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Executive Summary of the Thesis

Displacement detection of an oscillator based on a nano-mechanical qubit

Laurea Magistrale in Engineering Physics - Ingegneria Fisica

Author: Andrea Amico

Advisor: Prof. Andrea Crespi

Co-advisor: Prof. Fabio Pistolesi

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1. Introduction

In recent years mechanical oscillators have been proposed as an alternative for the implementation of quantum computing. One of the proposal for a nanomechanical qubit is a suspended carbon nanotube in which a double quantum dot is integrated using gate voltages. In this work we study one of the possible read-out methods of this mechanical oscillator, exploiting a single electron transistor (SET). The SET, composed of a metallic island isolated by tunneling junctions from the rest of a circuit, allows the tunneling of one electron at a time, thus in principle being suitable for high sensitivity detection of the oscillator displacement. Here, we compute the back action of the SET on the oscillator, presenting the oscillator excitation in terms of an effective temperature. We show the exponential dependence of the effective temperature on the voltage bias applied at the SET electrodes. The dependence of the back action on the energy level of the SET metallic island is displayed too. Then, we compute the expression of the current of the SET, retrieving the dependence on the oscillator displacement. This computation is done both semiclassically and with a full quantum description of the oscillator-SET coupling. We formulate the sensitivity of the SET with the derivative of the SET conductance with respect to the oscillator displacement. We express this figure of merit as a function of the metallic island energy level and the tunneling rates, thus allowing to estimate the behaviour of the device in different operational regimes, defined by parameters such as the applied voltage on the SET and by the physical values of the oscillator-SET system.

2. Oscillator description

Mechanical oscillators are attractive for encoding information due to a variety of features.Since any force leads to a mechanical displacement, they show to be sensitive to a wide range of fields. Also, they show long coherence times. This opens up to the possibility of developing circuits of multiple qubits, as usually the coherence time collapses when multiple qubit are entangled. The scaling problem is indeed one, if not the biggest, challenge in the physical implementation of quantum circuits. In this work we begin from the structure proposed in [1], consisting of a suspended carbon nanotube whose flexural modes are coupled to the charge state of an integrated double quantum dot. The double

dot is obtained thanks to multiple gates generating electrostatic potential along the nanotube and is a key element, introducing the anharmonicity necessary to isolate energetically the two lower oscillating modes. These are the two states needed for the realization of the qubit.
The harmonic oscillator has discrete modes of oscillations all separated by the same energy. By introducing anharmonicity, exploiting the double dot in this case, the energy separations between the levels are changed. In this way, the two first modes of oscillation are made accessible to be read-out an manipulated, thus allowing the encoding of information in their superposition. Note that the stronger the coupling between the double dot and the oscillator, the stronger the anharmonicity.
The anharmonicity also takes part in driving the coupling between the oscillator and the read-out method proposed by this paper. Being a charge present in the double dot, it is suitable to interact trough Coulomb's force, coupling capacitively with the SET performing the read-out.

2.1. Read-out device

A SET is composed by a metallic island, separated by two tunneling junctions from the rest of the circuit. Here the island is micrometric, thus obtaining a quantum dot whose energy levels are going to be discrete. The dimensions of the dot can be adjusted in a way that only one level is accessible to the electrons, thus allowing for single electron tunneling, because of the Pauli exclusion principle. Other than the size of the island, one can act on the SET by changing the gate voltage that will change the energy level of the island. Furthermore, bias voltage can be applied to the leads of the SET inducing different current regimes. The SET tunneling rates describing the transport properties are indeed dependent on the gate and bias voltages, and on the energy level of the central dot. One can start from the Fermi golden rule describing the transition rate from an initial to a final state corresponding to a one electron transfer from the left contact to the island, and obtain the following tunneling rate:

$$\Gamma_L^+ = \frac{2\pi}{\hbar}\rho|M|^2 f_F[\epsilon_1 - ec_g(V_g - V_L) - ec_R(V_R - V_L)] \tag{1}$$

Here ρ is the density of states at the Fermi level, assumed constant and equal for both the metal leads and M is the matrix element associated to tunneling from the left contact to the island. f_F is the Fermi distribution function that incorporated the dependence on the energy level of the island trough $\epsilon_1 = \epsilon_D + \frac{e^2-2Qe}{2C_\Sigma}$, where the first term is the island energy level and the second term is the difference in Coulomb energy of the SET before and after the tunneling event. The Fermi distribution also contains the dependence on the voltage bias and the gate voltage. c_g and c_R are adimensional constant relative to the capacitance in the circuit, V_g is the gate voltage,V_R and V_L are the right and left lead voltages with their difference being the bias voltage. In the following we are going to group the constants in $\Gamma = \frac{2\pi}{\hbar}\rho|M|^2$.
The single electron transistor is assumed to be working in sequential tunneling regime i.e. with the charges allowed only to go from the left lead to the central island and then to the right lead. To realize this configuration the gate and bias voltage can be chosen in a suitable way.
As already stated, due to capacitive coupling between the charge in the island and the charge in the double quantum dot, one can expect that the device could be very sensitive to any change in the charge of the double dot. Since any displacement of the oscillator leads to a variation of the balance between the charge of the double quantum dot, one can detect in this way the displacement of the carbon nanotube. The Hamiltonian of the oscillator will be:

$$H_S = \epsilon_D\hat{n} - F_0\hat{n}\hat{x} + g_c\hat{n}\sigma_z + \frac{\epsilon_{DD}}{2}\sigma_z + \frac{t}{2}\sigma_x + \hbar\omega_m a^\dagger a + g\sigma_z\hat{x} \tag{2}$$

where a is the destruction operator for the displacement of the mechanical oscillator, ω_m its frequency, $\hat{n}$ the charge operator for the single dot, ϵ_D is the energy level of metallic island, ϵ_{DD} the energy of the double dot σ_z and σ_x are the Pauli matrices associated to the two level system composed by the double dot, with t the hopping energy needed for the transition from one to the other via tunneling. Furthermore F_0 and g_c are the coupling of the charge state in the SET dot with the mechanical displacement and the charge of the single dot respectively. Lastly g is the coupling between the double dot and the displacement.

3. SET back action

To study the back action of the SET on the oscillator we started from the Langevin equation of the oscillator [2] taking in to account the oscillator interaction with the SET. Exploiting the results of the quantum linear response theory [3],

we write the quantum noise, w.r.t. the oscillator, associated to the charge state of the dot as:

$$S_n(\omega) = \int_{-\infty}^{+\infty} \langle n(t)n(0)\rangle e^{-i\omega t} dt. \quad (3)$$

From this we retrieve the expression for the damping and the fluctuations induced in the motion of the oscillator by the charge state $\hat{n}$ of the island. Once again via quantum linear response theory we put together these results to obtain an expression for the effective temperature provoked by the back action on the oscillator.
We considered two leads being at a voltage difference V, leading to the Fermi levels being at an energy difference $-eV$. Fermi distributions in the leads will read:

$$f_R(\epsilon_D, V) = \frac{1}{e^{\beta(\epsilon_D + eV/2)} + 1} = \frac{1}{e^R + 1} \quad (4)$$

$$f_L(\epsilon_D, V) = \frac{1}{e^{\beta(\epsilon_D - eV/2)} + 1} = \frac{1}{e^L + 1} \quad (5)$$

Note that in this paper we are going to consider implicit the dependence of the Fermi distributions on energy. With this assumptions we retrieved the following equations, respectively for damping, fluctuations and effective temperature:

$$\gamma = \frac{\gamma_1}{2}(f_L(1-f_L) + f_R(1-f_R)), \quad (6)$$

$$S_F = \frac{S_1}{2}(f_L + f_R)(2 - f_L - f_R), \quad (7)$$

$$\frac{T_{eff}}{T} = 1 + \frac{1}{2}\frac{(f_L - f_R)^2}{f_L + f_R - f_L^2 - f_R^2}. \quad (8)$$

where $S_1 = F_0^2/4\Gamma m k_B T$ and $\gamma_1 = F_0^2/4\Gamma$, with m the mass of the oscillator and T the temperature. The damping is characterized by two peaks coinciding with the Fermi levels of the two metal leads, i.e. when at $\epsilon_D = \pm eV/2$, displaying a local minimum for $\epsilon_D = 0$. This local minimum

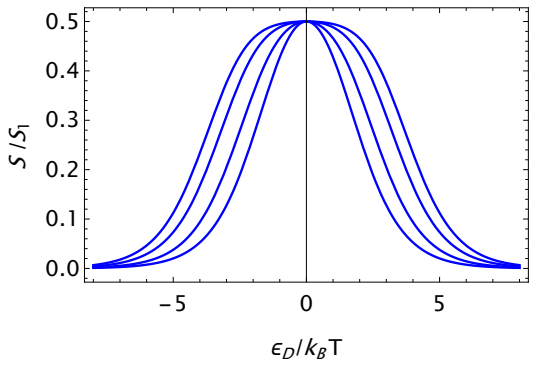

Figure 1: Fluctuations for $V_1 = 2, 3.5, 5, 6$. FWHM is proportional to eV

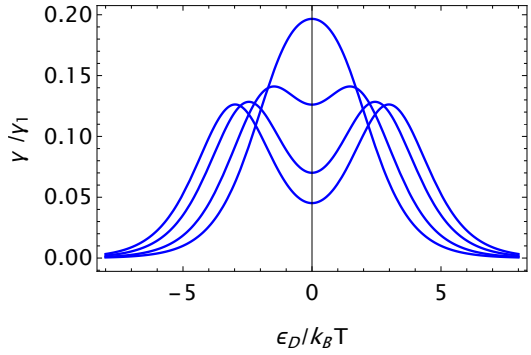

Figure 2: Damping for $V_1 = 2, 3.5, 5, 6$

appears just below a certain value of the bias voltage. Both in Figures 2 and 3 dependence of damping and effective temperature on the energy of the dot is displayed for different values of the adimensional parameter $V_1 = eV/k_BT$ related to the bias voltage. We see that for high voltage the damping narrows and loses the local minimum, which turns in to a maximum. On the other hand the effective temperature broadens and its maximum increases exponentially.
Both damping and fluctuations go to zero for $|\epsilon_D| >> eV/2$. One can see that the fluctuations max value does not depend on the bias voltage applied (see Figure 1), while the FWHM is eV. The function increases like $1 - (f_R)^2$ for $\epsilon_D \to -(eV/2)^-$ and decreases like $f_L(2 - f_L)$ for $\epsilon_D \to (eV/2)^+$. Near the zero this term increases more or less slowly depending on V, reaching the maximum for $\epsilon_D = 0$. The effective temperature starts from the limiting value of T and reaches a maximum for $\epsilon_D = 0$ showing exponential behaviour around $\pm eV/2$. We un-

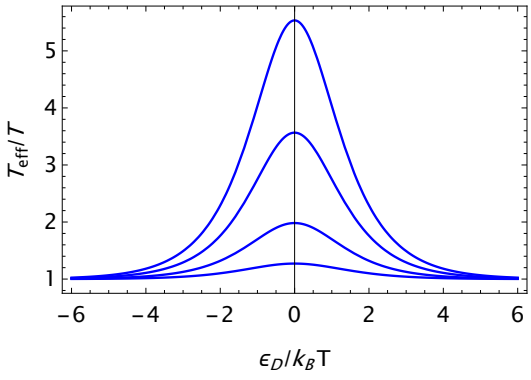

Figure 3: T_{eff}/T for $V_1 = 2, 3.5, 5, 6$. FWHM here is given by k_BT

derline that the dependence of the effective temperature on the voltage has an exponential nature, therefore in experiments one would want to limit the bias voltage. For $eV > k_BT$ the effective temperature rapidly increases reaching high value at its maximum, since for energy among zero the damping term reaches very low values, while the fluctuation for $\epsilon_D = 0$ is constant.

4. Sensitivity to displacement

4.1. Current expression

In order to evaluate the device sensitivity to displacement we first had to evaluate the current trough the SET as a function of displacement. In doing so we carried out both a semiclassical computation and a full quantum treatment of the problem, both actually yielding in the end the same result:

$$I = \frac{g\langle x\rangle}{k_BT}\int \frac{d\omega}{2\pi}\frac{\Gamma_L\Gamma_R}{\omega^2+\frac{\Gamma}{4}^2}(f_L(\omega)(f_L(\omega)-1)- f_R(\omega)(f_R(\omega)-1)). \tag{9}$$

Here Γ_L and Γ_R are the left and right lead tunneling rates, $\langle x\rangle$ is the expectation value of the position operator associated to the oscillator. Hereafter we are going to consider $f_L(\omega) = f_R(\omega) = f(\epsilon_D - \mu_0 - \omega)$ where μ_0 is the chemical potential of the unperturbed right and left electrodes.

4.2. Sensitivity

From Equation 9 it was possible to evaluate the conductance obtaining:

$$G = G_0 - xg\int \frac{d\omega}{2\pi}\frac{\Gamma_L\Gamma_R}{\omega^2+\frac{\Gamma}{4}^2}f''(\epsilon_D-\mu_0-\omega) \tag{10}$$

where $G_0 = \int \frac{d\omega}{2\pi}\frac{\Gamma_L\Gamma_R}{\omega^2+\frac{\Gamma}{4}^2}f'(\epsilon_D-\mu_0-\omega)$, and f' and f'' are respectively the first and the second derivative of the Fermi distribution function. To have a measure of sensitivity we proceeded by analyzing the derivative of conductance with respect to displacement, i.e.:

$$\frac{\partial G}{\partial x} = -F_0\int \frac{d\omega}{2\pi}\frac{\Gamma_L\Gamma_R}{\omega^2+\frac{\Gamma}{4}^2}f''(\epsilon_D-\mu_0-\omega) \tag{11}$$

Applying the following changes in variables: $y = \frac{\omega}{\Gamma/2}, \epsilon_1 = \frac{\epsilon_D-\mu_0}{k_BT}, \Gamma_1 = \frac{\Gamma}{k_BT}$, so to have adimensional parameters, we got:

$$\frac{\partial G}{\partial x} = -\frac{F_0\Gamma_L\Gamma_R}{\pi\Gamma^3}\Gamma_1^2\int dy\frac{1}{y^2+1}f''(\epsilon_1-\frac{\Gamma_1}{2}y). \tag{12}$$

In this way have an expression composed by a constant, $S_0 = -\frac{F_0\Gamma_L\Gamma_R}{\pi\Gamma^3}$ (which has the dimensions of a conductance divided by a displacement) times an adimensional function $\frac{\partial G}{\partial x} = S_0S(\epsilon_1,\Gamma_1)$, with :

$$S(\epsilon_1,\Gamma_1) = \Gamma_1^2\int dy\frac{1}{y^2+1}f''(\epsilon_1-\frac{\Gamma_1}{2}y). \tag{13}$$

Via numerical calculation we obtained the results displayed in Figure 4, where the sensitivity is normalized on S_0. Also, we show just half of the function since it is anti symmetric. The maximum value of the function increases with higher values of Γ_1. This means that the lower is the temperature the higher will be the sensitivity in detecting the fluctuations of the current. Nonetheless, we note that for values of Γ_1 higher than 10 the increase in the maximum sensitivity steadily slows down. To better understand this behaviour we approach the problem analytically distinguishing between two cases. It can be calculated that for high temperatures the above expression will be reduced to:

$$\left.\frac{\partial G}{\partial x}\right|_{\Gamma_1\ll 1} = S_0\pi\Gamma_1^2f''(\epsilon_1), \tag{14}$$

in these conditions the Fermi distribution functions appearing in the integral can be considered smooth with respect to the Lorentzian. The maximum of this function is obtained for

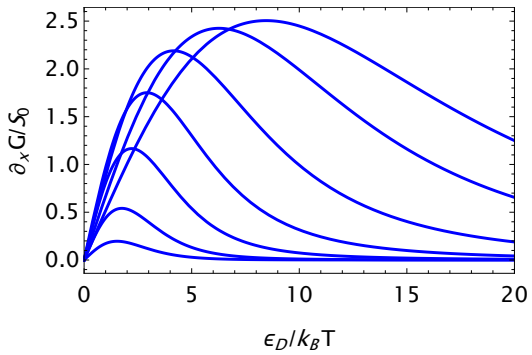

Figure 4: Device Sensitivity as a function of the dot energy, for values of $\Gamma_1 = 1, 2, 4, 7, 12, 20, 28$

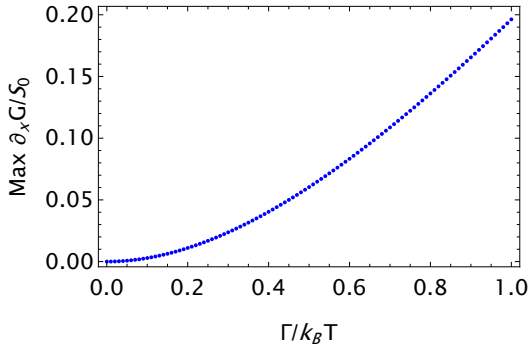

Figure 5: Max sensitivity of the device for values of Γ_1 from 0 to 1 with a 0.01 step

$\epsilon_1 = \log(2-\sqrt{3})$, therefore:

$$\left.\frac{\partial G}{\partial x}\right|_{Max} = F_0 \frac{\Gamma_L \Gamma_R}{\Gamma^3} \frac{\Gamma_1^2}{6\sqrt{3}} = \frac{\Gamma_L \Gamma_R}{\Gamma(k_B T)^2} \frac{F_0}{6\sqrt{3}}, \tag{15}$$

Note that for very high temperature the absolute value of the max sensitivity follows a quadratic relation with respect to Γ_1 (see Figure 5). This case in interesting because it corresponds to conditions realized experimentally.

For very low temperatures, on the other hand, the Fermi distribution functions could be considered peaked and, integrating per parts, we obtain:

$$\left.\frac{\partial G}{\partial x}\right|_{\Gamma_1 \gg 1} = S_0 \frac{8(\epsilon_1/\frac{\Gamma_1}{2})}{(1+(\epsilon_1/\frac{\Gamma_1}{2})^2)^2} \tag{16}$$

A scent of the behaviour of sensitivity for low temperature can be obtained from Figure 6, in the region for which $\Gamma_1 > 10$. This expression

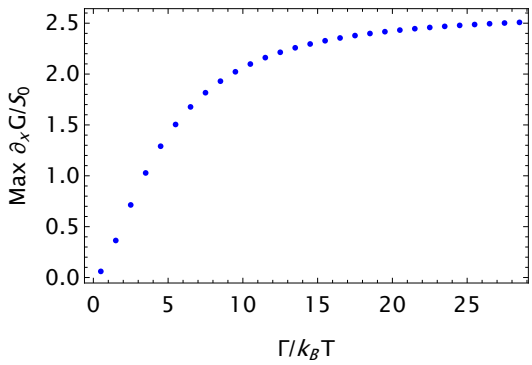

Figure 6: Max sensitivity of the device for values of Γ_1 from .5 to 28.5 with a 1 step

has its global maximum for $\epsilon_1 = \Gamma_1/2\sqrt{3}$, leading to:

$$\left.\frac{\partial G}{\partial x}\right|_{Max} = \frac{3\sqrt{3}}{2} S_0 = \frac{3\sqrt{3}}{2\pi} F_0 \frac{\Gamma_L \Gamma_R}{\Gamma^3} \tag{17}$$

The value of the maximum sensitivity does not depend on temperature in agreement with the behaviour of the sensitivity function for high values of Γ observed in numerical results. The maximum sensitivity achievable therefore depends on physical values characteristic of the system i.e. the coupling F_0 and the tunneling rates. As expected, the max sensitivity is also proportional to the coupling constant in both the limiting conditions.

5. Conclusions

In conclusion we studied the sensitivity and the back-action, of the oscillator-SET system. This will allow for the tuning of the main parameters, so to obtain any configuration at will. Future developments could include the study of the noise associated to conductance. Another possible development is the study of the second order quantum treatment which may lead to a different expression for current with respect to the semi classical one.

6. Acknowledgements

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