



POLITECNICO
MILANO 1863

**SCUOLA DI INGEGNERIA INDUSTRIALE
E DELL'INFORMAZIONE**

EXECUTIVE SUMMARY OF THE THESIS

The ε -regularity theorem for perimeter minimizers via viscosity methods

LAUREA MAGISTRALE IN MATHEMATICAL ENGINEERING - INGEGNERIA MATEMATICA

Author: MICHELE TOFFOLI

Advisor: PROF. MARCO POZZETTA

Co-advisor: PROF. MARCO CAROCCIA

Academic year: 2025-2026

1. Introduction

The study and classification of area minimizing surfaces is a classical and highly fruitful topic in mathematics. One of the first pioneers in the field was Joseph Plateau, who experimentally investigated soap films in the nineteenth century. The mathematical question he posed, now known as the Plateau problem, is the following: given a closed boundary, does there exist an area minimizing surface spanning it?

Since then, the theory has undergone substantial development, and one of its central questions concerns the regularity of area minimizing surfaces. As we shall discuss below, there exist area minimizing (hyper)surfaces in $\mathbb{R}^8$ that are not regular: in fact, they may possess singular points. This naturally leads to the following question: which conditions guarantee that an area minimizing surface is smooth?

A fundamental answer was provided by Ennio De Giorgi, one of the leading figures in the calculus of variations and geometric measure theory. Roughly speaking, he proved that if an area minimizing surface is locally trapped inside a sufficiently thin slab, then it must be smooth.

In this thesis, drawing inspiration from recent ideas of Ovidiu Savin, we provide a new ap-

proach to this celebrated result.

The perimeter of a measurable set $E \subset \mathbb{R}^n$ can be regarded as a suitable weak notion of the $(n-1)$ -dimensional measure of its boundary. When ∂E is sufficiently smooth, the perimeter coincides with the $(n-1)$ -dimensional Hausdorff measure of the boundary, namely $\mathcal{H}^{n-1}(\partial E)$. In the following, we introduce the distributional notion of perimeter based on the Gauss-Green formula, which remains meaningful even for sets with non-smooth boundary. Within this framework, one gets the natural class of objects of the so-called sets of locally finite perimeter, namely sets whose perimeter may be infinite globally but is finite inside every bounded region. In particular, in this thesis we will focus on local perimeter minimizers, namely sets for which any local modification can only increase the perimeter or leave it unchanged. See (2) for the precise definition.

In dimensions $n \geq 8$, perimeter minimizers may possess singularities, and their singular set can have dimension as large as $n-8$. A classical example is the Simons cone

$$K = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x| < |y|\},$$

which is a perimeter minimizer in $\mathbb{R}^8$ and is singular at the origin (see [3, 6]). One way to rule out such irregular behavior is to impose a flatness condition on the boundary. The main goal of this thesis is precisely to prove the following theorem of De Giorgi, commonly referred to as the ε -regularity theorem for perimeter minimizers. It shows that sufficiently flat perimeter minimizers are regular, thereby excluding the presence of singularities under the sole assumption of ε -flatness of the boundary.

Theorem 1.1 (*ε -regularity of perimeter minimizers, De Giorgi [4]*). *Let $n \in \mathbb{N}$, $n \geq 2$. Then there exists $\varepsilon_n > 0$, depending only on n , such that if $E \subset \mathbb{R}^n$ is a perimeter minimizer such that*

$$\partial E \cap B_1 \subset \{|x \cdot v| \leq \varepsilon\},$$

for some $v \in S^{n-1}$ (that is, E is ε -flat in the direction v), and if $\varepsilon \leq \varepsilon_n$, then $\partial E \cap B_{1/2}$ coincides with the graph of a smooth function.

After [4], a classical strategy for the proof of these ε -regularity theorems has been established. We refer the reader to [8] for all the details. The classical proof roughly proceeds as follows.

- Under the ε -flatness assumption, it is possible to show that a large part of the boundary of the set can be covered by the graph of a Lipschitz function f .
- Since the minimal surface equation linearizes in the Laplacian equation, one prove that this function f is close to a harmonic function. Hence a-priori estimates on harmonic functions can be transferred to f . Eventually, an iteration argument leads to the conclusion.

The ε -regularity theorem for perimeter minimizers mentioned above was first proved by De Giorgi [4]. Subsequently, partial ε -regularity results in codimension greater than one were obtained by Almgren in [2] for rectifiable sets of arbitrary codimension minimizing sufficiently smooth elliptic functionals, and later by Allard in [1] for general rectifiable sets under suitable summability assumptions on the distributional mean curvature. The setting of these latter works is considerably more general than that of sets of finite perimeter. As of today, the above

mentioned techniques have been employed and generalized to prove a number of ε -regularity results in many different situations.

The aim of this thesis is to give a proof of theorem 1.1 following a strategy much more recently devised by Savin [9, 10], which relies on a substantially different approach. Here we use a viscosity-type method, often also referred to as the Aleksandrov-Bakelman-Pucci (ABP) method, to establish a geometric Harnack inequality for a perimeter minimizer as in the assumptions. Once this is achieved, one can more classically derive a so-called improvement of flatness result: ε -flatness in a direction v at scale 1 implies $\frac{\varepsilon}{2}$ -flatness in a (possibly different) direction at a smaller scale $r_0 < 1$. Once again, an iteration argument shows that the boundary of a perimeter minimizer can be written, in a ball of radius $1/2$, as the graph of a function $u \in C^{1,\alpha}$ for some $\alpha \in (0, 1)$. Once this is established, standard elliptic regularity theory implies that u is actually smooth.

This viscosity method applied to perimeter minimizers can be seen as the geometric counterpart of the result obtained by Savin in [10], which is, instead, an ε -regularity result for solutions to certain PDEs. More precisely, given a viscosity solution u to the fully nonlinear equation

$$F(D^2u, Du, u, x) = 0,$$

assuming that $u_0 \equiv 0$ is a solution and F is smooth and uniformly elliptic, if $\|u\|_{L^\infty}$ is sufficiently small, then u is smooth.

A similar viscosity approach was employed by Savin in [11] to prove a version of De Giorgi's conjecture asserting that level sets of suitable critical points of the Allen-Cahn functional

$$J(u, \Omega) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - u^2)^2 \, dx.$$

are flat hyperplanes in dimensions $n \leq 8$.

At the end of [10], Savin claims that these viscosity methods can be applied to derive the ε -regularity theorem for perimeter minimizers (theorem 1.1). In fact, geometric ε -regularity results that rely on the viscosity method have appeared for example in [5, 7].

However, the proof of the ε -regularity theorem 1.1 is only sketched in [9], and several arguments are left implicit. In this thesis, we

provide a complete and rigorous presentation of the proof within the framework of sets of finite perimeter. We have worked through all the arguments only outlined by Savin, supplied the missing details, and filled the gaps in the original exposition. We hope that this manuscript will make the ideas and techniques behind [9] considerably more accessible to the reader.

2. Preliminaries in Geometric Measure Theory

In this chapter, we present the fundamental results of geometric measure theory without proof. We define the perimeter of a measurable set $E \subset \mathbb{R}^n$ in an open set $A \subset \mathbb{R}^n$ as

$$P(E; A) = \sup \left\{ \int_E \operatorname{div} T(x) dx : \right. \\ \left. T \in C_c^\infty(A; \mathbb{R}^n), \sup_{\mathbb{R}^n} |T| \leq 1 \right\}. \quad (1)$$

It can be proved that the function $P(E; \cdot)$ extends to a Borel measure $P(E; \cdot)$, called perimeter measure of E .

Moreover, if E is a Lebesgue measurable set in $\mathbb{R}^n$, then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$ -valued Radon measure μ_E on $\mathbb{R}^n$ such that

$$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n).$$

$|\mu_E| = P(E, \cdot)$ and E is a set of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$.

We describe the tangential properties that characterize the measure theoretic boundary of a set of locally finite perimeter, namely its reduced boundary $\partial^* E$, that yields the set on which μ_E is concentrated. We then state one of the most celebrated results in the theory: De Giorgi's Structure Theorem.

Theorem 2.1 (De Giorgi's Structure Theorem). *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then the generalized Gauss-Green formula holds true:*

$$\int_E \operatorname{div}(T) = - \int_{\partial^* E} T \cdot \nu_E d\mathcal{H}^{n-1}, \quad \forall T \in C_c^1(\mathbb{R}^n).$$

Moreover, there exists countably many C^1 -hypersurfaces M_h in $\mathbb{R}^n$, compact sets $K_h \subset M_h$,

and a Borel set F with $\mathcal{H}^{n-1}(F) = 0$, such that

$$\partial^* E = F \cup \bigcup_{h \in \mathbb{N}} K_h,$$

and, for every $x \in K_h$, $\nu_E(x)^\perp = T_x M_h$, the tangent space to M_h at x . In particular, this implies that $\partial^* E$ is a rectifiable set.

We state two fundamental properties of the perimeter: the lower semicontinuity of the perimeter on open sets with respect to L^1 -convergence, and the compactness theorem for sequences of sets with uniformly bounded perimeter in balls of arbitrary radius.

At this point, we introduce the notion of local perimeter minimizer: given an open bounded set A , and a set of locally finite perimeter E in $\mathbb{R}^n$, we say that E is a perimeter minimizer in A if, whenever $E \Delta F \subset \subset A$, there holds

$$P(E; A) \leq P(F; A). \quad (2)$$

We prove the density estimates satisfied by such sets which prevent the perimeter minimizers from pathological behaviors.

Theorem 2.2 (Density estimates for perimeter minimizers). *For every $n \geq 2$, there exists a positive constant $c(n)$ with the following property. If A is an open bounded set in $\mathbb{R}^n$ and E is a perimeter minimizer in A , then for every ball $B(x, r) \subset \subset A$ with $x \in A \cap \operatorname{spt} \mu_E$, we have*

$$\frac{1}{2^n} \leq \frac{|E \cap B(x, r)|}{\omega_n r^n} \leq 1 - \frac{1}{2^n},$$

$$c(n) \leq \frac{P(E; B(x, r))}{r^{n-1}} \leq n\omega_n$$

In particular,

$$\mathcal{H}^{n-1}(A \cap (\partial E \setminus \partial^* E)) = 0.$$

Finally, we introduce the notion of mean curvature and show that the boundary of a perimeter minimizer has vanishing mean curvature (possibly in a distributional sense).

Theorem 2.3. *A set of locally finite perimeter E is stationary for perimeter in the open set A if and only if*

$$\int_{\partial^* E} \operatorname{div}_E T d\mathcal{H}^{n-1} = 0, \quad \forall T \in C_c^1(A; \mathbb{R}^n).$$

In particular, E has vanishing distributional mean curvature in A .

Using this fact, we prove the Monotonicity Theorem for the density ratio of a set that is stationary with respect to the perimeter.

Theorem 2.4 (Monotonicity of density ratios). *If E is stationary for perimeter in the open set A and $x_0 \in A$, then the density ratios*

$$\frac{P(E; B(x_0, r))}{\omega_{n-1} r^{n-1}}$$

are non-decreasing with respect to $r \in (0, \text{dist}(x_0, \partial A))$. Moreover, if that ratio is constant with respect to r , then E is a cone.

3. Perimeter minimizers, convergence and the distance function

In the first section of this chapter, we prove several basic properties of perimeter minimizers: their precompactness in L^1_{loc} , the rigidity of perimeter-minimizing cones contained in a half-space, and the fact that a perimeter minimizer whose boundary is trapped in a thin slab cannot simultaneously occupy both sides of the slab. Otherwise, the boundary would essentially appear twice, leading to an increase in perimeter; see fig. 1.

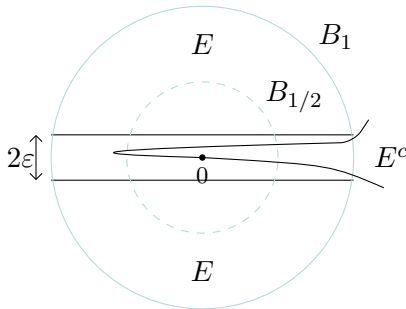


Figure 1

After introducing the notions of Hausdorff and Kuratowski convergence and establishing their equivalence, we prove the main result of this section: the L^1_{loc} -convergence of a sequence of perimeter minimizers implies the local Hausdorff convergence of their boundaries.

Next, we show that any function whose graph is tangent from one side to the boundary of a perimeter minimizer must have non-positive mean curvature.

Theorem 3.1. *Consider a perimeter minimizer*

$E \subset \mathbb{R}^n$ such that $0 \in \partial E$, and let

$$\varphi : \mathcal{B}_r(0) \rightarrow \mathbb{R}$$

be a smooth function, $\varphi \in C^\infty(\mathcal{B}_r)$ with $|\varphi| < L$ for some $r, L > 0$, such that

$$\varphi(0) = 0, \quad E \cap (\mathcal{B}_r \times [-L, L]) \subset \{x_n > \varphi(x')\}.$$

Then

$$H(\varphi(0)) = \text{div} \left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) (0) \leq 0.$$

Moreover, we prove that the distance function from a perimeter minimizer has non-positive Laplacian in the viscosity sense throughout its complement.

Theorem 3.2. *Consider a perimeter minimizer $E \subset \mathbb{R}^n$ and call $\bar{d}$ the distance from it. Then*

$$\Delta \bar{d} \leq 0 \text{ in the viscosity sense on } \mathbb{R}^n \setminus \bar{E}.$$

In the last section, we present useful geometric conditions that guarantees $C^{1,\alpha}$ -regularity.

4. The ε -regularity theorem for perimeter minimizers

It is mostly in the first lemmas of the proof that one sees applications of the viscosity-type method. The first step in the proof is to show that if a perimeter minimizer is contained in a half-space and one of its boundary points is close to the bounding hyperplane, then the boundary remains close to that hyperplane in a neighborhood of the point. More precisely, for every $\delta \in (0, \frac{1}{3})$, there exist constants $\varepsilon(\delta)$ and $C(\delta) > 0$ such that if a perimeter minimizer $E \subset \mathbb{R}^n$ in B_1 satisfies

- $\partial E \cap B_1 \subset \{x_n \geq 0\}$,
- $(0', \varepsilon) \in \partial E$ for some $\varepsilon \leq \varepsilon(\delta)$,

then

$$\partial E \cap (\mathcal{B}_\delta(x') \times [0, C(\delta)\varepsilon]) \neq \emptyset \quad \forall x' \in \mathcal{B}_{1/3},$$

where $0', x', y'$ denote points in $\{x_n = 0\}$, and $\mathcal{B}_r(y')$ is the open ball of radius r and center y' in $\{x_n = 0\}$.

We prove this first lemma by contradiction. Suppose that a cylinder of radius δ and height of order ε does not intersect the boundary of the

perimeter minimizer. Then it is possible to place the graph of a volcano-shaped function φ inside this empty region and move it upward until its outer slopes eventually make first contact with ∂E . See fig. 2 for an explanatory picture.

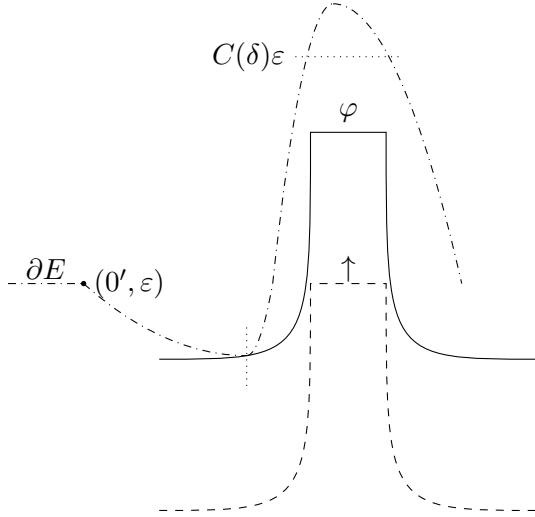


Figure 2: We reach a contradiction combining the perimeter minimality of E with the presence of a cavity as in the picture.

This is where the viscosity approach plays a crucial role. We do not know whether the boundary of the perimeter minimizer can be represented as the graph of a function, let alone whether such a function would be sufficiently smooth to satisfy the minimal surface equation in the classical sense. However, exploiting the perimeter minimizing property of E , by Theorem 3.1 the mean curvature of the graph of a function touching ∂E from one side must be non-positive. On the other hand, the contradicting assumption allows to set up φ so that its graph has strictly positive mean curvature at the touching point, leading to the contradiction. This contradiction rules out the existence of the empty cavity, and proves the first lemma.

Equipped with this powerful tool, and building again upon analogous geometric viscosity-type arguments, we can establish a Harnack inequality for ∂E . More precisely, assuming that E is a perimeter minimizer in B_1 such that

$$\partial E \cap B_1 \subset \{|x_n| \leq \varepsilon\},$$

with ε sufficiently small, then there exists $\eta \in (0, 1)$, depending only on the dimension, such

that

$$\partial E \cap B_{1/2} \subset \{x_n \geq -\varepsilon(1 - \eta)\}$$

or

$$\partial E \cap B_{1/2} \subset \{x_n \leq \varepsilon(1 - \eta)\}.$$

In other words, upon restricting our attention to the smaller ball $B_{1/2}$, one can exclude either the upper or the lower portion of the original slab from the possible location of the boundary. Thus, the boundary is confined to a strictly thinner slab at scale $1/2$.

Once the Harnack inequality is proved, the proof follows a more classical path to the conclusion. First we prove a so-called Improvement of Flatness Theorem: if a perimeter minimizer is ε -flat in a given direction at scale 1, then it is $\frac{\varepsilon}{2}$ -flat in a ball B_{r_0} , for some $r_0 < 1$, with respect to a possibly new direction, that is quantitatively close to the original one.

Finally, exploiting this quantitative control on the oscillation of the boundary, we show that ∂E can be represented as the graph of a $C^{1,\alpha}$ function. In the thesis, we carry out explicitly all the details behind these various steps. In the end, standard elliptic regularity theory improves the previously obtained $C^{1,\alpha}$ regularity into smoothness.

References

- [1] William K. Allard. On the first variation of a varifold. *Ann. of Math. (2)*, 95:417–491, 1972.
- [2] F. J. Almgren, Jr. Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure. *Ann. of Math. (2)*, 87:321–391, 1968.
- [3] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Invent. Math.*, 7:243–268, 1969.
- [4] Ennio De Giorgi. *Frontiere orientate di misura minima*. Editrice Tecnico Scientifica, Pisa, 1961. Seminario di Matematica della Scuola Normale Superiore di Pisa, 1960-61.
- [5] Guido De Philippis, Carlo Gasparetto, and Felix Schulze. A short proof of Allard’s and

- Brakke's regularity theorems. *Int. Math. Res. Not. IMRN*, (9):7594–7613, 2024.
- [6] Guido De Philippis and Emanuele Paolini. A short proof of the minimality of simons cone. *Rendiconti del Seminario Matematico della Università di Padova*, 121:233–241, 2009.
- [7] Carlo Gasparetto. Epsilon-regularity for the Brakke flow with boundary. *Anal. PDE*, 18(4):857–894, 2025.
- [8] F. Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.
- [9] O. Savin. Minimal surfaces and minimizers of the Ginzburg-Landau energy. In *Symmetry for elliptic PDEs*, volume 528 of *Contemp. Math.*, pages 43–57. Amer. Math. Soc., Providence, RI, 2010.
- [10] Ovidiu Savin. Small perturbation solutions for elliptic equations. *Comm. Partial Differential Equations*, 32(4-6):557–578, 2007.
- [11] Ovidiu Savin. Regularity of flat level sets in phase transitions. *Ann. of Math. (2)*, 169(1):41–78, 2009.