POLITECNICO MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE

Executive Summary of the Thesis

A Fourier Series-Based Semi-Analytical Model for 3D Low Thrust Collision Avoidance

Laurea Magistrale in Space Engineering

Author: Alessio Bocci

Advisor: Prof. Juan Luis Gonzalo Gómez

Co-advisor: Camilla Colombo

Academic year: 2021-2022

1. Introduction

The number of satellites is growing continuously year after year. This, coupled with the problem of space debris, leads to possible hazardous impacts with resident orbiting objects. For this reason, collision avoidance manoeuvres (CAMs) are planned to mitigate the risk. The increasing number of close approaches and objects makes analysis more complex and operator time-demanding, thus came the need of computationally efficient models for preliminary analysis. The aim of this work is to develop a semi-analytical mathematical model for the 3D low thrust collision avoidance problem. Some relevant research papers investigate such a problem. In Gonzalo et al. [2] the analytical expression for the variations of the Keplerian elements as a function of the eccentric anomaly is obtained by means of incomplete elliptic integrals of first and second kind. In Gonzalo et al. [4] those solutions are decomposed into a sum of two contributions: a mean value and an oscillatory term. Such a decoupling yields both to an easier way to handle such solutions and a faster evaluation of the time law. This is somehow a precursor of what we will do in this work. The main assumption in these two works is the approximation at zero order of the temporal derivative of the eccentric anomaly. In Gonzalo et al. [3] this limitation is overcome with a first order approximation. In Gao [1] the problem is analysed in terms of three types of control laws: the perigee centered tangential steering, the apogee centered inertial steering and the piecewise constant yaw steering. It is mostly the only one who tries to build up a complete model relying its argumentation on a sort of *superposition principle* suitably adapted for non linear equations. In our work instead, starting from Gauss Planetary Equations in absence of any environmental perturbation, we first passed from the time-derivative formulation to a true anomaly-derivative one. Then, assuming a small variation of the Keplerian parameters after the application of the thrust action, we performed a Taylor expansion in the neighbourhood of the reference condition. From this point on, we developed two different methods. The first, denoted as *full model*, consists of a direct integration of the system obtained after the expansion. On the contrary, the second, denoted as *small thrust model*, has in addition a MacLaurin expansion of the previous equations with the aim of making explicit the dependence from the small thrust parameters. In both cases the integrations are performed by means of the Fourier Series tool, allowing not only to carry out the integrations in an easier manner, but also to take apart the constant and oscillatory contributions of the solutions. In the case of the *small thrust model* the expressions of the Fourier Series coefficients are provided in closed form in terms of complete elliptic integrals and series expansions involving Gauss Hypergeometric function. Finally, different simulations with various test cases are provided to assess the accuracy and the effectiveness of the method.

2. Preliminary definitions

Let $f(\theta)$ be a 2π-periodic function, then its Fourier Series expansion $\mathfrak{F}[f](\theta)$ is:

$$\mathfrak{F}[f](\theta) = \frac{\alpha_0[f]}{2} + \mathcal{P}_{2\pi}\left(\begin{array}{c} \alpha_n[f] \\ \beta_n[f] \end{array} \middle| \theta \right) \tag{1}$$

where $\mathcal{P}_{2\pi}$ is the Periodic P function of period 2π defined as:

$$\mathcal{P}_{2\pi}\left(\begin{array}{c} \alpha_n[f] \\ \beta_n[f] \end{array} \middle| \theta \right) = \\ = \sum_{n=1}^{+\infty} \{\alpha_n[f]\cos(n\theta) + \beta_n[f]\sin(n\theta)\}$$

and $\alpha_0[f]$, $\alpha_n[f]$ and $\beta_n[f]$ are the coefficients of the Fourier Series expansion defined as:

$$\alpha_0[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\mathrm{d}\theta$$
$$\alpha_n[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\mathrm{d}\theta$$
$$\beta_n[f] = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\mathrm{d}\theta$$

The main advantage of expanding f in the form of Equation (1) is that the operations of derivation and integration becomes straightforward. In particular:

Definition 2.1. The θ-integral of $\mathfrak{F}[f](\theta)$ is:

$$\int \mathfrak{F}[f](\theta)\mathrm{d}\theta = \frac{\alpha_0[f]}{2}\theta + \mathcal{P}_{2\pi}\left(\begin{array}{c} -\beta_n[f]/n \\ \alpha_n[f]/n \end{array} \middle| \theta \right)$$

In a similar fashion other integrals such as the one of $\theta f(\theta)$ or $\exp(-k\theta)f(\theta)$ can be obtained. We are not reporting those here for brevity.

3. Gauss Planetary Equations overview

In general, the Gauss Planetary Equations is a time domain set of ODEs of the kind:

$$\begin{aligned} \mathrm{d}a/\mathrm{d}t &= \alpha(a, e, \theta; a_t) \\ \mathrm{d}e/\mathrm{d}t &= \beta(a, e, \theta; a_t, a_n) \\ \mathrm{d}i/\mathrm{d}t &= \gamma(a, e, \omega, \theta; a_h) \\ \mathrm{d}\Omega/\mathrm{d}t &= \delta(a, e, i, \omega, \theta; a_h) \\ \mathrm{d}\omega/\mathrm{d}t &= \epsilon(a, e, i, \omega, \theta; a_t, a_n, a_h) \\ \mathrm{d}\theta/\mathrm{d}t &= \zeta(a, e, \theta; a_t, a_n) \end{aligned} \tag{2}$$

In the system (2), the **shape problem** $\{a, e, \theta\}$ is decoupled from the **orientation problem** $\{i, \Omega, \omega\}$ because, since $\{a_t, a_n, a_h\}$ are constant, the functions α, β and ζ form a self consisting subsystem of ODEs. Assuming as independent variable the true anomaly θ and indicating with $()'$ all the θ-derivatives, the system (2) is led to:

$$\begin{aligned} a' &= \alpha/\zeta = R(a, e, \theta; a_t, a_n) \\ e' &= \beta/\zeta = S(a, e, \theta; a_t, a_n) \\ t' &= 1/\zeta = \tau(a, e, \theta; a_t, a_n) \\ i' &= \gamma/\zeta = \mathfrak{I}(a, e, \omega, \theta; a_t, a_n, a_h) \\ \Omega' &= \delta/\zeta = \mathfrak{O}(a, e, i, \omega, \theta; a_t, a_n, a_h) \\ \omega' &= \epsilon/\zeta = \mathcal{O}(a, e, i, \omega, \theta; a_t, a_n, a_h) \end{aligned} \tag{3}$$

The constant small thrust parameters are present inside these functions in terms of complicated algebraic forms. They are not referred into the functional dependencies in the further computations to have a simpler notation.

4. The *full model*

The solution procedure consists of expanding in Taylor series all the previous functions near to the reference condition $\boldsymbol{x}_0 = \{a_0, e_0, i_0, \omega_0\}$. All functions are expanded up to first order with the exception of S. We also assume that i is affected in its variation from a and e much more than from ω. This assumptions are done coherently with the results of different numerical simulations. The system (3) reduces to:

$$a' = R_0 + R_a\bar{a} + R_e\bar{e} \tag{4a}$$
$$e' = S_0 \tag{4b}$$
$$t' = \tau_0 + \tau_a\bar{a} + \tau_e\bar{e} \tag{4c}$$
$$i' = \mathfrak{I}_0 + \mathfrak{I}_a\bar{a} + \mathfrak{I}_e\bar{e} \tag{4d}$$
$$\Omega' = \mathfrak{O}_0 + \mathfrak{O}_a\bar{a} + \mathfrak{O}_e\bar{e} + \mathfrak{O}_i\bar{i} + \mathfrak{O}_\omega\bar{\omega} \tag{4e}$$
$$\omega' = \mathcal{O}_0 + \mathcal{O}_a\bar{a} + \mathcal{O}_e\bar{e} + \mathcal{O}_i\bar{i} + \mathcal{O}_\omega\bar{\omega} \tag{4f}$$

Where, if $f(\boldsymbol{x},\theta)$ is a generic function, then:

$$f_0 = f(\boldsymbol{x}_0,\theta), \quad f_{x_j} = \left.\frac{\partial f}{\partial x_j}\right|_{(\boldsymbol{x}_0,\theta)}$$

and, if x is the generic Keplerian element, then:

$$\bar{x} = x - x_0$$

4.1. Zero order solution for the orbital shape problem

Assuming that both variations of a and e are driven by the zero order terms, Equation (4a) and Equation (4b) become:

$$a' = R_0(\theta) \quad \text{and} \quad e' = S_0(\theta)$$

and integrating we get :

$$\boxed{a(\theta) - a_0 = \frac{\alpha_0[R_0]}{2}\theta + \Xi_a(\theta)} \tag{5}$$

$$\boxed{e(\theta) - e_0 = \frac{\alpha_0[S_0]}{2}\theta + \Xi_e(\theta)} \tag{6}$$

with:

$$\Xi_a(\theta) = -\frac{\alpha_0[R_0]}{2}\theta_0 + \left[\mathcal{P}_{2\pi}\left(\begin{array}{c} -\beta_n[R_0]/n \\ \alpha_n[R_0]/n \end{array}\middle|\, \xi\right)\right]_{\theta_0}^{\theta}$$

$$\Xi_e(\theta) = -\frac{\alpha_0[S_0]}{2}\theta_0 + \left[\mathcal{P}_{2\pi}\left(\begin{array}{c} -\beta_n[S_0]/n \\ \alpha_n[S_0]/n \end{array}\middle|\, \xi\right)\right]_{\theta_0}^{\theta}$$

Plugging Equation (5) and Equation (6) into Equation (4c) and integrating the corresponding time law assumes the form:

$$\boxed{t - t_0 = A\theta^2 + B(\theta)\theta + C(\theta) - k_t} \tag{7}$$

where k_t is a constant.

4.2. First order approximation for the semi-major axis

The general statement is expressed by Equation (4a) and Equation (4b). Taking the θ-integral of Equation (4b) as in Equation (6) and plugging that solution into Equation (4a) we obtain:

$$a' = \mathfrak{R}_0(\theta) + R_a(\theta)(a - a_0)$$

where:

$$\mathfrak{R}_0(\theta) = \mathfrak{R}_{01}(\theta) + \mathfrak{R}_{02}(\theta)\theta$$

with:

$$\mathfrak{R}_{01} = R_0 + R_e\Xi_e, \quad \mathfrak{R}_{02} = \frac{\alpha_0[S_0]}{2}R_e$$

Then with the change of variable:

$$\bar{a} = \exp[\mathfrak{R}_a(\theta)]v(\theta), \quad \mathfrak{R}_a(\theta) = \int R_a(\theta)\mathrm{d}\theta$$

the ODE reduces to:

$$v' = \exp[-\mathfrak{R}_a(\theta)]\mathfrak{R}_0(\theta), \quad v(\theta_0) = 0$$

Hence the overall solution is given by:

$$\bar{a} = \exp[\mathfrak{R}_a(\theta)]\int_{\theta_0}^{\theta}\exp[-\mathfrak{R}_a(\xi)]\mathfrak{R}_0(\xi)\mathrm{d}\xi \tag{8}$$

The relevant integral to be performed in Equation (8) is:

$$\mathfrak{I}_t(\theta) = \int \exp[-\mathfrak{R}_a(\theta)]\mathfrak{R}_0(\theta)\mathrm{d}\theta \tag{9}$$

where $\mathfrak{R}_a$ is defined as:

$$\mathfrak{R}_a(\theta) = k\theta + \mathcal{P}_{2\pi}\left(\begin{array}{c} -\beta_n[R_a]/n \\ \alpha_n[R_a]/n \end{array}\middle|\, \theta\right)$$

with $k = \alpha_0[R_a]/2$. Then its exponential can be conveniently decomposed as:

$$\exp[\pm\mathfrak{R}_a(\theta)] = \exp[\pm k\theta]\,\Xi_{\pm\mathfrak{R}_a}(\theta)$$

where $\Xi_{+\mathfrak{R}_a}$ and $\Xi_{-\mathfrak{R}_a}$ are defined as:

$$\Xi_{\pm\mathfrak{R}_a}(\theta) = \exp\left[\pm\mathcal{P}_{2\pi}\left(\begin{array}{c} -\beta_n[R_a]/n \\ \alpha_n[R_a]/n \end{array}\middle|\, \theta\right)\right]$$

Then the integral in Equation (9) reduces to:

$$\mathfrak{I}_t(\theta) = \int\left\{\hat{\mathfrak{R}}_{01}(\theta) + \hat{\mathfrak{R}}_{02}(\theta)\theta\right\}\exp[-k\theta]\mathrm{d}\theta$$

with:

$$\hat{\mathfrak{R}}_{01}(\theta) = \Xi_{-\mathfrak{R}_a}(\theta)[R_0(\theta) + R_e(\theta)\Xi_e(\theta)]$$
$$\hat{\mathfrak{R}}_{02}(\theta) = \Xi_{-\mathfrak{R}_a}(\theta)\alpha_0[S_0]R_e(\theta)/2$$

whose solution assumes the form:

$$\mathfrak{I}_t(\theta) = [\mathfrak{B}_0(\theta) + \mathfrak{B}_1(\theta)\theta]\exp[-k\theta] \tag{10}$$

So finally, plugging Equation (10) into Equation (8) we get:

$$\boxed{\bar{a} = \tilde{\mathfrak{B}}_0(\theta) + \tilde{\mathfrak{B}}_1(\theta)\theta + \tilde{\mathfrak{B}}_2(\theta)\exp[k\theta]} \tag{11}$$

with:

$$\tilde{\mathfrak{B}}_0 = \Xi_{+\mathfrak{R}_a}\mathfrak{B}_0, \quad \tilde{\mathfrak{B}}_1 = \Xi_{+\mathfrak{R}_a}\mathfrak{B}_1$$
$$\tilde{\mathfrak{B}}_2 = -\Xi_{+\mathfrak{R}_a}\mathfrak{I}_t(\theta_0)$$

Plugging Equation (11) and Equation (6) into Equation (4c) and integrating, the time law assumes the form:

$$\boxed{\begin{aligned} t - t_0 = \hat{A}\theta^2 + \hat{B}(\theta)\theta + \hat{C}(\theta) + \hat{D}(\theta)\exp[k\theta] + \\ - \left\{\hat{A}\theta_0^2 + \hat{B}(\theta_0)\theta_0 + \hat{C}(\theta_0) + \hat{D}(\theta_0)\exp[k\theta_0]\right\} \end{aligned}}$$

4.3. Orientation problem

Once the shape problem has been solved, we can proceed to evaluate the remaining three Keplerian elements. Starting from Equation (4d), the ODE for the inclination can be rewritten as:

$$i' = A_{i'}(\theta) + \frac{1}{2}B_{i'}(\theta)\theta \tag{12}$$

with:

$$A_{i'}(\theta) = \mathfrak{I}_0(\theta) + \mathfrak{I}_a(\theta)\Xi_a(\theta) + \mathfrak{I}_e(\theta)\Xi_e(\theta)$$
$$B_{i'}(\theta) = \alpha_0[R_0]\mathfrak{I}_a(\theta) + \alpha_0[S_0]\mathfrak{I}_e(\theta)$$

Integrating Equation (12) in the same way of Equation (7) we get:

$$\boxed{i(\theta) - i_0 = A_i\theta^2 + B_i(\theta)\theta + C_i(\theta) - k_i} \tag{13}$$

where k_i is a constant. Plugging Equation (13) into Equation (4f) we obtain the form:

$$\omega' = A_{\omega'}(\theta)\theta^2 + B_{\omega'}(\theta)\theta + C_{\omega'}(\theta) + \mathcal{O}_\omega(\theta)\bar{\omega}$$

with:

$$A_{\omega'}(\theta) = A_i\mathcal{O}_i(\theta)$$
$$B_{\omega'}(\theta) = \frac{1}{2}\alpha_0[R_0]\mathcal{O}_a(\theta) + B_i(\theta)\mathcal{O}_i(\theta) + \\ + \frac{1}{2}\alpha_0[S_0]\mathcal{O}_e(\theta)$$
$$C_{\omega'}(\theta) = \Xi_a(\theta)\mathcal{O}_a(\theta) + [C_i(\theta) - k_i]\mathcal{O}_i(\theta) + \\ + \Xi_e(\theta)\mathcal{O}_e(\theta) + \mathcal{O}_0(\theta)$$

Similarly to Equation (11) we get a solution in the form:

$$\boxed{\begin{aligned} \omega(\theta) - \omega_0 = A_\omega(\theta)\theta^2 + B_\omega(\theta)\theta + \\ + C_\omega(\theta) + D_\omega(\theta)\exp[k\theta] \end{aligned}} \tag{14}$$

Finally plugging Equation (14) into Equation (4e) we get:

$$\Omega' = A_{\Omega'}(\theta)\theta^2 + B_{\Omega'}(\theta)\theta + \\ + C_{\Omega'}(\theta) + D_{\Omega'}(\theta)\exp[k\theta]$$

with:

$$A_{\Omega'}(\theta) = A_i\mathfrak{O}_i(\theta) + A_\omega(\theta)\mathfrak{O}_\omega(\theta)$$
$$B_{\Omega'}(\theta) = \frac{1}{2}\alpha_0[R_0]\mathfrak{O}_a(\theta) + B_i(\theta)\mathfrak{O}_i(\theta) + \\ + B_\omega(\theta)\mathfrak{O}_\omega(\theta) + \frac{1}{2}\alpha_0[S_0]\mathfrak{O}_e(\theta)$$
$$C_{\Omega'}(\theta) = \Xi_a(\theta)\mathfrak{O}_a(\theta) + \mathfrak{O}_i(\theta)(C_i(\theta) - k_i) + \\ + C_\omega(\theta)\mathfrak{O}_\omega(\theta) + \Xi_e(\theta)\mathfrak{O}_e(\theta) + \mathfrak{O}_0(\theta)$$
$$D_{\Omega'}(\theta) = D_\omega(\theta)\mathfrak{O}_\omega(\theta)$$

and integrating the solution assumes the form:

$$\boxed{\begin{aligned} \Omega(\theta) - \Omega_0 = A_\Omega\theta^3 + B_\Omega(\theta)\theta^2 + C_\Omega(\theta)\theta + \\ + D_\Omega(\theta) + E_\Omega(\theta)\exp[k\theta] - k_\Omega \end{aligned}}$$

5. The *small thrust model*

For $a_t \to 0$, $a_n \to 0$ and $a_h \to 0$, performing a McLaurin expansion truncated at first order we pass from the form in system (4) to:

$$a' = \{\bar{R}_{a_t,0} + \bar{R}_{a_t,a}\bar{a} + \bar{R}_{a_t,e}\bar{e}\}\tilde{a}_t \tag{15a}$$
$$e' = \bar{S}_{a_t,0}\tilde{a}_t + \bar{S}_{a_n,0}\tilde{a}_n \tag{15b}$$
$$i' = \{\bar{\mathfrak{I}}_{a_h,0} + \bar{\mathfrak{I}}_{a_h,a}\bar{a} + \bar{\mathfrak{I}}_{a_h,e}\bar{e}\}\tilde{a}_h \tag{15c}$$
$$\begin{aligned} t' = \{\bar{\tau}_{a_t,0} + \bar{\tau}_{a_t,a}\bar{a} + \bar{\tau}_{a_t,e}\bar{e}\}\tilde{a}_t + \\ + \{\bar{\tau}_{a_n,0} + \bar{\tau}_{a_n,a}\bar{a} + \bar{\tau}_{a_n,e}\bar{e}\}\tilde{a}_n + \\ + \tau_{00} + \tau_{0,a}\bar{a} + \tau_{0,e}\bar{e} \end{aligned} \tag{15d}$$
$$\begin{aligned} \omega' = \{\bar{\mathcal{O}}_{a_t,0} + \bar{\mathcal{O}}_{a_t,a}\bar{a} + \bar{\mathcal{O}}_{a_t,e}\bar{e}\}\tilde{a}_t + \\ + \{\bar{\mathcal{O}}_{a_n,0} + \bar{\mathcal{O}}_{a_n,a}\bar{a} + \bar{\mathcal{O}}_{a_n,e}\bar{e}\}\tilde{a}_n + \\ + \{\bar{\mathcal{O}}_{a_h,0} + \bar{\mathcal{O}}_{a_h,a}\bar{a} + \bar{\mathcal{O}}_{a_h,e}\bar{e} + \\ + \bar{\mathcal{O}}_{a_h,i}\bar{i} + \bar{\mathcal{O}}_{a_h,\omega}\bar{\omega}\}\tilde{a}_h \end{aligned} \tag{15e}$$
$$\begin{aligned} \Omega' = \{\bar{\mathfrak{O}}_{a_h,0} + \bar{\mathfrak{O}}_{a_h,a}\bar{a} + \bar{\mathfrak{O}}_{a_h,e}\bar{e} + \\ + \bar{\mathfrak{O}}_{a_h,i}\bar{i} + \bar{\mathfrak{O}}_{a_h,\omega}\bar{\omega}\}\tilde{a}_h \end{aligned} \tag{15f}$$

where we defined the adimentional thrust parameters:

$$\tilde{a}_t = \frac{a_0^2}{\mu}a_t, \quad \tilde{a}_n = \frac{a_0^2}{\mu}a_n, \quad \tilde{a}_h = \frac{a_0^2}{\mu}a_h$$

and if $y(\theta)$ is the generic function then:

$$\bar{y} = \frac{\mu}{a_0^2}y(\theta)$$

5.1. *small thrust model* main results

We will now go through the main results obtained. Let be:

$$\mathcal{R}(e,\theta) = \sqrt{1 + e^2 + 2e\cos(\theta)}$$
$$\mathcal{S}(e,\theta) = (e\cos(\theta) + 1)^2 \tag{16}$$

The ODE for the semi-major axis is given by Equation (15a) and has a solution in the form:

$$\bar{a} = k_0[1 - \{1 + \tilde{a}_t Q_a\} \exp(k_a\theta + k_1)] \tag{17}$$

where k_0, k_a and k_1 are constants and:

$$Q_a(\theta) = 6\lambda \mathcal{P}_{2\pi} \left(\begin{matrix} 0 \\ \alpha_n[g]/n \end{matrix} \middle| \theta \right)$$

with $\lambda = 1 - e_0^2$ and $g(\theta) = \mathcal{R}(e_0, \theta)/\mathcal{S}(e_0, \theta)$. In a similar fashion, for the eccentricity we get:

$$\bar{e} = \tilde{a}_t[k_e\theta + Q_{1e}] + \tilde{a}_n Q_{2e} \tag{18}$$

where k_e is a constant and:

$$Q_{1e}(\theta) = -k_e\theta_0 + 2\lambda^2 \left[\mathcal{P}_{2\pi} \left(\begin{matrix} 0 \\ \alpha_n[\bar{\bar{S}}_{a_t,0}]/n \end{matrix} \middle| \xi \right) \right]_{\theta_0}^{\theta}$$

$$Q_{2e}(\theta) = -\lambda^3 \left[\mathcal{P}_{2\pi} \left(\begin{matrix} -\beta_n[\bar{\bar{S}}_{a_n,0}]/n \\ 0 \end{matrix} \middle| \xi \right) \right]_{\theta_0}^{\theta}$$

with:

$$\bar{\bar{S}}_{a_t,0}(\theta) = \frac{e_0 + \cos(\theta)}{\mathcal{S}(e_0,\theta)\mathcal{R}(e_0,\theta)}$$

$$\bar{\bar{S}}_{a_n,0}(\theta) = \frac{\sin(\theta)}{\mathcal{S}^{3/2}(e_0,\theta)\mathcal{R}(e_0,\theta)}$$

Rewriting Equation (15d) in matrix form notation, substituting into that the expressions of $\bar{a}$ and $\bar{e}$ given by Equation (17) and Equation (18) respectively and neglecting all terms higher than first order we get:

$$t' = \boldsymbol{w} \begin{bmatrix} \mathfrak{T}_1(\theta) + \mathfrak{T}_2(\theta)\exp[k_a\theta + k_1] \\ \mathfrak{T}_3(\theta) + \mathfrak{T}_4(\theta)\theta + \mathfrak{T}_5(\theta)\exp[k_a\theta + k_1] \\ \mathfrak{T}_6(\theta) + \mathfrak{T}_7(\theta)\exp[k_a\theta + k_1] \end{bmatrix}$$

where $\boldsymbol{w} = [1, \tilde{a}_t, \tilde{a}_n]$ and $\mathfrak{T}_1$ to $\mathfrak{T}_7$ are obtained by combination of the previous solutions.
For the inclination, starting from Equation (15c) and recalling Equation (17) and Equation (18) we get:

$$\begin{aligned} i' = &\frac{1}{3}\bar{\mathfrak{I}}_{a_h,0}(\theta)\{1 + 2\exp[k_a\theta + k_1]\}\tilde{a}_h + \\ &+ \Big\{\frac{2}{3}\bar{\mathfrak{I}}_{a_h,0}(\theta)Q_a(\theta)\exp[k_a\theta + k_1] + \\ &+ k_e\bar{\mathfrak{I}}_{a_h,e}(\theta)\theta + \bar{\mathfrak{I}}_{a_h,e}(\theta)Q_{1e}(\theta)\Big\}\tilde{a}_t\tilde{a}_h + \\ &+ \bar{\mathfrak{I}}_{a_h,e}(\theta)Q_{2e}(\theta)\tilde{a}_n\tilde{a}_h \end{aligned}$$

Thus at the first order:

$$i' = \frac{1}{3}\bar{\mathfrak{I}}_{a_h,0}(\theta)\{1 + 2\exp[k_a\theta + k_1]\}\tilde{a}_h$$

For the argument of perigee, leading the problem to the quadratures we obtain:

$$\bar{\omega} = \frac{1}{3}\exp[k_\omega\theta] \cdot \hat{\boldsymbol{w}} \int_{\theta_0}^{\theta} h(\xi) \begin{bmatrix} \bar{\mathcal{O}}_{a_t,0}(\xi) \\ \bar{\mathcal{O}}_{a_n,0}(\xi) \\ \bar{\mathcal{O}}_{a_h,0}(\xi) \end{bmatrix} d\xi$$

where $h(\theta) = \exp[-k_\omega\theta] + 2\exp[k_{a\omega}\theta + k_1]$, $\hat{\boldsymbol{w}} = [\tilde{a}_t, \tilde{a}_n, \tilde{a}_h]$ and $k_{a\omega} = k_a - k_\omega$ is a constant.
Finally, for the RAAN we make the assumption of neglecting the small variations of i and ω. This is done for two reasons mainly: first they are both second order terms; furthermore both variations are really small over one thrust arc. Then a similar expression like that of the inclination is achieved:

$$\begin{aligned} \Omega' = &\frac{1}{3}\bar{\mathfrak{O}}_{a_h,0}(\theta)\{1 + 2\exp[k_a\theta + k_1]\}\tilde{a}_h + \\ &+ \Big\{\frac{2}{3}\bar{\mathfrak{O}}_{a_h,0}(\theta)Q_a(\theta)\exp[k_a\theta + k_1] + \\ &+ k_e\bar{\mathfrak{O}}_{a_h,e}(\theta)\theta + \bar{\mathfrak{O}}_{a_h,e}(\theta)Q_{1e}(\theta)\Big\}\tilde{a}_t\tilde{a}_h + \\ &+ \bar{\mathfrak{O}}_{a_h,e}(\theta)Q_{2e}(\theta)\tilde{a}_n\tilde{a}_h \end{aligned}$$

And at the first order:

$$\Omega' = \frac{1}{3}\bar{\mathfrak{O}}_{a_h,0}(\theta)\{1 + 2\exp[k_a\theta + k_1]\}\tilde{a}_h$$

5.2. Fourier series coefficients

A relevant result is the close form solution for the Fourier Series coefficients for all the functions involved in the *small thrust model*. Here for shortness we report only the expressions for the function $g(\theta)$, the others being analogous. It can be proved that :

$$g(\theta) = \frac{1}{2}(\pi v)^{1/2} \sum_{m=0}^{+\infty} q_m(e_0)\cos^m(\theta)$$

where $v = 1 + e_0^2$ and q_m is given by:

$$q_m(e_0) = \frac{(2e_0/v)^m}{\Gamma(3/2 - m)\, m!}\, {}_2F_1 \left(\begin{matrix} 2 \quad -m \\ 3/2 - m \end{matrix} \middle| \frac{v}{2} \right)$$

where Γ is the complete Gamma Function and ${}_2F_1$ is the Gauss Hypergeometric function. It follows that the relevant integral to be performed is:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^m(\theta)\cos(n\theta)d\theta = p_m(n)$$

with:

$$p_m(n) = -\frac{\pi(-2)^m(n+m-1)m!}{w_m(n)(-n+m)!(n+m)!}$$

and:

$$w_m(n) = \Gamma\left((-n-m+3)/2\right)\Gamma\left((n-m+1)/2\right)$$

So, the coefficients final expression is:

$$\boxed{\begin{aligned} \alpha_0[g] &= \frac{2}{\pi}\left[\frac{1}{1+e_0}\mathbf{E}(\kappa) + \frac{1}{1-e_0}\mathbf{K}(\kappa)\right] \\ \alpha_n[g] &= \frac{1}{2}(\pi v)^{1/2}\sum_{m=0}^{+\infty} q_m(e_0)p_m(n) \\ \beta_n[g] &= 0 \end{aligned}} \tag{19}$$

Where $\mathbf{K}(\kappa)$ and $\mathbf{E}(\kappa)$ are the complete elliptic integrals of first and second kind respectively with elliptic modulus $\kappa = -4e_0/(e_0-1)^2$.

6. Conclusions

The number of spacecraft and orbiting objects is so growing that came the need of computationally efficient models for collision avoidance. In this work we developed a semi-analytical mathematical model for the 3D low thrust collision avoidance problem. The Gauss Planetary Equations are reduced to a simpler form by means of a Taylor expansion in the neighbourhood of the reference condition. Two different methods are developed: the first, denoted as *full model*, where the main functions are complicated non linear relations of the thrust accelerations and the second, denoted as *small thrust model*, which, on the contrary, has an explicit dependency from the small thrust parameters. All the integrations are performed by means of the Fourier Series tool and for the *small thrust model* the expressions of the Fourier Series coefficients are provided in closed form also involving the Gauss Hypergeometric function. Different simulations with various test cases to assess the effectiveness of the method are provided. Looking to the numerical outputs, both models are capable of reproducing accurately the solution of the Gauss Planetary Equations solved by means of Adams-Bashforth-Moulton method. In particular, for not too high eccentricity reference values, the solution provided by the *full model* has by far a greater accuracy for the semi-major axis and the true anomaly, specially if the first period is assumed as time span. On the contrary, the *small thrust model* even if less accurate, is nevertheless capable of granting an acceptable precision and a remarkable reduction of computational time. And this becomes even more evident considering higher eccentricities. In this case we would have a lower performance as it concerns accuracy: nevertheless knowing the closed form expressions of Fourier coefficients, the *small thrust model* succeeds to be more efficient in accuracy and elapsed time with respect to the *full model*. Then, the trade off chose the *small thrust model* as the winner. For both methods the greater computational expense occurs in time law inversion when solving a non linear root finding problem. One of the best computational qualities of these methods is that they allow to evaluate the orbital parameters (and then the state vector) at the wished instant of time without passing through the previous ones, what is convenient in PoC computing and in CAMs design. Two sample tests have been provided. In the first, only the tangential thrust is active; in the second there is also the normal component too. In both cases, beyond a good PoC evaluation, the *small thrust model* has been seen to provide a satisfactory approximation of orbital parameters.

Future developments

This thesis opens to a variety of future developments. First, a possible technique to provide the time law inversion could be investigated. This would lead to a remarkable gain in computational time because the non linear root finding problem would be substituted by a convergent series providing $\theta = \theta(t)$. Moreover, some techniques for series manipulation could be used to deal with the problem of machine precision in the computation of the Fourier Series coefficients when their expression is in the form of Equation (19). Both *full model* and *small thrust model* could be adopted for approximating a generic low thrust problem, not necessary a CAM one. For both models, some simulations show that there are regions where the error is slightly higher with relative low thrust action and higher length of the thrust arc. All these proposed refinements are the preamble for the biggest challenge as future development: optimize the method such that it could be implemented on-board. Finally, one of the most important assumptions we made is that no environmental perturbations are acting on the spacecraft. This is not true in reality. Typically, the satellite motion is perturbed by the solar radiation pressure, the drag due to Earth atmosphere and the effect of earth oblateness (i.e. the J_2 effect). These should be added to include non Keplerian orbits in the model.

7. Acknowledgements

This work was part of the COMPASS project: “Control for Orbit Manoeuvring through Perturbations for Application to Space Systems" (Grant agreement No 679086). This project is a European Research Council (ERC) funded project under the European Unions Horizon 2020 research.

References

[1] Yang Gao. Near-optimal very low-thrust earth-orbit transfers and guidance schemes. *Journal of Guidance, Control, and Dynamics*, 30(2):529–539, 2007.

[2] Juan L Gonzalo, C Colombo, and P Di Lizia. A semi-analytical approach to low-thrust collision avoidance manoeuvre design. In *70th International Astronautical Congress (IAC 2019)*, pages 1–9, 2019.

[3] Juan L Gonzalo, C Colombo, and P Di Lizia. Computationally efficient approaches for low-thrust collision avoidance activities. In *72nd International Astronautical Congress (IAC 2021)*, pages 1–10, 2021.

[4] Juan L Gonzalo and Camilla Colombo. Lightweight algorithms for collision avoidance applications. In *11th International ESA Conference on Guidance, Navigation & Control Systems, ESA GNC & ICATT 2021*, pages 1–15, 2021.