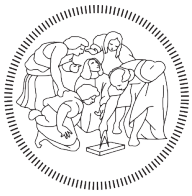

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School of Industrial and Information Engineering
Master of Science in Mathematical Engineering

Discretization Methods: an Application of Nearly Exact Discretization Scheme to an Economical Model

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Abstract

This thesis considers the issue of applying data collected with discrete frequency to continuous time model. This rough approach could lead to misleading dynamics since discrete data are applied to models that consider continuous parameter values.
The most direct solution may be to prefer discrete time dynamical systems, however, for some values of model's parameters, they could have different dynamics with respect to continuous time dynamical systems. So it is crucial to develop methods, known as *discretization methods*, which transform continuous time dynamical systems into discrete ones and also pay particular attention to safeguard the original stability.
In the continuation of the thesis, after presenting main characteristics and differences of continuous and discrete time dynamical models, several discretization methods will be presented with a particular attention to *Nearly Exact Discretization Scheme (NEDS)*, a discretization method used for 1-dimensional biological continuous time models.
Hereafter, NEDS will be applied to a 2-dimensional economical model, its stability will be discussed. Finally, numerical simulations of the economical model will be performed and moreover we will consider data collected at discrete frequency regarding the price of a chosen commodity and from that, we will calibrate the parameters of continuous and NEDS-discretized models. Lastly we will compare the data trend and the simulations of the continuous and discretized models.

Keywords: dynamical systems, continuous time, discrete time, discretization methods, NEDS method

Sommario

Questa tesi tratta il problema dell'applicazione di dati raccolti con frequenza discreta ad un modello a tempo continuo. Questo approccio superficiale può portare a dinamiche fuorvianti dato che i dati discreti sono applicati ad un modello che considera valori di parametri continui.
La soluzione immediata potrebbe essere quella di preferire sistemi dinamici a tempo discreto, tuttavia, per alcuni valori dei parametri del modello, essi possono avere dinamiche differenti rispetto a sistemi dinamici a tempo continuo. È quindi molto importante sviluppare dei metodi, noti come *metodi di discretizzazione*, che trasformano sistemi dinamici a tempo continuo in a tempo discreto ed inoltre porre particolare attenzione a preservare la stabilità del modello originario.
Nel prosieguo della tesi, dopo aver presentato le principali caratteristiche e differenze dei sistemi a tempo continuo e discreto, saranno mostrati anche numerosi metodi di discretizzazione con un particolare focus al *Nearly Exact Discretization Scheme (NEDS)*, un metodo di discretizzazione usato per modelli biologici a tempo continuo con 1 variabile.
In seguito, il NEDS verrà applicato ad un modello economico a 2 variabili e la sua stabilità verrà discussa. Infine, verranno mostrate delle simulazioni numeriche del modello economico ed inoltre verranno considerati dei dati raccolti con frequenza discreta riguardanti il prezzo di una commodity precedentemente scelta. Da lì, verranno poi calibrati i parametri del modello continuo e di quello discretizzato con il metodo NEDS. L'ultima analisi sarà la comparazione dell'andamento temporale dei dati con le simulazioni del modello continuo e di quello discretizzato.

Parole chiave: sistemi dinamici, tempo continuo, tempo discreto, metodi di discretizzazione, metodo NEDS

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Introduction

Real systems (physical, biological, mechanical, economical etc.) are usually very complex to manage in their own totality, in this regard a mathematical description is needed to describe them. When evolution over time is considered, these mathematical descriptions are known as *dynamical systems* and their biggest advantage is that, through a vector of mathematical variables, known as *state variables*, they give a sufficient knowledge of the dynamics of the real system. More briefly, in order to obtain knowledge from a real system, it is required to model it through mathematical variables which give the "state" of the considered system, so that the real system, which is complex and would be impossible to consider in each of its aspects, is approximated by a mathematical model.

Usually, the most used dynamical systems in practice consider time as a continuous variable, i.e. the set of all admissible values for time is $\mathbb{R}$.
However, data are not collected with continuous frequency; take for instance a economical model on electricity, its price is not calculated every single moment but it is available only at established time; consider also an ecological model on the population of a certain type of fish in the Adriatic Sea, the number of fishes is given only every year since it would not be possible to compute it with a higher frequency.
Arguing in this way, it is clear that using continuous time dynamical systems with data available only at discrete time is an incoherent choice. Nevertheless, it is not just a matter of ethic, indeed, we will see that continuous time and discrete time dynamical systems could have completely different behaviours, so using discrete dynamical systems with discrete time data is crucial in order to obtain result which could be safely applied in the practice.

Another problem is the fact that sometimes it is not possible to choose between a continuous time and a discrete time dynamical system, since the latter are not so common in practical uses.
The relative solution is to study a method which transforms a continuous model into a discrete one, the above-mentioned *discretization methods*. We will show some of them with a particular focus on *Nearly Exact Discretization Schemes (NEDS)*, presented in [5].

Once showed the different types of dynamical systems and the discretization methods, the last step will be studying a continuous time economical model and apply NEDS method, that will be also the aim of the thesis.
A relative simple 2-dimensional continuous time model, presented in [6] will be taken into account and it will be performed its discretization. The result will not be as we previously expected since a tiny additional hypothesis will be added in order to preserve the original stability.
Finally, numerical simulations will be performed and a comparison, between real data related to the price of a chosen commodity versus continuous time model and its discretized version will be showed.

The thesis will be divided into 3 main chapters and final conclusions:

Chapter (1) will be devoted to present dynamical systems, their most important bifurcations and some definitions that will be useful also in next chapters. The chapter will be focused on *continuous time* and *discrete time* systems showing similarities and differences through simple but explicative examples. Indeed, at the end of each section, a numerical example (in the majority of the sections, the example will be the well known *logistic model*) will be showed with a greater attention on the different results obtained according to the number of variables considered in the model.

Chapter (2) will present the process of discretization that allows to pass from a continuous time system to a discrete one. Some different types of discretization methods will be showed, starting from the simpler, like *Euler* and *Kahan* methods, following *Non-Standard Finite Different (NSFD)* scheme will be showed and finally, it will be paid specific attention to the *Nearly Exact Discretization Scheme (NEDS)* that will be used also in chapter (3) for the numeric application.
As done for chapter (1), after each section regarding a discretization method, a numerical example (the *logistic model*, coherently) will be performed and great attention will be paid to differences between different discretizations and their pros and cons.

Chapter (3) will be the aim of the thesis, indeed an application of NEDS model will be performed for a continuous time energy model presented in [6], we will observe that a small additional hypothesis will be introduced in order to preserve the stability of the continuous time model.
A numerical simulation for the price of soft wheat in two European countries (Italy and Austria) will be performed and the theoretical notions regarding the consistency of NEDS scheme will be confirmed also for a generic economical model like the one used here in this thesis.

Finally a comparison between data regarding the price of the benchmark for Arabica coffee and the simulations of the continuous and discretized model

will be showed. In particular before simulating the two models, in order to give solidity to our analysis, we will perform a calibration for model's 5 parameters. We will observe that, for optimal initial values, the discretized model will perform better than the original one, with a smaller error than the continuous time model.

Chapter (4), the conclusion, will summarize the thesis work, highlighting the result obtained, showing the processes used in order to obtain them and the difficulties that arose in the third chapter.
Finally, some ideas for future works will be given and the potential of NEDS method for economy will be discussed.

Chapter 1

Dynamical Systems

1.1 General Definitions

A *dynamical system* is a mathematical model i.e. a mathematical description through variables and equations whose state changes over time, for these reasons, these models are called dynamical. *Constant systems*, i.e. systems in which the state does not change over time, can be considered a particular case of dynamical systems.
The first step in order to give a characterization of the "state" of the system: therefore, measurable quantities, called *state variables* are defined in the following way

$$x_1, ..., x_n$$

where $x_i \in \mathbb{R}, i = 1, .., n$.
In general, in order to describe a real system, many variables x_i are necessary; in some particular cases, only one variable is enough and its notation will be just x. In next chapters, a focus will done especially for systems described by one and two variables.
In an economic framework x_i may represent the price of a stock or the inflation of a given country, in a thermodynamic system x_i may represent the pressure or the temperature and so on.
Sometimes x_i may have a value that is not admissible for the system considered, for example in ecology, where x_i represents the density of a certain species, x_i cannot be negative. For this reason the following definition is given:

Definition 1. *The state space (or phase space) $M \subseteq \mathbb{R}^n$ is the set of the admissible values of the state variables x_i, $i = 1, .., n$.*

In dynamical systems the model changes over time therefore state variables are function of time: $x_i = x_i(t), i = 1, .., n$. The time t may be a real number or a natural one.
In the first case systems are known as *continuous time systems*: $x_i : \mathbb{R} \to \mathbb{R}$,

in the other case they are called *discrete time systems*: $x_i : \mathbb{N} \to \mathbb{R}$.
Usually, the state of the system at a certain time t_0 is given and the aim is to compute the state at a different time t, in other words the purpose is the knowledge of the *evolution operator* $\mathbf{G}$: $\mathbf{G}(\cdot) = (G_1(\cdot), ..., G_n(\cdot)) : M \to M$, such that $\mathbf{x}(t) = \mathbf{G}(t, \mathbf{x}(t_0))$, where $\mathbf{x}(t) = (x_1(t), ..., x_n(t)) \in M$.
Knowing the evolution operator $\mathbf{G}$ and the initial condition $\mathbf{x}(t_0)$, it is possible to compute the future state of the system for $t > t_0$ (or past state for $t < t_0$).
The vector function of the state variables $\mathbf{x}(t)$ is the parametric equations of a *trajectory*, as t varies. If $t \in \mathbb{R}$ (continuous time system) the trajectory is a curve in $\mathbb{R}^n$ otherwise if $t \in \mathbb{N}$ (discrete time), the trajectory is countable set of points.

A particular trajectory is given when all the state variables are constant: $\mathbf{x}(t) = \mathbf{G}(t, \mathbf{x^*}) = \mathbf{x^*}$ for $t > t_0$. In this case $\mathbf{x^*} = (x_1^*, ..., x_n^*)$ is called *equilibrium (or fixed point)*. A consequence is that any trajectory which is inside the equilibrium will stay in it in the future i.e. if $x(t_0) = \mathbf{x^*}$ then $x(t) = \mathbf{x^*}$ for $t > t_0$.
The definition of equilibrium can be extended to a subset of the state space:

Definition 2. *A set $A \subseteq M$ is trapping if $\boldsymbol{x}(\boldsymbol{t_0}) \in A$ then $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{G}(t, \boldsymbol{x}(\boldsymbol{t_0})) \in A$ for any $t > t_0$.*

Commonly speaking, any trajectory that starts from a point inside a trapping set will always stay inside it.
A stronger property is *invariance*:

Definition 3. *A closed set $A \subseteq M$ is invariant if $\boldsymbol{G}(t, A) = A$.*

From the previous definition, we can affirm that a set A is invariant if each subset $A' \subset A$ is not trapping. In other words, let A be a trapping set, then any trajectory starting inside A will always remains in A and moreover all the points of A are reached by trajectories starting inside it.
A reasonable question could now be: what happens for a trajectory that starts from a point belonging to a neighbourhood of the invariant set?
We have the following definitions for the concept of *stable set*:

Definition 4 (Liapunov stability). *An invariant set A is stable if for each neighbourhood U of A there exists another neighbourhood V of A, with $V \subseteq U$, such that any trajectory that starts from V remains inside U.*

Definition 5 (Asymptotic stability or attractor). *An invariant set A is asymptotically stable (or called attractor) if:*

(i) A is stable w.r.t. Liapunov stability, def. (4)

(ii) $\lim_{t \to +\infty} \boldsymbol{G}(t, \boldsymbol{x}) \in A$, $\forall$ initial condition $x \in V$.

These two definitions can be expressed in the following way: a set A is *Liapunov stable* if for any initial condition outside and close to A, all the possible trajectories will stay around A.
Consequently a set A is *asymptotically stable* if it is Liapunov stable and all the trajectories starting "near" it will tend to A, in the long run.
Finally, from definition (4), we obtain a characterization of an *instable (or unstable)* set A as a set such that exists a neighbourhood $U \supset A$ such as initial conditions taken sufficiently close to A generate trajectories that leave U.
These definitions can be rephrased substituting neighbourhoods with norms in $\mathbb{R}^n$ with no substantial difference.

Previous definitions concerned the behaviour of the dynamical system when the initial conditions belong to a sufficient small neighbourhood of an invariant set, i.e. these definitions are *local*. The question now could be: what happens if the initial conditions are far from the invariant set?
Therefore we are interested in the *global* behaviour of the system, i.e. how far can be the perturbation shift of the state of a system from an equilibrium, in order to have a trajectory that goes back to the original equilibrium in the long run. For this reason it is given the following definition:

Definition 6 (**Basin of attraction**). *Let A be an attractor, the basin of attraction of A, denoted as B(A), is the set of all points* $\boldsymbol{x} \in M$ *such that* $\lim_{t\to+\infty} \boldsymbol{G}(t, \boldsymbol{x}) \in A$.

In the particular case $B(A) = M$ (where M is the set of all admissible values of the state variables), A is called *global attractor*.
A very rough measure of robustness of a system (i.e. the capacity of absorbing small exogenous perturbations without significant changes of the equilibrium) can be the extension of the basin of attraction, despite so, there is no evidence of a positive relation between large basins of attraction and the absence of vulnerable equilibria, due to the fact that the shape of a basin can influence the robustness of the system.

Usually the evolution operator **G** (that describes the state of the system) is not known or sometimes it has such a complex structure that it cannot be used in practice.
To this end, dynamical systems are described in terms of *local evolution equations* (or *dynamic equations*) that gives information how the dynamical state changes with small time steps.
If the dynamical system is in continuous time then the evolution equations are described by *Ordinary Differential Equations (ODE)*:

$$\begin{cases} \frac{dx_i(t)}{dt} = f_i(x_1(t), ..., x_n(t); \boldsymbol{\alpha}), \quad i = 1, ..., n \\ x_i(t_0) = \bar{x}_i \end{cases} \tag{1.1}$$

where the time derivative represents the speed of change of the state variable $x_i(t)$ with respect to the time variations. The right-hand side describes the influence of the same state variable $x_i(t)$ and the other state variables $x_j(t), j \neq i$. The vector $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_m), \alpha_i \in \mathbb{R}$, represents m real parameters that can assume different values, they stand for the exogenous influences of the dynamical system. If these parameters vary with respect to time, i.e. $\alpha_i = \alpha_i(t)$, the model is *non-autonomous*.

In the case of discrete time, the evolution equations are expressed by *Difference Equations (DE)* that describes the evolution of the system:

$$\begin{cases} x_i(t+1) = f_i(x_1(t), ..., x_n(t); \boldsymbol{\alpha}), \quad i = 1, ..., n \\ x_i(0) = \bar{x}_i \end{cases} \tag{1.2}$$

The study of equations (1.1) and (1.2) gives the local behaviour of the system that changes with time. This is just a qualitative analysis but it can be enough to represent properly the system, indeed solving ODE (or DE in case of discrete time) gives information about existence of attracting sets, their basins and substantial changes in case of variation of parameters α_i, $i = 1, ..., m$.

In next sections, continuous time dynamical systems will be presented and following discrete time models will be analyzed, moreover similarities and main differences will be highlighted.

1.2 Continuous Time Dynamical Systems

This chapter is devoted to dynamic equation of the form expressed in (1.1). It will be at first present the simplest case: 1-dimensional model ($n = 1$), then the case with 2 dynamic equation and eventually the case with $n > 2$. There are many theorems regarding existence and uniqueness of a solution of an ODE; here it is reported a theorem that has "strong" assumptions but it is enough for the aim of thesis to give a general framework for dynamical systems.

Theorem 1 (**Existence and Uniqueness**). *If the functions f_i have continuous partial derivatives $\frac{\partial f_i}{\partial x_k}$ in M and $x(t_0) \in M$, then there exists a unique solution $x_i(t), i = 1, ..., n$ of the system (1.1) such that $x(t_0) = \bar{\boldsymbol{x}}$ and each $x_i(t)$ is a continuous function.*

1.2.1 1 - Dimensional Continuous Time Dynamical Systems

The simplest continuous time model is the when the only dynamic equation is linear, it has the following form:

$$\dot{x} = \alpha x, \qquad with \quad x(t_0) = x_0. \tag{1.3}$$

α is the parameter that relates the rate of growth of x to itself (indeed, the sign of the derivative of x represents the increase or decrease of the variable). In particular if $\alpha > 0$, x will increase, on the contrary if $\alpha < 0$, x will decrease (here we are assuming that x is positive, the reasoning can be reversed for x negative). For equation (1.3) there is an explicit solution:

$$x(t) = x_0 e^{\alpha(t-t_0)} \tag{1.4}$$

Nonetheless explicit solutions are very infrequent, especially with nonlinear differential equations; in order to obtain information regarding the behaviour of the system, qualitative methods are very helpful. Suppose to have the differential equation:

$$\dot{x} = f(x, \alpha) \tag{1.5}$$

where again α is a parameter of the system.
The procedure in order to find the *equilibrium points* is to solve the equation $\dot{x} = 0$, i.e. $f(x, \alpha) = 0$.
Suppose now that x^* is a fixed point for (1.5) and $f'(x^*) \neq 0$. Then the right-hand side of (1.5) can be approximated by Taylor expansion (first order approximation) as follows:

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + o(x - x^*)$$

This is a linear and local approximation for the system in the neighbourhood of x^*.
The following result is a cornerstone for stability of 1-dimensional systems in continuous time.

Proposition 1 (1-dim. local asymptotic stability in continuous time). *Let x^* be a fixed point of (1.5). If $f'(x^*) < 0$ then x^* is a locally asymptotically stable equilibrium, if $f'(x^*) > 0$ then x^* is unstable.*

We can also give an important definition for equilibria:

Definition 7. *An equilibrium point x^* is called hyperbolic if $f'(x^*) \neq 0$, otherwise if $f'(x^*) = 0$ then x^* is a non-hyperbolic equilibrium.*

Proposition (1) gives a result for *hyperbolic* equilibrium points, what happens for *non-hyperbolic* fixed points?

The answer is not as immediate as in the previous case; nothing can be said about the stability of x^* with just a linear approximation, higher derivatives should be taken into account and simulations should be done in order to give a local characterization of the stability of the equilibrium point.

In the previous part, the role of parameters was sidelined, we now focus on what happens to the stability of the system if the value of a parameter is changed.

Definition 8 (**Bifurcation**). *A variation of parameter's value that leads to a qualitatively different dynamic scenario is known as bifurcation.*

In other words, when an arbitrarily small modification of the parameter's value causes a qualitative change in the stability properties of the equilibria then a bifurcation arises at the boundary between the two equivalence classes. There are 3 possible bifurcations in a 1-dimensional continuous time model:

Definition 9 (**Fold Bifurcation**). *Fold bifurcation is characterized by a creation of two equilibrium points (one stable and one unstable) as the parameter varies.*

The canonical example of fold bifurcation is the dynamical system: $\dot{x} = \mu - x^2$.
The two fixed points are: $x^*_{1,2} = \pm\sqrt{\mu}$ (of course the two equilibrium points exist only for $\mu \geq 0$, when $\mu < 0$ there are no fixed points). In the canonical example, the value $\mu = 0$ is the bifurcation point, indeed at $\mu = 0$ the equilibrium points are coincident and non-hyperbolic ($x^*_{1,2} = 0$ and $f'(x^*_{1,2}) = -2x = 0$). We now focus on the stability of the two equilibria: the first one is unstable since the derivative is positive, indeed $f'(x^*_1) = f'(-\sqrt{\mu}) = 2\sqrt{\mu} > 0$. On the contrary, the second equilibrium is stable, due to the negativity of the derivative computed for the fixed point, in fact $f'(x^*_2) = f'(+\sqrt{\mu}) = -2\sqrt{\mu} < 0$.
The bifurcation diagram is showed in fig. 1.1, where the dotted red line stands for an unstable equilibrium and the green line for a stable one, and the relative phase portrait in fig. 1.2, where the direction of the arrow represents the sign of $\dot{x}$ in that zone of the state space, with the usual convention: $\dot{x} < 0$ implies a left arrow, $\dot{x} > 0$ a right arrow.

Another type of bifurcation that could arise in a 1-dimensional continuous time model is the so-called transcritical bifurcation.

Definition 10 (**Transcritical Bifurcation**). *Transcritical bifurcation is characterized by the existence of two equilibrium points (one stable and one unstable), that merge at the bifurcation point and after the bifurcation the pre-existent two equilibria have swapped their stability.*

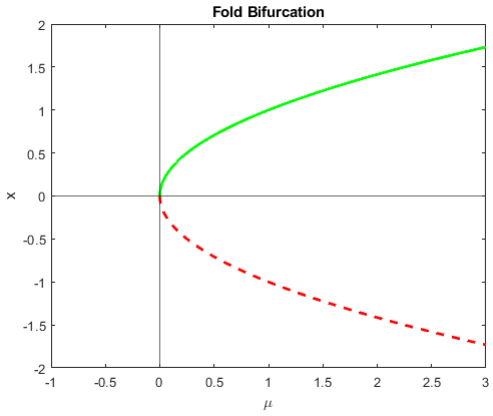
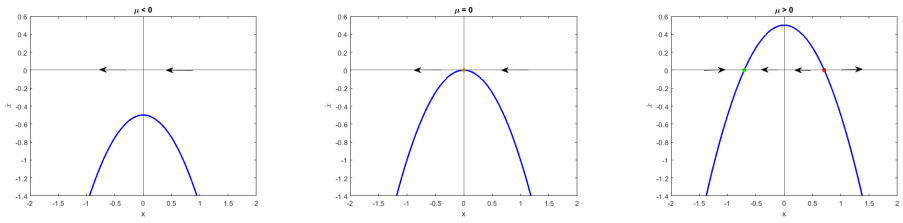


Figure 1.1: Fold bifurcation diagram for the system $\dot{x} = \mu - x^2$.



(a) (b) (c)

Figure 1.2: Fold bifurcation phase portrait for the system $\dot{x} = \mu - x^2$.

The canonical example of transcritical bifurcation is the system $\dot{x} = \mu x - x^2$.
The two fixed points are $x_1^* = 0$ and $x_2^* = \mu$, they merge at $\mu = 0$, that is the bifurcation value, where the equilibrium is non-hyperbolic ($f'(x_{1,2}^*) = \mu - 2x = 0$ for $\mu = 0$ and $x_{1,2}^* = 0$). Studying the stability of x_1^*, it is immediate to deduce that its derivative is $f'(x_1^*) = f'(0) = \mu$; so for $\mu < 0$, x_1^* is a stable equilibrium, for $\mu > 0$, it is unstable. Regarding x_2, $f'(x_2^*) = f'(\mu) = -\mu$, so its stability is the complementary of x_1^*'s (i.e. unstable for $\mu < 0$ and stable for $\mu > 0$).
In figures 1.3 and 1.4, it is presented the bifurcation diagram for the system in question and its phase portrait with the same meaning for red and green colors and the previous notation for the direction of the arrows.

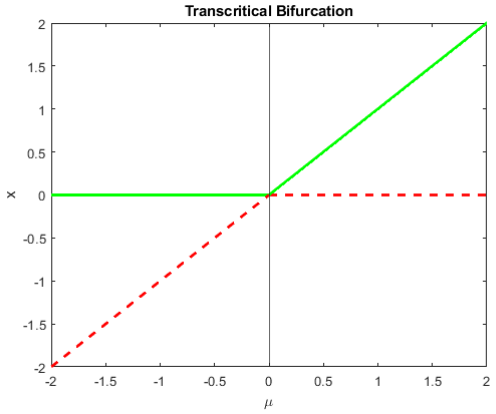
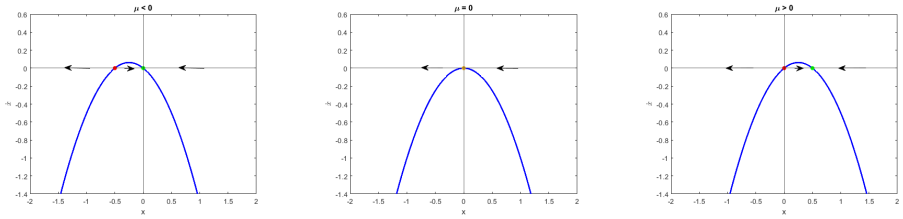


Figure 1.3: Transcritical bifurcation diagram for the system $\dot{x} = \mu x - x^2$.

The third possible bifurcation for a 1-dim. continuous time dynamical system is the pitchfork bifurcation.

Definition 11 (**Pitchfork Bifurcation**). *Pitchfork bifurcation is characterized by a transition from one equilibrium to three equilibrium points, the one already existent changes its stability as the parameter crosses the bifurcation point while two equilibria are created.*

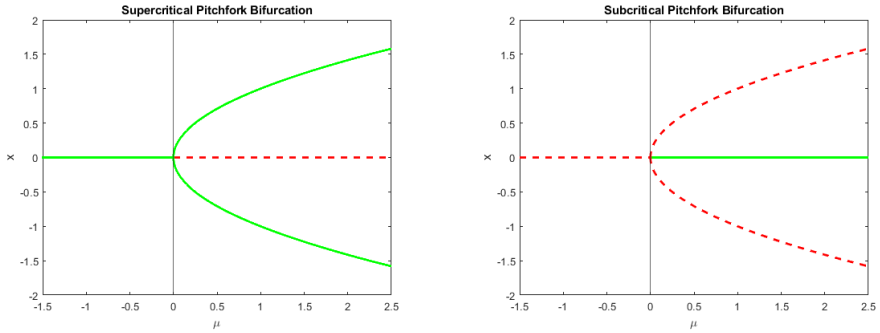
The canonical example is $\dot{x} = \mu x - x^3$, the fixed point $x_0^* = 0$ always exists, in particular for $\mu < 0$ it is stable and unstable for $\mu > 0$, since its derivative is $f'(x_0^*) = f'(0) = \mu$. The other two equilibria are $x_{1,2}^* = \pm\sqrt{\mu}$, of course $x_{1,2}^*$ exist only if $\mu \geq 0$, they are stable since $f'(x_{1,2}^*) = f'(\pm\sqrt{\mu}) = -\mu < 0$.
Three equilibrium are coincident for $\mu = 0$, which is the bifurcation value.
This type of pitchfork bifurcation is known as *supercritical*, in order to distinguish it from the *subcritical* pitchfork bifurcation where the unique unstable equilibrium becomes stable at the bifurcation value with the creation of two unstable equilibria. The canonical example for the subcritical version is the



(a) (b) (c)

Figure 1.4: Transcritical bifurcation phase portrait for the system $\dot{x} = \mu x - x^2$.

dynamical system $\dot{x} = x^3 - \mu x$. The results for this model are obtained in the same way of supercritical case's.
In figure 1.5 we show both type of pitchfork bifurcation and in figures 1.6-1.7 their phase portraits, with the usual meaning for colors and arrows.



(a) $\dot{x} = \mu x - x^3$ (b) $\dot{x} = x^3 - \mu x$

Figure 1.5: Supercritical and Subcritical Pitchfork Bifurcation.

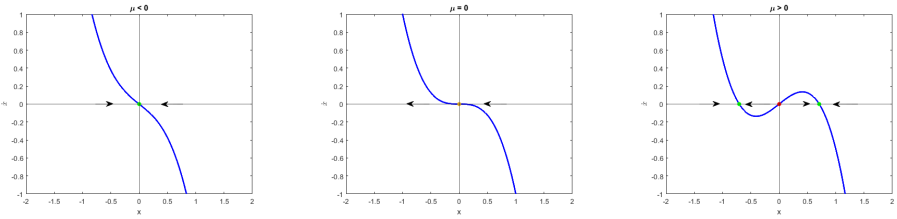
Example (The Logistic Growth Equation)

Consider the logistic growth equation by Verhulst:

$$\dot{x} = rx\left(1 - \frac{x}{k}\right) \quad =: f(x) \tag{1.6}$$

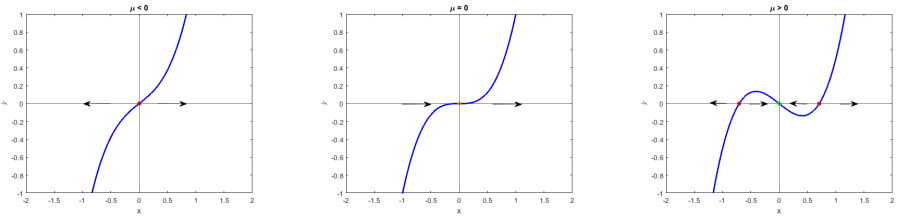
where r and k are positive parameters.

Imposing $f(x) = 0$, we find the two fixed points: $x_1^* = 0$ and $x_2^* = k$. Since (1.6) is a nonlinear 1-dimensional continuous time dynamical model, we can apply proposition (1), which allows us to get stability results by studying the derivative of f in x_1^* and x_2^* (checking that the two equilibria are hyperbolic). In particular we obtain: $f'(x_1^*) = f'(0) = r > 0$ and



(a) (b) (c)

Figure 1.6: Supercritical pitchfork bifurcation phase portrait for the system $\dot{x} = \mu x - x^3$.



(a) (b) (c)

Figure 1.7: Subcritical pitchfork bifurcation phase portrait for the system $\dot{x} = x^3 - \mu x$.

$f'(x_2^*) = f'(k) = -r < 0$. So, from the previously cited proposition, x_1^* is unstable equilibrium and x_2^* is stable.
We can also draw a simple state state space where the direction of arrows indicates the positivity (or negativity) of $\dot{x}$, with the convention that $\dot{x} < 0$ is represented by left arrow and $\dot{x} > 0$ by right arrow. It is showed in fig. 1.8

1.2.2 2 - Dimensional Continuous Time Dynamical Systems

The next step is to consider a 2-dimensional continuous time model, for example:

$$\begin{cases} \dot{x}_1 = f_1(x_1(t), x_2(t)) \\ \dot{x}_2 = f_2(x_1(t), x_2(t)) \end{cases} \tag{1.7}$$

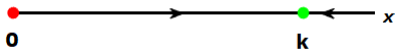


Figure 1.8: State space for model (1.6) with unstable and stable equilibria (in red and green, respectively).

Usually, a good graphical method in order to have a qualitative view of the model is to represent in the phase space (x_1, x_2) the two curves: $f_1(x_1(t), x_2(t)) = 0$ and $f_2(x_1(t), x_2(t)) = 0$. These curves are called *nullclines* and their intersections define the fixed points of the model.
Moreover, the utility of drawing nullclines lies in the fact that the phase plane is divided into zones where the sign of the time derivatives $(\dot{x_1}, \dot{x_2})$ gives the resulting direction (obtained by the rule of vector sum) of the trajectory in that part of the plane. In this way, after computing the direction of the trajectory in every zone of the phase plane, it is possible to draw a useful qualitative sketch for trajectories.
An example, of what we said above for a 2-dimensional continuous time model, is the prey-predator model, a famous model used in ecology that describes the population of two species:

$$\begin{cases} \dot{x_1} = rx_1(1 - \frac{x_1}{k}) - \frac{ax_1}{1+ahx_1}x_2 & =: f_1(x_1, x_2) \\ \dot{x_2} = e\frac{ax_1}{1+ahx_1}x_2 - dx_2 & =: f_2(x_1, x_2) \end{cases} \tag{1.8}$$

In particular, x_1 represents the number of a species (the prey) while x_2 represents the quantity of their predators. In fig. 1.9, we draw, at first, the nullclines of f_1 and f_2 (in magenta and cyan, respectively), the three fixed points are their intersections (yellow dots) and then for each zone of the phase plane we computed the sign of $\dot{x_1}$ and $\dot{x_2}$, drawing a black arrow with the following convention, $\dot{x_1} < 0$ implies left arrow, $\dot{x_1} > 0$ right arrow, $\dot{x_2} < 0$ down arrow, $\dot{x_2} > 0$ up arrow. Then we summed the arrows with the rule of vector sum, computing the direction of the trajectory in each zone, defined by the red arrow.
Of course, since the considered model is ecological, it would be meaningless to consider trajectories and equilibria outside the first quadrant.

It is immediate to observe that the "central" fixed point (the equilibrium which does not lies on the Cartesian axes) attracts the trajectories generating a sort of "vortex", we will see in next pages that it is called *stable focus*.

We now consider a generic linear 2-dimensional continuous time model:

$$\begin{cases} \dot{x}_1 = a_{11}x_1(t) + a_{12}x_2(t) \\ \dot{x}_2 = a_{21}x_1(t) + a_{22}x_2(t) \end{cases} \tag{1.9}$$

Model (1.9) can be also rewritten in matrix form as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}; \quad \dot{\mathbf{x}}(\mathbf{t}) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$$

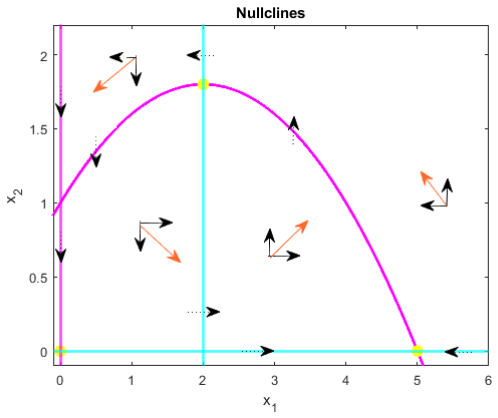


Figure 1.9: Nullclines and qualitative sketch for trajectories' directions for model (1.8), with $k = 5, a = 1, h = 1, d = 1, e = 2$, fixed points are in yellow dots and trajectories are represented by red arrows.

In order to study the stability of system (1.9), the characteristic equation, for the eigenvalues $\lambda_{1,2}$ of the system, should be computed:

$$\lambda^2 - Tr(A)\lambda + Det(A) = 0 \tag{1.10}$$

Many types of phase portraits can arise with respect to the value (the sign in particular) of the two eigenvalues of matrix $\mathbf{A}$.

1. If $\lambda_2 < \lambda_1 < 0$ (i.e. two distinct, real, negative eigenvalues). The general solution is:

   $$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} \tag{1.11}$$

   This case can be reformulated with these conditions for trace and determinant of matrix $\mathbf{A}$:

   $$Tr(A)^2 - 4Det(A) > 0, \quad Tr(A) < 0, \quad Det(A) > 0$$

   The asymptotically stable equilibrium is called *stable node* (or *sink*).

2. If $\lambda_1 > \lambda_2 > 0$ (i.e. two distinct, real, positive eigenvalues) that can also be rewritten as:

   $$Tr(A)^2 - 4Det(A) > 0, \quad Tr(A) > 0, \quad Det(A) > 0$$

   The general solution is analogous to (1.11). In this case, the unstable equilibrium is known as *unstable node* (or *source*).

3. If $\lambda_2 < 0 < \lambda_1$ (i.e. two distinct, real eigenvalues, one positive and one negative), that can be rephrased as

   $$Tr(A)^2 - 4Det(A) > 0 \quad Det(A) < 0$$

In this case the unstable fixed point is called *saddle*, since the trajectory at first approaches the equilibrium point along the invariant line (known as *stable manifold*) associated to the eigenvector related to the negative eigenvalue (λ_2, in the example) and then moves away following the invariant line (*unstable manifold*) related to the positive eigenvalue (λ_1).

4. If $\lambda_2 = \lambda_1 < 0$ (i.e. two coincident, real, negative eigenvalues) or

$$Tr(A)^2 - 4Det(A) = 0, \quad Tr(A) < 0$$

The equilibrium is called *stable improper node*. The general solution is:

$$\mathbf{x}(t) = c_1\mathbf{v}e^{\lambda t} + c_2\mathbf{v}te^{\lambda t}$$

with $\lambda = \lambda_1 = \lambda_2$ and $v = v_1 = v_2$.

5. If $\lambda_2 = \lambda_1 > 0$ (i.e. two coincident, real, negative eigenvalues) or

$$Tr(A)^2 - 4Det(A) = 0, \quad Tr(A) > 0$$

The equilibrium is called *unstable improper node* and the general solution is the same of the stable improper node.

6. If $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ (i.e. two complex conjugate eigenvalues) or

$$Tr(A)^2 - 4Det(A) < 0$$

In this case the general solution is given by:

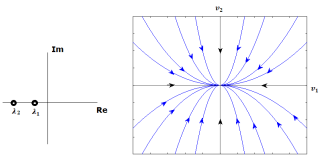
$$\mathbf{x}(t) = e^{at}[c_1(Re(\mathbf{v}_1))cos(bt) + c_2(Im(\mathbf{v}_1))sin(bt)]$$

The sign of the trace of matrix A, gives information regarding the expanding or contracting nature of the oscillations around the equilibrium point.

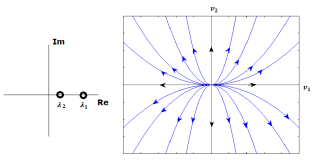
   (a) If $Tr(A) < 0$ then the equilibrium is a *stable focus* (negative eigenvalues since $a < 0$).
   (b) If $Tr(A) > 0$ the equilibrium is a *unstable focus* (positive eigenvalues since $a > 0$).
   (c) Finally if the eigenvalues are purely imaginary (i.e. $Tr(A) = 0$) the equilibrium is a *centre*.

Local phrase portraits of previously presented equilibrium points are showed in figures 1.10 and 1.11.

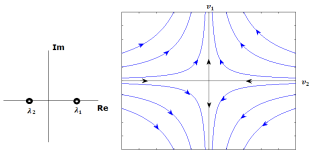
The goal is now to understand what happens when the 2-dimensional continuous time model is nonlinear. As for 1-dimensional version, there are results



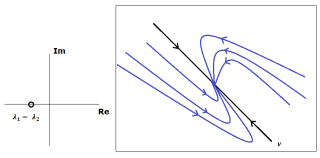
(a) *Stable node*



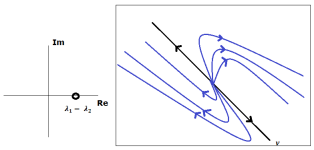
(b) *Unstable node*



(c) *Saddle*

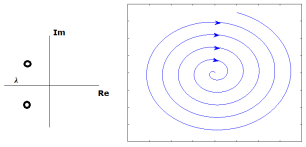


(d) *Stable improper node*

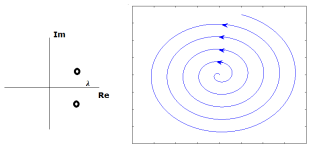


(e) *Unstable improper node*

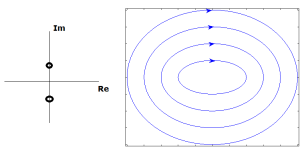
Figure 1.10: Local phase portraits with real eigenvalues.



(a) *Stable focus*



(b) *Unstable focus*



(c) *Centre*

Figure 1.11: Local phase portraits with complex eigenvalues.

that give information regarding the local behaviour of the system (i.e. a qualitative result for a neighbourhood of the equilibrium point).
Consider again system (1.7), the next theorem gives a powerful tool for qualitative information regarding the behaviour of the system.

Theorem 2 (**Hartman-Grobman, 1964**). *Give a nonlinear system, as (1.7). Let* $\mathbf{x}^* \in \mathbb{R}^2$ *be an equilibrium point, if* $\mathbf{x}^*$ *is hyperbolic, i.e. all the eigenvalues of the Jacobian matrix* $J(\mathbf{x}^*)$ *have non-null real part, then the local phase in a neighbourhood of* $\mathbf{x}^*$ *is qualitatively equivalent to the one related to the linear approximation.*

This theorem can be generalized for $n > 2$ and using definitions of homeomorphisms and neighbourhoods but the relevant part is identical: any hyperbolic equilibrium can be classified as stable or unstable just observing the linear approximation through the Jacobian matrix.
Of course, if the equilibrium is non-hyperbolic, Hartman-Grobman theorem cannot be applied, so other results should be used or numerical simulation could be enough to explain the local stability.

In a 2-dimensional continuous time system invariant sets are not just equilibrium points.

Definition 12. *A limit cycle is an invariant closed orbit such that a trajectory starting from a point of the invariant set will cross the original point after time* T*, i.e.* $x(t) = x(t+T)$*.* T *is the period of the cycle*

A limit cycle may be attractive, repellent or neutrally stable (centre).
An example of a limit cycle is presented here in figure 1.12 (where stability for the limit cycle is denoted by the color blue and instability by red):

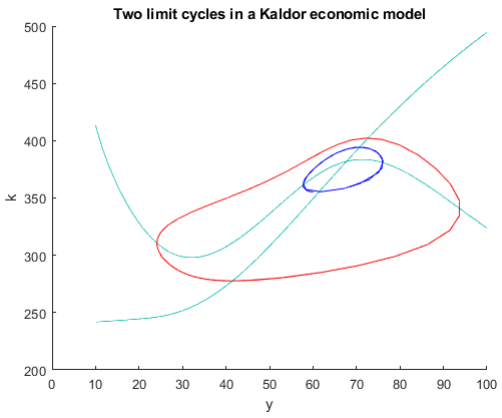


Figure 1.12: Limit cycles for the Kaldor business cycle model [3].

There are several ways in order to find out if a closed invariant orbit exists:

Theorem 3 (**Poincaré-Bendixson**). *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a set of two ordinary differential equations defined in an open set $G \subseteq \mathbb{R}$, and let $D \in G$ be a compact trapping set that does not contain any equilibrium point. Then D must contain at least one closed invariant orbit.*

Poincaré-Bendixson is applicable only in a 2-dimensional system. It gives an existence result but it does not specify whether the limit cycle is stable or not. Dulac proposed a method to rule out limit cycles:

Proposition 2 (**Dulac Criterion**). *Let $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ be a continuously differentiable vector field defined on a simply connected subset R of the plane. If there exists a continuously differentiable, real-valued function $g(\dot{\mathbf{x}})$ such that $\nabla \cdot (g\dot{\mathbf{x}})$ has one sign throughout R , then there are no closed orbits lying entirely in R.*

The drawback of Dulac criterion is that there is not an algorithm for searching $(g\dot{x})$. Usually good candidates are $g = 1, 1/x^a y^b, e^{ax}$ and e^{ay} [11]. Another method used in order to exclude limit cycles is to find a *Liapunov function.*

Definition 13 (**Liapunov function**). *Consider a fixed point $\mathbf{x}^*$, a Liapunov function V is a continuously diffrentiable, real-valued function such that:*

(i) $V(x) > 0$ *for all* $\mathbf{x} \neq \mathbf{x}^*$ *and* $V(\mathbf{x}^*) = \mathbf{0}$ *(V is positive definite);*

(ii) $\dot{V} < 0$ *for all* $\mathbf{x} \neq \mathbf{x}^*$.

From this definition the homonymous criterion is:

Proposition 3 (**Liapunov Criterion**). *If it is possible to find a Liapunov function then $\mathbf{x}^*$ is asymptotically stable and in particular closed orbits are not present.*

The problem of the method is analogous to Dulac criterion, in practice finding a Liapunov function can be very complicated since no algorithms exists.

The last useful criterion for excluding the presence of limit cycles that we present is:

Proposition 4 (**Bendixson Criterion**). *Let A be a closed and bounded region of $\mathbb{R}^2$, if the divergence of the system does not change its sign (or at most it is null of the axis) then there are not limit cycles in A.*

We conclude this section, regarding 2-dimensional continuous time dynamical systems, showing possible local bifurcations.
In addition to the ones presented in definitions (9), (10) and (11), there is now

the possibility that two conjugate complex eigenvalues cross the the imaginary axis. We have already seen that eigenvalues with non-null imaginary part create an oscillatory behaviour (stable and unstable focus or centre). The "new-entry" bifurcation in 2-dimension is the so called *Andronov-Hopf* (or simply *Hopf*).

Theorem 4 (**Andronov-Hopf bifurcation**). *Consider the 2-dimensional dynamic model:*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu), \qquad \mathbf{x} \in \mathbb{R}^{\mathbf{2}}, \quad \mu \in \mathbb{R}$$

let $\mathbf{x}^*(\mu)$ *be an isolated equilibrium point. Assume that the eigenvalues,* $\lambda_{1,2} = \alpha(\mu) \pm i\omega(\mu)$*, are complex for* μ *in a neighbourhood of* μ_0 *and for* $\mu = \mu_0$ *there holds that* $\alpha(\mu_0) = 0$ *and* $\omega(\mu_0) > 0$*. If* $\frac{\partial \mathbf{Re}\lambda_{1,2}}{\partial \mu} > 0$ *(for* $\mu = \mu_0$*). Then* $\mathbf{x}^*$ *is a stable focus for* $\mu < \mu_0$ *and an unstable focus for* $\mu > \mu_0$*, for* $\mu = \mu_0$ *a closed invariant orbit* Γ *is created around* $\mathbf{x}^*$ *such that one of the following holds:*

(i) Γ *exists for* $\mu > \mu_0$ *and is a stable limit cycle;*

(ii) Γ *exists for* $\mu < \mu_0$ *and is a unstable limit cycle;*

(iii) infinitely many closed invariant curves exist for $\mu = \mu_0$*.*

Moreover, the period of trajectories is $T(\mu) = \frac{2\pi}{\omega_0} + o(|\mu - \mu_0|)$ *and in cases (i) and (ii) the amplitude of* Γ *increases, as* μ *moves away from the bifurcation value* (μ_0)*, proportionally to* $\sqrt{|\mu - \mu_0|}$*.*

It is immediate to observe that (as in Pitchfork bifurcation), case (i) in theorem (4) is known as *supercritical Hopf bifurcation* and case (ii) is the *subcritical* version. A visual representation is shown below in fig. 1.13:

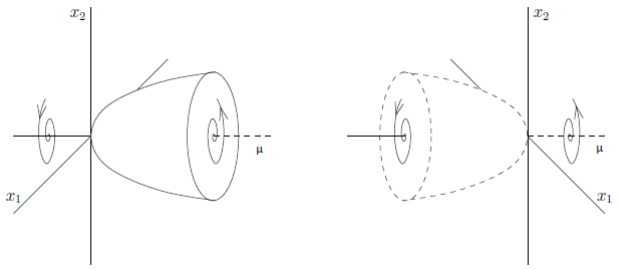


Figure 1.13: Hopf bifurcation (left supercritical, right subcritical).

Another way of seeing Hopf bifurcation is the following: the loss of stability in the supercritical case is "soft" since the fixed point becomes unstable, however there a stable limit cycle is created around it. This implies that trajectories remains nearby the fixed point. A completely opposite scenario

is the subcritical case, where the stable fixed point is at first surrounded by an unstable limit cycle (so trajectories starting inside the cycle will converge to the equilibrium) then when the bifurcation threshold is reached, the limit cycle collapses to the equilibrium point and disappears, moreover the equilibrium looses its stability. The result is an unstable fixed point where even trajectories starting from an arbitrarily small neighbourhood of it, will diverge. This is an example of *hard* stability loss.

Example (2-dim. Continuous Time Model)

Consider the dynamical model:

$$\begin{cases} \dot{x_1} = -px_1 + x_2 & =: f_1(x_1, x_2) \\ \dot{x_2} = \frac{x_1^2}{1+x_1^2} - x_2 & =: f_2(x_1, x_2) \end{cases} \tag{1.12}$$

where $p > 0$.

First of all, by applying Bendixson Criterion, proposition (4), it is immediate to deduce that there cannot be limit cycles in the first quadrant since the divergence of the system is equal to $-p-1 < 0$, for every value of p (observing that p is positive by hypothesis of the system).
Another interesting observation can be done by looking at fig. 1.14, where we draw $f_2(x_1, x_2)$ and $f_1(x_1, x_2)$ for specific values of p. We observe that, if p is greater than a threshold ($\bar{p} = 0.5$), then the intersection of nullclines gives only one fixed point (x_0^*, y_0^*), which is in origin of the axis. On the contrary if p is smaller than the threshold, the slope of f_1 allows the presence of 3 fixed points.

The result of this qualitative observation could also be done analytically, by imposing both $f_1(x_1, x_2)$ and $f_1(x_1, x_2)$ equal to 0. The result are 3 fixed points:

$$(x_0^*, y_0^*) = (0, 0), \quad (x_{1,2}^*, y_{1,2}^*) = \left(\frac{1 \pm \sqrt{1 - 4p^2}}{2p}, \frac{1 \pm \sqrt{1 - 4p^2}}{2} \right)$$

Of course, $(x_{1,2}^*, y_{1,2}^*)$ exist only if $1 - 4p^2 > 0$, i.e. if $p < \bar{p} = 0.5$. We saw that varying parameter p, there could be one or three equilibria; the first question one may have is: is it a pitchfork bifurcation or a fold?

The answer comes from the study of the stability of (x_0^*, y_0^*). By applying theorem (2), we can study the linear approximation via the Jacobian matrix and the results are qualitatively equivalent to the original ones, since the equilibrium is hyperbolic.
The Jacobian matrix for (x_0^*, y_0^*) is:

$$J = \begin{bmatrix} -p & 1 \\ 0 & -1 \end{bmatrix}$$

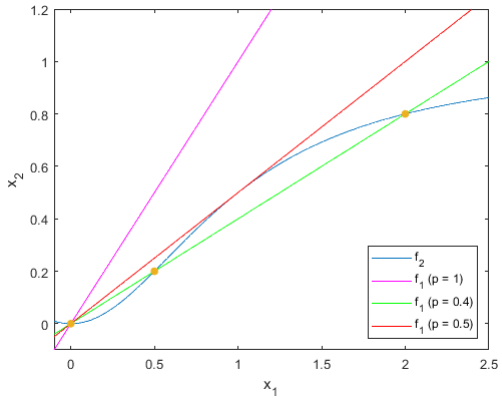


Figure 1.14: Nullclines for model (1.12), yellow dots represent fixed point in the case $p = 0.4$.

Its eigenvalues are: $-p$ and -1 and they are always both negative for every value of p (remembering again that by hypothesis of model (1.12) p is positive), which means that the equilibrium (x_0^*, y_0^*) is always stable.
So the answer to the previous question is that $\bar{p} = 0.5$ is the bifurcation value related to a fold bifurcation, since the origin never looses its stability.

Let us move to the analytical part of the exercise; in order to have simple computation we divide it into two cases: $p = 1$ and $p = 0.4$.

The first one is almost already done, since we saw previously that there is only one equilibrium point in the origin and it is stable. Its eigenvalues are equal both to -1, so they are coincident. It is a *stable improper node*. We only have to compute the eigenvector for $\lambda = -1$, which is equal to $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Its general solution will be:

$$\mathbf{x}(t) = c_1 \mathbf{v} e^{\lambda t} + c_2 \mathbf{v} t e^{\lambda t}$$

We now move to the more complex case $p = 0.4$. As said before there are 2 other fixed points in addition to the origin of the axes, more precisely they are $(x_1^*, y_1^*) = (2, 0.8)$ and $(x_2^*, y_2^*) = (0.5, 0.2)$. Consider at first (x_1^*, y_1^*), applying again Hartman-Grobman theorem (2), we study the linearization, which gives the Jacobian matrix:

$$J = \begin{bmatrix} -0.4 & 1 \\ 0.16 & -1 \end{bmatrix}$$

which gives the two eigenvectors $\lambda_1 = -0.2, \lambda_2 = -1.2$ and the correspondent eigenvectors:$\mathbf{v_1} = \begin{bmatrix} 0.98 \\ 0.20 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -0.78 \\ 0.62 \end{bmatrix}$.
There are two distinct, real and negative eigenvalues, so (x_1^*, y_1^*) is a *stable node*.
Its general solution will be:

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

Regarding (x_2^*, y_2^*), the Jacobian matrix is:

$$J = \begin{bmatrix} -0.4 & 1 \\ 0.64 & -1 \end{bmatrix}$$

which gives the two eigenvalues: $\lambda_1 = 0.15, \lambda_2 = -1.55$. The correspondent eigenvectors are:$\mathbf{v_1} = \begin{bmatrix} 0.87 \\ 0.48 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -0.65 \\ 0.75 \end{bmatrix}$.
It can be observed that here one eigenvalue is negative and one positive, so (x_2^*, y_2^*) is a *saddle*, where the stable manifold is $\mathbf{v_2}$ and the unstable manifold is $\mathbf{v_1}$.
The general solution is the same of (x_1^*, y_1^*).
Last but not least, we consider (x_0^*, y_0^*), its Jacobian matrix is:

$$J = \begin{bmatrix} -0.4 & 1 \\ 0 & -1 \end{bmatrix}$$

Its eigenvalues are $\lambda_1 = -0.4, \lambda_2 = -1$. For $p = 0.4$ the origin is a *stable node*, its eigenvectors are:$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -0.86 \\ 0.51 \end{bmatrix}$.
Again its general solution is analogous to the one of the other two fixed points.
Finally we report in figure 1.15 a sketch for the trajectories, where also the three fixed points are reported, as usual, in red if unstable (the saddle) and in green if stable (the two stable nodes).

It can be observed that a trajectory starting near the saddle node at first approaches it through the stable manifold and then suddenly moves away from (x_2^*, y_2^*) following the unstable manifold, indeed the name "saddle" comes from this peculiar characteristic of trajectories nearby it.

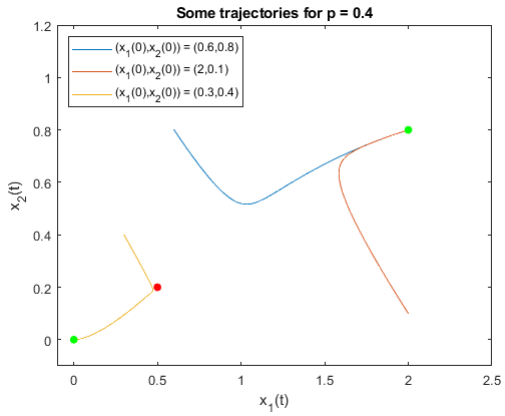


Figure 1.15: Different trajectories for model (1.12) for $p = 0.4$, stable and unstable equilibria are represented by green and red dots, respectively.

1.2.3 n - Dimensional Continuous Time Dynamical Systems

Many of the results obtained for 2-dimensional dynamical spaces in continuous time can be extended also for $n > 2$ dimensional spaces. However some behaviours can happen only if dimension of the system is grater than 2.
In particular, consider the system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu) \qquad \mathbf{x} \in \mathbb{R}^{\mathbf{n}}, \quad \mu \in \mathbb{R} \tag{1.13}$$

Then, as usual in order to find the equilibrium point $\mathbf{x}^*(\mu)$, we have to impose the system of n equations equal to 0, i.e. $\mathbf{f}(\mathbf{x}, \mu) = \mathbf{0}$.
In analogy to what claimed in chapters (1.2.1) and (1.2.2), if the equilibrium point is hyperbolic (i.e. the eigenvalues of the Jacobian matrix calculated in the fixed point $\mathbf{x}^*(\mu)$ have non vanishing real part), then the qualitative study of the the local stability of the fixed point is equivalent to study the stability of the linear approximation.
Again, varying the value of parameter μ, can lead to different scenarios for trajectories, that means that a bifurcation can arise.
If the eigenvalue that crosses the imaginary axis is real the bifurcations are the one presented in chapter (1.2.1), that means saddle-node, transcritical and pitchfork bifurcation.
If a pair of complex conjugate eigenvalues crosses the imaginary axis then the bifurcation that occurs is the Andronov-Hopf version, as mentioned in the previous chapter (1.2.2).

However, since now the dimension of the system is greater than 2, new scenarios may happen for trajectories. In particular, more complicated attractors than limit cycle can appear. They are called *strange attractors*.
The discover of these attractors lead to a particular field of study called

deterministic chaos. It might seem the union of two contrasting words without a meaning. However, we are dealing with continuous time deterministic models where also initial conditions are known but these attractors have such a complicated shape that induces the reader to compare the deterministic trajectories to the ones obtained using a stochastic model.
For this reason they juxtapose the "chaos" to the "determinism" of dynamical systems.
An example is the world-famous strange attractor of the Lorenz metereological model:

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) \\ \dot{x}_2 = \rho x_1 - x_2 - x_1 x_3 \\ \dot{x}_3 = x_1 x_2 - \beta x_3 \end{cases} \tag{1.14}$$

The chaotic trajectory is represented in figure 1.16:

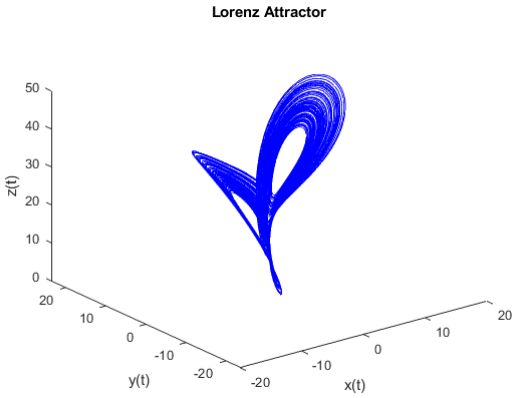
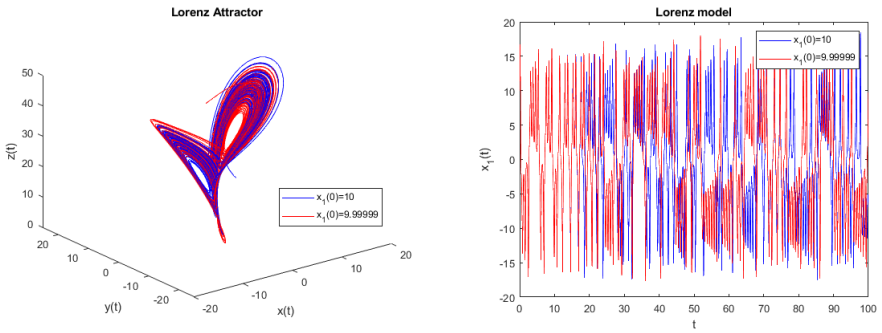


Figure 1.16: Lorenz attractor for $\sigma = 10$, $\rho = 28$, $\beta = 2.\bar{6}$ and $(x_1(0), x_2(0), x_3(0)) = (10, 15, 20)$.

Another characteristic of strange attractors is that if a trajectory starts from an initial condition inside the set then it remains there and covers all the points of the attractor (since it is an invariant set). If the trajectory starts from a point outside the attractor (but very close to it), then it will move towards the set and then it will have non-periodic trajectory.
Another characteristic is the sensitivity to initial conditions, also known as *butterfly effect*, it means that even a small difference from the initial conditions (a butterfly blink of wings, precisely) cause large-scale events (in Lorenz article, a butterfly flap may cause a tornado in another part of the planet).

This extreme sensitivity to initial condition can be well showed in fig. 1.17,

where a small variation to the initial condition for x_1 is imposed (from 10 to 9.99999). It can be observed that a tiny difference generates two completely different trajectories such that a long-time prediction is impossible; now it is clearer why the word "chaos" is used to describe these deterministic models:



(a) *"Lorenz's butterfly"* (b) *Time representations*

Figure 1.17: Lorenz attractor and time representation for $x_1(0) = 10$ and $x_1(0) = 9.99999$ (blue and red, respectively).

As said previously, these scenarios may happen only for continuous time systems with a dimension greater than 2 and with specific values for the parameters involved in the model.
In chapter (1.3), it will be showed that non-periodic trajectories and chaos can occur for bi-dimensional discrete time dynamical systems and even if the dimension is 1.

1.3 Discrete Time Dynamical Systems

This chapter is devoted to present discrete time dynamical systems, these models are characterized by a system of evolution equations of the form in (1.2), which are known as *Difference Equations*. Discrete time dynamical systems are models where changes in the state system occurs as a consequence of decisions which cannot be done continuously, in economy, for example, it is not possible to manage a portfolio buying and selling assets continuously, because of bank commissions that arise every time an asset is sold/bought, it would not be clever to modify very frequently the percentage of the assets that form the portfolio. Also in social models, it is frequent to deal with discrete time dynamical systems.
In this framework, the state of the system at next time $t+1$ is obtained by the application of the map $T : M \to M$ where, again as in definition (1), $M \subseteq \mathbb{R}^n$ is the state space which contains all the admissible values of the state variables. In general, the single application of map T is the "unit time advancement" of the state of the dynamical system. A graphical representation of the application of map T in a discrete system is presented in figure 1.18.

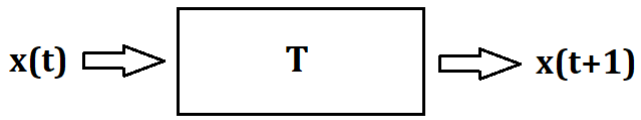


Figure 1.18: Graphical representation of the application of map T in a discrete time dynamical model.

The inductively iteration of map T defines the trajectory of the system and in general $\mathbf{x}(n) = T^n(\mathbf{x}(0))$.

The first part will be for mono-dimensional discrete time systems, then the next sub-chapter will be devoted to bi-dimensional systems.
One of the objectives of this chapter, apart from presenting discrete time system, is also to highlight main differences between continuous and discrete time systems; in this regard some examples will be showed.

1.3.1 1 - Dimensional Discrete Time Dynamical Systems

The simplest example is when the linear map is homogeneous, i.e. the involved model is:

$$x(t+1) = ax(t), \quad x(0) = x_0 \tag{1.15}$$

The solution of model (1.15), obtained by induction, is:

$$x(t) = x_0 a^t$$

where $t \in \mathbb{N}$.

If $|a| < 1$ then the trajectory of the model convergence to the unique fixed point $x^* = 0$, in particular, if $-1 < a < 0$ then the converges has an oscillatory behaviour, if $0 < a < 1$ the convergence is monotone.
On the other side if $|a| > 1$ then the equilibrium cannot be attractive, in particular if $a > 1$ system (1.15) will monotonically diverge while it as well diverge with oscillations in the case where $a < -1$.
The two final cases are when $a = 1$ and $a = -1$. In the first one, an identity map is obtained (i.e. $x(t+1) = x(t)$) so the result will be a constant sequence (i.e. $x(t) = x_0$). In the second case, the result will be the oscillating sequence $x(t) = (-1)^t x_0$.
It is important to observe that, even if we are dealing with the simplest discrete time model with one dimension, it is possible to obtain oscillating trajectories; this cannot happen for 1-dimensional continuous time models.

A similar reasoning, to the one done for the previous model, can be done for the linear non-homogeneous discrete time model: $x(t+1) = ax(t) + b$.

We now introduce a generic 1-dimensional model:

$$x(t+1) = f(x(t)), \quad x(0) = x_0 \tag{1.16}$$

In order to find the equilibria of the system (i.e. points where $x(t+1) = x(t)$), it is necessary to impose the right-hand side of equation (1.16) equal to x (i.e. $f(x) = x$).
Suppose that x^* is a fixed point for system (1.16), then we have the following definition:

Definition 14. *An equilibrium point x^* is hyperbolic if $|f'(x^*)| \neq 1$, otherwise if $|f'(x^*)| = 1$ then x^* is a non-hyperbolic equilibrium.*

It can be observed that for discrete time systems the condition of non-hyperbolicity is divided into two cases: $f'(x^*) = 1$, that corresponds to the continuous time analogous condition presented in definition (7) and $f'(x^*) = -1$ that has no correspondence in continuous time.

Proposition 5. *Let x^* be an equilibrium point for system (1.16). If $|f'(x^*)| < 1$ then x^* is an asymptotic stable equilibrium, if $|f'(x^*)| > 1$ then x^* is unstable.*

The previous proposition comes from the fact that, in case of an hyperbolic equilibrium, the stability results obtained from a linear approximation are qualitatively equivalent to the stability of the original model. Indeed consider the linear approximation of (1.16):

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + o(x - x^*)$$

that leads to:

$$x(t+1) = x^* + f'(x^*)(x - x^*)$$

Again, as for continuous time models, there is the possibility that a variation of a model's parameter causes a qualitative different dynamic scenario. We already saw that there is the possibility for 1-dimensional discrete time systems to have oscillatory trajectories but this chapter will show that a new bifurcation can happen for discrete time models.
First of all, consider the system:

$$x(t+1) = (x(t), \alpha) \tag{1.17}$$

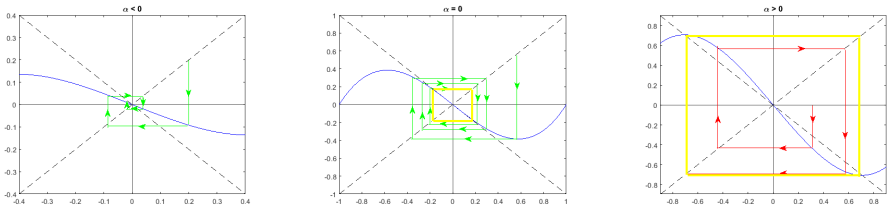
and let $x^*(\alpha)$ be a fixed point.
We saw the stability conditions in proposition (5) and now there are two possibilities for a system to loose stability, indeed the quantity $f'(x^*(\alpha))$ can cross the bifurcation value at ± 1.
In particular, when $f'(x^*(\alpha)) = +1$ three types of bifurcation may occur: *fold, transcritical* or *pitchfork (supercritical and subcritical)*. They are analogous to the one seen in continuous time, the only difference is that the tangency occurs along the diagonal of the first and third quadrant whilst it comes on the abscissa in the continuous time case. For this reason bifurcation diagrams are equivalent to the ones in fig. (1.1), (1.3) and (1.5).

The new bifurcation is related to the case $f'(x^*(\alpha)) = -1$.

Definition 15 (Flip Bifurcation). *Flip bifurcation is characterized by the creation of a cycle of period* 2 *at the bifurcation value.*

As for the pitchfork bifurcation, the flip has a *supercritical* case and a *subcritical* one. The first one coincides with the creation of a stable 2-periodic cycle (i.e. trajectories starting inside it will approach the periodic cycle). On the contrary, the subcritical case coincides with the presence of an unstable 2-periodic cycle.
In fig. 1.19, it is showed the phase plane for the supercritical flip bifurcation

when the considered function f is the canonical example $f(x,\alpha) = -(1+\alpha)x + x^3$ and the bifurcation parameter is $\alpha = 0$. We observe that the stable equilibrium looses stability and a stable periodic cycle is created around it. Again the red color indicates an unstable trajectory while green a stable one, the stable period cycle is denoted by the thick yellow line, finally function f is locally represented in blue.



(a) (b) (c)

Figure 1.19: Flip bifurcation (supercritical) phase portrait for the system $x(t+1) = -(1+\alpha)x + x^3$, stable and unstable trajectories are represented in green and red, respectively. The yellow line represents the stable periodic cycle.

After the flip bifurcation, two new fixed points of the periodic cycle arises for the map $f^2(x,\alpha) = f(f(x,\alpha))$, moreover the flip bifurcation for the map f coincides with the pitchfork bifurcation of map f^2. Indeed if $f'(x^*) = -1$ then $\frac{df^2}{dx}(x^*) = f'(f(x^*))f'(x^*) = f'(x^*)f'(x^*) = +1$.
Bifurcations are not just for fixed points, they can be seen also in k-periodic cycles (since a point of the k-periodic cycle is a fixed point for the map f^k). Indeed the *fold bifurcation* arises when two k-periodic cycles (one stable, one unstable) are created or destroyed at the bifurcation value, the *transcritical bifurcation* happens when two k-periodic cycles with opposite stability merge and then exchange their stability and finally, the *supercritical (or subcritical) pitchfork bifurcation* is characterized by the creation (or destruction) or two k-periodic cycles.

Example (The Logistic Map)

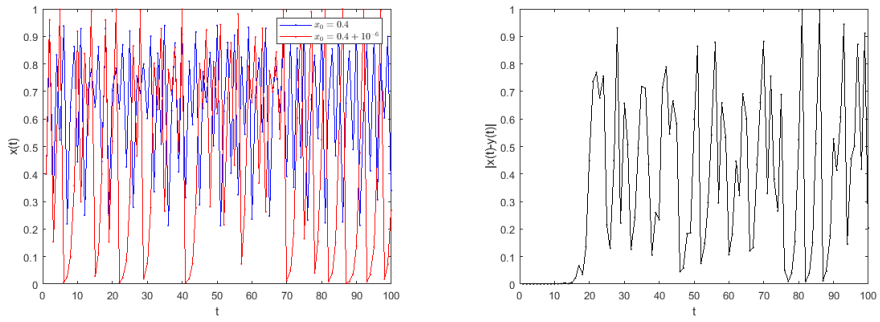
As for the continuous time case, the logistic growth equation (1.6) has discrete version:

$$x(t+1) = \mu x(t)(1 - x(t)), \quad \mu > 0 \tag{1.18}$$

In order to find the fixed points, the procedure involves setting equal to x the logistic map $f(x) := \mu x(t)(1-x(t))$; the resulting equilibrium points are $x_0^* = 0$ and $x_1^* = 1 - \frac{1}{\mu}$.

The objective now is to show that a chaotic behaviour can arise for particular values of parameter μ, this fact would highlight a huge difference to the continuous time version of the model (1.6).
Before that, we study the local stability of x_0^* and x_1^*. Proposition (5) gives the possibility to analyze the stability by computing the derivative of $f(x)$, which is equal to $f'(x) = \mu(1-2x)$, so $f'(x_0^*) = f'(0) = \mu$ and $f'(x_1^*) = f'\left(1-\frac{1}{\mu}\right) = 2-\mu$. The stability conditions expressed in prop. (5) are $|\mu| < 1$ and $|2-\mu| < 1$. So x_0^* is asymptotically stable if $\mu < 1$ and x_1^* when $1 < \mu < 3$. The local bifurcation at $\mu = 1$ is trascritical, since the two equilibria merge and then exchange their stability.
We now want to observe what happens for $\mu = 3$, in particular $f'(x_1^*) = 2-\mu = -1$. We saw that the condition $f'(x^*) = -1$ is typical and unique of discrete time models and the arising local bifurcation is the *flip*, in which a stable cycle, formed by two points, is created and the fixed point x_1^* looses its stability. In this case the flip bifurcation is supercritical, like the one in fig. 1.19. Moreover we saw that points of the 2-periodic cycle are fixed points for the map $f^2(x)$, the computation gives two periodic points α and β, which exist only if $\mu > 3$, $\alpha, \beta = \frac{\mu+1\pm\sqrt{(\mu-3)(\mu+1)}}{2\mu}$.
Again, we observe that the flip bifurcation for f coincides with a supercritical pitchfork bifurcation for f^2.
The next step is to check that the two points of the periodic cycle are stable; in order to do that it is necessary to compute the derivative of $F := f^2$ and evaluate it in α and β. The result is that $F'(\alpha) = F'(\beta)$ is less in modulus than 1 until the bifurcation value $\mu = 1+\sqrt{6} \simeq 3.449$ where $F'(\alpha) = F'(\beta) = -1$, i.e. another flip bifurcation arises, this time not for the map f but for f^2; in particular the 2-periodic cycle looses stability and another stable cycle is created with period 4.
Increasing μ, the stable cycle of period 4 becomes unstable and another stable cycle with period 8 is created through a flip bifurcation for map f^4.
We can iterate the procedure until, intuitively, the distance between two consecutive bifurcation points will tend to 0, the limit point $\mu_\infty \simeq 3.5699$. After the limit point, periodic trajectories of any period appear as aperiodic trajectories, since the iteration of flip bifurcations, also known as *doublig cascade*, creates such complicated cycles that a trajectory never hits an already visited point of the k-period cycle. This trajectories are called *chaotic*.
A well known consequence of chaotic trajectories is the sensitivity to initial conditions, i.e. two trajectories starting from two arbitrarily close initial conditions may assume completely different paths.
This property is highlighted in fig. 1.20.

This concept was already present in continuous time for systems with a dimension grater than 2, like the Lorenz model (1.14) in fig. 1.17.
In figure 1.21, we report the bifurcation diagram of the logistic map.



(a) *Trajectories for different initial conditions*

(b) *Difference (in absolute value) of the two trajectories*

Figure 1.20: Trajectories for model (1.18) for close different initial values.

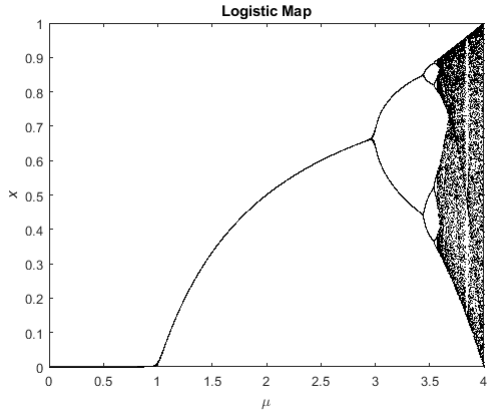


Figure 1.21: Bifurcation diagram for model (1.18).

The important message the logistic map is that, even if it is just a 1-dimensional model (so it might be a "simple" model at first glance), for some values of μ a flip bifurcation occurs and increasing the value of the parameter a doubling cascade arises until the complexity of periodic cycles is so high that trajectories can be seen as chaotic (without periodicity). We saw in chapter (1.2.1) that for 1-dimensional continuous time models, the trajectories cannot be chaotic and the local bifurcations that can arise are fold, transcritical and pitchfork.

The last property of this important example, that we would like to highlight, is the *periodic windows*

Definition 16 (**Periodic Windows**). *Periodic windows are strips in the bifurcation diagram where chaos seems to disappear temporarily, instead an attracting periodic cycle is present.*

This curious behaviour is due to a fold bifurcation of f^k caused by a tangency between f^k and the diagonal of the first quadrant.
Slightly increasing the value of μ, the stable cycle looses stability via a flip bifurcation and and a doubling cascade leads to a new chaotic condition.
Looking at fig. 1.22, which is nothing else than a zoom of the bifurcation diagram in fig. 1.21, it can be observed that other small periodic windows are still present (they are infinitely many), the result is that the zoom of the bifurcation diagram present the same qualitative characteristics of the general diagram, the phenomenon of observing the same qualitative behaviour by taking into account a smaller part of the original object is known as *fractals*. Many books were written about this very interesting argument, indeed fractal structures can be seen in nature like clouds, broccoli, pineapples, etc. In particular the bifurcation diagram of the logistic map creates a fractal structure for $\mu \gtrsim 3.5$.

Again, it is necessary to highlight the main difference between mono-dimensional continuous and discrete time dynamical models, here we observed in the logistic map that it is possible to obtain the so-called "deterministic chaos" simply varying the value of the parameter μ. In 1-dimensional continuous time models (1.2.1), this behaviour is impossible to obtain, moreover not even flip bifurcation can arise in continuous time.
This observation makes clear why continuous time models have always been preferred to discrete ones in practical uses, since the latter could easily generate a chaotic behaviour which is difficult to conciliate to the choice of using a deterministic model which, sometimes naively, gives the expectation of getting relatively simple trajectories.

1.3.2 2 - Dimensional Discrete Time Dynamical Systems

A general discrete time model with two variables has the form:

$$\begin{cases} x_1(t+1) = T_1(x_1(t), x_2(t)) \\ x_2(t+1) = T_2(x_1(t), x_2(t)) \end{cases} \tag{1.19}$$

In order to find the equilibrium points the previous model, it is necessary to create a system of equation where the right-hand side of (1.19) is equal to x_1 and x_2. That means:

$$\begin{cases} T_1(x_1, x_2) = x_1 \\ T_2(x_1, x_2) = x_2 \end{cases} \tag{1.20}$$

Again, the stability of fixed points and periodic cycles can be qualitatively determined by the linearization of the map $\mathbf{T} = (T_1, T_2)$ in an arbitrarily small neighbourhood of the fixed point.

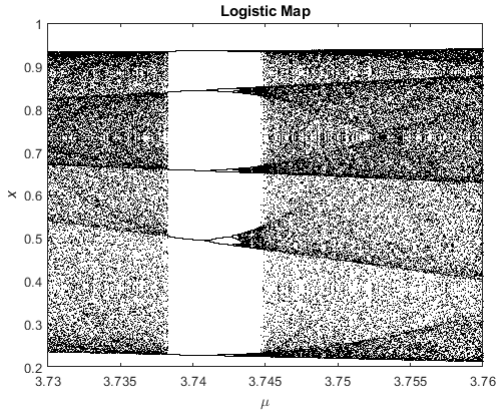


Figure 1.22: Periodic windows for model (1.18).

We now focus on the simplest 2-dimensional discrete time model: the linear system:

$$\begin{cases} x_1(t+1) = a_{11}x_1(t) + a_{12}x_2(t) \\ x_2(t+1) = a_{21}x_2(t) + a_{22}x_2(t) \end{cases} \tag{1.21}$$

which, like for the continuous time case, can be written in the matrix form:

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t)$$

where:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

The characteristic equation is used to study the stability of the linear model (1.21) and it is imposed equal to 0:

$$P(\lambda) = \lambda^2 - Tr(A)\lambda + Det(A) = 0$$

We can analyze the results by looking at the value of $\Delta = Tr(A)^2 - 4Det(A)$.

1. If $\Delta > 0$ then there are two real, distinct eigenvalues and the general solution is:
$$\mathbf{x}(t) = c_1\mathbf{v}_1\lambda_1^t + c_2\mathbf{v}_2\lambda_2^t$$
where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the eigenvectors related to λ_1 and λ_2 and c_1 and c_2 two real constants determined by applying the initial condition given in the original model

2. If $\Delta = 0$ then there are two real and coincident eigenvalues, the general solution is:
$$\mathbf{x}(t) = c_1\mathbf{v}\lambda^t + c_2\mathbf{v}t\lambda^t$$
where $\lambda = \lambda_1 = \lambda_2$ and $\mathbf{v}$ is the eigenvector

3. If $\Delta < 0$ then there are two complex, distinct eigenvalues, the general solution is:

$$\mathbf{x}(t) = |\lambda|^t[(c_1\mathbf{v}_1 - c_2\mathbf{v}_2)sin(\theta t) + (c_1\mathbf{v}_1 + c_2\mathbf{v}_2)cos(\theta t)]$$

where: $\lambda_{1,2} = -\frac{Tr(A)}{2} \pm i\frac{\sqrt{-\Delta}}{2} = |\lambda|(cos\theta \pm sin\theta), |\lambda| = \sqrt{Re(\lambda)^2 + Im(\lambda)^2} = \sqrt{Det(A)}$ and $\theta = arctan\left(\frac{Im(\lambda)}{Re(\lambda)}\right)$.

Of course, the local dynamic is stable if both the eigenvalues are inside the unit circle in the imaginary plane.
Depending on the the value of λ_1 and λ_2, the local phase portrait is essentially analogous to a linear 2-dimensional continuous time model.
Here in figures 1.23 - 1.24, we report the local phase portraits for real and complex eigenvalues, respectively.

An important necessary and sufficient condition for discrete time models is reported below:

Proposition 6 (**Stability Criterion**). *All eigenvalues of model (1.21) are less than* 1 *in modulus if:*

$$\begin{cases} P(1) = 1 - Tr(A) + Det(A) > 0 \\ P(-1) = 1 + Tr(A) + Det(A) > 0 \\ Det(A) < 1 \end{cases} \tag{1.22}$$

Equations in system (1.22) form the so called *Stability Triangle* because they can be represented by a triangle in the Trace-Determinant plane. In fig. 1.25, we report the graphical interpretation of the Stability Criterion.

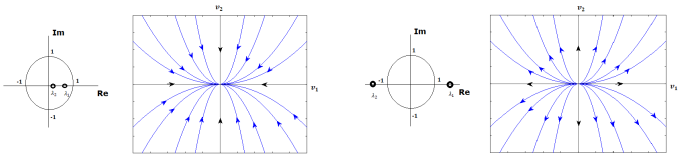
When the point, whose coordinates are determined by the trace and the determinant of matrix $\mathbf{A}$, is inside the stability triangle then the fixed point $\mathbf{x} = \mathbf{0}$ is globally asymptotically stable, when it is outside then trajectories will diverge.
Of course, if the dynamical system has parameters, then bifurcations will arise when the point will exit from the stability triangle.

Consider now a nonlinear 2-dimensional discrete time model (1.19) and a fixed point for it: $\mathbf{x}^* = (x_1^*, x_2^*)$, obtained by solving system (1.20).
Linearizing in a neighbourhood of the fixed point the obtained approximation is:

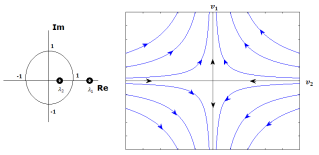
$$\mathbf{x}(t+1) - \mathbf{x}^* = \mathbf{J}_T(\mathbf{x}^*)(\mathbf{x}(t) - \mathbf{x}^*)$$

where $\mathbf{J}_T$ is the Jacobian matrix.

Proposition 7 (**Sufficient condition for stability**). *A sufficient condition for local asymptotic stability of a fixed point* $\boldsymbol{x}^*$ *of the nonlinear model*

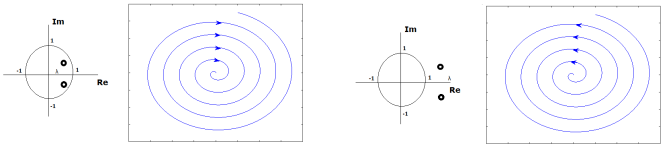


(a) *Stable node* (b) *Unstable node*

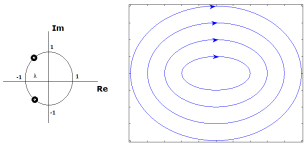


(c) *Saddle*

Figure 1.23: Local phase portraits with real eigenvalues.



(a) *Stable focus* (b) *Unstable focus*



(c) *Centre*

Figure 1.24: Local phase portraits with complex eigenvalues.

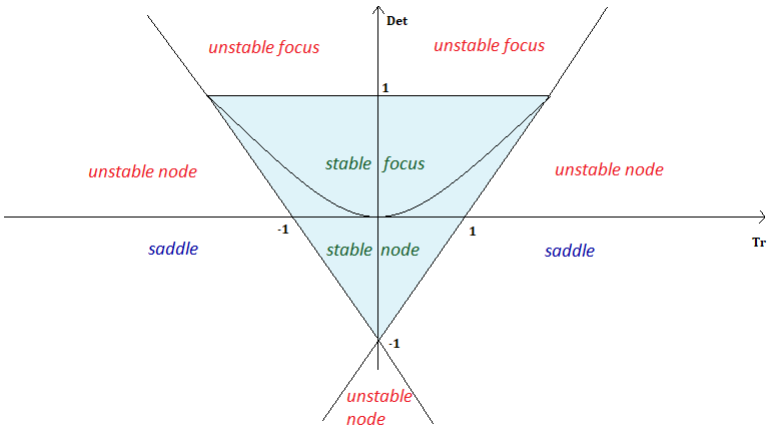


Figure 1.25: Stability Triangle (1.22) in Trace-Determinant plane.

(1.19) is represented by the Stability Criterion (6) where the Jacobian matrix $\boldsymbol{J}_T(\boldsymbol{x}^*)$ *is substituted in place of matrix A in the stability triangle (1.22).*

An important observation, which it is useful to remark, is that proposition (6) is a necessary and sufficient condition for linear models while in case of a generic nonlinear model, proposition (7) is just a sufficient condition.

We now deal with possible local bifurcations for model (1.19): we have already seen in chapter (1.3.1) that, varying the value of an exogenous parameter, a real eigenvalue can exit from the stability region (i.e. the unit circle in the imaginary plane) crossing the value $\lambda = 1$, generating a fold, transcritical and pitchfork bifurcation, or $\lambda = -1$, generating a 2-periodic cycle via a flip bifurcation (that can be iterated forming a doubling cascade until chaos is reached as in the logistic map example).
Since now, we are dealing with bi-dimensional models, the Jacobian matrix $\mathbf{J}_T(\mathbf{x}^*)$ can generate a pair of complex conjugate eigenvalues. Again, varying the value of parameter μ this pair can exit from the stability region causing the discrete time analogous of the Andronov-Hopf bifurcation for continuous time models, called *Neimark-Sacker bifurcation.*

Theorem 5 (**Neimark-Sacker bifurcation**). *Consider a nonlinear 2-dimensional discrete time dynamical model. Let* $T(x, \mu) : \mathbb{R}^2 \to \mathbb{R}^2$ *be the map with a family of fixed points* $x^*(\mu)$ *at which the eigenvalues are complex conjugates:* $\lambda(\mu)$ *and* $\bar{\lambda}(\mu)$*. When the parameter* μ *is equal to the bifurcation value* μ_0*, if the following conditions are true:*

(i) $|\lambda(\mu_0)| = 1$*, but* $\lambda^j(\mu_0) \neq 1$ *for* $j = 1, 2, 3, 4$*;*

(ii) $\frac{d|\lambda(\mu)|}{d\mu}(\mu_0) = d \neq 0.$

Then in a neighbourhood of $x^*(\mu_0)$ the map T is conjugate to the map (expressed in polar coordinates) $T_e(r,\theta) = (r(1+d(\mu-\mu_0)+ar^2), \theta+c+br^2)+o(r,\theta)$.
Moreover, if:

(iii) $a \neq 0$,

then there exist a simple closed invariant curve in a neighbourhood of $x^*(\mu_0)$.

The sign of the coefficient a determines the stability of the bifurcation, indeed if $a < 0$, the Neimark-Sacker bifurcation is in the *supercritical* case. On the contrary, if $a > 0$, it is the *subcritical* case.
the stability of the bifurcation can be studied by looking at the fixed point at the bifurcation value μ_0; if it is stable then the Neimark-Sacker is supercritical; viceversa, if the equilibrium is unstable then the bifurcation is subcritical.
In fig. 1.26, the two cases of the Neimark-Sacker bifurcation are reported.

In particular, since the discrete time behaviour involves jumps along the closed invariant curve, the trajectory may never hit an already visited point of the curve (*quasi-periodic trajectory*) or after n iterations may hit a visited point of the curve entering in a n-cycle (the related behaviour is known as *frequency locking*).

Finally, summing up, in fig. 1.27, the three way of possible bifurcations are showed with respect to the Stability Triangle (1.22) in the Trace-Determinant plane.

Example (Financial Market with Heterogeneous Agents)

In this simplified example, let consider a world where there are only two types of market agents: fundamentalist and chartists. The first ones assume that they know the value of a risky asset, they suppose it is equal to F,

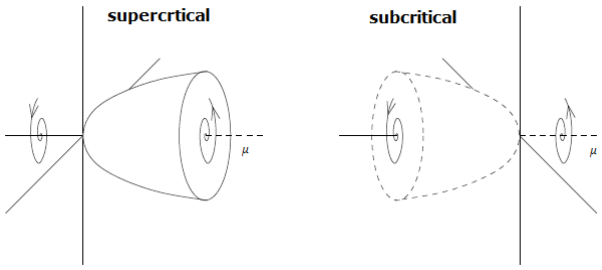


Figure 1.26: Neimark-Sacker bifurcation.

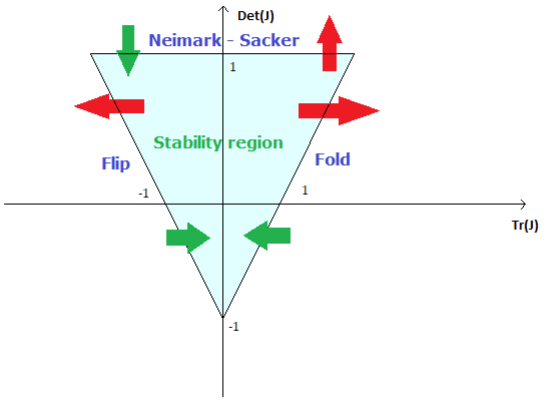


Figure 1.27: Bifurcations in Trace-Determinant plane.

so whenever the market price of the asset is less than F, fundamentalist will buy the asset; on the contrary when the price $p(t)$ is greater than the threshold F, fundamentalists will sell the stock because in their's opinion it is overestimated. Chartists (also called technical traders) use a strategy that take into account the performance of the stock, so when the chart shows an increasing behaviour of the stock they will buy it, believing that the price will increase again; on the contrary, when the stock price decreases, chartists will sell it, relying on the fact that this negative trend will continue also in the future.
Suppose that the price p of the risky asset at the future time $t+1$ depends on the price now at time t and on the excess of demand ΔD at time t, ($\Delta D(t)$ can be positive or negative). This assumptions form the law of the discrete dynamics for the price in the future:

$$p(t+1) = p(t) + \gamma \Delta D(t) \tag{1.23}$$

where γ is a positive constant that measures market's reactivity (the speed of adjustment)

Suppose that chartists compute the price p of the stock at time $t+1$ by considering the trend of p at time t and $t-1$. Their excess of demand for the stock will be (if they are prudent):

$$\Delta D^C(t) = \beta \arctan(p(t) - p(t-1)) \tag{1.24}$$

where β is a positive coefficient that measures the weight of chartists in the market.
Supposing that the excess of demand for fundamentalist is connected to their expected value of the stock price F and it is:

$$\Delta D^F(t) = \alpha(F - p(t)) \tag{1.25}$$

where α is the positive coefficient that measures the weight of fundamentalists in the market ($\alpha + \beta = 1$).
Plugging in (1.23) expressions (1.24) and (1.25) and thanks to the auxiliary variable $z(t) = p(t-1)$ we obtain the nonlinear system of order one:

$$\begin{cases} z(t+1) = p(t) & =: T_1(z(t), p(t)) \\ p(t+1) = (1-\gamma\alpha)p(t) + \gamma\beta \arctan(p(t) - z(t)) + \gamma\alpha F & =: T_2(z(t), p(t)) \end{cases} \tag{1.26}$$

with initial conditions:

$$\begin{cases} z(0) = p_{-1} \\ p(0) = p_0 \end{cases}$$

By imposing the system:

$$\begin{cases} T_1(z, p) = z \\ T_2(z, p) = p \end{cases}$$

We obtain the unique fixed point of model (1.26): $(z^*, p^*) = (F, F)$.
In order to study the stability of the system, we apply proposition (6), the relative Jacobian matrix $\mathbf{J} := \mathbf{J}_{(z^*, p^*)}$ is:

$$\mathbf{J} = \begin{bmatrix} 0 & 1 \\ -\gamma\beta & 1 + \gamma(\beta - \alpha) \end{bmatrix}$$

We can apply the sufficient condition of the stability triangle (7) and obtain:

$$\begin{cases} 1 - Tr(\mathbf{J}) + Det(\mathbf{J}) = \gamma\alpha > 0 \\ 1 + Tr(\mathbf{J}) + Det(\mathbf{J}) = 2 + 2\gamma\beta - \gamma\alpha > 0 \\ Det(\mathbf{J}) = \gamma\beta < 1 \end{cases}$$

Since α, β and γ are strictly positive coefficients, the first condition is always fulfilled.
Regarding the second and third conditions, they can be summarized by the inequality for β:

$$\frac{\alpha}{2} - \frac{1}{\gamma} < \beta < \frac{1}{\gamma}$$

When β is increased (i.e. increasing the weight of chartists in the market) and reaches the value $\frac{1}{\gamma}$ a Neimark-Sacker bifurcation occurs, this is also confirmed by looking at fig. 1.27, since, crossing the bifurcation value, the

stable focus becomes unstable and a stable closed invariant curve is created around the unique fixed point.
On the other hand, increasing the value of α (i.e. the weight of fundamentalists increases) leads to a flip bifurcation, where a stable periodic cycle is created.
These bifurcations are summed up in fig. 1.28, where the stability region is colored in light green.

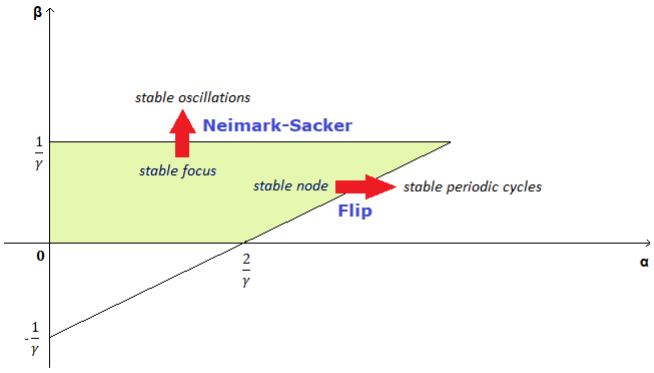


Figure 1.28: Stability region (in light green) and possible bifurcations for model (1.26) in α-β plane.

Here, for our purposes, the performed analysis is enough in order to show possible behaviours for a 2-dimensional discrete time model. Of course, in order to have a more complete view of the model's trajectories, it is unavoidable to perform numerical simulations, which could also validate the correctness of results obtained by our qualitative analysis, but, as previously said, it goes beyond the goal of the example.

Chapter 2

Discretization

2.1 General Introduction

In finance and economics, the majority of models are described using *Ordinary Differential Equations (ODE)*, which means that these models are such that time is a continuous independent variable. However, data are available in discrete time and in a very few cases it is possible to obtain such frequent data that they could be considered, with a reasonable approximation, as continuous time data.
For this reason, practitioners use to convert continuous time models into discrete time ones. There are many discretization methods, some of them are considered naïve. The biggest problem of discretizing a continuous time model is the arise of complex dynamics in the discretized model. For example, in chapter (1.2.1), we observed that only pitchfork, saddle or transcritical bifurcation may appear in a 1-dimensional continuous time model; on the contrary, in chapter(1.3.1), we saw the 1-dimensional logistic map (1.6) could present the flip bifurcation (which cannot happen in a continuous time dynamic system) and even chaos may arise for particular values of the parameter μ.
The natural consequence is a bloom of discretization models. In particular, Ronald Mickens proposed the so called *Non-Standard Finite Difference schemes (NSFD)*, a discretization model that preserves the original dynamics of the continuous time model.
In this thesis, in chapter (3), we will apply a similar discretization method, known as *Nearly Exact Discretization Scheme (NEDS)*, proposed in [5], which is a sort of evolution of a NSFD model, to an economic continuous time model. Here, in the following part of the chapter, different discretization method will be presented and eventually NEDS model will be showed.
The first method presented may be considered the simplest one, known as *Euler method*.

2.2 Euler Method

The Euler method is largely used, even in professional environments, mostly due to its simplicity, however its biggest drawback is that, in some cases, it does not preserve the stability of the original continuous time model.

Consider a system of differential equations (with n variables):

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) \tag{2.1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)^T$.
Euler method transforms model (2.1) into:

$$\frac{\mathbf{x}(t+h) - \mathbf{x}(t)}{h} = f(\mathbf{x}(t)), \quad h > 0$$

where h is the step-size.

Substituting t by n and $t + h$ by $n + 1$, we have a system of difference equations:

$$\mathbf{x}(n+1) = \mathbf{x}(n) + hf(\mathbf{x}(n)), \quad n = 1, 2, ... \tag{2.2}$$

with the usual initial condition: $x(0) = x_0$.

Thanks to Euler discretization method, it is possible to state the following proposition:

Proposition 8 (Fixed points for Euler method). *If p_0 is a fixed point for model (2.1) then $f(p_0) = 0$ and the same is true for the discretized model (2.2). Moreover the Jacobian matrix calculated in p_0 is respectively $J(f(p_0))$ and $I + hJ(f(p_0))$ and if z is an eigenvalue of $J(f(p_0))$ of the continuous time model (2.1) then $w = 1+hz$ is an eigenvalue of the matrix $I+hJ(f(p_0))$ of the discretized model (2.2).*

In addition to that, [9] discussed a relation between z and $w = 1 + hz$ in the following lemma.

Lemma 1. *For any $h > 0$; the map $w = 1 + hz$ maps the left half plane $Re(z) < 0$ onto the left half plane $Re(w) < 1$ and the disk $|z + 1/h| < 1/h$ onto the unit disk $|w| < 1$.*

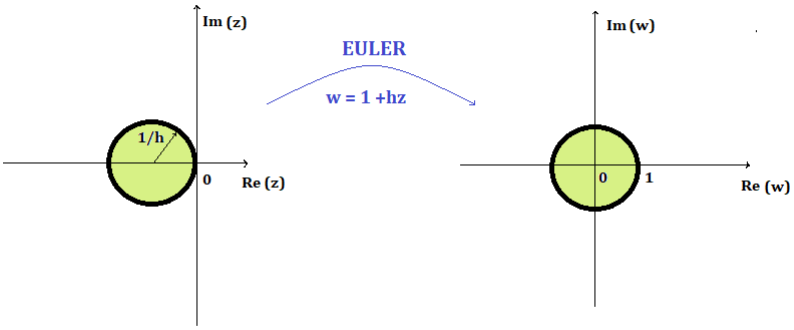


Figure 2.1: Euler's mapping $w = 1 + hz$ maps the disk $|z + 1/h| < 1/h$ into the unit circle $|w| < 1$.

Figure 2.1 shows graphically Lemma (1).

The straight-forward implication, regarding the Euler method, from Lemma (1) is that some eigenvalues of the continuous time model that are on the left-half plane might not be inside the disk $|z + 1/h| < 1/h$ and thus they will not be inside the unit circle $|w| < 1$.
The result is the following proposition:

Proposition 9 (**Stability of Fixed Points**). *Let p_0 be a fixed point for the continuous time model (2.1), after that we apply Euler method, the stability of equilibrium p_0, for the model (2.2), will present 3 possible cases:*

1. *p_0 is unstable if p_0 is unstable for the continuous time model.*
2. *p_0 is stable if p_0 is stable for the continuous time model and all the eigenvalues of $J(f(p_0))$ are inside the disk $|z + 1/h| < 1/h$.*
3. *p_0 is unstable if p_0 is stable for the continuous time model but unlike before, there is at least one eigenvalue of $J(f(p_0))$ which is outside the disk $|z + 1/h| < 1/h$.*

Of course, with a small step size h, the possibility for the Euler discretization method to preserve the stability of the starting model increases.

Another analogous problem, always presented in [9], for Euler method is related to the *Hopf bifurcation*, that we presented in theorem (4).

Proposition 10. *Consider the fixed point p_0 for model (2.1) and suppose that a Hopf bifurcation occurs at p_0. Then p_0 will be always unstable for the discretized model (2.2).*

Proof. Consider the Hopf bifurcation for the equilibrium point p_0 for the parameter $r = r_0$ (r_0 is the bifurcation value), then the differential equation in the model (2.1) presents a pair of purely imaginary eigenvalues, i.e. $z(r) = \alpha(r) + i\beta(r)$ with $\alpha(r_0) = 0$ and $\beta(r_0) \neq 0$.
Now, if we consider the complex eigenvalues for the discretized model (2.2), we obtain:

$$w(r) = 1 + hz(r) = 1 + h\alpha(r) + ih\beta(r)$$

Then for the bifurcation value $r = r_0$, we have:

$$|w(r_0)| = \sqrt{(1 + h\alpha(r_0))^2 + h^2\beta^2(r_0)} = \sqrt{1 + h^2\beta^2(r_0)} > 1$$

So, because of what we saw in chapter (1.3.2) regarding the stability of discrete models, we have that p_0 is an unstable equilibrium for the discretized model. □

Summing up, Euler's main drawback is that, in case of a big step size h, likely it will not preserve the stability of the continuous time model and moreover, as seen above, in case of a Hopf bifurcation Euler method will always produce an unstable fixed point.

2.3 Kahan Method

Another discretization scheme, similar to the Euler's, is the Kahan reflexive method, named after its Canadian author William Kahan.
As said in [9], this method is applicable only in the case the function $f(\mathbf{x})$ is linear or at most quadratic in $\mathbf{x}$.
If, again, we consider a system of differential equation and we impose that f is at most a quadratic function, we have:

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) = A(\mathbf{x}, \mathbf{x}) + B\mathbf{x} + \mathbf{b} \tag{2.3}$$

where $\mathbf{x} = (x_1, x_2, ..., x_n)^T$, $A(\mathbf{x}, \mathbf{x})$ is the symmetric vector of quadratic terms, $B\mathbf{x}$ is a (nxn)-matrix and $\mathbf{b}$ is the vector of n constant terms.
Equation (2.3) can be rewritten in the following way:

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}) = \frac{1}{2}J_A(\mathbf{x})\mathbf{x} + B\mathbf{x} + \mathbf{b} \tag{2.4}$$

where $J_A(\mathbf{x})$ is the Jacobian matrix of $A(\mathbf{x}, \mathbf{x})$.

Kahan method generates the discrete time model:

$$\frac{\mathbf{X}-\mathbf{x}}{h} = \frac{1}{2}J_A(\mathbf{x})\mathbf{x} + B\mathbf{x} + \mathbf{b} = \frac{1}{2}J(f(\mathbf{x}))(\mathbf{X}-\mathbf{x}) + f(\mathbf{x}) \tag{2.5}$$

where h is, as usual, the step-size and $\mathbf{X} = \mathbf{x} + h$.

The difference equation (2.5) can be rewritten, making it explicit for the term $\mathbf{X}$:

$$\mathbf{X} = \mathbf{x} + h\left(I - \frac{h}{2}J(f(\mathbf{x}))\right)^{-1} f(\mathbf{x}) \tag{2.6}$$

From [9], we have two very useful lemmas that we report here:

Lemma 2. *Consider the fixed point p_0 for model (2.4). Suppose the step-size h is small enough that the quantity $\left(I - \frac{h}{2}J(f(\boldsymbol{x}))\right)$ is invertible. Now consider z an eigenvalue of the Jacobian matrix of $f(p_0)$ (i.e. $J(f(p_0))$). Then $\frac{2+hz}{2-hz}$ is an eigenvalue of $\left(I - \frac{h}{2}J(f(p_0))\right)^{-1}\left(I + \frac{h}{2}J(f(p_0))\right)$.*

The map $w = \frac{2+hz}{2-hz}$ maps circle onto circles and relates z and w (remember that z is an eigenvalue for $J(f(p_0))$).

Lemma 3. *Let the step-size h be positive, the map $w = \frac{2+hz}{2-hz}$ maps the left half plane $Re(z) < 0$ onto the unit disk $|w| < 1$ and the imaginary axis $Re(z) = 0$ onto the unit circle $|w| = 1$.*

Proof. $|w| < 1 \Leftrightarrow |2+hz|^2 < |2-hz|^2 \Leftrightarrow 4h(z+\bar{z}) < 0 \Leftrightarrow 8hRe(z) < 0 \Leftrightarrow Re(z) < 0$. □

Figure 2.2 represents graphically Lemma (3).

An important theorem that testifies the preserving of the stability of the Kahan method (including also in case of a Hopf bifurcation) is reported below:

Theorem 6 (**Kahan local stability**). *Let p_0 be an equilibrium point for the differential equation (2.4) and the step-size h is such that $\|J(f(p_0))\| < \frac{2}{h}$. Then p_0 is an unstable equilibrium for the continuous time model (2.4) if and only if p_0 is an unstable equilibrium for the discretized model (2.6). Moreover if an Hopf bifurcation occurs for p_0 in the starting model (2.4) then it will also occur for the discretized model (2.6).*

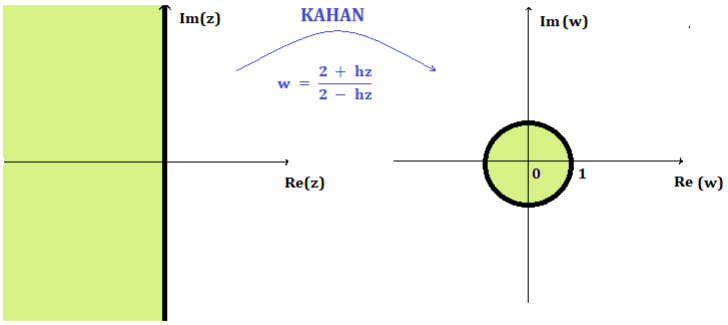


Figure 2.2: Kahan's mapping $w = \frac{2+hz}{2-hz}$ maps the left half plane $Re(z) < 0$ onto the unit circle $|w| < 1$.

It can observed that Kahan discretization method preserves better the stability of a continuous time model with respect to the Euler method, especially in the case of a *Hopf bifurcation*, where the Euler method always produce an unstable discrete time model. For this reason the Kahan method is *locally consistent*.
Nonetheless, it cannot be affirmed that Kahan method does not have any drawback: again, as for the Euler method, in order to have an acceptable discretization, h should be small and one of the hypothesis of theorem (6) impose that the norm of the Jacobian matrix must be smaller than $\frac{h}{2}$. Finally, it is crucial to remark that the Kahan method presented in [9] is only applicable in the case where the function $f(\mathbf{x})$ is linear or at most quadratic in $\mathbf{x}$; this fact implies that impossibility to apply Kahan to more complex dynamic models.

We now describe an example of a well known continuous time dynamic system and we discretize it through the Euler and the Kahan methods and finally we highlight main differences between them.

Example: Comparison Between Euler and Kahan Methods

We consider an already presented 1-dimensional continuous time model: the *logistic growth model*.
We saw in chapter (1.2.1) that the equation:

$$\frac{dx}{dt} = f(x) = x(1-x) \tag{2.7}$$

Note that, the two positive parameters r and k in equation (1.6) are both equal to 1.
We already know that the two fixed points of model (2.7) are 0 and 1. Since $f'(0) = 1 > 0$ then 0 is an unstable equilibrium; on the contrary, $f'(1) = -1 < 0$ implies that 1 in a stable fixed point.

Euler method produces the following discrete time model:

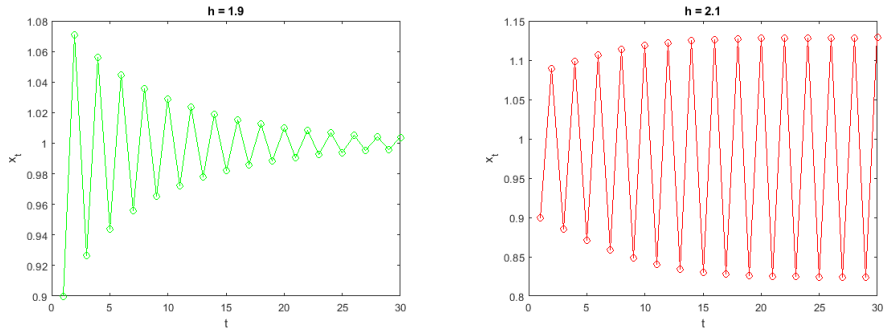
$$x(t+1) = x(t)(1 + h - hx(t)) \tag{2.8}$$

Kahan discretization method transforms the starting model into:

$$x(t+1) = x(t)\left(\frac{2+h}{2-h+2hx(t)}\right) \tag{2.9}$$

Regarding the Euler method, we observed in Lemma (1) that, in order to preserve local stability of equilibrium 1, $|z+1/h| < 1/h$ i.e. $|-1+1/h| < 1/h$ which implies that the step-size h must be grater than 0 and less than 2 (i.e. $0 < h < 2$). For $h \geq 2$ we should obtain a non-consistent discretization method.

In figures 2.3, Euler method is showed graphically: on the left the step-size h is 1.9, so it satisfies the hypothesis of Lemma (1), on the contrary, on the right side, the value of h exceeds the threshold imposed by the Lemma, indeed the dynamics of the discretized model shows that the equilibrium 1 is unstable.



(a) (b)

Figure 2.3: Euler discretization method applied to the logistic growth model (2.8) with $h = 1.9$ (left) and $h = 2.1$ (right), $x(0) = 0.9$.

Regarding Kahan's, we observed in theorem (6) that the condition for pre-

serving the local stability is $|f'(1)| < 2/h$ which implies $0 < h < 2$.
We show in figure 2.4, a numerical application of theorem (6), which shows that, if the hypothesis is fulfilled, then the result that Kahan method is consistent.

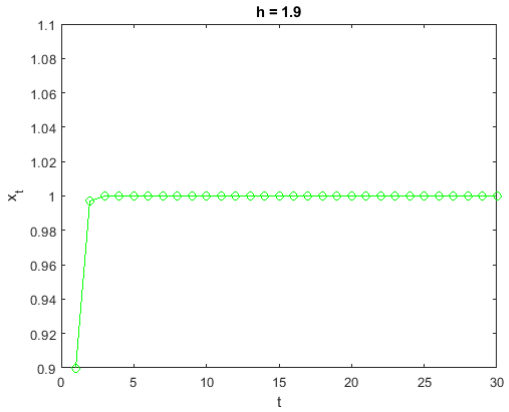


Figure 2.4: Kahan discretization method applied to the logistic growth model (2.9) with $h = 1.9$, $x(0) = 0.9$.

Finally, comparing fig. 2.3(a) and 2.4, it is evident that Kahan's produces a much faster method than the Euler's, for time-step h ceteris paribus.

2.4 Non-Standard Finite Difference (NSFD) Schemes

Ronald Mickens started to think about *Non-Standard Finite Difference (NEDS) Schemes* in 1989, during the years he refined the scheme in order to obtain a discretization method that does not have the drawbacks of Euler and Kahan methods. Of course, his aim was to reduce as much as possible unwanted behaviours in the discretized model like chaos and instability that were not present in the continuous time model.
A discretization method, in order to be called a Non-Standard Finite Difference method must be created following particular rules, described by Mickens in [7].

Remark 1. ***NSFD Method's Constructing Rules:***

1. *The orders of the discrete derivatives must be equal to the orders of the corresponding derivatives in the continuous time model.*
2. *The discrete representation for a derivative should not have a trivial denominator (i.e. let $\phi(h)$ be the denominator, then $\phi(h) = h+O(h^2)$). For instance, a trivial denominator is $\phi(h) = 1$.*

3. *Nonlinear terms should be substituted by nonlocal discrete representation, which means that a (continuous time) term of order greater than* 1 *must be substituted by a (discrete time) term of order greater than* 1 *or by the multiplication of two or more (discrete time) terms.*
This rule will be explained more in detail through a numerical example below.

4. *Particular conditions that hold for the differential equation in the continuous time model must hold also in the difference equation in the discretized model.*

It is worth to remark that the second rule can be written in the following way, consider the first order derivative of $x(t)$, then its discretization is:

$$\frac{dx}{dt} \to \frac{x_{k+1} - \psi(h)x_k}{\phi(h)}$$

where ψ and ϕ are functions of the step-size h. Moreover $x(t) \to x_k$, $t_k = hk$, $\psi(h) = 1 + O(h)$ and $\phi(h) = h + O(h^2)$.

Regarding the third rule, a typical application is related to the logistic equation:

$$\frac{dx}{dt} = x(1 - x) \tag{2.10}$$

We have previously seen, in chapter (1.2.1), that the two fixed points are 0 and 1. They are, respectively, unstable and stable equilibrium.
If we transform the logistic equation into the following discrete time version:

$$z_{k+1} = (1 + h)z_k(1 - z_k)$$

The above discrete time dynamical system may cause unwanted behaviours. Indeed, in chapter (1.3.1), we observed that this discrete time equation presents a behaviour different from the original continuous time logistic equation. This is due to the fact that flip bifurcations and even chaotic behaviour may arise in the discretized model.
Mickens proposed a nonlocal representation for terms with an order greater than 1.
In particular it lead to a possible non-local representation for x^2:

$$x^2 \to x_{k+1}x_k$$

Which leads to the discretized model:

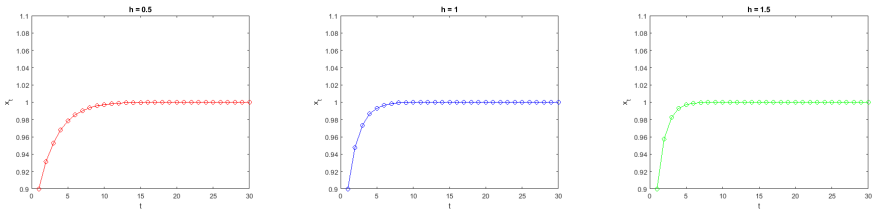
$$x_{k+1} = \left(\frac{1+h}{1+hx_k} \right) x_k \tag{2.11}$$

In figure 2.5, we are able to confirm the dynamical consistence (also included in the fourth rule) of NSFD method for the logistic growth model.
It can be observed how, contrary to the case of Kahan and Euler, increasing the time-step h leads to a model which reaches faster the equilibrium point 1.

From these rules Mickens provided the formal definition for NSFD schemes:

Definition 17 (**NSFD Scheme**). *A discretization method, whose construction is based on the rules in remark (1), is called a Non-Standard Finite Difference (NSFD) Scheme.*

However, in [8], the author presented a list of drawbacks of Non-Standard Finite Difference methods.
Firstly, a very difficult scenario to apply NSFD method is the case of fixed points with neutral stability (i.e. complex conjugate eigenvalues with null real part, for example $\lambda_{1,2} = \pm i$), we observed that this scenario could lead to unwanted dynamics also for Euler method.
The author also expressed that few work has done for a continuous time dynamical system with 2 dimensions (or more); moreover, increasing the number of variables leads to a more complicated structure for the discretized model. Until the year of publication of [8], Ronald Mickens did not developed a Non-Standard Finite Difference method which can be applied also for 2-dimensional dynamical systems.
For this reason we report here, from [8], a numerical example of a NSFD method application with just one variable.



(a) (b) (c)

Figure 2.5: NSFD discretization method applied to the logistic growth model (2.11) with $h = 0.5$ (left), $h = 1$ (center) and $h = 1.5$ (right), $x(0) = 0.9$.

Example: An application of NSFD Method

We consider a 1-dimensional continuous time dynamical system (quite similar to the known logistic growth):

$$\frac{du}{dt} = f(u) = u^2(1-u) \tag{2.12}$$

The above model is a basic model for combustion in physics.
Recalling chapter (1.2.1), there are 2 fixed points: $u_1^* = 0$ and $u_2^* = 1$. Moreover, equilibrium u_2^* is stable, since $f'(u_2^*) = f'(1) = -1 < 0$ while equilibrium u_1^* is non-hyperbolic since $f'(u_1^*) = f'(0) = 0$, so nothing can be said regarding its stability only looking at its first derivative (see definition (7)). However, using a numerical approach, it can be showed that equilibrium $u_1^* = 0$ is unstable.
Applying NSFD to the left-hand side of (2.12) the result is:

$$\frac{du}{dt} \to \frac{u_{k+1} - \psi(h)u_k}{\phi(h)}$$

In this case, the author suggested $\psi(h) = 1$ and $\phi(h) = 1 - e^{-h}$.

Regarding the right-hand side of model (2.12), we have to be careful since we have non-linear terms (i.e. u^2 and u^3), so they must be substituted by a nonlocal discrete representations (see Rule 3).
The author proposed the following:

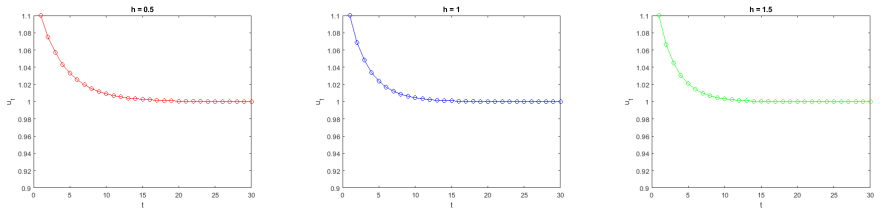
$$u^2 \to 2u_k^2 - u_{k+1}u_k^2; \qquad u^3 \to u_{k+1}u_k^2$$

So the resulting discretized model with NSFD method is:

$$u_{k+1} = \frac{u_k(1 + 2\phi(h)u_k)}{1 + \phi(h)(u_k + u_k^2)} \tag{2.13}$$

Where, as before, $\phi(h) = 1 - e^{-h}$.
It can be observed that the fixed points are the same of the continuous time model (2.12), and the discretization method preserves the stability, so in this case, NSFD method is *locally consistent*.

In figure 2.6, we show the stability of fixed point $u_2^* = 1$, for different values of the step-size h. Again, we observe that increasing h leads to a model which converges faster to the equilibrium.



(a) (b) (c)

Figure 2.6: NSFD discretization method applied to combustion model (2.13) with $h = 0.5$ (left), $h = 1$ (center) and $h = 1.5$ (right), $x(0) = 1.1$.

2.5 Nearly Exact Discretization Scheme (NEDS)

The *Nearly Exact Discretization Scheme*, also known with its acronym *NEDS*, is a discretization method presented by its four authors in [5], predominantly used in ecological dynamical systems.
This method can be seen as an evolution of NSFD methods, indeed, as for the method presented by Mickens, the objective of the authors is always the same: develop a discretization method which preserves the properties of the continuous time model.
As usually done before, we start by considering the continuous time model described by the ODE:

$$\frac{dx}{dt} = f(x(t)) \tag{2.14}$$

Before giving the definitions of NEDS and its discretization scheme, we have to show several useful definitions in order to construct it.

Definition 18 (**Dynamical consistency**). *A discretization scheme is dynamically consistent if these conditions hold:*

- ***A1:*** *The stability of the considered continuous time model (ODE) coincides with the stability of the discrete time model (DE).*
- ***A2:*** *The bifurcations of ODE coincide with the bifurcation of DE.*
- ***A3:*** *If two ODEs are equivalent after a re-parametrization then the resulting DEs (obtained by a discretization of the two ODEs) must be equivalent by the same re-parametrization.*

Definition 19 (**NEDS**). *A discretization scheme is called NEDS if it is dynamically consistent (through definition (18)) and the trajectories of the discretized model are the same or "nearly the same" of the starting continuous time model.*

After presenting the last definitions we now give the principles in order to perform the NEDS discretization scheme, before that, in oder to simplify these steps, we make some assumptions on the continuous time model (2.14).

Definition 20. *Consider the function f in the right-hand side of (2.14); f is T_1 if $f(x) = rx + g(x)x + b$, f is T_2 if $f(x) = rx - g(x)x + b$.*

Thanks to definition (20), we are supposing that f has a particular shape, where r is a non-zero linear constant and $g(x)$ a function which takes as input only real values.

Remark 2. ***NEDS Method's Constructing Rules:***

1. *The left-hand side of ODE (2.14) is discretized in the following way:*

$$\frac{dx}{dt} \to \frac{x_{t+1} - x_t}{\phi(h)}$$

where h is the step-size and $\phi(h) = h + O(h^2)$ as $h \to 0^+$.

2. *ϕ has the following form:*

$$\phi(h) = \frac{e^{rh} - 1}{r}$$

Moreover, the resulting DE of the right-hand side of (2.14) is:

$$\frac{x_{t+1} - x_t}{\phi(h)} = \begin{cases} rx_t + g(x_t)x_t + b, & \text{if } f \text{ is } T_1 \\ rx_t - g(x_t)x_t + b, & \text{if } f \text{ is } T_2 \end{cases}$$

Finally, the resulting discretized model is:

$$x_{t+1} = \begin{cases} e^{rh}x_t + \phi(h)g(x_t)x_t + \phi(h)b, & \text{if } f \text{ is } T_1 \\ \frac{e^{rh}x_t + \phi(h)b}{1 + \phi(h)g(x_t)}, & \text{if } f \text{ is } T_2 \end{cases} \tag{2.15}$$

3. *If the right-hand side of (2.14) is $f(x) = r \pm g(x)x$ then a change of variable is required to bring it back to the known case; in particular we define $u = x - x^*$ where x^* is the non-zero fixed point of model (2.14), for this reason, the right-hand side will be $f(x) = rx \pm g(x)x$.*

4. *If the structure of f cannot be re-parametrized as in step 3, we impose:*

$$f(x) = rx - g(x)x$$

where:

$$r = k(-x^*)^{k-1} f^{(k)}(x^*), \qquad with \quad k = \min(\alpha > 0 : f^{(\alpha)}(x^*) \neq 0)$$

An equivalent definition of (17) is the following:

Definition 21 (**NSFD Scheme**). *A discretization scheme is called Non-Standard Finite Difference (NSFD) Scheme if it satisfies conditions **A1** of definition (18) and the first step of NEDS Method's constructing rules of remark (2).*

It is immediate to observe from the previous definitions that NEDS and NSFD schemes are closely related and that NEDS is a sort of refinement of NSFD.
Indeed consider the case where f is T_1, we have already seen the NEDS formula in (2.15); regarding NSFD, the result is:

$$x_{t+1} = \frac{e^{rh}x_t}{1 - \phi(h)g(x_t)}$$

A drawback of NSFD is that x_{t+1} could be negative if $g(x_t) \leq \frac{1}{\phi(h)} = \frac{r}{e^{rh}-1}$, this could lead to problems if we are considering an ecological model, where negative values for the variables may not make sense. On the contrary NEDS, in this case, returns a non-negative value for x_{t+1}.

One of the virtues of NEDS discretization scheme, proclaimed by the authors in [5], is the fact that the method, unlike other previously seen discretization methods like Euler and Kahan, it allows large time steps. However, always in the same paper, the authors admitted that there could be models where the ODE and DE have the same dynamics only for particular values of the step-size h and also for this reason the research on discretization methods cannot be absolutely considered finished.

We now move to the comparison between the continuous time model and its NEDS-discretized version, the objective is to find fixed points and study their stability.
In order to obtain the same results as in [5] we slightly change the assumptions made in model (2.15), indeed here we do not consider the constant term b. The result is that the right-hand side of (2.14) will be:

$$f(x) = \begin{cases} rx + g(x)x, & \text{if } f \text{ is } T_1 \\ rx - g(x)x, & \text{if } f \text{ is } T_2 \end{cases}$$

Consider now the fixed points:

Lemma 4 (**Fixed Points**). *The fixed points of the continuous time model (2.14) and its discretized version are:*

$$x^* = 0, \qquad g(x^*) = \pm r \quad (+r, \quad \textit{if } f \textit{ is } T_1; -r, \quad \textit{if } f \textit{ is } T_2)$$

Consider now the stability of the ODE; the authors of NEDS stated this theorem:

Theorem 7 (**Asymptotic Stability of the Continuous Time Model**). *Consider the fixed point $x^* = 0$, x^* is asymptotically stable if and only if:*

$$\begin{cases} r + g(0) < 0, & \text{if } f \text{ is } T_1 \\ r - g(0) < 0, & \text{if } f \text{ is } T_2 \end{cases}$$

Consider now a non-zero fixed point x^*, it is asymptotically stable if and only if:

$$x^* g'(x^*) > 0$$

We report below the theorem regarding the stability of the DE:

Theorem 8 (**Asymptotic Stability of the Discretized Model**). *Consider the fixed point $x^* = 0$, x^* is asymptotically stable if and only if:*

$$r > \frac{e^{rh} - 1}{e^{rh} + 1} g(0) \quad \text{and} \quad \begin{cases} r + g(0) < 0, & \text{if } f \text{ is } T_1 \\ r - g(0) < 0, & \text{if } f \text{ is } T_2 \end{cases}$$

Consider now a non-zero fixed point x^*, it is asymptotically stable if and only if:

$$x^* g'(x^*) > 0 \quad \text{and} \quad \begin{cases} \phi(h) g'(x^*) x^* > -2, & \text{if } f \text{ is } T_1 \\ \phi(h) e^{-rh} g'(x^*) x^* < 2, & \text{if } f \text{ is } T_2 \end{cases}$$

Finally, as done for the previous discretization methods, we report an example, in order to show the potential of NEDS. We report here 2 examples, one regarding the case when f is T_1 and the other one related to f equal to T_2.

Example: An application of NEDS Method (T_1 case)

We introduce the model:

$$\frac{dx}{dt} = f(x) = rx + x^3 \tag{2.16}$$

where r is non-zero.
It is immediate to observe that the equilibria are: $x_1^* = 0$ and $x_{2,3}^* = \mp\sqrt{-r}$ where $x_{2,3}^*$ exist only if $r < 0$.
In particular since $f'(x_1^*) = f'(0) = r$ we obtain that the equilibrium point x_1^* is stable if $r < 0$ and unstable if $r > 0$. Regarding $x_{2,3}^*$, we have always

instability since $f'(x^*_{2,3}) = f'(\mp\sqrt{-r}) = -2r > 0$.

In order to redirect to a known case (i.e. f is T_1) for model (2.16), just consider $g(x) = x^2$ so that it is immediate to perform NEDS discretization in the case where $f(x) = rx + g(x)x$.
The result of NEDS, applied to model (2.16), is the discrete time model:

$$x_{t+1} = e^{rh}x_t + \frac{e^{rh} - 1}{r}x_t^3 \tag{2.17}$$

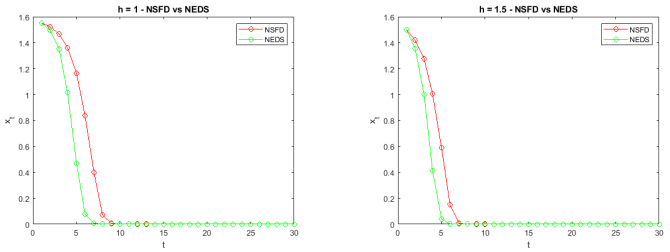
The obtained fixed point are in common to the continuous time model, indeed we have: $x^*_1 = 0$ and $x^*_{2,3} = \mp\sqrt{-r}$ where $x^*_{2,3}$ exist only in the case $r < 0$.
Moreover, by applying rules showed in theorem (8), we obtain the same local dynamics, of the continuous time model, for $x^*_1 = 0$ and $x^*_{2,3}$.
We proved that NEDS discretization, in this case, is dynamically consistent.

A remark is now required, we want to compare the NEDS discretization model (2.17) with a possible NSFD discretization; a problem could arise with discretization of the non-linear term x^3 in model (2.16), a possible way to solve it could be discretize x^3 as $2x_t^3 - x_{t+1}x_t^2$. The result model would be:

$$x_{t+1} = \frac{e^{rh}x_t + 2\left(\frac{e^{rh}-1}{r}\right)x_t^3}{1 + \frac{e^{rh}-1}{r}x_t^2} \tag{2.18}$$

It is immediate to observe that, unlike the NEDS discretization, the NSFD method gives birth to a model which could have some problems. Indeed, here in figure 2.7, we show a numerical example where, due to its velocity and robustness, NEDS is preferable with respect to NSFD.

We observe that, in both cases where we changed the value of step-size h,



(a) (b)

Figure 2.7: NEDS and NSFD discretization methods applied to model (2.16) with $h = 1$ (left), $h = 1.5$ (right), $x(0)$ 1.5 and $r = -2.5$.

NEDS is faster than NSFD.

Example: An application of NEDS Method (T_2 case)

Now we want to deal with an application of T_2-case for NEDS. The simplest example is the *logistic growth model*. We have already seen this example several times and its possible discretizations with Euler, Kahan and NSFD methods. It is now reasonable to study its discretization with NEDS.

Consider the continuous time logistic model (also seen in (1.6)):

$$\frac{dx}{dt} = f(x) = rx\left(1 - \frac{x}{k}\right) \tag{2.19}$$

where $r, k > 0$ are positive, real constants.
We have already seen in chapter (1.2.1) that the fixed points are $x_1^* = 0$ and $x_2^* = k$. In particular x_1^* is unstable, on the contrary x_2^* is locally stable.

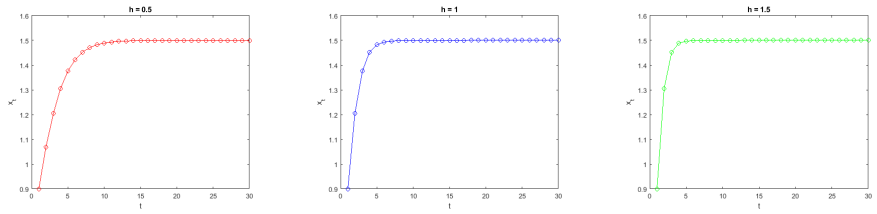
In oder to perform NEDS discretization in the case T_2, from definition (20), we have to consider $g(x) = r\frac{x}{k}$. The result is the discrete time model:

$$x_{t+1} = \frac{ke^{rh}x_t}{k + (e^{rh} - 1)x_t} \tag{2.20}$$

where we considered, as usually done in NEDS discretizations, $\phi(h) = \frac{e^{rh}-1}{r}$.

The two fixed points are the same of continuous time model (2.19), i.e. $x_1^* = 0$ and $x_2^* = k$.
We now want to study the local stability of the discretized model, in order to do that, we apply theorem (8).
Consider $x_1^* = 0$, it immediate that the necessary and sufficient conditions for asymptotic stability are $r > 0$ and $r < 0$, the last condition cannot be satisfied since by initial hypothesis the logistic growth model has two strictly positive parameters: r and k.
Regarding the non-zero equilibrium $x_2^* = k$, we verify that the necessary and sufficient conditions in theorem (8) are satisfied.
The first one is $x_2^* g'(x_2^*) > 0$ that becomes $r > 0$, so it is satisfied. The second condition is $\phi(h)e^{-rh}g'(x^*)x^* < 2$ that, after some computations becomes, $1 - e^{-rh} < 1 < 2$.
From these two results we have that the stability of the continuous time model is preserved through NEDS discretization method.
This result confirms the goodness of NEDS for ecology models, as said by its authors in [5].

We now show some graphs regarding a numerical simulation of model (2.20) and a comparison of the logistic growth model with NEDS and NSFD discretization, respectively in figures 2.8 and 2.9.



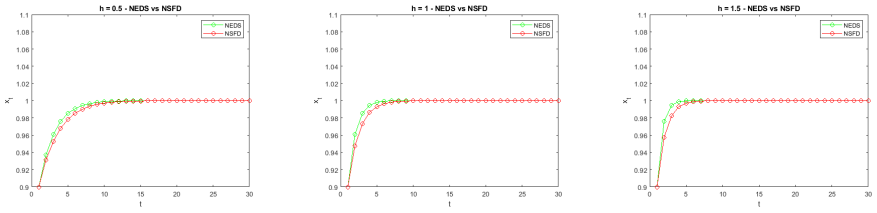
(a) (b) (c)

Figure 2.8: NEDS discretization method applied to logistic growth model (2.20) with $h = 0.5$ (left), $h = 1$ (center) and $h = 1.5$ (right), $x(0) = 0.9$, $k = 1.5$ and $r = 1$.

In fig. 2.8, it can be observed that, increasing the value of the step-size h, implies a greater velocity of reaching the stable fixed point $x_2^* = k = 1.5$.

Again, in fig. 2.9, it can be observed that NEDS is faster than NSFD, so it preferable to discretize the logistic model with NEDS in order to have a *fast* and a *dynamically consistent* discrete time model.

In next chapter we want to make a step forward, indeed we will introduce a model with *2 independent variables* used in economy and the objective will be to verify if it is possible to apply NEDS and if addition hypothesis are needed in order to obtain a dynamically consistent model.



(a) (b) (c)

Figure 2.9: Comparison between *NEDS* (equation (2.20), in green) and *NSFD* (equation (2.11), in red) discretization methods applied to the logistic model, with $h = 0.5$ (left), $h = 1$ (center) and $h = 1.5$ (right), $x(0) = 0.9$, $k = 1$ and $r = 1$.

Chapter 3

Application

The aim of this chapter is to study a continuous time model and its discretized version through the NEDS discretization method presented in section (2.5).
As said in the introduction, the overwhelming majority of economical models are of continuous time type. This is due to the fact that they are well studied in economic literature and they are, in general, simpler to use than discrete time ones, since a 1 or 2-dimensional discrete time model can generate chaotic trajectories while it is impossible for a continuous one. Moreover, many economical models exploit stochastic calculus and this was a great advantage for the spread of continuous time model in economics in the past.

The problem arises when data are collected; indeed data are gathered in discrete time (even if, for example oil price data are available at high frequency, data cannot be considered in continuous time). Discretization methods could be difficult to use in practice, indeed in chapter (2) we saw that some discretization schemes does not preserve all stability features (fixed points and their stability) that were present in the continuous time model. A simplistic solution to this problem could be using a continuous time model with discrete time collected data, in other words: use the classical models for data collected with hourly, daily, monthly or yearly frequency. However, this procedure can be very dangerous, indeed not expected behaviours can arise using a continuous time model with discrete data.
Another less naïf solution can be using a relatively simple discretization method in order to convert a continuous time model into a discrete one. Nonetheless, sometimes the characteristics of continuous time model are not preserved in the discretization, as showed, for instance, in Euler (chapter (2.2)) and Kahan (chapter (2.3)) methods.
The goal of this section is to apply NEDS discretization scheme and to study its goodness and its negative parts, following the idea used in [4].

The first step is to introduce the continuous time model and show its stability

and bifurcations.

3.1 The Continuous Model

3.1.1 Model presentation

The chosen model is the one discussed in [6]. The author of the article proposed this model for the dynamics of oil price; actually since it is a very generic model for energy it could be used in order to describe trajectories of prices of many other commodities. The author himself showed how this very generic model could be used to describe variation of price of cereals or grains produced in a specific country.
Another possible application of the model is the one proposed by [12]. The Chinese authors presented this model in order to characterize the energy price in Jiangsu province, known for high energy consumption and very little production; so import is crucial in order to fulfill the energy requirement of the province.
In this thesis the model will generically describe the energy or another commodity price as showed in [12], given that the aim of this work is mainly to apply the discretization method, study fixed points, their stability and repeat the procedure for the discretized model version.

The proposed model is the result of assumptions made in [12] and [6], that we report here in order to offer a complete model's comprehension.

First of all, the rising of price dP/dt is proportional to the difference between the quantity of energy stocked Q and the determined quantity stocked $\bar{Q}$. dP/dt is also proportional to the quantity of energy imported I, always assuming that $Q, \bar{Q}$ and I are non-negative.
The obtained equation is:

$$\frac{dP}{dt} = u(\bar{Q} - Q) + rI \tag{3.1}$$

where u and r are constant that determines adjustment velocity of energy price. These two parameters are chosen positive ($u > 0$, $r > 0$).
In agreement with [12] and [6], the quantity stocked Q at time t is the sum of the initial quantity Q_0 and the difference between energy supply and demand at each time $s \in [0, t]$; the used notation is respectively S and D where $S, D \geq 0$. This is expressed in the following equation:

$$Q(t) = Q_0 + \int_0^t [S(s) - D(s)]ds \tag{3.2}$$

Energy demand is chosen as positively correlated to energy price if the latter does not overcome the threshold that purchasing power can tolerate. If

the threshold is surpassed, an increase of energy price might decrease the purchasing power. For this reason the equation that controls energy demand is:

$$D = D_0 - \beta P - \Psi(P)\frac{dP}{dt} \tag{3.3}$$

where D_0 is, intuitively, the energy demand at time 0 and it is chosen positive (i.e. $D_0 > 0$), $\beta > 0$ is the so-called marginal energy demand quantity. $\Psi(P) \geq 0$ is the function of the price and it reflects the dependence of energy demand on the rise of energy price. In [12] the authors decided to assume $\Psi(P)$ as linear function ($\Psi(P(t)) = b(P(t)) + c$ with b, c real constants). Here, in order to have the model as general as possible, $\Psi(P)$ will be a generic non-negative function of price P.
Energy supply is assumed to be linear with respect to the price, indeed a power plant is keen to sell more energy as the price increases.

$$S = S_0 + \alpha P \tag{3.4}$$

$S_0 \geq 0$ is the energy supply at time $t = 0$ and $\alpha > 0$ is the marginal energy supply quantity.
Lastly, the import I is assumed as a linear function of the price:

$$I = n - mP \tag{3.5}$$

where n and m are positive constants (i.e. $n > 0, m > 0$).

Combining (3.1)-(3.2)-(3.3)-(3.4)-(3.5), the result is equation (3.6) with the notation $x = P$ and $y = \frac{dP}{dt} = \frac{dx}{dt}$.

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -u(\alpha + \beta)x - [u(\Psi(x)) + rm]y + u(D_0 - S_0) \end{cases} \tag{3.6}$$

3.1.2 Fixed Points Computation and Stability of the Model

Once presented the continuous time model, the next step is to find the equilibrium points (or fixed points). As showed in chapter (1.2.2), in order to find the fixed points of the model we have to impose the two equations, that defines the model, equal to 0; that means:

$$\begin{cases} \frac{dx}{dt} = y = 0 \\ \frac{dy}{dt} = -u(\alpha + \beta)x - [u(\Psi(x)) + rm]y + u(D_0 - S_0) = 0 \end{cases}$$

The result is the only fixed point with coordinates:

$$(x^*, y^*) = \left(\frac{D_0 - S_0}{\alpha + \beta}, 0\right) \tag{3.7}$$

In [6], the author affirmed that in order to make a meaningful model it is mandatory to impose $D_0 - S_0 > 0$, since it is not reasonable for oil price be negative.
However, unlike the cited article, in this thesis the model presented is not referred just to oil price but to a larger set of commodities like energy, cereals, etc. As said before, the aim is to use a generic and simplified model also for energy, so the condition $D_0 - S_0 > 0$ adopted in [6], will not be imposed here. Indeed, as claimed in [1], the price of electricity can be even negative. This is due to the fact that electricity cannot be stored (more precisely, only an almost negligible quantity of electricity can be stored) and sometimes it is more convenient for coal-fired plants to run even if fuel cost is higher than electricity price because of huge starting and shutting costs for power plants.

After a small digression on electricity, we now focus on the stability of the only fixed point (3.7). Coherently with what was done in [6], we assume that Ψ is differentiable in x_0.
The Jacobian matrix calculated in the fixed point for the system (3.6) is:

$$J(x_0, y_0) = \begin{bmatrix} 0 & 1 \\ -u(\alpha + \beta) & -u\Psi(x_0) - rm \end{bmatrix} \tag{3.8}$$

In [6], the author affirmed that, when $\Psi(x_0) = 0$ it could happen, for specific values of parameters, that the eigenvectors associated to matrix (3.8) are real and positive.
This statement is untrue, indeed, once computed trace and determinant of $J(x_0, y_0)$ it is immediate to observe that the trace is always negative and the determinant is always positive, for any value of the parameters involved in the model.

$$\begin{cases} \mathbf{Tr}(J(x_0, y_0)) = -u\Psi(x_0) - rm < 0 \\ \mathbf{Det}(J(x_0, y_0)) = u(\alpha + \beta) > 0 \end{cases}$$

This observation leads to a local stable scenario for the continuous time system. The eigenvectors associated to (3.8) could be real or conjugate complex but in any case, the real part is always negative since by hypothesis of the model: $r > 0, m > 0, u > 0$ and $\Psi(x_0) \geq 0$.

The explicit formula for the eigenvectors is:

$$\lambda_{1,2} = \frac{-u\Psi(x_0) - rm \pm \sqrt{(u\Psi(x_0) + rm)^2 - 4u(\alpha + \beta)}}{2}$$

Then we can distinguish 3 cases for them:

1. if $(u\Psi(x_0)+rm)^2 < 4u(\alpha+\beta)$: 2 conjugate complex eigenvector with negative real part (i.e. $Re(\lambda_{1,2}) < 0$);
2. if $(u\Psi(x_0)+rm)^2 = 4u(\alpha+\beta)$: 2 real negative coincident eigenvalues;
3. if $(u\Psi(x_0)+rm)^2 > 4u(\alpha+\beta)$: 2 real negative distinct eigenvalues.

According to the definition of bifurcation (8), the local stability of the unique fixed point does not change with respect to the value of parameters so in the continuous time model there are no bifurcations. Indeed, a stable node, a stable focus and a stable star are qualitatively equivalent since their local phases can be identical by applying a continuous local deformation.

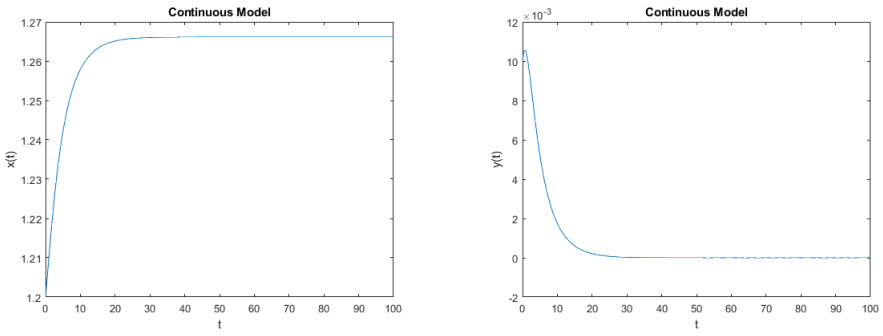
3.1.3 Numerical Simulations

In order to perform a numerical simulation, it is preferable to give a value to the parameters of model (3.6) such that they have a practical meaning. Unfortunately neither in [6] nor in [12], there is a deep discussion regarding the practical meaning of the parameters used in the model. In order not to perform numerical simulation without practical significance and also in order to do a thesis work as complete as possible, we perform a calibration of the parameters.
In particular, we used a dataset downloaded from Eurostat [1] related to the selling price of soft wheat in each country of the European Union for the last 11 years. From this data, using the Matlab function `fmincon`, which general presentation is given in the Appendix (A.1), we found the calibrated values for the parameters u, α, β, r and m, after that, we computed the continuous time trajectory for the system using the Matlab function `ode45`, also presented in (A.2).

The first calibration (and then numerical simulation) is performed for Italy, which has the highest soft wheat price on average from 2008 to 2019.
The chosen function for $\Psi(x)$ is a cubic one, i.e. $\Psi(x) = x^3$.
In fig. 3.1 it is represented the price and its speed, in fig. 3.2 the trajectory.

It is possible to observe that the two eigenvalues are distinct, negative and with null imaginary part (i.e. real eigenvalues), since $(u\Psi(x_0)+rm)^2 >$

[1] Dataset available at: `https://ec.europa.eu/eurostat/databrowser/bookmark/1e1cea95-23ab-49bd-af65-45da1dea8631?lang=en`



(a) *Price:* $x(t)$ (b) *Speed of price:* $y(t)$

Figure 3.1: Italy, soft wheat price and speed of price for $D_0 = 2, S_0 = 1, u = 0.23, \alpha = 0.39, \beta = 0.39, r = 1.10, m = 0.56, \Psi = x^3$. Initial data: $x(0) = 1.2, y(0) = 0.01$.

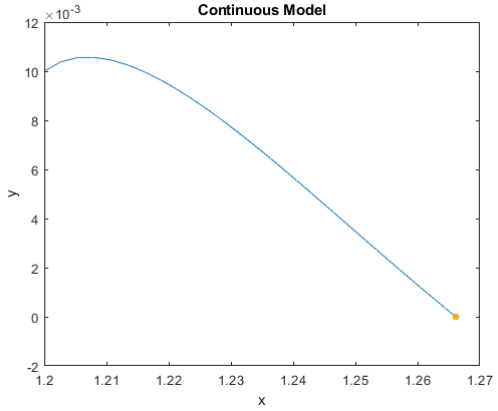
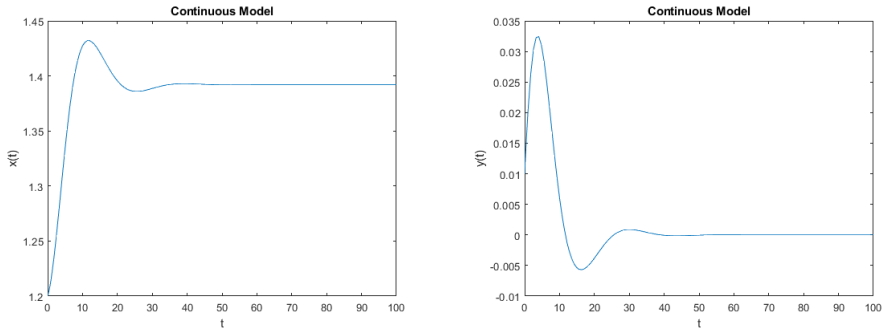


Figure 3.2: Italy, trajectory for $D_0 = 2, S_0 = 1, u = 0.23, \alpha = 0.39, \beta = 0.39, r = 1.10, m = 0.56, \Psi = x^3$. Initial data: $x(0) = 1.2, y(0) = 0.01$.

$4u(\alpha + \beta)$.
Observing the trajectory in fig. 3.2 it can be noticed that the system converges to the fixed point (3.7).

The second calibration and numerical simulation is done for Austria, which, unlike Italy, has on average the lowest soft wheat price among the European Union countries. As done for Italy, three figures are presented below in fig. 3.3 and 3.4:



(a) *Price:* $x(t)$ (b) *Speed of price:* $y(t)$

Figure 3.3: Austria, soft wheat price and speed of price for $D_0 = 2, S_0 = 1, u = 0.10, \alpha = 0.36, \beta = 0.36, r = 0.10, m = 10^{-3}, \Psi = x^3$. Initial data: $x(0) = 1.2, y(0) = 0.01$.

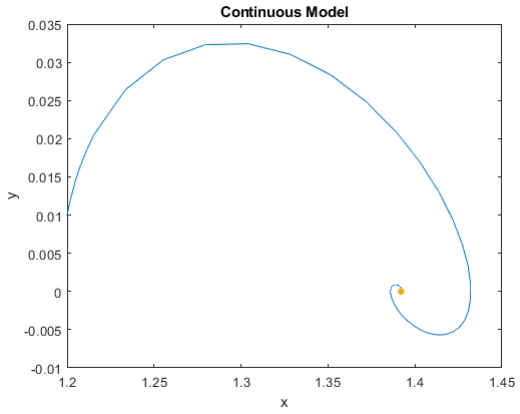


Figure 3.4: Austria, trajectory for $D_0 = 2, S_0 = 1, u = 0.10, \alpha = 0.36, \beta = 0.36, r = 0.10, m = 10^{-3}, \Psi = x^3$. Initial data: $x(0) = 1.2, y(0) = 0.01$.

Again, it can be observed the nature of the eigenvalues related to the continuous time system. This time, the values of the parameters obtained

through calibration for Austria, lead to a oscillatory dynamics as it can be seen, for example, in fig. 3.4. This guess is confirmed by the fact that $(u\Psi(x_0) + rm)^2 < 4u(\alpha + \beta)$ and, from the previous chapter, it is immediate to affirm that the continuous time 2-dimensional model has 2 conjugate complex eigenvalues with negative real part: for this reason the trajectory oscillates towards the unique fixed point (3.7), which is a stable focus.

3.2 The Discrete Model

3.2.1 NEDS discretization

The next step is to apply NEDS discretization scheme to the continuous time model. The result will be a discretized model. As previously mentioned, data are not published continuously so using a continuous time model could lead to unexpected dynamical scenarios that could be a problem for the reliability of a economical model as the one presented here.

We apply NEDS scheme (2.15) to the continuous time model (3.6), that we rewrite it here for convenience:

$$\begin{cases} \frac{dx}{dt} = y =: F_1(x,y) \\ \frac{dy}{dt} = -u(\alpha+\beta)x - [u(\Psi(x)) + rm]y + u(D_0 - S_0) =: F_2(x,y) \end{cases} \tag{3.6}$$

For the same convenience reason here the NEDS discretization scheme (2.15) is re-proposed:

$$x_{t+1} = \begin{cases} e^{rh}x_t + \phi(h)g(x_t)x_t + \phi(h)b, & \text{if } f \text{ is } T_1 \\ \frac{e^{rh}x_t + \phi(h)b}{1+\phi(h)g(x_t)}, & \text{if } f \text{ is } T_2 \end{cases} \tag{2.15}$$

Since the aim is to have a T_1-form equation as in (2.15), equation $F_1(x,y)$ is rewritten as $rx + \left(\frac{y}{x} - r\right)x$.
So in particular the function $g(x)$ is in this case equal to $\left(\frac{y}{x} - r\right)$. The first discretized equation obtained is: $x_{t+1} = x_t + hy_t$.

Regarding $F_2(x,y)$, it could be useful to rewrite the second equation as:

$$\frac{dy}{dt} = -rmy + u\left(-\Psi(x) - \frac{(\alpha+\beta)x}{y} + \frac{(D_0 - S_0)}{y}\right)y$$

It is immediate to observe that it is possible again to apply NEDS discretization scheme since $F_2(x,y)$ can be identified with a T_1-type equation (2.15); where: $r^* = -rm$ and $g(y) = u\left(-\Psi(x) - \frac{(\alpha+\beta)x}{y} + \frac{(D_0-S_0)}{y}\right)$.

The result, after some computations, is:

$$y_{t+1} = e^{-rmh}y_t + \left(\frac{1-e^{-rmh}}{rm}\right)u\left(-\Psi(x_t)y_t + (D_0 - S_0) - (\alpha+\beta)x_t\right)$$

The discretized model obtained is reported in (3.9):

$$\begin{cases} x_{t+1} = x_t + hy_t \\ y_{t+1} = e^{-rmh} y_t + \left(\frac{1-e^{-rmh}}{rm}\right) u\left(-\Psi(x_t)y_t + (D_0 - S_0) - (\alpha+\beta)x_t\right) \end{cases} \tag{3.9}$$

3.2.2 Fixed Points Computation and Stability of the Model

In chapter (1.3.2), we saw that for a discrete system, in order to find its fixed points, the equilibrium condition is $x(t+1) = x(t)$ and $y(t+1) = y(t)$; so the first step is to impose the first equation of (3.9) equal to x and the second one equal to y.
The result, after some simplifications, is the same unique fixed point that was found in continuous time:

$$(x^*, y^*) = \left(\frac{D_0 - S_0}{\alpha+\beta}, 0\right) \tag{3.7}$$

So, the first important result is that the discretized model (3.9) has the identical fixed point of the continuous model (3.6).

The next step is to study the stability of the discretized model, we also saw in chapter (1.3.2) that it is fundamental to compute the Jacobian matrix of system (3.9), which it is presented here:

$$J = \begin{bmatrix} 1 & h \\ j_{21} & j_{22} \end{bmatrix} \tag{3.10}$$

where: $j_{21} = \left(\frac{1-e^{-rmh}}{rm}\right) u(-\Psi'(x_t)y_t - (\alpha+\beta))$
and $j_{22} = e^{-rmh} - \left(\frac{1-e^{-rmh}}{rm}\right) u\Psi(x_t)$.

From the previous matrix, substituting the coordinates of the unique fixed point (3.7), we get:

$$J(x_0, y_0) = \begin{bmatrix} 1 & h \\ j_{21}(x_0, y_0) & j_{22}(x_0, y_0) \end{bmatrix} \tag{3.11}$$

where: $j_{21} = -\left(\frac{1-e^{-rmh}}{rm}\right) u(\alpha+\beta)$
and $j_{22} = e^{-rmh} - \left(\frac{1-e^{-rmh}}{rm}\right) u\Psi(x_0)$.

From this result, hereafter the values of trace and determinant of the matrix just calculated are reported below:

$$\begin{cases} \mathbf{Tr}(J(x_0, y_0)) = 1 + e^{-rmh} - \left(\frac{1-e^{-rmh}}{rm}\right) u\Psi(x_0) \\ \mathbf{Det}(J(x_0, y_0)) = e^{-rmh} + \left(\frac{1-e^{-rmh}}{rm}\right) u(-\Psi(x_0) + h(\alpha + \beta)) \end{cases} \tag{3.12}$$

We now focus on the sufficient condition for stability in a discrete system. From chapter (1.3.2) we saw the conditions that defines the so called stability triangle (1.22); these three conditions give local asymptotic stability of a fixed point.
For convenience, we report here these conditions:

$$\begin{cases} P(1) = 1 - Tr(J(x_0, y_0)) + Det(J(x_0, y_0)) > 0 \\ P(-1) = 1 + Tr(J(x_0, y_0)) + Det(J(x_0, y_0)) > 0 \\ Det(J(x_0, y_0)) < 1 \end{cases} \tag{3.13}$$

In order to have a clearer view of the results we divide the analysis into two cases:

Case 1: $\Psi(x_0) = 0$

Substituting the values of (3.12) in the first condition of (3.13) the result is:

$$\left(\frac{1 - e^{-rmh}}{rm}\right) uh(\alpha + \beta) \quad > 0$$

and since the above term is positive for any value of the parameters (taking into account that by hypothesis of the model $r, m, u, \alpha, \beta > 0$), the first condition of the stability triangle is satisfied.

Repeating the same procedure for the second condition, the result is:

$$2 + 2e^{-rmh} + \left(\frac{1 - e^{-rmh}}{rm}\right) uh(\alpha + \beta) \quad > 0$$

and again the quantity is positive for any value of the parameters.
This result can also be obtained geometrically: we can express the previous inequality in terms of the step size h, it becomes:

$$h > \left(\frac{-2rm}{u(\alpha + \beta)}\right) \left(\frac{1 + e^{-rmh}}{1 - e^{-rmh}}\right) \tag{3.14}$$

The limit, as $h \to +\infty$, of the right-hand side of inequality (3.14) is $\frac{-2rm}{u(\alpha+\beta)}$ which is always a negative quantity, this allows us to affirm that the second condition of the stability triangle is always verified.

We now focus on the last inequality of (3.13), the computation state that this condition is satisfied when:

$$(1 - e^{-rmh}) \left(-1 + \frac{uh(\alpha + \beta)}{rm} \right) < 0$$

which leads to the final result:

$$h < \frac{rm}{u(\alpha + \beta)} \tag{3.15}$$

So the unique condition, in the case $\Psi(x_0) = 0$, in order to have a discretized model that is locally asymptotically stable, is (3.15).

Case 2: $\Psi(x_0) > 0$

We now deal with a more generic assumption than Case 1 (i.e. we consider $\Psi(x_0) > 0$). Substituting the values of (3.12) in the first condition of (3.13) the result is again:

$$\left(\frac{1 - e^{-rmh}}{rm} \right) uh(\alpha + \beta) \quad > 0$$

and we previously checked that the quantity is strictly positive.

While, the second condition is satisfied in the case where this inequality is true:

$$u(-2\Psi(x_0) + h(\alpha + \beta) > -\frac{(2 + 2e^{-rmh})}{\left(\frac{1 - e^{-rmh}}{rm} \right)}$$

which can be rewritten in terms of the step size h:

$$h > \left(\frac{-2rm(e^{-rmh} + 1)}{1 - e^{-rmh}} + 2u\Psi(x_0) \right) \frac{1}{u(\alpha + \beta)} \tag{3.16}$$

However, differently from case $\Psi(x_0) = 0$, inequality (3.16) is not always satisfied, for particular values of the parameters the inequality does not hold;for example, it can be observed that for $u = 5, \alpha = \beta = r = 0.5, m = D_0 = 1, S_0 = 2$ and initial conditions $x(0) = 1.2, y(0) = 0.01$, the previous inequality is not satisfied for a step-size like $h = 0.9$ while it is verified for a smaller value of h (which leads to a more precise discretized model).
This geometrically description is also represented graphically in fig. 3.5, where it can observed that for particular values of h ($0.6 \lesssim h \lesssim 1.2$) the condition is not satisfied.

Finally, the third condition of the stability triangle is satisfied when:

$$(1 - e^{-rmh}) \left(\frac{u}{rm}(-\Psi(x_0) + h(\alpha + \beta)) - 1 \right)$$

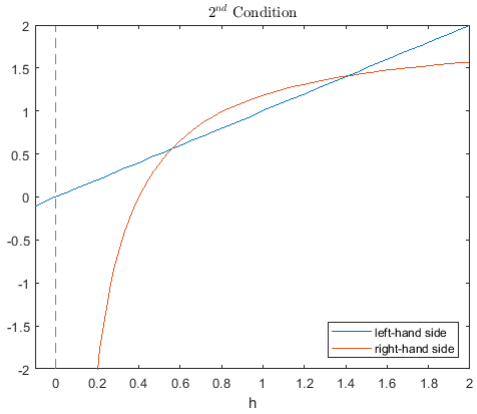


Figure 3.5: Condition (3.16), with respect to the step size h, with $u = 5, \alpha = \beta = r = 0.5, m = D_0 = 1, S_0 = 2, x(0) = 1.2, y(0) = 0.01$.

which result, in terms of h, is:

$$h < \left(\frac{rm}{u} + \Psi(x_0)\right) \frac{1}{(\alpha + \beta)} \tag{3.17}$$

From these considerations for the most generic case $\Psi(x_0) > 0$, it is possible to summarize the obtained results in the next proposition:

Proposition 11. *Continuous time model (3.6) and its NEDS discretized version (3.9) can be compared, in particular:*

(i) The discretized time model has the same unique fixed point as the original continuous time model.

(ii) Necessary conditions for the fixed point $(x^*, y^*) = \left(\frac{D_0 - S_0}{\alpha+\beta}, 0\right)$ *to be locally asymptotically stable, both for continuous and discrete time model, are defined by inequalities (3.16) - (3.17), i.e.:*

$$\begin{cases} h > \left(\frac{-2rm(e^{-rmh}+1)}{1-e^{-rmh}} + 2u\Psi(x_0)\right) \frac{1}{u(\alpha+\beta)} \\ h < \left(\frac{rm}{u} + \Psi(x_0)\right) \frac{1}{(\alpha+\beta)} \end{cases}$$

Proof. Results were computed in this section (3.2.2). □

It is important to observe that, besides an extra assumption on the step-size h described in proposition (11), NEDS discretization model works also for the economical model taken into exam in this section. This result is

less obvious than it might seem, since NEDS model was presented by its authors in [5] as a discretization scheme useful in ecological systems (like logistic and exponential growth model), while here, we showed, that adding few hypothesis on h, it works also for an economical system.

On the other hand, proposition (11) gives a counterexample of what written in the article that presented NEDS method [5]; indeed, the authors affirmed that in the most of the cases, NEDS method preserves the stability of the continuous time model; here, in order to retrieve the property, we had to add an extra assumption on the step-size h.

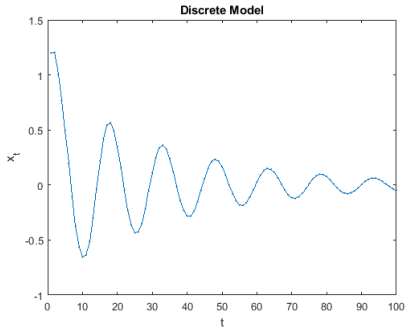
Finally, resuming the theoretical obtained result, we can comment it in two ways: the first one (the optimistic one) affirms than a simple additional hypothesis gives power to NEDS to be applicable not only to ecological model (like the logistic growth model) but also to simple economical models with two variables (like the considered model (3.6)). The second way (the pessimistic one) of commenting proposition (11) is to sadly admit that NEDS discretization method is not infallible, indeed when we consider a dynamical model with more than 1 variable or a model related to economy, with different characteristics of 1-dimensional ecology models, some consistency problems could arise and some additional hypothesis, like the one we introduced, must be added in order to obtain a discrete time model with same local stability of the starting model.

3.2.3 Bifurcations

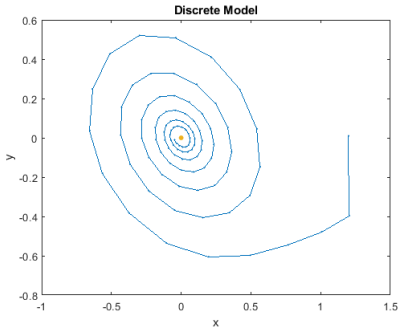
According to the results obtained in the previous sections, there are different stability scenarios with respect to the value of the parameters.
For convenience, again, we divide the analysis into the two previous cases:

Case 1: $\Psi(x_0) = 0$

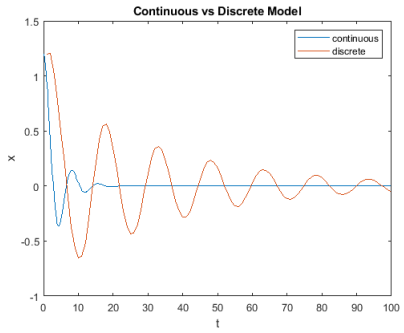
In particular we observed that the third condition of the stability triangle (3.13) may not be satisfied. More precisely, from (3.15) if $h < \frac{rm}{u(\alpha+\beta)}$ the discretized model follows a stable condition, when h reaches the bifurcation threshold the two complex conjugate eigenvalues exit the stability region (i.e. the unit circle) and an invariant curve is created.
This behaviour is typical of the *Neimark-Sacker bifurcation*, where the fixed point looses stability and a stable limit cycle is born around it. This particular case is the *subcritical Neimark-Sacker*.
We want to show two simulation of the discretized model; in fig. 3.6 it is considered the case where condition (3.15) is satisfied, in fig. 3.7, it is showed the Neimark-Sacker bifurcation that creates a stable limit cycle around the fixed point.



(a) *Price*

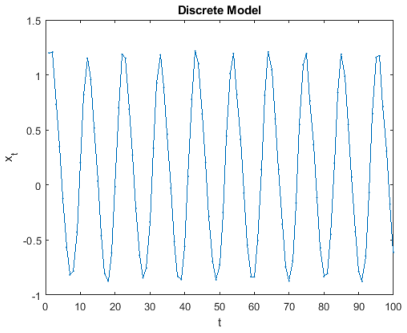


(b) *Simulated trajectory*

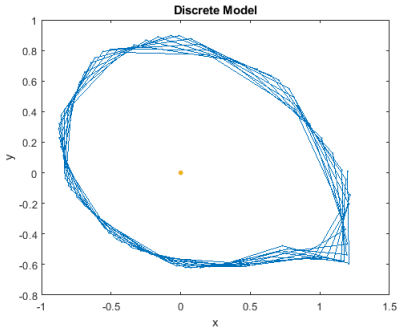


(c) *Continuous model vs discretized*

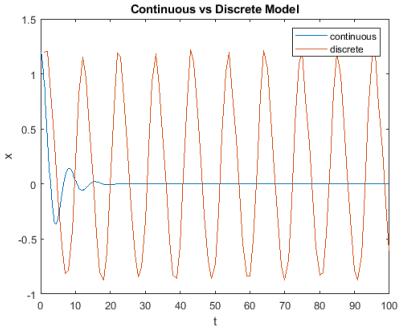
Figure 3.6: Case $\Psi(x_0) = 0$. Simulation of price x, trajectory for model (3.9) and comparison with the continuous one (3.6)with $r = \alpha = \beta = 0.5, m = S_0 = D_0 = 1, u = 0.75, x(0) = 1.2, y(0) = 0.01$ and $h = 0.5$. Condition (3.15) is satisfied.



(a) *Price*



(b) *Simulated trajectory*



(c) *Continuous model vs discretized*

Figure 3.7: Case $\Psi(x_0) = 0$. Simulation of price x, trajectory for model (3.9) and comparison with the continuous one (3.6)with $r = \alpha = \beta = 0.5, m = S_0 = D_0 = 1, u = 0.75, x(0) = 1.2, y(0) = 0.01$ and $h = 0.8$. Condition (3.15) is not satisfied.

Case 2: $\Psi(x_0) > 0$

This case is obviously more delicate than case 1, since also the function $\Psi(x_0)$ is considered in the stability conditions.
In chapter (1.3.2) it was pointed out that, at most, 3 types of bifurcation may show up in a 2-dimensional discrete time dynamic system: we previously observed that if two complex conjugate eigenvalues exit the unit circle a *Neimark-Sacker* bifurcation occurs, moreover if an eigenvalue λ is such that $\lambda = +1$, we have a *transcritical, pitchfork* or *saddle-node* bifurcation. Finally if $\lambda = -1$, a *flip* may occur; in the latter case, the oscillatory behaviour of the trajectory creates a periodic cycle and, increasing the bifurcation parameter, it might lead to a doubling cascade that after some period flips provokes an aperiodic trajectory, also called chaotic.

In the considered model, from condition (3.17), a *Neimark-Sacker* bifurcation is present for particular values of the step size h. This is analogous to the scenario of case $\Psi(x_0) = 0$, where two complex eigenvalues exit the stability circle.
Also for the case $\Psi(x_0) > 0$ two indicative simulations are performed in order to show the arise of the Neimark-Sacker bifurcation as h increases. They are reported in figures 3.8 and 3.9:

3.2.4 Numerical Simulations

As done for the continuous time model, some numerical simulations are performed also for the discretized model. Moreover, in order to make a significant comparison, the same commodity (price of soft wheat) is used for the simulations. The countries involved (Italy and Austria) and of course their own calibrated parameters are both used in the simulations.

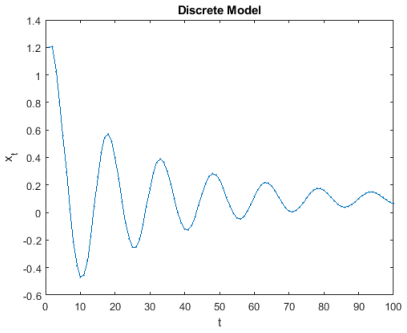
The graphics for Italy are presented in figures 3.10 and 3.11:

Regarding Italy, it can be observed how the discretization model works well in this case, indeed as the time horizon is increased, the discretized model well approximates the originally continuous time model, as showed in the right panel of fig. 3.11.

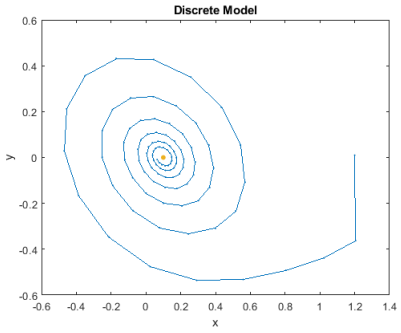
The result of the numerical simulation for the Austrian case are reported in figures 3.12 and 3.13:

Regarding this last simulation, as for the Italian version, NEDS originates a qualitatively identical discrete time model, in fact also the discretized model has an oscillatory behaviour, and as showed in the right panel of fig. 3.13, it is qualitatively equivalent to the continuous version.

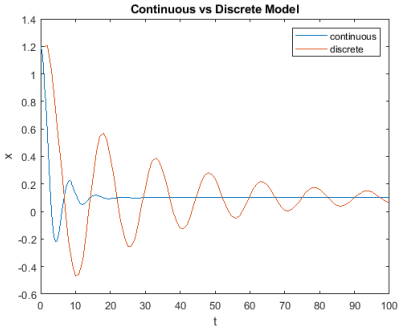
It is worth to notice that conditions (3.16) and (3.17) are satisfied both for



(a) *Price*

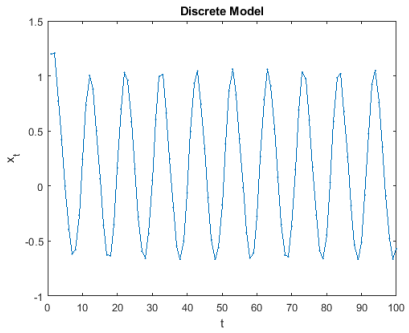


(b) *Simulated trajectory*

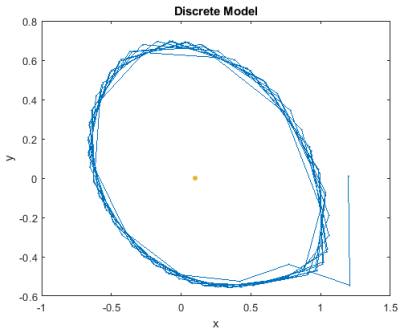


(c) *Continuous model vs discretized*

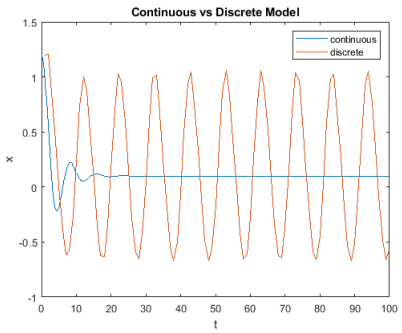
Figure 3.8: Case $\Psi(x_0) > 0$. Simulation of price x, trajectory for model (3.9) and comparison with the continuous one (3.6)with $r = \alpha = \beta = 0.5, m = S_0 = D_0 = 1.1, u = 0.75, x(0) = 1.2, y(0) = 0.01$ and $h = 0.5$. Condition (3.17) is satisfied.



(a) *Price*

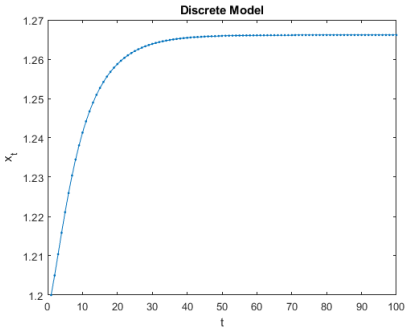


(b) *Simulated trajectory*

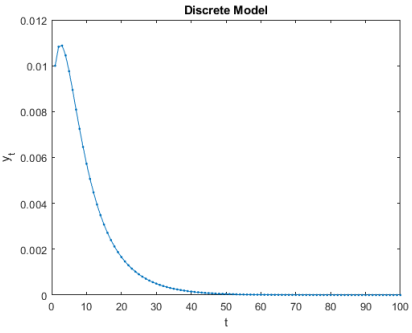


(c) *Continuous model vs discretized*

Figure 3.9: Case $\Psi(x_0) > 0$. Simulation of price x, trajectory for model (3.9) and comparison with the continuous one (3.6)with $r = \alpha = \beta = 0.5, m = S_0 = D_0 = 1.1, u = 0.75, x(0) = 1.2, y(0) = 0.01$ and $h = 0.8$. Condition (3.17) is not satisfied.

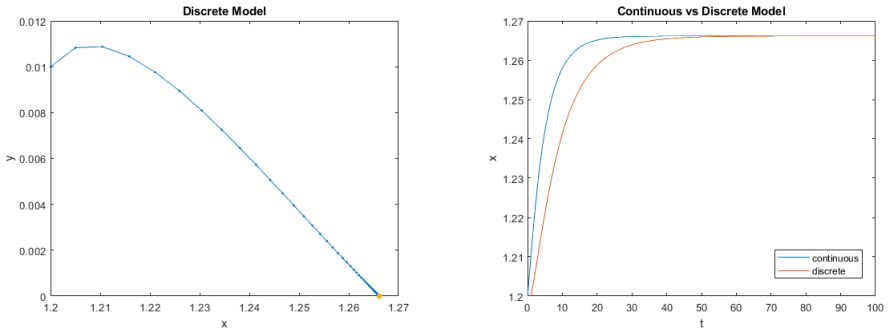


(a) *Price:* $x(t)$



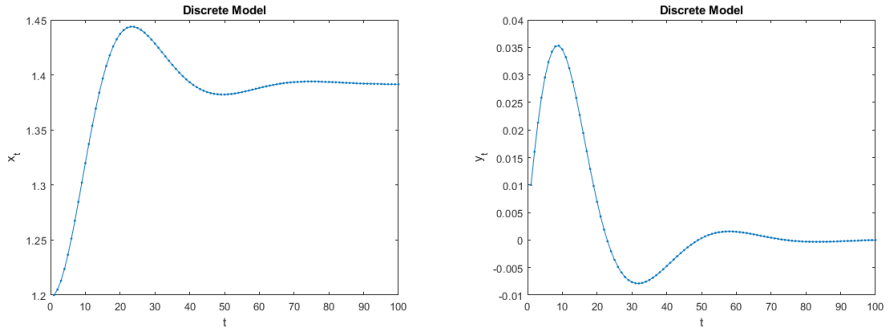
(b) *Speed of price:* $y(t)$

Figure 3.10: Italy, soft wheat price and speed of price for $D_0 = 2, S_0 = 1, u = 0.23, \alpha = 0.39, \beta = 0.39, r = 1.10, m = 0.56, \Psi = x^3, h = 0.5$. Initial data: $x(0) = 1.2, y(0) = 0.01$.



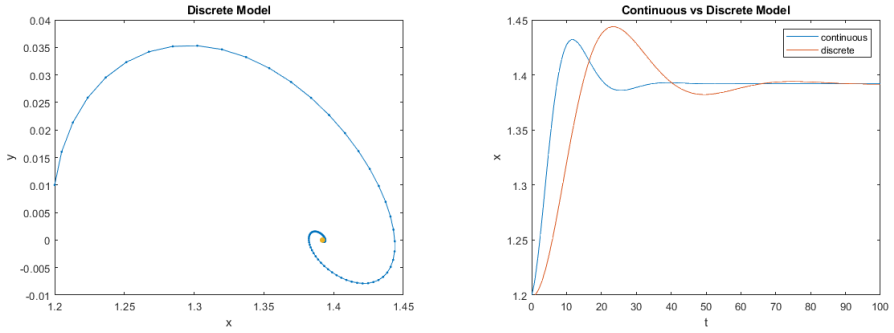
(a) *Trajectory* (b) *Continuous vs Discrete*

Figure 3.11: Italy, discrete trajectory and comparison for $D_0 = 2, S_0 = 1, u = 0.23, \alpha = 0.39, \beta = 0.39, r = 1.10, m = 0.56, \Psi = x^3, h = 0.5$. Initial data: $x(0) = 1.2, y(0) = 0.01$.



(a) *Price:* $x(t)$ (b) *Speed of price:* $y(t)$

Figure 3.12: Austria, soft wheat price and speed of price for $D_0 = 2, S_0 = 1, u = 0.10, \alpha = 0.36, \beta = 0.36, r = 0.10, m = 10^{-3}, \Psi = x^3, h = 0.5$. Initial data: $x(0) = 1.2, y(0) = 0.01$.



(a) *Trajectory* (b) *Continuous vs Discrete*

Figure 3.13: Austria, discrete trajectory and comparison for $D_0 = 2, S_0 = 1, u = 0.10, \alpha = 0.36, \beta = 0.36, r = 0.10, m = 10^{-3}, \Psi = x^3, h = 0.5$. Initial data: $x(0) = 1.2, y(0) = 0.01$.

Italy and Austria, so proposition (11) can be applied and the result, confirmed by the simulations, is that the discretized model and the continuous original version are *qualitatively equivalent*. This result might not be so relevant at first glance but, on the contrary, shows the great power of NEDS discretization scheme: its flexibility that allows us to use it for different kind of models, ecological, physical, economical, etc.

3.2.5 Comparison between Continuous and Discrete Time Models and Real Data

The last objective of this thesis work is to investigate on the relationship between the continuous time model (3.6), its discretization through NEDS model (3.9) and "real data" obtained in the market.
It is quite evident that, even before testing it, the considered model (3.6) cannot be enough complex to predict and estimate the price of any commodity present in official markets, indeed the model is too elementary and it does not take into account several aspects that may influence the price of a generic commodity, for instance: seasonality, geopolitical decisions (like wars, embargoes, imposition of duties, etc), lack of raw materials (for oil and other fossil fuels), climate changes, etc.
However our aim is to study a particular case where the discretized model has an error less than the continuous time one's; for the described case, it can be affirmed that NEDS discretization produced a model which of course cannot reply the behaviour of the price of the considered commodity for the above reasons, but at least, it produced a discrete time model better than the starting one.

We considered the price of *coffee futures* at 1 month (ICE-US Coffee C Fu-

tures Electronic Commodity Future Continuation 1), downloaded from Eikon (by Thompson-Reuters) from 06/01/2012 to 09/07/2013 with a 1-week frequency in US Dollars. In particular, this data refers to United States coffee futures prices for Arabica coffee beans and it is considered as the world benchmark for Arabica coffee. It prices the physical delivery of green beans from 20 countries of America, Asia, Africa and Oceania. These beans are harvested and brought to official warehouses located in USA, Germany, Belgium and Spain. This index is managed by ICE (Intercontinental Exchange). Unfortunately, since Eikon is a pay-per-use financial platform, it is not possible to report here coffee futures prices.

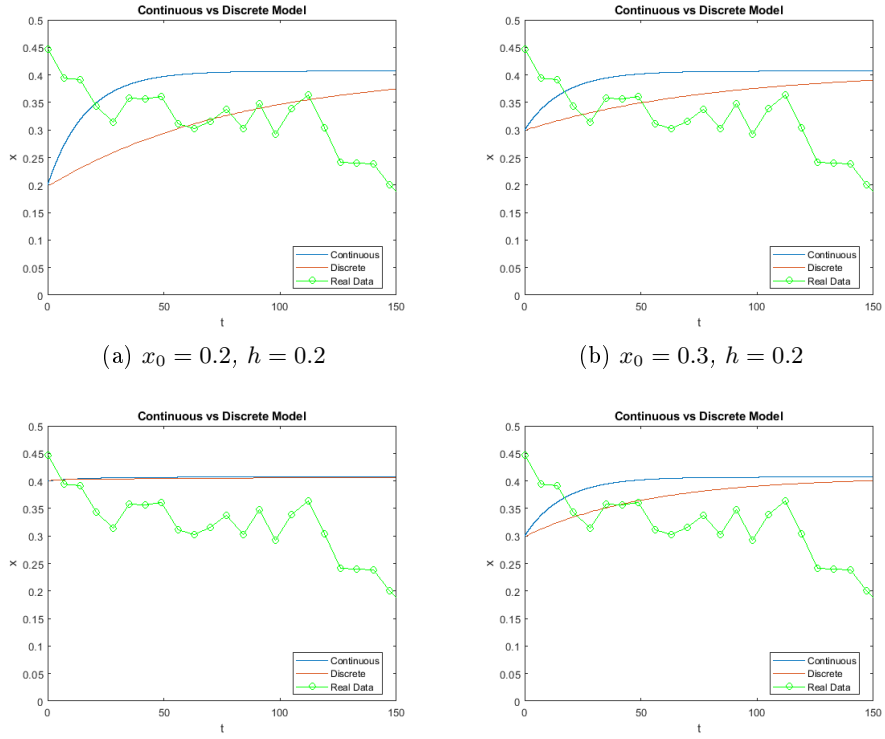
We subdivided the data into two parts, the first was used for calibrating the 5 parameters of model (3.6) (u, α, β, r and m), starting from the initial point $x_0 = 0.81$ and $y_0 = 0.01$; the second part was partly used as a "test" for the models (continuous and discrete) and we used it as "real data" in oder to make the comparison between market data and what the models predicted. Moreover we used the log-price in order to better manage real data.
Matlab code is reported in Appendix (B), it is worth to highlight the use of Matlab functions `fcn2optimexpr`, `optimproblem` and `solve`, their use in explained in Appendix (A).
We observed that unexpectedly, for optimal values of initial conditions and step-size, the discretized model performs better than the continuous time one; indeed the error of the discretized model, calculated as the sum of the squares of the difference at each time between real data and values obtained from the model, is less than in the continuous version.

In figure 3.14 we report some examples where the discrete time model better approximates the behaviour of real data with respect to the continuous time model. We report also in table 3.1 the respective errors.

Figure	x_0	h	Continuous Error	Discrete Error
3.14 (a)	0.2	0.2	0.2879	0.2352
3.14 (b)	0.3	0.2	0.2394	0.1638
3.14 (c)	0.4	0.2	0.2256	0.2200
3.14 (d)	0.3	0.3	0.2394	0.1931

Table 3.1: Error (calculated with the euclidean distance) for continuous time model (3.6) and for discrete time (3.9), related to figure 3.14.

It is immediate to appreciate from table 3.1 how, for optimal initial values, the error of the discrete time model is less than the error of the continuous time error. However, observing figure 3.14, it would be a big mistake



(a) $x_0 = 0.2$, $h = 0.2$

(b) $x_0 = 0.3$, $h = 0.2$

(c) $x_0 = 0.4$, $h = 0.2$

(d) $x_0 = 0.3$, $h = 0.3$

Figure 3.14: Representation of coffee futures log-prices (in green), continuous time (in blue) and discrete time model (in red) for $u = 5.9996, \alpha = 1.2287, \beta = 1.2287, r = 18.1767, m = 13.2694, D_0 = 2, S_0 = 1$ and $y_0 = 0.01$. The considered time horizon is 150 days.

affirming that the continuous time model or its discretization predicts the price of coffee futures, characterized by peaks and lows, since their behaviours are really different.

Finally, it is important to observe that the discrete time dynamical system performs better than the continuous version, this fact has huge relevance since the data used in the model (coffee futures prices) has discrete frequency (1 week) and so, using a discrete time model is a coherent choice, which, moreover, gives also an appreciable result showed in table 3.1, where the discrete time model has a lower error than the continuous system.

Chapter 4

Conclusions

This thesis was written with the aim of describing main differences between continuous and discrete time dynamical systems and discretization methods. In particular the latter could be a practical solution for the common problem of considering continuous time models with data collected with discrete frequency.

The first two chapters were devoted to introduce dynamical systems (continuous and discrete) and the discretization methods, respectively.
In chapter (3), we did a step forward: we applied *NEDS* method, that was introduced by its authors in [5] as a discretization method for *1-dimensional biological* models, to a *2-dimensional* continuous time *economical* model. The result was a little unexpected since we had to introduce an additional hypothesis, contained in proposition (11), in order to preserve the stability of the unique fixed point also in the discretized model.
It is worth to highlight again that NEDS model was applied to a different type of models from the ones expressed by its authors, moreover the rise of the independent variables (from 1 to 2) was well managed by just adding a small hypothesis on the step-size h.

We performed a calibration for the 5 parameters of the model using the price of soft wheat for Italy and Austria from 2008 to 2019, taken from Eurostat, and we confirmed that, for admissible values of h, the behaviour of the continuous time model is replicated by its NEDS-discretized version.

Finally we considered the price of futures of Arabica coffee, it particular we took from Eikon platform the *ICE-US Coffee C Futures Electronic Commodity Future Continuation 1* which is considered a world benchmark for Arabica coffee. We performed, as before, a parameters' calibration and then we calculated the price using the continuous time model and the discrete one and, at the end, we compared them with real data.
The result was that, for particular initial values and step-size h, the dis-

cretized model had a smaller error than the continuous time model. This fact confirmed our initial idea: since data is available only at discrete time, it is a better choice to use discrete time models.

Since the literature on discretization methods for n-dimensional dynamical systems, where $n \geq 2$, is in development, there is space for more deep analysis; indeed, using *NEDS* method for very complex models could be tricky in some cases. It might happen that some constraints on the step-size h or model's parameters should be added in order to preserve the consistency of the discretization method.

The economical model (3.6) considered in this thesis is basic and it would be impossible for it to describe the performance of any commodity since it does not take into account several aspects that may influence the commodity price.
However it would be interesting to verify if, taking into account the price of a chosen commodity and after deleting disturbances on the behaviour like seasonality or sudden economical and geopolitical crisis, the discretized model has still a smaller error than the continuous time model for any initial point and the majority of the values of the step-size h and so the discretization could be a way of better approximating the behaviour of a commodity.

Another spark for possible future analysis is to consider a much more complex continuous time model (with more than 2 independent variables and many parameters) used in practice for pricing an asset and then discuss the application of NEDS and finally verify if, maybe with the addition of other constraints, the discretization is dynamical consistent as for simpler starting models.

Appendix A

Matlab Functions

Here, in this Appendix the description of Matlab functions used in the code, available in Appendix (B), is reported:

A.1 fmincon

`fmincon` is a Matlab function used for finding the minimum of a constrained nonlinear multivariable function. In particular the Matlab function is a nonlinear solver which finds the minimum of function f, that means:

$$\min_x f(x)$$

such that:

$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$

where $\mathbf{A}$ is the matrix that takes into account the coefficients of the left-hand side of the linear constraints system. The vector $\mathbf{b}$ is the vector of the constant terms of the linear constraint system, i.e. it is the right-hand side of the system:

$$\begin{cases} 3x + 4y \leq 10 \\ -5y \leq 3 \end{cases}$$

In this example $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 0 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$.

In addition to $\mathbf{A}$ and $\mathbf{b}$, input parameters of `fmincon` are initial conditions $\mathbf{x}_0$ and, of course, the function f that has to be minimized.

The output is the vector $\mathbf{x}$ of coefficients of the variables that minimize the function f under the applied constraints.

Possible limitations are related to cases when the objective and constraint functions are not continuous and do not have continuous first derivatives because `fmincon` is gradient-based method.

A.2 ode45

`ode45` is the most general Matlab ODE solver.

Input arguments are: the the function handle which defines the function to be integrated, the integration window, which is a vector and the initial condition y_0 for each equation of the integrating function.
Output arguments are: the vector of integration points t, which contains the internal evaluation points used to perform the integration and the vector y which contains the solutions of the ODE.
There are many ODE solvers used in Matlab and, as said previously, the version used in the code is the most general between them and it solves non-stiff differential equation with medium order, as expressed in [10]. `ode45` solves the differential equation $y' = f(t, y)$ and it is also based on the Runge-Kutta explicit formula (the Dormand-Prince pair [2]), it is a single step solver, i.e. in order to compute $y(t_n)$, it only needs the knowledge of the previous step $y(t_{n-1})$.

A.3 fcn2optimexpr

`fcn2optimexpr` is used in order to convert a generic function into an optimization expression.
Sometimes it could be useful to include, as input, the desired size of the output. In this work, it was useful to obtain as output a matrix with 2 rows, where the first one represents the price and the second row its variation.

A.4 optimproblem

`optimproblem` is used for creating optimization problems.
The function we want to use as the objective function must be inserted in input. The output will be the optimization problem as *OptimizationProblem* object.

A.5 solve

Matlab function `solve` is used in the code in the last step of calibration, indeed it solves an optimization problem.
Input arguments are: the problem we want to optimize (obtained from the use of function `optimproblem`) and an initial guess which is a vector of the same size of the vector that represents the parameters that we want to optimize.

Appendix B

Matlab Code

Here below, it is presented the Matlab code used in the calibration of soft wheat price for Italy and Austria and the following simulations of the continuous and discrete time dynamical model showed in chapters (3.1.3) and (3.2.4).

Moreover, it is showed the Matlab code used for the calibration of coffee futures price and the following analysis between real data and the result of continuous and discrete time model for the calibrated parameters, previously discussed in chapter (3.2.5).

Objective Function: obj_fun.m:

```
% objective function for soft wheat (Italy and Austria)
function cost = obj_fun_1(param)
% read Excel file
load('soft_wheat_EUROSTAT.mat')
data = softwheatEUROSTATS2;
nation = table2array(data(12,2:end)); % find Italy and
    Austria in the table (12th and 20th row, respectively)
price = zeros(2,length(nation));
price(1,:) = nation;
for i = 2:size(price,2)
    price(2,i) = price(1,i) - price(1,i-1); % vector of
        derivatives (incremental ratio with t=1y)
end
price = price';
time_short = [0:1:size(price,1)-1]'; % vector of number of
    years
D0 = 2; % initial demand
S0 = 1; % initial supply
y0 = [1.2  0.01]; % initial conditions for the ODE
[time_long , y] = ode45(@(t,y) lara_1(t,y,param,D0,S0), [0
    time_short(end)], y0);
n = size(y,1);
```

```
if (size(price,1) ~= n)
    price_new = interp1(time_short,price(:,1),time_long);
% figure()
% plot(time_short,price(:,1),'o',time_long,price_new,':.')
end
cost = 0;
% the objective will be to minimize "cost"
for i = 1:length(price_new)
    cost = cost + (y(i,1) - price_new(i))^2;
end
end
```

Psi Function: Psi.m:

```
% Function Psi (cubic)
function val = Psi(x)
val = x.^3;
end
```

Run Script: run_1.m:

```
% run for soft wheat (Italy and Austria)
clear all
close all
clc
%% Calibration
x0 = [0.75  0.5  0.5  0.5  1]'; % initial guess (= of Teodoro
    Lara simulation)
A = [-1  0  0  0  0; % u > 0
      0 -1  0  0  0; % alpha > 0
      0  0 -1  0  0; % beta > 0
      0  0  0 -1  0; % r > 0
      0  0  0  0 -1; % m > 0
      1  0  0  0  0; % u < 10
      0  1  0  0  0; % alpha < 1
      0  0  1  0  0; % beta < 1
      0  0  0  1  0; % r < 1.5 for Austria (r < 10 for Italy)
      0  0  0  0  1]; % m < 1.5
b = [0-eps; % u > 0
      0-eps; % alpha > 0
      0-eps; % beta > 0
      0-eps;     % r > 0
      0-eps;     % m > 0
      10*(1-eps); % u < 10
      1*(1-eps); % alpha < 1
      1*(1-eps); % beta < 1
```

```
      10*(1-eps); % r < 1.5 for Austria (r < 10 for Italy)
      1.5*(1-eps)]; % m < 1.5
param = fmincon(@obj_fun_1, x0, A , b);
u = param(1)
alpha = param(2)
beta = param(3)
r = param(4)
m = param(5)

D0 = 2;
S0 = 1;

%% Continuous Time
T_max = 100;
t_span = [0 T_max];
y0 = [1.2  0.01];
[t, y] = ode45(@(t,y) lara(t, y, u, alpha, beta, r, m, D0,
    S0), t_span,y0);
x_cont = y(:,1);
y_cont = y(:,2);

% Plots
figure()
plot(t,x_cont)
title('Continuous Model')
xlabel('t')
ylabel('x(t)')
figure()
plot(t,y_cont)
title('Continuous Model')
xlabel('t')
ylabel('y(t)')
figure()
plot(x_cont,y_cont)
title('Continuous Model')
xlabel('x')
ylabel('y')
hold on
plot((D0-S0)/(alpha+beta), 0, 'Color', '#EDB120','Marker', '
    .',...
     'MarkerSize',15)

% Condition negative real or negative complex conjugate
    eigenvalues
(u*Psi((D0-S0)/(alpha+beta))+r*m)^2
4*u*(alpha+beta)

%% Discrete Time
h = 0.5;
```

```
x_discr = zeros(T_max,1);
y_discr = zeros(T_max,1);
x_discr(1) = 1.2;
y_discr(1) = 0.01;
for i = 1:T_max-1
    x_discr(i+1) = x_discr(i) + h*y_discr(i);
%    NEDS (T1 case)
    y_discr(i+1) = exp(-r*m*h)*y_discr(i) + ((1-exp(-r*m*h))
        /(r*m))*u*...
        (-Psi(x_discr(i))*y_discr(i)+(D0-S0)-(alpha+beta)*
            x_discr(i));

end

% Plots
figure()
plot(1:T_max, x_discr, '.-')
title('Discrete Model')
xlabel('t')
ylabel('x_t')
figure()
plot(1:T_max, y_discr, '.-')
title('Discrete Model')
xlabel('t')
ylabel('y_t')
figure()
plot(x_discr,y_discr, '.-')
title('Discrete Model')
xlabel('x')
ylabel('y')
hold on
plot((D0-S0)/(alpha+beta), 0, 'Color', '#EDB120','Marker', '
    .',...
    'MarkerSize',15)

figure()
plot(t,x_cont)
hold on
plot(1:T_max, x_discr)
title('Continuous vs Discrete Model')
xlabel('t')
ylabel('x')
legend('continuous','discrete')

% Condition 3 if Psi(x0)=0
(r*m)/(h*(alpha+beta))
% Condition 2 if Psi(x0)>0
u*(-2*Psi((D0-S0)/(alpha+beta))+h*(alpha+beta))
-(2+2*exp(-r*m*h))/((1-exp(-r*m*h))/(r*m))
```

```
% Condition 3 if Psi(x0)>0
(u*(-Psi((D0-S0)/(alpha+beta))+h*(alpha+beta))-r*m)/(r*m)
```

Function used in the parameters' calibration: PARAMtoODE.m:

```
% function used in the calibration model
function solpts = PARAMtoODE(param,tspan,y0,D0,S0)
sol = ode45(@(t,y) lara_2(t,y,param,D0,S0),tspan,y0);
solpts = deval(sol,tspan);
end
```

Run Script: run_2.m:

```
% run for coffee calibration
clear all
close all
clc
%% Calibration
% load coffee data 4 (weekly)
load('COFFEE_data_4.mat')
data = COFFEEdata4;
nation_old = table2array(data);
nation_old = flipud(nation_old);
nation_old = log(nation_old); % consider log-prices
nation = nation_old(1:ceil(end/2));
price_real = zeros(2,length(nation));
price_real(1,:) = nation;
% compute the derivative as the difference of price at time
    t and (t-1)
for i = 2:size(price_real,2)
    price_real(2,i) = price_real(1,i)-price_real(1,i-1); %
        vector of derivatives (incremental ratio with t=1y)
end
time_short = [0:1:size(price_real,2)-1]; % vector of number
    of years
% for i = 1:2
%     subplot(1,2,i)
%     plot(time_short,price_real(i,:),'-o')
%     title(['y(',num2str(i),') '])
% end
D0 = 2;
S0 = 1;
y0 = [0.81 0.01]; % initial conditions for the ODE
% Create optimization variable
param = optimvar('param',5, "LowerBound", 0.01, "UpperBound
    ", [40 20 20 40 35]);
```

```
% Convert function to an optmization expression
myfcn = fcn2optimexpr(@PARAMtoODE, param, time_short, y0, D0, S0,
    'OutputSize',[2, size(price_real,2)]);
% the objective function is the sum of squared differences
    between the ODE
% and the real prices
obj = sum((myfcn(1,:) - price_real(1,:)).^2); % take only
    first row
% Create an optimization problem with the objective function
     obj
prob = optimproblem('Objective',obj);
% View the problem
showproblem(prob)
% Initial guess for the parameters (u, alpha, beta, r, m)
param0.param = [0.75  0.5  0.5  0.5  1]; % initial guess
% Solve the optimization problem
[paramsol,sumsq] = solve(prob,param0)
% Show calibrated parameters
disp(paramsol.param)

u = paramsol.param(1)
alpha = paramsol.param(2)
beta = paramsol.param(3)
r = paramsol.param(4)
m = paramsol.param(5)

D0 = 2;
S0 = 1;

%% Continuous Time
T_max = 150;
t_span = [0 T_max];
y0 = [0.3  0.01];
[t, y] = ode45(@(t,y) lara(t, y, u, alpha, beta, r, m, D0,
    S0), t_span,y0);
x_cont = y(:,1);
y_cont = y(:,2);

% Plots
figure()
plot(t,x_cont)
title('Continuous Model')
xlabel('t')
ylabel('x(t)')
figure()
plot(t,y_cont)
title('Continuous Model')
xlabel('t')
ylabel('y(t)')
```

```
figure()
plot(x_cont,y_cont)
title('Continuous Model')
xlabel('x')
ylabel('y')
hold on
plot((D0-S0)/(alpha+beta), 0, 'Color', '#EDB120','Marker', '
    .',...
     'MarkerSize',15)

% Condition negative real or negative complex conjugate
    eigenvalues
(u*Psi((D0-S0)/(alpha+beta))+r*m)^2
4*u*(alpha+beta)

%% Discrete Time
h = 0.3;
x_discr = zeros(T_max,1);
y_discr = zeros(T_max,1);
x_discr(1) = 0.3;
y_discr(1) = 0.01;
for i = 1:T_max-1
     x_discr(i+1) = x_discr(i) + h*y_discr(i);
%    NEDS (T1 case)
     y_discr(i+1) = exp(-r*m*h)*y_discr(i) + ((1-exp(-r*m*h))
         /(r*m))*u*...
          (-Psi(x_discr(i))*y_discr(i)+(D0-S0)-(alpha+beta)*
              x_discr(i));

end

% Plots
figure()
plot(1:T_max, x_discr, '.-')
title('Discrete Model')
xlabel('t')
ylabel('x_t')
figure()
plot(1:T_max, y_discr, '.-')
title('Discrete Model')
xlabel('t')
ylabel('y_t')
figure()
plot(x_discr,y_discr, '.-')
title('Discrete Model')
xlabel('x')
ylabel('y')
hold on
plot((D0-S0)/(alpha+beta), 0, 'Color', '#EDB120','Marker', '
```

```
    '.',...
119     'MarkerSize',15)
120
121 figure()
122 plot(t,x_cont)
123 hold on
124 plot(1:T_max, x_discr)
125 title('Real Data vs Continuous vs Discrete Model')
126 xlabel('t')
127 ylabel('x')
128
129 % Condition 3 if Psi(x0)=0
130 (r*m)/(h*(alpha+beta))
131 % Condition 2 if Psi(x0)>0
132 u*(-2*Psi((D0-S0)/(alpha+beta))+h*(alpha+beta))
133 -(2+2*exp(-r*m*h))/((1-exp(-r*m*h))/(r*m))
134 % Condition 3 if Psi(x0)>0
135 (u*(-Psi((D0-S0)/(alpha+beta))+h*(alpha+beta))-r*m)/(r*m)
136
137
138 %% Graphical Comparison with real coffee data
139 % coffee 4 (weekly)
140 load('COFFEE_data_4.mat')
141 data = COFFEEdata4;
142 nation_old = table2array(data);
143 nation_old = flipud(nation_old);
144 nation_old = log(nation_old);
145 rd = nation_old(ceil(end/2)+1:end);
146 rt = 0:7:(length(rd)-1)*7;
147 plot(rt, rd, 'g-o')
148 axis([0 150 0 0.5])
149 legend('Continuous', 'Discrete', 'Real Data', 'Location', '
    Best')
150
151 %% Continuous model error vs Discretized model error
152 % continuous model
153 [~, y_err] = ode45(@(t,y) lara(t, y, u, alpha, beta, r, m,
    D0, S0), rt,y0);
154 x_cont_err = y_err(:,1);
155 % discretized model
156 x_discr_err = interp1(1:T_max, x_discr, rt);
157 x_discr_err(1) = y0(1);
158 x_discr_err = x_discr_err';
159 % compute error
160 index = 0;
161 for i = 1:length(x_discr_err)
162     if(isnan(x_discr_err(i)) == 0)
163         index = i;
164     end
```

```
end
x_cont_err_new = x_cont_err(1:index);
x_discr_err_new = x_discr_err(1:index);
rd_new = rd(1:index);
err_cont = sum((rd_new - x_cont_err_new).^2)
err_discr = sum((rd_new - x_discr_err_new).^2)
```

Economical model: lara.m:

```
% economic model: Teodoro Lara ("On the Dynamics of Oil
    Price Model")
function dy = lara(t, y, u, alpha, beta, r, m, D0, S0)
dy = zeros(2,1);
dy(1) = y(2);
dy(2) = -u*(alpha+beta)*y(1)-(u*Psi(y(1))+r*m).*y(2)+u*(D0-
    S0);
end
```

Functions `lara_1.m` and `lara_2.m` are equivalent to previous function `lara.m`.

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