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Adaptive Rational Equilibrium Dynamics: stability bifurcations and deterministic chaos

Tesi di Laurea Magistrale in
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Abstract

Adaptive Rational Equilibrium Dynamics models were introduced in the late 90s.
In these capital asset pricing models, investors have the option to choose between a fundamental trading strategy, incurring a positive cost, and a cost-free trend-following strategy. The distinction between fundamentalists and trend followers is then introduced. A Market Maker balances the (positive or negative) excess demand for the risky asset, updating its price accordingly. More sophisticated models account for the presence of other types of investors, such as contrarians and bias traders.
Under the Efficient Market Hypothesis and the Rational Expectation Hypothesis, investors are assumed to be rational, aiming to adopt the best performing strategy. Analytical and numerical results demonstrate that financial markets are stable, and prices tend to converge to the fundamental value when agents are not inclined to change their beliefs at each time instant. Multiple non fundamental equilibria may arise as the tendency to update strategies increases. As the intensity of choice further rises, market dynamics become chaotic, leading to massive price fluctuations.
From a mathematical perspective, financial instability arises due to homoclinic bifurcations of stable and unstable manifolds of the fundamental steady state, giving rise to strange and chaotic attractors. Pessimistic phases, characterized by price fluctuations below the fundamental value, can be mitigated by constraining traders' possibilities. The uptick rule serves as an example.

Keywords: complex systems, stability, bifurcations, deterministic chaos, Adaptive Rational Equilibrium Dynamics

Abstract in lingua italiana

I modelli Adaptive Rational Equilibrium Dynamics furono introdotti a fine anni 90.
In questi capital asset pricing models gli investitori possono scegliere tra una strategia d'investimento fondamentale, pagando un costo positivo, e una strategia basata sull' osservazione dei trend che non ha un costo. La distinzione tra fondamentalisti e seguitori di trend è quindi introdotta. Un Market Maker bilancerà l'eccesso di domanda (positivo o negativo) del titolo rischioso e il suo prezzo verrà aggiornato di conseguenza. Modelli più sofisticati permettono la presenza di ulteriori tipi di investitori. Per esempio, investitori in controtendenza e bias traders.
Sotto l'Ipotesi di Mercato Efficiente e l'Ipotesi di Attesa Razionale gli investitori sono considerati razionali: il loro obiettivo è adottare la strategia più performante.
Risultati analitici e numerici mostrano come i mercati finanziari sono stabili e i prezzi tendono al valore fondamentale quando gli agenti non sono così inclini a cambiare le loro convinzioni ad ogni istante di tempo. Più di un equilibrio non fondamentale può apparire al crescere della tendenza ad aggiornare strategia. Al crescere ulteriore dell'intensità di scelta le dinamiche di mercato sono caotiche portando a consistenti fluttuazioni dei prezzi. Da un punto di vista matematico, l'instabilità finanziaria è dovuta alle biforcazioni omocline delle varietà stabile e instabile dell'equilibrio fondamentale, dalle quali compaiono attratori strani e caotici.
Fasi pessimistiche caratterizzate da fluttuazioni dei prezzi al di sotto del valore fondamentali possono essere contrastate limitando le possibilità degli investitori. L'uptick rule rappresenta un esempio.

Parole chiave: sistemi complessi, stabilità, biforcazioni, caos deterministico, Adaptive Rational Equilibrium Dynamics

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Introduction

Adaptive Rational Equilibrium Dynamics models, introduced by William A. Brock and Cars H. Hommes in 1997, are commonly referred to as Brock-Hommes models (BH models for short) in the literature. These models quickly garnered attention for their effectiveness in capturing various aspects of financial markets: investors' tendencies to buy and sell risky assets, price fluctuations, endogenous shifts between pessimistic (undervalued prices) and optimistic (overvalued prices) phases, as well as financial instability and chaos.

Traders in the model can choose from different investment strategies based on predictions of future prices, each strategy incurring a cost. Investors are assumed to be rational, adhering to the Efficient Market Hypothesis and Rational Expectation Hypothesis. This implies that agents are mean-variance maximizers, learning from the past. A Market Maker balances the excess demand for the risky asset, updating its price accordingly. It is reasonable to assume that the best performing strategy in the recent past will attract more followers, with strategy performance measured by the last observed net realized profit. The intensity of choice parameter plays a crucial role in the market dynamics, influencing the global behavior of the stock market. For small values of the intensity of choice parameter, the market is stable, and prices tend to the fundamental value, while non fundamental equilibria appear, and large price fluctuations are possible as it increases.

The interaction between prices and investor beliefs is modeled through a discrete time dynamical system, representing a complex system governed by nonlinear functions. The theory of complex systems is suitable for explaining market behavior concerning the parameter values, particularly the intensity of choice. Concepts such as equilibrium, stability, bifurcation, chaotic orbits, and strange attractors play a crucial role in understanding why financial markets become unstable and prices fluctuate when investors are likely to adapt their strategy. Chaotic dynamics result from the presence of at least one strange attractor arising from homoclinic bifurcations of stable and unstable manifolds of the fundamental steady state. Instability and chaos become intrinsic properties of the system when the intensity of choice parameter is sufficiently large and cannot be avoided unless the possibilities of the investors are limited by regulators.

This thesis is organized as follows. Chapter 1. provides an introduction to the Efficient Market Hypothesis, where markets are assumed to be efficient and traders are rational. Section 1.1 presents the historical economic context, while section 1.2 develops the mathematics of the EMH and introduces two tools used by fundamentalists and trend followers: Fundamental Analysis (subsection 1.2.1) and Technical Analysis (subsection 1.2.2). Chapter 2. discusses the ARED model proposed by Hommes et al. (section 2.2) following a brief historical introduction to these models (section 2.1). Analytical and numerical results are presented in sections 2.3 and 2.4, respectively. Section 2.5 covers the most recent and potential future developments, and section 2.6 concludes the chapter with the application of the uptick rule to the considered adaptive system. All proofs of the theorems stated in this work are contained in Appendix B, while Appendix A provides a brief guide to the mathematics of complex systems.

1 | Efficient Market Hypothesis

1.1. Historical context

The formalization of efficient market theory occurred in the 1960s and 1970s, pioneered by Eugene Fama and Paul Anthony Samuelson. They independently and simultaneously developed the theory, laying the foundation for what would become the Efficient Market Hypothesis (EMH).

Even in the late 19th century, economists had intuitions that financial agents should react rationally to market information. George Gibson's 1889 publication, "The stock markets of London, Paris, and New York" marked the early appearance of the idea of efficient markets in economic and financial literature. Gibson emphasized that publicly known information in an open market determines the value of shares.

Albert Einstein's 1905 formalization [1] of Brownian motion [2] came after Louis Bachelier's 1900 PhD thesis, "The Theory of Speculation" which proposed Arithmetic Brownian Motion for asset price dynamics [3]. Although Bachelier's work was overlooked for decades, it was rediscovered in the 1950s and 60s [4].

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[1]Albert Einstein developed the mathematical tools for Brownian motion only in 1905, as detailed in his paper titled "On the Movement of Small Particles Suspended in Stationary Liquids Required by the Molecular-Kinetic Theory of Heat" citeEinstein. Remarkably, this occurred five years after Louis Bachelier's groundbreaking work. It's noteworthy that Einstein was unaware of Bachelier's contributions at the time.

[2]A Brownian motion (BM) W_t is a stochastic process such that:

1. $W_0 = 0$ a.s. ;
2. Increments are independent i.e. $W_t - W_s \perp\!\!\!\perp \mathcal{F}_s$ for any $t > s$, and stationary, namely $W_{t+h} - W_t \sim W_h$ for any $t, h > 0$;
3. for any $t > s$ it follows $W_t - W_s \sim N(0, t - s)$.

[3]In his thesis Bachelier proposed an Arithmetic Brownian Motion (ABM) for asset dynamics, namely $S_t = S_0 + \sigma W_t$. "The Theory of Speculation" can be considered the genesis of financial modeling.

[4]Bachelier and his work were essentially rediscovered around 1955. Leonard Jimmie Savage stumbled upon some of Bachelier's publications in the Yale library, prompting him to inquire among his colleagues about anyone familiar with Bachelier. Among those approached was Paul Anthony Samuelson. As a

In 1961, John Fraser Muth in his paper "Rational expectations and the theory of price movements" introduced the Rational Expectations Hypothesis (REH), asserting that market agents are rational decision-makers who base their actions on the best available information. Eugene Fama, in 1965, released "The behaviour of stock-market prices" outlining key features of efficient markets.
Paul Anthony Samuelson ("Proof that properly anticipated prices fluctuate randomly") provided a formal economic argument for efficient markets in the same year, modelling asset prices dynamics in terms of martingales [5] rather than in terms of random walks [6].

The term "Efficient Market Hypothesis" was first used in 1967 by Harry Roberts in "Statistical versus clinical prediction of the stock market" Eugene Fama's influential 1970 paper, "Efficient capital markets: A review of theory and empirical work" provided the first definition of an efficient market: a market is said to be efficient with respect to an information set if the price fully reflects that information set.
In 1992, Burton Malkiel, in his work "Efficient Market Hypothesis" , presented his unique perspective on market efficiency, a definition essentially equivalent to Fama's (1970): a market is said to be efficient if the price would be unaffected by revealing the information set to all market participants.

A famous quote by Jensen ("Some anomalous evidence regarding market efficiency",1978): "I believe there is no other proposition in economics which has more solid empirical evidence supporting it than the Efficient Market Hypothesis". In his opinion "a market is efficient with respect to an information set if it is impossible to make economic profits by trading on the basis of that information set."

Critiques of the EMH emerged in the 1980s and 90s, challenging the theory's assumptions. In 1980 Grossman and Stiglitz ("On the impossibility of informationally efficient markets") argued the impossibility of perfectly informationally efficient markets, as information gathering incurs costs. In 1985, De Bondt and Thaler ("Does the stock market overreact?") presented evidence of asset prices overreaction, contributing to the birth of behavioral finance.

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student at MIT, Samuelson subsequently located a copy of Bachelier's Ph.D. thesis in the MIT library.

[5] A stochastic process X_t is a martingale if and only if $\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}_s(X_t) = X_s$ for any $t > s$. Namely the expected value of X_t conditional to the information available at time s is X_s. As a consequence, a martingale is such that $\mathbb{E}(X_t) = X_0$ for any $t > 0$.

[6] A random walk is a stochastic process with independent and identically (IID) increments with zero mean. The Brownian motion is a classical example of random walk. If asset prices follow random walks, they exhibit a random movement, uninfluenced by their past history. Further details on this concept will be explored in the following section.

The 2007-2008 financial crisis [7], though seemingly a challenge to market efficiency, was interpreted by some, like Ray Ball ("The global financial crisis and the efficient market hypothesis: What have we learned?", 2009) , as a failure to heed the lessons of efficient markets rather than evidence against them.

Despite ongoing debates, the impact of efficient market theory on real financial markets is substantial. The following sections will delve into the mathematics of efficient markets, introducing two tools derived from the EMH: Fundamental and Technical Analysis.

1.2. Efficient Market Hypothesis

As introduced in the previous section, the Efficient Market Hypothesis (EMH) posits that markets are efficient. However, before delving into the definition of what constitutes an "efficient market," let's address the question of why a market should be efficient.

The answer is straightforward: markets need to be efficient; otherwise, the information available in the markets becomes inconsequential for decision-making. In an inefficient market, firms may engage in speculation or market timing [8], attempting to capitalize on market inefficiencies. This can lead to distortions in investment decisions, particularly when company managers are incentivized by high asset returns, creating opportunities for companies to exploit mispricing of securities.

As previously emphasized, following Fama (1970), a market is deemed efficient concerning an information set if the price fully reflects that information set. In 1992, Malkiel provided an alternate definition, asserting that a market is efficient if the price remains unaffected upon revealing the information set to all market participants.

These expressions may seem somewhat enigmatic, prompting further inquiries: What constitutes the information within the information set? What does it mean for the price to fully reflect that information set? And, importantly, why should prices exhibit such behavior?

[7]The financial crisis, originating with the Subprime crisis in 2006, led to the collapse of numerous banks and companies. This collapse was primarily attributed to the lack of adequate liquidity, preventing the mitigation of insolvency scenarios. Initially occurring in the U.S., the crisis swiftly spread to infect the world economy, and its repercussions are still evident today.

[8]Market timing is an investing strategy wherein traders attempt to forecast future prices movements.

Certainly, there are no definitive answers to these questions, and diverse interpretations give rise to different iterations of market efficiency, as will become evident shortly. If prices fully reflect the information set, trading assets at market prices would yield a null Net Present Value [9] activity. Deviations from this scenario could potentially open avenues for arbitrage.

The information set in the market is categorized into three main types:

1. Past prices and time series. This category includes historical prices and other time series data, such as past interest rates;
2. Public information. Public information encompasses a wide range of data that is available to the general public. This includes time series data, the latest economic and financial news, and other publicly accessible information;
3. Private information owned by insiders. Private information refers to data that is not publicly disclosed and is known only to a select group of individuals, often insiders within a company.

These categories reflect the different levels of information incorporated into the Efficient Market Hypothesis (EMH), each associated with a specific form of market efficiency: weak; semi-strong; and strong. The efficiency of a market is determined by how quickly and accurately prices adjust to the information within these categories. This distinction was first introduced by Michael C. Jensen in 1978.

A market is considered weak form efficient if it is impossible to forecast future prices by observing their past values. This implies that all historical information, specifically past prices and other time series such as interest rates, is already reflected in current market prices. The logic behind this form of efficiency is that if investors could consistently predict future price movements based on historical data, they would exploit these patterns for arbitrage opportunities. In a frictionless stock market, meaning no trans-

[9] Indeed, the Net Present Value (NPV) serves as a metric for evaluating the desirability of an investment. It is computed by summing up all the future cash flows generated by the investment and then discounting this sum back to its present value. If the NPV is calculated to be zero, the investment is termed "fair." In practical terms, this implies that the present value of expected future cash flows equals the initial investment outlay. When NPV is positive, the investment is generally considered attractive, indicating potential profitability. Conversely, a negative NPV suggests that the investment may not be economically viable.

action costs, no short selling constraints [10], and no capital controls [11], investors would buy assets expected to rise and sell those expected to fall, taking advantage of the anticipated price movements. To prevent such arbitrage opportunities and ensure weak form efficiency, market prices must adjust quickly to new information, making it challenging to profit from historical price patterns. This concept leads to the idea that future prices follow a random walk, as they are not systematically influenced by past price movements or patterns.

(a) FTSE MIB index in a one year window updated to the 15^{th} of October 2023 (source: Yahoo! Finance)

(b) simulated random walk starting from S_0=1

Figure 1.1: Comparison between asset prices (a) and random walks (b)

A market is considered semi-strong form efficient if it is impossible to forecast future prices by studying public information. Public information encompasses time series data, such as prices, macroeconomic indicators like GDP, inflation, and interest rates, and news related to companies, including balance sheets, earnings, losses, investment plans, or announcements of cost reductions. Generally, this information is readily available at no cost. Naturally, asset prices react positively to good news and negatively to bad news, reaching a new equilibrium. Following this adjustment, prices should not exhibit predictable patterns; otherwise, arbitrage opportunities would arise.

A market is deemed strong form efficient if it is impossible to forecast future prices by utilizing both public and private information. Private information refers to data accessible only to a restricted group, such as employees of companies or central banks and regulators.

[10] Financial regulators may impose rules against short selling to prevent aggressive speculative behavior and potential market manipulation. In section 2.6 the example of the uptick rule is provided.

[11] Measures taken by national governments or monetary authorities to regulate the flow of capital in and out of a country's economy. These controls are often implemented to maintain stability in financial markets, protect the domestic currency, and prevent excessive speculation or abrupt capital flight. Capital controls can take various forms, including: transaction takes; currency pegs or bands; limits on foreign investment and liquidity requirements.

This particular definition of market efficiency, involving private information, is controversial for two main reasons. Firstly, in numerous countries, insider trading (trading based on private information) is illegal and strictly opposed. Secondly, acquiring private information incurs a cost, leading to a paradox known in the literature as the Grossman & Stiglitz paradox [12].

Weak form efficiency implies the unpredictability of future price movements. In other words, if there are identifiable trends or recurring patterns in time series data, it suggests that market prices can be predicted, allowing individuals to capitalize on this knowledge and generate profits. This forms the foundation of Technical Analysis.

In a market characterized by strong form efficiency, where prices reflect both public and private information, analyzing how past prices responded to market information can be useful for investors. This approach aligns with the objectives of Fundamental Analysis.

1.2.1. Fundamental Analysis

Fundamental Analysis can be defined as the study of asset prices and their determinants.

As a consequence of the strong form efficient market hypothesis, the objective of Fundamental Analysis is to uncover the relationship between prices, returns, and dividends and their determinants, which are the fundamental information or signals available in the market. Individuals who employ Fundamental Analysis are colloquially referred to as fundamentalists.

The primary focus of fundamentalists is to calculate the intrinsic value of an asset and make trading decisions based on this valuation. Fundamental Analysis can also be utilized to forecast the future performances of companies by ranking them according to various economic indicators. One example of such a ranking system is the F-score [13].

[12] According to Grossman and Stiglitz, if prices accurately reflect all available information, then the information becomes worthless. In their view, if there's a cost associated with obtaining information, rational individuals would refrain from paying for it. Consequently, if nobody is willing to pay for information, it follows that information does not impact prices. For a more in-depth examination, refer to the already cited Grossman and Stiglitz (1980)

[13] The Piotroski F-score, named after Joseph Piotroski, an accounting professor at Stanford, assigns an integer score F between 1 and 9 to each company. Based on this score, companies are categorized into three groups: low financial performance ($F \in$1, 2, 3), medium financial performance ($F \in$4, 5, 6), and high financial performance ($F \in$7, 8, 9). For a more in-depth examination, refer to "Value Investing: The Use of Historical Financial Statement Information to Separate Winners from Losers" by Piotroski.

The intrinsic value of an asset, often referred to as the fundamental price, is essentially the present value of future cash flows discounted back to the present day using the appropriate risk-adjusted interest rate. When the actual price is lower than the calculated fundamental value, it is considered advantageous to buy the asset. Conversely, if the actual price exceeds the fundamental value, the recommendation is to sell. This investment approach aligns with the principles followed by fundamentalists in the Adaptive Rational Equilibrium Dynamics model, which is explored in detail in the subsequent chapter.

In a simple Capital Asset Pricing Model (CAPM) with N risky assets and a risk-free asset with interest rate r_f, the Market portfolio (M) is considered, then the risk adjusted discount factor k for the asset is given by:

$$k = \mathbb{E}(\tilde{r}) = r_f + \beta[\mathbb{E}(r_M) - r_f]$$

where β is the systematic risk and $\mathbb{E}(r_M) - r_f$ the risk premium of the Market portfolio.

The intrinsic value V_0 is then calculated as the sum of the expected future dividends d properly discounted back to the present day:

$$V_0 = \sum_{i}^{\infty} \frac{\mathbb{E}(d_i)}{(1+k)^i}$$

1.2.2. Technical Analysis

Technical Analysis involves the search for predictable patterns in asset prices.

As discussed earlier, Technical Analysis is a consequence of the weak form definition of market efficiency, where traders believe that prices react slowly to new information. Therefore, they look for predictable trends in asset dynamics. The main difference with fundamentalists is that traders opting for Technical Analysis only focus on prices since they consider them the manifestation of all available fundamental information. Investors who rely on this kind of analysis are called trend followers or chartists, and they will be considered in the adaptive system proposed in Chapter 2.

First of all, let's enumerate the possible different types of trends. Trends can be differentiated based on their duration in time or their direction, or both. For trends in time, we can further distinguish between:

- Primary trend: it lasts for a consistent period;
- secondary trend: it is registered inside a primary trend;
- minor trend: short term trend observed inside a secondary trend;

Figure 1.2: Primary trends (purple arrows), secondary trends (blue arrows) and minor trends (yellow arrows) spotted in the FTSE MIB index (source: Yahoo! Finance)

Regarding price movements, trends can be classified as follows:

- Up-trend: characterized by a sequence of higher highs and higher lows.[14] and higher lows;
- Down-trend: marked by a sequence of lower highs and lower lows;
- Consolidation: a phase in which neither up-trends nor down-trends are evident.

Alongside asset prices, chartists delve into the realm of traded volumes. Traded volume denotes the quantity of stocks exchanged during a given period. These volumes are typically categorized as either high or low, with the distinction drawn from the average volume observed over the specified time frame. In figure 1.2 and 1.3 daily volumes are displayed alongside prices, manifesting as green (if positive) and red (if negative) peaks at the bottom of the graphs. In essence, the traded volume, when correlated with trends, becomes a pivotal element for trend followers, offering insights to predict potential scenarios in the immediate future.

[14]Highs and lows of prices in the context of trends are relative to the specific monitoring window, which can be defined based on different timeframes such as daily, weekly, monthly, or yearly.

Figure 1.3: UP-trends (green arrows), down-trends (red arrows) and consolidations (yellow arrows) spotted in the FTSE MIB index (source: Yahoo! Finance)

An extremely simplified guideline is presented in the following table. This guideline aims to provide a straightforward approach to understanding the interplay between trends and volumes in technical analysis.

trend	volume	interpretation
up	high	the up-trend may persist
up	low	there is a lack of conviction in the up-trend: possible stagnation
down	high	the down-trend may persist
down	low	there is a lack of conviction in the down-trend: possible stagnation

Table 1.1: Possible scenarios and their interpretation

In the upcoming chapter, trend followers will base their actions on this guideline.

Indicators play a crucial role in Technical Analysis, serving as formulas and indexes to measure trends, momentum [15], volatility, and more. The Simple Moving Average (SMA) [16], the Relative Strength Index (RSI) [17], and the Average Direction Index (ADX) [18] are among the most widely considered indicators.

Figure 1.4: SMA (orange line), RSI (purple line) and ADX (black line) for the FTSE MIB index (source: Yahoo! Finance). In this case a time window of 14 days is considered

[15]It indicates the strength of a market by measuring the rate of change of prices with respect to their effective values.

[16]It is the usual moving average applied to closing prices over the considered period. Since registered prices are assumed to be daily, the moving average will take into account 5 observations for the weekly or shorter-term progress, 20 observations for the monthly or medium-term progress, and 250 observations for the yearly or longer-term progress.

[17]Introduced by Welles Wilder in 1978 in his book "New Concepts in Technical Trading Systems", it measures the speed and velocity of price movements for stocks, futures, and bonds over a specified period of time, making it useful for spotting trends. For a more in-depth understanding, refer to "New Concepts in Technical Trading Systems" by Welles Wilder.

[18]Introduced by Welles Wilder in 1978, measures the strength of a trend by quantifying the velocity of a price, regardless of its movement. For a more comprehensive treatment, refer to "New Concepts in Technical Trading Systems" by Welles Wilder.

2 | Adaptive Rational Equilibrium Dynamics

2.1. Brief history of ARED models

The concept of Adaptive Rational Equilibrium (ARE) was initially introduced by William A. Brock and Cars H. Hommes in 1997 in their paper "A rational route to randomness".

The fundamental assumption underlying the Adaptive Rational Equilibrium (ARE) is that agents in financial markets make decisions based on future predictions of relevant variables, such as the price of a traded asset. Investors can choose from a finite set of predictors, denoted as P, which are functions of past values of the variables of interest, essentially incorporating all available information at the decision time. However, obtaining information comes at a cost, and each predictor, being an expected value, carries a specific cost. More sophisticated and comprehensive predictors typically incur higher costs compared to simpler, naive predictors.

Choosing a strategy in this context is equivalent to selecting a predictor. Each strategy is associated with a utility function, and a "performance" function takes into consideration both the utility of the strategy and its associated cost. According to the Efficient Market Hypothesis, particularly the Rational Expectation Hypothesis, at each time instant (given that the model operates in discrete time), agents will select their strategy based on the best performance observed up to that point. This could involve choosing the strategy that has led to the highest profit or the smallest prediction error.

The parameter β in the model, referred to as the "intensity of choice", represents the tendency of agents to change their predictor type. Its value influences the overall dynamics of the system, encompassing the asset price and the fractions of investors following each strategy.

The term "Adaptive Rational Equilibrium Dynamics" (ARED) is employed to describe the dynamics generated by this system, and its name reflects two primary considerations, elucidated by Brock and Hommes:

1. Deliberate economic act: the selection of a predictor is considered a deliberate economic act, where agents make rational decisions regarding their strategy. This intentional decision making process is integral to the concept of equilibrium within the economic framework;

2. Reciprocal interaction: the choice of a predictor actively contributes to the dynamics of the market equilibrium. Simultaneously, the market equilibrium, in its dynamics, influences the subsequent selection of predictors. This mutual interaction forms a feedback loop, emphasizing the interconnected nature of the choice of predictors and the overall equilibrium dynamics in the market.

Brock and Hommes designed this model with a specific focus on two key features:

1. Realistic dynamics replication: the model should possess the capability to replicate the dynamics observed in actual financial markets. This includes the characteristic behavior of investors, who regularly adjust their strategies based on the available information in the market. Consequently, these strategy changes should induce corresponding fluctuations in prices;

2. Complex and chaotic dynamics: despite its simplicity, the model should have the capacity to generate intricate and possibly chaotic dynamics. The aim is to illustrate that even from a straightforward conceptual framework, complex and non linear behaviors can emerge [1], akin to the unpredictability often observed in real financial markets.

In their initial exploration of ARED models, Brock and Hommes concentrated on a scenario where only one asset is traded, and investors are limited to choosing between two predictors: a sophisticated predictor, denoted as H_1, incurring an information cost C, and a costless naive predictor, H_2. While this represents the simplest case, it is crucial to note that the set of potential predictors, P, can be arbitrarily large.

[1] Chaotic dynamics manifests when nonlinearities are introduced into a dynamical system, a phenomenon recognized as early as the late 19th century by Henri Poincaré. While grappling with the "three-body problem" in celestial mechanics, Poincaré noticed that the overall trajectory of three interacting bodies is highly sensitive to their initial positions: a small alteration in the starting conditions can lead to a significant divergence in the final behavior of the system. This characteristic, now termed "high sensitivity to initial conditions," is a hallmark of complex systems and forms the basis of deterministic chaos. Poincaré's groundbreaking insights have established him as the pioneer of the mathematical field known as deterministic chaos. Refer to Appendix A for a more comprehensive exploration of this topic.

This specific case yields a 2-dimensional discrete dynamical system that, despite its simplicity, encapsulates key features of asset price dynamics and trader behaviors, including chaotic dynamics, high sensitivity to initial conditions, and parameter values (particularly the intensity of choice β) [2]. Moreover, the bidimensional case provides analytical stability and bifurcation results, a feat that is challenging or impossible in higher-dimensional scenarios.

In their subsequent research "Heterogeneous beliefs and route to chaos in a simple asset pricing model" (1998) Brock and Hommes introduced a tractable form of evolutionary dynamics known as Adaptive Belief Systems (ABS) in a simple Present Discounted Value (PDV) asset pricing model.

In this paper, new types of investors are introduced: rational agents, fundamentalists, trend followers, contrarians [3], and biased traders [4]. All traders have future expectations that are linear in the variables (information) available at decision time. Along with the risky asset, a risk-free asset with a risk-free rate r_f is introduced, and the agents aim to maximize expected wealth.

The introduction of these simple predictors may lead, especially when the tendency to change strategy is high, to dynamics observed in real financial markets where price fluctuations are characterized by sudden changes of phases. Asset prices can either be stable around their fundamental value (under the Efficient Market Hypothesis) or experience phases of optimism ("castle in the air") [5], making investors excited and causing prices

[2]Another characteristic of complex systems is the dependency of trajectories on the parameter values. In complex systems with non linearities, the overall dynamics are influenced by the values of the parameters. Equilibria, cycles, and tori are all functions of the parameters, and their stability is parameter dependent. Some parameter values may yield equilibria without cycles, while others may introduce cycles, and so forth. The specific values of parameters where the qualitative behavior of the system undergoes a change are referred to as "bifurcations." For a more comprehensive treatment, refer to Appendix A.

[3]A contrarian is an investor who goes against the prevailing market trends. Contrarians typically buy assets that are currently out of favor or sell assets that are currently popular. The underlying philosophy is to take positions opposite to the crowd, assuming that market sentiment may lead to overreactions and mispricing of assets. Contrarian investing is based on the belief that markets are not always efficient and that opportunities for profit exist when investor sentiment diverges from fundamental values. Contrarians often seek to capitalize on market reversals and corrections.

[4]A biased trader is an investor who makes decisions based on a perceived bias or periodic repetition in asset dynamics. This means that the investor believes there is a pattern or trend in the market that repeats over time, and they adjust their trading strategy accordingly. The bias could be related to various factors, such as seasonal patterns, economic cycles, or other recurring trends that the investor identifies in the market. Investors employing biased trading strategies aim to capitalize on these perceived patterns to make profitable investment decisions.

[5]The phrase "castle in the air" is used by John Maynard Keynes to convey the idea that investors often act optimistically, even when market data contradicts their beliefs or when there's a lack of fundamental

to rise, or phases characterized by a crisis, making investors depressed or nervous and resulting in falling prices.

In 2000, Hommes published "Financial Markets as Nonlinear Adaptive Evolutionary Systems" in which he described the main features of the Adaptive Beliefs Systems and presented a possible extension of the model formulated by Hommes himself and Andrea Gaunersdorfer. The latest version of the model incorporates volatility clustering and fat tails in the distribution of stock dividends. The key concept behind this extension is the consideration of risk in assessing profits, assuming that agents are risk averse. The risk adjusted profits serve as a measure for the performance of the strategies in this updated model.

In 2002, Carl Chiarella and Xue-Zhong He extended the model proposed by Brock and Hommes (with the additional contribution of Gaunersdorfer) in their work titled "Asset Pricing and Wealth Dynamics - An Adaptive Model with Heterogeneous Agents". In this extension, traders are allowed to have different attitudes toward risk and varying expectations about the expected value of an asset's price. The researchers demonstrated that incorporating these diverse perspectives brings the model dynamics even closer to those observed in the real world.

In 2003, Chiarella and He, in their work titled "Heterogeneous Beliefs, Risk and Learning in a Simple Asset Pricing Model with a Market Maker", introduced a new version of the model that incorporates the presence of a Market Maker (MM). The Market Maker's role involves clearing the market by buying or selling the risky asset based on whether there is a negative or positive excess demand for the asset, respectively. After completing all transactions, the Market Maker adjusts the asset price for the next period in the same direction as the excess demand. The adjustment process is linear in the excess demand and is governed by a positive constant parameter known as μ, representing the "speed of adjustment." This introduces an additional parameter alongside the intensity of choice, influencing the dynamics of the system.

As a result of the various updates to the original model proposed in 1997, in 2005, Hommes, along with Hai Huang and Duo Wang, explored the possibilities offered by these updated models in their paper titled "A robust rational route to randomness in

support for certain asset prices. In this context, it suggests that investors may engage in speculative behavior, placing value on assets based more on expectations and perceptions rather than on a careful analysis of intrinsic value or underlying economic fundamentals.

a simple asset pricing model". Additionally, the authors introduced the possibility for agents to change their strategies asynchronously. Up to that point, in the proposed models, all agents were obliged to update their beliefs simultaneously. This asynchronous updating introduces more complexity to the system dynamics. A new parameter, α, is introduced, representing the fraction of agents that do not change their strategies and remain convinced of their beliefs. The authors note that the dynamics generated by the original model proposed in 1997 is surprisingly close to the one generated by the latest version of 2005, indicating that the original model serves as a good approximation of the more complete version.

Given the significance of Hommes' 2005 model as the culmination of nearly a decade of research on dynamical systems applied to asset pricing, and considering that subsequent studies on Adaptive Rational Equilibrium Dynamics (ARED) are primarily grounded in this version, the upcoming section will present the key aspects of the 2005 model. Following the introduction of the model, analytical and numerical findings regarding the stability and bifurcations of trajectories will be detailed.

2.2. Model setup

As previously indicated, the objective of this section is to outline the ARED model as presented in Hommes et al. (2005).

Agents have the option to allocate their funds between a risky asset or a risk-free asset, such as a bank account, with a risk-free rate denoted as r_f. Let's introduce the gross return $R = 1 + r_f$. The variable p_t represents the ex-dividend price of the risky asset, and y_t represents the stochastic process of dividends. Denoted as h the trader's type, the wealth of an investor of type h at time $t + 1$ is given by:

$$W_{h,t+1} = RW_{h,t} + (p_{t+1} + y_{t+1} - Rp_t)z_{h,t}$$

where $z_{h,t}$ is the number of shares of the risky asset purchased at time t by traders of type h.

$\mathbb{E}h, t$ and $\mathbb{V}h, t$ represent the beliefs or forecasts at time t concerning the conditional expectation and volatility of wealth for the next period for the type h investor. These expected values and volatilities are conditioned on the information available at time t, which includes past prices and dividends.

Agents are assumed to be myopic mean-variance maximizer, meaning that $z_{h,t}$ (h=1,2) solves the optimization problem:

$$\max_{z_h}\{\mathbb{E}_{h,t}(W_{h,t+1}) - \frac{a}{2}\mathbb{V}_{h,t}(W_{h,t+1})\}$$

With $a>0$ representing the aversion to risk, which is assumed to be constant over time and equal for all investors (Chiarella and He (2002) also treated the case in which the aversion to risk is a priori different for each type of trader).

The demand for the risky asset by the type h investor is given by:

$$z_{h,t} = \frac{\mathbb{E}_{h,t}(p_{t+1} + y_{t+1} - Rp_t)}{a\mathbb{V}_{h,t}(p_{t+1} + y_{t+1} - Rp_t)}$$

Assuming that the conditional variance of excess returns is uniform across all agents, $\mathbb{V}_{h,t} \equiv \sigma^2$, the expression for $z_{h,t}$ takes on a simplified form:

$$z_{h,t} = \frac{\mathbb{E}_{h,t}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2}$$

Consider the scenario in which all investors share identical thoughts and beliefs regarding the conditional expectation $\mathbb{E}_{h,t}(pt + 1 + y_{t+1} - Rp_t)$, the quantity z_s can be introduced:

$$z_s = \frac{\mathbb{E}_{h,t}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2} = \frac{\mathbb{E}_t(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2}$$

It represents the fixed and constant supply of shares per agent in the market.

The equation governing the market equilibrium can be reformulated as:

$$Rp_t = \mathbb{E}_t(p_{t+1} + y_{t+1} - Rp_t) - a\sigma^2 z_s$$

The quantity $a\sigma^2 z_s$ represents the market risk premium that investors demand for holding the risky asset instead of investing in the risk-free asset.

Under the assumption that the "transversality condition" [6] holds, i.e.

$$\lim_{k\to\infty} \frac{\mathbb{E}_t(p_{t+k})}{R^k} = 0$$

[6] The transversality condition, a crucial aspect in optimal control theory, serves as a boundary condition for the terminal values of costate variables. This condition is deemed necessary for infinite horizon optimal control problems without endpoint constraints on state variables. In financial contexts, it is often referred to as the "no bubble condition".

by iteratively applying the market equilibrium equation over time, it becomes possible to uniquely determine the "fundamental rational expectations (RE) price," commonly referred to as the fundamental price:

$$p_t^* = \sum_{k=1}^{\infty} \frac{\mathbb{E}_t(y_{t+1}) - a\sigma^2 z_s}{R^k}$$

As previously discussed in Chapter 1. under the Efficient Market Hypothesis (and the Rational Expectations Hypothesis), the fundamental price is entirely determined by economic fundamentals (the macroeconomic information available in the market). It is the sum of discounted expected future dividends adjusted by the market risk premium.

For simplicity, dividends over time are assumed to be independent and identically distributed (IID) with a constant mean $\mathbb{E}(y_t) = \bar{y}$, meaning $y_t = \bar{y} + \epsilon_t$. In this manner, the fundamental price becomes independent of time:

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y} - a\sigma^2 z_s}{R^k} = \frac{\bar{y} - a\sigma^2 z_s}{R - 1} = \frac{\bar{y} - a\sigma^2 z_s}{r_f}$$

In this model, as previously introduced by Chiarella et al. (2002) and Chiarella and He (2003), there are three types of investors: the Market Maker (MM) responsible for maintaining market order; the fundamentalists (h=1); and trend followers (h=2). All traders share the same expectation for future dividends, denoted by $\mathbb{E}_{h,t}(y_{t+1}) = \mathbb{E}_t(y_t) = \bar{y}$, while they diverge in their predictions for the future price of the risky asset.

A fundamentalist believes that the price tends toward its fundamental value, namely:

$$\mathbb{E}_{1,t}(p_{t+1}) = \mathbb{E}_t(p_{t+1}^*) = p^*$$

While trend followers believe that the trend observed in the antecedent period will persist in the following one:

$$\mathbb{E}_{2,t}(p_{t+1}) = \mathbb{E}_t(p_{t+1}^*) + g(p_t - p^*) = p^* + g(p_t - p^*)$$

$g > 0$ represents the “trend extrapolation” parameter.

At the start of each period t, the price p_t is revealed to everyone and the demand $z_{h,t}$ ($h = 1, 2$) is updated substituting the expected dividend $\bar{y}$ and the expected price in $t+1$ for fundamentalists and trend followers in its expression.

The aggregate excess of demand for the period t is given by:

$$z_{e,t} = \sum_{h=1}^{2} n_{h,t} z_{h,t} - z_s$$

Where $n_{h,t}$ represents the fraction of type h investors.

The goal of the MM is to provide liquidity. When the excess of demand is positive, the MM sells shares of the risky asset from his inventory, which occurs when the price is low. When the price is high, and the excess of demand is negative, the MM buys shares and adds them to his inventory. After all transactions are completed at the end of period t, the price is updated proportionally to the recorded excess of demand:

$$p_{t+1} = p_t + \mu z_{e,t}$$

Where the parameter $\mu > 0$ denotes the "speed of adjustment".

Thanks to this adjustment procedure, it is possible to observe the phenomenon known as the "law of demand and supply": prices rise when there is an excess of demand and fall otherwise.

The key point of this model is the mechanism by which fundamentalists can choose to change their minds and become trend followers, and vice versa. Namely, how the fractions $n_{h,t}$ evolve over time. Traders are assumed to be rational (EMH, REH), and they tend to adopt a strategy that has performed well in the last period.

Let's define an utility function at time t for each type of investor h:

$$U_{h,t} = (p_{t+1} + y_{t+1} - Rp_t) z_{h,t} + \omega U_{h,t-1}$$

Where ω is a weight parameter.

The utility function represents a weighted average of realized profits, with exponentially declining weights. For simplicity and analytical tractability, the weight parameter ω is set to 0. Consequently, the utility simplifies to the last realized profit.

A cost C_h is assigned to each type of investor, representing the expense incurred to gather information and form conclusions.

It is widely acknowledged that $C_1 > C_2$, as fundamentalists must acquire more information (economic fundamentals) compared to trend followers. Without loss of generality, C_2 can be set to 0, implying no information cost for trend followers.

After introducing utility functions and information costs, performance functions $\Pi_{h,t}$ can be defined to assess the performance of each strategy h. Trivially, $\Pi_{h,t} = U_{h,t} - C_h$. The strategy with the highest performance function at time t will be regarded as the most favorable, leading traders to adopt it.

The equation governing the evolution of $n_{h,t}$ is given by:

$$n_{h,t+1} = \alpha n_{h,t} + (1-\alpha)\frac{e^{\beta(U_{h,t}-C_h)}}{\sum_{h=1}^{2} e^{\beta(U_{h,t}-C_h)}} = \alpha n_{h,t} + (1-\alpha)\frac{e^{\beta\Pi_{h,t}}}{\sum_{h=1}^{2} e^{\beta\Pi_{h,t}}}$$

Two additional parameters appear in the equation above: α and β.

The parameter α, which falls within the range $[0,1]$, represents the fraction of investors that do not change their thoughts and beliefs. As discussed in the previous section, when $0 < \alpha < 1$, agents can change their decisions in an asynchronous way—a possibility introduced by Hommes et al. in 2005. If α=1, traders will never change their type, while the case with α=0 is the one that has been studied since the formulation of these models in 1997. In this case, investors update their strategy in a synchronized way.

The parameter β is the most crucial parameter in the system as it quantifies the tendency of traders to adopt the strategy with the best performance.

There are two extreme scenarios: β=0 and $\beta = \infty$. When β=0, agents are entirely indifferent to which strategy is the best, and both $n_{1,t}$ and $n_{2,t}$ tend to $\frac{1}{2}$ as t tends to infinity. On the other hand, if $\beta = \infty$, at each time instant, (1-α) of the investors adopt the most performing strategy, representing the extreme scenario in which agents are extremely anxious and impatient.

Finally, all the equations governing the evolutionary adaptive belief system have been introduced:

$$\begin{cases} p_{t+1} = p_t + \mu(n_{1,t}z_{1,t} + n_{2,t}z_{2,t} - z_s) \\ n_{h,t+1} = \alpha n_{h,t} + (1-\alpha)\frac{e^{\beta((p_{t+1}+y_{t+1}-Rp_t)z_{h,t}-C_h)}}{\sum_{h=1}^{2} e^{\beta((p_{t+1}+y_{t+1}-Rp_t)z_{h,t}-C_h)}} \end{cases}$$

Changing variables, transitioning from p_t to $x_t = p_t - p^*$ (deviation of the price from its fundamental value) and from $n_{1,t}$ and $n_{2,t}$ to $m_t = n_{1,t} - n_{2,t}$ (difference of fractions), it is possible to obtain a 2-dimensional discrete dynamical system:

$$\begin{cases} x_{t+1} = (1 - \frac{\mu R}{a\sigma^2} + \frac{\mu g}{2a\sigma^2}(1 - m_t))x_t \\ m_{t+1} = \alpha m_t + (1-\alpha)tanh[\frac{\beta}{2}(R - (1 - \frac{\mu R}{a\sigma^2} + \frac{\mu g}{2a\sigma^2}(1 - m_t))x_t^2 - gz_s x_t - C - \frac{g}{a\sigma^2}x_t\epsilon_{t+1})] \end{cases}$$

where $\epsilon_{t+1} = y_{t+1} - \bar{y}$ is a white noise [7] and $C = C_1 - C_2$ (hence C>0).

The system is composed of two parts: a deterministic part and a stochastic part introduced by the white noise term ϵ_t. To analyze the stability and bifurcations of the system, it is common to focus on the deterministic part, often referred to as the "deterministic skeleton". This deterministic skeleton is obtained by setting the white noise term to zero in the original system.

2.3. Analytical results

In this section, we delve into analytical results related to the stability and bifurcations of trajectories. The presented theorems and their corresponding proofs are based on the work of Hommes et al. (2005).

The deterministic skeleton of the system is a 2-dimensional discrete dynamical model:

$$\mathcal{F} : \begin{pmatrix} x_t \\ m_t \end{pmatrix} \to \begin{pmatrix} x_{t+1} \\ m_{t+1} \end{pmatrix} = \begin{pmatrix} F_1(x_t, m_t) \\ F_2(x_t, m_t) \end{pmatrix}$$

where $F_1(x,m) = v(m)x$ with $v(m) = (1 - \frac{\mu R}{a\sigma^2} + \frac{\mu g}{2a\sigma^2}) - \frac{\mu g}{2a\sigma^2}m$
and $F_2(x,m) = \alpha m + (1-\alpha)tanh[\beta w(x.m)]$ with $w(x,m) = \frac{1}{2}(\frac{g}{a\sigma^2}(R - v(m))x^2 - gz_s x - C)$

For simplicity and without loss of generality, we exclude the case where $\alpha = 1$ (investors never change strategy), set $a\sigma^2$ equal to 1 for normalization [8], and consider R in the range (1, 1.2) (corresponding to $0 < r_f < 0.2$, as it makes no financial sense to have a risk-free interest rate higher than 0.2).

[7] A white noise (WN) is a random variable with a zero expected value. It is commonly used to model the random interference or variability observed in time series data.

[8] In the general case, we can apply the following transformations: $\frac{\mu}{a\sigma^2} \to \bar{\mu}$; $\frac{\beta}{a\sigma^2} \to \bar{\beta}$; $a\sigma^2 z_s \to \bar{z}_s$; $a\sigma^2 C \to \bar{C}$. After these transformations, we have $a\sigma^2$=1.

The equilibrium of the system must be found in that (possibly unique) couple of values $E = (\bar{x}, \bar{m})$ satisfying $\mathcal{F}(\bar{x}, \bar{m}) = (F_1(\bar{x}, \bar{m}), F_2(\bar{x}, \bar{m})) = (\bar{x}, \bar{m})$, leading to a nonlinear system of two equations in two unknowns. From the first equation $F_1(\bar{x}, \bar{m}) = \bar{x}$ one gets $\bar{x} = 0$, namely $p = p^*$: at the equilibrium, the price of the asset corresponds to its fundamental value. For this reason E is known as "fundamental steady state". Imposing $F_2(\bar{x}, \bar{m}) = \bar{m}$, knowing that $\bar{x} = 0$, it is possible to find the value at equilibrium of m, $\bar{m} = -tanh(\frac{\beta C}{2})$. Therefore $E = (0, -tanh(\frac{\beta C}{2}))$

In the preceding section, we assumed that $C{>}0$ (setting C_2 to 0 without loss of generality), indicating that being a fundamentalist requires a higher cost than being a trend follower. It is evident that when $\beta{=}0$, resulting in agents being indifferent to which strategy is better, $E = (0, 0)$ (both fractions are the same and equal to $\frac{1}{2}$). On the other hand, as β approaches infinity, E tends to $(0, -1)$, signifying that all traders lean towards becoming trend followers.

A logical explanation underlies this phenomenon: when $p_t = p^*$, both types of investors forecast the same value for p_{t+1}, leading to $E(p_{t+1}) = p^*$. Consequently, there is no incentive to be a fundamentalist and incur a higher cost for gathering information. As β increases (approaching infinity in the extreme case), all agents swiftly tend to adopt the trend-following strategy.

After computing the fundamental steady state E, the next step involves studying its stability first with $z_s = 0$. This scenario corresponds to a situation where there is no supply of outside shares, akin to the analysis in Brock and Hommes (1997).

Theorem 2.1. *(Stability and bifurcations of fundamental steady state for $z_s = 0$)*
Let $z_s = 0$. Let $\beta_{pd} = \frac{2}{C}tanh^{-1}(\frac{2R}{g} - 1 - \frac{4}{\mu g})$, $\beta^* = \frac{2}{C}tanh^{-1}(\frac{2R}{g} - 1)$, $E = (0, -tanh(\frac{\beta C}{2}))$ be the fundamental steady state. Then

1. *For $0 < g < R$:*
    - $0 < \mu < \frac{2}{R}$*: E is globally stable for all $\beta \geq 0$*
    - $\frac{2}{R} < \mu < \frac{4}{2R-g}$*: E is locally but not globally stable for all $\beta \geq 0$ (for $\mu = \frac{2}{R}$ a 2-cycle emerges at infinity)*
    - $\frac{4}{2R-g} < \mu < \frac{2}{R-g}$*: E is unstable for $0 \leq \beta < \beta_{pd}$ and locally stable for $\beta > \beta_{pd}$; E undergoes a (subcritical) period-doubling bifurcation [9] at $\beta = \beta_{pd}$*

[9]This bifurcation occurs when a small change in the parameters leads to the emergence of a new periodic trajectory from an existing one, with the period doubling compared to the original trajectory.

- *$\mu > \frac{2}{R-g}$: E is unstable for any $\beta \geq 0$*

2. For $R < g < 2R$:

- *$0 < \mu < \frac{2}{R}$: E is globally stable for $0 \leq \beta < \beta^*$ and unstable for $\beta > \beta^*$; E undergoes a (supercritical) pitchfork bifurcation [10] at $\beta = \beta^*$*
- *$\frac{2}{R} < \mu < \frac{4}{2R-g}$: E is locally stable for $0 < \beta < \beta^*$ and unstable for $\beta > \beta^*$; E undergoes a (supercritical) pitchfork bifurcation at $\beta = \beta^*$*
- *$\mu > \frac{4}{2R-g}$: E is unstable for $0 \leq \beta < \beta_{pd}$, locally stable for $\beta_{pd} < \beta < \beta^*$ and unstable for $\beta > \beta^*$; E undergoes a (subcritical) period-doubling bifurcation in $\beta = \beta_{pd}$ and a (supercritical) pitchfork bifurcation in $\beta = \beta^*$*

3. For $g > 2R$: E is unstable for any $\mu > 0$ and for any $\beta \geq 0$

The proof is provided in Appendix B., along with the proofs for all the theorems presented in this work.

The length of the statement and the consideration of multiple cases are not surprising, given that the dynamics depend on three parameters: g, μ and β. The theorem distinguishes between three cases:

1. $0 < g < R$ (weak trend extrapolation);
2. $R < g < 2R$ (strong trend extrapolation);
3. $g > 2R$ (very strong trend extrapolation);

In the case of weak trend extrapolation, E is globally stable for a small adjustment speed ($0<\mu<\frac{2}{R}$), while it is globally unstable when $\mu > \frac{2}{R}$ (large adjustment speed), leading to the emergence of an unstable 2-cycle. This phenomenon is influenced by the adjustment mechanism, where large values of the speed of adjustment result in an extreme form of overshooting [11], causing prices to fluctuate significantly. As overshooting dynamics are not of interest, the case $\mu > \frac{2}{R}$ will be disregarded.

For a more comprehensive treatment, refer to Appendix A.

[10] This bifurcation occurs when, after a small change in the parameters, the system transitions from having one equilibrium to three distinct equilibria. For a more comprehensive treatment, refer to Appendix A.

[11] The term "overshooting" refers to the phenomenon in which economic variables, such as prices and rates, temporarily exceed their long-run values as a result of the adjustment process following an exogenous shock. The overshooting model was initially introduced by the German economist Rüdiger Dornbusch in the 1970s to explain the significant fluctuations observed in the dynamics of exchange rates.

In the presence of very strong trend extrapolation, E is always unstable for any choice of μ and β. The case $R < g < 2R$ is particularly interesting, as through a pitchfork bifurcation E becomes unstable, and two non fundamental steady states can emerge.

Theorem 2.2. *(Stability and bifurcation of non fundamental steady states for $z_s = 0$)* *Let $z_s = 0$. For $R < g < 2R$ and $0 < \mu < \frac{2}{R}$, let $x^* = \sqrt{\frac{C-\frac{2}{\beta}tanh^{-1}(\frac{2R}{g}-1)}{g(R-1)}}$, $\beta_{NS} = \frac{2}{C}tanh^{-1}(\frac{2R}{g} - 1) + \frac{g(R-1)}{\mu R(g-R)(2R-1)C}$. Then:*

1. *For $0 \leq \beta < \beta^*$ the fundamental steady state E is the unique steady state of the system*
2. *For $\beta > \beta^*$ two non fundamental steady states exist: $E_l = (x_l, 1-\frac{2R}{g})$ with $x_l = -x^*$ and $E_r = (x_r, 1-\frac{2R}{g})$ with $x_r = x^*$. Moreover:*
   - *E_l and E_r are locally stable for $\beta^* < \beta < \beta_{NS}$ and unstable for $\beta > \beta_{NS}$*
   - *Both E_l and E_r undergo a supercritical Neimark-Sacker bifurcation [12] in $\beta = \beta_{NS}$ for $\alpha < 0.6$*

Following their creation through a supercritical pitchfork bifurcation, the two non fundamental steady states are initially locally stable. However, they become unstable through a supercritical Neimark-Sacker bifurcation as the parameter β increases.

Up to this point, z_s has been set equal to 0 to make the system symmetrical with respect to the x-axis. Consequently, E_l and E_r are symmetrical and emerge for the same value of $\beta = \beta_{NS}$. However, when $z_s > 0$, the system is no longer symmetrical. Therefore, z_s is referred to as the "symmetry-breaking parameter" of the system. In the following theorem, this case is discussed, with the additional assumption of strong trend extrapolation.

Theorem 2.3. *(Stability and bifurcations of fundamental/non fundamental steady states for $z_s > 0$)* *Let $z_s > 0$. For $R < g < 2R$ and $0 < \mu < \frac{2}{R}$, let $\beta_{sn} = \frac{8tanh^{-1}(\frac{2R}{g}-1)(R-1)}{4(R-1)C+gz_s^2}$, $x_l = \frac{z_s}{2(R-1)} - \sqrt{\frac{C-\frac{2}{\beta}tanh^{-1}(\frac{2R}{g}-1)}{g(R-1)} + \frac{z_s^2}{4(R-1)^2}}$ and $x_r = \frac{z_s}{2(R-1)} + \sqrt{\frac{C-\frac{2}{\beta}tanh^{-1}(\frac{2R}{g}-1)}{g(R-1)} + \frac{z_s^2}{4(R-1)^2}}$*

[12] Bifurcation consisting in the origin of a closed invariant curve from a fixed point, when the fixed point changes stability due to the presence of a pair of complex eigenvalues with unit modulus. See Appendix A. for a more exhaustive treatment.

Then:

1. $0 \leq \beta < \beta_{sn}$*: the fundamental steady state* E *is globally stable*
2. $\beta = \beta_{sn}$*: a saddle-node bifurcation* [13] *occurs and it exists one and only one non fundamental steady state*
3. $\beta_{sn} < \beta < \beta^*$*:* E *is locally stable and there exist two non fundamental steady states* $E_l = (x_l, 1 - \frac{2R}{g})$ *and* $E_r = (x_r, 1 - \frac{2R}{g})$ *with* $0 < x_l < x_r$
4. $\beta = \beta^*$*: a trans-critical bifurcation* [14] *occurs and* E_l *coincides with* E
5. $\beta > \beta^*$*:* E *is unstable and there exist two non fundamental steady states* $E_l = (x_l, 1 - \frac{2R}{g})$ *and* $E_r = (x_r, 1 - \frac{2R}{g})$ *with* $x_l < 0 < x_r$*. Moreover, when* z_s *is positive but small, there exists* $\beta_r > \beta_{sn}$ *and* $\beta_l > \beta^*$ *such that*
6. E_r *is locally stable for* $\beta_{sn} < \beta < \beta_r$ *and unstable for* $\beta > \beta_r$*.* E_r *undergoes a supercritical Neimark-Sacker bifurcation at* $\beta = \beta_r$ *for* $\alpha < 0.6$
7. E_l *is unstable for* $\beta_{sn} < \beta < \beta^*$*, locally stable for* $\beta^* < \beta < \beta_l$ *and unstable for* $\beta > \beta_l$*.* E_l *undergoes a trans-critical bifurcation at* $\beta = \beta^*$*, for which* $x_l = 0$*, and a Neimark-Sacker bifurcation occurs in* $\beta = \beta_l$ *for* $\alpha < 0.6$

The introduction of z_s>0 in the system breaks the symmetry, leading to non-coinciding values for β_l and β_r (specifically, β_r<β_l). This asymmetry results in the occurrence of the Neimark-Sacker bifurcations for E_l and E_r not happening simultaneously. For detailed expressions of β_l and β_r, refer to Appendix B.

In the context of strong trend extrapolation ($g > R$), where trend followers anticipate prices to rise faster than the risk-free rate, the fundamental steady state E becomes unstable for higher values of β.

An additional significant constraint on the permissible values of g is $g < R^2$ (noting that $R^2 < 2R$ for the given R values). This condition is crucial as it ensures that trajectories remain bounded. Without this restriction, for higher g values, trend followers could generate substantial profits, leading to the asset price diverging to infinity.

[13]This expression, also referred to as tangential or fold or blue sky bifurcation, denotes a scenario in bifurcation theory where two equilibria, one stable and the other unstable, collide and both vanish. For a more comprehensive exploration of this concept, please refer to the Appendix A.

[14]This bifurcation, also known as fold bifurcation, represents a scenario in bifurcation theory where two equilibria, one stable and the other unstable, collide and exchange their stability. Following the collision, the initially stable equilibrium becomes unstable, and vice versa. For a more detailed exploration of this concept, refer to Appendix A.

As highlighted in Hommes et al. (2005), it is conceivable to expand the model to accommodate larger g values without compromising trajectory boundedness. This extension might involve incorporating a stabilizing mechanism to prevent prices from diverging to infinity. However, such modifications would introduce additional complexity to the model, making the analysis much more challenging, if not impractical.

Upon assuming $R < g < R^2$, the system, for certain values of the intensity of choice, may undergo a homoclinic bifurcation [15].

Theorem 2.4. *(Homoclinic bifurcation of fundamental steady state for $z_s = 0$)*
Let $z_s = 0$, $R < g < R^2$, $\frac{1}{R} < \mu < min\{\frac{2}{R}, \frac{R-1}{g-R}\}$, $0 < \alpha < \sqrt{\frac{\mu R-1}{\mu g}}$. Then the fundamental steady state E is a dissipative saddle point [16] for any $\beta > \beta^*$ (after the pitchfork bifurcation). Moreover:

1. *for $\beta > \beta^*$ with $|\beta - \beta^*|$ small, there is no intersection between $W^s(E)$ and $W^u(E)$[17] (the unstable manifold of E is attracted by E_l and E_r).*
2. *for some $\beta_h > \beta^*$ a homoclinic bifurcation between the stable and unstable manifold of E occurs.*
3. *for $\beta > \beta_h$ transversal homoclinic orbits [18] of E always exist.*

The presence of homoclinic orbits implies the existence of strange (or chaotic) attractors[19]. The following theorem can also be seen as a corollary of the previous one.

Theorem 2.5. *(Existence of strange attractors)*
Let the parameters be fixed as in the previous theorem, then it exists a positive Lebesgue measure [20] set of values for β in the neighborhood of β_h, say $(\beta_h - \epsilon, \beta_h + \epsilon)$, for which the dynamics generated by $\mathcal{F}$ admits a strange attractor.

[15] It refers to a scenario in which a limit cycle collides with a saddle point, resulting in an infinite period cycle. For a more comprehensive treatment, refer to Appendix A.

[16] Equilibrium point that attracts certain initial conditions while repelling others. In other words, it can act as both an attractor and a repellor, depending on the initial conditions. For a more detailed treatment, refer to Appendix A.

[17] The stable manifold W^s comprises the set of initial conditions for which trajectories tend toward the equilibrium, while the unstable manifold W^u is the set of initial conditions for which the equilibrium serves as a repellor for trajectories. For further details, refer to Appendix A.

[18] It a trajectory that connects a saddle point to itself, representing the intersection between the stable and unstable manifolds of the saddle equilibrium. For further details, refer to Appendix A.

[19] These attractors are termed "strange" due to their irregular shape, displaying self-similar patterns and non integer dimensions, characteristics common to fractal sets. The presence of homoclinic orbits contributes to the emergence of chaotic trajectories. For a more comprehensive discussion, refer to Appendix A.

[20] Regarding the set of real numbers $\mathbb{R}$, the Lebesgue measure of a set corresponds to its length.

Finally, for some sufficiently large values of the intensity of choice chaotic orbits [21] appear.

Also the theorem below can be seen a consequence of the theorem 2.4.

Theorem 2.6. *(Existence of chaotic orbits)*
Let the parameters be fixed as in theorem 2.4. , if β is sufficiently large then it exists an invariant Cantor set [22] containing an uncountable set [23] of initial conditions for which chaotic orbits arise.

In the next section, numerical simulations of the ARED model will be presented along with visual representations of the equilibria as a function of β. From a financial perspective, the focus will be on analyzing the most interesting cases, considering various combinations of the parameters R, g, μ, and β.

2.4. Numerical simulations

This section of Chapter 2. is dedicated to the numerical simulations of the ARED model across various values of the intensity of choice parameter. The aim is to illustrate how the dynamics significantly depend on β and how complex and intricate behaviors can emerge from a system governed by "simple" rules.

As previously mentioned in the preceding section, only specific values of R, g, μ, and α will be considered, focusing on the most significant ones from both financial and numerical perspectives. Let's briefly discuss each selection:

- $R = 1.1$: it is reasonable to assume r_f very close to 0 when dealing with daily risk-free interest rates. However, R must be greater than 1 otherwise, the fundamental price of the asset p^* wouldn't be well defined as it needs to be finite and positive;
- $g = 1.15$: this configuration ensures that the trend extrapolator satisfies both the condition $R < g < 2R$ (strong trend extrapolator), leading to the most interesting cases shown in Theorems 2.1. 2.2. and 2.3. and $R < g < R^2$, a sufficient condition for Theorems 2.4. 2.5. and 2.6. (existence of chaotic orbits and strange attractors);

[21] Orbits of the system exhibiting strong dependence on the initial conditions. For a more comprehensive discussion, refer to Appendix A.

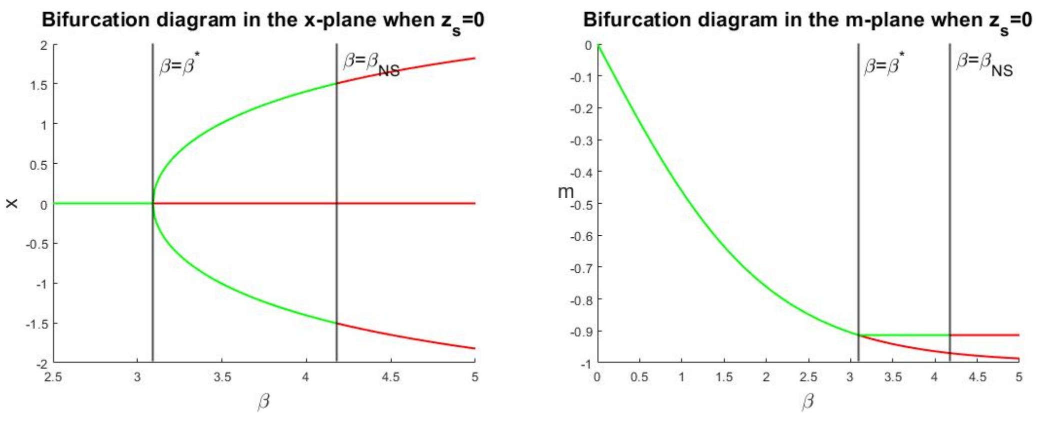
[22] One of the most popular examples of a fractal set, it represents a perfect set that is nowhere dense. For a more comprehensive discussion, refer to Appendix A.

[23] There are various definitions of an uncountable set. One of them states that a set X is (infinite) uncountable if its cardinality is neither finite nor equal to $\aleph_0$, where $\aleph_0$ represents the cardinality of the set of natural numbers—a classical example of a countable infinite set. In simpler terms, an uncountable set contains too many elements to be countable.

- $\mu = 1.6$: It is an appropriate value for the speed of adjustment since it is contained in the range $(\frac{1}{R}, \min\{\frac{2}{R}, \frac{R-1}{g-R}\})$. This choice is manageable and ensures the presence of a chaotic regime due to the existence of homoclinic orbits, as stated in Theorems 2.4. 2.5. and 2.6. ;
- $\alpha = 0.5$: this choice of α ensures that 50% of agents (eventually) change their strategy according to the adaptive mechanism. It also satisfies the inequalities $0 < \alpha < \sqrt{\frac{\mu R-1}{\mu g}}$ necessary for Theorems 2.4. 2.5. and 2.6. to hold. It is noteworthy that α is an eigenvalue of the Jacobian matrix computed in the fundamental steady state E (see the proofs of the theorems in Appendix B.), representing the speed of convergence when the price moves toward the fundamental value;
- $C = 1$: without loss of generality C_1 can be set equal to 1 and C_2 equal to 0, simplifying the computations;
- $a\sigma^2 = 1$: as previously discussed, this term can be simplified without loss of generality;
- $z_s = 0.01$: when it is not null (otherwise the symmetric case is considered), the outside supply of shares per trader must be positive and "small".

The symmetric ($z_s = 0$) and the asymmetric case ($z_s > 0$) will be discussed separately.

2.4.1. Symmetric case ($z_s = 0$)



(a) Bifurcation diagram in the x-plane (b) Bifurcation diagram in the m-plane

Figure 2.1: Steady states and their stability (green stable, red unstable) depending on β

When $z_s = 0$, with the other parameters held constant, E undergoes a pitchfork bifurcation, occurring approximately at $\beta^* = 3.09104$. This bifurcation leads to the emergence of two non fundamental steady states, denoted as E_l and E_r.

Subsequently, both E_l and E_r experience a Neimark-Sacker bifurcation at the same critical value of β, approximately $\beta_{NS} = 4.18006$.

The fundamental steady state $E = (0, -\tanh(\frac{\beta C}{2}))$ is globally stable until the pitchfork bifurcation takes place. Following the bifurcation, the two non fundamental steady states, $E_l = (-x, 1-\frac{2R}{g})$ and $E_r = (x, 1-\frac{2R}{g})$, are locally stable until the Neimark-Sacker bifurcation occurs.

	$\beta < \beta^*$	$\beta = \beta^*$	$\beta^* < \beta < \beta_{NS}$	$\beta = \beta_{NS}$	$\beta > \beta_{NS}$
E	globally stable	pitchfork	unstable	unstable	unstable
E_l	$\times$	pitchfork	locally stable	Neimark-Sacker	unstable
E_r	$\times$	pitchfork	locally stable	Neimark-Sacker	unstable

Table 2.1: Stability of the steady states and bifurcations as β increases.

It is evident that chaotic dynamics may emerge when β is large and all steady states (both fundamental and non fundamental) are unstable. Additionally, Theorems 2.4. 2.5. and 2.6. guarantee the presence of strange attractors, and hence chaotic orbits, for a certain value of the intensity of choice $\beta_h > \beta^*$. This occurs through a homoclinic bifurcation involving the stable and unstable manifolds of E.

In the following, examples for each possible scenario are provided. But before delving into that, let's briefly discuss how the adaptive system functions in practice, considering the following example.

Consider β=2.5 and the initial state (2, 0.8): a scenario where the initial price is overvalued, and 90% of the investors are fundamentalists. In contrast to the (sole, as $\beta < \beta^*$) fundamental steady state E=(0,$-\tanh(\frac{\beta C}{2})$), approximately (0,-0.8483), the initial state represents an "extreme" situation.

At time t=0, the initial price $p_0 = p^* + x_0$ is revealed to all investors. As the price surpasses its fundamental value, fundamentalists expect a decline in the next period:

$$\mathbb{E}_{1,0}(p_1) = p^* < p^* + x_0$$

while trend followers believe that it will rise again:

$$\mathbb{E}_{2,0}(p_1) = p^* + g(p_0 - p^*) = p^* + gx_0 > p^* + x_0$$

So the demand for the risky asset will be:

$$z_{1,0} = p^* + \bar{y} - R(p^* + x_0) = p^*(1 - R) + \bar{y} - Rx_0 = -Rx_0 = -2.2$$

for fundamentalists and

$$z_{2,0} = p^* + gx_0 + \bar{y} - R(p^* + x_0) = p^*(1 - R) + \bar{y} + (g - R)x_0 = (g - R)x_0 = 0.1$$

for trend followers.

We recall the fact that, by definition, when $z_s = 0$ it follows $p^*(1 - R) + \bar{y}$=0.

Notice that $z_{1,0} < 0$, indicating that fundamentalists will sell the risky asset as they anticipate its future value to decrease, while $z_{2,0} > 0$ reflects the interest of trend followers in buying the asset.

Given the fractions of fundamentalists $n_{1,0} = 0.9$ and trend followers $n_{2,0} = 0.1$, it is evident that the aggregate excess demand for the current period t=0 is negative:

$$z_{e,0} = n_{1,0}z_{1,0} + n_{2,0}z_{2,0} = 0.9 * (-2.2) + 0.1 * 0.1 = -1.97$$

Hence, due to the adjustment mechanism:

$$p_1 = p_0 + \mu z_{e,0} = p^* + x_0 + \mu z_{e,0} = p^* + 2 + 1.6 * (-1.97) = p^* - 1.152$$

Not only will the price in t=1 be lower than the initial price, but it will also fall below the fundamental value. It is essential to note that this substantial drop is a consequence of fundamentalists representing the majority (90%) of investors: the positive demand from trend followers (10% of traders) cannot offset the massive negative demand from fundamentalists, resulting in a significant price decline.

In this scenario, fundamentalists were correct: the price has indeed decreased.

Now, considering the performance functions for both fundamentalists and trend followers:

$$\begin{cases} \Pi_{1,0} = U_{1,0} - C_1 = (p_1 - Rp_0)z_{1,0} - 1 = (-1.152 - 1.1 * 2) * (-2.2) - 1 = 6.3744 \\ \Pi_{2,0} = U_{2,0} - C_2 = (p_1 - Rp_0)z_{2,0} = -0.3352 \end{cases}$$

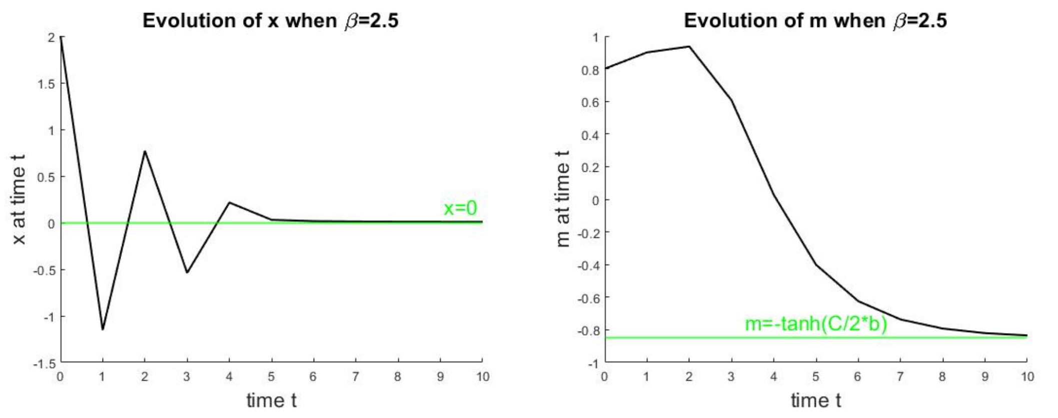
The fraction of fundamentalists at time t=1 will be higher than in the previous period, as determined by the formula:

$$\begin{cases} n_{1,1} = \alpha n_{1,0} + (1-\alpha)\frac{e^{\beta \Pi_{1,0}}}{\sum_{h=1}^{2} e^{\beta \Pi_{h,0}}} = 0.5 * 0.9 + 0.5 * \frac{e^{2.5*6.3744}}{e^{2.5*6.3744} + e^{2.5*(-0.3352)}} = 0.9499 \\ n_{2,1} = 1 - n_{1,1} = 0.0501 \end{cases}$$

As a consequence, m at time t=1 will increase:

$$m_1 = n_{1,1} - n_{2,1} = 0.9499 - 0.0501 = 0.8998$$

Therefore, the map $\mathcal{F}$ transforms the initial state (2, 0.8) into (-1.152, 0.8998). This procedure is repeated at each step, converging to the fundamental steady state E



(a) Evolution of x

(b) Evolution of m

Figure 2.2: Dynamics in the symmetric case when β=2.5 with initial state (2, 0.9)

• $\beta < \beta^*$: This scenario, as illustrated in the previous example, is the simplest possible. The fundamental steady state E is the unique equilibrium of the system, and it is globally stable. Investors exhibit a reluctance to change their beliefs, causing the price to fluctuate around the fundamental value with decreasing amplitude over time. As the deviation x approaches 0,as previously discussed in this chapter, there's no advantage in being a fundamentalist and incurring the cost C for gathering information. The difference in fractions, m, tends to the equilibrium value $-\tanh(\frac{\beta C}{2})$, which is negative for any $\beta > 0$ (indicating that there are more trend followers than fundamentalists).

• $\beta = \beta^*$: The dynamics in this scenario are equivalent to the previous one. The fundamental steady state has an associated eigenvalue with a unit modulus, coinciding with the two non-fundamental steady states E_r and E_l, both of which are stable.

• $\beta^* < \beta < \beta_{NS}$: The system exhibits three distinct equilibria: the unstable fundamental steady state E and the stable non fundamental steady states E_r and E_l. The presence of multiple equilibria is a key aspect of ARED models, making them of continued interest. E is a saddle node with one associated eigenvalue ($\lambda_2 = \alpha$) having modulus strictly less than one, while the other ($\lambda_1 = v(-\tanh(\frac{\beta C}{2}))$, see Appendix B.) is unstable. The stable manifold $W^s(E)$, representing the states attracted to E, is defined by the lines $x = 0$ and $m = 1 - \frac{2R}{g} + \frac{2}{\mu g}$. The system's behavior is explained by a financial perspective: when the price is at its fundamental, both types of investors predict the same future price, leading to no demand for the risky asset. As time passes, the fraction of trend followers increases, approaching the equilibrium $m = -\tanh(\frac{\beta C}{2})$. The stable manifolds $W^s(E_r)$ and $W^s(E_l)$ exhibit complex geometries, contributing to high sensitivity to initial conditions for $\beta > \beta^*$. For instance, a neighborhood of $(0, 1 - \frac{2R}{g} + \frac{2}{\mu g})$ contains initial states attracted by E, E_r, or E_l.

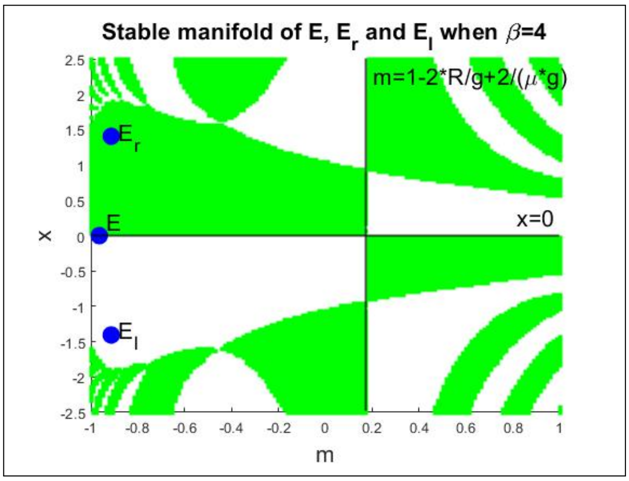
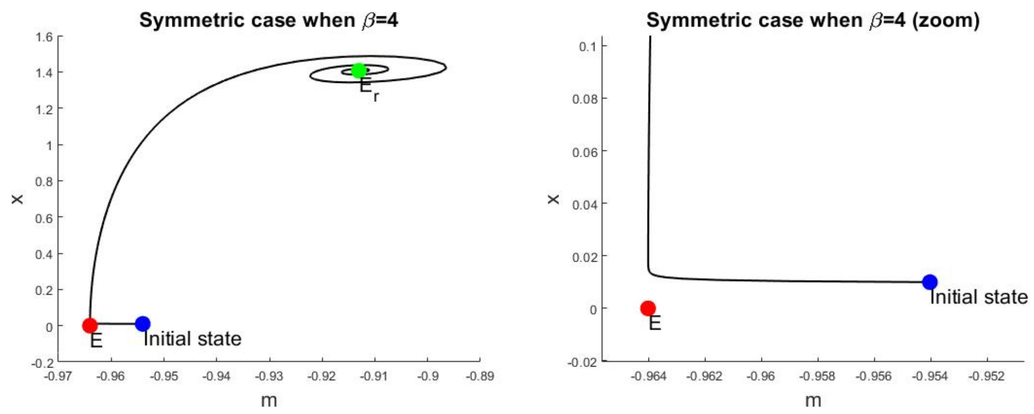


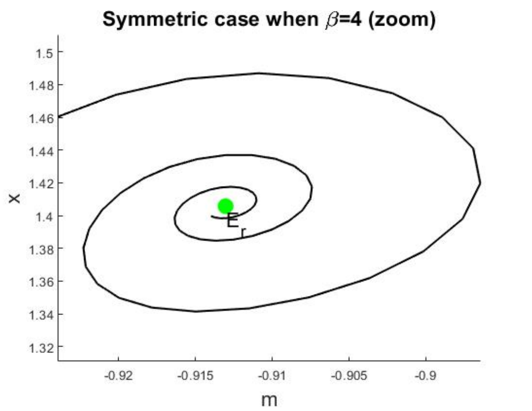
Figure 2.3: Stable manifold of E (lines $x = 0$ and $m = 1 - \frac{2R}{g} + \frac{2}{\mu g}$), E_r (green region) and E_r (white region) when β=4. The manifolds are symmetric with respect to the x-axis

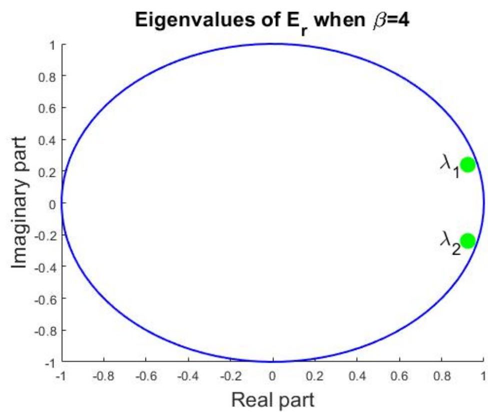
Considering an initial state in a neighborhood of E with $x_0 \neq 0$, it is initially attracted by the fundamental steady state. The projection of the state onto the stable manifold $W^s(E)$ is predominant in the early stages. However, as time progresses, the state will eventually be drawn towards either E_r when $x_0 > 0$ or E_l when $x_0 < 0$. In these cases, the projection onto $W^s(E_r)$ or $W^s(E_l)$ becomes predominant. The spiral trend toward E_r (or E_l) is a result of the complex conjugate eigenvalues associated with E_r (or E_l), as detailed in Appendix B.



(a) Orbit converging to E_r (b) Orbit converging to E_r (zoom)

Figure 2.4: Orbit converging to E_r starting from an initial state close to E when β=4





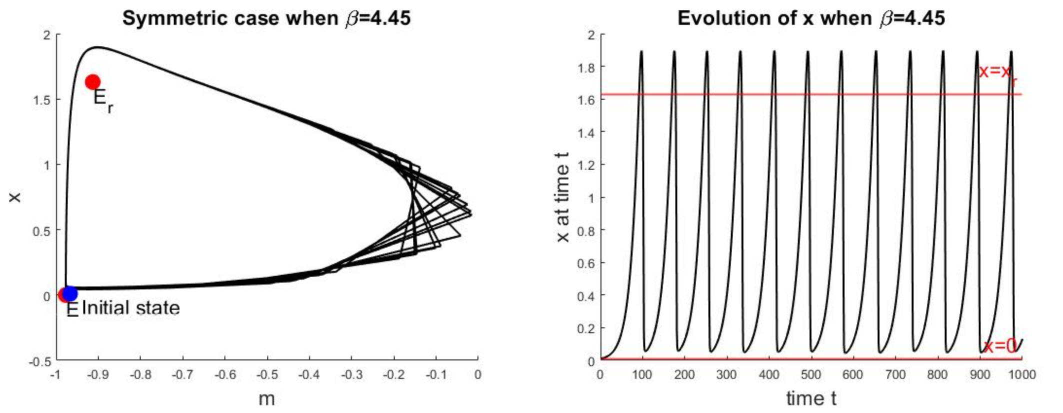
(a) Orbit converging to E_r (zoom) when β=4

(b) Eigenvalues of E_r when β=4

Figure 2.5: Spiral trend toward E_r when β=4

- $\beta = \beta_{NS}$: both non-fundamental steady states undergo a Neimark-Sacker bifurcation since their eigenvalues (complex conjugates) have unit modulus. However, they remain (locally) stable because the Neimark-Sacker bifurcation is supercritical (see Appendix A.). This means that the convergence to either E_r or E_l is slower than in the previous case where $\beta < \beta_{NS}$. The stable manifold of E is still represented by the lines $x = 0$ and $m = 1 - \frac{2R}{g} + \frac{2}{\mu g}$ (and this remains true for any value of the intensity of the choice parameter), while the stable manifold of E_r and E_l does not show substantial differences compared to the previous case.

• $\beta > \beta_{NS}$: This case is particularly intriguing as it gives rise to complex and chaotic dynamics. Investors are highly likely to adopt the best-performing strategy, becoming anxious and impatient as the inclination to change their ideas increases. Chaotic orbits emerge because all the equilibria in the system are unstable, and strange attractors manifest due to homoclinic bifurcations when the stable and unstable manifolds of the steady states intersect. When $|\beta - \beta_{NS}|$ is sufficiently small, two strange attractors coexist: one encompasses the positive non-fundamental steady state E_r and is situated in the positive semi-plane $x>0$, while the other encompasses the negative non-fundamental steady state E_l and is situated in the negative semi-plane $x<0$. Orbits around the positive strange attractor represent an optimistic phase of the market, where prices consistently surpass the fundamental value. In contrast, a pessimistic phase is observed when prices persistently fall below the fundamental threshold. Both attractors are tangent to the stable manifold of E.



(a) Orbit converging to the positive strange attractor when β=4.45

(b) Evolution of x when the orbit is attracted to the positive strange attractor when β=4.45

Figure 2.6: Strange attractor on the positive semi-plane $x>0$ when β=4.45

Once again, the stable manifolds of the two strange attractors exhibit a non trivial geometry, and the dependence on initial conditions is notably high. The regimes are quasi periodic, signifying that the system will never pass through the same state twice. These attractors are characterized by non integer (fractal) dimensions; they form Cantor like sets consisting of uncountable sets of isolated points, meaning they have no length.

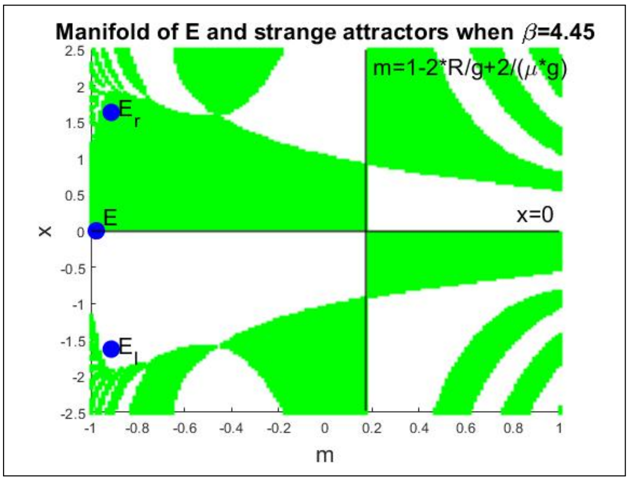
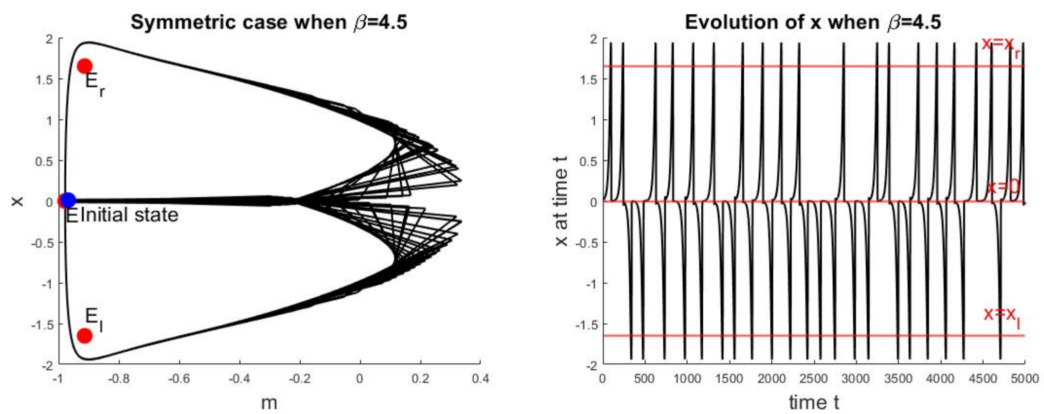


Figure 2.7: Manifold of E (lines $x = 0$ and $m = 1 - \frac{2R}{g} + \frac{2}{\mu g}$), the positive strange attractor (green region) and the negative one (white region) (β=4.45). The manifolds are symmetric

As β increases, a single strange attractor emerges, encompassing both E_r and E_l: an endogenous switch between optimistic and pessimistic phases occurs, leading to marked price fluctuations. The attractor remains tangent to the stable manifold of E and is almost globally stable (with the caveat that only orbits starting from $W^s(E)$ will not be attracted by this attractor).

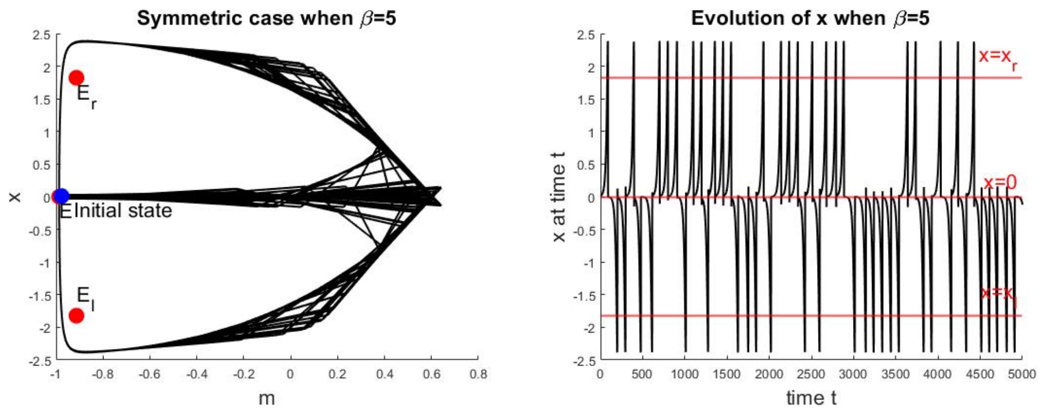


(a) Orbit converging to the unique strange attractor when β=4.5

(b) Evolution of x when β=4.5

Figure 2.8: Global strange attractor when β=4.5

These fluctuations arise because trend followers consistently act to perpetuate the current trend (whether upward or downward), while fundamentalists aim to guide the price toward its fundamental value. When trend followers dominate, the price rises (or falls) continuously until it reaches a peak, where the demand from fundamentalists becomes sufficient to pull the price back toward its fundamental. This mechanism becomes more pronounced as the intensity of choice increases, leading to larger price fluctuations.

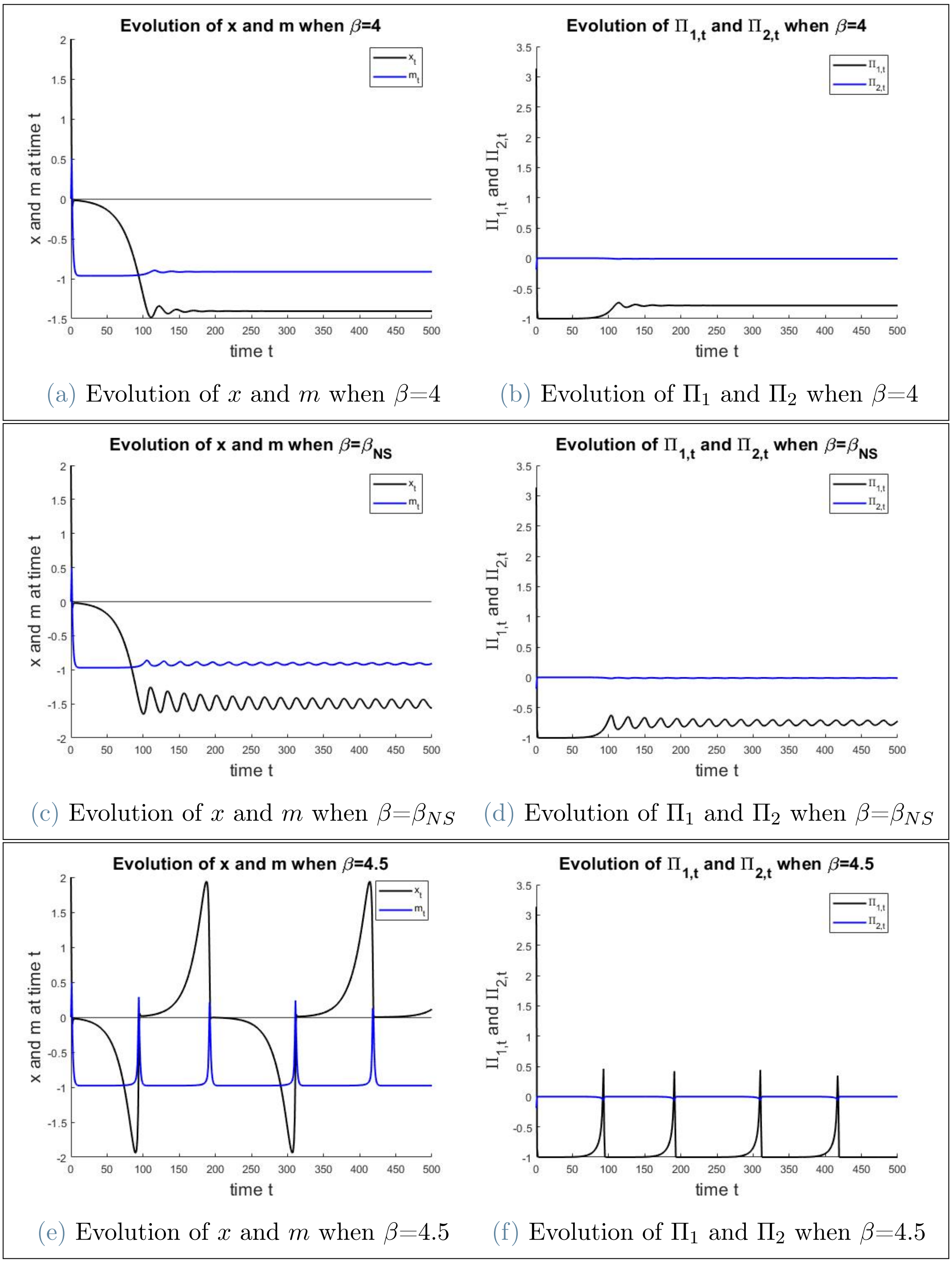


(a) Orbit converging to the unique strange attractor when β=5

(b) Evolution of x when β=5

Figure 2.9: Global strange attractor when β=5

In Figure 2.10 the initial state (2, 0) (indicating a price above the fundamental value and an equal distribution of investors) is considered in all simulations.



(a) Evolution of x and m when β=4 (b) Evolution of Π_1 and Π_2 when β=4

(c) Evolution of x and m when β=β_{NS} (d) Evolution of Π_1 and Π_2 when β=β_{NS}

(e) Evolution of x and m when β=4.5 (f) Evolution of Π_1 and Π_2 when β=4.5

Figure 2.10: Evolution of x (in black), m (in blue), Π_1 (in black) and Π_2 (in blue) for various values of β: β=4 (panel (a) and (b)); β=β_{NS} (panel (c) and (d)); β=4.5 (panel (e) and (f))

2.4.2. Asymmetric case ($z_s > 0$)

When $z_s > 0$, with the other parameters fixed, the two non-fundamental steady states E_l and E_r arise from a saddle-node bifurcation at β_{sn}=3.09015. In the asymmetric case, E_r undergoes a Neimark-Sacker bifurcation at β_r=4.11408, while E_l becomes unstable at β_l=4.24672.

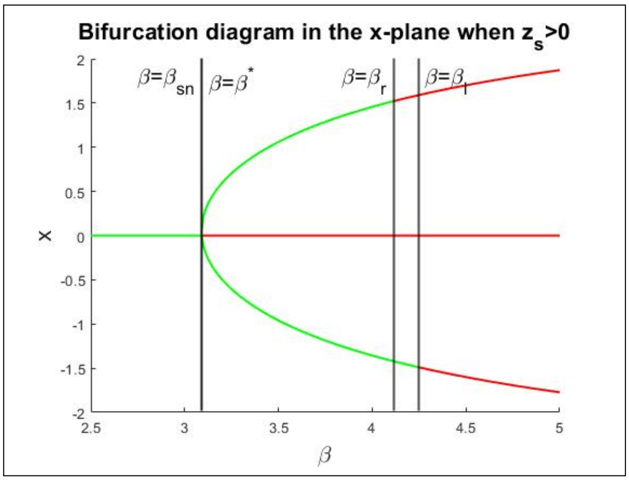
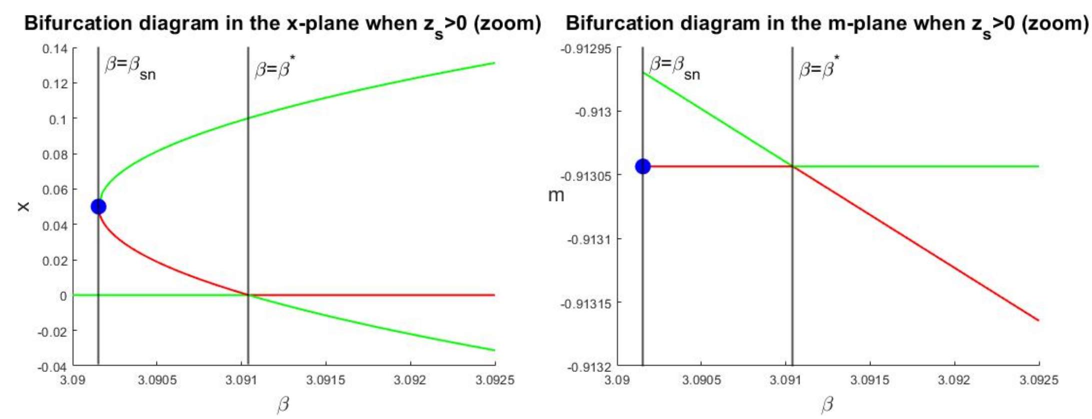


Figure 2.11: Bifurcation diagram in the x-plane

Notice how the dynamics of the system is highly sensitive to the value of the intensity of the choice parameter when it is in the vicinity of (approximately) 3.09. At β_{sn}=3.09015, E_r and E_l are generated, while at β^*=3.09104, E and E_l collide in a trans-critical bifurcation, subsequently changing their stability.



(a) Bifurcation diagram in the x-plane (b) Bifurcation diagram in the m-plane

Figure 2.12: Bifurcation diagram in a neighborhood of x=3.09

	$\beta < \beta_{sn}$	$\beta = \beta_{sn}$	$\beta_{sn} < \beta < \beta^*$	$\beta = \beta^*$	$\beta_* < \beta < \beta_r$
E	globally stable	locally stable	locally stable	trans-critical	unstable
E_l	$\times$	saddle-node	unstable	trans-critical	locally stable
E_r	$\times$	saddle-node	locally stable	locally stable	locally stable

	$\beta = \beta_r$	$\beta_r < \beta < \beta_l$	$\beta = \beta_l$	$\beta > \beta_l$
E	unstable	unstable	unstable	unstable
E_l	locally stable	locally stable	Neimark-Sacker	unstable
E_r	Neimark-sacker	unstable	unstable	unstable

Table 2.2: Stability of the steady states and bifurcations as β increases.

As in the symmetric case, strange attractors and chaotic dynamics around them will emerge for large values of the intensity of choice parameter, especially for $\beta > \beta_l$ when all the steady states are unstable, allowing for the possibility of homoclinic orbits.

• $\beta < \beta_r$: this case doesn't generate significant interest, as either the fundamental steady state or the non-fundamental steady states are stable, making the global dynamics of the system equivalent to the one presented in the symmetric case. The case where β is within the range (β_{sn}, β^*) is notable, as E and E_r are stable while E_l (located in the positive semi-plane $x > 0$) is unstable. The lines x=0 and $m = 1 - \frac{2R}{g} + \frac{2}{\mu g}$ continue to belong to the stable manifold of the fundamental steady state.

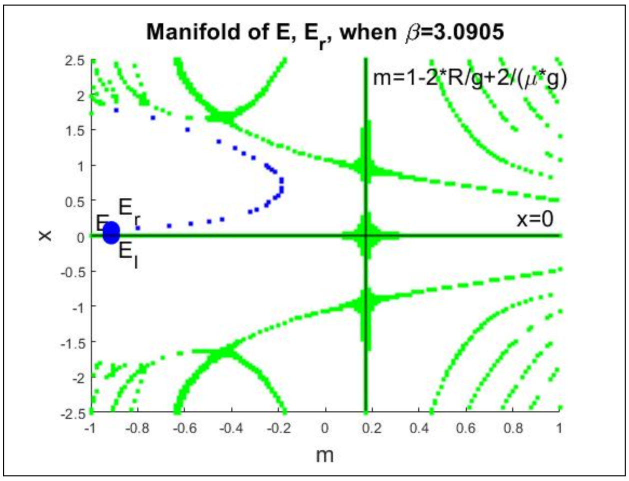
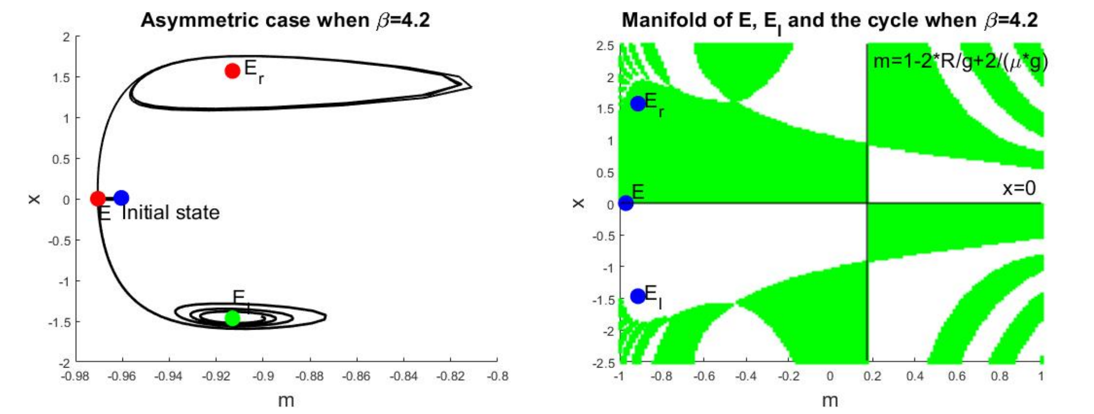


Figure 2.13: Manifold of E (green region) and E_r (blue region) when β=3.0905

• $\beta_r < \beta < \beta_l$: the positive non fundamental steady state E_r is unstable due to the Neimark-Sacker bifurcation occurring at β_r, while E_l remains stable. Consequently, the system exhibits three different types of attractors: the saddle node in E; the stable fixed point E_l; and the stable cycle around E_r.

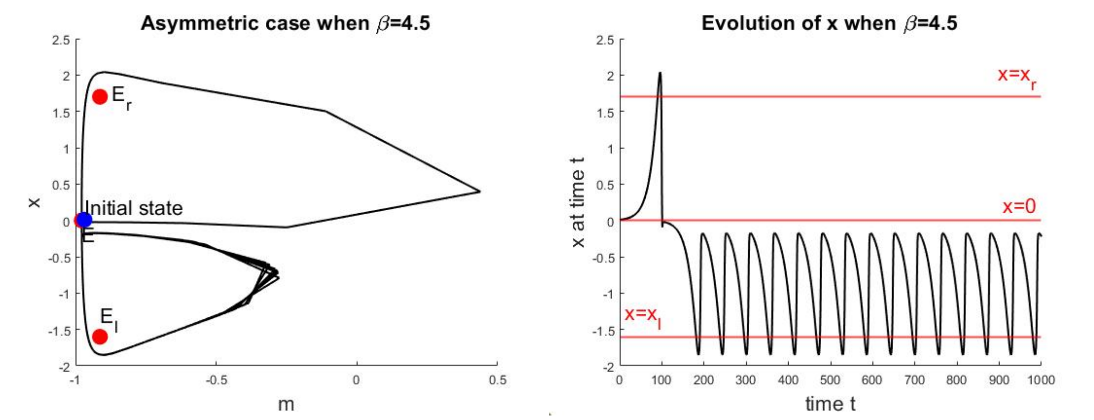


(a) Orbit converging to E_l and orbit converging to the stable cycle around E_r when β=4.2

(b) Manifold of E (lines x=0 and $m = 1 - \frac{2R}{g} + \frac{2}{\mu g}$), cycle around E_r (green region) and E_l (white region)

Figure 2.14: Dynamics and manifolds of the asymmetric system when β=4.2

- $\beta > \beta_l$: both E_r and E_l are unstable, allowing the system to potentially exhibit the presence of strange attractors and chaotic orbits around them. Up to (approximately) β=4.39288, a chaotic attractor encompasses the positive non-fundamental steady state E_r. Afterward, it disappears. For 4.39288<β<4.55327, there exists a unique strange attractor contained in the negative semi-plane $x < 0$. In this range, any initial state (not belonging to $W^s(E)$) will lead the market into a never-ending pessimistic phase characterized by significant price fluctuations below the fundamental threshold.

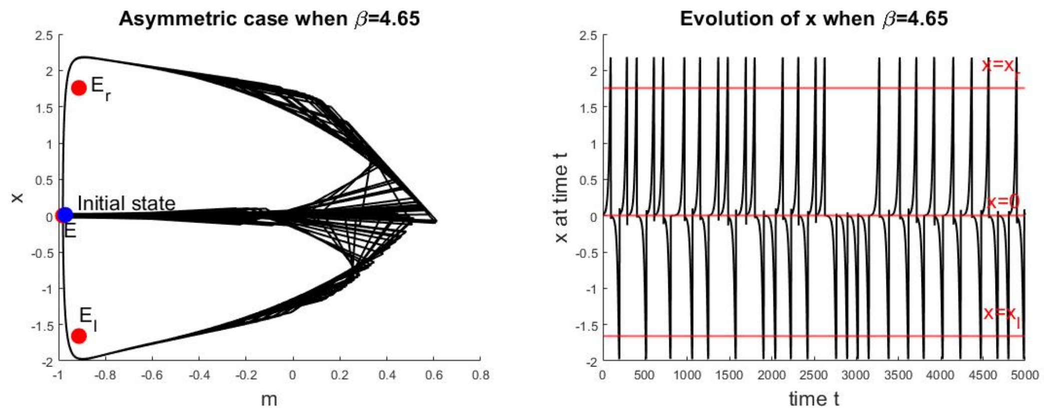


(a) Orbit converging to the strange attractor around E_l when β=4.5

(b) Evolution of x when β=4.5

Figure 2.15: Pessimistic phase when β=4.5

When $\beta > 4.55327$, the strange attractor encompasses both E_r and E_l, signifying an endogenous switch between optimistic (overvalued prices) and pessimistic (undervalued prices) phases.

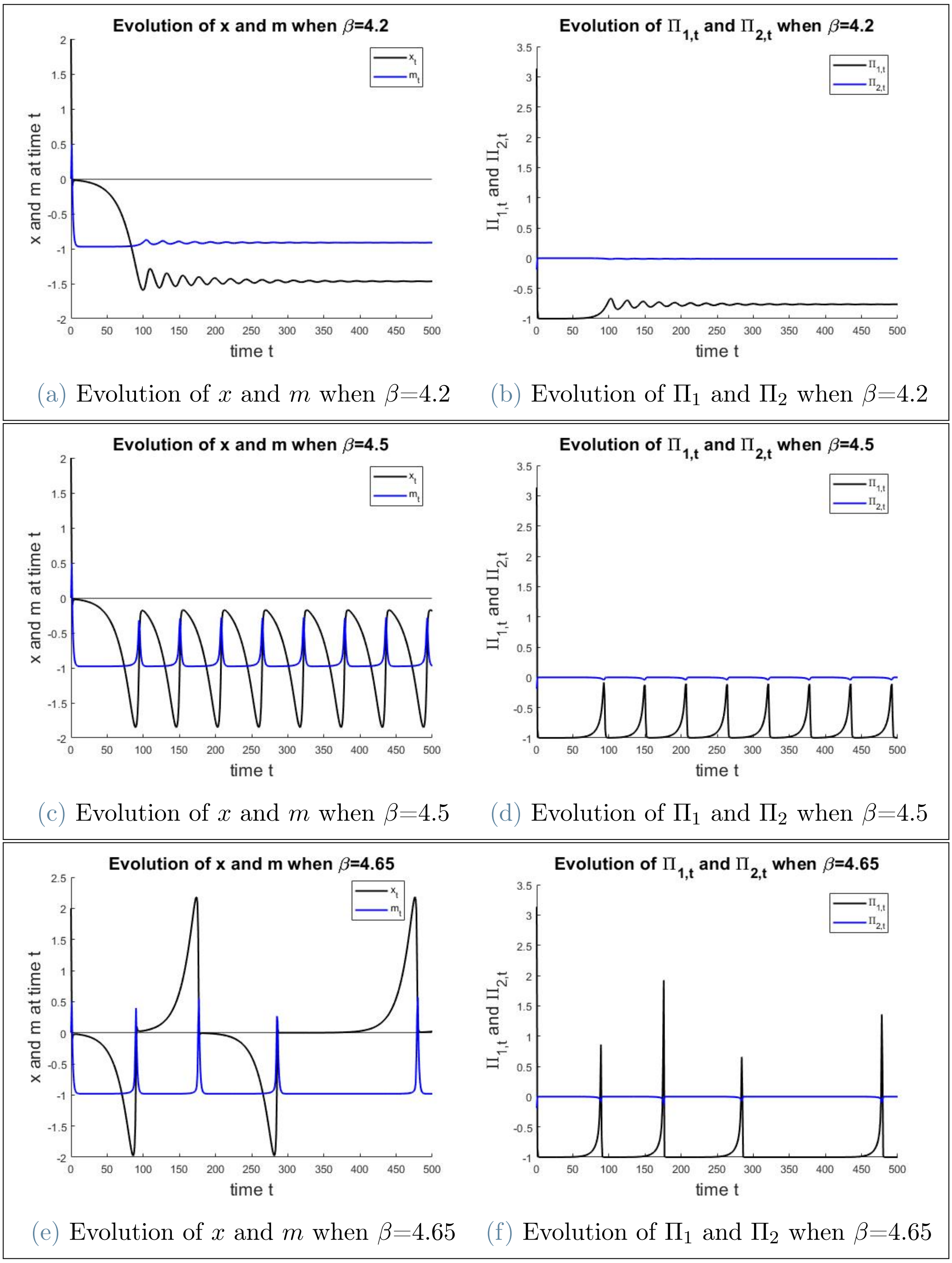


(a) Orbit converging to the strange attractor when β=4.65

(b) Evolution of x when β=4.65

Figure 2.16: Optimistic and pessimistic phases when β=4.65

In Figure 2.17 the initial state (2, 0) (indicating a price above the fundamental value and an equal distribution of investors) is considered in all the simulations.



(a) Evolution of x and m when β=4.2 (b) Evolution of Π_1 and Π_2 when β=4.2

(c) Evolution of x and m when β=4.5 (d) Evolution of Π_1 and Π_2 when β=4.5

(e) Evolution of x and m when β=4.65 (f) Evolution of Π_1 and Π_2 when β=4.65

Figure 2.17: Evolution of x (in black), m (in blue), Π_1 (in black) and Π_2 (in blue) for various values of β: β=4.2 (panel (a) and (b)); β=4.5 (panel (c) and (d)); β=4.65 (panel (e) and (f))

2.5. Further developments

Recent studies have focused attention on the distribution of quoted asset returns and prediction functions.

It is well-known that return distributions are not symmetrical, contrary to what is assumed in many models for asset dynamics, such as the Black & Scholes model [24]. Notably, negative returns are more common than positive ones, suggesting that 'bad news' is more prevalent than 'good news.' To replicate the skewness of return distributions, it is recommended to consider the semivariance [25] of returns instead of the variance in the strategy optimization problem. Consequently, investors should shift from being mean-variance maximizers to being mean-semivariance maximizers.

In his work 'Mean-semivariance behavior: Downside risk and capital asset pricing' (2005), Javier Estrada highlighted reasons why an optimization problem based on mean and semivariance is preferable to one based on mean and variance.

Apart from being more correct from a probabilistic and statistical perspective, considering semivariance allows us to capture the phenomenon in which traders, when going long, do not dislike upside volatility, but instead, they dislike downside volatility. The opposite holds when investors go short.

Consequently, it is more accurate to develop a model in which agents react positively to higher-than-expected returns and negatively to lower-than-expected returns, as opposed to a model in which both returns contribute equally to the investors' risk aversion.

In the updated model, the demand for the risky asset by the type h investor is determined by solving the optimization problem:

[24] In the Black & Scholes model proposed in 1973, the dynamics for a risky asset is described by a Geometric Brownian Motion (GBM), and its Stochastic Differential Equation (SDE) is given by $dSt = rStdt + \sigma StdWt$. In this framework, logarithmic returns in a certain time interval are expected to be distributed as normal random variables and should not depend on the past. However, real logarithmic returns deviate from a precisely normal distribution, exhibiting skewness and fat tails. Additionally, they are autocorrelated over time.

[25] For a given (real) random variable X, we distinguish between positive semivariance $\mathbb{V}^+(X) = \mathbb{E}((X - \mathbb{E}(X))^2 1_{X \geq \mathbb{E}(X)})$ and negative semivariance $\mathbb{V}^-(X) = \mathbb{E}((X - \mathbb{E}(X))^2 1_{X < \mathbb{E}(X)})$. The positive semivariance represents the expected square deviation from the mean for values of X greater than or equal to the mean, while the negative semivariance is the expected square deviation from the mean for values of X less than the mean.

$$z^*_{h,t} = \arg\max_{z_{h,t}} \begin{cases} \mathbb{E}_{h,t}(W_{h,t+1}) - \frac{a}{2}\mathbb{V}^-_{h,t}(W_{h,t+1}), z_{h,t} \geq 0 \\ \mathbb{E}_{h,t}(W_{h,t+1}) - \frac{a}{2}\mathbb{V}^+_{h,t}(W_{h,t+1}), z_{h,t} < 0 \end{cases}$$

Obviously, traders are exposed to downside risk when they want to buy the risky asset and to upside risk otherwise.

Assuming that the conditional semivariances of the excess returns are the same for all the agents, i.e. $\mathbb{V}^-_{h,t} \equiv \sigma^2_-$ and $\mathbb{V}^+_{h,t} \equiv \sigma^2_+$, a simplified expression for $z^*_{h,t}$ is available:

$$z^*_{h,t} = \begin{cases} z^-_{h,t} = \frac{\mathbb{E}_{h,t}(p_{t+1})+\bar{y}-Rp_t}{a\sigma^2_-}, \mathbb{E}_{h,t}(p_{t+1}) + \bar{y} \geq Rp_t \\ z^+_{h,t} = \frac{\mathbb{E}_{h,t}(p_{t+1})+\bar{y}-Rp_t}{a\sigma^2_+}, \mathbb{E}_{h,t}(p_{t+1}) + \bar{y} < Rp_t \end{cases}$$

It is reasonable to assume $\sigma^2_- \neq \sigma^2_+$, otherwise there would be no distinction between upside and downside risk, rendering this model equivalent to the original one. In particular, the case $\sigma^2_- > \sigma^2_+$ represents a realistic scenario where negative returns (and thus downside risk) are predominant [26].

With regards to prediction functions, a possible extension of the model is represented by the case in which trend followers adopt a less naive prediction function. Specifically, this includes the scenario where chartists consider the so known price rate of change (ROC) averaged over the last $n \geq 1$ time instants.

Introduced in 1993 by Alexander Elder in his book "Trading for a Living: Psychology, Trading Tactics, Money Management", these predictors were first applied in ARED models in 2010 in "A new stock market model with adaptive rational equilibrium dynamics" by Cecchetto and Dercole.

The expected future price at time $t+1$ conditional to the information available at time t is given by:

$$\mathbb{E}_{2,t}(p_{t+1}) = p_t ROC^2$$

where the rate of change ROC is defined as

$$ROC = (\frac{p_t}{p_{t-n}})^{\frac{1}{n}} = (\frac{p^* + x_t}{p^* + x_{t-n}})^{\frac{1}{n}}$$

[26] We express our gratitude to Fabio Dercole and Davide Radi for the implementation of this model.

ROC predictors are particularly useful when trend followers are skeptical of extreme price rates.

An extension of the model can be considered where trend followers, also known as non linear technical analysts or *ROC* traders, incorporate the mean between the *ROC* index and the unit rate.

$$\mathbb{E}_{2,t}(p_{t+1}) = p_t(\alpha_{R0C}ROC + (1 - \alpha_{ROC}))^2$$

2.6. Counteracting low prices: the uptick rule

It is evident that ARED models attract interest as they reproduce some empirical features observed in real financial markets, allowing for the presence of multiple equilibria: the fundamental steady state and the two non fundamental steady states.

The first one is consistently present and remains stable when the intensity of the choice parameter is small, signifying that investors are not inclined to change their strategy frequently. The emergence of the two non-fundamental steady states occurs when traders are more likely to update their beliefs, and chaotic dynamics ensue when financial agents are exceedingly anxious and change their strategy very frequently.

However, the adaptive system analyzed in this chapter, despite being grounded in the Efficient Market Hypothesis and the Rational Expectation Hypothesis, does not preclude investors from speculating and influencing the pricing process through their investment decisions. Notably, fundamentalists consistently invest to guide the price towards its fundamental value, although their strategy is tempered by the cost C incurred to gather information. Conversely, trend followers predict more significant price deviations.

When the intensity of the choice parameter is sufficiently large, analytical and numerical analyses have demonstrated that the system can enter pessimistic phases, leading to undervalued prices falling below the fundamental. During such phases, the deviation from the fundamental price can increase as β becomes larger. This is notably observed when the system is attracted to the negative non-fundamental steady state or the chaotic attractor encompassing it.

Obviously, pessimistic phases are not desirable, and financial regulators take action to prevent aggressive speculative strategies where investors massively go short.

While short selling is not inherently malicious and can enhance market liquidity and pricing efficiency, its improper use can lead to uncontrolled price declines and accelerate market declines, as witnessed in 1937 [27] and 2007. One regulatory tool available to counter speculative short selling is the so called uptick rule.

Introduced for the first time by the U.S. Securities and Exchange Commission (SEC)[28] in 1934 (Rule 10a-1, Security Exchange Act) and implemented in 1938 during the 1937-38 recession, the uptick rule allows traders to open short positions only when the stock price has risen with respect to the last registered value (uptick).
An alternative version is represented by the zero-plus-tick rule: short selling is allowed even when the price is equal to the last traded one, but the last observed increment must be positive.
The uptick rule fell into disuse in the U.S. markets in 2007 but was reintroduced in 2010 (Rule 201) and implemented in 2011 in a new version: if an asset loses at least 10% of its value in one day, short selling is allowed only if the price is above the current best bid. In other words, investors may go short only at a higher price than the previous traded.

ARED models provide an excellent framework to assess the (at least theoretical) effectiveness of the uptick rule: it is sufficient to incorporate the rule into the adaptive mechanism at each time step.

An exemplary study in this regard is presented by Fabio Dercole and Davide Radi in their work 'Does the “uptick rule” stabilize the stock market? Insights from adaptive rational equilibrium dynamics", published in 2020. However, the two authors tested the uptick rule using the first ARED model introduced by Brock and Hommes in the 90s.

In the following lines, the uptick rule, in its basic version, will be tested considering the ARED model proposed by Hommes in 2005, which introduces variations such as the incorporation of a parameter α representing the portion of investors who will update their strategy, and a slightly different price adjustment mechanism involving the Market Maker.

[27]Economic downturn occurred during the Great Depression in the U.S.
[28]Founded by Franklin Delano Roosevelt in 1934 in response to the Great Depression, it is the agency in charge of enforcing the law against market manipulation in the U.S. markets.

2.6.1. Model setup

At each time step t, the current values of the fractions $n_{1,t}$, $n_{2,t}$, and the deviation x_t are known. Additionally, the deviation at the previous time instant, x_{t-1}, must be considered. If $x_t > x_{t-1}$, indicating a price increase, investors face no restrictions on their strategies, and the standard adaptive mechanism can be applied. However, if $x_t \leq x_{t-1}$, implying a price decrease, both fundamentalists and trend followers are restricted from going short. In other words, $z_{h,t+1}$ cannot be negative for any type of trader h=1,2. The expression for $z_{h,t}$ under these conditions is as follows:

$$z_{h,t} = \max\{0, \frac{\mathbb{E}_{h,t}(p_{t+1} + y_{t+1} - Rp_t)}{a\sigma^2}\}$$

Then at time t three scenarios are possible:

1. $x_t > x_{t-1}$ with no imposed restrictions;
2. $x_t \leq x_{t-1}$, and fundamentalists are prohibited from going short (this occurs when x_t>0, leading to $z_{1,t}$<0);
3. $x_t \leq x_{t-1}$, and trend followers are restricted from going short (this occurs when x_t<0, resulting in $z_{2,t}$<0);

It is necessary to expand the state space by transitioning from (x_t, m_t) to (x_t, x_{t-1}, $z_{1,t}$, $z_{2,t}$, m_t). Within the expanded state space, four distinct regions can be identified:

1. $U = \{(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) : x_t > x_{t-1}\}$ (no restrictions);
2. $Z_0 = \{(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) : x_t \leq x_{t-1}, z_{1,t} \geq 0, z_{2,t} \geq 0\}$ (negligible restrictions);
3. $Z_1 = \{(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) : x_t \leq x_{t-1}, z_{1,t} < 0, z_{2,t} \geq 0\}$ (restrictions only for fundamentalists);
4. $Z_2 = \{(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) : x_t \leq x_{t-1}, z_{1,t} \geq 0, z_{2,t} < 0\}$ (restrictions only for trend followers);

By applying the uptick rule and thereby restricting investors' possibilities, the ARED model transforms into a piece-wise smooth dynamical model. The state space is partitioned, and the function governing the system's evolution depends on the current state. In particular:

$$z_{1,t} = \begin{cases} \frac{\mathbb{E}_{1,t}(p_{t+1}+y_{t+1}-Rp_t)}{a\sigma^2}, (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in U \cup Z_0 \cup Z_2 \\ 0, (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in Z_1 \end{cases}$$

and

$$z_{2,t} = \begin{cases} \frac{\mathbb{E}_{2,t}(p_{t+1}+y_{t+1}-Rp_t)}{a\sigma^2}, (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in U \cup Z_0 \cup Z_1 \\ 0, (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in Z_2 \end{cases}$$

The ARED model, enhanced with the uptick rule, is expressed as follows:

$$(x_{t+1}, m_{t+1}) = \begin{cases} G_0(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t), (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in U \cup Z_0 \\ G_1(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t), (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in Z_1 \\ G_2(x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t), (x_t, x_{t-1}, z_{1,t}, z_{2,t}, m_t) \in Z_2 \end{cases}$$

where G_0, G_1, G_2 represent the functions governing the adaptive systems when the state belongs to $U \cup Z_0$, Z_1, and Z_2, respectively.

Considering the projections of U, Z_0, Z_1, and Z_2 onto the semi-plane $(x_t,\ x_{t-1})$, it is evident that:

- The projection of U is $\{(x_t, x_{t-1}) \in \mathbb{R}^2 : x_t > x_{t-1}\}$;
- The projection of Z_0 is $\{(x_t, x_{t-1}) \in \mathbb{R}^2 : x_t = 0, x_{t-1} \geq 0\}$;
- The projection of Z_1 is $\{(x_t, x_{t-1}) \in \mathbb{R}^2 : x_t \leq x_{t-1}, x_t > 0\}$;
- The projection of Z_2 is $\{(x_t, x_{t-1}) \in \mathbb{R}^2 : x_t \leq x_{t-1}, x_t < 0\}$;

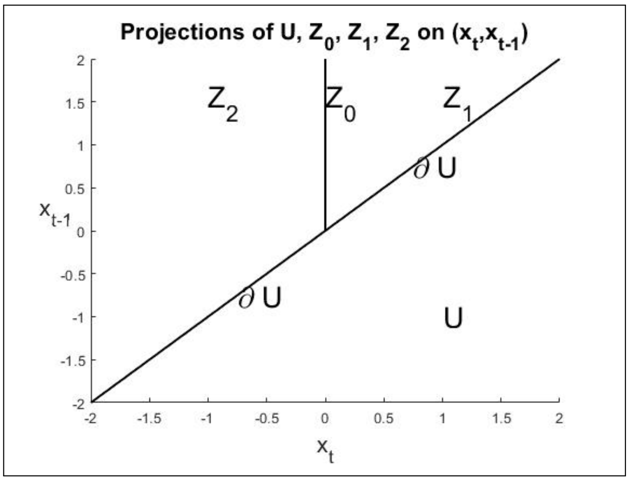


Figure 2.18: Projections of U, Z_0, Z_1 and Z_2 on $(x_t,\ x_{t-1})$

Note that Z_0 serves as the boundary between Z_1 and Z_2. Denoting ∂U as the boundary between U and $Z_1 \cup Z_2$, it is evident that the function generating the considered dynamical system is discontinuous in ∂U, while in Z_0 it is continuous but not differentiable.

Each function G_i, where $i = 0, 1, 2$, has its own fixed points, which must be categorized as admissible or virtual. An equilibrium of G_0 is termed admissible if it lies in $U \cup Z_0$ and virtual otherwise. A fixed point of G_1 is admissible if it belongs to Z_1 and virtual otherwise, while a stationary point of G_2 is admissible if it is contained in Z_2 and virtual otherwise. Virtual fixed points are not equilibria of the system, and their study is beyond the scope of this analysis. The fundamental steady state E (belonging to $U \cup Z_0$) is considered admissible.

2.6.2. Numerical simulations

Numerical simulations are conducted to test the ARED model enhanced with the uptick rule, aiming to determine the rule's capability to counteract the occurrence and magnitude of undervalued prices (i.e., negative deviations). It is reasonable to infer that the efficacy of the rule will be contingent on the value of the intensity of choice parameter β.

All other parameters will be set equal to the values considered in the previous section where the original adaptive system has been analyzed. This choice facilitates a direct comparison between the two cases. Additionally, it is necessary to assume that:

- The parameter z_s must be positive (and relatively small). If z_s equals zero, short selling would be imperative for conducting exchanges. However, when z_s is greater than zero, short selling is not mandatory and can be restricted by the uptick rule;
- During the first iteration, the uptick rule cannot be enforced since there is no prior deviation available for comparison with the current one, x_0. Given the initial pair (x_1, x_0), the imposition of the uptick rule begins from the second time step onward. An alternative approach, as suggested by Dercole and Radi (2020), is to randomly draw a deviation for comparison with x_0—implementing the uptick rule from the second time step.

The primary consequence of implementing the uptick rule is that the fundamental steady state E becomes unstable for any admissible value of β. For small values of β, the fundamental steady state E transforms into a saddle node coexisting with a stable 2-cycle in the positive semi-plane where $x > 0$. The line $x = 0$ (and consequently the line $m = 1 - \frac{R}{g} + \frac{2}{\mu g}$, which is mapped to $x = 0$), remains invariant. However, in the presence of two consecutive null deviations, the uptick rule is invoked, limiting the actions of investors. Nevertheless, in this scenario, both types of traders request the same amount of shares (z_s), which is positive, thereby rendering the impact of the uptick rule negligible. Consequently, the excess demand is null, and the price does not deviate from the fundamental value.

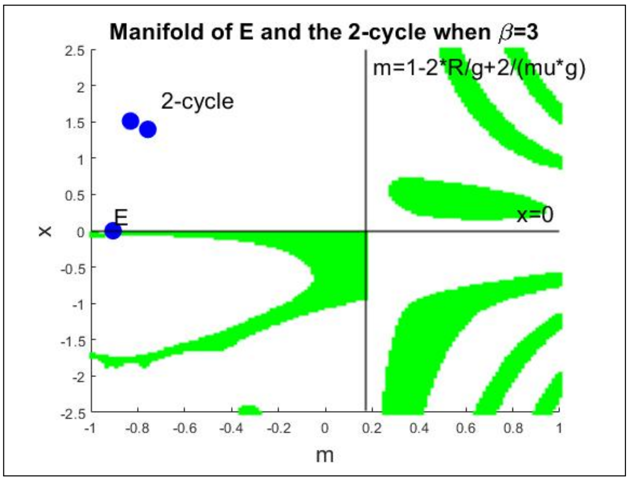
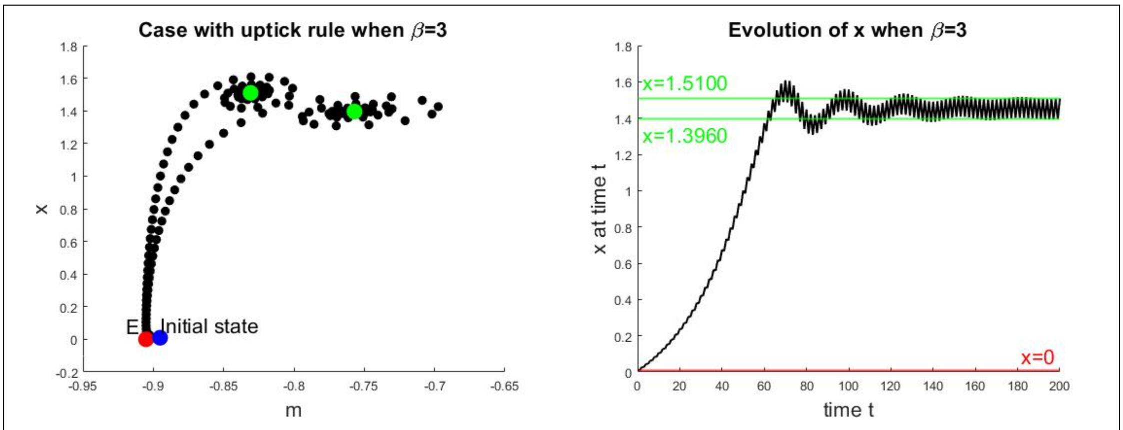


Figure 2.19: Manifold of E (green region) and the 2-cycle (white region) when β=3



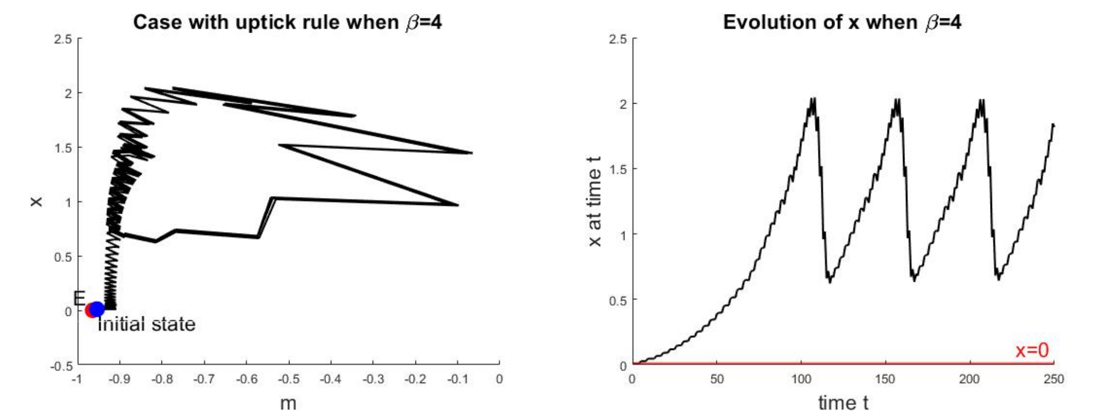
(a) Orbit converging to the 2-cycle when β=3

(b) Evolution of x when β=3

Figure 2.20: Dynamics under the uptick rule starting close to E when β=3

With an increase in the intensity of choice parameter, a global strange attractor emerges, contained within the positive semi-plane where $x>0$. Starting from any initial state, the system enters a perpetual optimistic phase.

The price doesn't rise indefinitely due to the fact that, at a certain point, it fails to increase as trend followers expect, leading to losses on their part. Consequently, the fraction of fundamentalists increases until the negative demand for the risky asset by fundamentalists (when they can open short positions) becomes sufficient to induce a price decline.



(a) Orbit converging to the strange attractor when β=4

(b) Evolution of x when β=4

Figure 2.21: Dynamics under the uptick rule starting close to E when β=4

As β continues to increase, the positive fluctuations become larger.

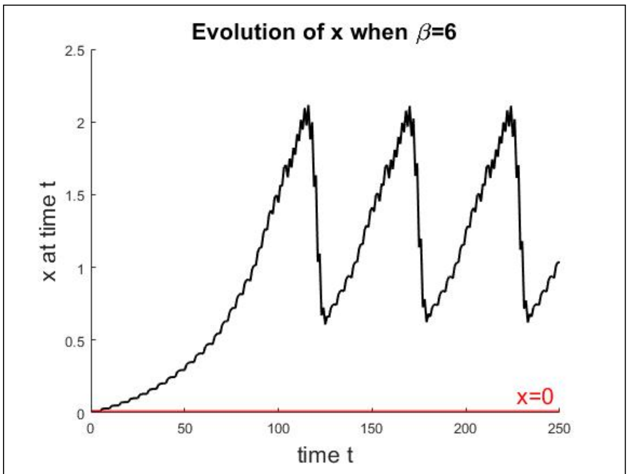
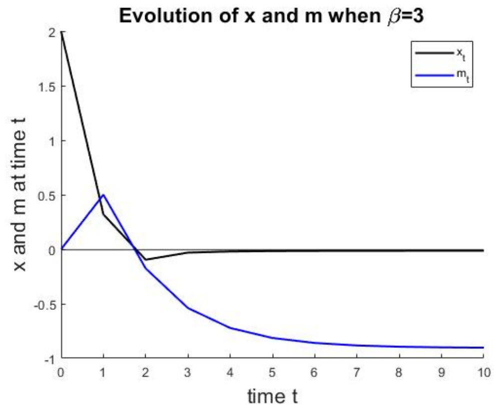


Figure 2.22: Evolution of x when β=6

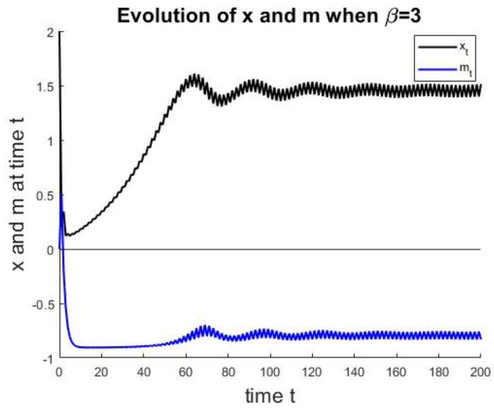
In summary, the uptick rule proves effective in preventing undervalued prices and pessimistic phases. However, it does not prevent significant positive price fluctuations, particularly as investors exhibit a heightened inclination to alter their beliefs and strategies.

In figures 2.23 and 2.24, all simulations consider the initial state (2, 0) (indicating a price above the fundamental price and an equal distribution of investors).

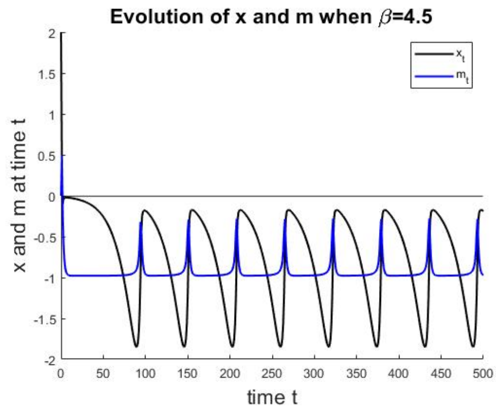
Primarily, only fundamentalists are constrained by the uptick rule because they consistently attempt to go short, anticipating the price to fall towards the fundamental value.



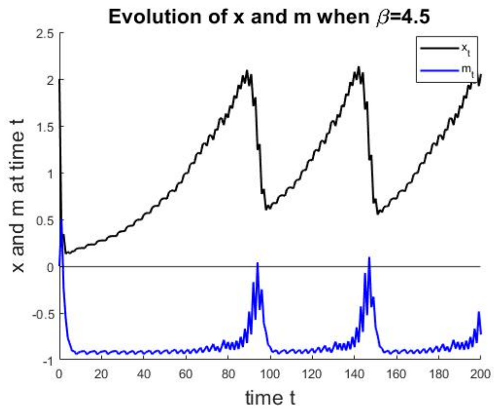
(a) Evolution of x and m (without uptick rule) when β=3



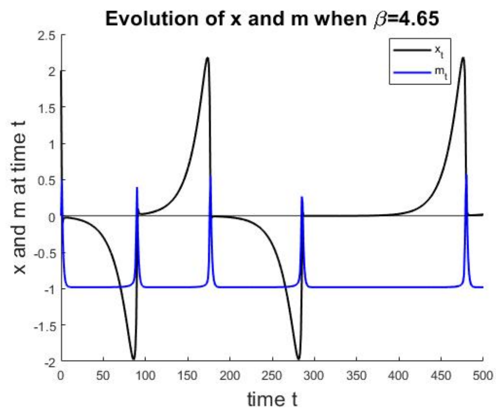
(b) Evolution of x and m (with uptick rule) when β=3



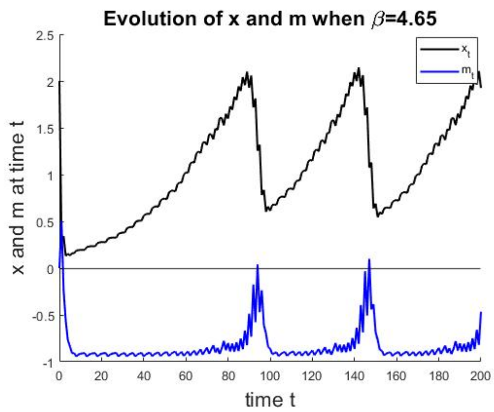
(c) Evolution of x and m (without uptick rule) when β=4.5



(d) Evolution of and m (with uptick rule) when β=4.5

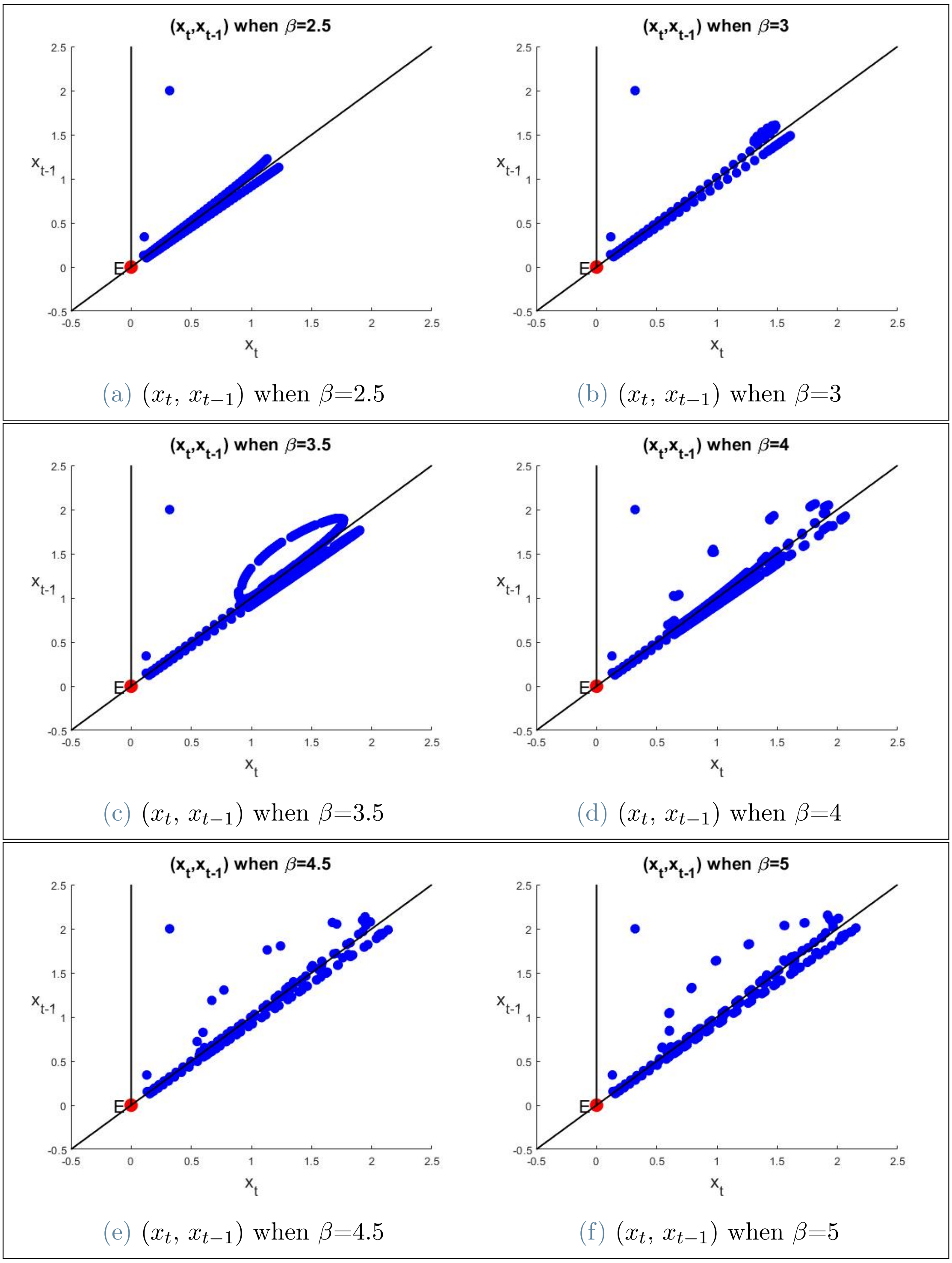


(e) Evolution of x and m (without uptick rule) when β=4.65



(f) Evolution of x and m (with uptick rule) when β=4.65

Figure 2.23: Comparison between the evolution of x (in black) and m (in blue) in the asymmetric case with (right column) and without (left column) uptick rule for various values of β: β=3 (panel (a) and (b)); β=4.5 (panel (c) and (d)); β=4.65 (panel (e) and (f))



(a) $(x_t,\, x_{t-1})$ when β=2.5 (b) $(x_t,\, x_{t-1})$ when β=3

(c) $(x_t,\, x_{t-1})$ when β=3.5 (d) $(x_t,\, x_{t-1})$ when β=4

(e) $(x_t,\, x_{t-1})$ when β=4.5 (f) $(x_t,\, x_{t-1})$ when β=5

Figure 2.24: Evolution of the adaptive system under the uptick rule in the semi-plane $(x_t,\, x_{t-1})$ for various values of β: β=2.5 (a); β=3 (b); β=3.5 (c); β=4 (d); β=4.5 (e); β=5 (f).

3 Conclusions

The proposed and analyzed Adaptive Rational Equilibrium Dynamics model incorporates three main features:

- Investors adapt their strategy based on recent past performance;
- A Market Maker is present. The Market Maker balances the excess demand for the risky asset, updating its price proportionally to that excess;
- Agents update their beliefs asynchronously. At each time instant, only a portion of traders are willing to change strategy if necessary.

The adaptive system introduces several parameters, including R (gross return), g (trend extrapolation), μ (speed of adjustment), and α (portion of investors who do not change their ideas). Only the most plausible financial scenarios have been considered in the numerical simulations.

The intensity of choice parameter, β, plays a crucial role. The system transitions from stability, with prices tending to the fundamental value for small β, to instability characterized by large price fluctuations, leading to pessimistic or optimistic phases for larger β. As β increases, the unique and globally stable fundamental steady state becomes unstable, giving rise to two stable non fundamental steady states. These states become unstable due to a Neimark-Sacker bifurcation, leading to the emergence of two stable cycles around them. Strange attractors arise from these closed invariant curves as the intensity of choice parameter further increases, causing homoclinic bifurcations of stable and unstable manifolds of the fundamental steady state. In this scenario, market dynamics become chaotic, resulting in massive price fluctuations.

When a positive (but "small") outside outside supply of shares, z_s, is introduced investors require a premium for holding the risky asset instead of investing on the risk-free asset. z_s impacts the fundamental price of the asset, p^*, influencing agents' utilities measures and the adaptive mechanism itself.

The main conclusion is that instability and chaos cannot be avoided when β is high, indicating that investors are too sensitive to utility variations. Limiting the possibilities of agents represents a potential solution. This work considers the enforcement of the uptick rule as an example of how ARED models are applied in practice. Empirical evidence from numerical simulations demonstrates that allowing short selling only at a higher price than the last traded one prevents markets from entering pessimistic phases characterized by price fluctuations below the fundamental value.

Despite financial agents being limited to either Fundamental or Technical Analysis, the proposed model recreates several aspects typical of real financial markets. While more than two types of investors can be considered, such as contrarians and bias traders, this would require a system in more than two dimensions, making it challenging, if not impossible, to treat analytically and numerically.

In more sophisticated models, agents can choose among less naive predictors, such as non-linear predictors. *ROC* predictors have been mentioned. Finally, ARED models can be extended to consider risk measures based on semi-variance of expected returns rather than variance, incorporating the distinction between upside (higher than expected returns) and downside (lower than expected returns) risk.

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A | Complex Systems

The objective of this work is to exemplify how complex systems and their theory can be utilized to model financial phenomena. The presented model explores the interplay between prices and investors' strategies, analyzing their mutual influence.

Complex systems are essentially dynamical systems where the variations in the variables of interest depend on their current values through non linear relationships. The introduction of non linearities results in complex dynamics characterized by multiple equilibria, cycles, tori, and strange attractors, along with high sensitivity to initial conditions and parameters, leading to bifurcations.

This appendix serves as a concise yet comprehensive guide to (a priori non-linear) dynamical systems, providing all the necessary notions and tools to better comprehend the proposed model. Dynamical systems are categorized into continuous and discrete systems. In continuous systems, a relation between the instantaneous variation of variables and their values is established ($\dot{x}(t) = f(x(t))$), while in discrete systems, the value of variables for the next period depends on their current values ($x_{t+1} = f(x_t)$).

For simplicity, only autonomous systems are considered, meaning the variation of variables depends solely on their current values and not on time. Since the model in question is a discrete dynamical system (DDS), definitions, theorems, and bifurcations will be presented in the discrete time framework. It is important to note that identical results hold in the continuous case.

For further reference, the books "Discrete Dynamical Systems" (2014) by Salinelli and Tomarelli, and "Elements of Applied Bifurcation Theory" (1998) by Kuznetsov are recommended.

A.1. Discrete Dynamical Systems

This section serves as a concise introduction to discrete dynamical systems, introducing key concepts and fundamental notions.

Definition A.1.1. *(State space)*
The state space of a system, say $\mathbb{X}$, is the set of all the possible states a system can assume.

In nearly every system, $\mathbb{X}$ is assumed to be a closed and bounded (and therefore compact) subset of $\mathbb{R}^n$.

Definition A.1.2. *(Evolution operator)*
The evolution operator is a continuous function $f : \mathbb{X} \to \mathbb{X}$ that maps the state of the system at time t, $x_t \in \mathbb{X}$, to the state at the subsequent time step, $x_{t+1} \in \mathbb{X}$.

Certainly, it is imperative for the evolution operator to be continuous and to map $\mathbb{X}$ into itself. Notably, it is evident that f does not depend on time.

Definition A.1.3. *(Discrete dynamical system)*
A discrete dynamical system (DDS) is a couple $\{\mathbb{X}, f\}$, where $\mathbb{X} \subseteq \mathbb{R}^n$ is closed and bounded, $f : \mathbb{X} \to \mathbb{X}$ is continuous. The couple $\{\mathbb{X}, f\}$ is called DDS on $\mathbb{X}$, of the first order, autonomous, in normal form.

From the definition, it is straightforward to comprehend that the system is entirely deterministic. In other words, given the initial state $x_0 \in \mathbb{X}$, it is possible to determine all future states of the system through the evolution operator f: $x_1 = f(x_0)$, $x_2 = f(x_1)$, and so forth.

We use f^n to denote the evolution operator applied n times, namely $x_n = f^n(x_0)$.
The evolution operator must satisfy two fundamental properties:

1. $f^0(x) = x$ for any $x \in \mathbb{X}$;
2. $f^{m+n}(x) = f^m(f^n(x))$ for any $x \in \mathbb{X}$ and $m, n \in \mathbb{N}$;

Definition A.1.4. *(Linear system, Affine system, Non linear system)*
A DDS is said to be linear if the evolution operator f is linear [1], affine is f is (linear) affine [2], non linear otherwise.

[1] Function of the form $f(x) = Ax + b$ with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$

[2] Restricted to functions defined on $\mathbb{R}$, function of the form $f(x) = \frac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$, of course $c \neq 0$ and $x \neq -\frac{d}{c}$

Definition A.1.5. *(Orbit or Trajectory)*
Given a DDS $\{\mathbb{X}, f\}$ *and an initial state* $x_0 \in \mathbb{X}$*, the sequence*

$$Or(x_0) = \{x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), ...\} = \{x \in \mathbb{X} : x = f^n(x_0), n \in \mathbb{N}\}$$

is called orbit or trajectory corresponding to the initial state x_0.

Definition A.1.6. *(Phase portrait or Trajectories portrait)*
The phase (or trajectories) portrait of a DDS $\{\mathbb{X}, f\}$ *is the collection of all the orbits of the system varying* $x_0 \in \mathbb{X}$.

Of all the possible orbits, stationary and periodic orbits are of particular interest.

Definition A.1.7. *(Stationary orbit)*
Given a DDS $\{\mathbb{X}, f\}$*, a stationary orbit is an orbit of the type* $Or(x) = \{x, x, x, ...\}$

$x \in \mathbb{X}$ is termed an equilibrium of the system. This designation implies that if the system reaches an equilibrium state, it will remain at that equilibrium indefinitely.

Definition A.1.8. *(Equilibrium or Fixed point or Stationary point)*
Given a DDS $\{\mathbb{X}, f\}$*, a state* $x \in \mathbb{X}$ *is called equilibrium (or fixed point, or stationary point) for the system if* $x = f(x)$.

Calculating a fixed point is straightforward for linear [3] or affine [4] systems, while it is less trivial and sometimes impossible to compute analytically for nonlinear systems. In such cases, numerical methods, such as the Newton method [5], are required. However, before implementing a numerical algorithm, it is crucial to determine the potential existence of fixed points. The following theorem can be applied in specific cases.

Theorem A.1. *(Contraction Mapping Theorem)*
Let $(\mathbb{X}, d)$ *a complete metric space* [6] *and* $f : \mathbb{X} \to \mathbb{X}$ *a contraction* [7]*. Then it exists one and only one* $x \in \mathbb{X}$ *fixed point of* f*. Moreover, for any* $x_0 \in \mathbb{X}$ *it holds that* $\lim_{n\to\infty} f^n(x_0) = x$

[3] In this scenario, the unique fixed point is obtained as the solution to the equation $x = Ax + b$. Therefore, $x = (I - A)^{-1}b$, where it's important to note that the inverse of $(I - A)$ must be well defined.

[4] In this scenario, two equilibria arise as solutions to the equation $x = \frac{ax+b}{cx+d}$. Specifically, the equilibria are given by $x_{1,2} = \frac{a-d\pm\sqrt{(d-a)^2+4cb}}{2c}$.

[5] Numerical method commonly employed for finding the roots of a function. In this context, the method is applied to the function $g(x) = f(x) - x$. A fixed point of f corresponds to a root of g.

[6] A metric space $(\mathbb{X}, d)$ is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ (for any $\epsilon > 0$ there exist $m, n \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$) is also a convergent sequence (it exists $x \in \mathbb{X}$ such that $x_n \to x$).

[7] A contraction is a Lipschitz function with Lipschitz constant less than 1: for any $x \neq y$ belonging to $\mathbb{X}$, $d(f(x), f(y)) < d(x, y)$. Precisely, every pair of elements "approaches" each other through the map f.

In this context, $\mathbb{X}$ is assumed to be a closed and bounded (therefore compact) subset of $\mathbb{R}^n$. Additionally, it is a complete metric space when considering the usual Euclidean distance in $\mathbb{R}^n$. If it can be demonstrated that f is a contraction, then not only does an equilibrium exist for the DDS, and it is unique, but furthermore, it can be found (at least approximately) by iteratively applying the map f infinitely starting from any initial state.

Before delving into the definition of a periodic orbit, it is beneficial to introduce the concept of definitely stationary orbits.

Definition A.1.9. *(Definitely stationary orbit)*
Given a DDS $\{\mathbb{X}, f\}$, if there exists a state $y \neq x$ such that $f(y) = f(x) = x$ then the orbit $Or(y) = \{y, x, x, ...\}$ is said to be definitely periodic.

Definition A.1.10. *(Period orbit or Cycle)*
Given a DDS $\{\mathbb{X}, f\}$, if there exists a set of s distinct states $\{x_0, x_1, ..., x_{s-1}\} \subset \mathbb{X}$ satisfying

$$x_1 = f(x_0); x_2 = f(x_1); ...; x_0 = f(x_{s-1});$$

then such a set is called periodic orbit of (minimum) period s or cycle of period s or s-cycle. s is the period of the orbit, also known as order of the cycle.

Commencing from the state $x_0 \in \mathbb{X}$ that belongs to the s-cycle, the trajectory is given by $Or(x_0) = \{x_0, x_1, \ldots, x_{s-1}, x_0, x_1, \ldots\}$. Naturally, $f^{n+s}(x) = f^n(x)$ for any $n \in \mathbb{N}$. The states within the s-cycle can be perceived as the s distinct fixed points of the map f^s that are not fixed points of f.

Given that the emergence of a limit cycle is discussed in Chapter 2. its definition is provided below.

Definition A.1.11. *(Limit cycle)*
Given a DDS $\{\mathbb{X}, f\}$, a limit cycle is a cycle in a neighborhood of which there are no other cycles.

Similar to stationary orbits, definitely periodic orbits are introduced.

Definition A.1.12. *(Definitely periodic orbit)*
Given a DDS $\{\mathbb{X}, f\}$, if there exists $y \neq x_{s-1}$ such that $f(y) = x_0$ where x_0 belongs to a s-cycle, then the orbit $Or(y) = \{y, x_0, x_1, ..., x_{s-1}, x_0, ...\}$ is said to be definitely periodic of period s.

An illustrative example can be employed to demonstrate all the concepts introduced above.

Example A.1.1. *(Logistic map)*
Let's consider the DDS $\{\mathbb{X}, f\}$*, with* $\mathbb{X}$=$[0, 1]$ *and* $f(x) = 3.2x(1 - x)$.

This system is also recognized as the logistic map (with parameter $a = 3.2$ *in this instance), commonly applied in biology to model the evolution over time of the proportion of a population. A proportion of 1 indicates that the population is at its maximum, while a proportion of 0 signifies that the population has become extinct.*

From the graph below, it is evident that the system has two equilibria: $x = 0$ *and (approximately)* $x = 0.688$*. Subsequently, two stationary orbits, commencing from the fixed points, are present. Moreover, a definitely stationary orbit exists, as the state* $x = 1$ *will be mapped into the equilibrium* $x = 0$*. In terms of periodic orbits, a 2-cycle is formed by the states* $x = 0.513$ *and* $x = 0.799$ *(approximately).*

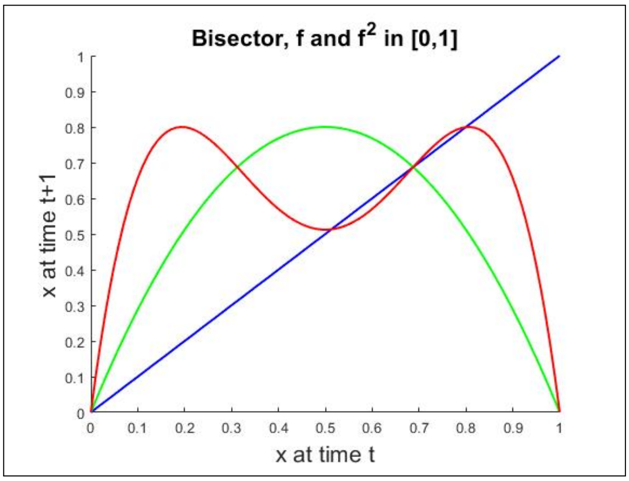

Figure A.1: Bisector (blue line), f (green line) and f^2 (red line) in [0,1]

The two equilibria arise from the intersection of f *and the bisector, while the states constituting the 2-cycle are the points of intersection between* f^2 *and the bisector that are not also intersection points of* f *and the bisector.*

A.2. Stability analysis

Stationary and periodic orbits represent the simplest trajectories a dynamical system can generate. However, obtaining these kinds of orbits is often impractical since, in most cases, they require consideration of an exact initial state (such as the equilibrium or a state belonging to the cycle), which may be challenging due to numerical approximations and errors. For example, if the equilibrium in a system is an irrational number, computers can only handle a rational approximation of it, despite their high precision.

Thus, it becomes crucial to understand the system's behavior when the initial state is "close" to the desired position. A desirable dynamic is one in which the state tends toward the equilibrium or cycle. Consequently, the concept of stability (second to Lyapunov) is introduced.

Definition A.2.1. *(Stable and Unstable equilibria)*
Given a DDS $\{\mathbb{X}, f\}$*, an equilibrium* $x \in \mathbb{X}$ *is said to be stable if for any* $\epsilon > 0$ *it exists* $\delta > 0$ *such that* $|x_0 - x| < \delta$ *implies* $|f^n(x_0) - x| < \epsilon$ *for any* $n \in \mathbb{N}$*. Otherwise the equilibrium is said to be unstable.*

Given the definition, when a fixed point is stable, orbits starting close to the equilibrium will persist in proximity to it. If the orbits are attracted by the stationary point, then:

Definition A.2.2. *(Locally and Globally attractive equilibria)*
Given a DDS $\{\mathbb{X}, f\}$*, an equilibrium* $x \in \mathbb{X}$ *is said to be globally attractive if for any* $x_0 \in \mathbb{X}$ *it follows* $\lim_{n\to\infty} f^n(x_0) = x$*. The equilibrium is said to be locally stable if it exists* $\eta > 0$ *such that for any* $x_0 \in \mathbb{X} \cap B_\eta(x)$ [8] *it follows that* $\lim_{n\to\infty} f^n(x_0) = x$*.*

In summary,

Definition A.2.3. *(Locally and Globally asymptotically stable equilibria)*
Given a DDS $\{\mathbb{X}, f\}$*, an equilibrium* $x \in \mathbb{X}$ *is said to be globally asymptotically stable if it is stable and globally attractive, locally asymptotically stable if it is stable and locally attractive.*

When an equilibrium is both stable and attractive, orbits starting close to it not only remain close, but the state also tends toward the fixed point. The distinction between global and local is contingent upon the dimension of the set collecting the initial states for which the attractiveness property holds.

[8] With $B_\eta(x)$ it is denoted the set of all states x_0 which are distant from x less than η. In symbols, $B_\eta(x) = \{x_0 \in \mathbb{X} : |x_0 - x| < \eta\}$

Notice that if the assumptions of the Contraction Mapping Theorem are satisfied, then the unique equilibrium is globally asymptotically stable.

Regarding cycles, similar definitions apply: it suffices to consider that states belonging to an s-cycle are fixed points for the map f^s. For instance, here is the definition of a globally asymptotically stable cycle.

Definition A.2.4. *(Globally asymptotically stable cycle)*
Given a DDS $\{\mathbb{X}, f\}$ *and an* s*-cycle* $x_0, x_1, ..., x_{s-1}$*, then the cycle is said to be globally stable if the set* $\{x_0, x_1, ..., x_{s-1}\}$ *is globally asymptotically stable for the DDS* $\{\mathbb{X}, f^s\}$*.*

In general, attractive equilibria and cycles represent specific instances of sets that attract certain initial states, commonly referred to as "attractors" or "attractive sets".

Definition A.2.5. *(Attractor or Attractive set)*
Given a DDS $\{\mathbb{X}, f\}$*, a set* $A \subset \mathbb{X}$ *is called attractor or (locally) attractive set if*

1. *It is closed;*
2. *It is invariant* [9]*;*
3. *It exists an open set* $B \subset \mathbb{X}$ *such that* $A \subset B$ [10] *and any orbit starting from a state belonging to* B *will be attracted by* A [11]*;*
4. *It is minimal, i.e. it doesn't exist a proper subset of* A *satisfying 1) 2) and 3).*

Definition A.2.6. *(Basin of attraction)*
Given a DDS $\{\mathbb{X}, f\}$ *and an attractor* $A \subset \mathbb{X}$*, the basin of attraction is defined as*

$$B(A) = \{x \in \mathbb{X} : \lim_{n\to\infty} dist(f^n(x), A) = 0\}$$

An attractor (such as an equilibrium) is globally asymptotically stable if the basin of attraction encompasses the entire space $\mathbb{X}$, almost globally asymptotically stable if the basin of attraction includes $\mathbb{X}$ except for a set with zero measure [12].

A repulsor (or repulsive set) can be considered as the opposite of an attractor.

[9]Namely, $f(A) = A$. In simpler terms, all the orbits starting from A will persist within A.
[10]Now it is evident why A must be closed and B open: B has to completely cover A.
[11]It is necessary to define the distance between a set and a point external to it.: $dist(x, A) = min_{a\in A} d(x, a) = min_{a\in A}|x - a|$. Notice that, to be well defined, A must be a closed set.
[12]For instance, points in $\mathbb{R}$, points, lines in $\mathbb{R}^2$, points, lines, surfaces in $\mathbb{R}^3$ and so on.

Definition A.2.7. *(Repulsor or Repulsive set)*
Given a DDS $\{\mathbb{X}, f\}$, a set $R \subset \mathbb{X}$ is called repulsor or repulsive set if

1. *It is closed;*
2. *It is invariant;*
3. *It exists an open neighborhood U of R* [13] *such that for any neighborhood V of R it exists $x_0 \in V/R$ such that $f^n(x_0) \notin U$ for infinite many values of n;*
4. *It is minimal.*

A repulsive set R is characterized by the property that orbits starting even close to R will be moved away from it for infinitely many values of n.

To analyze the stability of an equilibrium, say x, it is helpful to study the dynamics in a neighborhood of the fixed point. In this neighborhood, the system's dynamics are equivalent to those of the linearized system:

$$f(\bar{x}) = f(x) + J_f(x)\dot{(}\bar{x} - x)$$

where $J_f(x)$ is the Jacobian matrix [14] of f computed at the equilibrium x.

The criteria outlined in the following lines involve the eigenvalues of the Jacobian matrix and are known as "first order stability conditions" since they necessitate the computation of the first order derivatives of f. These criteria will be extensively employed in the subsequent appendix to analyze the stability of the equilibria of the adaptive system introduced in Chapter 2.

Theorem A.2. *(First order stability condition)*
Given a DDS $\{\mathbb{X}, f\}$ where f is at least C^1 and a fixed point $x \in \mathbb{X}$, let $J_f(x)$ be the Jacobian matrix of f computed in x. Then if all the eigenvalues of $J_f(x)$ $\lambda_1, \lambda_2, ..., \lambda_n$ have modulus strictly less than 1 (i.e. they all stay inside the unit circle on the complex plane) then the equilibrium is (at least) locally asymptotically stable, otherwise if at least one eigenvalue has modulus strictly higher than 1 the equilibrium is unstable.

Extreme scenarios arise when the eigenvalues have unit modulus, requiring the use of the so known "second order stability conditions" involving second order derivatives.

[13] Namely, $R \subset U$ and $\mathbb{X}/U$ is closed
[14] Given a generic function $f : \mathbb{R}^n \to \mathbb{R}^m$ at least C^1 the Jacobian matrix is given by the collection of all possible partial derivatives of f, namely $[J_f]_{i,j} = \frac{\partial f_i}{\partial x_j}$ with $i = 1, ..., m$ and $j = 1, ..., n$

However, the computation of the Hessian matrix [15] of f computed at the equilibrium is anything but trivial. Therefore, only systems for which first order stability conditions are sufficient will be considered.

For the continuation of the section, it is preferable to introduce the integers n_0, n_-, and n_+. They represent, respectively, the number of eigenvalues with unit modulus, with modulus strictly less than 1, and with modulus strictly greater than 1.

Definition A.2.8. *(Hyperbolic equilibrium and Saddle point)*
Given a DDS $\{\mathbb{X}, f\}$, an equilibrium is said to be hyperbolic if $n_0 = 0$. A hyperbolic equilibrium is said to be a saddle point if $n_- n_+ \neq 0$.

Saddle points have the peculiarity that they can be either attractive or repulsive depending on the initial state.

Definition A.2.9. *(Stable and Unstable manifold)*
Given a DDS $\{\mathbb{X}, f\}$ and a fixed point $x \in \mathbb{X}$, the stable manifold of x $W^s(x)$ is the set of all initial states attracted by the equilibrium, while the unstable manifold of x $W^u(x)$ is the set of all initial states rejected by the stationary point. In symbols

$$W^s(x) = \{x_0 \in \mathbb{X} : lim_{n \to \infty} f^n(x_0) = x\}$$

$$W^u(x) = \{x_0 \in \mathbb{X} : lim_{n \to -\infty} f^n(x_0) = x\}$$

Notice that the two manifold are invariant.

The following theorem is remarkable.

Theorem A.3. *(Local stable manifold)*
Given a DDS $\{\mathbb{X}, f\}$ and $x \in \mathbb{X}$ a hyperbolic equilibrium (namely $n_0 = 0$), then the intersections of $W^s(x)$ and $W^u(x)$ with a sufficiently small neighborhood of x contain sub-manifolds $W^s_{loc}(x)$ and $W^u_{loc}(x)$ of dimension n_- and n_+ respectively.
Moreover, $W^s_{loc}(x)$ is tangent in x to T^s, where T^s is the generalized eigenspace corresponding to the union of all eigenvalues of J_f with modulo strictly less than 1. Similarly $W^u_{loc}(x)$ is tangent in x to T^u, union of all eigenvalues of J_f with modulo strictly higher than 1.

[15] Given a generic function $f : \mathbb{R}^n \to \mathbb{R}^m$ at least C^2 the Hessian is given by the collection of all possible partial second order derivatives of f, namely $[H_f]i, j, k = \frac{\partial^2 fi}{\partial x_j \partial x_k}$ with $i = 1, ..., m$ $j = 1, ..., n$ and $k = 1, ..., n$. It is clear that generally the Hessian is a tensor of order 3.

Example A.2.1. *(Logistic map)*
The same DDS as in Example A.1.1. is considered.

It is straightforward to demonstrate, either through the first order stability condition or a visual representation, that both equilibria $x = 0$ *and* $x = 0.688$ *are unstable. The 2-cycle* $\{0.513, 0.799\}$ *is almost globally asymptotically stable, signifying that any orbit starting from an initial condition different from the two fixed points tends to the cycle.*

In the figure below, a state close to the unstable equilibrium $x = 0$ *(red line) serves as the starting point for an orbit (black curve) that tends toward the almost globally stable 2-cycle (green lines).*

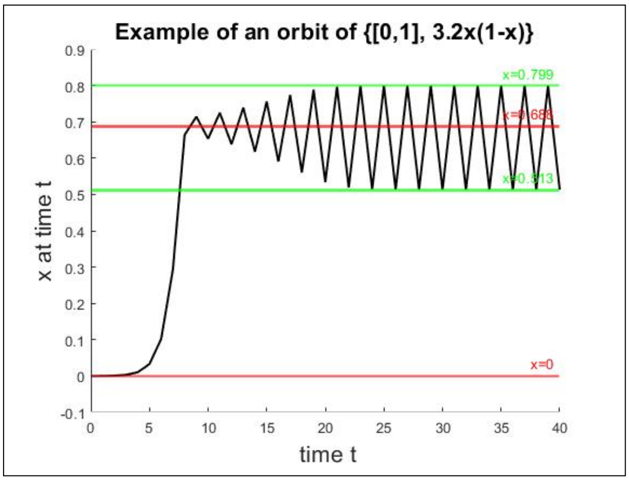

Figure A.2: Example of an orbit of the logistic map tending to the stable 2-cycle

A.3. Bifurcations analysis

Complex systems are dynamical systems characterized by:

1. High sensitivity to initial conditions: trajectories starting from "close" initial states may significantly differ over the long term;
2. High sensitivity to parameters: evolution operators are continuous in the parameters, but often even imperceptible changes in the parameters can completely alter the entire dynamics of the system;
3. Presence of strange (or chaotic) attractors (or repellors): attractive (or repulsive) sets may be characterized by the self-similarity property and non integer (fractal) dimension.

The first feature has already been discussed in the previous section. For example, an orbit starting very close to an unstable equilibrium will diverge from the stationary orbit. In the presence of a saddle point, two closely situated states may belong, one to the stable manifold and the other to the unstable one. One trajectory will tend towards the saddle, while the other will be rejected.

Strange attractors will be introduced and analyzed in the following section.

This section serves as a brief introduction to bifurcation theory. In simple terms, a bifurcation occurs when a slight change in the parameters leads to a qualitative shift in the behavior of a dynamical system. In common jargon, the phrase "the straw that broke the camel's back" provides a naive example of a bifurcation. There are various possible bifurcations depending on the dimension of the system, the number of parameters, and whether time is considered continuous or discrete. In this section, we will focus on analyzing one-parameter discrete-time bifurcations within the adaptive model.

Definition A.3.1. *(Parameter space)*
Given a dynamical system, the parameter space $A \subseteq \mathbb{R}^n$ is the collection of all possible admissible parameters.

The parameter set is denoted by A, and $a \in A$ represents an n-tuple of parameters. It is essential to explicitly express the dependence of the evolution operator f on these parameters. Thus, we denote the evolution operator with parameters a as f_a.

To enhance the visualization of bifurcations, consider using a bifurcation diagram.

Definition A.3.2. *(Bifurcation diagram)*
Given a dynamical system, a bifurcation diagram is a stratification of its parameter space together with representative phase portraits for each stratum.

For the subsequent definitions, it is necessary to introduce the concepts of the topological type of a dynamical system and the topological equivalence between two dynamical systems.

Definition A.3.3. *(Topological equivalence)*
Two dynamical systems $\{\mathbb{X}, f\}$ and $\{\mathbb{Y}, g\}$ with $\mathbb{X}, \mathbb{Y} \subseteq \mathbb{R}^n$, $f : \mathbb{X} \to \mathbb{X}$ and $g : \mathbb{Y} \to \mathbb{Y}$ are topologically equivalent if there exists a homeomorphism [16] $h : \mathbb{R}^n \to \mathbb{R}^n$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.

[16] A homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function admitting a continuous inverse function.

Definition A.3.4. *(Conjugate function)*
Two functions $f : \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying* $f(x) = h^{-1}(g(h(x)))$ *for any* $x \in \mathbb{R}^n$ *for some homeomorphism* $h : \mathbb{R}^n \to \mathbb{R}^n$ *are called conjugate.*

Finally the definition of bifurcation can be stated.

Definition A.3.5. *(Bifurcation)*
A bifurcation is the appearance of a non equivalent phase portrait under a (even small) change of the parameters.

Roughly speaking, bifurcations arise from the dependence of equilibria, cycles, and their stability on the system parameters. Bifurcations are classified as super-critical when a stable fixed point always exists regardless of parameter values, and sub-critical when this is not the case. An illustrative example of a trans-critical bifurcation will be presented.

In general, bifurcation diagrams consist of a finite number of regions in $\mathbb{R}^n$ where the phase portraits are topologically equivalent within each region. These regions are delineated by bifurcation boundaries, which are submanifolds of $\mathbb{R}^n$. Introducing the concept of bifurcation boundaries allows for the definition of the codimension of a bifurcation.

Definition A.3.6. *(Codimension of a bifurcation)*
The codimension of a bifurcation (codim for short) is the difference between the dimension of the parameter space A *of a dynamical system and the dimension of the corresponding bifurcation boundary.*

Illustrative examples of all the introduced concepts will be provided, along with a discussion of the most significant and common bifurcations. However, before proceeding, one final definition is necessary.

Definition A.3.7. *(Normal form of a bifurcation)*
The normal form of a bifurcation is a dynamical system that exhibits that bifurcation, characterized by a polynomial evolution operator and by the fact that any dynamical system exhibiting that bifurcation is (at least locally, namely in a neighborhood of the equilibrium) topologically equivalent to it.

In essence, the normal form of a bifurcation represents the 'simplest' system that demonstrates that bifurcation. Any other system in which the same bifurcation occurs can be reduced, via a homeomorphism, to its corresponding normal form.

Example A.3.1. *(Saddle-node bifurcation)*
A saddle-node bifurcation occurs when, as the parameter $a \in \mathbb{R}$ varies, two equilibria, one stable and the other unstable,collide and subsequently annihilate. This phenomenon is associated with the appearance of an eigenvalue $\lambda = 1$, and it possesses a codimension of 1. Such a bifurcation can manifest in a system of any dimension. In one dimension, the normal form is given by $x_{t+1} = a + x_t \pm x_t^2$, with $a \in \mathbb{R}$.
In the case of $x_{t+1} = a + x_t - x_t^2$, the two equilibria $x_{1,2} = \pm\sqrt{a}$ are well defined and distinct when $a > 0$. They coincide when $a = 0$, and then for $a < 0$ they disappear. The derivative of the evolution operator is trivially $\frac{df}{dx}(x) = 1 - 2x$. Consequently, the positive equilibrium $x_1 = \sqrt{a}$ is locally stable in a right neighborhood of $a = 0$, while the other equilibrium $x_2 = -\sqrt{a}$ is unstable. When $a = 0$, the unique equilibrium $x = 0$ becomes a local saddle node.

Example A.3.2. *(Trans-critical bifurcation)*
A trans-critical bifurcation occurs as the parameter $a \in \mathbb{R}$ varies, leading to the collision of two equilibria—one stable and the other unstable. Subsequently, their stabilities exchange: the initially stable fixed point becomes unstable, and vice versa. This phenomenon is associated with the appearance of an eigenvalue $\lambda = -1$, and it possesses a codimension of 1. A saddle-node bifurcation can manifest in a system of any dimension. In one dimension, the normal form is given by $x_{t+1} = (1 + a)x_t \pm x_t^2$, with $a \in \mathbb{R}$.
Consider the specific case of $x_{t+1} = (1 + a)x_t - x_t^2$. Two equilibria always exist: $x_1 = 0$ and $x_2 = a$. When $a = 0$, the two equilibria coincide. Given that $\frac{df}{dx}(x) = 1 + a - 2x$, the first equilibrium is associated with the eigenvalue $\lambda_1 = 1 + a$, while the second one is associated with $\lambda_2 = 1 - a$. Consequently, for $a < 0$, x_1 is locally stable, and x_2 is unstable. The situation reverses when $a > 0$. When $a = 0$, the coinciding equilibria at $x = 0$ form a saddle node.

Example A.3.3. *(Pitchfork bifurcation)*
A pitchfork bifurcation occurs as the parameter $a \in \mathbb{R}$ varies, giving rise to two equilibria from an existing one, and they assume its stability while the initial fixed point changes it. This bifurcation has a codimension of 1 and can manifest in a system of any dimension. In one dimension, the normal form is given by $x_{t+1} = (1 + a)x_t \pm x_t^3$, with $a \in \mathbb{R}$.
In the super-critical case of $x_{t+1} = (1+a)x_t - x_t^3$, the equilibrium $x_0 = 0$ is always present. It is locally stable for $a < 0$ and unstable for $a > 0$, as its eigenvalue is $\lambda_0 = 1 + a$. When $a > 0$, two additional equilibria arise: $x_{1,2} = \pm\sqrt{a}$. They are stable, at least in a right neighborhood of $a = 0$, as they are associated with the eigenvalue $\lambda_{1,2} = 1 - a$. Obviously, when $a = 0$, the three equilibria coincide, resulting in a locally stable equilibrium.

Example A.3.4. *(Period-doubling or Flip bifurcation)*
A period-doubling bifurcation occurs as the parameter $a \in \mathbb{R}$ varies, leading to the emergence of a cycle with a period double that of the existing one. This phenomenon is associated with the appearance of an eigenvalue $\lambda = -1$, and it has a codimension of 1. Such a bifurcation can manifest in a system of any dimension. In one dimension, the normal form is given by $x_{t+1} = -(1+a)x_t \pm x_t^3$, with $a \in \mathbb{R}$.
Consider the specific case of $x_{t+1} = -(1+a)x_t + x_t^3$. For the equilibrium $x_0 = 0$, the associated eigenvalue is $\lambda_0 = -(1+a)$, which equals -1 when $a = 0$. Furthermore, $x_0 = 0$ is also a fixed point of the map f^2. Graphically, it can be observed that two other fixed points of f^2 emerge for $a > 0$, and they collide with x_0 when $a = 0$. This implies that at $a = 0$, a 2-cycle arises from a 1-cycle in $x_0 = 0$ (the stationary orbit).

Example A.3.5. *(Homoclinic bifurcation)*
A global homoclinic bifurcation occurs as the parameter $a \in \mathbb{R}$ varies, resulting in the expansion of a periodic orbit until it collides with a saddle point, causing its period to become infinite. During this bifurcation, the stable and unstable manifolds of the saddle point intersect, forming the cycle. Subsequently, after the bifurcation, the cycle disappears. This type of bifurcation has a codimension of 2 and can be present in a system of dimension at least 2

Example A.3.6. *(Neimark-Sacker bifurcation)*
A Neimark-Sacker occurs as the parameter $\alpha \in \mathbb{R}$ varies, leading to the emergence of a closed invariant curve from a fixed point when it changes its stability via a pair of complex conjugate eigenvalues with unit modulus not belonging to $-1, 1$: $\lambda_{1,2} = e^{i\theta}$ with $0 < \theta < \pi$. This bifurcation has a codimension of 2 and can be present in a system of dimension at least 2. In dimension 2, the normal form is given by

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \end{pmatrix} = (1+\alpha) \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix} + (x_{1,t}^2 + x_{2,t}^2) \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$$

where θ, a and b are smooth functions [17] of α satisfying $0 < \theta(0) < \pi$ and $a(0) \neq 0$.
The equilibrium at the origin has eigenvalues $\lambda_{1,2} = (1+\alpha)e^{i\theta(\alpha)}$. In a neighborhood of 0, it is locally stable for $\alpha < 0$ and unstable for $\alpha > 0$. For $\alpha = 0$, the origin is stable if $a(0) < 0$. The case where $a(\alpha) < 0$ is considered super-critical, as for $\alpha > 0$, a locally stable cycle emerges: a centered circumference with radius $\rho_\alpha = \sqrt{-\frac{\alpha}{a(\alpha)}}$.

[17] In simple terms, a function is considered smooth at a point if it is differentiable an infinite number of times at that point. The notation $C^\infty(X)$ denotes the set of all functions that are smooth at every point in the domain X.

A.4. Strange (or Chaotic) Attractors

Attractors introduced in previous sections, such as stable fixed points, cycles, and tori, exhibit simple and well known geometries in $\mathbb{R}^n$: points (equilibria), closed lines (periodic orbits), and bounded volumes (tori).

However, certain dynamical systems admit attractive sets with anything but trivial geometries, a phenomenon observed in the so called "strange attractors". These attractors are characterized by fractal geometry. The term "strange attractors" was coined in 1971 by David Ruelle and Floris Takens in their work "On the Nature of Turbulence" , where they aimed to describe the nature of attractors in fluid dynamical systems. When strange attractors also exhibit a high sensitivity to initial conditions, commonly referred to as the "butterfly effect", they are termed chaotic. To delve deeper:

Definition A.4.1. *(Chaotic system)*
A DDS $\{\mathbb{X}, f\}$ is called chaotic (or it admits chaotic orbits) if:

1. *Cycles of any period are dense, namely any open interval contained in $\mathbb{X}$ contains at least one state belonging to a periodic orbit;*
2. *f is topologically transitive, meaning that for any pair of non empty open sets $U, V \subset \mathbb{X}$ it exist $u \in U$ and $n \in \mathbb{N}$ such that $f^n(u) \in V$;*
3. *The system exhibits a high dependence on the initial conditions, i.e. it exists $\delta > 0$ such that for any $x \in \mathbb{X}$ and $\epsilon > 0$ there exist $z \in \mathbb{X}$ and $n \in \mathbb{N}$ such that $|z - x| < \epsilon$ and $|f^n(z) - f^n(x)| > \delta$.*

The following theorem provides sufficient conditions that a 2-dimensional dynamical system must satisfy to admit a strange attractor. These conditions will be applied in the proof of Theorem 2.5. as presented in the second appendix. More comprehensive treatments on this topic can be found in Mora and Viana's "Abundance of Strange Attractors" (1993) and Palis and Takens' "Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations" (1993) .

Theorem A.4. *(Strange Attractor Theorem)*
Let $\mathcal{F}_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be a 2-dimensional map with real parameter α and let p be a dissipative saddle point. If the map $\mathcal{F}_\alpha$ exhibits a generic homoclinic bifurcation between the stable and the unstable manifoldof the saddle point at $\alpha = \alpha_h$, then there exists a positive Lebesgue measure set $\Omega \subset (\alpha_h - \epsilon, \alpha_h + \epsilon)$ such that for any $\alpha \in \Omega$ the map $\mathcal{F}_\alpha$ has a strange attractor.

In essence, the occurrence of a homoclinic bifurcation implies the emergence of strange attractors with positive probability in the parameter space.

Example A.4.1. *(Lorenz attractor)*
Introduced in 1963 by the mathematician and meteorologist Edward Norton Lorenz in his article "Deterministic Nonperiodic Flow", it is also known as "Lorenz butterfly" due to its distinctive shape. It is the chaotic attractor arising from the 3-dimensional dynamical system in continuous time:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}$$

Generally σ *is set equal to 10 and* β *to* $\frac{8}{3}$

As will become evident shortly, the dimension of the attractor falls between 2 and 3.

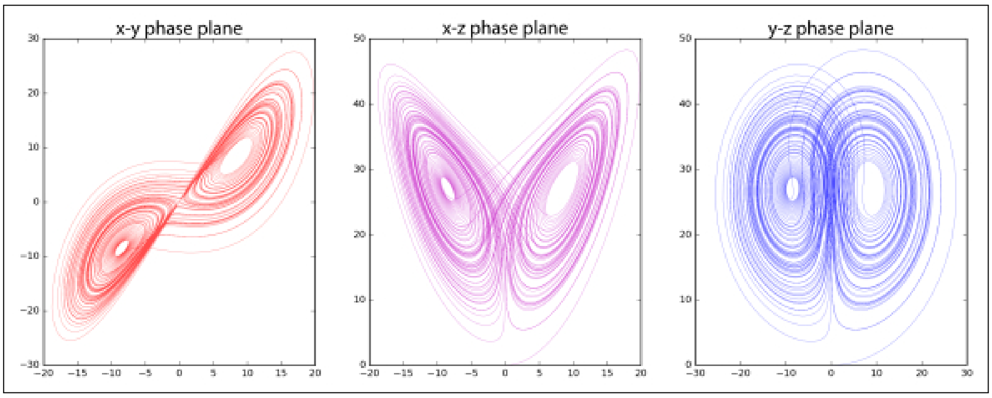

Figure A.3: Lorenz attractor visualized in the $x-y$ plane (left), $x-z$ plane (middle) and $y-z$ plane (right). The orbit around the attractor is non periodic, meaning the trajectory will not pass twice through the same state, indicating an infinite period.

The French mathematician Benoit Mandelbrot is universally recognized as the founder of fractal geometry. In his 1967 seminal work "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension", Mandelbrot introduced the concepts of self-similarity and fractal dimension, illustrating how natural shapes, such as coastlines, exhibit these characteristics [18]. He further developed this theory in his book "Fractals: Form, Chance and Dimension" (1975).

[18] In his article, Mandelbrot tackled the challenge of measuring the length of the Coast of Britain. The jagged shape, typical of the coastline, is preserved at any scale: a characteristic feature of fractal objects.

Mandelbrot is also widely known, even outside the mathematical community, for his study of the fractal set named after him: the Mandelbrot set, arguably the most famous and evocative fractal.

Example A.4.2. *(Mandelbrot set)*
In 1980, Mandelbrot, in his work "Fractal Aspects of the Iteration $z \to \lambda z(1-z)$ *for complex* λ*,*z*" , obtained the first detailed image of the Mandelbrot set. Notably, two years earlier, Robert Brooks and Peter Matelski, in "The Dynamics of 2-generator Subgroups of PSL(2,*$\mathbb{C}$*)" , had obtained a simplified version of it. What's remarkable is that this intricate object emerges from a seemingly simple mathematical concept: the Mandelbrot set is the set of all* $c \in \mathbb{C}$ *for which the trajectories generated by the system* $z_{n+1} = z_n^2 + c$*, starting from the origin of the complex plane, remain bounded. In Mandelbrot's 1980 work, it is demonstrated that by applying the transformations* $z = a(\frac{1}{2} - x)$ *and* $c = \frac{a}{2}(1 - \frac{a}{2})$ *to the system* $z_{n+1} = z_n^2 + c$*, one can obtain the logistic map* $x_{n+1} = ax_n(1 - x_n)$ *with* $a \in [0, 4]$*.*

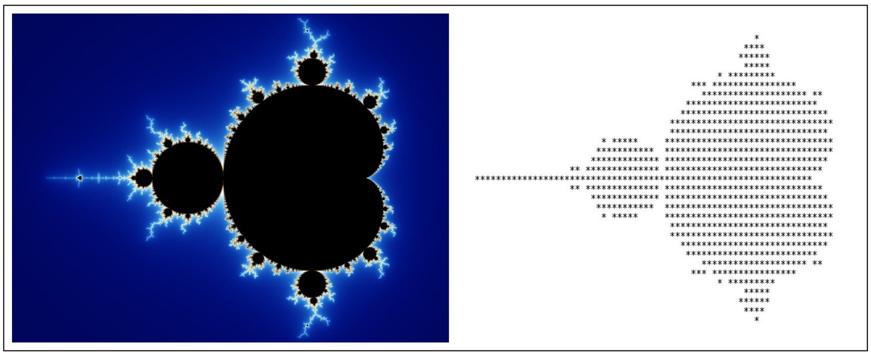

Figure A.4: The Mandelbrot set (left) and its first visualization in 1978 (right).

The self-similarity property and fractal dimension are intricately linked: fractals preserve their shape at any arbitrary scale precisely because they lack an integer dimension, and vice versa. Here, three examples illustrate this connection.

Example A.4.3. *(Cantor set)*
Consider the following transformation of the interval $[0, 1]$*: divide it into three thirds and remove the middle one. The result is the set* $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$*. Apply the same transformation again to the two segments and repeat this process infinitely many times. The result is the well known Cantor set* $\mathcal{C}$*. At iteration* n*, this set is composed of* 2^n *intervals, each with a length of* $(\frac{1}{3})^n$*. It is immediate to notice that the Cantor set has infinitely many points (it has the same cardinality as* $\mathbb{R}$*) and it has no length. Thus, its dimension isn't 0 (otherwise, it would have a finite number of points) nor 1 (because its length is 0).*

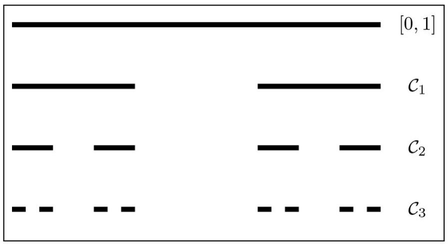

Figure A.5: First steps of the construction of the Cantor set.

Example A.4.4. *(Von Koch curve)*
Similar to the previous example, the middle third is removed from the interval $[0, 1]$*. However, in this case, it is replaced by the two sides of an equilateral triangle. This process is repeated infinitely many times, resulting in the Von Koch curve or snowflake. At iteration* n*, this curve has a length of* $(\frac{4}{3})^n$*, so its length tends to infinity. Nevertheless, as it is the countable union of segments, this fractal has no area. Therefore, it doesn't fit into dimension 1 (infinite length) nor dimension 2 (no area).*

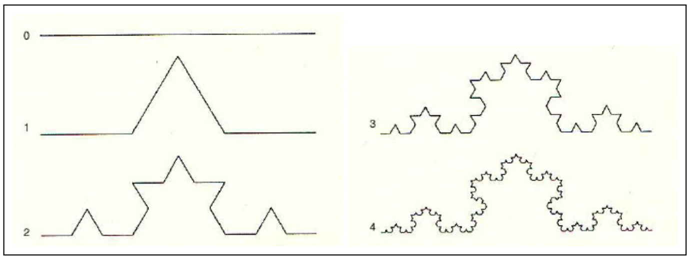

Figure A.6: First steps of the construction of the Von Koch curve.

Example A.4.5. *(Menger sponge)*
In this case, a cube with a unit side is divided into 27 sub-cubes, and only the 20 cubes sharing a side with the original cube are retained. The Menger sponge is the fractal generated by applying this transformation infinitely many times. It is evident that it has an infinite surface (hence its dimension is not 2) but no volume (thus its dimension is not 3).

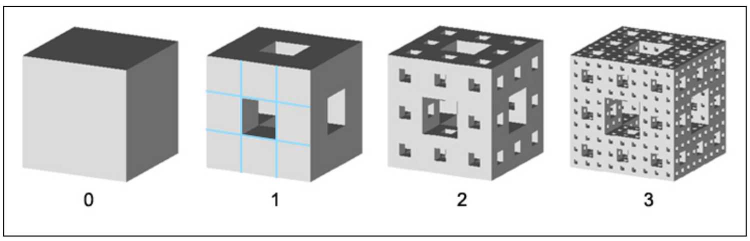

Figure A.7: First steps of the construction of the Menger sponge.

To determine the proper dimension of these sets, it is necessary to introduce a measure for sets in $\mathbb{R}^n$ that generalizes the common notions of length, area, volume, and so forth.

Definition A.4.2. *(s-dimensional Hausdorff measure)*
Given a set $F \subset \mathbb{R}^n$ and a non negative real number s, the s-dimensional Hausdorff measure of F is given by

$$\mathcal{H}^s(F) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(F) = \sup_{\delta > 0} \mathcal{H}^s_\delta(F)$$

where $0 < \delta < \infty$ and $\mathcal{H}^s_\delta(F)$ is defined as

$$\mathcal{H}^s_\delta(F) = \frac{\omega^s}{2^s} \inf\{\sum_{j=1}^{\infty}(diamU_j)^s : F \subset \bigcup_{j=1}^{\infty} U_j, diamU_j \leq \delta\}$$

with $diamU_j = \sup\{|x - y| : x, y \in U_j\}$ and $\omega^s = \pi^{\frac{s}{2}}(\int_0^\infty e^{-x} x^{\frac{s}{2}} dx)^{-1}$

With this definition, it becomes possible to define the Hausdorff dimension of a set.

Definition A.4.3. *(Hausdorff dimension)*
Given a set $F \subset \mathbb{R}^n$, its Hausdorff dimension is defined as

$$dim_{\mathcal{H}}(F) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}$$

Before stating a theorem that provides sufficient conditions for the existence of a set characterized by non integer dimension and the self-similarity property, it is necessary to explain what a collection of similarities is.

Definition A.4.4. *(Collection of similarities)*
A collection of similarities is a map $\Psi : \mathbb{K} \to \mathbb{K}$ mapping the space of compact sets $\mathbb{K}$ into itself consisting in a collection of $I \in \mathbb{N}$ similarities $\{\Psi_i\}_{i=1,..,I}$.
Namely, for any compact set $K \in \mathbb{K}$ we have $\Psi(K) = \Psi_1(K) \cup \Psi_2(K) \cup ... \cup \Psi_I(K)$ where $\Psi_i(K) = p_i + \rho_i \mathbb{M}_i K$ where $p_i \in \mathbb{R}^n$, $0 < \rho_i < 1$ and $\mathbb{M}_i \in \mathbb{R}^{n \times n}$ is a rotation matrix [19].

A collection of similarities satisfies the so known "open set condition" when there exists a bounded open set, denoted as U, such that $\Psi_i(U) \subset U$ for any $i = 1, ..., I$ and $\Psi_i(U) \cap \Psi_j(U) = \emptyset$ for any $i \neq j$.

[19] When Ψ_i is applied to a compact set K, it produces the compact set obtained by rotating K through $\mathbb{M}_i$, scaling it by the factor ρ_i, and then translating it by p_i.

Theorem A.5. *(Hutchinson theorem)*
Let Ψ be a collection of similarities satisfying the open set condition. Then it exists an unique compact set $K \in \mathbb{K}$ invariant with respect to that collection, namely $\Psi(K) = K$. K is called self-similar fractal (Hutchinson fractal) whose Hausdorff dimension $dim_{\mathcal{H}}(K)$ is given by the unique real value s satisfying the equation $\sum_{i=1}^{I} \rho_i^s = 1$.

The theorem can be proven by defining the Kuratowski distance between two sets. With this distance, $\mathbb{K}$ becomes a complete metric space, and Ψ acts as a contraction. Hence, the Contraction Mapping Theorem can be applied, and the fractal set is the result of applying the transformation Ψ infinitely many times to any compact set in $\mathbb{K}$.

Definition A.4.5. *(Kuratowski distance)*
Given two compact sets A, B their Kuratowski distance is given by

$$dist_{\mathcal{K}}(A, B) = \max\{\max_{a \in A} dist(a, B), \max_{b \in B} dist(b, A)\}$$

An easier measure of the dimension of a set is provided by the box-counting dimension, also known as the Minkowski–Bouligand dimension.

Definition A.4.6. *(Box-counting dimension)*
Given a set $F \subset \mathbb{R}^n$, let $N_\delta(F)$ be the minimum possible number of sets with diameter at most δ required to fully cover F, then the box-counting dimension of F is defined as

$$dim_{\mathcal{B}}(F) = \lim_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta}$$

However, this limit, and consequently the box-counting dimension, is not always well defined, while the Hausdorff dimension is always well defined. Nevertheless, when the box-counting dimension of a set can be determined, it coincides with the Hausdorff dimension.

Example A.4.6. *The Cantor set $\mathcal{C}$ is invariant under two similarities, both with a coefficient $\rho = \frac{1}{3}$: $\Psi_1(x) = \frac{1}{3}x$ and $\Psi_2(x) = \frac{2}{3} + \frac{1}{3}x$. Consequently, the Hausdorff dimension s satisfies $2(\frac{1}{3})^s = 1$, leading to $dim_{\mathcal{H}}(\mathcal{C}) = \frac{\log 2}{\log 3} \approx 0.6309$.*

Let $\mathcal{C}n$ be the transformed interval $[0, 1]$ after n iterations. It is easy to observe that 2^n subintervals of length $(\frac{1}{3})^n$ are required to fully cover $\mathcal{C}n$. Therefore, the box-counting dimension is given by $dim_{\mathcal{B}}(\mathcal{C}) = \lim_{\delta \to 0^+} \frac{\log N_\delta(\mathcal{C})}{-\log \delta} = \lim_{n \to \infty} \frac{\log 2^n}{-\log (\frac{1}{3})^n} = \frac{\log 2}{\log 3} \approx 0.6309$.

Example A.4.7. *(Von Koch curve)*
The same reasoning as for the Cantor set holds: both the Hausdorff and the box-counting dimensions are defined, and they coincide. After n iterations, the resulting set can be covered with 4^n *subintervals of length* $(\frac{1}{3})^n$*. Thus, the fractal dimension of the Von Koch snowflake is given by* $\frac{\log 4}{\log 3} \approx 1.2618$*. One would obtain the same result by noticing that the Von Koch curve is invariant under a collection of four similarities with a coefficient* $\rho = \frac{1}{3}$.

Example A.4.8. *(Menger sponge)*
Similarly, the Menger sponge is invariant under a collection of twenty similarities with a coefficient $\rho = \frac{1}{3}$*. Consequently, the Hausdorff dimension, which coincides with the box-counting dimension, is equal to* $\frac{\log 20}{\log 3} \approx 2.7268$.

B | Proofs of the theorems

In this second appendix, the proofs for all the theorems presented in Chapter 2. are provided.

Proof. Theorem 2.1.
the Jacobian matrix J_β, which depends on the parameter β, is given by:

$$J_\beta(x,m) = \begin{pmatrix} \frac{\partial F_1}{\partial x}(x,m) & \frac{\partial F_1}{\partial m}(x,m) \\ \frac{\partial F_2}{\partial x}(x,m) & \frac{\partial F_2}{\partial m}(x,m) \end{pmatrix}$$

The Jacobian matrix computed at the fundamental steady state E is then

$$J_\beta(E) = \begin{pmatrix} v(-tanh(\frac{\beta C}{2})) & 0 \\ 0 & \alpha \end{pmatrix}$$

since it is diagonal, the two eigenvalues are given by$\lambda_1 = v(-tanh(\frac{\beta C}{2}))$ and $\lambda_2 = \alpha$. Considering that $0 \leq \alpha < 1$, the stability of E depends solely on λ_1. E is stable if $|\lambda_1| < 1$.

On the real line $-1 < \lambda_1 < 1$ is equivalent to $\frac{2R}{g} - 1 - \frac{4}{\mu g} < \tanh(\frac{\beta C}{2}) < \frac{2R}{g} - 1$, or more explicitly $\frac{2}{C}\tanh^{-1}(\frac{2R}{g} - 1 - \frac{4}{\mu g}) < \beta < \frac{2}{C}\tanh^{-1}(\frac{2R}{g} - 1)$. For well defined conditions, it is necessary to ensure that $\frac{2R}{g} - 1 - \frac{4}{\mu g} > -1$ and $\frac{2R}{g} - 1 < 1$, leading to the parameter restrictions: $\mu > \frac{2}{R}$; $g > R$.

For $0 < g < R$ and $0 < \mu < \frac{2}{R}$, the equilibrium state E is always stable. This is because $\frac{2R}{g} - 1 - \frac{4}{\mu g} < \tanh(\frac{\beta C}{2}) < \frac{2R}{g} - 1$ holds true for any β.

To determine stability in the range $R < g < 2R$ and $0 < \mu < \frac{2}{R}$, we observe that points below the line $m = 1 - \frac{2R}{g}$ will, after a finite number of iterations, be mapped above this line and subsequently attracted to the equilibrium state E.

Let C_m be the collection of all the points (x, m) such that $F_2(x, m) = m$, in symbols:

$$C_m = \{(x, m) \in \mathbb{R} \times [-1, 1] : F_2(x, m) = m\}$$

The creation of a 2-cycle at infinity for $\mu = \frac{2}{R}$ and the occurrence of a doubling-period bifurcation at $\beta = \beta_{pd}$ for $\mu > \frac{4}{2R-g}$ can be easily verified. This is evident by noting that the points belonging to the 2-cycle are precisely the two intersections between the line $m = 1 - \frac{2R}{g} + \frac{4}{\mu g}$ and the collection C_m. Furthermore, it is possible to demonstrate the instability of the cycle through direct computation.

In the range $R < g < 2R$ and $0 < \mu < \frac{4}{2R-g}$, it is evident that (x, m) with $x \neq 0$ forms a non-fundamental steady state if and only if (x, m) represents the intersection between C_m and the line $m = 1 - \frac{2R}{g}$, the value of m for which $v(m =$ and subsequently $F_1(x, m)$ are zero. The first value of β for which this intersection is possible, marking the occurrence of the pitchfork bifurcation, is denoted as β^*.

□

Proof. Theorem 2.2.
As $z_s = 0$ and the system is symmetric, it is possible to focus exclusively on the stability of E_r, the positive non fundamental steady state. The analysis of E_l follows a similar reasoning.

Fixing the parameters g (between R and $2R$) and μ (between 0 and $\frac{2}{R}$), as β ranges from β^* to ∞ E_r shifts from $(0{,}1 - \frac{2R}{g})$ to $(\sqrt{\frac{C}{g(R-1)}}, 1 - \frac{2R}{g})$ along the line $m = 1 - \frac{2R}{g}$.

The Jacobian computed at E_r is given by:

$$J_\beta(E_r) = \begin{pmatrix} 1 & -\frac{\mu g}{2} x_r \\ \frac{4\beta}{g}(1-\alpha)(R-1)(g-R)x_r & \alpha + \beta\mu R(1-\alpha)(g-R)x_r^2 \end{pmatrix}$$

The stability of E_r can be determined using the trace-determinant criterion [1]

[1]Method for deducing the position of eigenvalues in the complex plane, and consequently, determining the stability of an equilibrium. It involves studying the signs of the trace and determinant of the Jacobian. It is important to note that the method is applicable in a 2-dimensional system, where $\lambda_1 + \lambda_2 = trJ$ and $\lambda_1\lambda_2 = detJ$.

The characteristic equation is

$$Q(\lambda) = \lambda^2 - trJ\lambda + detJ = 0$$

with
$trJ = 1 + \alpha + \beta\mu R(1-\alpha)(g-R)x_r^2$
$detJ = \alpha + \beta\mu R(1-\alpha)(g-R)(2R-1)x_r^2.$

Let λ_1 and λ_2 be the eigenvalues.

For $\beta > \beta^*$, where $x_r > 0$, it is easy to verify that both trJ and $detJ$ are positive. It is well-known that $\lambda_1\lambda_2 = detJ > 0$, indicating that the two eigenvalues remain in the same part of the complex plane. Moreover, $\lambda_1 + \lambda_2 = trJ > 1 + \alpha > 1$, so the eigenvalues belong to the right part of $\mathbb{C}$.

When $\beta = \beta^*$, then $\lambda_1 = 1$ and $\lambda_2 = \alpha < 1$. Notice that $Q(1) > 0$ and $\frac{dQ(1)}{d\beta} > 0$ for any $\beta > \beta^*$, implying that $0 < \lambda_2 < \lambda_1 < 1$ when $\beta > \beta^*$ with $|\beta - \beta^*|$ small.

This implies that 1 is never an eigenvalue when $\beta > \beta^*$.

It is readily apparent that $\frac{d(\lambda_1\lambda_2)}{d\beta} = \frac{d(detJ)}{d\beta} > 0$ for $\beta > \beta^*$, and as $detJ \to \infty$ as $\beta \to \infty$. Consequently, there exists a critical value $\beta_{NS} > \beta$ such that E_r is locally stable for $\beta^* < \beta < \beta_{NS}$ and unstable for $\beta > \beta_{NS}$.

Let x_0 represent the x-coordinate where the Neimark-Sacker bifurcation occurs, and let $J_0 = J_{\beta_{NS}}(E_r)$. The values of β_{NS} and x_0 must satisfy:

$$\begin{cases} detJ_0 = \lambda_1\lambda_2 = 1 \\ |\lambda_1| = |\lambda_2| = 1 \end{cases}$$

It is possible to verify that:

$$\begin{cases} \beta_{NS} = \frac{2}{C} tanh^{-1}(\frac{2R}{g} - 1) + \frac{g(R-1)}{\mu R(g-R)(2R-1)C} \\ x_0 = \sqrt{\frac{C}{g(R-1) + 2\mu R tanh^{-1}(\frac{2R}{g} - 1)(g-R)(2R-1)}} \end{cases}$$

Now the characteristic equation is:

$$Q(\lambda) = \lambda^2 - trJ_0 + 1 = 0$$

where $tr J_0 = \frac{2R+2\alpha(R-1)}{2R-1}$.

Consequently, the two eigenvalues are given by:

$$\lambda_{1,2} = \frac{1}{2} tr J_0 \pm i\sqrt{1 - (\frac{1}{2} tr J_0)^2}$$

Notice that the two eigenvalues are complex conjugates and can be expressed in polar form, i.e., $\lambda_{1,2} = e^{\pm i\theta}$.

It is straightforward to deduce that $0 < \theta < \frac{\pi}{3}$; therefore, $\lambda_{1,2}$ satisfy the non-resonance condition [2]:

$$\lambda_{1,2} \notin \{e^{i2\pi\frac{p}{q}} : p, q = 1, ..., 6\}$$

Moreover, it is clear that $\frac{d|\lambda_1|}{d\beta} = \frac{d|\lambda_2|}{d\beta} > 0$ for $\beta = \beta_{NS}$.

To ascertain the supercritical nature of the Neimark-Sacker bifurcation, it is necessary to compute the normal form (refer to Chapter 4.7 in Kuznetsov (1998) for a more comprehensive treatment). The first step involves determining the eigenvector q that satisfies $J_0 q = e^{i\theta} q$, where

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mu g x_0 \\ 1 - e^{i\theta} \end{pmatrix}.$$

The adjoint eigenvector p, satisfying $J_0^T p = e^{-i\theta} p$, is given by

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{(g-R)R(2R-1)^2}{g(3R-1+\alpha(R-1))}\beta_{NS} x_0 (1 + e^{i\theta}) \\ \frac{(2R-1)^2}{4(1-\alpha)(R-1)(3R-1+\alpha(R-1))}(e^{-i\theta} - e^{i\theta}) \end{pmatrix}.$$

It can be easily verified that $< q, p >= \bar{q}_1 p_1 + \bar{q}_2 p_2 = 1$.

Then

$$\begin{pmatrix} x \\ m \end{pmatrix} = \begin{pmatrix} x_0 + q_1 z + \bar{q}_1 \bar{z} \\ 1 - \frac{2R}{g} + q_2 z + \bar{q}_2 \bar{z} \end{pmatrix}$$

Let's evaluate in $\beta = \beta_{NS}$ the function:

$$H(z, \bar{z}) = \bar{p}_1(F_1(x_0 + q_1 z + \bar{q}_1 \bar{z}, 1 - \tfrac{2R}{g} + q_2 z + \bar{q}_2 \bar{z}) - x_0) + \bar{p}_2(F_2(x_0 + q_1 z + \bar{q}_1 \bar{z}, 1 - \tfrac{2R}{g} + q_2 z + \bar{q}_2 \bar{z}) - (1 - \tfrac{2R}{g}))$$

[2] Condition on the eigenvalues to avoid the occurrence of resonance in nonlinear systems.

Its Taylor expansion in (0,0) is:

$$H(z,\bar{z}) = e^{i\theta z} + \sum_{2\leq j+k\leq 3} \frac{1}{j!k!} g_{jk} z^j \bar{z}^k + O(|z|^4)$$

With further computation, it becomes evident that the critical real part $a(\beta_{NS})$ is negative:

$$a(\beta_{NS}) = Re(\frac{e^{i\theta} g_{21}}{2}) - Re(\frac{(1-2e^{i\theta})e^{-2i\theta}}{2(1-e^{i\theta})} g_{20} g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^4 < 0$$

This holds for $1 < R < 1.2$, $R < g < 2R$ ans $0 < \alpha < 0.6$.

This implies that the Neimark-Sacker bifurcation at $\beta = \beta_{NS}$ is supercritical and a unique stable closed invariant curve bifurcates from E_r for $\beta > \beta_{NS}$ □

Proof. Theorem 2.3.
For $z_s > 0$ the Jacobian at E is given by:

$$J_\beta(E) = \begin{pmatrix} v(-tanh(\frac{\beta C}{2})) & 0 \\ -\frac{(1-\alpha)\beta g z_s}{2}(sech(\frac{\beta C}{2}))^2 & \alpha \end{pmatrix}$$

Also in this scenario (refer to the proof of Theorem 2.1.), the eigenvalues of the Jacobian at E for $z_s > 0$ are $\lambda_1 = v(-\tanh(\frac{\beta C}{2}))$ and $\lambda_2 = \alpha < 1$. The stability of E can be ascertained through direct computation.

E is globally stable for $0 \leq \beta < \beta_{sn}$: observe that the points below the line $m = 1 - \frac{2R}{g}$ will, after a finite number of iterations, be mapped above that line and subsequently attracted by E.

Obviously (x, m) with $x \neq 0$ constitutes a non fundamental steady state precisely when it lies at the intersection of the line $m = 1 - \frac{2R}{g}$ and C_m. This verification establishes the outcome of the saddle-node bifurcation at $\beta = \beta_{sn}$.

As β evolves from β_{sn} to infinity, E_r traverses the path from $(\frac{z_s}{2(R-1)}, 1 - \frac{2R}{g})$ to $(\frac{z_s}{2(R-1)} + \sqrt{\frac{C}{g(R-1)} + (\frac{z_s}{2(R-1)})^2}, 1 - \frac{2R}{g})$ along the line $m = 1 - \frac{2R}{g}$.

The Jacobian at E_r is:

$$J_\beta(E_r) = \begin{pmatrix} 1 & -\frac{\mu g}{2}x_r \\ \frac{4\beta}{g}(1-\alpha)R(R-1)(g-R)(x_r - \frac{z_s}{2(R-1)}) & \alpha + \beta\mu R(1-\alpha)(g-R)x_r^2 \end{pmatrix}$$

where
$trJ = 1 + \alpha + \beta\mu R(1-\alpha)(g-R)x_r^2$
$detJ = \alpha + \beta\mu R(1-\alpha)(g-R)((2R-1)^2 x_r^2 - z_s x_r)$

Let λ_1 and λ_2 be the two eigenvalues.

As discussed in the earlier proof, when $x_r > \frac{z_s}{2(R-1)}$, it follows that $\lambda_1\lambda_2 = detJ > 0$, signifying that the two eigenvalues remain in the same part of the complex plane.

Furthermore, $\lambda_1 + \lambda_2 = trJ > 1 + \alpha > 1$, indicating that the eigenvalues belong to the right part of $\mathbb{C}$.

For $\beta = \beta_{sn}$, direct computation reveals that $\lambda_1 = 1$ and $\lambda_2 = detJ$.

z_s is considered small when $\frac{tanh^{-1}(\frac{2R}{g}-1)\mu R(g-R)z_s^2}{(R-1)(4C(R-1)+gz_s^2)} < 1$. In this scenario it can be demonstrated that $\lambda_2 < 1$ for $\beta = \beta_{sn}$.

Notice that $Q(1) > 0$ and $\frac{dQ(1)}{d\beta} > 0$ for $\beta > \beta_{sn}$, implying that $0 < \lambda_2 < \lambda_1 < 1$ when $\beta > \beta_{sn}$ with $|\beta - \beta_{sn}|$ small. This, in turn, implies that 1 is never an eigenvalue when $\beta > \beta_{sn}$.

Since $\frac{d(\lambda_1\lambda_2)}{d\beta} = \frac{d(\det J)}{d\beta} > 0$ for $\beta > \beta_{sn}$, and $\det J \to \infty$ for $\beta \to \infty$, there exists $\beta_r > \beta_{sn}$ such that E_r is locally stable for $\beta_{sn} > \beta > \beta_r$ and unstable for $\beta > \beta_r$.

Similar reasoning holds for E_l.

Analytical expressions for β_l and β_r are available:

$$\begin{cases} \beta_l = \frac{2tanh^{-1}(\frac{2R}{g}-1)\mu R(g-R)(2R-1)+g(R-1)}{\mu R(g-R)((2R-1)C+gRz_s x_l)} \\ \beta_r = \frac{2tanh^{-1}(\frac{2R}{g}-1)\mu R(g-R)(2R-1)+g(R-1)}{\mu R(g-R)((2R-1)C+gRz_s x_r)} \end{cases}$$

with $-x_0 < x_l < 0 < x_0 < x_r$ where The value x_0 represents the x-coordinate of equilibrium E during the Neimark-Sacker bifurcation when z_s=0.

Therefore $\beta_r < \beta_{NS} < \beta_l$. Given the normal form of the Neimark-Sacker bifurcation depends on z_s with continuity, it follows that both E_l and E_r undergo a supercritical Neimark-Sacker bifurcation, occurring respectively in $\beta = \beta_l$ and $\beta = \beta_r$. □

Proof. Theorem 2.4.
It can be demonstrated that the fundamental steady state E is a dissipative saddle fixed point through direct computation, considering the eigenvalues $\lambda_1 = v(-\tanh(\frac{\beta C}{2}))$ and $\lambda_2 = \alpha$.

E will be a dissipative saddle equilibrium for $\beta > \beta^*$ if $\alpha < \frac{1}{1+\mu(g-R)}$.

This inequality arises from the assumptions on g, R, and μ:

$$\alpha < \sqrt{\frac{\mu R - 1}{\mu g}} < \sqrt{\frac{R - 1/\mu}{g}} < \sqrt{\frac{R/2}{g}} < \frac{1}{2} < \frac{1}{R} < \frac{1}{1+\mu(g-R)}$$

After the pitchfork bifurcation, the unstable manifold of E is promptly attracted by E_l and E_r. Since $0 < \lambda_2 = \alpha < 1$ for $\beta = \beta^*$, the dynamical system near the pitchfork bifurcation can be effectively reduced to the 1-dimensional center manifold. At $\beta = \beta^*$, the two branches of $W^u(E)$ coincide with a portion of the center manifold, forming connections between E and E_l as well as E_r. Consequently, for $\beta > \beta^*$ with $|\beta - \beta^*|$ small, there is no intersection between the stable and unstable manifolds of E.

To gain deeper insights into the properties of the unstable manifold of the fundamental steady state when β is large, it is insightful to initially consider the extreme scenario of $\beta = \infty$.

In this case F_1 remains defined as $F_1(x, m) = v(m)x$ while F_2 takes the following form:

$$F_2(x, m) = \begin{cases} \alpha m + (1-\alpha) & w(x, m) > 0 \\ \alpha m & w(x, m) = 0 \\ \alpha m - (1-\alpha) & w(x, m) < 0 \end{cases}$$

Consider C_r and C_l as the collections of points in the $x-m$ plane where $w(x,m)=0$ on the right and left sides, respectively.

$$\begin{cases} C_l = \{(x,m) \in \mathbb{R}^- \times [-1,1] : x(m) = -\sqrt{\frac{C}{g(R-v(m))}}\} \\ C_r = \{(x,m) \in \mathbb{R}^+ \times [-1,1] : x(m) = \sqrt{\frac{C}{g(R-v(m))}}\} \end{cases}$$

Recall that in the extreme scenario as β approaches infinity, the fundamental steady state E takes the form $E=(0,-1)$.

In the definition of C_r, $x(m)$ is a decreasing function of m. Since $\mu < \frac{R-1}{g-R}$, C_r intersects with the line $m=-1$ at the point $P_0 = \left(\sqrt{\frac{C}{g(R-v(m))}}, -1\right)$. The segment EP_0 lies on the unstable manifold. Let $P_{-1} = \left(\frac{\sqrt{C}}{v(-1)\sqrt{g(R-v(-1))}}, -1\right)$, then $\mathcal{F}(P_{-1}) = P_0$. The first iteration of the segment $P_{-1}P_0$ (excluding the endpoints) is the segment P_0P_1, where $P_1 = \left(\frac{\sqrt{C}v(-1)}{\sqrt{g(R-v(-1))}}, -1\right)$. The second iteration is the segment $P_2'P_2$ with $P_2' = \left(\frac{\sqrt{C}v(-1)}{\sqrt{g(R-v(-1))}}, 1-2\alpha\right)$ and $P_2 = \left(\frac{\sqrt{C}(v(-1))^2}{\sqrt{g(R-v(-1))}}, 1-2\alpha\right)$.

It is evident that for $0 < \alpha < \frac{\mu R-1}{\mu g}$, the segment $P_2'P_2$ lies above the line $m=m_0$, while for $\frac{\mu R-1}{\mu g} < \alpha < \sqrt{\frac{\mu R-1}{\mu g}}$, $P_2'P_2$ lies between the lines $m=m_0$ and $m=m_1=1-\frac{2R}{g}$. Recall that the line $m=m_0$ belongs to the stable manifold of E.

Concerning the second considered case, let's examine the third iterate, yielding a segment $P_3'P_3$ where $P_3' = \left(\frac{\sqrt{C}v(-1)v(1-2\alpha)}{\sqrt{g(R-v(-1))}}, 1-2\alpha^2\right)$ and $P_3 = \left(\frac{\sqrt{C}v(-1)^2v(1-2\alpha)}{\sqrt{g(R-v(-1))}}, 1-2\alpha^2\right)$. It is evident that $P_3'P_3$ lies above the line $m=m_0$.

For β large but finite, the unstable manifold of E will closely resemble the unstable manifold in the extreme case $\beta=\infty$.

Let's consider the case $\frac{\mu R-1}{\mu g} < \alpha < \sqrt{\frac{\mu R-1}{\mu g}}$ (similar reasoning holds for $0 < \alpha < \frac{\mu R-1}{\mu g}$). When $\beta=\infty$, the iteration of P_1P_2' through the map $\mathcal{F}$ results in the segment P_2P_3', which intersects the line $m=m_0$ (the stable manifold of E) at the point $\mathcal{Q}^\infty = \left(\frac{\sqrt{C}v(-1)(\mu R-1)}{\sqrt{g(R-v(-1))}}\left(\frac{1}{\alpha}-1\right), m_0\right)$.

For β large but finite, the unstable manifold of E closely follows the polyline $P_0P_1P_2'P_2P_3'P_3$, and it transversely intersects the stable manifold $m=m_0$ at some point $\mathcal{Q}$ near $\mathcal{Q}^\infty$.

This establishes the existence of a transversal homoclinic point between the stable and unstable manifolds of E when β is sufficiently large.

Thus, by continuity, there exists a critical value $\beta_h > \beta^*$ at which the homoclinic bifurcation occurs. □

Proof. Theorem 2.5.
The application of the Strange Attractor Theorem (see Appendix A) is essential. To fulfill this requirement, three conditions must be met, as elucidated by Floris Takens in his paper titled "Abundance of generic homoclinic tangencies in real-analytic families of diffeomorphisms" .

In particular:

1. $\mathcal{F}$ must be real and analytical;
2. The function $\mathcal{H}(\beta) = -\frac{\log|\lambda_1(\beta)|}{\log|\lambda_2(\beta)|}$ must be not constant;
3. The homoclinic tangency is inevitable, signifying a transition in the system from having no homoclinic points to a configuration with transversal homoclinic points.

In this case, all conditions are satisfied.

Furthermore, the theorem is applied to a (local) diffeomorphism [3]. Thus, it is imperative to demonstrate that $\mathcal{F}$ is a (local) diffeomorphism in the region where the homoclinic bifurcation takes place.

Let $S = \{(x, m) \in \mathbb{R} \times [-1, 1] : -1 < m < m_0\}$ where $m = m_0$ belongs to the stable manifold of the fundamental steady state E and let $S' = \mathcal{F}(S)$.

It is essential to establish that $\mathcal{F} : S \to S'$ is a diffeomorphism.

Verification through direct computation confirms that for any $p \in S$, the determinant of the Jacobian of $\mathcal{F}$ is positive. Furthermore, for two distinct points p and q, it is straightforward to demonstrate that $\mathcal{F}(p) \neq \mathcal{F}(q)$, under the condition $\mu < \frac{R-1}{g-R}$.

[3] A diffeomorphism is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable.

Consider now the region above the line $m = m_0$.

Let J^0 be the collection of all the points where the determinant of the Jacobian of $\mathcal{F}$ vanishes. In symbols:

$$J^0 = \{(x,m) \in \mathbb{R} \times [-1,1] : \alpha v(m) + (1-\alpha)\frac{\mu g^2 \beta}{4}(sech[\beta w(x,m)])^2(2R - v(m))x^2 = 0\}$$

Verification reveals that J^0 exhibits three branches in the strip $m_0 \leq m \leq 1$, symmetric with respect to the m-axis and strictly above the line $m = m_0$, except at the point $(0, m_0)$. Consequently, in a neighborhood where the homoclinic bifurcation occurs, $\mathcal{F}$ is locally a diffeomorphism, allowing the application of the Strange Attractor Theorem. □

Proof. Theorem 2.6.
It is necessary to demonstrate that for sufficiently large β, the map $\mathcal{F}$ forms a local diffeomorphism in proximity to the homoclinic orbit.

From the proof of Theorem 2.4. it is established that for sufficiently large β, a transversal homoclinic point Q exists. The mapping $\mathcal{F}$ transforms Q into a point on the x-axis, situated strictly below the line $m = \alpha m_0 + (1-\alpha)$. Additionally, the second iterate of Q remains on the x-axis, positioned strictly below the line $m = m_0$ for $0 < \alpha < \sqrt{\frac{\mu R - 1}{\mu g}}$.

Now, J^0 reduces to a single point, namely $(0, m_0)$. Moreover, as β approaches infinity, J^0 converges toward the line $m = m_0$ and the segments of the curves C_l and C_r situated between the lines $m = m_0$ and $m = 1$.

It can be established that, except for possibly one value of α (the one leading to $\mathcal{F}(Q^\infty) = (0, m_0)$ as β approaches infinity), $\mathcal{F}$ is locally invertible in the vicinity of every point on the homoclinic orbit passing through Q.

Hence, the unstable manifold intersects the stable manifold at every point along the homoclinic orbit. This establishes the existence of chaotic orbits for large values of β attributed to the presence of transversal homoclinic orbits. □

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