiplex participation coefficient. It is computed using equation (2.18). It is interesting to understand which are the nodes with multiplex participation coefficient equal to 1 and 0.
The nodes with $P_i = 1$ are only 2. They are: Banca Profilo and Tesmec. These nodes have the links equally distributed among the layers. Despite this property, they do not appear in Table 5.1.
The nodes with $P_i = 0$ are 35. These nodes are active only in one layer. Despite they appear only in one layer, they can have a central role in the multiplex network. In particular, there are two companies, Juventus Football Club and Ross SPA, which are present in Table 5.1. More precisely: Juventus Football Club is sixth in betweenness centrality and Ross SPA is tenth in eigenvector centrality. So, the number of layers in which a node comes to view is not relevant for the importance that this node has in the multiplex network.

Finally, we concentrate on the projection network of the first multiplex.
The degree distribution is built using (2.1) to compute the degree of the nodes. It is

represented in Figure 5.1 and it is very different from the ones of the single layers. It cannot be compared to the degree distribution of a scale-free network because there are not many scarcely connected nodes; indeed, there are only 23 nodes with degree smaller or equal than 3. The maximum degree reached in this network is 49 and it is reached by OVS SPA. However, the degree of OVS SPA is different from the value it reached in degree centrality computed before, because previously we use formula (2.12) to compute it. Moreover, there are 6 isolated nodes. These are the nodes that have been cited when we were talking about the centrality. Indeed, the projection network is a representation of the multiplex network and so the isolated nodes must be the same.

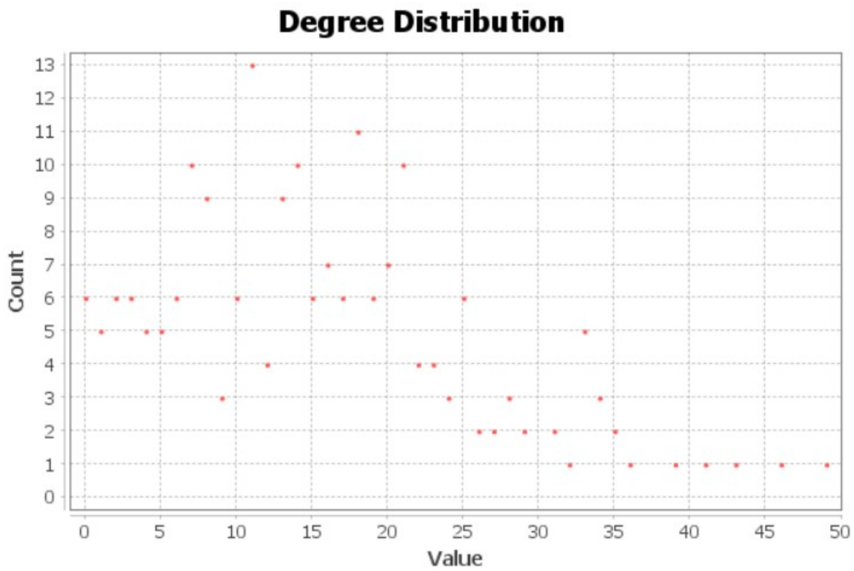

Figure 5.1: Degree distribution of the first projection network

Applying (2.2) we can compute the average degree of the network which is: 15, 39. Moreover, applying equation (2.3) and equation (2.4), we can compute the average weighted degree; we obtain the value: 2, 86. These values are bigger than the ones obtained for the single layers because we are considering all the layers together.
Also the density of the links is bigger than before. Using equation (2.9) we obtain: 0, 077. This value is not so high with respect to the total links that could be in the network but it is almost equal to the sum of the densities of the single layers: 0, 08. This confirms what we said before: interlocking and governance mechanism have not strong relationships because their links are complementary, i.e. tend to connect nodes which the other phenomenon does not link.
Particularly interesting is the giant component which comes up from this network. The giant component is composed by 193 nodes and 1546 links; they correspond to the 96, 02%

of all the nodes and to the 99, 94% of all the links. There is only one link which is not in the giant component and it connects Garofalo Healthcare SPA and Azimut Holding. These two nodes are the unique vertices which do not belong to the giant component together with the isolated nodes.
We can compute the modularity of the network using equation (2.10). It is interesting to understand if the situation is improved with respect than before when we consider the single layers. However, the situation does not improve and the modularity coefficient is negative. Also for the multiplex network, the partition in sector is not a good partition. The best partition we can obtain for this network has a modularity coefficient equal to 0, 476. This is not a very good value but it is the best we are able to obtain. This partition is constituted by 15 components: 6 are the isolated nodes, 1 is constituted by only two nodes and the other are constituted by more nodes. The component with only 2 nodes is formed by Azimut Holding and Garofalo Healthcare SPA which are the unique nodes which do not belong to the giant component, except for the isolated nodes; the biggest component is constituted by 37 nodes. From these comments we can grasp how the components are different from the sectors of the market.
Finally, the contribution that each layer gives to the multiplex is computed using (2.21) and the result is the following: 15, 61% Layer 1, 49, 95% Layer 2 and 34, 44% Layer 3. The majority of links depends on Layer 2 while the first layer is not so important. If we consider the direct and the indirect interlocking together, then it is possible to see as, in this network, the main communication channels, through which companies are linked, are the directors on more than one boards. So, we can deduce that the single directors are more important than the governance mechanism.

5.2. Second multiplex

The second multiplex is composed by the direct interlocking layer, the indirect interlocking layer and the second governance layer with threshold equal to 0, 945. In this multiplex there are only intra-layer links and there are not inter-layer links. Notice that this network differs from the previous one only for the third layer which has been constructed in a different way than before. In particular, we give a biggest weight to the variable with the biggest variance and we want to see if this relation influences the whole network.

We start considering the centrality of the nodes. In this multiplex there are 4 isolated nodes: D'Amico, Italian Exhibition Group, Landi Renzo SPA and Lazio SPA. We immediately see that these nodes are the same of the previous network except for Conafi and Gpi SPA which increase their degree. Particularly, Gpi SPA has a degree equal to 6. The 4 isolated nodes are the least important nodes of the multiplex network but we want to concentrate on the most important vertices of the network which are described in Table 5.2. We compare these nodes with the vertices in the top ten of the single layers and of the previous multiplex.

To compute degree centrality we use equation (2.12). In the top ten of degree centrality of Network 2 there are: 7 companies which are also in the top ten of Layer 1, 6 companies which are in the top ten of Layer 2 and 1 company which is in the top ten of Layer 3; the 6 nodes which appear in Layer 2 are already in Layer 1, they are: OVS SPA, Erg SPA, Atlantia SPA, Cerved Group SPA, Tamburi Investment Partners SPA and Amplifon. As for Network 1, the first two layers heavily influence this measure of centrality while the third layer does not play a big role. Indeed, La Doria SPA, which is the unique company of Layer 3, occupies the ninth position. There are 2 companies which do not belong to any top ten of the layers, they are: Brembo and Nexi SPA. Even if they are not in any top ten, in Table 5.2 occupy the fifth and sixth position and are in front of some agencies, as Cerved Group SPA, which play a central role in the single layers. Finally, we can make a comparison with Network 1: the first four nodes are the same and are in the same order; Brembo acquires centrality also in the second multiplex while Cerved Group SPA loses some positions; Nexi SPA acquires centrality at expense of Buzzi Unicem SPA and Elica which do not appear in the top ten of Network 2.

Now, we concentrate on betweenness centrality. To compute this quantity, we build the network as explained in Section 2.2 and then we apply (2.6) on this network. With respect to the single layers there are: 1 node which appears also in Table 4.1, 2 nodes which appear in Table 4.2 (one of these nodes is the same of Layer 1: OVS SPA) and 2 nodes which come to view in Table 4.4. The nodes which appear also in the interlocking layers are: Unipol Gruppo Finanziario SPA and OVS SPA. They occupy the first positions in the betweenness centrality of their layers but they lose some positions in Table 5.2. Instead, regarding the governance layer, we have the opposite situation: La Doria SPA and Luve SPA do not occupy the first positions in the betweenness centrality of their layer but acquire centrality in the multiplex. There are 6 companies which do not appear in any top ten but use the structure of the multiplex to play a central role in this network. An example can be Aedes Siiq SPA which occupies the second position. Finally, the situation is different with respect to Network 1. Indeed, even if the first position is always taken up by La Doria SPA, all the other positions are different. In particular, there are agencies, as OVS SPA, which lose some of their centrality. Moreover, there are other companies as Aedes Siiq SPA and Recordati Industria Chimica e Farmaceutica SPA which use the

different notion of distance to acquire importance in the second network.
Now, we concentrate on closeness centrality. To compute this value, we create the new network as explained in Section 2.2 and then we apply equation (2.7) on this network. As Network 1, all the nodes are very close. There are: 1 company which appears in the top ten of Layer 1, 0 companies which appear in the top ten of Layer 2 and 2 companies which come up to view in the top ten of Layer 3. So, there are 7 companies which do not appear in any top ten. This means that the layers do not influence this notion of centrality while the multiplex plays a great role. Indeed, agencies which are not close to the others in the single layers are close in the multiplex. In particular, Banca Generali SPA and Caltagirone Editore acquire great centrality. However, the first and third position are occupied by vertices which are also in the top ten of the layers. This is a particular situation and differs from what we have in Network 1. Indeed, there are companies, as Caltagirone Editore and Tamburi Investment Partners SPA, which do not appear in Network 1 and acquire importance in Network 2 while others, as Banca Farmafactoring SPA, lose their centrality.
Finally, we concentrate on eigenvector centrality. To compute this value, we apply equation (2.15). In Network 2 there is a situation very similar to the one in Network 1. Indeed, the first 4 nodes are the same that there are in the top ten of eigenvector centrality of Layer 1 and Layer 2 even if they are in a different order; five of the last six positions are occupied by companies which are also in the top ten of eigenvector centrality of Layer 3 even if they are not in the same order. There is only one vertex which is a new entry in this notion of centrality and it is Alkemy SPA, it is able to acquire importance because it is linked to all the nodes which come from the governance layer. So, the first two layers influence this notion of centrality but we cannot say that they have a bigger influence than the governance layer. Finally, we can compare the order of the nodes with the first network. The first six positions are the same of Network 1 highlighting how the new similarity index does not influence the eigenvector centrality in the first positions. The differences can be seen only from the seventh position. Here there are some agencies which take advantage of the new index, as Tiscali and Alkemy SPA. Moreover, we can make the same considerations of eigenvector centrality of the first multiplex: these nodes earn their relevance by taking advantage one of the other; moreover, the first four nodes are not related with the last six vertices just as the last six vertices are not related with the first four nodes.
In conclusion, also in this network, there is no node which appears in all the top ten. A node as Erg SPA, which is in the second place regarding degree centrality, completely disappears in all the other measures of centrality. Another interesting node is OVS SPA, which has great importance when we speak about degree, betweenness and closeness cen-

trality but does not come into view in eigenvector centrality. Another interesting node is Compagnia Immobiliare Azionaria which appears only in betweenness and eigenvector centrality. This is an interesting result because the company is connected with the most important nodes in the network and it is an intermediary between them acquiring great centrality and importance in the network despite it is not the most connected and close vertex of the multiplex. As for the first multiplex, it is possible to see the different inclination of the nodes in the links' creation.

Degree	**Betweenness**
OVS SPA	LA DORIA SPA
ERG SPA	AEDES SIIQ SPA
TAMBURI INVESTMENT SPA	UNIPOL GRUPPO FINANZIARIO
ATLANTIA SPA	RECORDATI SPA
BREMBO	SAES GETTERS SPA
NEXI SPA	BANCA GENERALI SPA
CERVED GROUP SPA	LUVE SPA
INTERPUMP GROUP SPA	COMPAGNIA IMMOBILIARE AZIONARIA
LA DORIA SPA	OVS SPA
AMPLIFON	TRIBOO SPA

Closeness	**Eigenvector**
OVS SPA	COMPAGNIA IMMOBILIARE AZIONARIA
BANCA GENERALI SPA	CLASS EDITORI
LA DORIA SPA	INTEK GROUP
CALTAGIRONE EDITORE	TOD'S
DATALOGIC	PIQUADRO SPA
TAMURI INVESTMENT SPA	FULLSIX
BREMBO	TISCALI
ILLIMITY BANK	JUVENTUS FOOTBALL CLUB
JUVENTUS FOOTBALL CLUB	ALKEMY SPA
AEFFE SPA	SOFTLAB

Table 5.2: Ten most important companies of network 2 for each centrality

Now, we concentrate on local and global edge overlap and we use, respectively, formula (2.17) and formula (2.16) to compute them.
Layer 1 and Layer 2 are the same of Network 1 and so the situation is the same described in the previous section.
So, we consider Layer 1 and Layer 3. There are 15 overlapping links. They correspond to the $5,56\%$ of the links of Layer 1 and to the $1,89\%$ of the links of Layer 3. The percentage of overlapped links is low both for Layer 1 and for Layer 3. This means that these layers are somehow complementary, i.e. one index connects nodes which are not connected by the other index. To see if this relation holds for the interlocking layers in general, we study what happens between Layer 2 and Layer 3.
The relation between Layer 2 and Layer 3 is interesting for the reason we have just exposed but also because the links of these two layers are comparable. Indeed, in Layer 2 there are 802 links while in Layer 3 there are 792 links. There are 40 links which overlap, they correspond to the $4,99\%$ of all the links of Layer 2 and to the $5,05\%$ of all the links of Layer 3. Despite the percentage of Layer 3 is bigger than before, we cannot say that there is a big portion of links which overlap. So, it is confirmed what we found also in Network 1: the interlocking layers connect nodes which are not connected by the governance layer, and viceversa.
Finally, we want to see what happens when we consider all the layers together. As for Network 1, there are only 5 links which overlap. They are the same we have in Network 1: Buzzi Unicem SPA - Erg SPA; Caltagirone Editore - Caltagirone SPA; Interpump Group SPA - OVS SPA; Interpump Group SPA - Tamburi Investment Partners SPA; OVS SPA - Tamburi Investment Partners SPA. They are few links with respect to the possible overlap. The three layers together describe different types of links between the nodes and so describe different relationships which do not overlap one with the other.

Another property is the multiplex participation coefficient computed using equation (2.18). Also for this network we want to understand the nodes for which the multiplex participation coefficient has value 0 or 1.
The nodes with $P_i = 1$ are 4: Banca Mediolanum SPA, Banca Profilo, Banca Sistema SPA and Enav. They are 2 more of the previous network and we can notice that Banca Profilo have links equally distributed among the layers in both the networks. However, none of these companies belongs to Table 5.2, so none of these agencies is in the top ten of any measure of centrality.
The nodes with $P_i = 0$ are 36, one more than the previous network. Between these companies, 33 are in common with the previous network. Moreover, some of these companies,

Alkemy SPA, Juventus Football Club and Tiscali appear in Table 5.2. More precisely: Alkemy SPA is ninth in eigenvector centrality, Juventus Football Club is eight in eigenvector centrality and Tiscali is seventh in eigenvector centrality. So, being central in only one layer can help to maintain the centrality in the multiplex network.

Finally, we concentrate on the projection network of the second multiplex.
The degree distribution is built using (2.1) to compute the degree of the nodes. It is represented in Figure 5.2 and it is very different from the one obtained by the single layers.

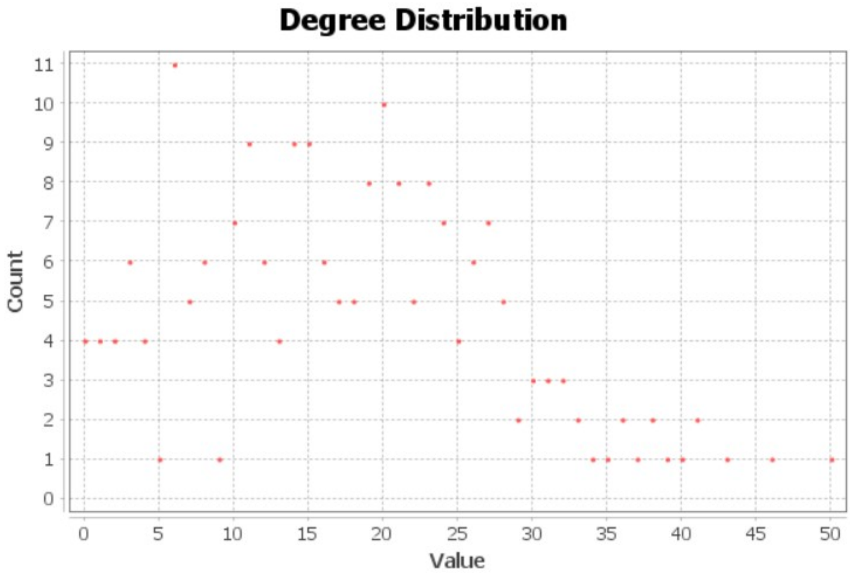

Figure 5.2: Degree distribution of the second projection network

As in the first network, it cannot be compared to the degree distribution of a scale-free network because there are many nodes with a degree in $[10, 30]$. The maximum degree is 50 and it is reached by OVS SPA which continues to be the most connected companies in the network. As we noticed before, the degrees of the nodes in the projection network are different from the ones obtained using the previous aggregation function. Moreover, there are 4 isolated nodes which are the same found before. Finally, there are more links than the projection network computed for the first multiplex; indeed, the third layer of this second network is bigger than the previous one.
Applying equation (2.2) we can compute the average degree of the network, which is $17,70$. Applying equation (2.3) and equation (2.4), we can compute the average weighted degree of the network, which is equal to $3,65$. The values are bigger than the single layers and are also bigger than the first projection network.

Also the density of the links is bigger than the single layers and than the first multiplex. Using equation (2.9) we obtain: $0,089$. This value is always small but it is almost equal to the sum of the densities of the single layers: $0,092$. As before, this confirms what we said for the edge overlap: the layers do not have a relevant amount of overlapping links because otherwise the density would not be so near to the sum of the densities of the single layers.
Particularly interesting is the giant component of this projection network. The giant component is composed by 197 nodes and 1779 links which correspond respectively to the $98,01\%$ of all the nodes and to the 100% of all the links. The unique nodes which are not in the giant component are the isolated nodes. This is an interesting result because all the other nodes are connected and so there is a path which links every node. There is no other component in the network except for the ones constituted by the isolated nodes. This giant component is even bigger than the giant component of the previous multiplex. This proves that the interlocking layers and the governance relation are even more complementary than the previous multiplex network.
Finally, we compute the modularity of the projection network using equation (2.10). The situation does not improve with respect to the single layers, indeed the modularity coefficient is always negative highlighting how the partition in sectors is not good for this network. The maximum modularity reached by this network is $0,522$ and in that partition there are 11 communities. These communities do not coincide with the sectors. There are 4 communities constituted by one node, the isolated ones; the other 7 communities are constituted by a minimum number of 15 nodes to a maximum number of 45 nodes. With respect to the first multiplex there are less components but they are bigger. However, the single components are very different from the sectors of the market.
Finally, the contribution that each layer gives to create the multiplex is computed using (2.21) and the results are the following: $13,55\%$ Layer 1, $43,00\%$ Layer 2 and $43,45\%$ Layer 3. In this second multiplex network, the importance and the influence of the third layer is bigger than the other layers. In the second multiplex the governance mechanism has the same influence on the network that has the indirect interlocking. Contrary to the first network, there is not a dominant phenomenon but there are two phenomena which exercise the same role. This could be very interesting because companies which have great importance in only one of the two phenomena contribute in the same way to the multiplex network. However, if we consider the direct and indirect interlocking together, interlocking has always a predominant role but it is no more marked as before. This linear combination increases the importance of the governance mechanism.

5.3. Sectors

In this section we describe some of the previous results for each sector. In particular, we want to see what happens regarding global overlap between two layers and between all the layers and what is the multiplex participation coefficient of the nodes in each sector. These quantities are computed using respectively (2.16) and (2.18). Moreover, we compare each result with the results of the other sectors and with the results of the corresponding multiplex network.
Then we concentrate also on the centrality measures. We do not report all the tables with the top ten nodes for every notion of centrality. We want to see if the nodes in the top ten of each notion of centrality for the whole multiplex network maintain their central role even in their sector.
Finally, we create the projection network for each sector and we analyze the results we obtain. We report in a Table the main results we obtain: Average degree, Maximum degree, the number of links in the network (L) and the Density of the network. To compute the average degree we use equation (2.2) and to compute the density of the network we use formula (2.9).

5.3.1. First multiplex

We consider the first multiplex built for all the 11 sectors in our dataset.
There are some sectors for which the multiplex does not exist. Indeed, for Raw materials, Energy and Telecommunications there is no link for any layer (it can be seen in Table 4.5, Table 4.6 and Table 4.7). In these sectors the nodes are all isolated and so we do not consider them in the following.
We start computing the global overlap in all the remaining multiplex networks. For each sector, the global overlap is always null. So, there is no overlap neither between Layer 1 and Layer 2 nor between Layer 1 and Layer 3 nor between Layer 2 and Layer 3 nor between all the layers. This is a result which was expected particularly when it is considered Layer 2; indeed, looking at Table 4.6, we can notice that there are many sectors with no link inside. So, all the overlaps previously found in Section 5.1 are between nodes of different sectors.
Another property we want to study is the multiplex participation coefficient. This result is influenced by Layer 2; indeed, many sectors (8: Real estate, Healthcare, Finance, Raw materials, Technology, Commodities, Energy and Telecommunications) have not links in Layer 2 and so for the nodes in these sectors the multiplex participation coefficient cannot

be 1. We concentrate on the remaining 3 sectors to see if there are nodes which have links equally distributed in all the layers. There are 2 nodes with edges equally distributed in all the layers: Salcef Group in Industry and Brembo in Luxury goods. The nodes are in the two biggest sectors of the network; indeed, they are composed respectively by 45 and 49 nodes. Moreover, we try to see if there are nodes which are active in only layer. Now, we consider also the sectors for which there is no link in Layer 2. There are many nodes which are active in only one layer for every sector; in particular: 3 nodes in Utilities, 12 nodes in Luxury goods, 4 in Real estate, 17 in Industry, 2 in Healthcare, 16 in Finance, 6 in Technology and 2 in Commodities. The sectors in which there are more nodes in only one layer are the biggest sectors: Luxury goods, Industry and Finance. Summing all these nodes we obtain a value bigger than the one obtained in the whole multiplex because we are not considering all the links between nodes of different sectors which constitute a relevant part of the network.

Then, we can briefly concentrate on centrality.
We start from degree centrality. In Table 5.1, there are: 1 company which belongs to Utilities, 1 company which belongs to Finance, 1 company which belongs to Healthcare, 3 companies which belong to Luxury goods and 4 companies which belong to Industry. We concentrate on these sectors and we try to see if these companies maintain their central role. In Finance, Tamburi Investment Partners SPA is not the most central node, indeed it is overcome by Banca Generali SPA which does not appear in the top ten of degree centrality of Multiplex 1; in Healthcare, Amplifon is an isolated node and so loses all the centrality it has in the multiplex; in Utilities, there is the same situation of Healthcare: Erg SPA loses all its centrality and becomes an isolated node; in Luxury goods, the nodes lose their centrality; in Industry, Interpump Group SPA and Buzzi Unicem SPA lose their centrality while Atlantia SPA and Cerved Group SPA maintain their central role. So, except for Atlantia SPA and Cerved Group SPA, in their sector the nodes tend to lose the degree centrality they have in the multiplex.
Now, we concentrate on betweenness centrality. In Table 5.1 there are: 1 node in Commodities, 2 nodes in Real estate, 2 nodes in Luxury goods and 5 nodes in Finance. In Commodities and Real estate, all the companies have betweenness centrality equal to 0; in Luxury goods, Juventus Football Club and OVS SPA lose their centrality; in Finance, only Tamburi Investment Partners SPA and Generali SPA maintain their centrality while the other nodes lose it. So, the situation is the same as degree centrality: except for few nodes, the others tend to lose their betweenness centrality in their sector.
Then, we consider closeness centrality. In Table 5.1 there are: 1 vertex in Industry, 1

vertex in Commodities, 2 vertices in Luxury goods, 2 vertices in Real estate and 4 vertices in Finance. In Luxury goods, OVS SPA and Aeffe SPA lose all their centrality; in Commodities, La Doria SPA maintains its centrality; in Real estate, Aedes Siiq SPA and Restart SPA maintain their centrality; in Industry, Datalogic loses its centrality; in Finance, all the companies lose their centrality. So, regarding closeness centrality there are some companies which maintain their centrality in their sector but the majority of the companies loses it.
Finally, we concentrate on eigenvector centrality. In Table 5.1 there are: 1 agency in Raw materials, 1 agency in Technology, 1 agency in Real estate, 1 agency in Utilities, 1 agency in Telecommunications and 5 agencies in Luxury goods. In Telecommunications and Raw materials, the nodes are isolated and so lose their centrality; in Real estate, Technology and Utilities, all the companies lose their centrality because they have the same value of many other agencies which do not appear in Table 5.1; in Luxury goods, only Netweek SPA and Ross SPA maintain their centrality. So, also in this notion of centrality the companies tend to lose the centrality they have in the whole multiplex.
In conclusion, except for a little number of companies as Netweek SPA, Ross SPA, Banca Generali SPA, in their sector, the companies tend to lose the centrality they have in the whole network. Obviously, for companies that lose some of their centrality there are other firms that acquire importance. Once again, this highlights how the sectors' partition is not so relevant for the network and how different is the situation in the sectors with respect to the whole market.

Finally, we concentrate on the projection network of each sector. We report in Table 5.3 the measures introduced before and already used in Section 4.4.
There are some sectors, Real estate, Healthcare, Technology and Commodities, which have the same characteristics of Table 4.5, these sectors have links only in Layer 1 and so maintain the same characteristics. Moreover, Raw materials, Energy and Telecommunications do not have any link in the projection network; indeed, the multiplex network is constituted by three layers with no link inside.
The average degree of all the sectors is very low with respect to the average degree of the whole projection network. The same happens for the maximum degree: the maximum degree between nodes of the same sector is 6 and it is reached by three different companies: Cairo Communication SPA, RCS Mediagroup SPA and Tod's. Neither of the three companies appear in Table 5.1, and so the centrality measures in the sector are different from the one in the whole network.
Moreover, the number of links which are inside the sectors is equal to 88 which correspond

to the $5,89\%$ of the links in the multiplex. So, the links between nodes of different sectors are very important and this testifies how the partition in sector is not a good partition also for the multiplex network.

Properties	Utilities	Luxury goods	Real estate	Industry
Avg. Degree	0,5714	1,7959	0,5000	0,8444
Maximum Degree	2	6	1	4
L	4	44	2	19
Density	0,0440	0,0374	0,0714	0,0192

Properties	Healthcare	Finance	Raw materials	Technology
Avg. Degree	0,2000	0,7778	0	0,4615
Maximum Degree	1	3	0	1
L	1	14	0	3
Density	0,0222	0,0222	0	0,0385

Properties	Commodities	Energy	Telecommunications
Avg. Degree	0,2000	0	0
Maximum Degree	1	0	0
L	1	0	0
Density	0,0222	0	0

Table 5.3: Projection network 1 properties per sector

Finally, the density of each sector is smaller than the density of the whole projection network. This is due to the fact that almost any sector has at least a layer in which there is no link. There are only two sectors with links in all the layers: Luxury goods and Industry, the biggest sectors in the partition. They increase all their properties but are not able to have properties similar to the whole projection network. The unique network which have a similar value of density is Real estate; however, this is due to the fact that the number of nodes which compose this sector is very small and so a little number of links is sufficient to obtain high density. Moreover, in each sector the density of the multiplex is equal to the sum of the densities of the single layers because in the sectors there is no overlapping link.

5.3.2. Second multiplex

We consider the second multiplex built for all the 11 sectors in our dataset.
There are 2 sectors for which the multiplex does not exist. Indeed, in Raw materials and Energy there is no link for any layer (it can be seen in Table 4.5, Table 4.6 and Table 4.8). So, in these sectors the nodes are all isolated and so we do not consider them in the following.
Now, we consider the global overlap in all the remaining multiplex. For each sector the global overlap is null. So, there is no overlap neither between Layer 1 and Layer 2 nor between Layer 1 and Layer 3 nor between Layer 2 and Layer 3 nor between all the layers. This is the same situation we found in the previous section for the first multiplex. Indeed, this result is always influenced by Layer 2 where there are many sectors which have no link inside (Table 4.6). So, all the overlaps found in Section 5.2 are between nodes of different sectors.
Another property we can study is the multiplex participation coefficient. As in Section 5.3.1, the result is influenced by Layer 2 which is the same in both the networks. Indeed, in Real estate, Healthcare, Finance, Raw materials, Technology, Commodities, Energy and Telecommunications there cannot be nodes with links equally distributed in all the layers because in Layer 2 of these sectors there is no link. We concentrate on the other sectors. There are 2 sectors in which there are nodes with $P_i = 1$: in Utilities there is Terna Rete Elettrica Nazionale SPA and in Luxury goods there are B&C Speakers SPA, Brembo and SanLorenzo SPA. The unique node in common with the first multiplex is Brembo while all the others are different. In particular, in the second multiplex there is no node with $P_i = 1$ in Industry but there is in Utilities. Moreover, we check, for each sector, if there are nodes which are active only in one layer. There are: 2 vertices in Commodities, 20 in Finance, 4 in Healthcare, 19 in Industry, 14 in Luxury goods, 4 in Real estate, 6 in Technology, 2 in Telecommunications and 2 in Utilities. There are more nodes with $P_i = 0$ than the previous multiplex. This is due to the fact that we are assigning the maximum weight to the variable with the biggest variance.

Then, we concentrate on centrality. We do the same discussion made before in Section 5.3.1 and we see if, also in the second multiplex, the nodes lose their centrality in their sectors.

We start from degree centrality. In Table 5.2 there are: 1 company of Utilities, 1 company of Commodities, 1 company of Finance, 1 company of Healthcare, 2 companies of Luxury goods and 4 companies of Industry. In Utilities, Commodities, Finance, Healthcare and Luxury goods the companies lose their centrality; in Industry only Atlantia SPA maintains its central role while the other companies lose their centrality. So, also for degree centrality of the second multiplex, the nodes tend to lose their centrality when their action is restricted only to their sector.

Now, we consider betweenness centrality. In Table 5.2 there are: 1 node of Commodities, 1 node of Healthcare, 2 nodes of Real estate, 2 nodes of Finance, 2 nodes of Industry and 2 nodes of Luxury goods. In the second multiplex are represented more sectors in the top ten with respect to the first multiplex where the rankings is dominated by companies belonging to Finance. In Commodities, La Doria SPA loses its centrality at the detriment of Orsero SPA which does not come to view in the top ten of the multiplex; in Healthcare and Real estate, there is no node with a betweenness centrality different from 0; in Luxury goods and Industry, the companies lose their centrality; in Finance, Unipol Gruppo Finanziario SPA loses all its centrality while Banca Generali SPA maintains its centrality. So, in general, the nodes tend to lose their centrality when they are restricted to their sector.

Then, we concentrate on closeness centrality. In Table 5.2 there are: 1 vertex of Industry, 1 vertex of Commodities, 3 vertices of Finance and 5 vertices of Luxury goods. In the second multiplex, Luxury goods acquires more importance than the one it has in the first multiplex. In Commodities, La Doria SPA loses its centrality at the expense of Orsero SPA; in Industry, Finance, and Luxury goods the companies lose their centrality. So, the nodes tend to lose their centrality when they are restricted to their sector.

Finally, we concentrate on eigenvector centrality. In Table 5.2 there are: 1 agency of Real estate, 1 agency of Raw materials, 1 agency of Technology, 2 agencies of Telecommunications and 5 agencies of Luxury goods. In Raw materials, Real estate, Technology and Telecommunications the nodes lose their centrality because all the companies have the same eigenvector centrality; in Luxury goods all the nodes lose their centrality at the detriment of other nodes which do not appear in any top ten.

In conclusion, the most important nodes of the multiplex tend to lose their central position when their action is restricted only to their sector. The result obtained is the same of the first multiplex, so we can conclude by saying that the relationships between companies of different sectors play a relevant role in the market.

Finally, we concentrate on the projection network of each sector. We report in Table 5.4 the measures introduced before.

Properties	Utilities	Luxury goods	Real estate	Industry
Avg. Degree	0,8571	2,1633	0,5000	1,0667
Maximum Degree	3	7	1	4
L	6	53	2	24
Density	0,0659	0,0451	0,0714	0,0242

Properties	Healthcare	Finance	Raw materials	Technology
Avg. Degree	0,4000	1,1111	0	0,7692
Maximum Degree	1	4	0	2
L	2	20	0	5
Density	0,0444	0,0317	0	0,0641

Properties	Commodities	Energy	Telecommunications
Avg. Degree	0,4000	0	0,2857
Maximum Degree	2	0	1
L	2	0	1
Density	0,0444	0	0,0476

Table 5.4: Projection network 2 properties per sector

There are two sectors, Real estate and Technology, which maintain the properties of one of their layers because the other two have no link inside. Moreover, there are other two sectors, Energy and Raw materials, which do not have any link in the projection network because neither of the three layers have links inside. All the other sectors have more links inside than their single layers because they have at least two layers with edges inside. In this second multiplex there are more sectors which increase their properties because in Layer 3 there are more links than the previous one.

The average degree of each sector is very low with respect to the average degree of the whole network. The same happens for the maximum degree. Indeed, the maximum degree between all the sectors is 7 and it is reached by Cairo Communication SPA which belongs to Luxury goods. Cairo Communication SPA has a great importance also in the first projection network and so we can make the same considerations as before. Moreover, this firm has great importance in its sector, more than companies that occupy the first positions in the whole projection network.

The total links which are inside the layers are 115 which correspond to the $6,46\%$ of the total links in the network. So, the links between companies of different sectors are very important and make the difference also in this second projection network. The percentage of links inside the sectors is slightly bigger than the one of the first multiplex but this difference is not so significant to allow different considerations between the two projection networks.

Finally, we concentrate on the density of links in every sector. All the sectors have a density lower than the one of the whole network. This underlines again the importance of the links between companies of different sectors and how this partition is not significant for this multiplex network. Moreover, in each sector the density of the multiplex network is equal to the sum of the densities of the single layers because there is no overlapping link.

6 | Discussion

After having analyzed both the single layers and the multiplex networks, we discuss the obtained results and we compare the two multiplex networks to see what changes when we consider different coefficients for the linear combination. Our objective is to understand if the interlocking phenomenon affects or is affected by the governance mechanism of the firms.

We start considering the first network.
First of all, we want to understand if the edge overlap between the layers of the network is consistent. To do so, we study if the values obtained in our network can be obtained also by considering random networks with the same densities of the layers: if this happens then the results we obtained are not significant because they are got also by random networks, instead if this does not happen we have a relevant property for these layers. So we decide to build random networks with the same density of the considered layers and compute the edge overlap between these networks. To obtain a more consistent result, we make this simulation 100 times and then we make a mean of the obtained overlap between the random networks. Finally, we compute the ratio between the edge overlap of the random networks and the edge overlap of the layers. We concentrate on the relations between Layer 1 and Layer 3 and between Layer 2 and Layer 3 because we want to concentrate on the relations between interlocking layers and governance layer and also because the two interlocking layers are somehow related. The ratio between the edge overlap of two random networks with the same density of Layer 1 and Layer 3 and between the edge overlap of these layers is equal to $0,5654$. The ratio between the edge overlap of two random networks with the same density of Layer 2 and Layer 3 and the edge overlap of these layers is equal to $0,7355$. Both the ratios are not near 0, but the first ratio, the one between Layer 1 and Layer 3, is sufficiently low to say that there is a remarkable number of links overlapping between the layers. This means that there is a relation between Layer 1 and Layer 3 which is not found in random networks. Direct interlocking and good governance mechanism are related. However, this influence is not a strong influence because there are not many links of the layers which coincide. Indeed, as it is noticed

in Section 5.1, a small percentage of links of Layer 1 is in common with Layer 3, and viceversa. We can conclude that these two layers weakly influence one with the other. The same consideration cannot be made for indirect interlocking: the ratio is too high to say that there is a remarkable number of links overlapping between Layer 2 and Layer 3. We conclude that there is not a remarkable influence between these two layers.

Moreover, we can investigate the results obtained in the projection network of the multiplex (Figure B.1). All the nodes, except the isolated vertices and Azimut Holding and Garofalo Healthcare SPA, belong to the giant component meaning that they are all connected. The density of the projection network is almost the double of the biggest density of the single layers, meaning that there are many more links in the projection network with respect to the single layers. The degree and the strength of the nodes are bigger than the one obtained in the single layers. All these properties underline how low is the overlap between interlocking layers and governance layer and how this two phenomena follow different patterns to create links. This confirms the low influence between interlocking and governance mechanism: they do not influence one with the other except for a weak effect that has been underlined between direct interlocking and corporate governance.

Regarding the single nodes, there is no predominant node in the multiplex network. This can be seen in the centrality (Table 5.1). Indeed, there is no node which belong to the top ten of all the notions of centrality. Moreover, by the changes in the rankings, we can understand how the companies use different strategies to achieve importance in the market.

Now, we concentrate on the second multiplex.

As done for the first multiplex, we want to understand if the edge overlap between the layers of the network is consistent. We use the procedure used before and, for the same reasons as before, we concentrate on the relations between Layer 1 and Layer 3 and between Layer 2 and Layer 3. The ratio between the edge overlap of two random networks with the same density of Layer 1 and Layer 3 and the edge overlap between these layers is equal to $0,7213$. The ratio between the edge overlap of two random networks with the same density of Layer 2 and Layer 3 and the edge overlap between these layers is equal to $0,7988$. Both these ratios are too high to consider one of these relations significant. So, it is possible to say that none of the interlocking layers influences the governance layer and its links in a significant way. In this case, interlocking and governance mechanism are independent phenomena which develop without being condition by the other phenomenon. This is a first difference with the relation between direct interlocking and governance we found for the first network.

Moreover, we can investigate the results obtained in the projection network of the multiplex (Figure B.2). All the nodes, except for the isolated vertices, belong to the same giant component. The density of the projection network is more than the double of the biggest density of the single layers, meaning that there are many more links in the projection network with respect to the single layers. The degree and the strength of the nodes are bigger than the one obtained in the single layers. So the situation is very similar to the one of the previous network and we can make the same considerations as before: the overlap between interlocking layers and governance layer is very low and these two phenomena follow different patterns to create links. This confirms the low influence between the layers: both direct and indirect interlocking have no influence towards the governance mechanism and viceversa.

Regarding the single nodes, we can make the same considerations of the first multiplex. Despite there are some differences between Table 5.1 and Table 5.2, because some nodes acquire importance while others lose it, there is no node in the top ten of all the measures of centrality. As before, there is no predominant company and also here it is possible to see the different way in which companies try to achieve importance in the market.

We can make some other remarks about the sectors' partition of the network.

As highlighted both for the single layers and also for the two multiplex networks, the partition in sector is not a good partition for the network. Indeed there are more inter-community links than intra-community links and the centrality of the nodes in their sector does not represent the centrality they have in the whole networks.

However, this result could be expected for the interlocking layers. Indeed, in Chapter 1, we already highlighted how the interlocking phenomenon in the Italian context mainly depends on the connection that the families who control the companies want to have. So, it is not mentioned a partition in sector and we verified that the sectors are not important for ID. Noteworthy this behaviour can be observed also in the governance layer. Indeed, also for the governance layer the sectors' partition is not relevant.

The partition in sectors is an important partition in the market to classify the field in which each company is active but it is not important to identify the links of the company when we speak about interlocking and governance mechanism. The networks prove how the companies are more interested in tying links with firms of different sectors. In the first projection network only the $5,89\%$ of all the links are between nodes of the same sector; in the second projection network only the $6,46\%$ of all the links are between nodes of the same sector. We can conclude that both the multiplex networks do not follow the sectors' partition and both interlocking and good governance mechanism facilitate the communication between companies of different sectors.

In conclusion, there are some differences between the two multiplex introduced by the different weights of the linear combination: the edge overlap between direct interlocking and governance mechanism and the contribution of the corporate governance in the creation of the projection network. But, in general, the properties of the projection network and of the nodes does not change so much, noteworthy is that the overlap between all the three layers gives the same result in both the multiplex. We can conclude that: the interlocking layers neither affect nor are affected by the governance layer, actually they tend to create link between nodes that are not connected by the other phenomenon; there is no predominant company in the market and each company has different ways to obtain importance in the network; the sectors' partition is not relevant in the market, on the contrary the companies prefer to tie links with firms of different sectors.

7 | Conclusions

In the thesis we want to study the relationships between Interlocking Directorates and governance mechanism: we want to see if they follow the same pattern to link the nodes or if they are somehow complementary, i.e. if they connect nodes which are not linked by the other phenomenon. To study these relationships, we decide to perform a multiplex network analysis. We build two different multiplex networks, each one made by three layers.

The first two layers are the same in both the multiplex networks and they are built using, respectively, direct and indirect interlocking. The direct interlocking is the situation in which one or more directors affiliated with one organization sits on the board of directors of another organization; the indirect interlocking is a situation in which there are two directors, one on the board of one organization and one on the board of the other organization, who sit together on the board of directors of a third organization. Both the layers we create using these phenomena are weighted undirected networks.

The third layer is built using a governance index. The governance index is composed by some characteristics of the board of directors of the companies: board size, board independence, 'busy-ness' of corporate directors, gender diversity and minority directors. We use a linear combination of these quantities to create the similarity index between two firms; the differences between the two multiplex networks arise exactly in the third layer because we consider different weights in the linear combination. In particular, in the first multiplex network there is a third layer where all the weights of the linear combination are equal; in the second multiplex network there is a third layer where the biggest weight is associated to the variable with the biggest variance, the second biggest weight is associated to the variable with the second biggest variance and so on. After having build the layers we find the thresholds under which we do not consider the links because we want a network comparable with the first two layers and because we want to consider only the relevant links. Curiously, we use the same threshold for both the third layers. Finally, we normalize and obtain two weighted undirected networks. Due to the different linear combinations, the third layers have different properties: there are more links in the third layer of the second multiplex, the degrees and the nodes' strength are bigger in the third layer of the second multiplex, the notions of centrality give different results and so on. One last notation: both the third layers have different nodes in the top ten of each notion of centrality with respect to the first two layers, there are only few nodes which appear in all the rankings. This highlights how the nodes which have an important role in the two phenomena are different, suggesting that the patterns followed by the layers to link the nodes are different.

So, to better compare the layers, we perform a study on the multiplex networks. In the first multiplex we obtain a weak relation between direct interlocking and governance mechanism, in particular these two layers have a small but relevant number of overlapping links. However, despite this small overlapping, the interlocking and the governance layer are somehow complementary. Indeed, all the properties of the projection network highlight how the two phenomena follow different patterns to connect the nodes: the density of the links is very similar to the sum of the densities of the single layers; the average degree and average weighted degree are sensibly higher than the one of the single layers; the giant component is composed by the great majority of the non-isolated nodes. In the second multiplex there are the same conclusions with only a difference: the complementarity of the two phenomena is bigger than before because there is no relation between direct interlocking and governance mechanism and the giant component connects all the non-isolated nodes. So, the linear combination influences individually the third layer but, despite there are some differences in the properties of the multiplex, it allows to obtain the same conclusions about the relationships between Interlocking Directorates and governance mechanism.

Finally, we study these phenomena for each sector of the market. The Italian market is divided in 11 sectors: Utilities, Luxury goods, Real estate, Industry, Healthcare, Finance, Raw materials, Technology, Commodities, Energy and Telecommunications. For each sector we built the two multiplex networks and we study their properties. In each sector there is no edge overlap and the two phenomena are completely complementary. Moreover, the partition in sector of the network is not a good partition of our network. Both in the single layers and in the multiplex networks the modularity coefficient is very low and the partition in sector is not the best partition of the network. Moreover, there are many sectors that in some layers have no link inside; in particular, Raw materials and Energy have no link in both the multiplex while Telecommunications has no link only in the first multiplex. So, we conclude that the partition in sector is not relevant for these two phenomena, on the contrary the companies prefer to tie links with firms of different sectors.

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A | Layers' representation

Here it is possible to see an image of all the considered layers. In particular, it is possible to see the differences that there are between the layers and that have been described in Chapter 4.
These images have been obtained using *Gephi*.

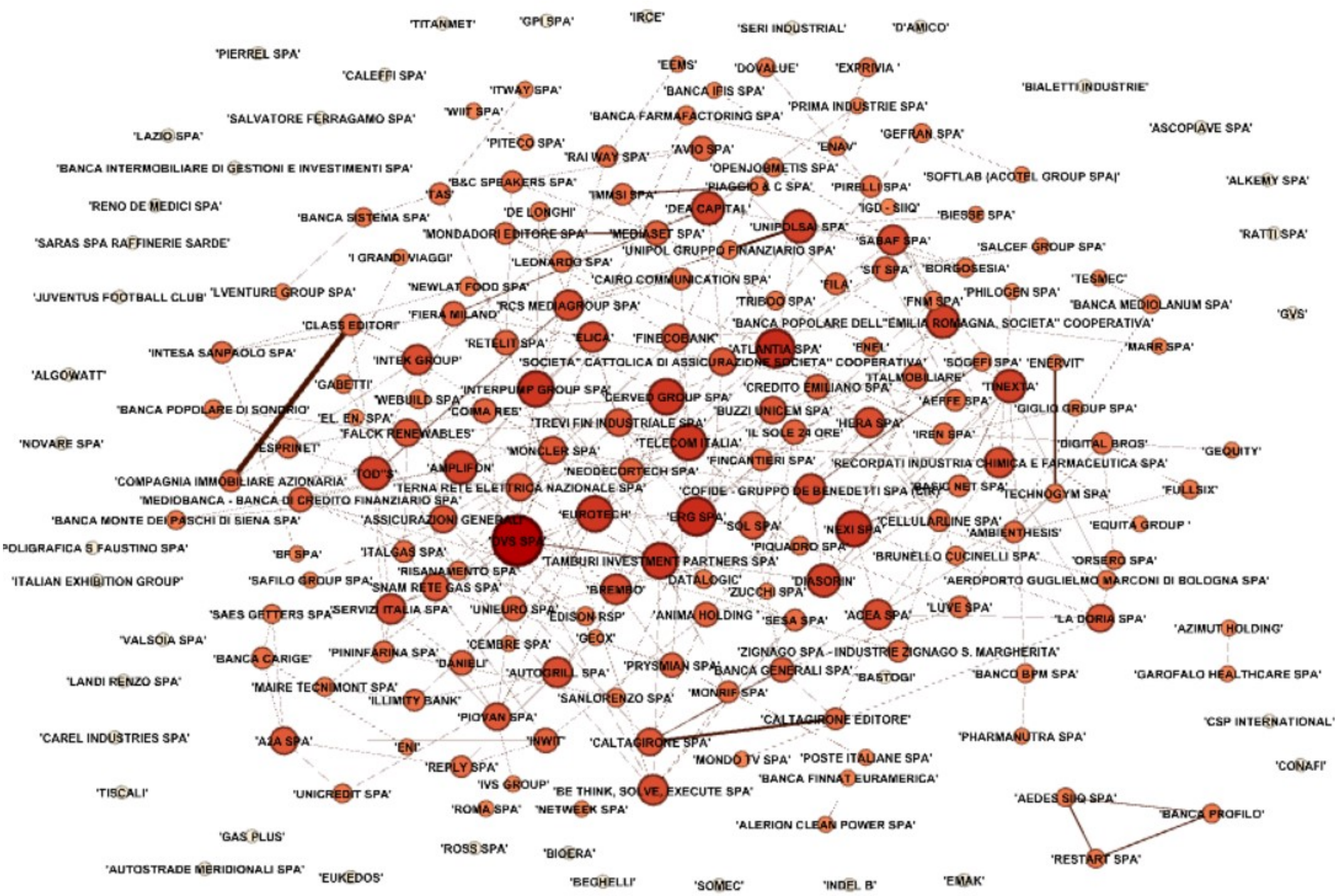

Figure A.1: Layer 1

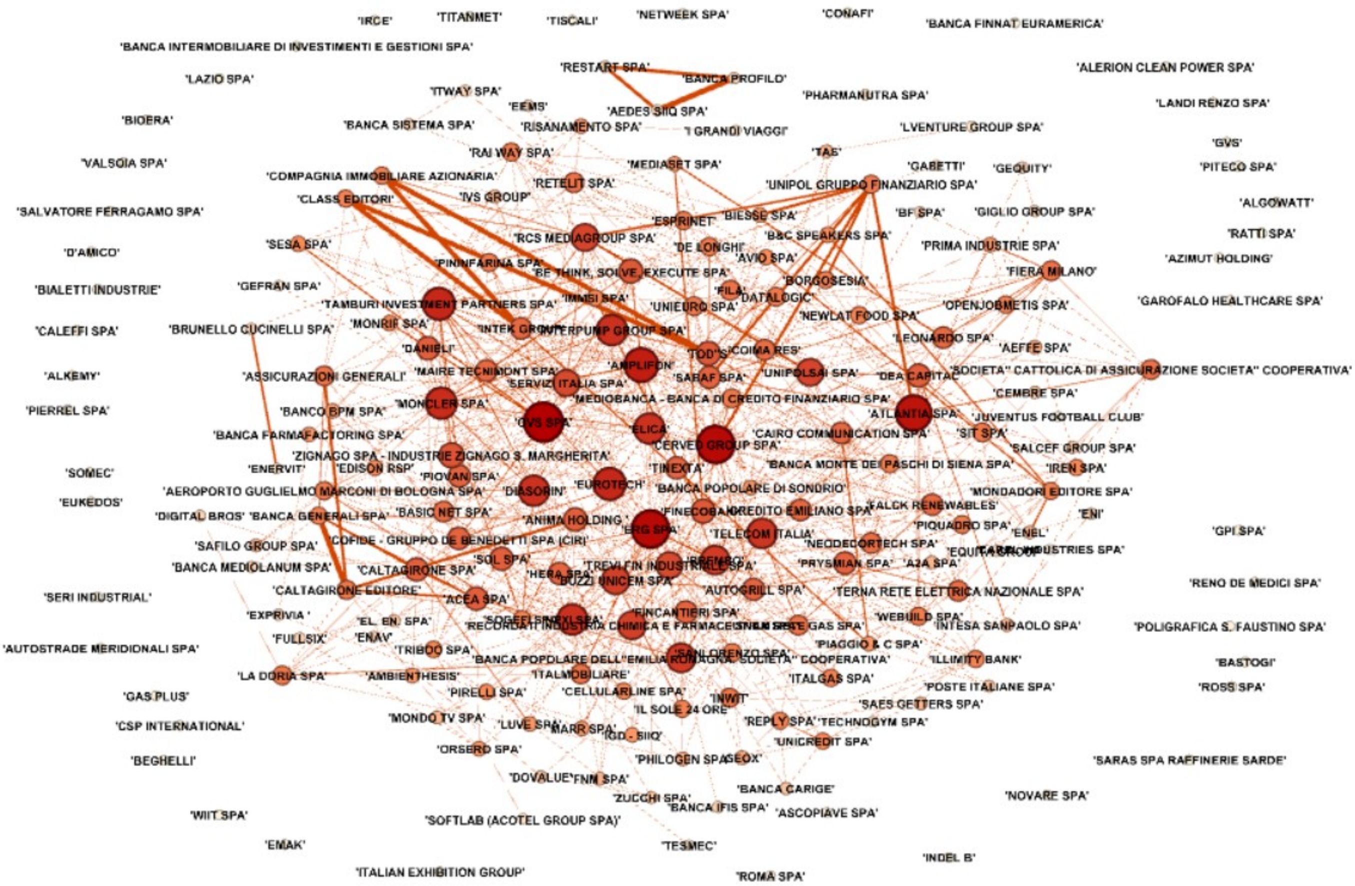

Figure A.2: Layer 2

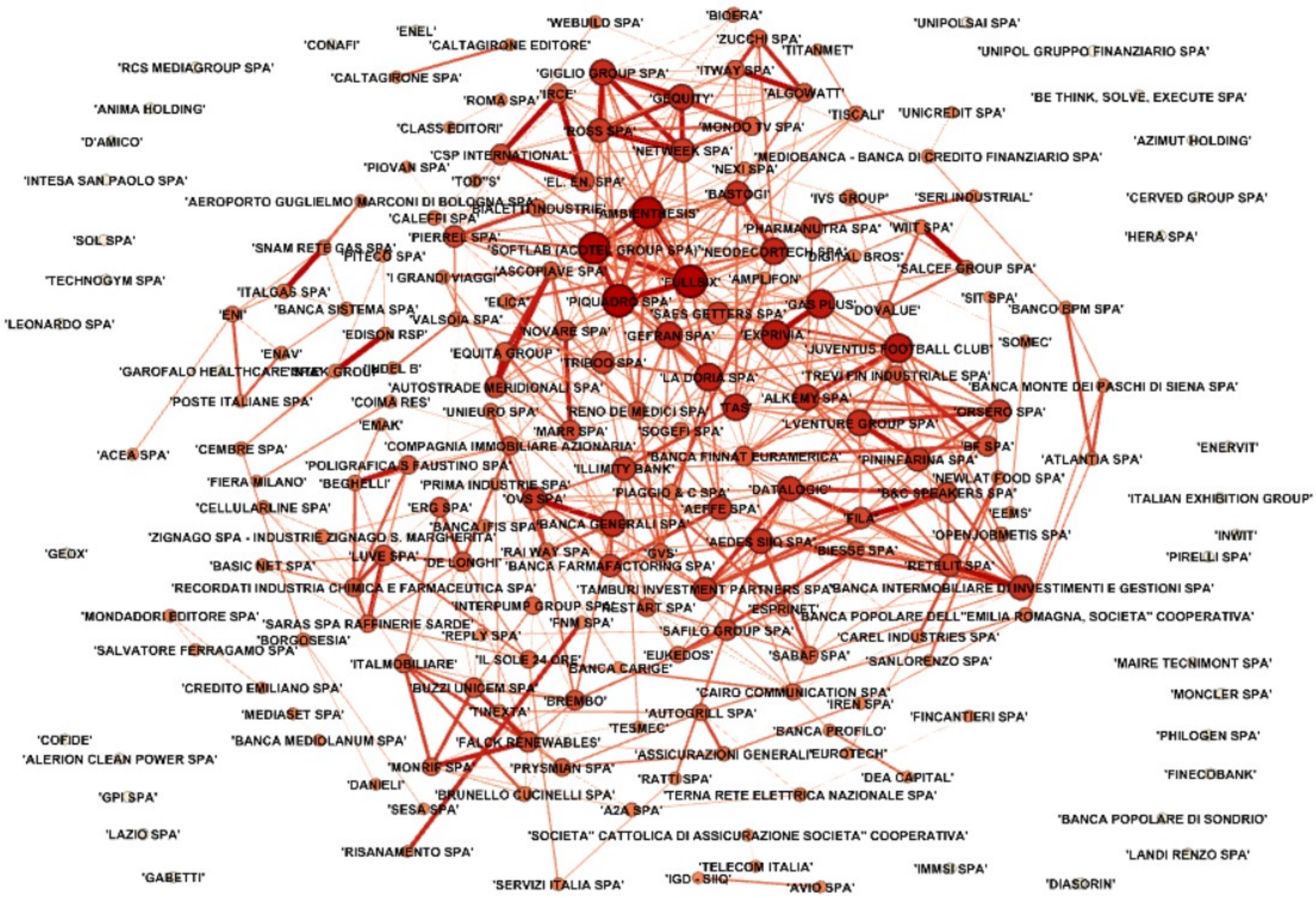

Figure A.3: Layer 3 Network 1

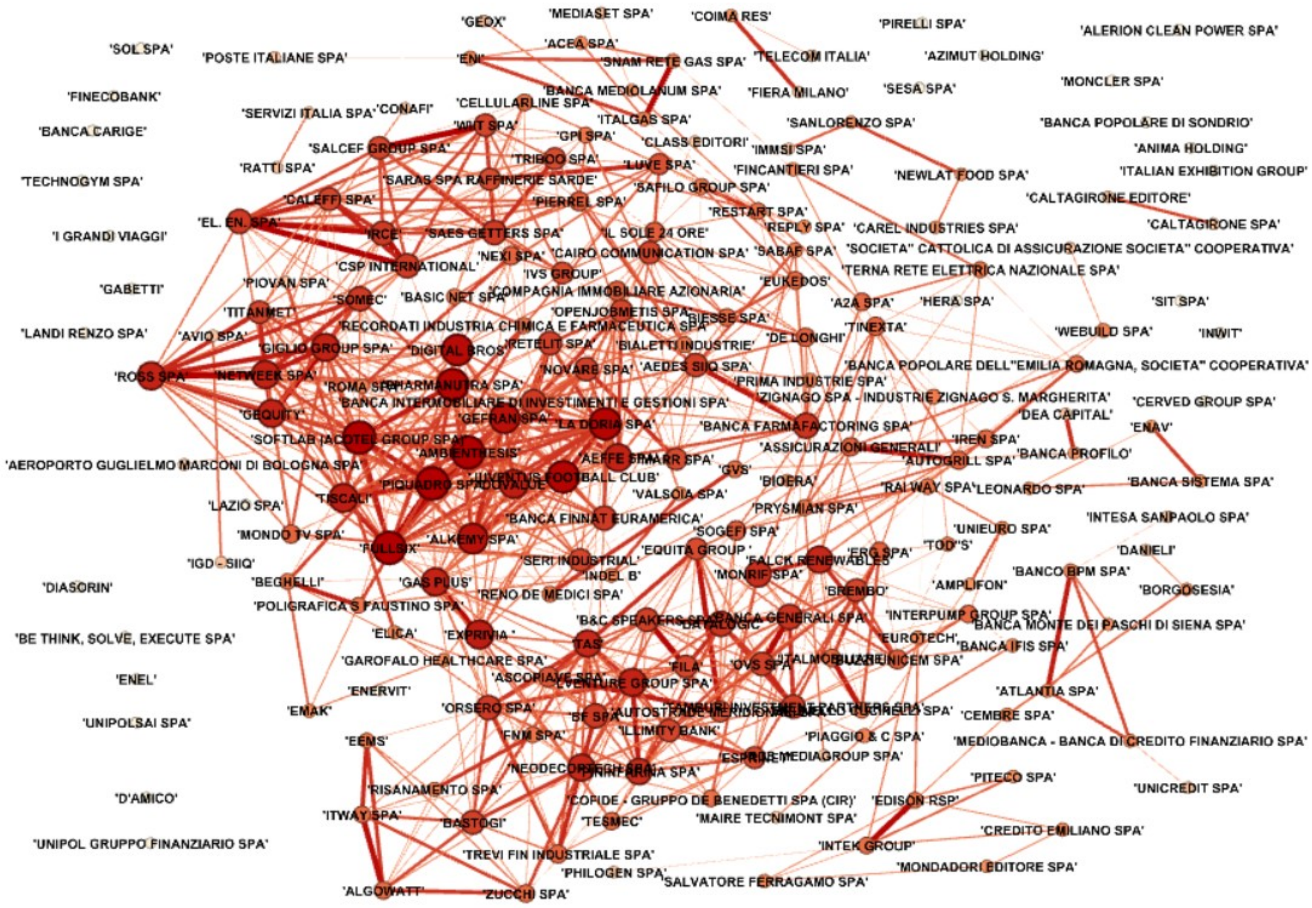

Figure A.4: Layer 3 Network 2

B | Projection networks' representation

Here it is possible to see an image of the two projection networks. These networks are described in Chapter 5. These images have been obtained using *Gephi*.

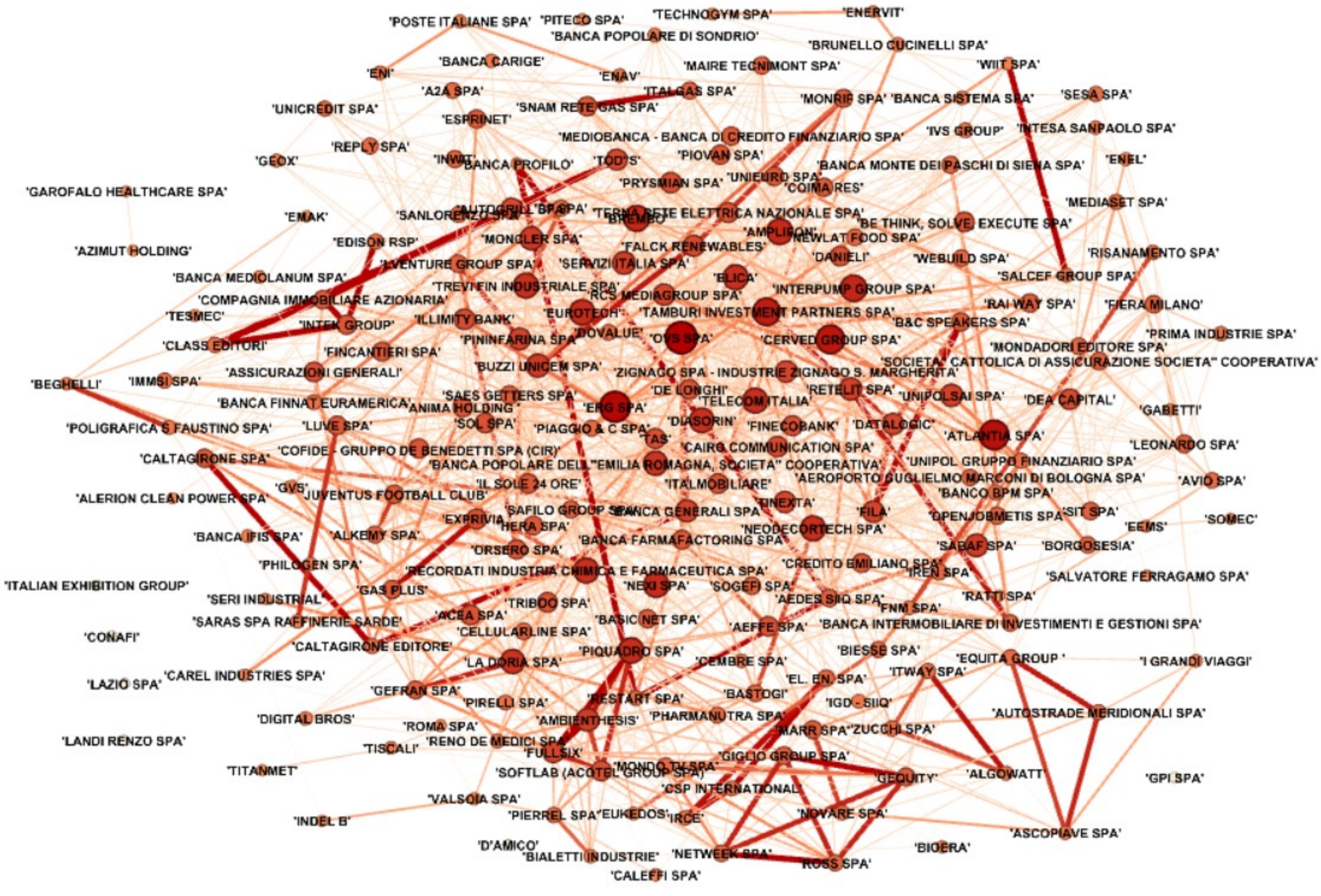

Figure B.1: Projection network 1

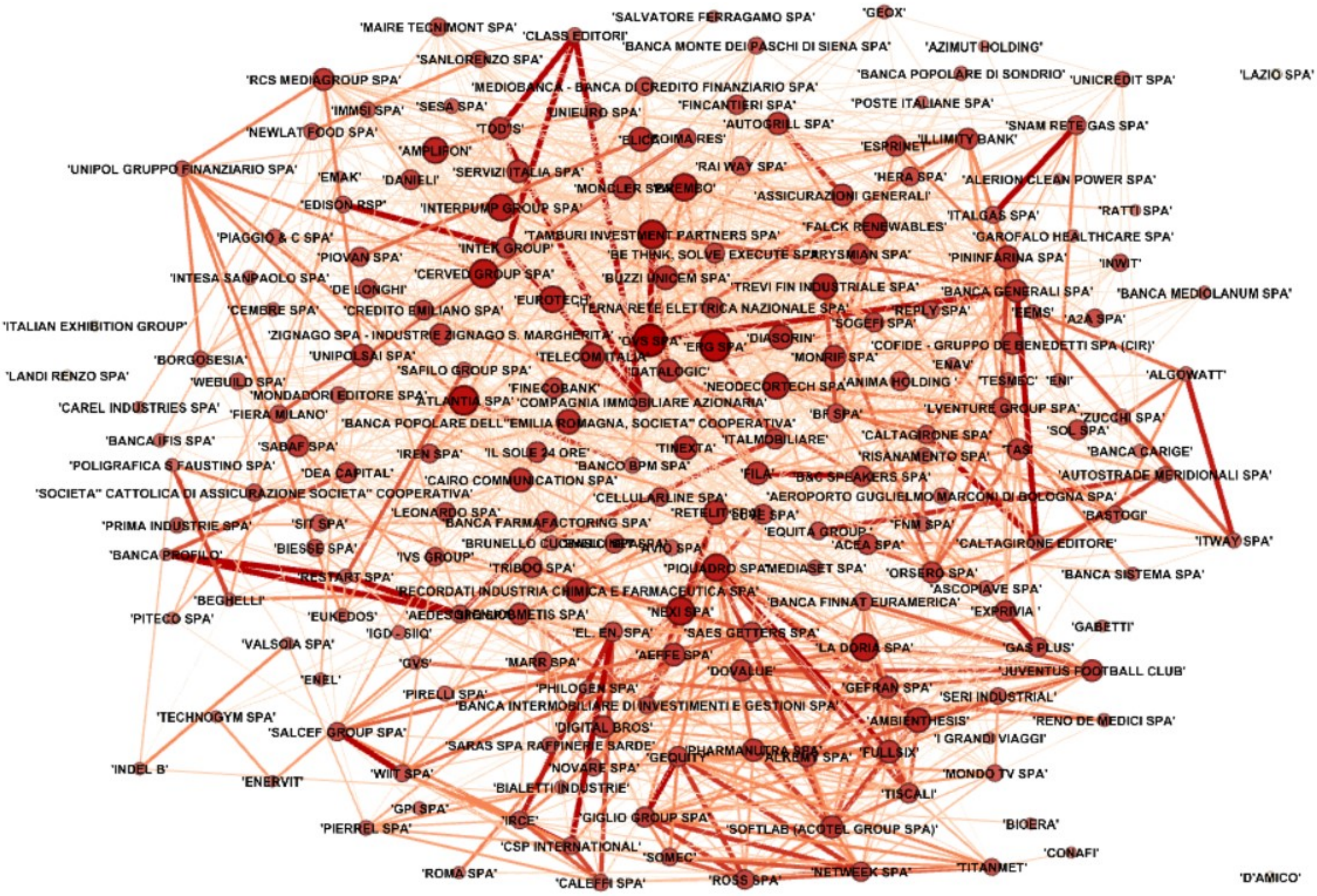

Figure B.2: Projection network 2

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