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Reproducing kernel techniques and polyanalytic function theory in Hypercomplex analysis

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Abstract

In this dissertation, we present some mathematical methods and techniques involving reproducing kernel Hilbert spaces (RKHSs) and associated integral transforms in the setting of complex, quaternions and Clifford analysis. The focus will be on some particular examples such as Fock spaces and Segal-Bargmann theory, Bergman spaces, Hardy spaces and Gabor spaces. These models are very important in several areas of mathematics including complex analysis, functional analysis, operator theory, etc. They have some important applications in mathematical physics, more precisely, in quantum mechanics, signal processing and time frequency analysis. It turns out that such spaces are relevant also to develop support vector machines (SVMs) kernel methods. In particular, Fock spaces are related to the radial basis function (RBF) kernels that are popular kernels used in machine learning.

In the first part of the thesis, we study Fock spaces of slice hyperholomorphic functions in the Hilbert and Banach cases. We obtain new quaternionic approximation results both in the first and second kind theory. We develop also Segal-Bargmann transforms in the noncommutative case of quaternions and give descriptions in terms of generalized versions of the creation and annihilation operators. In particular, we deal with an extension of the Cholewinski-Fock space in this setting. Moreover, based on the quaternionic Bargmann transform we introduce and study a quaternionic short-time Fourier transform QSTFT with a Gaussian window that can be computed for hypercomplex signals.

In the second part of the thesis, we introduce a special Clifford-Appell system which can be obtained using the Fueter mapping theorem. We study the behaviour of such system of polynomials with respect to the classical Cauchy-Kowalevski product. Then, we present some new QRKHS of Fueter hyperholomorphic functions based on this Clifford-Appell system. We study in this case different kernel techniques and integral transforms concerning the Fock, Hardy and Bergman spaces and associated operators. We compute also the Bergman

kernel function on different quaternionic domains.

Finally in the last part of this thesis, we introduce a new theory of polyanalytic functions in hypercomplex analysis. It turns out that this theory contains an interesting subclass of special monogenic functions of axial type and we prove a poly Cauchy formula. Then, we relate different polyanalytic function theories in hypercomplex analysis by providing two extended versions of the famous Fueter-Sce-Qian mapping theorem. We prove also an integral representation of this result as a direct application of the poly Cauchy formula.

The results obtained in this dissertation open various questions and research problems to investigate in the future, that are discussed in the last section.

Summary

In this thesis, we study different reproducing kernel function spaces and associated integral transforms in the setting of complex, quaternions and Clifford analysis. In particular, we focus on some specific examples such as Segal-Bargmann-Fock spaces, Bergman spaces, Hardy spaces and Gabor spaces. These models are very important in complex analysis, operator theory and have several applications in mathematical physics, especially in quantum mechanics, and also in signal processing and time frequency analysis thanks to the link with the short-time Fourier transform. As it is well-known, in quantum mechanics physical quantities such as position, momentum and energy are represented by operators acting on some complex Hilbert spaces. In 1961, Bargmann constructed a Hilbert space of entire functions on which the creation and annihilation operators are adjoints of each other and satisfy the classical commutation rules. This space is known as Fock or Segal-Bargmann space. Moreover, to any particle moving on the real line is associated a wave function which defines some unit vector of the classical Schrödinger Hilbert space. This unit vector is mapped onto a special holomorphic function making use of a particular exponential kernel. The new resulting complex function is the so-called Segal-Bargmann transform. In the last years, this subject attracted several mathematicians and physicists working in the field of Clifford analysis and related topics. As a consequence, many results and research problems were considered and developed in this direction. In the hypercomplex setting, we investigated some new reproducing kernels and associated Hilbert spaces using different techniques and tools from complex and Clifford analysis motivated by such special integral transforms involved in several applications in mathematical physics, like Segal-Bargmann transforms in quantum mechanics and Gabor or short-time Fourier transforms in signal analysis.

Another main contribution that we achieved during this work is that we initiated exploring a new research path by extending the theory of slice regular or slice monogenic functions to higher order and considering the so-called slice poly-analytic or slice poly-monogenic function theory on which several questions are open now. These functions can be considered from different points of view. A first approach consists of considering the space of quaternions as union of complex planes and to see these functions as null solutions of the n-th power of the Cauchy-Riemann operator with respect to each complex plane. A second approach is based on the so-called poly-decomposition, which makes a slice poly-analytic function of order n obtained as a sum of n slice regular functions multiplied on the left by some conjugate powers. A third approach consists in considering slice poly-analytic functions in the kernel of the n-th power of a certain global operator with non-constant coefficients. We note also that a generalized version of the poly-Cauchy formula and the famous Fueter-Sce-Qian mapping theorem in Clifford analysis were also introduced and proved in this framework. This new construction allowed to relate the different poly function theories in hypercomplex analysis. Furthermore, an important fact that was observed is that this slice polyanalytic function theory contains one of the most important subclasses of the Cauchy-Fueter hyperholomorphic functions, namely the class of Fueter hyperholomorphic functions of axial type. A very natural and interesting problem that has to be conisdered now and which is still under investigation is to develop a natural S-functional calculus associated to this new poly-analytic function theory in hypercomplex analysis.

In this thesis, we dealt with different problems touching several topics including: slice hyperholomorphic and monogenic function theories, reproducing kernel theory, quaternionic approximation theory, Fock and Bergman spaces, poly-analytic function theory, Dirac operator in Clifford analysis, quaternionic Segal-Bargmann and Fourier transforms, poly-Fueter mapping theorems and their applications, Clifford-Appell systems, Short-time Fourier transforms and reproducing kernel Gabor spaces, hypercomplex Hardy spaces and Schur analysis, etc. We give here a brief overview on the different results obtained:

- In [62], jointly with Prof. Sabadini and Prof. Gal, we introduced the Banach Fock spaces of slice hyperholomorphic functions on the quaternions, both of the first and of the second kind. In particular, we proved several approximation results on these different spaces, some of them are based on constructive methods making use of the Taylor expansion and the convolution polynomials. The techniques used in these two cases are different. Moreover, for the second kind theory, we discussed also some density results of reproducing kernels. This paper extends some classical results of complex analysis contained in the famous book of Kehe Zhu titled *"Analysis on Fock spaces"*.

- The Cholewinski-Fock space in the slice hyperholomorphic setting was studied in [61]. It presents an extension of the classical slice hyperholomorphic Fock space introduced in 2014 by Alpay, Colombo, Sabadini and Salomon. This was possible by considering on the space of slice entire functions a specific weight involving a modified Bessel function of the third kind, namely the Macdonald function. We gave a complete description of this quaternionic Hilbert space. Then, its reproducing kernel is obtained making use of the slice hyperholomorphic extension of the classical complex Dunkl kernel. We introduced also an associated unitary integral transform, and studied some specific quaternionic operators on the slice hyperholomorphic Cholewinski-Fock space. This construction follows an approach by Cholewinski in 1984.

- In [56], with my colleague De Martino, we introduced a new quaternionic short-time Fourier transform QSTFT with a Gaussian window. We proved there several results about this QSTFT like Moyal formula, reconstruction formula and Lieb uncertainty principle. This construction was possible thanks to the use of the quaternionic Segal-Bargmann transform. Moreover, we computed the reproducing kernel associated to the Gabor space considered in this framework.

- The results obtained in [63], joint with Prof. Sabadini and Prof. Krausshar can be considered as applications of the famous Fueter-Sce-Qian mapping theorem. It is well-known in the literature that this theorem relates two main theories in Clifford analysis, namely the recent theory of slice-monogenic functions and the classical one of monogenic functions (i.e: solutions of Dirac operator). More precisely, making use of the Fueter-Sce-Qian mapping theorem we constructed and studied some special integral transforms of Bargmann-Fock type in the setting of quaternion slice hyperholomorphic and Cauchy-Fueter regular functions. In particular, starting with the normalized Hermite functions we got an Appell system of quaternionic regular polynomials. We obtained also some new integral representations and generating functions in both the Fock and Bergman cases. In this article, we computed also the explicit expressions of the slice hyperlomorphic Bergman kernels on the quaternionic unit half ball and the fractional wedge domain. We discussed also the Bergman-Fueter transforms and presented some of its consequences.

- The paper [8] is a joint work with Prof. Sabadini and Prof. Alpay. It deals with a specific system of Clifford-Appell polynomials and in particular their Cauchy-Kowalevski product. We first study how this Clifford-Appell system behave with respect to the CK product. We gave also a characterization of axially Fueter regular functions in terms of this Clifford-Appell system. We introduced there a new family of quaternionic reproducing kernel Hilbert spaces in the framework of Fueter regular functions. This construction is based on a general idea which allows to obtain various function spaces, by specifying a suitable sequence of real numbers. We focused more on the Fock and Hardy cases and associated operators like creation, annihilation, shift and backward shift operators. We studied also the action of the Fueter mapping and its range.

- In [5], jointly with Prof. Sabadini, Prof. Colombo and Prof. Alpay, we started a new research direction to begin the study of Schur analysis and de Branges-Rovnyak spaces in the framework of Fueter hyperholomorphic functions. In this paper we treated several problems related to Hardy space, Schur multipliers, Blaschke functions, Herglotz multipliers and their associated kernels and Hilbert spaces, based on the Cauchy-Kowalesvkay product and the notion of Appell-like polynomials.

- In [17], jointly with Prof. Sabadini and Prof. Alpay, we proposed a new definition extending to higher order the theory of slice hyperholomorphic functions on the quaternions originally introduced by Gentili and Struppa in 2007. This definition extends the notion of complex polyanalytic functions to quaternions. We studied some basic properties of such functions and proved the counterparts of the following results : Splitting Lemma, Identity Principle, Representation Formula, Extension Lemma, Refined Splitting Lemma and presented some of their consequences. We proved also a very important characterization of slice polyanalytic functions, namely the so-called poly-decomposition. Then, we considered the Fock and Bergman spaces in this new setting and computed explicit expressions of their reproducing kernels.

- In [9], jointly with Prof. Sabadini and Prof. Alpay, we proved that slice polyanalytic functions of order $n \geq 1$ on quaternions can be considered as null solutions of the n-th power of some special global operator with nonconstant coefficients as it happens in the case of slice hyperholomorphic functions. We investigated also some extension versions of the Fueter mapping theorem in this polyanalytic setting. In particular, we showed that under axially symmetric conditions it is always possible to construct both Fueter regular and poly-Fueter regular functions through slice polyanalytic ones using what we call the poly-Fueter mappings. This allows to present two different extended formulations of the poly-Fueter mapping theorem. Furthermore, we proved a new poly-Cauchy formula that suggests to start several new interesting research problems. In particular, as a first application of this poly-Cauchy formula we gave the integral representation of the poly-Fueter mapping theorem, extending a very important result obtained in 2010 by Colombo, Sabadini and Sommen.

As avenues for further research, we already started some new projects that are still under progress. We plan to develop them more in the future and to start new research investigations in some recent related topics such as:

1. Wiener algebra on quaternions: The Fueter variables case.
2. PS and PF functional calculus and their applications.
3. Poly-Bergman-Fueter transforms.
4. Fischer decomposition in the space of slice hyperholomorphic functions.
5. Short-time Fourier transforms with Hermite windows: hypercomplex poly-analytic framework and applications in time-frequency analysis.
6. Quaternionic support vector machines, reproducing kernel methods in machine learning and stochastic processes.

List of symbols

- $\mathbb{H}$ the space of quaternions
- $\mathbb{S}$ the sphere of imaginary units
- $\mathbb{B}$ the quaternionic unit ball
- $\mathbb{C}_I$ the complex plane corresponding to the imaginary unit I
- $\overline{\partial_I}$ the Cauchy-Riemann operator on the slice $\mathbb{C}_I$
- ∂_S the slice derivative
- V the global operator
- $\mathcal{D}$ or ∂ the Cauchy-Fueter operator
- $\mathcal{SR}(\Omega)$ the space of slice regular functions on Ω
- $\mathcal{SP}_n(\Omega)$ the space of slice polyanalytic functions of order n on Ω
- $\mathcal{R}(\Omega)$ or $\mathcal{FR}(\Omega)$ the space of Cauchy-Fueter regular functions on Ω
- $\mathcal{R}_n(\Omega)$ or $\mathcal{FR}_n(\Omega)$ the space of polyanalytic Cauchy-Fueter regular functions of order n on Ω
- $\mathcal{F}_{Slice}(\mathbb{H})$ the quaternionic Fock space of slice hyperholomorphic functions
- $\mathcal{F}^p_\alpha(\mathbb{H})$ the quaternionic Fock spaces of the first kind
- $\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$ the quaternionic Fock spaces of the second kind
- $\mathcal{A}_{Slice}(\mathbb{B})$ the quaternionic Bergman space of slice hyperholomorphic functions of the second kind
- Δ the Laplace operator on $\mathbb{R}^4$

- τ the Fueter mapping
- Q_n the quaternionic Appell polynomials
- P_n the quaternionic Appell-like polynomials
- τ_n the poly-Fueter mapping of order n (form I)
- $\mathcal{C}_n$ the poly-Fueter mapping of order n (form II)
- ζ_l the Fueter variables for $l = 1, 2, 3$
- $\mathbb{R}_n$ the Clifford algebra over n-imaginary units
- QRKHS stands for quaternionic reproducing kernel Hilbert space
- QSTFT stands for quaternionic short-time Fourier transform
- SVMs stands for support vector machines
- RBF stands for radial basis function

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CHAPTER 1

Introduction

In the noncommutative setting, the main function theories that extend complex analysis, operator theory and their mathematical physics applications to higher dimensions are the so-called monogenic and slice monogenic functions with values in a Clifford algebra. In the case of a quaternionic variable these two theories are known respectively as Fueter regular (or Fueter hyperholomorphic) and slice regular (or slice hyperholomorphic) functions, see [28,35,47,75,83]. It is interesting to investigate any possible relations and intersections between these two different function theories. For example, we note that it is always possible to construct Fueter hyperholomorphic functions starting from slice regular ones using different techniques such as the Fueter mapping theorem and its inverse [48, 49], or using the Radon and dual Radon transforms, see [44]. But in general, the slice monogenicity does not imply, nor is implied by monogenicity.

In 1931, the work of Moisil [95] was at the origin of extending the classical theory of holomorphic functions in complex analysis to quaternions by generalizing the classical Cauchy-Riemann operator. Then, in 1935, Fueter developed this approach by Moisil and introduced a new theory of quaternionic regular functions generalizing the classical one of holomorphic functions, see [67]. This theory is based on the well-known Cauchy-Fueter operator defined by

$$\mathcal{D} := \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Thus, a quaternionic valued function is said to be Fueter regular or Fueter hyperholomorphic if it solves the equation

$$\mathcal{D}f \equiv 0.$$

It turns out that Fueter's theory of quaternionic regular functions generalized several complex analysis results using new techniques, in particular it is possible to consider the counterparts of different notions, such as a Cauchy kernel, Cauchy formula, identity principle, Liouville theorem, etc. Unfortunately, the elementary functions like the quaternionic polynomials and power series are not regular with respect to the Fueter theory. We will revise some facts in Chapter 2 about this theory, but for more details we suggest the reader to consult the books [47, 83].

In 2006-2007 a new function theory of a quaternionic variable extending the classical theory of complex analysis to the quaternionic setting has been introduced by Gentili and Struppa in [76, 77]. Then, it was also extended to the Clifford valued functions, by considering the so-called slice monogenic functions, see [36]. Both these theories include all the elementary transcendental functions. This new class of functions was extensively developed in the last years and found several applications in different topics in mathematics and physics, including for example Schur analysis and quaternionic operator theory, see [7,35]. Moreover, this new noncommutative function theory is now considered more suitable for applications in quaternionic quantum mechanics thanks to the discovery of the new notion of the S-spectrum which allowed to develop a new S-functional calculus for quaternionic operators, see the books [35, 37, 38]. Furthermore, this theory is useful also to develop the formalism for quaternionic quantum mechanics, see [97] and the classical book [4].

In general, positive definite functions and reproducing kernel Hilbert spaces appear in several areas of mathematics, physical sciences and engineering. They are relevant not only in operator theory and coherent states in quantum mechanics but also in the study of support vector machines and kernel methods in machine learning. Thus, a current intresting topic in hypercomplex analysis is related to quaternionic reproducing kernel Hilbert spaces like Hardy, Bergman, Besov, Dirichlet and Fock spaces in the new quaternionic and slice monogenic setting. Thanks to their use in quantum mechanics and signal processing such reproducing kernel Hilbert spaces, especially Fock spaces and associated Segal-Bargmann integral transforms attracted recently the attention of several mathematicians and physicists from different points of views, see for example [15, 34, 46, 53, 60, 88, 96, 101]. In particular, we present in this research project several results related to this topic. Indeed, we considered different quaternionic reproducing kernel Hilbert spaces and associated integral transforms, like Segal-Bargmann transforms and Gabor or short-time Fourier transforms in the noncommutative framework both in slice and Fueter hyperholomorphic theories. So, we would like to present briefly in this introduction the state of the art related to such topics in classical complex analysis. We explain also some interactions with mathematical physics and present some quantum mechanics interpretations of such mathematical objects.

Indeed, contrary to classical physics, in quantum mechanics physical quantities such as position, momentum and energy are represented by operators acting on some complex Hilbert space. We note that in 1961 Bargmann introduced in his original paper [23] a Hilbert space of entire functions on which the creation and annihilation operators, namely

$$M_z f(z) := zf(z) \quad \text{and} \quad D_z f(z) := \frac{d}{dz} f(z)$$

are closed, densely defined operators that are adjoints of each other and satisfy the classical commutation rule

$$[D_z, M_z] = \mathcal{I}$$

where $[.,.]$ and $\mathcal{I}$ are respectively the commutator and the identity operator. In addition to that, it turns out that the creation and annihilation operators are unitary equivalent to the classical position and momentum operators of quantum mechanics trough the well-known Segal-Bargmann transform which was also introduced in the same paper [23].

The latter is an integral transform mapping unitary the classical Schrödinger Hilbert space of wave functions $L^2(\mathbb{R}^n)$ onto a space of holomorphic functions. In the literature, this output space is known as Fock or Segal-Bargmann space and sometimes called also the bosonic Fock space with n degrees of freedom. It consists of entire functions that are square integrable on the complex plane with respect to the normalized Gaussian measure. We refer to [85, 98, 115] for more detailed explanations.

It turns out that the Bargmann-Fock spaces and associated Segal-Bargmann transforms are important mathematical models used in classical (complex) quantum mechanics and signal processing. Indeed, from one side this integral transform allows to construct a bridge between the Schrödinger Hilbert space and a special Hilbert space of holomorphic functions on the whole plane, namely entire functions. From the other side, this transform can be obtained as the short-time Fourier transform with a specific Gaussian window. Moreover, by taking Hermite windows it is possible to find connections with polyanalytic function theory.

It was explained in chapter 3 of [85] that it is impossible to predict the result of an experiment in quantum mechanics. Only the probabilities of the outcome of a measurement can be predicted and these probabilities are encoded in a *wave function* that is a function of the real variable $x \in \mathbb{R}^n$. Then, we note that to any particle moving in the real line is associated a wave function $\psi : \mathbb{R} \longrightarrow \mathbb{C}$. The square modules of this function

$$x \mapsto |\psi(x)|^2$$

is interpreted as the probability density for the position of this particle. More precisely, taking a region $A \subset \mathbb{R}$ the quantity

$$\int_A |\psi(x)|^2 dx$$

is defined to be the probability that the given particle belongs to the set A. Obviously, the probability that its position is on the whole real line $\mathbb{R}$ should be equal to 1 so that we have

$$\|\psi\|^2 := \int_{\mathbb{R}} |\psi(x)|^2 dx = 1.$$

The wave function is said to be a unit vector of the Hilbert space $L^2(\mathbb{R})$ in this case. Then, to each unit vector in the standard Hilbert space on the real line is associated a holomorphic function which is also a unit vector of the Fock space. The new output complex function is the Segal-Bargmann transform.

Such spaces have been considered in some higher dimensional extensions of complex analysis, namely the analysis based on functions with values in a Clifford algebra or, in particular, quaternions. For some recent works, we refer the reader to e.g. [101] which is the framework of monogenic functions, to [15, 60] in the framework of slice hyperholomorphic functions and to [88], which makes use of slice hyperholomorphic functions and in which the authors point out the link with the study of quantum systems with internal, discrete degrees of freedom corresponding to nonzero spins. We note also that the class of slice hyperholomorphic functions [35,50,75] has attracted interest in the past decade for its various applications especially in operator theory. One of its features is that it contains power series (despite what happens for other theories of hypercomplex variables) thus it is natural to consider functions which are "entire" in this class and, in particular, Fock spaces. In these directions, we obtained several results related to different topics such as: reproducing kernels, Fock spaces, Bergman spaces, Segal-Bargmann transforms, quaternionic approximation theory, short-time Fourier transform, Fueter mapping theorem, Dirac operator, etc.

In this thesis it was also very useful for us to understand the so-called Appell systems. Actually, in 1880, the French mathematician Appell introduced a new class of polynomial sequences generalizing the well-known property satisfied by the classical monomials with respect to the real derivative, namely

$$\frac{d}{dx} P_n = n P_{n-1},$$

see [20]. So, a polynomial sequence $\{P_n\}_{n\geq 0}$ of degree n satisfying such an identity with respect to a derivative is called an Appell set or an Appell sequence. In [32, 107] the authors followed a different approach to define an Appell set by requesting the identity

$$\frac{d}{dx} P_n = P_{n-1}.$$

In the classical case, where x is interpreted as a real or complex variable, the standard monomials $P_n(x) = x^n$ form an Appell set, but also the famous Hermite, Bernoulli and Euler polynomials are examples of Appell sets. The importance of such polynomials in various settings is well known, and we mention here, with no pretense of completeness their relevance in probability theory and stochastic process since they can be connected to random variables, see [111], they were used also to study optimal stopping problems related to Lévy process in [105]. Moving to the hypercomplex analysis setting, we have various function theories, associated with different differential operators.
In the slice hyerholomorphic setting, Appell systems can be obtained by simply extending the variable in use to become hypercomplex, and so we have that, for example, the standard monomials in the quaternionic variable are among them with respect to the slice derivative. But these sets of polynomials were studied also in the setting of quaternionic and Clifford analysis with respect to the hypercomplex derivative, see [29, 30, 63, 93, 99]. It turns out that the Appell systems in this framework play a similar role as the complex monomials do to define elementary functions in terms of their power series like cosine, sine, exponential, etc. This fact opens a variety of questions about such Appell systems also in relation to various function spaces including Fock, Hardy, Bergman, Dirichlet spaces, etc. Various questions arise also about their associated operators such as creation, annihilation, shift and backward shift operators. Actually, what makes Appell systems in quaternionic and Clifford analysis rather peculiar, is the fact that the function theory has been developed using the so-called Fueter polynomials, see [28, 83], and these polynomials do not satisfy the Appell property in general. However, a series expansion for hyperholomorphic functions is possible using both the approaches. In this dissertation, we present aslo some results in this direction.

One of the main achievements that we made also in this work is that we introduced in [9, 17] a new research direction which is opening several interesting questions to investigate. Indeed, we extended the notion of slice regular functions to higher order by considering the so-called slice polyanalytic functions. In particular, this gives two different directions of the extension, from one side the new theory that we proposed extended the complex polyanalytic function theory to the noncommutative setting. On the other side, it can be seen also as an extension of the original quaternionic function theory introduced by Gentili and Struppa to higher orders. We got also a counterpart of the poly-Cauchy formula for such functions. Futhermore, we gave two possible poly extensions of the famous Fueter-Sce-Qian mapping theorem and proved an integral representation of this result based on a certain global operator with nonconstant coefficients. These slice polyanayltic functions can be seen from three different points of view. The first approach consists of considering the space of quaternions $\mathbb{H}$ as union of complex planes and to see these functions as null solutions of the n-th power of the Cauchy-Riemann operator with respect to each complex plane, i.e if on any complex plane f satisfies the equation

$$\overline{\partial_I}^n f(u+vI) := \frac{1}{2^n}\left(\frac{\partial}{\partial u}+I\frac{\partial}{\partial v}\right)^n f_I(u+Iv)=0.$$

The second approach is based on the so-called poly-decomposition which allows to consider such functions as sums of the form

$$\sum_{k=0}^{n-1} \overline{x}^k f_k(x), \qquad x\in\mathbb{H}$$

with all the components f_k which are slice regular functions and n is the order of poly-analyticity. The third approach consists in considering slice polyanalytic functions as elements in the kernel of the n-th power of the global operator with nonconstant coefficients V, see [9]. In this sense, we have

$$V^n(f)(x) := \left(\partial_{x_0}+\frac{\vec{x}}{|\vec{x}|^2}\sum_{l=1}^{3} x_l\partial_{x_l}\right)^n f(x)=0.$$

Furthermore, using the Fueter mapping theorem it was possible to introduce some special Appell polynomials $(Q_n(x))_{n\geq 0}$ where

$$Q_n(x)=\sum_{j=0}^{n} T_j^n \bar{x}^j x^{n-j}, \qquad n\geq 0,$$

that are at the same time Fueter hyperholomorphic and slice polyanalytic functions of order $n+1$, for suitable real coefficients T_j^n, see chapters 7, 8 and 9 for more explanations and bibliography notes related to such coefficients. These polynomials are very special since they belong to the intersection of two different noncommutative function theories, namely the classical Fueter theory and the slice polyanalytic theory, moreover they have nice properties with respect to the CK product and hypercomplex derivative. Another important feature, see Theorem 3.10 in [8], is that we proved that any Fueter hyperholomorphic function f of axial type admits a power series expansion in terms of the polynomials

Q_n of the form

$$f(x) = \sum_{n=0}^{\infty} Q_n(x) u_n, \qquad u_n \in \mathbb{H}.$$

This fact allows to embed the space of Fueter hyperholomorphic functions of axial type, denoted by $\mathcal{AR}$, into a space consisting of series of slice polynalytic functions that we denote here by

$$\mathcal{SP}_\infty := \mathcal{SP}_1 + \mathcal{SP}_2 + ... + \mathcal{SP}_{n+1} + ...,$$

where $\mathcal{SP}_n$ denotes the space of slice polyanalytic functions of order n. More precisely we consider the subspaces of slice polyanalytic functions associated to the polynomials $(Q_n)_{n\geq 0}$ defined by

$$\mathcal{Q}_n := \{Q_n(x)\lambda, \ \lambda \in \mathbb{H}\}$$

and

$$\mathcal{Q}_\infty := \bigoplus_{n=0}^{\infty} \mathcal{Q}_n.$$

We can show that the space of hyperholomorphic functions of axial type $\mathcal{AR}$ corresponds to the space $\mathcal{Q}_\infty$, i.e.

$$\mathcal{AR} = \mathcal{Q}_\infty.$$

The previous subspaces of slice polyanalytic functions $\mathcal{Q}_n$ were already considered before but from a different point of view and using a different terminology, namely they were called spaces of homogeneous special monogenic polynomials of degree n, see for example Lemma 1 in [3]. Using these ideas and identifications we can show also that it is always possible to embed this interesting subclass of special monogenic functions in a more general framework of slice polyanalytic functions. In particular, we can use techniques from slice polyanalytic function theory to prove results on such special monogenic functions. For example in Chapter 9 we can prove a Representation Formula in the monogenic setting using a slice polyanalytic approach. Furthermore, we note that these slice polyanalytic (and Fueter hyperholomorphic) polynomials $(Q_n)_{n\geq 0}$ are just a particular case of a more general interesting construction which makes use of the classical Cauchy-Kovalevskaya extension theorem. We explained more in details this general construction thanks to some new Appell-like polynomials and the classical CK product in Clifford analysis, see [5].

Description of the contents

This thesis is divided into 11 chapters besides this introduction. The second and third chapters revise briefly the state of the art and main backgrounds. We present there some very well-known results in the litterature related to positive definite functions, reproducing kernel Hilbert spaces and associated operators. We discuss also the different hypercomplex function theories and their connections based on the Fueter-Sce-Qian mapping theorems. The main results and contributions of our research are presented from chapter 4 to chapter 12. We give now a brief description of the contents following the chapters order in the manuscript:

- *Chapter 4:* This chapter is based on [62]. It continues the study of slice hyperholomorphic Fock spaces over the quaternions started in [15] with the purpose of providing some approximation results, specifically our goal is to extend results on the density of polynomials in the complex case to the slice hyperholomorphic setting. We shall show that in this context one may define two types of Fock spaces, which are called of the first and of the second kind, and for which the approximation results require different techniques. The plan of the chapter is the following: Section 2 review the Hilbert slice hyperholomorphic Fock space and quaternionic Segal-Bargmann transform. Section 3 introduces Banach Fock spaces of the first and second kind. In Section 4 we study the approximation in Fock spaces of the first kind. In Section 5 we study the approximation in Fock spaces of the second kind, obtaining a result of general validity. We obtain quantitative estimates in terms of higher order moduli of smoothness and of best approximation quantity. Finally we discuss type and order of functions in the Fock spaces of the second kind.

- *Chapter 5:* This chapter is based on [61]. Its purpose is to continue this exploration of generalized Fock spaces following an approach by Cholewinski, Sifi and Soltani in order to present a study of a quaternionic Hilbert space of slice entire functions weighted by a modified Bessel function that we shall call the quaternionic slice hyperholomorphic Cholewinski-Fock space or the slice Cholewinski-Fock space for short. This will allow us to extend some results obtained in [15, 60] on the slice hyperholomorphic Fock space and the quaternionic analogue of the Segal-Bargmann transform. Moreover, we study there some specific quaternionic operators associated to the slice Cholewinski-Fock space. In a particular case, we show that the slice derivative and the quaternionic multiplication are adjoints of each other and satisfy the classical commutation rule on the slice hyperholomorphic Fock space.

  The chapter has the following structure: we first give some motivations for this study. Then, in Section 2 we collect some basic facts about the Macdonald function as it will be needed in the sequel. In Section 3, we define the slice Cholewinski-Fock space and we introduce an orthonormal basis. Moreover, we show that it is a quaternionic reproducing kernel Hilbert space. Section 4 is devoted to the study of a quaternionic unitary isomorphism between the slice Cholewinski-Fock space and a suitable quaternionic Hilbert space on the real line. This quaternionic isomorphism will be connected also to what we call the slice Dunkl transform. Then, Section 5 deals with two right quaternionic linear operators that are proved to be adjoint of each other and satisfy a specific commutation rule on the slice Cholewinski-Fock space. Finally, the last section explains how the results obtained in the slice quaternionic setting could be extended in a similar way to the slice monogenic setting for Clifford algebras valued functions.

- *Chapter 6:* This Chapter is based on [56]. We introduce an extension of the short-time Fourier transform in a quaternionic setting in dimension one. To this end, we fix a property that relates the complex short-time Fourier transform and the complex Segal-Bargmann transform:

$$V_\varphi f(x,\omega) = e^{-\pi i x\omega} Gf(\bar{z}) e^{\frac{-\pi|z|^2}{2}}, \tag{1.0.1}$$

  where V_φ is the complex short-time Fourier transform with respect to the Gaussian window φ and $Gf(z)$ denotes the complex version of the Segal-Bargmann transform according to [78]. To achieve our aim we use the quaternionic Segal-Bargmann transform studied in [60]. In order to present these results, we adopt the following structure: After a brief motivation to the topic, in Section 2 we prove some new properties of the quaternionic Segal-Bargmann transform. In particular we deal with an unitary property and give a characterization of the range of the Schwartz space. Moreover, we provide some calculations related to the position and the momentum operators. In Section 3, we give a brief overview of the 1D Fourier transform [65] and show a Plancherel theorem in this framework. In Section 4, we define the 1D QSTFT and prove an isometric relation, a Moyal formula, a reconstruction formula, etc. From this, it follows that the adjoint operator defines a left inverse. Furthermore, it gives the possibility to write the 1D QSTFT using the reproducing kernel associated to the Gabor space

$$\mathcal{G}_{\mathbb{H}}^{\varphi} := \{\mathcal{V}_\varphi f,\ f \in L^2(\mathbb{R},\mathbb{H})\}.$$

  Finally, we show that the 1D QSTFT follows a Lieb's uncertainty principle.

- *Chapter 7:* This Chapter is based on [63]. We construct a Clifford-Appell system of spherical monogenics in the right Fueter-Bargmann space over quaternions, denoted by $\mathcal{RB}(\mathbb{H})$, and consisting of quaternionic Fueter regular functions that are square integrable with respect to the Gaussian measure. The main tool that we use is the Fueter mapping theorem which relates slice hyperholomorphic functions to Fueter regular ones through the Laplacian. More precisely, we apply the Fueter mapping on a special basis of the slice hyperholomorphic Fock space constructed in [15] and obtain a set of homogeneous monogenic polynomials in the right monogenic Bargmann space over the quaternions. This allows us to construct

on the standard Hilbert space on the real line the so called Bargmann-Fock-Fueter integral transform whose range is a quaternionic reproducing kernel Hilbert space of Cauchy-Fueter regular functions. In particular, this may give a partial answer to Remark 4.6 in [88] about Clifford coherent state transforms using the Fueter mapping theorem in the setting of quaternions.
In order to present these results, we collect some basic definitions and preliminaries in Section 2. In Section 3, we study how the Fueter mapping acts on special basis elements of the slice hyperholomorphic Fock space. Then, we show that the obtained polynomials constitute an Appell set in the Bargmann space of Cauchy-Fueter regular functions. In Section 4, the Fock-Fueter kernel is discussed and the Bargmann-Fock-Fueter integral transform is introduced and studied. Section 5 presents a factorization of the Bargmann-Fock-Fueter transform.

- *Chapter 8:* This Chapter is a continuation of the previous one. It is based also on [63], we first deal with some explicit formulas for the slice hyperholomorphic Bergman kernels on some different quaternionic domains. We consider the case of the quaternionic unit half ball and the fractional wedge domain. Then, we treat an integral transform similar to the one considered in the previous chapter in the case of the Bergman spaces on the unit ball, on the half space and on the unit half-ball on quaternions.

- *Chapter 9:* This chapter is based on [5, 8]. In order to define and study quaternionic reproducing kernel Hilbert spaces the approach that makes use of the Appell systems looks very promising and allows to define the associated operators. We will show that using a special set of Clifford Appell polynomials, denoted by $\{Q_n\}_{n\geq 0}$, we can introduce various functions spaces denoted by $\mathcal{HM}_b$ whose elements are converging series of the form $\sum Q_n a_n$, where the quaternionic coefficients a_n satisfy suitable conditions which depend on a given sequence $b=(b_n)_{n\geq 0}$ of real (in fact rational) numbers. This approach is rather general, and it is used also in the slice hyperholomorphic setting in which the series under consideration are of the form $\sum q^n a_n$, where q denotes the quaternionic variable and give rise to spaces denoted by $\mathcal{HS}_c$, $c=(c_n)_{n\geq 0}$. In this chapter we treat the case of the quaternionic Fock and the Hardy spaces which have been already studied in the slice setting but are new in the Fueter regular framework combined with the Appell polynomials. For this reason, these spaces are called Clifford-Appell-Fock space and Clifford-Appell-Hardy space, respectively. One problem of the system $\{Q_n\}_{n\geq 0}$ is that if we multiply two such polynomials we do no obtain an element in the system. This is expected provided the non-commutative setting and in fact hyperholomorphic functions can be multiplied using the so-called CK product. With the polynomials Q_n there is the additional problem of remaining within the Appell system and in fact we show how this can be achieved. This

technical result opens the possibility to prove several results and also to introduce creation, annihilation and shift operators. An advantage of this description is that we can prove that the function spaces $\mathcal{HM}_b$ and $\mathcal{HS}_c$ for suitable choices of b, c, can be related using the Fueter mapping theorem.
The structure of the chapter is the following: in Section 2 we revise some notations and preliminary results that we need. In Section 3 we introduce some quaternionic reproducing kernel Hilbert spaces (QRKHS) based on a specific Appell system, and prove different properties on such kind of polynomials. We show also that, under suitable conditions, any axially Fueter regular function can be expanded in terms of these Appell polynomials. In Section 4 we focus more on the Fock space in this setting. In particular, we study different properties related to the notions of creation, annihilation operators and Segal-Bargmann transforms. In Section 5 we treat the Hardy space case, and study different properties related to the shift and backward shift operators. Section 6 shows how the Fueter mapping acts by sending spaces of slice hyperholomorphic functions into spaces of Fueter regular functions. Moreover, we prove that in some special cases the Fueter mapping acts as an isometric isomorphism up to a constant. Finally, in the last section we briefly present Appell-like polynomials and discuss a bit some results related to Schur analysis in this framework.

- *Chapter 10:* This chapter is based on [17]. We extend the definition of slice hyperholomorphic functions to higher order and define the slice polyanalytic functions of a quaternionic variable. Then, we shall use the obtained results to introduce and study the Fock and Bergman spaces of quaternionic slice polyanalytic functions and give explicit formulas for their reproducing kernels. Note that by considering polyanalytic functions with respect to the classical Cauchy-Fueter regularity on quaternions, it turns out that even the simple example given by $F(q, \overline{q}) = |q|^2$ is not polyanalytic of order 2. However, a natural question that may arise in this direction is about a possible extension of the well-known Fueter mapping theorem on quaternions allowing to construct Cauchy-Fueter polyanalytic functions starting from slice polyanalytic functions of the same order. The chapter has the following structure: in Section 2 we introduce the quaternionic slice polyanalytic functions and prove the poly-decomposition. In particular, on slice domains we show that a slice polyanalytic function is the sum of the quaternionic conjugate powers multiplied by slice regular functions, thus extending the analogous result for complex functions. We prove also the counterparts of the Splitting Lemma, Identity Principle, Representation Formula, Extension Lemma and the Refined Splitting Lemma in this framework. We also discuss slice polyanalytic functions as a subclass of slice functions on axially symmetric domains. In particular, we prove a version of the identity principle in this situation. In Section 3, we introduce and study the Fock space of slice polyanalytic functions on quaternions and we give the formula of its reproducing kernel. We treat also the case of the Bergman theory of the second kind in the quaternionic slice polyanalytic setting in the case of the unit ball.

- *Chapter 11:* This chapter is based on [9]. It proposes a bridge between two theories: the one of slice polyanalytic functions and the one of poly-Fueter regular functions. It is interesting to note that the class of slice hyperholomorphic functions is related with the class of functions considered by Fueter to construct regular functions and thus there is a bridge between them, specifically the so-called Fueter mapping, in fact by applying the Laplacian to a slice hyperhomolorphic function one obtains a regular function, i.e. a function in the kernel of the Cauchy-Fueter operator, see for example [48]. Also the theory of polyanalytic functions can be extended to the slice setting by considering a suitable definition, as we explained in chapter 10. Thus it is a natural question to ask whether there is an analog of the Fueter map in this more general setting. The answer is positive and it is one of the main results here: we show that by applying the Laplacian composed with the $(n-1)$ power of a global operator with non-constant coefficients to any slice polyanalytic function of order n we obtain a Cauchy-Fueter regular function. A second approach to extend the Fueter mapping to the polyanalytic setting consists to apply the standard Fueter mapping on each component associated to the poly-decomposition. This construction allows to generate poly-Fueter regular functions starting from slice polyanalytic ones of the same order.

  This Chapter has the following structure: in Section 2 we set up basic notations and revise some preliminary results. Section 3 contains some results on the powers of the global operator V and the main statements and proofs of the poly-Fueter mapping theorems. In Section 4 we prove a poly-Cauchy formula in this framework. Then, in Section 5 we study an integral representation of the poly-Fueter mapping theorem on the quaternionic unit ball. In Section 6, we rewrite our results in the slice polymonogenic case.

- *Chapter 12:* In this chapter we give a conclusion of this work. We present also some new research directions and perspectives that are still under investigations.

CHAPTER 2

Reproducing kernel Hilbert spaces in complex analysis

Positive definite functions and reproducing kernel Hilbert spaces play an important role in different areas of mathematics such as complex analysis, operator theory, Schur analysis, etc. They are used also to define coherent states in quantum mechanics and appear in the field of support vector machines and kernel methods in machine learning. In this chapter, we will revise the main properties of positive definite functions, reproducing kernel Hilbert spaces, and their associated operators. We will consider specific examples such as Hardy and Bergman spaces on the unit disk $\mathbb{D}$ and Fock spaces on the whole complex plane $\mathbb{C}$. Their polyanalytic counterparts will be briefly discussed also. The material revised in this chapter is based on the following references [10, 22, 85, 86, 98, 115]. Since most of the results presented in this chapter are very well-known and classical we have omitted to give proofs.

2.1 Positive definite kernels and RKHS

We start by recalling the notion of a positive definite kernel.

Definition 2.1.1. *Let Ω be a set. The function $K(z,w)$ from $\Omega\times\Omega$ into $\mathbb{C}$ is called a positive definite kernel if for every $N\in\mathbb{N}$, every choice of $w_1,\ldots,w_N\in\Omega$, and* *every choice of* $c_1, ..., c_N \in \mathbb{C}$*, we have*

$$\sum_{j,k=1}^{N} \overline{c_i} K(w_j, w_k) c_k \geq 0. \tag{2.1.1}$$

Remark 2.1.1. *We note that the condition* (2.1.1) *is equivalent to saying that the* $N \times N$ *matrix with* (j, k) *entry* $K(w_j, w_k)$ *is positive.*

The next examples are all positive definite kernels associated to some famous reproducing kernel Hilbert spaces.

Examples 2.1.2.

1. *Cauchy kernel:* $K_1(z, w) = \dfrac{1}{1 - z\overline{w}}, \quad \forall (z, w) \in \mathbb{D} \times \mathbb{D}.$

2. *Bergman kernel:* $K_2(z, w) = \dfrac{1}{(1 - z\overline{w})^2}, \quad \forall (z, w) \in \mathbb{D} \times \mathbb{D}.$

3. *Fock kernel:* $K_3(z, w) = e^{z\overline{w}}, \quad \forall (z, w) \in \mathbb{C} \times \mathbb{C}.$

4. *Poly-Fock kernel of order* 2*:* $K_4(z, w) = e^{z\overline{w}}(1 - |z - w|^2), \quad \forall (z, w) \in \mathbb{C} \times \mathbb{C}.$

We recall some basic facts on positive definite kernels:

Proposition 2.1.3. *Let* Ω *be a set in* $\mathbb{C}$*. Then, we have*

1. *If* K *is a positive definite kernel then* $K(z, w)$ *is Hermitian, that is*

$$K(z, w) = \overline{K(w, z)}, \quad \textit{for all } z, w \in \Omega.$$

2. *The sum of two positive definite kernels is still positive definite.*

3. *The product of two complex-valued positive definite kernels is still positive definite.*

4. *If* $K(z, w)$ *is positive definite so is* $F(z, w) = \overline{K(z, w)}$*.*

A very useful way to check that a given function is positive definite is to express it as an inner product. This observation is the idea behind *the kernel trick* and *feature mapping* terminology which are used in machine learning

Proposition 2.1.4. *Let* Ω *be some set and* $(\mathcal{H}, \langle ., . \rangle_{\mathcal{H}})$ *be a Hilbert space. Let* $z \longmapsto h_z$ *be a function from* Ω *into* $\mathcal{H}$*. Then, the function defined by*

$$K(z, w) = \langle h_w, h_z \rangle_{\mathcal{H}}$$

is positive definite. In general, the function $\varphi(z) = h_z$ *is called a feature map.*

Definition 2.1.2. *A Hilbert space $\mathcal{H}$ of functions defined on a set Ω is called a reproducing kernel Hilbert space if the point evaluations*

$$\Lambda_w : f \longmapsto f(w), \quad w \in \Omega$$

are bounded. Then, by Riesz representation theorem there exists a uniquely determined function $K(z, w)$ defined on $\Omega \times \Omega$, satisfying the two following properties:

i) For every $w \in \Omega$, the function

$$K_w : z \longmapsto K(z, w)$$

belongs to $\mathcal{H}$.

ii) Reproducing property: for every $f \in \mathcal{H}$ and $w \in \Omega$, we have

$$\langle K_w, f \rangle_{\mathcal{H}} = f(w).$$

The function $K(z, w)$ is positive definite and is called the reproducing kernel of $\mathcal{H}$.

Conversely, we have the following fundamental result which is known as Moore-Aronszajn theorem, see for example [10]:

Theorem 2.1.5. *Associated to a function $K(z, w)$ positive definite on a set Ω is uniquely determined a Hilbert space $\mathcal{H}(K)$, whose elements are functions on Ω, and with reproducing kernel $K(z, w)$.*

By the end of this section, we give two important kernel examples that are used in machine learning algorithms, see [110, 112].

1. **Radial basis function kernel (RBF kernel):** in machine learning, the RBF kernel is a popular kernel function used in various kernelized learning algorithms. In particular, it is used in support vector machines (SVMs) classification. For two samples x and x', sometimes called also feature vectors, the RBF kernel is defined by

$$K(x, x') = \exp\left(-\frac{||x - x'||^2}{2\sigma^2}\right),$$

   where $||x - x'||^2$ is the the square Euclidean distance between the two feature vectors and $\sigma > 0$ is a free parameter.

2. **The polynomial kernel:** in machine learning, the polynomial kernel is a kernel function used with SVMs and other kernelized models, that represents similarity of vectors (training samples) in a feature spaces over polynomials of the original variables, allowing learning of non-linear models. For polynomials of degree d, the polynomial kernel is defined as

$$K(x, y) = (x^T y + c)^d,$$

   where x and y are vectors in the input space and $c \geq 0$ is a free parameter.

Althought the RBF kernel is more popular in SVM classification than the polynomial kernel, the latter is quite popular in natural language processing (NLP).

2.2 Hardy and Bergman spaces

In this section, we revise briefly some classical reproducing kernel Hilbert spaces on the complex unit disk, in particular we recall the Hardy and Bergman spaces based on [10].
The Hardy space $\mathbf{H}_2(\mathbb{D})$ (see Definition below) provides a convenient setting to describe shift-invariant subspaces of $\ell^2(\mathbb{N})$, and this is one of the main motivations for introducing this space. It has applications in several other problems in analysis and digital signal processing. Indeed, a sequence in $\ell^2(\mathbb{N})$ represents a finite energy discrete signal, and its associated power series belongs to $\mathbf{H}_2(\mathbb{D})$. This allows to transform different problems in signal processing into problems in the setting of function theory in the open unit disk. We recall that, given a function f analytic in the open unit disk, the function

$$M_2(r) = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad r \in (0,1)$$

is increasing. Then, we recall the Hardy space definition

Definition 2.2.1. *The Hardy space $\mathbf{H}_2(\mathbb{D})$ is the set of analytic functions in $\mathbb{D}$ such that*

$$\sup_{r\in(0,1)} M_2(r) = \sup_{r\in(0,1)} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty.$$

An equivalent characterization of the Hardy space is given by

Theorem 2.2.1. *A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined on the unit disk, belongs to the Hardy space $\mathbf{H}_2(\mathbb{D})$ if and only if it holds that*

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Theorem 2.2.2. *The Hardy space $\mathbf{H}_2(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel given by the kernel function,*

$$K_{\mathbf{H}_2(\mathbb{D})}(z,w) = \frac{1}{1-z\overline{w}}, \quad \textit{for all } z,w \in \mathbb{D}. \tag{2.2.1}$$

Two important operators that appear in Hardy spaces theory are the shift and backward shift operators, defined and denoted respectively by

$$M_z : f \longmapsto M_z(f)(z) = zf(z)$$

and

$$R_a : f \longmapsto R_a(f)(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a. \end{cases}$$

We note that the shift operator M_z defines an isometric operator on the Hardy space $\mathbf{H}_2(\mathbb{D})$, its adjoint there is given by the backward shift R_0 under which the Hardy space is invariant. Moreover, it holds that

$$||R_0(f)||^2_{\mathbf{H}_2(\mathbb{D})} = ||f||^2_{\mathbf{H}_2(\mathbb{D})} - |f(0)|^2. \tag{2.2.2}$$

One of the main important results associated to the shift operator on the Hardy space, is the famous Beurling, or Lax-Beurling theorem which allows to characterize invariant subspaces of the shift operator, see [10] and references therein.

Another important example of reproducing kernel Hilbert spaces on the unit disk, is the Bergman space that we recall here

Definition 2.2.2. *The Bergman space $A_2(\mathbb{D})$ is the set of analytic functions in $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

with $dA(z) = \frac{1}{\pi} dxdy$ is the Lebesgue measure with respect to $z = x + iy$.

An equivalent characterization of the Bergman space is given by

Theorem 2.2.3. *A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined on the unit disk, belongs to the Bergman space $A_2(\mathbb{D})$ if and only if it holds that*

$$\sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

Theorem 2.2.4. *The Bergman space $A_2(\mathbb{D})$ is a reproducing kernel Hilbert space with reproducing kernel given by the kernel function,*

$$K_{A_2(\mathbb{D})}(z, w) = \frac{1}{(1 - z\overline{w})^2}, \quad \text{for all } z, w \in \mathbb{D}. \tag{2.2.3}$$

It is also possible to study weighted Bergman spaces $A_2^{\alpha}(\mathbb{D})$ with $\alpha > -1$, see [86]. In this situation, the kernels are of the form

$$K_{A_2^{\alpha}(\mathbb{D})}(z, w) = \frac{1}{(1 - z\overline{w})^{\alpha+2}}, \quad \text{for all } z, w \in \mathbb{D}. \tag{2.2.4}$$

We note that the case $\alpha = 0$ corresponds to the standard Bergman space. The Bergman kernels can be considered also on different domains like the annulus, ellipse, half space, etc [10]. In particular, we recall briefly here the cases of half space and half unit disk, since this will be used after in chapter 8. Let $\mathbb{C}_+$ denote the half space defined by the conditions $z \in \mathbb{C}$ and $Re(z) > 0$. Then, the complex Bergman kernel on $\mathbb{C}_+$ is given by

$$K_{\mathbb{C}_+}(z,w) = \frac{1}{\pi(z+\overline{w})^2}, \quad \text{for all } z,w \in \mathbb{C}_+. \tag{2.2.5}$$

The reproducing kernel on the half unit disk $\mathbb{D}_+$ is obtained as the sum of the Bergman kernels of both the complex unit disk and the half plane. In particular, we have

$$K_{\mathbb{D}_+}(z,w) := \frac{1}{\pi(1-z\overline{w})^2} + \frac{1}{\pi(z+\overline{w})^2} \quad \text{for all } z,w \in \mathbb{D}_+, \tag{2.2.6}$$

where the first term corresponds to the Bergman kernel of the unit disk $K_{\mathbb{D}}$ while the second one is the Bergman kernel of the complex half plane $K_{\mathbb{C}_+}$, (see, e,g., p. 812 in [52]).

2.3 Fock spaces and Segal-Bargmann transforms

In this section we review basic notions related to Fock spaces on $\mathbb{C}^n$. We recall also the Segal-Bargmann transform and discuss its behavior with respect to some classical operators like the creation and annihilation operators, Fourier transform and Weyl operator. The material revised here is based mainly on the following references [23, 85, 98, 115].

For any positive parameter $\alpha > 0$, we consider the Gaussian measure on $\mathbb{C}^n$ defined by

$$d\lambda_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{2}} e^{-\alpha|z|^2} d\lambda(z)$$

where $z = (z_1, ..., z_n)$, $|z|^2 = \displaystyle\sum_{k=1}^{n} |z_k|^2$, $z_k = x_k + iy_k$ $\forall k = 1, ..., n$ and $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$ given by

$$\prod_{k=1}^{n} d\lambda(z_k) = \prod_{k=1}^{n} dx_k dy_k.$$

The Fock or Segal-Bargmann space on $\mathbb{C}^n$ denoted by $\mathcal{F}^{2,\alpha}(\mathbb{C}^n)$ or $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ is the space consisting of all entire functions $f(z)$ on $\mathbb{C}^n$ satisfying the condition

$$\int_{\mathbb{C}^n} |f(z)|^2 d\lambda_\alpha(z) < \infty.$$

According to the book [85], the constant α is related to some physics quantities associated to the quantum particle. It is in general taken to be equal to $\dfrac{1}{\hbar}$ where $\hbar$ stands for the reduced Planck's constant.

The Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ can be defined as the intersection of holomorphic functions on $\mathbb{C}^n$ with the Hilbert space $L^{2,\alpha}(\mathbb{C}^n) = L^2(\mathbb{C}^n, d\lambda_\alpha)$. Then, we can consider on $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ the scalar product induced from $L^{2,\alpha}(\mathbb{C}^n)$ and defined by

$$\langle f, g\rangle_\alpha = \int_{\mathbb{C}^n} f(z)\overline{g(z)}d\lambda_\alpha(z).$$

According to [98], $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ is called the *boson* Fock space with n degrees of freedom. We call it simply Fock space or Segal-Bargmann space since we do not consider the *fermion* Fock space. For $k = (k_1, ..., k_n) \in \mathbb{N}^n$, $w = (w_1, ..., w_n) \in \mathbb{C}^n$, we use the following notations :

$$z^k = z_1^{k_1}...z_n^{k_n}, \quad k! = k_1!...k_n! \quad \text{and} \quad |k| = \sum_{t=1}^n k_t.$$

Then, a first important result that we know on this space is the following

Theorem 2.3.1. *The set $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ is a Hilbert space with respect to the scalar product $\langle ., .\rangle_\alpha$. Moreover, the monomials defined by*

$$f_k(z) = z^k = z_1^{k_1}...z_n^{k_n}, \quad \forall k = (k_1, ..., k_n) \in \mathbb{N}^n; \forall z = (z_1, ..., z_n) \in \mathbb{C}^n$$

form an orthogonal basis in $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ and using the polar coordinates for all $k = (k_1, ..., k_n) \in \mathbb{N}^n$, we have

$$\|f_k\|^2_{\mathcal{F}^2_\alpha(\mathbb{C}^n)} = \frac{k!}{\alpha^{|k|}}.$$

An interesting characterization of the Fock space making use of power series is given by

Proposition 2.3.2. *A function $f(z) = \sum_{k\in\mathbb{N}^n} a_k z^k$ belongs to the Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ if and only if the following condition is satisfied*

$$\|f\|^2_{\mathcal{F}^2_\alpha(\mathbb{C}^n)} = \sum_{k\in\mathbb{N}^n} \frac{k!}{\alpha^{|k|}}|a_k|^2 < \infty.$$

For $z = (z_1, ..., z_n) \in \mathbb{C}^n$, we have the following growth condition

Proposition 2.3.3. *For every $f \in \mathcal{F}^2_\alpha(\mathbb{C}^n)$, it holds that*

$$|f(z)| \leq e^{\alpha\frac{|z|^2}{2}}\|f\|_{\mathcal{F}^2_\alpha(\mathbb{C}^n)}.$$

We consider the evaluation mapping on the Fock space defined by

$$\Lambda_z : f \in \mathcal{F}^2_\alpha(\mathbb{C}^n) \longmapsto f(z) \in \mathbb{C}$$

The previous proposition shows that all the evaluation mappings Λ_z on the Fock space are continuous. Hence, by the Riesz representation theorem it can be proved that $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ is a reproducing kernel Hilbert space whose reproducing kernel is given by the following exponential function

$$K_\alpha(z,w) = e^{\alpha z\overline{w}} \quad \forall z, w \in \mathbb{C}^n.$$

The normalized reproducing kernel of the Fock space is given by

$$k_a^\alpha(z) := \frac{K^\alpha(a,z)}{\sqrt{K_a^\alpha(a)}}$$

Furthermore, any function $f(z)$ of the Fock space maybe reproduced thanks to the following integral representation formula

$$f(z) = \int_{\mathbb{C}^n} f(z) e^{\alpha\bar{z}w} d\lambda_\alpha(w); \quad z \in \mathbb{C}^n.$$

For $z \in \mathbb{C}^n$ fixed, sometimes the functions

$$K_z^\alpha : w \mapsto K_z^\alpha(w) = K_\alpha(z,w)$$

are called coherent states.

Recall that every closed subspace F of a Hilbert space $\mathcal{H}$ uniquely determines an orthogonal projection $Proj : F \longrightarrow \mathcal{H}$.
In this case, the ortogonal projection is described by the following

Proposition 2.3.4. *The orthogonal projection*

$$P_\alpha : L^{2,\alpha}(\mathbb{C}^n) \longrightarrow \mathcal{F}^2_\alpha(\mathbb{C}^n)$$

is an integral operator. More specifically,

$$P_\alpha f(z) = \int_{\mathbb{C}^n} K_\alpha(z,w) f(w) d\lambda_\alpha(w)$$

for all $f \in L^{2,\alpha}(\mathbb{C}^n)$ *and all* $z \in \mathbb{C}^n$.

The Segal-Bargmann transform is a natural unitary operator associated to the Fock space, it was introduced in [23]. It identifies the standard Hilbert space $L^2(\mathbb{R}^n)$ and the Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$.
In fact, $L^2(\mathbb{R}^n)$ denote the classical Hilbert space on the n-real space $\mathbb{R}^n$ with respect to its standard Lebesgue measure $dx = dx_1 \cdots dx_n$. An orthogonal basis of $L^2(\mathbb{R}^n; dx)$ is given by the multi-dimensional Hermite functions

$$h_m^\alpha(x) := (-1)^{|m|} e^{\frac{\alpha}{2}x^2} \frac{\partial^{|m|}}{\partial x^{m_1} \cdots \partial x^{m_n}} \left(e^{-\alpha x^2}\right) = \prod_{\ell=1}^{d} h_{m_\ell}^\alpha(x_\ell), \tag{2.3.1}$$

for varying $m = (m_1, \cdots, m_d) \in (\mathbb{Z}^+)^d$, where $h^\alpha_{m_\ell}(x_\ell)$ is the one-dimensional Hermite function (see [90]). Their norm is known to be given explicitly by

$$\|h^\alpha_m\|^2_{L^2(\mathbb{R}^n;dx)} = 2^{|m|}\alpha^{|m|}m! \left(\frac{\pi}{\alpha}\right)^{d/2}. \tag{2.3.2}$$

Then, taking a wave function $\psi \in L^2(\mathbb{R}^n, dx)$, $\psi : \mathbb{R}^n \longrightarrow \mathbb{C}$ the Bargmann transform B_n is defined as follows

$$B_n\psi(z) = \int_{\mathbb{R}^n} A_n(z,x)\psi(x)dx.$$

The kernel function of this transform is given by the following

$$A_n(z,x) = A_z(x) := \left(\frac{\alpha}{\pi}\right)^{\frac{n}{4}} e^{\frac{-\alpha}{2}(z^2+x^2)+\alpha\sqrt{2}zx}$$

where for $z = (z_1, .., z_n) \in \mathbb{C}^n$, $x = (x_1, .., x_n) \in \mathbb{R}^n$ we have the following notations

$$z^2 := \sum_{i=1}^n z_i^2, x^2 := \sum_{i=1}^n x_i^2 \quad \text{and} \quad zx := \sum_{i=1}^n z_i x_i.$$

It is known that the last kernel could be obtained making use of the generating function of Hermite polynomials. Namely, we have the following formula

$$\sum_{k\in\mathbb{N}^n} \frac{z^k}{\|z^k\|_{\mathcal{F}^2_\alpha(\mathbb{C}^n)}} \frac{h^\alpha_k(x)}{\|h^\alpha_k\|_{L^2(\mathbb{R}^n)}} = A_n(z,x).$$

In 1961, Bargmann proved this important result

Theorem 2.3.5. *The Segal-Bargmann transform B_n is an isometric isomorphism from the standard Hilbert space $L^2(\mathbb{R}^n)$ to the Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$. Moreover, for a fixed $z \in \mathbb{C}^n$ we have the following*

$$B_n A^z_n(w) = \left(\frac{\alpha}{\pi}\right)^{-\frac{n}{4}} K^z_\alpha(w) \quad \forall w \in \mathbb{C}^n.$$

Another important property of this transform B_n is that it maps the Hermite functions h^α_m to the standard orthogonal basis of $\mathcal{F}^{2,\alpha}(\mathbb{C}^n)$, constituted by the complex monomials. More exactly, we have (see [23])

Proposition 2.3.6. *For all $m \in \mathbb{N}^n$ and $z \in \mathbb{C}^n$,*

$$[B_n h^\alpha_m](z) = \left(\frac{\alpha}{\pi}\right)^{\frac{n}{4}} 2^{\frac{|m|}{2}}\alpha^{|m|}z^m.$$

Since B_n is a unitary operator sending a basis to a basis. Then the Segal-Bargmann transform admits an inverse and we have $B_n^{-1} = B_n^*$. More exactly, the inverse of a function $f(z)$ in the Fock space $\mathcal{F}^2_\alpha(\mathbb{C}^n)$ is given by the following formula

$$B_n^{-1}f(x) = \int_{\mathbb{C}^n} \overline{A_n(z,x)}f(z)d\lambda_\alpha(z).$$

Remark 2.3.7. *Without lost of generality, for the rest of this chapter we suppose that the dimension $n = 1$ and the Segal-Bargmann transform B_1 will be denoted simply by B.*

The creation and annihilation operators on the Fock space $\mathcal{F}^2_\alpha(\mathbb{C})$ are defined respectively by

$$M_z f(z) = z f(z); \quad D_z f(z) = \frac{d}{dz} f(z).$$

Notice that M_z and D_z are unbounded operators on the Fock space and satisfy the canonical commutation relations namely

$$[M_z, D_z] = -\mathcal{I}.$$

Moreover, the operator D_z is adjoint to M_z, i.e

$$M_z^* = D_z.$$

We have the following

Proposition 2.3.8. *Let $\varphi \in L^2(\mathbb{R})$ be such that $x\varphi, \dfrac{d}{dx}\varphi \in L^2(\mathbb{R})$ then we have*

1. $\left(z + \dfrac{d}{dz}\right) B[\varphi](z) = \sqrt{2} B[x\varphi](z)$.
2. $\sqrt{2} z B[\varphi](z) = B\left[\left(x - \dfrac{d}{dx}\right)\varphi\right](z)$.

Corollary 2.3.9. *The Segal-Bargmann transform B maps the dimensionless raising and lowering operators*

$$a^\dagger = \left(x - \frac{d}{dx}\right) \quad \textit{and} \quad a = \left(x + \frac{d}{dx}\right)$$

on $L^2(\mathbb{R})$ onto the respective raising and lowering operators

$$b^\dagger = \sqrt{2} M_z \quad \textit{and} \quad b = \sqrt{2} M_z^*$$

on the Fock space $\mathcal{F}^2_\alpha(\mathbb{C})$.

A second interesting approach to introduce Segal-Bargmann transforms is by considering the convolution product of the function $\varphi \in L^2(\mathbb{R})$ with the continuous extension of the fundamental solution of the heat equation. Indeed, let $\rho_t(x)$ denote the fundamental solution of the heat equation

$$\frac{\partial}{\partial t}\rho_t(x) = \frac{1}{2}\Delta_x \rho_t(x),$$

i.e.

$$\rho_t(x) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2t}}$$

where Δ is the usual Laplacian on $\mathbb{R}$. Then, the Segal-Bargmann transform of the function $\varphi \in L^2(\mathbb{R})$ can be defined by setting $C\varphi(z) = \rho_1 * \varphi(z)$ where ρ_t has been analytically continued to $\mathbb{C}$. Explicitly, we have

$$C\varphi(z) = \int_{\mathbb{R}} \rho_1(z-x)\varphi(x)dx.$$

Such construction was used in [88] to study some extensions of Segal-Bargmann or coherent state transforms in the setting of Clifford analysis.

We end this section by turning the attention to the fact that Segal-Bargmann transform can be seen also as a particular example of a more general result known as *The Stone-von Neumann Theorem.* This construction involves mainly the Schrödinger Hilbert space $H = L^2(\mathbb{R})$ combined together with the classical position and momentum operators X and P satisfying the canonical commutations relations namely,

$$X : \varphi \mapsto X\varphi(x) := x\varphi(x), \quad P : \varphi \mapsto P\varphi(x) := -\frac{i}{\alpha}\frac{d}{dx}\varphi(x)$$

are defined such that we have

$$[X, P] = \frac{i}{\alpha}\mathcal{I},$$

where the symbol $[,]$ denote the commutator of two operators and $\mathcal{I}$ is the identity.

The Fourier transform and Weyl operator

We review also the Fourier transform and Weyl operator once connected to the Segal-Bargmann transform. The Fourier transform of a signal $f : \mathbb{R} \longrightarrow \mathbb{C}$ is defined by

$$\mathcal{F}_\alpha(f)(\xi) := \sqrt{\frac{\alpha}{2\pi}} \int_{\mathbb{R}} f(x)e^{-\alpha i x\xi}dx.$$

Thanks to the Plancherel-Theorem, it turns out that the Fourier transform maps unitary $L^2(\mathbb{R}; dx)$ onto itself. The following diagram is commutative

$$\begin{tikzcd}
\mathcal{F}^{2,\alpha}(\mathbb{C}) \arrow[r, "S_\alpha"] \arrow[d, "B^{-1}"'] & \mathcal{F}^{2,\alpha}(\mathbb{C}) \\
L^2(\mathbb{R}) \arrow[r, "\mathcal{F}_\alpha"'] & L^2(\mathbb{R}) \arrow[u, "B"']
\end{tikzcd}$$

Then, we may consider the operator

$$S_\alpha := B\mathcal{F}_\alpha B^{-1}.$$

We note that S_α maps isometrically $\mathcal{F}^{2,\alpha}(\mathbb{C})$ into $\mathcal{F}^{2,\alpha}(\mathbb{C})$. Moreover, we have the following

Proposition 2.3.10. *For a given $f \in \mathcal{F}^{2,\alpha}(\mathbb{C})$, we have*

$$S_\alpha(f)(z) = f(-iz)$$

for all $z \in \mathbb{C}$.

A well-known fact is that the classical Fourier transform corresponding to $\alpha = 1$ admits the normalized Hermite functions $\varphi_n = \dfrac{h_n}{\|h_n\|}$ as eigenfunctions. Indeed, we have

$$\mathcal{F}(\varphi_n) = (-i)^n \varphi_n.$$

The Weyl operators form a family of isometric operators on the Fock space that can be defined using the normalized reproducing kernel combined with the translation operator on the Fock space. They are of particular interest for quantum mechanics and have a semi-group property with respect to a complex parameter. For more details about this subject see for example [85, 115]. We recall quickly this notion to stress the importance of Fock spaces.

Definition 2.3.1. *Let a be a fixed complex number. The Weyl operator is defined to be the operator $\mathcal{W}_a : \mathcal{F}^{2,\alpha}(\mathbb{C}) \longrightarrow \mathcal{F}^{2,\alpha}(\mathbb{C})$ such that*

$$\begin{aligned}\mathcal{W}_a f(z) &:= k_a^\alpha(z) f(z-a)\\ &= e^{\alpha(z\bar{a} - \frac{|a|^2}{2})} f(z-a).\end{aligned}$$

for every $f \in \mathcal{F}^{2,\alpha}(\mathbb{C})$ and $z \in \mathbb{C}$, where k_a^α is the normalized reproducing kernel of the Bargmann-Fock space.

An important fact on the Weyl operators is given by the following

Theorem 2.3.11. *Let $a, b \in \mathbb{C}$. The Weyl operator $\mathcal{W}_a$ is a unitary operator on the Fock space $\mathcal{F}^{2,\alpha}(\mathbb{C})$. Moreover, we have the semi-group property*

$$\mathcal{W}_a \mathcal{W}_b f(z) = e^{i\alpha \Im(a\bar{b})} \mathcal{W}_{a+b} f(z)$$

for any $f \in \mathcal{F}^{2,\alpha}(\mathbb{C})$ and $z \in \mathbb{C}$.

Now, fix c a real number and consider the translation operator on the standard Hilbert space defined by $T_c : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), T_c\varphi(x) := \varphi(x-c)$. Then, the following diagram is also commutative

$$\begin{tikzcd} \mathcal{F}^{2,\alpha}(\mathbb{C}) \arrow[r, "U_c"] \arrow[d, "B^{-1}"'] & \mathcal{F}^{2,\alpha}(\mathbb{C}) \\ L^2(\mathbb{R}) \arrow[r, "T_c"'] & L^2(\mathbb{R}) \arrow[u, "B"'] \end{tikzcd}$$

and we can consider the operator

$$U_c := BT_cB^{-1}.$$

Observe that U_c maps isometrically $\mathcal{F}^{2,\alpha}(\mathbb{C})$ into $\mathcal{F}^{2,\alpha}(\mathbb{C})$. Moreover, it holds that

Proposition 2.3.12. *For a given $f \in \mathcal{F}^{2,\alpha}(\mathbb{C})$, we have*

$$U_c(f)(z) = \mathcal{W}_{\frac{c}{\sqrt{2}}} f(z)$$

for all $z \in \mathbb{C}$.

2.4 Polyanalytic functions and associated reproducing kernels

In this section, we revise the needed material concerning complex polyanalytic functions. The reader interested in more details, may consult the book [22].

Definition 2.4.1. *Let Ω be a domain of $\mathbb{C}$. A function $f : \Omega \longrightarrow \mathbb{C}$ is said to be polyanalytic of order n or $n-$analytic if*

$$\left(\frac{\partial}{\partial \overline{z}}\right)^n f(z) = 0, \ \forall z \in \Omega.$$

The space of all polyanalytic functions of order n is denoted by $H_n(\Omega)$.

Example. *The function $F(z) = 1 - z\overline{z}$ is polyanalytic of order 2 on $\mathbb{C}$.*

Proposition 2.4.1. *Let Ω be a domain of $\mathbb{C}$ and $f : \Omega \longrightarrow \mathbb{C}$. Then, the two following conditions are equivalent*

1. *f is polyanalytic of order n.*
2. *$f(z) = \displaystyle\sum_{k=0}^{n-1} \overline{z}^k a_k(z), \forall z \in \Omega$ where $a_0, ..., a_{n-1}$ are analytic on Ω.*

Proposition 2.4.2. *Let f and g be two polyanalytic functions of order n on a domain Ω. If Ω_0 is a subdomain of Ω such that f and g coincide on Ω_0, then f and g coincide everywhere in Ω.*

We recall also the polyanalytic Cauchy formula in complex analysis, see Theorem 2.1 in [57].

Theorem 2.4.3. *For $k \geq 1$, we set*

$$\psi_k(z) = \frac{1}{2\pi i}\frac{\bar{z}}{|z|^2}\frac{Re(z)^{k-1}}{(k-1)!}.$$

For $z = x + iy$, set $d\sigma = dx \wedge dy$. If f is polyanalytic of order n, then for all $z \in \mathbb{D}$ we have

$$f(z) = \int_{\partial\mathbb{D}} \sum_{j=0}^{n-1} (-2)^j \psi_{j+1}(u-z)\frac{\partial^j}{\partial \bar{u}^j} f(u) d\sigma.$$

In the book of Balk [22], the Fock space $\mathcal{F}_n(\mathbb{C})$ of polyanalytic functions of order n is defined by

$$\mathcal{F}_n(\mathbb{C}) = \{f \in H_n(\mathbb{C}); \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} d\lambda(z) < \infty\},$$

where $d\lambda(z)$ denotes the usual Lebesgue measure on the complex plane. Note that, $\mathcal{F}_n(\mathbb{C})$ is a reproducing kernel Hilbert space whose reproducing kernel is

$$F_n(z,w) = e^{\overline{w}z} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \frac{1}{k!} |z-w|^{2k} \tag{2.4.1}$$

Moreover, for all $f \in \mathcal{F}_n(\mathbb{C})$ and $z \in \mathbb{C}$ we have

$$|f(z)| \leq \sqrt{n} e^{\frac{|z|^2}{2}} \|f\|_{\mathcal{F}_n(\mathbb{C})}.$$

On the other hand, the Bergman space $A^2_n(\mathbb{D})$ of polyanalytic functions of order n in the unit disc is given by

$$A^2_n(\mathbb{D}) = \{f \in H_n(\mathbb{D}); \int_{\mathbb{D}} |f(z)|^2 d\lambda(z) < \infty\}.$$

Also $A^2_n(\mathbb{D})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$B_n(z,w) = \frac{n}{\pi(1-\overline{w}z)^{2n}} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n+k}{n} |1-\overline{w}z|^{2(n-1-k)} |z-w|^{2k} \tag{2.4.2}$$

for any $z, w \in \mathbb{D}$. Moreover, for all $f \in A^2_n(\mathbb{D})$ and $z \in \mathbb{D}$, we have the following

$$|f(z)| \leq \frac{n}{\sqrt{\pi}} \frac{\|f\|_{A^2_n(\mathbb{D})}}{(1-|z|^2)}.$$

CHAPTER 3

Preliminaries on hypercomplex analysis

In this chapter, we present the quaternions and revise the main results that will be needed in the sequel about different hypercomplex function theories. In particular, slice regular and Fueter hyperholomorphic functions. Then, we review a fundamental result in Clifford analysis that allows to connect both the function theories, namely the so-called Fueter-Sce-Qian mapping theorem. The material revised in this chapter is based mainly on the following references [7, 28, 35–38, 47, 75, 77, 83]. Since most of the results presented in this chapter are very well-known and classical we have omitted to give proofs.

3.1 Slice hyperholomorphic function theory

The non-commutative field of quaternions is defined to be

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k \quad ; \, x_0, x_1, x_2, x_3 \in \mathbb{R}\}$$

where the imaginary units satisfy the multiplication rules

$$i^2 = j^2 = k^2 = -1 \quad \text{and} \quad ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

On $\mathbb{H}$ the conjugate and the modulus of q are defined respectively by

$$\overline{q} = Re(q) - Im(q) \quad \text{where} \quad Re(q) = x_0, \quad Im(q) = x_1 i + x_2 j + x_3 k$$

and

$$|q| = \sqrt{q\overline{q}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

We can use also $\vec{q}$ to denote the vector part of the quaternionic variable. The imaginary units can be denoted sometimes as e_1, e_2 and $e_3 = e_1e_2$. We note that the quaternionic conjugation satisfy the property $\overline{pq} = \overline{q}\,\overline{p}$ for any $p, q \in \mathbb{H}$. Moreover, the unit sphere

$$\{q = x_1i + x_2j + x_3k;\ x_1^2 + x_2^2 + x_3^2 = 1\}$$

coincides with the set of all imaginary units given by

$$\mathbb{S} = \{q \in \mathbb{H}; q^2 = -1\}.$$

Any quaternion $q \in \mathbb{H} \setminus \mathbb{R}$ can be written in a unique way as $q = x + Iy$ for some real numbers x and $y > 0$, and imaginary unit $I \in \mathbb{S}$, in fact we have

$$q = x_0 + \frac{x_1i + x_2j + x_3k}{|x_1i + x_2j + x_3k|}|x_1i + x_2j + x_3k|.$$

Then, for every given $I \in \mathbb{S}$, the slice $\mathbb{C}_I$ is defined to be $\mathbb{R} + \mathbb{R}I$ and it is isomorphic to the complex plane $\mathbb{C}$ so that it can be considered as a complex plane in $\mathbb{H}$ passing through 0, 1 and I. The semi-slice $\mathbb{C}_I^+$ is given by the set

$$\{x + yI; x, y \in \mathbb{R}, y \geq 0\}.$$

If $q = x_0 \in \mathbb{R}$ then $q \in \mathbb{C}_I$ for all $I \in \mathbb{S}$. It is immediate that we have

$$\mathbb{H} = \underset{I \in \mathbb{S}}{\cup} \mathbb{C}_I.$$

We denote by $\mathbb{B}_r$ the open ball in $\mathbb{H}$ of radius $r > 0$, i.e.

$$\mathbb{B}_r = \{q = x_0 + ix_1 + jx_2 + kx_3, \text{ such that } x_0^2 + x_1^2 + x_2^2 + x_3^2 < r^2\}.$$

To introduce convolution operators of a quaternion variable, we need a suitable exponential function of quaternion variable. For any $I \in \mathbb{S}$, we choose the following well-known definition for the exponential:

$$e^{It} = \cos(t) + I\sin(t),\ t \in \mathbb{R},$$

see [83]. The Euler's kind formula holds :

$$(\cos(t) + I\sin(t))^k = \cos(kt) + I\sin(kt),$$

and therefore we can write

$$(e^{It})^k = e^{Ikt}.$$

For any $q \in \mathbb{H} \setminus \mathbb{R}$, let $r := |q|$; then, see [83], there exists a unique $a \in (0, \pi)$ such that $\cos(a) := \frac{x_1}{r}$ and a unique $I_q \in \mathbb{S}$, such that

$$q = re^{I_q a},\ \text{ with } I_q = iy + jv + ks,\ y = \frac{x_2}{r\sin(a)},\ v = \frac{x_3}{r\sin(a)},\ s = \frac{x_4}{r\sin(a)}.$$

Now, if $q \in \mathbb{R}$, then we choose $a = 0$, if $q > 0$ and $a = \pi$ if $q < 0$, and as I_q we choose an arbitrary fixed $I \in \mathbb{S}$. So that if $q \in \mathbb{R} \setminus \{0\}$, then again we can write $q = |q|(\cos(a) + I\sin(a))$ (but with a non unique I). The above is called the trigonometric form of the quaternion number $q \neq 0$. For $q = 0$ we do not have a trigonometric form for q (exactly as in the complex case).
In [77], the authors proposed a new definition to extend the classical theory of holomorphic functions in complex analysis to the quaternionic setting. This leads to the new theory of slice hyperholomorphic or slice regular functions on quaternions:

Definition 3.1.1. *A real differentiable function $f : \Omega \longrightarrow \mathbb{H}$, on a given domain $\Omega \subset \mathbb{H}$, is said to be a (left) slice regular function if, for every $I \in \mathbb{S}$, the restriction f_I to the slice $\mathbb{C}_I$ satisfies*

$$\overline{\partial_I} f(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x+Iy) = 0,$$

on Ω_I. The slice derivative $\partial_S f$ of f is defined by :

$$\partial_S(f)(q) := \begin{cases} \partial_I(f)(q) & \text{if } q = x+Iy, y \neq 0 \\ \dfrac{\partial}{\partial x}(f)(x) & \text{if } q = x \text{ is real.} \end{cases}$$

In addition we introduce the following terminology

1. A quaternionic valued function on a domain Ω is said to be (quaternionic) intrinsic if $f(\Omega_I) \subset \mathbb{C}_I$ for any $I \in \mathbb{S}$.
2. A function which is slice regular on the whole space of quaternions $\mathbb{H}$ is said to be entire.

We will refer to left slice regular functions as slice regular functions, for short. The set of these functions is denoted by $\mathcal{SR}(\Omega)$. It turns out that $\mathcal{SR}(\Omega)$ is a right vector space over the noncommutative field $\mathbb{H}$.

Remark 3.1.1. *The multiplication and composition of slice regular functions are not slice regular, in general. Moreover, the slice derivative does not satisfy the Leibniz rule with respect to the pointwise multiplication. However, the composition $f \cdot g$ of two slice regular functions is slice regular if g is intrinsic and the pointwise product fg is slice regular if f is intrinsic, see [35].*

According to this definition, the basic polynomials in q with quaternionic coefficients on the right are slice regular. Moreover, for any power series $\sum_n q^n a_n$, there exists $0 \leq R \leq \infty$, called the radius of convergence such that the power series is a slice regular function on $B(0,R) := \{q \in \mathbb{H};\ |q| < R\}$. The space of slice regular functions is endowed with the natural topology of uniform convergence on compact sets. The characterization of slice regular functions on a ball $B = B(0,R)$ centered at the origin is given by

Theorem 3.1.2 (Series expansion). *An $\mathbb{H}$-valued function f is slice regular on $B(0,R) \subset \mathbb{H}$ if and only if it has a series expansion of the form:*

$$f(q) = \sum_{n=0}^{+\infty} q^n \frac{1}{n!} \partial_S^{(n)}(f)(0)$$

converging on $B(0,R) = \{q \in \mathbb{H}; \mid q \mid < R\}$.

Definition 3.1.2. *A domain $\Omega \subset \mathbb{H}$ is said to be a slice domain (or just s-domain) if $\Omega \cap \mathbb{R}$ is nonempty and for all $I \in \mathbb{S}$, the set $\Omega_I := \Omega \cap \mathbb{C}_I$ is a domain of the complex plane $\mathbb{C}_I$. If moreover, for every $q = x + Iy \in \Omega$, the whole sphere*

$$[q] := \{x + Jy;\ J \in \mathbb{S}\},$$

is contained in Ω, we say that Ω is an axially symmetric slice domain.

Example. *The whole space $\mathbb{H}$ and the Euclidean ball $B = B(0,R)$ of radius R centered at the origin are axially symmetric slice domains.*

The following properties of slice regular functions are fundamental and very useful to develop this theory, see [35, 75].

Lemma 3.1.3 (Splitting Lemma). *Let f be a slice regular function on a domain Ω. Then, for every $I, J \in \mathbb{S}$ with $I \perp J$, there exist two holomorphic functions $F, G : \Omega_I \longrightarrow \mathbb{C}_I$ such that for all $z = x + Iy \in \Omega_I$, we have*

$$f_I(z) = F(z) + G(z)J,$$

where $\Omega_I = \Omega \cap \mathbb{C}_I$ and $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$.

Theorem 3.1.4 (Identity Principle). *Let f and g be two slice regular functions on a slice domain Ω. If, for some $I \in \mathbb{S}$, f and g coincide on a subset of Ω_I having an accumulation point in Ω_I, then $f = g$ on the whole domain Ω.*

Theorem 3.1.5. *Let Ω be an axially symmetric slice domain and $f \in \mathcal{SR}(\Omega)$. Then, for any $I, J \in \mathbb{S}$, we have the formula*

$$f(x + Jy) = \frac{1}{2}(1 - JI) f_I(x + Iy) + \frac{1}{2}(1 + JI) f_I(x - Iy)$$

for all $q = x + Jy \in \Omega$.

In other words, we have

Theorem 3.1.6 (Representation Formula). *Let Ω be an axially symmetric slice domain, $f \in \mathcal{SR}(\Omega)$ and $I, J \in \mathbb{S}$. Then, for all $q = x + yI \in \mathbb{H}$, we have*

$$f(x + yI) = \alpha(x,y) + I\beta(x,y)$$

where

$$\alpha(x,y) = \frac{1}{2}[f(x+yJ) + f(x-yJ)]$$

and

$$\beta(x,y) = \frac{J}{2}[f(x-yJ) - f(x+yJ)].$$

Moreover, α and β are $\mathbb{H}$-valued differentiable functions satisfying the Cauchy-Riemann conditions. We have also $\alpha(x,-y) = \alpha(x,y)$ and $\beta(x,-y) = -\beta(x,y)$.

Lemma 3.1.7 (Extension Lemma). *Let Ω_I be a domain in $\mathbb{C}_I$ symmetric with respect to the real axis and such that $\Omega_I \cap \mathbb{R} \neq \emptyset$. Let $h : \Omega_I \longrightarrow \mathbb{H}$ be a holomorphic function. Then, the function $ext(h)$ defined by*

$$ext(h)(x+Jy) := \frac{1}{2}[h(x+Iy)+h(x-Iy)]+\frac{JI}{2}[h(x-Iy)-h(x+Iy)]; \quad J \in \mathbb{S},$$

extends h to a slice regular function $ext(h)$ on $\widetilde{\Omega} = \underset{x+Iy\ ;\ x+Jy\in\Omega}{\cup} x+Iy$, the symmetric completion of Ω_I. Moreover, $ext(h)$ is the unique slice regular extension of h.

It is also possible to introduce the notion of Cauchy kernel for slice hyperholomorphic functions, see for example [35, 48].

Definition 3.1.3. *Let $q, s \in \mathbb{H}$ be such that $sq \neq qs$. The series expansion given by*

$$S^{-1}(s,q) := \sum_{n=0}^{\infty} q^n s^{-1-n}, \quad |q| < |s|$$

is called a noncommutative Cauchy kernel series or shortly Cauchy kernel series.

Theorem 3.1.8. *Let $q, s \in \mathbb{H}$ be such that $q \notin [s]$. Then, we have*

$$S^{-1}(s,q) = -(q^2 - 2Re(s)q + |q|^2)^{-1}(q - \overline{s}).$$

An important fact in this theory, is that the slice hyperholomorphic Cauchy kernel can be written in two different forms thanks to the following identity.

Proposition 3.1.9. *Let $q, s \in \mathbb{H}$ be such that $q \notin [s]$. Then, we have*

$$-(q^2 - 2Re(s)q + |q|^2)^{-1}(q - \overline{s}) = (s - \overline{q})(s^2 - 2Re(q)s + |s|^2)^{-1}.$$

Then, two formulations of the Cauchy kernel can be introduced in this framework

Definition 3.1.4. *Let $q, s \in \mathbb{H}$ be such that $q \notin [s]$.*

- *We say that $S^{-1}(s,q)$ is written in the form I if*

$$S^{-1}(s,q) := -(q^2 - 2Re(s)q + |q|^2)^{-1}(q - \overline{s}).$$

- *We say that $S^{-1}(s,q)$ is written in the form II if*

$$S^{-1}(s,q) := (s-\overline{q})(s^2 - 2Re(q)s + |s|^2)^{-1}.$$

The previous notion of Cauchy kernel allows to introduce a Cauchy formula for slice hyperholomorphic functions.

Theorem 3.1.10 (Cauchy Formula). *Let $W \subset \mathbb{H}$ be an open set and f a left slice regular function . Let U be a bounded axially symmetric open set such that $U \subset W$. Suppose that theboundary of $U \cap \mathbb{C}_I$ consists of a finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $q \in U$, we have*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} f(s) ds_I S^{-1}(s,q),$$

where $ds_I = ds/I$ and the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

Another interesting approach to define slice hyperholomorphic functions is to consider them as solutions of a special global operator with non constant coefficients that was introduced and studied in [40, 51, 80]. This leads to the following definition

Definition 3.1.5. *Let Ω be an open set in $\mathbb{H}$ and $f : \Omega \longrightarrow \mathbb{H}$ a function of class $\mathcal{C}^1$. We define the global operator $G(f)$ by*

$$G(f)(q) := |\vec{q}|^2 \partial_{x_0} f(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l} f(q),$$

for any $q = x_0 + \vec{q} \in \Omega$.

It was proved in [40] that any slice hyperholomorphic function belongs to $\ker(G)$ on axially symmetric slice domains. Other interesting properties of the global operator G were studied in [42]. We recall some of them that will be helpful for our purposes:

Proposition 3.1.11. *Let Ω be an open set in $\mathbb{H}$ and $f, g : \Omega \longrightarrow \mathbb{H}$ two functions of class $\mathcal{C}^1$. Then, for $q = x_0 + \vec{q} \in \Omega$ we have*

1. $G(fg) = G(f)g + fG(g) + (\vec{q}f - f\vec{q}) \sum_{l=1}^{3} x_l \partial_{x_l} g.$

   In particular, it holds:

2. $G(f\lambda + g) = G(f)\lambda + G(g), \forall \lambda \in \mathbb{H}.$

3. $G(x_0 f) = |\vec{q}|^2 f + x_0 G(f)$ *and* $G(\vec{q}f) = -|\vec{q}|^2 f + \vec{q}G(f).$

4. $G(q^k f) = q^k G(f), \forall k \in \mathbb{N}.$

3.2 Quaternionic intrinsic functions

Let us consider the subclass of $\mathcal{SR}(\Omega)$ defined on an open set $\Omega \subset \mathbb{H}$ by

$$\mathcal{N}(\Omega) := \{f \in \mathcal{SR}(\Omega) : f(\Omega \cap \mathbb{C}_I) \subset \mathbb{C}_I; \forall I \in \mathbb{S}\}.$$

If Ω is axially symmetric, functions of this class are called quaternionic intrinsic in analogies with the complex case thanks to the following property

Proposition 3.2.1. *A slice regular function f belongs to the class $\mathcal{N}(\Omega)$ if and only if it satisfies $f(\overline{q}) = \overline{f(q)}$ for all $q \in \Omega$.*

If one considers the ball $\Omega = B(0, R)$ with center at the origin it is clear that a slice regular function is belonging to $\mathcal{N}(\Omega)$ if and only if its power series expansion has real coefficients. These functions are also called real, in a more general case we have

Proposition 3.2.2. *Let $\Omega \subset \mathbb{H}$ be an axially symmetric open set and consider the slice regular function $f(x + yI) = \alpha(x, y) + I\beta(x, y)$. Then $f \in \mathcal{N}(\Omega)$ if and only if*

$$f(x + yI) = \alpha(x, y) + I\beta(x, y)$$

with α, β are real valued functions satisfying the Cauchy-Riemann conditions.

Remark 3.2.3. *All elementary transcendal functions are belonging to the class $\mathcal{N}(\mathbb{H})$. These functions coincide with the analogous complex function on any complex plane $\mathbb{C}_I$.*

1. $\exp(q) = e^q = \displaystyle\sum_{n=0}^{\infty} \frac{q^n}{n!}$.
2. $\sin(q) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1}}{(2n+1)!}$.
3. $\cos(q) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{q^{2n}}{(2n)!}$.

Another version of the splitting lemma involving complex intrinsic functions is the following

Lemma 3.2.4 (Refined Splitting Lemma)**.** *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let f be a slice regular function on U. For any $I, J \in \mathbb{S}$ with J orthogonal to I, there exist four holomorphic intrinsic functions $h_l : U \cap \mathbb{C}_I \longrightarrow \mathbb{C}_I$, $l = 0, ..., 3$ such that*

$$f_I(x + yI) = h_0(x + yI) + h_1(x + yI)I + h_2(x + yI)J + h_3(x + yI)K,$$

where $K = IJ$.

An important fact is that the class of slice regular functions on axially symmetric slice domains can be obtaind from the subclass of quaternionic intrinsic functions. This is explained thanks to the following

Proposition 3.2.5. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and $\{1, I, J, IJ\}$ a basis of $\mathbb{H}$. Then,*

$$\mathcal{SR}(U) = \mathcal{N}(U) \oplus \mathcal{N}(U)I \oplus \mathcal{N}(U)J \oplus \mathcal{N}(U)IJ$$

Because of the noncommutativity of $\mathbb{H}$ the composition and multiplication of two slice regular functions are not slice regular in general. Consider the following example, set $f(q) = q - i$ we have

$$f(q)^2 = (q-i)(q-i) = q^2 - qi - iq - 1$$

The product $f^2 = ff$ is not slice regular because of the term iq. However, we know that

Proposition 3.2.6. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let f and g be two slice regular functions on U belonging to $\mathcal{N}(U)$. Then, the point wise multiplication fg belongs also to $\mathcal{N}(U)$.*

On the other hand, we consider the function $g(q) = q^2$. Clearly the composition $g \circ f = f^2$ which is not slice regular. To preserve the slice regularity of the composition, we have the following

Proposition 3.2.7. *Let U be an axially symmetric quaternionic slice domain and V an open set in $\mathbb{H}$. Let g and f be respectively two slice regular functions on U and V such that $g(U) \subset V$ and $g \in \mathcal{N}(U)$. Then, the composition $f \circ g$ is slice regular on U.*

For $I \in \mathbb{S}$ fixed, we define another subclass of slice regular which is larger than $\mathcal{N}(U)$. Namely, we consider

$$\mathcal{V}_I(U) := \{f \in \mathcal{SR}(U) : f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I\}$$

Remark 3.2.8. *We have the following*

1. $\mathcal{N}(U) = \bigcap\limits_{I \in \mathbb{S}} \mathcal{V}_I(U)$.

2. *Let $I \in \mathbb{S}$ fixed and $J \in \mathbb{S}$ such that $I \perp J$. Then, direct computations using the extension lemma shows the following*

$$\mathcal{SR}(U) = \mathcal{V}_I(U) \oplus \mathcal{V}_I(U)J.$$

3.3 Hardy, Bergman and Fock spaces of slice hyperholomorphic functions

The Hardy space of slice hyperholomorphic functions on the quaternionic unit ball $\mathbb{B}$ was introduced first in [11,12] and is denoted by $\mathbf{H}_2(\mathbb{B})$. See also [55] for Hardy spaces $\mathbf{H}_p(\mathbb{B})$. We recall that the Hardy space is defined to be the space of slice regular power series given by

$$\mathbf{H}_2(\mathbb{B}) = \left\{ f = \sum_{k=0}^{\infty} q^k a_k;\ a_k \in \mathbb{H} : ||f||^2 = \sum_{k=0}^{\infty} |a_k|^2 < \infty \right\}.$$

We note that $\mathbf{H}_2(\mathbb{B})$ is a quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by

$$K_{\mathbf{H}_2(\mathbb{B})}(q,r) := \sum_{n=0}^{\infty} q^n \overline{r}^n = (1 - q\overline{r})^{-*}, \tag{3.3.1}$$

where $*$ denotes the classical star product of slice hyperholomorphic functions. Furthermore, we have the following charaterization of the quaternionic Hardy space:

Theorem 3.3.1. *Let $f \in \mathbf{H}_2(\mathbb{B})$. Then, the norm of f can be written as*

$$||f||^2 = \sup_{0<\rho<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{I\theta}|^2 d\theta \right)^{\frac{1}{2}}.$$

The slice Bergman space of the first and second kind were introduced in [41,43]. In particular, we focus on the case of the Bergman space of the second kind of the quaternionic unit ball $\mathbb{B}$. For $I \in \mathbb{S}$, the slice hyperholomorphic Bergman space of the second kind is defined to be

$$\mathcal{A}_{Slice}(\mathbb{B}) := \{ f \in \mathcal{SR}(\mathbb{B}); \int_{\mathbb{B}_I} |f_I(p)|^2 d\lambda_I(p) < \infty \}.$$

Note that, $\mathcal{A}_{Slice}(\mathbb{B})$ is a right quaternionic Hilbert space which does not depend on the choice of the imaginary unit I. Its reproducing kernel is obtained by extending the classical kernel in complex analysis; in closed form it can be written as follows, see [43]:

$$B_S(q,r) = \frac{1}{\pi}(1 - 2\overline{q}\overline{r} + \overline{q}^2\overline{r}^2)(1 - 2Re(q)\overline{r} + |q|^2\overline{r}^2)^{-2}. \tag{3.3.2}$$

We note that this kernel can be written also in the following form

$$B_S(q,r) = \frac{1}{\pi}(1 - 2Re(r)q + |r|^2 q^2)^{-2}(1 - 2qr + q^2 r^2). \tag{3.3.3}$$

The paper [15] studies the slice hyperholomorphic quaternionic Fock space $\mathcal{F}_{Slice}(\mathbb{H})$, defined for a given $I \in \mathbb{S}$ to be

$$\mathcal{F}_{Slice}(\mathbb{H}) := \left\{ f \in \mathcal{SR}(\mathbb{H});\ \frac{1}{\pi} \int_{\mathbb{C}_I} |f_I(p)|^2 e^{-|p|^2} d\lambda_I(p) < \infty \right\},$$

where $f_I = f|_{\mathbb{C}_I}$ and $d\lambda_I(p) = dxdy$ for $p = x + yI$. The definition of this space does not depend on the choice of I. This quaternionic Fock space can be characterized in terms of the slice hyperholomorphic power series as follows

$$\mathcal{F}_{Slice}(\mathbb{H}) = \left\{ \sum_{k=0}^{\infty} q^k a_k;\ a_k \in \mathbb{H} : \sum_{k=0}^{\infty} k! |a_k|^2 < \infty \right\}.$$

It was proved also that $\mathcal{F}_{Slice}(\mathbb{H})$ is a right quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by

$$e_*(p\overline{q}) = \sum_{n=0}^{\infty} \frac{p^n \overline{q}^n}{n!}. \tag{3.3.4}$$

Equivalently, the reproducing kernel of the slice hyperholomorphic Fock space could be obtained also by taking the slice regular extension of the complex function $e^{z\overline{q}}$ where z and q belong to the same slice. This means that

$$e_*(p\overline{q}) = ext(e^{z\overline{q}})(p). \tag{3.3.5}$$

Its associated Segal-Bargmann transform was studied in [60] by considering the slice hyperholomorphic kernel obtained making use of the normalized Hermite functions $(\eta_n)_{n\geq 0}$. The explicit expression of this kernel is given by

$$\mathcal{A}^S_{\mathbb{H}}(q,x) := \sum_{k=0}^{\infty} \frac{q^k}{\sqrt{k!}} \eta_k(x) = e^{-\frac{1}{2}(q^2+x^2)+\sqrt{2}qx},\ \forall (q,x) \in \mathbb{H} \times \mathbb{R}. \tag{3.3.6}$$

Then, for any quaternionic valued function φ in $L^2(\mathbb{R},\mathbb{H})$ the slice hyperholomorphic Segal-Bargmann transform is defined by

$$\mathcal{B}^S_{\mathbb{H}}(\varphi)(q) = \int_{\mathbb{R}} \mathcal{A}^S_{\mathbb{H}}(q,x)\varphi(x)dx. \tag{3.3.7}$$

In the same spirit different famous spaces of slice hyperholomorphic functions such as Hardy, Besov, Bloch, Dirichlet and Bergman spaces were studied in [13, 43, 113].

3.4 Fueter hyperholomorphic function theory and Fueter mapping theorem

We recall the classical notion of Fueter hyperholomorphic or Fueter regular functions, for more details one can see [47, 83].

Definition 3.4.1. *Let $U \subset \mathbb{H}$ be an open set and $f : U \longrightarrow \mathbb{H}$ a real differentiable function. We say that f is (left) Fueter regular or regular for short if*

$$\mathcal{D}f(q) := \left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} \right) f(q) = 0, \forall q \in U.$$

The right linear space of Fueter regular functions is denoted by $\mathcal{R}(U)$.

Sometimes the Cauchy-Fueter operator can be denoted simply by ∂. The right Fueter regular functions can be defined just by taking the imaginary units on the right of the derivatives of the function f. The quaternionic monomials $P_n(q) = q^n$ are not Fueter regular. However, there exist some other important functions in this theory, the so-called Fueter variables, defined by

$$\zeta_l(x) = x_l - e_l x_0, \ l = 1, 2, 3. \tag{3.4.1}$$

These functions play the same role that complex monomials play in complex analysis. For example, a series expansion for Fueter regular functions is obtained using these Fueter variables. A suitable product that allows to preserve the regularity in this setting is the so-called C-K product, denoted $\odot$. Given two Fueter regular functions f and g, we take their restriction to $x_0 = 0$ and consider their pointwise multiplication. Then, we take the Cauchy-Kowalevskaya extension of this pointwise product, which exists and is unique, to define $f \odot g$, see [83]. As customary, a Fueter regular polynomial of degree k is called a quaternionic spherical monogenic of degree k. For more details about the theory of quaternionic Fueter regular functions we refer the reader to, e.g. [47, 83].

The two theories of slice hyperholomorphic and Fueter regular functions are related by the Fueter mapping theorem, see [48]. We briefly recall below the variation of this result that we will use later and we refer the reader to [102] for more details. We recall below the variations of the Fueter mapping theorem that we will use later in the next chapters and refer the reader to [48, 102] for several extensions.

Theorem 3.4.1 (Fueter mapping theorem [48])**.** *Let U be an axially symmetric set in $\mathbb{H}$ and let $f : U \subset \mathbb{H} \longrightarrow \mathbb{H}$ be a slice hyperholomorphic function of the form $f(x+yI) = \alpha(x, y)+I\beta(x, y)$, where $\alpha(x, y)$ and $\beta(x, y)$ are quaternionic-valued functions such that $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$ and satisfying the Cauchy-Riemann system. Then, the function*

$$\tilde{f}(x_0 + \vec{q}) = \Delta \left(\alpha(x_0, |\vec{q}|) + \frac{\vec{q}}{|\vec{q}|}\beta(x_0, |\vec{q}|) \right)$$

extends to a Fueter regular function on the whole U.

Remark 3.4.2. *If U is an axially symmetric slice domain in $\mathbb{H}$, then every slice hyperholomorphic function $f : U \subset \mathbb{H} \longrightarrow \mathbb{H}$ is of the form $f(x + Iy) =$*

$\alpha(x, y) + I\beta(x, y)$, where α and β have the properties mentioned in the preceding statement. This is an immediate consequence of the Representation formula observed in Lemma 2.2 in [45].

Remark 3.4.3. *Below, we can consider the Fueter mapping defined by*

$$\tau : \mathcal{SR}(U) \to \mathcal{FR}(U),\ f \longmapsto \tau(f) = \Delta(f).$$

Theorem 3.4.4 ([48])**.** *Given a quaternion $s \in \mathbb{H}$, we define*

$$[s] = \{p \in \mathbb{H} :\ p = Re(s) + I|\vec{s}|, I \in \mathbb{S}\}.$$

Let $S^{-1}(s, q)$ be the Cauchy kernel defined by:

$$S^{-1}(s, q) = (s - \overline{q})(s^2 - 2Re(q)s + |q|^2)^{-1},\ q \notin [s].$$

Then the function

$$\mathcal{F}(s, q) := \Delta S^{-1}(s, q) = -4(s - \overline{q})(s^2 - 2Re(q)s + |q|^2)^{-2},$$

is a Cauchy-Fueter regular function in the variable q, and it is right slice regular in the variable s for $q \notin [s]$.

Theorem 3.4.5 (The Fueter mapping theorem in integral form [48])**.** *Let $W \subset \mathbb{H}$ be an axially symmetric open set and let f be slice hyperholomorphic in W. Let U be a bounded axially symmetric open set such that $\overline{U} \subset W$. Suppose that the boundary of $U_I = U \cap \mathbb{C}_I$ consists of finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $q \in U$, the Cauchy-Fueter regular function given by*

$$\tau(f)(q) = \Delta f(q)$$

has the integral representation

$$\tau(f)(q) = \frac{1}{2\pi} \int_{\partial U_I} \Delta S^{-1}(s, q) ds_I f(s),\ ds_I = ds/I,$$

and the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

We will need also these useful results in our computations

Proposition 3.4.6 ([24])**.** *For all $n \geq 2$, we have*

$$\mathcal{D}[q^n] = -2 \sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1}.$$

Proposition 3.4.7 ([63])**.** *For all $n \geq 2$, we have*

$$\tau[q^n] = -4 \sum_{k=1}^{n-1} (n-k) q^{n-k-1}\overline{q}^{k-1}.$$

3.5 Clifford monogenic case

Let $\{e_1, e_2, ..., e_m\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^m$ in which we introduce a non-commutative product defined by the following multiplication law

$$e_k e_s + e_s e_k = -2\delta_{k,s}, k, s = 1, ..., m$$

where $\delta_{k,s}$ is the Kronecker symbol. The set

$$\{e_A : A \subset \{1, ..., m\} \text{ with } e_A = e_{h_1} e_{h_2} ... e_{h_r}, 1 \leq h_1 < ... < h_r \leq m, e_\emptyset = 1\}$$

forms a basis of the 2^m-dimensional Clifford algebra $\mathbb{R}_m$ over $\mathbb{R}$. Let $\mathbb{R}^{m+1}$ be embedded in $\mathbb{R}_m$ by identifying $(x_0, x_1, ..., x_m) \in \mathbb{R}^{m+1}$ with the paravector $x = x_0 + \underline{x} \in \mathbb{R}_m$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x_0^2 + ... + x_m^2$. For $m \geq 1$, the Euclidean Dirac operator on $\mathbb{R}^m$ is given by

$$\partial_{\underline{x}} = \sum_{k=1}^{m} e_k \partial_{x_k}.$$

The generalized Cauchy-Riemann operator (also known as Weyl operator) and its conjugate in $\mathbb{R}^{m+1}$ are given respectively by

$$\partial := \partial_{x_0} + \partial_{\underline{x}} \text{ and } \overline{\partial} := \partial_{x_0} - \partial_{\underline{x}}.$$

Notice that

$$\overline{\partial}\partial = \partial\overline{\partial} = \Delta_{\mathbb{R}^{m+1}}$$

where $\Delta_{\mathbb{R}^{m+1}}$ stands for the usual Laplacian on the Euclidean space $\mathbb{R}^{m+1}$. Real differentiable functions on an open subset of $\mathbb{R}^{m+1}$ taking their values in $\mathbb{R}_m$ that are in the kernel of the generalized Cauchy-Riemann operator are called left monogenic or monogenic, for short. Moreover, for a monogenic function f we have the following Leibniz rule, see e.g. [100]

$$\partial_{\underline{x}}(\underline{x}f) = -mf - \underline{x}\partial_{\underline{x}}f - 2\sum_{l=1}^{m} x_l \partial_{x_l} f. \tag{3.5.1}$$

The latter formula will be very important for our calculations. In the particular case of quaternions the generalized Cauchy-Riemann operator in $\mathbb{R}^{m+1}$ becomes the Cauchy-Fueter operator and this leads to the theory of quaternionic Fueter regular functions. The $(n-1)$ dimensional sphere of units $1-$vectors in $\mathbb{R}^n$ is denoted by

$$\mathbb{S}^{m-1} = \{x = x_1 e_1 + ... + x_m e_m; x_1^2 + ... x_m^2 = 1\}.$$

Note that if $I \in \mathbb{S}^{n-1}$, then $I^2 = -1$. Based on these notations, in [36] the theory of slice regular functions on quaternions was extended to the slice monogenic setting thanks to the following :

Definition 3.5.1. *A real differentiable function $f : \Omega \subset \mathbb{R}^{m+1} \longrightarrow \mathbb{R}_m$ on a given open set is said to be a slice (left) monogenic function if, for very $I \in \mathbb{S}^{m-1}$, the restriction f_I to the slice $\mathbb{C}_I$, with variable $x = u + Iv$, satisfies the following equation on $\Omega_I = \Omega \cap \mathbb{C}_I$*

$$\overline{\partial_I} f(u+Iv) := \frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u+vI) = 0.$$

The space of all slice monogenic functions on Ω is denoted by $\mathcal{SM}(\Omega)$.

Finally, we state the famous Fueter-Sce-Qian mapping theorem in the Clifford monogenic case

Theorem 3.5.1 (Fueter-Sce-Qian mapping theorem). *Let Ω be an axially symmetric slice domain of $\mathbb{R}^{m+1}$. If f is an s-polymonogenic function. Then, the function defined by*

$$\tau_m(f)(x) = \Delta_{\mathbb{R}^{m+1}}^{\frac{m-1}{2}} f(x)$$

is monogenic.

CHAPTER 4

Approximation in slice hyperholomorphic Fock spaces

In this chapter we introduce two Fock spaces of slice regular functions. These spaces can be of two different kinds since they are equipped with different inner products and contain different functions. Then, we show that the set of quaternionic polynomials is dense in both Fock spaces of the first and of the second kind. Several proofs are presented, including constructive methods based on the Taylor expansion and on the convolution polynomials. In the last case, quantitative estimates in terms of higher order moduli of smoothness and of best approximation quantity are obtained. The results obtained in this part of the thesis are based on [62].

4.1 Motivation

Fock spaces have been introduced in quantum mechanics via tensor products to describe the quantum states space of variables belonging to a same Hilbert space. Then, it was realized that this description corresponds in fact to the Segal-Bargmann spaces, i.e. spaces of holomorphic functions in several variables which are square integrable with respect to a Gaussian measure. These spaces are important also in other settings, like in infinite dimensional analysis and in free analysis, since these spaces are related to the white noise space and to the theory of stochastic distributions, see [114]. For an account on the theory of Fock spaces one may consult for example the book [115]. Here we continue the study of slice hyperholomorphic Fock spaces over the quaternions started in [15] with the purpose of providing some approximation results, specifically our goal is to extend results on the density of polynomials in the complex case to this setting. We shall show that in this context one may define two types of Fock spaces, which are called of the first and of the second kind, and for which the approximation results require different techniques.

We review results on the Hilbert quaternionic Fock space and Segal-Bargmann transform. We introduce also the definition of Fock spaces of the first and second kind. First, we study the approximation result in Fock spaces of the first kind. Then, we move to prove the approximation result in Fock spaces of the second kind, obtaining a result of general validity. We obtain quantitative estimates in terms of higher order moduli of smoothness and of best approximation quantity. Finally we discuss the density of reproducing kernels, type and order of functions in the Fock spaces of the second kind.

4.2 The slice hyperholomorphic Fock space and Segal-Bargmann transform

The Bargmann-Fock space of slice hyperholomorphic functions in the Hilbert case was first introduced by Alpay, Colombo, Sabadini and Salomon in [15]. In this section, we briefly review this notion and recall some results on the quaternionic Segal-Bargmann transform introduced in [60] that will be useful in the sequel.

For given $I\in\mathbb{S}$ and $\nu>0$, we set

$$\mathcal{F}_I^{2,\nu}(\mathbb{H}):=\{f\in\mathcal{SR}(\mathbb{H});\ \frac{\nu}{\pi}\int_{\mathbb{C}_I}|f_I(q)|^2e^{-\nu|q|^2}d\lambda_I(q)<\infty\},$$

where $f_I=f|_{\mathbb{C}_I}$ and $d\lambda_I(q)=dxdy$ for $q=x+yI$. The set $\mathcal{F}_I^{2,\nu}(\mathbb{H})$ is called the slice hyperholomorphic Fock space. The right $\mathbb{H}$-vector space $\mathcal{F}_I^{2,\nu}(\mathbb{H})$ is endowed with the inner product

$$\langle f,g\rangle_{\mathcal{F}_I^{2,\nu}(\mathbb{H})}=\frac{\nu}{\pi}\int_{\mathbb{C}_I}\overline{g_I(q)}f_I(q)e^{-\nu|q|^2}d\lambda_I(q) \tag{4.2.1}$$

for $f,g\in\mathcal{F}_I^{2,\nu}(\mathbb{H})$, so that the associated norm is given by

$$\|f\|^2_{\mathcal{F}_I^{2,\nu}(\mathbb{H})}=\frac{\nu}{\pi}\int_{\mathbb{C}_I}|f_I(q)|^2e^{-\nu|q|^2}d\lambda_I(q).$$

It was shown in [15] that the monomials $f_n(q):=q^n;\ n=0,1,2,\cdots$, form an orthogonal basis of $\mathcal{F}_I^{2,\nu}(\mathbb{H})$ with

$$\langle f_n,f_n\rangle_{\mathcal{F}_I^{2,\nu}(\mathbb{H})}=\frac{m!}{\nu^m}\delta_{m,n}. \tag{4.2.2}$$

Moreover, for any $f = \sum_{n=0}^{\infty} q^n a_n$ and $g = \sum_{n=0}^{\infty} q^n b_n$ in $\mathcal{F}_I^{2,\nu}(\mathbb{H})$, we have

$$\langle f, g \rangle_{\mathcal{F}_I^{2,\nu}(\mathbb{H})} = \sum_{n=0}^{\infty} \frac{n!}{\nu^n} \overline{b_n} a_n, \tag{4.2.3}$$

Thus, a given series $f(q) = \sum_{n=0}^{\infty} q^n a_n$ belongs to $\mathcal{F}_I^{2,\nu}(\mathbb{H})$ if and only if the sequence of quaternions $(a_n)_{n\geq 0}$ satisfies the growth condition

$$\|f\|^2_{\mathcal{F}_I^{2,\nu}(\mathbb{H})} = \sum_{n=0}^{\infty} \frac{n!}{\nu^n} |a_n|^2 < \infty. \tag{4.2.4}$$

The definition of the quaternionic Bargmann-Fock space $\mathcal{F}_I^{2,\nu}(\mathbb{H})$ does not depend on the choice of the complex plane thanks to this observation:

Proposition 4.2.1. *Let f be slice entire function and $I, J \in \mathbb{S}$. Then, we have*

$$\frac{1}{2}\|f\|_{\mathcal{F}_I^{2,\nu}(\mathbb{H})} \leq \|f\|_{\mathcal{F}_J^{2,\nu}(\mathbb{H})} \leq 2\|f\|_{\mathcal{F}_I^{2,\nu}(\mathbb{H})}.$$

Remark 4.2.2. *According to the previous comment, we will denote in general the slice hyperholomorphic Fock space by $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$.*

We note that for a fixed $q \in \mathbb{H}$, the evaluation map $\delta_q : \mathcal{F}_{slice}^{2,\nu}(\mathbb{H}) \longrightarrow \mathbb{H}$; $\delta_q(f) := f(q)$, is a continuous linear form. More precisely, we have

Lemma 4.2.3. *For every $f \in \mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$, we have the estimate*

$$|\delta_q(f)| \leq \exp\left(\frac{\nu}{2}|q|^2\right)\|f\|_{\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})}.$$

Thus, by Riesz' representation theorem for quaternionic Hilbert spaces, there exists a unique element K_q^ν in $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$ such that:

$$\left\langle f, K_q^\nu \right\rangle_{\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})} = \delta_q(f) = f(q)$$

for all $f \in \mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$. The reproducing kernel function $K_\nu : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$; $(p, q) \longmapsto K_\nu(p, q) = K_q^\nu(p)$ is then given by

$$K_\nu(p, q) = \sum_{n=0}^{\infty} \frac{\nu^n p^n \overline{q}^n}{n!} = \overline{K_\nu(q, p)}. \tag{4.2.5}$$

We have also

Proposition 4.2.4. *For every $q, q' \in \mathbb{H}$, we have*

$$\left\langle K_q^\nu, K_{q'}^\nu \right\rangle_{\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})} = K_\nu(q', q)$$

and in particular

$$\left\|K_q^\nu\right\|^2_{\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})} = e^{\nu|q|^2}.$$

The reproducing kernel of $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$ is given by

$$K_\nu(p,q) = K_q^\nu(p) = \sum_{n=0}^{\infty} \frac{\nu^n p^n \overline{q}^n}{n!} = e_*(\nu p\overline{q}), \quad \forall (p,q) \in \mathbb{H}\times\mathbb{H}.$$

Now, we turn our attention to the quaternionic Segal-Bargmann transform. It can be defined from the quaternionic Hilbert space $L^2(\mathbb{R};dx) = L^2(\mathbb{R};\mathbb{H})$, consisting of all the square integrable $\mathbb{H}$-valued functions with respect to

$$\langle \varphi, \psi\rangle_{L^2(\mathbb{R};dx)} := \int_{\mathbb{R}} \overline{\psi(x)}\varphi(x)dx, \tag{4.2.6}$$

onto the slice hyperholomorphic Bargmann-Fock space $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$. For this, we consider the kernel function

$$A(q;x) := \left(\frac{\nu}{\pi}\right)^{3/4} e^{\frac{-\nu}{2}(q^2+x^2)+\nu\sqrt{2}qx}; \quad (q,x)\in\mathbb{H}\times\mathbb{R}, \tag{4.2.7}$$

obtained as the slice hyperholomorphic extension of the kernel function of the classical Segal-Bargmann transform. This is closely connected with the fact that $A(q;x)$ can be seen as the generating function of the real weighted Hermite functions

$$h_n^\nu(x) := (-1)^n e^{\frac{\nu}{2}x^2} \frac{d^n}{dx^n}\left(e^{-\nu x^2}\right)$$

that form an orthogonal basis of $L^2(\mathbb{R};dx)$, with norm given explicitly by

$$\|h_n^\nu\|^2_{L^2(\mathbb{R};dx)} = 2^n \nu^n n! \left(\frac{\pi}{\nu}\right)^{1/2}. \tag{4.2.8}$$

In particular, we have the expansion

Proposition 4.2.5. *For all $q\in\mathbb{H}$ and $x\in\mathbb{R}$, we have*

$$A(q;x) = \sum_{n=0}^{\infty} \frac{h_n^\nu(x)}{\|h_n^\nu\|_{L^2(\mathbb{R};dx)}} \frac{q^n}{\|q^n\|_{\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})}}.$$

Another property concerns the partial function of the above kernel function defined on $\mathbb{R}$ by $A_q : x \mapsto A_q(x) := A(q;x)$ for every fixed $q\in\mathbb{H}$. It connects the norm of A_q in $L^2(\mathbb{R},\mathbb{H})$ to the one of the reproducing kernel function K_q^ν in $\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})$. In fact, we have

Proposition 4.2.6. *For every fixed $q\in\mathbb{H}$, the function A_q is an element of $L^2(\mathbb{R},\mathbb{H})$ and satisfies*

$$\|A_q\|_{L^2(\mathbb{R},\mathbb{H})} = e^{\frac{\nu}{2}|q|^2} = \left\|K_q^\nu\right\|_{\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})}. \tag{4.2.9}$$

Associated to the kernel function $A(q;x)$ given by (4.2.7), we consider the integral transform defined by

$$\mathcal{B}_{\mathbb{H}}^{\nu}(\psi)(q)=\int_{\mathbb{R}}A(q;x)\psi(x)dx=\left(\frac{\nu}{\pi}\right)^{\frac{3}{4}}\int_{\mathbb{R}}e^{\frac{-\nu}{2}(q^2+x^2)+\nu\sqrt{2}qx}\psi(x)dx \quad (4.2.10)$$

for $q\in\mathbb{H}$ and $\psi:\mathbb{R}\longrightarrow\mathbb{H}$, provided that the integral exists. We will call it the quaternionic Segal-Bargmann transform. The following shows that $\mathcal{B}_{\mathbb{H}}^{\nu}$ is well defined on $L^2(\mathbb{R};dx)$.

Proposition 4.2.7. *For every $q\in\mathbb{H}$ and every $\psi\in L^2(\mathbb{R};dx)$, we have*

$$|\mathcal{B}_{\mathbb{H}}^{\nu}(\psi)(q)|\leq\left(\frac{\nu}{\pi}\right)^{1/2}e^{\frac{\nu}{2}|q|^2}\|\psi\|_{L^2(\mathbb{R};dx)}. \quad (4.2.11)$$

The explicit expression of the Segal-Bargmann transform acting on the Hermite functions h_n^{ν} is given by Namely, we have

Lemma 4.2.8. *For every quaternion $q\in\mathbb{H}$ and nonnegative integer n, we have*

$$\mathcal{B}_{\mathbb{H}}^{\nu}(h_n^{\nu})(q)=\left(\frac{\nu}{\pi}\right)^{\frac{1}{4}}2^{\frac{n}{2}}\nu^n q^n$$

and

$$\|\mathcal{B}_{\mathbb{H}}^{\nu}(h_n^{\nu})\|_{\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})}=\|h_n^{\nu}\|_{L^2(\mathbb{R},\mathbb{H})}.$$

An important fact that was proved in [60] is given by

Theorem 4.2.9. *The quaternionic Segal-Bargmann transform $\mathcal{B}_{\mathbb{H}}^{\nu}$ realizes a surjective isometry from the Hilbert space $L^2(\mathbb{R},\mathbb{H})$ onto the slice hyperholomorphic Bargmann-Fock space $\mathcal{F}_{slice}^{2,\nu}(\mathbb{H})$.*

The following properties hold for the quaternionic Segal-Bargmann transform

Proposition 4.2.10. *For all $\varphi\in L^2_{\mathbb{H}}(\mathbb{R})$ such that $x\varphi,\dfrac{d}{dx}\varphi\in L^2_{\mathbb{H}}(\mathbb{R})$ we have*

1. $(\partial_S+\nu q)\,\mathcal{B}_{\mathbb{H}}^{\nu}\left[\varphi\right](q)=\nu\sqrt{2}\mathcal{B}_{\mathbb{H}}^{\nu}[x\varphi](q)$.
2. $\mathcal{B}_{\mathbb{H}}^{\nu}\left[\left(x-\dfrac{d}{dx}\right)\varphi\right](q)=\nu\sqrt{2}q\mathcal{B}_{\mathbb{H}}^{\nu}[\varphi](q)$.

4.3 Banach Fock spaces of slice hyperholomorphic functions

In the framework of slice regular functions, one may consider two kinds of function spaces. In the papers [68, 73], the properties of density for quaternionic polynomials in these kinds of spaces were obtained for Bergman, Bloch and Besov spaces. See also the recent book [72] about a general quaternionic approximation theory. Let us mention here that in the complex case, convolution polynomials were used to obtain constructive approximation results in complex Bergman spaces, see [70]. In this chapter we continue this type of study for the quaternionic slice regular functions in the so-called Fock spaces. Before to introduce them, we firstly recall some known facts about Fock spaces in the complex case

Definition 4.3.1. *(see, e.g., [115], p. 36) Let $0 < p < \infty$ and $\alpha > 0$. The Fock space $F_{\alpha}^{p}(\mathbb{C})$ is defined as the space of all entire functions in $\mathbb{C}$ with the property that $\frac{\alpha p}{2\pi}\int_{\mathbb{C}}\left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z) < +\infty$, where $dA(z) = dxdy = rdrd\theta$, $z = x + iy = re^{i\theta}$, is the area measure in the complex plane.*

Remark 4.3.1. Endowed with

$$\|f\|_{p,\alpha}^{p} = \frac{\alpha p}{2\pi}\int_{\mathbb{C}}\left|f(z)e^{-\alpha|z|^2/2}\right|^p dA(z),$$

it is known (see, e.g., [115], p. 36) that F_{α}^{p} is a Banach space for $1 \le p < \infty$, and a complete metric space for $\|\cdot\|_{p,\alpha}^{p}$ with $0 < p < 1$. Also, if $p = +\infty$, then endowed with $\|f\|_{\infty,\alpha} = esssup\{|f(z)|e^{-\alpha|z|^2/2}; z \in \mathbb{C}\}$, F_{α}^{∞} is a Banach space.

Remark 4.3.2. Concerning the approximation by polynomials in Fock spaces, qualitative results without any quantitative estimates were obtained. For any $0 < p < \infty$, and $f \in F_{\alpha}^{p}$, there exists a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\|f - P_n\|_{p,\alpha} = 0$ (see, e.g., Proposition 2.9, p. 38 in [115]). The proof of this result is not constructive and consists in two steps : at step 1, one approximates $f(z)$ by its dilations $f(rz)$ with $r \to 1^-$ and at step 2 one approximates each f_r by its attached Taylor polynomials. If $1 < p < \infty$, then one can construct P_n as the Taylor polynomials attached to f (see, e.g., Exercise 5, p. 89 in [115]) but if $0 < p \le 1$, then there exists $f \in F_{\alpha}^{p}$ which cannot be approximated by its associated Taylor polynomials (see, e.g., Exercise 6, p. 89 in [115]). However, if $f \in F_{\alpha}^{\infty}$ is such that $\lim_{z\to\infty} f(z)e^{\alpha|z|^2/2} = 0$, then f can be approximated by polynomials in the norm $\|\cdot\|_{\infty,\alpha}$ (see, e.g., Exercise 8, p. 89 in [115]).

We now consider Fock spaces in the quaternionic setting and we begin with the following definition. We note that this notion has not been previously considered in the literature in this generality.

Definition 4.3.2. *Let $0 < p < +\infty$ and $0 < \alpha < +\infty$. The Fock space of the first kind $\mathcal{F}_{\alpha}^{p}(\mathbb{H})$ is defined as the space of entire slice regular functions $f \in \mathcal{SR}(\mathbb{H})$, such that*

$$\|f\|_{p,\alpha}^{p} := \left(\frac{\alpha p}{2\pi}\right)^2 \int_{\mathbb{H}} |f(q)|^p (e^{-\alpha|q|^2/2})^p dm(q) < +\infty,$$

with $dm(q)$ representing the Lebesgue volume element in $\mathbb{R}^4$.

Remark 4.3.3. With the same techniques used in the complex case, one may verify that for $1 \leq p < +\infty$, $\| \cdot \|_{p,\alpha}$ has the properties of a norm, while for $0 < p < 1$, $\|f-g\|_{p,\alpha}^p$ has the properties of quasi-norm.

To introduce the Fock spaces of the second kind, we need the following definition:

Definition 4.3.3. *For $I \in \mathbb{S}$, $0 < \alpha < +\infty$ and $0 < p < +\infty$, let us denote*

$$\|f\|_{p,\alpha,I}^p = \frac{\alpha p}{2\pi} \int_{\mathbb{C}_I} |f(q)|^p (e^{-\alpha|q|^2/2})^p dm_I(q),$$

where $dm_I(q)$ represents the area measure on $\mathbb{C}_I$.
The space of all entire functions f with the property that $\|f\|_{p,\alpha,I} < +\infty$ will be denoted by $\mathcal{F}^p_{\alpha,I}(\mathbb{H})$.

We are now in position to introduce the following:

Definition 4.3.4. *The Fock space of the second kind $\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$ is defined as the space of all $f \in \mathcal{SR}(\mathbb{H})$ with the property that for some $I \in \mathbb{S}$ we have $f \in \mathcal{F}^p_{\alpha,I}(\mathbb{H})$. In order to make the norm independent of the choice of the imaginary unit, we set*

$$\|f\|_{\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})} = \sup_{I \in \mathbb{S}} \|f\|_{p,\alpha,I}.$$

4.4 Approximation by polynomials in Fock spaces of the first kind

In the sequel, we consider the quaternionic Fock spaces of the first kind $\mathcal{F}^p_\alpha(\mathbb{H})$ introduced in the previous section. First, we start by proving the following estimate:

Lemma 4.4.1. *Let $f \in \mathcal{F}^p_\alpha(\mathbb{H})$. Then, there exists a constant $c > 0$ such that for all $q \in \mathbb{H}$, we have*

$$|f(q)| \leq c e^{\frac{\alpha}{2}|q|^2} \|f\|_{p,\alpha},$$

where $c = 4\left(\dfrac{2\pi}{\alpha p}\right)^{\frac{1}{p}}$.

Proof. Let $I \in \mathbb{S}$, since f is slice regular on $\mathbb{H}$, then making use of the Splitting Lemma we have that for all $z \in \mathbb{C}_I$,

$$f_I(z) = F(z) + G(z)J,$$

where $J \in \mathbb{S}$ is orthogonal to I, and F, G are two holomorphic functions on the slice $\mathbb{C}_I$. Note that since $f \in \mathcal{F}^p_\alpha(\mathbb{H})$ it is easy to see that F and G belong to the classical Fock space $\mathcal{F}^p_\alpha(\mathbb{C}_I)$. Thus, by the classical complex analysis the following estimates are satisfied for any $z \in \mathbb{C}_I$

$$|F(z)| \leq e^{\frac{\alpha}{2}|z|^2} \|F\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} \text{ and } |G(z)| \leq e^{\frac{\alpha}{2}|z|^2} \|G\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)}.$$

Then,

$$\begin{aligned}|f(z)| &\leq |F(z)| + |G(z)| \\ &\leq e^{\frac{\alpha}{2}|z|^2}(\|F\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} + \|G\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)}).\end{aligned}$$

However, since $|F(z)| \leq |f(z)|$ for any $z \in \mathbb{C}_I$, we have

$$\begin{aligned}\|F\|^p_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} &= \frac{\alpha p}{2\pi}\int_{\mathbb{C}_I} |F(z)|^p (e^{-\alpha|z|^2/2})^p dm_I(z) \\ &\leq \frac{\alpha p}{2\pi}\int_{\mathbb{C}_I} |f(z)|^p (e^{-\alpha|z|^2/2})^p dm_I(z) \\ &\leq \frac{\alpha p}{2\pi}\int_{\mathbb{H}} |f(q)|^p (e^{-\alpha|q|^2/2})^p dm(q) \\ &= \frac{2\pi}{\alpha p}\|f\|^p_{p,\alpha}.\end{aligned}$$

By similar arguments we get also $\|G\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} \leq \left(\dfrac{2\pi}{\alpha p}\right)^{\frac{1}{p}} \|f\|_{p,\alpha}$. So, for any $z \in \mathbb{C}_I$ we have the following estimate

$$|f(z)| \leq 2\left(\frac{2\pi}{\alpha p}\right)^{\frac{1}{p}} e^{\frac{\alpha}{2}|z|^2}\|f\|_{p,\alpha}.$$

Finally, for $q = x + Jy \in \mathbb{H}$ by the Representation Formula we have

$$f(q) = \frac{1}{2}\left[f(z) + f(\overline{z})\right] + J\frac{I}{2}\left[f(\overline{z}) - f(z)\right]; z = x + Iy \in \mathbb{C}_I$$

Thus,

$$|f(q)| \leq |f(z)| + |f(\overline{z})|.$$

Hence, the last inequality combined with the estimate on $\mathbb{C}_I$ give

$$|f(q)| \leq 4\left(\frac{2\pi}{\alpha p}\right)^{\frac{1}{p}} e^{\frac{\alpha}{2}|q|^2}\|f\|_{p,\alpha},$$

for all $q \in \mathbb{H}$. □

Now, we can state and prove the main result of the polynomial approximation in this setting.

Theorem 4.4.2. *Let $\alpha > 0$ and $0 < p < \infty$. The set of all quaternionic polynomials is included in $\mathcal{F}^p_\alpha(\mathbb{H})$ and for every $f \in \mathcal{F}^p_\alpha(\mathbb{H})$, there exists a sequence of quaternionic polynomials $(p_n)_{n\in\mathbb{N}}$ such that $\|p_n - f\|_{p,\alpha} \to 0$ as $n \to +\infty$.*

Proof. First of all, we observe that any quaternionic polynomial belongs to $\mathcal{F}_\alpha^p(\mathbb{H})$. This follows easily from the fact that for any $k = 0, 1, \ldots.$, we have

$$\int_{\mathbb{H}} |q^k|^p (e^{-\alpha|q|^2/2})^p dm(q) < +\infty.$$

We then divide the proof in two steps.

Step 1. Let $0 < r < 1$, $f \in \mathcal{F}_\alpha^p(\mathbb{H})$ and define $f_r(q) = f(rq)$. Evidently f_r is an entire slice regular function.

Firstly, we will prove that $\lim_{r\to 1^-} \|f_r - f\|_{p,\alpha} = 0$. We will reason similar to the complex case in the proof of Proposition 2.9, p. 38 in [115], taking into account that $f : \mathbb{H} \to \mathbb{H}$ can be written componentwise as

$$f(q) = f_1(q) + if_2(q) + jf_3(q) + kf_4(q),$$

$q = x_1 + ix_2 + jx_3 + kx_4$ and that applying the Lemma 3.17, p. 66 in [86] to f is equivalent to apply it to each real-valued function of four real variables $f_k(q)$, $k = 1, 2, 3, 4$.

By using the componentwise form, since f is entire slice regular it follows that it is continuous on $\mathbb{H}$ and it is immediate that $\lim_{r\to 1^{-1}} f(rq) = f(q)$, for all $q \in \mathbb{H}$.

Now, for $f \in \mathcal{F}_\alpha^p(\mathbb{H})$, changing the variable $rq = w$ and taking into account that as in the proof of Theorem 2.1 in [68], we have $dm(q) = \frac{1}{r^4} dm(w)$, we obtain

$$\begin{aligned}
\|f_r\|_{p,\alpha}^p &= \left(\frac{\alpha p}{2\pi}\right)^2 \int_{\mathbb{H}} |f(rq)e^{-\alpha|q|^2/2}|^p dm(q) \\
&= \left(\frac{\alpha p}{2\pi}\right)^2 \frac{1}{r^4} \cdot \int_{\mathbb{H}} |f(w)e^{-\alpha|w|^2/2}|^p \cdot e^{-p\alpha|w|^2(r^{-2}-1)/2} dm(w).
\end{aligned}$$

Since for all $w \in \mathbb{H}$ and $0 < r < 1$ we have $e^{-p\alpha|w|^2(r^{-2}-1)/2} \leq 1$, by applying the dominated convergence theorem in the above mentioned Lemma 3.17 in [86], we are lead to $\lim_{r\to 1^{-1}} \|f_r - f\|_{p,\alpha} = 0$.

Step 2. The proof is terminated if we can show that for every $r \in (0, 1)$, the function f_r can be approximated by some quaternionic polynomials in the norm topology of $\mathcal{F}_\alpha^p(\mathbb{H})$. To this end, let $0 < r < 1$ and $\alpha r^2 < \beta < \alpha$. On one hand, note that f_r is slice regular on $\mathbb{H}$. Moreover, according to Lemma 4.4.1 there exists $c > 0$ such that for any $q \in \mathbb{H}$ we have

$$|f_r(q)| = |f(rq)| \leq c e^{\frac{\alpha}{2} r^2 |q|^2} \|f\|_{p,\alpha}.$$

Thus, since $\alpha r^2 - \beta < 0$ we get

$$\int_{\mathbb{H}} |f_r(q)|^2 e^{-\beta|q|^2} dm(q) \leq c^2 \|f\|_{p,\alpha}^2 \int_{\mathbb{H}} e^{(\alpha r^2 - \beta)|q|^2} dm(q) < \infty.$$

In particular, this shows that f_r belongs to $\mathcal{F}_\beta^2(\mathbb{H})$. Furthermore, since $\beta - \alpha < 0$ we can see also that $\mathcal{F}_\beta^2(\mathbb{H})$ is continuously embedded in $\mathcal{F}_\alpha^p(\mathbb{H})$. Indeed, for

$h \in \mathcal{F}^2_\beta(\mathbb{H})$, applying Lemma 4.4.1 there exists $C > 0$, such that for any $q \in \mathbb{H}$, we have
$$|h(q)| \leq Ce^{\frac{\beta}{2}|q|^2}\|h\|_{2,\beta}.$$
Thus,
$$\int_{\mathbb{H}} |h(q)|^p e^{-\frac{\alpha p}{2}|q|^2} dm(q) \leq C^p \|h\|^p_{2,\beta} \int_{\mathbb{H}} e^{\frac{(\beta-\alpha)p}{2}|q|^2} dm(q) < \infty.$$
Hence, this shows that $\|h\|_{p,\alpha} \leq K\|h\|_{2,\beta}$ where $K = K(\alpha, \beta, p) > 0$. On the other hand, it is clear that the quaternionic monomials $(q^n)_n$ are contained and generate any element of the quaternionic Hilbert space $\mathcal{F}^2_\beta(\mathbb{H})$ but they do not form an orthogonal basis of the Hilbert Fock space of the first kind. So, using the orthonormalization process we can obtain an orthonormal total family $(p_n)_n$ of quaternionic polynomials in $\mathcal{F}^2_\beta(\mathbb{H})$. Therefore, f_r can be approximated by $(p_n)_n$ since $f_r \in \mathcal{F}^2_\beta(\mathbb{H})$. Moreover,there exists $K > 0$ such that
$$\|f_r - p_n\|_{p,\alpha} \leq K\|f_r - p_n\|_{2,\beta}.$$
Finally, the previous inequality shows that f_r can be approximated by a sequence of quaternionic polynomials in the norm topology of $\mathcal{F}^p_\alpha(\mathbb{H})$. This ends the proof. □

Remark 4.4.3. The approximation in Fock spaces of the first kind is not based on the Taylor expansion since the quaternionic monomials do not form an orthogonal basis of the Hilbert Fock space of the first kind.

4.5 Approximation by polynomials in Fock spaces of the second kind

In this section we prove the density of polynomials in Fock spaces of the second kind, including a result with quantitative estimates in terms of higher order moduli of smoothness and in terms of the best approximation quantity.
Before to state our main result, we need to prove some technical results. The following proposition has a rather standard proof that we write for the sake of completeness.

Proposition 4.5.1. *Let $p \geq 1$ (resp. $0 < p < 1$) and let $\|\cdot\|_{p,\alpha,I}$ be the norm (resp. quasi-norm) in $\mathcal{F}^p_{\alpha,I}(\mathbb{H})$. Then $\|\cdot\|_{p,\alpha,I}$ and $\|\cdot\|_{p,\alpha,J}$ are equivalent for any $I, J \in \mathbb{S}$.*

Proof. From the representation formula we easily get
$$|f(x + yI)| \leq |f(x + yJ)| + |f(x - yJ)|.$$
Then, by taking $|.|^p$ in the above formula, and using the inequalities $(a + b)^p \leq 2^{p-1}(a^p + b^p)$, if $1 \leq p < +\infty$, and $(a + b)^p \leq a^p + b^p$, if $0 < p < 1$, for all $a, b \geq 0$, we obtain
$$|f(x + yI)|^p \leq 2^{p-1} [|f(x + yJ)|^p + |f(x - yJ)|^p], \text{ if } 1 \leq p < \infty$$

and

$$|f(x+yI)|^p \leq [|f(x+yJ)|^p + |f(x-yJ)|^p], \text{ if } 0<p<1.$$

Multiplying both terms in the above inequalities with $e^{-p\alpha|x+yI|^2/2}$, integrating on $\mathbb{C}_I$ with respect to $dm_I(q)$, then multiplying the corresponding obtained inequality with $e^{-p\alpha|x+yJ|^2/2}$, integrating on $\mathbb{C}_J$ with respect to $dm_J(q)$ and taking into account that $|x+yI|^2 = |x+yJ|^2 = |x-yJ|^2 = x^2+y^2$, we obtain an inequality of the form $\|f\|_{p,\alpha,I} \leq C_p\|f\|_{p,\alpha,J}$, with C_p independent of I and J.

Interchanging now I with J and repeating the above reasonings, we get the desired conclusion. □

Corollary 4.5.2. *Given any $I, J \in \mathbb{S}$ the slice hyperholomorphic Fock spaces $\mathcal{F}^p_{\alpha,I}(\mathbb{H})$ and $\mathcal{F}^p_{\alpha,J}(\mathbb{H})$ contain the same elements and have equivalent norms.*

Remark 4.5.3. The notion of Fock space of the second kind given in Definition 4.3.4 is independent of the choice of the imaginary unit, and this justifies the notation $\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$.

Lemma 4.5.4. *Let $0<p<\infty, \alpha>0$ and $f \in \mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$. Then, for any $q \in \mathbb{H}$ we have*

$$|f(q)| \leq 4e^{\frac{\alpha}{2}|q|^2}\|f\|_{\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})}.$$

Proof. Let $f \in \mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$ and let $I \in \mathbb{S}$. Then, choose J in $\mathbb{S}$ perpendicular to I. In particular, f is slice regular on $\mathbb{H}$, then by the Splitting Lemma there exist $F, G : \mathbb{C}_I \longrightarrow \mathbb{C}_I$ two holomorphic functions such that we have

$$f_I(z) = F(z) + G(z)J;\ \forall z \in \mathbb{C}_I.$$

Then, we use similar arguments as in the Lemma 4.4.1 to see that for any $z \in \mathbb{C}_I$, we have

$$|f(z)| \leq e^{\frac{\alpha}{2}|z|^2}(\|F\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} + \|G\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)}).$$

However, note that

$$\|F\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} \leq \|f\|_{\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})} \text{ and } \|G\|_{\mathcal{F}^p_\alpha(\mathbb{C}_I)} \leq \|f\|_{\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})}.$$

Thus, for any $z \in \mathbb{C}_I$ we get

$$|f(z)| \leq 2e^{\frac{\alpha}{2}|z|^2}\|f\|_{\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})}.$$

Finally, we apply the Representation Formula in order to prove the estimate for any $q \in \mathbb{H}$ and this completes the proof. □

The first main result of this section is the following.

Theorem 4.5.5. *Let $0<p<+\infty$, $0<\alpha<+\infty$ and $f \in \mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$. There exists a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ such that for any $I \in \mathbb{S}$ we have $\|P_n - f\|_{p,\alpha,I} \to 0$ as $n \to +\infty$.*

Proof. We divide the proof in two steps.

Step 1. Fix $I_0 \in \mathbb{S}$. For $0 < r < 1$, we define $f_r(q) = f(rq)$, $q \in \mathbb{H}$. By hypothesis, we know that $f \in \mathcal{F}^p_{\alpha,I_0}(\mathbb{H})$, i.e. f_r is an entire function on $\mathbb{C}_{I_0}$.

Firstly, we prove that $\lim_{r\to 1^-} \|f_r - f\|_{p,\alpha,I_0} = 0$. To this end, we note that since the restriction of f_r is entire on $\mathbb{C}_{I_0}$, it is continuous, which evidently implies that pointwise we have $\lim_{r\to 1^-} f(rq) = f(q)$, for all $q \in \mathbb{C}_{I_0}$.

Now, for $f \in \mathcal{F}^p_{\alpha,I_0}(\mathbb{H})$, by setting $rq = w$ and taking into account that as in the proof of Theorem 2.1 in [68] (see also [26]), we have $dm_{I_0}(q) = \frac{1}{r^2} dm_{I_0}(w)$, we obtain

$$\begin{aligned}
\|f_r\|^p_{p,\alpha,I_0} &= \frac{\alpha p}{2\pi} \int_{\mathbb{C}_{I_0}} |f(rq)e^{-\alpha|q|^2/2}|^p dm_{I_0}(q) \\
&= \frac{\alpha p}{2\pi} \frac{1}{r^2} \cdot \int_{\mathbb{C}_{I_0}} |f(w)e^{-\alpha|w|^2/2}|^p \cdot e^{-p\alpha|w|^2(r^{-2}-1)/2} dm_{I_0}(w).
\end{aligned}$$

Since for all $w \in \mathbb{C}_{I_0}$ and $0 < r < 1$ we have

$$e^{-p\alpha|w|^2(r^{-2}-1)/2} \leq 1,$$

by applying the dominated convergence theorem we easily obtain

$$\lim_{r\to 1^{-1}} \|f_r\|^p_{p,\alpha,I_0} = \|f\|^p_{p,\alpha,I_0}.$$

Therefore, an application of the above mentioned Lemma 3.17 in [86] (see the proof of Theorem 2.1) leads to $\lim_{r\to 1^{-1}} \|f_r - f\|_{p,\alpha,I_0} = 0$.

Step 2. This part is exactly the same as in the case of complex variable, proof of part (b) in Proposition 2.9, p. 39 in [115], but reasoning on $\mathbb{C}_{I_0}$. Indeed, let $0 < r < 1$ and choose $r^2\alpha < \beta < \alpha$. Lemma 4.5.4 allows to see that $f_r \in \mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})$ since $\alpha r^2 < \beta$. On the other hand, the condition $\beta < \alpha$ combined with the Lemma 4.5.4 show that $\mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})$ is continuously embedded in $\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$. Moreover, for any $h \in \mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})$ there exists $c = c(p,\alpha,\beta) > 0$ such that we have the following estimate

$$\|h\|_{\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})} \leq c\|h\|_{\mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})}. \tag{4.5.1}$$

Note that the family of functions given by

$$e_k(q) := \sqrt{\frac{\beta^k}{k!}} q^k,$$

forms an orthonormal basis of $\mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})$ according to [15] and $f_r \in \mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})$ for any $0 < r < 1$. Thus, there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of quaternionic polynomials with right coefficients such that $\|f_r - P_n\|_{\mathcal{F}^{\beta,2}_{Slice}(\mathbb{H})} \longrightarrow 0$ when $n \to \infty$. Therefore, we just need to use the estimate (4.5.1) to conclude that the polynomials $(P_n)_{n\in\mathbb{N}}$ approximate f_r in $\mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$ for any $0 < r < 1$.

Finally, according to Proposition 3.1, any other norm (or quasi-norm, according to p), $\|\cdot\|_{p,\alpha,I}$ with $I \in \mathbb{S}$, is equivalent to the norm (quasi-norm) $\|\cdot\|_{p,\alpha,I_0}$, it follows that the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ converges to f in any norm (quasi-norm) $\|\cdot\|_{p,\alpha,I}$, which proves the theorem. □

Remark 4.5.6. The approximation in Fock spaces of the second kind is based on the Taylor expansion since the quaternionic monomials form an orthogonal basis of the Hilbert Fock space of the second kind.

In what follows, for $1 \le p < +\infty$ we present a constructive proof for the density result in Theorem 4.5.5, with quantitative estimates in terms of higher order moduli of smoothness and in terms of the best approximation quantity.

For this end, we introduce the following definition, in which we keep the notations from Section 2.

Definition 4.5.1. *Let $0 < p < +\infty$, $I \in \mathbb{S}$ and $f \in \mathcal{F}^p_{\alpha,I}(\mathbb{H})$. The higher order L^p-moduli of smoothness of k-th order is defined by*

$$\begin{aligned}\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,I}} &= \sup_{0\le|h|\le\delta}\left\{\int_{\mathbb{C}_I}|\Delta^k_h f(z)|^p\cdot[e^{-\alpha|z|^2/2}]^p dm_I(z)\right\}^{1/p}\\ &= \sup_{0\le|h|\le\delta}\|w_\alpha\Delta^k_h f\|_{L^p(\mathbb{C}_\mathbb{I})},\end{aligned}$$

where $k \in \mathbb{N}$, $w_\alpha(z) = e^{-\alpha|z|^2/2}$,

$$\Delta^k_h f(z) = \sum_{s=0}^{k}(-1)^{k+s}\binom{k}{s}f(ze^{Ish}) \text{ and } \|f\|_{L^p(\mathbb{C}_I)} = \left(\int_{\mathbb{C}_I}|f(z)|^p dm_I(z)\right)^{1/p}.$$

(In other words, $\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,I}} = \omega_k(f;\delta)_{w_\alpha,L^p(\mathbb{C}_I)}$ is a weighted modulus of smoothness.)
The best approximation quantity is defined by

$$E_n(f)_{p,\alpha,I} = \inf\{\|f-P\|_{p,\alpha,I}; P\in\mathcal{P}_n\} = \inf\{\|w_\alpha(f-P)\|_{L^p(\mathbb{C}_I)}; P\in\mathcal{P}_n\},$$

where $\mathcal{P}_n$ denotes the set of all polynomials of degree $\le n$.

Note that exactly as in the case of the L^p-moduli of smoothness for functions of real variable (see, e.g., [59], pp. 44-45), it can be proved that

$$\lim_{\delta\to 0}\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,I}} = 0,$$

$$\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,I}} \le (\lambda+1)^k\cdot\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,I}}, \text{ if } 1\le p<+\infty \tag{4.5.2}$$

and

$$[\omega_k(f;\lambda\cdot\delta)_{\mathcal{F}^p_{\alpha,I}}]^p \le (\lambda+1)^k\cdot[\omega_k(f;\delta)_{\mathcal{F}^p_{\alpha,I}}]^p, \text{ if } 0<p<1. \tag{4.5.3}$$

Indeed, this is immediate from the fact that denoting (for fixed z) $g(x) = f(ze^{ix})$, we get $\Delta_h^k f(z) = \overline{\Delta}_h^k g(0)$, where

$$\overline{\Delta}_h^k g(x_0) = \sum_{s=0}^{k} (-1)^{s+k} \binom{k}{s} g(x_0 + sh).$$

Now, for any $1 \leq p < +\infty$ and $f \in \mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$, let us define the convolution operators

$$L_n(f)(q) = \int_{-\pi}^{\pi} f(qe^{I_q t}) \cdot K_n(t)dt, \; q \in \mathbb{H}.$$

Here $K_n(t)$ is a positive and even trigonometric polynomial with the property $\int_{-\pi}^{\pi} K_n(t)dt = 1$.

In particular, we can consider the Fejér kernel

$$K_n(t) = \frac{1}{2\pi n} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^2,$$

and in this case we will denote $L_n(f)(q)$ by $F_n(f)(q)$.

For

$$K_{n,r}(t) = \frac{1}{\lambda_{n,r}} \cdot \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^{2r},$$

where r will be chosen as the smallest integer with $r \geq \frac{p(m+1)+2}{2}$, $m \in \mathbb{N}$ and the constants $\lambda_{n,r}$ are chosen such that $\int_{-\pi}^{\pi} K_{n,r}(t)dt = 1$, let us define

$$I_{n,m,r}(f)(q) = -\int_{-\pi}^{\pi} K_{n,r}(t) \sum_{k=1}^{m+1} (-1)^k \binom{m+1}{k} f(qe^{I_q kt})dt, \; q \in \mathbb{H}.$$

Also, let us define $V_n(f)(q) = 2F_{2n}(f)(q) - F_n(f)(q)$, $q \in \mathbb{H}$.

According to the reasonings in [74], [71], for fixed $I \in \mathbb{S}$, if $q \in \mathbb{C}_I$ then $L_n(f)(q)$, $I_{n,m,r}(f)(q)$ and $V_n(f)(q)$ are polynomials in q on $\mathbb{C}_I$, with coefficients independent of I and depending only on the series development of f. Therefore, as functions of q, $L_n(f)(q)$, $I_{n,m,r}(f)(q)$ and $V_n(f)(q)$ represent polynomials on the whole $\mathbb{H}$.

The second main result of this section is the following.

Theorem 4.5.7. *Let $1 \leq p < +\infty$, $0 < \alpha < +\infty$, $m \in \mathbb{N} \cup \{0\}$ and $f \in \mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$ be arbitrary but fixed.*

(i) $I_{n,m,r}(f)(q)$ is a quaternionic polynomial of degree less than $r(n-1)$, which for any $I \in \mathbb{S}$ satisfies the estimate

$$\|I_{n,m,r}(f) - f\|_{p,\alpha,I} \leq C_{p,m,r} \cdot \omega_{m+1}\left(f; \frac{1}{n}\right)_{\mathcal{F}^p_{\alpha,I}}, n \in \mathbb{N},$$

where $m \in \mathbb{N}$, r is the smallest integer with $r \geq \frac{p(m+1)+2}{2}$ and $C(p,m,r) > 0$ is a constant independent of f, n and I.

(ii) $V_n(f)(q)$ is a quaternionic polynomial of degree $\leq 2n-1$, satisfying for any $I \in \mathbb{S}$ the estimate

$$\|V_n(f) - f\|_{p,\alpha,I} \leq [2^{(p-1)/p} \cdot (2^p+1)^{1/p} + 1] \cdot E_n(f)_{p,\alpha,I},\ n \in \mathbb{N}.$$

Proof. For the fact that the convolution operators $I_{n,m,r}(f)(q)$ and $V_n(f)(q)$ are polynomials of the corresponding degrees see [74], [71].

(i) In the sequel we will apply the following well known Jensen type inequality for integrals: if $\int_{-\pi}^{+\pi} G(u)du = 1$, $G(u) \geq 0$ for all $u \in [-\pi,\pi]$ and $\varphi(t)$ is a convex function on the range of the measurable function of real variable F, then

$$\varphi\left(\int_{-\pi}^{+\pi} F(u)G(u)du\right) \leq \int_{-\pi}^{+\pi} \varphi(F(u))G(u)du.$$

Let $m \in \mathbb{N}$ and r be the smallest integer such that $r \geq \frac{p(m+1)+2}{2}$.

Now, by choosing $\varphi(t) = t^p$, $1 \leq p < \infty$, $q \in \mathbb{C}_I$, we get

$$\begin{aligned}|f(q) - I_{n,m,r}(f)(q)|^p &= \left|\int_{-\pi}^{\pi} \Delta_t^{m+1} f(q) K_{n,r}(t)dt\right|^p \\ &\leq \left[\int_{-\pi}^{\pi} |\Delta_t^{m+1} f(q)| K_{n,r}(t)dt\right]^p \\ &\leq \int_{-\pi}^{\pi} |\Delta_t^{m+1} f(q)|^p K_{n,r}(t)dt.\end{aligned}$$

Multiplying above by $[e^{-\alpha|q|^2/2}]^p$, integrating on $\mathbb{C}_I$ with respect to $dm_I(q)$ and taking into account the Fubini's theorem, we obtain

$$\begin{aligned}&\int_{\mathbb{C}_I} |I_{n,m,r}(f)(q) - f(q)|^p \cdot [e^{-\alpha|q|^2/2}]^p dm_I(q) \\ &\leq \int_{-\pi}^{\pi} \left[\int_{\mathbb{C}_I} |\Delta_t^{m+1} f(q)|^p \cdot [e^{-\alpha|q|^2/2}]^p dm_I(q)\right] K_{n,r}(t)dt \\ &\leq \int_{-\pi}^{\pi} \omega_{m+1}(f;|t|)^p_{\mathcal{F}^p_{\alpha,I}} \cdot K_{n,r}(t)dt \\ &\leq \int_{-\pi}^{\pi} \omega_{m+1}(f;1/n)^p_{\mathcal{F}^p_{\alpha,I}} (n|t|+1)^{(m+1)p} \cdot K_{n,r}(t)dt.\end{aligned}$$

By [92], p. 57, relation (5), for $r \in \mathbb{N}$ with $r \geq \frac{p(m+1)+2}{2}$, we get

$$\int_{-\pi}^{\pi} (n|t|+1)^{(m+1)p} \cdot K_{n,r}(t)dt \leq C_{p,m,r} < +\infty, \tag{4.5.4}$$

which proves the estimate in (i).

(ii) Now, let $f, g \in \mathcal{F}^{\alpha,p}_{Slice}(\mathbb{H})$ and $1 \leq p < +\infty$. By the convexity of $\varphi(t) = t^p$ we get the obvious inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, valid for all $a, b \geq 0$, which for all $q \in \mathbb{C}_I$ implies

$$|V_n(f)(q) - V_n(g)(q)| \leq 2|F_{2n}(f)(q) - F_{2n}(g)(q)| + |F_n(f)(q) - F_n(g)(q)|$$

$$\leq 2\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_{2n}(t)dt + \int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_n(t)dt$$

and

$$|V_n(f)(q) - V_n(g)(q)|^p \leq 2^{p-1}\left[\left(2\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_{2n}(t)dt\right)^p + \left(\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})| \cdot K_n(t)dt\right)^p\right]$$

$$\leq 2^{p-1}\left[2^p\int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})|^p \cdot K_{2n}(t)dt + \int_{-\pi}^{\pi} |f(qe^{It}) - g(qe^{It})|^p \cdot K_n(t)dt\right].$$

Multiplying above with $[e^{-\alpha|q|^2/2}]^p = [e^{-\alpha|qe^{It}|^2/2}]^p$, integrating this inequality on $\mathbb{C}_I$ with respect to $dm_I(q)$ and reasoning as at the above point (i), we obtain

$$\|V_n(f) - V_n(g)\|^p_{p,\alpha,I} \leq 2^{p-1}\left[2^p\int_{-\pi}^{\pi}\left(\int_{\mathbb{C}_I} |f(qe^{It}) - g(qe^{It})|^p[e^{-\alpha|qe^{It}|^2/2}]^p dm_I(q)\right) \times K_{2n}(t)dt + \int_{-\pi}^{\pi}\left(\int_{\mathbb{C}_I} |f(qe^{It}) - g(qe^{It})|^p[e^{-\alpha|qe^{It}|^2/2}]^p dm_I(q)\right) K_n(t)dt\right].$$

Setting $F(q) = |f(q) - g(q)|^p[e^{-\alpha|q|^2/2}]^p$, $q \in \mathbb{C}_I$, writing $q = r\cos(\theta) + Ir\sin(\theta)$ and taking into account that

$$dm_I(q) = \frac{1}{\pi} r dr d\theta,$$

simple calculations lead to the equality

$$\int_{\mathbb{C}_I} |F(qe^{It})|^p dm_I(q) = \int_{\mathbb{C}_I} |F(q)|^p dm_I(z), \text{ for all } t,$$

which replaced in the above inequality immediately implies

$$\|V_n(f) - V_n(g)\|^p_{p,\alpha,I} \leq 2^{p-1}[2^p\|f-g\|^p_{p,\alpha} + \|f-g\|^p_{p,\alpha}] = 2^{p-1}(2^p+1)\|f-g\|^p_{p,\alpha},$$

that is

$$\|V_n(f) - V_n(g)\|_{p,\alpha,I} \leq 2^{(p-1)/p} \cdot (2^p+1)^{1/p}\|f - g\|_{p,\alpha,I}.$$

Now, let us denote by P^*_n a polynomial of best approximation by elements in $\mathcal{P}_n$ in the norm in $\|\cdot\|_{p,\alpha,I}$, that is

$$E_n(f)_{p,\alpha,I} = \inf\{\|f - P\|_{p,\alpha,I}; P \in \mathcal{P}_n\} = \|f - P^*_n\|_{p,\alpha,I}.$$

Note that since $\mathcal{P}_n$ is finite dimensional (for fixed n), this polynomial P_n^* exists.

Since by similar reasonings with those in [69], p. 425 we get $V_n(P_n^*) = P_n^*$, for all $q \in \mathbb{C}_I$, it follows

$$\begin{aligned}\|f - V_n(f)\|_{p,\alpha,I} &\leq \|f - P_n^*\|_{p,\alpha,I} + \|V_n(P_n^*) - V_n(f)\|_{p,\alpha,I} \\ &\leq E_n(f)_{p,\alpha,I} + 2^{(p-1)/p} \cdot (2^p + 1)^{1/p} \|P_n^* - f\|_{p,\alpha,I} \\ &= [2^{(p-1)/p} \cdot (2^p + 1)^{1/p} + 1] \cdot E_n(f)_{p,\alpha,I},\end{aligned}$$

which proves (ii) and the theorem. $\square$

Remark 4.5.8. The result in Theorem 4.5.7 evidently holds also in the complex Fock spaces. In this context, the result is new.

In [15] the authors proved that the quaternionic Fock space of the second kind $\mathcal{F}_{Slice}^{\alpha,2}(\mathbb{H})$ is a right quaternionic Hilbert space whose reproducing kernel is given for all $(r, q) \in \mathbb{H}^2$ by

$$\begin{aligned}K_\alpha(r, q) &:= e_*(\alpha r \overline{q}) \\ &= \sum_{k=0}^{\infty} \frac{\alpha^k r^k \overline{q}^k}{k!}.\end{aligned}$$

Then, we denote by $\mathcal{R}$ the set of all functions of the form

$$f(r) = \sum_{k=1}^{n} K_\alpha(r, q_k) b_k, \forall r \in \mathbb{H}$$

where $(b_k)_k, (q_k)_k \in \mathbb{H}$ for all $k = 1, .., n$. As a consequence of the Theorem 4.5.5 we obtain the following result:

Theorem 4.5.9. *Let $\alpha > 0$ and $0 < p < \infty$. The set $\mathcal{R}$ is dense in the quaternionic Fock spaces of the second kind $\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$.*

Proof. i) The result is clear in the Hilbert case when $p = 2$. Indeed, we only use the reproducing kernel property to see that the orthogonal of $\mathcal{R}$ is reduced to zero.

ii) For $p > 0$, let f be a quaternionic polynomial with right coefficients. Then, there exists $0 < \beta < \alpha$ such that $\mathcal{F}_{Slice}^{\beta,2}(\mathbb{H})$ is continuously embedded in $\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$. Note that since f is a polynomial, by the Hilbert case it can be approximated by a sequence of $\mathcal{R}$ in the topology norm of $\mathcal{F}_{Slice}^{\beta,2}(\mathbb{H})$. Thus, let $q_1, ..., q_n \in \mathbb{H}$ and $(a_k)_{k=1,...,n} \subset \mathbb{H}$ be such that $\|f - \sum_{k=1}^{n} K_\beta^{\frac{\alpha}{\beta} q_k} a_k, \|_{\mathcal{F}_{Slice}^{\beta,2}(\mathbb{H})}$ tends to zero as $n \to \infty$. However, there exists $c > 0$ such that we have the following estimate

$$\|f - \sum_{k=1}^{n} K_\alpha^{q_k} a_k, \|_{\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})} \leq c\|f - \sum_{k=1}^{n} K_\alpha^{q_k} a_k, \|_{\mathcal{F}_{Slice}^{\beta,2}(\mathbb{H})}$$
$$\leq c\|f - \sum_{k=1}^{n} K_\beta^{\frac{\alpha}{\beta} q_k} a_k, \|_{\mathcal{F}_{Slice}^{\beta,2}(\mathbb{H})}.$$

This shows that $\|f - \sum_{k=1}^{n} K_\alpha^{q_k} a_k, \|_{\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})}$ tends to zero as $n \to \infty$, for any quaternionic polynomial f. However, in Theorem 4.5.5 we proved that the set of quaternionic polynomials is dense in any quaternionic Fock space of the second kind. Hence, $\mathcal{R}$ is dense in the quaternionic Fock spaces of the second kind $\mathcal{F}_\alpha^p(\mathbb{H})$.

□

The order and type of slice regular entire functions on quaternions were introduced in Chapter 5 of the book [50]. In the setting of the Fock spaces $\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$, we have:

Proposition 4.5.10. *Let $0 < p < \infty$ and $f \in \mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$. Then, f is of order less or equal than 2. Moreover, if f is of order 2, then it is of type $\sigma(f) \leq \frac{\alpha}{2}$.*

Proof. Note that $f \in \mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})$, then by Lemma 4.5.4, we have

$$|f(q)| \leq ce^{\frac{\alpha}{2}|q|^2} \|f\|_{\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})}.$$

In particular, we have

$$M_f(r) = \max_{|q|=r} |f(q)| \leq ce^{\frac{\alpha}{2}r^2} \|f\|_{\mathcal{F}_{Slice}^{\alpha,p}(\mathbb{H})}.$$

Therefore,

$$\rho(f) = \lim_{r\to\infty} \frac{\log(\log M_f(r))}{\log r} \leq 2.$$

Moreover, if $\rho(f) = 2$, then we have

$$\sigma(f) = \lim_{r\to\infty} \frac{\log M_f(r)}{r^2} \leq \frac{\alpha}{2}.$$

□

CHAPTER 5

The Cholewinski-Fock space in the slice hyperholomorphic setting

Inspired from the Cholewinski approach see [33], we investigate a family of Fock spaces in the quaternionic slice hyperholomorphic setting as well as some associated quaternionic linear operators. In a particular case, we reobtain the slice hyperholomorphic Fock space introduced and studied in [15]. The results obtained in this chapter are based on [61].

5.1 Motivation

We recall that in 1961 Bargmann introduced the Bargmann-Fock space on which the creation and annihilation operators, namely

$$M_z f(z) := z f(z) \quad \text{and} \quad Df(z) := \frac{d}{dz} f(z)$$

are closed, densely defined operators that are adjoints of each other and satisfy the classical commutation rule

$$[D, M_z] = \mathcal{I}$$

where $[.,.]$ and $\mathcal{I}$ are respectively the commutator and the identity operator. Furthermore, the standard Schrödinger Hilbert space on the real line is unitary equivalent to the Fock space via the so-called Segal-Bargmann transform. A few years later, in [33] Cholewinski extended this construction by studying a

Hilbert space of even entire functions weighted by a modified Bessel function of the third kind sometimes also called Macdonald function. His construction generalized the original one of Bargmann so that in a particular case the weight is exactly the classical normalized Gaussian measure. He also proved in [33] some commutator relations between the Schrödinger radial kinetic energy operator and the operator M_{z^2}. Then, in 2002 based on the approach of Cholewinski, Sifi and Soltani considered and studied in [108] a Hilbert space of entire functions that are not necessarily even with a weight involving the Macdonald function. As we already discussed before the topic of Segal-Bargmann-Fock spaces and associated integral transforms in this new quaternionic and slice monogenic setting was interesting from several points of view, see [15, 34, 46, 53, 60, 88, 96, 101]. The purpose of this chapter is to continue this exploration and present the study of a quaternionic Hilbert space of slice entire functions weighted by a modified Bessel function that we shall call here the quaternionic slice hyperholomorphic Cholewinski-Fock space or the slice Cholewinski-Fock space for short. This will allow us to extend some results obtained in [15, 60] on the slice hyperholomorphic Fock space and the quaternionic analogue of the Segal-Bargmann transform. Moreover, we study some specific quaternionic operators associated to the slice Cholewinski-Fock space. In a particular case, we show that the slice derivative and the quaternionic multiplication are adjoints of each other and satisfy the classical commutation rule on the slice Fock space introduced in [15].

The chapter has the following structure: in the next section we briefly review some useful properties of the Macdonald function as it will be needed in the sequel. Then, we define the slice Cholewinski-Fock space and we introduce an orthonormal basis. Moreover, we show that it is a quaternionic reproducing kernel Hilbert space. Section 4 is devoted to the study of a quaternionic unitary isomorphism between the slice Cholewinski-Fock space and a suitable quaternionic Hilbert space on the real line. This quaternionic isomorphism will be connected also to what we call the slice Dunkl transform. Then, we deal with two right quaternionic linear operators that are proved to be adjoint of each other and satisfy a specific commutation rule on the slice Cholewinski-Fock space. Finally, the last section explains how the results obtained in the quaternionic setting could be extended in a similar way to the slice monogenic setting with Clifford algebras valued functions.

5.2 Some properties of Bessel and modified Bessel functions

For more details about the subject of Bessel functions and related topics we refer the reader to [66, 90].

To any complex number ν is associated the so-called Bessel differential equation

$$x^2\frac{d^2}{dx^2}y+x\frac{d}{dx}y+(x^2-\nu^2)y=0 \tag{5.2.1}$$

Using the Frobenius method, a solution of the last equation is given by the Bessel function of the first kind, namely

$$J_\nu(x):=\left(\frac{x}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}.$$

The second linear independent solution of the Bessel equation is the Bessel function of the second kind Y_ν which is defined by

$$Y_\nu(x)=\frac{\cos(\nu\pi)J_\nu(x)-J_{-\nu}(x)}{\sin(\nu\pi)} \quad \text{if} \quad \nu\notin\mathbb{Z}$$

and

$$Y_n(x)=\lim_{\nu\to n}Y_\nu(x) \quad \text{if} \quad \nu=n\in\mathbb{Z}.$$

The same reasoning is adopted to construct a modified Bessel function of the third kind sometimes called also the Macdonald function and denoted by $K_\nu(x)$. To this end, we consider the modified Bessel equation given by

$$x^2\frac{d^2}{dx^2}y+x\frac{d}{dx}y-(x^2+\nu^2)y=0 \tag{5.2.2}$$

Analogously, the modified Bessel function of the first kind is defined by

$$I_\nu(x):=\left(\frac{x}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{1}{\Gamma(k+1)\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}$$

and the Macdonald function is defined by

$$K_\nu(x)=\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu\pi)} \quad \text{if} \quad \nu\notin\mathbb{Z}$$

and

$$K_n(x)=\lim_{\nu\to n}K_\nu(x) \quad \text{if} \quad \nu=n\in\mathbb{Z}.$$

The Macdonald function is of a particular interest for our study since it will appear in the next section as a weight of the quaternionic Hilbert space of entire slice regular functions instead of the classical Gaussian measure.

So, we summarize in the following Proposition some interesting properties of this function that will be useful in the sequel, see [66, 90].

Proposition 5.2.1. *Let $x > 0$ and $\delta, \nu \in \mathbb{R}$ such that $\delta + \nu > 0$ and $\delta - \nu > 0$. Then, we have the following formulas*

1. $K_\nu(x) = \displaystyle\int_0^\infty \exp(-x \cosh t) \cosh(\nu t) dt.$

2. $K_{\frac{1}{2}}(x) = K_{-\frac{1}{2}}(x) = \sqrt{\dfrac{\pi}{2x}} e^{-x}.$

3. $\displaystyle\int_0^\infty t^{\delta-1} K_\nu(t) dt = 2^{\delta-2} \Gamma\left(\frac{\delta}{2} + \frac{\nu}{2}\right) \Gamma\left(\frac{\delta}{2} - \frac{\nu}{2}\right).$

5.3 The slice hyperholomorphic Cholewinski-Fock space $\mathcal{F}^\alpha_{Slice}(\mathbb{H})$

Any quaternionic entire function may be written as

$$f = f^e + f^o$$

f^e and f^o are respectively even and odd functions where

$$f^e(q) := \frac{f(q) + f(-q)}{2} \quad \text{and} \quad f^o(q) := \frac{f(q) - f(-q)}{2}.$$

Then, thanks to the series expansion theorem for slice regular functions we have

$$f(q) = \sum_{n=0}^\infty q^n a_n \quad \text{with} \quad a_n \in \mathbb{H}$$

so that,

$$f^e(q) = \sum_{n=0}^\infty q^{2n} a_{2n} \quad \text{and} \quad f^o(q) = \sum_{n=0}^\infty q^{2n+1} a_{2n+1}.$$

Now, let $\alpha \geq -\dfrac{1}{2}$ and I be any imaginary unit in the sphere $\mathbb{S}$. Then, for $p = x + yI$ in the slice $\mathbb{C}_I$ we consider the following probability measure

$$d\lambda_{\alpha,I}(p) := \frac{|p|^{2\alpha+2}}{\pi 2^\alpha \Gamma(\alpha+1)} K_\alpha(|p|^2) d\lambda_I(p)$$

where K_α is the Macdonald function and $d\lambda_I(p)$ is the usual Lebesgue measure on the slice $\mathbb{C}_I$. In [108] the complex generalized Fock space $\mathcal{F}^\alpha(\mathbb{C})$ was defined to be the space consisting of complex entire functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying:

$$\int_{\mathbb{C}} |f^e(z)|^2 d\lambda_\alpha(z) + 2(\alpha+1) \int_{\mathbb{C}} |f^o(z)|^2 |z|^{-2} d\lambda_{\alpha+1}(z) < \infty.$$

Then, we consider the following definition

Definition 5.3.1. *A slice entire function $f : \mathbb{H} \longrightarrow \mathbb{H}$ is said to be in the slice Cholewinski-Fock space or the generalized slice Fock space if, for $I \in \mathbb{S}$ it satisfies the following condition*

$$\int_{\mathbb{C}_I} |f_I^e(p)|^2 d\lambda_{\alpha,I}(p) + 2(\alpha+1) \int_{\mathbb{C}_I} |f_I^o(p)|^2 |p|^{-2} d\lambda_{\alpha+1,I}(p) < \infty.$$

The space containing all such functions will be denoted $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$.

Remark 5.3.1. *Notice that if $\alpha = -\dfrac{1}{2}$ then thanks to (2) in Proposition 5.2.1 we can see that $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ is exactly the slice hyperholomorphic Fock space introduced and studied in [15]. Indeed, for $\alpha = -\dfrac{1}{2}$ we get*

$$d\lambda_{\alpha,I}(p) := \frac{1}{2\pi} e^{-|p|^2} d\lambda_I(p).$$

In particular, in this case f belongs to $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ if and only if it belongs to the classical slice hyperholomorphic Fock space.

For $f, g \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ we define the following inner product

$$\langle f, g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} := \int_{\mathbb{C}_I} \overline{g_I^e(p)} f_I^e(p) d\lambda_{\alpha,I}(p) + 2(\alpha+1) \int_{\mathbb{C}_I} \overline{g_I^o(p)} f_I^o(p) |p|^{-2} d\lambda_{\alpha+1,I}(p)$$

We shall see later that this definition is well posed since it does not depend on the choice of the imaginary unit I. We have :

Proposition 5.3.2. *$\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ is a right quaternionic Hilbert space with respect to $\langle \cdot, \cdot \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}$.*

Proof. Let (f_n) be a Cauchy sequence in $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. Take $I, J \in \mathbb{S}$ such that $I \perp J$. Then, since f_n are slice regular we can use the Splitting Lemma to write

$$f_{n,I} := F_n + G_n J \quad \forall n \in \mathbb{N}$$

where F_n and G_n are holomorphic functions on the slice $\mathbb{C}_I$ belonging to the generalized complex Fock space $\mathcal{F}^{\alpha}(\mathbb{C}_I)$. It is easy to see that $(F_n)_n$ and $(G_n)_n$ are Cauchy sequences in $\mathcal{F}^{\alpha}(\mathbb{C}_I)$. Hence, there exists two functions F and G belonging to $\mathcal{F}^{\alpha}(\mathbb{C}_I)$ such that the sequences $(F_n)_n$ and $(G_n)_n$ are converging respectively to F and G. Let $f_I = F + GJ$ and consider $f = ext(f_I)$ we have then $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. Moreover, the sequence (f_n) converges to f with respect to the norm of $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. This ends the proof. $\square$

For any $m, n \geq 0$, we set

$$E_{m,n}(\alpha) := \int_{\mathbb{C}_I} \overline{q^{2m}} q^{2n} d\lambda_{\alpha,I}(q)$$

and

$$O_{m,n}(\alpha) := \int_{\mathbb{C}_I} \overline{q^{2m+1}} q^{2n+1} |q|^{-2} \lambda_{\alpha+1,I}(q).$$

Then, the following formulas hold

Lemma 5.3.3. *For all $m, n \geq 0$, we have*

i) $E_{m,n}(\alpha) = \delta_{m,n} 2^{2n} n! \dfrac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)}$.

ii) $O_{m,n}(\alpha) = E_{m,n}(\alpha+1)$.

Proof. i) We write $q = re^{I\theta}$ using the polar coordinates. This leads to

$$\begin{aligned} E_{m,n}(\alpha) &= \frac{1}{2^\alpha \pi \Gamma(\alpha+1)} \int_0^\infty \int_0^{2\pi} e^{2(n-m)\theta I} r^{2(m+n+\alpha+1)} K_\alpha(r^2) r dr d\theta \\ &= \frac{1}{2^\alpha \pi \Gamma(\alpha+1)} \int_0^{2\pi} e^{2(n-m)\theta I} d\theta \int_0^\infty r^{2(m+n+\alpha+1)} K_\alpha(r^2) r dr \\ &= \frac{2\delta_{m,n}}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty r^{2(2n+\alpha+1)} K_\alpha(r^2) r dr. \end{aligned}$$

Making use of the change of variable $t = r^2$, we obtain

$$E_{m,n}(\alpha) = \frac{\delta_{m,n}}{2^\alpha \Gamma(\alpha+1)} \int_0^\infty t^{2n+\alpha+1} K_\alpha(t) dt$$

Then, the proof of i) ends thanks to the property 3 in Proposition 5.2.1 by taking $\delta = 2n + \alpha + 2$.

ii) This is obvious from the definition of $O_{m,n}(\alpha)$.

□

Thanks to the last lemma we have the two following propositions :

Proposition 5.3.4. *Let $f(q) = \sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{\infty} q^n b_n$ be two slice regular functions belonging to $\mathcal{F}^\alpha_{Slice}(\mathbb{H})$. Then, we have*

$$\langle f, g \rangle_{\mathcal{F}^\alpha_{Slice}(\mathbb{H})} = \sum_{n=0}^{\infty} \overline{b_n} a_n \beta_n(\alpha)$$

where

$$\beta_n(\alpha) := 2^n \left[\frac{n}{2}\right]! \frac{\Gamma\left(\left[\frac{n+1}{2}\right] + \alpha + 1\right)}{\Gamma(\alpha+1)}.$$

Here the symbol $[.]$ stands for the integer part.

Proof. We have

$$\langle f,g\rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} := \int_{\mathbb{C}_I} \overline{g^e_I(p)} f^e_I(p) d\lambda_{\alpha,I}(p) + 2(\alpha+1)\int_{\mathbb{C}_I} \overline{g^I_o(p)} f^o_I(p)|p|^{-2} d\lambda_{\alpha+1,I}(p)$$

Then, we set

$$A := \int_{\mathbb{C}_I} \overline{g^e_I(p)} f^e_I(p) d\lambda_{\alpha,I}(p) \quad \text{and} \quad B := 2(\alpha+1)\int_{\mathbb{C}_I} \overline{g^o_I(p)} f^o_I(p)|p|^{-2} d\lambda_{\alpha+1,I}(p)$$

Notice that

$$f^e(q) = \sum_{n=0}^{\infty} q^{2n} a_{2n} \quad \text{and} \quad g^e(q) = \sum_{m=0}^{\infty} q^{2m} b_{2m}.$$

Thus,

$$\begin{aligned} A &= \lim_{R\to\infty} \int_{\{|p|<R\}} \left(\sum_{m=0}^{\infty} \overline{b_{2m} p^{2m}}\right)\left(\sum_{n=0}^{\infty} p^{2n} a_{2n}\right) d\lambda_{\alpha,I}(p) \\ &= \sum_{m,n=0}^{\infty} \overline{b_{2m}} E_{m,n}(\alpha) a_{2n}. \end{aligned}$$

Hence, making use of the Lemma 5.3.3 we get

$$A = \sum_{k=0}^{\infty} \overline{b_{2k}} a_{2k} \beta_{2k}(\alpha).$$

Similarly, by writing

$$f^o(q) = \sum_{n=0}^{\infty} q^{2n+1} a_{2n+1} \quad \text{and} \quad g^o(q) = \sum_{m=0}^{\infty} q^{2m+1} b_{2m+1}$$

we obtain

$$B = \sum_{k=0}^{\infty} \overline{b_{2k+1}} a_{2k+1} \beta_{2k+1}(\alpha).$$

This leads to

$$\langle f,g\rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} = \sum_{n=0}^{\infty} \overline{b_n} a_n \beta_n(\alpha).$$

□

Remark 5.3.5. *Proposition 5.3.4 shows that the scalar product is independent of the choice of the imaginary unit I.*

Proposition 5.3.6. *For any $n \in \mathbb{N}$, we consider the functions*

$$\phi_n^\alpha(q) = \frac{q^n}{\sqrt{\beta_n(\alpha)}}.$$

Then, $\{\phi_n^\alpha\}_n$ form an orthonormal basis of $\mathcal{F}_{Slice}^\alpha(\mathbb{H})$.

Proof. Lemma 5.3.3 allows us to easily check that

$$\langle \phi_m^\alpha, \phi_n^\alpha \rangle_{\mathcal{F}_{Slice}^\alpha(\mathbb{H})} = \delta_{m,n} \quad \forall m, n \in \mathbb{N}.$$

Let us prove now that these functions form a basis of $\mathcal{F}_{Slice}^\alpha(\mathbb{H})$. Indeed, take f in $\mathcal{F}_{Slice}^\alpha(\mathbb{H})$ such that $\langle f, \phi_k^\alpha \rangle_{\mathcal{F}_{Slice}^\alpha(\mathbb{H})} = 0, \forall k$. Then, since f is entire slice regular it admits a series expansion so that $f = \displaystyle\sum_{n=0}^{\infty} \phi_n^\alpha c_n$ where $(c_n)_n \subset \mathbb{H}$. Notice that from the Proposition 5.3.4 we have

$$\langle f, \phi_k^\alpha \rangle_{\mathcal{F}_{Slice}^\alpha(\mathbb{H})} = c_k, \forall k \geq 0.$$

This shows that f is identically zero. □

An immediate consequence is :

Corollary 5.3.7. *An entire function of the form $f(q) = \displaystyle\sum_{n=0}^{\infty} q^n a_n$ belongs to $\mathcal{F}_{Slice}^\alpha(\mathbb{H})$ if and only if it satisfies the following growth condition*

$$\sum_{n=0}^{\infty} |a_n|^2 \beta_n(\alpha) < \infty.$$

Lemma 5.3.8. *For all $n \in \mathbb{N}$, we have $n! \leq \beta_n(\alpha)$.*

Proof. This is a consequence of the Duplication formula for the Gamma function given by

$$\frac{\Gamma(x)\Gamma(x + \frac{1}{2})}{\Gamma(2x)} = \frac{\sqrt{\pi}}{2^{2x-1}},$$

combined with the fact that the function β_n is increasing for $\alpha \geq -\dfrac{1}{2}$. Indeed, by treating both cases of $n = 2k$ and $n = 2k + 1$ with $k \in \mathbb{N}$ using the Duplication formula we get

$$\beta_n\left(-\frac{1}{2}\right) = n! \leq \beta_n(\alpha).$$

□

Remark 5.3.9. *$\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ is continuously embedded in the slice Fock space $\mathcal{F}_{Slice}(\mathbb{H})$. Indeed, we have*

$$\|f\|^2_{\mathcal{F}_{Slice}(\mathbb{H})} = \sum_{n=0}^{\infty} |a_n|^2 n! \leq \sum_{n=0}^{\infty} |a_n|^2 \beta_n(\alpha) = \|f\|^2_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}.$$

The equality holds if $\alpha = -\dfrac{1}{2}$.

In the sequel, we shall prove that $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ is a quaternionic reproducing kernel Hilbert space and give an expression of its reproducing kernel. To this end, let us fix $q \in \mathbb{H}$ and consider the evaluation mapping

$$\Lambda_q : \mathcal{F}^{\alpha}_{Slice}(\mathbb{H}) \longrightarrow \mathbb{H}; f \mapsto \Lambda_q(f) = f(q).$$

Then, we have the following estimate on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$:

Proposition 5.3.10. *For any $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$, there exists $0 < C(|q|) \leq e^{\frac{|q|^2}{2}}$ such that*

$$|\Lambda_q(f)| = |f(q)| \leq C(|q|) \, \|f\|_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} .$$

Proof. The series expansion theorem for slice regular functions asserts that

$$f(q) = \sum_{n=0}^{\infty} q^n a_n \quad \text{with} \quad (a_n)_n \subset \mathbb{H}.$$

Then using the Cauchy-Schwarz inequality we have the following estimates,

$$\begin{aligned} |\Lambda_q(f)| = |f(q)| &\leq \sum_{n=0}^{\infty} |q|^n |a_n| \\ &\leq \sum_{n=0}^{\infty} \frac{|q|^n}{\sqrt{\beta_n(\alpha)}} \sqrt{\beta_n(\alpha)} |a_n| \\ &\leq \left(\sum_{n=0}^{\infty} \frac{|q|^{2n}}{\beta_n(\alpha)} \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \beta_n(\alpha) |a_n|^2 \right)^{\frac{1}{2}} \\ &\leq C(|q|) \|f\|_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}. \end{aligned}$$

Notice that thanks to Lemma 5.3.8 this constant could be also estimated so that we have

$$C(|q|) = \left(\sum_{n=0}^{\infty} \frac{|q|^{2n}}{\beta_n(\alpha)} \right)^{\frac{1}{2}} \leq e^{\frac{|q|^2}{2}}.$$

$\square$

Remark 5.3.11. *Proposition 5.3.10 shows that all the evaluation mappings on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ are continuous. Then, the Riesz representation theorem for quaternionic Hilbert spaces, see [28] asserts that $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ is a quaternionic reproducing kernel Hilbert space.*

For $p, q \in \mathbb{H}$, we consider the function

$$L_\alpha(p,q) = L^q_\alpha(p) := \sum_{n=0}^{\infty} \frac{p^n q^n}{\beta_n(\alpha)}$$

where

$$\beta_n(\alpha) := 2^n \left[\frac{n}{2}\right]! \frac{\Gamma\left(\left[\frac{n+1}{2}\right] + \alpha + 1\right)}{\Gamma(\alpha+1)}.$$

If $q = x \in \mathbb{R}$ we use the following notation $L_\alpha(p,x) = L_\alpha(px)$ since $px = xp$.

The function

$$K_\alpha(p,q) = K^q_\alpha(p) := L^{\overline{q}}_\alpha(p)$$

satisfies the following properties

Proposition 5.3.12. *Let $q, s \in \mathbb{H}$ fixed,*

(a) $K^\alpha_q \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$.

(b) For all $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$, *we have* $f(q) = \left\langle f, K^\alpha_q \right\rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}$.

(c) $\left\langle K^\alpha_q, K^\alpha_s \right\rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} = K^\alpha_q(s)$.

Proof. 1. We have by definition

$$K^q_\alpha(p) := \sum_{n=0}^{\infty} p^n a_n \quad \text{where} \quad a_n = \frac{\overline{q}^n}{\beta_n(\alpha)}.$$

Thus,

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n(\alpha)|a_n|^2 &= \sum_{n=0}^{\infty} \frac{|q|^{2n}}{\beta_n(\alpha)} \\ &= C^2(|q|) \\ &\leq e^{|q|^2} < \infty. \end{aligned}$$

2. Let $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$; since f is slice regular on $\mathbb{H}$ we can write $f(p) = \sum_{n=0}^{\infty} p^n b_n$ with $(b_n)_n \subset \mathbb{H}$. Then, using the expression of K^α_q combined with Proposition 5.3.4 we obtain

$$\left\langle f, K^\alpha_q \right\rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} = \sum_{n=0}^{\infty} q^n b_n = f(q).$$

3. We just need to write

$$K^{\alpha}_{q}(p)=\sum_{n=0}^{\infty}\frac{\overline{q}^{n}}{\beta_n(\alpha)}p^n \quad \text{and} \quad K^{\alpha}_{s}(p)=\sum_{n=0}^{\infty}\frac{\overline{s}^{n}}{\beta_n(\alpha)}p^n$$

and then use the Proposition 5.3.4.

□

The results obtained in this section may be summarized in the following

Theorem 5.3.13. *$\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ is a quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by the following formula*

$$K_{\alpha}(p,q)=\sum_{n=0}^{\infty}\frac{p^n\overline{q^n}}{\beta_n(\alpha)} \quad \textit{for all } (p,q)\in\mathbb{H}\times\mathbb{H}.$$

Proof. It is a consequence of the Propositions 5.3.10 and 5.3.12. □

Remark 5.3.14. *For $\alpha=-\dfrac{1}{2}$, it turns out that $K_{\alpha}(.,.)$ is the reproducing kernel of the slice Fock space $\mathcal{F}_{Slice}(\mathbb{H})$ obtained in [15] and given by*

$$e_{*}(p\overline{q})=\sum_{n=0}^{\infty}\frac{p^n\overline{q}^n}{n!}.$$

5.4 A unitary integral transform associated to $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$

In this section we introduce an integral operator T_{α} and show that it defines an isometric isomorphism between the quaternionic slice Cholewinski Fock space introduced in the last section and a specific quaternionic Hilbert space on the real line, namely $\mathcal{H}_{\alpha}=L^2_{\mathbb{H}}(\mathbb{R},d\mu_{\alpha})$ where

$$d\mu_{\alpha}(x):=\frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}dx.$$

Note that the quaternionic Hilbert space $\mathcal{H}_{\alpha}$ is endowed by the inner product

$$\langle\psi,\phi\rangle_{\mathcal{H}_{\alpha}}:=\int_{\mathbb{R}}\overline{\phi(x)}\psi(x)d\mu_{\alpha}(x).$$

Note that $\mathcal{H}_{-\frac{1}{2}}$ is the standard Hilbert space $L^2_{\mathbb{H}}(\mathbb{R})$. In this case, the isomorphism T_{α} is the quaternionic Segal-Bargmann transform introduced and studied in [60].

Let us consider the kernel

$$\mathcal{C}_{\alpha}(p,x):=2^{\frac{\alpha+1}{2}}e^{-\frac{1}{2}(p^2+x^2)}L_{\alpha}(\sqrt{2}px) \quad \forall(p,x)\in\mathbb{H}\times\mathbb{R}$$

so that, for $\varphi \in \mathcal{H}_\alpha$ and $q \in \mathbb{H}$ we define

$$T_\alpha\varphi(q) := \int_{\mathbb{R}} \mathcal{C}_\alpha(q,x)\varphi(x)d\mu_\alpha(x).$$

In the sequel, we study the integral transform T_α. To this end, let us recall some properties of the so called generalized Hermite polynomials, see [103, 104, 108]. These generalized Hermite polynomials are defined by

$$H_n^\alpha(x) := \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k\beta_n(\alpha)}{k!\beta_{n-k}(\alpha)}(2x)^{n-2k}; x \in \mathbb{R}$$

their generating function in the classical complex case is given by

$$e^{-\frac{z^2}{4}}L_\alpha(zx) = \sum_{n=0}^{\infty} \frac{H_n^\alpha(x)}{2^n\beta_n(\alpha)}z^n; \quad z \in \mathbb{C}, x \in \mathbb{R}.$$

Moreover, it turns out that the Hermite functions

$$h_n^\alpha(x) := \frac{2^{-\frac{(n-\alpha-1)}{2}}}{\sqrt{\beta_n(\alpha)}}e^{-\frac{x^2}{2}}H_n^\alpha(x)$$

associated to these polynomials form an orthonormal basis of the Hilbert space $\mathcal{H}_\alpha$.

Then, similarly to the complex case we prove the following

Lemma 5.4.1. *Let $p \in \mathbb{H}$ and $x \in \mathbb{R}$. Then, we have*

$$e^{-\frac{p^2}{4}}L_\alpha(px) = \sum_{n=0}^{\infty} \frac{H_n^\alpha(x)}{2^n\beta_n(\alpha)}p^n.$$

Proof. Set

$$f_i(z) = e^{-\frac{z^2}{4}}L_\alpha(zx) \quad \text{and} \quad g_i(z) = \sum_{n=0}^{\infty} \frac{H_n^\alpha(x)}{2^n\beta_n(\alpha)}z^n; \quad \forall z \in \mathbb{C}_i.$$

Notice that f_i and g_i are two entire functions. Then, they could be extended into two slice entire regular functions denoted respectively $ext(f_i)$ and $ext(g_i)$. On the other hand, the functions

$$F(q) = e^{-\frac{q^2}{4}}L_\alpha(qx) \quad \text{and} \quad G(q) = \sum_{n=0}^{\infty} \frac{H_n^\alpha(x)}{2^n\beta_n(\alpha)}q^n$$

are entire slice regular on $\mathbb{H}$ since $q \mapsto e^{-\frac{q^2}{4}}$ is quaternionic intrinsic. It follows then from the uniqueness in the Lemma 3.1.7 that $F = ext(f_i)$ and $G = ext(g_i)$. Finally, since F and G coincide on the slice $\mathbb{C}_i$ we use the identity principle to conclude the proof. $\square$

As a consequence of the last lemma we have

Proposition 5.4.2. *Let $p \in \mathbb{H}$ and $x \in \mathbb{R}$. Then,*

$$\mathcal{C}_\alpha(p,x) = \sum_{n=0}^{\infty} h_n^\alpha(x)\frac{p^n}{\sqrt{\beta_n(\alpha)}}.$$

Proof. Let $(p,x) \in \mathbb{H} \times \mathbb{R}$, and write

$$\sum_{n=0}^{\infty} h_n^\alpha(x)\frac{p^n}{\sqrt{\beta_n(\alpha)}} = 2^{\frac{\alpha+1}{2}} e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{H_n^\alpha(x)}{\beta_n(\alpha)}\left(\frac{\sqrt{2}p}{2}\right)^n$$

Then, by taking $q = \sqrt{2}p \in \mathbb{H}$ we can apply the last lemma to get

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^\alpha(x)\frac{p^n}{\sqrt{\beta_n(\alpha)}} &= 2^{\frac{\alpha+1}{2}} e^{-\frac{1}{2}(x^2+p^2)} L_\alpha(\sqrt{2}px) \\ &= \mathcal{C}_\alpha(p,x).\end{aligned}$$

□

Another interesting property of the kernel $\mathcal{C}_\alpha$ is given by

Lemma 5.4.3. *Let $p, q \in \mathbb{H}$. Then,*

$$\int_{\mathbb{R}} \mathcal{C}_\alpha(p,x)\mathcal{C}_\alpha(q,x)d\mu_\alpha(x) = L_\alpha(p,q).$$

Proof. Let $p, q \in \mathbb{H}$, making use of the Proposition 5.4.2 we can write

$$\mathcal{C}_\alpha(p,x) = \sum_{n=0}^{\infty} h_n^\alpha(x)\frac{p^n}{\sqrt{\beta_n(\alpha)}} \quad \text{and} \quad \mathcal{C}_\alpha(q,x) = \sum_{n=0}^{\infty} h_n^\alpha(x)\frac{q^n}{\sqrt{\beta_n(\alpha)}}.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{C}_\alpha(p,x)\mathcal{C}_\alpha(q,x)d\mu_\alpha(x) &= \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} h_n^\alpha(x)\frac{p^n}{\sqrt{\beta_n(\alpha)}}\right)\left(\sum_{m=0}^{\infty} h_m^\alpha(x)\frac{q^m}{\sqrt{\beta_m(\alpha)}}\right)d\mu_\alpha(x) \\ &= \sum_{n,m=0}^{\infty} \frac{p^n q^m}{\sqrt{\beta_n(\alpha)}\sqrt{\beta_m(\alpha)}} \int_{\mathbb{R}} h_n^\alpha(x)h_m^\alpha(x)d\mu_\alpha(x).\end{aligned}$$

Then, since $\{h_n^\alpha\}_{n\geq 0}$ form an orthonormal set in $\mathcal{H}_\alpha$ we have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{C}_\alpha(p,x)\mathcal{C}_\alpha(q,x)d\mu_\alpha(x) &= \sum_{n,m=0}^{\infty} \frac{p^n q^m}{\sqrt{\beta_n(\alpha)}\sqrt{\beta_m(\alpha)}}\delta_{n,m} \\ &= L_\alpha(p,q).\end{aligned}$$

□

Therefore, we have

Proposition 5.4.4. *Let $q \in \mathbb{H}$ and $\varphi \in \mathcal{H}_\alpha$. Then,*

1. $\|\mathcal{C}_\alpha^q\|_{\mathcal{H}_\alpha} = \sqrt{L_\alpha(q,\overline{q})} = \sqrt{L_\alpha(|q|^2)}$.
2. $|T_\alpha \varphi(q)| \leq \sqrt{L_\alpha(|q|^2)}\|\varphi\|_{\mathcal{H}_\alpha}$.

Proof. 1. It is a direct consequence of the Lemma 5.4.3 combined with the identity $\overline{\mathcal{C}_\alpha(q,x)} = \mathcal{C}_\alpha(\overline{q},x)$ for all $q \in \mathbb{H}$.

2. We start by writing

$$|T_\alpha \varphi(q)| \leq \int_{\mathbb{R}} |\mathcal{C}_\alpha(q,x)||\varphi(x)|d\mu_\alpha(x)$$

Then, we use the Cauchy-Schwarz inequality to complete the proof.

□

Remark 5.4.5. *For $n \in \mathbb{N}$, we have $T_\alpha h_n^\alpha = \phi_n^\alpha$ and $\|T_\alpha h_n^\alpha\|_{\mathcal{F}^\alpha_{Slice}(\mathbb{H})} = \|h_n^\alpha\|_{\mathcal{H}_\alpha}$. Moreover, if $\varphi = \displaystyle\sum_{n=0}^{N} h_n^\alpha c_n$ then $\|T_\alpha \varphi\|_{\mathcal{F}^\alpha_{Slice}(\mathbb{H})} = \displaystyle\sum_{n=0}^{N} |c_n|^2 = \|\varphi\|_{\mathcal{H}_\alpha}$.*

Finally, we prove the following

Theorem 5.4.6. *The integral transform T_α is an isometric isomorphism mapping the quaternionic Hilbert space $\mathcal{H}_\alpha$ onto $\mathcal{F}^\alpha_{Slice}(\mathbb{H})$.*

Proof. Let $\varphi \in \mathcal{H}_\alpha$ and set $\varphi_N = \displaystyle\sum_{n=0}^{N} h_n^\alpha a_n$ with $(a_n) \subset \mathbb{H}$ such that φ_N converges to φ in $\mathcal{H}_\alpha$. Then, making use of the second estimate in Proposition 5.4.4 we can show that $T_\alpha \varphi_N$ is a Cauchy sequence in the quaternionic Hilbert space $\mathcal{F}^\alpha_{Slice}(\mathbb{H})$. Thus, there exists $f \in \mathcal{F}^\alpha_{Slice}(\mathbb{H})$ such that $T_\alpha \varphi_N \underset{\mathcal{F}^\alpha_{Slice}(\mathbb{H})}{\longrightarrow} f$. Consequently, it will exist a subsequence $(T_\alpha \varphi_{N_k})_{N_k}$ converging to f pointwise almost everywhere. Moreover, according to the Proposition 5.4.4 we have the following

$$|T_\alpha \varphi(p) - T_\alpha \varphi_N(p)| \leq C(|p|)\|\varphi - \varphi_N\|_{\mathcal{H}_\alpha}$$

Then, by letting N goes to infinity we can see that $(T_\alpha \varphi_N)_N$ converges pointwise to $T_\alpha \varphi$. In particular, the pointwise convergence shows that $T_\alpha \varphi = f$. However, by definition we have $f := \displaystyle\lim_{N \to \infty, \mathcal{F}^\alpha_{Slice}(\mathbb{H})} T_\alpha \varphi_N$.

Therefore, it follows that

$$\|T_\alpha \varphi\|_{\mathcal{F}^\alpha_{Slice}(\mathbb{H})} = \|f\|_{\mathcal{F}^\alpha_{Slice}(\mathbb{H})} = \lim_{N \to \infty} \|T_\alpha \varphi_N\|_{\mathcal{F}^\alpha_{Slice}(\mathbb{H})} = \|\varphi\|_{\mathcal{H}_\alpha}.$$

Hence, T_α is a quaternionic isometric integral operator which is one-to-one. Moreover, since $T_\alpha h_n^\alpha = \phi_n^\alpha$ it is also surjective and this ends the proof. □

As a consequence we have

Proposition 5.4.7. *Let $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ and $x \in \mathbb{R}$. Then, for any imaginary unit $I \in \mathbb{S}$ we have*

$$T_{\alpha}^{-1} f(x) = \int_{\mathbb{C}_I} \mathcal{C}^{e}_{\alpha,I}(\overline{p},x) f^{I}_{e}(p) d\lambda_{\alpha,I}(p) + 2(\alpha+1) \int_{\mathbb{C}_I} \mathcal{C}^{o}_{\alpha,I}(\overline{p},x) f^{I}_{o}(p) |p|^{-2} d\lambda_{\alpha+1,I}(p).$$

Proof. First, note that T_{α} is a surjective isometry. Then, it defines a quaternionic unitary operator which means that its inverse is given by $T_{\alpha}^{-1} = T_{\alpha}^{*}$ where T_{α}^{*} is the adjoint operator of T_{α}. Now, let $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$, then since $T_{\alpha}^{-1} = T_{\alpha}^{*}$ we have

$$\left\langle T_{\alpha}^{-1} f, g \right\rangle_{\mathcal{H}_{\alpha}} = \langle f, T_{\alpha} g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} \quad \forall g \in \mathcal{H}_{\alpha}.$$

Observe that $T_{\alpha} g$ is given by

$$T_{\alpha} g(q) = \int_{\mathbb{R}} \mathcal{C}_{\alpha}(q,x) g(x) d\mu_{\alpha}(x).$$

Then, we have

$$\begin{aligned}
\left\langle T_{\alpha}^{-1} f, g \right\rangle_{\mathcal{H}_{\alpha}} &= \langle f, T_{\alpha} g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} \quad \forall g \in \mathcal{H}_{\alpha}. \\
&= \int_{\mathbb{C}_I} \overline{(T_{\alpha} g)^{e}_{I}(p)} f^{e}_{I}(p) d\lambda_{\alpha,I}(p) + 2(\alpha+1) \int_{\mathbb{C}_I} \overline{(T_{\alpha} g)^{o}_{I}(p)} f^{o}_{I}(p) |p|^{-2} d\lambda_{\alpha+1,I}(p)
\end{aligned}$$

However, we have

$$(T_{\alpha} g)^{e}_{I}(p) = \int_{\mathbb{R}} \mathcal{C}^{e}_{\alpha}(p,x) g(x) d\mu_{\alpha}(x) \text{ and } (T_{\alpha} g)^{o}_{I}(p) = \int_{\mathbb{R}} \mathcal{C}^{o}_{\alpha}(p,x) g(x) d\mu_{\alpha}(x).$$

Thus, making use of Fubini's theorem we get that for any $g \in \mathcal{H}_{\alpha}$ we have

$$\begin{aligned}
\left\langle T_{\alpha}^{-1} f, g \right\rangle_{\mathcal{H}_{\alpha}} &= \int_{\mathbb{R}} \overline{g(x)} \psi_f(x) d\mu_{\alpha}(x) \\
&= \langle \psi_f, g \rangle_{\mathcal{H}_{\alpha}},
\end{aligned}$$

where we have set

$$\psi_f(x) = \int_{\mathbb{C}_I} \mathcal{C}^{e}_{\alpha,I}(\overline{p},x) f^{I}_{e}(p) d\lambda_{\alpha,I}(p) + 2(\alpha+1) \int_{\mathbb{C}_I} \mathcal{C}^{o}_{\alpha,I}(\overline{p},x) f^{I}_{o}(p) |p|^{-2} d\lambda_{\alpha+1,I}(p).$$

Since the last equality holds for all $g \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ it follows that $T_{\alpha}^{-1} f = \psi_f$. □

Remark 5.4.8. *The integral on the right hand side in Proposition 5.4.7 does not depend on the choice of the imaginary unit since the scalar product does not depend on the choice of the slice.*

We finish this section by connecting this unitary transform T_α to what we call the (right) slice Dunkl transform. Indeed, we define the (right) slice Dunkl transform of a function $\varphi \in \mathcal{H}_\alpha$ with respect to a slice $\mathbb{C}_I$ to be

$$D_\alpha^I \varphi(x) := \int_{\mathbb{R}} L_\alpha(-Ixt)\varphi(t)d\mu_\alpha(t)$$

More properties of the classical complex Dunkl transform can be found for example in [109]. Actually, this transform generalizes the classical Fourier transform on the real line. It satisfies a version of the Plancherel theorem since it extends uniquely to a unitary operator from the Hilbert space $\mathcal{H}_\alpha$ onto itself. Then, we prove

Lemma 5.4.9. *Let I be any imaginary unit in $\mathbb{S}$ and $\psi \in \mathcal{H}_\alpha$. Then,*

$$T_\alpha D_\alpha^I \psi(x) = T_\alpha \psi \circ g(x) \quad \textit{where} \quad g(x) = -xI \quad \textit{for all } x \in \mathbb{R}.$$

Proof. Let $x \in \mathbb{R}$ and $\psi \in \mathcal{H}_\alpha$, and consider the function

$$\varphi(s) = D_\alpha^I \psi(s) := \int_{\mathbb{R}} L_\alpha(-Ist)\psi(t)d\mu_\alpha(t)$$

Then, thanks to the Plancherel and Fubini's theorems $\varphi \in \mathcal{H}_\alpha$ we can write

$$\begin{aligned} T_\alpha\varphi(x) &= \int_{\mathbb{R}} \mathcal{C}_\alpha(x,s)\varphi(s)d\mu_\alpha(s) \\ &= \int_{\mathbb{R}} \mathcal{C}_\alpha(x,s) \left(\int_{\mathbb{R}} L_\alpha(-Ist)\psi(t)d\mu_\alpha(t) \right) d\mu_\alpha(s) \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{C}_\alpha(x,s)L_\alpha(-Ist)d\mu_\alpha(s) \right) \psi(t)d\mu_\alpha(t) \end{aligned}$$

Note that, we have

$$\phi(x,t) := \int_{\mathbb{R}} \mathcal{C}_\alpha(x,s)L_\alpha(-Ist)d\mu_\alpha(s).$$

Thus, we get

$$\phi(x,t) := 2^{\frac{\alpha+1}{2}} e^{-\frac{x^2}{2}} \int_{\mathbb{R}} e^{-\frac{s^2}{2}} L_\alpha(\sqrt{2}xs)L_\alpha(-Ist)d\mu_\alpha(s).$$

The last integral can be evaluated as in Theorem 3.4 in [109] since $x \in \mathbb{R}$. Then, we get

$$\phi(x,t) = \mathcal{C}_\alpha(-Ix,t).$$

Therefore, we obtain

$$T_\alpha D_\alpha^I \psi(x) = \int_{\mathbb{R}} \mathcal{C}_\alpha(-Ix,t)\psi(t)d\mu_\alpha(t).$$

Finally, this shows that
$$T_\alpha D^I_\alpha \psi(x) = T_\alpha \psi(-xI).$$
□

As a consequence of the Lemma 5.4.9 we have the following :

Proposition 5.4.10. *For any $I \in \mathbb{S}$ and $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ we have*
$$T_\alpha D^I_\alpha T^{-1}_\alpha (f)(x) = f(-xI) \quad \forall x \in \mathbb{R}.$$

Proof. We just have to take $\psi = T^{-1}_\alpha f \in \mathcal{H}_\alpha$ and then apply the Lemma 5.4.9. □

5.5 Some quaternionic operators on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$

In this section, we shall consider the two following operators on the slice Cholewinski-Fock space defined by
$$\mathcal{M}f(q) = qf(q) \quad \text{and} \quad \mathcal{D}_{\alpha,S}f(q) = \partial_S f(q) + (2\alpha+1)q^{-1}f^o(q)$$
with domains given respectively by
$$D(\mathcal{M}) = \{f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H}); \mathcal{M}f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})\}$$
and
$$D(\mathcal{D}_{\alpha,S}) = \{f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H}); \mathcal{D}_{\alpha,S}f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})\}.$$
Note that $\mathcal{M}$ and $\mathcal{D}_{\alpha,S}$ are quaternionic right linear operators densely defined on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ since $\{\phi^\alpha_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of this quaternionic Hilbert space.

In the sequel, we present some properties of these right quaternionic operators on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$

Proposition 5.5.1. *$\mathcal{M}$ and $\mathcal{D}_{\alpha,S}$ are two closed quaternionic operators on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$.*

Proof. We consider the graph of $\mathcal{M}$ defined by
$$\mathcal{G}(\mathcal{M}) := \{(f, \mathcal{M}f); f \in D(\mathcal{M})\}.$$
Let us show that $\mathcal{G}(\mathcal{M})$ is closed. Indeed, let ϕ_n be a sequence in $D(\mathcal{M})$ such that ϕ_n and $\mathcal{M}\phi_n$ converge to ϕ and ψ respectively on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. Then, thanks to the Proposition 5.4.4 we have
$$|\phi_n(q) - \phi(q)| \leq C_q\|\phi_n - \phi\| \quad \text{and} \quad |\mathcal{M}\phi_n(q) - \psi(q)| \leq C_q\|\mathcal{M}\phi_n - \psi\|;$$
it follows that ϕ_n and $\mathcal{M}\phi_n$ converge respectively to ϕ and ψ pointwise. This leads to $\psi(q) = \mathcal{M}\phi(q)$ which ends the proof. The same technique could be adopted to prove the closeness of the operator $\mathcal{D}_{\alpha,S}$ on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. □

Proposition 5.5.2. *Let $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. Then, $\mathcal{M}f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$ if and only if $\mathcal{D}_{\alpha,S}f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. In particular this means that $D(\mathcal{M}) = D(\mathcal{D}_{\alpha,S})$.*

Proof. Let $f(q) = \sum_{n=0}^{\infty} q^n a_n$ be an entire slice regular function belonging to $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$; we shall compute $\|\mathcal{M}f\|$ and $\|\mathcal{D}_{\alpha,S}f\|$. We have

$$\mathcal{M}f(q) = \sum_{n=0}^{\infty} q^{n+1}a_n \quad \text{and} \quad \|\mathcal{M}f\|^2 = \sum_{n=0}^{\infty} \beta_{n+1}(\alpha)|a_n|^2.$$

On the other hand,

$$\begin{aligned}
\mathcal{D}_{\alpha,S}f(q) &= \sum_{k=1}^{\infty} 2kq^{2k-1}a_{2k} + \sum_{k=1}^{\infty} 2(\alpha+k+1)q^{2k}a_{2k+1} \\
&= \sum_{k=1}^{\infty} \frac{\beta_{2k}(\alpha)}{\beta_{2k-1}(\alpha)} q^{2k-1}a_{2k} + \sum_{k=1}^{\infty} \frac{\beta_{2k+1}(\alpha)}{\beta_{2k}(\alpha)} q^{2k}a_{2k+1} \\
&= \sum_{n=1}^{\infty} \frac{\beta_n(\alpha)}{\beta_{n-1}(\alpha)} q^{n-1}a_n
\end{aligned}$$

Thus we have

$$\mathcal{D}_{\alpha,S}f(q) = \sum_{n=0}^{\infty} q^n c_n \quad \text{where} \quad c_n = \frac{\beta_{n+1}(\alpha)}{\beta_n(\alpha)} a_{n+1}.$$

Hence, making use of Proposition 5.3.4 we obtain

$$\|\mathcal{D}_{\alpha,S}f\|^2 = \sum_{n=0}^{\infty} \frac{\beta_{n+1}(\alpha)}{\beta_n(\alpha)} \beta_{n+1}(\alpha)|a_{n+1}|^2.$$

Now, we use the fact that

$$\frac{\beta_{n+1}(\alpha)}{\beta_n(\alpha)} = n+1+\frac{2\alpha+1}{2}(1+(-1)^n)$$

and setting $k = n+1$ we get

$$\|\mathcal{D}_{\alpha,S}f\|^2 = \sum_{k=0}^{\infty} \left(k + \frac{2\alpha+1}{2}(1-(-1)^k)\right) \beta_k(\alpha)|a_k|^2$$

This leads to

$$\|\mathcal{D}_{\alpha,S}f\|^2 = \|\mathcal{M}f\|^2 - \|f\|^2 - (2\alpha+1)\sum_{k=0}^{\infty}(-1)^k\beta_k(\alpha)|a_k|^2$$

Last equality concludes the proof and shows that $\mathcal{M}$ and $\mathcal{D}_{\alpha,S}$ have the same domain on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$. □

Proposition 5.5.3. *For $f \in D(\mathcal{D}_{\alpha,S})$ and $g \in D(\mathcal{M})$, we have*

$$\langle \mathcal{D}_{\alpha,S} f, g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} = \langle f, \mathcal{M} g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}$$

and

$$\langle \mathcal{M} g, f \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} = \langle g, \mathcal{D}_{\alpha,S} f \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}.$$

Proof. Take $f(q) = \displaystyle\sum_{n=0}^{\infty} q^n a_n$ and $g(q) = \displaystyle\sum_{n=0}^{\infty} q^n b_n$. Then, as we have seen before we have

$$\mathcal{D}_{\alpha,S} f(q) = \sum_{n=0}^{\infty} q^n c_n \quad \text{with} \quad c_n = \frac{\beta_{n+1}(\alpha)}{\beta_n(\alpha)} a_{n+1}$$

and by taking $b_{-1} = 0$ we have

$$\mathcal{M} g(q) = \sum_{n=0}^{\infty} q^n b_{n-1}$$

Therefore, it follows from the Proposition 5.3.4 that

$$\begin{aligned} \langle \mathcal{D}_{\alpha,S} f, g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})} &= \sum_{n=0}^{\infty} \beta_{n+1}(\alpha) \overline{b_n} a_{n+1} \\ &= \langle f, \mathcal{M} g \rangle_{\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})}. \end{aligned}$$

Then, we just need to apply $\overline{\langle h, l \rangle} = \langle l, h \rangle$ to get the second formula. □

Proposition 5.5.4. *The commutator of the operators $\mathcal{D}_{\alpha,S}$ and $\mathcal{M}$ satisfies*

$$[\mathcal{D}_{\alpha,S}; \mathcal{M}] = \mathcal{I} + (2\alpha + 1) A$$

where $\mathcal{I}$ is the identity operator and $Af(q) = f(-q)$ on $\mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$.

Proof. Let $f \in \mathcal{F}^{\alpha}_{Slice}(\mathbb{H})$, then we have

$$\mathcal{M} f(q) = q f(q) \quad \text{and} \quad \mathcal{D}_{\alpha,S} f(q) = \partial_S f(q) + (2\alpha + 1) q^{-1} \left(\frac{f(q) - f(-q)}{2} \right).$$

Thus,

$$\mathcal{M} \mathcal{D}_{\alpha,S} f(q) = q \partial_S f(q) + (2\alpha + 1) \left(\frac{f(q) - f(-q)}{2} \right)$$

Moreover, since the identity is an intrinsic entire slice regular function then the slice derivative satisfies the Leibniz formula so that we have

$$\mathcal{D}_{\alpha,S} \mathcal{M} f(q) = f(q) + q \partial_S f(q) + (2\alpha + 1) \left(\frac{f(q) + f(-q)}{2} \right).$$

Hence, by substituting the two last equations we get the desired result. □

Finally, all the previous properties could be summarized in the following main result

Theorem 5.5.5. *$\mathcal{M}$ and $\mathcal{D}_{\alpha,S}$ are closed densely defined right quaternionic linear operators adjoints of each other on the slice Cholewinski-Fock space. Moreover, they satisfy the commutation rule*

$$[\mathcal{D}_{\alpha,S};\mathcal{M}]=\mathcal{I}+(2\alpha+1)A.$$

Remark 5.5.6. *If $\alpha=-\dfrac{1}{2}$ the last theorem states that the slice derivative ∂_S and the quaternionic multiplication operator M_q are adjoints one of each other and satisfy the classical commutation rule $[\partial_S;M_q]=\mathcal{I}$ on the slice hyperholomorphic Fock space introduced in [15].*

5.6 The slice monogenic Cholewinski-Fock spaces

Let $\{e_1,e_2,...,e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ with a non-commutative product defined by the following multiplication law

$$e_ke_s+e_se_k=-2\delta_{k,s};\quad k,s=1,...,n$$

where $\delta_{k,s}$ is the Kronecker symbol. The set $\{e_A:A\subset\{1,...,m\}\}$ with $e_A=e_{h_1}e_{h_2}...e_{h_r}, 1\leq h_1<...<h_r\leq n, e_\emptyset=1$ forms a basis of the Clifford algebra $\mathbb{R}_n$. Let $\mathbb{R}^{n+1}$ be embedded in $\mathbb{R}_n$ by identifying $(x_0,x_1,...,x_n)\in\mathbb{R}^{n+1}$ with the para-vector $x=x_0+\underline{x}\in\mathbb{R}_n$. The conjugate of x is given by $\bar{x}=x_0-\underline{x}$ and the norm of x is defined by $|x|^2=x_0^2+...+x_n^2$. Furthermore, the $(n-1)$ dimensional sphere of units $1-$vectors in $\mathbb{R}^n$ is denoted by

$$\mathbb{S}^{n-1}=\{x=x_1e_1+...+x_ne_n;x_1^2+...+x_n^2=1\}.$$

Note that if $I\in\mathbb{S}^{n-1}$, then $I^2=-1$. Based on these notations, in [36] the theory of slice regular functions on quaternions was extended to the slice monogenic setting where the space of all slice monogenic functions on Ω is denoted by $\mathcal{SM}(\Omega)$. Then, by analogy with the quaternionic setting, to $f\in\mathcal{SM}(\mathbb{R}^{n+1})$ such that

$$f(x)=f^e(x)+f^o(x)$$

we consider

$$\|f\|^2_{\alpha,n}:=\int_{\mathbb{C}_I}|f_I^e(x)|^2d\lambda_{\alpha,I}(x)+2(\alpha+1)\int_{\mathbb{C}_I}|f_I^o(x)|^2|x|^{-2}d\lambda_{\alpha+1,I}(x)$$

where for the para-vector $x=u+vI\in\mathbb{C}_I$ we have

$$d\lambda_{\alpha,I}(x):=\frac{|x|^{2\alpha+2}}{\pi2^\alpha\Gamma(\alpha+1)}K_\alpha(|x|^2)d\lambda_I(x).$$

Hence, we define the slice monogenic Cholewinski-Fock space on $\mathbb{R}^{n+1}$ to be

$$\mathcal{F}^{\alpha}_{Slice}(\mathbb{R}^{n+1}) := \{f \in \mathcal{SM}(\mathbb{R}^{n+1}); \|f\|_{n,\alpha} < \infty\}.$$

The monomials given by paravectors $(x^n)_n$ form an orthogonal basis of $\mathcal{F}^{\alpha}_{Slice}(\mathbb{R}^{n+1})$. Moreover, note that if $\alpha = -\dfrac{1}{2}$ this space is the slice hyperholomorhic Clifford Fock space introduced in section 5 of [15].
Finally, we conclude this by this comment

Remark 5.6.1. *The theory of slice monogenic functions with Clifford valued functions [35, 36] extends following the same spirit the one of slice regular functions on quaternions so that we have the same extended versions of : Splitting Lemma, series expansion theorem, Representation Formula, etc. Hence, most of the results obtained in this paper in the quaternionic setting could be rewritten in the slice monogenic setting.*

CHAPTER 6

A quaternionic Short-time Fourier transform QSTFT

In this chapter, we study a special one dimensional quaternion short-time Fourier transform (QSTFT). Its construction is based on the slice hyperholomorphic Segal-Bargmann transform. We discuss some of its basic properties and prove different results on this QSTFT such as Moyal formula, reconstruction formula and Lieb's uncertainty principle. We provide also the reproducing kernel associated to the Gabor space considered in this case. The results obtained here are based on [56]

6.1 Motivation

There has been an increased interest in the generalization of integral transforms to the quaternionic and Clifford settings in the last years. Such transforms are widely studied, since they help in analysis of vector-valued signals and images. In [31] it was explained that some hypercomplex signals can be useful tools for extracting intrinsically 1D-features from images. The reader can find other motivations for studying the extension of the time frequency-analysis to quaternions in [31]. In the survey [54] the author states that this research topic is based on three main approaches: the eigenfunction approach, the generalized roots of -1 approach and the spin group approach. In particular, using the second approach a quaternionic short-time Fourier transform in dimension 2 is studied in [21]. In the paper [94] the same transform is defined in a Clifford setting for even dimension more than two. We introduce here an extension of the short-time Fourier transform in the quaternionic setting in dimension one. To this end, we fix a property that relates the complex short-time Fourier transform and the complex Segal-Bargmann transform:

$$V_\varphi f(x,\omega) = e^{-\pi i x\omega} Gf(\bar{z}) e^{\frac{-\pi|z|^2}{2}}, \tag{6.1.1}$$

where V_φ is the complex short-time Fourier transform with respect to the Gaussian window φ (see [78, Def. 3.1]) and $Gf(z)$ denotes the complex version of the Segal-Bargmann transform according to [78]. To achieve our aim we use the quaternionc analogue of the Segal-Bargmann transform studied in [60]. This integral transform was used also in [63] to study some quaternionic Hilbert spaces of Cauchy-Fueter regular functions.

In order to present our results, we adopt the following structure: in the next section, we prove some new properties of the quaternionic Segal-Bargmann transform that will be useful for our purpose. In particular we deal with an unitary property and give a characterization of the range of the Schwartz space. Moreover, we provide some calculations related to the position and the momentum operators. After that, we give a brief overview of the 1D Fourier transform [65] and show a Plancherel theorem in this framework. Then, we will define the 1D QSTFT and prove an isometric relation for the 1D QSTFT and a Moyal formula using the Segal-Bargmann techniques. We show also the following reconstruction formula

$$f(y) = 2^{-\frac{1}{4}} \int_{\mathbb{R}^2} e^{2\pi I\omega y} \mathcal{V}_\varphi f(x,\omega) e^{-\pi(y-x)^2} dx d\omega, \ \forall y \in \mathbb{R}.$$

From this follows that the adjoint operator of the QSTFT defines a left inverse. Furthermore, it gives the possibility to write the 1D QSTFT using the reproducing kernel associated to the Gabor space

$$\mathcal{G}_{\mathbb{H}}^{\varphi} := \{\mathcal{V}_\varphi f, \ f \in L^2(\mathbb{R},\mathbb{H})\}.$$

Finally, we prove that the 1D QSTFT considered here follows a Lieb's uncertainty principle.

6.2 Further properties of the quaternionic Segal-Bargmann transform

In this section we prove some new properties of the quaternionic Segal-Bargmann transform.

6.2.1 A unitary property

We start from an unitary property which is not found in literature in the following explicit form.

Proposition 6.2.1. *Let $f, g \in L^2(\mathbb{R}, \mathbb{H})$. Then, we have*

$$\langle \mathcal{B}_{\mathbb{H}}^S(f), \mathcal{B}_{\mathbb{H}}^S(g) \rangle_{\mathcal{F}^{2,\nu}_{slice}(\mathbb{H})} = \langle f, g \rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{6.2.1}$$

Proof. Any $f, g \in L^2(\mathbb{R}, \mathbb{H})$ can be expanded as

$$f(x) = \sum_{k \geq 0} h_k^\nu(x) \alpha_k,$$

$$g(x) = \sum_{k \geq 0} h_k^\nu(x) \beta_k,$$

where $(\alpha_k)_{k \in \mathbb{N}}, (\beta_k)_{k \in \mathbb{N}} \subset \mathbb{H}$.

$$\begin{aligned} \langle f, g \rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{g(x)} f(x)\, dx = \sum_{k \geq 0} \int_{\mathbb{R}} \overline{h_k^\nu(x) \beta_k} h_k^\nu(x) \alpha_k \, dx \\ &= \sum_{k \geq 0} \overline{\beta_k} \left(\int_{\mathbb{R}} \overline{h_k^\nu(x)} h_k^\nu(x)\, dx \right) \alpha_k \\ &= \sum_{k \geq 0} \| h_k^\nu(x) \|^2_{L^2(\mathbb{R},\mathbb{H})} \overline{\beta_k} \alpha_k. \end{aligned} \tag{6.2.2}$$

On the other way, since

$$\langle f, h_k^\nu \rangle_{L^2(\mathbb{R},\mathbb{H})} = \sum_{j \geq 0} \left(\int_{\mathbb{R}} \overline{h_k^\nu(x)} h_j^\nu(x)\, dx \right) \alpha_j = \| h_k^\nu(x) \|^2_{L^2(\mathbb{R},\mathbb{H})} \alpha_k.$$

We have by [60]

$$\begin{aligned} \mathcal{B}_{\mathbb{H}}^S(f)(q) &= \sum_{k \geq 0} e_k(q) \frac{\langle f, h_k^\nu \rangle_{L^2(\mathbb{R},\mathbb{H})}}{\| h_k^\nu(x) \|_{L^2(\mathbb{R},\mathbb{H})} \| e_k \|_{\mathcal{F}^{2,\nu}_{Slice}}} \\ &= \sum_{k \geq 0} e_k(q) \frac{\| h_k^\nu(x) \|_2^2}{\| h_k^\nu(x) \|_{L^2(\mathbb{R},\mathbb{H})} \| e_k \|_{\mathcal{F}^{2,\nu}_{Slice}}} \alpha_k \\ &= \sum_{k \geq 0} e_k(q) \frac{\| h_k^\nu(x) \|_{L^2(\mathbb{R},\mathbb{H})}}{\| e_k \|_{\mathcal{F}^{2,\nu}_{Slice}}} \alpha_k. \end{aligned} \tag{6.2.3}$$

Using the same calculus we obtain

$$\overline{\mathcal{B}_{\mathbb{H}}^S(g)(q)} = \sum_{k \geq 0} \frac{\| h_k^\nu(x) \|_{L^2(\mathbb{R},\mathbb{H})}}{\| e_k \|_{\mathcal{F}^{2,\nu}_{Slice}}} \overline{e_k(q) \beta_k}. \tag{6.2.4}$$

By putting together (6.2.3) and (6.2.4) we obtain

$$\begin{aligned}
\langle \mathcal{B}^S_{\mathbb{H}}(f), \mathcal{B}^S_{\mathbb{H}}(g)\rangle_{\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})} &= \sum_{k\geq 0}\int_{\mathbb{C}_I} \|h^\nu_k(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\frac{\overline{e_k(q)}}{\|e_k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\cdot \\
&\quad \cdot\frac{e_k(q)}{\|e_k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\alpha_k e^{-\nu|q|^2}\, d\lambda_I(q) \\
&= \sum_{k\geq 0}\|h^\nu_k(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\bigg(\int_{\mathbb{C}_I}\frac{\overline{e_k(q)}}{\|e_k\|_{\mathcal{F}^{2,\nu}_{Slice}}}\cdot \\
&\quad \cdot\frac{e_k(q)}{\|e_k\|_{\mathcal{F}^{2,\nu}_{Slice}}}e^{-\nu|q|^2}\, d\lambda_I(q)\bigg)\alpha_k \\
&= \sum_{k\geq 0}\|h^\nu_k(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\frac{1}{\|e_k\|^2_{\mathcal{F}^{2,\nu}_{Slice}}}\cdot \\
&\quad \cdot\bigg(\int_{\mathbb{C}_I}\overline{e_k(q)}e_k(q)e^{-\nu|q|^2}\, d\lambda_I(q)\bigg)\alpha_k \\
&= \sum_{k\geq 0}\|h^\nu_k(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\frac{1}{\|e_k\|^2_{\mathcal{F}^{2,\nu}_{Slice}}}\|e_k\|^2_{\mathcal{F}^{2,\nu}_{Slice}}\alpha_k \\
&= \sum_{k\geq 0}\|h^\nu_k(x)\|^2_{L^2(\mathbb{R},\mathbb{H})}\overline{\beta_k}\alpha_k
\end{aligned} \tag{6.2.5}$$

Finally, since (6.2.2) and (6.2.5) are equal we obtain the thesis. □

Remark 6.2.2. *If $f = g$ in* (6.2.1) *we have that the quaternionic Segal-Bargmann transform realizes an isometry from $L^2(\mathbb{R},\mathbb{H})$ onto the slice hyperholomorphic Bargmann-Fock space $\mathcal{F}^{2,\nu}_{Slice}(\mathbb{H})$, as proved in a different way in [60, Thm. 4.6]*

6.2.2 Range of the Schwartz space

We characterize the range of the Schwartz space under the Segal-Bargmann transform with parameter $\nu = 1$ in the slice hyperholomorphic setting of quaternions. We consider also some equivalence relations related to the position and momentum operators in this setting. The quaternionic Schwartz space on the real line that we are considering in this framework is defined by

$$\mathcal{S}_{\mathbb{H}}(\mathbb{R}) := \{\psi : \mathbb{R} \longrightarrow \mathbb{H}\ :\ \sup_{x\in\mathbb{R}}\left|x^\alpha\frac{d^\beta}{dx^\beta}(\psi)(x)\right| < \infty,\ \forall\alpha,\beta\in\mathbb{N}\}.$$

For $I \in \mathbb{S}$, the classical Schwartz space is given by

$$\mathcal{S}_{\mathbb{C}_I}(\mathbb{R}) := \{\varphi : \mathbb{R} \longrightarrow \mathbb{C}_I;\ :\ \sup_{x\in\mathbb{R}}\left|x^\alpha\frac{d^\beta}{dx^\beta}(\varphi)(x)\right| < \infty,\ \forall\alpha,\beta\in\mathbb{N}\}.$$

Clearly, we have that

$$\mathcal{S}_{\mathbb{C}_I}(\mathbb{R}) \subset \mathcal{S}_{\mathbb{H}}(\mathbb{R}) \subset L^2_{\mathbb{H}}(\mathbb{R}).$$

Moreover, we prove the following

Lemma 6.2.3. *Let $\psi : x \longmapsto \psi(x)$ be a quaternionic valued function. Let $I, J \in \mathbb{S}$ be such that $I \perp J$. Then, $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$ if and only if there exist $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}_I}(\mathbb{R})$ such that we have*

$$\psi(x) = \varphi_1(x) + \varphi_2(x)J, \ \forall x \in \mathbb{R}.$$

Proof. Let $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$. Then, we can write

$$\psi(x) = \varphi_1(x) + \varphi_2(x)J,$$

where φ_1 and φ_2 are $\mathbb{C}_I-$valued functions. Note that for all $\alpha, \beta \in \mathbb{N}$ we have

$$\left|x^\alpha \frac{d^\beta}{dx^\beta}(\psi)(x)\right|^2 = \left|x^\alpha \frac{d^\beta}{dx^\beta}(\varphi_1)(x)\right|^2 + \left|x^\alpha \frac{d^\beta}{dx^\beta}(\varphi_2)(x)\right|^2.$$

In particular, this implies that $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$ if and only if $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}_I}(\mathbb{R})$. □

Let us now denote by $\mathcal{SF}(\mathbb{H})$ the range of $\mathcal{S}_{\mathbb{H}}(\mathbb{R})$ under the quaternionic Segal-Bargmann transform $\mathcal{B}^S_{\mathbb{H}}$. Therefore, we have the following characterization of $\mathcal{SF}(\mathbb{H})$:

Theorem 6.2.4. *A function $f(q) = \displaystyle\sum_{k=0}^{\infty} q^k c_k$ belongs to $\mathcal{SF}(\mathbb{H})$ if and only if*

$$\sup_{k \in \mathbb{N}} |c_k| k^p \sqrt{k!} < \infty, \forall p > 0.$$

i.e,

$$\mathcal{SF}(\mathbb{H}) = \{\sum_{k=0}^{\infty} q^k c_k, \ c_k \in \mathbb{H} \text{ and } \sup_{k \in \mathbb{N}} |c_k| k^p \sqrt{k!} < \infty, \forall p > 0\}.$$

Proof. Let $f \in \mathcal{SF}(\mathbb{H})$, then by definition $f = \mathcal{B}^S_{\mathbb{H}}\psi$ where $\psi \in \mathcal{S}_{\mathbb{H}}(\mathbb{R})$. Let $I, J \in \mathbb{S}$, be such that $I \perp J$. Thus, Lemma 6.2.3 implies that

$$\psi(x) = \varphi_1(x) + \varphi_2(x)J,$$

where $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}_I}(\mathbb{R})$. Therefore, we have

$$\mathcal{B}^S_{\mathbb{H}}(\psi)(q) = \mathcal{B}^S_{\mathbb{H}}(\varphi_1)(q) + \mathcal{B}^S_{\mathbb{H}}(\varphi_2)(q)J.$$

Then, we take the restriction to the complex plane $\mathbb{C}_I$ and get:

$$\mathcal{B}^S_{\mathbb{H}}(\psi)(z) = \mathcal{B}_{\mathbb{C}_I}(\varphi_1)(z) + \mathcal{B}_{\mathbb{C}_I}(\varphi_2)(z)J, \ \forall z \in \mathbb{C}_I,$$

where the complex Bargmann transform (see [23]) is given by

$$\mathcal{B}_{\mathbb{C}_I}(\varphi_l)(z) = \frac{1}{\pi^{\frac{3}{4}}} \int_{\mathbb{R}} e^{-\frac{1}{2}(z^2+x^2)+\sqrt{2}zx} \varphi_l(x)dx, \ l = 1, 2.$$

In particular, we set $f_I := \mathcal{B}^S_{\mathbb{H}}(\psi)$, $f_1 := \mathcal{B}_{\mathbb{C}_I}(\varphi_1)$ and $f_2 := \mathcal{B}_{\mathbb{C}_I}(\varphi_2)$. Then, we have $f_1, f_2 \in \mathcal{SF}(\mathbb{C}_I)$. Thus, by applying the classical result in complex analysis, see [98] we have

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } f_2(z) = \sum_{n=0}^{\infty} b_n z^n, \ \forall z \in \mathbb{C}_I.$$

Moreover, for all $p > 0$ the following conditions hold

$$\sup_{n\in\mathbb{N}} |a_n| n^p \sqrt{n!} < \infty \text{ and } \sup_{n\in\mathbb{N}} |b_n| n^p \sqrt{n!} < \infty.$$

In particular, we have then

$$f_I(z) = \sum_{n=0}^{\infty} a_n z^n + (\sum_{n=0}^{\infty} a_n z^n) J, \ \forall z \in \mathbb{C}_I.$$

Therefore,

$$f_I(z) = \sum_{n=0}^{\infty} z^n c_n \text{ with } c_n = a_n + b_n J, \text{ for all } z \in \mathbb{C}_I.$$

Thus, by taking the slice hyperholomorphic extension we get

$$f(q) = \sum_{n=0}^{\infty} q^n c_n, \ \forall q \in \mathbb{H}.$$

Moreover, note that $c_n = a_n + b_n J, n \in \mathbb{N}$. Then, $|c_n| \leq |a_n| + |b_n|, \forall n \in \mathbb{N}$. Thus, for all $p > 0$, we have

$$\sup_{n\in\mathbb{N}} |c_n| n^p \sqrt{n!} \leq \sup_{n\in\mathbb{N}} |a_n| n^p \sqrt{n!} + \sup_{n\in\mathbb{N}} |b_n| n^p \sqrt{n!} < \infty.$$

Finally, we conclude that

$$\mathcal{SF}(\mathbb{H}) = \{f(q) = \sum_{k=0}^{\infty} q^k c_k, \ c_k \in \mathbb{H} \text{ and } \sup_{k\in\mathbb{N}} |c_k| k^p \sqrt{k!} < \infty, \forall p > 0\}.$$

$\square$

6.2.3 Position and momentum operators

Now, let us consider on $L^2(\mathbb{R}, \mathbb{H}) = L^2_{\mathbb{H}}(\mathbb{R})$ the position and momentum operators defined by

$$X : \varphi \mapsto X\varphi(x) = x\varphi(x) \text{ and } D : \varphi \mapsto D\varphi(x) = \frac{d}{dx}\varphi(x).$$

Their domains are given respectively by

$$\mathcal{D}(X) := \{\varphi \in L^2_{\mathbb{H}}(\mathbb{R});\ X\varphi \in L^2_{\mathbb{H}}(\mathbb{R})\} \text{ and } \mathcal{D}(D) := \{\varphi \in L^2_{\mathbb{H}}(\mathbb{R});\ D\varphi \in L^2_{\mathbb{H}}(\mathbb{R})\}.$$

First, let us prove the following

Lemma 6.2.5. *For all* $(q, x) \in \mathbb{H} \times \mathbb{R}$, *we have*

$$\partial_S \mathcal{A}^S_{\mathbb{H}}(q, x) = (-q + \sqrt{2}x)\mathcal{A}^S_{\mathbb{H}}(q, x).$$

Proof. Let $(q, x) \in \mathbb{H}\times\mathbb{R}$. Then, by definition of the quaternionic Segal-Bargmann kernel we can write

$$\mathcal{A}^S_{\mathbb{H}}(q, x) := \pi^{-\frac{3}{4}} e^{-\frac{x^2}{2}} e^{-\frac{q^2}{2}} e^{\sqrt{2}qx}.$$

In this case, we can apply the Leibnitz rule with respect to the slice derivative and get

$$\partial_S \mathcal{A}^S_{\mathbb{H}}(q, x) = \pi^{-\frac{3}{4}} e^{-\frac{x^2}{2}} \left(e^{-\frac{q^2}{2}} \partial_S(e^{\sqrt{2}xq}) + \partial_S(e^{-\frac{q^2}{2}}) e^{\sqrt{2}xq} \right).$$

However, using the series expansion of the exponential function and applying the slice derivative we know that

$$\partial_S(e^{-\frac{q^2}{2}}) = -qe^{-\frac{q^2}{2}} \text{ and } \partial_S(e^{\sqrt{2}xq}) = \sqrt{2}xe^{\sqrt{2}xq}.$$

Therefore, we obtain

$$\partial_S \mathcal{A}^S_{\mathbb{H}}(q, x) = (-q + \sqrt{2}x) A^S_{\mathbb{H}}(q, x).$$

□

Theorem 6.2.6. *Let* $\varphi \in \mathcal{D}(X)$. *Then, we have*

$$(\partial_S + q)\, \mathcal{B}^S_{\mathbb{H}}(\varphi)(q) = \sqrt{2}\mathcal{B}^S_{\mathbb{H}}(x\varphi)(q),\ \forall q \in \mathbb{H}.$$

Proof. Let $\varphi \in \mathcal{D}(X)$ and $q \in \mathbb{H}$. Then, we have

$$\partial_S \mathcal{B}^S_{\mathbb{H}}(\varphi)(q) = \int_{\mathbb{R}} \partial_S \mathcal{A}^S_{\mathbb{H}}(q, x)\varphi(x)dx.$$

Therefore, using Lemma 6.2.5 we obtain

$$\partial_S \mathcal{B}_{\mathbb{H}}^S(\varphi)(q) = \sqrt{2}\mathcal{B}_{\mathbb{H}}^S(x\varphi)(q) - q\mathcal{B}_{\mathbb{H}}^S(\varphi)(q).$$

Finally, we get

$$(\partial_S + q)\, \mathcal{B}_{\mathbb{H}}^S(\varphi)(q) = \sqrt{2}\mathcal{B}_{\mathbb{H}}^S(x\varphi)(q),\ \forall q \in \mathbb{H}.$$

□

As a quick consequence, we have

Corollary 6.2.7. *The position operator X on $L^2_{\mathbb{H}}(\mathbb{R})$ is equivalent to the operator $\frac{1}{\sqrt{2}}(\partial_S+q)$ on the space $\mathcal{F}^{2,1}_{Slice}(\mathbb{H})$ via the quaternionic Segal-Bargmann transform $\mathcal{B}_{\mathbb{H}}^S$. In other words, for all $\varphi \in \mathcal{D}(X)$ we have*

$$X(\varphi) = (\mathcal{B}_{\mathbb{H}}^S)^{-1}\frac{(\partial_S + q)}{\sqrt{2}}\mathcal{B}_{\mathbb{H}}^S(\varphi).$$

On the other hand, we have also the following

Theorem 6.2.8. *We denote by $M_q : \varphi \longmapsto M_q\varphi(q) = q\varphi(q)$ the creation operator on $\mathcal{F}^{2,1}_{Slice}(\mathbb{H})$. Then, we have*

$$(\mathcal{B}_{\mathbb{H}}^S)^{-1}M_q\mathcal{B}_{\mathbb{H}}^S = \frac{1}{\sqrt{2}}(X - D) \textit{ on } \mathcal{D}(X) \cap \mathcal{D}(D).$$

Proof. Let $\varphi \in \mathcal{D}(X) \cap \mathcal{D}(D)$. Then, we have

$$\begin{aligned}\mathcal{B}_{\mathbb{H}}^S(D\varphi)(q) &= \int_{\mathbb{R}} \mathcal{A}_{\mathbb{H}}^S(q,x)\frac{d}{dx}\varphi(x)dx \\ &= -\int_{\mathbb{R}} \frac{d}{dx}\mathcal{A}_{\mathbb{H}}^S(q,x)\varphi(x)dx.\end{aligned}$$

However, note that for all $(q,x) \in \mathbb{H} \times \mathbb{R}$, we have

$$\frac{d}{dx}\mathcal{A}_{\mathbb{H}}^S(q,x) = (-x + \sqrt{2}q)\mathcal{A}_{\mathbb{H}}^S(q,x).$$

Therefore,

$$\mathcal{B}_{\mathbb{H}}^S(D\varphi)(q) = \mathcal{B}_{\mathbb{H}}^S(x\varphi)(q) - \sqrt{2}q\mathcal{B}_{\mathbb{H}}^S(\varphi)(q).$$

Thus, we obtain

$$M_q\mathcal{B}_{\mathbb{H}}^S(\varphi) = \mathcal{B}_{\mathbb{H}}^S\left(\frac{1}{\sqrt{2}}(X - D)\right)(\varphi).$$

Finally, we just need to apply $(\mathcal{B}_{\mathbb{H}}^S)^{-1}$ to complete the proof. □

6.3 1D quaternion Fourier transform

In this section, we study the one dimensional quaternion Fourier transforms (QFT). Namely, we are considering here the 1D left sided QFT studied in chapter 3 of the book [65]. In order to have less problems with computations we add -2π to the exponential.

Definition 6.3.1. *The left sided 1D quaternionic Fourier transform of a quaternion valued signal $\psi : \mathbb{R} \longrightarrow \mathbb{H}$ is defined on $L^1(\mathbb{R}; dx) = L^1(\mathbb{R}; \mathbb{H})$ by*

$$\mathcal{F}_I(\psi)(\omega) = \int_{\mathbb{R}} e^{-2\pi I\omega t}\psi(t)dt$$

for a given $I \in \mathbb{S}$. Its inverse is defined by

$$\widetilde{\mathcal{F}}_I(\phi)(t) = \int_{\mathbb{R}} e^{2\pi I\omega t}\phi(\omega)d\omega.$$

Let $J \in \mathbb{S}$ be such that $J \perp I$. We can split the signal ψ via symplectic decomposition into simplex and perplex parts with respect to I such that we have:

$$\psi(t) = \psi_1(t) + \psi_2(t)J$$

where $\psi_1(t), \psi_2(t) \in \mathbb{C}_I$. The left sided 1D QFT of ψ becomes

$$\mathcal{F}_I(\psi)(\omega) = \int_{\mathbb{R}} e^{-2\pi I\omega t}\psi_1(t)dt + \int_{\mathbb{R}} e^{-2\pi I\omega t}\psi_2(t)dtJ$$

so that

$$\mathcal{F}_I(\psi)(\omega) = \mathcal{F}_I(\psi_1)(\omega) + \mathcal{F}_I(\psi_2)(\omega)J.$$

According to [65], most of the properties may be inherited from the classical complex case thanks to the equivalence between $\mathbb{C}_I$ and the standard complex plane and the fact that QFT can be decomposed into a sum of complex subfield functions.

Now, we define two fundamental operators for the time-frequency analysis.

Translation

$$\tau_x\psi(t) := \psi(t-x) \qquad x \in \mathbb{R}.$$

Modulation

$$M_\omega\psi(t) = e^{2\pi I\omega t}\psi(t), \qquad \omega \in \mathbb{R}.$$

As in the classical case we have a commutative relation between the two operators.

Lemma 6.3.1. *Let ψ be a function in $L^2(\mathbb{R}, \mathbb{H})$ then we have*

$$\tau_x M_\omega\psi(t) = e^{-2\pi I\omega x}M_\omega\tau_x\psi(t), \qquad \omega, x \in \mathbb{R}. \tag{6.3.1}$$

Proof. It is just a matter of computations

$$\begin{aligned}\tau_x M_\omega \psi(t) &= M_\omega \psi(t-x) = e^{2\pi I\omega(t-x)}\psi(t-x) \\ &= e^{2\pi I\omega t}e^{-2\pi I\omega x}\psi(t-x) \\ &= e^{-2\pi I\omega x}e^{2\pi I\omega t}\psi(t-x) \\ &= e^{-2\pi I\omega x}M_\omega \tau_x \psi(t).\end{aligned}$$

□

From [65, Table 3.2] we have the following properties

$$\mathcal{F}_I(\tau_x\psi) = M_{-x}\mathcal{F}_I(\psi), \tag{6.3.2}$$

$$\mathcal{F}_I(M_\omega\psi) = \tau_\omega\mathcal{F}_I(\psi). \tag{6.3.3}$$

From (6.3.2) and (6.3.3) follow easily that

$$\mathcal{F}_I(M_\omega\tau_x\psi) = \tau_\omega M_{-x}\mathcal{F}_I(\psi). \tag{6.3.4}$$

Then, we prove a version of the Plancherel theorem for 1D QFT.

Theorem 6.3.2. *Let $\phi, \psi \in L^2(\mathbb{R}, \mathbb{H})$. Then, we have*

$$\langle \mathcal{F}_I(\phi), \mathcal{F}_I(\psi)\rangle_{L^2(\mathbb{R},\mathbb{H})} = \langle \phi, \psi\rangle_{L^2(\mathbb{R},\mathbb{H})}.$$

In particular, for any $\phi \in L^2(\mathbb{R}, \mathbb{H})$ we have

$$||\mathcal{F}_I(\phi)||_{L^2(\mathbb{R},\mathbb{H})} = ||\phi||_{L^2(\mathbb{R},\mathbb{H})}.$$

Proof. Let $\phi, \psi \in L^2(\mathbb{R}, \mathbb{H})$. By inversion formula for the 1D QFT, see [65], we have

$$\phi(\omega) = \widetilde{\mathcal{F}_I}(\mathcal{F}_I(\phi))(\omega), \ \forall \omega \in \mathbb{R}.$$

Thus, direct computations using Fubini's theorem lead to

$$\begin{aligned}\langle \phi, \psi\rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{\psi(\omega)}\left(\int_{\mathbb{R}} e^{2\pi I\omega t}\mathcal{F}_I(\phi)(t)dt\right)d\omega \\ &= \int_{\mathbb{R}}\left(\overline{\int_{\mathbb{R}} e^{-2\pi I\omega t}\psi(\omega)d\omega}\right)\mathcal{F}_I(\phi)(t)dt \\ &= \int_{\mathbb{R}} \overline{\mathcal{F}_I(\psi)(t)}\mathcal{F}_I(\phi)(t)dt \\ &= \langle \mathcal{F}_I(\phi), \mathcal{F}_I(\psi)\rangle_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

As a direct consequence, we have for any $\phi \in L^2(\mathbb{R}, \mathbb{H})$

$$\begin{aligned}||\mathcal{F}_I(\phi)||^2_{L^2(\mathbb{R},\mathbb{H})} &= \langle \mathcal{F}_I(\phi), \mathcal{F}_I(\phi)\rangle_{L^2(\mathbb{R},\mathbb{H})} \\ &= \langle \phi, \phi\rangle_{L^2(\mathbb{R},\mathbb{H})} \\ &= ||\phi||^2_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

□

The following remark may be of interest in some other contexts.

Remark 6.3.3. *The formal convolution of two given signals $\phi, \psi : \mathbb{R} \longrightarrow \mathbb{H}$ when it exists is defined by*

$$(\phi * \psi)(t) := \int_{\mathbb{R}} \phi(\tau)\psi(t-\tau)d\tau.$$

In particular, if the window function ϕ is real valued the 1D QFT satisfies the classical property

$$\mathcal{F}_I(\phi * \psi) = \mathcal{F}_I(\phi)\mathcal{F}_I(\psi).$$

6.4 Quaternion short-time Fourier transform with a Gaussian window

The idea of the short-time Fourier transform is to obtain information about local properties of the signal f. In order to achieve this aim the signal f is restricted to an interval and after its Fourier transform is evaluated. However, since a sharp cut-off can introduce artificial discontinuities and can create problems, it is usually chosen a smooth cut-off function φ called "window function". The aim of this section is to propose a quaternionic analogue of the short-time Fourier transform in dimension one with a Gaussian window function $\varphi(t) = 2^{1/4}e^{-\pi t^2}$. For this, we consider the following formula [78, Prop. 3.4.1]

$$V_\varphi f(x,\omega) = e^{-\pi i x\omega} Gf(\bar{z}) e^{\frac{-\pi|z|^2}{2}}, \tag{6.4.1}$$

where the variables $(x,\omega) \in \mathbb{R}^2$ have been converted into a complex vector $z = x + i\omega$, and $Gf(z)$ is the complex version of the Segal-Bargmann transform according to [78]. Therefore, we want to extend (6.4.1) to the quaternionic setting. To this end, we use the quaternionic analogue of the Segal-Bargmann transform [60] and the slicing representation of the quaternions $q = x + I\omega$, where $I \in \mathbb{S}$. If the signal is complex we denote the short-time Fourier transform as V_φ, while if the signal is $\mathbb{H}$-valued we identify the short-time Fourier transform as $\mathcal{V}_\varphi$.

Definition 6.4.1. *Let $f : \mathbb{R} \to \mathbb{H}$ be a function in $L^2(\mathbb{R}, \mathbb{H})$. We define the 1D quaternion short time Fourier transform of f with respect to $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ as*

$$\mathcal{V}_\varphi f(x,\omega) = e^{-I\pi x\omega} \mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right) e^{-\frac{|q|^2\pi}{2}}, \tag{6.4.2}$$

where $q = x + I\omega$ and $\mathcal{B}^S_{\mathbb{H}}(f)(q)$ is the quaternionic Segal-Bargmann transform.

Fixing $\nu = 2\pi$, we can write (6.4.2) in the following way

$$\mathcal{V}_\varphi f(x,\omega) = 2^{\frac{3}{4}} \int_{\mathbb{R}} e^{-\pi\left(\frac{\bar{q}^2}{2}+t^2\right)+2\pi\bar{q}t - I\pi x\omega - \frac{|q|^2\pi}{2}} f(t)\, dt. \tag{6.4.3}$$

From this formula we are able to put in relation the 1D quaternion short-time Fourier transform and the 1D quaternion Fourier transform defined in the previous section.

Lemma 6.4.1. *Let f be a function in $L^2(\mathbb{R},\mathbb{H})$ and $\varphi(t) = 2^{1/4}e^{-\pi t^2}$, recalling the 1D quaternion Fourier transform we have*

$$\mathcal{V}_\varphi f(x,\omega) = \sqrt{2}\mathcal{F}_I(f \cdot \tau_x\varphi)(\omega). \tag{6.4.4}$$

Proof. By putting $q = x + I\omega$ in (6.4.3) we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= 2^{\frac{3}{4}}e^{-I\pi x\omega}e^{-\frac{x^2\pi}{2}}e^{-\frac{\omega^2\pi}{2}}\int_{\mathbb{R}} e^{-\pi t^2}e^{-\frac{\pi}{2}\left(x^2-\omega^2-2x\omega I\right)}\cdot \\
&\quad \cdot e^{2\pi(x-I\omega)t}f(t)\,dt \\
&= 2^{\frac{3}{4}}\int_{\mathbb{R}} e^{-\pi t^2-\pi x^2+2\pi xt}e^{-2\pi I\omega t}f(t)\,dt \\
&= \sqrt{2}\int_{\mathbb{R}} e^{-2\pi I\omega t}f(t)2^{\frac{1}{4}}e^{-\pi(t-x)^2}\,dt \\
&= \sqrt{2}\int_{\mathbb{R}} e^{-2\pi I\omega t}f(t)\varphi(t-x)\,dt = \sqrt{2}\mathcal{F}_I(f\cdot\tau_x\varphi)(\omega).
\end{aligned}$$

□

Now, we prove a formula which relates the 1D quaternion Fourier transform and its signal through the 1D short-time Fourier transform.

Proposition 6.4.2. *If φ is a Gaussian function $\varphi(t) = 2^{1/4}e^{-\pi t^2}$ and $f \in L^2(\mathbb{R},\mathbb{H})$ then*

$$\mathcal{V}_\varphi f(x,\omega) = \sqrt{2}e^{-2\pi I\omega x}\mathcal{V}_\varphi\mathcal{F}_I(f)(\omega,-x). \tag{6.4.5}$$

Proof. Recalling the definition of modulation and of inner product on $L^2(\mathbb{R},\mathbb{H})$, by Lemma 6.4.1 we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= \sqrt{2}\int_{\mathbb{R}} \overline{e^{2\pi I\omega t}\varphi(t-x)}f(t)\,dt \qquad (6.4.6) \\
&= \sqrt{2}\int_{\mathbb{R}} \overline{M_\omega\tau_x\varphi(t)}f(t)\,dt = \sqrt{2}\left\langle f, M_\omega\tau_x\varphi\right\rangle.
\end{aligned}$$

Using the Plancherel theorem for the 1D quaternion Fourier transform, the property (6.3.4) and the fact that $\mathcal{F}_I(\varphi) = \varphi$ we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x,\omega) &= \sqrt{2}\left\langle\mathcal{F}_I(f),\mathcal{F}_I(M_\omega\tau_x\varphi)\right\rangle \\
&= \sqrt{2}\left\langle\mathcal{F}_I(f),\tau_\omega M_{-x}\mathcal{F}_I(\varphi)\right\rangle \\
&= \sqrt{2}\left\langle\mathcal{F}_I(f),\tau_\omega M_{-x}\varphi\right\rangle
\end{aligned}$$

Finally, from (6.3.1) and (6.4.6) we get

$$\mathcal{V}_\varphi f(x,\omega) = \sqrt{2}e^{-2\pi I\omega x}\left\langle\mathcal{F}_I(f),M_{-x}\tau_\omega\varphi\right\rangle = \sqrt{2}e^{-2\pi I\omega x}\mathcal{V}_\varphi\mathcal{F}_I(f)(\omega,-x).$$

□

6.4.1 Moyal fromula

Now, we prove the Moyal formula and an isometric relation for the 1D quaternion short-time Fourier transform in two ways. In the first way we use the properties of the quaternionic Segal- Bargmann transform, whereas in the second way we use Lemma 6.4.1 and some basic properties of 1D quaternion Fourier transform.

Proposition 6.4.3. *For any $f \in L^2(\mathbb{R}, \mathbb{H})$*

$$\|\mathcal{V}_\varphi f\|_{L^2(\mathbb{R}^2,\mathbb{H})} = \sqrt{2}\|f\|_{L^2(\mathbb{R},\mathbb{H})}. \tag{6.4.7}$$

Proof. We use the slicing representation of the quaternions $q = x + I\omega$ and formula (6.4.2) to get

$$\begin{aligned}\|\mathcal{V}_\varphi f\|^2_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^2\, d\omega\, dx \\ &= \int_{\mathbb{R}} |e^{-I\pi x\omega}|^2 \left|\mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)\right|^2 e^{-|q|^2\pi}\, d\omega\, dx.\end{aligned}$$

Now, using the change of variable $p = \frac{\bar{q}}{\sqrt{2}}$ we have that $dA(p) = \frac{1}{2}\, d\omega\, dx$, hence by [60, Thm. 4.6]

$$\begin{aligned}\|\mathcal{V}_\varphi f\|^2_{L^2(\mathbb{R},\mathbb{H})} &= 2\int_{\mathbb{R}^2} |\mathcal{B}^S_{\mathbb{H}}(f)(p)|^2 e^{-2\pi|q|^2}\, dA(p) \\ &= 2\|\mathcal{B}^S_{\mathbb{H}}(f)\|^2_{\mathcal{F}^{2,2\pi}_{Slice}} = 2\|f\|^2_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

Therefore

$$\|\mathcal{V}_\varphi f\|_{L^2(\mathbb{R},\mathbb{H})} = \sqrt{2}\|f\|_{L^2(\mathbb{R},\mathbb{H})}.$$

□

Thus, the 1D quaternionic short-time Fourier transform is an isometry from $L^2(\mathbb{R}, \mathbb{H})$ into $L^2(\mathbb{R}^2; \mathbb{H})$.

Proposition 6.4.4 (Moyal formula)**.** *Let f, g be functions in $L^2(\mathbb{R}, \mathbb{H})$. Then we have*

$$\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{6.4.8}$$

Proof. From (6.4.2) we get

$$\begin{aligned}\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} &= \int_{\mathbb{R}^2} \overline{\mathcal{V}_\varphi g(x,\omega)}\mathcal{V}_\varphi f(x,\omega)\, d\omega\, dx \\ &= \int_{\mathbb{R}^2} \overline{e^{-I\pi x\omega}\mathcal{B}^S_{\mathbb{H}}(g)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}e^{-I\pi x\omega}}\, .\end{aligned}$$

$$\cdot \mathrm{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}}\,d\omega\,dx$$

$$= \int_{\mathbb{R}^2} \overline{\mathcal{B}^S_{\mathbb{H}}(g)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{I\pi x\omega}e^{-I\pi x\omega}}\,\cdot$$

$$\cdot \mathrm{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-|q|^2\pi}\,d\omega\,dx$$

$$= \int_{\mathbb{R}^2} \overline{\mathcal{B}^S_{\mathbb{H}}(g)\left(\frac{\bar{q}}{\sqrt{2}}\right)\mathcal{B}^S_{\mathbb{H}}(f)\left(\frac{\bar{q}}{\sqrt{2}}\right)e^{-|q|^2\pi}\,d\omega\,dx}.$$

Using the same change of variables as before $p = \frac{\bar{q}}{\sqrt{2}}$ and from (6.2.1) we obtain

$$\begin{aligned}\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} &= 2\int_{\mathbb{R}^2} \overline{\mathcal{B}^S_{\mathbb{H}}(g)(p)}\mathcal{B}^S_{\mathbb{H}}(f)(p)e^{-2|q|^2\pi}\,d\omega\,dx \\ &= 2\langle \mathcal{B}^S_{\mathbb{H}}(f), \mathcal{B}^S_{\mathbb{H}}(g)\rangle_{\mathcal{F}^{2,2\pi}_{Slice}(\mathbb{H})} = 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

□

Remark 6.4.5. *If we put $f = \frac{h_k^{2\pi}(t)}{\|h_k^{2\pi}(t)\|_2^2}$ in* (6.4.2) *by [60, Lemma 4.4] we get*

$$\mathcal{V}_\varphi f(x,\omega) = e^{-I\pi x\omega}e^{-\frac{\pi}{2}|q|^2}\frac{2^{3/4}}{2^k k!}\bar{q}^k.$$

Remark 6.4.6. *From* (6.4.4) *we can prove* (6.4.8) *in another way. This proof may be of interest in some other contexts.*

Let us assume $f, g \in L^2(\mathbb{R},\mathbb{H})$ and recall $\varphi(t) = 2^{1/4}e^{-\pi t^2}$, by Lemma 6.4.1 and Plancherel theorem for the 1D quaternion Fourier transform we have

$$\begin{aligned}\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= \int_{\mathbb{R}^2} \overline{\mathcal{V}_\varphi g(x,\omega)}\mathcal{V}_\varphi f(x,\omega)\,d\omega\,dx \\ &= 2\int_{\mathbb{R}^2} \overline{\mathcal{F}_I(g\cdot\tau_x\varphi)(\omega)}\mathcal{F}_I(f\cdot\tau_x\varphi)(\omega)\,d\omega\,dx \\ &= 2\int_{\mathbb{R}^2} \overline{g(\omega)\cdot\tau_x\varphi(\omega)}f(\omega)\cdot\tau_x\varphi(\omega)\,d\omega\,dx.\end{aligned}$$

Now, by Fubini's theorem and the fact that $\|\varphi\|_2^2 = 1$ *we get*

$$\begin{aligned}
\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} &= 2\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \overline{g(\omega)\cdot \tau_x\varphi(\omega)} f(\omega)\cdot \tau_x\varphi(\omega)\, dx\right) d\omega \\
&= 2\int_{\mathbb{R}}\left(\int_{\mathbb{R}} \overline{g(\omega)} f(\omega)\varphi^2(x-\omega)\, dx\right) d\omega \\
&= 2\int_{\mathbb{R}} \overline{g(\omega)} f(\omega)\left(\int_{\mathbb{R}} \varphi^2(x-\omega)\, dx\right) d\omega \\
&= 2\int_{\mathbb{R}} \overline{g(\omega)} f(\omega)\|\varphi\|_2^2\, d\omega = 2\int_{\mathbb{R}} \overline{g(\omega)} f(\omega)\, d\omega \\
&= 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}.
\end{aligned}$$

Hence

$$\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi g\rangle_{L^2(\mathbb{R}^2;\mathbb{H})} = 2\langle f, g\rangle_{L^2(\mathbb{R},\mathbb{H})}. \tag{6.4.5}$$

If we put $f = g$ *in* (6.4.5) *we obtain* (6.4.7).

6.4.2 Inversion formula and adjoint of QSTFT

The 1D QSTFT with Gaussian window φ satisfies a reconstruction formula that we prove in the following.

Theorem 6.4.7. *Let* $f \in L^2(\mathbb{R},\mathbb{H})$*. Then, we have*

$$f(y) = 2^{-\frac{1}{4}}\int_{\mathbb{R}^2} e^{2\pi I\omega y}\mathcal{V}_\varphi f(x,\omega)e^{-\pi(y-x)^2}dxd\omega,\ \forall y \in \mathbb{R}.$$

Proof. For all $y \in \mathbb{R}$, we set

$$g(y) = 2^{-\frac{1}{4}}\int_{\mathbb{R}^2} e^{2\pi I\omega y}\mathcal{V}_\varphi f(x,\omega)e^{-\pi(y-x)^2}dxd\omega.$$

Let $h \in L^2(\mathbb{R},\mathbb{H})$. Fubini's theorem combined with Moyal formula for QSTFT leads to

$$\begin{aligned}
\langle g, h\rangle_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \overline{h(y)}g(y)dy \\
&= 2^{-\frac{1}{4}}\int_{\mathbb{R}^3} \overline{h(y)}e^{2\pi I\omega y}\mathcal{V}_\varphi f(x,\omega)e^{-\pi(y-x)^2}dxd\omega dy \\
&= 2^{-1}\sqrt{2}\int_{\mathbb{R}^2}\left(\overline{\int_{\mathbb{R}} e^{-2\pi I\omega y}2^{\frac{1}{4}}e^{-\pi(y-x)^2}h(y)dy}\right)\mathcal{V}_\varphi f(x,\omega)dxd\omega \\
&= 2^{-1}\int_{\mathbb{R}^2}\overline{\mathcal{V}_\varphi h(x,\omega)}\mathcal{V}_\varphi f(x,\omega)dxd\omega \\
&= 2^{-1}\left\langle \mathcal{V}_\varphi f, \mathcal{V}_\varphi h\right\rangle_{L^2(\mathbb{R}^2)} \\
&= \left\langle f, h\right\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

Hence, we have

$$f(y) = g(y) = 2^{-\frac{1}{4}} \int_{\mathbb{R}^2} e^{2\pi I\omega y} \mathcal{V}_\varphi f(x,\omega) e^{-\pi(y-x)^2} dx d\omega.$$

This ends the proof. □

We note that the QSTFT admits a left side inverse that we can compute as follows

Theorem 6.4.8. *Let φ denote the Gaussian window $\varphi(t) = 2^{1/4} e^{-\pi t^2}$ and let us consider the operator $\mathcal{A}_\varphi : L^2(\mathbb{R}^2, \mathbb{H}) \longrightarrow L^2(\mathbb{R}, \mathbb{H})$ defined for any $F \in L^2(\mathbb{R}^2, \mathbb{H})$ by*

$$\mathcal{A}_\varphi(F)(y) = 2^{\frac{3}{4}} \int_{\mathbb{R}^2} e^{2\pi I\omega y} F(x,\omega) e^{-\pi(y-x)^2} dx d\omega, \ \forall y \in \mathbb{R}.$$

Then, $\mathcal{A}_\varphi$ is the adjoint of $\mathcal{V}_\varphi$. Moreover, the following identity holds

$$\mathcal{V}_\varphi^* \mathcal{V}_\varphi = 2Id. \tag{6.4.6}$$

Proof. Let $F \in L^2(\mathbb{R}^2, \mathbb{H})$ and $h \in L^2(\mathbb{R}, \mathbb{H})$. We use some calculations similar to the previous result and get

$$\begin{aligned}
\langle \mathcal{A}_\varphi(F), h \rangle_{L^2(\mathbb{R},\mathbb{H})} &= \int_{\mathbb{R}} \overline{h(y)} \mathcal{A}_\varphi(F)(y) dy \\
&= 2^{\frac{3}{4}} \int_{\mathbb{R}^3} \overline{h(y)} e^{2\pi I\omega y} F(x,\omega) e^{-\pi(y-x)^2} dx d\omega dy \\
&= \int_{\mathbb{R}^2} \sqrt{2} \left(\overline{\int_{\mathbb{R}} e^{-2\pi I\omega y} 2^{\frac{1}{4}} e^{-\pi(y-x)^2} h(y) dy} \right) F(x,\omega) dx d\omega \\
&= \int_{\mathbb{R}^2} \overline{\mathcal{V}_\varphi h(x,\omega)} F(x,\omega) dx d\omega \\
&= \langle F, \mathcal{V}_\varphi h \rangle_{L^2(\mathbb{R}^2,\mathbb{H})}.
\end{aligned}$$

In particular, this shows that

$$\mathcal{A}(\varphi)(F) = \mathcal{V}_\varphi^*(F), \ \forall F \in L^2(\mathbb{R}^2, \mathbb{H}).$$

From reconstruction formula we obtain (6.4.6).

□

Remark 6.4.9. *We note that the identity $\mathcal{V}_\varphi^* \mathcal{V}_\varphi = 2Id$ provides another proof for the fact that QSTFT is an isometric operator and the adjoint $\mathcal{V}_\varphi^*$ defines a left inverse.*

6.4.3 The eigenfunctions of the 1D quaternion Fourier transform

Through the 1D QSTFT we can prove in another way that the eigenfunctions of the 1D quaternion Fourier transform are given by the Hermite functions.

Proposition 6.4.10. *The Hermite functions $h_k^{2\pi}(t)$ are eigenfunctions of the 1D quaternion Fourier transform :*

$$\mathcal{F}_I(h_k^{2\pi})(t) = 2^{-1/2}(-I)^k h_k^{2\pi}(t), \qquad t \in \mathbb{R}.$$

Proof. By identity (6.4.2) and [60, Lemma 4.4] we have

$$\begin{aligned} \mathcal{V}_\varphi(h_k^{2\pi})(x,-\omega) &= e^{I\pi x\omega}\mathcal{B}_{\mathbb{H}}^S(h_k^{2\pi})\big(\frac{q}{\sqrt{2}}\big)e^{-\frac{\pi|q|^2}{2}} \qquad (6.4.7)\\ &= e^{I\pi x\omega}2^{1/4}2^{k/2}(2\pi)^k 2^{-k/2}q^k e^{-\frac{\pi|q|^2}{2}} \\ &= e^{I\pi x\omega}2^{1/4}(2\pi)^k q^k e^{-\frac{\pi|q|^2}{2}}. \end{aligned}$$

Recalling that $q = x + I\omega$ and using (6.4.5) we obtain

$$\begin{aligned} \mathcal{V}_\varphi\mathcal{F}_I(h_k^{2\pi})(x,-\omega) &= 2^{-1/2}e^{2\pi I\omega x}\mathcal{V}_\varphi h_k^{2\pi}(\omega,x) \\ &= 2^{-1/2}e^{2\pi I\omega x}e^{-I\pi\omega x}\mathcal{B}_{\mathbb{H}}^S(h_k^{2\pi})\left(\frac{\omega - Ix}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{-1/2}e^{\pi I\omega x}\mathcal{B}_{\mathbb{H}}^S(h_k^{2\pi})\left(\frac{-Iq}{\sqrt{2}}\right)e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{-1/2}e^{\pi I\omega x}2^{1/4}2^{k/2}(2\pi)^k(-I)^k 2^{-k/2}q^k e^{-\frac{|q|^2\pi}{2}} \\ &= 2^{-1/2}(-I)^k e^{I\pi\omega x}2^{1/4}(2\pi)^k q^k e^{-\frac{|q|^2\pi}{2}}. \end{aligned}$$

Combining with (6.4.7)

$$\mathcal{V}_\varphi\mathcal{F}_I(h_k^{2\pi})(x,-\omega) = 2^{-1/2}(-I)^k\mathcal{V}_\varphi h_k^{2\pi}(x,-\omega).$$

From (6.4.6) we know that V_φ is injective, hence we have the thesis. ☐

6.4.4 Reproducing kernel property

The inversion formula gives us the possibility to write the 1D QSTFT using the reproducing kernel associated to the quaternion Gabor space, introduced in [2], with a Gaussian window that is defined by

$$\mathcal{G}_{\mathbb{H}}^\varphi := \{\mathcal{V}_\varphi f,\ f \in L^2(\mathbb{R},\mathbb{H})\}.$$

Theorem 6.4.11. *Let f be in $L^2(\mathbb{R},\mathbb{H})$ and $\varphi(t) = 2^{1/4}e^{-\pi t^2}$. If*

$$\mathbb{K}_\varphi(\omega,x;\omega',x') = \int_{\mathbb{R}} e^{-2\pi I\omega' t}\varphi(t-x')\overline{e^{-2\pi I\omega t}\varphi(t-x)}\,dt,$$

then $\mathbb{K}_\varphi(\omega, x; \omega', x')$ *is the reproducing kernel i.e.*

$$\mathcal{V}_\varphi f(x', \omega') = \int_{\mathbb{R}^2} \mathbb{K}_\varphi(\omega, x; \omega', x') \mathcal{V}_\varphi f(x, \omega)\, dx d\omega.$$

Proof. By Lemma 6.4.1 and the reconstruction formula we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x', \omega') &= 2^{3/4} \int_{\mathbb{R}} e^{-2\pi I \omega' t} f(t) e^{-\pi(t-x')^2}\, dt \\
&= 2^{3/4} \int_{\mathbb{R}} e^{-2\pi I \omega' t} e^{-\pi(t-x')^2} 2^{-\frac{1}{4}} \cdot \\
&\quad \cdot \left(\int_{\mathbb{R}^2} e^{2\pi I \omega t} e^{-\pi(t-x)^2} \mathcal{V}_\varphi f(x, \omega)\, dx\, d\omega \right) dt \\
&= \sqrt{2} \int_{\mathbb{R}^3} e^{-2\pi I (\omega' - \omega) t} e^{-\pi(t-x')^2} e^{-\pi(t-x)^2} \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x, \omega)\, dx\, d\omega\, dt.
\end{aligned}$$

Using Fubini's theorem we have

$$\begin{aligned}
\mathcal{V}_\varphi f(x', \omega') &= \sqrt{2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} e^{-2\pi I (\omega' - \omega) t} e^{-\pi(t-x')^2} e^{-\pi(t-x)^2}\, dt \right) \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x, \omega)\, dx\, d\omega \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} e^{-2\pi I \omega' t} 2^{1/4} e^{-\pi(t-x')^2} \overline{2^{1/4} e^{-2\pi I \omega t} e^{-\pi(t-x)^2}}\, dt \right) \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x, \omega)\, dx\, d\omega \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} e^{-2\pi I \omega' t} \varphi(t - x') \overline{e^{-2\pi I \omega t} \varphi(t - x)}\, dt \right) \cdot \\
&\quad \cdot \mathcal{V}_\varphi f(x, \omega)\, dx\, d\omega \\
&= \int_{\mathbb{R}^2} \mathbb{K}_\varphi(\omega, x; \omega', x') \mathcal{V}_\varphi f(x, \omega)\, dx d\omega.
\end{aligned}$$

□

6.4.5 Lieb's uncertainty principle for QSTFT

The QSTFT follows the Lieb's uncertainty principle with some weak differences comparing to the classical complex case. Indeed, we first study the weak uncertainty principle which is the subject of this result

Theorem 6.4.12 (Weak uncertainty principle). *Let* $f \in L^2(\mathbb{R}, \mathbb{H})$ *be a unit vector (i.e* $||f|| = 1$*),* U *an open set of* $\mathbb{R}^2$ *and* $\varepsilon \geq 0$ *such that*

$$\int_U |\mathcal{V}_\varphi f(x, \omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

Then, we have

$$|U| \geq \frac{1-\varepsilon}{2},$$

where $|U|$ denotes the Lebesgue measure of U.

Proof. We note that using Definition of QSTFT and [60, Prop. 4.3] we obtain

$$\begin{aligned}|\mathcal{V}_\varphi f(x,\omega)| &= |\mathcal{B}_{\mathbb{H}}^S f(\bar{q}/\sqrt{2})|e^{-\frac{|q|^2}{2}\pi} \\ &= |\mathcal{B}_{\mathbb{H}}^S f(p)|e^{-\pi|p|^2};\ p=\bar{q}/\sqrt{2} \\ &\leq \sqrt{2}||f||_{L^2(\mathbb{R})}.\end{aligned}$$

Thus, by hypothesis we get

$$1-\varepsilon \leq \int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega \leq ||\mathcal{V}_\varphi f||_\infty^2 |U| \leq 2|U|.$$

Hence, we have

$$|U| \geq \frac{1-\varepsilon}{2}.$$

□

Theorem 6.4.13 (Lieb's inequality). *Let $f \in L^2(\mathbb{R},\mathbb{H})$ and $2 \leq p < \infty$. Then, we have*

$$\int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^p dx d\omega \leq \frac{2^{p+1}}{p} ||f||^p_{L^2(\mathbb{R},\mathbb{H})}$$

Proof. Let $I, J \in \mathbb{S}$ be such that I is orthogonal to J. Then, for $f \in L^2(\mathbb{R},\mathbb{H})$, there exist $f_1, f_2 \in L^2(\mathbb{R},\mathbb{C}_I)$ such that

$$f(t) = f_1(t) + f_2(t)J,\ \forall t \in \mathbb{R}$$

and for which the classical Lieb's inequality [91] holds , i.e:

$$\int_{\mathbb{R}^2} |V_\varphi f_l(x,\omega)|^p dx d\omega \leq \frac{2}{p}||f_l||^p_{L^2(\mathbb{R},\mathbb{C}_I)};\ l=1,2.$$

In particular, by definition of QSTFT we have

$$\mathcal{V}_\varphi f(x,\omega) = \mathcal{V}_\varphi f_1(x,\omega) + V_\varphi f_2(x,\omega)J,\ \forall (x,\omega) \in \mathbb{R}^2.$$

Thus,

$$\begin{aligned}|\mathcal{V}_\varphi f(x,\omega)|^p &\leq (|V_\varphi f_1(x,\omega)| + |V_\varphi f_2(x,\omega)|)^p \\ &\leq 2^{p-1}\left(|V_\varphi f_1(x,\omega)|^p + |V_\varphi f_2(x,\omega)|^p\right).\end{aligned}$$

We use the classical Lieb's inequality on each component combined with the fact that $||f_l||_p \leq ||f||_p$ for $l = 1, 2$ and get

$$\begin{aligned}\int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^p dx d\omega &\leq \frac{2^p}{p}\left(||f_1||^p_{L^2(\mathbb{R})} + ||f_2||^p_{L^2(\mathbb{R})}\right)\\ &\leq \frac{2^{p+1}}{p}||f||^p_{L^2(\mathbb{R},\mathbb{H})}.\end{aligned}$$

This ends the proof. □

The next result improves the weak uncertainty principle in the sense that it gives a best sharper estimate for $|U|$.

Theorem 6.4.14. *Let $f \in L^2(\mathbb{R},\mathbb{H})$ be a unit vector, U an open set of $\mathbb{R}^2$ and $\varepsilon \geq 0$ such that*

$$\int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega \geq 1 - \varepsilon.$$

Then, we have

$$|U| \geq c_p(1-\varepsilon)^{\frac{p}{p-2}},$$

where $|U|$ denotes the Lebesgue measure of U and $c_p = \left(\frac{2^{p+1}}{p}\right)^{-\frac{2}{p-2}}$.

Proof. Let $f \in L^2(\mathbb{R},\mathbb{H})$ be such that $||f||_{L^2(\mathbb{R},\mathbb{H})} = 1$. We first apply Holder inequality with exponents $q = \dfrac{p}{2}$ and $q' = \dfrac{p}{p-2}$. Then, using Lieb's inequality for QSTFT we get

$$\begin{aligned}\int_U |\mathcal{V}_\varphi f(x,\omega)|^2 dx d\omega &= \int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^2 \chi_U(x,\omega) dx d\omega\\ &\leq \left(\int_{\mathbb{R}^2} |\mathcal{V}_\varphi f(x,\omega)|^p dx d\omega\right)^{\frac{2}{p}} |U|^{\frac{p-2}{p}}\\ &\leq \left(\frac{2^{p+1}}{p}\right)^{\frac{2}{p}} |U|^{\frac{p-2}{p}}.\end{aligned}$$

Hence, by hypothesis we obtain

$$|U| \geq c_p(1-\varepsilon)^{\frac{p}{p-2}}$$

where $c_p = \left(\frac{2^{p+1}}{p}\right)^{-\frac{2}{p-2}}$.

□

6.5 Concluding remarks

In this chapter, we studied a quaternion short-time Fourier transform (QSTFT) with a Gaussian window. This window function corresponds to the first normalized Hermite function given by $\psi_0(t) = \varphi(t) = 2^{1/4}e^{-\pi t^2}$. Based on the quternionic Segal-Bargmann transform we proved several results including different versions of Moyal formula, reconstruction formula, Lieb's principle, etc. A more general problem in this framework is to consider a QSTFT associated to some generic quaternion valued window ψ. For a given quaternion $q = x + I\omega$ we plan to investigate in our future research works the properties of the QSTFT defined for any $f \in L^2(\mathbb{R}, \mathbb{H})$ by

$$\mathcal{V}_\psi f(x,\omega) = \int_{\mathbb{R}} e^{-2\pi I t\omega}\overline{\psi(t-x)}f(t)dt.$$

In particular, studying such transforms with normalized Hermite functions $\{\psi_n(t)\}_{n\geq 0}$ that are real valued windows will be related to the theory of slice poly-analytic functions on quaternions considered in [17].

CHAPTER 7

A Clifford-Appell system and Bargmann-Fock-Fueter transform

This chapter deals with some special integral transforms of Bargmann-Fock type in the setting of quaternionic valued slice hyperholomorphic and Cauchy-Fueter regular functions. The construction is based on the well-known Fueter mapping theorem. In particular, starting with the normalized Hermite functions we can construct an Appell system of quaternionic regular polynomials. The ranges of such integral transforms are quaternionic reproducing kernel Hilbert spaces of regular functions. New integral representations and generating functions in this quaternionic setting are obtained in the Fock case. The results obtained in this chapter are based on [63].

7.1 Motivation

The study of Appell sequences has been performed in the setting of Clifford analysis with respect to the hypercomplex derivative, see for example [29, 93, 99]. In [46, 101] the authors introduced some special modules of monogenic functions of Bargmann-type in Clifford analysis. This line of research opens some new research directions on Bargmann-Fock spaces and associated transforms in the setting of Clifford analysis. In this chapter, we construct an Appell sequence of spherical monogenics in the right Fueter-Bargmann space over quaternions, denoted by $\mathcal{RB}(\mathbb{H})$, and consisting of quaternionic Fueter regular functions that are square integrable with respect to the Gaussian measure. The main tool that

we use is the Fueter mapping theorem which relates slice hyperholomorphic functions to Fueter regular ones through the Laplacian. More precisely, we apply the Fueter mapping on a special basis of the slice hyperholomorphic Fock space constructed in [15] and obtain a set of homogeneous monogenic polynomials in the right monogenic Bargmann space over the quaternions. This allows us to construct on the standard Hilbert space on the real line the so called Bargmann-Fock-Fueter integral transform whose range is a quaternionic reproducing kernel Hilbert space of Cauchy-Fueter regular functions. In particular, we give a partial answer to Remark 4.6 in [88] about Clifford coherent state transforms using the Fueter mapping theorem in the setting of quaternions.
In section 3 of [46] the real monogenic Bargmann module on the Euclidean space $\mathbb{R}^m$ was defined to be the module consisting of solutions of the s-th power of the Dirac operator that are square integrable on $\mathbb{R}^m$ with respect to a Gaussian measure. In this work we use a similar definition for the quaternions by replacing the s-th power of the Dirac operator by the Cauchy-Fueter operator. So, we call the right Fueter-Bargmann space on quaternions the space defined by

$$\mathcal{RB}(\mathbb{H}) := \{f \in \mathcal{R}(\mathbb{H});\ \frac{1}{\pi^2}\int_{\mathbb{H}} |f(q)|^2 e^{-|q|^2} d\lambda(q) < \infty\},$$

where $d\lambda$ denotes the usual Lebesgue measure on the Euclidean vector space $\mathbb{R}^4$.

In order to present our results, we first study how the Fueter mapping acts on a special basis elements of the slice hyperholomorphic Fock space. Then, we show that the obtained polynomials constitute an Appell set of the Bargmann space of Cauchy-Fueter regular functions over the quaternions. Then, we discuss the notion of Fock-Fueter kernel. We use the previous notion to introduce and study the Bargmann-Fock-Fueter integral transform iand characterize the Fueter mapping range. Some new integral representations and generating functions in this quaternionic setting are obtained in the Fock space case.

7.2 A Clifford-Appell system based on the Fueter mapping

The main goal of this section is to apply the Fueter mapping on the quaternionic monomials forming an orthogonal basis of the slice hyperholomorphic Fock space $\mathcal{F}_{Slice}(\mathbb{H})$ and to get an Appell set of $\mathcal{RB}(\mathbb{H})$. A different proof of this result using Cauchy-Kowalevski extension arguments can be found in [84].

First, we need a lemma that describes the action of the Cauchy-Fueter operator on the quaternionic monomials $f_n(q) = q^n$:

Lemma 7.2.1 (see [24]). *For all $n \geq 2$, we have*

$$\partial f_n(q) = -2\sum_{k=1}^{n} q^{n-k}\overline{q}^{k-1}.$$

Then, we prove the following

Theorem 7.2.2. *For all $n \geq 2$, we have*

$$\tau[f_n](q) = \widetilde{f}_n(q) = -4\sum_{k=1}^{n-1}(n-k)q^{n-k-1}\overline{q}^{k-1}.$$

Proof. Let $q = x_0 + x_1 i + x_2 j + x_3 k$, thanks to the quaternions multiplication rules we have

$$f_2(q) = q^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 + 2x_0(x_1 i + x_2 j + x_3 k)$$

It is easy to check that $\widetilde{f}_2(q) = -4$, so the formula holds for $n = 2$.
Let $n \geq 2$. We suppose the proposition is true for n and we show that

$$\widetilde{f}_{n+1}(q) = -4\sum_{l=1}^{n}(n+1-k)q^{n-k}\overline{q}^{k-1}.$$

Indeed, we have $\widetilde{f}_{n+1} = \Delta f_{n+1}$ and $f_{n+1}(q) = qf_n(q)$. Therefore, applying the classical Leibniz rule we get the following system

$$\begin{cases} \dfrac{\partial^2}{\partial x_0^2} f_{n+1}(q) = 2\dfrac{\partial}{\partial x_0} f_n(q) + q\dfrac{\partial^2}{\partial x_0^2} f_n(q) \\ \dfrac{\partial^2}{\partial x_1^2} f_{n+1}(q) = 2i\dfrac{\partial}{\partial x_1} f_n(q) + q\dfrac{\partial^2}{\partial x_1^2} f_n(q) \\ \dfrac{\partial^2}{\partial x_2^2} f_{n+1}(q) = 2j\dfrac{\partial}{\partial x_2} f_n(q) + q\dfrac{\partial^2}{\partial x_2^2} f_n(q) \\ \dfrac{\partial^2}{\partial x_3^2} f_{n+1}(q) = 2k\dfrac{\partial}{\partial x_3} f_n(q) + q\dfrac{\partial^2}{\partial x_3^2} f_n(q) \end{cases}$$

Thus, by adding both sides of the system we obtain

$$\widetilde{f}_{n+1} = \Delta f_{n+1} = 2\partial f_n + q\Delta f_n = 2\partial f_n + q\widetilde{f}_n$$

Then, thanks to Lemma 11.2.6 combined with the induction hypothesis we obtain

$$\widetilde{f}_{n+1}(q) = -4\sum_{k=1}^{n}(n+1-k)q^{n-k}\overline{q}^{k-1}.$$

This completes the proof. □

Proposition 7.2.3. *For all $n \geq 2$, we have $\widetilde{f}_n \in \mathcal{RB}(\mathbb{H})$.*

Proof. First of all, by the Fueter mapping theorem the functions $\widetilde{f}_n$ are monogenic. We now show that for $n \geq 2$, we have

$$\int_{\mathbb{H}} |\widetilde{f}_n(q)|^2 e^{-|q|^2} d\lambda(q) < \infty.$$

Indeed, we have

$$\begin{aligned}\left|\widetilde{f}_n(q)\right| &= 4\left|\sum_{k=1}^{n-1}(n-k)q^{n-k-1}\overline{q}^{k-1}\right| \\ &\leq 4\sum_{k=1}^{n-1}(n-k)\,|q|^{n-2}.\end{aligned}$$

And since $\displaystyle\sum_{k=1}^{n-1}(n-k)=\frac{(n-1)n}{2}$ we get the following estimate

$$|\widetilde{f}_n(q)| \leq 2(n-1)n|q|^{n-2}$$

Hence, for all $n \geq 2$, we have $\|\widetilde{f}_n\|_{\mathcal{RB}(\mathbb{H})} \leq 2n(n-1)\|f_{n-2}\|_{L^2(\mathbb{H})}$. The proof is completed since the quaternionic monomials are square integrable with respect to the Gaussian measure on $\mathbb{H}$.

□

Remark 7.2.4. *Let $n \geq 2$ and $k \geq 0$. Then*

1. *The functions $\widetilde{f}_n$ are spherical monogenics of degree $n-2$.*
2. $\widetilde{f}_{n+1} = 2\partial f_n + q\widetilde{f}_n$.
3. $\widetilde{f}_{k+2}(q) = -4\displaystyle\sum_{j=0}^{k}(k+1-j)q^{k-j}\overline{q}^j$.

As a consequence we obtain an Appell set of spherical monogenics in $\mathcal{RB}(\mathbb{H})$. To prove this fact we need some preliminary lemmas.

Lemma 7.2.5. *Let $f : \mathbb{H} \longrightarrow \mathbb{H}$ be a Fueter regular function. Then,*

$$\overline{\partial}(qf) = 4f + 2\sum_{l=0}^{3} x_l\partial_{x_l}f.$$

Proof. Notice that for the particular case of quaternions the Leibniz rule given by (3.5.1) correspond to $m=3$. Then, if we write $q = x_0 + \underline{x}$ with $\underline{x} = x_1 i + x_2 j + x_3 k$ we obtain

$$\partial_{\underline{x}}(\underline{x}f) = -3f - \underline{x}\partial_{\underline{x}}f - 2\sum_{l=1}^{3} x_l\partial_{x_l}f. \tag{7.2.1}$$

Morever, we have

$$\overline{\partial}(qf) = f + x_0\overline{\partial}f + \underline{x}\partial_{x_0}f - \partial_{\underline{x}}(\underline{x}f).$$

So, thanks to (7.2.1) we obtain

$$\overline{\partial}(qf) = 4f + x_0\overline{\partial}f + \underline{x}\partial f + 2\sum_{l=1}^{3} x_l\partial_{x_l}f.$$

It is easy to see that $\overline{\partial}f = 2\partial_{x_0}f$. Moreover, if f is regular then $\partial f = 0$ which completes the proof. $\square$

Let us consider the Euler operator

$$E_q := \sum_{l=0}^{3} x_l\partial_{x_l}.$$

We have:

Lemma 7.2.6. *Let $h \geq 2$ and $0 \leq s \leq h$. Then*

$$E_q(q^{h-s}\overline{q}^s) = hq^{h-s}\overline{q}^s.$$

Proof. Note that for all $l \geq 0$, we have

$$\frac{d}{dx_1}(q^l) = iq^{l-1} + qiq^{l-2} + q^2iq^{l-3} + ... + q^{l-1}i \tag{7.2.2}$$

and

$$\frac{d}{dx_1}(\overline{q}^l) = -i\overline{q}^{l-1} - \overline{q}i\overline{q}^{l-2} - \overline{q}^2i\overline{q}^{l-3} - ... - \overline{q}^{l-1}i. \tag{7.2.3}$$

We have analogous relations for $\frac{d}{dx_2}(q^l)$, $\frac{d}{dx_3}(q^l)$ and $\frac{d}{dx_2}(\overline{q}^l)$, $\frac{d}{dx_3}(\overline{q}^l)$. Now observe that by the classical Leibniz rule we have

$$\frac{d}{dx_0}(q^{h-s}\overline{q}^s) = sq^{h-s}\overline{q}^{s-1} + (h-s)q^{h-s-1}\overline{q}^s.$$

On the other hand, applying the Leibniz rule we also have

$$\frac{d}{dx_1}(q^{h-s}\overline{q}^s) = q^{h-s}\frac{d}{dx_1}(\overline{q}^s) + \frac{d}{dx_1}(q^{h-s})\overline{q}^s.$$

Therefore, we use the formulas (7.2.2), (7.2.3) and those ones with respect to all other derivatives to compute $\frac{d}{dx_1}(q^{h-s}\overline{q}^s)$, $\frac{d}{dx_2}(q^{h-s}\overline{q}^s)$ and $\frac{d}{dx_3}(q^{h-s}\overline{q}^s)$. Then, by standard computations we obtain the result. $\square$

Lemma 7.2.7. *For all $k \geq 1$,*

$$\overline{\partial}\widetilde{f}_{k+2} = 2(k+2)\widetilde{f}_{k+1}.$$

Proof. Direct computations show that the formula holds for $k = 1$ and $k = 2$. Let $k \geq 2$, we can just prove $\overline{\partial}\widetilde{f}_{k+3} = 2(k+3)\widetilde{f}_{k+2}$. Indeed, we have

$$\widetilde{f}_{k+3} = 2\partial f_{k+2} + q\widetilde{f}_{k+2}$$

Then, we apply the conjugate of the Cauchy-Fueter operator on both sides of the latter equality and we use the fact that $\overline{\partial}\partial = \partial\overline{\partial} = \Delta$ to get

$$\overline{\partial}\widetilde{f}_{k+3} = 2\widetilde{f}_{k+2} + \overline{\partial}(q\widetilde{f}_{k+2}). \tag{7.2.4}$$

Let us calculate $\overline{\partial}(q\widetilde{f}_{k+2})$.

Since $\widetilde{f}_{k+2}$ is Fueter regular and in view of Lemma 7.2.5 we have

$$\overline{\partial}(q\widetilde{f}_{k+2}) = 4\widetilde{f}_{k+2} + 2E_q\widetilde{f}_{k+2}, \tag{7.2.5}$$

and

$$E_q\widetilde{f}_{k+2} = -4\sum_{s=0}^{k}(k+1-s)E_q(q^{k-s}\overline{q}^s). \tag{7.2.6}$$

Hence, we apply Lemma 7.2.6 to obtain

$$E_q(q^{k-s}\overline{q}^s) = kq^{k-s}\overline{q}^s.$$

Therefore, by replacing in (7.2.6) we get

$$E_q\widetilde{f}_{k+2} = k\widetilde{f}_{k+2}.$$

Finally, we conclude from the equations (7.2.4) and (7.2.5) that

$$\overline{\partial}\widetilde{f}_{k+3} = 2(k+3)\widetilde{f}_{k+2}.$$

This concludes the proof. □

For $k \geq 0$, let us consider the sequence of polynomials defined by

$$P_k(q) := \frac{\widetilde{f}_{k+2}(q)}{(k+2)!}.$$

We prove the following

Theorem 7.2.8. *The polynomials $\{P_k\}_{k\geq 0}$ form an Appell set of spherical monogenics of degree k in the quaternionic vector space $\mathcal{RB}(\mathbb{H})$.*

Proof. Any homogeneous monogenic polynomial of the sequence $\{P_k\}_{k\geq 0}$ is exactly of degree k and belongs to $\mathcal{RB}(\mathbb{H})$ since $\widetilde{f}_{k+2}$ is for all $k \geq 0$. Furthermore, thanks to Lemma 7.2.7 we can easily see that for all $k \geq 1$ we have $\overline{\partial} P_k = 2P_{k-1}$. It follows that this sequence forms an Appell set in $\mathcal{RB}(\mathbb{H})$, in the sense of [32, 107], with respect to the hypercomplex derivative $\frac{1}{2}\overline{\partial}$. □

Remark 7.2.9. *Let $k \geq 0$ and set $Q_k := -\frac{k!}{2}P_k$. Then, we have*

$$Q_k = -\frac{\widetilde{f}_{k+2}}{2(k+1)(k+2)} \tag{7.2.7}$$

and

$$\frac{1}{2}\overline{\partial} Q_k = kQ_{k-1}.$$

Moreover, we can see that the obtained family of polynomials may be expressed in terms of the coefficients used in formulas (5) and (6) in the paper [29]. Namely, we have

$$Q_k(q) = \sum_{j=0}^{k} T_j^k q^{k-j}\overline{q}^j \tag{7.2.8}$$

where

$$T_j^k := T_j^k(3) = \frac{k!}{(3)_k}\frac{(2)_{k-j}(1)_j}{(k-j)!j!} = \frac{2(k-j+1)}{(k+1)(k+2)}$$

and $(a)_n = a(a+1)...(a+n-1)$ is the Pochhammer symbol.

7.3 The Bargmann-Fock-Fueter transform

In this section, we study the Bargman-Fock-Fueter transform on the space of quaternions. A similar integral transform was introduced in [39] making use of the theory of slice hyperholomorphic Bergman spaces on the quaternionic unit ball and the Fueter mapping theorem.

7.3.1 Fock-Fueter kernel and Fock-Fueter transform

To this end, we introduce the Fock-Fueter kernel on the quaternions. Indeed, in [15], the authors proved that the slice hyperholomorphic Fock space $\mathcal{F}_{Slice}(\mathbb{H})$ is a right quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by the formula

$$K_{\mathbb{H}}(p,q) := e_*(p\overline{q}) = \sum_{k=0}^{\infty} \frac{p^k\overline{q}^k}{k!}, \qquad \forall (p,q) \in \mathbb{H}\times\mathbb{H}.$$

Then, we consider the following

Definition 7.3.1 (Fock-Fueter kernel). *The Fock-Fueter kernel $K_{\mathcal{F}}(q,p)$ is defined by*

$$K_{\mathcal{F}}(q,p) := \tau_q K_{\mathbb{H}}(q,p) = \Delta K_{\mathbb{H}}(q,p), \qquad \forall (q,p) \in \mathbb{H} \times \mathbb{H},$$

where Laplacian Δ is taken with respect to the variable q.

We prove the following

Proposition 7.3.1. *For all $(q,p) \in \mathbb{H} \times \mathbb{H}$, we have*

$$K_{\mathcal{F}}(q,p) = -2 \sum_{k=0}^{\infty} \frac{Q_k(q)}{k!} \overline{p}^{k+2},$$

where $Q_k(q)$ are the quaternionic monogenic polynomials defined in Remark 7.2.9.

Proof. Let $(q,p) \in \mathbb{H} \times \mathbb{H}$, by definition of the Fock-Fueter kernel we have

$$\begin{aligned} K_{\mathcal{F}}(q,p) &= \Delta K_{\mathbb{H}}(q,p) \\ &= \Delta \left(\sum_{k=0}^{\infty} \frac{q^k \overline{p}^k}{k!} \right) \\ &= \sum_{k=2}^{\infty} \frac{\Delta(q^k) \overline{p}^k}{k!}. \end{aligned}$$

However, thanks to Remark 7.2.9 we observe that

$$\Delta(q^k) = -2(k-1)kQ_{k-2}(q); \ \forall k \geq 2.$$

Therefore, we get

$$\begin{aligned} K_{\mathcal{F}}(q,p) &= -2 \sum_{k=2}^{\infty} \frac{Q_{k-2}(q)}{(k-2)!} \overline{p}^k \\ &= -2 \sum_{k=0}^{\infty} \frac{Q_k(q)}{k!} \overline{p}^{k+2}. \end{aligned}$$

□

Remark 7.3.2. *For $s \in \mathbb{H}$, let*

$$\mathrm{Exp}(s) := \sum_{k=0}^{\infty} \frac{Q_k(s)}{k!}$$

be the generalized Cauchy-Fueter regular exponential function considered in the paper [29]. Then, we have

$$K_{\mathcal{F}}(q,p) = -2p^2 \mathrm{Exp}(pq), \ \forall (q,p) \in \mathbb{H} \times \mathbb{R}.$$

Proposition 7.3.3. *The Fock-Fueter kernel $K_{\mathcal{F}}(q,p)$ is Cauchy-Fueter regular on $\mathbb{H}$ with respect to the variable q and anti-slice entire regular with respect to the variable p.*

Proof. Note that $K_{\mathcal{F}}(q,p)$ is Cauchy-Fueter regular on $\mathbb{H}$ with respect to the first variable thanks to the Fueter-mapping theorem. On the other hand, for all $p\in\mathbb{H}$ we have

$$K_{\mathcal{F}}(q,p)=f_q(p)=\sum_{k=0}^{\infty}a_k(q)\overline{p}^{k+2} \text{ where } a_k(q)=-2\frac{Q_k(q)}{k!}.$$

Then, it is clear by the series expansion theorem for slice hyperholomorphic functions that the Fock-Fueter kernel is slice anti-regular with respect to the variable p. ☐

The Fock-Fueter kernel admits the following estimate

Proposition 7.3.4. *For all $(q,p)\in\mathbb{H}\times\mathbb{H}$, we have*

$$|K_{\mathcal{F}}(q,p)|\leq 2|p|^2e^{|qp|}.$$

Proof. First, observe that for all $k\geq 0$ and $q\in\mathbb{H}$ we have

$$\begin{aligned}|Q_k(q)| &\leq \sum_{j=0}^{k}T_j^k(3)|q|^k\\ &= |q|^k\frac{2}{(k+1)(k+2)}\sum_{j=0}^{k}(k+1-j)\\ &= |q|^k.\end{aligned}$$

Hence, making use of Proposition 7.3.1 we obtain

$$\begin{aligned}|K_{\mathcal{F}}(q,p)| &\leq 2\sum_{k=0}^{\infty}\frac{|Q_k(q)|}{k!}|p|^{k+2}\\ &\leq 2|p|^2\sum_{k=0}^{\infty}\frac{|qp|^k}{k!}\\ &= 2|p|^2e^{|qp|}.\end{aligned}$$

☐

In this case we introduce the following definition

Definition 7.3.2 (Fock-Fueter transform). *Let $f\in\mathcal{F}_{Slice}(\mathbb{H})$. We define the Fock-Fueter transform of f by*

$$\breve{f}(q):=\int_{\mathbb{C}_I}K_{\mathcal{F}}(q,p)f(p)d\mu_I(p);$$

where $K_{\mathcal{F}}$ is the Fock-Fueter kernel, $d\mu_I(p)=\dfrac{1}{\pi}e^{-|p|^2}d\lambda_I(p)$ and $I\in\mathbb{S}$.

Let $L^2_{\mathbb{H}}(\mathbb{R})$ denote the space of functions $\psi : \mathbb{R} \longrightarrow \mathbb{H}$ so that

$$\|\psi\|^2_{L^2_{\mathbb{H}}(\mathbb{R})} = \int_{\mathbb{R}} |\psi(x)|^2 dx < \infty.$$

Then, for any $\varphi \in L^2_{\mathbb{H}}(\mathbb{R})$ its quaternionic Segal-Bargmann transform is defined by

$$\mathcal{B}_{\mathbb{H}}\varphi(q) := \int_{\mathbb{R}} A(q,x)\varphi(x)dx;$$

where the kernel function $A(q,x)$ is given by the formula

$$A(q,x) = \pi^{-\frac{1}{4}} e^{-\frac{1}{2}(q^2+x^2)+\sqrt{2}qx}, \qquad \forall (q,x) \in \mathbb{H} \times \mathbb{R}.$$

It was shown in [60] that $\mathcal{B}_{\mathbb{H}}$ defines an isometry on $L^2_{\mathbb{H}}(\mathbb{R})$ with range $\mathcal{F}_{Slice}(\mathbb{H})$. Then, for any $\varphi \in L^2_{\mathbb{H}}(\mathbb{R})$ we set

$$f_\varphi := \mathcal{B}_{\mathbb{H}}\varphi \in \mathcal{F}_{Slice}(\mathbb{H})$$

and consider the associated Fock-Fueter transform $\breve{f}_\varphi$ that we call Bargmann-Fock-Fueter transform. We can easily check the following

Proposition 7.3.5. *Let $\varphi \in L^2_{\mathbb{H}}(\mathbb{R})$, $q \in \mathbb{H}$ and $I \in \mathbb{S}$. Then, we have*

$$\breve{f}_\varphi(q) := \int_{\mathbb{R}} \Phi(q,x)\varphi(x)dx;$$

where

$$\Phi(q,x) = \int_{\mathbb{C}_I} K_{\mathcal{F}}(q,p)A(p,x)d\mu_I(p).$$

Proof. This follows directly from the quaternionic Segal-Bargmann transform and Fock-Fueter transform definitions making use of the Fubini's theorem. □

7.3.2 Fueter mapping range of the slice hyperholomorphic Fock space

Now, let us consider the quaternionic regular polynomials defined in Remark 7.2.9 and which may be written as :

$$Q_k(q) = \sum_{j=0}^{k} T_j^k q^{k-j}\overline{q}^j; \forall q \in \mathbb{H}. \tag{7.3.1}$$

Then, we denote the range of the Fueter mapping on the slice hyperholomorphic Fock space by

$$\mathcal{A}(\mathbb{H}) := \{\tau(f);\ f \in \mathcal{F}_{Slice}(\mathbb{H})\}.$$

We have the following sequential characterization of this vector space:

Theorem 7.3.6. *Let $g \in \mathcal{R}(\mathbb{H})$. Then, g belongs to $\mathcal{A}(\mathbb{H})$ if and only if the following conditions are satisfied:*

i) $\forall q \in \mathbb{H},\ g(q) = \displaystyle\sum_{k=0}^{\infty} Q_k(q)\alpha_k$ *where* $(\alpha_k)_{k\geq 0} \subset \mathbb{H}$.

ii) $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)}|\alpha_k|^2 < \infty.$

Proof. The Fueter mapping theorem gives $\mathcal{A}(\mathbb{H}) \subset \mathcal{R}(\mathbb{H})$. Then, we suppose that $g \in \mathcal{A}(\mathbb{H})$, thus $g = \tau(f)$ where $f \in \mathcal{F}_{Slice}(\mathbb{H})$. Then, according to [15] we have

$$f(q) = \sum_{k=0}^{\infty} q^k c_k, \text{ with } (c_k) \subset \mathbb{H} \text{ and } \|f\|^2_{\mathcal{F}_{Slice}(\mathbb{H})} = \sum_{k=0}^{\infty} k!|c_k|^2 < \infty.$$

However, we know that

$$\tau(1) = \tau(q) = 0 \text{ and } \tau(q^k) = -2(k-1)kQ_{k-2}(q), \qquad \forall k \geq 2.$$

Therefore, we get

$$g(q) = \sum_{k=0}^{\infty} Q_k(q)\alpha_k \text{ with } \alpha_k = -2(k+1)(k+2)c_{k+2}, \qquad \forall k \geq 0,$$

moreover,

$$\sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)}|\alpha_k|^2 = 4\sum_{k=0}^{\infty}(k+2)!|c_{k+2}|^2 \leq 4\|f\|^2_{\mathcal{F}_{Slice}(\mathbb{H})} < \infty.$$

Conversely, let us suppose that the conditions i) and ii) hold. Then, we consider the function

$$h(q) = \sum_{k=2}^{\infty} q^k\beta_k, \text{ where } \beta_k = -\frac{\alpha_{k-2}}{2(k-1)k}; \forall k \geq 2.$$

Thus, we get $g = \tau(h)$ since

$$Q_k(q) = -\frac{\tau(q^{k+2})}{2(k+1)(k+2)}, \qquad \forall k \geq 0.$$

Moreover, note that we have

$$\|h\|^2_{\mathcal{F}_{Slice}(\mathbb{H})} = \sum_{k=2}^{\infty} k!|\beta_k|^2 = \frac{1}{4}\sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)}|\alpha_k|^2 < \infty.$$

Hence, $g = \tau(h)$ with $h \in \mathcal{F}_{Slice}(\mathbb{H})$. In particular, it shows that $g \in \mathcal{A}(\mathbb{H})$. This completes the proof. $\square$

Remark 7.3.7. *As a direct consequence of Theorem 7.3.6 we have*

$$\mathcal{A}(\mathbb{H})=\{\sum_{k=0}^{\infty}Q_k(q)\alpha_k;\ (\alpha_k)_{k\geq 0}\subset\mathbb{H} \text{ and } \sum_{k=0}^{\infty}\frac{k!}{(k+1)(k+2)}|\alpha_k|^2<\infty\}.$$

Given $f(q)=\sum_{k=0}^{\infty}Q_k(q)\alpha_k$ and $g(q)=\sum_{k=0}^{\infty}Q_k(q)\beta_k$ in $\mathcal{A}(\mathbb{H})$ we define their inner product by

$$\langle f,g\rangle_{\mathcal{A}(\mathbb{H})}:=\sum_{k=0}^{\infty}\frac{k!}{(k+1)(k+2)}\overline{\beta_k}\alpha_k,$$

so that the associated norm is

$$\|f\|^2=\langle f,f\rangle_{\mathcal{A}(\mathbb{H})}:=\sum_{k=0}^{\infty}\frac{k!}{(k+1)(k+2)}|\alpha_k|^2.$$

Then, one can easily check the following properties

Proposition 7.3.8. *Let $f,g,h\in\mathcal{A}(\mathbb{H})$ and $\lambda\in\mathbb{H}$. Then, we have:*

i) $\overline{\langle f,g\rangle_{\mathcal{A}(\mathbb{H})}}=\langle g,f\rangle_{\mathcal{A}(\mathbb{H})}$.

ii) $\|f\|^2=\langle f,f\rangle_{\mathcal{A}(\mathbb{H})}>0$ *unless* $f=0$.

iii) $\langle f,g+h\rangle_{\mathcal{A}(\mathbb{H})}=\langle f,g\rangle_{\mathcal{A}(\mathbb{H})}+\langle f,h\rangle_{\mathcal{A}(\mathbb{H})}$.

iv) $\langle f\lambda,g\rangle_{\mathcal{A}(\mathbb{H})}=\langle f,g\rangle_{\mathcal{A}(\mathbb{H})}\lambda$ *and* $\langle f,g\lambda\rangle_{\mathcal{A}(\mathbb{H})}=\overline{\lambda}\langle f,g\rangle_{\mathcal{A}(\mathbb{H})}$.

Proof. This statement follows using classical arguments. □

Now, for all $k\geq 0$, we consider the quaternionic regular polynomials defined by

$$T_k(q)=\sqrt{\frac{(k+1)(k+2)}{k!}}Q_k(q),\qquad \forall q\in\mathbb{H}, \tag{7.3.2}$$

and we introduce the following:

Definition 7.3.3. *For all $(p,q)\in\mathbb{H}\times\mathbb{H}$, we define the function*

$$G(p,q)=G_q(p):=\sum_{k=0}^{\infty}T_k(p)\overline{T_k(q)}. \tag{7.3.3}$$

Note that, for any $(q,p)\in\mathbb{H}\times\mathbb{H}$ we have:

i) $|G(p,q)|\leq\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{k!}|pq|^k<\infty.$

ii) $\overline{G(p,q)} = G(q,p)$.

iii) $G(q,q) = \sum_{k=0}^{\infty} |T_k(q)|^2 < \infty$.

Let us prove that all the evaluation mappings are continuous on $\mathcal{A}(\mathbb{H})$. Indeed, we have

Proposition 7.3.9. *Let $q, q' \in \mathbb{H}$, then we have:*

i) The function $G_q : p \longmapsto G_q(p) = G(p,q)$ belongs to $\mathcal{A}(\mathbb{H})$.

ii) The evaluation mapping $\Lambda_q : f \longmapsto \Lambda_q(f) = f(q)$ is a continuous linear functional on $\mathcal{A}(\mathbb{H})$. Moreover, for any $f \in \mathcal{A}(\mathbb{H})$ we have

$$|\Lambda_q(f)| = |f(q)| \leq \|G_q\|_{\mathcal{A}(\mathbb{H})} \|f\|_{\mathcal{A}(\mathbb{H})}.$$

iii) $\langle G_{q'}, G_q \rangle_{\mathcal{A}(\mathbb{H})} = G(q,q')$.

Proof. i) Note that by definition of the polynomials $(T_k(q))_{k\geq 0}$, for any fixed $q \in \mathbb{H}$ we have

$$G_q(p) = \sum_{k=0}^{\infty} Q_k(p)\alpha_k(q) \text{ with } \alpha_k(q) = \frac{(k+1)(k+2)}{k!}\overline{Q_k(q)} \in \mathbb{H}, \ \ \forall k \geq 0. \tag{7.3.4}$$

Moreover, observe that

$$\begin{aligned} \|G_q\|^2_{\mathcal{A}(\mathbb{H})} &= \sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)} |\alpha_k(q)|^2 \\ &= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{k!} |Q_k(q)|^2 \\ &\leq \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{k!} |q|^{2k} < \infty. \end{aligned} \tag{7.3.5}$$

This shows that $G_q \in \mathcal{A}(\mathbb{H})$ for any $q \in \mathbb{H}$.

ii) If $f \in \mathcal{A}(\mathbb{H})$, then by definition we have

$$f(q) = \sum_{k=0}^{\infty} Q_k(q)\alpha_k \text{ and } \|f\|^2_{\mathcal{A}(\mathbb{H})} = \sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)} |\alpha_k|^2 < \infty.$$

Therefore, making use of the Cauchy-Schwarz inequality we get

$$\begin{aligned}|\Lambda_q(f)| &= |f(q)|\\ &\leq \sum_{k=0}^{\infty} |Q_k(q)||\alpha_k|\\ &\leq \left(\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{k!}|Q_k(q)|^2\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)}|\alpha_k|^2\right)^{\frac{1}{2}}\\ &= \|G_q\|_{\mathcal{A}(\mathbb{H})}\|f\|_{\mathcal{A}(\mathbb{H})}.\end{aligned}$$

iii) Let $q, q' \in \mathbb{H}$ be such that

$$G_q(p) = \sum_{k=0}^{\infty} Q_k(p)\alpha_k(q) \text{ and } G_{q'}(p) = \sum_{k=0}^{\infty} Q_k(p)\alpha_k(q')$$

where we have set $\alpha_k(w) = \dfrac{(k+1)(k+2)}{k!}\overline{Q_k(w)}$ for any $w \in \mathbb{H}$ and $k \geq 0$. Therefore, we get

$$\begin{aligned}\langle G_{q'}, G_q\rangle_{\mathcal{A}(\mathbb{H})} &= \sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)}\overline{\alpha_k(q)}\alpha_k(q')\\ &= \sum_{k=0}^{\infty} T_k(q)\overline{T_k(q')}\\ &= G_{q'}(q) = G(q, q').\end{aligned}$$

□

As a consequence we prove the following result

Theorem 7.3.10. *The set $\mathcal{A}(\mathbb{H})$ is a right quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by the kernel function $G : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$ defined in* (7.3.3). *Moreover, for any $q \in \mathbb{H}$ and $f \in \mathcal{A}(\mathbb{H})$ we have*

$$f(q) = \langle f, G_q\rangle_{\mathcal{A}(\mathbb{H})}.$$

Proof. According to Proposition 7.3.9 we know that all the evaluation mappings are continuous on $\mathcal{A}(\mathbb{H})$ and $G_q \in \mathcal{A}(\mathbb{H})$ for any $q \in \mathbb{H}$. So, we only need to prove the reproducing kernel property. Indeed, let $q \in \mathbb{H}$ and $f \in \mathcal{A}(\mathbb{H})$ be such that $f(p)=\sum_{k=0}^{\infty}Q_k(p)\beta_k$, for any $p\in\mathbb{H}$. Using (7.3.4) we obtain

$$\begin{aligned}\langle f,G_q\rangle_{\mathcal{A}(\mathbb{H})}&=\sum_{k=0}^{\infty}\frac{k!}{(k+1)(k+2)}\overline{\alpha_k(q)}\beta_k\\&=\sum_{k=0}^{\infty}Q_k(q)\beta_k\\&=f(q).\end{aligned}$$

This completes the proof. □

7.4 Factorization of the Bargmann-Fock-Fueter transform and consequences

We can factorize the Bargmann-Fock-Fueter transform thanks to the following:

Theorem 7.4.1. *The Bargmann-Fock-Fueter transform can be realized by the commutative diagram*

$$\mathcal{S}_{\mathbb{H}}:\quad \begin{array}{ccc} L^2_{\mathbb{H}}(\mathbb{R}) & \longrightarrow & \mathcal{A}(\mathbb{H}) \\ {\scriptstyle\mathcal{B}_{\mathbb{H}}}\downarrow & & \uparrow{\scriptstyle\tau} \\ \mathcal{F}^2_{Slice}(\mathbb{H}) & \xrightarrow[Id]{} & \mathcal{SR}(\mathbb{H}) \end{array}$$

so that

$$\mathcal{S}_{\mathbb{H}}:=\tau\circ Id\circ\mathcal{B}_{\mathbb{H}}.$$

More precisely, for any $\varphi\in L^2_{\mathbb{H}}(\mathbb{R})$, and $q\in\mathbb{H}$, we have

$$\mathcal{S}_{\mathbb{H}}\varphi(q)=\breve{f}_{\varphi}(q)=\int_{\mathbb{R}}\Phi(q,x)\varphi(x)dx;$$

where

$$\Phi(q,x)=-\frac{1}{\pi^{\frac{1}{4}}}\sum_{k=0}^{\infty}\frac{Q_k(q)h_{k+2}(x)}{2^{\frac{k}{2}}k!};\ \forall(q,x)\in\mathbb{H}\times\mathbb{R}.$$

Proof. Let $\varphi\in L^2_{\mathbb{H}}(\mathbb{R})$ and $q\in\mathbb{H}$, observe that

$$\begin{aligned}\mathcal{S}_{\mathbb{H}}[\varphi](q)&=\tau\circ Id\circ\mathcal{B}_{\mathbb{H}}[\varphi](q)\\&=\int_{\mathbb{R}}\Delta A(q,x)\varphi(x)dx.\end{aligned}$$

Thus, by Proposition 7.3.5 we only need to prove that

$$\Delta A(q,x)=\Phi(q,x)\ \forall(q,x)\in\mathbb{H}\times\mathbb{R},$$

where

$$\Phi(q,x)=\int_{\mathbb{C}_I}K_{\mathcal{F}}(q,p)A(p,x)d\mu_I(p).$$

Indeed, note that according to Proposition 4.1 in [60] for all $(q,x)\in\mathbb{H}\times\mathbb{R}$ we have the following expansion of the Segal-Bargmann kernel

$$A(q,x)=\sum_{k=0}^{\infty}\frac{q^k}{\|q^k\|}\frac{h_k(x)}{\|h_k\|},$$

where $\{h_k\}_{k\geq 0}$ stands for the well-known Hermite functions forming an orthogonal basis of $L^2_{\mathbb{H}}(\mathbb{R})$. Therefore, on the one hand we have

$$\begin{aligned}\Delta A(q,x)&=\frac{1}{\pi^{\frac{1}{4}}}\sum_{k=2}^{\infty}\frac{\Delta(q^k)h_k(x)}{k!2^{\frac{k}{2}}}\\&=-\frac{2}{\pi^{\frac{1}{4}}}\sum_{k=2}^{\infty}\frac{Q_{k-2}(q)h_k(x)}{(k-2)!2^{\frac{k}{2}}}\\&=-\frac{1}{\pi^{\frac{1}{4}}}\sum_{k=0}^{\infty}\frac{Q_k(q)h_{k+2}(x)}{k!2^{\frac{k}{2}}}.\end{aligned}$$

On the other hand, making use of Proposition 7.3.1 combined with the expansion of the Segal-Bargmann kernel we get

$$\begin{aligned}\Phi(q,x)&=\int_{\mathbb{C}_I}K_{\mathcal{F}}(q,p)A(p,x)d\mu_I(p)\\&=-\frac{2}{\pi^{\frac{1}{4}}}\int_{\mathbb{C}_I}\left(\sum_{k=0}^{\infty}\frac{Q_k(q)}{k!}\bar{p}^{k+2}\right)\left(\sum_{j=0}^{\infty}\frac{p^j}{j!2^{\frac{j}{2}}}h_j(x)\right)d\mu_I(p)\\&=-\frac{2}{\pi^{\frac{1}{4}}}\sum_{k,j=0}^{\infty}\frac{Q_k(q)}{k!j!}\frac{h_j(x)}{2^{\frac{j}{2}}}\left\langle p^j,p^{k+2}\right\rangle_{\mathcal{F}_{Slice}(\mathbb{H})}\\&=-\frac{2}{\pi^{\frac{1}{4}}}\sum_{l=2,j=0}^{\infty}\frac{Q_{l-2}(q)}{(l-2)!j!}\frac{h_j(x)}{2^{\frac{j}{2}}}\left\langle p^j,p^{l}\right\rangle_{\mathcal{F}_{Slice}(\mathbb{H})}\\&=-\frac{2}{\pi^{\frac{1}{4}}}\sum_{l=2,j=0}^{\infty}\frac{Q_{l-2}(q)}{(l-2)!}\frac{h_j(x)}{2^{\frac{j}{2}}}\delta_{l,j}\\&=-\frac{2}{\pi^{\frac{1}{4}}}\sum_{l=2}^{\infty}\frac{Q_{l-2}(q)}{(l-2)!}\frac{h_l(x)}{2^{\frac{l}{2}}}\\&=-\frac{1}{\pi^{\frac{1}{4}}}\sum_{k=0}^{\infty}\frac{Q_k(q)h_{k+2}(x)}{k!2^{\frac{k}{2}}}.\end{aligned}$$

This completes the proof. □

Proposition 7.4.2. *For all* $(q,p)\in\mathbb{H}\times\mathbb{H}$ *we have*

$$\int_{\mathbb{R}}\Phi(q,x)\Phi(p,x)dx=4\sum_{k=0}^{\infty}T_k(q)T_k(p).$$

Proof. Let $(q,p)\in\mathbb{H}\times\mathbb{H}$, then

$$\begin{aligned}\int_{\mathbb{R}}\Phi(q,x)\Phi(p,x)dx&=\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}}\left(\sum_{k=0}^{\infty}\frac{Q_k(q)h_{k+2}(x)}{2^{\frac{k}{2}}k!}\right)\left(\sum_{j=0}^{\infty}\frac{Q_j(p)h_{j+2}(x)}{2^{\frac{j}{2}}j!}\right)\\&=\frac{1}{\sqrt{\pi}}\sum_{k,j=0}^{\infty}\frac{Q_k(q)Q_j(p)}{k!j!2^{\frac{k+j}{2}}}\int_{\mathbb{R}}h_{k+2}(x)h_{j+2}(x)dx\\&=\frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{Q_k(q)Q_k(p)}{(k!)^2 2^k}\|h_{k+2}\|^2.\end{aligned}$$

Therefore, making use of the orthogonality of Hermite functions we get

$$\begin{aligned}\int_{\mathbb{R}}\Phi(q,x)\Phi(p,x)dx&=4\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{k!}Q_k(q)Q_k(p)\\&=4\sum_{k=0}^{\infty}T_k(q)T_k(p).\end{aligned}$$

□

Remark 7.4.3. *Recalling that* $L^2_{\mathbb{H}}(\mathbb{R})$ *is endowed with the scalar product*

$$\langle\phi_1,\phi_2\rangle=\int_{\mathbb{R}}\overline{\phi_2(x)}\phi_1(x)dx,\ \forall\phi_1,\phi_2\in L^2_{\mathbb{H}}(\mathbb{R}),$$

as a consequence of Proposition 7.4.2 and of (7.3.3) *we get*

$$G(q,\overline{p})=\frac{1}{4}\left\langle\Phi_p,\Phi_{\overline{q}}\right\rangle,\ \forall(q,p)\in\mathbb{H}\times\mathbb{H}.$$

Corollary 7.4.4. *For all* $q\in\mathbb{H}$, *the function* $\Phi_q : x\longmapsto\Phi_q(x):=\Phi(q,x)$ *belongs to* $L^2_{\mathbb{H}}(\mathbb{R})$ *and*

$$\|\Phi_q\|_{L^2_{\mathbb{H}}(\mathbb{R})}=2\left(\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{k!}|Q_k(q)|^2\right)^{\frac{1}{2}}<\infty.$$

Moreover, for any $\varphi\in L^2_{\mathbb{H}}(\mathbb{R})$ *we have*

$$|\mathcal{S}_{\mathbb{H}}\varphi(q)|\leq\|\Phi_q\|_{L^2_{\mathbb{H}}(\mathbb{R})}\|\varphi\|_{L^2_{\mathbb{H}}(\mathbb{R})}.$$

Proof. Let $q \in \mathbb{H}$, then we have

$$\begin{aligned}\|\Phi_q\|^2_{L^2_{\mathbb{H}}(\mathbb{R})} &= \int_{\mathbb{R}} \Phi(q,x)\overline{\Phi(q,x)}dx \\ &= \int_{\mathbb{R}} \Phi(q,x)\Phi(\bar{q},x)dx.\end{aligned}$$

Thus, by Proposition 7.4.2 we get

$$\begin{aligned}\|\Phi_q\|^2_{L^2_{\mathbb{H}}(\mathbb{R})} &= 4\sum_{k=0}^{\infty} T_k(q)T_k(\bar{q}) \\ &= 4\sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{k!}|Q_k(q)|^2.\end{aligned}$$

However, since $|Q_k(q)|^2 \leq |q|^{2k}$ for all $k \geq 0$, using (7.3.5) and the Cauchy-Schwarz inequality we conclude the proof. □

The action of the Bargmann-Fock-Fueter transform on the normalized Hermite functions is given by

Proposition 7.4.5. *For all $n \geq 0$, set*

$$\xi_n(x) = \frac{h_n(x)}{\|h_n\|_{L^2_{\mathbb{H}}(\mathbb{R})}}. \tag{7.4.1}$$

Then, we have

$$\mathcal{S}_{\mathbb{H}}\xi_n = \breve{f}_{\xi_n} = 0; \text{ for } n = 0, 1$$

and

$$\mathcal{S}_{\mathbb{H}}\xi_n(q) = \breve{f}_{\xi_n}(q) = -2T_{n-2}(q); \text{ for all } n \geq 2.$$

Proof. To prove this fact we only need to use the definition of $\mathcal{S}_{\mathbb{H}}$ as a composition of the Fueter mapping τ and the quaternionic Segal-Bargmann transform $\mathcal{B}_{\mathbb{H}}$. Then, by Lemma 4.4 in [60] we know that

$$\mathcal{B}_{\mathbb{H}}(\xi_n)(q) = \frac{q^n}{\sqrt{n!}}; \forall n \geq 2.$$

Finally, we apply Remark 3.8 to conclude the proof. □

Then, we have

Proposition 7.4.6. *The Bargmann-Fock-Fueter transform*

$$\mathcal{S}_{\mathbb{H}} : L^2_{\mathbb{H}}(\mathbb{R}) \longrightarrow \mathcal{A}(\mathbb{H})$$

is a quaternionic right linear bounded surjective operator such that for any $\varphi \in L^2_{\mathbb{H}}(\mathbb{R})$, we have

$$\|\mathcal{S}_{\mathbb{H}}\varphi\|_{\mathcal{A}(\mathbb{H})} \leq 2\|\varphi\|_{L^2_{\mathbb{H}}(\mathbb{R})}.$$

Proof. Let $\varphi \in L^2(\mathbb{R})$. Since $(\xi_k)_k$ as in (7.4.1) form an orthonormal basis of $L^2(\mathbb{R})$ we then have

$$\varphi = \sum_{k=0}^{\infty} \xi_k \alpha_k \text{ with } (\alpha_k)_k \subset \mathbb{H} \text{ and such that } \|\varphi\|^2 = \sum_{k=0}^{\infty} |\alpha_k|^2 < \infty.$$

Hence, since $\mathcal{B}_{\mathbb{H}}$ is an isometric isomorphism we use Proposition 7.4.5 to get

$$\mathcal{S}_{\mathbb{H}}\varphi(q) = \sum_{k=0}^{\infty} Q_k(q)\beta_k \text{ where } \beta_k = -2\sqrt{\frac{(k+1)(k+2)}{k!}}\alpha_{k+2}.$$

In particular, this implies that

$$\begin{aligned} \|\mathcal{S}_{\mathbb{H}}\varphi\|^2_{\mathcal{A}(\mathbb{H})} &= \sum_{k=0}^{\infty} \frac{k!}{(k+1)(k+2)} |\beta_k|^2 \\ &= 4\sum_{k=2}^{\infty} |\alpha_k|^2 \leq 4\|\varphi\|^2. \end{aligned}$$

Finally, $\mathcal{S}_{\mathbb{H}} : L^2_{\mathbb{H}}(\mathbb{R}) \longrightarrow \mathcal{A}(\mathbb{H})$ is surjective by construction. This completes the proof. ☐

For any $k \geq 0$, we consider the subspaces of $L^2_{\mathbb{H}}(\mathbb{R})$ defined by

$$\mathcal{H}_k := \xi_k\mathbb{H} = \{\xi_k\alpha; \alpha \in \mathbb{H}\},$$

where ξ_k denote the normalized Hermite functions. It is clear that we have the orthogonal decomposition

$$L^2_{\mathbb{H}}(\mathbb{R}) = \oplus_{k=0}^{\infty}\mathcal{H}_k.$$

Then, we consider $\mathcal{H} = \oplus_{k=2}^{\infty}\mathcal{H}_k$ as a subspace of $L^2_{\mathbb{H}}(\mathbb{R})$, endowed with the induced norm and prove

Proposition 7.4.7. *Let $\varphi, \psi \in \mathcal{H}$, then we have*

$$\langle \mathcal{S}_{\mathbb{H}}\varphi, \mathcal{S}_{\mathbb{H}}\psi \rangle_{\mathcal{A}(\mathbb{H})} = 4 \langle \varphi, \psi \rangle_{\mathcal{H}}.$$

In particular,

$$\|\mathcal{S}_{\mathbb{H}}\varphi\|_{\mathcal{A}(\mathbb{H})} = 2\|\varphi\|_{\mathcal{H}}.$$

Proof. Let $\varphi = \sum_{k=2}^{\infty} \xi_k\alpha_k$ and $\psi = \sum_{k=2}^{\infty} \xi_k\beta_k$ be two functions belonging to $\mathcal{H}$. Thus, by Proposition 7.4.5 we get

$$\mathcal{S}_{\mathbb{H}}\varphi = \sum_{k=0}^{\infty} Q_k\alpha'_k \text{ and } \mathcal{S}_{\mathbb{H}}\psi = \sum_{k=0}^{\infty} Q_k\beta'_k,$$

where we have set

$$\alpha'_k = -2\sqrt{\frac{(k+1)(k+2)}{k!}}\alpha_{k+2} \text{ and } \beta'_k = -2\sqrt{\frac{(k+1)(k+2)}{k!}}\beta_{k+2}.$$

Therefore, we obtain

$$\begin{aligned}
\langle \mathcal{S}_{\mathbb{H}}\varphi, \mathcal{S}_{\mathbb{H}}\psi\rangle_{\mathcal{A}(\mathbb{H})} &= 4\sum_{k=0}^{\infty}\frac{k!}{(k+1)(k+2)}\overline{\beta'_k}\alpha'_k \\
&= 4\sum_{k=2}^{\infty}\overline{\beta_k}\alpha_k \\
&= 4\,\langle\varphi,\psi\rangle_{\mathcal{H}}\,.
\end{aligned}$$

Thus, in particular, for $\psi = \varphi$ we obtain

$$\|\mathcal{S}_{\mathbb{H}}\varphi\|_{\mathcal{A}(\mathbb{H})} = 2\|\varphi\|_{\mathcal{H}}.$$

□

Finally, we finish this section by giving some integral representations of the quaternionic regular polynomials $(Q_k)_{k\geq 0}$ in terms of the Fock-Fueter kernel $K_{\mathcal{F}}(q,p)$ and the Segal-Bargmann-Fueter kernel $\Phi(q,x)$, respectively. Indeed,

Proposition 7.4.8. *Let $I\in\mathbb{S}$ and $q\in\mathbb{H}$. Then, we have*

i) $Q_k(q) = -\dfrac{1}{2(k+1)(k+2)}\displaystyle\int_{\mathbb{C}_I} K_{\mathcal{F}}(q,p)p^{k+2}d\mu_I(p), \forall k\geq 0.$

ii) $Q_k(q) = -\dfrac{1}{4\pi^{\frac{1}{4}}2^{\frac{k}{2}}(k+1)(k+2)}\displaystyle\int_{\mathbb{R}} \Phi(q,x)h_{k+2}(x)dx, \forall k\geq 0.$

Proof. i) When $k\geq 0$, Proposition 7.3.1 yields

$$K_{\mathcal{F}}(q,p) = -2\sum_{k=0}^{\infty}\frac{Q_k(q)}{k!}\bar{p}^{k+2}, \qquad \forall (q,p)\in\mathbb{H}\times\mathbb{H}.$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{C}_I} K_{\mathcal{F}}(q,p)p^{k+2}d\mu_I(p) &= -2\sum_{j=0}^{\infty}\frac{Q_j(q)}{j!}\left\langle p^{k+2}, p^{j+2}\right\rangle_{\mathcal{F}_{Slice}(\mathbb{H})} \\
&= -2\frac{Q_k(q)}{k!}\|p^{k+2}\|^2_{\mathcal{F}_{Slice}(\mathbb{H})} \\
&= -2(k+1)(k+2)Q_k(q).
\end{aligned}$$

ii) This assertion follows reasoning in the same way we did for i) using Theorem 7.4.1 combined with the fact that Hermite functions form an orthogonal basis of $L^2_{\mathbb{H}}(\mathbb{R})$.

□

As a consequence we have this special identity

Corollary 7.4.9. *For any $x \in \mathbb{R}, I \in \mathbb{S}$ and $k \geq 0$, we have*

$$\int_{\mathbb{C}_I} p^k |p|^4 e^{-|p|^2 + x\overline{p}} d\lambda_I(p) = \pi(k+1)(k+2)x^k,$$

where $I \in \mathbb{S}$ and $d\lambda_I$ is the Lebesgue measure on $\mathbb{C}_I$.

Proof. We only need to apply Proposition 7.4.8 combined with the expression of the Fock-Fueter kernel for $x \in \mathbb{R}$, which is given by

$$K_{\mathcal{F}}(x,p) = -2\overline{p}^2 e^{x\overline{p}}, \qquad \forall (x,p) \in \mathbb{R} \times \mathbb{H}.$$

□

CHAPTER 8

The Bergman kernel and Bergman-Fueter transform on different quaternionic domains

In this chapter, we continue the study related to the Clifford-Appell polynomials constructed using the Fueter mapping theorem. In particular, we calculate the Bergman kernels on some different quaternionic domains. We treat also the so-called Bergman-Fueter integral transform in the cases of the unit ball, the half space and the unit half-ball on quaternions. As a consequence of this construction some new integral representations and generating functions related to the Clifford-Appell system are obtained. The results presented in this chapter are also based on [63].

8.1 The slice hyperholomorphic Bergman kernels

In this section, we compute the explicit expression of the slice hyperholomorphic Bergman kernel on the quaternionic unit half ball and the fractional wedge domain. The case of the quarter-ball could be treated also using similar techniques. For the study of the Bergman kernel function in the setting of monogenic or Cauchy Fueter regular functions one may consult for example [52, 106].

8.1.1 The quaternionic unit half ball $\mathbb{B}^+$ case

Let $\mathbb{B}^+$ denote the quaternionic half ball defined by the conditions $q \in \mathbb{B}$ and $Re(q) > 0$. For a fixed $I \in \mathbb{S}$, let $\mathbb{B}^+_I := \mathbb{B}^+ \cap \mathbb{C}_I$ be the half disk of the complex plane $\mathbb{C}_I$. Then, the classical complex Bergman space on $\mathbb{B}_I^+$ is defined by

$$\mathcal{A}(\mathbb{B}_I^+) := \{f \in Hol(\mathbb{B}_I^+), \frac{1}{\pi}\int_{\mathbb{B}_I^+} |f_I(z)|^2 dA(z) < \infty\}$$

where $Hol(\mathbb{B}_I^+)$ denotes the space of holomorphic functions on the half disk $\mathbb{B}_I^+$, $z = x + Iy$ and $dA(z) = dxdy$. Note that the space $\mathcal{A}(\mathbb{B}_I^+)$ is a complex reproducing kernel Hilbert space. Furthermore, its reproducing kernel $K_{\mathbb{B}_I^+}$ is obtained as the sum of the Bergman kernels of both the complex unit disk and half plane. In particular, we have

$$K_{\mathbb{B}_I^+}(z,w) := \frac{1}{(1-z\overline{w})^2} + \frac{1}{(z+\overline{w})^2}; \; \forall (z,w) \in \mathbb{B}_I^+ \times \mathbb{B}_I^+ \tag{8.1.1}$$

where the first term corresponds to the Bergman kernel of the unit disk $K_{\mathbb{B}_I}$ while the second one is the Bergman kernel of the complex half plane $K_{\mathbb{C}_I^+}$, (see, e,g., p. 812 in [52]). Now, let us fix an imaginary unit $I \in \mathbb{S}$ and consider on the quaternionic half ball $\mathbb{B}^+$ the set defined by

$$\mathcal{A}_{Slice}(\mathbb{B}^+) := \{f \in \mathcal{SR}(\mathbb{B}^+), \frac{1}{\pi}\int_{\mathbb{B}_I^+} |f_I(p)|^2 d\sigma_I(p) < \infty\} \tag{8.1.2}$$

where for $p = x + Iy$ we have set $d\sigma_I(p) = dxdy$. The set $\mathcal{A}_{Slice}(\mathbb{B}^+)$ is a right quaternionic vector space and may be endowed with the inner product:

$$\langle f, g\rangle_{\mathcal{A}_{Slice}(\mathbb{B}^+)} := \frac{1}{\pi}\int_{\mathbb{B}_I^+} \overline{f_I(p)} g_I(p) d\sigma_I(p). \tag{8.1.3}$$

Moreover, since the quaternionic half-ball is a bounded axially symmetric slice domain it turns out that $\mathcal{A}_{Slice}(\mathbb{B}^+)$ is the slice hyperholomorphic Bergman space of the second kind on $\mathbb{B}^+$. These spaces were introduced and studied in a more general setting on axially symmetric slice domains in [43]. In particular we have:

Proposition 8.1.1. *The set $\mathcal{A}_{Slice}(\mathbb{B}^+)$ defined in* (8.1.2) *is a right quaternionic Hilbert space which does not depend on the choice of the imaginary unit $I \in \mathbb{S}$.*

Note that in this framework the evaluation mapping

$$\delta_q : f \longmapsto \delta_q(f) = f(q)$$

is a right quaternionic bounded linear form on $\mathcal{A}_{Slice}(\mathbb{B}^+)$ for any $q \in \mathbb{B}^+$. Moreover, the slice hyperholomorphic Bergman kernel of the second kind associated with $\mathbb{B}^+$ or slice Bergman kernel for short, is the function

$$K_{\mathbb{B}^+} : \mathbb{B}^+ \times \mathbb{B}^+ \longrightarrow \mathbb{B}^+, \qquad (q,r) \longmapsto K_{\mathbb{B}^+}(q,r)$$

which is defined making use of the slice hyperholomorphic extension operator, i.e.

$$\begin{aligned} K_{\mathbb{B}^+}(q,r) &:= K^r_{\mathbb{B}^+}(q) \\ &:= ext[K^r_{\mathbb{B}^+_J}(z)](q), \qquad \text{for } r \in \mathbb{B}^+ \cap \mathbb{C}_J,\ q \in \mathbb{B}^+. \end{aligned}$$

The next result relates the slice Bergman kernel on the quaternionic half ball to the slice Bergman kernels in the case of the quaternionic unit ball and of the half space.

Theorem 8.1.2. *The slice hyperholomorphic Bergman space $\mathcal{A}_{Slice}(\mathbb{B}^+)$ is a right quaternionic reproducing kernel Hilbert space. Moreover, for all $(q,r) \in \mathbb{B}^+ \times \mathbb{B}^+$ we have:*

$$K_{\mathbb{B}^+}(q,r) = K_{\mathbb{B}}(q,r) + K_{\mathbb{H}^+}(q,r),$$

where $K_{\mathbb{B}}$ and $K_{\mathbb{H}^+}$ are, respectively, the slice Bergman kernels of the quaternionic unit ball and half space.

Proof. The first assertion follows from the general theory.
Then, let us fix $r \in \mathbb{B}^+$ such that r belongs to the slice $\mathbb{C}_J$ with $J \in \mathbb{S}$. Then, we consider the function ψ_r defined by

$$\psi_r(q) := K_{\mathbb{B}}(q,r) + K_{\mathbb{H}^+}(q,r), \qquad \forall q \in \mathbb{B}^+.$$

Clearly ψ_r belongs to $\mathcal{A}_{Slice}(\mathbb{B}^+)$ since $\mathbb{B}^+$ is contained in both $\mathbb{B}$ and $\mathbb{H}^+$ and since by definition $K_{\mathbb{B}}$ and $K_{\mathbb{H}^+}$ are the slice Bergman kernels of the quaternionic unit ball and half space. Then, we only need to prove the reproducing kernel property. Indeed, let $f \in \mathcal{A}_{Slice}(\mathbb{B}^+)$. In particular, by the Splitting Lemma we can write $f_J(z) = F(z) + G(z)J'$ for any $z \in \mathbb{B}^+_J$ with $J' \in \mathbb{S}$ is orthogonal to J and $F, G : \mathbb{B}^+_J \longrightarrow \mathbb{C}_J$ belong to the complex Bergman space $\mathcal{A}(\mathbb{B}^+_J)$. Therefore, we have

$$\begin{aligned} \langle \psi_r, f \rangle_{\mathcal{A}_{Slice}(\mathbb{B}^+)} &= \int_{\mathbb{B}^+_J} \overline{\psi_r(z)} f_J(z) d\sigma_J(z) \\ &= \left(\int_{\mathbb{B}^+_J} \overline{\psi_r(z)} F(z) d\sigma_J(z) \right) + \left(\int_{\mathbb{B}^+_J} \overline{\psi_r(z)} G(z) d\sigma_J(z) \right) J' \\ &= \left(\int_{\mathbb{B}^+_J} \overline{K_{\mathbb{B}^+_J}(z,r)} F(z) d\sigma_J(z) \right) + \left(\int_{\mathbb{B}^+_J} \overline{K_{\mathbb{B}^+_J}(z,r)} G(z) d\sigma_J(z) \right) J'. \end{aligned}$$

Thus, by applying the results from the classical complex setting we get

$$\begin{aligned} \langle \psi_r, f \rangle_{\mathcal{A}_{Slice}(\mathbb{B}^+)} &= F(r) + G(r)J' \\ &= f(r). \end{aligned}$$

So, it follows that the function ψ_r belongs and reproduces any element of the space $\mathcal{A}_{Slice}(\mathbb{B}^+)$ for any $r \in \mathbb{B}^+$. Hence, by the uniqueness of the reproducing kernel we get

$$K_{\mathbb{B}^+}(q,r) = K_{\mathbb{B}}(q,r) + K_{\mathbb{H}^+}(q,r), \qquad \forall (q,r) \in \mathbb{B}^+ \times \mathbb{B}^+.$$

This completes the proof. $\square$

The explicit expression of the slice Bergman kernel of the quaternionic half-ball is given by the following

Theorem 8.1.3. *For all $(q,r) \in \mathbb{B}^+ \times \mathbb{B}^+$, we have:*

$$K_{\mathbb{B}^+}(q,r) = (1+q^2)\left[(1-q\overline{r}) * (q+\overline{r})\right]^{-*2}(1+\overline{r}^2),$$

where the $*$-product is taken with respect to the variable q.

Proof. Let $(q,r) \in \mathbb{B}^+ \times \mathbb{B}^+$ and assume that r belongs to a slice $\mathbb{C}_J$. First, observe that

$$K_{\mathbb{B}_J^+}(z,r) = \frac{(1+z^2)(1+\overline{r}^2)}{((1-z\overline{r})(z+\overline{r}))^2}; \forall z \in \mathbb{B}_J^+.$$

Let $\Phi^r : \mathbb{B}^+ \longrightarrow \mathbb{H}$ be the function defined by

$$\Phi^r(q) := ext\left[\frac{1}{(1-z\overline{r})^2(z+\overline{r})^2}\right](q); \forall q \in \mathbb{B}^+.$$

Then, we consider the function

$$\Psi^r(q) = (1+q^2)\Phi^r(q)(1+\overline{r}^2); \forall q \in \mathbb{B}^+.$$

Note that, Ψ^r is slice regular on $\mathbb{B}^+$ as a multiplication of the intrinsic slice regular function $q \longmapsto 1+q^2$ with $q \longmapsto \Phi^r(q)(1+\overline{r}^2)$ which is also slice regular on the quaternionic half ball by construction. Moreover, for any $z \in \mathbb{B}_J^+$ we have

$$\Psi^r(z) = \frac{(1+z^2)(1+\overline{r}^2)}{((1-z\overline{r})(z+\overline{r}))^2} = K_{\mathbb{B}_J^+}(z,r).$$

Therefore, by the Identity Principle for slice regular functions we get

$$\Psi^r(q) = K_{\mathbb{B}^+}(q,r); \forall (q,r) \in \mathbb{B}^+ \times \mathbb{B}^+.$$

Finally, we use the definition of the $*$ product to see that, for all $(q,r) \in \mathbb{B}^+ \times \mathbb{B}^+$ we have

$$\Phi^r(q) = (q+\overline{r})^{-*2} * (1-q\overline{r})^{-*2} = \left[(1-q\overline{r}) * (q+\overline{r})\right]^{-*2}.$$

$\square$

8.1.2 The fractional wedge domain case

For $I \in \mathbb{S}$, let us now consider the wedge domain defined by

$$\mathcal{W}^n_{\mathbb{C}_I} := \{z \in \mathbb{C}_I,\ Re(z) > 0 \text{ and } Re(\alpha^{\frac{1}{2}} z \alpha^{\frac{1}{2}}) < 0 \text{ with } \alpha = e^{\frac{I\pi}{n}}\}.$$

In particular, in the complex case the Bergman kernel is given in [52] by

$$K_{\mathcal{W}^n_{\mathbb{C}_I}}(z,w) = (-1)^n n^2 \frac{z^{n-1}\overline{w}^{n-1}}{(z^n - (-1)^n \overline{w}^n)^2}.$$

Let $\mathcal{W}^n_{\mathbb{H}}$ denotes the axially symmetric completion of $\mathcal{W}^n_{\mathbb{C}_I}$. In the next result, we compute the quaternionic slice hyperholomorphic Bergman kernel on $\mathcal{W}^n_{\mathbb{H}}$:

Theorem 8.1.4. *For all $(q,r) \in \mathcal{W}^n_{\mathbb{H}} \times \mathcal{W}^n_{\mathbb{H}}$, we have*

$$\begin{aligned} K_{\mathcal{W}^n_{\mathbb{H}}}(q,r) &= (-1)^n n^2 q^{n-1}(\bar{q}^{2n} - 2(-1)^n \bar{q}^n \bar{r}^n + \bar{r}^{2n})\bar{r}^{n-1} \\ &\quad \cdot(|q|^2 - 2(-1)^n Re(q^n)\bar{r}^n + \bar{r}^{2n})^{-2}. \end{aligned}$$

Proof. Let $q,r \in \mathcal{W}^n_{\mathbb{H}}$ be such that r belongs to $\mathbb{C}_J$ where $J \in \mathbb{S}$. Then, for $q = x + I_q y$ and $z = x + Jy$ thanks to the extension operator we have that

$$K_{\mathcal{W}^n_{\mathbb{H}}}(q,r) = \frac{1}{2}\left(K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(z) + K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(\bar{z})\right) + \frac{I_q J}{2}\left(K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(\bar{z}) - K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(z)\right).$$

Thus, using the complex case formula we get

$$\begin{aligned} K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(z) + K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(\bar{z}) &= 2(-1)^n n^2 \frac{Re(z^{n-1}\bar{z}^{2n})\bar{r}^{n-1} + Re(z^{n-1})\bar{r}^{3n-1}}{(|z|^{2n} + \bar{r}^{2n} - 2(-1)^n Re(z^n)\bar{r}^n)^2} \\ &\quad -2(-1)^n n^2 \frac{2(-1)^n Re(z^{n-1}\bar{z}^n)\bar{r}^{2n-1}}{(|z|^{2n} + \bar{r}^{2n} - 2(-1)^n Re(z^n)\bar{r}^n)^2}. \end{aligned}$$

and

$$\begin{aligned} K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(\bar{z}) - K^r_{\mathcal{W}^n_{\mathbb{C}_J}}(z) &= 2(-1)^n n^2 \frac{-Im(z^{n-1}\bar{z}^{2n})J\bar{r}^{n-1} - Im(z^{n-1})J\bar{r}^{3n-1}}{(|z|^{2n} + \bar{r}^{2n} - 2(-1)^n Re(z^n)\bar{r}^n)^2} \\ &\quad +2(-1)^n n^2 \frac{2(-1)^n Im(z^{n-1}\bar{z}^n)J\bar{r}^{2n-1}}{(|z|^{2n} + \bar{r}^{2n} - 2(-1)^n Re(z^n)\bar{r}^n)^2}. \end{aligned}$$

Therefore, developing the computations we obtain

$$\begin{aligned} K_{\mathcal{W}^n_{\mathbb{H}}}(q,r) &= (-1)^n n^2 \left(q^{n-1}\bar{q}^{2n}\bar{r}^{n-1} - 2(-1)^n q^{n-1}\bar{q}^n \bar{r}^{2n-1} + q^{n-1}\bar{r}^{3n-1}\right) \\ &\quad \times \left(|q|^{2n} + \bar{r}^{2n} - 2(-1)^n Re(q^n)\bar{r}^n\right)^{-2}. \end{aligned}$$

Hence, we finally get

$$K_{\mathcal{W}^n_{\mathbb{H}}}(q,r) = (-1)^n n^2 q^{n-1}(\bar{q}^{2n} - 2(-1)^n \bar{q}^n \bar{r}^n + \bar{r}^{2n})\bar{r}^{n-1}(|q|^2 - 2(-1)^n Re(q^n)\bar{r}^n + \bar{r}^{2n})^{-2}.$$

This completes the proof. $\square$

Remark 8.1.5. *Observe that for the case $n = 1$ in Theorem 8.1.4 the Bergman kernel function coincide with the result obtained on the quaternionic half space in [43].*

8.2 The Bergman-Fueter transform and consequences

In this section, we study the Bergman-Fueter integral transform on different axially symmetric slice domains U on the quaternions, namely we deal with the unit ball, the half space and the unit half ball. In particular, we obtain some new generating functions and integral representations of the quaternionic regular polynomials $(Q_k)_{k\geq 0}$ obtained in the previous chapter. We give also the sequential characterization of the range of the Fueter mapping on the slice hyperholomorphic Bergman space on the quaternionic unit ball. First, associated to U we recall from [39] the following

Definition 8.2.1 (Bergman-Fueter transform associated to U)**.** *Let $f : U \longrightarrow \mathbb{H}$ be in the slice hyperholomorphic Bergman space of the second kind $\mathcal{A}_{Slice}(U)$. Then, we define the Bergman-Fueter transform of f associated to U to be*

$$\breve{f}(q) := \int_{U\cap\mathbb{C}_I} K^U_{BF}(q,r)f(r)d\sigma(r),$$

where K^U_{BF} is the Bergman-Fueter kernel on U defined through the following formula

$$K^U_{BF}(q,r) := \Delta K_U(q,r),\ \forall(q,r)\in U\times U.$$

The Laplacian Δ is taken with respect to the variable q and $d\sigma(r)$ defines the restriction of the normalized Lebesgue measure on $U_I = U\cap\mathbb{C}_I$.

8.2.1 The quaternionic unit ball case $U=\mathbb{B}$

In [43] an explicit expression of the Fueter-Bergman kernel was obtained when U is the quaternionic unit ball $\mathbb{B}$. More precisely, we have the following result originally proved in [43]:

Theorem 8.2.1. *For all $(q,r)\in\mathbb{B}\times\mathbb{B}$, we have*

$$\begin{aligned} K^{\mathbb{B}}_{BF}(q,r) = -4\left(1-2Re(q)\bar{r}+|q|^2\bar{r}^2\right)^{-2}\bar{r}^2 + 2(1-2\bar{q}\bar{r}+\bar{q}\bar{r}^2) \\ \times\left(1-2Re(q)\bar{r}+|q|^2\bar{r}^2\right)^{-3}\bar{r}^2. \end{aligned} \tag{8.2.1}$$

Furthermore, if we set

$$R(q,r) = \left(1-2Re(q)\bar{r}+|q|^2\bar{r}^2\right)^{-1},$$

then

$$K^{\mathbb{B}}_{BF}(q,r) = -4\left[R(q,r)+2K_{\mathbb{B}}(q,r)\right]R(q,r)\bar{r}^2.$$

We prove the following

Proposition 8.2.2. *Let $(q,r)\in\mathbb{B}\times\mathbb{B}$, we have*

$$K^{\mathbb{B}}_{BF}(q,r) = -2\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)Q_k(q)\bar{r}^{k+2}.$$

Proof. Let $(q,r)\in\mathbb{B}\times\mathbb{B}$, making use of the slice hyperholomorphic extension operator it is clear that the slice Bergman kernel on $\mathbb{B}$ is given by the series expansion

$$K_{\mathbb{B}}(q,r)=\sum_{k=0}^{\infty}(k+1)q^k\overline{r}^k.$$

Therefore, by definition of the Bergman-Fueter kernel we obtain:

$$\begin{aligned}K_{BF}^{\mathbb{B}}(q,r)&=\tau_q K_{\mathbb{B}}(q,r)\\&=-2\sum_{k=2}^{\infty}(k-1)k(k+1)Q_{k-2}(q)\overline{r}^k.\\&=-2\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)Q_k(q)\overline{r}^{k+2}.\end{aligned}$$

□

As a consequence of the latter result, we obtain the following generating function associated to the quaternionic regular polynomials $(Q_k)_{k\geq 0}$:

Theorem 8.2.3. *For all $(q,r)\in\mathbb{B}\times\mathbb{B}$, we have*

$$\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)Q_k(q)\overline{r}^k=2R^2(q,r)+4K_{\mathbb{B}}(q,r)R(q,r);$$

where

$$R(q,r)=\left(1-2Re(q)\overline{r}+|q|^2\overline{r}^2\right)^{-1} \text{ and } K_{\mathbb{B}}(q,r)=(1-2\overline{q}\overline{r}+\overline{q}^2\overline{r}^2)R(q,r)^2.$$

Proof. Note that Theorem 8.2.1 gives

$$K_{BF}^{\mathbb{B}}(q,r)=-4\left[R(q,r)+2K_{\mathbb{B}}(q,r)\right]R(q,r)\overline{r}^2.$$

This result combined with Proposition 8.2.2 leads to

$$\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)Q_k(q)\overline{r}^k=2R^2(q,r)+4K_{\mathbb{B}}(q,r)R(q,r).$$

This completes the proof. □

In particular, we get the following series representation

Corollary 8.2.4. *Let $-1<q<1$ and $r\in\mathbb{B}$. Then, we have*

$$\sum_{k=0}^{\infty}\frac{(k+1)(k+2)(k+3)}{6}q^k\overline{r}^k=(1-q\overline{r})^{-4}.$$

Proof. We only need to observe that if $q \in \mathbb{R}$ then for all $k \geq 0$ we have $Q_k(q) = q^k$ thanks to the identity $\sum_{j=0}^{k} T_j^k = 1$. Moreover, since $-1 < q < 1$ we have

$$R(q,r) = K_{\mathbb{B}}(q,r) = (1 - q\bar{r})^{-2}.$$

Finally, the proof is concluded by making use of Theorem 8.2.3. □

Remark 8.2.5. *As a consequence of Corollary 8.2.4 we observe that for all $s, t > 1$ we have*

$$\sum_{k=0}^{\infty} \frac{(k+1)(k+2)(k+3)}{s^k t^k} = 6\left(\frac{st}{1-st}\right)^4.$$

Note also that using the fact

$$\forall n \geq 0 : \sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3},$$

we have

$$\sum_{j=1}^{\infty} \sum_{k=1}^{j} k(k+1) q^{j-1} \bar{r}^{j-1} = 2(1 - q\bar{r})^{-4}.$$

The right Bergman-Fueter space $\mathcal{B}(\mathbb{B})$ is the range of the slice hyperholomorphic Bergman space through the Fueter mapping. Indeed, it is defined by

$$\mathcal{B}(\mathbb{B}) := \{\tau(f);\ f \in \mathcal{A}_{Slice}(\mathbb{B})\}.$$

Then, the next result gives the sequential characterization of the Bergman-Fueter space $\mathcal{B}(\mathbb{B})$:

Theorem 8.2.6. *Let $g \in \mathcal{R}(\mathbb{B})$. Then, $g \in \mathcal{B}(\mathbb{B})$ if and only if the following conditions are satisfied:*

i) $\forall q \in \mathbb{B},\ g(q) = \displaystyle\sum_{k=0}^{\infty} Q_k(q)\alpha_k$ *where* $(\alpha_k)_{k \geq 0} \subset \mathbb{H}$.

ii) $\displaystyle\sum_{k=0}^{\infty} \frac{|\alpha_k|^2}{(k+1)^2(k+2)^2(k+3)} < \infty.$

Proof. First, note that by the Fueter mapping theorem we have $\mathcal{B}(\mathbb{B}) \subset \mathcal{R}(\mathbb{B})$. Let $g \in \mathcal{B}(\mathbb{B})$, thus $g = \tau(f)$ where $f \in \mathcal{A}_{Slice}(\mathbb{B})$ such that we have

$$f(q) = \sum_{k=0}^{\infty} q^k c_k, \text{ with } (c_k) \subset \mathbb{H} \text{ and } \|f\|^2 = \sum_{k=0}^{\infty} \frac{|c_k|^2}{k+1} < \infty.$$

However,

$$\tau(1) = \tau(q) = 0 \text{ and } \tau(q^k) = -2(k-1)kQ_{k-2}(q), \qquad \forall k \geq 2.$$

Therefore, we get

$$g(q)=\sum_{k=0}^{\infty}Q_k(q)\alpha_k \text{ with } \alpha_k=-2(k+1)(k+2)c_{k+2}, \qquad \forall k\geq 0.$$

Moreover, we have

$$\sum_{k=0}^{\infty}\frac{|\alpha_k|^2}{(k+1)^2(k+2)^2(k+3)}=4\sum_{k=2}^{\infty}\frac{|c_k|^2}{k+1}\leq 4\|f\|^2<\infty.$$

Conversely, let us suppose that the conditions i) and ii) hold. Then, we consider the function

$$h(q)=\sum_{k=2}^{\infty}q^k\beta_k, \text{ where } \beta_k=-\frac{\alpha_{k-2}}{2(k-1)k}, \qquad \forall k\geq 2.$$

Thus, we get $g=\tau(h)$ thanks to the formula

$$Q_k(q)=-\frac{\tau(q^{k+2})}{2(k+1)(k+2)}, \qquad \forall k\geq 0.$$

Moreover, note that we have

$$\|h\|^2=\sum_{k=2}^{\infty}\frac{|\beta_k|^2}{k+1}=\frac{1}{4}\sum_{k=0}^{\infty}\frac{|\alpha_k|^2}{(k+1)^2(k+2)^2(k+3)}<\infty.$$

Hence, $g=\tau(h)$ with $h\in\mathcal{A}_{Slice}(\mathbb{B})$. In particular, it shows that $g\in\mathcal{B}(\mathbb{B})$. This completes the proof. □

Remark 8.2.7. *We observe that*

$$\mathcal{B}(\mathbb{B}):=\{f(q)=\sum_{k=0}^{\infty}Q_k(q)\alpha_k, \forall q\in\mathbb{B},\ \alpha_k\in\mathbb{H};\sum_{k=0}^{\infty}\frac{|\alpha_k|^2}{(k+1)^2(k+2)^2(k+3)}<\infty\}.$$

As we have seen in Section 4 for the Fock case, it is also possible to endow the Fueter-Bergman space $\mathcal{B}(\mathbb{B})$ with the inner product

$$\langle f,g\rangle_{\mathcal{B}(\mathbb{B})}:=\sum_{k=0}^{\infty}\frac{\overline{\alpha_k}\beta_k}{(k+1)^2(k+2)^2(k+3)},$$

for any $f=\sum_{k=0}^{\infty}Q_k\alpha_k$ and $g=\sum_{k=0}^{\infty}Q_k\beta_k$. It is also possible to show that $\mathcal{B}(\mathbb{B})$ is a right quaternionic reproducing kernel Hilbert space whose reproducing kernel function is given by

$$L(q,r):=L_r(q)=\sum_{k=0}^{\infty}(k+1)^2(k+2)^2(k+3)Q_k(q)Q_k(\bar{r}), \forall (q,r)\in\mathbb{B}\times\mathbb{B}.$$

So that, for any $f \in \mathcal{B}(\mathbb{B})$ and $p \in \mathbb{B}$ we have

$$\langle f, L_p \rangle_{\mathcal{B}(\mathbb{B})} = f(p).$$

An integral representation of the polynomials $(Q_k)_{k\geq 0}$ on the quaternionic unit ball $\mathbb{B}$ in terms of the Bergman-Fueter kernel is given in the following:

Proposition 8.2.8. *Let $I \in \mathbb{S}$, $q \in \mathbb{B}$ and $k \geq 0$. Then, we have*

$$Q_k(q) = -\frac{1}{2(k+1)(k+2)} \int_{\mathbb{B}_I} K_{BF}^{\mathbb{B}}(q,r) r^{k+2} d\sigma_I(r).$$

Proof. This follows with direct computations making use of Proposition 8.2.2. □

As a result we get this special identity

Corollary 8.2.9. *For any $-1 < q < 1, I \in \mathbb{S}$ and $k \geq 0$, we have*

$$\int_{\mathbb{B}_I} r^k |r|^4 (1 - q\bar{r})^{-4} d\lambda_I(r) = \frac{\pi}{6}(k+1)(k+2)q^k,$$

where $I \in \mathbb{S}$ and $d\lambda_I$ is the Lebesgue measure on $\mathbb{B}_I$.

Proof. We only need to apply Proposition 8.2.8 combined with the expression of the Bergman-Fueter kernel when $-1 < q < 1$.

□

8.2.2 The Bergman-Fueter transform on $\mathbb{H}^+$ and $\mathbb{B}^+$

The next result gives the explicit expression of the Bergman-Fueter kernel on the quaternionic half space $\mathbb{H}^+$:

Theorem 8.2.10. *For all $(q,r) \in \mathbb{H}^+ \times \mathbb{H}^+$, we have*

$$K_{BF}^{\mathbb{H}^+}(q,r) = -4\left[\left(|q|^2 + 2Re(q)\overline{r} + \overline{r}^2\right)^{-2} + 2(\overline{q}^2 + 2\overline{q}\overline{r} + \overline{r}^2)\left(|q|^2 + 2Re(q)\overline{r} + \overline{r}^2\right)^{-3}\right].$$

Moreover, if we set

$$P(q,r) := \left(|q|^2 + 2Re(q)\overline{r} + \overline{r}^2\right)^{-1},$$

then

$$K_{BF}^{\mathbb{H}^+}(q,r) = -4\left[P(q,r) + 2K_{\mathbb{H}^+}(q,r)\right] P(q,r).$$

Proof. First, note that by Theorem 4.4 in [43] we have

$$K_{\mathbb{H}^+}(q,r) = \frac{1}{\pi}\left(\overline{q}^2 + 2\overline{q}\overline{r} + \overline{r}^2\right)\left(|q|^2 + 2Re(q)\overline{r} + \overline{r}^2\right)^{-2}.$$

However, the Bergman-Fueter kernel is obtained by computing the Laplacian of the slice Bergman kernel with respect to the variable q, so that we have

$$K_{BF}^{\mathbb{H}^+}(q,r) := \Delta K_{\mathbb{H}^+}(q,r), \qquad \forall (q,r) \in \mathbb{H}^+ \times \mathbb{H}^+.$$

Then, direct computations using the formula of $K_{\mathbb{H}^+}(q,r)$ show that

$$\begin{aligned}\frac{d^2}{dx_0^2} K_{\mathbb{H}^+}(q,r) &= 2(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-2} - 16(\bar{q}+\bar{r})(x_0+\bar{r})(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-3} \\ &\quad - 4(\bar{q}^2 + 2\bar{q}\bar{r} + \bar{r}^2)(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-3} + 24(\bar{q}^2 + 2\bar{q}\bar{r} + \bar{r}^2)(x_0+\bar{r})^2 \\ &\quad \times (|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-4}\end{aligned}$$

and also

$$\begin{aligned}\frac{d^2}{dx_1^2} K_{\mathbb{H}^+}(q,r) &= -2(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-2} + 8x_1(\bar{q}i + i\bar{q} + 2i\bar{r})(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-3} \\ &\quad - 4(\bar{q}^2 + 2\bar{q}\bar{r} + \bar{r}^2)(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-3} + 24x_1^2(\bar{q}^2 + 2\bar{q}\bar{r} + \bar{r}^2) \\ &\quad \times (|q|^2 + 2Re(q)\bar{r} + \bar{r}^2)^{-4}.\end{aligned}$$

Similarly we calculate $\dfrac{d^2}{dx_2^2} K_{\mathbb{H}^+}(q,r)$ and $\dfrac{d^2}{dx_3^2} K_{\mathbb{H}^+}(q,r)$. Then, with some computations, we get

$$K_{BF}^{\mathbb{H}^+}(q,r) = -\frac{4}{\pi}\left[\left(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2\right)^{-2} + 2(\bar{q}^2 + 2\bar{q}\bar{r} + \bar{r}^2)\left(|q|^2 + 2Re(q)\bar{r} + \bar{r}^2\right)^{-3}\right].$$

Finally, by replacing the function $P(q,r)$ in the previous formula we obtain

$$K_{BF}^{\mathbb{H}^+}(q,r) = -4\left[P(q,r) + 2K_{\mathbb{H}^+}(q,r)\right]P(q,r).$$

□

Proposition 8.2.11. *The Bergman-Fueter kernel $K_{BF}^{\mathbb{H}^+}(q,r)$ is Fueter regular in q and slice anti-regular in r on $\mathbb{H}^+$.*

Proof. Note that on the one hand the Fueter mapping theorem implies that $K_{BF}^{\mathbb{H}^+}(q,r)$ is Fueter regular in q since $K_{\mathbb{H}^+}$ is slice regular in q. On the other hand, the function $P^{-1}(q,r)$ is an anti-slice regular function with real coefficients with respect to r and so is the function $P(q,r)$. Finally, the result follows since $K_{\mathbb{H}^+}$ is also anti-slice regular in r. □

Concerning the Fueter-Bergman kernel of the quaternionic half unit ball $\mathbb{B}^+$ we have the following:

Theorem 8.2.12. *For all $(q,r) \in \mathbb{B}^+ \times \mathbb{B}^+$, the following formula holds*

$$K_{BF}^{\mathbb{B}^+}(q,r) = K_{BF}^{\mathbb{B}}(q,r) + K_{BF}^{\mathbb{H}^+}(q,r).$$

Furthermore, the Bergman-Fueter kernel $K_{BF}^{\mathbb{B}^+}$ is Fueter regular in q and slice anti-regular in r on $\mathbb{B}^+$.

Proof. For the first statement, we only need to use the result obtained in Theorem 8.1.2 combined with the definition of the Fueter-Bergman kernel. Then, since $\mathbb{B}^+$ is contained in both of $\mathbb{B}$ and $\mathbb{H}^+$, we have that $K_{BF}^{\mathbb{B}^+}(q,r)$ is Fueter regular in q and slice anti-regular in r as the sum of $K_{BF}^{\mathbb{B}}(q,r)$ and $K_{BF}^{\mathbb{H}^+}(q,r)$. □

CHAPTER 9

Fock and Hardy spaces: the Clifford-Appell case

In this chapter, we study the Clifford-Appell polynomials and in particular their CK product. Moreover, we introduce a new family of quaternionic reproducing kernel Hilbert spaces in the framework of Fueter regular functions. The construction is based on a general idea which allows to obtain various function spaces, by specifying a suitable sequence of real numbers. We focus on the Fock and Hardy cases in this setting, and we study the action of the Fueter mapping and its range. The results presented in this chapter are based on [8]

9.1 Motivation

As we have already seen before, we recall that a set of polynomials $\{P_n\}_{n\in\mathbb{N}}$ satisfying an identity with respect to the real derivative that takes P_n to nP_{n-1} is called an Appell system [20]. The importance of such systems in various settings is well known, and we mention here, with no pretense of completeness their relevance in probability theory and stochastic process since they can be connected to random variables. In hypercomplex analysis, we have various function theories, associated with different differential operators. We will treat the quaternionic case in this dissertation. Indeed, in the slice hyerholomorphic setting, Appell systems can be obtained by simply extending the variable in use to become hypercomplex, and so we have that, for example, the standard monomials in the quaternionic variable define an Appell system with respect to the slice derivative. But these sets of polynomials were studied also in the setting of quaternionic and Clifford analysis with respect to the hypercomplex derivative,

see [29, 30, 63, 93, 99]. It turns out that these Clifford-Appell systems play a similar role as the complex monomials do to define elementary functions in terms of their power series like cosine, sine, exponential, etc. This fact opens a variety of questions also in relation to various function spaces including Fock, Hardy, Bergman, Dirichlet spaces, etc. Moreover, various questions arise about their associated operators such as creation, annihilation, shift and backward shift operators. In addition to that, what makes Appell systems in quaternionic and Clifford analysis rather peculiar, is the fact that the function theory has been developed using the so-called Fueter polynomials, see [28], [83], and these polynomials do not satisfy the Appell property in general. However, a series expansion for hyperholomorphic functions is possible using both the approaches.
In order to define and study quaternionic reproducing kernel Hilbert spaces, the approach that makes use of the Appell systems looks very promising and allows to define the associated operators. We will show that using a special set of Clifford Appell polynomials, denoted by $\{Q_n\}_{n\geq 0}$, we can introduce various functions spaces denoted by $\mathcal{HM}_b$ whose elements are converging series of the form $\sum Q_n a_n$, where the quaternionic coefficients a_n satisfy suitable conditions which depend on a given sequence $b = (b_n)_{n\geq 0}$ of real (in fact rational) numbers. This approach is rather general, and it is used also in the slice hyperholomorphic setting in which the series under consideration are of the form $\sum q^n a_n$, where q denotes the quaternionic variable and give rise to spaces denoted by $\mathcal{HS}_c$, $c = (c_n)_{n\geq 0}$.
We treat the case of the quaternionic Fock and the Hardy spaces which have been already studied in the slice setting but are new in the Fueter regular framework combined with the Appell polynomials. For this reason, these spaces are called Clifford-Appell Fock space and Clifford-Appell Hardy space, respectively. One problem of the system $\{Q_n\}_{n\geq 0}$ is that if we multiply two such polynomials we do no obtain an element in the system. This is expected provided the non-commutative setting and in fact hyperholomorphic functions can be multiplied using the so-called CK product. With the polynomials Q_n there is the additional problem of remaining within the Appell system and in fact we show how this can be achieved. This technical result opens the possibility to prove several results and also to introduce creation, annihilation and shift operators.
An advantage of our description is that we can prove that the function spaces $\mathcal{HM}_b$ and $\mathcal{HS}_c$ for suitable choices of b, c, can be related using the Fueter mapping theorem.
The structure of the chapter is the following: we first revise some notations and preliminary results that we need in the sequel. Then, we introduce some quaternionic reproducing kernel Hilbert spaces (QRKHS) based on a specific Appell system, and prove different properties on such kind of polynomials. We show also that, under suitable conditions, any axially Fueter regular function can be expanded in terms of these Appell polynomials. We will focus more on the Fock space in this setting. In particular, we study different properties related to the notions of creation, annihilation operators and Segal-Bargmann transforms.

Then, we move to treat the Hardy space case, and study different properties related to the shift and backward shift operators. Finally, we show how the Fueter mapping acts by sending spaces of slice hyperholomorphic functions into spaces of Fueter regular functions. Moreover, we show that in some special cases the Fueter mapping acts as an isometric isomorphism up to a constant.

9.2 Notations

First we recall some basic facts about Cauchy-Fueter regular functions, Fueter variables and their CK product.

We note that the quaternionic monomials $P_n(q) = q^n$ are not Fueter regular. However, there exist some other important functions in this theory, the so-called Fueter variables, defined by

$$\zeta_l(x) = x_l - e_l x_0, \ l = 1, 2, 3. \tag{9.2.1}$$

These functions play the same role that complex monomials play in complex analysis. For example, a series expansion for Fueter regular functions is obtained using these Fueter variables. A suitable product that allows to preserve the regularity in this setting is the so-called C-K product, denoted $\odot$. Given two Fueter regular functions f and g, we take their restriction to $x_0 = 0$ and consider their pointwise multiplication. Then, we take the Cauchy-Kowalevskaya extension of this pointwise product, which exists and is unique, to define $f \odot g$, see [83].

We recall also the slice hyperholomorphic quaternionic Fock space $\mathcal{F}_{Slice}(\mathbb{H})$ (see chapter 4), defined for a given $I \in \mathbb{S}$ to be

$$\mathcal{F}_{Slice}(\mathbb{H}) := \left\{ f \in \mathcal{SR}(\mathbb{H});\ \frac{1}{\pi}\int_{\mathbb{C}_I} |f_I(p)|^2 e^{-|p|^2} d\lambda_I(p) < \infty \right\},$$

where $f_I = f|_{\mathbb{C}_I}$ and $d\lambda_I(p) = dxdy$ for $p = x + yI$. This quaternionic Fock space can be characterized in terms of the slice hyperholomorphic power series as follows

$$\mathcal{F}_{Slice}(\mathbb{H}) = \left\{ \sum_{k=0}^{\infty} q^k a_k;\ a_k \in \mathbb{H} : \sum_{k=0}^{\infty} k!|a_k|^2 < \infty \right\}.$$

Its associated Segal-Bargmann transform was studied in [60] by considering the slice hyperholomorphic kernel obtained making use of the normalized Hermite functions $(\eta_n)_{n\geq 0}$. The explicit expression of this kernel is given by

$$\mathcal{A}^S_{\mathbb{H}}(q,x) := \sum_{k=0}^{\infty} \frac{q^k}{\sqrt{k!}}\eta_k(x) = e^{-\frac{1}{2}(q^2+x^2)+\sqrt{2}qx},\ \forall (q,x) \in \mathbb{H}\times\mathbb{R}. \tag{9.2.2}$$

Then, for any quaternionic valued function φ in $L^2(\mathbb{R},\mathbb{H})$ the slice hyperholomorphic Segal-Bargmann transform is defined by

$$\mathcal{B}^S_{\mathbb{H}}(\varphi)(q) = \int_{\mathbb{R}} \mathcal{A}^S_{\mathbb{H}}(q,x)\varphi(x)dx. \tag{9.2.3}$$

In the same spirit different famous spaces of slice hyperholomorphic functions such as Hardy, Besov, Bloch, Dirichlet and Bergman spaces were studied in [13, 43, 113].

9.3 A new family of hyperholomorphic QRKHS: General setting

Let us consider the quaternionic polynomials defined by

$$Q_k(q) = \sum_{j=0}^{k} T_j^k q^{k-j} \overline{q}^j, q \in \mathbb{H}, \ k \geq 0 \tag{9.3.1}$$

where

$$T_j^k := \frac{k!}{(3)_k} \frac{(2)_{k-j}(1)_j}{(k-j)!j!} = \frac{2(k-j+1)}{(k+1)(k+2)} \tag{9.3.2}$$

and $(a)_n = a(a+1)...(a+n-1)$ is the Pochhammer symbol.

Remark 9.3.1. *Notice that the polynomials* $(Q_k)_{k\geq 0}$ *given by* (9.3.1) *are Fueter regular on* $\mathbb{H}$*. Moreover, they form an Appell system with respect to the hypercomplex derivative* $\frac{\overline{\partial}}{2}$*. i.e, for all* $k \geq 1$ *we have the Appell property*

$$\frac{\overline{\partial}}{2} Q_k = k Q_{k-1}. \tag{9.3.3}$$

For $s \in \mathbb{H}$, let

$$\text{Exp}(s) := \sum_{k=0}^{\infty} \frac{Q_k(s)}{k!} \tag{9.3.4}$$

be the generalized Fueter regular exponential function considered in the paper [29]. Then, we introduce the following

Definition 9.3.1. *Let* Ω *be a domain in* $\mathbb{H}$*. Let* $c = (c_k)_{k\in\mathbb{N}}$ *and* $b = (b_k)_{k\in\mathbb{N}}$ *be two non decreasing sequences with* $c_0 = b_0 = 1$*. Then, associated to* b *and* c *we define*

1. *The subspace of Fueter regular functions defined by*

$$\mathcal{HM}_b(\Omega) = \left\{ \sum_{k=0}^{\infty} Q_k \alpha_k; \ \alpha_k \in \mathbb{H} : \sum_{k=0}^{\infty} b_k |\alpha_k|^2 < \infty \right\}.$$

2. *The subspace of slice hyperholomorphic functions defined by*

$$\mathcal{HS}_c(\Omega) = \left\{ \sum_{k=0}^{\infty} q^k f_k; \ f_k \in \mathbb{H} : \sum_{k=0}^{\infty} c_k |f_k|^2 < \infty \right\}.$$

Given $f = \sum_{k=0}^{\infty} Q_k \alpha_k$ and $g = \sum_{k=0}^{\infty} Q_k \beta_k$ in $\mathcal{HM}_b(\Omega)$ we define the Hermitian inner product given by

$$\langle f, g \rangle_{\mathcal{H}_b} = \sum_{k=0}^{\infty} b_k \overline{\alpha_k} \beta_k.$$

Remark 9.3.2. *We note that, by specifying the sequence c, $\mathcal{HS}_c$ include different spaces of slice hyperholomorphic functions such as Fock, Hardy, Dirichlet and generalized Fock spaces. Such spaces are the quaternionic counterpart of the complex version introduced in [14].*

We are interested in two main problems in this setting:

Problem 9.3.3. *Study the counterparts of the spaces introduced in Definition 9.3.1 by suitably chosing the sequence b in order to include in this framework of Cauchy-Fueter regularity : Fock, Bergman, Hardy, Dirichlet spaces, etc.*

In this paper, we will treat the Fock and Hardy cases that correspond, respectively, to the sequences $b_k = k!$ and $b_k = 1, \forall k \geq 0$.

Problem 9.3.4. *Study the range of the Fueter mapping on $\mathcal{HS}_c$ and see when it is possible to obtain spaces of regular functions of the form $\mathcal{HM}_b$. More in general, we ask if using the Fueter mapping it is possible to get information on the sequence (b_k) in terms of the given datum (c_k) ?*

Remark 9.3.5. *We note that the answer to Problem 9.3.4 for Fock and Bergman cases were considered in [63]. See also [15,43] for the slice hyperholomorphic setting. The answer in these two cases is given by:*

1. *The Fock case:*

$$c_k = k! \text{ and } b_k = \frac{k!}{(k+1)(k+2)}, \ \forall k \geq 0.$$

2. *The Bergman case:*

$$c_k = \frac{1}{k+1} \text{ and } b_k = \frac{1}{(k+1)^2(k+2)^2(k+3)}, \ \forall k \geq 0.$$

We will show that, under suitable conditions, for some special choices of the sequence b in Definition 9.3.1 we have the estimate:

$$|f(q)| \leq \left(\sum_{k=0}^{\infty} \frac{|q|^{2k}}{b_k} \right)^{\frac{1}{2}} \|f\|_{\mathcal{HM}_b}, \ f \in \mathcal{HM}_b(\Omega), \ q \in \Omega. \tag{9.3.5}$$

In these cases, we can also prove that $\mathcal{HM}_b(\Omega)$ are right quaternionic reproducing kernel Hilbert spaces with reproducing kernel given by

$$K_{\mathcal{HM}_b(\Omega)}(q,p)=\sum_{k=0}^{\infty}\frac{Q_k(q)\overline{Q_k(p)}}{b_k},\ \forall(q,p)\in\Omega\times\Omega. \tag{9.3.6}$$

Furthermore, in such situations $\left\{\frac{Q_k}{\sqrt{b_k}}\right\}_{k\geq 0}$ form an orthonormal basis of $\mathcal{HM}_b(\Omega)$.

Now, we will prove an interesting result on the Appell polynomials $(Q_k)_{k\geq 0}$ useful to compute their C-K product.

Proposition 9.3.6. *Let $k,s\geq 0$. Then, for any $q=x_0+\vec{x}\in\mathbb{H}$ we have*

$$Q_k\odot Q_s(q)=\frac{c_kc_s}{c_{k+s}}Q_{k+s}(q),$$

where $\odot$ is the C-K product and $c_l:=\sum_{j=0}^{l}(-1)^jT_j^l,\ \forall l\geq 0.$

Proof. Since Q_k and Q_s are Fueter regular functions on $\mathbb{H}$, their C-K product $Q_k\odot Q_s$ is also Fueter regular. Then, we use the formula of the C-K extension, see [28], given by

$$CK[h(\vec{q})](q)=\exp\left(-x_0\partial_{\vec{q}}\right)[h(\vec{q})](q).$$

We write the explicit series expression using the fact that $Q_l(\vec{q})=c_l\vec{q}^{\,l}$ for all $l\geq 0$ and obtain

$$\begin{aligned}Q_k\odot Q_s(q)&=\sum_{j=0}^{\infty}\frac{(-1)^jx_0^j}{j!}\partial_{\vec{q}}^j\left(Q_k(\vec{q})Q_s(\vec{q})\right)\\&=c_kc_s\sum_{j=0}^{\infty}\frac{(-1)^jx_0^j}{j!}\partial_{\vec{q}}^j\left(\vec{q}^{\,k+s}\right).\end{aligned}$$

In particular, we get

$$Q_k\odot Q_s(q)=c_kc_sCK\left(\vec{q}^{\,k+s}\right)(q),\ q\in\mathbb{H},k,s\geq 0, \tag{9.3.7}$$

with $c_l:=\sum_{j=0}^{l}(-1)^jT_j^l,\ \forall l\geq 0$. On the other hand, we observe that Q_{k+s} is also Fueter regular on $\mathbb{H}$. Moreover, it is restriction to $x_0=0$ gives

$$Q_{k+s}(\vec{q})=c_{k+s}\vec{q}^{\,k+s}.$$

Therefore, by uniqueness of the C-K extension we get

$$Q_{k+s}(q)=c_{k+s}CK\left(\vec{q}^{\,k+s}\right)(q),\ \forall q\in\mathbb{H}. \tag{9.3.8}$$

Hence, we combine (9.3.7) and (9.3.8) to conclude that

$$Q_k \odot Q_s(q) = \frac{c_k c_s}{c_{k+s}} Q_{k+s}(q), \forall q \in \mathbb{H}, \forall k, s \geq 0.$$

□

Remark 9.3.7. *If we consider the Fueter regular polynomials given by* $P_k = \dfrac{Q_k}{c_k}$, $\forall k \geq 0$. *Then, the classical multiplication rule holds, in the sens that we have*

$$P_k \odot P_s = P_{k+s}, \ \forall k, s \geq 0. \tag{9.3.9}$$

Corollary 9.3.8. *Let* $k, s \geq 0$. *Then, for any* $q = x_0 + \vec{q} \in \mathbb{H}$ *we have*

$$Q_k \odot Q_s(q) = c_k c_s \lambda_0^{k+s} r^{k+s} \left(C^1_{k+s}(\frac{x_0}{r}) + \frac{2}{k+s+2} C^2_{k+s-1}(\frac{x_0}{r}) \frac{\vec{q}}{r} \right),$$

where C^ν_t *are the Gegenbauer polynomials,* λ_0 *is a constant and* $r^2 = |q|^2$.

Proof. Proposition 9.3.6 gives

$$Q_k \odot Q_s(\vec{q}) = c_k c_s \vec{q}^{k+s}, \ k, s \geq 0,$$

thus, by the regularity of the C-K product $Q_k \odot Q_s$ and uniqueness of the C-K extension we have that

$$Q_k \odot Q_s(q) = c_k c_s CK[\vec{q}^{k+s}], \ q \in \mathbb{H}, k, s \geq 0.$$

Hence, the result follows as a direct application of Theorem 2.2.1 in [58] that gives the expression of the C-K extension for the vector part powers in terms of Gegenbauer polynomials. □

Remark 9.3.9. *We note that the Appell polynomials given by* (9.3.1) *define a family of Fueter regular functions of axial type (or axially Fueter regular functions), in the sense that if we write* $q = x_0 + \omega|\vec{q}| \in \Omega$ *with* $\omega \in \mathbb{S}$ *there exist two quaternionic valued functions* $A = A(x_0, |\vec{q}|)$ *and* $B = B(x_0, |\vec{q}|)$ *independent of* ω *such that we have*

$$Q_k(q) = A(x_0, |\vec{q}|) + \omega B(x_0, |\vec{q}|), \ \forall k \geq 0. \tag{9.3.10}$$

We end this section by proving a converse result of the previous remark. This allows to characterize axially Fueter regular functions on quaternionic axially symmetric slice domains in terms of the Appell system $(Q_k)_{k\geq 0}$.

Theorem 9.3.10. *Let* $\Omega \subseteq \mathbb{H}$ *be an axially symmetric slice domain. Let* g *be an axially Fueter regular function on* Ω. *Then, there exist some quaternion coefficients* $(\alpha_k)_{k\geq 0}$ *such that we have the expansion*

$$g(q) = \sum_{k=0}^{\infty} Q_k(q)\alpha_k, \ \forall q \in \Omega. \tag{9.3.11}$$

Proof. We note that g is an axially Fueter regular function on Ω. Thus, by the inverse Fueter mapping theorem proved in [49] there will exist $f \in \mathcal{SR}(\Omega)$ such that we have

$$g = \tau(f), \tag{9.3.12}$$

where $\tau = \Delta_{\mathbb{R}^4}$ is the Fueter mapping. Then, using the series expansion theorem for slice hyperholomorphic functions there exist some quaternion coefficients $(a_k)_{k\geq 0}$ so that we can write

$$f(q) = \sum_{k=0}^{\infty} q^k a_k, \ \forall q \in \Omega. \tag{9.3.13}$$

In particular, we apply the Fueter mapping τ on (9.3.13) and get

$$\tau(f)(q) = \sum_{k=0}^{\infty} \tau(q^k) a_k.$$

However, we know by [63] that

$$\tau(q^k) = -2(k-1)kQ_{k-2}, \forall k \geq 2.$$

Therefore, we continue the calculations and obtain

$$\tau(f) = \sum_{k=0}^{\infty} Q_k \alpha_k, \tag{9.3.14}$$

where we have set $\alpha_k = -2(k+1)(k+2)a_{k+2}, \ \forall k \geq 0$. Hence, comparing (9.3.12) with (9.3.14) we conclude that

$$g(q) = \sum_{k=0}^{\infty} Q_k(q)\alpha_k, \ \forall q \in \Omega.$$

This ends the proof. □

9.4 The Fock space case

In this section, we consider the Clifford-Appell Fock space defined by

$$\mathcal{F}(\mathbb{H}) := \left\{ \sum_{k=0}^{\infty} Q_k \alpha_k; \ \alpha_k \in \mathbb{H} \ : \ \sum_{k=0}^{\infty} k! |\alpha_k|^2 < \infty \right\}.$$

This space corresponds to the space $\mathcal{HM}_b$ in Definition 9.3.1 associated with the sequence $b = k!$, $k \geq 0$ on the domain $\Omega = \mathbb{H}$. Let $f = \sum_{k=0}^{\infty} Q_k \alpha_k$ and $g =$

$\sum_{k=0}^{\infty} Q_k\beta_k$ in $\mathcal{F}(\mathbb{H})$ we can equip $\mathcal{F}(\mathbb{H})$ with the scalar product

$$\langle f, g\rangle_{\mathcal{F}(\mathbb{H})} = \sum_{k=0}^{\infty} k!\overline{\alpha_k}\beta_k.$$

Then, we can see that all the evaluation mappings on $\mathcal{F}(\mathbb{H})$ are continuous. Indeed, we prove the following estimate

Proposition 9.4.1. *For any $f \in \mathcal{F}(\mathbb{H})$ and $q \in \mathbb{H}$, we have*

$$|f(q)| \leq e^{\frac{|q|^2}{2}} \|f\|_{\mathcal{F}(\mathbb{H})}. \tag{9.4.1}$$

Proof. We write $f(q) = \sum_{k=0}^{\infty} Q_k(q)\alpha_k$. Thus, we have

$$|f(q)| \leq \sum_{k=0}^{\infty} \frac{|Q_k(q)|}{\sqrt{k!}} |\alpha_k| \sqrt{k!}.$$

Then, by the Cauchy-Schwarz inequality we obtain

$$|f(q)| \leq \left(\sum_{k=0}^{\infty} \frac{|Q_k(q)|^2}{k!} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} k! |\alpha_k|^2 \right)^{\frac{1}{2}}$$

However, we know that $|Q_k(q)| \leq |q|^k$ for all $q \in \mathbb{H}$. Hence, we get

$$|f(q)| \leq e^{\frac{|q|^2}{2}} \|f\|_{\mathcal{F}(\mathbb{H})}.$$

□

As a consequence, we have the following result

Theorem 9.4.2. *The set $\mathcal{F}(\mathbb{H})$ is a right quaternionic Hilbert space of Cauchy-Fueter regular functions whose reproducing kernel is given by*

$$K_{\mathcal{F}(\mathbb{H})}(q, p) = \sum_{k=0}^{\infty} \frac{Q_k(q)\overline{Q_k(p)}}{k!}, \ \forall (q, p) \in \mathbb{H} \times \mathbb{H}.$$

Moreover, if we set $\psi_k(q) = \dfrac{Q_k(q)}{\sqrt{k!}}$, $k \geq 0$, then, the family $\{\psi_k\}_{k\geq 0}$ form an orthonormal basis of $\mathcal{F}(\mathbb{H})$.

Proof. For a fixed $p \in \mathbb{H}$, we consider the function defined by

$$K_p(q) = \sum_{k=0}^{\infty} Q_k(q)\beta_k(p), \ \forall q \in \mathbb{H}, \text{ where } \beta_k(p) = \frac{\overline{Q_k(p)}}{k!}.$$

We observe that

$$\sum_{k=0}^{\infty} k!|\beta_k(p)|^2 = \sum_{k=0}^{\infty} \frac{|Q_k(p)|^2}{k!} \leq e^{|q|^2} < \infty.$$

So, the function K_p belongs to $\mathcal{F}(\mathbb{H})$ for all $p \in \mathbb{H}$. Now, let $f = \sum_{k=0}^{\infty} Q_k\alpha_k$ be any function in $\mathcal{F}(\mathbb{H})$. Then

$$\langle K_p, f \rangle_{\mathcal{F}(\mathbb{H})} = \sum_{k=0}^{\infty} k!\overline{\beta_k(p)}\alpha_k = \sum_{k=0}^{\infty} Q_k(p)\alpha_k = f(p), \ \forall p \in \mathbb{H},$$

therefore, the reproducing kernel of the space $\mathcal{F}(\mathbb{H})$ is given by

$$K_{\mathcal{F}(\mathbb{H})}(q,p) = \sum_{k=0}^{\infty} \frac{Q_k(q)\overline{Q_k(p)}}{k!}, \ \forall (q,p) \in \mathbb{H} \times \mathbb{H}.$$

It is clear by definition of the scalar product that

$$\langle \psi_k, \psi_j \rangle_{\mathcal{F}(\mathbb{H})} = \delta_{k,j}, \ \forall k, j \in \mathbb{N}.$$

Furthermore, let $f = \sum_{k=0}^{\infty} Q_k\alpha_k$ in $\mathcal{F}(\mathbb{H})$ be such that

$$\langle \psi_k, f \rangle_{\mathcal{F}(\mathbb{H})} = 0, \ \forall k \in \mathbb{N}.$$

We have

$$\sqrt{k!}\alpha_k = \langle \psi_k, f \rangle_{\mathcal{F}(\mathbb{H})} = 0, \ \forall k \in \mathbb{N},$$

so, $f = 0$ for any $q \in \mathbb{H}$. In particular, this proves that $\{\psi_k\}_{k \geq 0}$ form an orthonormal basis of $\mathcal{F}(\mathbb{H})$. □

Remark 9.4.3. *We note that*

i) $K_{\mathcal{F}(\mathbb{H})}(\vec{q}, \vec{p}) = \sum_{k=0}^{\infty} (-1)^k \frac{c_k^2}{k!} \vec{q}^{\,k} \vec{p}^{\,k}, \ \forall (q,p) \in \mathbb{H}_0 \times \mathbb{H}_0.$

ii) $K_{\mathcal{F}(\mathbb{H})}(x,y) = e^{xy}, \ \forall (x,y) \in \mathbb{R} \times \mathbb{R}.$

Now we turn our attention to the notion of creation operator associated with the Clifford-Appell Fock space $\mathcal{F}(\mathbb{H})$. For this, we consider a sequence of real numbers $\gamma = (\gamma_k)_{k\geq 0}$ that allows to define a weighted shift operator by

$$T_\gamma(Q_k) := \gamma_k Q_{k+1},\ \forall k \geq 0. \tag{9.4.2}$$

We would like to preserve in this setting the main properties of adjoint and commutation rules satisfied by the standard creation and annihilation operators on the Fock space. First, we deal with the following

Proposition 9.4.4. *Let γ be a sequence with $\gamma_0 = 1$ and such that* (9.4.2) *is well defined. Then, we have*

$$\left[\frac{\overline{\partial}}{2}T_\gamma, T_\gamma\frac{\overline{\partial}}{2}\right] = \mathcal{I}_{\mathcal{F}(\mathbb{H})},$$

if and only if

$$\gamma_k = \frac{1+k\gamma_{k-1}}{1+k},\ \forall k \geq 1.$$

Proof. Let $f = \displaystyle\sum_{k=0}^{\infty} Q_k\alpha_k$ be a function in $\mathcal{F}(\mathbb{H})$. Then, we have

$$T_\gamma(f) = \sum_{k=0}^{\infty} \gamma_k Q_{k+1}\alpha_k \text{ and } \frac{\overline{\partial}}{2}(f) = \sum_{k=1}^{\infty} kQ_{k-1}\alpha_k.$$

Thus, we obtain

$$\frac{\overline{\partial}}{2}T_\gamma(f) = \sum_{k=0}^{\infty}(k+1)\gamma_k Q_k\alpha_k \text{ and } T_\gamma\frac{\overline{\partial}}{2}(f) = \sum_{k=1}^{\infty} k\gamma_{k-1}Q_k\alpha_k.$$

Therefore, it follows that

$$\left[\frac{\overline{\partial}}{2}T_\gamma, T_\gamma\frac{\overline{\partial}}{2}\right](f) = \gamma_0 Q_0\alpha_0 + \sum_{k=1}^{\infty}[(k+1)\gamma_k - k\gamma_{k-1}]Q_k\alpha_k \tag{9.4.3}$$

We can see that if

$$\gamma_k = \frac{1+k\gamma_{k-1}}{1+k},\ \forall k \geq 1,$$

we have then

$$(k+1)\gamma_k - k\gamma_{k-1} = 1,\ \forall k \geq 1.$$

Therefore, using the condition $\gamma_0 = 1$ and formula (9.4.3) we obtain

$$\left[\frac{\overline{\partial}}{2}T_\gamma, T_\gamma\frac{\overline{\partial}}{2}\right](f) = Q_0\alpha_0 + \sum_{k=1}^{\infty} Q_k\alpha_k = f.$$

For the converse, if we assume that

$$\left[\frac{\overline{\partial}}{2}T_\gamma, T_\gamma\frac{\overline{\partial}}{2}\right](f) = f,$$

we apply (9.4.3) and get

$$\gamma_0 Q_0(q)\alpha_0 + \sum_{k=1}^{\infty}[(k+1)\gamma_k - k\gamma_{k-1}]Q_k(q)\alpha_k = \sum_{k=0}^{\infty} Q_k(q)\alpha_k, \ \forall q \in \mathbb{H}.$$

In particular, using the fact that $Q_k(t) = t^k, \forall t \in \mathbb{R}$ and $\gamma_0 = 1$ we observe that

$$\alpha_0 + \sum_{k=1}^{\infty}[(k+1)\gamma_k - k\gamma_{k-1}]t^k\alpha_k = \sum_{k=0}^{\infty} t^k\alpha_k, \ \forall t \in \mathbb{R}.$$

Therefore, comparing the coefficients of the same degree we obtain

$$(k+1)\gamma_k - k\gamma_{k-1} = 1, \ \forall k \geq 1.$$

Hence, we have the condition

$$\gamma_k = \frac{1 + k\gamma_{k-1}}{1+k}, \ \forall k \geq 1.$$

□

Furthermore, we can prove the following

Proposition 9.4.5. *Let γ be a sequence with $\gamma_0 = 1$ and such that* (9.4.2) *holds. If one of the following properties is satisfied*

i) $\left[\frac{\overline{\partial}}{2}T_\gamma, T_\gamma\frac{\overline{\partial}}{2}\right] = \mathcal{I}_{\mathcal{F}(\mathbb{H})}$;

ii) T_γ *is the adjoint of the hypercomplex derivative* $\frac{\overline{\partial}}{2}$;

then, we have

$$\gamma_k = 1, \ \forall k \geq 0.$$

Proof. We observe that condition i) and Proposition 9.4.5 show that

$$\gamma_k = \frac{1 + k\gamma_{k-1}}{1+k}, \ \forall k \geq 1.$$

Thus, since $\gamma_0 = 1$ a simple induction reasoning allows to prove that if i) holds then $\gamma_k = 1$, for all $k \geq 1$. On the other hand, the condition ii) implies in particular that we have

$$\left\langle \frac{\overline{\partial}}{2}(Q_k), Q_j \right\rangle_{\mathcal{F}(\mathbb{H})} = \langle Q_k, T_\gamma(Q_j)\rangle_{\mathcal{F}(\mathbb{H})}, \ \forall k, j \geq 1.$$

So, we conclude

$$k(k-1)!\delta_{k-1,j} = \gamma_j k!\delta_{k,j+1},\ \forall k, j \geq 1,$$

where $\delta_{m,n}$ is the Kronecker symbol. In particular, this leads to the same conclusion that $\gamma_j = 1$, $j \geq 1$. $\square$

Remark 9.4.6. *We note that thanks to Proposition 9.4.5 the only operator T_γ that can play the role of the creation operator with respect to the Clifford-Appell system should act as follows*

$$T_\gamma(Q_k) = Q_{k+1},\ \forall k \geq 0. \tag{9.4.4}$$

We now introduce the notion of creation operator associated with the quaternionic Hilbert space $\mathcal{HM}_b$ in terms of the C-K product that allows to have the property (9.4.4) . To this end, let $k \geq 0$, and we define first the family of operators given by

$$\mathcal{S}_k(f) := \frac{c_{1+k}}{c_1 c_k} Q_1 \odot f,\ \forall f \in \mathcal{HM}_b \tag{9.4.5}$$

where $\odot$ denote the C-K product and $c_l := \displaystyle\sum_{j=0}^{l} (-1)^j T_j^l,\ \forall l \geq 0.$

Then, for $f = \displaystyle\sum_{k=0}^{\infty} Q_k \alpha_k$ in $\mathcal{HM}_b$ we consider the operator $\mathcal{S}$ defined by applying $\mathcal{S}_k$ on each component with the corresponding degree as follows

$$\mathcal{S}(f) := \sum_{k=0}^{\infty} \mathcal{S}_k(Q_k)\alpha_k. \tag{9.4.6}$$

Therefore, we have the explicit expression given by

$$\mathcal{S}(f) := \frac{1}{c_1} \sum_{k=0}^{\infty} \frac{c_{1+k}}{c_k} [Q_1 \odot Q_k]\alpha_k. \tag{9.4.7}$$

We note that the operator $\mathcal{S}$ acts like the classical shift operator with respect to the Clifford-Appell system $(Q_k)_{k\geq 0}$. This can be seen in the following

Proposition 9.4.7. *For all $k \geq 0$, we have*

$$\mathcal{S}(Q_k)(q) = Q_{k+1}(q),\ \forall q \in \mathbb{H}.$$

Proof. Let $k \geq 0$. Then, for all $q \in \mathbb{H}$ we have

$$\begin{aligned} \mathcal{S}(Q_k)(q) &= \mathcal{S}_k(Q_k)(q) \\ &= \frac{c_{1+k}}{c_1 c_k} Q_1 \odot Q_k(q). \end{aligned}$$

Now, we apply Proposition 9.3.6 and get

$$Q_1 \odot Q_k = \frac{c_1 c_k}{c_{1+k}} Q_{k+1}.$$

Hence, we obtain

$$\mathcal{S}(Q_k) = Q_{k+1}.$$

□

As a consequence of Proposition 9.4.7 we note that the creation operator on $\mathcal{F}(\mathbb{H})$ given by (9.4.6) acts as follows

$$\mathcal{S}(\sum_{k=0}^{\infty} Q_k \alpha_k) = \sum_{k=0}^{\infty} Q_{k+1} \alpha_k.$$

The annihilation operator corresponds to the hypercomplex derivative

$$\frac{\overline{\partial}}{2} := \frac{1}{2}\left(\frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}\right).$$

It is known by the Appell property that

$$\frac{\overline{\partial}}{2}(Q_k) = kQ_{k-1}, \ \forall k \geq 1.$$

The domains of $\mathcal{S}$ and $\frac{\overline{\partial}}{2}$ in $\mathcal{F}(\mathbb{H})$ are denoted respectively by

$$D(\mathcal{S}) := \{f \in \mathcal{F}(\mathbb{H});\ \mathcal{S}(f) \in \mathcal{F}(\mathbb{H})\}$$

and

$$D(\frac{\overline{\partial}}{2}) := \{f \in \mathcal{F}(\mathbb{H});\ \frac{\overline{\partial}}{2}(f) \in \mathcal{F}(\mathbb{H})\}.$$

We note that the creation operator $\mathcal{S}$ and the hypercomplex derivative $\frac{\overline{\partial}}{2}$ are quaternionic right linear operators densely defined on $\mathcal{F}(\mathbb{H})$ since $\left\{\frac{Q_k}{\sqrt{k!}}\right\}_{k\geq 0}$ is an orthonormal basis of the quaternionic Fock Hilbert space. In the sequel, we shall prove some different properties of these operators:

Proposition 9.4.8. *$\mathcal{S}$ and $\frac{\overline{\partial}}{2}$ are two closed quaternionic operators on $\mathcal{F}(\mathbb{H})$.*

Proof. We consider the graph of $\mathcal{S}$ defined by

$$\mathcal{G}(\mathcal{S}) := \{(f, \mathcal{S}f); f \in D(\mathcal{S})\}.$$

Let us show that $\mathcal{G}(\mathcal{S})$ is closed. Indeed, let ϕ_n be a sequence in $D(\mathcal{S})$ such that ϕ_n and $\mathcal{S}\phi_n$ converge to ϕ and ψ respectively on $\mathcal{F}(\mathbb{H})$. Then, thanks to Proposition 9.4.1 we have

$$|\phi_n(q) - \phi(q)| \leq e^{\frac{|q|^2}{2}} \|\phi_n - \phi\|_{\mathcal{F}(\mathbb{H})}$$

and

$$|\mathcal{S}\phi_n(q) - \psi(q)| \leq e^{\frac{|q|^2}{2}} \|\mathcal{S}\phi_n - \psi\|_{\mathcal{F}(\mathbb{H})}.$$

Therefore, it follows that ϕ_n and $\mathcal{S}\phi_n$ converge pointwise to ϕ and ψ, respectively. This leads to $\psi = \mathcal{S}\phi$ which ends the proof. The same technique could be adapted to prove the closedness of the hypercomplex derivative on $\mathcal{F}(\mathbb{H})$. □

Furthermore, we prove also the following

Proposition 9.4.9. *Let $f \in \mathcal{F}(\mathbb{H})$. Then, $\mathcal{S}(f)$ belongs to $\mathcal{F}(\mathbb{H})$ if and only if $\frac{\overline{\partial}}{2} f$ belongs to $\mathcal{F}(\mathbb{H})$. In particular, this means that we have*

$$D(\mathcal{S}) = D(\frac{\overline{\partial}}{2}).$$

Proof. We write $f = \displaystyle\sum_{k=0}^{\infty} Q_k \alpha_k$ in $\mathcal{F}(\mathbb{H})$. Then, we have

$$\mathcal{S}(f) = \sum_{h=1}^{\infty} Q_h \alpha_{h-1}.$$

In particular, we have

$$||\mathcal{S}(f)||^2_{\mathcal{F}(\mathbb{H})} = \sum_{h=1}^{\infty} h! |\alpha_{h-1}|^2. \tag{9.4.8}$$

On the other hand, using the Appell property with respect to the hypercomplex derivative we have

$$\frac{\overline{\partial}}{2}(f) = \sum_{h=0}^{\infty} Q_h \beta_h, \ \beta_h = (h+1)\alpha_{h+1}, \forall h \geq 0.$$

Some calculations lead to

$$||\frac{\overline{\partial}}{2}(f)||^2_{\mathcal{F}(\mathbb{H})} = \sum_{h=1}^{\infty} h(h!)|\alpha_h|^2. \tag{9.4.9}$$

We note that by (9.4.8) we have

$$\begin{aligned} ||\mathcal{S}f||^2_{\mathcal{F}(\mathbb{H})} &= \sum_{h=0}^{\infty} (h+1)! |\alpha_h|^2 \\ &= \sum_{h=0}^{\infty} (h+1) h! |\alpha_h|^2 \\ &= \sum_{h=0}^{\infty} h(h)! |\alpha_h|^2 + \sum_{h=0}^{\infty} h! |\alpha_h|^2. \end{aligned}$$

Therefore, we use (9.4.9) in order to get

$$||\mathcal{S}f||^2_{\mathcal{F}(\mathbb{H})} = ||\frac{\overline{\partial}}{2}f||^2_{\mathcal{F}(\mathbb{H})} + ||f||^2_{\mathcal{F}(\mathbb{H})}. \tag{9.4.10}$$

Hence, formula (9.4.10) shows that $||\mathcal{S}f||_{\mathcal{F}(\mathbb{H})} < \infty$ if and only if $||\frac{\overline{\partial}}{2}f||_{\mathcal{F}(\mathbb{H})} < \infty$ which ends the proof. □

Now, we prove the adjoint property

Proposition 9.4.10. *Let $f \in D(\frac{\overline{\partial}}{2})$ and $g \in D(\mathcal{S})$. Then, we have*

$$\left\langle \frac{\overline{\partial}}{2}f, g \right\rangle_{\mathcal{F}(\mathbb{H})} = \langle f, \mathcal{S}(g)\rangle_{\mathcal{F}(\mathbb{H})}.$$

Proof. Let $f = \sum_{k=0}^{\infty} Q_k\alpha_k$ in $D(\frac{\overline{\partial}}{2})$ and $g = \sum_{k=0}^{\infty} Q_k\beta_k$ in $D(\mathcal{S})$. Thus, we have

$$\begin{aligned}\frac{\overline{\partial}}{2}f &= \sum_{k=0}^{\infty}\frac{\overline{\partial}}{2}(Q_k)\alpha_k \\ &= \sum_{k=1}^{\infty} kQ_{k-1}\alpha_k \\ &= \sum_{h=0}^{\infty}(h+1)Q_h\alpha_{h+1}.\end{aligned}$$

On the other hand, making use of Proposition 9.4.7 we have

$$\begin{aligned}\mathcal{S}(g) &= \sum_{k=0}^{\infty}\mathcal{S}(Q_k)\beta_k \\ &= \sum_{k=0}^{\infty} Q_{k+1}\beta_k \\ &= \sum_{k=1}^{\infty} Q_k\beta_{k-1}.\end{aligned}$$

Therefore, we obtain

$$\left\langle \frac{\overline{\partial}}{2}f, g \right\rangle_{\mathcal{F}(\mathbb{H})} = \sum_{k=0}^{\infty}(k+1)!\overline{\alpha_{k+1}}\beta_k = \langle f, \mathcal{S}(g)\rangle_{\mathcal{F}(\mathbb{H})}.$$

This ends the proof. □

Proposition 9.4.11. *Let $f \in \mathcal{D}(\frac{\overline{\partial}}{2}) \cap \mathcal{D}(\mathcal{S})$. Then, we have*

$$\left[\frac{\overline{\partial}}{2}\mathcal{S}, \mathcal{S}\frac{\overline{\partial}}{2}\right](f) = f.$$

Proof. Let $f = \sum_{k=0}^{\infty} Q_k \alpha_k$ be in $\mathcal{D}(\frac{\overline{\partial}}{2}) \cap \mathcal{D}(\mathcal{S})$. Thus, computations using Proposition 9.4.7 and the Appell property give

$$\frac{\overline{\partial}}{2}\mathcal{S}(f) = \sum_{k=0}^{\infty}(k+1)Q_k\alpha_k \quad \text{and} \quad \mathcal{S}\frac{\overline{\partial}}{2}(f) = \sum_{k=0}^{\infty} kQ_k\alpha_k.$$

In particular, it shows that

$$\frac{\overline{\partial}}{2}\mathcal{S}(f) - \mathcal{S}\frac{\overline{\partial}}{2}(f) = f.$$

This ends the proof. □

Remark 9.4.12. *Note that the creation and annihilation operators denoted respectievly by $\mathcal{S}$ and $\frac{\overline{\partial}}{2}$ are adjoint of each other and satisfy the classical commutation rules on the Fock space of Fueter regular functions $\mathcal{F}(\mathbb{H})$ like in the classical complex case. Moreover, observe that we have also $\mathcal{S}\frac{\overline{\partial}}{2}(Q_k) = kQ_k$, for any $k \geq 1$. This property is related to the notion of number operators that appears in quantum mechanics.*

Let $(\eta_n)_{n \in \mathbb{N}}$ denote the normalized Hermite functions. In order to study the Segal-Bargmann transform notion in this framework we introduce the Fueter regular kernel function given by

$$\mathcal{A}_{\mathbb{H}}^F(q,x) := \sum_{k=0}^{\infty} \frac{Q_k(q)}{\sqrt{k!}}\eta_k(x), \ \forall (q,x) \in \mathbb{H} \times \mathbb{R}. \tag{9.4.11}$$

Then, for any quaternionic valued function φ in $L^2(\mathbb{R}, \mathbb{H})$ and $q \in \mathbb{H}$ we define

$$\mathcal{B}_{\mathbb{H}}^F(\varphi)(q) = \int_{\mathbb{R}} \mathcal{A}_{\mathbb{H}}^F(q,x)\varphi(x)dx. \tag{9.4.12}$$

We shall prove the following result:

Theorem 9.4.13. *The integral transform $\mathcal{B}_{\mathbb{H}}^F$ defines an isometric isomorphism mapping the standard Hilbert space $L^2(\mathbb{R}, \mathbb{H})$ onto the Clifford-Appell Fock space $\mathcal{F}(\mathbb{H})$.*

Proof. Let $\varphi \in L^2(\mathbb{R},\mathbb{H})$. We write $\varphi = \sum_{j=0}^{\infty} \eta_j(x)\beta_j$ such that $\|\varphi\|^2_{L^2(\mathbb{R},\mathbb{H})} = \sum_{j=0}^{\infty} |\beta_j|^2 < \infty$. Then, note that we have

$$\mathcal{B}^F_{\mathbb{H}}(\varphi)(q) = \sum_{k=0}^{\infty} \frac{Q_k(q)}{\sqrt{k!}} \int_{\mathbb{R}} \eta_k(x)\varphi(x)dx.$$

So, by setting $\alpha_k = \frac{1}{\sqrt{k!}} \int_{\mathbb{R}} \eta_k(x)\varphi(x)dx$ for all $k \geq 0$, we get

$$\begin{aligned} \|\mathcal{B}^F_{\mathbb{H}}(\varphi)\|^2_{\mathcal{F}(\mathbb{H})} &= \sum_{k=0}^{\infty} k!|\alpha_k|^2 \\ &= \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}} \eta_k(x)\varphi(x)dx \right|^2. \end{aligned}$$

However, by definition of φ and using the orthogonality of Hermite functions we obtain

$$\int_{\mathbb{R}} \eta_k(x)\varphi(x)dx = \sum_{j=0}^{\infty} \beta_j \int_{\mathbb{R}} \eta_k(x)\eta_j(x)dx = \beta_k, \ \forall k \geq 0.$$

Hence, we conclude that

$$\|\mathcal{B}^F_{\mathbb{H}}(\varphi)\|^2_{\mathcal{F}(\mathbb{H})} = \sum_{j=0}^{\infty} |\beta_j|^2 = \|\varphi\|^2_{L^2(\mathbb{R},\mathbb{H})}.$$

Moreover, observe that

$$\mathcal{B}^F_{\mathbb{H}}(\eta_k) = \frac{Q_k}{\sqrt{k!}}, \ \forall k \geq 0.$$

In particular, this allows to prove that $\mathcal{B}^F_{\mathbb{H}}$ is an isometric isomorphism mapping the standard Hilbert space $L^2(\mathbb{R},\mathbb{H})$ onto the Fock space $\mathcal{F}(\mathbb{H})$ on the quaternions. ☐

Now, we consider the following:

Problem 9.4.14. *Is it possible to map $\mathcal{F}_{Slice}(\mathbb{H})$ onto $\mathcal{F}(\mathbb{H})$ without using the Fueter mapping, see [63], and keeping the isometry property ?*

To answer the question, we will compute $\mathcal{B}^F_{\mathbb{H}}$ composed with the slice hyperholomorphic Segal-Bargmann transform.

In order to answer this problem, we need the slice hyperholomorphic Segal-Bargmann transform given by (9.2.3).

Notice that thanks to these integral transforms $\mathcal{B}_{\mathbb{H}}^S$ and $\mathcal{B}_{\mathbb{H}}^F$ it is possible to relate the two notions of Fock spaces on the quaternions, namely the slice hyperholomorphic $\mathcal{F}_{Slice}(\mathbb{H})$ and the Cauchy-Fueter regular one $\mathcal{F}(\mathbb{H})$. Indeed, for a fixed $i \in \mathbb{S}$, $f \in \mathcal{F}_{Slice}(\mathbb{H})$ and $q \in \mathbb{H}$ we define the integral transform given by

$$\Upsilon(f)(q) := \int_{\mathbb{C}_i} \mathcal{L}(q,z) f_i(z) d\mu_i(z),$$

where $d\mu_i(z) := \dfrac{1}{\pi} e^{-|z|^2} dA_i(z)$ and the kernel function is obtained by taking the series

$$\mathcal{L}(q,z) = \sum_{k=0}^{\infty} \frac{Q_k(q)}{k!} \overline{z}^k, \ \forall (q,z) \in \mathbb{H} \times \mathbb{C}_i.$$

Then, we prove:

Theorem 9.4.15. *The quaternionic integral transform Υ does not depend on the choice of the imaginary unit $i \in \mathbb{S}$. Furthermore, it defines an isometric isomorphism mapping the slice hyperholomrphic Fock space $\mathcal{F}_{Slice}(\mathbb{H})$ onto the Clifford-Appell Fock space $\mathcal{F}(\mathbb{H})$.*

Proof. Let $f \in \mathcal{F}_{Slice}(\mathbb{H})$, by Proposition 3.11 in [15] we have

$$f(q) = \sum_{k=0}^{\infty} q^k a_k \text{ and } \sum_{k=0}^{\infty} |a_k|^2 k! < \infty.$$

In particular, by definition of Υ we have

$$\begin{aligned}\Upsilon(f)(q) &= \int_{\mathbb{C}_i} \left(\sum_{k=0}^{\infty} \frac{Q_k(q)}{k!} \overline{z}^k \right) \left(\sum_{j=0}^{\infty} z^j a_j \right) d\mu_i(z) \\ &= \sum_{k,j=0}^{\infty} \frac{Q_k(q)}{k!} \left(\int_{\mathbb{C}_i} \overline{z}^k z^j d\mu_i(z) \right) a_j.\end{aligned}$$

However, it is known that

$$\int_{\mathbb{C}_i} \overline{z}^k z^j d\mu_i(z) = k! \delta_{k,j}.$$

Therefore, we get

$$\Upsilon(f)(q) = \sum_{k=0}^{\infty} Q_k(q) a_k.$$

Hence, since the coefficients $(a_k)_{k \geq 0}$ do not depend on the choice of the imaginary unit i we conclude that $\Upsilon(f)$ is well defined and does not depend on the choice of the imaginary unit. Now, we observe that the operator Υ can be obtained thanks to the commutative diagram such that we have

$$\Upsilon = \mathcal{B}_{\mathbb{H}}^{F} \circ (\mathcal{B}_{\mathbb{H}}^{S})^{-1}.$$

Indeed, to prove this fact. Let $f \in \mathcal{F}_{Slice}(\mathbb{H})$ and set

$$\phi(x) = (\mathcal{B}_{\mathbb{H}}^{S})^{-1}(f)(x) = \int_{\mathbb{C}_i} \mathcal{A}_{\mathbb{H}}^{S}(\overline{z}, x) f_i(z) d\mu_i(z).$$

Thus, for any $q \in \mathbb{H}$ we have:

$$\begin{aligned}
\mathcal{B}_{\mathbb{H}}^{F}(\phi)(q) &= \int_{\mathbb{C}_i} \mathcal{A}_{\mathbb{H}}^{F}(q, x)\phi(x)dx \\
&= \int_{\mathbb{C}_i} \mathcal{A}_{\mathbb{H}}^{F}(q, x) \left(\int_{\mathbb{C}_i} \mathcal{A}_{\mathbb{H}}^{S}(\overline{z}, x) f_i(z) d\mu_i(z) \right) dx \\
&= \int_{\mathbb{C}_i} \left(\int_{\mathbb{R}} \mathcal{A}_{\mathbb{H}}^{F}(q, x)\mathcal{A}_{\mathbb{H}}^{S}(\overline{z}, x)dx \right) f_i(z) d\mu_i(z).
\end{aligned}$$

Then, we set

$$H(q, z) = \int_{\mathbb{R}} \mathcal{A}_{\mathbb{H}}^{F}(q, x)\mathcal{A}_{\mathbb{H}}^{S}(\overline{z}, x)dx, \ \forall (q, z) \in \mathbb{H} \times \mathbb{C}_i.$$

So, for all $(q, z) \in \mathbb{H} \times \mathbb{C}_i$ we have

$$\begin{aligned}
H(q, z) &= \int_{\mathbb{C}_i} \left(\sum_{k=0}^{\infty} \frac{Q_k(q)}{\sqrt{k!}} \eta_k(x) \right) \left(\sum_{j=0}^{\infty} \frac{\overline{z^j}}{\sqrt{j!}} \eta_j(x) \right) dx \\
&= \sum_{k,j=0}^{\infty} \frac{Q_k(q)}{\sqrt{k!}} \left(\int_{\mathbb{R}} \eta_k(x)\eta_j(x)dx \right) \frac{\overline{z^j}}{\sqrt{j!}}
\end{aligned}$$

Then, using the fact that Hermite functions form an orthonormal basis of $L^2(\mathbb{R}, \mathbb{H})$ we get

$$H(q, z) = \sum_{k=0}^{\infty} \frac{Q_k(q)}{k!} \overline{z}^k = \mathcal{L}(q, z), \ \forall (q, z) \in \mathbb{H} \times \mathbb{C}_i.$$

At this stage, we replace $H(q, z)$ by its expression and conclude that we have

$$\Upsilon = \mathcal{B}_{\mathbb{H}}^{F} \circ (\mathcal{B}_{\mathbb{H}}^{S})^{-1}.$$

Therefore, since both of $\mathcal{B}_{\mathbb{H}}^{F}$ and $\mathcal{B}_{\mathbb{H}}^{S}$ are isometric isomorphisms mapping $L^2(\mathbb{R}, \mathbb{H})$ respectievly onto $\mathcal{F}(\mathbb{H})$ and $\mathcal{F}_{Slice}(\mathbb{H})$. This ends the proof. □

This quaternionic operator satisfies also the following properties :

Proposition 9.4.16. *For all $n \geq 0$, we set $f_n(q) = \dfrac{q^n}{\sqrt{n!}}$ and $\phi_n(q) = \dfrac{Q_n(q)}{\sqrt{n!}}$, $q \in \mathbb{H}$. Then, we have*

i) $\Upsilon(f_n) = \phi_n, \forall n \geq 0.$

ii) $\displaystyle\int_{\mathbb{C}_i} \mathcal{L}(q,z)\overline{\mathcal{L}(p,z)}d\mu_i(z) = K_{\mathcal{F}(\mathbb{H})}(q,p),\ \forall (q,p) \in \mathbb{H} \times \mathbb{H}.$

Proof. The first statement is a direct consequence of the fact that

$$\Upsilon = \mathcal{B}_{\mathbb{H}}^F \circ (\mathcal{B}_{\mathbb{H}}^S)^{-1}.$$

This is combined with the two following relations

$$(\mathcal{B}_{\mathbb{H}}^S)^{-1}(\eta_n) = f_n \text{ and } \mathcal{B}_{\mathbb{H}}^F(f_n) = \phi_n,\ \forall n \geq 0.$$

Now, let $(q,p) \in \mathbb{H} \times \mathbb{H}$. Then, we have

$$\begin{aligned}
\int_{\mathbb{C}_i} \mathcal{L}(q,z)\overline{\mathcal{L}(p,z)}d\mu_i(z) &= \sum_{k,j=0}^{\infty} \frac{Q_k(q)}{k!}\left(\int_{\mathbb{C}_i} \overline{z}^k z^j d\mu_i(z)\right)\frac{\overline{Q_j(p)}}{j!} \\
&= \sum_{k=0}^{\infty} \frac{Q_k(q)\overline{Q_k(p)}}{k!}, \\
&= K_{\mathcal{F}(\mathbb{H})}(q,p).
\end{aligned}$$

□

Corollary 9.4.17. *Let $i \in \mathbb{S}$. Then, for all $x, y \in \mathbb{R}$ and $n \geq 0$, we have the following identities*

i) $\displaystyle\int_{\mathbb{C}_i} e^{x\overline{z}} z^n d\mu_i(z) = x^n.$

ii) $\displaystyle\int_{\mathbb{C}_i} e^{x\overline{z}+yz} d\mu_i(z) = e^{xy}.$

Proof. Observe that we have

$$\mathcal{L}(t,z) = e^{t\overline{z}},\ \forall (t,z) \in \mathbb{R} \times \mathbb{C}_i. \tag{9.4.13}$$

The first identity follows from i) of Proposition 9.4.16 combined with (9.4.13).

The second statement is also a consequence of (9.4.13) combined with ii) of Proposition 9.4.16 and the fact that

$$K_{\mathcal{F}(\mathbb{H})}(x,y) = e^{xy},\ \forall (x,y) \in \mathbb{R} \times \mathbb{R}.$$

□

9.5 The Hardy space case

In this section, we study on the quaternionic unit ball $\Omega = \mathbb{B}$ the spaces associated to some sequence b as considered in Definition 9.3.1. First, we give some general proofs related to these spaces $\mathcal{HM}_b(\mathbb{B})$. Then, we will give more specific results on the Clifford-Appell Hardy space in this framework that corresponds to the sequence $b_k = 1, \forall k \geq 0$. In all this part, we take $\Omega = \mathbb{B}$ and $b = (b_k)_{k\geq 0}$ a non decreasing sequence with $b_0 = 1$. Then, we have

Proposition 9.5.1. *The following estimate holds*

$$|f(q)| \leq \left(\sum_{k=0}^{\infty} \frac{|q|^{2k}}{b_k}\right)^{\frac{1}{2}} \|f\|_{\mathcal{HM}_b},\ f \in \mathcal{HM}_b(\mathbb{B}),\ q \in \mathbb{B}.$$

Proof. Let us consider $f(q) = \sum_{k=0}^{\infty} Q_k(q)\alpha_k$ in $\mathcal{HM}_b(\mathbb{B})$. Thus, we have

$$|f(q)| \leq \sum_{k=0}^{\infty} \frac{|Q_k(q)|}{\sqrt{b_k}} |\alpha_k| \sqrt{b_k}.$$

Then, by the Cauchy-Schwarz inequality we have

$$|f(q)| \leq \left(\sum_{k=0}^{\infty} \frac{|Q_k(q)|^2}{b_k}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} b_k|\alpha_k|^2\right)^{\frac{1}{2}}$$

However, we know that $|Q_k(q)| \leq |q|^k$. Hence, we get

$$|f(q)| \leq \left(\sum_{k=0}^{\infty} \frac{|q|^{2k}}{b_k}\right)^{\frac{1}{2}} \|f\|_{\mathcal{HM}_b}.$$

□

As a consequence, we get this result

Theorem 9.5.2. *The sets $\mathcal{HM}_b(\mathbb{B})$ are right quaternionic reproducing kernel Hilbert spaces. Their reproducing kernel functions are given by*

$$K_{\mathcal{H}_b(\mathbb{B})}(q,p) = \sum_{k=0}^{\infty} \frac{Q_k(q)\overline{Q_k(p)}}{b_k},\ \forall (q,p) \in \mathbb{B} \times \mathbb{B}. \tag{9.5.1}$$

Furthermore, the family $\{\psi_k^b := \frac{Q_k}{\sqrt{b_k}},\ k \geq 0\}$ forms an orthonormal basis of $\mathcal{H}_b(\mathbb{B})$.

Proof. For a fixed $p \in \mathbb{B}$, we consider the function defined by

$$K_p(q) = \sum_{k=0}^{\infty} Q_k(q)\beta_k(p), \ \forall q \in \mathbb{B}, \text{ where } \beta_k(p) = \frac{\overline{Q_k(p)}}{b_k}.$$

Thanks to the d'Alembert ratio test for power series, we have

$$\sum_{k=0}^{\infty} b_k|\beta_k(p)|^2 = \sum_{k=0}^{\infty} \frac{|Q_k(p)|^2}{b_k} \leq \sum_{k=0}^{\infty} \frac{|q|^{2k}}{b_k} < \infty.$$

So, the function K_p belongs to $\mathcal{H}_b(\mathbb{B})$ for any $p \in \mathbb{B}$. Now, let $f = \sum_{k=0}^{\infty} Q_k\alpha_k \in \mathcal{H}_b(\mathbb{B})$. Then, we have

$$\langle K_p, f\rangle_{\mathcal{H}(\mathbb{B})} = \sum_{k=0}^{\infty} b_k\overline{\beta_k(p)}\alpha_k = \sum_{k=0}^{\infty} Q_k(p)\alpha_k = f(p), \ \forall p \in \mathbb{B}.$$

Therefore, the reproducing kernel of the space $\mathcal{H}_b(\mathbb{B})$ is given by

$$K_{\mathcal{H}_b(\mathbb{B})}(q,p) = \sum_{k=0}^{\infty} \frac{Q_k(q)\overline{Q_k(p)}}{b_k}, \ \forall (q,p) \in \mathbb{B} \times \mathbb{B}.$$

It is clear by definition of the scalar product that

$$\left\langle \psi_k^b, \psi_j^b \right\rangle_{\mathcal{H}_b(\mathbb{B})} = \delta_{k,j}, \ \forall k, j \in \mathbb{N}.$$

Furthermore, let $f = \sum_{k=0}^{\infty} Q_k\alpha_k$ in $\mathcal{H}_b(\mathbb{B})$ be such that

$$\left\langle \psi_k^b, f \right\rangle_{\mathcal{H}_b(\mathbb{B})} = 0, \ \forall k \in \mathbb{N}.$$

Thus, we have

$$\sqrt{b_k}\alpha_k = \left\langle \psi_k^b, f \right\rangle_{\mathcal{H}_b(\mathbb{B})} = 0, \ \forall k \in \mathbb{N}.$$

So, $f = 0$ for any $q \in \mathbb{B}$. In particular, this proves that $\{\psi_k^b\}_{k\geq 0}$ form an orthonormal basis of $\mathcal{HM}_b(\mathbb{B})$. □

Remark 9.5.3. *The Clifford-Appell Hardy space corresponds to the sequence b with all the terms equal to 1, and will be denoted simply $\mathcal{H}(\mathbb{B})$. In this case, the previous results of this section read as follows*

i) $|f(q)| \leq \dfrac{\|f\|_{\mathcal{H}(\mathbb{B})}}{(1-|q|^2)^{\frac{1}{2}}}, \ \forall f \in \mathcal{H}(\mathbb{B}), \ \forall q \in \mathbb{B}.$

ii) $K_{\mathcal{H}(\mathbb{B})}(q,p) = \sum_{k=0}^{\infty} Q_k(q)\overline{Q_k(p)}, \ \forall (q,p) \in \mathbb{B} \times \mathbb{B}.$

iii) $K_{\mathcal{H}(\mathbb{B})}(\vec{q},\vec{p}) = \sum_{k=0}^{\infty} (-1)^k c_k^2 \vec{q}^{\,k} \vec{p}^{\,k}, \ \forall (q,p) \in \mathbb{B}_0 \times \mathbb{B}_0.$

iv) $K_{\mathcal{H}(\mathbb{B})}(x,y) = \dfrac{1}{1-xy}, \ \forall (x,y) \in (-1,1)^2.$

In the previous section we studied the notions of creation and annihilation operators associated to the Fock space in this framework. We do the same in this section for the Hardy case by studying the counterparts of the shift and backward shift operators. We keep the same definition and notation of the shift operator introduced in the expressions (9.4.6), (9.4.7) and Proposition 9.4.7. Then, we first prove the following

Proposition 9.5.4. *The shift operator $\mathcal{S}$ is a right quaternionic isometric operator from the Clifford-Appell Hardy space $\mathcal{H}(\mathbb{B})$ into itself.*

Proof. Let $f = \sum_{k=0}^{\infty} Q_k \alpha_k$ belongs to $\mathcal{H}(\mathbb{B})$. We apply Proposition 9.4.7 and get

$$\mathcal{S}(f)(q) = \sum_{k=1}^{\infty} Q_k(q)\alpha_{k-1}, \ \forall q \in \mathbb{B}.$$

Hence, we have

$$\begin{aligned} ||\mathcal{S}(f)||^2_{\mathcal{H}(\mathbb{B})} &= \sum_{k=0}^{\infty} |\alpha_k|^2 \\ &= ||f||^2_{\mathcal{H}(\mathbb{B})}. \end{aligned}$$

This shows that $\mathcal{S}$ defines an isometry on the Hardy space $\mathcal{H}(\mathbb{B})$. □

In order to calculate the adjoint operator of the shift on $\mathcal{H}(\mathbb{B})$ we first deal with the following observation

Proposition 9.5.5. *For all $k \geq 1$ and $q \in \mathbb{B}$ with $q \neq 0$ we have*

$$Q_1^{-1} \odot Q_k(q) = \frac{c_k}{c_1 c_{k-1}} Q_{k-1},$$

where $\odot$ is the C-K product and $c_l := \sum_{j=0}^{l} (-1)^j T_j^l, \ \forall l \geq 0.$

Proof. First, we observe that $(Q_1(\vec{q}))^{-1} = \dfrac{(\vec{q})^{-1}}{c_1}$ and $Q_{k-1}(\vec{q}) = c_{k-1}\vec{q}^{\,k-1}$. Then, we write the series expansion associated to the C-K product and use similar techniques we used to prove Proposition 9.3.6. ☐

For all $k \geq 1$, we introduce a family of operators defined by

$$\mathcal{M}_k(f) := \frac{c_1 c_{k-1}}{c_k} Q_1^{-1} \odot f, \ \forall f \in \mathcal{H}(\mathbb{B}). \tag{9.5.2}$$

Then, for any $f = \sum_{k=1}^{\infty} Q_k \alpha_k$ in $\mathcal{H}(\mathbb{B})$ we consider the operator obtained by applying $\mathcal{M}_k$ on each component with the corresponding degree, i.e

$$\mathcal{M}(f) := \sum_{k=1}^{\infty} \mathcal{M}_k(Q_k)\alpha_k. \tag{9.5.3}$$

Therefore, we have an explicit expression given by

$$\mathcal{M}(f) := c_1 \sum_{k=1}^{\infty} \frac{c_{k-1}}{c_k} [Q_1^{-1} \odot Q_k]\alpha_k. \tag{9.5.4}$$

We note that using Proposition 9.5.5 we can see that this operator $\mathcal{M}$ acts like the standard backwardshift with respect to the Appell system $(Q_k)_{k\geq 0}$, in the sens that we have

$$\mathcal{M}(Q_k) = Q_{k-1}, \ \forall k \geq 1. \tag{9.5.5}$$

The next result allows to compute the adjoint of the shift operator on the Hardy space $\mathcal{H}(\mathbb{B})$.

Proposition 9.5.6. *Let $f, g \in \mathcal{H}(\mathbb{B})$. Then, it holds that*

$$\langle \mathcal{M}(f), g \rangle_{\mathcal{H}(\mathbb{B})} = \langle f, \mathcal{S}(g) \rangle_{\mathcal{H}(\mathbb{B})}.$$

In other words, the adjoint of the shift on $\mathcal{H}(\mathbb{B})$ is given by

$$\mathcal{S}^* = \mathcal{M}.$$

Proof. Let $f = \sum_{k=0}^{\infty} Q_k \alpha_k$ and $g = \sum_{k=0}^{\infty} Q_k \beta_k$ in $\mathcal{H}(\mathbb{B})$. Thus, we have

$$\begin{aligned} \mathcal{M}(f) &= \sum_{k=1}^{\infty} \mathcal{M}_k(Q_k)\alpha_k \\ &= \sum_{k=1}^{\infty} Q_{k-1}\alpha_k = \sum_{k=0}^{\infty} Q_k \alpha_{k+1}. \end{aligned}$$

We know also by Proposition 9.4.7 that

$$\mathcal{S}(g) = \sum_{k=1}^{\infty} Q_k \beta_{k-1}.$$

Therefore, we can see that

$$\langle \mathcal{M}(f), g \rangle_{\mathcal{H}(\mathbb{B})} = \sum_{k=0}^{\infty} \overline{\alpha_{k+1}} \beta_k = \langle f, \mathcal{S}(g) \rangle_{\mathcal{H}(\mathbb{B})}.$$

This ends the proof. $\square$

In [18] the authors introduced a backward shift with respect to each Fueter variable using some integral operators. Inspired from this approach, we present now an equivalent way to deal with the backward shift operator in our situation. First, for all $\varepsilon > 0$ we consider on $\mathcal{H}(\mathbb{B})$ a family of operators $\mathcal{R}_\varepsilon : f \longmapsto R_\varepsilon(f)$ defined using the following expression

$$\mathcal{R}_\varepsilon(f)(q) := \int_\varepsilon^1 \frac{1}{t} \frac{\overline{\partial}}{2} \left[f(tq) \right] dt; \ q \in \mathbb{B} \setminus \{0\} \tag{9.5.6}$$

where $\frac{\overline{\partial}}{2}$ denote the hypercomplex derivative with respect to the variable q. Then, we consider the backward shift operator given by

$$\mathcal{R}(f)(q) := \lim_{\varepsilon \to 0} \mathcal{R}_\varepsilon(f)(q), \ q \in \mathbb{B} \setminus \{0\} \tag{9.5.7}$$

and

$$\mathcal{R} f(0) = \frac{\overline{\partial}}{2} f(0). \tag{9.5.8}$$

We note that the backward shift operator $\mathcal{R}$ acts by reducing the degree of the Appell system $(Q_k)_{k \geq 0}$ as follows

Proposition 9.5.7. *For all $k \geq 1$, it holds that*

$$\mathcal{R}(Q_k) = Q_{k-1}.$$

Proof. Let $k \geq 1$ and $\varepsilon > 0$. First, we note that

$$Q_k(qt) = t^k Q_k(q), \ \forall \varepsilon < t < 1.$$

Then, by definition of $\mathcal{R}_\varepsilon$ and Appell property of the system $(Q_k)_{k \geq 0}$ we have

$$\begin{aligned} \mathcal{R}_\varepsilon(Q_k)(q) &= \int_\varepsilon^1 \frac{1}{t} \frac{\overline{\partial}}{2} \left[Q_k(tq) \right] dt \\ &= \int_\varepsilon^1 \frac{t^k}{t} \frac{\overline{\partial}}{2} \left[Q_k(q) \right] dt \\ &= k Q_{k-1}(q) \int_\varepsilon^1 t^{k-1} dt. \end{aligned}$$

Therefore, we obtain

$$\mathcal{R}_\varepsilon(Q_k)(q) = Q_{k-1}(q)(1-\varepsilon^k), \forall \varepsilon > 0.$$

Hence, by letting $\varepsilon \longrightarrow 0$ we conclude that

$$\mathcal{R}(Q_k) = Q_{k-1}, \ \forall k \geq 1.$$

□

Remark 9.5.8. *We observe thanks to formula* (9.5.5) *and Proposition 9.5.7 that the two backward shift operators $\mathcal{M}$ and $\mathcal{R}$ coincide on the Clifford-Appell Hardy space $\mathcal{H}(\mathbb{B})$.*

We prove also another property related to the backward shift operator $\mathcal{R}$ on the spaces $\mathcal{HM}_b(\mathbb{B})$.

Proposition 9.5.9. *Let $b = (b_k)_{k\in\mathbb{N}}$ be a non decreasing sequence with $b_0 = 1$ and $f \in \mathcal{HM}_b(\mathbb{B})$. Then, the following inequality holds*

$$||\mathcal{R}(f)||^2_{\mathcal{HM}_b} \leq ||f||^2_{\mathcal{HM}_b} - |f(0)|^2. \tag{9.5.9}$$

The equality holds on the Clifford-Appell Hardy space $\mathcal{H}(\mathbb{B})$.

Proof. We write $f = \displaystyle\sum_{k=0}^{\infty} Q_k\alpha_k$ in $\mathcal{HM}_b(\mathbb{B})$. Thus, by Proposition 9.5.7 we can see that $\mathcal{R}(f) = \displaystyle\sum_{k=0}^{\infty} Q_k\alpha_{k+1}$. Therefore, using the fact that b is non decreasing we get

$$\begin{aligned} ||\mathcal{R}(f)||^2_{\mathcal{HM}_b(\Omega)} &= \sum_{k=0}^{\infty} b_k|\alpha_{k+1}|^2 \\ &\leq \sum_{k=0}^{\infty} b_{k+1}|\alpha_{k+1}|^2 \\ &= ||f||^2_{\mathcal{HM}_b(\Omega)} - |f(0)|^2. \end{aligned}$$

□

Remark 9.5.10. *We note that using Proposition 9.5.9 we can see that the QRKHS $\mathcal{HM}_b(\mathbb{B})$ are invariant under the backward shift $\mathcal{R}$ and they satisfy inequality 9.5.9. It would be intersting to investigate the relation with Schur functions and see if the converse holds also in this framework. If it is the case, it will present a counterpart of the structure result proved in Theorem 3.1.2 of [16].*

9.6 The Fueter mapping range

In this section we give an answer to Problem 9.3.4. Indeed, we give a characterisation of the Fueter mapping range related to the hypercomplex spaces introduced in Definition 9.3.1.

Theorem 9.6.1. *Let Ω be an axially symmetric slice domain and $c = (c_k)_{k\in\mathbb{N}}$ be a given non decreasing sequence with $c_0 = 1$. Then, there exists a sequence $b = (b_k)_{k\geq 0}$ such that we have*

$$\tau\left(\mathcal{HS}_c(\Omega)\right) = \mathcal{HM}_b(\Omega).$$

More precisely, we have

i) $b_k = \dfrac{c_{k+2}}{(k+1)^2(k+2)^2}, \ \forall k \geq 0.$

ii) For all $f \in \mathcal{HS}_c(\Omega)$, we have

$$||\tau(f)||_{\mathcal{HM}_b(\Omega)} = 2\sqrt{||f||^2_{\mathcal{HS}_c(\Omega)} - |f(0)|^2 - c_1|f'(0)|^2}.$$

Proof. Let $g \in \tau\left(\mathcal{HS}_c(\Omega)\right)$, thus there exists $f \in \mathcal{HS}_c$ such that $g = \tau(f)$. Then, we write the series expansion

$$f(q) = \sum_{k=0}^{\infty} q^k a_k, \ \forall q \in \Omega.$$

Thus, we have $g = \tau(f) = \displaystyle\sum_{k=0}^{\infty} Q_k\alpha_k$, with $\alpha_k = -2(k+1)(k+2)a_{k+2}, \ \forall k \geq 0$. Now, we set

$$b_k = \frac{c_{k+2}}{(k+1)^2(k+2)^2}, \ \forall k \geq 0.$$

Hence, since $a_0 = f(0)$ and $a_1 = f'(0)$ we obtain

$$\begin{aligned}
||\tau(f)||^2_{\mathcal{HM}_b(\Omega)} &= \sum_{k=0}^{\infty} b_k|\alpha_k|^2 \\
&= 4\sum_{k=2}^{\infty} c_k|a_k|^2 \\
&= 4\left(||f||^2_{\mathcal{HS}_c(\Omega)} - |f(0)|^2 - c_1|f'(0)|^2\right) < \infty.
\end{aligned}$$

This ends the proof. □

Corollary 9.6.2. *If we set $\mathcal{HS}_c^0 := \{f \in \mathcal{HS}_c, \ f(0) = f'(0) = 0\}$. Then, the Fueter mapping τ defines a right quaternionic isometric operator (up to constant) from $\mathcal{HS}_c^0$ onto $\mathcal{HM}_b$.*

Proof. We only have to apply ii) in Theorem 9.6.1 and get

$$||\tau(f)||_{\mathcal{HM}_b(\Omega)} = 2||f||_{\mathcal{HS}_c(\Omega)}, \ \forall f \in \mathcal{HS}_c^0.$$

□

Remark 9.6.3. *The generic calculations provided in Theorem 9.6.1 confirm the results obtained in [63] for the Fock and Bergman cases.*

Remark 9.6.4. *We note that in Theorem 9.6.1 even if the sequence b is not necessarily a non decreasing sequence but the corresponding spaces $\mathcal{HM}_b$ are QRKHS. For the Fock-Fueter space on $\mathbb{H}$ we refer to the calculation details provided in [63]. However, on the quaternionic unit ball $\mathbb{B}$ this fact results thanks to the convergence of a certain power series associated to the sequence b.*

Proposition 9.6.5. *Let c and b two sequences as in Theorem 9.6.1. Then, the power series given by*

$$\sum_{k=0}^{\infty} \frac{|q|^{2k}}{b_k} = \sum_{k=}^{\infty} \frac{(k+1)^2(k+2)^2}{c_{k+2}} |q|^{2k}, \tag{9.6.1}$$

is convergent on the quaternionic unit ball $\mathbb{B}$.

Proof. Let $q \in \mathbb{B}$ and set

$$s_k = \frac{(k+1)^2(k+2)^2}{c_{k+2}} |q|^{2k}, \ \forall k \geq 0.$$

We have

$$\frac{s_{k+1}}{s_k} = |q|^2 \frac{(k+3)^2 c_{k+2}}{(k+1)^2 c_{k+3}}, \ \forall k \geq 0.$$

Then, using the fact that the sequence $(c_k)_{k\geq 0}$ is non decreasing we can see that

$$\lim_{k\to\infty} \frac{s_{k+1}}{s_k} \leq |q|^2 < 1.$$

Hence, by the d'Alembert ratio test the thesis follows. □

Remark 9.6.6. *As a consequence of the previous Proposition it is not difficult to see that on $\mathbb{B}$ the hypercomplex space $\mathcal{HM}_b$ obtained in Theorem 9.6.1 is a QRKHS with a reproducing kernel given by*

$$K_{\mathcal{HM}_b}(q,p) = \sum_{k=0}^{\infty} \frac{(k+1)^2(k+2)^2}{c_{k+2}} Q_k(q)\overline{Q_k(p)}, \ \forall (q,p) \in \mathbb{B} \times \mathbb{B}. \tag{9.6.2}$$

In the following table we list some spaces of slice hyperholomorphic functions and their Fueter mapping ranges denoted respectively by $\mathcal{HS}_c$ and $\mathcal{HM}_b$, the associated sequences c and b and the Fueter mapping norms.

Table 9.1: *Some spaces $\mathcal{HM}_b$ obtained in Theorem 9.6.1*

$\mathcal{HS}_c$	c_k	b_k	$\|\|\tau(f)\|\|_{\mathcal{HM}_b}$
Hardy	1	$\dfrac{1}{(k+1)^2(k+2)^2}$	$2\sqrt{\|\|f\|\|^2_{\mathcal{HS}_c}-\|f(0)\|^2-\|f'(0)\|^2}$
Fock	$k!$	$\dfrac{k!}{(k+1)(k+2)}$	$2\sqrt{\|\|f\|\|^2_{\mathcal{HS}_c}-\|f(0)\|^2-\|f'(0)\|^2}$
Dirichlet	k	$\dfrac{1}{(k+1)^2(k+2)}$	$2\sqrt{\|\|f\|\|^2_{\mathcal{HS}_c}-\|f(0)\|^2-\|f'(0)\|^2}$
Bergman	$\dfrac{1}{k+1}$	$\dfrac{1}{(k+3)(k+1)^2(k+2)^2}$	$2\sqrt{\|\|f\|\|^2_{\mathcal{HS}_c}-\|f(0)\|^2-\frac{1}{2}\|f'(0)\|^2}$

9.7 Further related results

In [5], we started the study of Schur analysis and de Branges-Rovnyak spaces in the framework of Fueter hyperholomorphic functions. This allows to develop some new Schur analysis results in the Fueter hypercomplex setting. It was also possible to find connections with the recent theory of slice polyanalytic functions. Indeed, this is based on the notion of Appell-like polynomials and their nice properties with respect to the CK product. We briefly discuss such related results in this last section.

9.7.1 Appell-like polynomials

Let us set

$$\frac{Q_m(q)}{c_m} \overset{\text{def.}}{=} P_m(q),$$

where Q_m denotes the m-th quaternionic Appell polynomial

$$Q_m(q)=\sum_{j=0}^{m} T_j^m q^{m-j}\bar{q}^j \tag{9.7.1}$$

(see [8, (3.8)]), and where the coefficients c_m are given by

$$c_m=\sum_{j=0}^{m}(-1)^j T_j^m, \quad \text{and} \quad T_j^m=\frac{2(m-j+1)}{(m+1)(m+2)}, \quad m=0,1,\ldots.$$

The polynomials Q_m are called Appell since they satisfy the Appell property

$$\frac{1}{2}\overline{D}Q_m=mQ_{m-1}, \qquad m\geq 1;$$

the P_m do not respect such a property, since

$$\frac{1}{2}\overline{D}P_m=m\frac{c_{m-1}}{c_m}P_{m-1}, \qquad m\geq 1,$$

however, they behave better with respect to the CK-product, as we shall see below. In particular, for even indexes of the form $m = 2k$, the Appell property is still satisfied by the polynomials $(P_{2k})_{k\geq 0}$ since we have $c_{m-1} = c_m$ in this case.

In what follows, we are looking at a theory of hyperholomorphic functions of the variable

$$P_1(q) = \frac{Q_1(q)}{c_1} = \zeta_1(q)\mathbf{e}_1 + \zeta_2(q)\mathbf{e}_2 + \zeta_3(q)\mathbf{e}_3, \tag{9.7.2}$$

with the CK-product. Moreover, note that

$$P_1(x_0) = 3x_0. \tag{9.7.3}$$

We have the following characterization, see [17, Proposition 3.6]:

Proposition 9.7.1. *Let $f_0, ..., f_{N-1}$ be slice regular functions on a domain $\Omega \subseteq \mathbb{H}$. Then, the function defined by*

$$f(x) := \sum_{k=0}^{N-1} \overline{q}^k f_k(q) \tag{9.7.4}$$

is slice polyanalytic of order N on Ω.

As a consequence:

Corollary 9.7.2. *The polynomial P_m is slice polyanalytic of order $m+1$.*

Proof. For all $0 \leq k \leq m$, we set $f_k(q) = \dfrac{T_k^m}{c_m} q^{m-k}$. It is clear that all f_k are slice regular functions on Ω. Moreover, we note that

$$P_m(q) = \sum_{k=0}^{m} \overline{q}^k f_k(q), \forall q \in \Omega.$$

Hence, the thesis follows using Proposition 9.7.1. □

We observe that it is also possible to prove a Representation Formula in the setting of Fueter hyperholomorphic functions using techniques from slice polyanalytic function theory.

Proposition 9.7.3. *Let f be a Fueter hyperholomorphic function of axial type on some axially symmetric slice domain $\Omega \subset \mathbb{H}$. Let $J \in \mathbb{S}$, then for any $x = u + I_q v \in \Omega$ the following equality holds :*

$$f(u + I_q v) = \frac{1}{2}\left[f_J(u + Jv) + f_J(u - Jv)\right] + \frac{I_q J}{2}\left[f_J(u - Jv) - f_J(u + Jv)\right].$$

Proof. We note that the Fueter hyperholomorphic polynomials $(P_m)_{m\geq 0}$ are slice polyanalytic of order $m+1$ thanks to Corollary 9.7.2. Thus, we can apply Theorem 3.9 in [17] in order to justify that these polynomials satisfy the Representation Formula. In particular, for $z=u+Jv$ and $x=u+I_qv$ we have

$$P_m(q)=\frac{1}{2}\left[P_m(z)+P_m(\overline{z})\right]+\frac{I_qJ}{2}\left[P_m(\overline{z})-P_m(z)\right]. \tag{9.7.5}$$

However, by Theorem 3.10 in [8], any Fueter hyperholomorphic function f of axial type admits a power series expansion in terms of the polynomials P_n of the form

$$f(q)=\sum_{n=0}^{\infty}P_n(q)u_n, \qquad u_n\in\mathbb{H}.$$

Therefore, using (9.7.5) we obtain

$$\begin{aligned} f(q)&=\frac{1}{2}\left(\sum_{m=0}^{\infty}P_m(z)u_m+\sum_{m=0}^{\infty}P_m(\overline{z})u_m\right)+\frac{I_qJ}{2}\left(\sum_{m=0}^{\infty}P_m(\overline{z})u_m-\sum_{m=0}^{\infty}P_m(z)u_m\right)\\ &=\frac{1}{2}\left[f_J(z)+f_J(\overline{z})\right]+\frac{I_qJ}{2}\left[f_J(\overline{z})-f_J(z)\right] \end{aligned}$$

This ends the proof. □

Remark 9.7.4. *An alternative proof of the previous Representation Formula in the Fueter hyperholomorphic context consists to apply Proposition 3.13 in [17] to each polynomial P_m.*

9.7.2 Hardy space and intrinsic Fueter regular functions

In this section we introduce the Hardy space in this framework of Appell-like polynomials. To start with, we denote by $\mathcal{E}$ the ellipsoid

$$\mathcal{E}=\left\{q\in\mathbb{R}^4\ :\ 9x_0^2+x_1^2+x_2^2+x_3^2<1\right\} \tag{9.7.1}$$

and we note that

The function

$$k_{\mathcal{E}}(q,s)=\sum_{m=0}^{\infty}P_1^{m\odot}(q)\overline{P_1^{m\odot}(s)} \tag{9.7.2}$$

converges and is positive definite for $q,s\in\mathcal{E}$. We also note that

$$k_{\mathcal{E}}(x_0,y_0)=\frac{1}{1-9x_0y_0}, \quad x_0,y_0\in(-1/3,1/3) \tag{9.7.3}$$

and with (see [8, Remark 5.3])

$$K_Q(q,s)=\sum_{n=0}^{\infty}Q_n(q)\overline{Q_n(s)}$$

we have

$$K_Q(x_0, y_0) = \frac{1}{1 - x_0 y_0}, \quad x_0, y_0 \in (-1, 1).$$

Definition 9.7.1. *The reproducing kernel Hilbert space associated with* (9.7.2) *will be called the Hardy space, and denoted by* $\mathbf{H}_2(\mathcal{E})$.

Theorem 9.7.5. *The Hardy space* $\mathbf{H}_2(\mathcal{E})$ *consists of functions of the form*

$$f(q) = \sum_{m=0}^{\infty} (\zeta_1(q)\mathbf{e}_1 + \zeta_2(q)\mathbf{e}_2 + \zeta_3(q)\mathbf{e}_3)^{m\odot} f_m = \sum_{m=0}^{\infty} P_m(q) f_m, \tag{9.7.4}$$

where the coefficients f_m *belong to* $\mathbb{H}$ *and are such that*

$$\sum_{m=0}^{\infty} |f_m|^2 < \infty. \tag{9.7.5}$$

This expression is then the square of the norm of f *in the Hardy space.*

Proof. The proofs follows standard arguments, see [8]. □

From the form of the elements of the Hardy space $\mathbf{H}_2(\mathcal{E})$ and using the fact that the polynomials P_m are Fueter hyperholomorphic of axial type, see Remark 3.9 in [8], we deduce:

Corollary 9.7.6. *Elements of* $\mathbf{H}_2(\mathcal{E})$ *are Fueter hyperholomorphic of axial type, in particular are uniquely determined by their restriction to* $(-1/3, 1/3)$.

Lemma 9.7.7. *The operator* $\mathsf{S} : f \mapsto P_1 \odot f$ *is an isometry in the Hardy space, with adjoint given by*

$$\mathsf{S}^* \left(\sum_{n=0}^{\infty} P_n f_n \right) = \sum_{n=0}^{\infty} P_n f_{n+1}. \tag{9.7.6}$$

Furthermore

$$\mathsf{SS}^* f = f - f(0), \quad f \in \mathbf{H}_2(\mathcal{E}). \tag{9.7.7}$$

Proof. The proof is a consequence of

$$\begin{aligned} \mathsf{SS}^* f &= P_1 \odot \left(\sum_{n=0}^{\infty} P_n f_{n+1} \right) \\ &= \sum_{n=0}^{\infty} P_{n+1} f_{n+1} \\ &= f - f_0 \\ &= f - f(0). \end{aligned}$$

□

Let $Cf = f(0)$ be the point evaluation in $\mathbf{H}_2(\mathcal{E})$. Then $C^*u = k_{\mathcal{E}}(\cdot, 0)u = u$ and we get from the previous lemma

$$I - M_{P_1}M_{P_1}^* = C^*C. \tag{9.7.8}$$

This equation is really what makes several arguments work in [5].

The operator (9.7.6) will be called the backward-shift operator and denoted by R_0.

Now, we study quaternionic intrinsic Fueter hyperholomorphic functions. Let us recall that, given an hyperholomorphic function f on some axially symmetric open set Ω, we say that f is quaternionic intrinsic if it satisfies the relation

$$f(\overline{q}) = \overline{f(q)}, \; \forall q \in \Omega. \tag{9.7.9}$$

Proposition 9.7.8. *The family of polynomials $(P_n)_{n\geq 0}$ consists of axially hyperholomorphic quaternionic intrinsic functions on $\mathbb{H}$.*

Proof. We know that for all $n \geq 0$ the polynomials P_n are axially hyperholomorphic functions on $\mathbb{H}$. Furthermore, using the relation with the n-th quaternionic Appell polynomials Q_n, see [8, (3.8)], we have

$$\begin{aligned}
\overline{P_n(x)} &= \frac{\overline{Q_n(x)}}{c_n} \\
&= \sum_{j=0}^{n} \frac{T_j^n}{c_n} \overline{q}^{n-j} q^j \\
&= \frac{Q_n(\overline{q})}{c_n} \\
&= P_n(\overline{q}).
\end{aligned}$$

□

Proposition 9.7.9. *Let f be a hyperholomorphic function of axial type on some axially symmetric open set Ω. Then, f is quaternionic intrinsic if and only if it admits a power series representation with real coefficients with respect to the polynomials $(P_n)_{n\geq 0}$.*

Proof. We know by Theorem 3.10 in [8] that f admits a power series with respect to $(P_n)_{n\geq 0}$. So, we can write $f = \sum_{n=0}^{\infty} P_n f_n$ with $f_n \in \mathbb{H}$ for all $n \geq 0$. We assume that f is intrinsic, thus the formula (9.7.9) and Proposition 9.7.8 imply that

$$\begin{aligned}\overline{f(q)} = f(\overline{q}), \forall q \in \Omega &\Leftrightarrow \sum_{n=0}^{\infty} \overline{P_n(q) f_n} = \sum_{n=0}^{\infty} P_n(\overline{q}) f_n, \forall q \in \Omega \\ &\Leftrightarrow \sum_{n=0}^{\infty} \overline{f_n}\,\overline{P_n(q)} = \sum_{n=0}^{\infty} \overline{P_n(q)} f_n, \forall q \in \Omega \\ &\Leftrightarrow \sum_{n=0}^{\infty} \overline{f_n}(3x_0)^n = \sum_{n=0}^{\infty} (3x_0)^n f_n, \forall x_0 \in \mathbb{R} \\ &\Leftrightarrow \overline{f_n} = f_n, \forall n \geq 0 \\ &\Leftrightarrow f_n \in \mathbb{R}, \forall n \geq 0.\end{aligned}$$

The equivalence between the second and the third lines holds because P_n is the unique axially hyperholomorphic extension of $(3x_0)^n$. This ends the proof. □

Proposition 9.7.10. *Let S_1 and S_2 be two hyperholomorphic functions of axial type, defined on some axially symmetric open set Ω. If S_1 is quaternionic intrinsic, then $S_1 \odot S_2$ admits a power series expansion with respect to the polynomials $(P_n)_{n \geq 0}$.*

Proof. We note that S_1 and S_2 have power series expansions in terms of $(P_n)_{n \geq 0}$ that we can write $S_1 = \sum_{n=0}^{\infty} P_n a_n$ and $S_2 = \sum_{n=0}^{\infty} P_n b_n$. Since S_1 is quaternionic intrinsic we know by Proposition 9.7.9 that the coefficients $(a_n)_{n \geq 0}$ are real. Thus, we apply also the fact that $P_n \odot P_m = P_{n+m}$ in order to get

$$\begin{aligned}S_1 \odot S_2 &= \left(\sum_{n=0}^{\infty} P_n a_n\right) \odot \left(\sum_{m=0}^{\infty} P_m b_m\right) \\ &= \sum_{n,m=0}^{\infty} (P_n \odot P_m) a_n b_m \\ &= \sum_{n,m=0}^{\infty} P_{n+m} a_n b_m \\ &= \sum_{n=0}^{\infty} P_n \left(\sum_{k=0}^{n} a_k b_{n-k}\right).\end{aligned}$$

□

Proposition 9.7.11. *Let S be a hyperholomorphic function of axial type. If S is quaternionic intrinsic, then the operator M_S coincides with the multiplication operator $f \mapsto S \odot f$.*

Proof. We note that since S is quaternionic intrinsic, it has real coefficients. Thus, we have $P_n \odot S = S \odot P_n$ for all $n \geq 0$. Then, for any $f = \sum_{n=0}^{\infty} P_n u_n$, we have

$$\begin{aligned} M_S(f) &= \sum_{n=0}^{\infty} (P_n \odot S) u_n \\ &= \sum_{n=0}^{\infty} (S \odot P_n) u_n \\ &= S \odot \left(\sum_{n=0}^{\infty} P_n u_n \right) \\ &= S \odot f. \end{aligned}$$

□

Proposition 9.7.12. *Let S_1 and S_2 be two hyperholomorphic functions of axial type such that S_1 is quaternionic intrinsic. Then, we have*

$$M_{S_1} M_{S_2} = M_{S_1 \odot S_2}. \tag{9.7.10}$$

Proof. We know by Proposition 9.7.10 that $S_1 \odot S_2$ is well defined and admits a power series expansion in terms of $(P_n)_{n \geq 0}$ since S_1 is intrinsic. Therefore, using Proposition 9.7.11, we have

$$\begin{aligned} M_{S_1 \odot S_2}(f) &= (S_1 \odot S_2) \odot f \\ &= M_{S_1}(S_2 \odot f) \\ &= M_{S_1} M_{S_2}(f). \end{aligned}$$

□

CHAPTER 10

A new polyanalytic function theory in hypercomplex analysis

In this chapter, we introduce the quaternionic slice polyanalytic functions and prove some of their main properties including a poly-decomposition, identity principle, representation formula, etc. Then, we apply the obtained results and start the study of some quaternionic reproducing kernel Hilbert spaces in this new setting. In particular, we treat Fock and Bergman spaces and give explicit expressions of their reproducing kernels. The results obtained in this chapter are based on [17].

10.1 Motivation

The theory of polyanalytic functions is an interesting topic in complex analysis. It extends the concept of holomorphic functions to nullsolutions of higher order powers of the Cauchy-Riemann operator. An excellent reference on this subject is the book of Balk [22]. Some famous Hilbert spaces of holomorphic functions that were extended to the setting of polyanalytic functions are the Bergman and Fock spaces, see for example [2, 10, 22, 89] and the references therein. As we have already seen before, the classical theory of holomorphic functions in complex analysis was extended to obtain the theory of slice hyperholomorphic functions of a quaternionic variable, see [35, 75]. In this chapter, we extend slice hyperholomorphic functions to higher order and define the slice polyanalytic functions of a quaternionic variable. Then, we shall use the obtained results to introduce and study the Fock and Bergman spaces of quaternionic slice polyanalytic functions and give explicit formulas for their reproducing kernels. Note that by considering polyanalytic functions with respect to the classical Cauchy-Fueter regularity on quaternions, it turns out that even the simple example given by $F(q,\overline{q}) = |q|^2$ is not polyanalytic of order 2. A natural question that arises here is about a poly Fueter regular function theory [27,57] and associated Fueter mapping results. We shall discuss these points in more details in the next chapter by proving a new Fueter mapping theorem in the poly hypercomplex case.

This chapter has the following structure: we first introduce the quaternionic slice polyanalytic function theory and discuss some of its main results. In particular, on slice domains we show the so-called poly decomposition, that makes any slice polyanalytic function a sum of quaternionic conjugate powers multiplied by some unique slice regular functions, thus extending the analogous result for complex functions. We prove also the counterparts of the Splitting Lemma, Identity Principle, Representation Formula, Extension Lemma and the Refined Splitting Lemma in this framework. We also discuss slice polyanalytic functions as a subclass of slice functions on axially symmetric domains. In particular, we prove a version of the identity principle in this case also. We introduce two expamples of quaternionic reproducing kernel Hilbert spaces QRKHS in this framewor. In particular, we study Bergman and Fock spaces of slice polyanalytic functions on quaternions and give explicit expressions of their reproducing kernels.formula of its reproducing kernel. We conclude with a brief discussion about poly-Hardy spaces.

10.2 Slice polyanalytic functions of a quaternionic variable

In this section, we extend to higher order the theory of slice regular functions on quaternions. Most of the material presented here is based on the results we developed in [17].

10.2.1 Main properties of the function theory

First, we start by considering the following simple example

Example. *For any $q\in\mathbb{H}$, let $F(q)=1-\overline{q}qj$. Then, we have*

$$\overline{\partial}_I F_I(x+Iy) = -(x+Iy)j \textit{ and } \overline{\partial}_I^2 F_I(x+Iy)=0;\ \forall I\in\mathbb{S}.$$

We say that F is slice polyanalytic of order 2 on $\mathbb{H}$.

The slice polyanalytic functions of a quaternionic variable (or of a paravector variable, in the case of Clifford algebra-valued functions) have to be considered as a subclass of slice functions, see Definition 3.17 of [17].

Definition 10.2.1 (Slice polyanalytic functions). *Let $n \in \mathbb{N}$ and denote by $\mathcal{C}^n(U)$ the set of continuously differentiable functions with all their derivatives up to order n on an axially symmetric open set $U \subseteq \mathbb{H}$. We let $\mathcal{U} = \{(x, y) \in \mathbb{R}^2 \ : \ x + Iy \subset U\}$. A function $f : U \to \mathbb{H}$ is called a left slice function, if it is of the form*

$$f(q) = \alpha(x, y) + I\beta(x, y) \qquad \textit{for } q = x + Iy \in U$$

with the two functions $\alpha, \beta : \mathcal{U} \to \mathbb{H}$ that satisfy the compatibility conditions $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$. If in addition α and β are in $\mathcal{C}^n(U)$ and satisfy the poly Cauchy-Riemann equations of order $n \in \mathbb{N}$

$$\frac{1}{2^n}(\partial_x + I\partial_y)^n(\alpha(x, y) + I\beta(x, y)) = 0, \quad \text{for all } \ I \in \mathbb{S} \tag{10.2.1}$$

then f is called left slice polyanalytic function of order $n \in \mathbb{N}$.

The definition is easily adapted in the case of right slice polyanalytic functions. Note that a slice regular function is a function as in the previous definition, when $n = 1$.

Remark 10.2.1. *We note that when dealing with left slice polyanalytic functions we will refer to them simply as slice polyanalytic. Due to the lack of commutativity on $\mathbb{H}$, we can define in an analogous way the right slice polyanalytic functions on quaternions.*

The set of all slice polyanalytic functions of order n on a domain Ω is a right vector space over the noncommutative field of quaternions. It will be denoted $\mathcal{SP}_n(\Omega)$ or simply $\mathcal{SP}(\Omega)$ if no confusion can arise with respect to the order. Slice polyanalytic functions were considered also in [25]. A simple observation that will be needed in the sequel is the following

Proposition 10.2.2. *If f is an intrinsic, slice polyanalytic function of order m and g is a slice regular function on a domain Ω then the pointwise multiplication $h(q) = f(q)g(q)$ defines also a slice polyanalytic function of order m on Ω.*

Proof. This holds because f is intrinsic, thus we can use the Leibniz rule. Indeed, let $I \in \mathbb{S}$ and set $x = u + Iv$, we will prove that

$$\overline{\partial_I}^m(fg)(u + vI) = 0. \tag{10.2.2}$$

Indeed, first we note that f is intrinsic, we have $f(U \cap \mathbb{C}_I) \subset \mathbb{C}_I$. In particular, we have the Leibniz rule

$$\overline{\partial_I}(fg)(u + Iv) = f\overline{\partial}(g)(u + vI) + \overline{\partial_I}(f)g(u + vI).$$

We note that since f is poly slice hyperholomorphic of order m and g is slice hyperholomorphic we have $\overline{\partial_I}(g) = 0$ and $\overline{\partial_I}(f) \neq 0$. Thus, we obtain

$$\overline{\partial_I}(fg)(u + Iv) = \overline{\partial_I}(f)g(u + vI).$$

Then, since f is intrinsic we can use the Leibniz rule m times and get

$$\overline{\partial_I}^m(fg)(u+Iv) = \overline{\partial_I}^{m-1}(f)\overline{\partial_I}(g)(u+vI) + \overline{\partial_I}^m(f)g(u+vI).$$

Therefore, it follows that the formula (10.2.2) holds since $f \in \mathcal{SP}_L^m(U)$ and $g \in \mathcal{SP}_L(U)$. Moreover, since f is intrinsic we have that fg is a slice function. Hence, the pointwise product fg is poly slice hyperholomorphic of order m on U.

□

As a consequence of the previous proposition and the poly-decomposition that we shall see later (see Corollary 11.2.9), we can prove also this more general result

Theorem 10.2.3. *Let U be an axially symmetric slice domain and $n, m \geq 1$. Assume that $f : U \longrightarrow \mathbb{H}$ is an intrinsic left poly slice hyperholomorphic function of order n and $g : U \longrightarrow \mathbb{H}$ a left poly slice hyperholomorphic function of order m. Then, the pointwise product fg is poly slice hyperholomorphic of order $n+m-1$.*

Proof. Let $n \geq 1$ be fixed. We will use an induction process with respect to $m \geq 1$.

i) For $m = 1$, it is clear that the result holds in this case by Proposition 10.2.2.

ii) We suppose now the result holds for some $m \geq 1$ and let us prove it for $m+1$. Indeed, let f an intinsic function in $\mathcal{SP}_L^n(U)$ and g in $\mathcal{SP}_L^{m+1}(U)$. We shall prove that $fg \in \mathcal{SP}_L^{n+m}(U)$. Then, using the poly-decomposition for f and g we can write

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q) \text{ and } g(q) = \sum_{k=0}^{m} \overline{q}^k g_k(q), \text{ for all } q \in U,$$

with $(f_k)_{k=0,\dots,n-1}$ and $(g_k)_{k=0,\dots,m}$ are slice hyperholomorphic functions. Now, setting

$$\Psi_m(q) = \sum_{k=0}^{m-1} \overline{q}^k g_k(q), \text{ for all } q \in U.$$

We observe that $\Psi_m \in \mathcal{SP}_L^m(U)$, moreover we have

$$g(q) = \Psi_m(q) + \overline{q}^m g_m(q), \quad \forall q \in U.$$

Therefore, it follows that

$$(fg)(q) = f(q)\Psi_m(q) + f(q)\overline{q}^m g_m(q), \quad \forall q \in U. \tag{10.2.3}$$

We note that by induction hypothesis we have $(f\Psi_m) \in \mathcal{SP}_L^{n+m-1}(U)$, this holds because $\Psi_m \in \mathcal{SP}_L^m(U)$ and $f \in \mathcal{SP}_L^n(U)$.

On the other hand, since f is intrinsic we have $f(q)\overline{q}^m = \overline{q}^m f(q)$, for all $q \in U$. In particular, we obtain

$$f(q)\overline{q}^m g_m(q) = \sum_{k=0}^{n-1} \overline{q}^{k+m}(f_k g_k)(q), \quad \forall q \in U.$$

We note that using Proposition 10.2.4 we know that all the slice hyperholomorphic components $(f_k)_{k=0,...,n-1}$ are intrinsic slice hyperholomorphic functions. As a consequence, the pointwise product $f_k g_M$ are slice hyperholomorphic for all $k = 0, ..., n-1$. At this stage it is clear that $f\overline{q}^m g_m \in \mathcal{SP}_L^{n+m}(U)$. Hence, it follows by formula (10.2.3) that $fg \in \mathcal{SP}_L^{n+m}(U)$.

iii) We conclude by induction that the pointwise multiplication fg is poly slice hyperholomorphic of order $n + m - 1$.

□

Proposition 10.2.4 (Splitting Lemma). *Let f be a slice polyanalytic function of order n on a domain $\Omega \subseteq \mathbb{H}$. Then, for any imaginary units I and J with $I \perp J$ there exist $F, G : \Omega_I \longrightarrow \mathbb{C}_I$ polyanalytic functions of order n such that for all $z = x + Iy \in \Omega_I$, we have*

$$f_I(z) = F(z) + G(z)J.$$

Proof. Let $I, J \in \mathbb{S}$ be such that $I \perp J$, then $\{1, I, J, IJ\}$ forms an orthogonal basis of $\mathbb{H}$. Hence, for any $z = x + Iy$ we can write

$$f_I(z) = f_0(z) + f_1(z)I + f_2(z)J + f_3(z)IJ$$

where $f_0, .., f_3$ are real valued. This leads to

$$f_I(z) = F(z) + G(z)J$$

with $F(z) = f_0(z) + f_1(z)I$ and $G(z) = f_2(z) + f_3(z)I$. However, f is slice polyanalytic of order n which means that $\overline{\partial}_I^n f_I(x + Iy) = 0$ on Ω_I. Thus, by linearity of the operator $\overline{\partial}_I^n$ and linear independence of the basis elements we have $\overline{\partial}_I^n F(x + Iy) = 0$ and $\overline{\partial}_I^n G(x + Iy) = 0$ on Ω_I. This ends the proof. □

10.2.2 Poly-decomposition and Identity Principle

An immediate consequence of the Splitting Lemma for slice polyanalytic functions is

Remark 10.2.5. *A function f is slice polyanalytic of order n on a domain $\Omega \subseteq \mathbb{H}$ if and only if for any $I \in \mathbb{S}$, we have*

$$f_I(z) = \sum_{k=0}^{n-1} \overline{z}^k h_k(z)$$

where $h_k : \Omega_I \longrightarrow \mathbb{H}$ are holomorphic maps.

Proposition 10.2.6. *Let $f_0, ..., f_{n-1}$ be slice regular functions on a domain $\Omega \subseteq \mathbb{H}$. Then, the function defined by*

$$f(q) := \sum_{k=0}^{n-1} \overline{q}^k f_k(q) \tag{10.2.4}$$

is slice polyanalytic of order n on Ω.

Proof. Let $I \in \mathbb{S}$ and choose $J \in \mathbb{S}$ with $I \perp J$. The Splitting Lemma for slice regular functions yields

$$f_{k|_{\mathbb{C}_I}}(x+Iy) = F_k(x+Iy) + G_k(x+Iy)J; \ \forall k = 0, ..., n$$

where F_k and G_k are $\mathbb{C}_I$ valued holomorphic functions on Ω_I. Hence, we have

$$\begin{aligned} f_I(x+Iy) &= \sum_{k=0}^{n-1} (x-Iy)^k f_{k|_{\mathbb{C}_I}}(x+Iy) \\ &= \sum_{k=0}^{n-1} (x-Iy)^k F_k(x+Iy) + \sum_{k=0}^{n-1} (x-Iy)^k G_k(x+Iy)J \\ &= F(x+Iy) + G(x+Iy)J. \end{aligned}$$

It is immediate that F and G are polyanalytic of order n on Ω_I. Thus $\overline{\partial}_I^n f_I(x+Iy) = 0$ on Ω_I for any $I \in \mathbb{S}$. $\square$

Conversely, we have the following

Proposition 10.2.7. *If f is a slice polyanalytic function of order n defined on a slice domain $\Omega \subset \mathbb{H}$. Then,*

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q) \tag{10.2.5}$$

where $f_0, ..., f_{N-1}$ are slice regular functions on Ω.

Proof. $f(q) = f(x+Iy) = \alpha(x,y) + I\beta(x,y)$ is a left slice polyanalytic of order N. By fixing a basis $1, e_1, e_2, e_1e_2$ of $\mathbb{H}$, and writing $e_0 = 1$, $e_3 = e_1e_2$, we have $\alpha = \sum_{\ell=0}^{3} \alpha_\ell e_\ell$, $\beta = \sum_{\ell=0}^{3} \beta_\ell e_\ell$, where the functions α_ℓ, β_ℓ are real-valued and are, respectively, even and odd in the second variable. Since the basis elements e_ℓ are linear independent, the system expressing the slice polyanalyticity can be rewritten in terms of the real components f, in other words, each $\mathbb{C}_I$-valued function $F_\ell = \alpha_\ell + I\beta_\ell$ is polyanalytic and $\overline{F_\ell(x-Iy)} = F_\ell(x+Iy)$. By the classical result applied to each function F_ℓ, we have $F_\ell(x+Iy) = \sum_{k=0}^{N-1}(x-Iy)^k f_{k,\ell}(x+Iy)$ where the functions $f_{k,\ell}$ are $\mathbb{C}_I$-valued and satisfy the Cauchy-Riemann system. Since $\overline{F_\ell(x-Iy)} = \sum_{k=0}^{N-1}(x-Iy)^k \overline{f_{k,\ell}(x-Iy)} = \sum_{k=0}^{N-1}(x-$

$Iy)^k f_{k,\ell}(x+Iy)$ we have $\overline{f_{k,\ell}(x-Iy)} = f_{k,\ell}(x+Iy)$ and so each $f_{k,\ell}$ i s alsice function. We then deduce

$$f(x+Iy) = \sum_{\ell=0}^{3} F_\ell(x+Iy)e_\ell = \sum_{\ell=0}^{3}\sum_{k=0}^{N-1}(x-Iy)^k f_{k,\ell}(x+Iy)e_\ell$$
$$= \sum_{k=0}^{N-1}(x-Iy)^k f_k(x+Iy), \qquad f_k(x+Iy) = \sum_{\ell=0}^{3} f_{k,\ell}(x+Iy)e_\ell,$$

where the functions f_k are evidently left slice regular, and this concludes the proof. □

Therefore, we have the following characterization of slice polyanalytic functions on slice domains

Corollary 10.2.8. *A function f defined on a slice domain is slice polyanalytic of order n if and only if it has the form* (10.2.5).

Proof. This is a direct consequence of the Propositions 10.2.6 and 10.2.7. □

We point out that Theorem 2.16 in [79] also establishes the previous result. The next results of slice regular functions that we shall extend to a higher order in this section are the counterparts of the identity principle, representation formula, extension lemma and the refined splitting lemma for slice polyanalytic functions.

Theorem 10.2.9 (Identity Principle)**.** *Let f and g be two slice polyanalytic functions of order n on a slice domain $\Omega \subset \mathbb{H}$. If, for some $I \in \mathbb{S}$, f and g coincide on U a subdomain of Ω_I, then $f = g$ everywhere in Ω.*

Proof. Note that f and g are slice polyanalytic functions of order n on Ω. Thus, we can write

$$f(q) = \sum_{k=0}^{n-1}\overline{q}^k f_k(q) \text{ and } g(q) = \sum_{k=0}^{n-1}\overline{q}^k g_k(q); \forall q \in \Omega$$

where $(f_k)_{k=0,\dots,n-1}$ and $(g_k)_{k=0,\dots,n-1}$ are slice regular on Ω. Note that thanks to the Splitting Lemma for slice polyanalytic functions we have that $f_I = F_1 + F_2 J$ and $g_I = G_1 + G_2 J$ where $J \in \mathbb{S}$ such that $I \perp J$ and F_1, F_2, G_1, G_2 are four $\mathbb{C}_I-$valued polyanalytic functions on Ω_I. By hypothesis, we have $f_I = g_I$ on U, so $F_1 = G_1$ and $F_2 = G_2$ on U which is a subdomain of Ω_I. Thus, from classical complex analysis we know that $F_1 = G_1$ and $F_2 = G_2$ everywhere on Ω_I. In particular, we get that $f_I = g_I$ everywhere on Ω_I. Hence, $\overline{\partial}_I^{n-1} f_I = \overline{\partial}_I^{n-1} g_I$ on Ω_I which shows that f_{n-1} coincides with g_{n-1} on Ω_I. However, f_{n-1} and g_{n-1} are slice regular. Then, making use of the Identity Principle for slice regular functions we have that $f_{n-1} = g_{n-1}$ everywhere on Ω. Similarly, using the same arguments we show that $f_k = g_k$ on Ω for all $k = 0, ..., n-1$. This ends the proof. □

10.2.3 Representation Formula and Extension Lemma

Inspired from the proof proposed in [50] for slice regular functions, we can prove a representation formula for quaternionic slice polyanalytic functions:

Theorem 10.2.10 (Representation Formula)**.** *Let f be a slice polyanalytic function of order n defined on an axially symmetric slice domain $\Omega \subset \mathbb{H}$. Let $J \in \mathbb{S}$, then for any $q = x + Iy \in \Omega$ the following equality holds :*

$$f(x+Iy) = \frac{1}{2}\left[f_J(x+Jy) + f_J(x-Jy)\right] + I\frac{J}{2}\left[f_J(x-Jy) - f_J(x+Jy)\right]$$

Moreover, for all $x + yK \subset \Omega$, $K \in \mathbb{S}$, there exist two functions α, β independent of I, such that for any $K \in \mathbb{S}$ we have

$$\frac{1}{2}\left[f_K(x+yK) + f_K(x-yK)\right] = \alpha(x,y)$$

and

$$\frac{1}{2}K\left[f_K(x-yK) - f_K(x+yK)\right] = \beta(x,y).$$

Proof. The representation formula is valid since it is a consequence of the sliceness of a slice polyanalitic function, see Definition 10.2.1. ☐

Remark 10.2.11. *The proof of the second statement of Theorem 10.2.10 is similar to the one for slice regular functions which corresponds to $n = 1$, see [50].*

Some immediate consequences of the representation formula for slice polyanalytic functions are the following :

Corollary 10.2.12. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain, $D \subset \mathbb{R}^2$ such that $x + yI \in U$ whenever $(x, y) \in D$ and let $f : U \longrightarrow \mathbb{H}$. Then, $f \in \mathcal{SP}_n(\Omega)$ if and only if there exist $\alpha, \beta : D \longrightarrow \mathbb{H}$ satisfying $\alpha(x, y) = \alpha(x, -y), \beta(x, y) = -\beta(x, -y)$ and $\overline{\partial_I}^n(\alpha + I\beta) = 0$ such that*

$$f(x+yI) = \alpha(x,y) + I\beta(x,y).$$

Corollary 10.2.13. *Let $U \subset \mathbb{H}$ be an axially symmetric slice domain and let $f : U \longrightarrow \mathbb{H}$ be a slice polyanalytic function. Then, for all $x, y \in \mathbb{R}$ such that $x + yI \in U$ there exist $a, b \in \mathbb{H}$ such that*

$$f(x+yI) = a + Ib$$

for all $I \in \mathbb{S}$.

Inspired from the paper [19], we can show another version of the identity principle for slice polyanalytic functions without the hypothesis that the open set on which they are defined is a *slice domain.* First, note that slice functions can be recovered by their values on two semi-slices, see the Representation Formula given by Proposition 6 in [81]. We have the following

Proposition 10.2.14. *Let Ω be an axially symmetric domain and let $f : \Omega \longrightarrow \mathbb{H}$ be a slice polyanalytic function. Assume that there exist $J, K \in \mathbb{S}$, with $J \neq K$ and U_J, U_K two subdomains of Ω_J^+ and Ω_K^+ respectively where $\Omega_J^+ := \Omega \cap \mathbb{C}_J^+$ and $\Omega_K^+ := \Omega \cap \mathbb{C}_K^+$. If $f = 0$ on U_J and U_K, then $f = 0$ everywhere in Ω.*

Proof. Let f be a slice polyanalytic function on Ω such that $f = 0$ on U_J and U_K. Thus, since U_J and U_K are respectively subdomains of Ω_J^+ and Ω_K^+. It follows, from the Splitting Lemma for slice polyanalytic functions combined with the classical complex analysis that $f = 0$ everywhere on Ω_J^+ and Ω_K^+. Then, we just need to use the Representation Formula which allows to recover a slice function by its values on two semi-slices to complete the proof. □

Remark 10.2.15. *This last remark on slice functions allows to define slice polyanalytic functions on axially symmetric domains which do not necessarily intersect the real line.*

Another interesting fact that holds for slice polyanalytic functions is the Extension Lemma:

Proposition 10.2.16 (Extension). *Let Ω_I be a domain in $\mathbb{C}_I$ symmetric with respect to the real axis and such that $\Omega_I \cap \mathbb{R} \neq \emptyset$. If*

$$f(z) = \sum_{k=0}^{N-1} \overline{z}^k h_k(z)$$

with $h_k : \Omega_I \longrightarrow \mathbb{H}$ are holomorphic functions such that $\overline{h_k(\overline{z})} = h_k(z)$. Then the unique slice polyanalytic extension of f is

$$ext(f)(q) := \sum_{k=0}^{N-1} \overline{q}^k ext(h_k)(q); \forall q = x + I_q y \in \Omega.$$

Proof. Assume that f is polyanalytic of order n on Ω_I. Then, we have

$$f(z) = \sum_{k=0}^{n-1} \overline{z}^k h_k(z)$$

where $h_k : \Omega_I \longrightarrow \mathbb{H}$ are holomorphic functions. However, Ω_I is symmetric with respect to the real axis. Thus, according to the Extension Lemma for slice regular functions for any $k = 0, ..., n-1$ we can consider the slice regular functions defined by

$$f_k(x + I_q y) := \frac{1}{2}\left[h_k(z) + h_k(\overline{z})\right] + I_q \frac{I}{2}\left[h_k(\overline{z}) - h_k(z)\right]; z = x + Iy \in \Omega_I.$$

Let us consider

$$g(x + I_q y) = \sum_{k=0}^{n-1} (x - I_q y)^k f_k(x + I_q y),$$

we shall prove that

$$g(x+I_qy)=\frac{1}{2}\left[f(z)+f(\overline{z})\right]+I_q\frac{I}{2}\left[f(\overline{z})-f(z)\right];z=x+Iy\in\Omega_I.$$

Indeed, first note that we have the two following equalities

$$(x+I_qy)^k=\frac{1}{2}\left[(x+Iy)^k+(x-Iy)^k\right]+I_q\frac{I}{2}\left[(x-Iy)^k-(x+Iy)^k\right] \tag{10.2.6}$$

and

$$(x-I_qy)^k=\frac{1}{2}\left[(x-Iy)^k+(x+Iy)^k\right]+I_q\frac{I}{2}\left[(x+Iy)^k-(x-Iy)^k\right]. \tag{10.2.7}$$

Then, by definition of f_k we have

$$g(x+I_qy)=\frac{C_n(x,y)+D_n(x,y)}{2}$$

where we have set

$$C_n(x,y)=\sum_{k=0}^{n-1}(x-I_qy)^k\left[h_k(z)+h_k(\overline{z})\right]$$

and

$$D_n(x,y)=\sum_{k=0}^{n-1}I_q(x-I_qy)^kI\left[h_k(\overline{z})-h_k(z)\right].$$

We replace $(x-I_qy)^k$ by its expression using the formula (10.2.7) and get

$$\begin{aligned}C_n(x,y)=ext(f)(x+I_qy)+\frac{1}{2}\sum_{k=0}^{n-1}\left[z^kh_k(z)+\overline{z}^kh_k(\overline{z})\right]\\ +\frac{I_qI}{2}\sum_{k=0}^{n-1}\left[z^kh_k(z)-\overline{z}^kh_k(\overline{z})\right].\end{aligned} \tag{10.2.8}$$

On the other hand, after straightforward computations we obtain

$$\begin{aligned}D_n(x,y)=ext(f)(x+I_qy)-\frac{1}{2}\sum_{k=0}^{n-1}\left[z^kh_k(z)+\overline{z}^kh_k(\overline{z})\right]\\ +\frac{I_qI}{2}\sum_{k=0}^{n-1}\left[\overline{z}^kh_k(\overline{z})-z^kh_k(z)\right].\end{aligned} \tag{10.2.9}$$

Therefore, it follows that

$$g(x+I_qy)=ext(f)(x+I_qy)$$

this ends the proof. □

10.2.4 A generalized $\circledast-$product and intrinsic functions

We note that the pointwise multiplication of two slice polyanalytic functions is not slice polyanalytic, in general. In fact this problem appears already for $n = 1$, namely in the case of slice regular functions. However, the $*$-product preserves the structure in this case. For slice polyanalaytic functions of the same order we can introduce also a natural product denoted $\circledast_n$ in order to preserve the structure. Let f and g be two slice polyanalytic functions of order n on some axially symmetric slice domain Ω such that f and g have poly-decompositions given by

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q) \text{ and } g(q) = \sum_{k=0}^{n-1} \overline{q}^k g_k(q),$$

where f_k, g_k are slice regular for all $k = 0, .., n-1$. Then, we define

$$(f \circledast_n g)(q) := \sum_{k=0}^{n-1} \overline{q}^k (f_k * g_k)(q), \tag{10.2.10}$$

where $f_k * g_k$ stands for the classical $*$-product of slice regular functions. If there is no confusion on the order of polyanalyticity n, we simply denote $\circledast$. We note that the product $\circledast$ reduces to the standard $*$-product in the case of slice regular functions, namely when $n = 1$. Moreover, it turns out that the space of slice polyanlytic functions $\mathcal{SP}_n(\Omega)$ is a ring with respect to this new product $\circledast$. A slice polyanalytic function of order n on some slice domain Ω is quaternionic intrinsic if and only if all its slice regular components are also quaternionic intrinsic.

Proof. We use the poly-decomposition to write

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q),$$

where f_k are slice regular functions for all $k = 0, .., n-1$. First, we observe that if all the functions f_k are quaternionic intrinsic, thus f will preserve any complex plane $\Omega \cap \mathbb{C}_I$, that is to say that f is also quaternionic intrinsic. For the converse, we suppose that f is quaternionic intrinsic, that means $\overline{f(q)} = f(\overline{q})$, for all $q \in \Omega$. We write the series expansion of each slice regular component and justify that each of them has real coefficients. $\square$

Let f and g be two slice polyanalytic functions as in Definition 10.2.4. If we assume moreover that f is quaternionic intrinsic, then we have

$$(f \circledast g)(q) := \sum_{k=0}^{n-1} \overline{q}^k (f_k g_k)(q), \tag{10.2.11}$$

Proof. We note that the product $f \circledast g$ is slice polyanalytic of order n by construction. Furthermore, since f is quaternionic intrinsic we get from Proposition 10.2.4 that all the slice regular components are also quaternionic intrinsic. In particular, by classical results of slice hyperholomorphic theory we know that

$$f_k * g_k(q) = f_k(q)g_k(q), \quad \forall q \in \Omega.$$

Hence, we obtain

$$(f \circledast g)(q) := \sum_{k=0}^{n-1} \overline{q}^k (f_k g_k)(q), \quad \forall q \in \Omega.$$

□

Inspired from the book [50], we present the counterpart of the Refined Splitting Lemma for slice polyanalytic functions. First, let us consider the subclass of $\mathcal{SP}_n(\Omega)$ defined by

$$\mathcal{N}_n(\Omega) := \{f \in \mathcal{SP}_n(\Omega) : f(\Omega \cap \mathbb{C}_I) \subset \mathbb{C}_I, \forall I \in \mathbb{S}\}.$$

Then, we have

Proposition 10.2.17 (Refined Splitting Lemma). *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and f be a slice polyanalytic function of order n on Ω. Then, for any $I, J \in \mathbb{S}$ with $I \perp J$, there exist $\psi_\ell : \Omega_I \longrightarrow \mathbb{C}_I, \ell = 0, .., 3$ intrinsic polyanalytic such that:*

$$f_I(x+yI) = \psi_0(x+yI) + \psi_1(x+yI)I + \psi_2(x+yI)J + \psi_3(x+yI)K$$

where $K = IJ$.

Proof. If f is slice polyanalytic of order n, then we can write

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k h_k(q)$$

with $(h_k)_{k=0,..,n-1}$ are slice regular on Ω. In particular, making use of the Refined Splitting Lemma for slice regular functions we have that for all $k = 0, ..., n-1$:

$$h_k(x+yI) = h_k^0(x+yI) + h_k^1(x+yI)I + h_k^2(x+yI)J + h_k^3(x+yI)IJ$$

where $h_k^\ell : \Omega_I \longrightarrow \mathbb{C}_I$ are holomorphic intrinsic functions for all $\ell = 0, ..., 3$. We have,

$$f_I(z) = \sum_{k=0}^{n-1} \overline{z}^k h_k(z); \forall z \in \Omega_I.$$

Therefore, the thesis follows by considering the polyanalytic intrinsic functions defined by

$$\psi_\ell(x+yI)=\sum_{k=0}^{n-1}(x-yI)^k h_k^\ell(x+yI);\forall\ell=0,...,3.$$

$\square$

As a consequence of the Refined Splitting Lemma, we have the following

Theorem 10.2.18. *Let $\Omega\subset\mathbb{H}$ be an axially symmetric slice domain and $\{1,I,J,IJ\}$ a basis of $\mathbb{H}$. Then,*

$$\mathcal{SP}_n(\Omega)=\mathcal{N}_n(\Omega)\oplus\mathcal{N}_n(\Omega)I\oplus\mathcal{N}_N(\Omega)J\oplus\mathcal{N}_n(\Omega)IJ.$$

Proof. The Refined Splitting Lemma combined with the Extension Lemma for slice polyanalytic functions shows that we have

$$\mathcal{SP}_n(\Omega)=\mathcal{N}_n(\Omega)+\mathcal{N}_n(\Omega)I+\mathcal{N}_n(\Omega)J+\mathcal{N}_n(\Omega)IJ.$$

Moreover, we only need to use Proposition 2.7 in the book [50] and the characterization of slice polyanalytic functions obtained in corollary 11.2.9 to show that all the intersections between $\mathcal{N}_n(\Omega),\mathcal{N}_n(\Omega)I,\mathcal{N}_n(\Omega)J,\mathcal{N}_n(\Omega)IJ$ are reduced to zero. This ends the proof. $\square$

10.3 Two quaternionic reproducing kernel Hilbert spaces QRKHS of slice polyanalytic functions

10.3.1 The quaternionic slice polyanalytic Fock space

In this section, we introduce the Fock space of slice polyanalytic functions on quaternions. Let $N\geq 1$ and $I\in\mathbb{S}$ we define the space

$$\mathcal{F}_I^n(\mathbb{H}):=\{f\in\mathcal{SP}_n(\mathbb{H})/\int_{\mathbb{C}_I}|f_I(p)|^2e^{-|p|^2}d\lambda_I(p)<\infty\}.$$

This space is endowed with the following inner product

$$\langle f,g\rangle_{\mathcal{F}_I^n(\mathbb{H})}=\int_{\mathbb{C}_I}\overline{g_I(p)}f_I(p)e^{-|p|^2}d\lambda_I(p).$$

Then, we have the following:

Proposition 10.3.1. *The set $\mathcal{F}_I^n(\mathbb{H})$ is a right quaternionic Hilbert space.*

Proof. The proof is based on the Splitting Lemma for slice polyanalytic functions, see Proposition 11.2.8. Indeed, let (f_k) be a Cauchy sequence in $\mathcal{F}_I^n(\mathbb{H})$. Choose $J \in \mathbb{S}$ such that $I \perp J$. Then, since f_k are slice polyanalytic we have $f_{k,I} := F_k + G_k J \quad \forall n \in \mathbb{N}$ where F_k and G_k are polyanalytic functions on the slice $\mathbb{C}_I$ belonging to the classical polyanalytic Fock space $\mathcal{F}_n(\mathbb{C}_I)$. It is easy to see that $(F_k)_k$ and $(G_k)_k$ are Cauchy sequences in $\mathcal{F}_n(\mathbb{C}_I)$. Hence, there exists two functions F and G belonging to $\mathcal{F}_n(\mathbb{C}_I)$ such that the sequences $(F_k)_k$ and $(G_k)_k$ are converging respectively to F and G. Let $f_I = F + GJ$ and consider $f = ext(f_I)$ we have then $f \in \mathcal{F}_I^n(\mathbb{H})$ thanks to Proposition 10.2.16. Moreover, the sequence (f_k) converges to f with respect to the norm of $\mathcal{F}_I^n(\mathbb{H})$. This ends the proof. □

Proposition 10.3.2. *Let $f \in \mathcal{SP}_n(\mathbb{H})$ and $I, J \in \mathbb{S}$ two imaginary units. Then, we have the following*

$$\frac{1}{2}\|f\|_{\mathcal{F}_I^n(\mathbb{H})} \leq \|f\|_{\mathcal{F}_J^n(\mathbb{H})} \leq 2\|f\|_{\mathcal{F}_I^n(\mathbb{H})}.$$

Proof. This is a consequence of the Representation Formula, see Theorem 10.2.10. Indeed, since f is slice polyanalytic of order n on $\mathbb{H}$ we have

$$f(x+Iy) = \frac{1}{2}\left[f(x+Jy) + f(x-Jy)\right] + I\frac{J}{2}\left[f(x-Jy) - f(x+Jy)\right].$$

Then,

$$|f(x+Iy)| \leq |f(x+Jy)| + |f(x-Jy)|,$$

and therefore

$$\begin{aligned} |f(x+Iy)|^2 &\leq (|f(x+Jy)| + |f(x-Jy)|)^2 \\ &\leq 2\left(|f(x+Jy)|^2 + |f(x-Jy)|^2\right) \end{aligned}$$

because $(|f(x+Jy)| - |f(x-Jy)|)^2 \geq 0$. This implies that

$$\|f\|^2_{\mathcal{F}_I^n(\mathbb{H})} \leq 2\left(\|f\|^2_{\mathcal{F}_J^n(\mathbb{H})} + \|f\|^2_{\mathcal{F}_{-J}^n(\mathbb{H})}\right).$$

However, since $\|f\|_{\mathcal{F}_J^n(\mathbb{H})} = \|f\|_{\mathcal{F}_{-J}^n(\mathbb{H})}$ we get $\|f\|^2_{\mathcal{F}_I^n(\mathbb{H})} \leq 4\|f\|^2_{\mathcal{F}_J^n(\mathbb{H})}$. By interchanging the roles of I and J we get also $\|f\|^2_{\mathcal{F}_I^n(\mathbb{H})} \leq 4\|f\|^2_{\mathcal{F}_J^n(\mathbb{H})}$. Finally, it follows that

$$\frac{1}{2}\|f\|_{\mathcal{F}_I^n(\mathbb{H})} \leq \|f\|_{\mathcal{F}_J^n(\mathbb{H})} \leq 2\|f\|_{\mathcal{F}_I^n(\mathbb{H})}.$$

□

Corollary 10.3.3. *Given any $I, J \in \mathbb{S}$, the slice polyanalytic Fock spaces $\mathcal{F}_I^n(\mathbb{H})$ and $\mathcal{F}_J^n(\mathbb{H})$ contain the same elements and have equivalent norms.*

Remark 10.3.4. *By the previous Corollary, the quaternionic slice polyanalytic Fock space is independent of the choice of the imaginary unit. Thus, we shall use the notation $\mathcal{F}^n_{Slice}(\mathbb{H})$.*

Let us fix $q \in \mathbb{H}$ and consider the evaluation mapping

$$\Lambda_q : \mathcal{F}^n_{Slice}(\mathbb{H}) \longrightarrow \mathbb{H}; f \mapsto \Lambda_q(f) = f(q).$$

Then, we have the following estimate on $\mathcal{F}^n_{Slice}(\mathbb{H})$:

Proposition 10.3.5. *Let $f \in \mathcal{F}^n_{Slice}(\mathbb{H})$ and $q \in \mathbb{H}$. Then,*

$$|\Lambda_q(f)| \leq \sqrt{n} e^{\frac{|q|^2}{2}} \|f\|_{\mathcal{F}^n_{Slice}(\mathbb{H})}.$$

Proof. Let $I \in \mathbb{S}$ be such that $q \in \mathbb{C}_I$ and choose $J \in \mathbb{S}$ with $I \perp J$. Then, the Splitting Lemma yields

$$f_I(z) = F(z) + G(z)J; \ \forall z \in \mathbb{C}_I$$

where F and G belong to $\mathcal{F}_n(\mathbb{C}_I)$. In particular, we have

$$|f(q)|^2 = |F(q)|^2 + |G(q)|^2$$

However, we know from classical complex analysis that

$$|F(q)| \leq \sqrt{N} e^{\frac{|q|^2}{2}} \|F\|_{\mathcal{F}_n(\mathbb{C}_I)} \text{ and } |G(q)| \leq \sqrt{N} e^{\frac{|q|^2}{2}} \|G\|_{\mathcal{F}_n(\mathbb{C}_I)}.$$

Therefore,

$$|f(q)| \leq \sqrt{n} e^{\frac{|q|^2}{2}} \left(\|F\|^2_{\mathcal{F}_n(\mathbb{C}_I)} + \|G\|^2_{\mathcal{F}^n(\mathbb{C}_I)} \right)^{\frac{1}{2}} = \sqrt{n} e^{\frac{|q|^2}{2}} \|f\|_{\mathcal{F}^n_{Slice}(\mathbb{H})}.$$

□

Proposition 10.3.5 shows that all the evaluation mappings on $\mathcal{F}^n_{Slice}(\mathbb{H})$ are continuous. Then, the Riesz representation theorem for quaternionic right-linear Hilbert spaces, see [28] shows that for any $q \in \mathbb{H}$ there exists a unique function $K^q_n \in \mathcal{F}^n_{Slice}(\mathbb{H})$ such that for any $f \in \mathcal{F}^n_{Slice}(\mathbb{H})$ we have

$$f(q) = \langle f, K^q_n \rangle_{\mathcal{F}^n_{Slice}(\mathbb{H})}.$$

Let $J \in \mathbb{S}$ and $r \in \mathbb{C}_J$, then for $q = x + Iy$ and $z = x + Jy$ the corresponding reproducing kernel of the second kind is obtained by extending the kernel of the complex case. It is given by the following

$$K_n : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$K_n(q, r) := \frac{1}{2} \left[K_n(z, r) + K_n(\overline{z}, r) \right] + I \frac{J}{2} \left[K_n(\overline{z}, r) - K_n(z, r) \right].$$

In order to compute the kernel function we use the $*$-product of (left) slice functions with respect to the first variable, see [81].

Remark 10.3.6. *If $f \in \mathcal{SP}_n(\mathbb{H})$ and $g \in \mathcal{SP}_m(\mathbb{H})$, then we have*

$$f * g \in \mathcal{SP}_{n+m-1}(\mathbb{H}).$$

We observe that the $*$-product of (left) slice functions coincides with the convolution product related to the poly-decomposition considered in Definition 4.15 of [6].

As a consequence we can state the following result

Theorem 10.3.7. *The set $\mathcal{F}^n_{Slice}(\mathbb{H})$ is a right quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by*

$$K_n(q,r) = e_*(q\bar{r})*\left(\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{1}{k!}(\bar{q}q - q\bar{r} - \bar{q}r + \bar{r}r)^{k*}\right);\forall (q,r)\in\mathbb{H}\times\mathbb{H}.$$

Proof. Fix $r \in \mathbb{H}$ such that r belongs to the slice $\mathbb{C}_J$, we consider the function defined by

$$F^r_n(q) = e_*(q\bar{r}) * \varphi_n(q,r)$$

where

$$\varphi_n(q,r) := \sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\frac{1}{k!}(\bar{q}q - q\bar{r} - \bar{q}r + \bar{r}r)^{k*};\forall q\in\mathbb{H}.$$

Clearly $q \longmapsto e_*(q\bar{r})$ is slice regular on $\mathbb{H}$ with respect to the variable q. Moreover, thanks to Remark 10.3.6 we can see that $\varphi_n(q,r)$ is a slice polyanalytic function of order n on $\mathbb{H}$ with respect to q. Thus, F^r_n is a slice polyanalytic function of order n on $\mathbb{H}$ with respect to the variable q. Furthermore, the reproducing kernel of $\mathcal{F}^n_{Slice}(\mathbb{H})$ extends the classical one on the slice $\mathbb{C}_J$. In particular, $F^r_n(q)$ and $K_n(q,r)$ coincide on the slice $\mathbb{C}_J$ containing r. Hence, we have $K_n(q,r) = F^r_n(q)$ everywhere on $\mathbb{H}$ thanks to the Identity Principle for slice polyanalytic functions. This ends the proof. □

Remark 10.3.8. *For $n = 1$, the space $\mathcal{F}^n_{Slice}(\mathbb{H})$ is exactly the slice hyperholomorphic Fock space and the reproducing kernel obtained in Theorem 10.3.7 corresponds to the result obtained in [15].*

10.3.2 The quaternionic slice polyanalytic Bergman space

The slice polyanalytic Bergman space of the second kind on the quaternionic unit ball $\mathbb{B}$ is defined to be

$$\mathcal{A}^n_{Slice}(\mathbb{B}) := \{f \in \mathcal{SP}_N(\mathbb{B}) / \int_{\mathbb{B}_I} |f_I(p)|^2 d\lambda_I(p) < \infty\},$$

for $p = x + Iy$, $d\lambda_I(p) = dxdy$ is the usual Lebesgue measure on $\mathbb{B}_I = \mathbb{B} \cap \mathbb{C}_I$. This space is endowed with the following inner product

$$\langle f, g \rangle_{\mathcal{A}^n_{Slice}(\mathbb{B})} = \int_{\mathbb{C}_I} \overline{g_I(p)} f_I(p) d\lambda_I(p).$$

As we have seen in the previous section for the Fock space, we can use the same techniques involving the Splitting Lemma and Representation Formula for slice polyanalytic functions to prove that $\mathcal{A}^n_{Slice}(\mathbb{B})$ is a right quaternionic Hilbert space which does not depend on the choice of the slices. Furthermore, for any $q \in \mathbb{B}$ and $f \in \mathcal{A}^n_{Slice}(\mathbb{B})$ we have the following estimate

$$|f(q)| \leq \frac{n}{\sqrt{\pi}} \frac{\|f\|_{\mathcal{A}^n_{Slice}(\mathbb{B})}}{(1 - |q|^2)}.$$

Hence, the Riesz representation theorem for quaternionic right-linear Hilbert spaces shows that $\mathcal{A}^n_{Slice}(\mathbb{B})$ has a reproducing kernel. The theory of quaternionic Bergman spaces of the second kind introduced in [43] suggests that the expression of the reproducing kernel of $\mathcal{A}^n_{Slice}(\mathbb{B})$ denoted by $B^n_S(q, r)$ is obtained making use of the extension operator. Indeed, let $r \in \mathbb{B}$ be fixed such that $r \in \mathbb{C}_J$, the expression of the kernel in the slice $\mathbb{B}_J$ is given in [22] by

$$B^r_n(z) = \frac{n}{\pi(1 - \overline{r}z)^{2N}} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k+1} \binom{n+k}{n} |1 - \overline{r}z|^{2(n-1-k)} |z - r|^{2k}. \tag{10.3.1}$$

Then, by definition, for any $q = x + Iy \in \mathbb{B}$ we have

$$\begin{aligned} B^n_S(q, r) &= B^r_n(q) \\ &:= ext[B^r_n(z)](q). \end{aligned}$$

To give the explicit expression of $B^n_S(q, r)$, we consider first the function $f^r_n : \mathbb{B}_J \longrightarrow \mathbb{C}_J$, depending on r and defined by

$$f^r_n(z) = \frac{n}{\pi} \frac{1}{(1 - \overline{r}z)^{2n}}; \forall z \in \mathbb{B}_J.$$

We start by proving the following

Lemma 10.3.9. *For every fixed $r \in \mathbb{B}_J$, the slice regular extension of $f^r_n(z)$ to the quaternionic unit ball $\mathbb{B}$ is given by*

$$g^r_n(q) = P_n(q, r) Q_n(q, r); \ \forall q \in \mathbb{B}$$

where

$$P_n(q, r) = \frac{N}{\pi} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \overline{q}^k \overline{r}^k, \textit{ and } Q_n(q, r) = (1 - 2Re(q)\overline{r} + |q|^2 \overline{r}^2)^{-2n}.$$

Proof. Clearly, the function $f_n^r : z \mapsto f_n^r(z)$ is holomorphic on $\mathbb{B}_J$ for every fixed $r \in \mathbb{B}_J$. Then, by definition for $q = x + Iy$ and $z = x + Jy$ the slice regular extension of $f_N^r(z)$ to $\mathbb{B}$ is given by

$$g_n^r(q) = \frac{1}{2}[f_n^r(z) + f_n^r(\bar{z})] + \frac{IJ}{2}[f_n^r(\bar{z}) - f_n^r(z)].$$

We have

$$\begin{aligned}
\frac{f_n^r(z) + f_n^r(\bar{z})}{2} &= \frac{n}{2\pi}\left[\frac{1}{(1-\bar{r}z)^{2n}} + \frac{1}{(1-\bar{r}\bar{z})^{2n}}\right] \\
&= \frac{n}{\pi}\frac{\displaystyle\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\bar{r}^k\frac{(z^k+\bar{z}^k)}{2}}{(1-2Re(z)\bar{r}+|z|^2\bar{r}^2)^{2n}} \\
&= \frac{n}{\pi}\frac{\displaystyle\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\bar{r}^k Re(z^k)}{(1-2Re(z)\bar{r}+|z|^2\bar{r}^2)^{2n}}.
\end{aligned}$$

Similarly, we obtain

$$\frac{f_n^r(\bar{z}) - f_n^r(z)}{2} = \frac{n}{\pi}\frac{\displaystyle J\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\bar{r}^k Im(z^k)}{(1-2Re(z)\bar{r}+|z|^2\bar{r}^2)^{2n}}.$$

Since $(1 - 2Re(z)\bar{r} + |z|^2\bar{r}^2)^{-2n} = Q_n(q, r)$, it follows by the formula of the extension operator that

$$\begin{aligned}
g_n^r(q) &= \frac{n}{\pi}\left[\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}(Re(z^k) - Im(z^k)I)\bar{r}^k\right]Q_n(q,r) \\
&= \frac{N}{\pi}\left[\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\bar{q}^k\bar{r}^k\right]Q_n(q,r) \\
&= P_n(q,r)Q_n(q,r).
\end{aligned}$$

This ends the proof. $\square$

In order to have kernels that are slice functions we use the $*$-product of (left) slice functions in the first variable as we did in the previous section. We write the expression of the slice poly-Bergman kernel of the second kind of the quaternionic slice polyanalytic Bergman space $\mathcal{A}^n_{Slice}(\mathbb{B})$ as follows:

Theorem 10.3.10. *The set $\mathcal{A}^n_{Slice}(\mathbb{B})$ is a right quaternionic reproducing kernel Hilbert space whose reproducing kernel is given by*

$$B_S^n(q,r) = P_n(q,r)Q_n(q,r) * \psi_n(q,r); \ \forall (q,r) \in \mathbb{B} \times \mathbb{B}$$

where

$$P_n(q,r)=\frac{n}{\pi}\sum_{k=0}^{2n}(-1)^k\binom{2N}{k}\overline{q}^k\overline{r}^k,\ Q_n(q,r)=(1-2Re(q)\overline{r}+|q|^2\overline{r}^2)^{-2n}$$

and

$$\psi_n(q,r)=\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\binom{n+k}{n}(1-\bar{q}r-q\bar{r}+\bar{q}q\bar{r}r)^{(n-1-k)*}*(\bar{q}q-q\bar{r}-\bar{q}r+\bar{r}r)^{k*}.$$

Proof. For any $r\in\mathbb{B}$ we consider the function

$$h_n^r(q)=g_n^r(q)*\psi_n(q,r),$$

where $g_n^r(q)=P_n(q,r)Q_n(q,r)$ and $*$ is the product of slice functions. The polynomials $P_n(q,r)$ and $Q_n(q,r)$ are defined as in Lemma 10.3.9 and

$$\psi_n(q,r)=\sum_{k=0}^{n-1}(-1)^k\binom{n}{k+1}\binom{n+k}{n}(1-\bar{q}r-q\bar{r}+\bar{q}q\bar{r}r)^{(n-1-k)*}*(\bar{q}q-q\bar{r}-\bar{q}r+\bar{r}r)^{k*}.$$

According to Lemma 10.3.9 we note that g_n^r is slice regular by construction with respect to the variable q. We can see also that $\psi_n(q,r)$ is a slice polyanalytic function of order n on $\mathbb{B}$ with respect to q. Thus, h_n^r is a slice polyanalytic function of order n on $\mathbb{B}$ with respect to q. Moreover, $h_n^r(q)$ and $B_S^n(q,r)$ coincide on the slice $\mathbb{B}_J$ containing r. Hence, thanks to the Identity Principle for slice polyanalytic functions $B_S^n(q,r)=h_n^r(q)$ everywhere on $\mathbb{B}$. This completes the proof. □

Proposition 10.3.11. *The kernel $B_S^n(q,r)$ can be written also in this second form*

$$B_S^n(q,r)=R_n(q,r)L_n(q,r)*\psi_n(q,r);\ \forall(q,r)\in\mathbb{B}\times\mathbb{B}$$

with

$$R_n(q,r)=(1-2qRe(r)+q^2|r|^2)^{-2n}\ \textit{and}\ L_n(q,r)=\frac{n}{\pi}\sum_{k=0}^{2N}(-1)^k\binom{2n}{k}q^kr^k.$$

Proof. Set $\phi(q,r)=R_n(q,r)L_n(q,r)$ for all $q,r\in\mathbb{B}$. As a product of a rational function with real coefficients and a polynomial of order $2n$ with quaternionic coefficients on the right the function $\phi(.,r)$ is slice regular on $\mathbb{B}$ with respect to the variable q for every $r\in\mathbb{B}$. Moreover, if $r\in\mathbb{B}$ is fixed on a slice $\mathbb{C}_J$ we can see that the restriction of $\phi(.,r)$ on $\mathbb{B}_J$ coincides with the function $f_n^r(z)=\frac{n}{\pi}\frac{1}{(1-\overline{r}z)^{2n}}$. Then, the Identity Principle for slice regular functions gives

$$ext(f_n^r)(q)=R_n(q,r)L_n(q,r)\text{ for all }q,r\in\mathbb{B}.$$

The last equation leads to the desired result. □

Remark 10.3.12. *For the particular case $n=1$, the results obtained in this section coincide with the results of [43] concerning the theory of the second kind for the slice hyperholomorphic Bergman spaces.*

10.4 Further remarks

We finish this chapter with a brief discussion related to further developments of the theory of slice polyanalytic functions. First, we note that the pointwise multiplication of two slice polyanalytic functions is not slice polyanalytic, in general. In fact this fact appears also for $n = 1$, namely for the case of slice regular functions. However, the $*$-product preserves the structure of slice regular functions. For slice polyanalaytic functions of the same order we can consider also a natural product denoted $*_n$ in order to preserve the structure. Indeed, let f and g be two slice polyanalytic functions of order n on Ω such that

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q) \text{ and } g(q) = \sum_{k=0}^{n-1} \overline{q}^k g_k(q),$$

where f_k, g_k are slice regular for all $k = 0, .., n-1$. Then, we define

$$f *_n g(q) := \sum_{k=0}^{n-1} \overline{q}^k (f_k * g_k)(q)$$

where $f_k * g_k$ stands for the classical $*$-product of slice regular functions. It turns out that the set $(\mathcal{SP}_n(\Omega), +, *_n)$ is a ring, so we wish to study further properties of this product in future researches.

Furthermore, in the recent paper [87], the authors introduced and studied the poly-Hardy space on the unit ball in the monogenic setting. A natural problem would be to study the counterpart of the poly-Hardy space in this new slice poly-analytic setting. However, like in the classical complex case, this space would be trivial seen as subspace of $L^2(\mathbb{B})$.

CHAPTER 11

The global operator and Fueter mapping theorem for hypercomplex polyanalytic functions

In this chapter, we prove that slice polyanalytic functions on quaternions can be considered as solutions of a power of some special global operator with nonconstant coefficients as it happens in the case of slice hyperholomorphic functions. We investigate also an extension version of the Fueter mapping theorem in this polyanalytic setting. In particular, we show that under axially symmetric conditions it is always possible to construct Fueter regular and poly-Fueter regular functions through slice polyanalytic ones using what we call the poly-Fueter mappings. We study also some integral representations of these results on the quaternionic unit ball. The results presented in this chapter are based on [9].

11.1 Motivation

This chapter proposes a bridge between two theories: the one of slice polyanalytic functions and the one of poly-Fueter regular functions. To understand the framework, we recall that in classical complex analysis, n-analytic or polyanalytic functions are null-solutions of the n-power of the Cauchy-Riemann operator. In the quaternionic setting or, more in general, in the Clifford algebra setting, one can extend this notion by considering functions in the kernel of a generalized Cauchy-Riemann operator (thus obtaining the so-called regular or monogenic functions, see [47,83]) or of its n-power (thus obtaining poly-regular functions or poly-monogenic functions, see [87, 101]).

This was the first approach to extend holomorphic functions, and then polyanalytic functions, to a higher dimensional setting. It is interesting to note that the class of slice hyperholomorphic functions is related with the class of functions considered by Fueter to construct regular functions and thus there is a bridge between them, specifically the so-called Fueter mapping, in fact by applying the Laplacian to a slice hyperhomolorphic function one obtains a regular function, i.e. a function in the kernel of the Cauchy-Fueter operator, see for example [48]. Also the theory of polyanalytic functions can be extended to the slice setting by considering a suitable definition, as we did in [17]. Thus it is a natural question to ask whether there is an analog of the Fueter map in this more general setting. The answer is positive and it is one of the main results of this chapter: we show that by applying the Laplacian composed with the $(n-1)$ power of the global operator $V = 2\overline{\vartheta}$ (where $\overline{\vartheta}$ is the operator introduced in [80]) to any slice polyanalytic function of order n we obtain a Cauchy-Fueter regular function. A second approach to extend the Fueter mapping to the polyanalytic setting consists to apply the standard Fueter mapping on each component associated to the poly-decomposition. This constrution allows to generate poly-Fueter regular functions starting from slice polyanalytic ones of the same order.

To put our work in perspective, we recall that classical polyanalytic functions are important not only from the theoretical point of view, see [22], but also in the theory of signals since they allow to encode n independent analytic functions into a single polyanalytic one using a special decomposition. This idea is similar to the problem of multiplexing signals. This is related to the construction of the polyanalytic Segal-Bargmann transform mapping $L^2(\mathbb{R})$ onto the poly-Fock space, see [2]. In quantum physics these functions are relevant for the study of the Landau levels associated to Schrödinger operator, see [2, 64]. Polyanalytic functions were used also in [1] to study sampling and interpolation problems on Fock spaces using time frequency analysis techniques such as short-time Fourier transform (STFT) or Gabor transforms. This allows to extend Bargmann theory to the polyanalytic setting using Gabor analysis. The theory of signals is widely studied also with hypercomplex methods and for a list of references the reader may consult [31] and the references therein.

As we said, Fueter regular and slice hyperholomorphic functions are related by the famous Fueter mapping theorem. This result has some important consequences and allows to define the $\mathcal{F}$-functional calculus for quaternionic operators with commuting components. Recently, new several results for polyanalytic functions were proven in the slice hyperholomorphic context over the quaternions, see [17], and the counterparts of the Bergman and Fock spaces were also considered. We continue here the investigations in this direction. In particular, we prove a new version of the well-known Fueter mapping theorem that will relate slice and Cauchy-Fueter polyanalytic functions on quaternions and present an integral form of this result.

The chapter has the following structure: we set up first some basic notations and revise some preliminary results. Then, we present some new results on the powers of the global operator V and give the main statements and proofs of the poly-Fueter mapping theorems on quaternions. We study also an integral representation of these results based on the poly-Cauchy formula. Finally, we rewrite our results in the polymonogenic case.

11.2 Preliminary results

We revise different notions and results related to Cauchy-Fueter and slice hyperholomorphic functions and also the polyanalytic setting on quaternions. We first recall below the variations of the Fueter mapping theorem that we will use later in this work and refer the reader to [48, 102] for several extensions.

Theorem 11.2.1 (Fueter mapping theorem [48]). *Let U be an axially symmetric set in $\mathbb{H}$ and let $f : U \subset \mathbb{H} \longrightarrow \mathbb{H}$ be a slice hyperholomorphic function of the form $f(x+yI) = \alpha(x,y)+I\beta(x,y)$, where $\alpha(x,y)$ and $\beta(x,y)$ are quaternionic-valued functions such that $\alpha(x,-y) = \alpha(x,y)$, $\beta(x,-y) = -\beta(x,y)$ and satisfying the Cauchy-Riemann system. Then, the function*

$$\tilde{f}(x_0+\vec{q}) = \Delta\left(\alpha(x_0,|\vec{q}|) + \frac{\vec{q}}{|\vec{q}|}\beta(x_0,|\vec{q}|)\right)$$

extends to a Fueter regular function on the whole U.

Remark 11.2.2. *If U is an axially symmetric slice domain in $\mathbb{H}$, then every slice hyperholomorphic function $f : U \subset \mathbb{H} \longrightarrow \mathbb{H}$ is of the form $f(x+Iy) = \alpha(x,y) + I\beta(x,y)$, where α and β have the properties mentioned in the preceding statement. This is an immediate consequence of the Representation formula observed in Lemma 2.2 in [45].*

A function $f(x+yI) = \alpha(x,y)+I\beta(x,y)$, where α, β are $\mathbb{H}$ (or $\mathbb{R}_n$)-valued, $\alpha(x,-y) = \alpha(x,y)$, $\beta(x,-y) = -\beta(x,y)$ is called a slice function.

Remark 11.2.3. *We denote by $\mathcal{SR}(U)$ the space of slice regular functions which are slice functions. Below, we can consider the Fueter mapping defined by*

$$\tau : \mathcal{SR}(U) \to \mathcal{FR}(U),\ f \longmapsto \tau(f) = \Delta(f).$$

Theorem 11.2.4 ([48]). *Given a quaternion $s \in \mathbb{H}$, we define*

$$[s] = \{p \in \mathbb{H} :\ p = Re(s) + I|\vec{s}|, I \in \mathbb{S}\}.$$

Let $S^{-1}(s,q)$ be the Cauchy kernel defined by:

$$S^{-1}(s,q) = (s-\overline{q})(s^2 - 2Re(q)s + |q|^2)^{-1},\ q \notin [s].$$

Then the function

$$\mathcal{F}(s,q) := \Delta S^{-1}(s,q) = -4(s-\overline{q})(s^2 - 2Re(q)s + |q|^2)^{-2},$$

is a Cauchy-Fueter regular function in the variable q, and it is right slice regular in the variable s for $q \notin [s]$.

Theorem 11.2.5 (The Fueter mapping theorem in integral form [48]). *Let $W \subset \mathbb{H}$ be an axially symmetric open set and let f be slice hyperholomorphic in W. Let U be a bounded axially symmetric open set such that $\overline{U} \subset W$. Suppose that the boundary of $U_I = U \cap \mathbb{C}_I$ consists of finite number of rectifiable Jordan curves for any $I \in \mathbb{S}$. Then, if $q \in U$, the Cauchy–Fueter regular function given by*

$$\tau(f)(q) = \Delta f(q)$$

has the integral representation

$$\tau(f)(q) = \frac{1}{2\pi} \int_{\partial U_I} \Delta S^{-1}(s,q) ds_I f(s), \; ds_I = ds/I,$$

and the integral does not depend on U nor on the imaginary unit $I \in \mathbb{S}$.

We will need also these useful results in our computations

Proposition 11.2.6 ([24]). *For all $n \geq 2$, we have*

$$\mathcal{D}[q^n] = -2 \sum_{k=1}^{n} q^{n-k} \overline{q}^{k-1}.$$

Proposition 11.2.7 ([63]). *For all $n \geq 2$, we have*

$$\tau[q^n] = -4 \sum_{k=1}^{n-1} (n-k) q^{n-k-1} \overline{q}^{k-1}.$$

In [?] the theory of slice hyperholomorphic functions on quaternions is extended to higher order by considering:

Definition 11.1. *Let Ω be an axially symmetric open set in $\mathbb{H}$ and let $f : \Omega \longrightarrow \mathbb{H}$ a slice function of class $\mathcal{C}^n$. For each $I \in \mathbb{S}$, let $\Omega_I = \Omega \cap \mathbb{C}_I$ and let $f_I = f_{|\Omega_I}$ be the restriction of f to Ω_I. The restriction f_I is called (left) polyanalytic of order n if it satisfies on Ω_I the equation*

$$\overline{\partial_I}^n f(x+Iy) := \frac{1}{2^n} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right)^n f_I(x+Iy) = 0.$$

The function f is called left slice polyanalytic of order n, if for all $I \in \mathbb{S}$, f_I is left polyanalytic of order n on Ω_I. The right quaternionic vector space of slice polyanalytic functions of order n will be denoted by $\mathcal{SP}_n(U)$.

Note that slice regular functions are a special case of the definition of slice polyanalytic functions with $n = 1$. The right slice polyanalytic functions can be defined in a similar way just by taking the powers of the Cauchy-Riemann operator with imaginary unit on the right. Several results of these functions were studied and extended. In particular, we recall some properties that we need for our computations in the next sections.

Proposition 11.2.8 (Splitting Lemma). *Let f be a slice polyanalytic function of order n on a domain $\Omega \subseteq \mathbb{H}$. Then, for any imaginary units I and J with $I \perp J$ there exist $F, G : \Omega_I \longrightarrow \mathbb{C}_I$ polyanalytic functions of order n such that for all $z = x + Iy \in \Omega_I$, we have*

$$f_I(z) = F(z) + G(z)J.$$

We will be interested also by the following decomposition

Proposition 11.2.9 (Poly-decomposition). *A function $f : \Omega \longrightarrow \mathbb{H}$ defined on a slice domain is slice polyanalytic of order n if and only there exist $f_0, ..., f_{n-1}$ some unique slice hyperholomorphic functions on Ω such that we have the following decomposition:*

$$f(q) := \sum_{k=0}^{n-1} \overline{q}^k f_k(q);\ \forall q \in \Omega.$$

Finally, we consider the poly-Fueter regular functions that can be found for example in [87] for Clifford valued functions.

Definition 11.2.1. *Let $U \subset \mathbb{H}$ be an open set and let $f : U \longrightarrow \mathbb{H}$ be a function of class $\mathcal{C}^n$. We say that f is (left) poly-Fueter regular or poly-regular for short of order $n \geq 1$ on U if*

$$\mathcal{D}^n f(q) := \left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} \right)^n f(q) = 0, \forall q \in U.$$

The right quaternionic vector space of poly-Fueter regular functions will be denoted by $\mathcal{FR}_n(U)$.

The proof of the next result was communicated to us by Dan Volok, and appears earlier in section 6 and 7 of [27], see also [57] for the Clifford monogenic setting. We recall it for completeness

Proposition 11.2.10. *A function f is poly-Fueter regular of order n if and only if it can be decomposed in terms of some unique Fueter regular functions $\phi_0, ..., \phi_{n-1}$ such that we have*

$$f(q) = \sum_{k=0}^{n-1} x_0^k \phi_k(q).$$

11.3 The global operator and poly-Fueter mapping theorems

In this section, we show that slice polyanalytic functions of some order n are solutions of the n-th power of a certain global operator V. A new extension of the Fueter mapping theorem involving slice polyanalytic functions on quaternions will be proved also.

In [80], the author considered a modified version of the operator G which is defined by

$$V(f)(q) := \partial_{x_0} f(q) + \frac{\vec{q}}{|\vec{q}|^2} \sum_{l=1}^{3} x_l \partial_{x_l} f(q), \forall q \in \Omega \setminus \mathbb{R}.$$

Remark 11.3.1. *For suitable domains, we note that the operators G and V are related by the formula*

$$V(f)(q) = \frac{1}{|\vec{q}|^2} G(f)(q), \qquad \forall q \in \Omega \setminus \mathbb{R}.$$

In what follows, if $V(f)$ admits a (unique) continuous extension on the whole Ω, then we implicitly assume that $V(f)$ denotes such an extension. Given any $n \geq 2$, inductively we will say that $V^n(f)$ is a function on Ω if $V^{n-1}(f)$ is of class $\mathcal{C}^1$ on $\Omega \setminus \mathbb{R}$ and $V^n(f) := V(V^{n-1}(f))$ admits a continuous extension on Ω.

First, we prove some preliminary results on the global operators G and V that are needed in the sequel.

Lemma 11.3.2. *Let Ω be an open set in $\mathbb{H}$ and $\psi : \Omega \longrightarrow \mathbb{H}$ a function of class $\mathcal{C}^1$. Then, we have*

$$G(\overline{q}\psi)(q) = \overline{q} G(\psi)(q) + 2|\vec{q}|^2 \psi(q), \ \forall q = x_0 + \vec{q} \in \Omega.$$

Proof. Let ψ be a $\mathcal{C}^1$ function on Ω, we apply the definition of G and Leibniz rule with respect to the partial derivatives and we get

$$\begin{aligned} G(\overline{q}\psi)(q) &:= |\vec{q}|^2 \partial_{x_0}(\overline{q}\psi)(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l}(\overline{q}\psi)(q) \\ &= |\vec{q}|^2 \overline{q} \partial_{x_0} \psi(q) + |\vec{q}|^2 \psi(q) + \vec{q}\ \overline{q} \sum_{l=1}^{3} x_l \partial_{x_l} \psi(q) - \vec{q} \sum_{l=1}^{3} x_l e_l \psi(q). \end{aligned}$$

However, we know that

$$\vec{q} = \sum_{l=1}^{3} x_l e_l, \quad \vec{q}\ \overline{q} = \overline{q}\vec{q} \quad \text{and} \quad \vec{q}^2 = -|\vec{q}|^2.$$

Thus, for any $q \in \Omega$ we have

$$\begin{aligned} G(\overline{q}\psi)(q) &= \overline{q} \left(|\vec{q}|^2 \partial_{x_0} \psi(q) + \vec{q} \sum_{l=1}^{3} x_l \partial_{x_l} \psi(q) \right) + 2|\vec{q}|^2 \psi(q) \\ &= \overline{q} G(\psi)(q) + 2|\vec{q}|^2 \psi(q). \end{aligned}$$

This ends the proof. $\square$

Corollary 11.3.3. *Let $\Omega \subseteq \mathbb{H}$ be a domain and $f : \Omega \longrightarrow \mathbb{H}$ be a slice hyperholomorphic function. Then, we have*

$$G(\overline{q}f)(q) = 2|\vec{q}|^2 f(q), \ \forall q \in \Omega$$

and

$$V(\overline{q}f)(q) = 2f(q), \ \forall q \in \Omega. \tag{11.3.1}$$

Proof. The fact that f is slice hyperholomorphic on Ω implies that

$$G(f)(q) = 0, \ \forall q \in \Omega.$$

Hence, a direct application of Lemma 11.3.2 gives (11.3.1) on $\Omega \setminus \mathbb{R}$. However, since the right hand side of (11.3.1) extends the left hand side to all of Ω as a slice hyperholomorphic function, then (11.3.1) holds on Ω. □

Example. *To provide an example, let us consider the particular case $f \in \mathcal{SP}_2(\mathbb{H})$. Then, we have*

1. $V^2(f)(q) = 0, \ \forall q \in \mathbb{H}$.
2. $\Delta V(f)$ *is Cauchy-Fueter regular on* $\mathbb{H}$.
3. $\overline{\mathcal{D}}V(f)$ *is poly-Fueter regular of order* 2, *where* $\overline{\mathcal{D}}$ *is the conjugate of the Cauchy-Fueter operator.*

To see that (1) holds, we use the poly-decomposition that asserts the existence of some unique functions $f_0, f_1 \in \mathcal{SR}(\mathbb{H})$ such that

$$f(q) = f_0(q) + \overline{q}f_1(q), \forall q \in \mathbb{H}.$$

An application of corollary 11.3.3 combined with the fact that slice hyperholomorphic functions belong to $\ker(V)$ *show that (1) holds. The other two assertions follows similarly.*

Proposition 11.3.4. *Let Ω be an open set in $\mathbb{H}$ and $f : \Omega \longrightarrow \mathbb{H}$ a slice hyperholomorphic function. Let $n \geq 2$ and $1 \leq k \leq n-1$, then we have*

1. $G(\overline{q}^k f)(q) = 2k|\vec{q}|^2\overline{q}^{k-1} f(q), \ \forall q \in \Omega.$
2. $V(\overline{q}^k f)(q) = 2k\overline{q}^{k-1} f(q), \ \forall q \in \Omega.$

Proof. Let $f \in \mathcal{SR}(\Omega)$ and $n \geq 2$. We reason by induction with respect to n.

(1) First, we note that the result holds for $n = 2$ as a consequence of Corollary 11.3.3. Now, let $n \geq 2$ be such that we have

$$G(\overline{q}^k f)(q) = 2k|\vec{q}|^2\overline{q}^{k-1} f(q), \ \forall q \in \Omega, \forall 1 \leq k \leq n-1.$$

In order, to prove that the result holds for $n+1$, we only have to show that

$$G(\overline{q}^n f)(q) = 2n|\vec{q}|^2\overline{q}^{n-1} f(q), \forall q \in \Omega. \tag{11.3.2}$$

Indeed, we apply Lemma 11.3.2 and obtain

$$\begin{aligned}G(\overline{q}^n\psi)(q) &= G(\overline{q}\,\overline{q}^{n-1}f)(q)\\ &= \overline{q}G(\overline{q}^{n-1}f)(q) + 2|\vec{q}|^2\overline{q}^{n-1}f(q), \forall q \in \Omega.\end{aligned}$$

However, by induction hypothesis we know that

$$G(\overline{q}^{n-1}f)(q) = 2(n-1)|\vec{q}|^2\overline{q}^{n-2}f(q), \forall q \in \Omega.$$

Therefore, we get

$$\begin{aligned}G(\overline{q}^n f)(q) &= 2(n-1)|\vec{q}|^2\overline{q}^{n-1}f(q) + 2|\vec{q}|^2\overline{q}^{n-1}f(q)\\ &= 2n|\vec{q}|^2\overline{q}^{n-1}f(q), \forall q \in \Omega.\end{aligned}$$

Hence, the result holds by induction and this completes the proof.

(2) We know by Remark 11.3.1 that

$$V(f)(q) = \frac{1}{|\vec{q}|^2}G(f)(q), \forall q \in \Omega \setminus \mathbb{R}.$$

Then, since f is a slice hyperholomorphic function on Ω, the right hand side extends the left hand side as polyanalytic function of order k and so we get

$$V(\overline{q}^k f)(q) = 2k\overline{q}^{k-1}f(q),\ \forall q \in \Omega,\ \ 1 \leq k \leq n-1.$$

□

Proposition 11.3.5. *Let Ω be a slice domain in $\mathbb{H}$ and $f : \Omega \longrightarrow \mathbb{H}$ a slice polyanalytic function of order $n \geq 1$. Then, $V(f)$ is a slice polyanalytic function of order $n-1$ on Ω.*

Proof. We note that Ω is a slice domain. So, by poly-decomposition there exist some unique slice regular functions $\varphi_0, ..., \varphi_{n-1}$ such that we can write

$$f(q) = \sum_{k=0}^{n-1}\overline{q}^k\varphi_k(q), \forall q \in \Omega.$$

Thus, by Proposition 11.3.4 we know that for all $q \in \Omega$ we have

$$\begin{aligned}V(f)(q) &= \sum_{k=1}^{n-1}V(\overline{q}^k\varphi_k)(q)\\ &= 2\sum_{k=1}^{n-1}k\overline{q}^{k-1}\varphi_k(q)\\ &= \sum_{h=0}^{n-2}\overline{q}^h\zeta_h(q),\end{aligned}$$

where we have set $\zeta_h(q) = 2(h+1)\varphi_{h+1}$, $\forall 0 \leq h \leq n-2$ which are slice hyperholomorphic functions on the whole Ω by hypothesis. Hence, $V(f)$ extends as a slice polyanalytic function of order $n-1$ on Ω. □

Theorem 11.3.6. *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and $f : \Omega \longrightarrow \mathbb{H}$ a slice polyanalytic function of order $n \geq 1$. Then, f belongs to* $\ker(V^n)$, *i.e:*

$$V^n(f)(q) = 0, \ \forall q \in \Omega.$$

Proof. We apply Proposition 11.3.5 iteratively and obtain

$$V(f) \in \mathcal{SP}_{n-1}(\Omega), V^2(f) \in \mathcal{SP}_{n-2}(\Omega), ..., V^{n-1}(f) \in \mathcal{SP}_1(\Omega) = \mathcal{SR}(\Omega).$$

In particular, we deduce that $V^{n-1}(f)$ is a slice hyperholomorphic function on Ω. Therefore, it belongs to the kernel of the global operator V outside the real line. Hence, since $V^{n-1}(f)$ admits a continuous extension to the whole Ω, by Theorem 2.4 in [80], we conclude that

$$V^n(f)(q) = V(V^{n-1})(f)(q) = 0, \ \forall q \in \Omega.$$

This ends the proof. □

Theorem 11.3.7 (Poly-Fueter mapping theorem I)**.** *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and $f : \Omega \longrightarrow \mathbb{H}$ a slice polyanalytic function of order $n \geq 1$. Then the function given by*

$$\tau_n(f)(q) = \Delta \circ V^{n-1}(f)(q), \ \forall q \in \Omega$$

belongs to the kernel of the Cauchy-Fueter operator $\mathcal{D}$.

Proof. Using the same argument used to prove Theorem 11.3.6, we deduce that $V^{n-1}(f)$ is a slice hyperholomorphic function on Ω. Therefore, since Ω is an axially symmetric slice domain we can use Theorem 11.2.1 and Remark 11.2.2 to conclude that the function $\tau_n(f)$ is in the kernel of the Cauchy-Fueter operator $\mathcal{D}$ on Ω, i.e.,

$$\mathcal{D} \circ \tau_n(f)(q) = \mathcal{D} \circ \Delta \circ V^{n-1}(f)(q) = 0, \ \forall q \in \Omega.$$

□

Remark 11.3.8. *We note that the poly-Fueter mapping*

$$\tau_n := \Delta \circ V^{n-1}$$

takes the space of slice polyanalytic functions of order $n \geq 1$ into the space of Cauchy-Fueter regular functions $\mathcal{FR}(\Omega)$.

Theorem 11.3.9. *Let Ω be an axially symmetric slice domain of $\mathbb{H}$ and $f : \Omega \longrightarrow \mathbb{H}$ a slice hyperholomorphic function. Let $n \geq 1$ and consider the functions defined by*

$$\Psi_f^k(q) := \overline{q}^k f(q), \ \forall q \in \Omega, \forall 0 \leq k \leq n-1.$$

Then, the family $\{\Psi_f^k\}_{0 \leq k \leq n}$ forms an Appell system with respect to the operator $\frac{1}{2}V$, namely

$$\frac{1}{2}V(\Psi_f^0) = 0 \text{ and } \frac{1}{2}V(\Psi_f^k) = k\Psi_f^{k-1}, \forall 1 \leq k \leq n-1.$$

Proof. The function $\Psi_f^0 = f$ is slice hyperholomorphic on Ω. So, f belongs to the kernel of the global operator V on Ω. Thus, we have $\frac{1}{2}V(\Psi_f^0) = 0$. On the other hand, we know by Proposition 11.3.4 that

$$V(\overline{q}^k f)(q) = 2k\overline{q}^{k-1}f(q),\ \forall q \in \Omega, 1 \leq k \leq n-1.$$

Therefore, this combined with Theorem 2.4 in [80] allows to see that for all $q \in \Omega$ and $1 \leq k \leq n-1$ we have

$$\begin{aligned}\frac{1}{2}V(\Psi_f^k)(q) &= \frac{1}{2}V(\overline{q}^k f)(q)\\ &= k\overline{q}^{k-1}f(q)\\ &= k\Psi_f^{k-1}(q).\end{aligned}$$

This ends the proof. □

Corollary 11.3.10. *The sequence $\{\overline{q}^k\}_{k\geq 0}$ is an Appell system with respect to $\frac{1}{2}V$.*

Proof. If we take the constant function $f = 1$, we immediately obtain the result. □

Remark 11.3.11. *We note that for any slice hyperholomorphic function f the family $\{\Psi_f^k\}_{0\leq k\leq n}$ considered in Theorem 11.3.9 form also an Appell system with respect to the Cauchy-Riemann operator $\frac{1}{2}\overline{\partial_I}$ for all $I \in \mathbb{S}$.*

The next result allows to construct poly-Fueter regular functions starting from slice polyanalytic ones of the same order:

Theorem 11.3.12 (Poly-Fueter mapping theorem II). *Let $\Omega \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : \Omega \longrightarrow \mathbb{H}$ a slice polyanalytic function of order $n \geq 1$. Assume that f admits the decomposition*

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q), \forall q \in \Omega$$

where $f_0, ..., f_{n-1} \in \mathcal{SR}(\Omega)$. Then, the function defined by

$$\mathcal{C}_n(f)(q) = \sum_{k=0}^{n-1} x_0^k \Delta(f_k)(q), \forall q \in \Omega \tag{11.3.3}$$

is a poly-Fueter regular function of order n.

Proof. We note by Theorem 11.2.1 and Remark 11.2.2 that the functions $\phi_k = \Delta(f_k)$ are all Cauchy-Fueter regular on Ω for any $0 \leq k \leq n-1$. Hence, thanks to Proposition 11.2.10 we conclude that $\mathcal{C}_n(f)$ is poly-Fueter regular of order n. □

Let $n \geq 1$, the two poly-Fueter mappings τ_n and $\mathcal{C}_n$ can be related to each other so that we have

$$\tau_n := (2\mathcal{D})^{n-1} \circ \mathcal{C}_n,$$

in other words the diagram

$$\begin{tikzcd} \mathcal{SP}_n \arrow[r, "\tau_n"] \arrow[d, "\mathcal{C}_n"'] & \mathcal{FR} \\ \mathcal{FR}_n \arrow[ur, "(2\mathcal{D})^{n-1}"'] & \end{tikzcd}$$

is commutative.

The proof of this fact is contained in the next result:

Theorem 11.3.13. *Let $f : \Omega \longrightarrow \mathbb{H}$ be a slice polyanalytic function of order $n \geq 1$ on some axially symmetric slice domain. Then, we have*

$$\mathcal{D}^{n-1}\mathcal{C}_n(f)(q) = \frac{1}{2^{n-1}}\tau_n(f)(q), \forall q \in \Omega.$$

Proof. Since Ω is a slice domain, by the poly-decomposition for slice polyanalytic functions there exist $f_0, .., f_{n-1} \in \mathcal{SR}(\Omega)$ such that

$$f(q) = \sum_{k=0}^{n-1} \overline{q}^k f_k(q), \forall q \in \Omega.$$

Thus, by Proposition 11.3.4 gives

$$V(f)(q) = \sum_{k=1}^{n-1} 2k\overline{q}^{k-1} f_k(q), \forall q \in \Omega.$$

In a similar way, we apply $(n-1)$ times the global operator V and use Proposition 11.3.4 to get

$$V^{n-1}(f)(q) = 2^{n-1}(n-1)! f_{n-1}(q), \ \forall q \in \Omega.$$

As a direct consequence, by definition of τ_n we have

$$\tau_n(f)(q) = 2^{n-1}(n-1)!\Delta f_{n-1}(q), \ \forall q \in \Omega. \tag{11.3.4}$$

On the other hand, since $(f_k)_{0 \leq k \leq n-1}$ are all slice hyperholomorphic we know by the Fueter mapping theorem that

$$\mathcal{D}(\Delta f_k) = 0, \ \forall 0 \leq k \leq n-1.$$

Therefore, by Leibniz rule for the Cauchy-Fueter operator we have

$$\mathcal{D}(x_0^k \Delta f_k)(q) = k x_0^{k-1} \Delta f_k(q); \ \forall q \in \Omega, \forall 0 \leq k \leq n-1. \tag{11.3.5}$$

We know by definition of $\mathcal{C}_n$ that

$$\mathcal{C}_n(f)(q) = \sum_{k=0}^{n-1} x_0^k \Delta(f_k)(q), \forall q \in \Omega.$$

Thus, we use (11.3.5) and get

$$\mathcal{D}[\mathcal{C}_n(f)](q) = \sum_{k=1}^{n-1} k x_0^{k-1} \Delta f_k(q), \forall q \in \Omega.$$

Similarly, if we apply the Cauchy-Fueter operator $(n-1)$ times and use (11.3.5), with some computations we get

$$\mathcal{D}^{n-1}[\mathcal{C}_n(f)](q) = (n-1)! \Delta f_{n-1}(q), \forall q \in \Omega. \tag{11.3.6}$$

Finally, we combine the relations (11.3.4) and (11.3.6) to conclude that

$$\mathcal{D}^{n-1}\mathcal{C}_n(f)(q) = \frac{1}{2^{n-1}} \tau_n(f)(q), \forall q \in \Omega.$$

□

11.4 The poly-Cauchy integral theorem and poly-Cauchy formula

In this section, we prove a Cauchy integral theorem and Cauchy formula for slice polyanaytic functions.

First, we recall the polyanalytic Cauchy formula in complex analysis, see Theorem 2.1 in [57].

Theorem 11.4.1. *For $k \geq 1$, we set*

$$\psi_k(z) = \frac{1}{2\pi i} \frac{\bar{z}}{|z|^2} \frac{Re(z)^{k-1}}{(k-1)!}.$$

For $z = x + iy$, set $d\sigma = dx \wedge dy$. If f is polyanalytic of order n, then for all $z \in \mathbb{D}$ we have

$$f(z) = \int_{\partial \mathbb{D}} \sum_{j=0}^{n-1} (-2)^j \psi_{j+1}(u-z) \frac{\partial^j}{\partial \bar{u}^j} f(u) d\sigma.$$

First, we prove a version of the Cauchy's integral theorem for slice polyanalytic functions

Theorem 11.4.2 (Poly-Cauchy theorem). *Let f and g be a left and right slice polyanalytic functions of order n respectively on some axially symmetric slice domain Ω containing the closure of $\mathbb{B}$. Then, for any $I \in \mathbb{S}$ we have*

$$\int_{\partial \mathbb{B}_I} \sum_{j=0}^{n-1} (-1)^j g \overline{\partial_I}^{n-1-j} dw_I \overline{\partial_I}^j f = 0,$$

where $dw_I = -dwI$ for $w \in \mathbb{C}_I$.

Proof. Let $I \in \mathbb{S}$ and choose $J \in \mathbb{S}$ be such that $I \perp J$. Thus, by Splitting Lemma for slice polyanalytic functions proved in [17] we can write

$$f(w) = F_1(w) + F_2(w)J \text{ and } g(w) = G_1(w) + JG_2(w),$$

where $F_l, G_l : \mathbb{B}_I \longrightarrow \mathbb{C}_I$ for $l = 1, 2$ are complex polyanalytic functions of order n. In order to simplify the computations, we set

$$\Phi(f,g) := \int_{\partial\mathbb{B}_I} \sum_{j=0}^{n-1} (-1)^j g \overline{\partial_I}^{n-1-j} dw_I \overline{\partial_I}^j f$$

Then, direct computations lead to

$$\Phi(f,g) = \Phi(F_1, G_1) + \Phi(F_2, G_1)J + J\Phi(F_1, G_2) + J\Phi(F_2, G_2)J$$

At this stage, we apply the poly-Cauchy integral theorem proved in [57] to deduce that

$$\Phi(F_1, G_1) = \Phi(F_2, G_1) = \Phi(F_1, G_2) = \Phi(F_2, G_2) = 0.$$

This ends the proof. □

Now, let $n \geq 1$ and $w \in \mathbb{B}$ be such that $w \in \mathbb{C}_J$ with $J \in \mathbb{S}$. For all $0 \leq j \leq n-1$, we consider the function defined by

$$\phi_{j,w}(z) = \frac{1}{w-z} \frac{(Re(w-z))^j}{j!}; \ z \in \mathbb{B}_J, z \neq w.$$

Then, we have

Proposition 11.4.3 (Poly-Cauchy kernels). *For all $0 \leq j \leq n-1$, the slice polyanalytic extension of $\phi_{j,w}$ is given by*

$$\phi_{j,w}(q) = S^{-1}(w,q) \frac{(Re(w-q))^j}{j!} \ \forall q \in \mathbb{B}, q \notin [w],$$

where $S^{-1}(w,q)$ is the slice hyperholomorphic Cauchy kernel.

Proof. Let $0 \leq j \leq n-1$. We know that $S^{-1}(w,q)$ is left slice regular with respect to the variable q. Moreover, it is clear that $q \mapsto \dfrac{(Re(w-q))^j}{j!}$ is a real valued slice polyanalytic function of order n for all $0 \leq j \leq n-1$. So, we can apply Proposition 3.3 in [?] to see that the product $S^{-1}(w,q)\dfrac{(Re(w-q))^j}{j!}$ is slice polyanalytic of order n with respect to the variable q. And since it coincides with $\phi_{j,w}(z)$ on $\mathbb{B}_J$ the proof ends thanks to the identity principle (see [17]). □

Remark 11.4.4. *Another way to prove Proposition 11.4.3 consists of using the extension Lemma for slice polyanalytic functions, see [17]. Indeed, we note that* $z \mapsto \phi_{j,w}(z)$ *is polyanalytic of order* n *for any* $z \neq w$. *Thus, it admits a unique slice polyanalytic extension denoted by* $ext[\phi_{j,w}(z)](q)$. *By definition, for* $q = x + I_q y$ *and* $z = x + Jy$ *such that* $q \notin [w]$ *we have*

$$\begin{aligned} ext[\phi_{j,w}(z)](q) &= \frac{1}{2}[\phi_{j,w}(z) + \phi_{j,w}(\overline{z})] + \frac{I_q J}{2}[\phi_{j,w}(\overline{z}) - \phi_{j,w}(z)] \\ &= ext\left(\frac{1}{w-z}\right)\frac{(Re(w-q))^j}{j!} \\ &= S^{-1}(w,q)\frac{(Re(w-q))^j}{j!}, \end{aligned}$$

where $S^{-1}(w,q)$ *is the slice hyperholomorphic Cauchy kernel given by*

$$S^{-1}(w,q) = (w-\overline{q})(w^2 - 2Re(q)w + |q|^2)^{-1}.$$

Proposition 11.4.5. *Let* $q, w \in \mathbb{B}$ *be such that* $q \notin [w]$. *The function,* $\phi_{j,w}(q)$ *is right slice polynalytic of order* $j+1$ *in the variable* w.

Proof. The proof is easy using the fact that $S^{-1}(w,q)$ is right slice regular in w combined with the right version of Proposition 3.3 in [17]. □

Theorem 11.4.6 (Poly-Cauchy formula)**.** *Let* Ω *be an axially symmetric slice domain containing the closure of* $\mathbb{B}$ *and* $f : \Omega \longrightarrow \mathbb{H}$ *a slice polyanalytic function of order* $n \geq 1$. *For* $I \in \mathbb{S}$, *set* $dw_I = -dwI$. *The integral below*

$$\frac{1}{2\pi}\int_{\partial\mathbb{B}_I}\sum_{j=0}^{n-1}(-2)^j S^{-1}(w,q)\frac{(Re(w-q))^j}{j!}dw_I\overline{\partial_I}^j(f)(w),$$

does not depend on the choice of the imaginary unit $I \in \mathbb{S}$.

Moreover, for all $q \in \mathbb{B}$ *we have the integral representation*

$$f(q) = \frac{1}{2\pi}\int_{\partial\mathbb{B}_I}\sum_{j=0}^{n-1}(-2)^j S^{-1}(w,q)\frac{(Re(w-q))^j}{j!}dw_I\overline{\partial_I}^j(f)(w).$$

Proof. The independence of the choice of $I \in \mathbb{S}$ is a direct consequence of the poly-decomposition in Proposition 11.2.9 combined with the series expansion theorem for slice hyperholomorphic functions. To show the second part of the statement, let $J \in \mathbb{S}$ be such that $J \perp I$. We know that $f \in \mathcal{SP}_n(\mathbb{B})$, so by Proposition 3.4 in [17] there exist two polyanalytic functions $F, G : \mathbb{B}_J \longrightarrow \mathbb{C}_J$ of order n such that for any $w \in \mathbb{B}_J$ we have

$$f(w) = F(w) + G(w)J.$$

In particular,

$$\overline{\partial_I}^j f(w) = \overline{\partial_I}^j F(w) + \overline{\partial_I}^j G(w)J.$$

Then, we have on $\mathbb{B}_I$ the following reproducing property thanks to the complex poly-Cauchy formula applied to F and G

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial\mathbb{B}_I}\sum_{j=0}^{n-1}(-2)^j S^{-1}(w,q)\frac{(Re(w-q))^j}{j!}dw_I\overline{\partial_I}^j(f)(w) &= F(q)+G(q)J\\ &= f(q).\end{aligned}$$

Furthermore, in Proposition 11.4.3 we deal with a slice polyanalytic kernel. So, the function

$$\Psi(q)=\int_{\partial\mathbb{B}_I}\sum_{j=0}^{n-1}(-2)^j S^{-1}(w,q)\frac{(Re(w-q))^j}{j!}dw_I\overline{\partial_I}^j(f)(w),$$

is also slice polyanayltic of order n. Hence, we can conclude by Identity principle since Ψ coincides with f on $\mathbb{B}_I$.

□

Remark 11.4.7. *The case $n=1$ in the previous theorem gives the slice hyperholomorphic Cauchy formula that can be found in [35].*

11.5 The poly-Fueter mapping theorem in integral form

We shall study in this section an integral representation of the poly-Fueter mapping theorem on the quaternionic unit ball that will extend the results obtained in [48]. As a direct application of the slice poly Cauchy formula we will prove the poly-Fueter mapping theorem in its integral form. To this end, we need some technical lemmas. First, for every $n\geq 1$, $1\leq j\leq n-1$ and $w\in\partial\mathbb{B}$, denote by $\mathcal{F}_j(w,q)$ the quaternionic valued function on $\mathbb{B}$ sending q into

$$\mathcal{F}_j(w,q):=S^{-1}(w,q)\frac{Re^j(w-q)}{j!}, \tag{11.5.1}$$

where $Re^j(w-q):=(Re(w-q))^j$.

Lemma 11.5.1. *Let $w\in\partial\mathbb{B}$. Then, for every $q\in\mathbb{B}$, we have*

$$V(\mathcal{F}_0(w,q))=0 \text{ and } V(\mathcal{F}_j(w,q))=-\mathcal{F}_{j-1}(w,q), \forall j\geq 1.$$

Proof. First, we have $\mathcal{F}_0(w,q)=S^{-1}(w,q)$ is the slice hyperholomorphic Cauchy kernel. So, $q\longmapsto\mathcal{F}_0(w,q)$ is slice hyperholomorphic with respect to the variable q. Thus, we have $V(\mathcal{F}_0(w,q))=0$ for all q. On the other hand, for all $j\geq 1$ we have

$$G(\mathcal{F}_j(w,q))=G\left(S^{-1}(w,q)\frac{Re^j(w-q)}{j!}\right),\ \forall q\in\mathbb{B}.$$

Then, we apply Proposition 3.1.11 on which we see how the global operator G acts on the product keeping in mind that one of the functions is real valued and get

$$G(\mathcal{F}_j(w,q)) = S^{-1}(w,q)G\left(\frac{Re^j(w-q)}{j!}\right), \ \forall q \in \mathbb{B}. \tag{11.5.2}$$

However, we have

$$\begin{aligned} G\left(\frac{Re^j(w-q)}{j!}\right) &= |\vec{q}|^2\partial_{x_0}\left(\frac{Re^j(w-q)}{j!}\right) \\ &= -|\vec{q}|^2\frac{Re^{j-1}(w-q)}{(j-1)!} \end{aligned}$$

Then, we replace in (11.5.2) and get

$$G(\mathcal{F}_j(w,q)) = -|\vec{q}|^2 S^{-1}(w,q)\frac{Re^{j-1}(w-q)}{(j-1)!}, \ \forall q \in \mathbb{B}.$$

Hence, we use Remark 11.3.1 to see that the result holds outside the real line. Then, we apply again Theorem 2.4 in [80] which allows to extend the formula everywhere on $\mathbb{B}$. Finally, we conclude that for any $q \in \mathbb{B}$ we have

$$V(\mathcal{F}_j(w,q)) = -\mathcal{F}_{j-1}(w,q), \forall j \geq 1.$$

This ends the proof. □

Lemma 11.5.2. *Let $w \in \partial\mathbb{B}$. For any $n \geq 1$, we set*

$$\tau_n = \Delta \circ V^{n-1}.$$

Then, for every $q \in \mathbb{B}$, we have

1. $\tau_1(\mathcal{F}_0(w,q)) = \Delta S^{-1}(w,q)$.
2. *For all $n \geq 2$, we have*
   (a) $\tau_n(\mathcal{F}_j(w,q)) = 0, \forall 0 \leq j < n-1.$
   (b) $\tau_n(\mathcal{F}_{n-1}(w,q)) = (-1)^{n-1}\Delta S^{-1}(w,q).$

Proof. (1) It is immediate by the definition of the map $\tau_1 = \Delta$.
(2) We reason by induction. First, we note that for $n = 2$, $\mathcal{F}_0(w,q)$ is slice hyperholomorphic with respect to q so that

$$\tau_2(\mathcal{F}_0(w,q) = \Delta \circ V(\mathcal{F}_0(w,q)) = 0.$$

Moreover, we have

$$\tau_2(\mathcal{F}_1(w,q)) = \Delta\left(V(\mathcal{F}_1(w,q))\right).$$

Moreover, Lemma 11.5.1 yields

$$V(\mathcal{F}_1(w,q)) = -\mathcal{F}_0(w,q)$$

so we get

$$\tau_2(\mathcal{F}_1(w,q)) = -\Delta(\mathcal{F}_0(w,q)) = -\Delta S^{-1}(w,q).$$

We conclude that the result holds for $n=2$. Let us suppose by induction that the assertions (a), (b) in the statement hold for $n \geq 2$ and we prove them for $n+1$.

(a) Let $w \in \partial\mathbb{B}$. Then, for every $q \in \mathbb{B}$, it is clear that

$$\tau_{n+1}(\mathcal{F}_0(w,q)) = \Delta \circ V^n(\mathcal{F}_0(w,q)) = 0.$$

We observe that

$$\tau_{n+1} = \Delta \circ V^n = \Delta \circ V^{n-1} \circ V = \tau_n \circ V. \tag{11.5.3}$$

Then, for all $1 \leq j < n$ making use of Lemma 11.5.1 we have

$$\begin{aligned}\tau_{n+1}(\mathcal{F}_j(w,q)) &= \tau_n \circ V(\mathcal{F}_j(w,q)) \\ &= -\tau_n(\mathcal{F}_{j-1}(w,q)) \\ &= -\tau_n(\mathcal{F}_h(w,q));\ 0 \leq h = j-1 < n-1.\end{aligned}$$

Therefore, by induction hypothesis we conclude that

$$\tau_{n+1}(\mathcal{F}_j(w,q)) = 0, \forall 0 \leq j < n.$$

This shows that (a) holds.

(b) We use a second time the observation (11.5.3) combined with Lemma 11.5.1 and get by induction hypthesis

$$\begin{aligned}\tau_{n+1}(\mathcal{F}_n(w,q)) &= \tau_n \circ V(\mathcal{F}_n(w,q)) \\ &= -\tau_n(\mathcal{F}_{n-1}(w,q)) \\ &= (-1)^n \Delta S^{-1}(w,q).\end{aligned}$$

Hence, (b) also holds. This ends the proof. □

Theorem 11.5.3 (Poly-Fueter mapping integarl form)**.** *Let f be a slice polyanalytic function of order $n \geq 1$ on some axially symmetric slice domain Ω that contains the closure of $\mathbb{B}$. Then, the Fueter regular function $\tau_n(f)$ given by*

$$\tau_n(f)(q) = \Delta \circ V^{n-1}(f)(q)$$

has the integral representation

$$\tau_n(f)(q) = c(n,\pi) \int_{\partial\mathbb{B}_I} \Delta S^{-1}(w,q) dw_I \overline{\partial}_I^{n-1}(f)(w), \forall q \in \mathbb{B}$$

where $I \in \mathbb{S}$ and $c(n,\pi) = \dfrac{2^{n-1}}{2\pi}$.

Proof. Let $f \in \mathcal{SP}_n(\Omega)$, we know by the poly-Cauchy formula for slice polyanalytic functions (Theorem 11.4.6) that for all $q \in \mathbb{B}$ we have

$$f(q) = \frac{1}{2\pi} \int_{\partial\mathbb{B}_I} \sum_{j=0}^{n-1} (-2)^j \mathcal{F}_j(w,q) dw_I \overline{\partial}^j(f)(w).$$

Therefore, we apply the Fueter mapping $\tau_n = \Delta \circ V^{n-1}$ and obtain that

$$\tau_n(f)(q) = \frac{1}{2\pi} \int_{\partial\mathbb{B}_I} \sum_{j=0}^{n-1} (-2)^j \tau_n(\mathcal{F}_j(w,q)) dw_I \overline{\partial}^j(f)(w), \ \forall q \in \mathbb{B} \setminus \mathbb{R}.$$

However, by Lemma 11.5.1 we know that

$$\tau_n(\mathcal{F}_{n-1}(w,q)) = (-1)^{n-1} \Delta S^{-1}(w,q) \text{ and } \tau_n(\mathcal{F}_j(w,q)) = 0, \forall 0 \leq j < n-1$$

Hence, we obtain

$$\tau_n(f)(q) = \frac{2^{n-1}}{2\pi} \int_{\partial\mathbb{B}_I} \Delta S^{-1}(w,q) dw_I \overline{\partial}_I^{n-1}(f)(w), \forall q \in \mathbb{B} \setminus \mathbb{R}.$$

Finally, it is clear that the integral in the right hand side is Fueter regular with respect to q everywhere on $\mathbb{B}$ which allows to extend $\tau_n(f)$ to a Fueter regular function on the unit ball. This completes the proof. □

Corollary 11.5.4. *Under the same hypothesis of Theorem 11.5.3 we note that the poly-Fueter mapping has the explicit integral expression*

$$\tau_n(f)(q) = \frac{2^n}{\pi} \int_{\partial\mathbb{B}_I} (\overline{q} - w)(w^2 - 2Re(q)w + |q|^2)^{-2} dw_I \overline{\partial}_I^{n-1}(f)(w), \forall q \in \mathbb{B}.$$

Proof. We apply Theorem 11.5.3 and use the expression

$$\Delta S^{-1}(w,q) = -4(w - \overline{q})(w^2 - 2Re(q)w + |q|^2)^{-2},$$

that was proved in [48]. □

Remark 11.5.5.

1. *Thanks to Theorem 11.3.13 the integral formulation of the poly-Fueter mapping theorem can be expressed in terms of the map $\mathcal{C}_n$ as*

$$\mathcal{D}^{n-1}[\mathcal{C}_n(f)](q) = \frac{1}{2\pi} \int_{\partial\mathbb{B}_I} \Delta S^{-1}(w,q) dw_I \overline{\partial}_I^{n-1}(f)(w), \forall q \in \mathbb{B}.$$

2. *The case $n = 1$ in Theorem 11.5.3 is the Fueter mapping theorem in integral form proved in [48].*

11.6 The polymonogenic case: Fueter-Sce-Qian extension

In this section, we see how the results of quaternionic slice polyanalytic functions can be reformulated in the slice monogenic setting. We omit to write the proofs since they are similar to the quaternionic case. We recall first some basic notations, let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ satisfying the rule

$$e_k e_s + e_s e_k = -2\delta_{k,s}, k, s = 1, ..., n$$

where $\delta_{k,s}$ is the Kronecker symbol. The set

$$\{e_A : A \subset \{1, ..., n\} \text{ with } e_A = e_{h_1} e_{h_2} ... e_{h_r}, 1 \leq h_1 < ... < h_r \leq n, e_\emptyset = 1\}$$

forms a basis of the 2^n-dimensional Clifford algebra $\mathbb{R}_n$ over $\mathbb{R}$. Let $\mathbb{R}^{n+1}$ be embedded in $\mathbb{R}_n$ by identifying $(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1}$ with the paravector $x = x_0 + \underline{x} \in \mathbb{R}_n$. The conjugate of x is given by $\bar{x} = x_0 - \underline{x}$ and the norm $|x|$ of x is defined by $|x|^2 = x_0^2 + ... + x_n^2$. We denote also by $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional sphere of unit vectors in $\mathbb{R}^n$ given by

$$\mathbb{S}^{n-1} = \{\omega = x_1 e_1 + ... + x_n e_n : x_1^2 + ... + x_n^2 = 1\}, \ \omega^2 = -1.$$

The Euclidean Dirac operator on $\mathbb{R}^n$ is given by

$$D_{\underline{x}} = \sum_{j=1}^{n} e_j \partial_{x_j}.$$

The generalized Cauchy-Riemann operator (also known as Weyl operator) and its conjugate in $\mathbb{R}^{n+1}$ are given respectively by

$$D := \partial_{x_0} + D_{\underline{x}} \text{ and } \overline{D} := \partial_{x_0} - D_{\underline{x}}.$$

Real differentiable functions on some open subset of $\mathbb{R}^{n+1}$ taking their values in $\mathbb{R}_n$ that are in the kernel of D^k are called left k-monogenic or polymonogenic of order k, see [57]. We consider also the slice monogenic version given by

Definition 11.2. *Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$ and $f : U \longrightarrow \mathbb{R}_n$ be a slice function of class $\mathcal{C}^k$. We say that f is slice polymonogenic of order k or s-polymonogenic for short, if for any $I \in \mathbb{S}^{n-1}$, we have*

$$\overline{\partial_I}^k f_I(x + Iy) = 0.$$

The set of slice polymonogenic functions of order k is denoted $\mathcal{SM}_k(U)$.

Remark 11.6.1. 1. *The set $\mathcal{SM}_k(U)$ forms a right module on $\mathbb{R}_n$.*

2. *The case $k = 1$ corresponds to the slice monogenic functions considered in [?].*

We consider on $\Omega\subset\mathbb{R}^{n+1}$ the global operator on high dimensions defined by

$$V_n(f)(x):=\partial_{x_0}f(x)+\frac{\vec{x}}{|\vec{x}|^2}\sum_{l=1}^{n}x_l\partial_{x_l}f(x),\forall x\in\Omega\setminus\mathbb{R}.$$

Lemma 11.6.2 (Splitting Lemma). *Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$ and $f:U\longrightarrow\mathbb{R}_n$ be a slice polymonogenic function of order k. For every $I=I_1\in\mathbb{S}$ let $I_2,...,I_n$ be a completion to an orthonormal basis of $\mathbb{R}_n$. Then, there exists 2^{n-1} polyanayltic functions of order k denoted $F_A:U_I\longrightarrow\mathbb{C}_I$ such that for every $z=x+Iy$*

$$f_I(z)=\sum_{|A|=0}^{n-1}F_A(z)I_A,\ I_A=I_{i_1}...I_{i_l},$$

where $A=\{i_1,...,i_l\}$ is a subset of $\{2,...,n\}$, with $i_1<...<i_l$.

Theorem 11.6.3 (s-polymonogenic decomposition). *Let Ω be an axially symmetric slice domain of $\mathbb{R}^{n+1}$ and $f:\Omega\longrightarrow\mathbb{R}_n$. Then, $f\in\mathcal{SM}_k(\Omega)$ if and only if there exists unique $f_0,...,f_{k-1}\in\mathcal{SM}(\Omega)$ such that*

$$f(x)=f_0(x)+\overline{x}f_1(x)+...+\overline{x}^{k-1}f_{k-1}(x),\ \forall x\in\Omega.$$

Using similar calculations to the quaternions case, we can prove that

Theorem 11.6.4. *Let Ω be an axially symmetric slice domain of $\mathbb{R}^{n+1}$ and $f:\Omega\longrightarrow\mathbb{R}_n$ an s-polymonogenic function of order $k\geq1$. Then, f belongs to* $\ker(V_n^k)$*, i.e:*

$$V_n^k(f)(x)=0,\ \forall x\in\Omega.$$

For slice polymonogenic functions we state the poly-Sce-Fueter mapping theorems in the Clifford setting as follows

Theorem 11.6.5 (Poly-Fueter-Sce mapping theorem I). *Let n be an odd number and Ω an axially symmetric slice domain of $\mathbb{R}^{n+1}$. If f is an s-polymonogenic function of order k. Then, the poly-Fueter mapping defined by*

$$\tau_{n,k}(f)(x)=\Delta_{\mathbb{R}^{n+1}}^{\frac{n-1}{2}}V_n^{k-1}f(x)$$

is a monogenic function, in particular a polymonogenic of order k.

Theorem 11.6.6 (Poly-Fueter-Sce mapping theorem II). *Let Ω be an axially symmetric slice domain of $\mathbb{R}^{n+1}$ and $f:\Omega\longrightarrow\mathbb{R}_n$ a slice polyanalytic function of order $k\geq1$. Assume that f admits a poly-decomposition given by*

$$f(x)=\sum_{j=0}^{k-1}\overline{x}^jf_j(x),\forall x\in\Omega$$

where $f_0, ..., f_{n-1} \in \mathcal{SM}(\Omega)$. *Then, the function defined by*

$$\mathcal{C}_{n,k}(f)(q) = \sum_{j=0}^{k-1} x_0^j \Delta_{\mathbb{R}^{n+1}}^{\frac{n-1}{2}}(f_j)(x), \forall x \in \Omega \tag{11.6.1}$$

is a poly-monogenic function of order k.

CHAPTER 12

Conclusion and further research in progress

In this dissertation, we developed several mathematical methods and results about quaternionic reproducing kernel Hilbert spaces (QRKHSs), using different tools and techniques from complex and hypercomplex analysis. We considered different examples such as Hardy, Bergman and Fock spaces both in slice and Fueter hyperholomorphic settings. In particular, most of the results were obtained in the case of Fock spaces and Segal-Bargmann theory. We note that such mathematical models are relevant and used in several interesting applications including quantum mechanics, time-frequency analysis and machine learning methods. We explain a bit more how Fock spaces and Bargmann theory appear in these different areas:

- ***Quantum mechanics:*** Fock spaces and Segal-Bargmann transforms are important mathematical models used in quantum mechanics. Sometimes they are called *bosonic* Fock spaces of n degrees of freedom. They are related to several important operators there like creation, annihilation, position, momentum, Weyl operators, etc. They are used also to define coherent states in mathematical physics. For more details about such connections and applications we refer for example to [85] and references therein.

- ***Signal and time-frequency analysis:*** we note that the short-time Fourier transform corresponding to the Gaussian window is given by the Segal-Bargmann transform. Furthermore, the short-time Fourier associated to Hermite function windows lead to Fock spaces of polyanalytic functions. This explains how Fock spaces and Segal-Bargmann transforms are relevant also in signal and time-frequency analysis thanks to the link they have with the short-time Fourier transform. We refer to the classical book [78] for more details concerning such connections and applications.

- ***Machine learning methods:*** it is well-known that Gaussian radial basis function (RBF) kernels are one of the most used kernels in modern machine learning methods such as support vector machines (SVMs), see [112]. Actually, in [110] the reproducing kernel Hilbert spaces (RKHSs) corresponding to the (RBF) kernels were introduced and were used to analyze the learning performance of (SVMs). We can directly recongnize through such results how the Fock and Bargmann theory are related to machine learning kernel methods, in particular the case of (RBF) kernels. Moreover, we note that notions such as *kernel trick*, *feature map* and *feature spaces* can be obtained in this case just by computing the scalar product on $L^2(\mathbb{R})$ of the Segal-Bargmann kernels, which leads to the Fock kernel.

In chapter 4 and 5 we studied different Fock spaces of slice hyperholomorphic functions obtaining new approximation results in both the first and the second kind theories. We got also Segal-Bargmann type transforms in the noncommutative case of quaternions and gave different descriptions in terms of some generalized versions of the creation and annihilation operators.

In chapter 6, based on the quaternionic Bargmann transform we introduced a quaternionic short-time Fourier transform QSTFT with a Gaussian window that can be computed for hypercomplex signals. We proved different results there including a Moyal formula, a reconstruction formula and a Lieb's uncertainty principle.

In chapters 7, 8 and 9 we introduced a new vision of QRKHS of Fueter hyperholomorphic functions based on a specific Clifford-Appell system which can be obtained as an application of the Fueter mapping theorem. We studied in this framework also different kernel techniques and integral transforms related to Fock, Hardy and Bergman spaces. We studied also Bergman kernels and associated transforms on different quaternionic domains.

Finally in chapters 10 and 11, we introduced the basis for a new theory of polyanalytic functions in hypercomplex analysis that contains a very important subclass of special monogenic functions of axial type. Furthermore, we connect this noncommutative theory to the classical monogenic and poly monogenic function theories by constructing two extended versions of the Fueter-Sce-Qian mapping theorem in this generalized framework.

For perspectives and further research, we started working on some different problems that are still under progress. Such problems are related to the following topics:

1. Fischer decomposition in the space of slice hyperholomorphic functions.

2. Wiener algebra on quaternions: The Fueter variables case.

3. PS and PF functional calculus and their applications.
4. Poly-Bergman-Fueter transforms.
5. Short-time Fourier transforms with Hermite windows: hypercomplex poly-analytic framework and applications in time-frequency analysis.
6. Quaternionic support vector machines, reproducing kernel methods in machine learning and stochastic processes.

In the next lines we explain a bit more some examples of research problems that we are considering regarding the two first topics.

Fischer decomposition on slice entire functions

We would like to study an extension of the results obtained in [?] to the slice hyperholomorphic setting. Indeed, in 1985 Meril and Struppa proved the following

Theorem 12.0.1. *Let P and Q be two polynomials on $\mathbb{C}^n$ and consider the operator*

$$S : f \longmapsto S(f) := P(D)(Qf).$$

Then, the following conditions are equivalent:

1. *$S : \mathcal{H}(\mathbb{C}^n) \longrightarrow \mathcal{H}(\mathbb{C}^n)$ is a bijection.*
2. *$\mathcal{H}(\mathbb{C}^n) = \mathcal{I}(Q) \oplus \ker P(D)$, where $\mathcal{I}(Q) = \{Qg;\ g \in \mathcal{H}(\mathbb{C}^n)\}$.*

In order to extend Theorem 12.0.1 to the setting of quaternions, let us first introduce

Definition 12.1. *Let $P(X) := \sum_{k=0}^{N} X^k a_k$ with $(a_k)_{0\leq k\leq n} \subset \mathbb{H}$ be a right quaternionic polynomial. Then, associated to the slice derivative ∂_S, for every given slice regular function f we define*

$$P(\partial_S)(f) := \sum_{k=0}^{N} \partial_S^k(f)a_k.$$

First, we can prove two technical results

Lemma 12.0.2. *Let $u : \mathbb{C}_i \longrightarrow \mathbb{H}$ be a holomorphic function. Then, the following formula holds*

$$Ext(P(D_i(u_i)(z))(q) = P(\partial_S)(Ext(u_i))(q),\ \forall q \in \mathbb{H}.$$

Lemma 12.0.3. *Let* $P(X) := \sum_{k=0}^{N} X^k a_k$ *be a right quaternionic polynomial. Then,*

$$P(\partial_S) : \mathcal{SR}(\mathbb{H}) \longrightarrow \mathcal{SR}(\mathbb{H}), f \mapsto P(\partial_S)(f)$$

is a surjective mapping.

As a consequence we state the problem to extend Theorem 12.0.1:

Problem 12.0.4. *Let* P *and* Q *be two right quaternionic polynomials such that* $Q(p) \neq 0$ *for all* $p \in \mathbb{H}$*. Then, consider the mapping*

$$T : f \longmapsto T(f) := P(\partial_S)(f * Q).$$

Are the following conditions equivalent ?

1. $T : \mathcal{SR}(\mathbb{H}) \longrightarrow \mathcal{SR}(\mathbb{H})$ *is a bijection.*
2. $\mathcal{SR}(\mathbb{H}) = \mathcal{J}(Q) \oplus \ker P(\partial_S)$*, where* $\mathcal{J}(Q) = \{g * Q;\ g \in \mathcal{SR}(\mathbb{H})\}$.

We consider also plynomials of the form

$$P(X) := \sum_{k=0}^{N} a_k * X^k,\ a_k \in \mathbb{H}. \tag{12.0.1}$$

The associated quaternionic linear operator is defined by

$$P(\partial_S)(f) := \sum_{k=0}^{N} a_k * \partial_S^k(f).$$

Lemma 12.0.5. *Let* $u_i : \mathbb{C}_i \longrightarrow \mathbb{H}$ *be a holomorphic function and* P *be a quaternionic polynomial of the form* (12.0.1)*. Then, the following formula holds*

$$Ext(P(D_i(u_i)(z))(q) = P(\partial_S)(Ext(u_i))(q),\ \forall q \in \mathbb{H}.$$

Remark 12.0.6. *Following a similar reasoning as in Lemma 12.0.3 we can prove that* $P(\partial_S)$ *is a surjective operator for any quaternionic polynomial* P *of the form* (12.0.1).

Therefore, we can state the following problem for quaternionic polynomials of the form (12.0.1):

Problem 12.0.7. *Let* P *and* Q *be two quaternionic polynomials of the form* (12.0.1) *such that* $Q(p) \neq 0$ *for all* $p \in \mathbb{H}$*. Then, consider the mapping*

$$L : f \longmapsto L(f) := P(\partial_S)(f * Q).$$

Are the following conditions equivalent ?

1. $L : \mathcal{SR}(\mathbb{H}) \longrightarrow \mathcal{SR}(\mathbb{H})$ *is a bijection.*
2. $\mathcal{SR}(\mathbb{H}) = \mathcal{J}(Q) \oplus \ker P(\partial_S)$, *where* $\mathcal{J}(Q) = \{g * Q;\ g \in \mathcal{SR}(\mathbb{H})\}$.

Note that the notion of entire slice regular functions of expoential type was introduced in Chapter 5 of [50]. Indeed, we have the following

Definition 12.2. *An entire slice regular function f is said to be of exponential type if there exist some constants A, B such that we have*

$$|f(q)| \leq Be^{A|q|},\ \forall q \in \mathbb{H}.$$

The space of such functions will be denoted $\mathrm{Exp}(\mathbb{H})$.

Then, we consider also the following:

Problem 12.0.8. *Let P and Q be two right quaternionic polynomials such that $Q(p) \neq 0$ for all $p \in \mathbb{H}$. Then, consider the mapping*

$$T : f \longmapsto T(f) := P(\partial_S)(f * Q).$$

Are the following conditions equivalent ?

1. $T : \mathrm{Exp}(\mathbb{H}) \longrightarrow \mathrm{Exp}(\mathbb{H})$ *is a bijection.*
2. $\mathrm{Exp}(\mathbb{H}) = \mathcal{J}(\mathrm{Q}) \oplus \ker_{\mathrm{Exp}(\mathbb{H})} \mathrm{P}(\partial_{\mathrm{S}})$, *where* $\mathcal{J}(Q) = \{g * Q;\ g \in \mathrm{Exp}(\mathbb{H})\}$.

Wiener algebras and Lévy-Wiener theorems: the Fueter variables and Clifford-Appell cases

We would like to study new versions of the Lévy-Wiener theorem for the quaternions in the setting of Cauchy-Fueter regular functions. In particular, we are considering the Fueter variables and the Clifford-Appell cases. We are interested also by the continuous version of the Lévy-Wiener theorem in this framework.

The Fueter variables case

We recall that the so-called Fueter variables are defined by

$$\zeta_l(x) = x_l - e_l x_0,\ l = 1, 2, 3. \tag{12.0.2}$$

Then, let us consider the poly-disk on the quaternions with respect to the Fueter variables given by

$$\mathcal{B}_F = \{q \in \mathbb{H};\ |\zeta_l(q)| \leq 1 \text{ for } l = 1, 2, 3\}.$$

Definition 12.0.1. *We denote by $\mathcal{W}^F_{\mathbb{H}}$ the set of functions of the form*

$$f(q) = \sum_{\alpha \in \mathbb{N}^3} \zeta^\alpha(q) f_\alpha$$

where $(f_\alpha)_\alpha \subset \mathbb{H}$ are such that we have

$$\|f\| = \sum_{\alpha \in \mathbb{N}^3} |f_\alpha| < \infty.$$

We endow the algebra $\mathcal{W}_{\mathbb{H}}^{F}$ with the CK product denoted by $\odot$. Then, we can prove that $(\mathcal{W}_{\mathbb{H}}^{F}, \odot)$ is a non commutative Banach algebra.

We are studying a counterpart of the Lévy-Wiener theorem for the quternions in the special case of Fueter variables.

Problem 12.0.9 (Lévy-Wiener theorem: Fueter variables case)**.** *Let $f \in \mathcal{W}_{\mathbb{H}}^{F}$. Then, f is invertible in $\mathcal{W}_{\mathbb{H}}^{F}$ if and only if $f(q) \neq 0$ for all $q \in \mathcal{B}_F$.*

The Clifford-Appell case

Let us consider the Clifford-Appell polynomials considered in the previous chapters and defined by

$$Q_k(q) = \sum_{j=0}^{k} T_j^k q^{k-j}\overline{q}^j, \forall q \in \mathbb{H} \tag{12.0.3}$$

where

$$T_j^k := \frac{k!}{(3)_k}\frac{(2)_{k-j}(1)_j}{(k-j)!j!} = \frac{2(k-j+1)}{(k+1)(k+2)}$$

and $(a)_n = a(a+1)...(a+n-1)$ is the Pochhammer symbol. This family of polynomials form an Appell system with respect to the hypercomplex derivative. Moreover, for $s \in \mathbb{H}$, let

$$\operatorname{Exp}(s) := \sum_{k=0}^{\infty} \frac{Q_k(s)}{k!}$$

to be the generalized Cauchy-Fueter regular exponential function considered in the paper [29]. We are interested by the following problem.

Problem 12.0.10. *We would like to study the Wiener algebra $\mathcal{W}_{\mathbb{H}}'$ consisting of quaternionic valued functions of the form*

$$f = \sum_{k=0}^{\infty} Q_k c_k \text{ such that } \sum_{k=0}^{\infty} |c_k| < \infty.$$

A counterpart of the Lévy-Wiener theorem in this setting should be to justify that: A function $f \in \mathcal{W}_{\mathbb{H}}'$ is invertible in $\mathcal{W}_{\mathbb{H}}'$ if and only if $f(q) \neq 0$ for all $q \in \overline{\mathbb{B}}$.

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