POLITECNICO
MILANO 1863

POLITECNICO DI MILANO
DIPARTIMENTO DI ELETTRONICA, INFORMAZIONE E BIOINGEGNERIA
DOCTORAL PROGRAM IN INFORMATION TECHNOLOGY

REDUCING THE GAP BETWEEN THEORY AND APPLICATIONS IN ALGORITHMIC BAYESIAN PERSUASION

Doctoral Dissertation of:
Matteo Castiglioni

Supervisor:
Prof. Gatti Nicola

Tutor:
Prof. Cesare Alippi

The Chair of the Doctoral Program:
Prof. Barbara Pernici

2022 – Cycle XXXIV

Abstract

This thesis focuses on the following question: *is it possible to influence the behavior of self-interested agents through the strategic provision of information*? This 'sweet talk' is ubiquitous among all sorts of economics and non-economics activities. In this thesis, we model these multi-agent systems as games between an informed sender and one or multiple receivers. We study the computational problem faced by an informed sender that wants to use his information advantage to influence rational receivers with the partial disclosure of information. In particular, the sender faces an information structure design problem that amounts to deciding 'who gets to know what'.

Bayesian persuasion provides a formal framework to model these settings as asymmetric-information games. In recent years, much attention has been given to Bayesian persuasion in the economics and artificial intelligence communities due also to the applicability of this framework to a large class of scenarios like online advertising, voting, traffic routing, recommendation systems, security, and product marketing. However, there is still a large gap between the theoretical study of information in games and its applications in real-world scenarios. This thesis contributes to close this gap along two directions. First, we study the persuasion problem in real-world scenarios, focusing on voting, routing, and auctions. While the Bayesian persuasion framework can be applied to all these settings, the algorithmic problem of designing optimal information disclose polices introduces computational challenges related to the specific problem under study. Our goal is to settle the complexity of computing optimal sender's strategies, show-

ing when an optimal strategy can be implemented efficiently. Then, we relax stringent assumptions that limit the applicability of the Bayesian persuasion framework in practice. In particular, the classical model assumes that the sender has perfect knowledge of the receiver's utility. We remove this assumption initiating the study of an online version of the persuasion problem. This is the first step in designing adaptive information disclosure policies that deal with the uncertainty intrinsic in all real-world applications.

Contents

CHAPTER 1

Introduction

This thesis considers the following question: *is it possible to influence the behavior of self-interested agents through the strategic provision of information*? This 'sweet talk' is ubiquitous among all sorts of economic activities, and it was famously attributed to 30 per cent of the GDP in the United States (Antioch et al., 2013). Moreover, information is the foundation of any democratic election, as it allows voters for better choices. In many settings, uninformed voters have to rely on inquiries of third party entities to make their decision. With the advent of modern media environments, malicious actors have unprecedented opportunities to garble this information and influence the outcome of the election via misinformation and fake news (Allcott and Gentzkow, 2017). Reaching voters with targeted messages has never been easier. As another example, consider a multi-agent routing problem in which agents seek to minimize their own costs selfishly. In real-world problems, the state of the network may be uncertain, and not known to its users (e.g., drivers may not be aware of road works and accidents in a road network). A central authority or a navigation app may mitigate inefficiencies and reduce the social cost providing players with partial information about the state of the network.

Bayesian persuasion (Kamenica and Gentzkow, 2011) provides a framework to model the problem faced by an informed sender trying to influence the behavior of self-interested receivers. In particular, the sender faces an information structure design problem which amounts to deciding 'who gets to know what' about some exogenous parameters collectively termed state of nature. Since the seminal work of Kamenica and Gentzkow (2011), a large attention is been given to the Bayesian persuasion framework in the economics and artificial intelligence community due also to the applicability of this framework to a large class of scenarios like online advertising (Bro Miltersen and Sheffet, 2012; Emek et al., 2014; Badanidiyuru et al., 2018), voting (Alonso and Câmara, 2016; Cheng et al., 2015), traffic routing (Vasserman et al., 2015; Bhaskar et al., 2016), recommendation systems (Mansour et al., 2016), security (Rabinovich et al., 2015; Xu et al., 2016b), and product marketing (Babichenko and Barman, 2017; Candogan, 2019). However, there is a still large gap between the theoretical study of information in games and its applications in real-world scenarios. This thesis contributes to close this gap along two directions. First, we study the Bayesian persuasion framework in real-world scenarios, focusing on voting, routing, and auctions. While the Bayesian persuasion framework can be applied to all these settings, the algorithmic problem of designing optimal information disclose polices introduces computational challenges related to the specific problem under study. Then, we relax stringent assumptions that limit the applicability of the classical bayesian persuasion framework in practice. In particular, one of the most limiting assumption is, arguably, that the sender is required to know the receiver's utility function to compute an optimal signaling scheme. We remove this assumption by studying a repeated Bayesian persuasion problem in an online learning framework where, at each round, the receiver's type is adversarially chosen from a finite set of types. This is the first step in designing adaptive information disclosure policies that deals with the uncertainty intrinsic in all real-world applications.

1.1 The Bayesian Persuasion Framework

Bayesian persuasion (Kamenica and Gentzkow, 2011) studies the problem faced by an informed agent (the *sender*) trying to influence the behavior of other self-interested agents (the *receivers*) via the partial disclosure of payoff-relevant information. Agents' payoffs are determined by the actions played by the receivers and by an exogenous parameter represented as a *state of nature*, which is drawn by a known prior probability distribution and observed by the sender only. The sender commits to a public randomized information-disclosure policy, which is customarily called *signaling scheme*. In particular, it defines how the sender should send signals to the receivers. Depending on the application various types of signaling schemes have been introduced to represent the possible communication constraints between the sender and the receivers. In a private signaling scheme, the sender can use a private communication channel per receiver, in a public signaling scheme the sender can use a single communication channel for all the receivers, while in Chapter 6 we introduce semi-public signaling schemes in which the sender can use a single communication channel for a subset of the receivers.

Arguably, one of the most severe obstacle to the application of the classical bayesian persuasion model by Kamenica and Gentzkow (2011) to real-world scenarios is that the sender must know exactly the receiver's utility function to compute an optimal signaling scheme. This assumption is unreasonable in practice. However, only recently some works tries to relax this assumption. In particular, Babichenko et al. (2021) study a game with a single receiver and binary-actions in which the sender does not know the receiver utility, focusing on the problem of designing a signaling scheme that perform well for each possible receiver's utility. Zu et al. (2021) relax the perfect knowledge assumption, assuming that the sender and the receiver do not know the prior distribution over the states of nature. They study the problem of computing a sequence of persuasive signaling schemes that achieve small regret with respect to the optimal signaling scheme with the knowledge of the prior distribution. In this thesis, we follow a different approach and we deal with uncertainty about the receiver's utility by framing the Bayesian persuasion problem in an online learning framework.

1.2 Original Contributions

The goal of this thesis is to advance the state of the art on algorithmic Bayesian persuasion along two directions. First, we study Bayesian persuasion in games with structure, focusing on voting, routing, and auctions. Then, we initiate the study of Bayesian persuasion with payoff uncertainly. In the remaining of this section, we survey the original contributions of this thesis.

1.2.1 Exploiting the Problem Structure

In the first part of the thesis, we study persuasion in games with structure with a particular focus on voting scenarios. Information is the foundation of any democratic election, as it allows voters for better choices. In many settings, uninformed voters have to rely on inquiries of third party entities to make their decision. For example, in most trials, jurors are not given the possibility of choosing which tests to perform during the investigation or which questions are asked to witnesses. They have to rely on the prosecutor's investigation and questions. The same happens in elections, in which voters gather information from third-party sources. Hence, we pose the question: *can a malicious actor influence the outcome of a voting process only by the provision of information to voters who update their beliefs rationally?* We study majority voting, plurality voting and district-based elections, showing a sharp contrast in term of efficiency in manipulating elections and computational tractability between the case in which private signals are allowed and the more restrictive setting in which only public signals are allowed. In particular, we show that it is possible to compute an optimal private signaling scheme in polynomial time in all the elections that we considered, while the problem of approximating the optimal public signaling scheme is NP-hard even for majority voting. Moreover, we show that, assuming the Exponential Time Hypothesis (ETH), the problem of approximating the optimal public signaling scheme in majority voting requires quasi-polynomial time even relaxing persuasiveness. In doing so, we provide some insights on the complexity of general persuasion problems, such as the characterization of bi-criteria approximations in public signaling problems.

Then, we explore how information can be used to reduce the social cost in multi-agents systems, focusing on routing games. In particular, we study Bayesian games with atomic players, where network vagaries are modeled via a (random) state of nature which determines the costs incurred by the players. We focus on the problem of computing optimal ex-ante persuasive signaling schemes, showing that symmetry is a crucial property for its solution. Indeed, we show that an optimal ex-ante persuasive signaling scheme can be computed in polynomial time when players are symmetric and have affine cost functions. Moreover, the problem becomes NP-hard when players are asymmetric, even in non-Bayesian settings.

Finally, we study persuasion in posted price auctions in which the seller tries to sell an item by proposing take-it-or-leave-it prices to buyers arriving sequentially. Each buyer has to choose between declining the offer—without having the possibility of coming back—or accepting it, thus ending the auction. We study Bayesian posted price auctions, where the buyers valuations for the item depend on a random state of nature, which is known to the seller only. Thus, the seller does not only have to decide price proposals for the buyers, but also how to partially disclose information about the state so as to maximize revenue. Our model finds application in several real-world scenarios. For instance, in an e-commerce platform, the state of nature may reflect the condition (or quality) of the item being sold and/or some of its features. These are known to the seller only since the buyers cannot see the item given that the auction is carried out on the web. As a first negative result, we prove that, in both public and private signaling, the problem of computing an optimal seller's strategy does not admit an FPTAS unless P = NP. Indeed, the result holds for basic instances with a single buyer. Then, we provide tight positive results by designing a PTAS for each setting.

1.2.2 Facing the Uncertainty

In the second part of the thesis, we initiate the study of Bayesian persuasion with payoff uncertainty. First, we consider the setting with a single receiver and we deal with uncertainty about the receiver's type by framing the Bayesian persuasion problem in an online learning framework. We study a repeated Bayesian persuasion problem where, at each round, the receiver's type is adversarially chosen from a finite set of types. Our goal is the design of an online algorithm that recommends a signaling scheme at each round, guaranteeing an expected utility for the sender close to that of the best-in-hindsight signaling scheme. We study this problem under two models of feedback: in the full information model, the sender selects a signaling scheme and later observes the type of the receiver; in the partial information model, the sender only observes the actions taken by the receiver. We rule out the possibility of designing a no-regret algorithm with polynomial per-round running time. Then, we provide two no-regret algorithms for the full and partial information model which require exponential per-round running time. Finally, we show that, relaxing the persuasiveness constraints, we can design polynomial-time algorithms with small regret.

Then, we extend the online Bayesian persuasion framework to include multiple receivers. We focus on the case with no-externalities and binary actions. Moreover, to focus only on the receivers' coordination problem, we overcome the intractability of the single-receiver problem assuming that each receiver has a constant number of types. First, we prove a negative result: for any $0 < \alpha \leq 1$, there is no polynomial-time no-α-regret algorithm when the sender's utility function is supermodular or anonymous. Then, we focus on the case of submodular sender's utility functions and we show that, in this case, it is possible to design a polynomial-time no-$(1 - 1/e)$-regret algorithm, which is tight.

Both for the setting with a single and multiple receivers, we show that the design of polynomial-time no-regret algorithms is impossible due to the NP-Hardness of the underline offline problems in which the distribution over the types is known. Hence, the design of efficient algorithms for the offline problem is the bottleneck to the design of efficient online learning algorithms. In the last part of the thesis, we circumvent this issue by leveraging ideas from mechanism design. In particular, we introduce a type reporting step in which the receiver is asked to report her type to the sender, after the latter has committed to a menu defining a signaling scheme for each possible receiver's type. Surprisingly, we prove that, with a single receiver, the addition of this type reporting stage makes the sender's computational problem tractable. Then, we extend our Bayesian persuasion framework with type reporting to settings with multiple receivers, focusing on the widely-studied case of no-externalities and binary actions. In such setting, we show that it is possible to find a sender-optimal solution in polynomial-time for supermodular and anonymous sender's utility functions. As for the case of submodular sender's utility functions, we provide a $(1 - 1/e)$-approximation to the problem, which is tight.

1.3 Structure of the Work

In this section, we describe the structure of the thesis. In Chapter 2, we introduce the fundamental concepts related to algorithmic game theory. In particular, we provide a formal definition of game and we introduce some equilibrium concepts. Moreover, we described the online learning framework and other preliminaries results that are relevant for the rest of the dissertation. Chapter 3 formally defines the Bayesian persuasion framework, providing a definition of the persuasion problem with a single and multiple receivers. Moreover, it surveys the state of the art on Bayesian persuasion.

The first part of the thesis focuses on games with a structure. In particular, we study different games characterizing the computational complexity of the persuasion problem in various scenarios. Our contributions are organized as follows:

- Chapter 4 provides our results on persuasion with simple voting rules, including majority voting and plurality voting. The results in this chapter appeared in (Castiglioni et al., 2020a).
- Chapter 5 characterizes the computational complexity of bi-approximations with public signals, *i.e.*, that provides almost optimal and almost persuasive solutions in polynomial time, focusing on the general persua-

sion problem and k-voting (a generalization of majority voting). The results in this chapter appeared in (Castiglioni et al., 2020b).

- Chapter 6 extends the analysis of persuasion in voting scenarios to district-based elections. The results in this chapter appeared in (Castiglioni and Gatti, 2021)
- Chapter 7 focuses on the computation of ex-ante persuasive signaling schemes in routing games. The results in this chapter appeared in (Castiglioni et al., 2021a)
- Chapter 8 studies persuasion in posted price auctions. The results in this chapter appeared in (Castiglioni et al., 2022b).

In the second part of the thesis we relax the assumption that the sender perfectly knows the receivers payoffs. The contributions are organized as follows:

- In Chapter 9, we study online Bayesian persuasion with a single receiver. The results in this chapter appeared in (Castiglioni et al., 2020c).
- In Chapter 10, we study online Bayesian persuasion with multiple receiver. The results in this chapter appeared in (Castiglioni et al., 2021b).
- In Chapter 11, we extend the Bayesian persuasion framework with a type reporting step. The results in this chapter appeared in (Castiglioni et al., 2022a).

Finally Chapter 12 concludes the thesis with some possible directions for future research.

CHAPTER 2

Preliminaries

In this chapter, we provide an introduction to game theory, presenting some classes of games and equilibrium concepts. Moreover, we introduce the online learning framework and some concepts that we will use in the dissertation. Section 2.1 introduces the classical representation of finite games, *i.e.*, Normal-Form games and defines some of the classical solution concepts. Then, in Section 2.2, we introduce the online learning framework. In Section 2.3, we present a two-provers game that we will use in the dissertation. In Section 2.4, we define matroids and some classes of set functions. In Section 2.5, we introduce a result on error-correcting codes.

2.1 Games and Equilibria

Games provide a mathematical representation of the strategic interactions among rational agents. A game is defined by a set of players, a set of strategies for each player and the utilities of the players for each possible outcome. Formally, we can define a Normal-Form game as follows.

Definition 2.1 (Normal-Form game)**.** *A Normal-Form Game is a tuple* $(N, \mathcal{A}, U)$ *such that:*

- $N := \{1, \dots, \bar{n}\}$ *is a set of players;*
- $\mathcal{A} := \times_{p \in N} \mathcal{A}_p$ *is the set of action profiles, where* $\mathcal{A}_p$ *denotes the set of actions available to player* p *and* $\varrho_p := |\mathcal{A}_p|$ *denotes the number of actions available to player* p*;*
- $U := \{U_1, \dots, U_{\bar{n}}\}$ *is a set of matrices with* $U_p \in \mathbb{Q}^{\varrho_1 \times \cdots \varrho_{\bar{n}}}$*, where* U_p *represents the utility of player* p *and* $U_p^{a_1, \dots, a_{\bar{n}}}$ *correspond to the utility of player* p *when the players play action profile* $(a_1, \dots, a_{\bar{n}}) \in \mathcal{A}$*.*

Given an action profile $\mathbf{a}$, we will denote with $\mathbf{a}_{-p} \in \mathcal{A}_{-p} := \times_{p' \in N \setminus \{p\}} \mathcal{A}_{p'}$ the actions of all the players except p. We can represent player p mixed strategies as $\mathbf{x}_p \in \Delta_{\mathcal{A}_p} := \{\mathbf{x}_p \in [0,1]^{\varrho_p} : \sum_{a \in \mathcal{A}_p} x_{p,a} = 1\}$, where $x_{p,a}$ denotes the probability that player p plays action a. [1] When a player chooses an action deterministically, he is said to play in *pure* strategies, and, if he randomizes among actions, he is said to play in *mixed* strategies. We denote with $x = (\mathbf{x}_1, \dots, \mathbf{x}_n)$ a mixed strategy profile that specifies a mixed strategy $\mathbf{x}_p \in \Delta_{\mathcal{A}_p}$ for each player $p \in N$. We define $u_p(\mathbf{x}) := \sum_{\mathbf{a} \in \mathcal{A}} U_p^{\mathbf{a}} \prod_{p \in N} x_{p,a_p}$ the expected utility of player $p \in N$.

2.1.1 Solution Concepts

We assume that the players are rational and want to maximize their utilities. While in single agent problems, it is clear that the best solution is to optimize the objective (the player's utility), in games there are multiple agents with different objectives. In game theory can be defined various solution concepts. Usually, they represent an equilibrium, *i.e.*, a stable solution in which the players has no incentive to leave. The Nash Equilibrium (NE) introduced by Nash (1950) is the most famous and used solution concept. NE is based on a very simple idea: a strategy profile is a NE if no player has an incentive to deviate from his strategy. Formally:

Definition 2.2. *A mixed strategy profile* $\mathbf{x} = (\mathbf{x}_1, ..., \mathbf{x}_n) \in \times_{p \in N} \Delta_{\mathcal{A}_p}$ *is a Nash Equilibrium of a Normal-Form game* $(N, \mathcal{A}, U)$ *if for every player* $p \in N$ *and strategy* $\mathbf{x}'_p \in \Delta_{\mathcal{A}_p}$*:*

$$u_p(\mathbf{x}) \geq u_p(\mathbf{x}'_p, \mathbf{x}_{-p})$$

If players are allowed to play mixed strategies, then any Normal-Form game admits at least a Nash Equilibrium.

Theorem 2.1 ((Nash, 1950))**.** *Every Normal-Form game admits at least one Nash Equilibrium.*

[1] Vector are denoted by bold symbols. For any vector $\mathbf{x}$, the value of its i-th component is denoted by x_i

It is possible that the players can use some form of coordination during the game. This situation are usually model by the notion of Correlated Equilibrium (CE), introduced by Aumann (1974). In CEs there is an external mediator that can privately communicate to the agents which actions to play. Let $\Delta_{\mathcal{A}} := \{\mathbf{x} \in [0,1]^{|\mathcal{A}|} : \sum_{\mathbf{a} \in \mathcal{A}} x_{\mathbf{a}} = 1\}$ be the set of *correlated distributions*. Formally, a CE is defined as follows:

Definition 2.3. *Given a Normal-Form game $(N, \mathcal{A}, U)$, a correlated distribution $\mathbf{x} \in \Delta_{\mathcal{A}}$ is a correlated equilibrium if for every player p and pair of actions $a, a' \in \mathcal{A}_p$*

$$\sum_{\boldsymbol{a}_{-p} \in \mathcal{A}_{-p}} x_{a,\boldsymbol{a}_{-p}} (U_p^{a,\boldsymbol{a}_{-p}} - U_p^{a',\boldsymbol{a}_{-p}}) \geq 0$$

A Coarse Correlated Equilibria (CCE) relaxes the equilibrium constraints incentivizing the agents to follow the recommendations *a priori*, *i.e.* before receiving the recommendation (Moulin and Vial, 1978).

Definition 2.4. *Given a Normal-Form game $(N, \mathcal{A}, U)$, a correlated distribution $\mathbf{x} \in \Delta_{\mathcal{A}}$ is a coarse correlated equilibrium if for every player p and action $a' \in \mathcal{A}_p$ it holds:*

$$\sum_{\boldsymbol{a} \in \mathcal{A}} x_{\boldsymbol{a}} (U_p^{\boldsymbol{a}} - U_p^{a',\boldsymbol{a}_{-p}}) \geq 0$$

2.2 Online Learning Framework

We consider the following online setting. An agent plays a repeated game in which, at each round $t \in [T]$, she/he plays an action $y \in \mathcal{Y}$ while the environment selects an utility function u. [2] At each round $t \in [T]$, after selecting the action y^t, the agent observes an utility $u^t(y^t)$, where $u^t : \mathcal{Y} \to [0, 1]$.

We are interested in algorithms computing y^t at each round t. The performance of such algorithms is measured using the *regret* computed with respect to the best fixed action in hindsight. Formally:

$$R^T := \max_{y \in \mathcal{Y}} \sum_{t=1}^{T} u^t(y) - \mathbb{E}\left[\sum_{t=1}^{T} u^t(y^t)\right],$$

where the expectation is on the randomness of the online algorithm and T is the number of rounds. Ideally, we would like to find an algorithm that

[2] The set $\{1, \ldots, x\}$ is denoted by $[x]$.

generates a sequence $\{y^t\}_{t\in[T]}$ such that the regret is sublinear in T. An algorithm satisfying this property is usually called a *no-regret* algorithm. In the case in which requiring no-regret is too limiting, we use the following relaxed notion of regrets. Given an $\alpha \in [0, 1]$, the α-multiplicative-regret of an algorithm is defined as follows:

$$R^T_{\mathrm{M},\alpha} := \alpha \max_{y\in\mathcal{Y}} \sum_{t=1}^{T} u^t(y) - \mathbb{E}\left[\sum_{t=1}^{T} u^t(y^t)\right],$$

while the α-additive-regret of an algorithm is defined as follows:

$$R^T_{\mathrm{A},\alpha} := \max_{y\in\mathcal{Y}} \sum_{t=1}^{T} u^t(y) - \alpha - \mathbb{E}\left[\sum_{t=1}^{T} u^t(y^t)\right].$$

We call an algorithm that has α-multiplicative-regret or α-additive-regret sublinear in T a no-α-multiplicative-regret or no-α-additive-regret algorithm, respectively. The idea of no-α-regret is that the algorithm has no-regret with respect to an approximation of the optimal fixed action.

2.3 Two-provers Games

In this section, we summarize some key results on a class of *two-provers games*. In particular, we describe some of the results on two-prover games by Babichenko et al. (2015) and Deligkas et al. (2016).

A two-prover game $\mathcal{G}$ is a cooperative game played by two players (Merlin_1 and Merlin_2, respectively), and an adjudicator (verifier) called Arthur. At the beginning of the game, Arthur draws a pair of questions $(x, y) \in \mathcal{X} \times \mathcal{Y}$ according to a probability distribution $\mathcal{D}$ over the joint set of questions (*i.e.*, $\mathcal{D} \in \Delta_{\mathcal{X}\times\mathcal{Y}}$). Merlin_1 (resp., Merlin_2) observes x (resp., y) and chooses an answer p_1 (resp., p_2) from her finite set of answers $\mathcal{P}_1$ (resp., $\mathcal{P}_2$). Then, Arthur declares the Merlins to have *won* with a probability equal to the value of a *verification function* $\mathcal{V}(x, y, p_1, p_2)$. A *strategy* for Merlin_1 is a function $\eta_1 : \mathcal{X} \to \mathcal{P}_1$ mapping each possible question to an answer. Analogously, $\eta_2 : \mathcal{Y} \to \mathcal{P}_2$ is a strategy of Merlin_2. Before the beginning of the game, Merlin_1 and Merlin_2 can agree on their pair of (possibly mixed) strategies (η_1, η_2), but no communication is allowed during the games. The payoff of a game $\mathcal{G}$ to the Merlins under (η_1, η_2) is defined as: $u(\mathcal{G}, \eta_1, \eta_2) := \mathbb{E}_{(x,y)\sim\mathcal{D}}[\mathcal{V}(x, y, \eta_1(x), \eta_2(y))]$. The *value* of a two-prover game $\mathcal{G}$, denoted by $\omega(\mathcal{G})$, is the maximum expected payoff to the Merlins when they play optimally: $\omega(\mathcal{G}) := \max_{\eta_1} \max_{\eta_2} u(\mathcal{G}, \eta_1, \eta_2)$. The size of the game is $|\mathcal{G}| = |\mathcal{X} \times \mathcal{Y} \times \mathcal{P}_1 \times \mathcal{P}_2|$.

A two-prover game is called a *free game* if $\mathcal{D}$ is a uniform probability distribution over $\mathcal{X} \times \mathcal{Y}$. This implies that there is no correlation between the questions sent to Merlin_1 and Merlin_2. It is possible to build a family of free games mapping to 3SAT formulas arising from Dinur's PCP theorem. We say that the size n of a formula φ is the number of variables plus the number of clauses in the formula. Moreover, SAT(φ)$\in [0,1]$ is the maximum fraction of clauses that can be satisfied in φ. With this notation, the Dinur's PCP Theorem reads as follows:

Theorem 2.2 (Dinur's PCP Theorem (Dinur, 2007))**.** *Given any 3SAT instance φ of size n, and a constant $\rho \in (0, \frac{1}{8})$, we can produce in polynomial time a 3SAT instance φ' such that:*

1. *the size of φ' is $n\,\mathrm{polylog}(n)$;*
2. *each clause of φ' contains exactly 3 variables, and every variable is contained in at most $d = O(1)$ clauses;*
3. *if $\mathrm{SAT}(\varphi) = 1$, then $\mathrm{SAT}(\varphi') = 1$;*
4. *if $\mathrm{SAT}(\varphi) < 1$, then $\mathrm{SAT}(\varphi') < 1 - \rho$.*

A 3SAT formula can be seen as a bipartite graph in which the left vertices are the variables, the right vertices are the clauses, and there is an edge between a variable and a clause whenever that variable appears in that clause. Then, a such bipartite graph has constant degree since each vertex has constant degree. This holds because each clause has at most 3 variables and each variable is contained in at most d clauses. A useful result on bipartite graphs is the following.

Lemma 2.1 (Lemma 1 of (Deligkas et al., 2016))**.** *Let (V, E) be a bipartite graph with $|V| = n$, and U and W be the two disjoints and independent sets such that $V = U \cup W$, and where each vertex has a degree of at most ν. Suppose that U and W both have a constant fraction of the vertices,* i.e.*, $|U| = c\,n$ and $|W| = (1-c)\,n$ for some $c \in [0,1]$. Then, we can efficiently find a partition $\{S_i\}_{i=1}^{\sqrt{n}}$ of U, and a partition $\{T_j\}_{j=1}^{\sqrt{n}}$ of W, such that each set has a size of at most $2\sqrt{n}$, and for all i and j we have $|(S_i \times T_j) \cap E| \leq 2\,\nu^2$.*

Lemma 2.1 can be used to build the following free game.

Definition 2.5 (Definition 2 of (Deligkas et al., 2016))**.** *Given a 3SAT formula φ of size n, we define a free game $\mathcal{F}_\varphi$ as follows:*

1. *Arthur applies Theorem 2.2 to obtain formula φ' of size $n\,\mathrm{polylog}(n)$;*

2. *let $m = \sqrt{n}\,\mathrm{polylog}(n)$. Arthur applies Lemma 2.1 to partition the variables of φ' in sets $\{S_i\}_{i=1}^m$, and the clauses in sets $\{T_j\}_{j=1}^m$;*

3. *Arthur draws an index i uniformly at random from $[m]$, and independently an index j uniformly at random from $[m]$. Then, he sends S_i to Merlin$_1$ and T_j to Merlin$_2$;*

4. *Merlin$_1$ responds by choosing a truth assignment for each variable in S_i, and Merlin$_2$ responds by choosing a truth assignment to every variable that is involved with a clause in T_j;*

5. *Arthur awards the Merlins payoff 1 if and only if the following conditions are both satisfied:*

   - *Merlin$_2$'s assignment satisfies all clauses in T_j;*
   - *the two Merlins' assignments are* compatible*,* i.e.*, for each variable v appearing in S_i and each clause in T_j that contains v, Merlin$_1$'s assignment to v agrees with Merlin$_2$'s assignment to v;*

   Arthur awards payoff 0 otherwise.

When computing the Merlins' awards, the second condition is always satisfied when S_i and T_j share no variables. Moreover, when Merlin$_1$'s and Merlin$_2$'s assignments are not compatible, we say that they are *in conflict*.

The following lemma shows that, if φ is unsatisfiable, then the value of the corresponding free game $\mathcal{F}_\varphi$ is bounded away from 1.

Lemma 2.2 (Lemma 2 by (Deligkas et al., 2016))**.** *Given a 3SAT formula φ, the following holds:*

- *if φ is satisfiable then $\omega(\mathcal{F}_\varphi) = 1$;*
- *if φ is unsatisfiable then $\omega(\mathcal{F}_\varphi) \leq 1 - \rho/2\nu$.*

We define FREEGAME$_\delta$ as a specific problem within the class of *promise problems* (see, *e.g.*, (Even et al., 1984), (Goldreich, 2006)).

Definition 2.6 (FREEGAME$_\delta$)**.** *A* FREEGAME$_\delta$ *problem is defined as:*

- `INPUT`*: a free game $\mathcal{F}_\varphi$ and a constant $\delta \in (0, 1)$.*
- `OUTPUT`*:* YES*-instances: $\omega(\mathcal{F}_\varphi) = 1$;* NO*-instances: $\omega(\mathcal{F}_\varphi) \leq 1-\delta$.*

Finally, we will need to assume the *Exponential Time Hypothesis* (ETH), which conjectures that any deterministic algorithm solving 3SAT requires $2^{\Omega(n)}$ time.

Theorem 2.3. *(Theorem 2 by (Deligkas et al., 2016)) Assuming ETH, there exists a constant* $\delta = \rho/2\nu$ *such that* $\textsc{FreeGame}_\delta$ *requires time* $n^{\tilde{\Omega}(\log n)}$. [3]

2.4 Set Functions and Matroids

In this section we introduce some classes of set functions and matroids. Let $\mathcal{G}$ be a finite set and $f : 2^{\mathcal{G}} \to [0,1]$ be a function. We say that f is *submodular*, respectively *supermodular*, if for $I, I' \subseteq \mathcal{G}$: $f(I \cap I') + f(I \cup I') \leq f(I) + f(I')$, respectively $f(I \cap I') + f(I \cup I') \geq f(I) + f(I')$. The function f is *anonymous* if $f(I) = f(I')$ for all $I, I' \subseteq \mathcal{G} : |I| = |I'|$. A matroid $\mathcal{M} := (\mathcal{G}, \mathcal{I})$ is defined by a finite ground set $\mathcal{G}$ and a collection $\mathcal{I}$ of independent sets, *i.e.*, subsets of $\mathcal{G}$ satisfying some characterizing properties (see (Schrijver, 2003) for a detailed formal definition). We denote by $\mathcal{B}(\mathcal{M})$ the set of the *bases* of $\mathcal{M}$, which are the maximal sets in $\mathcal{I}$.

2.5 Error-Correcting Codes

In this section, we introduce error-correcting codes and some of their basic properties (see (Richardson and Urbanke, 2008) for further details). A *message* of length $k \in \mathbb{N}_+$ is encoded as a *block* of length $n \in \mathbb{N}_+$, with $n \geq k$. A *code* is a mapping $e : \{0,1\}^k \to \{0,1\}^n$. Moreover, let $\text{dist}(e(x), e(y))$ be the *relative Hamming distance* between $e(x)$ and $e(y)$, which is defined as the Hamming distance weighted by $1/n$. The *rate* of a code is defined as $R = \frac{k}{n}$. Finally, the *relative distance* $\text{dist}(e)$ of a code e is the maximum value d^{REL} such that $\text{dist}(e(x), e(y)) \geq d^{\text{REL}}$ for each $x, y \in \{0,1\}^k$.

In the following, we will need an infinite sequence of codes $\mathcal{E} := \{e_k : \{0,1\}^k \to \{0,1\}^{n(k)}\}_{k \in \mathbb{N}_+}$ containing one code e_k for each possible message length k. The following result, due to Gilbert (1952), can be used to construct an infinite sequence of codes with constant rate and distance.

Theorem 2.4 (Gilbert-Varshamov Bound)**.** *For every* $k \in \mathbb{N}_+$, $0 \leq d^{\text{REL}} < \frac{1}{2}$ *and* $n \geq \frac{k}{1-\mathcal{H}_2(d^{\text{REL}})}$, *there exists a code* $e : \{0,1\}^k \to \{0,1\}^n$ *with* $\text{dist}(e) = d^{\text{REL}}$, *where*

$$\mathcal{H}_2(d^{\text{REL}}) := d^{\text{REL}} \log_2\left(\frac{1}{d^{\text{REL}}}\right) + (1 - d^{\text{REL}}) \log_2\left(\frac{1}{1 - d^{\text{REL}}}\right).$$

Moreover, such a code can be computed in time $2^{O(n)}$.

[3] $\tilde{\Omega}$ hides polylogarithmic factors.

CHAPTER 3

Bayesian Persuasion Framework

This chapter introduce the framework of Bayesian persuasion. Section 3.1 defines the Bayesian persuasion game with a single receiver. Section 3.2 extends the framework to consider multiple receivers. Finally, Section 3.3 surveys the state of the art on Bayesian persuasion.

3.1 Bayesian Persuasion with a Single Receiver

Bayesian persuasion studies the problem faced by an informed sender trying to influence the behavior of a self-interested receiver via the strategic provision of payoff-relevant information. The receiver has a finite set of ϱ actions $\mathcal{A} := \{a_i\}_{i=1}^{\varrho}$. The receiver's payoff function is $u^r : \mathcal{A} \times \Theta \to [0, 1]$, where $\Theta := \{\theta_i\}_{i=1}^{d}$ is a finite set of d states of nature. For notational convenience, we denote by $u^r_\theta(a) \in [0, 1]$ the utility observed by the receiver when the realized state of nature is $\theta \in \Theta$ and she/he plays action $a \in \mathcal{A}$. The sender's utility when the state of nature is $\theta \in \Theta$ is described by the function $f_\theta : \mathcal{A} \to [0, 1]$. As it is customary in Bayesian persuasion, we assume that the state of nature is drawn from a common prior distribution $\boldsymbol{\mu} \in \Delta_\Theta$, which is explicitly known to both the sender and the receiver. Moreover, the sender can commit to a *signaling scheme* ϕ, which is a randomized mapping from states of nature to *signals* for the receiver. Formally $\phi : \Theta \to \Delta_{\mathcal{S}}$, where $\mathcal{S}$ is a finite set of signals. We denote by ϕ_θ the probability distribution employed by the sender when the state of nature is $\theta \in \Theta$, with $\phi_\theta(s)$ being the probability of sending signal $s \in \mathcal{S}$.

The interaction between the sender and the receiver goes on as follows:

(i) the sender commits to a publicly known signaling scheme ϕ and the receiver observes the commitment;

(ii) the sender observes the realized state of nature $\theta \sim \boldsymbol{\mu}$;

(iii) the sender draws a signal $s \sim \phi_\theta$ and communicates it to the receiver;

(iv) the receiver observes s and rationally updates her/his prior beliefs over Θ according to the *Bayes rule*;

(v) the receiver selects an action maximizing her/his expected utility.

Let $\Xi := \Delta_\Theta$ be the set of receiver's posterior beliefs over the states of nature. In step (iv), after observing $s \in \mathcal{S}$, the receiver performs a Bayesian update and infers a posterior belief $\boldsymbol{\xi} \in \Xi$ over the states of nature such that the component of $\boldsymbol{\xi}$ corresponding to state of nature $\theta \in \Theta$ is:

$$\xi_\theta := \frac{\mu_\theta \, \phi_\theta(s)}{\sum_{\theta' \in \Theta} \mu_{\theta'} \, \phi_{\theta'}(s)}. \tag{3.1}$$

After computing $\boldsymbol{\xi}$, the receiver solves a decision problem to find an action maximizing her/his expected utility given the current posterior. When the receiver is indifferent among multiple actions, we assume that the receiver breaks ties in favor of the sender. Letting $a \in \mathcal{A}$ be the receiver's choice, the receiver observes payoff $u^r_\theta(a)$, while the sender observes payoff $f_\theta(a)$.

A revelation-style argument shows that there always exists an optimal signaling scheme that is *direct* and *persuasive* (Kamenica and Gentzkow, 2011). A signaling scheme is direct if $\mathcal{S} = \mathcal{A}$, *i.e.*, signals can be interpreted as action recommendations and it is persuasive if the receiver has a incentive to follow the recommendations. The optimal direct and persuasive signaling scheme can be computed with the following LP.

$$\max_{\phi} \sum_{\theta \in \Theta, a \in \mathcal{A}} \mu_\theta \, \phi_\theta(a) \, f_\theta(a) \tag{3.2a}$$

$$\text{s.t.} \sum_{\theta \in \Theta} \mu_\theta \, \phi_\theta(a) \left(u^r_\theta(a) - u^r_\theta(a') \right) \geq 0 \qquad \forall a, a' \in \mathcal{A} \tag{3.2b}$$

$$\sum_{a \in \mathcal{A}} \phi_\theta(a) = 1 \qquad \forall \theta \in \Theta \qquad (3.2c)$$

$$\phi_\theta(a) \geq 0 \qquad \forall \theta \in \Theta, \forall a \in \mathcal{A} \qquad (3.2d)$$

where (3.2a) is the sender's utility, constraints (3.2b) force the signaling scheme to be persuasive, and constraints (3.2c) and (3.2d) force the signaling scheme to be feasible.

3.1.1 Working in the Space of Posterior Distributions

It is oftentimes useful to represent signaling schemes as convex combinations of posterior beliefs they can induce. First, we describe such interpretation (see (Kamenica, 2019) for further details). Then, we define the receiver's best response given an arbitrary posterior belief.

Given a signaling scheme ϕ, each signal realization $s \in \mathcal{S}$ leads to a posterior belief $\boldsymbol{\xi}^s \in \Xi$, whose components are defined as in Equation (3.1). Accordingly, each signaling scheme leads to a distribution over posterior beliefs. We denote a distribution over posteriors by $\boldsymbol{\gamma} \in \Delta_\Xi$. We say that a signaling scheme $\phi : \Theta \to \Delta_\mathcal{S}$ *induces* $\boldsymbol{\gamma} \in \Delta_\Xi$ if, for every $\boldsymbol{\xi} \in \Xi$, the component of $\boldsymbol{\gamma}$ corresponding to $\boldsymbol{\xi}$ is defined as follows:

$$\gamma_{\boldsymbol{\xi}} := \sum_{s \in \mathcal{S} : \boldsymbol{\xi}^s = \boldsymbol{\xi}} \sum_{\theta \in \Theta} \mu_\theta \, \phi_\theta(s). \qquad (3.3)$$

Intuitively, if ϕ induces $\boldsymbol{\gamma}$, then $\gamma_{\boldsymbol{\xi}}$ represents the probability that ϕ induces the posterior $\boldsymbol{\xi} \in \Xi$. We let supp$(\boldsymbol{\gamma}) := \{\boldsymbol{\xi} \in \Xi \mid \gamma_{\boldsymbol{\xi}} > 0\}$ be the set of posteriors induced with strictly positive probability. We say that a distribution over posteriors $\boldsymbol{\gamma} \in \Delta_\Xi$ is *consistent* (*i.e.*, intuitively, there exists a valid signaling scheme ϕ inducing $\boldsymbol{\gamma}$) if the following hods:

$$\sum_{\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}} \, \xi_\theta = \mu_\theta, \quad \text{for all } \theta \in \Theta. \qquad (3.4)$$

We let $\Gamma \subseteq \Delta_\Xi$ be the set of distributions over posteriors that are consistent according to Equation (3.4). In the remainder of the dissertation, we equivalently employ ϕ or $\boldsymbol{\gamma}$ to denote an arbitrary signaling scheme.

After observing a signal $s \in \mathcal{S}$ that induces a posterior $\boldsymbol{\xi} \in \Xi$, the receiver best responds by choosing an action that maximizes her/his expected utility (step (v)). The set of actions maximizing the receiver's expected utility given posterior $\boldsymbol{\xi}$ is defined as follows:

Definition 3.1 (BR-set)**.** *Given a posterior* $\boldsymbol{\xi} \in \Xi$*, the best-response set (BR-set) is:*

$$\mathcal{B}_{\boldsymbol{\xi}} := arg\,max_{a\in\mathcal{A}} \sum_{\theta\in\Theta} \xi_\theta\, u^r_\theta(a).$$

We denote by $b_{\boldsymbol{\xi}}$ the action belonging to the BR-set $\mathcal{B}_{\boldsymbol{\xi}}$ played by the receiver. When the receiver is indifferent among multiple actions for a given posterior $\boldsymbol{\xi}$, we assume that the receiver breaks ties in favor of the sender, *i.e.*, she/he chooses an action $b_{\boldsymbol{\xi}} \in arg\,max_{a\in\mathcal{B}_{\boldsymbol{\xi}}} \sum_\theta \xi_\theta\, f_\theta(a)$. [1]

Given an $\epsilon > 0$, a receiver plays an ϵ*-best response* when the selected action provides her an expected utility which is at most ϵ less than the optimal value. The set of ϵ-best responses is defined as follows.

Definition 3.2 (ϵ-BR-set)**.** *Given a posterior* $\boldsymbol{\xi} \in \Xi$*, the* ϵ*-best-response set (*ϵ*-BR-set) is the set* $\mathcal{B}_{\epsilon,\boldsymbol{\xi}}$ *of all the actions* $a \in \mathcal{A}$ *such that:*

$$\sum_{\theta\in\Theta} \xi_\theta u^r_\theta(a) \geq \sum_{\theta\in\Theta} \xi_\theta u^r_\theta(a') - \epsilon \quad \forall a' \in \mathcal{A}.$$

We denote by $b_{\epsilon,\boldsymbol{\xi}}$ the action in $\mathcal{B}_{\epsilon,\boldsymbol{\xi}}$ played by the receiver. When the receiver has multiple ϵ-best-response actions for a given posterior $\boldsymbol{\xi}$, we assume she breaks ties in favor of the sender, *i.e.*, she chooses an action $b_{\epsilon,\boldsymbol{\xi}} \in arg\,max_{a\in\mathcal{B}_{\epsilon,\boldsymbol{\xi}}} \sum_\theta \xi_\theta f_\theta(a)$.

Sometimes we will restrict attention to the subset of posteriors defined as follows.

Definition 3.3 (*q-uniform posteriors*)**.** *A posterior* $\boldsymbol{\xi} \in \Xi$ *is* q-uniform *if and only if it is the average of a multiset of* q *basis vectors in an* $|\Theta|$*-dimensional space.*

Equivalently, each entry ξ_θ of a q-uniform posterior has to be a multiple of $1/q$. We denote with $\Xi^q \subset \Xi$ the set of q-uniform posteriors.

3.2 Bayesian Persuasion with Multiple Receivers

In this section, we introduce the Bayesian persuasion problem with multiple receivers. We denote with $\mathcal{R} := \{r_i\}_{i=1}^{\bar{n}}$ the set of receivers. Similarly to the single receiver case, the sender can publicly commit to a signaling scheme which maps the realized state of nature to a signal for each player. Each receiver has a set of action $\mathcal{A}_r := \{a_{r,i}\}_{i=1}^{\varrho_r}$. We denote with

[1] This assumption is customary in settings involving commitments, such as Stackelberg games (Conitzer and Korzhyk, 2011; Conitzer and Sandholm, 2006; Paruchuri et al., 2008).

$\mathcal{A} := \bigtimes_{r \in \mathcal{R}} \mathcal{A}_r$ the set of action profiles. The utility of receiver $r \in \mathcal{R}$ is defined by an utility function $u^r : \Theta \times \mathcal{A} \to [0,1]$, where we denote with $u^r_\theta(\mathbf{a})$ the utility of receiver r when the state is $\theta \in \Theta$ and the action profile of the receivers is $\mathbf{a} \in \mathcal{A}$. The sender's utility in a state θ depends on the actions played by all the receivers, and it is defined by $f_\theta : \mathcal{A} \to [0,1]$. Sometimes, we restrict to games with binary actions, *i.e.*, such that $\mathcal{A}_r := \{a_0, a_1\}$ for each $r \in \mathcal{R}$. In this case, for the ease of presentation, we introduce the function $f_\theta : 2^{\mathcal{R}} \to [0,1]$ such that $f_\theta(R)$ represents the sender's utility when the state of nature is θ and all the receivers in $R \subseteq \mathcal{R}$ play action a_1, while the others play a_0. The sender can publicly commit to a signaling scheme ϕ which maps states of nature to signals for the receivers. A generic signal for receiver r is denoted by s_r, while the set of signals to each receiver r is denoted by $\mathcal{S}_r$. Moreover, $\mathcal{S} := \bigtimes_{r \in \mathcal{R}} \mathcal{S}_r$ denotes the set of signal profiles. In general, the sender can send different signals to each player through private communication channels. In this setting, a simple revelation-principle-style argument shows that it is enough to employ players' actions as signals (Arieli and Babichenko, 2019; Kamenica and Gentzkow, 2011). We call the signaling schemes that employ only action recommendations *direct*. Therefore, a private signaling scheme is a function $\phi : \Theta \to \Delta_{\mathcal{A}}$ which maps any state of nature to a probability distribution over action profiles (signals). For the ease of notation, the probability of recommending an action profile $\mathbf{a} \in \mathcal{A}$ given the state of nature $\theta \in \Theta$ is denoted by $\phi_\theta(\mathbf{a})$. Then, it has to hold $\sum_{\mathbf{a} \in \mathcal{A}} \phi_\theta(\mathbf{a}) = 1$, for each $\theta \in \Theta$. After observing the state of nature $\theta \in \Theta$, the sender draws an action profile $\mathbf{a} \in \mathcal{A}$ according to ϕ_θ and recommends action a_r to each player $r \in \mathcal{R}$. A signaling scheme is *persuasive* if following recommendations is an equilibrium of the underlying *Bayesian game* (Bergemann and Morris, 2016a,b).

Definition 3.4. *A signaling scheme $\phi : \Theta \to \Delta_{\mathcal{A}}$ is persuasive if, for each $r \in \mathcal{R}$ and $a, a' \in \mathcal{A}_r$, it holds:*

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{a} \in \mathcal{A}: a_r = a} \phi_\theta(\boldsymbol{a}) \Big(u^r_\theta(\boldsymbol{a}) - u^r_\theta(a'_r, \boldsymbol{a}_{-r}) \Big) \geq 0.$$

Notice that in the case in which there is only a state of nature a persuasive signaling scheme can be seen as a correlated equilibrium. Similarly, we say that a signaling scheme is ϵ-persuasive if the receivers have a small incentive ϵ not to follow the recommendations.

Definition 3.5. *For any $\epsilon > 0$, a signaling scheme $\phi : \Theta \to \Delta_{\mathcal{A}}$ is ϵ-*

persuasive *if, for each* $r \in \mathcal{R}$ *and* $a, a' \in \mathcal{A}^r$*, it holds:*

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{a} \in \mathcal{A}: a_r = a} \phi_\theta(\boldsymbol{a}) \Big(u^r_\theta(\boldsymbol{a}) - u^r_\theta(a', \boldsymbol{a}_{-r}) \Big) \geq -\epsilon.$$

A weaker solution concept is represented by *ex-ante persuasiveness* as defined by Xu (2020) and Celli et al. (2020).

Definition 3.6. *A signaling scheme* $\phi : \Theta \to \Delta_{\mathcal{A}}$ *is* ex ante persuasive *if, for each* $r \in \mathcal{R}$ *and* $a \in \mathcal{A}_r$*, it holds:*

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{a} \in \mathcal{A}} \phi_\theta(\boldsymbol{a}) \Big(u^r_\theta(\boldsymbol{a}) - u^r_\theta(a, \boldsymbol{a}_{-r}) \Big) \geq 0.$$

Then, a coarse correlated equilibrium (Moulin and Vial, 1978) may be seen as an *ex ante* persuasive signaling scheme in a game in which there is only a state of nature.

3.2.1 Bayesian Persuasion with No Externalities

In most of this dissertation, we assume that there are no inter-agent externalities among the receivers. Each receiver's payoff depends only on the action she takes and on a (random) state of nature $\theta \in \Theta$. In particular, with a slight abuse of notation, receiver r's utility is given by the function $u^r : \Theta \times \mathcal{A}_r \to [0, 1]$. We denote by $u^r_\theta(a) \in [0, 1]$ the utility observed by receiver r when the state of nature is $\theta \in \Theta$ and she plays $a \in \mathcal{A}_r$.

The interaction between the sender and the receivers goes as follows:

(i) the sender commits to a publicly known signaling scheme ϕ and the receivers observe the commitment;

(ii) the sender observes the realized state of nature $\theta \sim \boldsymbol{\mu}$;

(iii) the sender draws a signal profile $\mathrm{s} = (s_r)_{r=1}^{\bar{n}} \in \mathcal{S}$ for each receiver according to the signaling scheme ϕ_θ, and communicates to each receiver r the signal s_r;

(iv) each receiver r observes s_r and updates her prior beliefs over Θ following Bayes rule.

(v) each receiver selects an action maximizing her expected reward in the posterior $\boldsymbol{\xi}^{s_r}$.

Differently to the inter-agent-externalities setting, in step (v) each agent decision is not influenced by the actions played by the other receivers.

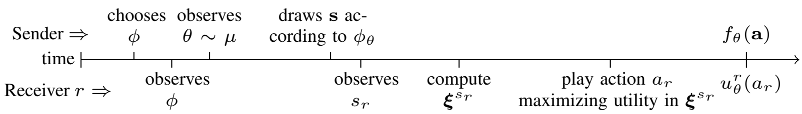


Figure 3.1: *Interaction between the sender and a receiver with no externalities.*

Given a signaling scheme ϕ, we denote with ϕ_r the *marginal* signaling scheme of receiver $r \in \mathcal{R}$. It is defined as $\phi_{r,\theta}(s_r) := \sum_{\mathbf{s}' \in \mathcal{S}: s_r = s'_r} \phi_\theta(\mathrm{s}')$ for each $\theta \in \Theta$ and $s_r \in \mathcal{S}_r$, *i.e.*, $\phi_{r,\theta}(s_r)$ represents the marginal probabilities that s_r is sent to r when the state of nature is θ. A receiver $r \in \mathcal{R}$ receiving a signal $s_r \in \mathcal{S}_r$ infers a posterior belief over states which we denote by $\boldsymbol{\xi}^{s_r} \in \Xi$, with $\xi^{s_r}_\theta$ being the posterior probability of state $\theta \in \Theta$. Formally,

$$\xi^{s_r}_\theta := \frac{\mu_\theta \phi_{r,\theta}(s_r)}{\sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{r,\theta'}(s_r)}. \tag{3.5}$$

3.2.2 Private Signaling Schemes with No Externalities

In general, each receiver can observe a different signal. We call this type of signaling schemes *private*. A simple revelation-principle style argument shows that there always exist an optimal signaling scheme that is direct and persuasive. We can encode the sender optimization problem using the following linear program of exponential size.

$$\max_{\phi} \sum_{\theta \in \Theta, \mathbf{a} \in \mathcal{A}} \mu_\theta \, \phi_\theta(\mathbf{a}) \, f_\theta(\mathbf{a}) \tag{3.6a}$$

$$\text{s.t.} \sum_{\theta \in \Theta} \mu_\theta \sum_{\mathbf{a} \in \mathcal{A}: a_r = a} \phi_\theta(\mathbf{a}) \Big(u^r_\theta(a) - u^r_\theta(a') \Big) \geq 0 \quad \forall r \in \mathcal{R}, a, a' \in \mathcal{A}_r \tag{3.6b}$$

$$\sum_{\mathbf{a} \in \mathcal{A}} \phi_\theta(\mathbf{a}) = 1 \qquad \forall \theta \in \Theta \tag{3.6c}$$

$$\phi_\theta(\mathbf{a}) \geq 0 \qquad \forall \theta \in \Theta, \forall \mathbf{a} \in \mathcal{A} \tag{3.6d}$$

where, we recall, $\mathbf{a} := (a_r)_{r=1}^{\bar{n}}$. The sender's goal is computing the signaling scheme maximizing her expected utility (Objective Function (3.6a)). Constraints (3.6b) force the private signaling scheme to be persuasive.

3.2.3 Public Signaling Schemes with No Externalities

Public signaling schemes are a class of constrained signaling schemes in which the sender is constrained to send the same signal to all the receivers.

Formally, a signaling scheme ϕ is public if for any θ and s $\sim \phi_\theta$, it holds $s_r = s_{r'}$ for each pair of receivers $r, r' \in \mathcal{R}$. With an overload of notation we write $s \in \mathcal{S}$ to denote the public signal received by all receivers. A public signaling scheme is *direct* when signals can be mapped to actions of the receivers, and interpreted as action recommendations, *i.e.*, $\mathcal{S} = \mathcal{A}$. Notice that each receiver is sent the same signal $\mathbf{a} \in \mathcal{A}$ specifying a (possibly different) action for each receiver. We write $\phi_\theta(\mathbf{a})$ to denote the probability with which the sender selects the action profile **a** when the realized state of nature is θ. A public signaling scheme is persuasive if when a receiver r receives a direct signal $\mathbf{a} \in \mathcal{A}$ following the recommendation a_r is an equilibrium of the underling game. Also in this setting, a simple revelation-principle style argument shows that there always exist an optimal signaling scheme that is direct and persuasive. Since the prior is common knowledge and all receivers observe the same signal, they all perform the same Bayesian update and have the same posterior belief regarding the realized state of nature.

The problem of determining an optimal public signaling scheme which is direct and persuasive can be formulated with the following (exponentially sized) LP:

$$\max_{\phi} \sum_{\theta \in \Theta, \mathbf{a} \in \mathcal{A}} \mu_\theta \, \phi_\theta(\mathbf{a}) \, f_\theta(\mathbf{a}) \tag{3.7a}$$

$$\text{s.t.} \sum_{\theta \in \Theta} \mu_\theta \, \phi_\theta(\mathbf{a}) \Big(u^r_\theta(a_r) - u^r_\theta(a') \Big) \geq 0 \; \forall r \in \mathcal{R}, \forall \mathbf{a} \in \mathcal{A}, a' \in \mathcal{A}_r \tag{3.7b}$$

$$\sum_{\mathbf{a} \in \mathcal{A}} \phi_\theta(\mathbf{a}) = 1 \qquad \forall \theta \in \Theta \tag{3.7c}$$

$$\phi_\theta(\mathbf{a}) \geq 0 \qquad \forall \theta \in \Theta, \forall \mathbf{a} \in \mathcal{A} \tag{3.7d}$$

where, we recall, $\mathbf{a} := (a_r)_{r=1}^{\bar{n}}$. The sender's goal is computing the signaling scheme maximizing her expected utility (Objective Function (3.7a)). Constraints (3.7b) force the public signaling scheme to be persuasive.

3.3 Previous Results on Bayesian Persuasion

Kamenica and Gentzkow (2011) introduce the model of Bayesian persuasion with a single sender and a single receiver. From a computational perspective, the single-receiver problem can be solved by a simple linear program of polynomial size. For this reason, most of the works focus on the multi-receiver problem. A notable exception is the work of Dughmi and Xu (2016) that study a class of succinctly represented games. Some works consider games with multiple receivers and inter-agent-externalities. The works of Bhaskar et al. (2016) and Rubinstein (2015) focus on the complexity of computing public signaling schemes. Bhaskar et al. (2016) and Rubinstein (2015) study public signaling problems in which two receivers play a zero-sum game. Bhaskar et al. (2016) rule out an additive PTAS assuming the planted-clique hardness. Moreover, Rubinstein (2015) proves that the problem of computing an ϵ-optimal signaling scheme requires at least quasi-polynomial time assuming the Exponential Time Hypothesis (ETH). This result is tight due to the quasi-polynomial approximation scheme designed by Cheng et al. (2015).

Some works simplify the problem assuming no inter-agent externalities. This assumption allows one to focus on the key problem of coordinating the receivers' behavior, without the additional complexity arising from externalities which have been shown to make the problem largely intractable (Bhaskar et al., 2016; Rubinstein, 2015). Arieli and Babichenko (2019) introduce the model of persuasion with multiple receivers and without inter-agent externalities, with a focus on private Bayesian persuasion. In particular, they study the setting with a binary action space for the receivers and a binary space of states of nature. They provide a characterization of the optimal signaling scheme in the case of supermodular, anonymous submodular, and supermajority sender's utility functions. Babichenko and Barman (2017) extend the work by Arieli and Babichenko (2019) providing a polynomial-time algorithm that computes a $(1 - 1/e)$-approximate signaling scheme for monotone submodular sender's utilities, which is tight. Moreover, they show that an optimal private signaling scheme for anonymous utility functions can be found efficiently. Dughmi and Xu (2017) generalize the previous model to settings with an arbitrary number of states of nature.

Other works focus on the public signaling problem with no inter-agent externalities. In particular, Dughmi and Xu (2017) rule out the existence of a PTAS even when receivers have binary action spaces and objectives are linear, unless P $=$ NP. Xu (2020) studies public persuasion with binary action spaces and an arbitrary number of states of nature, and he shows that no bi-criteria FPTAS is possible unless P $=$ NP. Furthermore, the author proposes a bi-criteria PTAS for monotone submodular sender's utility functions and shows that, when the number of states of nature is fixed and a non-degeneracy assumption holds, an optimal signaling scheme can be computed in polynomial time.

A recent line of work relaxes the assumption that the sender perfectly knows the receiver utility. Babichenko et al. (2021) study a game with a single receiver and binary actions in which the sender does not know the receiver's utility, focusing on the problem of designing a signaling scheme that performs well for each possible receiver's utility. Zu et al. (2021) relax the perfect knowledge assumption, assuming that the sender and the receiver do not know the prior distribution over the states of nature. They study the problem of computing a sequence of persuasive signaling schemes that achieve small regret with respect to the optimal persuasive signaling scheme with the knowledge of the prior distribution. Finally, Xu et al. (2016a); Gan et al. (2019) study a signaling problem in Bayesian Stackelberg games in which the receiver and the sender's have private information, *i.e.*, a type.

3.3.1 Previous Results on Games with Structure

Some works focus on problems with specific structures. Cheng et al. (2015) provide a polynomial-time tri-criteria approximation algorithm for the optimal public signaling scheme for k-voting. Arieli and Babichenko (2019) study a voting setting in which there are two state of nature and two possible candidates. They provide a characterization of the optimal private signaling scheme for majority voting.

Other works study congestion games. Bhaskar et al. (2016) study the inapproximability of finding optimal ex interim persuasive signaling schemes in non-atomic games. Liu and Whinston (2019) focus on atomic games with costs uncertainties and study ex-interim persuasion by placing stringent constraints on the network structure. Nachbar and Xu (2020) study how much the information designer can improve her objective using different types of signaling schemes. Zhou et al. (2021) study singleton congestion games, showing that for both the private and public signaling problem, the optimal signaling scheme can be computed in polynomial time when the number of resources is constant.

Finally, some works study signaling in auctions. These works are mainly relative to second-price auctions. Emek et al. (2014) study second-price auctions focusing on the known-valuation setting in which the sender knows the buyers' valuations. They provide a linear program to compute the optimal public signaling scheme. Moreover, they show that it is NP-hard to compute the optimal signaling scheme in the Bayesian valuation setting. Cheng et al. (2015) provide a PTAS for the Bayesian model. Badanidiyuru et al. (2018) focus on the design of algorithms with running time independent from the number of states of nature. Moreover they initiate the study of private signaling scheme showing that in second-price auctions private signaling schemes introduce non-trivial equilibrium selection problems.

Part I

Exploiting the Problem Structure

CHAPTER 4

Persuading Voters in Simple Elections

In this chapter, we focus on the computation of public and private signaling schemes with some simple voting rules, *i.e*, k-voting and plurality voting. In Section 4.1, we introduce the model. In Section 4.2 we compare the efficiency of public and private signaling schemes. In Section 4.3, we provide a polynomial-time algorithm for computing optimal private signaling schemes with k-voting. Section 4.4 presents a necessary and sufficient condition for the polynomial-time computation of private signaling schemes under a general class of sender's objective function (beyond voting). Section 4.5 provides some positive results on the computation of private signaling schemes. Among them, it shows that the optimal private signaling scheme can be computed in polynomial time for the plurality voting rule. Finally, in Section 4.6 we show that computing a public signaling scheme is NP-hard even for simple voting functions, *e.g.*, majority voting.

4.1 Model of Bayesian Persuasion in Elections

Our model comprises a *sender* and a finite set $\mathcal{R} := (r_i)_{i=1}^{\bar{n}}$ of *receivers* (voters) that must choose one alternative from a set $C := \{c_0, \dots, c_\varrho\}$ of candidates (*i.e.*, C is the set of voters' available actions). Each voter must choose a candidate from C. Each voter's utility depends only on her own action and the (random) state of nature, but not on the actions of other voters. In particular, we write $u^r : \Theta \times C \to [0,1]$, where $\Theta := \{\theta_i\}_{i=1}^d$ is the finite space of states of nature. The value of $u_\theta^r(c)$ is a measure of voter r's appreciation of candidate c when the state of nature is θ. A profile of votes (*i.e.*, one candidate for each voter) is denoted by $\mathbf{c} \in \mathcal{C} := \times_{r\in\mathcal{R}} C$. In general settings, beyond voting, we denote the sender's utility when the state of nature is θ with $f_\theta : \times_{r\in\mathcal{R}} C \to [0,1]$ (here C may be an arbitrary space of actions). Furthermore, we say that f is *anonymous* if its value depends only on θ and on the number of players selecting each action. In the specific context of voting, the sender's objective is maximizing the winning probability of c_0 (according to some voting rules). In this setting, instead of using f, we denote the sender's utility function by $W : \times_{r\in\mathcal{R}} C \to \{0,1\}$, where $W(\cdot) = 1$ if c_0 *wins*, and $W(\cdot) = 0$ otherwise. The state of nature influences the receivers' preferences but it does not affect the sender's payoff, which only depends on the final votes. [1] When the sender's signaling scheme ϕ is direct and persuasive we write $W(\phi)$ to denote the sender's expected utility. Finally, function $\delta : \mathcal{C} \times C \to \mathbb{N}$ is s.t. $\delta(\mathbf{c}, c)$ is the number of voters that are recommended c by $\mathbf{c}$.

We consider two commonly adopted voting rules: *k-voting rule* and *plurality voting rule* (see, *e.g.*, (Brandt et al., 2016)). In an election with a *k-voting rule* each voter chooses a candidate after observing the sender's signal. Candidate c_i is elected if it receives at least k votes, where $k \in [\bar{n}]$ is the established electoral rule. The problem of designing the optimal sender's persuasive signaling scheme under a k-voting rule is denoted by K-V. In an election with *plurality voting rule* the winner is determined as the candidate with a plurality (greatest number) of votes. The problem of finding an optimal persuasive signaling scheme for the sender with plurality voting is denoted by PL-V. In both settings, we focus on maximizing the winning probability of the sender. The problem can be written as the optimization problem: $\max_\phi \sum_{\theta\in\Theta,\mathbf{c}\in\mathcal{C}} \mu_\theta \phi_\theta(\mathbf{c}) W(\mathbf{c})$, subject to ϕ being persuasive for each voter.

To further clarify the election scenario, we provide the following simple example.

Example 4.1. *There are three voters $\mathcal{R} = \{r_1, r_2, r_3\}$ who must select one between two candidates $\{c_0, c_1\}$. The sender (*e.g.*, a politician or a lobbyist) observes the realized state of nature, drawn from the uniform probability distribution $(1/3, 1/3, 1/3)$ over $\Theta = \{\theta_A, \theta_B, \theta_C\}$, and exploits this*

[1] The sender's utility function is state-independent in many settings, *e.g.*, voting (Alonso and Câmara, 2016), and marketing (Candogan, 2019; Babichenko and Barman, 2017).

information to support the election of c_0*. The state of nature describes the position of* c_0 *on a matter of particular interest to the voters. Moreover, all the voters have a slightly negative opinion of candidate* c_1*, independently of the state of nature, while the opinion on candidate* c_0 *can be better or worse than the opinion on* c_1 *depending the state of nature. Table 4.1 describes the utility of the three voters.*

		State θ_A		State θ_B		State θ_C	
		c_0	c_1	c_0	c_1	c_0	c_1
Voters	1	1	3/8	0	3/8	0	3/8
	2	0	3/8	1	3/8	0	3/8
	3	0	3/8	0	3/8	1	3/8

Table 4.1: *Payoffs from voting different candidates in Example 4.1.*

We consider a k*-voting rule with* $k = 2$*. Without any form of signaling, all the voters would vote for* c_1 *because it provides an expected utility of* $3/8$*, against* $1/3$*, and the sender would get a utility of* 0*. If the sender discloses all the information regarding the state of nature (*i.e.*, with a fully informative signal), he would still get a utility of* 0*, since two out of three receivers would pick* c_1 *in each of the possible states. However, the sender can design a public signaling scheme guaranteeing herself a utility of* 1 *for each state of nature. Table 4.2 describes one such scheme with arbitrary signals. Suppose the observed state is* θ_A*, and that the signal sent by the sender is* not B*. Then, the posterior distribution over the states of nature is* $(1/2, 0, 1/2)$*. Therefore, receiver 1 and receiver 3 would vote for* a_0 *since their expected utility would be* $1/2$ *against* $3/8$*. Similarly, for any other signal, two receivers vote for* a_0*. Then, the sender's expected payoff is* 1*. We can recover an equivalent direct signaling scheme by sending a tuple with a candidates' suggestion for each voter. For example,* not A *would become* (c_1, c_0, c_0)*, and each voter would observe the recommendations given to the others.*

		Signals		
		not A	not B	not C
States	A	0	1/2	1/2
	B	1/2	0	1/2
	C	1/2	1/2	0

Table 4.2: *Optimal public signaling scheme in Example 4.1.*

4.2 Inefficiency of Public Persuasion

In this section, we provide an example of a majority voting election. This example show that restricting from private to public signaling can decrease the sender's utility by an arbitrarily large factor.

Example 4.2. *Consider a majority-voting election with seven voters* $\mathcal{R} = \{r_1, r_2, r_3, r_4, r_5, r_6, r_7\}$ *and two candidates* $C = \{c_0, c_1\}$*. The objective of the sender is to maximize the probability with which candidate* c_0 *is elected. Therefore, he needs to persuade at least half of the voters (*i.e., $\lceil |\mathcal{R}|/2 \rceil = 4$*) to make candidate* c_0 *be the winner. There are three states of nature, namely,* $\Theta = \{\theta_A, \theta_B, \theta_C\}$*, and each state is equally probable. Tab. 4.3 provides the parameters* u^r_θ *of the voters, defined as* $u^r_\theta = u^r_\theta(c_0) - u^r_\theta(c_1)$ *and capturing the net payoff of voter* r *from having candidate* c_0 *elected, in state of nature* θ*.*

		State θ_A	State θ_B	State θ_C
Voters	r_1,r_2	$+1/2$	-1	-1
	r_3,r_4	-1	$+1/2$	-1
	r_5,r_6	-1	-1	$+1/2$
	r_7	$+1/2$	$+1/2$	$+1/2$

Table 4.3: *Payoffs of the voters in Example 4.2.*

The sender can design a direct and persuasive private signaling scheme such that at least four voters prefer candidate c_0 *over* c_1 *for every signal profile* $\mathbf{s}$*. Hence, this scheme ensures that candidate* c_0 *is elected with a probability of 1. Specifically, in each state* θ *the scheme recommends candidate* c_0 *to every voter* r *with utility* $u^r_\theta \geq 0$ *and to one voter among those with* $u^r_\theta < 0$ *chosen randomly with uniform probability. It is easy to see that this private signaling scheme satisfies the incentive constraints. Consider, for example, voter* r_1*. The marginal probabilities with which he is recommended to vote for candidate* c_0 *are:* $\phi_{r_1,\theta_A}(c_0) = 1, \phi_{r_1,\theta_B}(c_0) = 1/4$ *and* $\phi_{r_1,\theta_C}(c_0) = 1/4$*. Therefore, when he receives the recommendation to vote for* c_0*, he has a posterior distribution* $\boldsymbol{\xi}$ *with* $\xi_{\theta_A} = \frac{\mu_{\theta_A} \cdot \phi_{r_1,\theta_A}(c_0)}{\sum_{\theta \in \Theta} \mu_\theta \cdot \phi_{r_1,\theta}(c_0)} = \frac{1/3}{1/3+1/3 \cdot 1/4+1/3 \cdot 1/4} = 2/3$ *and* $\xi_{\theta_B} = \xi_{\theta_C} = 1/6$*. Thus, the voter has expected utility* $u^{r_1}_{\theta_A}\xi_{\theta_A} + u^{r_1}_{\theta_B}\xi_{\theta_B} + u^{r_1}_{\theta_C}\xi_{\theta_C} = 0$ *and will follow the recommendation. Similarly, we can show that the incentive constraints associated with the other voters are satisfied.*

We switch to public signals and we show that we cannot design a public *signaling scheme that guarantees candidate c_0 to be elected with positive probability. Any public signaling scheme making candidate c_0 win the election with positive probability must assign a strictly positive probability to at least one signal that makes at least four voters prefer candidate c_0 over c_1. We show that we cannot design such a public signal. In particular, we show that there is no posterior $\boldsymbol{\xi} \in \Xi$ that provides an expected utility larger than or equal to zero to at least four voters.*[2] *Since receiver r_7 prefers candidate c_0 in every state of nature, he votes for c_0 independently from the posterior induced by the signal. Therefore, it is sufficient to persuade three voters among the first six. Suppose that voters r_1 and r_2 vote for c_0. This implies that $\xi_{\theta_A}/2 - \xi_{\theta_B} - \xi_{\theta_C} = \xi_{\theta_A}/2 - (1 - \xi_{\theta_A}) \geq 0$ and $\xi_{\theta_A} \geq 2/3$. Suppose, by contradiction, that also voters r_3 and r_4 vote for c_0. This requires that $-\xi_{\theta_A} + \xi_{\theta_B}/2 - \xi_{\theta_C} \geq 0$ and $\xi_{\theta_B} \geq 2/3$, reaching a contradiction with $\boldsymbol{\xi} \in \Xi$. It is easy to see that, by the symmetry of the instance, all the other sets of four voters cannot vote for c_0 at the same time.*

From the previous example, we can state the following:

Proposition 4.1. *There is an instance of majority-voting election in which the optimal private signaling scheme guarantees that candidate c_0 wins the election with a probability of 1, while the optimal public signaling scheme cannot guarantee a winning probability strictly larger than 0.*

4.3 Private Signaling with k-Voting Rules

In this section, we show that a solution to K-V (*i.e.*, finding an optimal persuasive signaling scheme under a k-voting rule) can be found in polynomial time when the sender can employ a private signaling scheme.

First, we show that the sender can restrict the choice of a signaling scheme to the set of the schemes ϕ whose marginal signaling schemes are Pareto efficient on the set $\{\phi_{r,\theta}(c_0)\}_{\theta \in \Theta, r \in \mathcal{R}}$ (Lemma 4.1), and recommend with positive probability either c_0 or the candidate giving r the best utility under θ (Lemma 4.2).

Lemma 4.1. *Given a signaling scheme ϕ' and a set of persuasive marginal signaling schemes $\{\phi_r\}_{r \in \mathcal{R}}$, if $\phi_{r,\theta}(c_0) \geq \phi'_{r,\theta}(c_0)$ for each $r \in \mathcal{R}$ and $\theta \in \Theta$, there exists a persuasive signaling scheme ϕ such that $W(\phi) \geq W(\phi')$.*

Proof. Given a signaling scheme ϕ', let $\{\phi_r\}_{r \in \mathcal{R}}$ be a set of persuasive marginal signaling schemes s.t. $\phi_{r,\theta}(c_0) \geq \phi'_{r,\theta}(c_0)$ for each $r \in \mathcal{R}$, $\theta \in \Theta$.

[2] Recall that in a public signaling scheme, all the receivers observe the same signal, perform the same update of the belief, and have the same posterior belief.

Intuitively, we show that it is possible to move probability mass to c_0 while guaranteeing persuasiveness with the following iterative procedure.

Let $\phi^0 = \phi'$. Then, we iterate over $r \in [\bar{n}]$, and update the signaling scheme with the following procedure. Let A_r be an arbitrary mapping from $[|\mathcal{C}_{-r}|]$ to $\mathcal{C}_{-r}$, which serves as an arbitrary ordering of elements in $\mathcal{C}_{-r}$ (*i.e.*, $A_r(i)$ returns the i-th element of $\mathcal{C}_{-r}$ in such ordering). Moreover, for each $\theta \in \Theta$, we define $\Delta_r^0(\theta) = \phi_{r,\theta}(c_0) - \phi'_{r,\theta}(c_0)$. For each r, we iterate over $i \in [|\mathcal{C}_{-r}|]$, and perform the following updates: $\mathbf{c}_{-r} = A_r(i)$,

$$\phi_\theta^r(c_0, \mathbf{c}_{-r}) = \min \Bigg\{ \phi_\theta^{r-1}(c_0, \mathbf{c}_{-r})) + \Delta_r^{i-1}(\theta), \sum_{c \in C} \phi_\theta^{r-1}(c_0, \mathbf{c}_{-r}) \Bigg\}, \quad (4.1)$$

and

$$\Delta_r^i(\theta) = \Delta_r^{i-1}(\theta) - \phi_\theta^r(c_0, \mathbf{c}_{-r}) + \phi_\theta^{r-1}(c_0, \mathbf{c}_{-r}),$$

where $\phi_\theta^r(c, \mathbf{c}_{-r})$ is the probability of recommending c to r and $\mathbf{c}_{-r}$ to the other receivers, under θ (at iteration r). Finally, for each $\mathbf{c}_{-r}$, and $c \neq c_0$, set:

$$\phi_\theta^r(c, \mathbf{c}_{-r}) = \frac{\phi_{r,\theta}(c) \Big(\sum_{c' \in C} \phi_\theta^{r-1}(c', \mathbf{c}_{-r}) - \phi_\theta^r(c_0, \mathbf{c}_{-r}) \Big)}{\sum_{c' \in C \setminus \{c_0\}} \phi_{r,\theta}(c')},$$

the numerator is well-defined because of the minimization in Equation (4.1). After having enumerated all the receivers, we obtain $\phi^{\bar{n}}$. We show that $\phi = \phi^{\bar{n}}$ is precisely the desired signaling scheme. First, we show that, at each iteration r, ϕ^r is well formed. For each iteration r, and pair $(\theta, \mathbf{c}_{-r})$, we show that $\sum_{c \in C} \phi_\theta^r(c, \mathbf{c}_{-r}) = \sum_{c \in C} \phi_\theta^{r-1}(c, \mathbf{c}_{-r})$. We have: $\sum_{c \in C} \phi_\theta^r(c, \mathbf{c}_{-r}) = \phi_\theta^r(c_0, \mathbf{c}_{-r}) + \sum_{c \in C \setminus \{c_0\}} \phi_\theta^r(c, \mathbf{c}_{-r})$. Then, by expanding $\phi_\theta^r(c, \mathbf{c}_{-r}))$ via the update rule, we obtain:

$$\sum_{c \in C} \phi_\theta^r(c, \mathbf{c}_{-r}) = \phi_\theta^r(c_0, \mathbf{c}_{-r}) + \sum_{c \in C} \phi_\theta^{r-1}(c, \mathbf{c}_{-r}) - \phi_\theta^r(c_0, \mathbf{c}_{-r}),$$

which is precisely $\sum_{c \in C} \phi_\theta^{r-1}(c, \mathbf{c}_{-r}))$. This implies that $\sum_{\mathbf{c} \in \mathcal{C}} \phi_\theta^r(\mathbf{c}) = 1$, and that receiver r's marginal probabilities are modified only at iteration r. Now, we show that receiver r's marginals are updated correctly. We distinguish the following two cases.

i) It is easy to see that, for candidate c_0,

$$\sum_{\mathbf{c}_{-r}\in\mathcal{C}_{-r}} \phi^r_\theta(c_0,\mathbf{c}_{-r}) =$$

$$= \Delta^0_r(\theta) + \sum_{\mathbf{c}_{-r}\in\mathcal{C}_{-r}} \phi^{r-1}_\theta(c_0,\mathbf{c}_{-r}) = \phi_{r,\theta}(c_0).$$

ii) For each candidate $c \neq c_0$, we have:

$$\sum_{\mathbf{c}_{-r}\in\mathcal{C}_{-r}} \phi^r_\theta(c,\mathbf{c}_{-r}) =$$

$$= \sum_{\mathbf{c}_{-r}\in\mathcal{C}_{-r}} \frac{\phi_{r,\theta}(c)\left(\sum_{c'\in C}\phi^{r-1}_\theta(c,\mathbf{c}_{-r}) - \phi^r_{\theta},(c_0,\mathbf{c}_{-r})\right)}{\sum_{c'\in C\setminus\{c_0\}}\phi_{r,\theta}(c')} =$$

$$= \frac{\phi_{r,\theta}(c)\left(\sum_{\mathbf{c}\in S}\phi^{r-1}_\theta(\mathbf{c}) - \sum_{\mathbf{c}_{-r}\in S_{-r}}\phi^r_\theta(c_0,\mathbf{c}_{-r})\right)}{\sum_{c'\in C\setminus\{c_0\}}\phi_{r,\theta}(c')} =$$

$$= \frac{\phi_{r,\theta}(c)(1-\phi_{r,\theta}(c_0))}{\sum_{c'\in C\setminus\{c_0\}}\phi_{r,\theta}(c')} = \phi_{r,\theta}(c).$$

Since $\{\phi_r\}_{r\in\mathcal{R}}$ are persuasive, also the new signaling scheme ϕ is persuasive. Finally, we show that the new signaling scheme does not decrease sender's expected utility. Let $\mathcal{C}^* = \{\mathbf{c}\in\mathcal{C}|\delta(\mathbf{c},c_0)\geq k\}$ be the set of joint signals recommending c_0 to more than k voters (under a k-voting rule). Then, $W(\phi) = \sum_{\theta\in\Theta}\mu_\theta\sum_{\mathbf{c}\in\mathcal{C}^*}\phi_\theta(\mathbf{c})$. It is enough to show that, for each iteration r, for each $\theta\in\Theta$, and, for each $\mathbf{c}_{-r}\in\mathcal{C}_{-r}$, it holds

$$\sum_{c\in C}\left(\phi^r_\theta(c,\mathbf{c}_{-r}) - \phi^{r-1}_\theta(c,\mathbf{c}_{-r})\right)\mathbb{I}[(c,\mathbf{c}_{-r})\in\mathcal{C}^*]\geq 0,$$

where $\mathbb{I}$ is the indicator function. We distinguish three cases:

- when $\delta(\mathbf{c}_{-r},c_0) < k-1$, a change in r's marginal probabilities does not affect the sender's winning probability, term $\mathbb{I}[(c,\mathbf{c}_{-r})\in\mathcal{C}^*]$ being always 0;
- when $\delta(\mathbf{c}_{-r},c_0) = k-1$, $\mathbb{I}[(c,\mathbf{c}_{-r})\in\mathcal{C}^*] = 1$ only if $c = c_0$, and $\phi^r_\theta(c_0,\mathbf{c}_{-r}) \geq \phi^{r-1}_\theta(c_0,\mathbf{c}_{-r})$;

- when $\delta(\mathbf{c}_{-r}, c_0) > k-1$, $\mathbb{I}[(c, \mathbf{c}_{-r}) \in \mathcal{C}^*]$ is always 1, and we already know that $\sum_{c \in C} \left(\phi^r_\theta(c, \mathbf{c}_{-r}) - \phi^{r-1}_\theta(c, \mathbf{c}_{-r})\right) = 0$.

This concludes the proof. □

We now state the next lemma.

Lemma 4.2. *There always exists a solution to* K-V *in which, for all* $r \in \mathcal{R}$ *and* $\theta \in \Theta$, $\phi_{r,\theta}(c) > 0$ *if and only if one of the following two conditions is satisfied:*

- $c = c_0$,
- $c \in arg\,max_{c' \in C} u^r_\theta(c')$.

Proof. Given a persuasive signaling scheme ϕ', we show that it is possible to build a collection $\{\phi_r\}_{r \in \mathbb{R}}$ with the property above, such that $\phi_{r,\theta}(c_0) \geq \phi'_{r,\theta}(c_0)$ for each $r \in \mathcal{R}$, $\theta \in \Theta$. This, together with Lemma 4.1, proves our result. We build ϕ iteratively. For each pair (θ, r), select a candidate $c^* \in arg\,max_{c \in C} u^r_\theta(c)$, and set $\phi_{r,\theta}(c^*) = 1 - \phi'_{r,\theta}(c_0)$, $\phi_{r,\theta}(c_0) = \phi'_{r,\theta}(c_0)$, and $\phi_{r,\theta}(c) = 0$ for each other $c \in C \setminus \{c_0, c^*\}$. It is immediate to see that, for each θ and r, $\sum_{c \in C} \phi_{r,\theta}(c) = 1$. Next, we show that each ϕ_r is persuasive, i.e., $\sum_{\theta \in \Theta} \mu_\theta \phi_{r,\theta}(c) \left(u^r_\theta(c) - u^r_\theta(c')\right) \geq 0$ for each $r \in \mathcal{R}$, and $c, c' \in C$. If $c = c_0$, we have $\phi_{r,\theta}(c_0) > \phi'_{r,\theta}(c_0)$ only if $c_0 \in arg\,max_{c \in C} u^r_\theta(c)$, which means $(u^r_\theta(c_0) - u^r_\theta(c')) \geq 0$, in the remaining cases we have $\phi_{r,\theta}(c_0) = \phi'_{r,\theta}(c_0)$. If $c \neq c_0$, $c \in arg\,max_{c' \in C} u^r_\theta(c')$ for each $\theta \in \Theta$ with $\phi_{r,\theta}(c) > 0$, which makes the incentive constraint satisfied. □

By exploiting Lemma 4.2, we show that an optimal persuasive signaling scheme under a k-voting rule can be computed in polynomial time via the following linear program (LP). Let $\beta_\theta \in \mathbb{R}$ be the probability with which k voters vote for c_0 with state θ. Then, we can compute an optimal solution to K-V as follows (the proof is provided below):

$$\max_{\substack{\boldsymbol{\beta} \in [0,1]^d, \mathbf{z} \in \mathbb{R}_-^{d \times k \times \bar{n}} \\ \mathbf{t} \in \mathbb{R}^{d \times k}, \boldsymbol{q} \in \mathbb{R}^{d \times k} \\ \phi_{\cdot,\cdot}(c_0) \in [0,1]^{\bar{n} \times d}}} \sum_{\theta \in \Theta} \mu_\theta \beta_\theta \tag{4.2a}$$

$$\text{s.t.} \sum_{\theta \in \Theta} \mu_\theta \phi_{r,\theta}(c_0) \left(u^r_\theta(c_0) - u^r_\theta(c)\right) \geq 0 \tag{4.2b}$$

$$\forall r \in \mathcal{R}, \forall c \in C \setminus \{c_0\}$$

$$\beta_\theta \leq \frac{1}{k-m} q_{\theta,m} \, \forall \theta \in \Theta, \forall m \in \{0, \dots, k-1\} \tag{4.2c}$$

$$q_{\theta,m} \leq (\bar{n} - m) t_{\theta,m} + \sum_{r \in \mathcal{R}} z_{\theta,r,m} \tag{4.2d}$$

$$\forall \theta \in \Theta, \forall m \in \{0, \ldots, k-1\}$$

$$\phi_{r,\theta}(c_0) \geq t_{\theta,m} + z_{\theta,m,r} \tag{4.2e}$$
$$\forall r \in \mathcal{R}, \forall \theta \in \Theta, \forall m \in \{0, \ldots, k-1\}.$$

This formulation allows us to state the following:

Theorem 4.1. *It is possible to compute an optimal persuasive private signaling scheme for* K-V *in* poly(d, ϱ, $\bar{n}$) *time.*

Proof. Formulation 6.8 has a polynomial number of variables and constraints. Then, proving Theorem 4.1 amounts to show that a solution to Formulation 6.8 is also a solution to K-V.

Let $c^*_{\theta,r} = arg\,max_{c \in C} u^r_\theta(c)$, for each θ and r. First, by Lemma 4.2, the space of available signals can be restricted to those in which, for each r and θ, only $\phi_{r,\theta}(c_0)$ and $\phi_{r,\theta}(c^*_{\theta,r})$ are > 0, and $\phi_{r,\theta}(c^*_{\theta,r}) = 1 - \phi_{r,\theta}(c_0)$. Constraints (4.2b) are the incentive constraints for action c_0. For any $c \neq c_0$, the incentive constraints are satisfied by construction. Objective (4.2a) is given by the sum over all $\theta \in \Theta$ of the prior of state θ, multiplied by the probability of having at least k vote for c_0 given θ. We need to show the correctness of β_θ. For each state of nature the maximum probability with which at least k receivers play c_0 is given by:

$$\beta_\theta = \min\left\{\min_{m \in \{0,\ldots,k-1\}} \frac{1}{k-m} q_{\theta,m}, 1\right\},$$

where $q_{\theta,m}$ is the sum of the lowest $\bar{n} - m$ elements in the set $\{\phi_{r,\theta}(c_0)\}_{r \in \mathcal{R}}$; for further details, see Lemma 3 of (Arieli and Babichenko, 2019). This is enforced via Constraints (4.2c). Constraints (4.2d) and (4.2e) ensure $q_{\theta,m}$'s consistency, and are derived from the dual of a simple LP of this kind: $\min_{\mathbf{y} \in \mathbb{R}^n} \mathbf{x}^\top \mathbf{y}$ s.t. $\mathbb{1}^\top \mathbf{y} = \mathbf{w}$ and $0 \leq \mathbf{y} \leq 1$ (where $\mathbf{x} \in \mathbb{R}^n$ is the vector from which we want to extract the sum of the smallest w entries). This concludes the proof. □

4.4 A Condition for Efficient Private Signaling

In the following, we provide a necessary and sufficient condition for the poly-time computation of persuasive private signaling schemes under a general class of sender's objective functions. In the next section, this result will be exploited when dealing with anonymous utility functions and plurality voting. We allow for general sender's utility functions of type $f_\theta : \times_{r \in \mathcal{R}} C \to [0, 1]$, which generalizes previous results by Dughmi and

Xu (2017) where the receivers' action space has to be binary. Given a collection of set functions $\mathcal{F}$, $P(\mathcal{F})$ denotes the class of persuasion instances in which, for each $\theta \in \Theta$, $f_\theta \in \mathcal{F}$. We can state the following.

Theorem 4.2. *Let $\mathcal{F}$ be any collection of set functions including $f_0(\cdot) = 0$. Given any instance in $P(\mathcal{F})$, there exists a polynomial-time algorithm for computing an optimal persuasive private signaling scheme if and only if there is a polynomial-time algorithm that computes*

$$\max_{\mathbf{c} \in \times_{r \in R} C} f(\mathbf{c}) + \sum_{r \in R} w_r(c_r), \tag{4.3}$$

for any $f \in \mathcal{F}$, and any weights $w_r(c_r) \in \mathbb{R}$, where c_r is the action chosen by r in $\mathbf{c}$.

Proof. Given a set $\{f_\theta\}_{\theta \in \Theta}$, the persuasion problem can be formulated with the following LP:

$$\max_{\mathbf{x} \in [0,1]^{|\Theta \times \mathcal{C}|}} \sum_{\theta \in \Theta, \mathbf{c} \in \mathcal{C}} x_{\theta,\mathbf{c}} f_\theta(\mathbf{c}) \tag{4.4a}$$

$$\sum_{\substack{\theta \in \Theta, \\ \mathbf{c}: c_r = c}} x_{\theta,\mathbf{c}} \left(u_\theta^r(c) - u_\theta^r(c')\right) \geq 0 \quad \forall r \in \mathcal{R}, \forall c, c' \in C \tag{4.4b}$$

$$\sum_{\mathbf{c} \in \mathcal{C}} x_{\theta,\mathbf{c}} = \mu_\theta \quad \forall \theta \in \Theta \tag{4.4c}$$

Note that Constraints (4.4b) force the signaling scheme to be persuasive, while Constraints (4.4c) force the signaling scheme to be feasible.

($\Longrightarrow$). Let $\mathbf{y} \in \mathbb{R}_-^{|\mathcal{R} \times C \times C|}$ be the dual variables relative to primal Constraints (4.4b) and $\boldsymbol{q} \in \mathbb{R}^{|\Theta|}$ be the dual variables of Constraints (4.4c). The dual of LP 4.4 has a polynomial number of variables and an exponential number of constraints, one for each pair $(\theta, \mathbf{c}) \in \Theta \times \mathcal{C}$, of type:

$$O(\theta, \mathbf{c}) \;=\; \left(-\sum_{\substack{r \in \mathcal{R}, \\ c \in C}} y_{r,c_r,c} \left(u_\theta^r(c_r) - u_\theta^r(c)\right) \right) - q_\theta + f_\theta(\mathbf{c}) \;\leq\; 0.$$

We show that, given a vector of dual variables $\bar{\mathbf{z}} = (\bar{\mathbf{y}}, \bar{\boldsymbol{q}})$, the problem of either finding a hyperplane separating $\bar{\mathbf{z}}$ from the set of feasible solutions to the dual or proving that no such hyperplane exists can be solved in polynomial time. The *separation problem* of finding an inequality of the dual which is maximally violated at $\bar{z}$ reads: $\max_{(\theta,\mathbf{c}) \in \Theta \times S} O(\theta, \mathbf{c})$. A pair $(\theta, \mathbf{c})$ yielding a violated inequality exists if and only if the separation problem admits an optimal solution of value > 0. One such pair (if any) can by found in polynomial time by enumerating over states in Θ. For each θ, the problem reduces to $\max_{\mathbf{c}} \sum_{r\in\mathcal{R}} v_r(\theta, c_r) + f_\theta(\mathbf{c})$, where $v_r(\theta, c_r) = -\sum_{c\in C} \bar{y}_{r.c_r,c}\,(u^r_\theta(c_r) - u^r_\theta(c))$. It is enough to take $w_r(c) = v_r(\theta, c)$ to complete the *if* part of the proof.

($\Longleftarrow$). Given a polynomial-time algorithm to determine an optimal signaling scheme for any instance of $P(\mathcal{F})$, we want to show that $\max_{\mathbf{c}\in\mathcal{C}} f(\mathbf{c}) - \sum_{r\in\mathcal{R}} w_r(c_r)$ can be solved efficiently for any $\{w_r(c)\}_{r,c}$, and $f \in \mathcal{F}$.

To reduce this problem to a signaling problem we employ a duality-based analysis introduced in (Dughmi and Xu, 2017), and later improved by Xu (2020). Our generalization to non-binary action spaces requires a more involved proof, as we will highlight in the following. Moreover, our proof completely diverges from Dughmi and Xu (2017)'s and Xu (2020)'s in the final construction of the mapping to a private signaling problem.

Given a set of weights $\{\bar{w}_r(c)\}_{r,c}$, and $f \in \mathcal{F}$, we are interested in the maximization of $\bar{f}(\mathbf{c}) = f(\mathbf{c}) + \sum_{r\in\mathcal{R}} \bar{w}_r(c_r)$ over S. First, we slightly modify weights by setting, for each $r \in \mathcal{R}$, $\bar{w}_r(c) \leftarrow \bar{w}_r(c) - \max_{c'} \bar{w}_r(c')$, for each $c \in C$. This modification preserves the set of optimal solutions of the maximization problem. After that, for each receiver r, it holds $\bar{w}_r \leq 0$, and there exists $\hat{c}^r \in C$ s.t. $\bar{w}_r(\hat{c}^r) = 0$. Let, for each $r \in \mathcal{R}$, $C_r = C \setminus \{\hat{c}^r\}$ ($\hat{c}^r$ can be selected arbitrarily from the actions s.t. $\bar{w}_r(c) = 0$). We show that $\max_{\mathbf{c}\in\mathcal{C}} \bar{f}(\mathbf{c})$ can be reduced to solving the following LP, for all possible linear coefficients α, $\{\ell_r(c)\}_{r\in\mathcal{R},c\in C_r}$:

$$\min_{\substack{\mathbf{z}\in\mathbb{R}^{|\mathcal{R}\times C_r|}\\ v\in\mathbb{R}}} \sum_{r\in\mathcal{R},c\in C_r} \ell_r(c) z_{r,c} + \alpha v \tag{4.5a}$$

$$\text{s.t.} \sum_{\substack{r\in\mathcal{R}\\ c_r\neq\hat{c}^r}} z_{r,c_r} + v \geq f(\mathbf{c}) \qquad \forall \mathbf{c} \in \mathcal{C} \tag{4.5b}$$

$$z_{r,c} \geq 0 \qquad \forall r \in \mathcal{R}, c \in C_r. \tag{4.5c}$$

To show this, we first argue that the maximization problem can be reduced to the separation problem for the feasible region of LP 4.5. Take $\bar{z}_{r,c} = -\bar{w}_r(c)$ for all r and $c \in C_r$. Constraints of family (4.5c) are satisfied by construction. Then, a pair $(\{\bar{\mathbf{z}}, v)$ is feasible if and only if $v \geq \max_{\mathbf{c}} f(\mathbf{c}) + \sum_{r,c_r\neq\hat{c}^r} \bar{w}_r(c_r)$. As a result, the optimal value v^* (which is the exact optimal objective of $\bar{f}(\mathbf{c})$) can be determined via binary search in $O(B)$ steps, where B is the bit complexity of the $f(\mathbf{c})$'s and $\mathbf{w}$'s. Then, by setting $\bar{v} = v^* - 2^{-B}$, we obtain an infeasible pair $(\bar{\mathbf{z}}, \bar{v})$. If the separation oracle is given in input $(\bar{\mathbf{z}}, \bar{v})$, it returns a separating hyperplane corresponding to the optimal solution of the maximization problem. The equivalence between optimization and separation implies that the maximization problem reduces to solving LP 4.5 for any linear coefficients $\{\ell_r(c)\}_{r\in\mathcal{R},c\in C_r}$ and α (Khachiyan, 1980; Grötschel et al., 1981).

A crucial difference between LP 4.5 and Xu (2020)'s analogous LP is that we modify the initial weights $\bar{\mathbf{w}}$ to make them ≤ 0 (simplifying the LP's structure), and, for each r, there is at least one $\bar{w}_r(c)$ equal to 0. This reduces the number of variables in LP 4.5, as variables $z_r(\hat{c}^r)$ are not included. This is fundamental for the last step of the proof.

The next step is showing that LP 4.5 can be solved *directly* for some parameters' values. Specifically:

- If $\alpha < 0$ the solution is unbounded (*i.e.*, the objective function tends to $-\infty$ as $v \to \infty$).
- If $\alpha = 0$ and there exists $(\bar{r}, \bar{c})$ s.t. $\ell_{\bar{r}}(\bar{c}) < 0$, then a feasible solution is obtained by setting: $z_{\bar{r},\bar{c}} = v$, and $z_{r,c} = 0$ for all $(r, c) \neq (\bar{r}, \bar{c})$. Again, for $v \to \infty$ the objective tends to $-\infty$.
- If $\alpha = 0$ and $\ell_r(c) \geq 0$ for all (r, c), then the objective is ≥ 0 for any feasible solution. By selecting a sufficiently large v we obtain a feasible and optimal solution with objective value 0.

Therefore, when $\alpha \leq 0$ the problem can be solved in polynomial time.

We focus on the case in which $\alpha > 0$. Since $\alpha > 0$, we can re-scale all coefficients of LP 4.5 by a factor $1/\alpha$ without affecting its optimal solutions, and obtain an equivalent LP with $\alpha = 1$. The dual of LP 4.5 with $\alpha = 1$ is:

$$\max_{\mathbf{p}\in\mathbb{R}_+^{|\mathcal{C}|}} \sum_{\mathbf{c}\in\mathcal{C}} p_{\mathbf{c}} f(\mathbf{c}) \tag{4.6a}$$

$$\text{s.t.} \sum_{\mathbf{c}:c_r=c} p(\mathbf{c}) \leq \ell_r(c) \quad \forall r \in \mathcal{R}, c \in C_r \tag{4.6b}$$

$$\sum_{\mathbf{c}\in\mathcal{C}} p_{\mathbf{c}} = 1 \tag{4.6c}$$

Finally, we show that finding an optimal solution to LP 4.6 reduces to finding an optimal signaling scheme in an instance of private persuasion with $|\Theta| = |C|$ states of nature, and $\mu_\theta = \frac{1}{|C|}$ for each θ. First, for each r we define an arbitrary one-to-one correspondence between elements of C_r, and elements of $\Theta \setminus \{\theta_0\}$. Let c_θ (θ_c) be the action (state) associated with θ (c).

Receiver r's utility function reads:

$$u_\theta^r(c) = \begin{cases} 1 & \text{if } \theta = \theta_0 \text{ and } c = \hat{c}^r \\ 0 & \text{if } \theta = \theta_0 \text{ and } c \neq \hat{c}^r \\ \ell_r(c) & \text{if } \theta \neq \theta_0 \text{ and } c = c_\theta \\ 0 & \text{if } \theta \neq \theta_0 \text{ and } c \neq c_\theta \end{cases}.$$

Let sender's utility be such that $f_\theta = f_0$, for each $\theta \neq \theta_0$, and $f_{\theta_0} = f$. We have that $f_\theta(\mathbf{c}) = 0$ for each $\theta \in \Theta \setminus \{\theta_0\}$ and $\mathbf{c} \in \mathcal{C}$. Then, there exists an optimal signaling scheme such that, in each state $\theta \neq \theta_0$, $\phi_\theta(\mathbf{c}_\theta) = 1$, where $\mathbf{c}_\theta$ is a signal recommending c_θ to each receiver (from an argument analogous to Lemma 4.2). Now, an optimal signaling scheme can be computed by focusing on θ_0 (*i.e.*, we employ the aforementioned signaling scheme for any $\theta \neq \theta_0$) via the following LP:

$$\max_{\phi_{\theta_0}(\cdot) \in [0,1]^{|\mathcal{C}|}} \mu_\theta \sum_{\mathbf{c} \in \mathcal{C}} \phi_{\theta_0}(\mathbf{c}) f_{\theta_0}(\mathbf{c}) \tag{4.7a}$$

$$\text{s.t.} \sum_{\theta \in \Theta} \sum_{\mathbf{c}: c_r = c} \mu(\theta) \phi_\theta(\mathbf{c}) (u_\theta^r(c) - u_\theta^r(c')) \geq 0$$

$$\forall r \in \mathcal{R}, \forall c, c' \in C \tag{4.7b}$$

$$\sum_{\mathbf{c} \in \mathcal{C}} \phi_{\theta_0}(\mathbf{c}) = 1. \tag{4.7c}$$

The incentive Constraints (4.7b) are trivially satisfied when $c = \hat{c}^r$. Moreover, for each $c \neq \hat{c}^r$, the incentive Constraints (4.7b) can be rewritten as follows: first, notice that it is enough to consider $c' = \hat{c}^r$. Then, for each $r \in \mathcal{R}$ and $c \in C_r$, we obtain:

$$\sum_{\mathbf{c}: c_r = c} \phi_{\theta_0}(\mathbf{c}) (u_{\theta_0}^r(c) - u_{\theta_0}^r(\hat{c}^r)) \geq u_{\theta_c}^r(\hat{c}^r) - u_{\theta_c}^r(c),$$

which can be rewritten as $\sum_{\mathbf{c}: c_r = c} \phi_{\theta_0}(\mathbf{c}) \leq \ell_r(c)$. The equivalence between LP 4.6 and LP 4.7 easily follows. □

The crucial difference with the result by Dughmi and Xu (2017) is that they consider set functions depending only on the set of players choosing the target action, between the two available. Theorem 4.2 generalizes this setting as it allows for functions taking as input any action profile $\mathbf{c}$. This is crucial in settings like plurality voting, where the sender is not only interested in votes favorable to c_0, but also in the distribution of the other preferences. Dughmi and Xu (2017)'s result cannot be applied to such settings.

4.5 Further Positive Results for Private Signaling

Despite Theorem 4.2, in the case of general utility functions the problem of determining an optimal persuasive private signaling scheme is still largely intractable. An intuition behind that is that there may be an exponential (in $|C|$) number of values of f (*e.g.*, in the case of anonymous utility functions, there are $\binom{|\mathcal{R}|+|C|-1}{|\mathcal{R}|}$ values of f). In order to identify tractable classes of the problem, we need to make some further assumptions on $\mathcal{F}$.

Anonymous Utility Functions. A reasonable (in the context of voting) restriction is to *anonymous utility functions* (see (Arieli and Babichenko, 2019)). Previous results on the computational complexity of private signaling with anonymous utility functions focus on the case of binary actions, which is shown to be tractable (Babichenko and Barman, 2017; Arieli and Babichenko, 2019; Dughmi and Xu, 2017). We generalize these results to a generic number of states of nature and receiver's actions with the following result.

Theorem 4.3. *Private Bayesian persuasion with anonymous sender's utility functions is fixed-parameter tractable in the number of receivers' actions.*

Proof. It is enough to provide an algorithm for the maximization problem in Theorem 4.2. We need to solve $\max_{\mathbf{c}\in\mathcal{C}} f(\mathbf{c}) + \sum_{r\in\mathcal{R}} w_r(c_r)$. Since f is anonymous, for any persuasive signal $\mathbf{c}$, f's value is determined by the vector $\mathbf{p} = (\delta(\mathbf{c}, c_0), \dots, \delta(\mathbf{c}, c_{|C|}))$. Let $P = \{\mathbf{p} = (k_0, \dots, k_{|C|}) \in \mathbb{N}_0^{|C|} \mid \sum_{i=0}^{|C|} k_i = |\mathcal{R}|\}$, and notice that $|P| = \binom{|\mathcal{R}|+|C|-1}{|\mathcal{R}|}$, which is polynomial in the input size once the $|C|$ has been fixed (see (Stanley, 2011)). In order to solve the maximization problem, we enumerate over all $\mathbf{p} \in P$. Once $\mathbf{p}$ has been fixed, we are left with the following maximization problem: $\max_{\mathbf{c}\in\mathcal{C}} \sum_{r\in\mathcal{R}} w_r(c_r)$, where $\mathbf{c}$ has to be such that $\delta(\mathbf{c}, c_i) = k_i$ for each $i \in \{0, \dots, |C|\}$. Specifically, the optimal assignment of receivers to actions can be found with the following LP:

$$\max_{\boldsymbol{\chi}\in\mathbb{R}_+^{|\mathcal{R}\times C|}} \sum_{(r,c)\in\mathcal{R}\times C} \chi_{r,c} w_r(c)$$

$$\text{s.t.} \sum_{r\in\mathcal{R}} \chi_{r,c_i} = k_i \qquad \forall i \in \{0, \dots, |C|\}$$

$$\sum_{c\in C} \chi_{r,c} = 1 \qquad \forall r \in \mathcal{R}.$$

We look for an integer solution of the problem, which always exists and can be found in polynomial time (see, e.g., (Orlin, 1997)). This is because the formulation is an instance of the *maximum cost flow problem*, which is, in its turn, a variation of the *minimum cost flow problem*. Once an integer solution has been found, an optimal solution of the original maximization problem is the signal obtained by assigning to each r the action c s.t. $\chi_{r,c} = 1$. □

Theorem 4.3 implies that, for any anonymous voting rule, the private Bayesian persuasion problem is fixed-parameter tractable in the number of candidates.

Plurality Voting. By further restricting our attention to specific voting rules, we can see the consequences of Theorem 4.2 to an even better extent. A simple and widespread voting rule is *plurality voting*. [3] In this setting $W(\mathbf{c}) = 1$ if and only if $\delta(\mathbf{c}, c_0) > \delta(\mathbf{c}, c)$ for any $c \neq c_0$, and $W(\mathbf{c}) = 0$ otherwise. We can state the following:

Theorem 4.4. *PL-V with private signaling can be solved in* $\text{poly}(d, \varrho, \bar{n})$ *time.*

Proof. We exploit Theorem 4.2, and show that the maximization Problem 4.3 can be solved efficiently. With an overload of notation, generic actions profiles are represented via signals. Then, the maximization problem reads: $\max_{\mathbf{c} \in \mathcal{C}} W(\mathbf{c}) + \sum_{r \in \mathcal{R}} w_r(c_r)$. We split the maximization problem in two steps. First, we consider the maximization over non-winning action profiles, *i.e.*, signals in the set $\bar{\mathcal{C}} = \{\mathbf{c} \in \mathcal{C} | \exists c \neq c_0 \text{ s.t. } \delta(\mathbf{c}, c) > \delta(\mathbf{c}, c_0)\}$. An upper bound to the optimal value of the maximization problem restricted to $\bar{\mathcal{C}}$ is given by $\max_{\mathbf{c} \in \mathcal{C}} \sum_r w_r(c_r)$. The latter problem can be solved independently for each receiver r, by choosing c maximizing $w_r(c)$. Once the relaxed problem has been solved, the objective function of the separation problem is adjusted by checking whether $\mathbf{c}$ is winning or not. The resulting value is then compared with the value from the following step.

We consider the maximization over winning action profiles, *i.e.*, signals in $\mathcal{C}^* = \mathcal{C} \setminus \bar{\mathcal{C}}$. For any $\mathbf{c} \in \mathcal{C}^*$, $W(\mathbf{c}) = 1$. Then, we have to maximize the same objective of the previous case with the following additional constraints: $\delta(\mathbf{c}, c_0) > \delta(\mathbf{c}, c)$, for all $c \neq c_0$. To determine an optimal solution to this problem, we enumerate over $k \in \{\lceil \frac{|\mathcal{R}|-1}{|C|} \rceil + 1, \dots, |\mathcal{R}|\}$, *i.e.*, the number of votes that make c_0 a potential winner of the election. Then, for each value of k, we consider action profiles such that $\delta(\mathbf{c}, c_0) = k$, and

[3] See, *e.g.*, its (*discussed*) adoption in direct presidential elections in a number of states (Blais et al., 1997).

$\delta(\mathbf{c}, c) < k$, for all $c \neq c_0$ (*i.e.*, winning signals where c_0 receives exactly k votes). An optimal solution for a fixed k can be determined with this LP:

$$\max_{\boldsymbol{\chi} \in \mathbb{R}_+^{|\mathcal{R} \times C|}} \sum_{(r,c) \in \mathcal{R} \times C} \chi_{r,c} w_r(c)$$
$$\text{s.t.} \sum_{r \in \mathcal{R}} \chi_{r,c_0} = k$$
$$\sum_{r \in \mathcal{R}} \chi_{r,c} \leq k-1 \quad \forall c \in C \setminus \{c_0\}$$
$$\sum_{c \in C} \chi_{r,c} = 1 \qquad \forall r \in \mathcal{R}.$$

We look for an integer solution of the problem, which always exists and can be found in polynomial time (see, *e.g.*, (Orlin, 1997)). This is because the formulation is an instance of the *maximum cost flow problem*, which is, in its turn, a variation of the *minimum cost flow problem*. Once an integer solution has been found, an optimal action profile of the original maximization problem is the one obtained by recommending to each r the candidate c s.t. $\chi_{r,c} = 1$. □

4.6 Public Signaling

In contrast with the results for private signaling problems, we show that public persuasion in the context of voting is largely intractable.

We reduce from MAXIMUM k-SUBSET INTERSECTION (MSI) (Clifford and Popa, 2011).

Definition 4.1 (MSI). *An instance of MAXIMUM k-SUBSET INTERSECTION is a tuple $(\mathcal{E}, A_1, \ldots, A_m, k, q)$, where $\mathcal{E} = \{e_1, \ldots, e_n\}$ is a finite set of elements, each A_i, $i \in [m]$, is a subset of $\mathcal{E}$, and k, q are positive integers. It is a "*yes*"-instance if there exist exactly k sets $A_{i_1}, \ldots, A_{i_k}$ such that $|\bigcap_{j\in[k]} A_{i_j}| \geq q$, and a "*no*"-instance otherwise.*

MSI has been recently shown to be NP-hard (Xavier, 2012; Elkind et al., 2015). Now, we prove the following negative result:

Theorem 4.5. *K-V with public signaling, even with two candidates, cannot be approximated in polynomial time to within* any *factor, unless P=NP.*

Proof. Given an instance of MSI, we build a public signaling problem with the following features.

Mapping. It has a voter r_i for each A_i, $i \in [m]$, and m voters $r_{e,j}$, $j \in [m]$, for each $e \in \mathcal{E}$. There is a state of nature θ_e for each $e \in \mathcal{E}$, and $\mu_{\theta_e} = 1/n$ for each θ_e. Finally, $C = \{c_0, c_1\}$. Receivers have the following utility functions: for each r_i, $i \in [m]$,

$$u^{r_i}_{\theta_e}(c) = \begin{cases} 1 & \text{if} \quad e \in A_i,\, c = c_0 \\ -n^2 & \text{if} \quad e \notin A_i,\, c = c_0 \\ 0 & \text{if} \quad c = c_1 \end{cases},$$

for each $r_{e,j}$, $e \in \mathcal{E}$, and $j \in [m]$,

$$u^{r_{e,j}}_{\theta_{e'}}(c) = \begin{cases} 1 & \text{if} \quad e = e',\, c = c_0 \\ -\frac{1}{q-1} & \text{if} \quad e \neq e',\, c = c_0 \\ 0 & \text{if} \quad c = c_1 \end{cases}.$$

The sender needs at least $k + mq$ votes (for c_0) in order to win the election (i.e., we are considering a $(k + mq)$-voting rule). We prove our theorem by showing that c_0 has a strictly positive probability of winning the election if and only if the corresponding MSI instance is satisfiable.

If. Suppose there exists a set $A^* = \{A_{i_1}, \ldots, A_{i_k}\}$ satisfying the MSI instance, and let $I = \cap_{j \in [k]} A_{i_j}$. Define a signaling scheme ϕ with two signals (s_0 and s_1) such that, for each $e \in I$, $\phi_{\theta_e}(s_0) = 1$, and, for each $e \notin I$, $\phi_{\theta_e}(s_1) = 1$, and it is equal to 0 otherwise. We show that such a signaling scheme guarantees a strictly positive winning probability for the sender. First, we show that, when the realized state of nature θ_e is such that $e \in I$ (*i.e.*, the sender recommends s_0), at least $k + mq$ receivers vote for c_0. Each receiver r_i such that $A_i \in A^*$ will choose c_0 when recommended s_0. Specifically, $\sum_{\theta_e} \frac{1}{n} \phi_{\theta_e}(s_0) u^{r_i}_{\theta_e}(c_0) = \frac{q}{n}$, while $\sum_{\theta_e} \frac{1}{n} \phi_{\theta_e}(s_0) u^{r_i}_{\theta_e}(c_1) = 0$. Receivers $r_{e,j}$ with $e \in I$ will vote for c_0 after observing s_0. This is because, for each $e \in I$ and $j \in [m]$, $r_{e,j}$ has expected utility $\frac{1}{n}\phi_{\theta_e}(s_0) - \sum_{\theta'_e : e' \neq e} \frac{1}{n} \frac{1}{q-1} \phi_{\theta_{e'}}(c_0) = 0$ for voting c_0, and expected utility 0 for voting c_1. Then, when the realized state of nature is θ_e with $e \in I$, there are at least $k + mq$ receivers voting for c_0. Therefore, the sender's winning probability is at least $\frac{k}{n}$ (*i.e.*, the probability of observing θ_e with $e \in I$ under a uniform prior).

Only if. Suppose, by contradiction, that MSI is not satisfiable, and that the sender's winning probability under the optimal signaling scheme is not null. This implies the existence of a signal s_0 such that, when recommended, a set of receivers $\mathcal{R}^*$ votes for c_0, and $|\mathcal{R}^*| \geq k + mq$. Then, there exist at least q states θ_e in which all voters $r_{e,j}$, $j \in [m]$, vote for

c_0. Each receiver $r_{e,j}$, having observed s_0, votes for c_0 only if $\phi_{\theta_e}(s_0) - \frac{1}{q-1}\sum_{\theta_{e'}:e'\neq e}\phi_{\theta_{e'}}(s_0) \geq 0$. This implies that $\phi_{\theta_e}(s_0) - \sum_{\theta_{e'}}\phi_{\theta_{e'}}(s_0) + \phi_{\theta_e}(s_0) \geq 0$ and $\phi_{\theta_e}(s_0) \geq \sum_{\theta_{e'}\in\Theta}\phi_{\theta_{e'}}(s_0)/q$. Then, there are exactly q states θ_e in which s_0 is played with probability $\sum_{\theta_{e'}\in\Theta}\phi_{\theta_{e'}}(s_0)/q$, while s_0 is never played in the remaining states. As a consequence, $\mathcal{R}^*$ includes exactly mq voters $r_{e,j}$, and at least k voters r_i.

Each voter $r_i \in \mathcal{R}^*$, after observing s_0, choose candidate c_0. Therefore, $\sum_{\theta_e\in\Theta}\mu_{\theta_e}\phi_{\theta_e}(s_0)(u^{r_i}_{\theta_e}(c_0) - u^{r_i}_{\theta_e}(c_1)) \geq 0$. We obtain $\sum_{e\in A_i}\phi_{\theta_e}(s_0) - n^2\sum_{e\notin A_i}\phi_{\theta_e}(s_0) \geq 0$. Then,

$$\sum_{e\in A_i}\phi_{\theta_e}(s_0) - n^2\sum_{e\in\mathcal{E}}\phi_{\theta_e}(s_0) + n^2\sum_{e\in A_i}\phi_{\theta_e}(s_0) \geq 0.$$

Moreover, we have

$$\sum_{e\in A_i}\phi_{\theta_e}(s_0) \geq \frac{n^2}{n^2+1}\sum_{e\in\mathcal{E}}\phi_{\theta_e}(s_0) \tag{4.10}$$

for each $i \in [m]$ such that $r_i \in \mathcal{R}^*$.

Let $\mathcal{E}^*$ be the set of elements e such that $r_{e,j} \in \mathcal{R}^*$, for all $j \in [m]$. In this case, since MSI is not satisfiable, there exists a pair $(r_{\hat{i}}, e) \in \mathcal{R} \times \mathcal{E}^*$ such that $r_{\hat{i}} \in \mathcal{R}^*$ and $e \notin A_{\hat{i}}$ (otherwise $\{A_i\}_{i:r_i\in\mathcal{R}^*}$ would be a feasible solution with intersection $\mathcal{E}^*$). We observed that, in each θ_e with $e \in \mathcal{E}^*$, s_0 is recommended with probability $\sum_{e\in\mathcal{E}}\phi_{\theta_e}(s_0)/q$. Then, $\sum_{e\in A_{\hat{i}}}\phi_{\theta_e}(s_0) = \sum_{e\in A_{\hat{i}}}\phi_{\theta_e}(s_0) \leq \frac{q-1}{q}\sum_{e\in\mathcal{E}}\phi_{\theta_e}(s_0)$. This leads to a contradiction with (4.10) since

$$\frac{q-1}{q}\sum_{e\in\mathcal{E}}\phi_{\theta_e}(s_0) \geq \frac{n^2}{n^2+1}\sum_{e\in\mathcal{E}}\phi_{\theta_e}(s_0)$$

has no solutions (since q and n are positive integers and $q \leq n$). This concludes our proof. □

Theorem 4.5 implies that the public signaling problem is intractable even with more general sender's utility functions. It is immediate to see that the same negative result holds for anonymous utility functions (a k-voting rule induces a sender's anonymous utility function), and we prove that the same hardness result also holds for plurality voting with two candidates and hence majority voting.

Corollary 4.1. *PL-V with public signaling, even with two candidates, cannot be approximated in polynomial time to within* any *factor, unless P=NP.*

Proof. PL-V with two candidates is equivalent to K-V with $k^* = \lfloor \frac{|\mathcal{R}|}{2} \rfloor + 1$. We show that K-V with arbitrary k reduces to K-V with $k = k^*$. Theorem 4.5 concludes the proof.

We distinguish two cases: i) Suppose $k > k^*$. We add $2k - |\mathcal{R}| - 1$ voters that prefer c_1 in any state. There are $|\mathcal{R}^*| = 2k - 1$ voters and candidate c_0 has $k = \lfloor \frac{|\mathcal{R}^*|}{2} \rfloor + 1$ votes only if k of the initial receivers vote for c_0. ii) Suppose $k < k^*$. We add $|\mathcal{R}| + 1 - 2k$ voters that prefer c_0 in any state. There are $|\mathcal{R}^*| = 2|\mathcal{R}| + 1 - 2k$ voters and candidate c_0 has $\lfloor \frac{|\mathcal{R}^*|}{2} \rfloor + 1 = |\mathcal{R}| - k + 1$ votes only if k of the initial receivers vote for c_0. □

CHAPTER 5

Bi-criteria Approximations in Public Bayesian Persuasion: Voting and Beyond

In this chapter, we characterize the complexity of bi-approximation with public signals. Even if the chapter focuses on a general bayesian persuasion problem, we take advantage of the simplicity of voting functions. In particular, our main result shows that computing bi-criteria approximation for public signaling in k-voting elections requires quasi-polynomial time assuming ETH, strengthening the result in Section 4.6. In Section 5.1, we define a bicriteria approximation for the Bayesian persuasion problem. In Section 5.2, we define an approximate problem related to finding the largest feasible subset of linear inequalities and characterize its computational complexity. In Section 5.3, we show that computing a bi-criteria approximation in k-voting elections requires at least quasi-polynomial time assuming ETH. Finally, in Section 5.4 we design a quasi-polynomial time algorithm that provides a bi-criteria approximation for general Bayesian persuasion problems, complementing the result in Section 5.3.

5.1 Approximations

We recall that a direct public signaling scheme is *ϵ-persuasive* if the following holds for any $r \in \mathcal{R}$, **a** $\in \mathcal{A}$, and $a' \in \mathcal{A}_r$:

$$\sum_{\theta \in \Theta} \mu_\theta \, \phi_\theta(\mathbf{a}) \left(u^r_\theta(a_r) - u^r_\theta(a') \right) \geq -\epsilon. \tag{5.1}$$

Throughout the chapter, we focus on the computation of approximately optimal signaling schemes. Let OPT be the optimal value of LP 3.7, *i.e.*, the best sender's expected revenue under public persuasion constraints. Since, for each state of nature θ, f_θ is a non-negative function, we have that $\text{OPT} \geq 0$. When a signaling scheme yields an expected sender utility of at least $\alpha \, \text{OPT}$, with $\alpha \in (0, 1]$, we say that the signaling scheme is *α-approximate* (that is, approximate in multiplicative sense). When a signaling scheme yields an expected sender utility of at least $\text{OPT} - \alpha$, with $\alpha \in [0, 1)$, we say that the scheme is *α-optimal* (that is, approximate in additive sense).

Finally, we consider approximations which relax both the optimality and the persuasiveness constraints. When a signaling scheme is both ϵ-persuasive and α-approximate (or α-optimal), we say it is a *bi-criteria approximation*. We say that one such signaling scheme is *(α, ϵ)-persuasive*.

5.2 Maximum ϵ-Feasible Subsystem of Linear Inequalities

As a first step, we prove the following auxiliary result on the two-prover game introduced in section 2.3.

Lemma 5.1. *Given a 3SAT formula φ, if φ is unsatisfiable, then for each (possibly randomized) Merlin_2's strategy η_2 there exists a set S_i such that each Merlin_1's assignment to variables in S_i is in conflict with Merlin_2's assignment with a probability of at least $\rho/2\nu$.*

Proof. Let $\omega(\mathcal{F}_\varphi, \eta_2 | S_i)$ be the probability with which Arthur accepts Merlin's answers when Merlin_1 receives S_i, and Merlin_2 follows strategy η_2. Formally:

$$\omega(\mathcal{F}_\varphi, \eta_2 | S_i) := \max_{\eta_1} \mathbb{E}_{T_i}[\mathcal{V}(S_i, T_i, \eta_1, \eta_2)].$$

By definition of the value of a free game, we have:

$$\omega(\mathcal{F}_\varphi) = \frac{1}{m} \max_{\eta_2} \sum_{S_i} \omega(\mathcal{F}_\varphi, \eta_2 | S_i) \geq \max_{\eta_2} \min_{S_i} \omega(\mathcal{F}_\varphi, \eta_2 | S_i).$$

Then, by Lemma 2.2, this results in:

$$\max_{\eta_2} \min_{S_i} \omega(\mathcal{F}_\varphi, \eta_2 | S_i) \leq 1 - \frac{\rho}{2\nu},$$

which proves the statement of the lemma. □

Now, we introduce the *maximum ϵ-feasible subsystem of linear inequalities* problem. Given a system of linear inequalities $A\,\mathbf{x} \geq 0$ with $A \in [-1, 1]^{n_{\text{row}} \times n_{\text{col}}}$ and $\mathbf{x} \in \Delta_{n_{\text{col}}}$, we study the problem of finding the largest subsystem of linear inequalities that violate the constraints of at most ϵ. As we will show in Section 5.3, this problem presents some deep analogies with the problem of determining *good* posteriors in persuasion problems. Let $\mathbb{I}$ denote the indicator function. Then, the problem of finding the maximum feasible subsystem of linear inequalities reads as follows.

Definition 5.1 (MFS). *Given a matrix $A \in [-1, 1]^{n_{\text{row}} \times n_{\text{col}}}$, the problem of finding the* maximum feasible subsystem of linear inequalities *(*MFS*) reads as follows:*

$$\max_{\mathbf{x}^* \in \Delta_{n_{\text{col}}}} \sum_{i \in [n_{\text{row}}]} \mathbb{I}[w_i^* \geq 0] \;\; \text{s.t. } \mathbf{w}^* = A\,\mathbf{x}^*.$$

We are interested in the problem of finding a vector $\mathbf{x}$ which results at least in the same number of feasible inequalities of MFS under a relaxation of the constraints with respect to Definition 5.1.

Definition 5.2 (ϵ-MFS). *Given a matrix $A \in [-1, 1]^{n_{\text{row}} \times n_{\text{col}}}$, let*

$$k := \max_{\mathbf{x}^* \in \Delta_{n_{\text{col}}}} \sum_{i \in [n_{\text{row}}]} \mathbb{I}[w_i^* \geq 0] \;\; \text{s.t. } \mathbf{w}^* = A\,\mathbf{x}^*.$$

Then, the problem of finding the maximum ϵ-feasible subsystem of linear inequalities *(*ϵ-MFS*) amounts to finding a probability vector* $\mathbf{x} \in \Delta_{n_{\text{col}}}$ *such that, by letting* $\mathbf{w} = A\,\mathbf{x}$*, it holds:* $\sum_{i \in [n_{\text{row}}]} \mathbb{I}[w_i \geq -\epsilon] \geq k$*.*

This problem is previously studied by Cheng et al. (2015). They design a PTAS for the ϵ-MFS problem guaranteeing the satisfaction of at least $k - \epsilon\, n_{\text{row}}$ inequalities. This results in a bi-criteria PTAS for the MFS problem.

Initially, we show that ϵ-MFS can be exactly solved in $n^{O(\log n)}$ steps for every fixed $\epsilon > 0$.

Theorem 5.1. ϵ-MFS *can be solved in $n^{O(\log n)}$ steps.*

Proof. Denote by $\mathbf{x}^*$ the optimal solution of ϵ-MFS. Let $\tilde{\mathbf{x}} \in \Delta_{n_{\text{col}}}$ be the empirical distribution of q i.i.d. samples drawn from probability distribution $\mathbf{x}^*$. Moreover, let $\mathbf{w}^* := A\,\mathbf{x}^*$ and $\tilde{\mathbf{w}} := A\,\tilde{\mathbf{x}}$. By Hoeffding's inequality we have

$$\Pr(w_i^* - \tilde{w}_i \geq \epsilon) \leq e^{-2q\epsilon^2}$$

for each $i \in [n_{\text{row}}]$. Then, by the union bound, we get

$$\Pr(\exists i \text{ s.t. } w_i^* - \tilde{w}_i \geq \epsilon) \leq n_{\text{row}}\, e^{-2q\epsilon^2}.$$

Finally, we can write

$$\Pr(w_i^* - \tilde{w}_i \leq \epsilon\ \forall i \in [n_{\text{row}}]) \geq 1 - n_{\text{row}}\, e^{-2q\epsilon^2}.$$

Thus, setting $q = \log n_{\text{row}}/\epsilon^2$ ensures the existence of a vector $\tilde{\mathbf{x}}$ guaranteeing that, if $w_i^* \geq 0$, then $\tilde{w}_i \geq -\epsilon$. Since $\tilde{\mathbf{x}}$ is q-uniform by construction, we can find it by enumerating over all the $O((n_{\text{col}})^q)$ q-uniform probability vectors where $q = \log n_{\text{row}}/\epsilon^2$. Trivially, this task can be performed in $n^{\log n_{\text{row}}/\epsilon^2}$ steps and, therefore, in $n^{O(\log n)}$ steps. □

Now we show that ϵ-MFS requires at least $n^{\tilde{\Omega}(\log n)}$ steps. [1] In doing so, we close the gap with the upper bound stated by Theorem 5.1 except for polylogarithmic factors of $\log n$ in the exponent.

Theorem 5.2. *Assuming ETH, there exists a constant $\epsilon > 0$ such that solving ϵ-MFS requires time $n^{\tilde{\Omega}(\log n)}$.*

Proof. OVERVIEW. We provide a polynomial-time reduction from the problem FREEGAME$_\delta$ (Def. 2.6) to ϵ-MFS, where $\epsilon = \frac{\delta}{26} = \frac{\rho}{52\nu}$ (see Section 2.3 for the definition of parameters δ, ρ, ν). We show that, given a free game $\mathcal{F}_\varphi$ instance, it is possible to build a matrix A s.t., for a certain value k, the following holds:

(i) if $\omega(\mathcal{F}_\varphi) = 1$, then there exists a vector $\mathbf{x}$ s.t.

$$\sum_{i \in [n_{\text{row}}]} \mathbb{I}[w_i \geq 0] = k, \tag{5.2}$$

where $\mathbf{w} = A\,\mathbf{x}$;

(ii) if $\omega(\mathcal{F}_\varphi) \leq 1 - \delta$, then all vectors $\mathbf{x}$ are s.t.

$$\sum_{i \in [n_{\text{row}}]} \mathbb{I}[w_i \geq -\epsilon] < k, \tag{5.3}$$

where $\mathbf{w} = A\,\mathbf{x}$;

[1] $\tilde{\Omega}$ hides polylogarithmic factors.

CONSTRUCTION. In the free game $\mathcal{F}_\varphi$, Arthur sends a set of variables S_i to Merlin$_1$ and a set of clauses T_j to Merlin$_2$, where $i, j \in [m]$, $m = \sqrt{n} \, \text{polylog}(n)$ (see Definition 2.5 for the definition of n). Then, Merlin$_1$'s (resp., Merlin$_2$'s) answer is denoted by $p_1 \in \mathcal{P}_1$ (resp., $p_2 \in \mathcal{P}_2$). The system of linear inequalities used in the reduction has a vector of variables $\mathbf{x}$ structured as follows.

1. *Variables corresponding to Merlin$_2$'s answers.* There is a variable x_{T_j,p_2} for each $j \in [m]$ and, due to Lemma 2.1 and assuming $|T_j| = 2m$, it holds $p_2 \in \mathcal{P}_2 = \{0,1\}^{6m}$ (if $|T_j| < 2m$, we extend p_2 with extra bits).

2. *Variables corresponding to Merlin$_1$'s answers.* We need to introduce some further machinery to augment the dimensionality of $\mathcal{P}_1$ via a viable mapping. Let $e : \{0,1\}^{2m} \to \{0,1\}^{8m}$ be the code stated in Theorem 2.4 with rate $1/4$ and relative distance $\text{dist}(e) \geq 1/5$. We can safely assume that $|S_i| = 2m$ and $p_1 \in \mathcal{P}_1 = \{0,1\}^{2m}$ (if $|S_i| < 2m$, we extend p_1 with extra bits). Then, $e(p_1)$ is the $8m$-dimensional encoding of answer p_1 via code e. Let $e(p_1)_j$ be the j-th bit of vector $e(p_1)$. We have a variable $x_{i,\ell}$ for each index $i \in [8m]$ and $\ell := \{\ell_j\}_{j \in [m]} \in \{0,1\}^m$. These $x_{i,\ell}$ variables can be interpreted as follows. Suppose to have an encoding of an answer for each of the possible set S_j. There are m such encodings, each of them having $8m$ bits. Then, it holds $x_{i,\ell} > 0$ if and only if the i-th bit of the encoding corresponding to S_j is ℓ_j.

There is a total of $m\,2^m\,(2^{5m} + 8)$ variables. Matrix A has a number of columns equal to the number of variables. We denote with $A_{\cdot,(T_j,p_2)}$ the entry in row $\cdot$ and column corresponding to variable x_{T_j,p_2}. Analogously, $A_{\cdot,(i,\ell)}$ is the entry in row $\cdot$ and column corresponding to variable $x_{i,\ell}$. Rows are grouped in four types, denoted by $\{\mathtt{t}_i\}_{i=1}^4$. We write $A_{\mathtt{t}_i,\cdot}$ when referring to an entry of *any* row of type $\mathtt{t}_i$. Further arguments may be added as a subscript to identify specific entries of A. Rows are structured as follows.

1. *Rows of type* $\mathtt{t}_1$: there are β (the value of β is specified later in the proof) rows of type $\mathtt{t}_1$ s.t. $A_{\mathtt{t}_1,(T_j,p_2)} = 1$ for each $j \in [m], p_2 \in \mathcal{P}_2$, and -1 otherwise.

2. *Rows of type* $\mathtt{t}_2$: there are β rows for each subset $\mathcal{T} \subseteq \{T_j\}_{j\in[m]}$ with cardinality $m/2$ (*i.e.*, there is a total of $\beta \binom{m}{m/2}$ rows of type $\mathtt{t}_2$). Then,

the following holds for each $\mathcal{T}$:

$$A_{(\mathtt{t}_2,\mathcal{T}),(T_j,p_2)} = \begin{cases} -1 & \text{if } T_j \in \mathcal{T}, p_2 \in \mathcal{P}_2 \\ 1 & \text{if } T_j \notin \mathcal{T}, p_2 \in \mathcal{P}_2 \end{cases} \quad \text{and}$$

$$A_{(\mathtt{t}_2,\mathcal{T}),(i,\ell)} = \quad 0 \quad \text{for each } i \in [8m], \ell \in \{0,1\}^m.$$

3. *Rows of type* $\mathtt{t}_3$: there are β rows of type $\mathtt{t}_3$ for each subset of $4m$ indices $\mathcal{I}$ drawn from $[8m]$, for a total of $\beta\binom{8m}{4m}$ rows. For each subset of indices $\mathcal{I}$ we have:

$$A_{(\mathtt{t}_3,\mathcal{I}),(T_j,p_2)} = \quad 0 \quad \text{for each } T_j, p_2 \text{ and}$$

$$A_{(\mathtt{t}_3,\mathcal{I}),(i,\ell)} = \begin{cases} -1 & \text{if } i \in \mathcal{I}, \ell \in \{0,1\}^m \\ 1 & \text{if } i \notin \mathcal{I}, \ell \in \{0,1\}^m \end{cases}.$$

4. *Rows of type* $\mathtt{t}_4$: there is a row of type $\mathtt{t}_4$ for each S_i and p_1. Each of these rows is such that:

$$A_{(\mathtt{t}_4,S_i,p_1),(T_j,p_2)} = \begin{cases} -1/2 & \text{if } \mathcal{V}(S_i,T_j,p_1,p_2) = 1 \\ -1 & \text{otherwise} \end{cases} \quad \text{and}$$

$$A_{(\mathtt{t}_4,S_i,p_1),(j,\ell)} = \begin{cases} 1/2 & \text{if } e(p_1)_j = \ell_i \\ -1 & \text{otherwise} \end{cases}.$$

Finally, we set $k = \left(1 + \binom{m}{m/2} + \binom{8m}{4m}\right)\beta + m$ and $\beta \gg m$ (*e.g.*, $\beta = 2^{10m}$). We say that row i satisfies ϵ-MFS condition for a certain $\mathbf{x}$ if $w_i \geq -\epsilon$, where $\mathbf{w} = A\mathbf{x}$ (in the following, we will also consider $w_i \geq 0$ as an alternative condition). We require at least k rows to satisfy the ϵ-MFS condition. Then, all rows of types $\mathtt{t}_1$, $\mathtt{t}_2$, $\mathtt{t}_3$ and at least m rows of type $\mathtt{t}_4$ must be s.t. w_i satisfies the ϵ-MFS condition.

COMPLETENESS. Given a satisfiable assignment of variables ζ to φ, we build vector $\mathbf{x}$ as follows. Let ζ_{T_j} be the partial assignment obtained by restricting ζ to the variables in the clauses of T_j (if $|T_j| < 2m$ we pad ζ_{T_j} with bits 0 until ζ_{T_j} has length $6m$). Then, we set $x_{T_j,\zeta_{T_j}} = 1/2m$. Moreover, for each $i \in [8m]$ and $\ell^i = (e(\zeta_{S_1})_i, \ldots, e(\zeta_{S_m})_i)$, we set $x_{i,\ell^i} = 1/16m$. We show that $\mathbf{x}$ is s.t. there are at least k rows i with $w_i \geq 0$ (Condition (5.2)). First, each row i of type $\mathtt{t}_1$ is s.t. $w_i = 0$ since $\sum_{T_j,p_2} x_{T_j,p_2} = \sum_{i,\ell} x_{i,\ell} = 1/2$. For each T_j, $\sum_{p_2} x_{T_j,p_2} = 1/2m$. Then, for each subset $\mathcal{T}$ of $\{T_j\}_{j\in[m]}$, we have $\sum_{p_2,T_j\in\mathcal{T}} x_{T_j,p_2} = 1/4$. This implies that each row i of type $\mathtt{t}_2$ is s.t. $w_i = 0$. A similar argument holds for rows of type $\mathtt{t}_3$. Finally, we show that for each S_i there is at least a row i of type $\mathtt{t}_4$ s.t. $w_i \geq 0$. Take the row corresponding to (S_i, ζ_{S_i}). For each $x_{b,\ell} > 0$ where $b \in [8m]$ and $\ell \in \{0,1\}^m$, it holds $e(\zeta_{S_i})_b = \ell_i$. Then, there are $8m$ columns played with probability $1/16\,m$ with value $1/2$, *i.e.*, $\sum_{b,\ell} A_{(\mathtt{t}_4, S_i, \zeta_{S_i}),(b,\ell)}\, x_{b,\ell} = 1/4$. Moreover, for each (T_j, ζ_{T_j}), it holds $\mathcal{V}(S_i, T_j, \zeta_{S_i}, \zeta_{T_j}) = 1$. Then, $\sum_{T_j,p_2} A_{(\mathtt{t}_4, S_i, \zeta_{S_i}),(T_j,\zeta_{T_j})}\, x_{T_j,p_2} = -1/4$. This concludes the proof of completeness.

SOUNDNESS. We show that, if $\omega(\mathcal{F}_\varphi) \leq 1 - \delta$, there is not any probability distribution $\mathbf{x}$ s.t.

$$\sum_{i \in n_{\text{row}}} \mathbb{I}[w_i \geq -\epsilon] \geq k, \tag{5.4}$$

with $\mathbf{w} = A\,\mathbf{x}$. Assume, by contradiction, that one such vector $\mathbf{x}$ exists. For the sake of clarity, we summarize the structure of the proof. (i) We show that the probability assigned by $\mathbf{x}$ to columns with index (T_j, p_2) has to be *close* to $1/2$, and the same has to hold for columns of type (i, ℓ). (ii) We show that $\mathbf{x}$ has to assign probability *almost* uniformly among T_js and indices i of the encoding of $\mathcal{P}_1$ (resp., Lemma 5.3 and Lemma 5.4 below). Intuitively, this resembles the fact that, in $\mathcal{F}_\varphi$, Arthur draws questions T_j according to a uniform probability distribution. (iii) For each S_i, there is at most one row $(\mathtt{t}_4, S_i, p_1)$ s.t. $w_{(\mathtt{t}_4, S_i, p_1)} \geq -\epsilon$ (Lemma 5.5). This implies, together with the hypothesis that at least m rows of type $\mathtt{t}_4$ satisfy the ϵ-MFS condition, that there exists exactly one such row for each S_i. (iv) Finally, we show that the above construction leads to a contradiction with Lemma 5.1 for a suitable free game.

Before providing the details of the four above steps, we introduce the following result, due to Babichenko et al. (2015).

Lemma 5.2 (Lemma 2 of Babichenko et al. (2015))**.** *Let $\mathbf{v} \in \Delta_n$ be a probability vector, and $\mathbf{u}$ be the n-dimensional uniform probability vector. If $||\mathbf{v} - \mathbf{u}||_1 > c$, then there exists a subset of indices $\mathcal{I} \subseteq [n]$ such that $|\mathcal{I}| = n/2$ and $\sum_{i \in \mathcal{I}} v_i > \frac{1}{2} + \frac{c}{4}$.*

Then, we proceed with the following steps:

1. Equation 5.4 requires all rows i of type $\mathtt{t}_1$, $\mathtt{t}_2$, $\mathtt{t}_3$ to be s.t. $w_i \geq -\epsilon$. This implies that, for rows of type $\mathtt{t}_1$, it holds

$$\sum_{T_j,p_2} x_{T_j,p_2} \geq \frac{1}{2}\left(1 - \epsilon\right). \tag{5.5}$$

If, by contradiction, this inequality did not hold, each row i of type $\texttt{t}_1$ would be s.t. $w_i < 1/2 - \epsilon/2 - (1/2 + \epsilon/2) = -\epsilon$, thus violating Equation 5.4. Moreover, Equation 5.4 implies that at least a row $(\texttt{t}_4, S_i, p_1)$ has $w_{(\texttt{t}_4, S_i, p_1)} \geq -\epsilon$. Therefore, it holds $\sum_{i,\ell} x_{i,\ell} \geq 1/2 - \epsilon$. Indeed, if, by contradiction, this condition did not hold, all rows of type $\texttt{t}_4$ would have $w_i < 1/2\,(1/2 - \epsilon) - 1/2\,(1/2 + \epsilon) = -\epsilon$.

2. Let $\mathbf{v}_1 \in \Delta_m$ be the probability vector defined as

$$v_{1,j} := \frac{\sum_{p_2} x_{T_j, p_2}}{\sum_{j, p_2} x_{T_j, p_2}},$$

and $\tilde{\mathbf{v}}$ be a uniform probability vector of suitable dimension. The following result shows that having a bounded element-wise difference between $\mathbf{v}_1$ and $\tilde{\mathbf{v}}$ is a necessary condition for Equation 5.4 to be satisfied.

Lemma 5.3. *If* $||\mathbf{v}_1 - \tilde{\mathbf{v}}||_1 > 16\epsilon$*, there exists a row* i *of type* $\texttt{t}_2$ *s.t.* $w_i < -\epsilon$.

Proof. Lemma 5.2 implies that, if $||\mathbf{v}_1 - \tilde{\mathbf{v}}||_1 > 16\epsilon$, there exists a subset $\mathcal{T} \subseteq \{T_j\}_{j \in [m]}$ such that

$$\sum_{T_j \in \mathcal{T}} \sum_{p_2} x_{T_j, p_2} > (1/2 + 4\epsilon) \sum_{j, p_2} x_{T_j, p_2} > 1/4 + \epsilon.$$

It follows that

$$\sum_{T_j \notin \mathcal{T}} \sum_{p_2} x_{T_j, p_2} < 1/2 + \epsilon - 1/4 - \epsilon = 1/4,$$

which implies that row $(\texttt{t}_2, \mathcal{T})$ is s.t.

$$w_{\texttt{t}_2, \mathcal{T}} < -1/4 - \epsilon + 1/4 < -\epsilon.$$

□

Let $\mathbf{v}_2 \in \Delta_{[8m]}$ be the probability vector defined as

$$v_{2,i} := \frac{\sum_{\ell} x_{i,\ell}}{\sum_{i,\ell} x_{i,\ell}},$$

and $\tilde{\mathbf{v}}$ be a suitable uniform probability vector. Moreover, the following holds.

Lemma 5.4. *If* $||\mathbf{v}_2 - \tilde{\mathbf{v}}||_1 > 16\epsilon$*, there exists a row* i *of type* $\mathtt{t}_3$ *s.t.* $w_i < -\epsilon$*.*

Proof. Lemma 5.2 implies that, if $||\mathbf{v}_2 - \tilde{\mathbf{v}}||_1 > 16\epsilon$, there exists a set $\mathcal{I} \subseteq [8m]$ such that

$$\sum_{i \in \mathcal{I}} \sum_{\ell} x_{i,\ell} > (1/2 + 4\epsilon) \sum_{i,\ell} x_{i,\ell} > 1/4 + \epsilon.$$

Then,

$$\sum_{i \notin \mathcal{I}} \sum_{\ell} x_{i,\ell} < 1/2 + \epsilon/2 - 1/4 - \epsilon = 1/4 - \epsilon/2.$$

It follows that there exists a row $(\mathtt{t}_3, \mathcal{I})$ such that

$$w_{\mathtt{t}_3,\mathcal{I}} < -1/4 - \epsilon + 1/4 - \epsilon/2 < -\epsilon.$$

□

In order to satisfy Equation (5.4), all rows i of type $\mathtt{t}_2$ and $\mathtt{t}_3$ have to be s.t. $w_i \geq -\epsilon$. Then, by Lemmas 5.3 and 5.4, it holds that $||\mathbf{v}_1 - \tilde{\mathbf{v}}||_1 \leq 16\,\epsilon$ and $||\mathbf{v}_2 - \tilde{\mathbf{v}}||_1 \leq 16\,\epsilon$.

3. We show that, for each S_i, there exists at most one row $(\mathtt{t}_4, S_i, p_1)$ for which $w_{(\mathtt{t}_4,S_i,p_1)} \geq -\epsilon$.

   Lemma 5.5. *For each* S_i*,* $i \in [m]$*, there exists at most one row* $(\mathtt{t}_4, S_i, p_1)$ *such that* $w_{(\mathtt{t}_4,S_i,p_1)} \geq -\epsilon$*.*

   Proof. Let $\mathfrak{f}(\mathbf{x}, p_1) := \sum_{j:\ell_i = e(p_1)_j} x_{j,\ell}$. Assume, by contradiction, that for a given S_i there exist two assignments p_1' and p_1'' such that $w_{(\mathtt{t}_4,S_i,p_1)} \geq -\epsilon$ for each $p_1 \in \{p_1', p_1''\}$. Then, $\mathfrak{f}(\mathbf{x}, p_1) \geq 1/2 - \epsilon$, for each $p_1 \in \{p_1', p_1''\}$. Otherwise, we would get $w_{(\mathtt{t}_4,S_i,p_1)} < 1/2(1/2 - \epsilon) - 1/2(1/2 + \epsilon) = -\epsilon$ for at least one $p_1 \in \{p_1', p_1''\}$. Let $\mathbf{x}'$ be the vector such that $x'_{i,\ell} := \frac{x_{i,\ell}}{\sum_{i,\ell} x_{i,\ell}}$. Then, $\mathfrak{f}(\mathbf{x}', p_1) \geq \frac{1/2-\epsilon}{1/2+\epsilon} \geq 1 - 4\epsilon$, for $p_1 \in \{p_1', p_1''\}$. By Lemmas 5.2 and 5.4, we have that $||\mathbf{v}_2 - \tilde{\mathbf{v}}||_1 \leq 16\epsilon$. Therefore, we can obtain a uniform vector $\tilde{\mathbf{x}}$ by moving at most 16ϵ probability from $\mathbf{x}'$. This results in a decrease of $\mathfrak{f}$ of at most 16ϵ, that is $\mathfrak{f}(\tilde{\mathbf{x}}, p_1) \geq 1 - 20\epsilon$ for each $p_1 \in \{p_1', p_1''\}$.

   By construction $\text{dist}(e) = 1/5$, which implies $\text{dist}(e(p_1'), e(p_1'')) \geq 1/5$. Then, there exists a set of indices $\mathcal{I}$, with $|\mathcal{I}| \geq 8m/5$, such that $e(p_1')_j \neq e(p_1'')_j$ for each $j \in \mathcal{I}$. Therefore, $\mathfrak{f}(\tilde{\mathbf{x}}, p_1') + \mathfrak{f}(\tilde{\mathbf{x}}, p_1'') \leq \sum_{j \in \mathcal{I}} 1/8m + \sum_{j \notin \mathcal{I}} 2/8m \leq 2 - 1/5$. This leads to a contradiction with $\mathfrak{f}(\tilde{\mathbf{x}}, p_1') + \mathfrak{f}(\tilde{\mathbf{x}}, p_1'') \geq 2 - 40\epsilon$. □

Then, there are at least m rows $(\mathtt{t}_4, S_i, p_1)$ s.t. $w_{(\mathtt{t}_4,S_i,p_1)} \geq -\epsilon$ and, by Lemma 5.5, we get that there exists exactly one such row for each S_i, $i \in [m]$. Therefore, for each S_i, there exists $p_1^i \in \mathcal{P}_1$ such that

$$\sum_{(T_j,p_2):\mathcal{V}(S_i,T_j,p_1^i,p_2)=1} x_{(T_j,p_2)} \geq \frac{1}{2} - 4\,\epsilon.$$

Notice that, if this condition did not hold, by Step (i) we would obtain

$$w_{\mathtt{t}_4,S_i,p_1^i} < -\frac{1}{2}\left(\frac{1}{2} - 4\,\epsilon\right) - \frac{7}{2}\epsilon + \frac{1}{2}\left(\frac{1}{2} + \frac{\epsilon}{2}\right) = -\epsilon,$$

which would go against the satisfiability of Equation (5.4).

4. Finally, let $\mathcal{F}_\varphi^*$ be a free game in which Arthur (*i.e.*, the verifier) chooses question T_j with probability $v_{1,j}$ as defined in Step (ii), and Merlin$_2$ (*i.e.*, the second prover) answers p_2 with probability $x_{T_j,p_2}/v_{1,j}$. In this setting (*i.e.*, $\mathcal{F}_\varphi^*$), given question S_i to Merlin$_1$, the two provers will provide compatible answers with probability

$$\Pr\left(\mathcal{V}^*(S_i, T_j, p_1^i, p_2) = 1 \mid S_i\right) = \frac{1/2 - 4\,\epsilon}{\sum_{j,p_2} x_{T_j,p_2}} \geq \frac{1/2 - 4\,\epsilon}{1/2 + \epsilon} \geq 1 - 10\,\epsilon,$$

where the first inequality holds for Equation 5.5 at Step (i). In a canonical (*i.e.*, as in Definition 2.5) free game $\mathcal{F}_\varphi$, Arthur picks questions according to a uniform probability distribution. Therefore, the main difference between $\mathcal{F}_\varphi$ and $\mathcal{F}_\varphi^*$ is that, in the latter, Arthur draws questions for Merlin$_2$ from $\mathbf{v}_1$ which may not be a uniform probability distribution. However, we know that differences between $\mathbf{v}_1$ and a uniform probability vector must be limited. Specifically, by Lemma 5.3, we have $||\mathbf{v}_1 - \tilde{\mathbf{v}}||_1 \leq 16\,\epsilon$. Then, if Merlin$_1$ and Merlin$_2$ applied in $\mathcal{F}_\varphi$ the strategies we described for $\mathcal{F}_\varphi^*$, their answers would be compatible with probability at least $\Pr(\mathcal{V}(S_i, T_j, p_1^i, p_2) = 1 \mid S_i) \geq 1 - 26\,\epsilon$, for each S_i. Finally, by picking $\epsilon = \rho/52\,\nu$, we reach a contradiction with Lemma 5.1. This concludes the proof.

□

5.3 Hardness of (α, ϵ)-persuasion

We show that a public signaling scheme approximating the value of the optimal one cannot be computed in polynomial time even if we allow it to be ϵ-persuasive (see Equation (5.1)). Specifically, assuming ETH, computing an (α, ϵ)-persuasive signaling scheme requires at least $n^{\tilde{\Omega}(\log n)}$, where the dimension of the instance is $n = O(\bar{n}\, d)$. We prove this result for the specific case of the k-voting problem with two candidates, as defined in Section 4.1. Besides its practical applicability, this problem is particularly instructive in highlighting the strong connection between the problem of finding suitable posteriors and the ϵ-MFS problem, as discussed in the following lemma.

Let $\sigma : \Delta_\Theta \to \mathbb{N}_0^+$ be a function returning, for a given posterior distribution $\boldsymbol{\xi} \in \Delta_\Theta$, the number of receivers such that $\sum_\theta \xi_\theta \left(u_\theta^r(a_0) - u_\theta^r(a_1)\right) \geq 0$, *i.e.*, the number of voters that will vote for a_0 with a persuasive signaling scheme. Analogously, $\sigma_\epsilon(\boldsymbol{\xi})$ is the number of receivers for which $\sum_\theta \xi_\theta \left(u_\theta^r(a_0) - u_\theta^r(a_1)\right) \geq -\epsilon$, *i.e.*, the number of voters that will vote for a_0 with an ϵ-persuasive signaling scheme. Then, we can prove the following.

Lemma 5.6. *Given a k-voting instance, the problem of finding a posterior $\boldsymbol{\xi} \in \Delta_\Theta$ such that $\sigma_\epsilon(\boldsymbol{\xi}) \geq k$ is equivalent to finding an ϵ-feasible subsystem of k linear inequalities over the simplex when $A \in [-1, 1]^{\bar{n} \times d}$ is such that:*

$$A_{r,\theta} = u_\theta^r(a_0) - u_\theta^r(a_1) \quad \text{for each } r \in \mathcal{R}, \theta \in \Theta. \tag{5.6}$$

Proof. By setting $\mathbf{x} = \boldsymbol{\xi}$, it directly follows that $\sum_{i \in [\bar{n}]} \mathbb{I}[A_i\, \mathbf{x} \geq -\epsilon] \geq k$ iff $\sigma_\epsilon(\boldsymbol{\xi}) \geq k$. □

The above lemma shows that deciding if there exists a posterior $\boldsymbol{\xi}$ such that $\sigma(\boldsymbol{\xi}) \geq k$ or if *all* the posteriors have $\sigma_\epsilon(\boldsymbol{\xi}) < k$ (*i.e.*, deciding if the utility of the sender can be greater than zero) is as hard as solving the ϵ-MFS problem. More precisely, if an ϵ-MFS instance does not admit any solution, then there does not exist any posterior guaranteeing a strictly positive winning probability for the sender's preferred candidate. On the other hand, if an ϵ-MFS instance admits a solution, there exists a signaling scheme where at least one of the induced posteriors guarantees strictly positive winning probability for the sender's preferred candidate. However, the above connection between the ϵ-MFS problem and the k-voting problem is not sufficient to prove the inapproximability of the k-voting problem, as the probability whereby this posterior is reached may be arbitrarily small.

Luckily enough, the next theorem shows that it is possible to strengthen the inapproximability result by constructing an instance in which, when 3SAT is satisfiable, there is a signaling scheme such that all the induced posteriors satisfy $\sigma(\boldsymbol{\xi}) \geq k$ (*i.e.*, the sender's preferred candidate wins with a probability of 1).

Theorem 5.3. *Given a k-voting instance and assuming ETH, there exists a constant $\epsilon^* > 0$ such that, for any $\epsilon \le \epsilon^*$, finding an (α, ϵ)-persuasive signaling scheme requires $n^{\tilde{\Omega}(\log n)}$ steps for any multiplicative or additive factor α.*

Proof. OVERVIEW. By following the proof of Theorem 5.1, we can provide a polynomial-time reduction from FREEGAME$_\delta$ to the problem of finding an ϵ-persuasive signaling scheme in k-voting, with $\epsilon = \delta/780 = \rho/1560\nu$. Specifically, if $\omega(\mathcal{F}_\varphi) = 1$, there exists a signaling scheme guaranteeing the sender an expected value of 1. Otherwise, if $\omega(\mathcal{F}_\varphi) \le 1 - \delta$, then all posteriors are such that $\sigma_\epsilon(\boldsymbol{\xi}) < k$ (*i.e.*, the sender cannot obtain more than 0).

CONSTRUCTION. The k-voting instance has the following possible states of nature.

1. $\theta_{(T_j, p_2)}$ for each set of clauses T_j, $j \in [m]$, and answer $p_2 \in \mathcal{P}_2 = \{0,1\}^{6m}$.
2. Let $e : \{0,1\}^{2m} \to \{0,1\}^{8m}$ be an encoding function with $R = 1/4$ and $\text{dist}(e) \ge 1/5$ (as in the proof of Theorem 5.1). We have a state $\theta_{(i,\ell)}$ for each $i \in [8m]$, and $\ell = (\ell_1, \ldots, \ell_m) \in \{0,1\}^m$.
3. There is a state $\theta_{\mathbf{d}}$ for each $\mathbf{d} \in \{0,1\}^{7m}$. It is useful to see vector $\mathbf{d}$ as the union of the subvector $\mathbf{d}_S \in \{0,1\}^m$ and the subvector $\mathbf{d}_T \in \{0,1\}^{6m}$.

The common prior μ is such that:

$$\mu_{\theta_{(T_j,p_2)}} = \frac{1}{m\, 2^{2+6m}} \qquad \text{for each } \theta_{(T_j,p_2)},$$
$$\mu_{\theta_{(i,\ell)}} = \frac{1}{m\, 2^{5+m}} \qquad \text{for each } \theta_{(i,\ell)},$$
$$\mu_{\theta_{\mathbf{d}}} = \frac{1}{2^{1+7m}} \qquad \text{for each } \theta_{\mathbf{d}}.$$

To simplify the notation, in the remaining of the proof, let $u^r_\theta := u^r_\theta(a_0) - u^r_\theta(a_1)$. The k-voting instance comprises the following receivers.

1. *Receivers of type* $\mathtt{t}_1$: there are β (the value of β is specified later in the proof) receivers of type $\mathtt{t}_1$, which are such that $u^{\mathtt{t}_1}_{\theta_{(T_j,p_2)}} = 1$ for each (T_j, p_2), and $-1/3$ otherwise.
2. *Receivers of type* $\mathtt{t}_2$*:* there are β receivers of type $\mathtt{t}_2$ such that $u^{\mathtt{t}_2}_{\theta_{(i,\ell)}} = 1$ for each (i, ℓ), and $-1/3$ otherwise.

3. *Receivers of type* $\mathtt{t}_3$*:* there are β receivers of type $\mathtt{t}_3$ for each subset $\mathcal{T} \subseteq \{T_j\}_{j\in[m]}$ of cardinality $m/2$. Each receiver corresponding to the subset $\mathcal{T}$ is such that:

$$u_{\theta_{(T_j,p_2)}}^{(\mathtt{t}_3,\mathcal{T})} = \begin{cases} -1 & \text{if } T_j \in \mathcal{T}, p_2 \in \mathcal{P}_2 \\ 1 & \text{if } T_j \notin \mathcal{T}, p_2 \in \mathcal{P}_2 \end{cases}$$

and

$$u_{\theta}^{(\mathtt{t}_3,\mathcal{T})} = 0 \text{ for every other } \theta.$$

4. *Receivers of type* $\mathtt{t}_4$*:* we have β receivers of type $\mathtt{t}_4$ for each subset $\mathcal{I}$ of $4m$ indices selected from $[8m]$. Each receiver corresponding to subset $\mathcal{I}$ is such that:

$$u_{\theta_{(i,\ell)}}^{(\mathtt{t}_4,\mathcal{I})} = \begin{cases} -1 & \text{if } i \in \mathcal{I}, \ell \in \{0,1\}^m \\ 1 & \text{if } i \notin \mathcal{I}, \ell \in \{0,1\}^m \end{cases}$$

and

$$u_{\theta}^{(\mathtt{t}_4,\mathcal{I})} = 0 \text{ for every other } \theta.$$

5. *Receivers of type* $\mathtt{t}_5$*:* there is a receiver of type $\mathtt{t}_5$ for each S_i, $p_1 \in \mathcal{P}_1$ and $\mathbf{d} \in \{0,1\}^{7m}$. Let $\oplus$ be the XOR operator. Then, for each receiver of type $\mathtt{t}_5$ the following holds:

$$u_{\theta}^{(\mathtt{t}_5,S_i,p_1,\mathbf{d})} = \begin{cases} -1/2 & \text{if } \theta = \theta_{(T_j,p_2)} \text{ and } \mathcal{V}(S_i,T_j,p_1,p_2 \oplus \mathbf{d}_T) = 1 \\ -1/2 & \text{if } \theta = \theta_{(i',\ell)} \text{ and } e(p_1)_{i'} = [\ell \oplus \mathbf{d}_S]_i \\ 1/2 & \text{if } \theta = \theta_{\mathbf{d}} \\ -1 & \text{otherwise} \end{cases}$$

Finally, we set $k = \left(2 + \binom{m}{m/2} + \binom{8m}{4m}\right)\beta + m$. By setting $\beta \gg m$ (*e.g.*, $\beta = 2^{10m}$), candidate a_0 can get at least k votes only if all receivers of type $\mathtt{t}_1$, $\mathtt{t}_2$, $\mathtt{t}_3$, $\mathtt{t}_4$ vote for her.

COMPLETENESS. Given a satisfiable assignment ζ to the variables in φ, let $[\zeta]_{T_j} \in \{0,1\}^{6m}$ be the vector specifying the variables assignment of each clause in T_j, and $[\zeta]_{S_i} \in \{0,1\}^{2m}$ be the vector specifying the assignment of each variable belonging to S_i. The sender has a signal for each $\mathbf{d} \in \{0,1\}^{7m}$. The set of signals is denoted by $\mathcal{S}$, where $|\mathcal{S}| = 2^{7m}$, and a signal is denoted by $s_{\mathbf{d}} \in \mathcal{S}$. We define a signaling scheme ϕ as follows. First, we set $\phi_{\theta_{\mathbf{d}}}(s_{\mathbf{d}}) = 1$ for each $\theta_{\mathbf{d}}$. If $|T_j| < 2m$ for some $j \in [m]$, we pad $[\zeta]_{T_j}$ with bits 0 util $|[\zeta]_{T_j}| = 6m$. Then, for each T_j,

$\phi_{\theta_{(T_j,[\zeta]_{T_j}\oplus \mathbf{d}_T)}}(s_{\mathbf{d}}) = 1/2^m$. For each $i \in [8m]$, set $\phi_{\theta_{(i,\ell\oplus \mathbf{d}_S)}} = 1/2^{6m}$, where $\ell = (e([\zeta]_{S_1})_i, \ldots, e([\zeta]_{S_m})_i)$. First, we prove that the signaling scheme is well-formed. For each state $\theta_{(T_j,p_2)}$, it holds that

$$\sum_{s_{\mathbf{d}}\in\mathcal{S}} \phi_{\theta_{(T_j,p_2)}}(s_{\mathbf{d}}) = \frac{1}{2^m}|\{\mathbf{d} : [\zeta]_{T_j} \oplus \mathbf{d}_T = p_2\}| = 1,$$

and, for each $\theta_{(i,\ell)}$, the following holds:

$$\sum_{s_{\mathbf{d}}\in\mathcal{S}} \phi_{\theta_{(i,\ell)}}(s_{\mathbf{d}}) = \frac{1}{2^{6m}}|\{\mathbf{d} : (e([\zeta]_{S_1})_i, \ldots, e([\zeta]_{S_m})_i \oplus \mathbf{d}_S = \ell\}| = 1.$$

Now, we show that there exist at least k voters that will choose a_0. Let $\boldsymbol{\xi} \in \Delta_\Theta$ be the posterior induced by a signal $s_{\mathbf{d}}$. All receivers of type $\mathtt{t}_1$ choose a_0 since it holds:

$$\begin{aligned}\sum_{(T_j,p_2)} \xi_{\theta_{(T_j,p_2)}} =& \frac{\sum_{(T_j,p_2)} \mu_{\theta_{(T_j,p_2)}} \phi_{\theta_{(T_j,p_2)}}(s_{\mathbf{d}})}{\sum_{\theta\in\Theta} \mu_\theta \phi_\theta(s_{\mathbf{d}})}\\ =& \frac{1}{2^{2+7m}}\left(\frac{1}{2^{1+7m}} + \frac{1}{2^{2+7m}} + \frac{1}{2^{2+7m}}\right)^{-1}\\ =& \frac{1}{4}.\end{aligned}$$

Analogously, all receivers of type $\mathtt{t}_2$ select a_0. Furthermore, for each T_j, it holds $\sum_{p_2} \xi_{\theta_{(T_j,p_2)}} = 1/4\,m$. Then, for each subset $\mathcal{T} \subseteq \{T_j\}_{j\in[m]}$ of cardinality $m/2$, it holds $\sum_{T_j\in\mathcal{T},p_2} \xi_{\theta_{(T_j,p_2)}} = m/2\ 1/4\ m = 1/8$. Therefore, each receiver of type $\mathtt{t}_3$ chooses a_0. An analogous argument holds for receivers of type $\mathtt{t}_4$.

Finally, we show that, for each S_i, the receiver $(\mathtt{t}_5, S_i, [\zeta]_{S_i}, \mathbf{d})$ chooses a_0. In particular, receiver $(\mathtt{t}_5, S_i, [\zeta]_{S_i}, \mathbf{d})$ has the following expected utility:

$$\frac{1}{2}\xi_{\theta_{\mathbf{d}}} - \frac{1}{2}\sum_{(T_j,p_2)} \xi_{\theta_{(T_j,p_2)}} - \frac{1}{2}\sum_{(i',\ell)} \xi_{\theta_{(i',\ell)}} = 0$$

since, for each $\xi_{\theta_{(T_j,p_2)}} > 0$, the following holds $p_2 \oplus \mathbf{d}_T = [\zeta]_{T_j} \oplus \mathbf{d}_T \oplus \mathbf{d}_T = [\zeta]_{T_j}$ and $\mathcal{V}(S_i, T_j, [\zeta]_{S_i}, p_2 \oplus \mathbf{d}_T) = \mathcal{V}(S_i, T_j, [\zeta]_{S_i}, [\zeta]_{T_j}) = 1$ for each T_j. Moreover, for each $\xi_{\theta_{(i',l)}} > 0$, it holds $[l \oplus d_S]_i = e([\zeta]_{S_i})_{i'} \oplus d_{S,i} \oplus d_{S,i} = e([\zeta]_{S_i})_{i'}$. This concludes the proof of completeness. [2]

[2] For the sake of presentation, in the proof, we employ indirect signals of type $s_{\mathbf{d}}$. However, it is possible to construct an equivalent direct signaling scheme. Let $\boldsymbol{\xi}^{s_{\mathbf{d}}} \in \Delta_\Theta$ be the posterior induced by $s_{\mathbf{d}}$. Then, it is enough to substitute each $s_{\mathbf{d}}$ with a direct signal recommending a_0 to all receivers such that $\sum_\theta \xi_\theta^{s_{\mathbf{d}}} u_\theta^r \geq 0$, and a_1 to all the others.

SOUNDNESS. We prove that, if $\omega(\mathcal{F}_\varphi) \leq 1-\delta$, there is no posterior in which a_0 is chosen by at least k receivers, thus implying that the sender's utility is equal to 0. Now, suppose, towards a contradiction, that there exists a posterior $\boldsymbol{\xi}$ such that at least k receivers select a_0. Let $\lambda := \sum_{(T_j,p_2)} \xi_{\theta_{(T_j,p_2)}} + \sum_{(i,\ell)} \xi_{\theta_{(i,\ell)}}$. Since all voters of types $\mathtt{t}_1$ and $\mathtt{t}_2$ vote for a_0, it holds that $\sum_{(T_j,p_2)} \xi_{\theta_{(T_j,p_2)}} \geq \frac{1}{4} - \epsilon$ and $\sum_{(i,\ell)} \xi_{\theta_{(i,\ell)}} \geq \frac{1}{4} - \epsilon$. Moreover, since at least a receiver $(\mathtt{t}_5, S_i, p_1, \mathbf{d})$ must play a_0, there exists a $\mathbf{d} \in \{0,1\}^{7m}$ and a state $\theta_{\mathbf{d}}$ with $\xi_{\theta_{\mathbf{d}}} \geq \frac{1}{2} - \epsilon$. This implies that $\frac{1}{2} - 2\epsilon \leq \lambda \leq \frac{1}{2} + \epsilon$.

Consider the reduction to ϵ'-MFS, with $\epsilon' = \rho/52\nu$ (Theorem 5.2). Let $x_{(T_j,p_2)} = \xi_{\theta_{(T_j,p_2 \oplus \mathbf{d}_T)}}/\lambda$, $x_{(i,\ell)} = \xi_{\theta_{(i,\ell \oplus \mathbf{d}_S)}}/\lambda$, and $\epsilon = \epsilon'/30$. All rows of type $\mathtt{t}_1$ of ϵ'-MFS are such that

$$w_{\mathtt{t}_1} = \frac{1}{\lambda}\left(\sum_{(T_j,p_2)} \xi_{\theta_{(T_j,p_2)}} - \sum_{(i,\ell)} \xi_{\theta_{(i,\ell)}}\right) \geq -\frac{3\epsilon}{\lambda} \geq -9\epsilon \geq -\epsilon'.$$

All voters of type $\mathtt{t}_3$ choose a_0. Then, for all $\mathcal{T} \subseteq \{T_j\}_{j\in[m]}$ of cardinality $m/2$, it holds:

$$\sum_{(T_j,p_2):T_j\in\mathcal{T}} \xi_{\theta_{(T_j,p_2)}} - \sum_{(T_j,p_2):T_j\notin\mathcal{T}} \xi_{\theta_{(T_j,p_2)}} \geq -\epsilon.$$

Then, all rows of type $\mathtt{t}_2$ of ϵ'-MFS are such that:

$$\begin{aligned} w_{(\mathtt{t}_2,\mathcal{T})} &= \frac{1}{\lambda}\left(\sum_{(T_j,p_2):T_j\in\mathcal{T}} \xi_{\theta_{(T_j,p_2)}} - \sum_{(T_j,p_2):T_j\notin\mathcal{T}} \xi_{\theta_{(T_j,p_2)}}\right) \\ &\geq -\frac{\epsilon}{\lambda} \geq -3\epsilon \geq -\epsilon'. \end{aligned}$$

A similar argument proves that all rows of type $\mathtt{t}_3$ of the instance of ϵ'-MFS have $w_{(\mathtt{t}_3,\mathcal{I})} \geq -\epsilon'$.

To conclude the proof, we prove that, for each voter $(\mathtt{t}_5, S_i, p_1, \mathbf{d})$ that votes for a_0, the corresponding row $(\mathtt{t}_4, S_i, p_1)$ of the instance ϵ'-MFS is such that $w_{(\mathtt{t}_4,S_i,p_1)} \geq -\epsilon'$. Let $\lambda' := \sum_{(T_j,p_2):\mathcal{V}(S_i,T_j,p_1,p_2)=1} x_{(T_j,p_2)}$ and $\lambda'' := \sum_{(i',\ell):e(p_1)_{i'}=\ell_i} x_{(i',\ell)}$. First, we have that $\lambda' \geq 1/4 - 7\epsilon$. If this did not hold, we would have

$$\sum_\theta \xi_\theta u_\theta^{(\mathtt{t}_5,S_i,p_1,\mathbf{d})} < -\frac{1}{2}\left(\frac{1}{4}-\epsilon\right) - \frac{1}{2}\left(\frac{1}{4}-7\epsilon\right) - 6\epsilon + \frac{1}{2}\left(\frac{1}{2}+2\epsilon\right) = \epsilon.$$

Similarly, it holds $\lambda'' \geq 1/4 - 7\epsilon$. Hence

$$\begin{aligned} w_{(\mathtt{t}_4, S_i, p_1)} &= -\frac{1}{2}\lambda' + \frac{1}{2}\lambda'' - (1 - \lambda' - \lambda'') \\ &= \frac{1}{2\lambda}\left(\sum_{(T_j, p_2): \mathcal{V}(S_i, T_j, p_1, p_2) = 1} \xi_{\theta_{(T_j, p_2 \oplus \mathbf{d}_T)}} + 3 \sum_{(i', \ell): e(p_1)_{i'} = \ell_i} \xi_{\theta_{(i', \ell \oplus \mathbf{d}_S)}} \right) - 1 \\ &\geq \frac{2(1/4 - 7\epsilon)}{1/2 + \epsilon} - 1 \geq -30\,\epsilon = -\epsilon'. \end{aligned}$$

Thus, there exists a probability vector $\mathbf{x}$ for the instance of ϵ'-MFS in which at least k rows satisfy the ϵ'-MFS condition (Equation 5.3), which is in contradiction with $\omega(\mathcal{F}_\varphi) \leq 1 - \delta$. This concludes the proof. □

Theorem 5.3 shows that, assuming ETH, computing an (α, ϵ)-persuasive signaling schemes requires at least a quasi-polynomial number of steps in the specific scenario of a k-voting instance. Therefore, the same holds in the general setting of arbitrary public persuasion problems with binary action spaces.

5.4 A Quasi-polynomial time algorithm for (α, ϵ)-persuasion

In this section, we prove that our hardness result (Theorem 5.3) is tight by devising a bi-criteria approximation algorithm. Our result extends the results by Cheng et al. (2015) and Xu (2020) for signaling problems with binary action spaces and state-independent sender's utility functions. Indeed, it encompasses scenarios with an arbitrary number of actions and state-dependent sender's utility functions.

In order to prove our result, we need some further machinery. Let $\mathcal{Z}_r := 2^{\mathcal{A}_r}$ be the power set of $\mathcal{A}_r$. Then, $\mathcal{Z} := \times_{r \in \mathcal{R}} \mathcal{Z}_r$ is the set of tuples specifying a subset of $\mathcal{A}_r$ for each receiver r. For a given probability distribution over the states of nature, we are interested in determining the set of best responses of each receiver r, *i.e.*, the subset of $\mathcal{A}_r$ maximizing her expected utility. Formally, we have the following generalization of Definition 3.1 to multiple receivers.

Definition 5.3 (BR-set). *Given $\boldsymbol{\xi} \in \Delta_\Theta$, the best-response set (BR-set) $\mathcal{M}_{\boldsymbol{\xi}} := (Z_1, \ldots, Z_{\bar{n}}) \in \mathcal{Z}$ is such that*

$$Z_r = arg\, max_{a \in \mathcal{A}_r} \sum_{\theta \in \Theta} \xi_\theta \, u_\theta^r(a) \qquad \text{for each } r \in \mathcal{R}.$$

Similarly, we define a notion of ϵ-BR-set which comprises ϵ-approximate best responses to a given distribution over the states of nature.

Definition 5.4 (ϵ-BR-set)**.** *Given $\boldsymbol{\xi} \in \Delta_\Theta$, the ϵ-best-response set (ϵ-BR-set) $\mathcal{M}_{\epsilon,\boldsymbol{\xi}} := (Z_1, \dots, Z_{\bar{n}}) \in \mathcal{Z}$ is such that, for each $r \in \mathcal{R}$, action a belongs to Z_r if and only if*

$$\sum_{\theta \in \Theta} \xi_\theta \, u_\theta^r(a) \geq \sum_{\theta \in \Theta} \xi_\theta \, u_\theta^r(a') - \epsilon \qquad \text{for each } a' \in \mathcal{A}_r.$$

We introduce a suitable notion of *approximability* of the sender's objective function. Our notion of α*-approximable function* is a generalization of (Xu, 2020, Definition 4.5) to the setting of arbitrary action spaces and state-dependent sender's utility functions.

Definition 5.5 (α-Approximability)**.** *Let $f := \{f_\theta\}_{\theta \in \Theta}$ be a set of functions $f_\theta : \mathcal{A} \to [0, 1]$. We say that f is α-approximable if there exists a function $g : \Delta_\Theta \times \mathcal{Z} \to \mathcal{A}$ computable in polynomial time such that, for all $\boldsymbol{\xi} \in \Delta_\Theta$ and $Z \in \mathcal{Z}$, it holds: $\boldsymbol{a} = g(\boldsymbol{\xi}, Z)$, $\boldsymbol{a} \in Z$ and*

$$\sum_{\theta \in \Theta} \xi_\theta \, f_\theta(\boldsymbol{a}) \geq \alpha \max_{\boldsymbol{a}^* \in Z} \sum_{\theta \in \Theta} \xi_\theta \, f_\theta(\boldsymbol{a}^*).$$

The k-voting sender's utility function is 1-approximable, while, *e.g.*, when the action space is binary a non-monotone submodular function is $1/2$-approximable. The α-approximability assumption is a natural requirement since, otherwise, even evaluating the sender's objective value in a given posterior would result in an intractable problem. When f is an α-approximable function, it is possible to find an approximation of the optimal receivers' actions profile when they are constrained to select actions profiles in Z.

We now provide an algorithm which computes in quasi-polynomial time, for any α-approximable f, a bi-criteria approximation of the optimal solution with an approximation on the objective value arbitrarily close to α. When f is 1-approximable our result yields a bi-criteria QPTAS for the problem. The key idea is showing that an optimal signaling scheme can be approximated by a convex combination of suitable q-uniform posteriors. As in previous works (Cheng et al., 2015; Xu, 2020), the key part of the proof is a decomposition lemma that proves that all the posteriors can be decomposed in a convex combination of q-uniform posteriors with a small loss in utility. However, the assumption of state-dependent sender's utility functions makes previous approaches ineffective in our setting. Therefore, we develop a completely new probabilistic analysis of the decomposition lemma. Our main positive result reads as follows.

Theorem 5.4. *Let the sender utility function f be α-approximate. Then, there exists a* poly $\left(n^{\frac{\log(n/\delta)}{\epsilon^2}}\right)$ *algorithm that outputs an $\alpha(1-\delta)$-approximate ϵ-persuasive public signaling scheme.*

Proof. Let $\varrho := \max_{r\in\mathcal{R}} \varrho_r$, $\bar{n} := |\mathcal{R}|$, and $d := |\Theta|$. We show that there exists a poly $\left(d^{\frac{\log(\bar{n}\varrho/\delta)}{\epsilon^2}}\right)$ algorithm that computes the given approximation. Let $q := \frac{32\log(4\bar{n}\varrho/\delta)}{\epsilon^2}$ and $\Xi^q \subset \Delta_\Theta$ be the set of q-uniform distributions over Θ (Def. 3.3). We prove that all posteriors $\boldsymbol{\xi}^* \in \Delta_\Theta$ can be decomposed as a convex combination of q-uniform posteriors without lowering too much the sender's expected utility. Formally, each posterior $\boldsymbol{\xi}^* \in \Delta_\Theta$ can be written as $\boldsymbol{\xi}^* = \sum_{\boldsymbol{\xi}\in\Xi^q} \gamma_{\boldsymbol{\xi}}\, \boldsymbol{\xi}$, with $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ such that

$$\sum_{\boldsymbol{\xi}\in\Xi^q} \gamma_{\boldsymbol{\xi}} \sum_{\theta\in\Theta} \xi_\theta\, f_\theta(g(p, \mathcal{M}_{\epsilon,\boldsymbol{\xi}})) \geq \alpha\,(1-\delta) \max_{\mathbf{a}^*\in\mathcal{M}_{\boldsymbol{\xi}^*}} \sum_{\theta\in\Theta} \xi^*_\theta\, f_\theta(\mathbf{a}^*).$$

Let $\tilde{\boldsymbol{\xi}} \in \Xi^q$ be the empirical distribution of q i.i.d. samples from $\boldsymbol{\xi}^*$, where each θ has probability ξ^*_θ of being sampled. Therefore, the vector $\tilde{\boldsymbol{\xi}}$ is a random variable supported on q-uniform posteriors with expectation $\boldsymbol{\xi}^*$. Moreover, let $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ be a probability distribution such as, for each $\boldsymbol{\xi} \in \Xi^q$, $\gamma_{\boldsymbol{\xi}} := \Pr(\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi})$. For each $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ and $\boldsymbol{\xi} \in \Xi^q$, we denote by $\gamma^{(\theta,i)}_{\boldsymbol{\xi}}$ the conditional probability of having observed posterior $\boldsymbol{\xi}$, given that the posterior must assign probability i/q to state θ. Formally, for each $\boldsymbol{\xi} \in \Xi^q$, if $\xi_\theta = i/q$, we have

$$\gamma^{(\theta,i)}_{\boldsymbol{\xi}} = \frac{\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}':\xi'_\theta=i/q} \gamma_{\boldsymbol{\xi}'}},$$

and $\gamma^{(\theta,i)}_{\boldsymbol{\xi}} = 0$ otherwise. The random variable $\tilde{\boldsymbol{\xi}}^{(\theta,i)} \in \Xi^q$ is such that, for each $\boldsymbol{\xi} \in \Xi^q$, $\Pr(\tilde{\boldsymbol{\xi}}^{(\theta,i)} = \boldsymbol{\xi}) = \gamma^{(\theta,i)}_{\boldsymbol{\xi}}$. Finally, let $\tilde{\Xi}^q \subseteq \Xi^q$ be the set of posteriors such that $\boldsymbol{\xi} \in \tilde{\Xi}^q$ if and only if $|\sum_\theta \xi_\theta u^r_\theta(a) - \sum_\theta \xi^*_\theta u^r_\theta(a)| \leq \frac{\epsilon}{2}$ for each $r \in \mathcal{R}$ and $a \in \mathcal{A} - r$.

We state the following intermediate result.

Lemma 5.7. *Given $\boldsymbol{\xi}^* \in \Delta_\Theta$, for each $\theta \in \Theta$ and for each $i \in [q]$ s.t. $|i/q - \xi^*_\theta| \leq \epsilon/4$, it holds:*

$$\sum_{\boldsymbol{\xi}\in\tilde{\Xi}^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} \geq \left(1-\frac{\delta}{2}\right) \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}},$$

where $\boldsymbol{\gamma}$ is the distribution of q i.i.d. samples from $\boldsymbol{\xi}^*$.

Proof. Fix $\bar\theta \in \Theta$ and $i \in [q]$ with $|i/q - \xi^*_{\bar\theta}| \le \epsilon/4$. Then, for each $r \in \mathcal{R}$ and $a \in \mathcal{A}_r$, let $\tilde t^r_a := \sum_\theta \tilde\xi^{(\bar\theta,i)}_\theta u^r_\theta(a)$ and $t^r_a := \sum_\theta \xi^*_\theta u^r_\theta(a)$. First, we show that $|\mathbb{E}[\tilde t^r_a] - t^r_a| \le \epsilon/4$. Equivalently, $|\sum_\theta u^r_\theta(a)\left(\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta\right)| \le \epsilon/4$. Assume $i/q \ge \xi^*_{\bar\theta}$. Then,

$$\sum_\theta |\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta| = \frac{i}{q} - \xi^*_{\bar\theta} + \sum_{\theta\neq\bar\theta}\left(\xi^*_\theta - \frac{\xi^*_\theta}{\sum_{\theta'\neq\bar\theta}\xi^*_{\theta'}}\cdot\left(1-\frac{i}{q}\right)\right) \tag{5.7a}$$

$$\le \frac{\epsilon}{4} + 1 - \xi^*_{\bar\theta} - 1 + \frac{i}{q} \le \frac{\epsilon}{2}. \tag{5.7b}$$

Analogously, if $i/q \le \xi^*_{\bar\theta}$, we get that $\sum_\theta |\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta| \le \epsilon/2$. Furthermore, let $M_1 := \left\{\theta \in \Theta \mid \mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta \ge 0\right\}$, and $M_2 := \Theta \setminus M_1$. Then,

$$\sum_{\theta\in M_1}\left(\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta\right) = -\sum_{\theta\in M_2}\left(\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta\right) \le \frac{\epsilon}{4}, \tag{5.8a}$$

where the equality comes from $\sum_\theta \mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] = \sum_\theta \xi^*_\theta = 1$ and the inequality follows from Eq. 5.7. Then,

$$\begin{aligned}&\sum_\theta u^r_\theta(a)\left(\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta\right)\\ &\quad= \sum_{\theta\in M_1} u^r_\theta(a)\left(\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta\right) + \sum_{\theta\in M_2} u^r_\theta(a)\left(\mathbb{E}[\tilde\xi^{(\bar\theta,i)}_\theta] - \xi^*_\theta\right)\\ &\quad\le \frac{\epsilon}{4},\end{aligned}$$

where we use both

$$\sum_{\theta\in M_2} u^r_\theta(a)\left(\mathbb{E}\left[\tilde\xi^{(\bar\theta,i)}_\theta\right] - \xi^*_\theta\right) \le 0$$

and

$$\sum_{\theta\in M_1} u^r_\theta(a)\left(\mathbb{E}\left[\tilde\xi^{(\bar\theta,i)}_\theta\right] - \xi^*_\theta\right) \le \frac{\epsilon}{4}$$

by Equation (5.8). Analogously, it is possible to show that

$$\sum_\theta u^r_\theta(a)\left(\mathbb{E}\left[\tilde\xi^{(\bar\theta,i)}_\theta\right] - \xi^*_\theta\right) \ge -\frac{\epsilon}{4}.$$

Then, $\Pr(|t_a^r - \tilde{t}_a^r| \geq \epsilon/2) \leq \Pr(|\tilde{t}_a^r - \mathbb{E}[\tilde{t}_a^r]| \geq \epsilon/4)$. Moreover, by the Hoeffding's inequality, we have that, for each $r \in \mathcal{R}$ and $a \in \mathcal{A}_r$, it holds:

$$\Pr(|\tilde{t}_a^r - \mathbb{E}[\tilde{t}_a^r]| \geq \epsilon/4) \leq 2e^{-2q(\frac{\epsilon}{4})^2} = 2e^{\frac{-4\epsilon^2 \log(4\bar{n}\varrho/\delta)}{\epsilon^2}} = 2\left(\frac{\delta}{4\bar{n}\varrho}\right)^4 \leq \frac{\delta}{2\bar{n}\varrho}.$$

The union bound yields the following:

$$\begin{aligned}\Pr\left(\bigcap_{r\in\mathcal{R},a\in\mathcal{A}^r} |\tilde{t}_a^r - t_a^r| \leq \frac{\epsilon}{2}\right) \geq & 1 - \sum_{r,a} \Pr\left(|\tilde{t}_a^r - t_a^r| \geq \frac{\epsilon}{2}\right) \\ \geq & 1 - \sum_{r,a} \Pr\left(|\tilde{t}_a^r - \mathbb{E}[\tilde{t}_a^r]| \geq \frac{\epsilon}{4}\right) \\ = & 1 - \frac{\delta}{2}.\end{aligned}$$

By the definition of $\tilde{\Xi}^q$, this implies that $\Pr(\tilde{\boldsymbol{\xi}}^{(\bar{\theta},i)} \in \tilde{\Xi}^q) \geq 1-\delta/2$. Finally,

$$\begin{aligned}\sum_{\boldsymbol{\xi}\in\tilde{\Xi}^q:\xi_{\bar{\theta}}=i/q} & \gamma_{\boldsymbol{\xi}} \Pr\left(\tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) \Pr\left(\tilde{\boldsymbol{\xi}} \in \tilde{\Xi}^q \mid \tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) \\ & = \Pr\left(\tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) \Pr\left(\tilde{\boldsymbol{\xi}}^{(\bar{\theta},i)} \in \tilde{\Xi}^q\right) \\ & \geq \left(1 - \frac{\delta}{2}\right) \Pr\left(\tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) \\ & = \left(1 - \frac{\delta}{2}\right) \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_{\bar{\theta}}=i/q} \gamma_{\boldsymbol{\xi}}.\end{aligned}$$

This concludes the proof. □

Then, we state the following auxiliary lemma:

Lemma 5.8. *Given $\boldsymbol{\xi}^* \in \Delta_\Theta$, for each $\theta \in \Theta$, it holds:*

$$\sum_{i:|i/q - \boldsymbol{\xi}_\theta^*| \geq \epsilon/4} \ \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} \leq \frac{\delta}{2}\xi_\theta^*,$$

where $\boldsymbol{\gamma}$ is the distribution of q i.i.d. samples from $\boldsymbol{\xi}^*$.

Proof. The random variable $\tilde{\xi}_\theta$ is drawn from a binomial distribution. We consider three possible cases. If $\xi_\theta^* \geq 1/8$, then, by Hoeffding's inequality, it holds

$$\Pr\left(|\tilde{\xi}_\theta - \xi_\theta^*| \geq \frac{\epsilon}{4}\right) \leq 2e^{-2q(\epsilon/4)^2}$$

$$
\begin{aligned}
&= 2e^{-4\log(4\bar{n}\varrho/\delta)} \\
&\le \delta/16 \\
&\le \frac{\delta}{2}\xi^*_\theta
\end{aligned}
$$

If $\xi^*_\theta \le 1/8$, then, by Chernoff's bound, it holds

$$
\begin{aligned}
\Pr\left(\tilde{\xi}_\theta - \xi^*_\theta \ge \frac{\epsilon}{4}\right) &\le e^{-q(\epsilon/4)^2 \frac{1}{1-2\xi^*_\theta}\log(\frac{1-\xi^*_\theta}{\xi^*_\theta})} && \text{(5.9a)}\\
&\le e^{-2\log(4\bar{n}\varrho/\delta)\log(\frac{7}{8\xi^*_\theta})} && \text{(5.9b)}\\
&\le \left(\frac{8}{7}\,\xi^*_\theta\right)^{2\log(4/\delta)} = && \text{(5.9c)}\\
&= \left(\frac{1}{e}\,\frac{8}{7}\,e\,\xi^*_\theta\right)^{2\log(4/\delta)} && \text{(5.9d)}\\
&\le (e)^{-2\log(4/\delta)}\frac{8}{7}\,e\,\xi^*_\theta && \text{(5.9e)}\\
&\le \frac{\delta}{16}\frac{8}{7}\,e\,\xi^*_\theta && \text{(5.9f)}\\
&\le \frac{\delta}{4}\,\xi^*_\theta && \text{(5.9g)}
\end{aligned}
$$

where, to get from (5.9d) to (5.9e), we use $\frac{8}{7}\,e\,\xi^*_\theta \le 1$ and $2\,\log(4/\delta) \ge 1$. Moreover

$$
\begin{aligned}
\Pr\left(\tilde{\xi}_\theta - \xi^*_\theta \le -\frac{\epsilon}{4}\right) &\le e^{-q(\epsilon/4)^2\frac{1}{2(1-\xi^*_\theta)\xi^*_\theta}} && \text{(5.10a)}\\
&\le e^{-\frac{\log(4\bar{n}\varrho/\delta)}{\xi^*_\theta}} = \left(e^{\frac{1}{\xi^*_\theta}}\right)^{\log\left(\frac{\delta}{4}\right)} && \text{(5.10b)}\\
&\le \left(\frac{1}{\xi^*_\theta}\,e\right)^{\log\left(\frac{\delta}{4}\right)} && \text{(5.10c)}\\
&\le \left(\frac{1}{\xi^*_\theta}\right)^{-1} e^{\log\left(\frac{\delta}{4}\right)} && \text{(5.10d)}\\
&= \frac{\delta}{4}\,\xi^*_\theta. && \text{(5.10e)}
\end{aligned}
$$

where in (5.10c) we use $e^x \ge ex$ and in (5.10d) that $\log(\delta/4) < -1$. Then,

$$
\sum_{i:|i/q-\boldsymbol{\xi}^*_\theta|\ge\epsilon/4}\;\sum_{\boldsymbol{\xi}\in\Xi^q:p_\theta=i/q}\gamma_{\boldsymbol{\xi}} = \Pr\left(|\tilde{\xi}_\theta - \xi^*_\theta| \ge \frac{\epsilon}{4}\right) \le \frac{\delta}{2}\,\xi^*_\theta,
$$

which concludes the proof of the lemma. □

Now we can prove that, given a $\boldsymbol{\xi}^* \in \Delta_\Theta$, $\sum_{\boldsymbol{\xi}\in\tilde{\Xi}^q} \gamma_{\boldsymbol{\xi}}\, \xi_\theta \geq (1-\delta)\, \xi^*_\theta$ for each θ.

Lemma 5.9. *Given a $\boldsymbol{\xi}^* \in \Delta_\Theta$, for each $\theta \in \Theta$, it holds:*

$$\sum_{\boldsymbol{\xi}\in\tilde{\Xi}^q} \gamma_{\boldsymbol{\xi}}\, \xi_\theta \geq (1-\delta)\, \xi^*_\theta,$$

where $\boldsymbol{\gamma}$ is the distribution of q i.i.d. samples from $\boldsymbol{\xi}^*$.

Proof. We show the following:

$$\begin{aligned}
\sum_{\boldsymbol{\xi}\in\tilde{\Xi}^q} \gamma_{\boldsymbol{\xi}}\xi_\theta \geq & \sum_{i:|i/q-\boldsymbol{\xi}^*_\theta|\leq\epsilon/4} \frac{i}{q} \sum_{\boldsymbol{\xi}\in\tilde{\Xi}^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} && \text{(5.11a)}\\
\geq & \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} \frac{i}{q} \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \left(1-\frac{\delta}{2}\right)\gamma_{\boldsymbol{\xi}} && \text{(5.11b)}\\
= & \left(1-\frac{\delta}{2}\right) \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} \frac{i}{q} \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} && \text{(5.11c)}\\
\geq & \left(1-\frac{\delta}{2}\right) \left(\xi^*_\theta - \sum_{i:|i/q-\xi^*_\theta|\geq\epsilon/4} \frac{i}{q} \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}}\right) && \text{(5.11d)}\\
\geq & \left(1-\frac{\delta}{2}\right) \left(\xi^*_\theta - \sum_{i:|i/q-\xi^*_\theta|\geq\epsilon/4} \sum_{\boldsymbol{\xi}\in\Xi^q:\boldsymbol{\xi}_\theta=i/q} \gamma_{\boldsymbol{\xi}}\right) && \text{(5.11e)}\\
\geq & \left(1-\frac{\delta}{2}\right)^2 \xi^*_\theta && \text{(5.11f)}\\
\geq & (1-\delta)\, \xi^*_\theta. && \text{(5.11g)}
\end{aligned}$$

Equation (5.11a) holds since we are restricting the set of posteriors; Equation (5.11b) holds by Lemma 5.7; Equation (5.11e) holds since $i/q \leq 1$; and Equation (5.11f) holds by Lemma 5.8. This concludes the proof of the lemma. □

We need to prove that all the posteriors in $\tilde{\Xi}^q$ guarantee to the sender at least the same expected utility of $\boldsymbol{\xi}^*$. Formally, we prove that the ϵ-BR-set of each $\boldsymbol{\xi} \in \tilde{\Xi}^q$ contains the BR-set of $\boldsymbol{\xi}^*$. This is shown via the following lemma.

Lemma 5.10. *Given $\boldsymbol{\xi}^* \in \Delta_\Theta$, for each $\boldsymbol{\xi} \in \tilde{\Xi}^q$, it holds: $\mathcal{M}_{\boldsymbol{\xi}^*} \subseteq \mathcal{M}_{\epsilon,\boldsymbol{\xi}}$.*

Proof. Let $Z^1 = \mathcal{M}_{\epsilon,\boldsymbol{\xi}}$ and $Z^2 = \mathcal{M}_{\boldsymbol{\xi}^*}$. Suppose $a \in Z^2_r$. Then, for all $a' \in \mathcal{A}_r$,

$$\sum_\theta \xi_\theta\, u^r_\theta(a) \geq \sum_\theta p^*_\theta\, u^r_\theta(a) - \frac{\epsilon}{2} \geq \sum_\theta \xi^*_\theta\, u^r_\theta(a') - \frac{\epsilon}{2} \geq \sum_\theta \xi_\theta\, u^r_\theta(a') - \epsilon.$$

Thus, $a \in Z^1_r$, which proves the lemma. □

Finally, we prove that we can represent each posterior $\boldsymbol{\xi}^*$ as a convex combination of q-uniform posteriors with a small loss in the sender's expected utility. For $\boldsymbol{\xi} \in \Xi^q$ and $Z \in \mathcal{Z}$, let $g^* : \Delta_\Theta \times \mathcal{Z} \to [0,1]$ be a function such that $g^*(\boldsymbol{\xi}, Z) := \max_{\mathbf{a} \in Z} \sum_\theta \xi_\theta f_\theta(\mathbf{a})$. Given $\boldsymbol{\xi}^* \in \Delta_\Theta$, we are interested in bounding the difference in the sender's expected utility when $\boldsymbol{\xi}^*$ is approximated as a convex combination γ of q-uniform posteriors, the sender exploits an α-approximation of f, and she allows receivers for ϵ-persuasive best-responses. Formally,

Lemma 5.11. *Given a $\boldsymbol{\xi}^* \in \Delta_\Theta$, it holds:*

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \sum_\theta \xi_\theta\, f_\theta(g(\boldsymbol{\xi}, \mathcal{M}_{\epsilon,\boldsymbol{\xi}})) \geq \alpha(1-\delta) f_\theta(g^*(\boldsymbol{\xi}^*, \mathcal{M}\boldsymbol{\xi}^*)),$$

where $\boldsymbol{\gamma}$ is the distribution of q i.i.d. samples from $\boldsymbol{\xi}^*$.

Proof. We prove the following:

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \sum_\theta \xi_\theta\, f_\theta(g(\boldsymbol{\xi}, \mathcal{M}_{\epsilon,\boldsymbol{\xi}})) \tag{5.12a}$$

$$\geq \alpha \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \sum_\theta \xi_\theta\, f_\theta(g^*(\boldsymbol{\xi}, \mathcal{M}_{\epsilon,\boldsymbol{\xi}})) \tag{5.12b}$$

$$\geq \alpha \sum_{\boldsymbol{\xi} \in \tilde{\Xi}^q} \gamma_{\boldsymbol{\xi}} \sum_\theta \xi_\theta\, f_\theta(g^*(\boldsymbol{\xi}, \mathcal{M}_{\epsilon,\boldsymbol{\xi}})) \tag{5.12c}$$

$$\geq \alpha \sum_{\boldsymbol{\xi} \in \tilde{\Xi}^q} \gamma_{\boldsymbol{\xi}} \sum_\theta \xi_\theta\, f_\theta(g^*(\boldsymbol{\xi}^*, \mathcal{M}_{\epsilon,\boldsymbol{\xi}})) \tag{5.12d}$$

$$\geq \alpha \sum_{\boldsymbol{\xi} \in \tilde{\Xi}^q} \gamma_{\boldsymbol{\xi}} \sum_\theta \xi_\theta\, f_\theta(g^*(\boldsymbol{\xi}^*, \mathcal{M}\boldsymbol{\xi}^*)) \tag{5.12e}$$

$$\geq \alpha\,(1-\delta) \sum_\theta \xi^*_\theta\, f_\theta(g^*(\boldsymbol{\xi}^*, \mathcal{M}\boldsymbol{\xi}^*)). \tag{5.12f}$$

Equation (5.12a) is the relaxed sender's expected utility; Equation (5.12b) holds by Definition 5.5; Equation (5.12c) holds by restricting the set of posteriors; Equation (5.12d) holds by the optimality of g^*; Equation (5.12e) holds by Lemma 5.10; and Equation (5.12c) holds by Lemma 5.9. This concludes the proof. □

Therefore, we can safely restrict to posteriors in Ξ^q. Since there are $|\Xi^q| = \text{poly}\left(d^{\frac{\log(\bar{n}\varrho/\epsilon)}{\epsilon^2}}\right)$ posteriors, the following linear program (LP 5.13) has $O(|\Xi^q|)$ variables and constraints and finds an $\alpha(1-\delta)$-approximation of the optimal signaling scheme.

$$\max_{\boldsymbol{\gamma}\in\Delta_{\Xi^q}} \sum_{\boldsymbol{\xi}\in\Xi^q} \gamma_{\boldsymbol{\xi}} \sum_{\theta\in\Theta} \xi_\theta \, f_\theta(g(\boldsymbol{\xi}, \mathcal{M}_\epsilon(\boldsymbol{\xi}))) \tag{5.13a}$$

$$\text{s.t.} \sum_{\boldsymbol{\xi}\in\Xi^q} \gamma_{\boldsymbol{\xi}} \, \xi_\theta = \mu_\theta \qquad \forall \theta \in \Theta \tag{5.13b}$$

Given the distribution on the q-uniform posteriors $\boldsymbol{\gamma}$, we can construct a direct signaling scheme ϕ by setting:

$$\phi_\theta(\mathbf{a}) = \sum_{\boldsymbol{\xi}\in\Xi^q:\mathbf{a}=g(\boldsymbol{\xi},\mathcal{M}_\epsilon(\boldsymbol{\xi}))} \gamma_{\boldsymbol{\xi}}\xi_\theta, \text{ for each } \theta \in \Theta \text{ and } \mathbf{a} \in \mathcal{A}.$$

This shows that such a ϕ is $\alpha(1-\delta)$-approximate and ϵ-persuasive, which are precisely our desiderata, thus concluding the proof. □

CHAPTER 6

Persuading Voters in District-based Elections

In this chapter, we study Bayesian persuasion in district-based elections. Section 6.1 introduces district-based elections and semi-public signaling schemes. Section 6.2 provides a polynomial-time algorithm to compute an optimal private signaling scheme. In Section 6.3, we employ some relaxations of the voting functions to provide two bi-criteria approximations for public and semi-public signaling.

6.1 Model of Bayesian Persuasion in District-based Elections

In this section, we introduce district-based elections. Moreover, we introduce semi-public signaling schemes. In a district-bases election, there is a set of candidates $C := \{c_0, c_1\}$ and a set of voters $\mathcal{R}$ divided in a set D of districts. The set of voters of district $\mathsf{d} \in D$ is denoted with $\mathcal{R}^{\mathsf{d}}$. Each voter casts a vote for one of the two candidates. Once the voters expressed their preferences, the election process proceeds in two steps. For the sake of simplicity, we study the basic case in which both steps follow a *majority-voting* rule. The election works as follows.

1. For each $\mathsf{d} \in D$, the votes expressed by all $r \in \mathcal{R}^{\mathsf{d}}$ are locally aggregated, and the candidate with the majority of the votes is elected as the winner of the district.

2. The outcomes of all the districts are aggregated, and the candidate that is the winner in the majority of the districts is chosen as the winner of the district-based election.

We assume that the manipulator prefers c_0 to be the winner of the election. Let $\mathbf{c} \in \mathcal{C}$ be a tuple composed by the votes of all the voters, where $\mathcal{C} := C^{|\mathcal{R}|}$. Similarly, $\mathbf{c}^{\mathsf{d}}$ is the tuple of the votes of the voters in district d. The manipulator's utility $\mathcal{W} : \mathcal{C} \to \{0, 1\}$ is defined as the composition of a collection of functions $W^{\mathsf{d}} : C^{|\mathcal{R}^{\mathsf{d}}|} \to C$, each representing the majority voting run in district d, and the function $\overline{W} : C^{|D|} \to \{0, 1\}$, representing the majority voting that aggregates the outcomes of all the districts. We define $K_D := \lceil |D|/2 \rceil$ and, for each district d, $K_{\mathsf{d}} := \lceil |\mathcal{R}^{\mathsf{d}}|/2 \rceil$. Then, $\mathcal{W}$ is defined as $\mathcal{W}(\mathbf{c}) := \overline{W}(W^1(\mathbf{c}^1), \dots, W^{|D|}(\mathbf{c}^{|D|}))$, where $W^{\mathsf{d}}(\mathbf{c}^{\mathsf{d}})$ assumes value c_0 if at least K_{d} of the voters in district d vote for candidate c_0, and $\overline{W}$ assumes value 1 if and only if c_0 wins in at least K_D districts.

We introduce some relaxations for the majority-voting rules W^{d} and $\overline{W}$. In the first relaxation, we allow the number of votes that the target candidate c_0 needs to win in each district d to be smaller than K_{d}. We denote with W^{d}_{δ} the resulting majority voting rule. Formally, $W^{\mathsf{d}}_{\delta} : C^{|\mathcal{R}^{\mathsf{d}}|} \to C$ assumes value c_0 if at least $\lceil (1-\delta)\, K_{\mathsf{d}} \rceil$ voters in district d vote for c_0 and c_1 otherwise. The manipulator's utility function of this first relaxed problem, denoted with $\mathcal{W}_{\delta}$, is defined as $\mathcal{W}_{\delta} := \overline{W}(W^1_{\delta}(\mathbf{c}^1), \dots, W^{|D|}_{\delta}(\mathbf{c}^{|D|}))$. In the second stronger relaxation, we also allow the number of districts that the target candidate c_0 needs to control to win the election to be smaller than K_D. We denote with $\overline{W}_{\delta}$ the resulting majority voting rule aggregating the outcomes of the districts. Formally, $\overline{W}_{\delta} : C^{|D|} \to \{0, 1\}$ assumes value 1 when c_0 wins in at least $\lceil (1-\delta)\, K_D \rceil$ districts. The manipulator's utility function of this second relaxed problem, denoted with $\mathcal{W}_{\delta\delta}$, is defined as $\mathcal{W}_{\delta\delta}(\mathbf{c}) := \overline{W}_{\delta}(W^1_{\delta}(\mathbf{c}^1), \dots, W^{|D|}_{\delta}(\mathbf{c}^{|D|}))$. Finally, we introduce a novel form of communication that suits our election model, where the sender has a communication channel toward each district d, and all the receivers in the same district receive the same signal, *i.e.*, $s_r = s'_r$ for all $r, r' \in \mathcal{R}^{\mathsf{d}}$. We call these signaling schemes semi-public.

As in the previus chapters, we assume no inter-agent externalities and we denote with $u^r_{\theta}(c)$ the utility of receiver r in state θ when voting for candidate c. Moreover, let $u^r_{\theta} := u^r_{\theta}(c_0) - u^r_{\theta}(c_1)$. A revelation-principle style argument shows that there always exists a signaling scheme that is *direct* and *persuasive*. In particular, the incentive constraints of a direct signaling scheme associated with a receiver r in a district d are:

- $\sum_{\theta,\mathbf{c}\in\mathcal{C}:c_r=c}\phi_\theta(\mathbf{c})(u^r_\theta(c)-u^r_\theta(c'))\geq 0\ \forall c,c'\in C$ (private signaling);
- $\sum_\theta\phi_\theta(\mathbf{c})(u^r_\theta(c_r)-u^r_\theta(c'))\geq 0\ \forall\mathbf{c}\in\mathcal{C},c'\in C$ (public signaling);
- $\sum_{\theta,\mathbf{c}\in\mathcal{C}:\mathbf{c}^{\mathsf{d}}=\bar{\mathbf{c}}}\phi_\theta(\mathbf{c})(u^r_\theta(\bar{c}_r)-u^r_\theta(c'))\ \geq\ 0\ \forall\bar{\mathbf{c}}\ \in\ C^{|\mathcal{R}^{\mathsf{d}}|},c'\ \in\ C$ (semi-public signaling).

Similarly, a direct signaling scheme is ϵ-persuasive if the incentive constraints are violated by at most ϵ.

Finally, we state the optimization problems we study in this chapter. PRIVATE-DBE is the problem of designing a private signaling scheme maximizing the probability of having candidate c_0 elected in district-based elections. PUBLIC-DBE and SEMIPUBLIC-DBE refer to the same problem with public and semi-public signaling, respectively.

6.2 Private Persuasion in District-based Elections

In this section, we show that an optimal private signaling scheme for district-based elections can be found in polynomial time. Our result is built upon the results in Chapter 4 on k-voting. Let $\sigma_{\mathsf{d},\theta}$ be the probability with which K_{d} voters vote for c_0 in district d when the state of nature is θ. Similarly, let α_θ be the probability that c_0 wins in at least K_D districts with state of nature θ. Finally, recall that given a direct private signaling scheme ϕ, we denote with $\phi_{r,\theta}(c)=\sum_{\mathbf{c}:c_r=c}\phi_\theta(\mathbf{c})$ the marginal probabilities of ϕ whereby c is recommended to r with state of nature θ. We can compute an optimal private signaling scheme by LP 6.1.

Theorem 6.1. *LP 6.1 computes an optimal solution of PRIVATE-DBE in polynomial time.*

Proof. LP 6.1 has a polynomial number of variables and constraints and, therefore, it can be solved in polynomial time. Thus, we just need to prove that LP 6.1 actually computes an optimal solution to PRIVATE-DBE. First, we remark that all the marginal probabilities $\phi_{r,\theta}(c_0)$ of the signaling scheme ϕ must satisfy the incentive Constraints (6.1b). $\sigma_{\mathsf{d},\theta}$ represents the probability of having at least K_{d} votes in district d, given state of nature θ. We need to show $\sigma_{\mathsf{d},\theta}$ is computed correctly given the other variables of LP 6.1. In particular, for every state of nature θ, the maximum probability with which at least K_{d} of the receivers in $\mathcal{R}^{\mathsf{d}}$ vote for c_0 given marginals probabilities $\phi_{r,\theta}(c_0)$ is:

$$\sigma_{\mathsf{d},\theta} = \min\left\{ \min_{m\in\{0,\dots,K_{\mathsf{d}}-1\}} \frac{1}{K_{\mathsf{d}}-m} v_{\theta,m};\ 1 \right\},$$

where $v_{\theta,m}$ is the sum of the lowest $|\mathcal{R}^{\mathsf{d}}|-m$ elements in the set $\{\phi_{r,\theta}(c_0)\}_{r\in\mathcal{R}^{\mathsf{d}}}$; further details are provided by Arieli and Babichenko (2019). This definition is encoded by Constraints (6.1f). Constraints (6.1g) and (6.1h) ensure the values $v_{\theta,m}$ are well defined and derived from the dual of a simple LP of this kind:

$$\begin{aligned} \min_{\mathbf{y}\in\mathbb{R}^n} & \mathbf{x}^\top \mathbf{y} \\ & \mathbf{1}^\top \mathbf{y} = w \\ & \mathbf{0} \le \mathbf{y} \le \mathbf{1} \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector from which we want to extract the sum of the smallest w entries. Finally, we prove that α_θ is computed correctly. The computation of the maximum probability α_θ with which at least K_D districts elect c_0 given probabilities $\sigma_{\mathsf{d},\theta}$ is similar to the computation of $\sigma_{\mathsf{d},\theta}$ given $\phi_{r,\theta}(c_0)$. For a similar argument as above, Constraints (6.1c), (6.1d), and (6.1e) correctly compute α_θ aggregating the marginal probabilities $\{\sigma_{\mathsf{d},\theta}\}_{\mathsf{d}\in D,\theta\in\Theta}$. Objective (6.1a) is given by the sum over all $\theta \in \Theta$ of the prior of state θ, multiplied by α_θ. Thus, by definition of α_θ, we are maximizing the probability of having c_0 locally elected in more than K_D districts.

Finally, we prove how to construct a signaling scheme ϕ' with the same objective function of LP 6.1. In particular, we find marginal signaling schemes ϕ'_r such that the incentive constraints relative to c_0 and c_1 are satisfied and $\phi'_{r,\theta}(c_0) \ge \phi_{r,\theta}(c_0)$ for all r and θ. Since we do not introduce the incentive constraint relative to action c_1, they could not be satisfied by ϕ. However, from the optimal marginal probabilities $\phi_{r,\theta}(c_0)$, it is straightforward to compute the marginal probabilities $\{\, \phi'_{r,\theta}(c_0),\, \phi'_{r,\theta}(c_1)\,\}_{r\in\mathcal{R},\theta\in\Theta}$. For each state of nature θ, let $\phi'_{r,\theta}(c_0) = 1$ if $u^r_\theta \ge 0$ and $\phi'_{r,\theta}(c_0) = \phi_{r,\theta}(c_0)$ otherwise. Then, $\phi'_{r,\theta}(c_1) = 1-\phi'_{r,\theta}(c_0)$. The marginal signaling scheme ϕ'_r is persuasive as c_1 is recommended only when it is the optimal action, while $\phi'_{r,\theta}(c_0) \ge \phi_{r,\theta}(c_0)$ if and only if $u^r_\theta \ge 0$. Formally, $\sum_{\theta\in\Theta} \mu_\theta\, \phi'_{r,\theta}(c_0)\, u^r_\theta \ge \sum_{\theta\in\Theta} \mu_\theta\, \phi_{r,\theta}(c_0)\, u^r_\theta \ge 0$ by constraints (6.1b). Finally, we can aggregate the marginal probabilities of the signaling scheme by using the same approach proposed by Arieli and Babichenko (2019). □

$$\max_{\substack{\boldsymbol{\alpha}\in[0,1]^{|\Theta|},\,\sigma\in[0,1]^{|D|\times|\Theta|}\\ \mathbf{i},\mathbf{l}\in\mathbb{R}^{|\Theta|\times K_D},\,\mathbf{o}\in\mathbb{R}^{|D|\times|\Theta|\times K_D}\\ t_{\mathsf{d},\theta,m},\,v_{\mathsf{d},\theta,m}\in\mathbb{R}\ \forall \mathsf{d}\in D,\theta\in\Theta,m\in\{1,\dots,K_{\mathsf{d}}\}\\ z_{\mathsf{d},\theta,r,m}\in\mathbb{R}\ \forall \mathsf{d}\in D,\theta\in\Theta,r\in\mathcal{R},m\in\{1,\dots,K_{\mathsf{d}}\}\\ \phi_{r,\cdot}(c_0)\in[0,1]^{|\Theta|}\ \forall r\in\mathcal{R}}} \sum_{\theta\in\Theta}\mu_\theta\,\alpha_\theta \tag{6.1a}$$

$$\text{s.t.} \sum_{\theta\in\Theta}\mu_\theta\,\phi_{r,\theta}(c_0)\,u_r(\theta)\geq 0 \qquad \forall r\in\mathcal{R} \tag{6.1b}$$

$$\alpha_\theta \leq \frac{1}{K_D-m} i_{\theta,m} \qquad \forall\theta\in\Theta,\forall m\in\{0,\dots,K_D-1\} \tag{6.1c}$$

$$i_{\theta,m}\leq (|D|-m)l_{\theta,m}+\sum_{\mathsf{d}\in D} o_{\mathsf{d},\theta,m} \qquad \forall\theta\in\Theta,\forall m\in\{0,\dots,K_D-1\} \tag{6.1d}$$

$$\sigma_{\mathsf{d},\theta}\geq l_{\theta,m}+o_{\mathsf{d},\theta,m} \qquad \forall \mathsf{d}\in D,\forall\theta\in\Theta,\forall m\in\{0,\dots,K_D-1\} \tag{6.1e}$$

$$\sigma_{\mathsf{d},\theta}\leq \frac{1}{K_{\mathsf{d}}-m} v_{d,\theta,m} \qquad \forall \mathsf{d}\in D,\forall\theta\in\Theta,\forall m\in\{0,\dots,K_d-1\} \tag{6.1f}$$

$$v_{\mathsf{d},\theta,m}\leq (|R^{\mathsf{d}}|-m)t_{d,\theta,m}+\sum_{r\in R^d} z_{\mathsf{d},\theta,r,m} \qquad \forall d\in D,\forall\theta\in\Theta,\forall m\in\{0,\dots,K_{\mathsf{d}}-1\} \tag{6.1g}$$

$$\phi_{r,\theta}(c_0)\geq t_{\mathsf{d},\theta,m}+z_{d,\theta,r,m} \qquad \forall \mathsf{d}\in D,\forall r\in\mathcal{R}^{\mathsf{d}},\forall\theta\in\Theta,\forall m\in\{0,\dots,K_{\mathsf{d}}-1\} \tag{6.1h}$$

6.3 Public and Semi-public Persuasion in District-based Elections

We turn our attention to the design of optimal public and semi-public signaling schemes. There is a sharp distinction between the nature of these problems and that one of private signaling. Indeed, in addition to being inefficient w.r.t. private signals (see Proposition 4.1), optimal (semi-)public signaling schemes are also inapproximable. The hardness follows from the results in Chapter 4. Specifically, we proved that it is NP-hard to approximate the optimal public signaling scheme within any factor in elections with majority voting. The extension of this hardness result to public and semi-public signaling in district-base elections is direct as a district-based election reduces to majority voting when there is only a single district. Thus, we focus on possible relaxations that make the problem computationally tractable.

Motivated by the fact that voters are somewhat biased to follow the sender's recommendations, several works relax the incentive constraints allowing the receivers to vote for the target candidate even if other candidates give them a slightly better expected utility (ϵ-persuasiveness). In Chapter 5, we prove that even allowing this relaxation the problem of designing an approximate public signaling scheme remains intractable with majority voting. Therefore, we focus on other different relaxations. In particular, Cheng et al. (2015) employ two forms of relaxation, adopting ϵ-persuasiveness and lowering the number of votes needed to win the election by an arbitrary small constant factor. With these two relaxations, they prove that an approximate public signaling scheme with majority voting can be computed efficiently. We prove that, adapting these two relaxations to our settings, both PUBLIC-DBE and SEMIPUBLIC-DBE admit a multi-criteria PTAS. As a preliminary step, we prove some results on the relation between the notion of stability and the design of approximately optimal signaling schemes that are of general interest in Bayesian persuasion beyond elections.

6.3.1 Comparative Stability and Public Signaling Schemes

We refer to the notion of stability of a function introduced by Xu (2020). In particular, a function is said stable if, for every action profile, the introduction of small perturbations leads to small changes in the value of the function. Here, we extend the notion of stability to pairs of functions, and we call it comparative. Our extension is such that comparative stability corresponds to (simple) stability in the degenerate case in which the two functions of the pair are the same. Furthermore, if function g satisfies the comparative stability property w.r.t. function h, we also say that g is β-stable compared with h. Initially, we introduce the notion of perturbation by the concept of α-noisy distribution.

Definition 6.1. *Let* $\boldsymbol{c} \in \mathcal{C}$ *be an action profile and* $\boldsymbol{y}$ *be a probability distribution supported on* $\Delta_{\mathcal{C}}$. *For any* $\alpha \in (0, 1]$, *we say that* $\boldsymbol{y}$ *is an* α*-noisy distribution around* $\boldsymbol{c}$ *if for all* $i \in \{1, \dots, |\mathcal{R}|\} : \Pr_{\tilde{\boldsymbol{y}} \sim \boldsymbol{y}}[\tilde{y}_i \neq c_i] \leq \alpha$.

Hence, an α-noisy distribution bounds the *marginal probability* of any single element of $\{1, \dots, \bar{n}\}$ to be corrupted. However, no assumption is made on how the corruptions of the elements correlate with each other. Now, we define our notion of comparative stability.

Definition 6.2. *Given two functions $g, h : \mathcal{C} \to [0,1]$ and a real number $\beta \geq 0$, we say that g is β-stable compared with h if and only if the following holds for all action profiles $\boldsymbol{c} \in \mathcal{C}$, $\alpha \in (0,1]$, and α-noisy distributions y around $\boldsymbol{c}$:*

$$\mathop{\mathbb{E}}_{\tilde{\boldsymbol{y}} \sim \boldsymbol{y}} [g(\tilde{\boldsymbol{y}})] \geq h(\boldsymbol{c})(1 - \alpha\beta).$$

Intuitively, if g satisfies the comparative stability property w.r.t. h, then, for every action profile, the value of h in that action profile is close to the value of g in the corresponding perturbed action profile.

We exploit the notion of comparative stability to design an efficient algorithm that computes approximate public signaling schemes. More precisely, we study a generic multi-agent Bayesian persuasion problem, where the sender faces a set of receivers $\mathcal{R}$, and each receiver needs to choose an action between a couple of alternatives. Let g, h be two sets of arbitrary functions depending on the state of nature θ and denoted with $g_\theta : \mathcal{C} \to [0,1]$ and $h_\theta : \mathcal{C} \to [0,1]$, respectively. According to Definition 6.2, we say that g is β-stable compared with h if g_θ is β-stable with respect to h_θ for all the states of nature $\theta \in \Theta$.

For the sake of clarity, in the following, we use indirect signaling schemes, and we express a signaling scheme as a weighted set of posteriors to which the receivers respond at best. Now, we describe the optimal behavior of the receivers. This is a extension to multiple receivers of Definition 3.1 when the receivers have only two actions.

Definition 6.3 (Receivers' behavior with persuasiveness)**.** *Given a set of functions $\{f_\theta\}_{\theta \in \Theta}$ such that $f_\theta : \mathcal{C} \to [0,1]$, the receivers' optimal behavior $\mathbf{b}_{\boldsymbol{\xi}} \in \mathcal{C}$ with persuasiveness given posterior $\boldsymbol{\xi} \in \Xi$ is as follows. Let:*

- $A = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u^r_\theta > 0\}$ *the set of receivers whose unique best response is action c_0,*
- $B = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u^r_\theta < 0\}$ *the set of receivers whose unique best response is action c_1,*
- $E = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u^r_\theta = 0\}$ *the set of receivers who are indifferent between action c_0 and c_1.*

Then, we have:

$$\mathbf{b}_{\boldsymbol{\xi}} = arg\,max_{\boldsymbol{c} \in \mathcal{C}: c_r = c_0 \forall r \in A, \, c_r = c_1 \forall r \in B} \sum_\theta p_\theta \, f_\theta(\boldsymbol{c}).$$

Notice that the previous definition is a characterization of the vector of best responses for the specific voting setting.

Similarly, we define the notion of ϵ-best response.

Definition 6.4 (Receivers' behavior with ϵ-persuasiveness)**.** *Given a set of functions $\{f_\theta\}_{\theta\in\Theta}$ such that $f_\theta : \mathcal{C} \to [0,1]$, the receivers' optimal behavior $\mathbf{b}_{\epsilon,\boldsymbol{\xi}} \in \mathcal{C}$ with ϵ-persuasiveness given posterior $\boldsymbol{\xi} \in \Xi$ is as follows. Let:*

- $A_\epsilon = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u_\theta^r > \epsilon\}$ *the set of receivers whose unique best response is action c_0,*
- $B_\epsilon = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u_\theta^r < -\epsilon\}$ *the set of receivers whose unique best response is action c_1,*
- $E_\epsilon = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u_\theta^r \in [-\epsilon, \epsilon]\}$ *the set of receivers who are indifferent between action c_0 and c_1.*

Then, we have:

$$\mathbf{b}_{\epsilon,\boldsymbol{\xi}} = arg\,max_{\boldsymbol{c}\in\mathcal{C}:c_r=c_0\forall r\in A_\epsilon\; c_r=c_1\forall r\in B_\epsilon} \sum_\theta \xi_\theta \, f_\theta(\boldsymbol{c}).$$

Now, we show that computing a direct public signaling scheme is equivalent to derive a Bayes plausible distribution of posteriors $\boldsymbol{\gamma} \in \Delta_\Xi$ that maximizes the sender's utility. Let $supp(\boldsymbol{\gamma})$ denote the set of posteriors induced with strictly positive probability. Similarly, let $supp(\phi)$ denote the set of posteriors induced by ϕ with strictly positive probability. Finding a public signaling scheme is equivalent to finding a probability distribution $\boldsymbol{\gamma} \in \Delta_\Xi$ on the set of posteriors Ξ such that $\sum_{\xi\in supp(\boldsymbol{\gamma})} \gamma_\xi \, \xi_\theta = \mu_\theta$ for every $\theta \in \Theta$. Given a well-defined distribution over posteriors $\boldsymbol{\gamma}$, we can recover a direct signaling schemes ϕ that induces such a probability distribution by setting $\phi_\theta(\mathbf{c}) = \sum_{\boldsymbol{\xi}\in\mathrm{supp}(\boldsymbol{\gamma}):\mathbf{c}=\mathbf{b}_{\boldsymbol{\xi}}} \gamma_{\boldsymbol{\xi}} \, \xi_\theta$. For this reason, in the following, we represent signaling schemes as probability distributions on the posteriors. We introduce some further notation. For every $\boldsymbol{\xi} \in \Xi$ and set of functions $f = \{f_\theta\}_{\theta\in\Theta}$, we define the sender's expected utility with persuasiveness as $f(\boldsymbol{\xi}) = \sum_\theta \boldsymbol{\xi}_\theta \, f_\theta(\mathbf{b}_{\boldsymbol{\xi}})$, and with ϵ-persuasiveness as $f_\epsilon(\boldsymbol{\xi}) = \sum_\theta \xi_\theta \, f_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}})$.

Our first result shows that we can decompose each posterior in a convex combination $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ of q-uniform posteriors (with q constant), such that $\sum_{\boldsymbol{\xi}\in\Xi^q} \gamma_p \, g_\epsilon(\boldsymbol{\xi})$ closely approximates $h(\boldsymbol{\xi}^*)$. This is a generalization of the result by Xu (2020) to state-dependent utility functions (and couples of functions), and it is crucial to prove the following results.

Lemma 6.1. *Let $\beta, \epsilon > 0, \eta \in (0,1]$ and set $q = 32 \log\left(\frac{4}{\eta \min\{1; 1/\beta\}}\right) / \epsilon^2$. Then, given a posterior $\boldsymbol{\xi}^* \in \Xi$ and two sets of functions g, h with g β-stable compared with h, there exists a $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ with $\sum_{p\in\Xi^q} \gamma_{\boldsymbol{\xi}} \, \boldsymbol{\xi} = \boldsymbol{\xi}^*$*

and

$$\sum_{p\in\Xi^q}\gamma_{\boldsymbol{\xi}}\sum_{\theta}\xi_\theta\, g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}})\geq(1-\eta)\sum_{\theta}\xi^*_\theta\, h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}).\tag{6.2}$$

Proof. Let $\tilde{\boldsymbol{\xi}}\in\Xi^q$ be the empirical distribution of q i.i.d. samples drawn from $\boldsymbol{\xi}^*$, where each θ has probability ξ^*_θ of being sampled. Therefore, the vector $\tilde{\boldsymbol{\xi}}$ is a random variable supported on q-uniform posteriors with expectation $\boldsymbol{\xi}^*$. Moreover, let $\boldsymbol{\gamma}\in\Delta_{\Xi^q}$ be a probability distribution such as, for every $\boldsymbol{\xi}\in\Xi^q$, it holds $\gamma_{\boldsymbol{\xi}}=\Pr(\tilde{\boldsymbol{\xi}}=\boldsymbol{\xi})$. It is easy to see that $\boldsymbol{\xi}^*=\sum_{\boldsymbol{\xi}\in\Xi^q}\gamma_{\boldsymbol{\xi}}\boldsymbol{\xi}$. We need to prove that Equation (6.2) holds. For every $\boldsymbol{\xi}\in\Xi^q$, we define with $\gamma^{(\theta,i)}_{\boldsymbol{\xi}}$ the conditional probability of having observed posterior $\boldsymbol{\xi}$ given that the posterior assigns a probability of i/q to state θ. Formally, for every $\boldsymbol{\xi}\in\Xi^q$, we have:

$$\gamma^{(\theta,i)}_{\boldsymbol{\xi}}=\begin{cases}\dfrac{\gamma_{\boldsymbol{\xi}}}{\sum\limits_{\boldsymbol{\xi}'\in\Xi^q:p'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}} & \text{if } \xi_\theta=i/q\\[2em] 0 & \text{otherwise}\end{cases}.$$

Then, the random variable $\tilde{\boldsymbol{\xi}}^{(\theta,i)}\in\Xi^q$ is such that, for every $\boldsymbol{\xi}\in\Xi^q$, it holds $\Pr(\tilde{\boldsymbol{\xi}}^{(\theta,i)}=\boldsymbol{\xi})=\gamma^{(\theta,i)}_{\boldsymbol{\xi}}$. For each $r\in\mathcal{R}$, we define $\mathcal{P}^r\subseteq\Xi^q$ as the set of posteriors that do not change the expected utility of r by more than ϵ with respect to $\boldsymbol{\xi}^*$. Formally, $\boldsymbol{\xi}\in\mathcal{P}^r$ if and only if $|\sum_\theta\xi_\theta\, u^r_\theta-\sum_\theta\xi^*_\theta u^r_\theta|\leq\epsilon$. Finally, let $\alpha=\eta\,\min\{1;\,1/\beta\}$.

To complete the proof, we introduce the following three lemmas. First, given a probability distribution $\boldsymbol{\xi}^*$ and a state of nature $\theta\in\Theta$, the following lemma bounds the maximum probability mass that $\boldsymbol{\gamma}$ assigns to posteriors $\boldsymbol{\xi}\in\Xi^q$ in which the probability assigned to state of nature θ deviates from the one prescribed by $\boldsymbol{\xi}^*$ by at least $\epsilon/4$.

Lemma 6.2. *Given $\boldsymbol{\xi}^*\in\Xi$, for each $\theta\in\Theta$, it holds:*

$$\sum_{i:|i/q-\xi^*_\theta|\geq\epsilon/4}\ \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q}\gamma_{\boldsymbol{\xi}}\leq\frac{\alpha}{2}\,\xi^*_\theta,$$

where $\boldsymbol{\gamma}$ is the probability distribution of q i.i.d samples drawn from $\boldsymbol{\xi}^*$.

Proof. We observe that the random variable $\tilde{\xi}_\theta$ is drawn from a binomial probability distribution. We consider two possible cases. If $\xi^*_\theta\geq 1/8$, then by Hoeffding's inequality we can write the following:

$$\Pr\left(|\tilde{\xi}_\theta-\xi^*_\theta|\geq\frac{\epsilon}{4}\right)\leq 2\,e^{-2\,q\,(\epsilon/4)^2}=\tag{6.3a}$$

$$= 2\, e^{-4\log(4/\alpha)} \leq \tag{6.3b}$$

$$\leq \alpha/16 \leq \frac{\alpha}{2}\, \xi^*_\theta. \tag{6.3c}$$

Instead, if $\xi^*_\theta \leq 1/8$, then by Chernoff's bound we can write the following:

$$\Pr\left(\tilde{\xi}_\theta - \xi^*_\theta \geq \frac{\epsilon}{4}\right) \leq e^{-q(\epsilon/4)^2 \frac{1}{1-2p^*_\theta} \log(\frac{1-\xi^*_\theta}{\xi^*_\theta})} \leq \tag{6.4a}$$

$$\leq e^{-2\log(4/\alpha)\log(\frac{7}{8\xi^*_\theta})} = \tag{6.4b}$$

$$= (\frac{8}{7}\,\xi^*_\theta)^{2\log(4/\alpha)} = \tag{6.4c}$$

$$= \left(\frac{1}{e}\,\frac{8}{7}\, e\, \xi^*_\theta\right)^{2\log(4/\alpha)} \leq \tag{6.4d}$$

$$\leq (e)^{-2\log(4/\alpha)}\frac{8}{7}\, e\, \xi^*_\theta \leq \tag{6.4e}$$

$$\leq \frac{\alpha}{16}\frac{8}{7}\, e\, \xi^*_\theta \leq \tag{6.4f}$$

$$\leq \frac{\alpha}{4}\, \xi^*_\theta, \tag{6.4g}$$

Moreover, we can write:

$$\Pr\left(\tilde{\xi}_\theta - \xi^*_\theta \leq -\frac{\epsilon}{4}\right) \leq e^{-q(\epsilon/4)^2 \frac{1}{2(1-\xi^*_\theta)\xi^*_\theta}} = \tag{6.5a}$$

$$= e^{-\frac{\log(4/\alpha)}{\xi^*_\theta}} = \tag{6.5b}$$

$$= \left(e^{\frac{1}{\xi^*_\theta}}\right)^{\log\left(\frac{\alpha}{4}\right)} \leq \tag{6.5c}$$

$$\leq \left(\frac{1}{\xi^*_\theta} e\right)^{\log\left(\frac{\alpha}{4}\right)} \leq \tag{6.5d}$$

$$\leq \left(\frac{1}{\xi^*_\theta}\right)^{-1} e^{\log\left(\frac{\alpha}{4}\right)} = \tag{6.5e}$$

$$= \frac{\alpha}{4}\, \xi^*_\theta, \tag{6.5f}$$

where in Equations (6.5d) and (6.5e) we use that $e^x \geq e\, x$ and $\log(\alpha/4) < -1$ as $\alpha \in (0,1]$. Hence, we obtain the following inequality:

$$\sum_{i:|i/q-\xi^*_\theta|>\epsilon/4} \;\sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} = \Pr\left(|\tilde{\xi}_\theta - \xi^*_\theta| > \frac{\epsilon}{4}\right) \leq \frac{\alpha}{2}\xi^*_\theta,$$

which concludes the proof. □

The second lemma we introduce proves that, when ξ_θ is close to ξ^*_θ, then the utility of every receiver is close to the utility in $\boldsymbol{\xi}^*$ with high probability.

Lemma 6.3. *Given $\boldsymbol{\xi}^* \in \Xi$, for each receiver $r \in \mathcal{R}$, each state $\theta \in \Theta$ and each $i : |i/q - \xi^*_\theta| \leq \epsilon/4$, it holds:*

$$\sum_{\boldsymbol{\xi} \in \mathcal{P}^r : \xi_\theta = i/q} \gamma_{\boldsymbol{\xi}} \geq \left(1 - \frac{\alpha}{2}\right) \sum_{\boldsymbol{\xi} \in \Xi^q : \xi_\theta = i/q} \gamma_{\boldsymbol{\xi}},$$

where $\boldsymbol{\gamma}$ is the distribution of q i.i.d samples from $\boldsymbol{\xi}^*$.

Proof. Fix $\bar{\theta} \in \Theta$, $r \in \mathcal{R}$ and i with $|i/q - \xi^*_{\bar{\theta}}| \leq \epsilon/4$. Then, let $\tilde{t} = \sum_\theta \tilde{\xi}^{(\bar{\theta},i)}_\theta u^r_\theta$ and $t = \sum_\theta p^*_\theta u^r_\theta$, where the notation $\tilde{\xi}^{(\bar{\theta},i)}_\theta$ is employed to denote the value of ξ_θ given that the random variable $\tilde{\boldsymbol{\xi}}^{(\bar{\theta},i)} \in \Xi^q$ assumes value $\boldsymbol{\xi}$. First, we show that $|\, \mathbb{E}[\,\tilde{t}\,] - t\,| \leq \epsilon/2$. This is equivalent to prove the following:

$$|\sum_\theta u^r_\theta \left(\mathbb{E}[\tilde{\xi}^{(\bar{\theta},i)}_\theta] - \xi^*_\theta\right)| \leq \sum_\theta |\mathbb{E}[\tilde{\xi}^{(\bar{\theta},i)}_\theta] - \xi^*_\theta| \leq \epsilon/2.$$

Assume $i/q \geq \xi^*_{\bar{\theta}}$, then,

$$\begin{aligned}
&\sum_\theta |\mathbb{E}[\,\tilde{\xi}^{(\bar{\theta},i)}_\theta\,] - \xi^*_\theta| = \\
&= \frac{i}{q} - \xi^*_{\bar{\theta}} + \sum_{\theta \neq \bar{\theta}} \left(\xi^*_\theta - \frac{\xi^*_\theta}{\sum_{\theta' \neq \bar{\theta}} \xi^*_{\theta'}} \left(1 - \frac{i}{q}\right)\right) \leq \\
&\leq \frac{\epsilon}{4} + 1 - \xi^*_{\bar{\theta}} - 1 + \frac{i}{q} \leq \frac{\epsilon}{2}.
\end{aligned}$$

Analogously, if $i/q \leq \xi^*_{\bar{\theta}}$, we get that $\sum_\theta |\, \mathbb{E}[\,\tilde{\xi}^{(\bar{\theta},i)}_\theta\,] - \xi^*_\theta\,| \leq \frac{\epsilon}{2}$. Now, we can exploit the fact that $|\, \mathbb{E}[\,\tilde{t}\,] - t\,| \leq \epsilon/2$ to show that: $\Pr(|\, t - \tilde{t}\,| \geq \epsilon) \leq \Pr(\,|\,\tilde{t} - \mathbb{E}[\,\tilde{t}\,]\,| \geq \epsilon/2)$ by the triangular inequality. Then, we use the Hoeffding's inequality to bound the last term:

$$\Pr(\,|\,\tilde{t} - \mathbb{E}[\,\tilde{t}\,]\,| \geq \epsilon/2) \leq 2e^{-\frac{2q}{4}(\frac{\epsilon}{2})^2} \leq 2e^{-\log(4/\alpha)} = \frac{\alpha}{2}$$

By definition of $\mathcal{P}^r$, this implies that $\Pr(\tilde{\boldsymbol{\xi}}^{(\bar{\theta},i)} \in \mathcal{P}^r) \geq 1 - \alpha/2$. Finally,

$$\sum_{\boldsymbol{\xi} \in \mathcal{P}^r : \xi_{\bar{\theta}} = i/q} \gamma_{\boldsymbol{\xi}} = \Pr\left(\tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) \Pr\left(\tilde{\boldsymbol{\xi}} \in \mathcal{P}^r \mid \tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) =$$

$$
\begin{aligned}
&= \Pr\left(\tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) \Pr\left(\tilde{\boldsymbol{\xi}}^{(\bar{\theta},i)} \in \mathcal{P}^r\right) \geq \\
&\geq \left(1 - \frac{\alpha}{2}\right) \Pr\left(\tilde{\xi}_{\bar{\theta}} = \frac{i}{q}\right) = \\
&= \left(1 - \frac{\alpha}{2}\right) \sum_{\boldsymbol{\xi} \in \Xi^q : \xi_{\bar{\theta}} = i/q} \gamma_{\boldsymbol{\xi}}.
\end{aligned}
$$

□

Before introducing the last lemma, we need some further notation. Following Definition 6.3, given a posterior, we partition the receivers in three sets, depending on their possible best-responses. We define the partition on the set of receivers induced by $\boldsymbol{\xi}^* \in \Xi$ as follows:

- $A = \{r \in \mathcal{R} : \sum_\theta \xi^*_\theta \, u^r_\theta > 0\}$,
- $B = \{r \in \mathcal{R} : \sum_\theta \xi^*_\theta \, u^r_\theta < 0\}$,
- $E = \{r \in \mathcal{R} : \sum_\theta \xi^*_\theta \, u^r_\theta = 0\}$.

Similarly, any q-uniform posterior $\boldsymbol{\xi} \in \Xi^q$ induces the following partition to the set of receivers when ϵ-persuasiveness is adopted:

- $A_\epsilon = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u^r_\theta > \epsilon\}$,
- $B_\epsilon = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u^r_\theta < -\epsilon\}$,
- $E_\epsilon = \{r \in \mathcal{R} : \sum_\theta \xi_\theta \, u^r_\theta \in [-\epsilon, \epsilon]\}$.

Then, we define an auxiliary variable $\mathbf{y}^{\boldsymbol{\xi}} \in \mathcal{C}$ as follows:

- For every $r \in A$, $y^{\boldsymbol{\xi}}_r = \begin{cases} c_0 & \text{if } r \in A_\epsilon \cup E_\epsilon \\ c_1 & \text{otherwise} \end{cases}$.
- For every $r \in B$, $y^{\boldsymbol{\xi}}_r = \begin{cases} c_1 & \text{if } r \in B_\epsilon \cup E_\epsilon \\ c_0 & \text{otherwise} \end{cases}$.
- For every $r \in E$, $y^{\boldsymbol{\xi}}_r = \begin{cases} c_r \text{ where } \mathbf{c} = \mathbf{b}_{\boldsymbol{\xi}^*} & \text{if } r \in E_\epsilon \\ c_0 & \text{if } r \in A_\epsilon \\ c_1 & \text{if } r \in B_\epsilon \end{cases}$.

Note that, by construction, $\mathbf{y}^{\boldsymbol{\xi}}$ is a valid action profile under ϵ-persuasiveness. Moreover, by the optimality of the ϵ-persuasive best-response, the following holds for every posterior $\boldsymbol{\xi}$:

$$
\sum_\theta \xi_\theta \, g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) \geq \sum_\theta \xi_\theta \, g_\theta(\mathbf{y}^{\boldsymbol{\xi}}). \tag{6.6}
$$

Finally, let $\tilde{\mathbf{y}}^{(\theta,i)} \in \mathcal{C}$ be the random variable such that:

$$\Pr(\,\tilde{\mathbf{y}}^{(\theta,i)} = \mathbf{c}\,) = \frac{\sum_{\boldsymbol{\xi}\in\Xi^q,\xi_\theta=i/q,\mathbf{y}^{\boldsymbol{\xi}}=\mathbf{c}}\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}}.$$

Now, we introduce the last lemma we use to complete the proof. This lemma proves that $\tilde{\mathbf{y}}^{\theta,i}$ are $\frac{\alpha}{2}$-noisy probability distributions around $\mathbf{b}_{\boldsymbol{\xi}^*}$.

Lemma 6.4. *Given $\boldsymbol{\xi}^* \in \Xi^q$, for each $\theta \in \Theta$ and $i : |i/q - \xi^*_\theta| \leq \epsilon/4$, $\tilde{\mathbf{y}}^{(\theta,i)} \in \mathcal{C}$ is a $\frac{\alpha}{2}$-noisy probability distribution around $\mathbf{b}_{\boldsymbol{\xi}^*}$.*

Proof. We need to prove that for every receiver r, it holds $\Pr(\tilde{y}_r^{(\theta,i)} = c^*_r) \geq 1 - \alpha/2$, where c^*_r is the action of receiver r in action profile $\mathbf{b}_{\boldsymbol{\xi}^*}$. It holds:

$$\begin{aligned}
\Pr(\tilde{y}_r^{(\theta,i)} = c^*_r) &= \frac{\sum_{\boldsymbol{\xi}\in\Xi^q:p_\theta=i/q,y_r^{\boldsymbol{\xi}}=c^*_r}\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}} \geq \\
&\geq \sum_{\boldsymbol{\xi}\in\mathcal{P}^r:\xi_\theta=i/q} \frac{\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}} \geq \\
&\geq \left(1-\frac{\alpha}{2}\right) \sum_{\boldsymbol{\xi}\in\Xi^q:\xi_\theta=i/q} \frac{\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}} = \\
&= \left(1-\frac{\alpha}{2}\right).
\end{aligned}$$

This concludes the proof. ☐

Now, we are ready to prove Equation (8.2).

$$\sum_\theta \sum_{\boldsymbol{\xi}\in\Xi^q} \gamma_{\boldsymbol{\xi}}\,\xi_\theta\, g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) \geq \tag{6.7a}$$

(By restricting the set of posteriors.)

$$\geq \sum_\theta \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} i/q \sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}}\, g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) = \tag{6.7b}$$

$$= \sum_\theta \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} i/q \left(\sum_{\boldsymbol{\xi}:\xi_\theta=i/q}\gamma_{\boldsymbol{\xi}}\right) \sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \frac{\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}':\xi'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}} g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) \geq \tag{6.7c}$$

(By Inequality (6.6).)

$$\geq \sum_{\theta} \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} i/q \left(\sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} \right) \sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \frac{\gamma_{\boldsymbol{\xi}}}{\sum_{\boldsymbol{\xi}':\xi'_\theta=i/q}\gamma_{\boldsymbol{\xi}'}} g_\theta(\mathbf{y}^{\boldsymbol{\xi}}) \geq \tag{6.7d}$$

(By stability of g compared to h and Lemma 6.4.)

$$\geq \sum_{\theta} \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} i/q \left(\sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} \right) \left(1-\frac{\alpha}{2}\beta\right) h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}) = \tag{6.7e}$$

$$= \left(1-\frac{\alpha}{2}\beta\right) \sum_{\theta} h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}) \sum_{i:|i/q-\xi^*_\theta|\leq\epsilon/4} i/q \sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} \geq \tag{6.7f}$$

$$\geq \left(1-\frac{\alpha}{2}\beta\right) \sum_{\theta} h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}) \left(\xi^*_\theta - \sum_{i:|i/q-\xi^*_\theta|\geq\epsilon/4} \sum_{\boldsymbol{\xi}:\xi_\theta=i/q} \gamma_{\boldsymbol{\xi}} \right) \geq \tag{6.7g}$$

(By Lemma 6.2.)

$$\geq \left(1-\frac{\alpha}{2}\beta\right) \sum_{\theta} h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}) \left(1-\frac{\alpha}{2}\right) \xi^*_\theta = \tag{6.7h}$$

$$= \left(1-\frac{\alpha}{2}\beta\right) \left(1-\frac{\alpha}{2}\right) \sum_{\theta} \xi^*_\theta \, h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}) \geq \tag{6.7i}$$

(By $\alpha = \eta \min\{1, 1/\beta\}$.)

$$\geq (1-\eta) \sum_{\theta} \xi^*_\theta \, h_\theta(\mathbf{b}_{\boldsymbol{\xi}^*}). \tag{6.7j}$$

This concludes the proof. □

Now, we can prove the main result of this section. Consider a couple of sets of functions g, h where g is β-stable compared with h. With abuse of notation, we define $g(\phi)$ and $h(\phi)$ as the functions which evaluate the expected sender's utility of a public signaling scheme ϕ with h and g, respectively. We can resort to Lemma 6.1 to state the following result. The proof is based on solving a linear program that works only with q-uniform posteriors.

Theorem 6.2. *Let $\beta, \epsilon > 0$ and $\eta \in (0,1]$. Consider two arbitrary state-dependent sets of functions g, h such that $g_\theta : \mathcal{C} \to [0,1]$ is β-stable compared with $h_\theta : \mathcal{C} \to [0,1]$ for all $\theta \in \Theta$. Then, there exists an algorithm with running time* $\text{poly}\left(|\mathcal{R}|\,|\Theta|^{\log(\frac{1}{\eta\,\min\{1;1/\beta\}})/\epsilon^2}\right)$ *that returns an ϵ-persuasive public signaling scheme ϕ^ϵ such that:*

$$g(\phi^\epsilon) \geq (1-\eta) \max_{\phi \in \Phi} h(\phi),$$

where Φ is the set of persuasive signaling schemes.

Proof. For every constant $\beta, \epsilon > 0$, $\eta \in (0,1]$, by Theorem 6.1, we know that any posterior $\boldsymbol{\xi}^* \in \Xi$ guaranteeing a value $h(\boldsymbol{\xi}^*)$ can be expressed as a convex combination of q-uniform posteriors such that $\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}}\, g_\epsilon(\boldsymbol{\xi}) \geq (1-\eta)\, h(\boldsymbol{\xi}^*)$. Therefore, given the optimal persuasive public signaling scheme ϕ^* optimizing h, we can decompose each posterior probability distribution $\boldsymbol{\xi} \in supp(\phi^*)$ into a convex combination of q-uniform posteriors and obtain an ϵ-persuasive public signaling scheme ϕ_ϵ maximizing g that satisfies the inequalities stated in the theorem. Let take $q := 32 \log\left(\frac{4}{\eta\,\min\{1;1/\beta\}}\right)/\epsilon^2$. Since, for a fixed number of samples q, the number of q-uniform probability distributions is at most $|\Theta|^q$, we can search for the ϵ-persuasive public signaling scheme maximizing g over probability distributions $\boldsymbol{\xi} \in \Xi^q$, by solving the following Linear Program composed of $\mathcal{O}(|\Xi^q|)$ variables and constraints:

$$\max_{\boldsymbol{\gamma} \in \Delta_{\Xi^q}} \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \sum_{\theta \in \Theta} \xi_\theta\, g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}})$$

$$\text{s.t.} \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}}\, \xi_\theta = \mu_\theta \qquad \forall \theta \in \Theta$$

Finally, given the probability distribution on the q-uniform posteriors $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$, it is easy to derive the corresponding public signaling scheme ϕ^ϵ by setting the following for every $\theta \in \Theta$ and $\mathbf{c} \in \mathcal{C}$:

$$\phi^\epsilon_\theta(\mathbf{c}) = \sum_{\boldsymbol{\xi} \in \Xi^q : \mathbf{b}_{\epsilon,\boldsymbol{\xi}} = \mathbf{c}} \gamma_{\boldsymbol{\xi}}\, \xi_\theta.$$

□

By setting $h = g$, we obtain a generalization of the result by Xu (2020) to state-dependent functions.

6.3.2 Comparative Stability of Voting Functions

We apply this novel concept of stability to voting problems. Our first result proves that the two relaxed majority-voting functions previously introduced satisfy the comparative stability property. This result is similar to that by Cheng et al. (2015). However, we use multiplicative factors (in place of additive factors) and prove a slightly stronger result than stability. In particular, we prove that the decrease in utility is small even if only the perturbations from action c_0 to c_1 are bounded. Recall that W represents the majority voting function and W_δ its relaxation. Then, we can prove the following.

Lemma 6.5. *W_δ is $1/\delta$-stable compared with W. Moreover, for all $\boldsymbol{c} \in \mathcal{C}$, $\alpha \in (0, 1]$, and $\boldsymbol{y} \in \Delta_{\mathcal{C}}$ such that $\Pr_{\boldsymbol{y}}(\tilde{y}_r = c_1 \wedge c_r = c_0) \leq \alpha$ for each $r \in \mathcal{R}$, it holds:*

$$\mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}}\left[W_\delta(\tilde{\boldsymbol{y}})\right] \geq W(\boldsymbol{c})\left(1 - \frac{\alpha}{\delta}\right).$$

Proof. To prove the first part of the lemma, we need to show that for every voting profile $\bar{\mathbf{c}} \in \mathcal{C}$ and α-noisy probability distribution $\mathbf{y}$ around $\bar{\mathbf{c}}$ with $\alpha \in (0, 1]$, the following inequality holds:

$$\mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}}\left[W_\delta(\tilde{\mathbf{y}})\right] = \sum_{\mathbf{c} \in \mathcal{C}} y_{\mathbf{c}} W_\delta(\mathbf{c}) \geq W(\bar{\mathbf{c}})\left(1 - \frac{\alpha}{\delta}\right). \tag{6.9}$$

Given that W and W_δ assume values exclusively in $\{0, 1\}$, Inequality (6.9) is satisfied, independently from the chosen distribution $\mathbf{y}$, for all the voting profiles $\bar{\mathbf{c}}$ such that $W(\bar{\mathbf{c}}) = 0$. Therefore, we can restrict our attention to the set of voting profiles such that $W(\bar{\mathbf{c}}) = 1$. Let $V_{c_0}(\mathbf{c}) = \{r \in \mathcal{R} : c_r = c_0\}$ and $C^- = \{\mathbf{c} : |V_{c_0}(\mathbf{c})| \leq \lceil (1-\delta)|\mathcal{R}|/2 \rceil - 1\}$. Then, for every $\mathbf{y}$, the following holds

$$\begin{aligned}
\alpha|V_{c_0}(\bar{\mathbf{c}})| &\geq \\
&\geq \sum_{r \in V_{c_0}(\bar{\mathbf{c}})} \sum_{\mathbf{c} \in \mathcal{C}: c_r = c_1} y_{\mathbf{c}} \geq \\
&\geq \sum_{\mathbf{c} \in \mathcal{C}^-} \sum_{r \in V_{c_0}(\bar{\mathbf{c}}): c_r = c_1} y_{\mathbf{c}} \geq \\
&\geq \left[|V_{c_0}(\bar{\mathbf{c}})| - \lceil (1-\delta)|\mathcal{R}|/2 \rceil - 1\right] \sum_{\mathbf{c} \in \mathcal{C}^-} y_{\mathbf{c}} \geq \\
&\geq \left[|V_{c_0}(\bar{\mathbf{c}})| - (1-\delta)|\mathcal{R}|/2\right] \sum_{\mathbf{c} \in \mathcal{C}^-} y_{\mathbf{c}} =
\end{aligned}$$

$$= [|V_{c_0}(\bar{\mathbf{c}})| - \lceil(1-\delta)|\mathcal{R}|/2\rceil](1 - \mathbb{E}_{\tilde{\mathbf{y}}\sim\mathbf{y}}[W_\delta(\tilde{\boldsymbol{y}})]).$$

This implies that

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbf{y}}\sim\mathbf{y}}[W_\delta(\tilde{\boldsymbol{y}})] &\geq 1 - \frac{\alpha|V_{c_0}(\bar{\mathbf{c}})|}{|V_{c_0}(\bar{\mathbf{c}})| - (1-\delta)|\mathcal{R}|/2} = \\ &= 1 - \frac{\alpha}{1-(1-\delta)\frac{|\mathcal{R}|/2}{|V_{c_0}(\bar{\mathbf{c}})|}} \geq \\ &\geq (1 - \frac{\alpha}{\delta})W(\bar{\mathbf{c}}),\end{aligned}$$

where the last inequality follows from

$$\frac{|\mathcal{R}|/2}{|V_{c_0}(\bar{\mathbf{c}})|} \leq \frac{|\mathcal{R}|/2}{\lceil|\mathcal{R}|/2\rceil} \leq 1$$

and from $W(\bar{\mathbf{c}}) = 1$ by assumption.

Finally, to prove the second part of the lemma, we can employ Algorithm 6.1 to show that for all $\bar{\mathbf{c}} \in \mathcal{C}$ and for all probability distributions $\mathbf{y}$ around $\bar{\mathbf{c}}$ such that $\Pr_{\tilde{\mathbf{y}}\sim\mathbf{y}}[\tilde{y}_r = c_1 \wedge \bar{c}_r = c_0] \leq \alpha$, there is an α-noisy probability distribution $\mathbf{y}'$ guaranteeing

$$\mathbb{E}_{\tilde{\mathbf{y}}\sim\mathbf{y}}[W_\delta(\tilde{\boldsymbol{y}})] \geq \mathbb{E}_{\tilde{\mathbf{y}}\sim\mathbf{y}'}[W_\delta(\tilde{\boldsymbol{y}})] \geq W(\bar{\mathbf{c}})\left(1 - \frac{\alpha}{\delta}\right).$$

It is easy to see that $\mathbf{y}'$ is α-noise: $\mathbf{y}'$ has null probability on all the voting profiles $\mathbf{c}$ with a $r \in \mathcal{R}$ such that $c_r = c_0 \wedge \bar{c}_r = c_1$, *i.e.*, $V_{c_0}(\mathbf{c}) \subsetneq V_{c_0}(\bar{\mathbf{c}})$, while, $\Pr_{\tilde{\mathbf{y}}\sim\mathbf{y}'}[\tilde{y}_r = c_1 \wedge \bar{c}_r = c_0] = \Pr_{\tilde{\mathbf{y}}\sim\mathbf{y}}[\tilde{y}_r = c_1 \wedge \bar{c}_r = c_0] \leq \alpha$. Moreover, since Algorithm 6.1 moves probability mass from an action profile $\mathbf{c}$ to an action profile $\mathbf{c}'$ with $V_{c_0}(\mathbf{c}') \subseteq V_{c_0}(\mathbf{c})$, it does not increase the expected value of W_δ. This concludes the proof. □

Algorithm 6.1

```
input: distribution y
for c ∈ C s.t V_{c0}(c) ⊊ V_{c0}(c̄) do
    Take c' : V_{c0}(c') = V_{c0}(c) ∩ V_{c0}(c̄)
    y'_{c'} ← y_{c'} + y_c
    y'_c ← 0
end for
return y'
```

We can use the result above to prove that $\mathcal{W}_{\delta\delta}$ satisfies the property of comparative stability with respect to $\mathcal{W}$. Intuitively, the result follows from the observation that $\mathcal{W}$ is the composition of two majority-voting steps.

Lemma 6.6. *$\mathcal{W}_{\delta\delta}$ is $\frac{1}{\delta^2}$-stable with respect to $\mathcal{W}$.*

Proof. We need to prove that the following inequality holds for all $\mathbf{c} \in \mathcal{C}$ and α-noisy distribution $\mathbf{y}$ around $\mathbf{c}$ with $\alpha \in (0, 1]$.

$$\mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left[\mathcal{W}_{\delta\delta}(\tilde{\mathbf{y}})\right] \geq \mathcal{W}(\mathbf{c}) \left(1 - \frac{\alpha}{\delta^2}\right).$$

The value of function $\mathcal{W}_{\delta\delta}$ depends on the values of all the district functions W^{d}_{δ}. Indeed, given a voting profile $\mathbf{c} \in \mathcal{C}$, the function $\mathcal{W}_{\delta\delta}$ assumes value $\mathcal{W}_{\delta\delta}(\mathbf{c}) = \bar{W}_{\delta}(\, W^1_{\delta}(\, \mathbf{c}^1), \ldots, W^{|D|}_{\delta}(\mathbf{c}^{|D|})\,)$. Therefore, when it is perturbed by an α-noisy probability distribution $\mathbf{y}$, its expected value can be expressed as:

$$\mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left[\mathcal{W}_{\delta\delta}(\tilde{\mathbf{y}})\right] = \mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left[\, \bar{W}_{\delta}(\, W^1_{\delta}(\tilde{\mathbf{y}}^1), \ldots, W^{|D|}_{\delta}(\tilde{\mathbf{y}}^{|D|})\,)\,\right].$$

Lemma 6.5 can be applied to all the couples of functions $W^{\mathsf{d}}, W^{\mathsf{d}}_{\delta}$, deriving the following inequality for every $d \in D$, $\mathbf{c} \in \mathcal{C}$, $\alpha \in (0, 1]$:

$$\Pr_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left(W^{\mathsf{d}}_{\delta}(\tilde{\mathbf{y}}^{\mathsf{d}}) = c_1 \wedge W^{\mathsf{d}}(\mathbf{c}^{\mathsf{d}}) = c_0\right) \leq \alpha/\delta.$$

If $W^{\mathsf{d}}(\mathbf{c}^{\mathsf{d}}) = c_1$, the above inequality is trivially satisfied, whereas, if $W^{\mathsf{d}}(\mathbf{c}^{\mathsf{d}}) = c_0$, we can write

$$\begin{aligned}
&\Pr_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left(W^{\mathsf{d}}_{\delta}(\tilde{\mathbf{y}}^{\mathsf{d}}) = c_1 \wedge W^{\mathsf{d}}(\mathbf{c}^{\mathsf{d}}) = c_0\right) = \\
&= \Pr_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left(W^{\mathsf{d}}_{\delta}(\tilde{\mathbf{y}}^{\mathsf{d}}) = c_1\right) = 1 - \mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}} \left[W_{\delta}(\tilde{\mathbf{y}}^{\mathsf{d}})\right] \leq \\
&\leq 1 - \left(1 - \frac{\alpha}{\delta}\right) W(\mathbf{c}^{\mathsf{d}}) = \alpha/\delta.
\end{aligned}$$

We can use the above inequality and the fact that $\bar{W}$ is a majority-voting function to apply Lemma 6.5 to the couple of functions $\bar{W}$ and $\bar{W}_{\delta}$, thus showing the following:

$$\begin{aligned}
\mathbb{E}_{\tilde{\mathbf{y}} \sim \mathbf{y}} &\left[\bar{W}_{\delta} \left(W^1_{\delta}(\tilde{y}^1), \ldots, W^{|D|}_{\delta}(\tilde{\mathbf{y}}^{|D|})\right)\right] \geq \\
&\geq \bar{W} \left(W^1(\mathbf{c}^1), \ldots, W^{|D|}(\mathbf{c}^{|D|})\right) \left(1 - \frac{\alpha}{\delta^2}\right).
\end{aligned}$$

This implies that $\mathcal{W}_{\delta\delta}$ is $1/\delta^2$ stable compared to $\mathcal{W}$. □

Finally, we derive a stronger decomposition lemma for majority-voting. Specifically, Lemma 6.1 shows that the decrease in the expected sender's utility when decomposing a posterior in q-uniform posteriors can be bounded. However, in generic settings, the sender's expected utility in a given state of nature can change arbitrarily. This is not the case in majority voting, where, instead, this decrease is bounded. In particular, we can show the following, that is crucial when addressing the SEMIPUBLIC-DBE problem.

Lemma 6.7. *Let $\epsilon > 0, \eta \in (0,1]$ and set $q = 32 \log\left(\frac{4}{\eta\delta}\right)/\epsilon^2$. Then, given a posterior $\boldsymbol{\xi}^* \in \Xi$, there exists a $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ with $\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}}\, \boldsymbol{\xi} = \boldsymbol{\xi}^*$ and*

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}}\, \xi_\theta\, W_\delta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) \geq (1-\eta)\, \xi^*_\theta\, W(\mathbf{b}_{\boldsymbol{\xi}^*})\ \forall \theta \in \Theta.$$

Proof. The proof follows the same steps of the proof of Lemma 6.1. In the following, we just highlight the differences between the two proofs. In the steps from Equation (6.7a) to Equation (6.7j), we remove the summation over the states of nature. All the other steps hold, except for Equation (6.7d). Indeed, since ϵ-best response is computed maximizing the expected utility of the sender, there are no guarantees that for each state of nature θ it holds $g_\theta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) \geq g_\theta(\mathbf{y}^{\boldsymbol{\xi}})$. However, since W_δ is state-independent and monotone non-decreasing in the number of receivers that vote for c_0, the best response $\mathbf{b}_{\epsilon,\boldsymbol{\xi}}$ is given by c_0 for all the voters with utility $u^r_\theta \geq -\epsilon$. Thus, we are guaranteed that it holds $W_\delta(\mathbf{y}^{\boldsymbol{\xi}}) \leq W_\delta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}})$ independently from the state of nature θ. Taking into account Lemma 6.5, the derivation is straightforward. □

6.3.3 Computing Public and Semi-public Signaling Schemes in District-based Elections

We present two multi-criteria PTASs for the PUBLIC-DBE problem and the SEMIPUBLIC-DBE problem, respectively, when our relaxations are adopted. First, we focus on the problem of designing public signaling schemes. We assume ϵ-persuasive signaling schemes, and we replace function $\mathcal{W}$ with $\mathcal{W}_{\delta\delta}$ (this corresponds to relaxing the majority voting within every single district and the majority voting aggregating the outcomes of all the districts). Let $\mathcal{W}(\phi)$ and $\mathcal{W}_{\delta\delta}(\phi)$ denote the functions returning the sender's expected utility provided by a public signaling scheme ϕ with voting rules $\mathcal{W}$ and $\mathcal{W}_{\delta\delta}$, respectively. We show that it is possible to compute efficiently an ϵ-persuasive public signaling scheme ϕ^ϵ that approximates the optimal persuasive signaling scheme with an approximation factor arbitrarily close to 1. Since the relaxed function $\mathcal{W}_{\delta\delta}$ is $1/\delta^2$-stable compared to the non-relaxed function $\mathcal{W}$ by Lemma 6.6, we can immediately apply Theorem 6.2 to these functions and then derive the following.

Corollary 6.1. *Let $\epsilon > 0$, $\delta \in (0,1)$ and $\eta \in (0,1]$, then there exists a poly $\left(|\mathcal{R}|\,|\Theta|^{\log\left(\frac{1}{\eta\delta^2}\right)/\epsilon^2}\right)$ time algorithm that returns an ϵ-persuasive public signaling scheme ϕ^ϵ such that:*

$$\mathcal{W}_{\delta\delta}(\phi^\epsilon) \geq (1-\eta)\max_{\phi\in\Phi}\mathcal{W}(\phi), \tag{6.14}$$

where Φ is the set of persuasive signaling schemes.

Then, we focus on the SEMIPUBLIC-DBE problem. As highlighted above, to overcome the intractability result, also in this setting, it is necessary to relax the problem. Specifically, we use ϵ-persuasive signaling schemes and we replace function $\mathcal{W}$ with $\mathcal{W}_\delta$ (this corresponds to relaxing the majority voting aggregating the outcomes of all the districts). We show that it is possible to compute efficiently an ϵ-persuasive semi-public signaling scheme ϕ^ϵ that approximates the optimal persuasive signaling scheme with an approximation factor arbitrarily close to 1. Computing a semi-public signaling scheme ϕ amounts to determining a collection $\{\phi_{\mathsf{d}}\}_{\mathsf{d}\in D}$ of $|D|$ public signaling schemes, one for each district, and correlate them. The crucial point concerns the computation of good marginal probabilities of the signaling scheme. Indeed, their aggregation is equivalent to computing a private signaling scheme in majority-voting elections, and this can be done efficiently (see LP 6.1 and Theorem 6.1). The main idea of our proof is that there are approximately optimal marginal probabilities of the signaling scheme that use only q-uniform posteriors (with q constant). Let α_θ be the probability that c_0 wins in at least K_D districts with state of nature θ, $\sigma^\delta_{\mathsf{d},\theta}$ be the probability that candidate c_0 receives at least $\lceil(1-\delta)\,K_{\mathsf{d}}\rceil$ votes in district d with state of nature θ, and $\boldsymbol{\gamma}^{\mathsf{d}}$ be a probability distribution over posteriors for the receivers in district d. Finally, let $\mathbb{I}[\mathcal{E}]$ denote the indicator function for the event $\mathcal{E}$. Then, the following formulation computes an approximately optimal signaling scheme in polynomial time.

$$\max_{\substack{\boldsymbol{\alpha}\in[0,1]^{|\Theta|},\,\sigma^\delta\in[0,1]^{|D|\times|\Theta|}\\ \mathbf{i},\mathbf{l}\in\mathbb{R}^{|\Theta|\times K_D},\,\mathbf{o}\in\mathbb{R}^{|D|\times|\Theta|\times K_D}\\ \boldsymbol{\gamma}^{\mathsf{d}}\in\Delta_{\Xi^q}\forall d\in D}} \sum_{\theta\in\Theta}\mu_\theta\alpha_\theta \tag{6.15a}$$

$$\text{s.t. } \alpha_\theta \leq \frac{1}{K_D-m} i_{\theta,m} \qquad \forall\theta\in\Theta, \forall m\in\{0,\ldots,K_D-1\} \tag{6.15b}$$

$$i_{\theta,m} \leq (|D|-m)l_{\theta,m} + \sum_{\mathsf{d}\in D} o_{\mathsf{d},\theta,m} \tag{6.15c}$$

$$\forall \theta \in \Theta, \forall m \in \{0, \dots, K_D - 1\}$$

$$\sigma^{\delta}_{\mathsf{d},\theta} \geq l_{\theta,m} + o_{d,\theta,m} \tag{6.15d}$$

$$\forall \mathsf{d} \in D, \forall \theta \in \Theta, \forall m \in \{0, \dots, K_D - 1\}$$

$$\sigma^{\delta}_{\mathsf{d},\theta} \leq \sum_{\boldsymbol{\xi} \in \Xi^q} \frac{\gamma^{\mathsf{d}}_{\boldsymbol{\xi}} \, \xi_\theta}{\mu_\theta} \mathbb{I}\left[W^d_\delta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) = c_0\right] \tag{6.15e}$$

$$\forall \mathsf{d} \in D, \forall \theta \in \Theta$$

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma^{\mathsf{d}}_{\boldsymbol{\xi}} \, \xi_\theta = \mu_\theta \qquad \forall \mathsf{d} \in D, \forall \theta \in \Theta \tag{6.15f}$$

Theorem 6.3. *Let $\epsilon > 0$, $\delta \in (0,1)$ and $\eta \in (0,1]$, then there exists a poly $\left(|\mathcal{R}| \, |\Theta|^{\log\left(\frac{1}{\eta\,\delta}\right)/\epsilon^2}\right)$ time algorithm that outputs an ϵ-persuasive semi-public signaling scheme ϕ^ϵ such that:*

$$\mathcal{W}_\delta(\phi^\epsilon) \geq (1-\eta) \max_{\phi \in \Phi} \mathcal{W}(\phi), \tag{6.16}$$

where Φ is the set of persuasive signaling schemes.

Proof. Let $q = 32 \log\left(\frac{4}{\eta\,\delta}\right)/\epsilon^2$ and $\Xi^q \subset \Delta_\Theta$ be the set of q-uniform probability distributions on Θ. We show that, given the optimal semi-public signaling scheme ϕ^*, there is a solution ϕ^ϵ to LP 6.15 with $\mathcal{W}_\delta(\phi^\epsilon) \geq (1-\eta)\mathcal{W}(\phi^*)$. Given the signaling scheme ϕ^*, let:

- $\sigma^*_{\mathsf{d},\theta}$ be the probability that c_0 wins in district d when the state of nature is θ and
- α^*_θ be the probability that c_0 wins in at least K_{d} when the state of nature is θ.

Then, as showed in Theorem 6.1, the probability such that c_0 wins in at least K_D districts with state of nature θ is:

$$\alpha^*_\theta = \min\left\{ \min_{m \in \{0,\dots,K_D-1\}} \frac{1}{K_D - m} v_{\theta,m};\ 1 \right\}, \tag{6.17}$$

where $v_{\theta,m}$ is the sum of the lowest $|\mathcal{R}^{\mathsf{d}}| - m$ elements in the set $\{\sigma^*_{\mathsf{d},\theta}\}_{\mathsf{d} \in D}$. We show that there is a solution to LP 6.15 with $\sigma^{\delta}_{\mathsf{d},\theta} \geq (1-\eta)\sigma^*_{\mathsf{d},\theta}$ for every d and θ. Since the value of each $\sigma_{d,\theta}$ is reduced by a multiplicative factor $(1-\eta)$, Equation (6.17) implies that $\alpha_\theta \geq (1-\eta)\alpha^*_\theta$ and $\sum_\theta \mu_\theta \alpha_\theta \geq (1-\eta)\sum_\theta \mu_\theta \alpha^*_\theta$.[1]

[1] See Theorem 6.1 for details on how LP 6.15 computes α_θ from σ^δ.

Hence, we conclude the proof showing that $\sigma^{\delta}_{\mathsf{d},\theta} \geq (1-\eta)\sigma^{*}_{\mathsf{d},\theta}$ for every d and θ. Let:

- ϕ^*_{d} be the marginal probabilities of the signaling scheme ϕ restricted to the receivers in district d,
- $\boldsymbol{\gamma}^* \in \Delta_{\Xi}$ be the probability distribution on posteriors induced by ϕ^*_{d},
- $\boldsymbol{\gamma}^{\boldsymbol{\xi}} \in \Delta_{\Xi}$ be the probability distribution on q-uniform posteriors obtained decomposing a posterior $\boldsymbol{\xi}$ as prescribed by Lemma 6.7, and
- $\boldsymbol{\gamma}^{\mathsf{d}} \in \Delta_{\Xi^q}$ be the distribution on q-uniform posteriors obtained by decomposing each posterior induced by ϕ^*_{d} as in Lemma 6.7, *i.e.*, $\gamma^{\mathsf{d}}_{\boldsymbol{\xi}} = \sum_{\boldsymbol{\xi}' \in supp(\phi^*)} \gamma^*_{\boldsymbol{\xi}'} \gamma^{\boldsymbol{\xi}'}_{\boldsymbol{\xi}}$ for every $\boldsymbol{\xi}$.

We conclude proving that $\boldsymbol{\gamma}^{\mathsf{d}}$ is a q-uniform distribution that induces a $\sigma^{\delta}_{\mathsf{d},\theta} \geq (1-\eta)\sigma^{*}_{\mathsf{d},\theta}$ for every θ.

$$
\begin{aligned}
&(1-\eta)\sigma^{*}_{\mathsf{d},\theta} = \\
&= (1-\eta) \sum_{\boldsymbol{\xi} \in supp(\phi^*_{\mathsf{d}})} \frac{\gamma^*_{\boldsymbol{\xi}}\, \xi_\theta}{\mu_\theta} \mathbb{I}\left[W^{\mathsf{d}}(\mathbf{b}_{\boldsymbol{\xi}}) = c_0\right] \leq \\
&\hspace{6cm} \text{(by Lemma 6.7)} \\
&\leq \sum_{\boldsymbol{\xi} \in supp(\phi^*_{\mathsf{d}})} \frac{\gamma^*_{\boldsymbol{\xi}}}{\mu_\theta} \sum_{\boldsymbol{\xi}' \in \Xi^q} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'} \xi'_\theta \mathbb{I}\left[W_\delta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}'}) = c_0\right] = \\
&= \sum_{\boldsymbol{\xi}' \in \Xi^q} \frac{\xi'_\theta}{\mu_\theta} \mathbb{I}\left[W_\delta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}'}) = c_0\right] \sum_{\boldsymbol{\xi} \in supp(\phi^*_{\mathsf{d}})} \gamma^*_{\boldsymbol{\xi}} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'} = \\
&= \sum_{\boldsymbol{\xi} \in \Xi^q} \frac{\gamma^{\mathsf{d}}_{\boldsymbol{\xi}} \xi_\theta}{\mu_\theta} \mathbb{I}\left[W_\delta(\mathbf{b}_{\epsilon,\boldsymbol{\xi}}) = c_0\right] = \\
&= \sigma^{\delta}_{\mathsf{d},\theta}.
\end{aligned}
$$

This concludes the proof. □

CHAPTER 7

Persuading in Network Congestion Games

In this chapter, we study atomic network congestion games where edge costs depend on a stochastic state of nature. In Section 7.1 we introduce Bayesian network congestion games (BNCGs) and the signaling problem. In Section 7.2, we design a polynomial-time algorithm to compute an optimal ex ante persuasive signaling scheme in *symmetric* BNCGs with affine cost functions. Finally, in Section 7.3, we show that the signaling problem is intractable for *asymmetric* network congestion games

7.1 Persuasion in Bayesian Network Congestion Games

In this section, we introduce the main elements of our model.

7.1.1 Network Congestion Games

A *network congestion game* (Fabrikant et al., 2004) is defined as a tuple $(N, G, \{c_e\}_{e\in E}, \{(s_p, t_p)\}_{p\in N})$, where:

- $N := \{1, \ldots, \bar{n}\}$ denotes the set of players;
- $G := (V, E)$ is the directed graph underlying the game, with V being

its set of nodes and each $e = (v, v') \in E$ representing a directed edge from v to v';

- $\{c_e\}_{e \in E}$ are the edge costs, with each $c_e : \mathbb{N} \to \mathbb{R}_+$ defining the cost of edge $e \in E$ as a function of the number of players traveling through e;
- $\{(s_p, t_p)\}_{p \in N}$, with $s_p, t_p \in V$, denote the source-destination pairs for all the players.

In an NCG, the set $\mathcal{A}_p$ of actions available to a player $p \in N$ is implicitly defined by the graph G, the source s_p, and the destination t_p. Formally, $\mathcal{A}_p$ is the set of all directed paths from s_p to t_p in the graph G. In this chapter, we use $a_p \in \mathcal{A}_p$ to denote a player p's path and we write $e \in a_p$ whenever the path contains the edge $e \in E$. An action profile $\mathbf{a} \in \mathcal{A}$, where $\mathcal{A} := \times_{p \in N} \mathcal{A}_p$, is a tuple of s_p to t_p directed paths $a_p \in \mathcal{A}_p$, one per player $p \in N$. For the ease of notation, given an action profile $\mathbf{a} \in \mathcal{A}$, we let $f_e^{\mathbf{a}}$ be the congestion of edge $e \in E$ in $\mathbf{a}$, *i.e.*, the number of players selecting a path passing thorough e in $\mathbf{a}$; formally, $f_e^{\mathbf{a}} := |\{p \in N \mid e \in a_p\}|$. Thus, $c_e(f_e^{\mathbf{a}})$ denotes the cost of edge e in $\mathbf{a}$. Finally, the cost incurred by player $p \in N$ in an action profile $\mathbf{a} \in \mathcal{A}$ is denoted by $c_p(\mathbf{a}) := \sum_{e \in a_p} c_e(f_e^{\mathbf{a}})$.

7.1.2 Bayesian Network Congestion Games

We define a *Bayesian network congestion game* (BNCG) as a tuple $(N, G, \Theta, \boldsymbol{\mu}, \{c_{e,\theta}\}_{e \in E, \theta \in \Theta}, \{(s_p, t_p)\}_{p \in N})$, where, differently from the basic setting, the edge cost functions $c_{e,\theta} : \mathbb{N} \to \mathbb{R}_+$ also depend on a state of nature θ drawn from a finite set of states Θ. Moreover, $\boldsymbol{\mu}$ encodes the prior beliefs that the players have over the states of nature, *i.e.*, $\boldsymbol{\mu} \in \Delta_\Theta$ is a probability distribution over the set Θ. All the other components are defined as in non-Bayesian NCGs. Notice that, in BNCGs, the cost experienced by player $p \in N$ in an action profile $\mathbf{a} \in \mathcal{A}$ also depends on the state of nature $\theta \in \Theta$, and, thus, it is defined as $c_{p,\theta}(\mathbf{a}) := \sum_{e \in a_p} c_{e,\theta}(f_e^{\mathbf{a}})$. A BNCG is *symmetric* if all the players share the same (s_p, t_p) pair, *i.e.*, whenever they all have the same set of actions (paths). For the ease of notation, in such settings we let $s, t \in V$ be the common source and destination. Moreover, we focus on BNCGs with *affine costs*, *i.e.*, for all $e \in E$ and $\theta \in \Theta$, there exist constants $\alpha_{e,\theta}, \beta_{e,\theta} \in \mathbb{R}_+$ such that the edge cost function can be expressed as $c_{e,\theta}(f_e^{\mathbf{a}}) := \alpha_{e,\theta} f_e^{\mathbf{a}} + \beta_{e,\theta}$. [1]

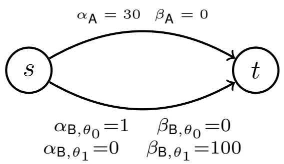
[1] We focus on affine costs since: *(i)* the assumption is reasonable in many applications (Vasserman et al., 2015), and *(ii)* the problem is trivially NP-hard when generic costs are allowed (see Section 7.3).

7.1.3 Signaling in BNCGs

Suppose that a BNCG is employed to model a road network subject to vagaries. It is reasonable to assume that third-party entities (*e.g.*, the road management company) may have access to the realized state of nature. We call one such entity *the sender*. We focus on the following natural question: *is it possible for an informed sender to mitigate the overall costs through the strategic provision of information to players who update their beliefs rationally?* The sender can publicly commit to a *signaling scheme* which maps the realized state of nature to a signal for each player. The sender can exploit general *private* signaling schemes, sending different signals to each player through private communication channels. We focus on the notion of *ex ante persuasiveness* as defined by Xu (2020) and Celli et al. (2020) (see Definition 3.6). Notice that a *coarse correlated equilibrium* (CCE) (see Definition 2.4) may be seen as an *ex ante* persuasive signaling scheme in non-Bayesian NCGs in which there are no states of nature, *i.e.*, when $|\Theta| = 1$. Finally, a sender's *optimal ex ante* persuasive signaling scheme ϕ^* is such that it minimizes the *expected social cost* of the solution, *i.e.*:

$$\phi^* \in arg\,min_\phi \sum_{\theta\in\Theta} \mu_\theta \sum_{\mathbf{a}\in\mathcal{A}} \phi_\theta(\mathbf{a}) \sum_{p\in N} c_{p,\theta}(\mathbf{a}).$$

The following example illustrates the interaction flow between the sender and the players (receivers).



	Signals			
	(B,B,B)	(A,A,B)	(A,B,A)	(B,A,A)
θ_0	1	0	0	0
θ_1	0	1/3	1/3	1/3

Figure 7.1: Left*: BNCG for Example 7.1.* Right*: An* ex ante *persuasive signaling scheme for the case with $\bar{n} = 3$. The table displays only those $\boldsymbol{a} \in \mathcal{A}$ such that $\phi_\theta(\boldsymbol{a}) > 0$ for some state of nature $\theta \in \Theta = \{\theta_0, \theta_1\}$.*

Example 7.1. *Figure 7.1 (Left) describes a simple BNCG modeling the road network between the JFK International Airport (node s), and Manhattan (node t). It is late at night and three lone researchers have to reach the AAAI venue. They are following navigation instructions from the same application, whose provider (the sender) has access to the current state of the roads (called* A *and* B*, respectively). Roads costs (i.e., travel times) are depicted in Figure 7.1 (Left). In normal conditions (state θ_0), road* B *is* *extremely fast* ($\alpha_{\mathsf{B}} = 1$ *and* $\beta_{\mathsf{B}} = 0$)*. However, it requires frequent road works for maintenance (state* θ_1*), which increase the travel time. Moreover, it holds* $\mu_{\theta_0} = \mu_{\theta_1} = 1/2$*. The interaction between the sender and the three players goes as follows: (i) the sender commits to a signaling scheme* ϕ*; (ii) the players observe* ϕ *and decide whether to adhere to the navigation system or not; (iii) the sender observes the realized state of nature and exploits this knowledge to compute recommendations. Figure 7.1 (Right) describes an* ex ante *persuasive signaling scheme. In this case, when the state of nature is* θ_1*, one of the players is randomly selected to take road* B*, even if it is undergoing maintenance. In expectation, following sender's recommendations is strictly better than congesting road* A.

A simple variation of Example 7.1 is enough to show that the introduction of signaling allows the sender to reach solutions with arbitrarily better expected social cost than what can be achieved via the optimal Bayes-Nash equilibrium in absence of signaling. Specifically, consider the BNCG in Figure 7.1 (Left) with the following modifications: $\bar{n} = 1$, β coefficients always equal to zero, $\alpha_{\mathsf{A},\theta_0} = \infty$, $\alpha_{\mathsf{A},\theta_1} = 0$, $\alpha_{\mathsf{B},\theta_0} = 0$, and $\alpha_{\mathsf{B},\theta_1} = \infty$. Without signaling, the optimal choice yields an expected social cost of ∞. However, a perfectly informative signal (*i.e.*, one revealing the realized state of nature) allows the player to avoid any cost.

7.2 The Power of Symmetry

We design a polynomial-time algorithm to compute an optimal *ex ante* persuasive signaling scheme in symmetric BNCGs with affine cost functions. Our algorithm exploits the ellipsoid method. We first formulate the problem as an LP (Problem 7.1) with polynomially many constraints and exponentially many variables. Then, we show how to find an optimal solution to the LP in polynomial time by applying the ellipsoid algorithm to its dual (Problem 7.2), which features polynomially many variables and exponentially many constraints. This calls for a polynomial-time separation oracle for Problem 7.2, which is not readily available since the problem has an exponential number of constraints. We prove that, in our setting, a polynomial-time separation oracle can be implemented by solving a suitably defined min-cost flow problem. The proof of this result crucially relies on the symmetric nature of the problem and the assumption that the costs are affine functions of the edge congestion.

The following lemma shows how to formulate the problem as an LP. [2]

[2] LPs analogous to Problem 7.1 and Problem 7.2 can also be derived for the asymmetric setting. However, the separation problem for the dual is solvable in polynomial time only in the symmetric case.

For the ease of presentation, we use $\mathbb{I}[e \notin a_p]$ to denote the indicator function for the event $e \notin a_p$, *i.e.*, it holds $\mathbb{I}[e \notin a_p] = 1$ if $e \notin a_p$, while $\mathbb{I}[e \notin a_p] = 0$ otherwise.

Lemma 7.1. *Given a symmetric BNCG, an optimal* ex ante *persuasive signaling scheme ϕ can be found with the LP:*

$$\min_{\phi \geq 0, \mathbf{x}} \sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{a} \in \mathcal{A}} \phi_\theta(\boldsymbol{a}) \sum_{p \in N} c_{p,\theta}(\boldsymbol{a}) \quad \text{s.t.} \tag{7.1a}$$

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{a} \in \mathcal{A}} c_{p,\theta}(\boldsymbol{a}) \phi_\theta(\boldsymbol{a}) \leq x_{p,s} \qquad \forall p \in N \tag{7.1b}$$

$$x_{p,v} \leq \sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{a} \in \mathcal{A}} c_{e,\theta}\left(f_e^{\boldsymbol{a}} + \mathbb{I}[e \notin a_p]\right) \phi_\theta(\boldsymbol{a}) + x_{p,v'} \qquad \forall p \in N, \forall e = (v, v') \in E \tag{7.1c}$$

$$x_{p,t} = 0 \qquad \forall p \in N \tag{7.1d}$$

$$\sum_{\boldsymbol{a} \in \mathcal{A}} \phi_\theta(\boldsymbol{a}) = 1 \qquad \forall \theta \in \Theta \tag{7.1e}$$

Proof. Clearly, Objective (7.1a) is equivalent to minimizing the social cost, while Constraints (7.1e) imply that ϕ is well formed. Constraints (7.1b) enforce *ex ante* persuasiveness for every player $p \in N$: the expression on the left-hand side represents player p's expected cost, while $x_{p,s}$ is the cost of her best deviation (*i.e.*, a cost-minimizing path given μ and ϕ). This is ensured by Constraints (7.1c) and (7.1d). In particular, for every player $p \in N$ and node $v \in V \setminus \{t\}$, the former guarantee that $x_{p,v}$ is the minimum cost of a path from v to t. This is shown by noticing that (given that $x_{p,t} = 0$) such cost can be inductively defined as follows:

$$\min_{\substack{v' \in V: \\ e=(v,v') \in E}} \left\{ \sum_{\theta \in \Theta} \mu_\theta \sum_{\mathbf{a} \in \mathcal{A}} c_{e,\theta}\left(f_e^{\mathbf{a}} + \mathbb{I}[e \notin a_p]\right) \phi_{\theta,\mathbf{a}} + x_{p,v'} \right\},$$

where $f_e^{\mathbf{a}} + \mathbb{I}[e \notin a_p]$ accounts for the fact that the congestion of edge e must be incremented by one if player p does not select a path containing e in the action profile $\mathbf{a}$. $\square$

Lemma 7.2. *The dual of Problem 7.1 reads as follows:*

$$\max_{\boldsymbol{y}} \sum_{\theta \in \Theta} y_\theta \quad \text{s.t.} \tag{7.2a}$$

$$\mu_\theta \left(\sum_{p \in N} c_{p,\theta}(\boldsymbol{a}) y_p - \sum_{p \in N} \sum_{e \in E} c_{e,\theta}\left(f_e^{\boldsymbol{a}} + \mathbb{I}[e \notin a_p]\right) y_{p,e} \right)$$

$$+ y_\theta \leq \mu_\theta \sum_{p \in N} c_{p,\theta}(\boldsymbol{a}) \qquad \forall \theta \in \Theta, \forall \boldsymbol{a} \in \mathcal{A} \tag{7.2b}$$

$$\sum_{v' \in V: e=(v,v') \in E} y_{p,e} - \sum_{v' \in V: e=(v',v) \in E} y_{p,e} = 0$$
$$\forall p \in N, \forall v \in V \setminus \{s,t\} \tag{7.2c}$$

$$\sum_{v \in V: e=(s,v) \in E} y_{p,e} - y_p = 0 \qquad \forall p \in N \tag{7.2d}$$

$$y_{p,t} - \sum_{v \in V: e=(v,t) \in E} y_{p,e} = 0 \qquad \forall p \in N \tag{7.2e}$$

$$y_p \leq 0 \qquad \forall p \in N \tag{7.2f}$$

$$y_{p,e} \leq 0 \qquad \forall p \in N, \forall e \in E \tag{7.2g}$$

Proof. It directly follows from LP duality, by letting y_p (for $p \in N$), $y_{p,e}$ (for $p \in N$ and $e \in E$), $y_{p,t}$ (for $p \in N$), and y_θ (for $\theta \in \Theta$) be the dual variables associated to, respectively, Constraints (7.1b), (7.1c), (7.1d), and (7.1e). □

Since $|\mathcal{A}|$ is exponential in the size of the game, Problem 7.1 features exponentially many variables, while its number of constraints is polynomial. Conversely, Problem 7.2 has polynomially many variables and exponentially many constraints, which enables the use of the ellipsoid algorithm to find an optimal solution to Problem 7.2 in polynomial time. This requires a polynomial-time separation oracle for Problem 7.2, *i.e.*, a procedure that, given a vector y of dual variables, it either establishes that **y** is feasible for Problem 7.2 or, if not, it outputs a hyperplane separating **y** from the feasible region. In the following, we focus on a particular type of separation oracles: those generating violated constraints of Problem 7.2.

Given that Problem 7.2 has an exponential number of constraints, a polynomial-time separation oracle is not readily available. It turns out that, in our setting, we can design one by leveraging the symmetry of the players and the fact that the cost functions are affine, as described in the following.

First, we prove that Problem 7.2 always admits an optimal player-symmetric solution, *i.e.*, a vector **y** such that, for each pair of players $p, q \in N$, it holds that $y_p = y_q$, $y_{p,e} = y_{q,e}$ for all $e \in E$, and $y_{p,t} = y_{q,t}$. This result allows us to restrict the attention to player-symmetric vectors **y**.

Lemma 7.3. *Problem 7.2 always admits an optimal player-symmetric solution.*

Proof. Given any optimal solution $\mathbf{y}$ to Problem 7.2, we can always recover, in polynomial time, a player-symmetric optimal solution $\tilde{\mathbf{y}}$. Specifically, for every $p \in N$, let $\tilde{y}_p = \frac{\sum_{p \in N} y_p}{n}$, $\tilde{y}_{p,e} = \frac{\sum_{p \in N} y_{p,e}}{n}$ for all $e \in E$, and $\tilde{y}_{p,t} = \frac{\sum_{p \in N} y_{p,t}}{n}$, while $\tilde{y}_\theta = y_\theta$ for every $\theta \in \Theta$. Let us remark that $\tilde{\mathbf{y}}$ is player-symmetric since: *(i)* for every $e \in E$, it holds $\tilde{y}_{p,e} = \tilde{y}_{q,e}$ for each pair of players $p, q \in N$; and *(ii)* $\tilde{y}_p = \tilde{y}_q$ and $\tilde{y}_{p,t} = \tilde{y}_{q,t}$ for each $p, q \in N$. First, notice that $\mathbf{y}$ and $\tilde{\mathbf{y}}$ provide the same objective value, as $\tilde{y}_\theta = y_\theta$ for all $\theta \in \Theta$. Thus, we only need to prove that $\tilde{\mathbf{y}}$ satisfies all the constraints of Problem 7.2. For $\mathbf{a} \in \mathcal{A}$ and $i \in [n]$, let us denote with $\pi_i(\mathbf{a})$ an action profile $\mathbf{a}' \in \mathcal{A}$ such that $a'_p = a_{((p+i) \mod n)}$, *i.e.*, a permutation of $\mathbf{a}$ in which each player $p \in N$ takes on the role of player $(p+i) \mod n$. Moreover, let $\pi(\mathbf{a}) := \bigcup_{i \in [n]} \pi_i(\mathbf{a})$. Constraints (7.2b) are satisfied by $\tilde{\mathbf{y}}$, since, for every $\theta \in \Theta$ and $\mathbf{a} \in \mathcal{A}$, it holds:

$$
\begin{aligned}
&\mu_\theta \left(\sum_{p \in N} c_{p,\theta}(\mathbf{a}) \tilde{y}_p - \sum_{p \in N} \sum_{e \in E} c_{e,\theta} \left(f_e^{\mathbf{a}} + \mathbb{I}[e \notin a_p] \right) \tilde{y}_{p,e} \right) + \tilde{y}_\theta \\
&= \frac{1}{n} \sum_{\mathbf{a}' \in \pi(\mathbf{a})} \mu_\theta \left(\sum_{p \in N} c_{p,\theta}(\mathbf{a}') y_p \right. \\
&\qquad \left. - \sum_{p \in N} \sum_{e \in E} c_{e,\theta} \left(f_e^{\mathbf{a}'} + \mathbb{I}[e \notin a'_p] \right) y_{p,e} \right) + y_\theta \\
&\leq \frac{1}{n} \sum_{\mathbf{a}' \in \pi(\mathbf{a})} \mu_\theta \sum_{p \in N} c_{p,\theta}(\mathbf{a}') = \mu_\theta \sum_{p \in N} c_{p,\theta}(\mathbf{a}).
\end{aligned}
$$

Similar arguments show that $\tilde{\mathbf{y}}$ satisfies all the other constraints, concluding the proof. □

Notice that any polynomial-time separation oracle for Problem 7.2 can explicitly check whether each member of the polynomially many Constraints (7.2c), (7.2d), and (7.2e) is satisfied for the given $\mathbf{y}$. Thus, we focus on the separation problem restricted to the exponentially many Constraints (7.2b), which, using Lemma 7.3, can be formulated as stated in the following lemma.

Lemma 7.4. *Given a player-symmetric* $\mathbf{y}$*, solving the separation problem for Constraints* (7.2b) *amounts to finding* $\theta \in \Theta$ *and* $\mathbf{a} \in \mathcal{A}$ *that are optimal* *for the following problem:*

$$\min_{\theta\in\Theta,\boldsymbol{a}\in\mathcal{A}} \mu_\theta \Bigg((1-\bar{y}) \sum_{p\in N} c_{p,\theta}(\boldsymbol{a}) - \\ - \sum_{p\in N}\sum_{e\in E} c_{e,\theta}\left(f_e^{\boldsymbol{a}} + \mathbb{I}[e \notin a_p]\right) \bar{y}_e \Bigg) - y_\theta, \tag{7.3}$$

where we let $\bar{y} = y_1$ *and* $\bar{y}_e = y_{1,e}$ *for all* $e \in E$.

Next, we show how Problem 7.3 can be equivalently formulated avoiding the minimization over the exponentially-sized set $\mathcal{A}$. Intuitively, we rely on the fact that, for a fixed $\theta \in \Theta$, we can exploit the symmetry of the players to equivalently represent action profiles $\mathbf{a} \in \mathcal{A}$ as integer vectors $\boldsymbol{q}$ of edge congestions $q_e \in [n]$, for all $e \in E$.

Lemma 7.5. *Problem 7.3 can be formulated as* $\min_{\theta\in\Theta} \chi(\theta)$, *where* $\chi(\theta)$ *is the optimal value of the following problem:*

$$\min_{\boldsymbol{q}\in\mathbb{Z}_+^{|E|}} (1-\bar{y}) \sum_{e\in E} \alpha_{e,\theta} q_e^2 + \beta_{e,\theta} q_e \\ - \sum_{e\in E} \bar{y}_e \Big(\bar{n}\alpha_{e,\theta} q_e + (\bar{n} - q_e)\alpha_{e,\theta} + \bar{n}\beta_{e,\theta} \Big) \tag{7.4a}$$

$$\text{s.t.} \sum_{v\in V: e=(s,v)\in E} q_e = \bar{n} \tag{7.4b}$$

$$\sum_{v\in V: e=(v,t)\in E} q_e = \bar{n} \tag{7.4c}$$

$$\sum_{\substack{v'\in V:\\ e=(v',v)\in E}} q_e = \sum_{\substack{v'\in V:\\ e=(v,v')\in E}} q_e \qquad \forall v \in V \setminus \{s,t\} \tag{7.4d}$$

Proof. First, given a state $\theta \in \Theta$, Problem 7.3 reduces to computing $\chi(\theta) := \min_{\mathbf{a}\in\mathcal{A}} (1-\bar{y}) \sum_{p\in N} c_{p,\theta}(\mathbf{a}) - \sum_{p\in N}\sum_{e\in E} c_{e,\theta}\left(f_e^{\mathbf{a}} + \mathbb{I}[e \notin a_p]\right) \bar{y}_e$, where the function to be minimized only depends on the number of players selecting each edge $e \in E$ in $\mathbf{a}$, rather than the identity of the players who are choosing e (since they are symmetric). Letting $q_e \in [\bar{n}]$ be the congestion level of edge $e \in E$ and using $c_{e,\theta} = \alpha_{e,\theta} q_e + \beta_{e,\theta}$ (affine costs), it holds $\sum_{p\in N} c_{p,\theta}(\mathbf{a}) = \sum_{e\in E} \alpha_{e,\theta} q_e^2 + \beta_{e,\theta} q_e$, and, for every $e \in E$, $\sum_{p\in N} c_{e,\theta}\left(f_e^a + \mathbb{I}[e \notin a_p]\right) = \bar{n}\alpha_{e,\theta} q_e + (\bar{n} - q_e)\alpha_{e,\theta} + \bar{n}\beta_{e,\theta}$. This gives Objective (7.4a). Moreover, Constraints (7.4b), (7.4c), and (7.4d) ensure that $\boldsymbol{q}$ is well defined. □

Let us remark that computing an optimal integer solution to Problem 7.4 is necessary in order to (possibly) find a violated constraint for a given **y**; otherwise, we would not be able to easily recover an action profile $\mathbf{a} \in \mathcal{A}$ from q.

Now, we show that an optimal integer solution to Problem 7.4 can be found in polynomial time by reducing it to an instance of integer min-cost flow problem. Intuitively, it is sufficient to consider a modified version of the original graph G in which each edge $e \in E$ is replaced with $\bar{n}$ parallel edges with unit capacity and increasing unit costs. This is possible given that the Objective (7.4a) is a convex function of $\boldsymbol{q}$, which is guaranteed by the fact that costs are affine.

Lemma 7.6. *An optimal integer solution to Problem 7.4 can be found in polynomial time by solving a suitably defined instance of integer min-cost flow problem.*

Proof. First, notice that Objective (7.4a) is a sum edge costs, in which the cost of each edge $e \in E$ is a convex function of the edge congestion q_e, as the only quadratic term is $(1 - \bar{y})a_{e,\theta}q_e^2$, where the multiplying coefficient is always positive, given $\bar{y} \leq 0$ and $\alpha_{e,\theta} \geq 0$. This allows us to formulate Problem 7.4 as an instance of integer min-cost flow problem. We build a new graph where each $e \in E$ is replaced with $\bar{n}$ parallel edges, say e_i for $i \in [\bar{n}]$. For $e \in E$ and $i \in [\bar{n}]$, let us define $g(e, i) := (1 - \bar{y})\left(\alpha_{e,\theta}i^2 + \beta_e i\right) - \bar{y}_e\left(\bar{n}\alpha_{e,\theta}i + (\bar{n} - i)\alpha_{e,\theta} + \bar{n}\beta_{e,\theta}\right)$. Each (new) edge e_i has unit capacity and a per-unit cost equal to $\delta(e_i) := g(e, i) - g(e, i - 1)$. Clearly, finding an integer min-cost flow is equivalent to minimizing Objective (7.4a). Notice that, since the original edge costs are convex, it holds $\delta(e_i) \geq \delta(e_j)$ for all $j < i \in [n]$. Thus, an edge e_i is used (*i.e.*, it carries a unit of flow) only if all the edges e_j, for $j < i \in [\bar{n}]$, are already used. This allows us to recover an integer vector $\boldsymbol{q}$ from a solution to the min-cost flow problem. Finally, let us recall that we can find an optimal solution to the integer min-cost flow problem in polynomial time by solving its LP relaxation. □

The last lemma allows us to prove our main result:

Theorem 7.1. *Given a symmetric BNCG, an optimal* ex ante *persuasive signaling scheme can be computed in poly-time.*

Proof. The algorithm applies the ellipsoid algorithm to Problem 7.2. At each iteration, we require that the vector of dual variables **y** given to the separation oracle be player-symmetric, which can be easily obtained by applying the symmetrization technique introduced in the proof of Lemma 7.3.

The separation oracle needs to solve an instance of integer min-cost flow problem for every $\theta \in \Theta$ (see Lemmas 7.4, 7.5, and 7.6). Notice that an integer solution is required in order to be able to identify a violated constraint. Finally, the polynomially many violated constraints generated by the ellipsoid algorithm can be used to compute an optimal ϕ. □

7.3 The Curse of Asymmetry

In this section, we provide our hardness result on asymmetric BNCGs. Our proof is split into two intermediate steps: *(i)* we prove a hardness result for a simple class of asymmetric non-Bayesian congestion games in which each player selects only one resource (Lemma 7.7); and *(ii)* we show that such games can be represented as NCGs with only a polynomial blow-up in the representation size (Lemma 7.8). Our main result reads as follows:

Theorem 7.2. *The problem of computing an optimal* ex ante *persuasive signaling scheme in BNCGs with asymmetric players is* NP*-hard, even with affine costs.* [3]

The proof of Theorem 7.2 is based on a reduction that maps an instance of 3SAT (a well-known NP-hard problem, see (Garey and Johnson, 1979)) to a game in the class of *singleton congestion games* (SCGs) (Ieong et al., 2005), where each player can select only one resource at a time. A (non-Bayesian) SCG is described by a tuple $(N, R, \{\mathcal{A}_p\}_{p \in N}, \{c_r\}_{r \in R})$, where R is a finite set of resources, each player $p \in N$ selects a single resource from the set $\mathcal{A}_p \subseteq R$ of available resources, and resource $r \in R$ has a cost $c_r : \mathbb{N} \to \mathbb{R}_+$. Another way of interpreting SCGs is as games played on parallel-link graphs, where each player can select only a subset of the edges.

First, let us provide the following definition and notation.

Definition 7.1 (3SAT). *Given a finite set C of three-literal clauses defined over a finite set V of variables, is there a truth assignment to the variables satisfying all the clauses?*

We denote with $l \in \varphi$ a literal (*i.e.*, a variable or its negation) appearing in a clause $\varphi \in C$. Moreover, we let m and s be, respectively, the number of clauses and variables, *i.e.*, $m := |C|$ and $s := |V|$. W.l.o.g., we assume that $m \geq s$.

[3] Without affine costs, computing an optimal *ex ante* persuasive signaling scheme is trivially NP-hard even in symmetric BNCGs. This directly follows from (Meyers and Schulz, 2012), which shows that even finding an optimal action profile (that is also an optimal Nash equilibrium) is NP-hard in symmetric (non-Bayesian) NCGs.

Lemma 7.7 introduces our main reduction, proving that finding a social-cost-minimizing CCE is NP-hard in SCGs with asymmetric players, *i.e.*, whenever the resource sets $\mathcal{A}_p$ are different among each other. [4] Notice that the games used in the reduction are *not* Bayesian; this shows that the hardness fundamentally resides in the asymmetry of the players.

Lemma 7.7. *The problem of computing a social-cost-minimizing CCE in SCGs with asymmetric players is* NP*-hard, even with affine costs.*

Proof. Our 3SAT reduction shows that the existence of a polynomial-time algorithm for computing a social-cost-minimizing CCE in SCGs would allow us to solve any 3SAT instance in polynomial time. Given (C, V), let $z := m^{40}$, $u := m^{12}$, and $\epsilon := \frac{1}{m^4}$. We build an SCG $\Gamma(C, V)$ admitting a CCE with social cost smaller than or equal to $\gamma := z^2 + (4us + s + 3m)(z - u) + \frac{3z}{m^9}$ iff (C, V) is satisfiable.

Mapping. $\Gamma(C, V)$ is defined as follows (for every $r \in R$, the cost c_r is an affine function with coefficients α_r and β_r).

- $N = \{p_v \mid v \in V\} \cup \{p_{\varphi,q} \mid \varphi \in C, q \in [3]\} \cup \{p_{v,j}, p_{\bar{v},j} \mid v \in V, j \in [2u]\} \cup \{p_i \mid i \in [z]\}$;
- $R = \{r_t\} \cup \{r_v, r_{\bar{v}}, r_{v,1}, r_{v,2}, r_{\bar{v},1}, r_{\bar{v},2} \mid v \in V\}$;
- $\mathcal{A}_{p_v} = \{r_v, r_{\bar{v}}, r_t\}\ \ \forall v \in V$;
- $\mathcal{A}_{p_{\varphi,q}} = \{r_l \mid l \in \varphi\}\ \ \forall\, \varphi \in C, \forall q \in [3]$;
- $\mathcal{A}_{p_{v,j}} = \{r_v, r_{v,1}, r_{v,2}\}\ \ \forall v \in V, \forall j \in [2u]$;
- $\mathcal{A}_{p_{\bar{v},j}} = \{r_{\bar{v}}, r_{\bar{v},1}, r_{\bar{v},2}\}\ \ \forall v \in V, \forall j \in [2u]$;
- $\mathcal{A}_{p_i} = \{r_t\}\ \ \forall i \in [z]$;
- $\alpha_{r_v} = \alpha_{r_{\bar{v}}} = \epsilon$ and $\beta_{r_v} = \beta_{r_{\bar{v}}} = z + 1 - \epsilon\ \ \forall v \in V$;
- $\alpha_{r_{v,1}} = \alpha_{r_{v,2}} = \alpha_{r_{\bar{v},1}} = \alpha_{r_{\bar{v},2}} = 1\ \ \forall v \in V$;
- $\beta_{r_{v,1}} = \beta_{r_{v,2}} = \beta_{r_{\bar{v},1}} = \beta_{r_{\bar{v},2}} = z + 1 - u\ \ \forall\, v \in V$;
- $\alpha_{r_t} = 1$ and $\beta_{r_t} = 0$.

Figure 7.2 shows a picture representing how the players' action sets are constructed in games $\Gamma(C, V)$, where, for simplicity, only the part referring to a single variable $v \in V$ and a single clause $\varphi \in C$ is reported.

[4] The reduction in Lemma 7.7 does *not* rely on standard constructions, as most of the reductions for congestion games only work with action profiles, while ours needs randomization. Indeed, in asymmetric SCGs, a social-cost-minimizing action profile can be computed in poly-time by solving an instance of min-cost flow problem. This also prevents the use of other techniques for proving the hardness of CCEs, *e.g.*, those by Barman and Ligett (2015).

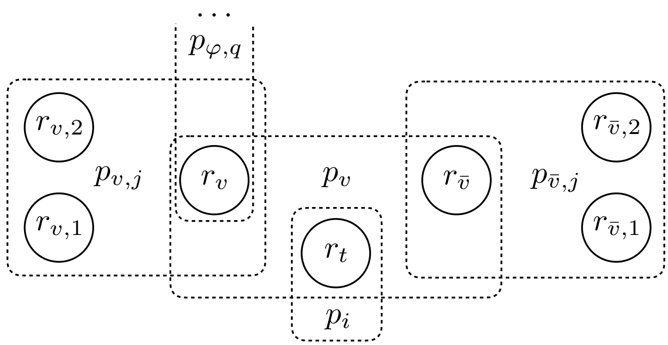


Figure 7.2: *Example of players' action sets in a game instance $\Gamma(C, V)$ used for the reduction in the proof of Lemma 7.7.*

Overview. Intuitively, in games $\Gamma(C, V)$ the social cost is small if players p_i (for $i \in [z]$) are the only ones selecting resource r_t. Then, each player p_v (for $v \in V$) must choose either r_v or $r_{\bar{v}}$ (rather than r_t), representing the fact that variable v is set to either false or true, respectively. At the same time, players p_v do not deviate to resource r_t only if they are the only players selecting their resources. This implies that all the players $p_{\varphi,q}$ (for $\varphi \in C$ and $q \in [3]$) must play a resource not selected by any player p_v. Hence, each player $p_{\varphi,q}$ plays a resource r_l whose corresponding literal l is true, which results in φ being satisfied. The action profile defined thus far does *not* constitute an equilibrium, as players $p_{\varphi,q}$ have an incentive to deviate to resources r_l with l evaluating to false. Players $p_{v,j}$ and $p_{\bar{v},j}$ are used to avoid such deviations. They are told to play resources r_l with very small probability, so that other players do not deviate to them.

If. Suppose (C, V) is satisfiable, and let $\tau : V \to \{\mathsf{T}, \mathsf{F}\}$ be a truth assignment satisfying all the clauses in C. For the ease of presentation, we let $\tau(l) \in \{\mathsf{T}, \mathsf{F}\}$ be the truth value of literal $l \in \{v, \bar{v} \mid v \in V\}$ under τ. Using τ, we recover a CCE $\phi \in \Delta_{\mathcal{A}}$ with social cost smaller than or equal to γ. This selects the action profiles $\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3\} \subseteq \times_{p \in N} \mathcal{A}_p$ defined in the following with probabilities $\phi_{\mathbf{a}^1} = \phi_{\mathbf{a}^2} = \frac{1}{2} - \frac{1}{2m^{10}}$ and $\phi_{\mathbf{a}^3} = \frac{1}{m^{10}}$. First, we determine actions for players $p_{\varphi,q}$ (the same in $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$). Each player $p_{\varphi,q}$ (for $\varphi \in C$ and $q \in [3]$) plays a resource r_l with $l \in \varphi$ such that $\tau(l) = \mathsf{T}$, so that none of these players has an incentive to deviate to another resource r_l with $\tau(l) = \mathsf{T}$. Moreover, players' actions are such that each r_l with $\tau(l) = \mathsf{T}$ has at least one player using it, which is useful to avoid that other players deviate on the resource. To formally define players $p_{\varphi,q}$' actions, we consider a congestion game Γ_{R} restricted to the players $\{p_{\varphi,q} \mid \varphi \in C, q \in [3]\}$ with action spaces limited to resources

$r_l \in A_{p_{\varphi,q}}$ with $\tau(l) = \mathsf{T}$ (since τ satisfies all clauses, each player has at least one action). Clearly, Γ_R admits a pure NE (Rosenthal, 1973). We show that, in any pure NE, each resource is selected by at least one player. By contradiction, suppose that there exists a resource r_l such that no player chooses it. Then, there must be at least two players $p_{\varphi,q}$ (with $l \in \varphi$) selecting some resource different from r_l. As a result, there must be one player with an incentive to deviate to the empty resource (as she would pay $z+1$ rather than something $\geq z+1+\epsilon$), contradicting the NE assumption. In conclusion, for every $\varphi \in C$ and $q \in [3]$, we let $a^1_{p_{\varphi,q}}$, $a^2_{p_{\varphi,q}}$, and $a^3_{p_{\varphi,q}}$ all be equal to the resource played by the corresponding player in some pure NE of Γ_R. Now, we define actions for players $p_{v,j}$ and $p_{\bar{v},j}$. Each player $p_{l,j}$ plays r_l in a^3 (drawn with a small probability of $\frac{1}{m^{10}}$) only if $\tau(l) = \mathsf{F}$, while this never happens in a^1 and a^2. Intuitively, this avoids that other players deviate to a resource r_l with $\tau(l) = \mathsf{F}$. Moreover, players $p_{l,j}$ are split into two groups alternating between resources $r_{l,1}$ and $r_{l,2}$ in action profiles $\mathbf{a}^1$ and $\mathbf{a}^2$. This prevents deviations to either $r_{l,1}$ or $r_{l,2}$ (as there are at least u players using the resource with high probability). Formally, for every $l \in \{v, \bar{v} \mid v \in V\}$:

- for $j \in [u]$, we let $a^1_{p_{l,j}} = r_{l,1}$, $a^2_{p_{l,j}} = r_{l,2}$, and $a^3_{p_{l,j}} = r_l$ if $\tau(l) = \mathsf{F}$, while $a^3_{p_{l,j}} = r_{l,1}$ if $\tau(l) = \mathsf{T}$;
- for $j \in [2u] : j > u$, we let $a^1_{p_{l,j}} = r_{l,2}$, $a^2_{p_{l,j}} = r_{l,1}$, and $a^3_{p_{l,j}} = r_l$ if $\tau(l) = \mathsf{F}$, while $a^3_{p_{l,j}} = r_{l,2}$ if $\tau(l) = \mathsf{T}$.

Finally, we introduce players p_v' actions. In $\mathbf{a}^1$ and $\mathbf{a}^2$ (selected with high probability $1 - \frac{1}{m^{10}}$), each player p_v uses r_v if $\tau(v) = \mathsf{F}$, while $r_{\bar{v}}$ otherwise. Instead, in a^3 (drawn with a small probability of $\frac{1}{m^{10}}$), player p_v selects r_t so as to keep the cost of players $p_{l,j}$ small. Thus, for every $v \in V$, we let $a^3_{p_v} = r_t$ and $a^1_{p_v} = a^2_{p_v} = r_v$ if $\tau(v) = \mathsf{F}$, while $a^1_{p_v} = a^2_{p_v} = r_{\bar{v}}$ if not. Next, we show that players have no incentive to defect from ϕ, *i.e.*, ϕ is a CCE. Given that player $p_{\varphi,q}$'s action (for $\varphi \in C$ and $q \in [3]$) is determined by a pure NE of Γ_R, she does not have any incentive to deviate to another resource $r_l \in A_{\varphi,q}$ with $\tau(l) = \mathsf{T}$ (as these resources are not selected by players not participating to Γ_R and the players in Γ_R are at an NE). Moreover, in ϕ, player $p_{\varphi,q}$'s expected cost is at most $z + 1 + 3\epsilon m$, while she would pay at least $(z+1+\epsilon)(1 - \frac{1}{m^{10}}) + (z+1+2u\epsilon)\frac{1}{m^{10}} \geq z+1+2\epsilon m^2$ by selecting a resource $r_l \in A_{\varphi,q}$ with $\tau(l) = \mathsf{F}$. Each player p_v (for $v \in V$) does not defect from ϕ, since her expected cost is $(z+1)(1 - \frac{1}{m^{10}}) + (z+s)\frac{1}{m^{10}}$, while she would pay:

- the same amount by switching to resource r_t;

- at least $z+1+\epsilon$ by playing r_l with $l \in \{v, \bar{v}\}$ and $\tau(l) = \mathsf{T}$ (as there is at least one player $p_{\varphi,q}$ on r_l);
- at least $(z+1)(1-\frac{1}{m^{10}})+(z+1+2u\epsilon)\frac{1}{m^{10}} = z+1+2\frac{1}{m^2}$ by selecting r_l with $l \in \{v, \bar{v}\}$ and $\tau(l) = \mathsf{F}$.

Each player $p_{l,j}$ (for $l \in \{v, \bar{v} \mid v \in V\}$ and $j \in [2u]$) with $\tau(l) = \mathsf{F}$ does not deviate, since her cost is $(z+1)(1-\frac{1}{m^{10}})+(z+1-\epsilon+2u\epsilon)\frac{1}{m^{10}}$, while she would pay:

- at least $(z+1)(\frac{1}{2}-\frac{1}{2m^{10}})+(z+2)(\frac{1}{2}-\frac{1}{2m^{10}})$ by switching to either $r_{l,1}$ or $r_{l,2}$;
- at least $(z+1+\epsilon)(1-\frac{1}{m^{10}})+(z+1-\epsilon+2u\epsilon)\frac{1}{m^{10}}$ by selecting resource $r_l \in A_{p_{l,j}}$.

Moreover, each player $p_{l,j}$ with $\tau(l) = \mathsf{T}$ does not deviate either, as her cost is $(z+1)$, while she would pay:

- at least $z+1+\epsilon$ by playing resource r_l;
- at least $(z+1)(\frac{1}{2}+\frac{1}{2m^{10}})+(z+2)(\frac{1}{2}-\frac{1}{2m^{10}})$ by switching to either $r_{l,1}$ or $r_{l,2}$.

Finally, players p_i must select resource r_t; thus, they experience a cost of $z(1-\frac{1}{m^{10}})+(z+s)\frac{1}{m^{10}}$. Moreover, since the maximum cost of a resource different from r_t is $z+1+u$, players p_v incur a cost at most of $(z+1+u)(1-\frac{1}{m^{10}})+(z+s)\frac{1}{m^{10}}$, while all the other players pay at most $z+1+u$. Then, the CCE ϕ provides a social cost smaller than or equal to

$$
\begin{aligned}
z\left[z(1-\frac{1}{m^{10}})+(z+s)\frac{1}{m^{10}}\right] + s\left[(z+1+u)(1-\frac{1}{m^{10}})+(z+s)\frac{1}{m^{10}}\right] \\
+ (4us+3m)(z+1+u) \\
\leq z^2 + \frac{zs}{m^{10}} + (z+1+u)(s+4us+3m) + s(z+s)\frac{1}{m^{10}} \\
= z^2 + (s+4us+3m)(z-u) + (2u+1)(s+4us+3m) \\
+ s(2z+s)\frac{1}{m^{10}} \leq \gamma,
\end{aligned}
$$

where the last inequality follows from

$$
\begin{aligned}
&(2u+1)(s+4us+3m) + s(2z+s)\frac{1}{m^{10}} \\
&\qquad \leq (2m^{12}+1)(m+4m^{13}+3m) + m(2z+m)\frac{1}{m^{10}} \leq \frac{3z}{m^9},
\end{aligned}
$$

for m large enough.

Only if. Suppose there exists a CCE $\phi \in \Delta_{\mathcal{A}}$ with social cost smaller than or equal to γ. First, we prove that, with probability at most $\frac{1}{m^8}$, at least one player p_v plays r_t. By contradiction, assume that this is not the case. Then, the social cost would be at least

$$(z^2 + (4us + s + 3m)(z - u))(1 - \frac{1}{m^8}) + ((z+1)^2 + (4us + s + 3m - 1)(z - u))\frac{1}{m^8}$$
$$\geq z^2 + (4us + s + 3m)(z - u) + (2z - z)\frac{1}{m^8} > \gamma.$$

This implies that each player p_v is playing either r_v or $r_{\bar{v}}$ with probability at least $1 - \frac{1}{m^8}$. Then, we prove that p_v is the only player on that resource with probability at least $1 - \frac{1}{m^8} - \frac{1}{m^2}$. Otherwise, by contradiction, her cost would be at least $z + 1 + \frac{\epsilon}{m^2} = z + 1 + \frac{1}{m^6}$, while by playing r_t she would pay at most $(z+1)(1 - \frac{1}{m^8}) + (z+s)\frac{1}{m^8} \leq z + 1 + \frac{1}{m^7}$. By a union bound, there exists an action profile $a \in \times_{p \in N} A_p$ played with probability at least $1 - s(\frac{1}{m^8} + \frac{1}{m^2}) > 0$ in which all the players p_v are alone on their resources (either r_v or $r_{\bar{v}}$). Let $\tau : V \rightarrow \{\mathsf{T}, \mathsf{F}\}$ be a truth assignment such that $\tau(v) = \mathsf{T}$ if $a_{p_v} = r_{\bar{v}}$ and $\tau(v) = \mathsf{F}$ if $a_{p_v} = r_v$. Then, τ satisfies all the clauses, since all the players $p_{\varphi,q}$ play r_l with $\tau(l) = \mathsf{T}$ and, thus, each clause has at least a true literal. □

The following lemma concludes the proof of Theorem 7.2.

Lemma 7.8. *Any SCG can be represented as an NCG of size polynomial in the size of the original SCG.*

Proof. Given an SCG $(N, R, \{\mathcal{A}_p\}_{p \in N}, \{c_r\}_{r \in R})$ we build an NCG $(N, G, \{c_e\}_{e \in E}, \{(s_p, t_p)\}_{p \in N})$ as follows. The graph $G = (V, E)$ has two nodes $v_{r,1}, v_{r,2} \in V$ for each resource $r \in R$, and, additionally, for every player $p \in N$, there is a source node $s_p \in V$ and a destination one $t_p \in V$. Moreover, there is an edge $(v_{r,1}, v_{r,2}) \in E$ for every $r \in R$ and, for every $p \in N$ and $r \in A_p$, there two edges $(s_p, v_{r,1}) \in E$ and $(v_{r,2}, t_p) \in E$. Finally, for the edges $e = (v_{r,1}, v_{r,2})$, we let $c_e = c_r$, while $c_e = 0$ for all the other edges. Clearly, the size of the NCG is polynomially bounded by that of the original SCG, proving the result. □

CHAPTER 8

Persuading in Posted Price Auctions

In this chapter, we focus on the problem of computing revenue-maximizing public and private signaling schemes in posted price auctions. Differently from the problems studied in the previous chapters, the sender has to deal with an additional challenge. In particular, the seller can coordinate the signaling scheme with actions, *i.e.*, the prices proposed to the buyers. In Section 8.1, we introduce Bayesian posted price auctions and the signaling problem. In Section 8.2, we show that there is no additive FPTAS for the problem of computing a revenue-maximizing signaling scheme, unless P=NP. In Section 8.3, we introduce a general framework that will be useful to compute both public and private signaling schemes. Section 8.4 presents some results on non-Bayesian posted price auction. Finally, Sections 8.5 and 8.6 present two PTASs for public and private signaling, respectively.

8.1 Model of Bayesian Persuasion in Posted Price auctions

In a *posted price auction*, the seller tries to sell an item to a finite set $\mathcal{R} := \{1, \dots, \bar{n}\}$ of buyers arriving sequentially according to a fixed ordering. W.l.o.g., we let buyer $i \in \mathcal{R}$ be the i-th buyer according to such ordering. The seller chooses a price proposal $p_i \in [0, 1]$ for each buyer $i \in \mathcal{R}$. Then, each buyer in turn has to decide whether to buy the item for the proposed price or not. Buyer $i \in \mathcal{R}$ buys only if their item valuation is at least the proposed price p_i.[1] In that case, the auction ends and the seller gets revenue p_i for selling the item, otherwise the auction continues with the next buyer.

We study *Bayesian* posted price auctions, characterized by a finite set of d states of nature, namely $\Theta := \{\theta_1, \ldots, \theta_d\}$. Each buyer $i \in \mathcal{R}$ has a valuation vector $v_i \in [0,1]^d$, with $v_i(\theta)$ representing buyer i's valuation when the state is $\theta \in \Theta$. Each valuation v_i is independently drawn from a probability distribution $\mathcal{V}_i$ supported on $[0,1]^d$. For the ease of presentation, we let $V \in [0,1]^{\bar{n} \times d}$ be the matrix of buyers' valuations, whose entries are $V(i,\theta) := v_i(\theta)$ for all $i \in \mathcal{R}$ and $\theta \in \Theta$.[2] Moreover, by letting $\mathcal{V} := \{\mathcal{V}_i\}_{i \in \mathcal{R}}$ be the collection of all distributions of buyers' valuations, we write $V \sim \mathcal{V}$ to denote that V is built by drawing each v_i independently from $\mathcal{V}_i$.

We consider the case in which the seller—having knowledge of the state of nature—acts as a *sender* by issuing signals to the buyers (the *receivers*). The seller *commits to a signaling scheme* ϕ, which is a randomized mapping from states of nature to signals for the receivers. Moreover, the seller commits to price proposals. Price proposals may depend on the signals being sent to the buyers. Formally, the seller *commits to a price function* $\mathsf{f} : \mathcal{S} \to [0,1]^{\bar{n}}$, with $\mathsf{f}(\mathrm{s}) \in [0,1]^{\bar{n}}$ being the price vector when the signal profile is $\mathrm{s} \in \mathcal{S}$. We assume that prices proposed to buyer i only depend on the signals sent to them, and *not* on the signals sent to other buyers. Thus, w.l.o.g., we can work with functions $\mathsf{f}_i : \mathcal{S}_i \to [0,1]$ defining prices for each buyer $i \in \mathcal{R}$ independently, with $\mathsf{f}_i(s_i)$ denoting the i-th component of $\mathsf{f}(\mathrm{s})$ for all $\mathrm{s} \in \mathcal{S}$ and $i \in \mathcal{R}$.[3]

The interaction involving the seller and the buyers goes on as follows (Figure 8.1): (i) the seller commits to a signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ and a price function $\mathsf{f} : \mathcal{S} \to [0,1]^{\bar{n}}$, and the buyers observe such commitments; (ii) the seller observes the state of nature $\theta \sim \boldsymbol{\mu}$; (iii) the seller draws a signal profile $\mathrm{s} \sim \phi_\theta$; and (iv) the buyers arrive sequentially, with each buyer $i \in \mathcal{R}$ observing their signal s_i and being proposed price $\mathsf{f}_i(s_i)$. Then, each buyer rationally updates their prior belief over states according to Bayes rule, and buys the item only if their expected valuation for the item is greater than or equal to the offered price. The interaction terminates

[1] As customary in the literature, we assume that buyers always buy when they are offered a price that is equal to their valuation.

[2] Sometimes, we also write $V_i := v_i^\top$ to denote the i-th row of matrix V, which is the valuation of buyer $i \in \mathcal{R}$.

[3] Let us remark that our assumption on the seller's price function ensures that a buyer does *not* get additional information about the state of nature by observing the proposed price, since the latter only depends on the signal which is revealed to them anyway.

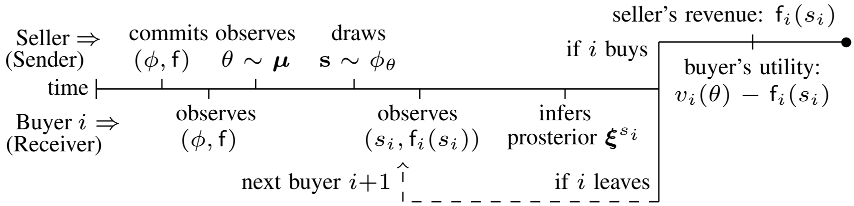


Figure 8.1: *Interaction between the seller and the buyers.*

whenever a buyer decides to buy the item or there no more buyers arriving.

In this chapter, we will work extensively in the space of the posteriors with multiple receivers. With abuse of notation, we denote with $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{\bar{n}})$ the tuple specifying a $\boldsymbol{\xi}_i \in \Xi$ for each $i \in \mathcal{R}$ and with $\xi_{i,\theta}$ the probability of state θ in posterior $\boldsymbol{\xi}_i$. Finally, we denote with $\boldsymbol{\xi}^{\mathrm{s}}$ the tuple of posteriors induced by signal profile $\mathrm{s} \in \mathcal{S}$ and with $\boldsymbol{\xi}^{s_i}$ the posterior induced by signal s_i. Since, given a signal profile $\mathrm{s} \in \mathcal{S}$, under a public signaling scheme all the buyers always share the same posterior (*i.e.*, $\boldsymbol{\xi}^{s_i} = \boldsymbol{\xi}^{s_j}$ for all $i, j \in \mathcal{R}$), we overload notation and sometimes use $\boldsymbol{\xi}^{\mathrm{s}} \in \Delta_\Theta$ to denote the unique posterior appearing in $\boldsymbol{\xi}^{\mathrm{s}} = (\boldsymbol{\xi}^{s_1}, \dots, \boldsymbol{\xi}^{s_{\bar{n}}})$. Similarly, in the public setting, given a posterior $\boldsymbol{\xi} \in \Delta_\Theta$ we sometimes write $\boldsymbol{\xi}$ in place of a tuple of $\bar{n}$ copies of $\boldsymbol{\xi}$.

8.1.1 Computational Problems

We focus on the problem of computing a signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ and a price function $\mathsf{f} : \mathcal{S} \to [0,1]^{\bar{n}}$ that maximize the seller's expected revenue, considering both public and private signaling settings.

We denote by $\textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$ the expected revenue of the seller when the distributions of buyers' valuations are given by $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$, the proposed prices are defined by the vector $\mathbf{p} \in [0,1]^{\bar{n}}$, and the buyers' posteriors are those specified by the tuple $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{\bar{n}})$ containing a posterior $\boldsymbol{\xi}_i \in \Delta_\Theta$ for each buyer $i \in \mathcal{R}$. Then, the seller's expected revenue is:

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\mathrm{s} \in \mathcal{S}} \phi_\theta(\mathrm{s}) \textsc{Rev}\left(\mathcal{V}, \mathsf{f}(\mathrm{s}), \boldsymbol{\xi}^{\mathrm{s}}\right).$$

In the following, we denote by OPT the value of the seller's expected revenue for a revenue-maximizing (ϕ, f) pair.

In this chapter, we assume that algorithms have access to a black-box oracle to sample buyers' valuations according to the probability distributions specified by $\mathcal{V}$ (rather than actually knowing such distributions). Thus, we look for algorithms that output pairs (ϕ, f) such that

$$\mathbb{E}\left[\sum_{\theta \in \Theta} \mu_\theta \sum_{\mathbf{s} \in \mathcal{S}} \phi_\theta(\mathbf{s}) \text{Rev}(\mathcal{V}, \mathsf{f}(\mathbf{s}), \boldsymbol{\xi}^{\mathbf{s}})\right] \geq OPT - \lambda,$$

where $\lambda \geq 0$ is an additive error. Notice that the expectation above is with respect to the randomness of the algorithm, which originates from using the black-box sampling oracle.

8.2 Hardness of Signaling with a Single Buyer

We start with a negative result: there is no FPTAS for the problem of computing a revenue-maximizing (ϕ, f) pair unless $\mathsf{P} = \mathsf{NP}$, in both public and private signaling settings. Our result holds even in the basic case with only one buyer, where public and private signaling are equivalent. Notice that, in the reduction that we use to prove our result, we assume that the support of the distribution of valuations of the (single) buyer is finite and that such distribution is perfectly known to the seller. This represents an even simpler setting than that in which the seller has only access to a black-box oracle returning samples drawn from the buyer's distribution of valuations. The result formally reads as follows:

Theorem 8.1. *There is no additive FPTAS for the problem of computing a revenue-maximizing* (ϕ, f) *pair unless* $\mathsf{P} = \mathsf{NP}$*, even when there is a single buyer.*

Proof. We employ a reduction from an NP-hard problem originally introduced by Khot and Saket (2012), which we formally state in the following. For any positive integer $k \in \mathbb{N}_{>0}$, integer $l \in \mathbb{N}$ such that $l \geq 2^k + 1$, and arbitrarily small constant $\epsilon > 0$, the problem reads as follows. Given an undirected graph $G := (U, E)$, distinguish between:

- *Case 1.* There exists a l-colorable induced subgraph of G containing a $1 - \epsilon$ fraction of all vertices, where each color class contains a $\frac{1-\epsilon}{l}$ fraction of all vertices.[4]
- *Case 2.* Every independent set of G contains less than a $\frac{1}{l^{k+1}}$ fraction of all vertices.[5]

[4]A *l-colorable induced subgraph* is identified by a subset of vertices such that it is possible to assign one among l different colors to each vertex, in such a way that there are *no* two adjacent vertices having the same color. Given some color, its associated color class is the subset of all vertices in the subgraph having that color.

[5]An *independent set* of G is a subset of vertices such that there are *no* two adjacent vertices.

We reduce from such problem for $k = 2, l = 5$, and $\epsilon = \frac{1}{2}$. Our reduction works as follows:

- *Completeness*. If *Case 1* holds, then there exists a signaling scheme, price function pair (ϕ, f) that provides the seller with an expected revenue at least as large as some threshold η (see Equation (8.1) below for its definition).
- *Soundness*. If *Case 2* holds, then the seller's expected revenue for any signaling scheme, price function pair (ϕ, f) is smaller than $\eta - \delta$ with $\delta := \frac{1}{m^2}$, where m denotes the number or vertices of the graph G.

This shows that it is NP-hard to approximate the optimal seller's expected revenue up to within an additive error δ. Thus, since δ depends polynomially on the size of the problem instance, this also shows that there is no additive FPTAS for the problem of computing a revenue-maximizing (ϕ, f) pair, unless $\mathsf{P} = \mathsf{NP}$.

Construction Given an undirected graph $G := (U, E)$, with vertices $U := \{u_1, \dots, u_m\}$, we build a single-buyer Bayesian posted price auction as follows.[6] There is one state of nature $\theta_u \in \Theta$ for each vertex $u \in U$, and the prior belief over states $\boldsymbol{\mu} \in \Delta_\Theta$ is such that $\mu_{\theta_u} = \frac{1}{m}$ for all $u \in U$. There is a finite set of possible buyer's valuations. For every vertex $u \in U$, there is a valuation vector $v_u \in [0, 1]^m$ such that:

- $v_u(\theta_u) = 1$;
- $v_u(\theta_{u'}) = \frac{1}{2}$ for all $u' \in U : (u, u') \notin E$; and
- $v_u(\theta_{u'}) = 0$ for all $u' \in U : (u, u') \in E$.

Each valuation v_u has probability $\frac{1}{m^2}$ of occurring according to the distribution $\mathcal{V}$. Moreover, there is an additional valuation vector $v_o \in [0, 1]^m$ such that $v_o(\theta) = \frac{1}{2} + \frac{l}{(1-\epsilon)2m}$ for all $\theta \in \Theta$, having probability $1 - \frac{1}{m}$.

Completeness Assume that a l-colored induced subgraph of G is given, and that it contains a fraction $1 - \epsilon$ of vertices, while each color class is made up of a fraction $\frac{1-\epsilon}{l}$ of all vertices. We let $L := \{1, \dots, l\}$ be the set of possible colors, with $j \in L$ denoting a generic color. In the following, we show how to build a signaling scheme, price function pair (ϕ, f) that provides the seller with an expected revenue greater than or equal to a suitably-defined

[6] In a single-buyer setting, we always omit the subscript i from symbols, as it is clear that they refer to the unique buyer. Moreover, with an overload of notation, we use buyer's signals as if they were signal profiles.

threshold η (Equation (8.1)). The seller has $l+1$ signals available, namely $\mathcal{S} := \{s_j\}_{j \in L} \cup \{s_o\}$. For every vertex $u \in U$, if u has been assigned some color $j \in L$ (that is, u belongs to the induced subgraph), then we set $\phi_{\theta_u}(s_j) = 1$ and $\phi_{\theta_u}(s) = 0$ for all $s \in \mathcal{S} : s \neq s_j$; otherwise, if u has no color (that is, u does not belong to the given subgraph), then we set $\phi_{\theta_u}(s_o) = 1$ and $\phi_{\theta_u}(s) = 0$ for all $s \in \mathcal{S} : s \neq s_o$. Moreover, the price function is such that

$$\mathsf{f}(s) = p_o := \frac{1}{2} + \frac{l}{(1-\epsilon)2m} \text{ for every signal } s \in \mathcal{S}.$$

Next, we prove that, after receiving a signal $s_j \in \mathcal{S}$ associated with some color $j \in L$, if the buyer has valuation v_u for a node $u \in U$ colored of color j, then they will buy the item. In particular, the buyer's posterior $\boldsymbol{\xi}^{s_j} \in \Delta_\Theta$ induced by signal s_j is such that only state θ_u and states $\theta_{u'}$ for $u' \in U : (u, u') \notin E$ have positive probability (since, when the seller sends signal s_j, it must be the case that the vertex corresponding to the actual state of nature is colored of color j). Moreover, such probabilities are equal to $\xi^{s_j}_{\theta_u} = \xi^{s_j}_{\theta_{u'}} = \frac{l}{(1-\epsilon)m}$ (by applying Equation (3.1) and using the fact that each color class has a fraction $\frac{1-\epsilon}{l}$ of vertices). Thus, since $v_u(\theta_u) = 1$ and $v_u(\theta_{u'}) = \frac{1}{2}$ for all $u' \in U : (u, u') \notin E$, the expected valuation of the buyer given the posterior $\boldsymbol{\xi}^{s_j}$ is

$$\sum_{\theta \in \Theta} v_u(\theta)\, \xi^{s_j}_\theta = \frac{1}{2}\left[1 - \frac{l}{(1-\epsilon)m}\right] + \frac{l}{(1-\epsilon)m} = \frac{1}{2} + \frac{l}{(1-\epsilon)2m} = p_o,$$

and the buyer will buy the item. Furthermore, when the seller sends signal s_o, their expected revenue is at least $(1 - \frac{1}{m})p_o$, as it is always the case that the buyer buys the item when they have valuation v_o. Since the total probability of sending signals $s_j \in \mathcal{S}$ for $j \in L$ is $1-\epsilon$ (given that the subgraph contains a fraction $1-\epsilon$ of vertices) and the probability of sending signal s_o is ϵ, we have that the seller's expected revenue is at least

$$\eta := (1-\epsilon)\left[\frac{1-\epsilon}{m\,l} + 1 - \frac{1}{m}\right] p_o + \epsilon\left(1 - \frac{1}{m}\right) p_o \tag{8.1a}$$

$$= \left[\frac{(1-\epsilon)^2}{m\,l} + \left(1 - \frac{1}{m}\right)\right] p_o. \tag{8.1b}$$

where the factor $\frac{1-\epsilon}{m\,l} + 1 - \frac{1}{m}$ represents the probability that the buyer buys when sending a signal s_j (this happens when either the buyer has valuation v_u for a vertex $u \in U$ colored of color j or the buyer has valuation v_o).

Soundness By contradiction, we show that, if there exists a signaling scheme, price function pair (ϕ, f) with seller's expected revenue exceeding $\eta - \delta$, then the graph G admits an independent set of size $\frac{1}{2l}(1-\epsilon)^2 m > \frac{m}{l^{k+1}}$ (recall the choice of values for k, l, and ϵ). If the seller's revenue is greater than $\eta - \delta$, by an averaging argument there must be at least one signal $s^* \in \mathcal{S}$ whose contribution to the revenue $\sum_{\theta \in \Theta} \mu_\theta \phi_\theta(s^*) \text{REV}\left(\mathcal{V}, \mathsf{f}(s^*), \boldsymbol{\xi}^{s^*}\right)$ is more than $\eta - \delta$, where $\boldsymbol{\xi}^{s^*} \in \Delta_\Theta$ is the buyer's posterior induced by signal s^*. Since the expected revenue cannot exceed the expected payment, the price $p^* := \mathsf{f}(s^*)$ that the seller proposes to the buyer when signal s^* is sent must be greater than

$$\begin{aligned}
\eta - \delta &= \left[\frac{(1-\epsilon)^2}{m\,l} + \left(1 - \frac{1}{m}\right)\right] p_o - \delta \\
&= \left[\frac{(1-\epsilon)^2}{m\,l} + \left(1 - \frac{1}{m}\right)\right] \left[\frac{1}{2} + \frac{l}{(1-\epsilon)2m}\right] - \delta \\
&\geq \left(1 - \frac{1}{m}\right) \left[\frac{1}{2} + \frac{l}{(1-\epsilon)2m}\right] - \delta \\
&\geq \frac{1}{2} + \frac{l}{(1-\epsilon)2m} - \frac{1}{m} - \frac{l}{(1-\epsilon)2m^2} - \delta > \frac{1}{2},
\end{aligned}$$

where the last inequality holds for $m \geq 2$ since we set $\epsilon = \frac{1}{2}$, $l = 5$, and $\delta = \frac{1}{m^2}$. Additionally, the price p^* must be smaller than p_o (see the completeness proof), otherwise, when the buyer has valuation v_o, they would never buy the item, resulting in a contribution to the seller's revenue at most of $\frac{1}{m}$ (recall that v_o happens with probability $1 - \frac{1}{m}$ and all other buyer's valuations do not exceed 1). As a result, it must be the case that $p^* \in \left(\frac{1}{2}, p_o\right]$. Next, we prove that, after receiving signal s^*, the buyer will buy the item in all the cases in which their valuation belongs to a subset of valuations v_u containing at least a fraction $\frac{1}{2l}(1-\epsilon)^2$ of all the valuations v_u. Indeed, if this is not the case, then the contribution to the seller's expected revenue due to signal s^* would be less than

$$\begin{aligned}
p^* \left[1 - \frac{1}{m} + \frac{(1-\epsilon)^2}{2m\,l}\right] &\leq p_o \left[1 - \frac{1}{m} + \frac{(1-\epsilon)^2}{2m\,l}\right] \\
&= \eta - p_o \frac{(1-\epsilon)^2}{2m\,l} \leq \eta - \delta,
\end{aligned}$$

where the last inequality holds since $\delta = \frac{1}{m^2} \leq p_o \frac{(1-\epsilon)^2}{2m\,l}$ for m large enough. Let $U^* \subseteq U$ be the subset of vertices $u \in U$ such that the buyer will buy the item for their corresponding valuations v_u. We have that $|U^*| \geq \frac{m}{2l}(1-\epsilon)^2$.

Next, we show that U^* constitutes an independent set of G. First, since $p^* > \frac{1}{2}$, when the buyer's valuation is v_u such that $u \in U^*$, then the buyer must value the item more than $\frac{1}{2}$, otherwise they would not buy. By contradiction, suppose that there is a couple of vertices $u, u' \in U^*$ such that $(u, u') \in E$. W.l.o.g., let us assume that $\xi^{s^*}_{\theta_u} \leq \xi^{s^*}_{\theta_{u'}}$. Then, the buyer's expected valuation induced by posterior $\boldsymbol{\xi}^{s^*}$ is

$$\xi^{s^*}_{\theta_u} + \frac{1}{2}\left(1 - \xi^{s^*}_{\theta_u} - \xi^{s^*}_{\theta_{u'}}\right) = \frac{1 + \xi^{s^*}_{\theta_u} - \xi^{s^*}_{\theta_{u'}}}{2} \leq \frac{1}{2},$$

which is a contradiction. Given that, by our initial assumption, the size of every independent set must be smaller than $\frac{m}{l^{k+1}} < \frac{1}{2l}(1-\epsilon)^2 m$, we reach the final contradiction proving the result. □

8.3 Unifying Public and Private Signaling

In this section, we introduce a general mathematical framework related to buyers' posteriors and distributions over them, proving some results that will be crucial in the rest of this chapter, both in public and private signaling scenarios.

For public signaling, a useful technique to reduce the space of available signals is to restrict the number of possible posteriors to q-uniform posteriors. For private signaling, the trick commonly used to reduce the space of signals when there are a finite number of valuations is to use direct signals, which explicitly specify action recommendations for each receiver's valuation (see, *e.g.*, Chapter 10). However, in our auction setting, this solution is *not* viable, since a direct signal for a buyer $i \in \mathcal{R}$ should represent a recommendation for every possible $v_i \in [0,1]^d$, and these are infinitely many. For this reason, in this section we provide a decomposition lemma that will be used to work on q-uniform posteriors for both public and private signaling.

Our main result (Theorem 8.2) is a decomposition lemma that is suitable for our setting. Before stating the result, we need to introduce some preliminary definitions.

Definition 8.1 ((α,ϵ)-decreasing distribution)**.** *Let* $\alpha, \epsilon > 0$. *A probability distribution* $\boldsymbol{\gamma}$ *over* Δ_Θ *is* (α,ϵ)-decreasing around a given posterior $\boldsymbol{\xi} \in \Delta_\Theta$ *if the following condition holds for every matrix* $V \in [0,1]^{\bar{n} \times d}$ *of buyers' valuations:*

$$\Pr_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}}\left\{V_i \tilde{\boldsymbol{\xi}} \geq V_i \boldsymbol{\xi} - \epsilon\right\} \geq 1 - \alpha \quad \forall i \in \mathcal{R}.$$

Intuitively, a probability distribution $\boldsymbol{\gamma}$ as in Definition 8.1 can be interpreted as a perturbation of the given posterior $\boldsymbol{\xi}$ such that, with high probability, buyers' expected valuations in $\boldsymbol{\gamma}$ are at most ϵ less than those in posterior $\boldsymbol{\xi}$.[7]

The second definition we need is about functions mapping vectors in $[0,1]^{\bar{n}}$—defining a valuation for each buyer—to seller's revenues. For instance, one such function could be the seller's revenue given price vector $\mathbf{p} \in [0,1]^{\bar{n}}$. In particular, we define the stability of a function g compared to another function h. Intuitively, g is stable compared to h if the value of g, in expectation over buyers' valuations and posteriors drawn from a probability distribution $\boldsymbol{\gamma}$ that is (α, ϵ)-decreasing around ξ, is "close" to the the value of h given $\boldsymbol{\xi}$, in expectation over buyers' valuations.[8] Formally:

Definition 8.2 ($(\delta, \alpha, \epsilon)$-stability)**.** *Let $\alpha, \epsilon, \delta > 0$. Given a posterior $\boldsymbol{\xi} \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$, and two functions $g, h : [0,1]^{\bar{n}} \to [0,1]$, g is $(\delta, \alpha, \epsilon)$*-stable compared to h for $(\boldsymbol{\xi}, \mathcal{V})$ *if, for every probability distribution $\boldsymbol{\gamma}$ over Δ_Θ that is (α, ϵ)-decreasing around $\boldsymbol{\xi}$, it holds:*

$$\mathbb{E}_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}, V \sim \mathcal{V}} \Big[g(V\tilde{\boldsymbol{\xi}}) \Big] \geq (1-\alpha) \mathbb{E}_{V \sim \mathcal{V}} \Big[h(V\boldsymbol{\xi}) \Big] - \delta\epsilon.$$

Now, we are ready to state our main result. We show that, for any buyer's posterior $\boldsymbol{\xi} \in \Delta_\Theta$, if a function g is stable compare to h, then there exists a suitable probability distribution over q-uniform posteriors such that the expected value of g given such distribution is "close" to that of h given $\boldsymbol{\xi}$.

Theorem 8.2. *Let $\alpha, \epsilon, \delta > 0$, and set $q := \frac{32}{\epsilon^2} \log \frac{4}{\alpha}$. Given a posterior $\boldsymbol{\xi} \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$, and two functions $g, h : [0,1]^{\bar{n}} \to [0,1]$, if g is $(\delta, \alpha, \epsilon)$-stable compared to h for $(\boldsymbol{\xi}, \mathcal{V})$, then there exists $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ such that, for every $\theta \in \Theta$, $\sum_{\tilde{\boldsymbol{\xi}} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\tilde{\boldsymbol{\xi}}} \tilde{\xi}_\theta = \xi_\theta$ and*

$$\mathbb{E}_{\substack{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma} \\ V \sim \mathcal{V}}} \Big[\tilde{\xi}_\theta g(V\tilde{\boldsymbol{\xi}}) \Big] \geq \xi_\theta \Big[(1-\alpha) \mathbb{E}_{V \sim \mathcal{V}} \Big[h(V\boldsymbol{\xi}) \Big] - \delta\epsilon \Big]. \tag{8.2}$$

Proof. The probability distribution $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ over q-uniform posteriors in the statement is defined as follows. Let $\boldsymbol{\xi}^q \in \Xi^q$ be a buyer's posterior defined as the empirical mean of q vectors built form q i.i.d. samples drawn from the given posterior $\boldsymbol{\xi}$. In particular, each sample is obtained by randomly drawing a state of nature, with each state $\theta \in \Theta$ having probability

[7] Definition 8.1 is similar to analogous ones in the literature (Xu, 2020), where the distance is usually measured in both directions, as $|V_i\tilde{\xi} - V_i\xi| \leq \epsilon$. We look only at the direction of decreasing values, since in a our setting, if a buyer's valuation increases, then the seller's revenue also increases.

[8] The notion of compared stability has been already used, see, *e.g.*, Chapter 6. However, previous works consider the case in which g is a relaxation of h. Instead, our definition is conceptually different, as g and h represent two different functions corresponding to different price vectors of the seller.

ξ_θ of being selected, and, then, a d-dimensional vector is built by letting all its components equal to 0, except for that one corresponding to θ, which is set to 1. Notice that $\boldsymbol{\xi}^q$ is a random vector supported on q-uniform posteriors, whose expected value is posterior $\boldsymbol{\xi}$. Then, $\boldsymbol{\gamma}$ is such that, for every $\tilde{\boldsymbol{\xi}} \in \Xi^q$, it holds $\gamma_{\tilde{\boldsymbol{\xi}}} = \Pr\left\{\boldsymbol{\xi}^q = \tilde{\boldsymbol{\xi}}\right\}$.

It is easy to check that $\xi_\theta = \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^q} \gamma_{\tilde{\boldsymbol{\xi}}} \tilde{\xi}_\theta = \mathbb{E}_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}}\left[\tilde{\xi}_\theta\right]$ for all $\theta \in \Theta$, proving the first condition needed.

Next, we prove that $\boldsymbol{\gamma}$ satisfies Equation (8.2). To do so, we first introduce some useful definitions. For every q-uniform posterior $\tilde{\boldsymbol{\xi}} \in \Xi^q$, with an overload of notation we let $\gamma_{\tilde{\boldsymbol{\xi}},\theta,j}$ be the conditional probability of having drawn $\tilde{\boldsymbol{\xi}}$ from $\boldsymbol{\gamma}$ given that the drawn posterior assigns probability $\frac{j}{q}$ to state $\theta \in \Theta$ with $j \in \{0, \dots, q\}$. Formally, for every $\tilde{\boldsymbol{\xi}} \in \Xi^q$:

$$\gamma_{\tilde{\boldsymbol{\xi}},\theta,j} := \begin{cases} \dfrac{\gamma_{\tilde{\boldsymbol{\xi}}}}{\sum_{\boldsymbol{\xi}' \in \Xi^q : \xi'_\theta = j/q} \gamma'_{\boldsymbol{\xi}}} & \text{if } \tilde{\xi}_\theta = \frac{j}{q} \\ 0 & \text{otherwise} \end{cases}.$$

Then, for every $\theta \in \Theta$ and $j \in \{0, \dots, q\}$, we let $\boldsymbol{\gamma}^{\theta,j}$ be a probability distribution over Δ_Θ supported on q-uniform posteriors such that $\boldsymbol{\gamma}^{\theta,j}_{\tilde{\boldsymbol{\xi}}} := \gamma_{\tilde{\boldsymbol{\xi}},\theta,j}$ for all $\tilde{\boldsymbol{\xi}} \in \Xi^q$. Moreover, for every buyer $i \in \mathcal{R}$ and matrix $V \in [0,1]^{\bar{n} \times d}$ of buyers' valuations, we let $\Xi^{i,V} \subseteq \Xi^q$ be the set of q-uniform posteriors that do *not* change buyer i's expected valuation by more than an additive factor ϵ with respect to their valuation in posterior ξ. Formally,

$$\Xi^{i,V} := \left\{\tilde{\boldsymbol{\xi}} \in \Xi^q \mid \left|V_i \boldsymbol{\xi} - V_i \tilde{\boldsymbol{\xi}}\right| \leq \epsilon\right\}.$$

In order to complete the proof, we introduce the following three lemmas. The first lemma shows that, for every state of nature $\theta \in \Theta$, it is possible to bound the cumulative probability mass that the distribution γ assigns to q-uniform posteriors $\tilde{\boldsymbol{\xi}} \in \Xi^q$ such that $\tilde{\xi}_\theta$ differs from ξ_θ by at least $\frac{\epsilon}{4}$ (in absolute terms). Formally:

Lemma 8.1 (Essentially Lemma 6.2). *Given $\boldsymbol{\xi} \in \Delta_\Theta$, for every $\theta \in \Theta$ it holds:*

$$\sum_{j : \left|\frac{j}{q} - \xi_\theta\right| \geq \frac{\epsilon}{4}} \; \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^q : \hat{\xi}_\theta = \frac{j}{q}} \gamma_{\tilde{\boldsymbol{\xi}}} \leq \frac{\alpha}{2} \xi_\theta,$$

where $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ is the probability distribution over q-uniform posteriors introduced at the beginning of the proof.

The second lemma, which is useful to prove Lemma 8.3, shows that, for q-uniform posteriors $\tilde{\boldsymbol{\xi}} \in \Xi^q$ such that $\tilde{\xi}_\theta$ is sufficiently close to ξ_θ for a state of nature $\theta \in \Theta$, the expected utility of each buyer is close to their utility in the given posterior $\boldsymbol{\xi}$ with high probability. Formally:

Lemma 8.2 (Essentially Lemma 6.3)**.** *Given $\xi \in \Delta_\Theta$, matrix $V \in [0,1]^{\bar{n} \times d}$ of buyers' valuations, state of nature $\theta \in \Theta$, and $j : \left|\frac{j}{q} - \xi_\theta\right| \leq \frac{\epsilon}{4}$, the following holds for every buyer $i \in \mathcal{R}$:*

$$\sum_{\tilde{\boldsymbol{\xi}} \in \Xi^{i,V} : \tilde{\xi}_\theta = \frac{j}{q}} \gamma_{\tilde{\boldsymbol{\xi}}} \geq \left(1 - \frac{\alpha}{2}\right) \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^q : \tilde{\xi}_\theta = \frac{j}{q}} \gamma_{\tilde{\boldsymbol{\xi}}},$$

where $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ is the probability distribution over q-uniform posteriors introduced at the beginning of the proof.

Finally, the third lemma that we need reads as follows:

Lemma 8.3. *Given $\boldsymbol{\xi} \in \Delta_\Theta$, for every state of nature $\theta \in \Theta$ and j : $\left|\frac{j}{q} - \xi_\theta\right| \leq \frac{\epsilon}{4}$, the probability distribution $\boldsymbol{\gamma}^{\theta,j}$ defined at the beginning of the proof is $\left(\frac{\alpha}{2}, \epsilon\right)$-decreasing around posterior $\boldsymbol{\xi}$.*

Proof. According to Definition 8.1 and by reversing the inequalities, we need to prove that, for every matrix $V \in [0,1]^{\bar{n} \times d}$ of buyers' valuations and buyer $i \in \mathcal{R}$, it holds $\Pr_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}^{\theta,j}} \left\{ V_i \tilde{\boldsymbol{\xi}} \geq V_i \boldsymbol{\xi} - \epsilon \right\} \geq 1 - \frac{\alpha}{2}$. By using the definition of the set $\Xi^{i,V}$, Lemma 8.2, and the definition of $\boldsymbol{\gamma}^{\theta,j}$, we can write the following:

$$\begin{aligned}
\Pr_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}^{\theta,j}} \left\{ V_i \tilde{\boldsymbol{\xi}} \geq V_i \boldsymbol{\xi} - \epsilon \right\} &= \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^{i,V}} \gamma^{\theta,j}_{\tilde{\boldsymbol{\xi}}} = \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^{i,V} : \tilde{\xi}_\theta = \frac{j}{q}} \frac{\gamma_{\tilde{\boldsymbol{\xi}}}}{\sum_{\boldsymbol{\xi}' \in \Xi^q : \xi'_\theta = \frac{j}{q}} \gamma_{\boldsymbol{\xi}'}} \\
&\geq \left(1 - \frac{\alpha}{2}\right) \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^q : \tilde{\xi}_\theta = \frac{j}{q}} \frac{\gamma_{\tilde{\boldsymbol{\xi}}}}{\sum_{\boldsymbol{\xi}' \in \Xi^q : \xi'_\theta = \frac{j}{q}} \gamma_{\boldsymbol{\xi}'}} = 1 - \frac{\alpha}{2},
\end{aligned}$$

which proves the lemma. □

Now, we are ready to prove the theorem, by means of the following inequalities:

$$\mathbb{E}_{\tilde{\boldsymbol{\xi}} \sim \gamma, V \sim \mathcal{V}} \left[\tilde{\xi}_\theta \, g(V \tilde{\boldsymbol{\xi}}) \right] = \sum_{\tilde{\boldsymbol{\xi}} \in \Xi^q} \gamma_{\tilde{\boldsymbol{\xi}}} \, \tilde{\xi}_\theta \mathbb{E}_{V \sim \mathcal{V}} \left[g(V \tilde{\boldsymbol{\xi}}) \right]$$

(By dropping terms from the sum)

$$
\begin{aligned}
&\geq \sum_{j:\left|\frac{j}{q}-\xi_\theta\right|\leq\frac{\epsilon}{4}} \frac{j}{q} \sum_{\tilde{\boldsymbol{\xi}}\in\Xi^q:\tilde{\xi}_\theta=\frac{j}{q}} \gamma_{\tilde{\boldsymbol{\xi}}}\, \mathbb{E}_{V\sim\mathcal{V}}\Big[g(V\tilde{\boldsymbol{\xi}})\Big] \\
&= \sum_{j:\left|\frac{j}{q}-\xi_\theta\right|\leq\frac{\epsilon}{4}} \frac{j}{q} \left(\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=\frac{j}{q}} \gamma_{\boldsymbol{\xi}'}\right) \sum_{\tilde{\boldsymbol{\xi}}\in\Xi^q:\tilde{\xi}_\theta=\frac{j}{q}} \frac{\gamma_{\tilde{\boldsymbol{\xi}}}}{\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=\frac{j}{q}}\gamma_{\boldsymbol{\xi}'}} \mathbb{E}_{V\sim\mathcal{V}}\Big[g(V\tilde{\boldsymbol{\xi}})\Big] \\
&= \sum_{j:\left|\frac{j}{q}-\xi_\theta\right|\leq\frac{\epsilon}{4}} \frac{j}{q} \left(\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=\frac{j}{q}} \gamma_{\boldsymbol{\xi}'}\right) \mathbb{E}_{\tilde{\boldsymbol{\xi}}\sim\boldsymbol{\gamma}^{\theta,j},V\sim\mathcal{V}}\Big[g(V\tilde{\boldsymbol{\xi}})\Big] \\
&\hspace{20em}\text{(By Def. 8.2 – Lem. 8.3)} \\
&= \sum_{j:\left|\frac{j}{q}-\xi_\theta\right|\leq\frac{\epsilon}{4}} \frac{j}{q} \left(\sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=\frac{j}{q}} \gamma_{\boldsymbol{\xi}'}\right) \left[\left(1-\frac{\alpha}{2}\right)\mathbb{E}_{V\sim\mathcal{V}}\Big[h(V\tilde{\boldsymbol{\xi}})\Big]-\delta\epsilon\right] \\
&= \left[\left(1-\frac{\alpha}{2}\right)\mathbb{E}_{V\sim\mathcal{V}}\Big[h(V\tilde{\boldsymbol{\xi}})\Big]-\delta\epsilon\right] \sum_{j:\left|\frac{j}{q}-\xi_\theta\right|\leq\frac{\epsilon}{4}} \frac{j}{q} \sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=\frac{j}{q}} \gamma_{\boldsymbol{\xi}'} \\
&= \left[\left(1-\frac{\alpha}{2}\right)\mathbb{E}_{V\sim\mathcal{V}}\Big[h(V\tilde{\boldsymbol{\xi}})\Big]-\delta\epsilon\right] \left(\xi_\theta - \sum_{j:\left|\frac{j}{q}-\xi_\theta\right|\geq\frac{\epsilon}{4}} \sum_{\boldsymbol{\xi}'\in\Xi^q:\xi'_\theta=\frac{j}{q}} \gamma_{\boldsymbol{\xi}'}\right) \\
&\hspace{12em}\text{(By Lemma 8.1), } (1-\alpha/2)^2\geq 1-\alpha, \text{ and } \alpha<1) \\
&\geq \xi_\theta\left[(1-\alpha)\,\mathbb{E}_{V\sim\mathcal{V}}\Big[h(V\tilde{\boldsymbol{\xi}})\Big]-\delta\epsilon\right].
\end{aligned}
$$

This concludes the proof. □

The crucial feature of Theorem 8.2 is that Equation (8.2) holds for every state. This is fundamental for proving our results in the private signaling scenario. On the other hand, with public signaling, we will make use of the following (weaker) corollary, obtained by summing Equation (8.2) over all $\theta \in \Theta$.

Corollary 8.1. *Let $\alpha, \epsilon, \delta > 0$, and set $q \coloneqq \frac{32}{\epsilon^2}\log\frac{4}{\alpha}$. Given a posterior $\boldsymbol{\xi} \in \Delta_\Theta$, some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{R}}$, and two functions $g, h : [0,1]^{\bar{n}} \to [0,1]$, if g is (δ,α,ϵ)-stable compared to h for $(\boldsymbol{\xi},\mathcal{V})$, then there exists $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ such that, for every $\theta \in \Theta$, $\sum_{\tilde{\boldsymbol{\xi}}\in\text{supp}(\boldsymbol{\gamma})} \gamma_{\tilde{\boldsymbol{\xi}}}\tilde{\xi}_\theta = \xi_\theta$ and*

$$
\mathbb{E}_{\tilde{\boldsymbol{\xi}}\sim\boldsymbol{\gamma},V\sim\mathcal{V}}\Big[g(V\tilde{\boldsymbol{\xi}})\Big] \geq (1-\alpha)\mathbb{E}_{V\sim\mathcal{V}}\Big[h(V\boldsymbol{\xi})\Big]-\delta\epsilon. \tag{8.3}
$$

8.4 Warming Up: Non-Bayesian Auctions

In this section, we focus on non-Bayesian posted price auctions, proving some results that will be useful in the rest of the chapter.[9] In particular, we study what happens to the seller's expected revenue when buyers' valuations are "slightly decreased", proving that the revenue also decreases, but only by a small amount. This result will be crucial when dealing with public signaling, and it also allows to design a polynomial-time algorithm for finding approximately-optimal price vectors in non-Bayesian auctions, as we show at the end of this section.

In the following, we extensively use distributions of buyers' valuations as specified in the definition below.

Definition 8.3. *Given $\epsilon > 0$, we denote by $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{R}}$ and $\mathcal{V}^\epsilon = \{\mathcal{V}^\epsilon_i\}_{i\in\mathcal{R}}$ two collections of distributions of buyers' valuations such that, for every price vector $\mathbf{p} \in [0,1]^{\bar{n}}$,*

$$\text{Pr}_{v_i\sim\mathcal{V}^\epsilon_i}\left\{v_i \geq p_i - \epsilon\right\} \geq \text{Pr}_{v_i\sim\mathcal{V}_i}\left\{v_i \geq p_i\right\} \quad \forall i \in \mathcal{R}.$$

Intuitively, valuations drawn from $\mathcal{V}^\epsilon$ are "slightly decreased" with respect to those drawn from $\mathcal{V}$, since the probability with which any buyer $i \in \mathcal{R}$ buys the item at the (reduced) price $[p_i - \epsilon]_+$ when their valuation is drawn from $\mathcal{V}^\epsilon_i$ is at least as large as the probability of buying at price p_i when their valuation is drawn from $\mathcal{V}_i$. [10]

Our main contribution in this section (Lemma 8.5) is to show that

$$\max_{\mathbf{p}\in[0,1]^{\bar{n}}} \text{REV}(\mathcal{V}^\epsilon, \mathbf{p}) \geq \max_{\mathbf{p}\in[0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p}) - \epsilon.$$

By letting $\mathbf{p}^* \in arg\,max_{\mathbf{p}\in[0,1]^{\bar{n}}}\text{REV}(\mathcal{V}, \mathbf{p})$ be any revenue-maximizing price vector under distributions $\mathcal{V}$, one may naïvely think that, since under distributions $\mathcal{V}^\epsilon$ and price vector $[\mathbf{p}^* - \epsilon]_+$ each buyer would buy the item at least with the same probability as with distributions $\mathcal{V}$ and price vector $\mathbf{p}^*$, while paying a price that is only ϵ less, then $\text{REV}(\mathcal{V}^\epsilon, [\mathbf{p}^* - \epsilon]_+) \geq \text{REV}(\mathcal{V}, \mathbf{p}^*) - \epsilon$, proving the result. However, this line of reasoning does *not* work, since, as shown by the following example, it has a major flaw.

[9] When we study non-Bayesian posted price auctions, we stick to our notation, with the following differences: valuations are scalars rather than vectors, namely $v_i \in [0,1]$; distributions $\mathcal{V}_i$ are supported on $[0,1]$ rather than $[0,1]^d$; the matrix V is indeed a column vector whose components are buyers' valuations; and the price function f is replaced by a single price vector $\mathbf{p} \in [0,1]^{\bar{n}}$, with its i-th component p_i being the price for buyer $i \in \mathcal{R}$. Moreover, we continue to use the notation REV to denote seller's revenues, dropping the dependence on the tuple of posteriors. Thus, in a non-Bayesian auction in which the distributions of buyers' valuations are $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{R}}$, the notation $\text{REV}(\mathcal{V}, \mathbf{p})$ simply denotes the seller's expected revenue by selecting a price vector $\mathbf{p} \in [0,1]^{\bar{n}}$.

[10] In this chapter, given $x \in \mathbb{R}$, we let $[x]_+ := \max\{x, 0\}$. We extend the $[\cdot]_+$ operator to vectors by applying it component-wise.

Example 8.1. *Consider a posted price auction with two buyers. In the first case (distributions $\mathcal{V}$), buyer 1 has valuation $v_1 = \frac{1}{2}$ and buyer 2 has valuation $v_2 = 1$. In such setting, an optimal price vector $\mathbf{p}^*$ is such that $p_1^* = \frac{1}{2}+\epsilon$ and $p_2^* = 1$, so that the revenue of the seller, namely $\text{REV}(\mathcal{V}, \mathbf{p}^*)$, is 1. In the second case (distributions $\mathcal{V}^\epsilon$), buyer 1 has valuations $v_1 = \frac{1}{2}$ and buyer 2 has valuation $v_2 = 1-\epsilon$. Thus, the revenue of the seller for the price vector $\mathbf{p}^{*,\epsilon}$ (with $p_1^{*,\epsilon} = \frac{1}{2}$ and $p_2^{*,\epsilon} = 1-\epsilon$), namely $\text{REV}(\mathcal{V}^\epsilon, \mathbf{p}^{*,\epsilon})$, is $\frac{1}{2}$, since buyer 1 will buy the item.*

The crucial feature of Example 8.1 is that there exists a $\mathbf{p}^*$ in which one buyer is offered a price that is too low, and, thus, the seller prefers *not* to sell the item to her, but rather to a following buyer. This prevents a direct application of the line of reasoning outlined above, as it shows that incrementing the probability with which a buyer buys is *not* always beneficial. One could circumvent this issue by considering a $\mathbf{p}^*$ such that the seller is never upset if some buyer buys. In other words, it must be such that each buyer is proposed a price that is at least as large as the seller's expected revenue in the posted price auction restricted to the following buyers. Next, we show that there always exists a $\mathbf{p}^*$ with such desirable property.

Letting $\text{REV}_{>i}(\mathcal{V}, \mathbf{p})$ be the seller's revenue for price vector $\mathbf{p} \in [0,1]^{\bar{n}}$ and distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{R}}$ in the auction restricted to buyers $j \in \mathcal{R} : j > i$, we prove the following:

Lemma 8.4. *For any $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{R}}$, there exists a revenue-maximizing price vector $\mathbf{p}^* \in arg\,max_{\mathbf{p}\in[0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p})$ such that $p_i^* \geq \text{REV}_{>i}(\mathcal{V}, \mathbf{p}^*)$ for every buyer $i \in \mathcal{R}$.*

Proof. In order to prove the lemma, we show an even stronger result: for every price vector $\mathbf{p} \in [0,1]^{\bar{n}}$, it is always possible to recover another price vector $\mathbf{p}' \in [0,1]^{\bar{n}}$ that provides the seller with an expected revenue at least as large as that provided by $\mathbf{p}$, and such that $p'_i \geq \text{REV}_{>i}(\mathcal{V}, \mathbf{p}')$ for every $i \in \mathcal{R}$. Let us assume that $\mathbf{p}$ does *not* satisfy the required condition for some buyer $i \in \mathcal{R}$. Then, let $\mathbf{p}'$ be such that $p'_i = \text{REV}_{>i}(\mathcal{V}, \mathbf{p}) > p_i$ and $p'_j = p_j$ for all $j \in \mathcal{R} : j \neq i$. Since by construction $\text{REV}_{>i}(\mathcal{V}, p') = \text{REV}_{>i}(\mathcal{V}, \mathbf{p})$, the condition $p'_i \geq \text{REV}_{>i}(\mathcal{V}, p')$ holds. Moreover, the seller's expected revenue for $\mathbf{p}'$ in the auction restricted to all buyers $j \in \mathcal{R} : j \geq i$, namely $\text{REV}_{\geq i}(\mathcal{V}, \mathbf{p}')$, is such that:

$$
\begin{aligned}
\text{REV}_{\geq i}(\mathcal{V}, \mathbf{p}') &= \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p'_i\}\, p'_i + (1 - \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p'_i\})\, \text{REV}_{>i}(\mathcal{V}, \mathbf{p}') \\
&= \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p'_i\}\, p'_i + (1 - \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p'_i\})\, \text{REV}_{>i}(\mathcal{V}, \mathbf{p}) \\
&= \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p_i\}\, p'_i + (1 - \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p_i\})\, \text{REV}_{>i}(\mathcal{V}, \mathbf{p})
\end{aligned}
$$

$$\geq \Pr_{v_i \sim \mathcal{V}_i} \{v_i \geq p_i\} p_i + (1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i \geq p_i\}) \text{Rev}_{>i}(\mathcal{V}, \mathbf{p})$$
$$= \text{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}),$$

where the first equality and the last one holds by definition of $\text{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}')$, the second one follows from $\text{Rev}_{>i}(\mathcal{V}, \mathbf{p}') = \text{Rev}_{>i}(\mathcal{V}, \mathbf{p})$, the third one holds since $p'_i = \text{Rev}_{>i}(\mathcal{V}, \mathbf{p})$, while the inequality follows from $p'_i \geq p_i$. As a result, we can conclude that $\text{Rev}(\mathcal{V}, \mathbf{p}') \geq \text{Rev}(\mathcal{V}, \mathbf{p})$. The lemma is readily proved by iteratively applying the procedure described above until we get a price vector $\mathbf{p}' \in [0,1]^{\bar{n}}$ such that $p'_i \geq \text{Rev}_{>i}(\mathcal{V}, \mathbf{p}')$ for every buyer $i \in \mathcal{R}$, starting from an optimal price vector

$$\mathbf{p}^* \in arg\,max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{Rev}(\mathcal{V}, \mathbf{p}).$$

□

The proof of the following lemma builds upon the existence of a revenue-maximizing price vector $\mathbf{p}^* \in [0,1]^{\bar{n}}$ as in Lemma 8.4 and the fact that, under distributions $\mathcal{V}^\epsilon$, the probability with which each buyer buys the item given price vector $[\mathbf{p}^* - \epsilon]_+$ is greater than that with which they would buy given $\mathbf{p}^*$. Since the seller's expected revenue is larger when a buyer buys compared to when they do *not* buy (as $p^*_i \geq \text{Rev}_{>i}(\mathcal{V}, \mathbf{p}^*)$), the seller's expected revenue decreases by at most ϵ.

Lemma 8.5. *Given $\epsilon > 0$, let $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$ and $\mathcal{V}^\epsilon = \{\mathcal{V}^\epsilon_i\}_{i \in \mathcal{R}}$ satisfying the conditions of Definition 8.3. Then,* $\max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{Rev}(\mathcal{V}^\epsilon, \mathbf{p}) \geq \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{Rev}(\mathcal{V}, \mathbf{p}) - \epsilon$.

Proof. Let $\mathbf{p}^* \in [0,1]^{\bar{n}}$ be a price vector such that

$$\mathbf{p}^* \in arg\,max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{Rev}(\mathcal{V}, \mathbf{p})$$

and $p^*_i \geq \text{Rev}_{>i}(\mathcal{V}, \mathbf{p}^*)$ for every $i \in \mathcal{R}$. Such price vector is guaranteed to exist by Lemma 8.4. We show by induction that $\text{Rev}_{\geq i}(\mathcal{V}^\epsilon, \mathbf{p}^{*,\epsilon}) \geq \text{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}) - \epsilon$, where $\mathbf{p}^{*,\epsilon} = [\mathbf{p} - \epsilon]_+$. As a base case, it is easy to check that

$$\begin{aligned}\text{Rev}_{\geq \bar{n}}(\mathcal{V}^\epsilon, \mathbf{p}^{*,\epsilon}) &= [p^*_n - \epsilon]_+ \Pr_{v_n \sim \mathcal{V}^\epsilon_n} \{v_n \geq [p^*_n - \epsilon]_+\} \\ &\geq (p^*_n - \epsilon) \Pr_{v_n \sim \mathcal{V}_n} \{v_n \geq p^*_n\} \\ &\geq p^*_n \Pr_{v_n \sim \mathcal{V}_n} \{v_n \geq p^*_n\} - \epsilon \\ &= \text{Rev}_{\geq n}(\mathcal{V}, \mathbf{p}^*) - \epsilon.\end{aligned}$$

By induction, assume that the above condition holds for $i+1$ (notice that $\text{Rev}_{>i}(\cdot,\cdot) = \text{Rev}_{\geq i+1}(\cdot,\cdot)$), then

$$\text{Rev}_{\geq i}(\mathcal{V}^\epsilon, \mathbf{p}^{*,\epsilon}) = [p^*_i - \epsilon]_+ \Pr_{v_i \sim \mathcal{V}^\epsilon_i} \{v_i \geq [p^*_i - \epsilon]_+\}$$

$$
\begin{aligned}
&\quad + \left(1 - \mathrm{Pr}_{v_i \sim \mathcal{V}_i^\epsilon}\left\{v_i \geq [p_i^* - \epsilon]_+\right\}\right) \textsc{Rev}_{>i}(\mathcal{V}^\epsilon, \mathbf{p}^{*,\epsilon}) \\
&\geq (p_i^* - \epsilon)\mathrm{Pr}_{v_i \sim \mathcal{V}_i^\epsilon}\left\{v_i \geq [p_i^* - \epsilon]_+\right\} \\
&\quad + \left(1 - \mathrm{Pr}_{v_i \sim \mathcal{V}_i^\epsilon}\left\{v_i \geq [p_i^* - \epsilon]_+\right\}\right) \left(\textsc{Rev}_{>i}(\mathcal{V}, \mathbf{p}^*) - \epsilon\right) \\
&= p_i^* \mathrm{Pr}_{v_i \sim \mathcal{V}_i^\epsilon}\left\{v_i \geq [p_i^* - \epsilon]_+\right\} \\
&\quad + \left(1 - \mathrm{Pr}_{v_i \sim \mathcal{V}_i^\epsilon}\left\{v_i \geq [p_i^* - \epsilon]_+\right\}\right) \textsc{Rev}_{>i}(\mathcal{V}, \mathbf{p}^*) - \epsilon \\
&\geq p_i^* \mathrm{Pr}_{v_i \sim \mathcal{V}_i}\left\{v_i \geq p_i^*\right\} + \left(1 - \mathrm{Pr}_{v_i \sim \mathcal{V}_i}\left\{v_i \geq p_i^*\right\}\right) \textsc{Rev}_{>i}(\mathcal{V}, \mathbf{p}^*) - \epsilon \\
&= \textsc{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}^*) - \epsilon,
\end{aligned}
$$

where the last inequality follows from $p_i^* \geq \textsc{Rev}_{>i}(\mathcal{V}, \mathbf{p}^*)$ and

$$
\mathrm{Pr}_{v_i \sim \mathcal{V}_i^\epsilon}\left\{v_i \geq [p_i^* - \epsilon]_+\right\} \geq \mathrm{Pr}_{v_i \sim \mathcal{V}_i}\left\{v_i \geq p_i^*\right\}.
$$

□

Lemma 8.5 will be useful to prove Lemma 8.6 and to show the compared stability of a suitably-defined function that is used to design a PTAS in the public signaling scenario.

8.4.1 Finding Approximately-Optimal Prices

Algorithm 8.1 computes (in polynomial time) an approximately-optimal price vector for any non-Bayesian posted price auction. For a discretization step $b \in \mathbb{N}_{>0}$, we let $P^b \subset [0, 1]$ be the set of prices multiples of $1/b$, while $\mathcal{P}^b := \bigtimes_{i \in \mathcal{R}} P^b$. The algorithm samples $K \in \mathbb{N}_{>0}$ matrices of buyers' valuations, each one drawn according to the distributions $\mathcal{V}$. Then, it finds an optimal price vector $\mathbf{p}$ in the discretized set $\mathcal{P}^b$, assuming that buyers' valuations are drawn according to the empirical distribution resulting from the sampled matrices. This last step can be done by backward induction, as it is well known in the literature (see, *e.g.*, (Xiao et al., 2020)).

The following Lemma 8.6 establishes the correctness of Algorithm 8.1, also providing a bound on its running time. The key ideas of its proof are: (i) the sampling procedure constructs a good estimation of the actual distributions of buyers' valuations; and (ii) even if the algorithm only considers discretized prices, the components of the computed price vector are at most $1/b$ less than those of an optimal (unconstrained) price vector. As shown in the proof, this is strictly related to reducing buyer's valuations by $\frac{1}{b}$. Thus, it follows by Lemma 8.5 that the seller's expected revenue is at most $1/b$ less than the optimal one.

Algorithm 8.1 FIND-APX-PRICES

Inputs: # of samples $K \in \mathbb{N}_{>0}$; # of discretization steps $b \in \mathbb{N}_{>0}$

for $i \in \mathcal{R}$ **do**
  for $k = 1, \dots, K$ **do**
    $v_i^k \leftarrow$ Sample buyer i's valuation using oracle for $\mathcal{V}_i$
  end for
  $\mathcal{V}_i^K \leftarrow$ Empirical distribution of the K i.i.d. samples v_i^K
end for
$\mathcal{V}^K \leftarrow \{\mathcal{V}_i^K\}_{i \in \mathcal{R}}$; $\mathbf{p} \leftarrow \mathbf{0}$; $r \leftarrow 0$
for $i = n, \dots, 1$ (in reversed order) **do**
  $p_i \leftarrow arg\,max_{p_i' \in P^b} \text{Pr}_{v_i \sim \mathcal{V}_i^K}\{v_i \geq p_i'\}\, p_i' + \left(1 - \text{Pr}_{v_i \sim \mathcal{V}_i^K}\{v_i \geq p_i'\}\right) r$
  $r \leftarrow p_i \text{Pr}_{v_i \sim \mathcal{V}_i^K}\{v_i \geq p_i\} + \left(1 - \text{Pr}_{v_i \sim \mathcal{V}_i^K}\{v_i \geq p_i\}\right) r$
end for
return $(\mathbf{p}, r)$

Lemma 8.6. *For any* $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$ *and* $\epsilon, \tau > 0$, *there exist* $b \in \text{poly}\left(\frac{1}{\epsilon}\right)$ *and* $K \in \text{poly}\left(\bar{n}, \frac{1}{\epsilon}, \log \frac{1}{\tau}\right)$ *such that, with probability at least* $1 - \tau$, *Algorithm 8.1 returns* $(\mathbf{p}, r)$ *satisfying* $\text{REV}(\mathcal{V}, \mathbf{p}) \geq \max_{\mathbf{p}' \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p}') - \epsilon$ *and* $r \in [\text{REV}(\mathcal{V}, \mathbf{p}) - \epsilon, \text{REV}(\mathcal{V}, \mathbf{p}) + \epsilon]$ *in time* $\text{poly}\left(\bar{n}, \frac{1}{\epsilon}, \log \frac{1}{\tau}\right)$.

Proof. Letting $b := \lceil \frac{2}{\epsilon} \rceil$ and $K := \frac{8}{\epsilon^2} \log \frac{2b^{\bar{n}}}{\tau} \in \text{poly}\left(\bar{n}, \frac{1}{\epsilon}, \log \frac{1}{\tau}\right)$, the proof unfolds in two steps.

The first step is to show that restricting price vectors to those in the discretized set $\mathcal{P}^b$ results in a small reduction of the seller's expected revenue. Formally, we prove that:

$$\max_{\mathbf{p} \in \mathcal{P}^b} \text{REV}(\mathcal{V}, \mathbf{p}) \geq \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p}) - \frac{\epsilon}{2}.$$

To do so, we define some modified distributions of buyers' valuations, namely $\mathcal{V}^b = \{\mathcal{V}_i^b\}_{i \in \mathcal{R}}$, which are supported on the discretized set $\mathcal{P}^b$ and are obtained by mapping each valuation $v_i \in [0, 1]$ in the support of $\mathcal{V}_i$ (for any $i \in \mathcal{R}$) to a discretized valuation $\frac{x}{b}$, where x is the greatest integer such that $\frac{x}{b} \leq v_i$. It is easy to see that, since an optimal price vector for distributions $\mathcal{V}^b$ must specify prices that are multiples of $\frac{1}{b}$, then

$$\max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}^b, \mathbf{p}) = \max_{\mathbf{p} \in \mathcal{P}^b} \text{REV}(\mathcal{V}^b, \mathbf{p}).$$

Moreover, by definition of b, distributions $\mathcal{V}^b$ are such that for every $i \in \mathcal{R}$ and price $p_i \in [0, 1]$ it holds $\text{Pr}_{v_i \sim \mathcal{V}_i^b}\left\{v_i \geq p_i - \frac{\epsilon}{2}\right\} \geq \text{Pr}_{v_i \sim \mathcal{V}_i}\{v_i \geq p_i\}$. Thus, by Lemma 8.5, $\max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}^b, \mathbf{p}) \geq \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p}) - \frac{\epsilon}{2}$, which implies that $\max_{p \in \mathcal{P}^b} \text{REV}(\mathcal{V}, \mathbf{p}) \geq \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p}) - \frac{\epsilon}{2}$.

This proves that we can restrict the attention to price vectors in $\mathcal{P}^b \subset [0,1]^{\bar{n}}$, loosing only an additive factor $\frac{\epsilon}{2}$ of the seller's optimal expected revenue.

The second step of the proof is to show that replacing distributions $\mathcal{V}$ with the empirical distributions $\mathcal{V}^K$ built by Algorithm 8.1 only reduces the seller's optimal expected revenue by a small amount, with high probability. For any price vector $\mathbf{p} \in [0,1]^{\bar{n}}$, by using Hoeffding's bound we obtain that

$$\Pr\left\{\left|\textsc{Rev}(\mathcal{V},\mathbf{p}) - \textsc{Rev}(\mathcal{V}^K,\mathbf{p})\right| \geq \frac{\epsilon}{4}\right\} \leq 2e^{-K\epsilon^2/8},$$

where the probability is with respect to the stochasticity of the algorithm (as a result of the sampling steps). Since the number of elements in the discretized set $\mathcal{P}^b$ is $b^{\bar{n}}$, by a union bound we get

$$\Pr\left\{\left|\textsc{Rev}(\mathcal{V},\mathbf{p}) - \textsc{Rev}(\mathcal{V}^K,\mathbf{p})\right| < \frac{\epsilon}{4} \;\; \forall \mathbf{p} \in \mathcal{P}^b\right\} \geq 1-2b^{\bar{n}}e^{-K\epsilon^2/8} = 1-\tau.$$

Letting $\mathbf{p} \in \mathcal{P}^b$ be the price vector returned by Algorithm 8.1, it is the case that $\mathbf{p} \in arg\,max_{\mathbf{p}' \in \mathcal{P}^b} \textsc{Rev}(\mathcal{V}^K, \mathbf{p}')$, given the correctness and optimality of the backward induction procedure with which the vector $\mathbf{p}$ is built (Xiao et al., 2020). Moreover, letting $\mathbf{p}^* \in arg\,max_{\mathbf{p}' \in \mathcal{P}^b} \textsc{Rev}(\mathcal{V}, \mathbf{p}')$ be an optimal price vector over the discretized set $\mathcal{P}^b$ for the actual distributions of buyers' valuations $\mathcal{V}$, with probability at least $1-\tau$ it holds that

$$\textsc{Rev}(\mathcal{V},\mathbf{p}) \geq \textsc{Rev}(\mathcal{V}^K,\mathbf{p}) - \frac{\epsilon}{4} \geq \textsc{Rev}(\mathcal{V}^K,\mathbf{p}^*) - \frac{\epsilon}{4} \geq \textsc{Rev}(\mathcal{V},\mathbf{p}^*) - \frac{\epsilon}{2}.$$

Hence, with probability at least $1-\tau$, it holds

$$\textsc{Rev}(\mathcal{V},\mathbf{p}) \geq \textsc{Rev}(\mathcal{V},\mathbf{p}^*) - \frac{\epsilon}{2} \geq \max_{\mathbf{p}' \in [0,1]^{\bar{n}}} \textsc{Rev}(\mathcal{V},\mathbf{p}') - \epsilon,$$

where the last step has been proved in the first part of the proof.

In order to conclude the proof, it is sufficient to notice that, with probability at least $1-\tau$, it also holds that

$$r = \textsc{Rev}(\mathcal{V}^K,\mathbf{p}) \in \left[\textsc{Rev}(\mathcal{V},\mathbf{p}) - \frac{\epsilon}{4}, \textsc{Rev}(\mathcal{V},\mathbf{p}) + \frac{\epsilon}{4}\right].$$

□

8.5 Public Signaling

In the following, we design a PTAS for computing a revenue-maximizing (ϕ, f) pair in the public signaling setting. Notice that this positive result is tight by Theorem 7.2.

As a first intermediate result, we prove the compared stability of suitably-defined functions, which are intimately related to the seller's revenue. In particular, for every price vector $\mathbf{p} \in [0,1]^{\bar{n}}$, we conveniently let $g_{\mathbf{p}} : [0,1]^{\bar{n}} \to [0,1]$ be a function that takes a vector of buyers' valuations and outputs the seller's expected revenue achieved by selecting $\mathbf{p}$ when the buyers' valuations are those specified as input. The following Lemma 8.7 shows that, given some distributions of buyers' valuations $\mathcal{V}$ and a posterior $\boldsymbol{\xi} \in \Delta_\Theta$, there always exists a price vector $\mathbf{p} \in [0,1]^{\bar{n}}$ such that $g_{\mathbf{p}}$ is stable compared with $g_{\mathbf{p}'}$ for every other $\mathbf{p}' \in [0,1]^{\bar{n}}$. This result crucially allows us to decompose any posterior $\boldsymbol{\xi} \in \Delta_\Theta$ by means of the decomposition lemma in Corollary 8.1, while guaranteeing a small loss in terms of seller's expected revenue.

Lemma 8.7. *Given $\alpha, \epsilon > 0$, a posterior $\boldsymbol{\xi} \in \Delta_\Theta$, and some distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$, there exists $\mathbf{p} \in [0,1]^{\bar{n}}$ such that, for every other $\mathbf{p}' \in [0,1]^{\bar{n}}$, the function $g_{\mathbf{p}}$ is $(1, \alpha, \epsilon)$-stable compared with $g_{\mathbf{p}'}$ for $(\boldsymbol{\xi}, \mathcal{V})$.*

Proof. As a first step, we prove the following: given any matrix $V \in [0,1]^{\bar{n} \times d}$ of buyers' valuations and any price vector $\mathbf{p}' \in [0,1]^{\bar{n}}$, for every distribution $\boldsymbol{\gamma}$ over Δ_Θ that is (α, ϵ)-decreasing around $\boldsymbol{\xi}$ (see Definition 8.1) it holds that

$$\mathbb{E}_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}} \left[g_{\mathbf{p}'}(V \tilde{\boldsymbol{\xi}}) \right] \geq \mathbb{E}_{\tilde{\boldsymbol{\xi}} \sim \boldsymbol{\gamma}} \left[g_{\mathbf{p}'} \left(\max \left\{ V \tilde{\boldsymbol{\xi}}, V \boldsymbol{\xi} - \epsilon \mathbf{1} \right\} \right) \right] - \alpha \, g_{\mathbf{p}' + \epsilon \mathbf{1}}(V \boldsymbol{\xi}). \tag{8.4}$$

W.l.o.g., let $i \in \mathcal{R}$ be the buyer that buys the item when buyers' valuations are specified by the vector $V\boldsymbol{\xi} - \epsilon\mathbf{1}$ and the proposed prices are those specified by $\mathbf{p}'$, that is, it must be the case that $p'_i \leq V_i \boldsymbol{\xi} - \epsilon$ and $p'_j > V_j \boldsymbol{\xi} - \epsilon$ for all $j \in \mathcal{R} : j < i$. Since $\boldsymbol{\gamma}$ is (α, ϵ)-decreasing around $\boldsymbol{\xi}$, by sampling a posterior $\tilde{\boldsymbol{\xi}} \in \Delta_\Theta$ according to $\boldsymbol{\gamma}$, with probability at least $1 - \alpha$ it holds that $V_i \tilde{\boldsymbol{\xi}} \geq V_i \boldsymbol{\xi} - \epsilon$ (see Definition 8.1). Moreover, let $\tilde{\Xi} := \{\tilde{\boldsymbol{\xi}} \in \Delta_\Theta \mid V_i \tilde{\boldsymbol{\xi}} \geq V_i \boldsymbol{\xi} - \epsilon\}$ be the set of posteriors which result in a buyer i's valuation that is at most ϵ less than that for $\boldsymbol{\xi}$ (notice that $\sum_{\tilde{\boldsymbol{\xi}} \in \tilde{\Xi}} \gamma_{\tilde{\boldsymbol{\xi}}} \geq 1 - \alpha$). Then, we split the posteriors in Δ_Θ into three groups, as follows:

- $\Xi^1 \subseteq \tilde{\Xi}$ is composed of all the posteriors $\tilde{\boldsymbol{\xi}} \in \tilde{\Xi}$ such that, for every $j \in \mathcal{R} : j < i$, it holds $V_j \tilde{\boldsymbol{\xi}} < p'_j$;
- $\Xi^2 \subseteq \tilde{\Xi}$ is composed of all the posteriors $\tilde{\boldsymbol{\xi}} \notin \tilde{\Xi}$ such that, for every $j \in \mathcal{R} : j < i$, it holds $V_j \tilde{\boldsymbol{\xi}} < p'_j$;

- $\Xi^3 \subseteq \Delta_\Theta$ is composed of all the posteriors $\tilde{\boldsymbol{\xi}} \in \Delta_\Theta$ for which there exists a buyer $j(\tilde{\boldsymbol{\xi}}) \in \mathcal{R} : j < i$ (notice the dependence on $\tilde{\boldsymbol{\xi}}$) such that $j(\tilde{\boldsymbol{\xi}}) = \min\{j \in \mathcal{R} \mid V_j \tilde{\boldsymbol{\xi}} \geq p'_j\}$

Next, we show that, for every posterior $\tilde{\boldsymbol{\xi}} \in \Xi^1 \cup \Xi^3$, it holds $g_{\mathbf{p}}(V\tilde{\boldsymbol{\xi}}) = g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\})$, while, for every $\tilde{\boldsymbol{\xi}} \in \Xi^2$, it holds $g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}) \leq g_{\mathbf{p}'+\epsilon\mathbf{1}}(V\boldsymbol{\xi})$. First, let us consider a posterior $\tilde{\boldsymbol{\xi}} \in \Xi^1$. For each $j \in \mathcal{R} : j < i$, it holds $V_j\tilde{\boldsymbol{\xi}} \leq \max\{V_j\tilde{\boldsymbol{\xi}}, V_j\boldsymbol{\xi} - \epsilon\} < p'_j$ (by definition of Ξ^1, and since buyer j does not buy the item for price p'_j). Moreover, since $V_i\tilde{\boldsymbol{\xi}} \geq V_i\boldsymbol{\xi} - \epsilon$, it holds $V_i\tilde{\boldsymbol{\xi}} = \max\{V_i\tilde{\boldsymbol{\xi}}, V_i\boldsymbol{\xi} - \epsilon\} \geq p'_i$. Hence, both when buyers' valuations are specified by the vector $V\tilde{\boldsymbol{\xi}}$ and when they are given by $\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}$ (with $\max$ applied component-wise), it is the case that buyer i buys the item at price p'_i, resulting in

$$g_{\mathbf{p}'}(V\tilde{\boldsymbol{\xi}}) = g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}).$$

Now, let us consider a posterior $\tilde{\boldsymbol{\xi}} \in \Xi^2$. In this case, $\max\{V_i\tilde{\boldsymbol{\xi}}, V_i\boldsymbol{\xi} - \epsilon\} = V_i\boldsymbol{\xi} - \epsilon \geq p'_i$, while $\max\{V_j\tilde{\boldsymbol{\xi}}, V_j\boldsymbol{\xi} - \epsilon\} < p'_j$ for every $j \in \mathcal{R} : j < i$. Thus, both when buyers' valuations are specified by $\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}$ and when they are given by $V\boldsymbol{\xi} - \epsilon\mathbf{1}$, it is the case that buyer i buys the item at price p'_i, resulting in

$$g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \mathbf{1}\}) = g_{\mathbf{p}'}(V\boldsymbol{\xi} - \epsilon\mathbf{1}) \leq g_{\boldsymbol{\xi}'+\epsilon\mathbf{1}}(V\boldsymbol{\xi}),$$

where the inequality holds since buyer i buys the item at price p'_i for valuations $V\boldsymbol{\xi} - \epsilon\mathbf{1}$ and price vector $\mathbf{p}'$, while the buyer would still buy the item, though at price $p'_i + \epsilon \geq p'_i$, for valuations $V\boldsymbol{\xi}$ and price vector $\mathbf{p}' + \epsilon\mathbf{1}$. Finally, let us consider a posterior $\tilde{\boldsymbol{\xi}} \in \Xi^3$. We have that, for every $j \in \mathcal{R} : j < j(\tilde{\boldsymbol{\xi}})$, it holds $V_j\tilde{\boldsymbol{\xi}} \leq \max\{V_j\tilde{\boldsymbol{\xi}}, V_j\boldsymbol{\xi} - \epsilon\} < p'_j$, while $\max\{V_{j(\tilde{\boldsymbol{\xi}})}\tilde{\boldsymbol{\xi}}, V_{j(\tilde{\boldsymbol{\xi}})}\boldsymbol{\xi} - \epsilon\} \geq V_{j(\tilde{\boldsymbol{\xi}})}\tilde{\boldsymbol{\xi}} \geq p'_{j(\tilde{\boldsymbol{\xi}})}$. As a result, both when buyers' valuations are specified by $V\tilde{\boldsymbol{\xi}}$ and when they are given by $\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}$, it is the case that buyer $j(\tilde{\boldsymbol{\xi}})$ buys the item at price $p'_{j(\tilde{\boldsymbol{\xi}})}$, resulting in

$$g_{\mathbf{p}'}(V\tilde{\boldsymbol{\xi}}) = g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}).$$

This allows us to prove Equation (8.4), as follows:

$$\mathbb{E}_{\tilde{\boldsymbol{\xi}}\sim\boldsymbol{\gamma}}\Big[g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\})\Big] - \mathbb{E}_{\tilde{\boldsymbol{\xi}}\sim\boldsymbol{\gamma}}\Big[g_{\mathbf{p}'}(V\tilde{\boldsymbol{\xi}})\Big] \leq \sum_{\tilde{\boldsymbol{\xi}}\in\Xi^2} \gamma_{\tilde{\boldsymbol{\xi}}}\, g_{\mathbf{p}'+\epsilon\mathbf{1}}(V\boldsymbol{\xi}) \leq \alpha g_{\mathbf{p}'+\epsilon\mathbf{1}}(V\boldsymbol{\xi}),$$

where the first inequality comes from the fact that, as previously proved, $g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}) = g_{\mathbf{p}'}(V\tilde{\boldsymbol{\xi}})$ for every posterior $\tilde{\boldsymbol{\xi}} \in \Xi^1 \cup \Xi^3$ and $g_{\mathbf{p}'}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}) \leq g_{\mathbf{p}'+\epsilon\mathbf{1}}(V\boldsymbol{\xi})$ for every posterior $\tilde{\boldsymbol{\xi}} \in \Xi^2$, while the second inequality is readily obtained by noticing that $\sum_{\tilde{\boldsymbol{\xi}}\in\Xi^2} \gamma_{\tilde{\boldsymbol{\xi}}} \leq \sum_{\tilde{\boldsymbol{\xi}}\notin\tilde{\Xi}} \gamma_{\tilde{\boldsymbol{\xi}}} \leq \alpha$.

Given any posterior $\tilde{\boldsymbol{\xi}} \in \Delta_\Theta$, the expression

$$\max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \mathbb{E}_{V\sim\mathcal{V}}\Big[\textsc{Rev}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\}, \mathbf{p}')\Big]$$

can be interpreted as the optimal seller's expected revenue when buyers' valuations are determined by distributions $\mathcal{V}^\epsilon = \{\mathcal{V}_i^\epsilon\}$ such that, for every buyer $i \in \mathcal{R}$, their valuation is sampled by first drawing a valuation $v_i \in [0,1]^d$ according to $\mathcal{V}_i$ and, then, taking $\max\{v_i^\top\tilde{\boldsymbol{\xi}}, v_i^\top\boldsymbol{\xi} - \epsilon\}$. Moreover, $\max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \mathbb{E}_{V\sim\mathcal{V}}\textsc{Rev}(V\boldsymbol{\xi}, \mathbf{p}')$ can be interpreted as the optimal seller's expected revenue when buyers' valuations are determined by distributions $\mathcal{V} = \{\mathcal{V}_i\}$ such that valuations are determined by first sampling a $v_i \in [0,1]^d$ from $\mathcal{V}_i$ and, then, taking $v_i^\top\boldsymbol{\xi}$. It is easy to see that $\Pr_{v_i\sim\mathcal{V}_i^\epsilon}\{v_i \geq p_i' - \epsilon\} \geq \Pr_{v_i\sim\mathcal{V}_i}\{v_i \geq p_i'\}$ for every price p_i', so that distributions $\mathcal{V}^\epsilon$ and $\mathcal{V}$ satisfy Definition 8.3. Then, by applying Lemma 8.5, we can conclude that there exists a price vector $\mathbf{p} \in [0,1]^{\bar{n}}$ such that $\textsc{Rev}(\mathcal{V}^\epsilon, \mathbf{p}) \geq \max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \textsc{Rev}(\mathcal{V}, \mathbf{p}') - \epsilon$. Thus, for every distribution $\boldsymbol{\gamma}$ over Δ_Θ that is (α, ϵ)-decreasing around $\boldsymbol{\xi}$, we get

$$\begin{aligned}
&\mathbb{E}_{\tilde{\boldsymbol{\xi}}\sim\boldsymbol{\gamma},V\sim\mathcal{V}}\Big[g_{\mathbf{p}}(V\tilde{\boldsymbol{\xi}})\Big] \\
&\quad\geq \mathbb{E}_{\tilde{\boldsymbol{\xi}}\sim\boldsymbol{\gamma},V\sim\mathcal{V}}\Big[g_{\mathbf{p}}(\max\{V\tilde{\boldsymbol{\xi}}, V\boldsymbol{\xi} - \epsilon\mathbf{1}\})\Big] - \mathbb{E}_{V\sim\mathcal{V}}\Big[\alpha\, g_{\mathbf{p}+\epsilon\mathbf{1}}(V\boldsymbol{\xi})\Big] \\
&\quad\geq \max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \mathbb{E}_{V\sim\mathcal{V}}\Big[\textsc{Rev}(V\boldsymbol{\xi}, \mathbf{p}')\Big] - \epsilon - \mathbb{E}_{V\sim\mathcal{V}}\Big[\alpha\, g_{\mathbf{p}+\epsilon\mathbf{1}}(V\boldsymbol{\xi})\Big] \\
&\quad\geq \max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \mathbb{E}_{V\sim\mathcal{V}}\Big[\textsc{Rev}(V\boldsymbol{\xi}, \mathbf{p}')\Big] - \epsilon - \max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \mathbb{E}_{V\sim\mathcal{V}}\Big[\alpha\, \textsc{Rev}(V\boldsymbol{\xi}, \mathbf{p}')\Big] \\
&\quad\geq (1-\alpha) \max_{\mathbf{p}'\in[0,1]^{\bar{n}}} \mathbb{E}_{V\sim\mathcal{V}}\Big[\textsc{Rev}(V\boldsymbol{\xi}, \mathbf{p}')\Big] - \epsilon,
\end{aligned}$$

where the first inequality holds by Equation (8.4), while the second one by Lemma 8.5. □

Our PTAS leverages the fact that public signaling schemes can be represented as probability distributions over buyers' posteriors (recall that, in the public signaling setting, all the buyers share the same posterior, as they all observe the same signal). In particular, the algorithm returns a pair $(\boldsymbol{\gamma}, \mathrm{f}^\circ)$, where γ is a probability distribution over Δ_Θ satisfying consistency constraints (see Equation (3.4)), while $\mathsf{f}^\circ : \Delta_\Theta \to [0,1]^{\bar{n}}$ is a function mapping each posterior to a price vector. In signaling problem in which the sender does not play an action, it is well known (see Subsection 3.1.1) that using distributions over posteriors rather than signaling schemes ϕ is without loss of generality. The following lemma shows that the same holds in our case, *i.e.*, given a pair $(\gamma, \mathsf{f}^\circ)$, it is always possible to obtain a pair (ϕ, f) providing the seller with the same expected revenue.

Lemma 8.8. *Given a pair* $(\gamma, \mathsf{f}^\circ)$*, where* γ *is a probability distribution over* Δ_Θ *with* $\sum_{\boldsymbol{\xi} \in \text{supp}(\gamma)} \gamma_{\boldsymbol{\xi}} \xi_\theta = \mu_\theta$ *for all* $\theta \in \Theta$ *and* $\mathsf{f}^\circ : \Delta_\Theta \to [0,1]^{\bar{n}}$*, there is a pair* (ϕ, f) *s.t.*

$$\sum_{\theta \in \Theta} \mu_\theta \sum_{\mathrm{s} \in \mathcal{S}} \phi_\theta(\mathrm{s}) \text{REV}(\mathcal{V}, \mathsf{f}(\mathrm{s}), \boldsymbol{\xi}^{\mathrm{s}}) = \sum_{\boldsymbol{\xi} \in \text{supp}(\gamma)} \gamma_{\boldsymbol{\xi}} \text{REV}(\mathcal{V}, \mathsf{f}^\circ(\boldsymbol{\xi}), \boldsymbol{\xi}).$$

Proof. The idea of the proof is to build a signaling scheme ϕ such that there is one-to-one correspondence between the buyers' posteriors induced by signal profiles $\mathrm{s} \in \mathcal{S}$ under ϕ and the posteriors in the support of the distribution γ, *i.e.*, the set of signal profiles is defined as $\mathcal{S} = \{\mathrm{s}^{\boldsymbol{\xi}}\}_{\boldsymbol{\xi} \in \text{supp}(\gamma)}$. Thus, in the following we can safely use the notation $\boldsymbol{\xi}^{\mathrm{s}}$ to the denote the posterior corresponding to signal profile $\mathrm{s} \in \mathcal{S}$. We define the signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ so that, for every state $\theta \in \Theta$, it holds $\phi_\theta(\mathrm{s}) = \frac{\gamma_{\boldsymbol{\xi}^{\mathrm{s}}} \xi^{\mathrm{s}}_\theta}{\mu_\theta}$ for all $\mathrm{s} \in \mathcal{S}$. Moreover, we define $\mathsf{f} : \mathcal{S} \to [0,1]^{\bar{n}}$ so that $\mathsf{f}(\mathrm{s}) = \mathsf{f}^\circ(\boldsymbol{\xi}^{\mathrm{s}})$ for all $\mathrm{s} \in \mathcal{S}$. First, notice that the signaling scheme ϕ is consistent, since, for every $\theta \in \Theta$, it holds $\sum_{\mathrm{s} \in \mathcal{S}} \phi_\theta(\mathrm{s}) = \sum_{\mathrm{s} \in \mathcal{S}} \frac{\gamma_{\boldsymbol{\xi}^{\mathrm{s}}} \xi^{\mathrm{s}}_\theta}{\mu_\theta} = \sum_{\boldsymbol{\xi} \in \text{supp}(\gamma)} \frac{\gamma_{\boldsymbol{\xi}} \xi_\theta}{\mu_\theta} = 1$, where the last two equalities follow from the correspondence between signal profiles and posteriors in $\text{supp}(\gamma)$ and the fact that $\sum_{\boldsymbol{\xi} \in \text{supp}(\gamma)} \gamma_{\boldsymbol{\xi}} \xi_\theta = \mu_\theta$. It is also easy to check that each signal profile $\mathrm{s} \in \mathcal{S}$ indeed induces its corresponding posterior $\boldsymbol{\xi}^{\mathrm{s}}$ under the signaling scheme ϕ. Finally, we have

$$\begin{aligned}
&\sum_{\theta \in \Theta} \mu_\theta \sum_{\mathrm{s} \in \mathcal{S}} \phi_\theta(\mathrm{s}) \text{REV}(\mathcal{V}, \mathsf{f}(\mathrm{s}), \boldsymbol{\xi}^{\mathrm{s}}) \\
&\qquad = \sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{\xi} \in \text{supp}(\gamma)} \frac{\gamma(\boldsymbol{\xi}) \xi_\theta}{\mu_\theta} \text{REV}(\mathcal{V}, \mathsf{f}^\circ(\boldsymbol{\xi}), \boldsymbol{\xi}) \\
&\qquad = \sum_{\boldsymbol{\xi} \in \text{supp}(\gamma)} \gamma_{\boldsymbol{\xi}} \text{REV}(\mathcal{V}, f^\circ(\boldsymbol{\xi}), \boldsymbol{\xi}),
\end{aligned}$$

which concludes the proof. □

Next, we show that, in order to find an approximately-optimal pair $(\gamma, \mathsf{f}^\circ)$, we can restrict the attention to q-uniform posteriors (with q suitably defined). First, we introduce the following LP that computes an optimal probability distribution restricted over q-uniform posteriors.

$$\max_{\gamma \in \Delta_{\Xi^q}} \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \text{ s.t.} \tag{8.5a}$$

$$\sum_{\xi \in \Xi^q} \gamma_{\boldsymbol{\xi}}\, \xi_\theta = \mu_\theta \qquad \forall \theta \in \Theta. \tag{8.5b}$$

The following Lemma 8.9 shows the optimal value of LP 8.5 is "close" to OPT. Its proof is based on the following core idea. Given the signaling scheme ϕ in a revenue-maximizing pair (ϕ, f), letting $\boldsymbol{\gamma}$ be the distribution over Δ_Θ induced by ϕ, we can decompose each posterior in the support of $\boldsymbol{\gamma}$ according to Corollary 8.1. Then, the obtained distributions over q-uniform posteriors are consistent according to Equation (3.4), and, thus, they satisfy Constraints (8.5b). Moreover, since such distributions are also decreasing around the decomposed posteriors, by Lemma 8.7 each time a posterior is decomposed there exists a price vector resulting in a small revenue loss. These observations allow us to conclude that the seller's expected revenue provided by an optimal solution to LP 8.5 is within some small additive loss of OPT.

Lemma 8.9. *Given $\eta > 0$ and letting $q = 128 \frac{1}{\eta^2} \log \frac{6}{\eta}$, an optimal solution to LP 8.5 has value at least $OPT - \eta$.*

Proof. Given a Bayesian posted price auction with prior $\boldsymbol{\mu} \in \Delta_\Theta$ and distributions of buyers' valuations $\mathcal{V}$, let (ϕ^*, f^*) be a revenue-maximizing signaling scheme, price function pair. Then, we define $\boldsymbol{\gamma}^*$ as the probability distribution over posteriors Δ_Θ induced by ϕ^*. Moreover, we define $\mathsf{f}^{\circ,*} : \text{supp}(\boldsymbol{\gamma}^*) \to [0,1]^{\bar{n}}$ in such a way that, for every posterior $\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma}^*)$, it holds $\mathsf{f}^{\circ,*}(\boldsymbol{\xi}) = \mathsf{f}^*(\mathsf{s})$, where $\mathsf{s} \in \mathcal{S}$ is the signal inducing $\boldsymbol{\xi}$, namely $\boldsymbol{\xi} = \boldsymbol{\xi}^s$.[11]

Let $\alpha = \epsilon = \frac{\eta}{2}$ and $q = \frac{32 \log \frac{4}{\alpha}}{\epsilon^2}$. Then, we build a probability distribution γ over posteriors in Δ_Θ by decomposing each posterior $\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma}^*)$ according to Corollary 8.1. Additionally, each time we decompose a posterior, for every newly-introduced posterior $\boldsymbol{\xi} \in \Delta_\Theta$ we define the function $\mathsf{f}^\circ : \Delta_\Theta \to [0,1]^{\bar{n}}$ so that $\mathsf{f}^\circ(\boldsymbol{\xi}) \in arg\,max_{\mathbf{p} \in [0,1]^{\bar{n}}} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$. Letting

[11] W.l.o.g., we can safely assume that there is a unique signal inducing $\boldsymbol{\xi}$. Indeed, if two signals $\mathsf{s} \in \mathcal{S}$ and $\mathsf{s}' \in \mathcal{S}$ induce the same posterior, then it is possible to build another signaling scheme, price function pair (ϕ^*, f^*) that joins the two signals in a new single signal $\mathsf{s}^* \in \mathcal{S}$, by setting $\phi^*_\theta(\mathsf{s}^*) \leftarrow \phi^*_\theta(\mathsf{s}) + \phi^*_\theta(\mathsf{s}')$ and $\mathsf{f}^*(\mathsf{s}^*) = \mathsf{f}(\mathsf{s})$ if $\textsc{Rev}(\mathcal{V}, \mathsf{f}^*(\mathsf{s}), \boldsymbol{\xi}) \geq \textsc{Rev}(\mathcal{V}, \mathsf{f}^*(\mathsf{s}'), \boldsymbol{\xi})$, while $\mathsf{f}^*(\mathsf{s}) = \mathsf{f}^*(\mathsf{s}')$ otherwise. It is easy to check that the new signaling scheme cannot decrease the seller's expected revenue.

$\gamma^{\xi} \in \Delta_{\Xi^q}$ be the probability distribution over q-uniform posteriors which is obtained by decomposing posterior $\xi \in \text{supp}(\gamma^*)$ according to Corollary 8.1, we define γ so that $\gamma_{\xi} = \sum_{\xi' \in \text{supp}(\gamma^*)} \gamma^*_{\xi'} \gamma^{\xi'}_{\xi}$ for every $\xi \in \Xi^q$.

First, let us notice that, for every $\theta \in \Theta$, it holds

$$\begin{aligned}\sum_{\xi \in \Xi^q} \gamma_{\xi} \xi_\theta &= \sum_{\xi' \in \text{supp}(\gamma^*)} \gamma^*_{\xi'} \sum_{\xi \in \Xi^q} \gamma^{\xi'}_{\xi} \xi_\theta \\ &= \sum_{\xi' \in \text{supp}(\gamma^*)} \gamma^*_{\xi'} \xi'_\theta = \mu_\theta,\end{aligned}$$

where the second equality follows from the property of the decomposition in Theorem 8.2, while the last one from the fact that γ^* is induced by a signaling scheme. Moreover, given any posterior $\xi \in \text{supp}(\gamma^*)$, let $\mathbf{p}^{\xi} \in [0,1]^{\bar{n}}$ be a price vector such that, for every $\mathbf{p} \in [0,1]^{\bar{n}}$, the function $g_{\mathbf{p}^{\xi}}$ is $(1, \alpha, \epsilon)$-stable compared with the function $g_{\mathbf{p}}$ in $(\mathcal{V}, \xi)$. such price vectors are guaranteed to exist by Lemma 8.7. Then, the pair $(\gamma, \mathsf{f}^\circ)$ provides the seller with an expected revenue of

$$\begin{aligned}\sum_{\xi \in \Xi^q} \gamma_{\xi} \text{REV}(\mathcal{V}, \mathsf{f}^\circ(\xi), \xi) &= \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \sum_{\xi' \in \Xi^q} \gamma^{\xi}_{\xi'} \text{REV}(\mathcal{V}, \mathsf{f}^\circ(\xi), \xi) \\ &= \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \sum_{\xi' \in \Xi^q} \gamma^{\xi}_{\xi'} \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \text{REV}(\mathcal{V}, \mathbf{p}, \xi) \\ &\geq \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \sum_{\xi' \in \Xi^q} \gamma^{\xi}_{\xi'} \text{REV}(\mathcal{V}, \mathbf{p}^{\xi}, \xi) \\ &\geq \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \left[(1-\alpha) \text{REV}(\mathcal{V}, \mathsf{f}^{\circ,*}(\xi), \xi) - \epsilon\right] \\ &= (1-\alpha) \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \text{REV}(\mathcal{V}, f^{\circ,*}(\xi), \xi) - \epsilon \\ &= \left(1 - \frac{\eta}{2}\right) \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \text{REV}(\mathcal{V}, \mathsf{f}^{\circ,*}(\xi), \xi) - \frac{\eta}{2} \\ &\geq \sum_{\xi \in \text{supp}(\gamma^*)} \gamma^*_{\xi} \text{REV}(\mathcal{V}, \mathsf{f}^{\circ,*}(\xi), \xi) - \eta \\ &= \sum_{\theta \in \Theta} \mu_\theta \sum_{s \in \mathcal{S}} \phi^*_\theta(s) \text{REV}(\mathcal{V}, f^*(s), \xi^{s}) - \eta,\end{aligned}$$

which allows us to conclude that there exists a pair $(\gamma, \mathsf{f}^\circ)$ that only uses q-uniform posteriors and provides the seller with an expected revenue arbitrary close to that of an optimal pair. □

Finally, we are ready to provide our PTAS. Its main idea is to solve LP 8.5 (of polynomial size) for the value of q in Lemma 8.9. This results in a small revenue loss. The last part missing for the algorithm is computing the terms appearing in the objective of LP 8.5, *i.e.*, a revenue-maximizing price vector (together with its revenue) for every q-uniform posterior. In order to do so, we can use Algorithm 8.1 (see also Lemma 8.6), which allows us to obtain in polynomial time good approximations of such price vectors, with high probability.

Theorem 8.3. *There exists an additive PTAS for computing a revenue-maximizing (ϕ, f) pair with public signaling.*

Proof. By Lemma 8.9, given any constant $\eta > 0$ and letting $q = \frac{128 \log \frac{6}{\eta}}{\eta^2}$, LP 8.5 has optimal value at least $OPT - \eta$. The polynomial-time algorithm that proves the theorem solves an approximated version of LP 8.5, which is obtained by replacing the terms $\max_{\mathbf{p} \in [0,1]^{\bar{n}}} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$ with suitable values $U(\boldsymbol{\xi})$. The latter are obtained by running Algorithm 8.1 (the values of ϵ and τ are defined in the following) for the (non-Bayesian) auctions in which the buyers' valuations are those resulting by multiplying samples drawn from distributions $\mathcal{V}_i$ by the posterior $\boldsymbol{\xi}$. We let $(\mathbf{p}^{\boldsymbol{\xi}}, U(\boldsymbol{\xi}))$ be the pair returned by Algorithm 8.1. By Lemma 8.6, for every q-uniform posterior $\boldsymbol{\xi} \in \Xi^q$, Algorithm 8.1 runs in polynomial time and the price vector $\mathbf{p}^{\boldsymbol{\xi}}$ is such that, with probability at least $1 - \tau$, it holds

$$\mathbb{E}_{V \sim \mathcal{V}}\left[g_{\mathbf{p}^{\boldsymbol{\xi}}}(V\boldsymbol{\xi})\right] \geq \max_{\mathbf{p} \in [0,1]^{\bar{n}}} \mathbb{E}_{V \sim \mathcal{V}}\left[g_{\mathbf{p}}(V\boldsymbol{\xi})\right] - \epsilon$$

and

$$U(\boldsymbol{\xi}) \in \left[\mathbb{E}_{V \sim \mathcal{V}}\left[g_{\mathbf{p}^{\boldsymbol{\xi}}}(V\boldsymbol{\xi})\right] - \epsilon, \mathbb{E}_{V \sim \mathcal{V}}\left[g_{\mathbf{p}^{\boldsymbol{\xi}}}(V\boldsymbol{\xi})\right] + \epsilon\right].$$

As a result, with probability at least $1 - \tau|\Xi^q|$, the previous conditions hold for every q-uniform posterior.

Next, we show that, with probability at least $1 - \tau|\Xi^q|$, an optimal solution to LP 8.5 is close to an optimal solution of the following LP obtained by replacing the $\max$ terms in the objective of LP 8.5 with the values $U(\boldsymbol{\xi})$:

$$\max_{\gamma \in \Delta_{\Xi^q}} \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) \quad \text{s.t.} \tag{8.6a}$$

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \, \xi_\theta = \mu_\theta \qquad \forall \theta \in \Theta. \tag{8.6b}$$

Notice that, for a constant $q \in \mathbb{N}_{>0}$, the number of q-uniform posteriors is at most d^q, so that LP 8.6 can be solved in polynomial time, as it involves $O(|\Xi^q|)$ variables and constraints.

Let $(\boldsymbol{\gamma}, \mathsf{f}^\circ)$ be such that $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ is an optimal solution to LP 8.6 and $\mathsf{f}^\circ : \Delta_\Theta \to [0,1]^{\bar{n}}$ is such that, for every $\boldsymbol{\xi} \in \Xi^q$, it holds $\mathsf{f}^\circ(\boldsymbol{\xi}) = \mathbf{p}^{\boldsymbol{\xi}}$, which is the price vector obtained by running Algorithm 8.1. Moreover, let $(\boldsymbol{\gamma}^*, \mathsf{f}^{\circ,*})$ be an optimal solution to LP 8.5. Then, with probability at least $1 - \tau|\Xi^q|$,

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \mathbb{E}_{V \sim \mathcal{V}} \left[g_{\mathsf{f}^\circ(\boldsymbol{\xi})}(V\boldsymbol{\xi}) \right] \geq \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) - \epsilon$$
$$\geq \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma^*_{\boldsymbol{\xi}} U(\boldsymbol{\xi}) - \epsilon \geq \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma^*_{\boldsymbol{\xi}} \mathbb{E}_{V \sim \mathcal{V}} \left[g_{\mathsf{f}^{\circ,*}(\boldsymbol{\xi})}(V\boldsymbol{\xi}, \mathsf{f}^{\circ,*}(\boldsymbol{\xi})) \right] - 2\epsilon.$$

In conclusion, since $\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma^*_{\boldsymbol{\xi}} \mathbb{E}_{V \sim \mathcal{V}} \left[g_{\mathsf{f}^{\circ,*}(\boldsymbol{\xi})}(V\boldsymbol{\xi}, \mathsf{f}^{\circ,*}(\boldsymbol{\xi})) \right] \geq OPT - \eta$ by Lemma 8.9, we have:

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \mathbb{E}_{V \sim \mathcal{V}} \left[g_{\mathbf{p}}(V\boldsymbol{\xi}) \right] \geq OPT - 2\epsilon - \eta$$

with probability at least $1 - \tau|\Xi^q|$. Hence,

$$\mathbb{E} \left[\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \text{REV}(\mathcal{V}, \mathsf{f}^\circ(\boldsymbol{\xi}), \boldsymbol{\xi}) \right] \geq (1 - \tau d^q)\, OPT - 2\epsilon - \eta,$$

where the expectation is over the randomness of the algorithm. Finally, Lemma 8.8 allows us to recover from $(\boldsymbol{\gamma}, \mathsf{f}^\circ)$ a signaling scheme with the same seller's expected revenue. For any additive approximation factor $\lambda > 0$, setting $\epsilon = \frac{\lambda}{6}$, $\eta = \frac{\lambda}{3}$, and $\tau = \frac{\lambda}{3d^q}$, we obtain the desired approximation bound. Moreover, the algorithm runs in polynomial time since η is constant and the running time of the algorithm is polynomial in ϵ, τ and the size of the problem instance. □

8.6 Private Signaling

With private signaling, computing a (ϕ, f) pair amounts to specifying a pair (ϕ_i, f_i) for each buyer $i \in \mathcal{N}$—composed by a marginal signaling scheme $\phi_i : \Theta \to \Delta_{\mathcal{S}_i}$ and a price function $\mathsf{f}_i : \mathcal{S}_i \to [0,1]$ for buyer i—, and, then, correlating the ϕ_i so as to obtain a (non-marginal) signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$. We leverage this fact to design our PTAS.

With abuse of notation, in this section we use Ξ_i^q to denote the (all equal) sets of q-uniform posteriors, one per buyer $i \in \mathcal{R}$, while $\Xi^q := \bigtimes_{i\in\mathcal{R}} \Xi_i^q$ is the set of tuples $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{\bar{n}})$ specifying a $\boldsymbol{\xi}_i \in \Xi_i^q$ for each $i \in \mathcal{R}$.

In Subsection 8.6.1, we first show that it is possible to restrict the set of marginal signaling schemes of a given buyer $i \in \mathcal{R}$ to those encoded as distributions over q-uniform posteriors, as we did with public signaling. Then, we provide an LP formulation for computing an approximately-optimal (ϕ, f) pair, dealing with the challenge of correlating marginal signaling schemes in a non-trivial way. Finally, in Subsection 8.6.2, we show how to compute a solution to the LP in polynomial time, which requires the application of the ellipsoid method in a non-trivial way, due to the features of the formulation.

8.6.1 LP for Approximate Signaling Schemes

Before providing the LP, we show that restricting marginal signaling schemes to q-uniform posteriors results in a buyer's behavior which is similar to the one with arbitrary posteriors. This amounts to showing that suitably-defined functions related to the probability of buying are comparatively stable.

For $i \in \mathcal{R}$ and $p_i \in [0,1]$, let $g_{i,p_i} : [0,1]^{\bar{n}} \to \{0,1\}$ be a function that takes as input a vector of buyers' valuations and outputs 1 if and only if $v_i \geq p_i$ (otherwise it outputs 0).

Lemma 8.10. *Given $\alpha, \epsilon > 0$ and some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathcal{R}}$, for every buyer $i \in \mathcal{R}$, posterior $\boldsymbol{\xi}_i \in \Delta_\Theta$, and price $p_i \in [0,1]$, the function $g_{i,[p_i-\epsilon]_+}$ is $(0,\alpha,\epsilon)$-stable compared with g_{i,p_i} for $(\boldsymbol{\xi}_i, \mathcal{V})$.*

Proof. Let us recall that $g_{i,p_i} : [0,1]^{\bar{n}} \to \{0,1\}$ is such that $g_{i,p_i}(\mathbf{x}) = \mathbb{I}[x_i \geq p_i]$ for any value $\mathbf{x} \in [0,1]^{\bar{n}}$. As a first step, we show that, for every valuation vector $v_i \in [0,1]^d$ and probability distribution $\boldsymbol{\gamma}$ over Δ_Θ that is (α,ϵ)-decreasing around $\boldsymbol{\xi}$, it holds $\mathbb{E}_{\tilde{\boldsymbol{\xi}}_i\sim\boldsymbol{\gamma}}\mathbb{I}[v_i^\top \tilde{\boldsymbol{\xi}}_i \geq [p_i - \epsilon]_+] \geq (1-\alpha)\mathbb{I}[v_i^\top \boldsymbol{\xi}_i \geq p_i]$. Two cases are possible. If $\mathbb{I}[v_i^\top \boldsymbol{\xi}_i \geq p_i] = 0$, then the inequality trivially holds. If $\mathbb{I}[v_i^\top \boldsymbol{\xi}_i \geq p_i] = 1$, by Definition 8.1 we have that, with probability at least $1-\alpha$, a posterior $\tilde{\boldsymbol{\xi}}_i \in \Delta_\Theta$ randomly drawn according to $\boldsymbol{\gamma}$ satisfies $v_i^\top \tilde{\boldsymbol{\xi}}_i \geq [v_i^\top \boldsymbol{\xi}_i - \epsilon]_+ \geq [p_i - \epsilon]_+$, which implies that $\mathbb{I}[v_i^\top \tilde{\boldsymbol{\xi}}_i \geq [p_i - \epsilon]_+] = 1$. Hence, $\mathbb{E}_{\tilde{\boldsymbol{\xi}}_i\sim\boldsymbol{\gamma}}\mathbb{I}[v_i^\top \tilde{\boldsymbol{\xi}}_i \geq [p_i - \epsilon]_+] \geq (1-\alpha) = (1-\alpha)\mathbb{I}[v_i^\top \boldsymbol{\xi}_i \geq p_i]$, as desired. Since $\mathbb{E}_{\tilde{\boldsymbol{\xi}}_i\sim\boldsymbol{\gamma}}\mathbb{I}[v_i^\top \tilde{\boldsymbol{\xi}}_i \geq [p_i - \epsilon]_+] \geq (1-\alpha)\mathbb{I}[v_i^\top \boldsymbol{\xi}_i \geq p_i]$ for every $v_i \in [0,1]^d$, by taking the expectation over vectors $v \sim \mathcal{V}$ we obtain $\mathbb{E}_{V\sim\mathcal{V}}\mathbb{E}_{\tilde{\boldsymbol{\xi}}_i\sim\boldsymbol{\gamma}}\mathbb{I}[V_i \tilde{\boldsymbol{\xi}}_i \geq [p_i - \epsilon]_+] \geq (1-\alpha)\mathbb{E}_{V\sim\mathcal{V}}\mathbb{I}[V_i \boldsymbol{\xi}_i \geq p_i]$, which fulfills the condition in Definition 8.2 and proves the result. $\square$

The following remark will be crucial for proving Lemma 8.12. It shows that, if for every $i \in \mathcal{R}$ we decompose buyer i's posterior $\boldsymbol{\xi}_i \in \Delta_\Theta$ by means of a distribution over q-uniform posteriors (α, ϵ)-decreasing around $\boldsymbol{\xi}_i$, then the probability with which buyer i buys only decreases by a small amount.

Remark 8.1. *Lemma 8.10 and Theorem 8.2 imply that, given a tuple of posteriors $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{\bar{n}}) \in \times_{i \in \mathcal{R}} \Delta_\theta$ and some distributions $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$, for every buyer $i \in \mathcal{R}$ and price $p_i \in [0,1]$, there exists $\boldsymbol{\gamma}^i \in \Delta_{\Xi_i^q}$ with $q = \frac{32}{\epsilon^2} \log \frac{4}{\alpha}$ s.t.*

$$\mathop{\mathbb{E}}_{\tilde{\boldsymbol{\xi}}_i \sim \boldsymbol{\gamma}^i} \left[\tilde{\xi}_{i,\theta} \Pr_{V \sim \mathcal{V}} \left\{ V_i \tilde{\boldsymbol{\xi}}_i \geq [p_i - \epsilon]_+ \right\} \right] \geq \xi_{i,\theta} (1 - \alpha) \Pr_{V \sim \mathcal{V}} \left\{ V_i \boldsymbol{\xi}_i \geq p_i \right\}$$

and $\sum_{\tilde{\boldsymbol{\xi}}_i \in \Xi_i^q} \gamma^i_{\tilde{\boldsymbol{\xi}}_i} \tilde{\xi}_{i,\theta} = \xi_{i,\theta}$ for all $\theta \in \Theta$.

Next, we show that an approximately-optimal pair (ϕ, f) can be found by solving LP 8.7 instantiated with suitably-defined $q \in \mathbb{N}_{>0}$ and $b \in \mathbb{N}_{>0}$. LP 8.7 employs:

- Variables $\gamma_{i,\boldsymbol{\xi}_i}$ (for $i \in \mathcal{R}$ and $\boldsymbol{\xi}_i \in \Xi_i^q$), which encode the distributions over posteriors representing the marginal signaling schemes $\phi_i : \Theta \to \Delta_{\mathcal{S}_i}$ of the buyers.
- Variables $t_{i,\boldsymbol{\xi}_i,p_i}$ (for $i \in \mathcal{R}$, $\boldsymbol{\xi}_i \in \Xi_i^q$, and $p_i \in P^b$), with $t_{i,\boldsymbol{\xi}_i,p_i}$ encoding the probability that the seller offers price p_i to buyer i and buyer i's posterior is $\boldsymbol{\xi}_i$.
- Variables $y_{\theta,\boldsymbol{\xi},\mathbf{p}}$ (for $\theta \in \Theta$, $\boldsymbol{\xi} \in \Xi^q$, and $\mathbf{p} \in \mathcal{P}^b$), with $y_{\theta,\boldsymbol{\xi},\mathbf{p}}$ encoding the probability that the state is θ, the buyers' posteriors are those specified by $\boldsymbol{\xi}$, and the prices that the seller offers to the buyers are those given by $\mathbf{p}$.

$$\max_{\boldsymbol{\gamma},\mathbf{t},\mathbf{y} \geq 0} \sum_{\theta \in \Theta} \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}} \, \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \quad \text{s.t.} \tag{8.7a}$$

$$\xi_{i,\theta} t_{i,\boldsymbol{\xi}_i,p_i} = \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'} \qquad \forall \theta \in \Theta, \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi_i^q, \forall p_i \in P^b \tag{8.7b}$$

$$\sum_{p_i \in P^b} t_{i,\boldsymbol{\xi}_i,p_i} = \gamma_{i,\boldsymbol{\xi}_i} \qquad \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi_i^q \tag{8.7c}$$

$$\sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \gamma^i_{\boldsymbol{\xi}_i} \, \xi_{i,\theta} = \mu_\theta \qquad \forall i \in \mathcal{R}, \forall \theta \in \Theta \tag{8.7d}$$

Variables $t_{i,\boldsymbol{\xi}_i,p_i}$ represent marginal signaling schemes, allowing for multiple signals inducing the same posterior. This is needed since signals may correspond to different price proposals.[12] One may think of marginal signaling schemes in LP 8.7 as if they were using signals defined as pairs $s_i = (\boldsymbol{\xi}_i, p_i)$, with the convention that $\mathsf{f}_i(s_i) = p_i$. Variables $y_{\theta,\boldsymbol{\xi},\mathbf{p}}$ and Constraints (8.7b) ensure that marginal signaling schemes are correctly correlated together, by directly working in the domain of the distributions over posteriors.

To show that an optimal solution to LP 8.7 provides an approximately-optimal (ϕ, f) pair, we need the following two lemmas. Lemma 8.11 proves that, given a feasible solution to LP 8.7, we can recover a pair (ϕ, f) providing the seller with an expected revenue equal to the value of the LP solution. Lemma 8.12 shows that the optimal value of LP 8.7 is "close" to OPT. These two lemmas imply that an approximately-optimal (ϕ, f) pair can be computed by solving LP 8.7.

Lemma 8.11. *Given a feasible solution to LP 8.7, it is possible to recover a pair (ϕ, f) that provides the seller with an expected revenue equal to the value of the solution.*

Proof. We define the set of signals for buyer $i \in \mathcal{R}$ as $\mathcal{S}_i := \Xi_i^q \times P^b$. Then, we set $\phi : \Theta \to \Delta_{\mathcal{S}}$ so that, for every $\theta \in \Theta$ and $\mathrm{s} \in \mathcal{S}$, it holds $\phi_\theta(\mathbf{s}) = \frac{y_{\theta,\boldsymbol{\xi},\mathbf{p}}}{\mu_\theta}$, where the pair $(\boldsymbol{\xi}, \mathbf{p})$ with $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n) \in \Xi^q$ and $\mathbf{p} \in \mathcal{P}^b$ is such that $(\boldsymbol{\xi}_i, p_i) = s_i$ for each $i \in \mathcal{R}$. Moreover, we set $f_i(s_i) = p_i$ for every buyer $i \in \mathcal{N}$ and signal $s_i = (\boldsymbol{\xi}_i, p_i) \in \mathcal{S}_i$. First, we show that ϕ is well defined, that is, for every state of nature $\theta \in \Theta$, it holds

$$\begin{aligned}\sum_{\mathbf{s} \in \mathcal{S}} \phi_\theta(\mathbf{s}) &= \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} \frac{y_{\theta,\boldsymbol{\xi},\mathbf{p}}}{\mu_\theta} \\ &= \sum_{\boldsymbol{\xi}_1 \in \mathcal{S}_1} \sum_{p_1 \in P^b} \frac{\xi_{1,\theta} t_{1,\boldsymbol{\xi}_1,p_1}}{\mu_\theta} = \sum_{\boldsymbol{\xi}_1 \in \Xi_1^q} \frac{\xi_{1,\theta} \gamma_{1,\boldsymbol{\xi}_1}}{\mu_\theta} = \frac{\mu_\theta}{\mu_\theta} = 1,\end{aligned}$$

where we use Constraints (8.7b) in the second equality, Constraints (8.7c) in the third one, and Constraints (8.7d) in the last one. Next, we show that,

[12] Notice that, in a classical setting in which the sender does *not* have to propose a price (or, in general, select some action after sending signals), there always exists a signaling scheme with no pair of signals inducing the same posterior. Indeed, two signals that induce the same posterior can always be joined into a single signal. This is *not* the case in our setting, where we can only join signals that induce the same posterior and correspond to the same price.

for any $\boldsymbol{\xi} \in \Xi^q$ and $\mathbf{p} \in \mathcal{P}^b$, it holds $\text{REV}(\mathcal{V}, f(\mathbf{s}), \boldsymbol{\xi}^{\mathbf{s}}) = \text{REV}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$, where the signal profile $\mathbf{s} \in \mathcal{S}$ is such that $s_i = (\boldsymbol{\xi}_i, p_i)$ for every $i \in \mathcal{R}$. Clearly, the prices coincide, namely $\mathsf{f}(\mathbf{s}) = \mathbf{p}$. Thus, it is sufficient to prove that each signal $s_i = (\boldsymbol{\xi}_i, p_i)$ induces posterior $\boldsymbol{\xi}_i$ for buyer $i \in \mathcal{R}$. For every $\theta \in \Theta$, it holds

$$\mu_\theta \sum_{\mathbf{s}' \in \mathcal{S}: s'_i = s_i} \phi_\theta(\mathbf{s}') = \sum_{\boldsymbol{\xi}' \in \Xi^q, \mathbf{p}' \in \mathcal{P}^b : (\boldsymbol{\xi}'_i, p'_i) = s_i} y_{\theta, \boldsymbol{\xi}', \mathbf{p}'} = \xi_{i,\theta} t_{i, \boldsymbol{\xi}_i, p_i}.$$

Hence, for every $\theta \in \Theta$,

$$\begin{aligned} \xi_\theta^{s_i} &= \frac{\mu_\theta \phi_{i,\theta}(s_i)}{\sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i)} \\ &= \frac{\mu_\theta \sum_{\mathbf{s}' \in \mathcal{S}: s'_i = s_i} \phi_\theta(\mathbf{s}')}{\sum_{\theta' \in \Theta} \mu_{\theta'} \sum_{\mathbf{s}' \in \mathcal{S}: s'_i = s_i} \phi_{\theta'}(\mathbf{s}')} = \frac{\xi_{i,\theta} t_{i, \boldsymbol{\xi}_i, p_i}}{t_{i, \boldsymbol{\xi}_i, p_i}} = \xi_{i,\theta}. \end{aligned}$$

Thus, the seller's expected revenue for the pair (ϕ, f) is

$$\begin{aligned} \sum_{\mathbf{s} \in \mathcal{S}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta(\mathbf{s}) \text{REV}(\mathcal{V}, \mathsf{f}(\mathbf{s}), \boldsymbol{\xi}^{\mathbf{s}}) &= \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} \sum_{\theta \in \Theta} \mu_\theta \frac{y_{\theta, \boldsymbol{\xi}, \mathbf{p}}}{\mu_\theta} \text{REV}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\ &= \sum_{\theta \in \Theta} \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} y_{\theta, \boldsymbol{\xi}, \mathbf{p}} \text{REV}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}), \end{aligned}$$

which proves the lemma. □

Lemma 8.12. *For every $\eta > 0$, there exist $b(\eta), q(\eta) \in \mathbb{N}_{>0}$ such that LP 8.7 has optimal value at least $OPT - \eta$.*

Proof. We show that, given a revenue-maximizing pair (ϕ, f) (with seller's revenue OPT), we can recover an optimal solution to LP 8.7 whose value is at least $OPT - \eta$ when the LP is instantiated with suitable constants $b(\eta) \in \mathbb{N}_{>0}$ and $q(\eta) \in \mathbb{N}_{>0}$ (depending on the approximation level η). Let $\alpha = \epsilon = \frac{\eta}{3}$, $b = \lceil \frac{3}{\eta} \rceil$, and $q = \frac{32 \log \frac{4}{\alpha}}{\epsilon^2}$. Recalling that $\boldsymbol{\xi}^{s_i} \in \Delta_\Theta$ denotes buyer i's posterior induced by signal $s_i \in \mathcal{S}_i$, we let $\boldsymbol{\gamma}^{s_i} \in \Delta_{\Xi_i^q}$ be the probability distribution over q-uniform posteriors obtained by decomposing $\boldsymbol{\xi}^{s_i}$ according to Theorem 8.2. By Lemma 8.10 and Theorem 8.2, it follows that, for every $p_i \in [0,1]$ and $\theta \in \Theta$,

$$\sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \gamma_{\boldsymbol{\xi}_i}^{s_i} \xi_{i,\theta} \text{Pr}_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \boldsymbol{\xi}_i \geq [p_i - \epsilon]_+ \right\} \geq \xi_\theta^{s_i} (1 - \alpha) \text{Pr}_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \boldsymbol{\xi}_i \geq p_i \right\}. \tag{8.8}$$

For every signal profile $\mathrm{s} \in \mathcal{S}$, we define a non-Bayesian posted price auction in which the distributions of buyers' valuations are $\mathcal{V}^{\mathrm{s}} = \{\mathcal{V}_i^s\}_{i\in\mathcal{N}}$, where each $\mathcal{V}_i^s$ is such that a valuation $v_i \sim \mathcal{V}_i^s$ is obtained by first sampling $\tilde{v}_i \sim \mathcal{V}_i$ and then letting $v_i = \tilde{v}_i^\top \boldsymbol{\xi}^{s_i}$. Moreover, we let $\mathbf{p}^{\mathrm{s}} \in [0,1]^{\bar{n}}$ be a price vector for the seller in such non-Bayesian auction, with $p_i^{\mathrm{s}} \geq \textsc{Rev}_{>i}(\mathcal{V}^{\mathrm{s}}, \mathbf{p}^{\mathrm{s}})$ for every $i \in \mathcal{R}$. By Lemma 8.4, such a vector always exists. Finally, given $\mathbf{p}^{\mathrm{s}}$, we let $\hat{\mathbf{p}}^{\mathrm{s}} \in [0,1]^{\bar{n}}$ be such that each price $\hat{p}_i^{\mathrm{s}}$ is the greatest price $p_i \in P^b$ (among discretized prices) satisfying the inequality $p_i \leq [p_i^{\mathrm{s}} - \epsilon]_+$; formally,

$$p_i^{\mathrm{s}} = \max\left\{p_i \in P^b \mid p_i \leq [p_i^{\mathrm{s}} - \epsilon]_+\right\}.$$

Next, we define the optimal solution to LP 8.7 that we need to prove the result:

- $\gamma_{i,\boldsymbol{\xi}_i} = \sum_{s_i\in\mathcal{S}_i}\sum_{\theta\in\Theta}\mu_\theta\phi_{i,\theta}(s_i)\gamma^{s_i}(\boldsymbol{\xi}_i)$ for every $i \in \mathcal{R}$ and $\boldsymbol{\xi}_i \in \Xi_i^q$.
- $t_{i,\boldsymbol{\xi}_i,p_i} = \sum_{s_i\in\mathcal{S}_i}\sum_{\theta\in\Theta}\mu_\theta\phi_{i,\theta}(s_i)\gamma_{\boldsymbol{\xi}_i}^{s_i}\mathbb{I}\left\{p_i = \hat{p}_i^{\mathrm{s}}\right\}$ for every $i \in \mathcal{R}$, $\boldsymbol{\xi}_i \in \Xi_i^q$, and $p_i \in P^b$.
- $y_{\theta,\boldsymbol{\xi},\mathbf{p}} = \sum_{\mathrm{s}\in\mathcal{S}}\mu_\theta\phi_\theta(\mathrm{s})\prod_{i\in\mathcal{R}}\frac{\xi_{i,\theta}\gamma_{\boldsymbol{\xi}_i}^{s_i}}{\xi_\theta^{s_i}}\mathbb{I}[p_i = \hat{p}_i^{\mathrm{s}}]$ for every $\theta \in \Theta$, $\boldsymbol{\xi} \in \Xi^q$, and $\mathbf{p} \in \mathcal{P}^b$.

The next step is to show that, for every signal profile $\mathrm{s} \in \mathcal{S}$, the seller's expected revenue obtained by decomposing each signal s_i according to Theorem 8.2 is "close" to the one for s. Formally, we show that, for every $s \in \mathcal{S}$ and $\theta \in \Theta$,

$$\sum_{\boldsymbol{\xi}\in\Xi^q}\prod_{i\in\mathcal{R}}\frac{\xi_{i,\theta}\gamma_{\boldsymbol{\xi}_i}^{s_i}}{\xi_\theta^{s_i}}\textsc{Rev}(\mathcal{V}, \hat{\mathbf{p}}^{\mathrm{s}}, \boldsymbol{\xi}) \geq \textsc{Rev}(\mathcal{V}, \mathsf{f}(\mathrm{s}), \boldsymbol{\xi}^{\mathrm{s}}) - \left(\alpha + \epsilon + \frac{1}{b}\right). \tag{8.9}$$

In order to do so, we relate the LHS of Equation (8.9) to the seller' revenue in a non-Bayesian posted price auction. In particular, we show that it is equivalent to the seller's revenue when employing price vector $\hat{p}^s$ in the auction defined by the distributions of buyers' valuations $\hat{\mathcal{V}}^{s,\theta} = \{\hat{\mathcal{V}}_i^{s,\theta}\}_{i\in\mathcal{R}}$, where each $\hat{\mathcal{V}}_i^{\mathrm{s},\theta}$ is such that a valuation $v_i \sim \hat{\mathcal{V}}_i^{\mathrm{s},\theta}$ is defined as $v_i = \tilde{v}_i^\top\tilde{\boldsymbol{\xi}}_i$, with $\tilde{v}_i \sim \mathcal{V}_i$ and $\tilde{\boldsymbol{\xi}}_i \in \Delta_\Theta$ sampled from a distribution such that

$$\Pr\left\{\tilde{\boldsymbol{\xi}}_i = \boldsymbol{\xi}_i\right\} = \frac{\xi_{i,\theta}}{\xi_\theta^{s_i}}\gamma_{\boldsymbol{\xi}_i}^{s_i}.$$

Notice that each $\hat{\mathcal{V}}_i^{s,\theta}$ is well defined, since $\mathcal{V}_i$ is by definition a probability distribution and $\sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \frac{\xi_{i,\theta}}{\xi_\theta^{s_i}} \gamma_{\boldsymbol{\xi}_i}^{s_i} = 1$ by Theorem 8.2, defining a probability distribution over the posteriors. Moreover, it is easy to check that valuations sampled from distributions $\hat{\mathcal{V}}_i^{s,\theta}$ are independent among each other. Finally, $\textsc{Rev}(\hat{\mathcal{V}}^{\mathbf{s},\theta}, \hat{\mathbf{p}}^{\mathbf{s}})$ is equal to to the LHS of Equation (8.9), since, by an inductive argument, for every $i \in \mathcal{R}$, it holds

$$\sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \prod_{j \in \mathcal{R}} \frac{\xi'_{j,\theta} \gamma_{\boldsymbol{\xi}'_j}^{s_j}}{\xi_\theta^{s_j}} = \frac{\xi_{i,\theta}}{\xi_\theta^{s_i}} \gamma_{\boldsymbol{\xi}_i}^{s_i},$$

where the equality comes from the fact that, for every $j \in \mathcal{N}$, it is the case that

$$\sum_{\boldsymbol{\xi}_j \in \Xi_j^q} \frac{\xi_{j,\theta} \gamma_{\boldsymbol{\xi}_j}^{s_j}}{\xi_\theta^{s_j}} = 1. \tag{8.10}$$

Let also notice that, in the auction defined above, the probability with which a buyer $i \in \mathcal{R}$ has a valuation greater than or equal to $\mathbf{p}_i^s$ is

$$\sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \frac{\xi_{i,\theta} \gamma_{\boldsymbol{\xi}_i}^{s_i}}{\xi_\theta^{s_i}} \text{Pr}_{\tilde{v}_i \sim \mathcal{V}_i}(\tilde{v}_i^\top \boldsymbol{\xi}_i \geq \mathbf{p}_i^s) \geq (1-\alpha) \text{Pr}_{\tilde{v}_i \sim \mathcal{V}_i}(\tilde{v}_i^\top \boldsymbol{\xi}^{s_i} \geq p_i^{\mathbf{s}}),$$

where the inequality holds by Equation (8.8). First, we compare the seller's revenue in the two non-Bayesian, namely $\textsc{Rev}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}})$ and $\textsc{Rev}(\hat{\mathcal{V}}^{\mathbf{s},\theta}, \hat{\mathbf{p}}^{\mathbf{s}})$. In particular, we show by induction that $\textsc{Rev}(\hat{\mathcal{V}}^{\mathbf{s},\theta}, \hat{\mathbf{p}}^{\mathbf{s}}) \geq \textsc{Rev}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}}) - \alpha - \epsilon - \frac{1}{b}$. Let $\textsc{Rev}_{\geq i}(\mathcal{V}, \mathbf{p})$ be the seller's expected revenue for $\mathbf{p}$ in the auction restricted to all buyers $j \in \mathcal{R} : j \geq i$. The base case is

$$\begin{aligned}
\textsc{Rev}_{\geq \bar{n}}(\hat{\mathcal{V}}^{\mathbf{s},\theta}, \hat{\mathbf{p}}^{\mathbf{s}}) &= \hat{p}_{\bar{n}}^{\mathbf{s}} \, \text{Pr}_{v_{\bar{n}} \sim \hat{\mathcal{V}}_{\bar{n}}^{\mathbf{s},\theta}} \{v_{\bar{n}} \geq \hat{p}_{\bar{n}}^{\mathbf{s}}\} \\
&\geq \hat{p}_{\bar{n}}^{\mathbf{s}} (1-\alpha) \text{Pr}_{v_{\bar{n}} \sim \mathcal{V}_{\bar{n}}^{\mathbf{s}}} \{v_{\bar{n}} \geq p_{\bar{n}}^{\mathbf{s}}\} \\
&\geq p_{\bar{n}}^{\mathbf{s}} \, \text{Pr}_{v_{\bar{n}} \sim \mathcal{V}_{\bar{n}}^{\mathbf{s}}} \{v_{\bar{n}} \geq p_{\bar{n}}^{\mathbf{s}}\} - \epsilon - \alpha - \frac{1}{b} \\
&= \textsc{Rev}_{\geq \bar{n}}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}}) - \epsilon - \alpha - \frac{1}{b}.
\end{aligned}$$

By induction, let us assume that the condition holds for $i+1$, then

$$\begin{aligned}
&\textsc{Rev}_{\geq i}(\hat{\mathcal{V}}^{\mathbf{s},\theta}, \hat{\mathbf{p}}^{\mathbf{s}}) \\
&\quad = \hat{p}_i^{\mathbf{s}} \text{Pr}_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \{v_i \geq \hat{p}_i^{\mathbf{s}}\} + \left(1 - \text{Pr}_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \{v_i \geq \hat{p}_i^{\mathbf{s}}\}\right) \textsc{Rev}_{> i}(\hat{\mathcal{V}}^{\mathbf{s},\theta}, \hat{\mathbf{p}}^{\mathbf{s}})
\end{aligned}$$

$$
\begin{aligned}
&\geq \left(p_i^{\mathbf{s}} - \epsilon - \frac{1}{b}\right) \Pr_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \left\{v_i \geq \hat{p}_i^{\mathbf{s}}\right\} \\
&\qquad + \left(1 - \Pr_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \left\{v_i \geq \hat{p}_i^{\mathbf{s}}\right\}\right) \left(\textsc{Rev}_{>i}(\mathcal{V}^s, p^s) - \epsilon - \alpha - \frac{1}{b}\right) \\
&= p_i^s \Pr_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \left\{v_i \geq \hat{p}_i^{\mathbf{s}}\right\} \\
&\qquad + \left(1 - \Pr_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \left\{v_i \geq \hat{p}_i^{\mathbf{s}}\right\}\right) \left(\textsc{Rev}_{>i}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}}) - \alpha\right) - \epsilon - \frac{1}{b} \\
&\geq p_i^{\mathbf{s}}(1-\alpha) \Pr_{v_i \sim \mathcal{V}_i^{\mathbf{s}}} \left\{v_i \geq p_i^{\mathbf{s}}\right\} \\
&\qquad + \left[1 - (1-\alpha)\Pr_{v_i \sim \mathcal{V}_i^{\mathbf{s}}} \left\{v_i \geq p_i^{\mathbf{s}}\right\}\right] \left(\textsc{Rev}_{>i}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}}) - \alpha\right) - \epsilon - \frac{1}{b} \\
&\geq \textsc{Rev}_{\geq i}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}}) - \epsilon - \alpha - \frac{1}{b},
\end{aligned}
$$

where the second to last inequality follows from $p_i^{\mathbf{s}} \geq \textsc{Rev}_{>i}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}})$ and $\Pr_{v_i \sim \hat{\mathcal{V}}_i^{\mathbf{s},\theta}} \left\{v_i \geq \hat{p}_i^{\mathbf{s}}\right\} \geq (1-\alpha)\Pr_{v_i \sim \mathcal{V}_i^{\mathbf{s}}} \left\{v_i \geq p_i^{\mathbf{s}}\right\}$. Hence, Equation (8.9) is readily proved, as follows

$$
\begin{aligned}
\sum_{\boldsymbol{\xi} \in \Xi^q} \prod_{i \in \mathcal{R}} \frac{\xi_{i,\theta} \gamma^{s_i}(\boldsymbol{\xi}_i)}{\xi_\theta^{s_i}} \textsc{Rev}(\mathcal{V}, \hat{\mathbf{p}}^{\mathbf{s}}, \boldsymbol{\xi}) &\geq \textsc{Rev}(\mathcal{V}^{\mathbf{s}}, \mathbf{p}^{\mathbf{s}}) - \left(\alpha + \epsilon + \frac{1}{b}\right) \\
&\geq \textsc{Rev}(\mathcal{V}, \mathsf{f}(\mathbf{s}), \boldsymbol{\xi}^{\mathbf{s}}) - \left(\alpha + \epsilon + \frac{1}{b}\right),
\end{aligned}
$$

Now, we are ready to bound the objective of LP 8.7, as follows:

$$
\begin{aligned}
&\sum_{\theta \in \Theta} \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} y_{\theta, \boldsymbol{\xi}, \mathbf{p}} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\
&\quad = \sum_{\theta \in \Theta} \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} \sum_{\mathbf{s} \in \mathcal{S}} \mu_\theta \phi_\theta(\mathbf{s}) \prod_{i \in \mathcal{R}} \frac{\xi_{i,\theta} \gamma^{s_i}(\boldsymbol{\xi}_i)}{\xi_\theta^{s_i}} \mathbb{I}[\hat{p}_i^{\mathbf{s}} = p_i] \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\
&\quad = \sum_{\mathbf{s} \in \mathcal{S}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta(\mathbf{s}) \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b s} \prod_{i \in \mathcal{R}} \frac{\xi_{i,\theta} \gamma^{s_i}(\boldsymbol{\xi}_i)}{\xi_\theta^{s_i}} \mathbb{I}[\hat{p}_i^{\mathbf{s}} = p_i] \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\
&\quad \geq \sum_{\mathbf{s} \in \mathcal{S}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta(\mathbf{s}) \left[\textsc{Rev}(\mathcal{V}, f(\mathbf{s}), \boldsymbol{\xi}^{\mathbf{s}}) - \left(\alpha + \epsilon + \frac{1}{b}\right)\right] \\
&\quad \geq OPT - \left(\alpha + \epsilon + \frac{1}{b}\right) \geq OPT - \eta.
\end{aligned}
$$

We conclude the proof showing that the defined solution is feasible for

LP 8.7. First, it prove that, for every $i \in \mathcal{R}$ and $\theta \in \Theta$,

$$\begin{aligned}\sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \xi_{i,\theta}\gamma_{i,\boldsymbol{\xi}_i} &= \sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \xi_{i,\theta} \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'}\phi_{i,\theta'}(s_i)\gamma_{\boldsymbol{\xi}_i}^{s_i} \\ &= \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'}\phi_{i,\theta'}(s_i) \sum_{\boldsymbol{\xi}_i \in \Xi_i^q} \xi_{i,\theta}\gamma_{\boldsymbol{\xi}_i}^{s_i} \\ &= \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'}\phi_{i,\theta'}(s_i)\xi_\theta^{s_i} \\ &= \sum_{s_i \in \mathcal{S}_i} \sum_{\theta' \in \Theta} \mu_{\theta'}\phi_{i,\theta'}(s_i)\frac{\mu_\theta\phi_{i,\theta}(s_i)}{\sum_{\theta' \in \Theta}\mu_{\theta'}\phi_{i,\theta'}(s_i)} \\ &= \sum_{s_i \in \mathcal{S}_i} \mu_\theta\phi_\theta(s_i) = \mu_\theta.\end{aligned}$$

Moreover, for every $i \in \mathcal{R}$ and $\boldsymbol{\xi}_i \in \Xi_i^q$, it holds

$$\begin{aligned}\sum_{p_i \in P^b} t_{i,\boldsymbol{\xi}_i,p_i} = \sum_{p_i \in P^b} \sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta\phi_{i,\theta}(s_i)\gamma_{\boldsymbol{\xi}_i}^{s_i}\mathbb{I}[p_i = \hat{p}_i^{\mathbf{s}}] = \\ \sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta\phi_{i,\theta}(s_i)\gamma_{\boldsymbol{\xi}_i}^{s_i} \sum_{p_i \in P^b} \mathbb{I}[p_i = \hat{p}_i^{\mathbf{s}}] = \\ \sum_{s_i \in \mathcal{S}_i} \sum_{\theta \in \Theta} \mu_\theta\phi_{i,\theta}(s_i)\gamma_{\boldsymbol{\xi}_i}^{s_i} = \gamma_{i,\boldsymbol{\xi}_i}.\end{aligned}$$

Finally, for every $\theta \in \Theta$, $i \in \mathcal{R}$, $\boldsymbol{\xi}_i \in \Xi_i^q$, and $p_i \in P^b$, it holds

$$\begin{aligned}&\sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'} \\ &= \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} \sum_{\mathbf{s} \in \mathcal{S}} \mu_\theta\phi_\theta(\mathbf{s}) \prod_{j \in \mathcal{R}} \frac{\xi'_{j,\theta}}{\xi_\theta^{s_j}}\gamma_{\boldsymbol{\xi}'_j}^{s_j}\mathbb{I}[p_j = \hat{p}_j^{\mathbf{s}}] \\ &= \sum_{\mathbf{s} \in \mathcal{S}} \mu_\theta\phi_\theta(\mathbf{s}) \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} \prod_{j \in \mathcal{R}} \frac{\xi'_{j,\theta}}{\xi_\theta^{s_j}}\gamma_{\boldsymbol{\xi}'_j}^{s_j}\mathbb{I}[p_j = \hat{p}_j^{\mathbf{s}}] \\ &= \sum_{\mathbf{s} \in \mathcal{S}} \mu_\theta\phi_\theta(\mathbf{s})\frac{\xi_{i,\theta}}{\xi_\theta^{s_i}}\gamma_{\boldsymbol{\xi}_i}^{s_i}\mathbb{I}[p_i = \hat{p}_i^{\mathbf{s}}] \qquad \text{(From Equation (8.10))} \\ &= \sum_{s_i \in \mathcal{S}_i} \mu_\theta\phi_{i,\theta}(s_i)\frac{\xi_{i,\theta}}{\xi_\theta^{s_i}}\gamma_{\boldsymbol{\xi}_i}^{s_i}\mathbb{I}[p_i = \hat{p}_i^{\mathbf{s}}] \\ &= \xi_{i,\theta} \sum_{s_i \in \mathcal{S}} \mu_\theta\phi_{i,\theta}(s_i)\frac{\sum_{\theta' \in \Theta}\phi_{i,\theta'}(s_i)}{\mu_\theta\phi_{i,\theta}(s_i)}\gamma_{\boldsymbol{\xi}_i}^{s_i}\mathbb{I}[p_i = \hat{p}_i^{\mathbf{s}}]\end{aligned}$$

$$= \xi_{i,\theta} \sum_{s_i \in \mathcal{S}} \sum_{\theta' \in \Theta} \mu_{\theta'} \phi_{i,\theta'}(s_i) \gamma^{s_i}_{\boldsymbol{\xi}_i} \mathbb{I}[p_i = \hat{p}^s_i] = \xi_{i,\theta} t_{i,\boldsymbol{\xi}_i,p_i}.$$

This concludes the proof. □

8.6.2 PTAS for Private Signaling

We provide an algorithm that approximately solves LP 8.7 in polynomial time, which completes our PTAS for computing a revenue-maximizing pair (ϕ, f) in the private setting. The core idea of our algorithm is to apply the ellipsoid method on the dual of LP 8.7.[13] In particular, our implementation of the ellipsoid algorithm uses an approximate separation oracle that needs to solve the following optimization problem.

Definition 8.4 (MAX-LINREV)**.** *Given some distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$ such that each $\mathcal{V}_i$ has finite support and a vector $\mathbf{w} \in [0,1]^{\bar{n} \times |\Xi^q_i| \times |P^b|}$, solve*

$$\max_{\boldsymbol{\xi} \in \Xi^q, \mathbf{p} \in \mathcal{P}^b} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) + \sum_{i \in \mathcal{R}} w_{i,\boldsymbol{\xi}_i,p_i}.$$

As a first step, we provide an FPTAS for MAX-LINREV using a dynamic programming approach. This will be the main building block of our approximate separation oracle.[14]

The FPTAS works as follows. Given an error tolerance $\delta > 0$, it first defines a step size $\frac{1}{c}$, with $c = \lceil \frac{\bar{n}}{\delta} \rceil$, and builds a set $A = \{0, \frac{1}{c}, \frac{2}{c}, \dots, \bar{n}\}$ of possible discretized values for the linear term appearing in the MAX-LINREV objective. Then, for every buyer $i \in \mathcal{R}$ (in reversed order) and value $a \in A$, the algorithm computes $M(i,a)$, which is an approximation of the largest seller's revenue provided by a pair $(\boldsymbol{\xi}, \mathbf{p})$ when considering buyers $i, \dots, \bar{n}$ only, and restricted to pairs $(\boldsymbol{\xi}, \mathbf{p})$ such that the inequality $\sum_{j \in \mathcal{R}: j \geq i} w_{j,\boldsymbol{\xi}_j,p_j} \geq a$ is satisfied. By letting $z_i := \Pr_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \boldsymbol{\xi}_i \geq p_i \right\}$, the value $M(i,a)$ can be defined by the following recursive formula:[15]

$$M(i,a) := \max_{\substack{\boldsymbol{\xi}_i \in \Xi^q_i, p_i \in P^b \\ a' \in A: w_{i,\boldsymbol{\xi}_i,p_i} + a' \geq a}} z_i p_i + (1 - z_i)\, M(i+1, a').$$

Finally, the algorithm returns $\max_{a \in A} \{M(1,a) + a\}$. Thus:

[13] To be precise, we apply the ellipsoid method to the dual of a relaxed version of LP 8.7, since we need an over-constrained dual.

[14] Notice that, since MAX-LINREV takes as input distributions with a *finite support*, we can safely assume that such distributions can be explicitly represented in memory. In our PTAS, the inputs to the dynamic programming algorithm are obtained by building empirical distributions thorough samples from the actual distributions of buyers' valuations, thus ensuring finiteness of the supports.

[15] Notice that, given a pair $(\boldsymbol{\xi}, \mathbf{p})$ with $\boldsymbol{\xi} \in \Xi^q$ and $\mathbf{p} \in \mathcal{P}^b$, it is possible to compute in polynomial time the probability with which a buyer $i \in \mathcal{R}$ buys the item.

Algorithm 8.2 Approximate Dynamic Programming algorithm for MAX-LINREV

Inputs: Discretization error tolerance $\delta > 0$; vector of linear components $\mathbf{w} \in [0,1]^{n \times |\Xi_i^q| \times |P^b|}$; finite-support distributions of buyers' valuations $\mathcal{V} = \{\mathcal{V}_i\}_{i \in \mathcal{R}}$

1: $c \leftarrow \lceil \frac{n}{\delta} \rceil$

2: $A \leftarrow \{0, \frac{1}{c}, \frac{2}{c}, \dots, \frac{n\delta}{c}\}$

3: Initialize an empty matrix M with dimension $n \times |A|$

4: **for** $a \in A$ **do**

5: $\quad M(n,a) \leftarrow \max\limits_{\boldsymbol{\xi}_n \in \Xi_n^q, p_n \in P^b : w_{n,\boldsymbol{\xi}_n,p_n} \geq a} \left\{ \Pr\limits_{v_n \sim \mathcal{V}_n} \left\{ v_n^\top \boldsymbol{\xi}_n \geq p_n \right\} p_n \right\}$

6: **end for**

7: **for** $i = n-1, \dots, 1$ (in reversed order) **do**

8: $\quad$ **for** $a \in A$ **do**

9: $\quad\quad M(i,a) \leftarrow \max\limits_{\substack{\boldsymbol{\xi}_i \in \Xi_i^q, p_i \in P^b, a' \in A: \\ w_{i,\boldsymbol{\xi}_i,p_i} + a' \geq a}} \left\{ \Pr\limits_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \boldsymbol{\xi}_i \geq p_i \right\} p_i + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \left\{ v_i^\top \boldsymbol{\xi}_i \geq p_i \right\}\right) M(i+1, a') \right\}$

10: $\quad$ **end for**

11: **end for**

12: **return** $\max_{a \in A} \{M(1,a) + a\}$

Lemma 8.13. *For any $\delta > 0$, there exists a dynamic programming algorithm that provides a δ-approximation (in the additive sense) to* MAX-LINREV. *Moreover, the algorithm runs in time polynomial in the size of the input and $\frac{1}{\delta}$.*

Proof. The algorithm is described in Algorithm 8.2. It works in polynomial time since the matrix M has $\bar{n}|A| = O(\frac{1}{\delta}\bar{n}^3)$ entries and each entry is computed in polynomial time. This proves the second part of the statement.

In the following, we denote with $\textsc{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$ the seller's expected revenue in the Bayesian posted price auction when they select price vector $\mathbf{p} \in \mathcal{P}^b$ and the buyers' posteriors are specified by the tuple $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{\bar{n}}) \in \Xi^q$.

Let $S(i,a) := \{(\boldsymbol{\xi}, \mathbf{p}) \in \Xi^q \times \mathcal{P}^b \mid \sum_{j \geq i} w_{j,\boldsymbol{\xi}_j,p_j} \geq a\}$ for every $i \in \mathcal{R}$ and $a \in A$. Moreover, for every $a' \in A$, let $\bar{S}(i,a,a') = \{(\boldsymbol{\xi}, \mathbf{p}) \in \Xi^q \times \mathcal{P}^b \mid w_{i,\boldsymbol{\xi}_i,p_i} \geq a' \wedge \sum_{j > i} w_{j,\boldsymbol{\xi}_j,p_j} \geq a - a'\}$. First, we prove by induction that $M(i, a - \frac{n-i}{c}) \geq \max_{(\boldsymbol{\xi},\mathbf{p}) \in S(i,a)} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$ for every $i \in \mathcal{R}$ and $a \in A$. For $i = \bar{n}$, the condition trivially holds by Line 5. For $i < \bar{n}$,

$$\max_{(\boldsymbol{\xi},\mathbf{p}) \in S(i,a)} \textsc{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$$

$$= \max_{a' \in [0,1]} \max_{(\boldsymbol{\xi},\mathbf{p}) \in \bar{S}(i,a,a')} \textsc{Rev}_{\geq i}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi})$$

$$
\begin{aligned}
&= \max_{a' \in [0,1]} \max_{(\boldsymbol{\xi},\mathbf{p}) \in \bar{S}(i,a,a')} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\} \\
&\quad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\}\right) \textsc{Rev}_{\geq i+1}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\
&\leq \max_{a' \in A} \max_{(\boldsymbol{\xi},\mathbf{p}) \in \bar{S}(i,a-\frac{1}{c},a')} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\} \\
&\quad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\}\right) \textsc{Rev}_{\geq i+1}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\
&= \max_{a' \in A} \max_{\boldsymbol{\xi}_i \in \Xi_i^q, p_i \in P^b : w_{i,\boldsymbol{\xi}_i,p_i} \geq a'} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\} \\
&\quad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\}\right) \max_{(\boldsymbol{\xi},\mathbf{p}) \in S(i+1,a-a'-\frac{1}{c})} \textsc{Rev}_{\geq i+1}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \\
&\leq \max_{a' \in A} \max_{\boldsymbol{\xi}_i \in \Xi_i^q, p_i \in P^b : w_{i,\boldsymbol{\xi}_i,p_i} \geq a'} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\} \\
&\quad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\}\right) M\left(i+1, a - a' - \frac{1}{c} - \frac{\bar{n} - i - 1}{c}\right) \\
&= \max_{a' \in A, \boldsymbol{\xi} \in \Xi^q, \mathbf{p} \in \mathcal{P}^b : w_{i,\boldsymbol{\xi}_i,p_i} + a' \geq a - \frac{\bar{n}-i}{c}} p_i \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\} \\
&\quad + \left(1 - \Pr_{v_i \sim \mathcal{V}_i} \{v_i^\top \boldsymbol{\xi}_i \geq p_i\}\right) M(i+1, a') \\
&= M\left(i, a - \frac{\bar{n} - i}{c}\right).
\end{aligned}
$$

In conclusion, let $OPT_{\textsc{Rev}}$ the revenue term in the value of an optimal solution to MAX-LINREV, while $a \in [0, \bar{n}]$ is the sum of the linear components in such optimal solution (the second term in the value of the solution). Let a^* be the greatest element in A such that $a^* \leq a - \frac{\bar{n}-1}{c}$. Notice that $a^* \geq a - \frac{\bar{n}}{c}$. Moreover, we have that $M(1, a^*) \geq \max_{(\boldsymbol{\xi},\mathbf{p}) \in S(i,a)} \textsc{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) = OPT_{\textsc{Rev}}$.[16] Hence, there exists a solution with value $M(1, a^*) + a^* \geq OPT_{\textsc{Rev}} + a - \frac{\bar{n}}{c} = OPT - \frac{\bar{n}}{c} \geq OPT - \delta$, concluding the proof. □

Now, we are ready to prove the main result of this section.

Theorem 8.4. *There exists an additive PTAS for computing a revenue-maximizing (ϕ, f) pair with private signaling.*

Proof. We start providing the following relaxation of LP 8.7:

$$
\max_{\boldsymbol{\gamma},\mathbf{x},\mathbf{y} \geq 0} \sum_{\theta \in \Theta} \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}} \textsc{Rev}(\mathcal{V}, \boldsymbol{\xi}, \mathbf{p}) \quad \text{s.t.} \tag{8.11a}
$$

[16] It is easy to see that, if $a < \frac{\bar{n}-1}{c}$, then the equality holds for $a = 0$.

$$\xi_{i,\theta} t_{i,\boldsymbol{\xi}_i,p_i} \geq \sum_{\boldsymbol{\xi}' \in \Xi : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'} \tag{8.11b}$$

$$\forall \theta \in \Theta, \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i, \forall p_i \in P^b \tag{8.11c}$$

$$\sum_{p_i \in P^b} t_{i,\boldsymbol{\xi}_i,p_i} = \gamma_{i,\boldsymbol{\xi}_i} \qquad \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i \tag{8.11d}$$

$$\sum_{\boldsymbol{\xi}_i \in \Xi^q} \gamma_{i,\boldsymbol{\xi}_i} \xi_{i,\theta} = \mu_\theta \qquad \forall i \in \mathcal{R}, \forall \theta \in \Theta \tag{8.11e}$$

The PTAS that we build in the rest of the proof works with the dual of LP 8.11 so as to take advantage of the fact that it is more constrained than that of the original LP 8.7. As a first step, the following lemma shows that LP 8.7 and LP 8.11 are equivalent.

Lemma 8.14. *LP 8.7 and LP 8.11 have the same optimal value. Moreover, given a feasible solution to LP 8.11, it is possible compute in polynomial time a feasible solution to LP 8.7 with a greater or equal value.*

Proof. To show the equivalence between the two LPs, it is sufficient to show that, given a feasible solution to LP 8.11, we can construct a solution to LP 8.7 with a greater or equal value. Let $(\mathbf{y}, \mathbf{t}, \boldsymbol{\gamma})$ be a solution to LP 8.11. For every $i \in \mathcal{R}$, $\boldsymbol{\xi}_i \in \Xi^q_i$, $p_i \in P^b$, let $\delta_{i,\boldsymbol{\xi}_i,p_i} := \xi_{i,\theta} t_{i,\boldsymbol{\xi}_i,p_i} - \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'}$. Moreover, let $\iota = \mu_\theta - \sum_{\boldsymbol{\xi} \in \Xi^q, \mathbf{p} \in \mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}}$. First, we show that $\sum_{\boldsymbol{\xi}_i \in \Xi^q_i, p_i \in P^b} \delta_{i,\boldsymbol{\xi}_i,p_i} = \iota$ for every $i \in \mathcal{R}$. For each $i \in \mathcal{R}$, it holds

$$\begin{aligned}
\sum_{\boldsymbol{\xi}_i \in \Xi^q_i} \sum_{p_i \in P^b} \delta_{i,\boldsymbol{\xi}_i,p_i} &= \sum_{\boldsymbol{\xi}_i \in \Xi^q_i} \sum_{p_i \in P^b} \left[\xi_{i,\theta} t_{i,\boldsymbol{\xi}_i,p_i} - \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'} \right] \\
&= \sum_{\boldsymbol{\xi}_i \in \Xi^q_i} \xi_{i,\theta} \gamma_{i,\boldsymbol{\xi}_i} - \sum_{\boldsymbol{\xi}' \in \Xi^q} \sum_{\mathbf{p}' \in \mathcal{P}^b} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'} \\
&= \mu_\theta - \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}} = \iota.
\end{aligned}$$

Next, we build a feasible solution $(\bar{\mathbf{y}}, \mathbf{t}, \boldsymbol{\gamma})$ to LP 8.7 with $\bar{y}_{\theta,\boldsymbol{\xi},\mathbf{p}} \geq y_{\theta,\boldsymbol{\xi},\mathbf{p}}$ for all $\theta \in \Theta$, $\boldsymbol{\xi} \in \Xi^q$, and $\mathbf{p} \in \mathcal{P}^b$. In particular, we set $\bar{y}_{\theta,\boldsymbol{\xi},\mathbf{p}} = y_{\theta,\boldsymbol{\xi},\mathbf{p}} + \frac{\prod_{i \in \mathcal{R}} \delta_{i,\boldsymbol{\xi}_i,p_i}}{\iota^{\bar{n}-1}}$. Since $\delta_{i,\boldsymbol{\xi}_i,p_i} \geq 0$ and $\iota \geq 0$ by the feasibility of $(\mathbf{y}, \mathbf{t}, \boldsymbol{\gamma})$, it holds that $\bar{y}_{\theta,\boldsymbol{\xi},\mathbf{p}} \geq y_{\theta,\boldsymbol{\xi},\mathbf{p}}$. Moreover, for each $i \in \mathcal{R}$, $\theta \in \Theta$, $\boldsymbol{\xi}_i \in \Xi^q_i$, and

$p_i \in P^b$, we have that

$$\begin{aligned}
&\sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} \bar{y}_{\theta, \boldsymbol{\xi}', \mathbf{p}'} \\
&\quad = \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta, \boldsymbol{\xi}', \mathbf{p}'} + \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} \frac{\prod_{j \in \mathcal{R}} \delta_{j, \boldsymbol{\xi}_j, p_j}}{\iota^{\bar{n}-1}} \\
&\quad = \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta, \boldsymbol{\xi}, \mathbf{p}'} + \delta_{i, \boldsymbol{\xi}_i, p_i} = t_{i, \boldsymbol{\xi}_i, p_i},
\end{aligned}$$

where the second equality follows from $\sum_{\boldsymbol{\xi}_j \in \Xi^q_j, p_j \in P^b} \delta_{j, \boldsymbol{\xi}_j, p_j} = \iota$ for every $j \in \mathcal{R}$. Since $\text{Rev}(\mathcal{V}, \mathbf{p}, \boldsymbol{\xi}) \geq 0$ for every $\mathbf{p} \in \mathcal{P}^b$ and $\boldsymbol{\xi} \in \Xi^q$, it follows that the value of $(\bar{\mathbf{y}}, \mathbf{t}, \boldsymbol{\gamma})$ is greater than or equal to the value of $(\bar{\mathbf{y}}, \mathbf{t}, \boldsymbol{\gamma})$. □

Our PTAS is described in Algorithm 8.3.

Algorithm 8.3 PTAS for the private signaling setting

Input: Error β, approximation factor of the approximation oracle δ, q defining the set of q-uniform posteriors, # of discretization steps b, number of samples K.

```
Initialization: ρ1 ← 0, ρ2 ← 1, H ← ∅, H* ← ∅.
obtain an empirical distribution of valuations V^K sampling K samples from V.
while ρ2 − ρ1 > β do
    ρ3 ← (ρ1 + ρ2)/2
    H ← {violated constraints returned by the ellipsoid method on (F)
      with objective ρ3 and approximation error δ}
    if unfeasible then
        ρ1 ← ρ3
        H* ← H
    else
        ρ2 ← ρ3
end if
return the solution to LP 8.16 with only constraints in H*
end while
```

Since we only have access to an oracle returning samples from distributions $\mathcal{V}$, our algorithm works with empirical distributions $\mathcal{V}^K$ built from K i.i.d samples, for a suitably-defined $K \in \mathbb{N}_{>0}$. The algorithm works with LP 8.11 for the values $b \in \mathbb{N}_{>0}$ and $q \in \mathbb{N}_{>0}$ defined in the following, finding an approximate solution to LP 8.11. Since LP 8.11 has an exponential number of variables, the algorithm works by applying the ellipsoid method to its dual formulation, as described in the following.

Let $\mathbf{a} \in \mathbb{R}^{\bar{n}\times|\Theta|}$, $\boldsymbol{\gamma} \in \mathbb{R}_{-}^{|\Theta|\times\bar{n}\times|\Xi_i^q|\times|P^b|}$, and $\mathbf{c} \in \mathbb{R}^{\bar{n}\times|\Xi_i^q|}$. Then, the dual of LP 8.11 reads as follow.

$$\min_{\mathbf{a},\boldsymbol{\gamma}\leq 0,\mathbf{c}} \sum_{i\in\mathcal{R}}\sum_{\theta\in\Theta} \mu_\theta a_{i,\theta} \quad \text{s.t.} \tag{8.12a}$$

$$\sum_{i\in\mathcal{R}} -w_{\theta,i,\boldsymbol{\xi}_i,p_i} \geq \textsc{Rev}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi}) \quad \forall\theta\in\Theta,\forall\boldsymbol{\xi}\in\Xi^q,\forall\mathbf{p}\in\mathcal{P}^b \tag{8.12b}$$

$$\sum_{\theta\in\Theta} \xi_{i,\theta} w_{\theta,i,\boldsymbol{\xi}_i,p_i} + c_{i,\boldsymbol{\xi}_i} \geq 0 \quad \forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q,\forall p_i\in P^b \tag{8.12c}$$

$$-c_{i,\boldsymbol{\xi}_i} + \sum_{\theta\in\Theta} \xi_{i,\theta} a_{i,\theta} \geq 0 \quad \forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q. \tag{8.12d}$$

Notice that, by using the dual of LP 8.11 instead of that of LP 8.7, we get the additional constraint $w \leq 0$. LP 8.12 has a polynomial number of variables and a polynomial number of Constraints (8.12c) and (8.12d). Hence, to solve the LP using the ellipsoid method we need a separation oracle for Constraints (8.12b), which are exponentially many. Instead of an exact separation oracle, we use an approximate separation oracle that employs Algorithm 8.2 with a suitably-defined $\delta > 0$. We use a binary search scheme to find a value $\rho^\star \in [0,1]$ such that the dual problem with objective $\rho^\star$ is unfeasible, while the dual with objective $\rho^\star + \beta$ is *approximately* feasible, for some $\beta \geq 0$ defined in the following. The algorithm requires $\log(\beta)$ steps and, at each step, it works by determining, for a given value ρ_3, whether there exists a feasible solution for the following feasibility problem that we call Ⓕ:

$$\sum_{i\in\mathcal{R}}\sum_{\theta\in\Theta} \mu_\theta a_{i,\theta} \leq \rho_3 \tag{8.13a}$$

$$\textsc{Rev}(\mathcal{V},p,\boldsymbol{\xi}) + \sum_{i\in\mathcal{R}} w_{\theta,i,\boldsymbol{\xi}_i,p_i} \leq 0 \quad \forall\theta\in\Theta,\forall\boldsymbol{\xi}\in\Xi^q,\forall\mathbf{p}\in\mathcal{P}^b \tag{8.13b}$$

$$\sum_{\theta\in\Theta} \xi_{i,\theta} w_{\theta,i,\boldsymbol{\xi}_i,p_i} + c_{i,\boldsymbol{\xi}} \geq 0 \quad \forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q,\forall p_i\in P^b \tag{8.13c}$$

$$-c_{i,\boldsymbol{\xi}_i} + \sum_{\theta\in\Theta} \xi_{i,\theta} a_{i,\theta} \geq 0 \quad \forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q \tag{8.13d}$$

$$w_{\theta,i,\boldsymbol{\xi}_i,p_i} \leq 0 \quad \forall\theta\in\Theta,\forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q,\forall p_i\in P^b \tag{8.13e}$$

At each iteration of the bisection algorithm, the feasibility problem Ⓕ is solved via the ellipsoid method. To do so, we need a separation oracle. We use an approximate separation oracle that returns a violated constraint that will be defined in the following. The bisection procedure terminates when it determines a value $\rho^\star$ such that on Ⓕ the ellipsoid method returns unfeasible for $\rho^\star$, while returning feasible for $\rho^\star + \beta$. Finally, the algorithm solves a modified primal LP 8.16 with only the subset of variables y in H^*, where H^* is the set of violated constraints returned by the ellipsoid method applied on the unfeasible problem with objective ρ^*. From this solution, we can use Lemma 8.14 to find a solution to LP 8.7 with the same value and Lemma 8.11 to find a signaling scheme with the same seller's revenue as the value of the solution.

Approximate Separation Oracle. Our separation oracle works as follow. Given a point $(\mathbf{a}, \mathbf{w}, \mathbf{c})$ in the dual space, we check if a constraint relative to the variables $\mathbf{t}$ and $\boldsymbol{\gamma}$ of the primal is violated. Since there are a polynomial number of these constraints, it can be done in polynomial time. If it is the case, we return that constraint. Otherwise, our idea is to use Algorithm 8.2 with a δ defined in the following to find if a constraint relative to variable $\mathbf{y}$ is violated. We apply Algorithm 8.2, once for each possible state $\theta \in \Theta$. In the following, we assume that θ is fixed and we denote $w_{\theta,i,\boldsymbol{\xi}_i,p_i}$ as $w_{i,\boldsymbol{\xi}_i,p_i}$. Algorithm 8.2 needs values such that $w_{i,\boldsymbol{\xi}_i,p_i} \in [0,1]$ for all $i \in \mathcal{R}$, $\boldsymbol{\xi}_i \in \Xi_i^q$, and $p_i \in P^b$. We show that we can restrict the inputs to $w_{i,\boldsymbol{\xi}_i,p_i} \in [-1,0]$.[17] By constraint $\mathbf{w} \le 0$, all $w_{i,\boldsymbol{\xi}_i,p_i}$ are non-positive. Otherwise, this constraint is violated and would have been returned in the first step. Moreover, given a vector $\mathbf{w}$, we give as input to the oracle a vector $\bar{\mathbf{w}}$ such that $\bar{w}_{i,\boldsymbol{\xi}_i,p_i} = -1$ whenever $w_{i,\boldsymbol{\xi}_i,p_i} < -1$.

If for at least one state θ a violated constraint is found by Algorithm 8.2, we return that constraint, otherwise we return feasible. Our separation oracle has two properties. When it returns a violated constraint, the constraint is actually violated. In particular, if $\sum_{i\in\mathcal{R}} \bar{w}_{\theta,i,\boldsymbol{\xi}_i,p_i} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}, \boldsymbol{\xi}) > 0$, then $\bar{w}_{\theta,i,\boldsymbol{\xi}_i,p_i} > -1$ for every $i \in \mathcal{R}$, implying $\bar{w}_{\theta,i,\boldsymbol{\xi}_i,p_i} = w_{\theta,i,\boldsymbol{\xi}_i,p_i}$ and $\sum_{i\in\mathcal{R}} w_{\theta,i,\boldsymbol{\xi}_i,p_i} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}, \boldsymbol{\xi}) = \sum_{i\in\mathcal{R}} \bar{w}_{\theta,i,\boldsymbol{\xi}_i,p_i} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}, \boldsymbol{\xi}) > 0$, Additionally, when the separation oracle returns feasible, then all the constraints relative to the variables y are violated by at most δ. Suppose by contradiction that a constraint for a triple $(\theta, \boldsymbol{\xi}, \mathbf{p})$ is violated by more than δ. Then, the separation oracle would have found $\theta^* \in \Theta$, $\boldsymbol{\xi}^* \in \Xi^q$, and $\mathbf{p}^* \in \mathcal{P}^b$ such that: $\sum_{i\in\mathcal{R}} \bar{w}_{\theta^*,i,\boldsymbol{\xi}_i^*,p_i^*} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}^*, \boldsymbol{\xi}^*) \ge \sum_{i\in\mathcal{R}} \bar{w}_{\theta,i,\boldsymbol{\xi}_i,p_i} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}, \boldsymbol{\xi}) - \delta \ge \sum_{i\in\mathcal{R}} w_{\theta,i,\boldsymbol{\xi}_i,p_i} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}, \boldsymbol{\xi}) - \delta > 0$, and, thus, it would have returned this violated constraint.

Approximation Guarantee. The algorithm finds a ρ^* such that the problem is unfeasible, *i.e.*, the value of ρ_1 when the algorithm terminates, and a value

[17] It is easy to see that summing 1 to all the elements of the vector $\mathbf{w}$ does not change the problem.

smaller than or equal to $\rho^* + \beta$ such that the ellipsoid method returns feasible, *i.e.*, the value of ρ_2 when the algorithm terminates. For each possible distribution of the samples $\mathcal{V}^K$, let $OPT^{\mathcal{V}^K}$ be the optimal value of LP 8.12. As a first step, we bound the value of $OPT^{\mathcal{V}^K}$. In particular, we show that $OPT^{\mathcal{V}^K} \leq \rho^* + \beta + \delta$. Since, the bisection algorithm returns that Ⓕ is feasible with objective $\rho^* + \beta$, it finds a solution $(\mathbf{a}, \mathbf{w}, \mathbf{c})$ such that all the constraints regarding variables $\mathbf{t}$ and $\boldsymbol{\gamma}$ of the primal are satisfied and the approximate separation oracle did not find a violated constraint for the constraints regarding variables y. We show that $(\mathbf{a}, \mathbf{w}, \mathbf{c})$ is a solution to the following LP.

$$\sum_{i \in \mathcal{R}} \sum_{\theta \in \Theta} \mu_\theta a_{i,\theta} \leq \rho^* + \beta \tag{8.14a}$$

$$\sum_{i \in \mathcal{R}} -w_{\theta,i,\boldsymbol{\xi}_i,p_i} \geq \textsc{Rev}(\mathcal{V}^k, \mathbf{p}, \boldsymbol{\xi}) - \delta \quad \forall \theta \in \Theta, \forall \boldsymbol{\xi} \in \Xi^q, \forall \mathbf{p} \in \mathcal{P}^b \tag{8.14b}$$

$$\sum_{\theta \in \Theta} \xi_{i,\theta} w_{\theta,i,\boldsymbol{\xi}_i,p_i} + c_{i,\boldsymbol{\xi}_i} \geq 0 \qquad \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i, \forall p_i \in P^b \tag{8.14c}$$

$$- c_{i,\boldsymbol{\xi}_i} + \sum_{\theta \in \Theta} \xi_{i,\theta} a_{i,\theta} \geq 0 \qquad \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i \tag{8.14d}$$

$$w_{\theta,i,\boldsymbol{\xi}_i,p_i} \leq 0 \qquad \forall \theta \in \Theta, \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i, \forall p_i \in P^b \tag{8.14e}$$

All the Constraints (8.14c) and (8.14d) are satisfied since the separation oracle checks them explicitly, while we have shown that, when the separation oracle return feasible, it holds $\sum_{i \in \mathcal{R}} w_{\theta,i,\boldsymbol{\xi}_i,p_i} + \textsc{Rev}(\mathcal{V}^K, \mathbf{p}, \boldsymbol{\xi}) \leq \delta$ for all $\theta \in \Theta, \boldsymbol{\xi} \in \Xi^q_i$, and $\mathbf{p} \in \mathcal{P}^b$, implying that all the Constraints (8.14b) are satisfied.

Then, by strong duality the value of the following LP is at most $\rho^* + \beta$.

$$\max_{\mathbf{y},\mathbf{t},\boldsymbol{\gamma}} \sum_{\theta \in \Theta} \sum_{\boldsymbol{\xi} \in \Xi^q} \sum_{\mathbf{p} \in \mathcal{P}^b} y_{\theta,\xi,\mathbf{p}} \left(\textsc{Rev}(\mathcal{V}^k, \mathbf{p}, \boldsymbol{\xi}) - \delta \right) \quad \text{s.t.} \tag{8.15a}$$

$$\xi_{i,\theta} t_{i,\boldsymbol{\xi}_i,p_i} \geq \sum_{\boldsymbol{\xi}' \in \Xi^q : \boldsymbol{\xi}'_i = \boldsymbol{\xi}_i} \sum_{\mathbf{p}' \in \mathcal{P}^b : p'_i = p_i} y_{\theta,\boldsymbol{\xi},\mathbf{p}}$$
$$\forall \theta \in \Theta, \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i, \forall p_i \in P^b \tag{8.15b}$$

$$\sum_{p \in P^b} t_{r,\boldsymbol{\xi}_i,p} = \gamma_{i,\boldsymbol{\xi}_i} \qquad \forall i \in \mathcal{R}, \forall \boldsymbol{\xi}_i \in \Xi^q_i \tag{8.15c}$$

$$\sum_{\boldsymbol{\xi}_i \in \Xi^q_i} \gamma_{i,\boldsymbol{\xi}_i} \xi_{i,\theta} = \mu_\theta \qquad \forall i \in \mathcal{R}, \forall \theta \in \Theta \tag{8.15d}$$

Notice that any solution to LP 8.11 is also a feasible solution to the previous modified problem. Since in any feasible solution it holds that $\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q,\mathbf{p}\in\mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}} = 1$ and LP 8.15 has value at most $\rho^*+\beta$, then $OPT^{\mathcal{V}^K} \leq \rho^*+\beta+\delta$.

Let H^* be the set of constraints regarding variables $\mathbf{y}$ returned by the ellipsoid method run with objective $\rho^\star$. Since the ellipsoid method with the approximate separation oracle returns unfeasible, by strong duality LP 8.11 with only the variables y relative to constraints in H^* has value at least ρ^*. Moreover, since the ellipsoid method guarantees that H^* has polynomial size, the LP can be solved in polynomial time. Hence, solving the following LP, *i.e.*, the primal LP 8.11 with only the variables $\mathbf{y}$ in H^*, we can find a solution with value at least ρ^*.

$$\max_{\boldsymbol{\gamma},\mathbf{t},\mathbf{y}\geq 0} \sum_{\theta\in\Theta}\sum_{(\boldsymbol{\xi},\mathbf{p}):(\theta,\boldsymbol{\xi},\mathbf{p})\in H^*} y_{\theta,\boldsymbol{\xi},\mathbf{p}}\text{REV}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi}) \quad \text{s.t.} \tag{8.16a}$$

$$\xi_{i,\theta}t_{i,\boldsymbol{\xi}_i,p_i} \geq \sum_{\boldsymbol{\xi}',\mathbf{p}':(\theta,\boldsymbol{\xi}',\mathbf{p}')\in H^*:\boldsymbol{\xi}_i'=\boldsymbol{\xi}_i,p_i'=p_i,} y_{\theta,\boldsymbol{\xi}',\mathbf{p}'}$$

$$\forall\theta\in\Theta,\forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q,\forall p_i\in P^b \tag{8.16b}$$

$$\sum_{p_i\in P^b} t_{i,\boldsymbol{\xi}_i,p_i} = \gamma_{i,\boldsymbol{\xi}_i} \qquad \forall i\in\mathcal{R},\forall\boldsymbol{\xi}_i\in\Xi_i^q \tag{8.16c}$$

$$\sum_{\boldsymbol{\xi}_i\in\Xi_i^q} \gamma_{i,\boldsymbol{\xi}_i}\xi_{i,\theta} = \mu_\theta \qquad \forall i\in\mathcal{R},\forall\theta\in\Theta \tag{8.16d}$$

To conclude the proof, we show that replacing the distributions $\mathcal{V}$ with $\mathcal{V}^K$, the expected revenue decreases by a small amount. Let $\mathbf{y}^{APX,\mathcal{V}^K}$ be the solution returned by the algorithm with distribution $\mathcal{V}^K$. Moreover, let $\mathbf{y}^{OPT,\mathcal{V}^K}$ be the optimal solution to LP 8.11 with distributions $\mathcal{V}^K$ and $\mathbf{y}^{OPT,\mathcal{V}}$ the optimal solution with distributions $\mathcal{V}$. Finally, let OPT be the value of the optimal private signaling scheme with distributions $\mathcal{V}$.

Let ϵ be a constant defined in the following and $K{=}8\log(2|\Xi^q||\mathcal{P}^b|/\epsilon)/\epsilon^2$. By Hoeffding bound, for every $\boldsymbol{\xi}\in\Xi^q$ and $\mathbf{p}\in\mathcal{P}^b$, with probability at least $1-e^{-2K/(\epsilon/4)^2} = 1-|\Xi^q||\mathcal{P}^b|\epsilon/4$,

$$|\text{REV}(\mathcal{V},p,\boldsymbol{\xi})| - |\text{REV}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi})| \leq \epsilon/4.$$

By the union bound, it implies that with probability at least $1-\epsilon/2$,

$$|\text{REV}(\mathcal{V},p,\boldsymbol{\xi})| - \text{REV}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi})| \leq \epsilon/2$$

for every $\boldsymbol{\xi} \in \Xi^q$ and $\mathbf{p} \in \mathcal{P}^b$. Then, with probability $1 - \epsilon/2$,

$$\begin{aligned}
&\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q}\sum_{\mathbf{p}\in\mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}}^{APX,\mathcal{V}^K}\textsc{Rev}(\mathcal{V},\mathbf{p},\boldsymbol{\xi}) \geq \\
&\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q}\sum_{\mathbf{p}\in\mathcal{P}^b} y_{\theta,\mathbf{p},\boldsymbol{\xi}}^{APX,\mathcal{V}^K}\textsc{Rev}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi}) - \epsilon/4 \geq \\
&\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q}\sum_{\mathbf{p}\in\mathcal{P}^b} y_{\theta,\mathbf{p},\boldsymbol{\xi}}^{OPT,\mathcal{V}^K}\textsc{Rev}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi}) - \epsilon/4 - \delta - \beta \geq \\
&\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q}\sum_{\mathbf{p}\in\mathcal{P}^b} y_{\theta,\mathbf{p},\boldsymbol{\xi}}^{OPT,\mathcal{V}}\textsc{Rev}(\mathcal{V}^K,\mathbf{p},\boldsymbol{\xi}) - \epsilon/4 - \delta - \beta \geq \\
&\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q}\sum_{\mathbf{p}\in\mathcal{P}^b} y_{\theta,\mathbf{p},\boldsymbol{\xi}}^{OPT,\mathcal{V}}\textsc{Rev}(\mathcal{V},\mathbf{p},\boldsymbol{\xi}) - \epsilon/2 - \delta - \beta \geq \\
&OPT - \epsilon/2 - \delta - \beta - \eta
\end{aligned}$$

Hence, with probability $1 - \epsilon/2$, the solution has value at least $OPT - \epsilon/2 - \delta - \beta - \eta$ and

$$\begin{aligned}
\mathbb{E}_{\mathcal{V}^K}&\left[\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\Xi^q}\sum_{\mathbf{p}\in\mathcal{P}^b} y_{\theta,\boldsymbol{\xi},\mathbf{p}}^{APX,\mathcal{V}^K} Rev(\mathcal{V},\mathbf{p},\boldsymbol{\xi})\right] \\
&\geq OPT - \epsilon/2 - \epsilon/2 - \delta - \beta - \eta \\
&= OPT - \epsilon - \delta - \beta - \eta,
\end{aligned}$$

where the expectation is on the sampling procedure.

To conclude the proof, to have an approximation error λ, we can set b and q such that the approximation error in Lemma 8.12 is $\eta = \lambda/4$ and $\epsilon = \delta = \beta = \lambda/4$. Finally, given an approximate solution to LP 8.16, Lemma 8.14 provides a solution to LP 8.7 with greater or equal value and Lemma 8.11 recover a signaling scheme with the same revenue. □

Part II

Facing the Uncertainty

CHAPTER 9

Online Single-receiver Bayesian Persuasion

In this chapter, we relax the assumption that the sender knows the utility of the receiver. In particular, we study the Bayesian persuasion problem in an online learning framework in which the sender repeatedly faces a receiver whose type is unknown and chosen adversarially at each round from a finite set of possible types. In section 9.1 we introduce the concept of receiver's type, while in Section 9.2 we introduce the online Bayesian persuasion framework. In Section 9.3, we show that designing no-regret algorithms is computationally intractable. Section 9.4 provides a no-regret algorithm with full information feedback, while Section 9.5 provides a no-regret algorithm with partial information feedback. Finally, in Section 9.6 we show that relaxing the persuasiveness constraints it is possible to design polynomial-time algorithms with small regret.

9.1 Bayesian Persuasion with Types

The receiver has a finite set of ϱ actions $\mathcal{A} := \{a_i\}_{i=1}^{\varrho}$ and a set of m possible types $\mathcal{K} := \{k_i\}_{i=1}^{m}$. For each type $k \in \mathcal{K}$, the receiver's payoff function is $u^k : \mathcal{A} \times \Theta \to [0,1]$, where $\Theta := \{\theta_i\}_{i=1}^d$ is a finite set of d states of nature. For notational convenience, we denote by $u_\theta^k(a) \in [0,1]$ the utility observed by the receiver of type $k \in \mathcal{K}$ when the realized state of nature is $\theta \in \Theta$ and she/he plays action $a \in \mathcal{A}$. The sender's utility when the state of nature is $\theta \in \Theta$ is described by the function $f_\theta : \mathcal{A} \to [0,1]$.

After observing a signal $s \in \mathcal{S}$ that induces a posterior $\boldsymbol{\xi} \in \Xi$, the receiver best responds by choosing an action that maximizes her/his expected utility. We extend the Definition of BR-set and ϵ-BR-set to accomodate multiple types. Formally,

Definition 9.1 (BR-set)**.** *Given posterior $\boldsymbol{\xi} \in \Xi$ and type $k \in \mathcal{K}$, the* best-response set (BR-set) *is:*

$$\mathcal{B}_{\boldsymbol{\xi}}^k := arg\, max_{a \in \mathcal{A}} \sum_{\theta \in \Theta} \xi_\theta \, u_\theta^k(a).$$

We denote by $b_{\boldsymbol{\xi}}^k$ the action belonging to the BR-set $\mathcal{B}_{\boldsymbol{\xi}}^k$ played by the receiver. When the receiver is indifferent among multiple actions for a given posterior $\boldsymbol{\xi}$, we assume that the receiver breaks ties in favor of the sender, *i.e.*, she/he chooses an action $b_{\boldsymbol{\xi}}^k \in arg\, max_{a \in \mathcal{B}_{\boldsymbol{\xi}}^k} \sum_\theta \xi_\theta \, f_\theta(a)$.

Similarly, the set of ϵ-best responses is defined as follows.

Definition 9.2 (ϵ-BR-set)**.** *Given $\boldsymbol{\xi} \in \Xi$ and $k \in \mathcal{K}$, the* ϵ-best-response set (ϵ-BR-set) *is the set $\mathcal{B}_{\epsilon,\boldsymbol{\xi}}^k$ of all the actions $a \in \mathcal{A}$ such that:*

$$\sum_{\theta \in \Theta} \xi_\theta u_\theta^k(a) \geq \sum_{\theta \in \Theta} \xi_\theta u_\theta^k(\hat{a}) - \epsilon \quad \forall \hat{a} \in \mathcal{A}.$$

We denote by $b_{\epsilon,\boldsymbol{\xi}}^k$ the action in $\mathcal{B}_{\epsilon,\boldsymbol{\xi}}^k$ played by the receiver. When the receiver has multiple ϵ-best-response actions for a given posterior $\boldsymbol{\xi}$, we assume she breaks ties in favor of the sender, *i.e.*, she chooses an action $b_{\epsilon,\boldsymbol{\xi}}^k \in arg\, max_{a \in \mathcal{B}_{\epsilon,\boldsymbol{\xi}}^k} \sum_\theta \xi_\theta f_\theta(a)$.

We conclude the section by introducing some additional notation. We denote by $f(\boldsymbol{\xi}, k) := \sum_\theta \xi_\theta \, f_\theta(b_{\boldsymbol{\xi}}^k)$ the sender's expected utility when she/he induces a posterior $\boldsymbol{\xi} \in \Xi$ and the receiver is of type $k \in \mathcal{K}$. Similarly, we let $f^\epsilon(\boldsymbol{\xi}, k) := \sum_\theta \xi_\theta f_\theta(b_{\epsilon,\boldsymbol{\xi}}^k)$ be the sender's expected utility with an ϵ-best-responding receiver. Moreover, we use $f(\phi, k)$ and $f^\epsilon(\phi, k)$ to denote the sender's expected utility achieved with the signaling scheme ϕ. Formally, $f(\phi, k) := \sum_{\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}} f(\boldsymbol{\xi}, k)$ and $f^\epsilon(\phi, k) := \sum_{\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}} f^\epsilon(\boldsymbol{\xi}, k)$, where $\boldsymbol{\gamma}$ is the distribution over posteriors induced by ϕ. Analogously, we write $f(\boldsymbol{\gamma}, k)$ and $f^\epsilon(\boldsymbol{\gamma}, k)$.

		State G ($\mu_G = .3$)		State I ($\mu_I = .7$)	
$\mathcal{A}$	A	0	0	0	1
	C	1	1	1	0

		Realized state State G	State I
$\mathcal{S}$	s_1	0	4/7
	s_2	1	3/7

	State of nature State G	State I	$\boldsymbol{\gamma}^\star$
$\boldsymbol{\xi}_1$	0	1	2/5
$\boldsymbol{\xi}_2$	1/2	1/2	3/5

Figure 9.1: ***Left:*** *The prosecutor/judge game. Rows represent the judge's actions. For each possible state of nature* $\{\mathrm{G}, \mathrm{I}\}$*, the first column is the prosecutor's payoff while the second is the judge's payoff.* ***Center:*** *The optimal signaling scheme* $\phi^\star$*. Each column describes the probability with which the two signals are drawn given the realized state of nature.* ***Right****: Representation of* $\phi^\star$ *as the convex combination of posteriors* $\boldsymbol{\gamma}^\star$*.*

Finally, letting OPT be the sender's optimal expected utility, we say that a signaling scheme is α-*optimal* (in the additive sense) if it provides the sender with a utility at least as large as $OPT - \alpha$.

9.1.1 Example of Bayesian Persuasion without Types

We illustrate the key notion of signaling scheme in a simple example with a single receiver type (*i.e.,* $|\mathcal{K}| = 1$). In Section 9.4.1 we will provide a more complex example that includes types. The example is inspired by Kamenica and Gentzkow (2011): a prosecutor (the sender) is trying to convince a rational judge (the receiver) that a defendant is guilty. The judge has two available actions: to *acquit* or to *convict* the defendant (denoted by A and C, respectively). There are two possible states of nature: the defendant is either *guilty* (denoted by G) or *innocent* (denoted by I). The prosecutor and the judge share a prior belief $\mu_G = 0.3$. Moreover, the prosecutor gets utility 1 if the judge convicts the defendant and 0 otherwise, regardless of the state of nature. The prosecutor gets to observe the realized state of nature (*i.e.,* whether the defendant is guilty or innocent). The she/he can exploit this information to select a signal from set $\mathcal{S} = \{s_1, s_2\}$ and send it to the judge. The judge has a unique type and she/he gets utility 1 for choosing the just action (convict when guilty and acquit when innocent) and utility 0 for choosing the unjust action (see fig. 9.1-Left).

Figure 9.1-Center depicts a sender-optimal signaling scheme $\phi^\star$ obtained via the following LP:

$$arg\,max_{\phi \geq 0} f(\phi, k) \quad \text{s.t.} \quad \sum_{s \in \mathcal{S}} \phi_\theta(s) = 1 \quad \forall \theta \in \Theta,$$

where k is the unique type of the judge. When the sender acts according to $\phi^\star$, signal s_1 (resp., s_2) originates posterior $\boldsymbol{\xi}_1$ (resp., $\boldsymbol{\xi}_2$; see Figure 9.1-Right). Applying Equation (3.4) yields the equivalent representation of $\phi^\star$

as a convex combination of posteriors, *i.e.*, $\gamma^\star_{\boldsymbol{\xi}_1} = 2/5$ and $\gamma^\star_{\boldsymbol{\xi}_2} = 3/5$.

By unpacking the objective function of the above LP (and dropping the dependency on k) we have: $\mathcal{B}_{\boldsymbol{\xi}_1} = \{\mathrm{A}\}$ and $\mathcal{B}_{\boldsymbol{\xi}_2} = \{\mathrm{A}, \mathrm{C}\}$. Therefore, if the posterior is $\boldsymbol{\xi}_1$, the judge will acquit the defendant, *i.e.,* $b_{\boldsymbol{\xi}_1} = \mathrm{A}$. Otherwise, if the posterior is $\boldsymbol{\xi}_2$, we have $b_{\boldsymbol{\xi}_2} = \mathrm{C}$ since the receiver breaks ties in favor of the sender. This highlights an intuitive interpretation of the signaling problem: the two signals may be interpreted as action recommendations. Signal s_1 (resp., s_2) is interpreted by the judge as a recommendation to play A (resp., C). Then, our definition of best-response set (definition 9.1) implies that it is in the receiver's best interest to follow the action recommendations. The best-response conditions can be formulated in terms of linear constraints on ϕ_θ as follows:

$$\sum_{\theta\in\Theta} \mu_\theta\, \phi_\theta(s_1)\Big(u_\theta(\mathrm{A}) - u_\theta(\hat{a})\Big) \geq 0 \quad \text{and}$$

$$\sum_{\theta\in\Theta} \mu_\theta\, \phi_\theta(s_2)\Big(u_\theta(\mathrm{C}) - u_\theta(\hat{a})\Big) \geq 0 \quad \forall \hat{a} \in \{\mathrm{A}, \mathrm{C}\}.$$

9.2 The Online Bayesian Persuasion Framework

We consider the following online setting. The sender plays a repeated game in which, at each round $t \in [T]$, she/he commits to a signaling scheme ϕ^t, observes a state of nature $\theta^t \sim \boldsymbol{\mu}$, and she/he sends signal $s^t \sim \phi^t_{\theta^t}$ to the receiver. Then, a receiver of unknown type updates her/his prior distribution and selects an action a^t maximizing her/his expected reward (in the *one-shot* interaction at round t). We focus on the problem in which the sequence of receiver's types $\boldsymbol{k} := \{k^t\}_{t\in[T]}$ is selected beforehand by an adversary. After the receiver plays a^t, the sender receives a *feedback* on her/his choice at round t. In the *full information* feedback setting, the sender observes the receiver's type k^t. Therefore, the sender can compute the expected payoff for any signaling scheme she/he could have chosen other than ϕ^t. Instead, in the *partial information* feedback setting, the sender only observes the action a^t played by the receiver at round t.

We are interested in algorithms computing ϕ^t at each round t. The performance of one such algorithm is measured using the average per-round *regret* computed with respect to the best signaling scheme in hindsight.

Formally:

$$R^T := \max_{\phi}\left\{\frac{1}{T}\sum_{t=1}^{T}\big(f(\phi,k^t)-\mathbb{E}\big[f(\phi^t,k^t)\big]\big)\right\},$$

where the expectation is on the randomness of the online algorithm (*i.e.*, the probability distribution which is used by the sender to draw the signaling scheme at round t) and T is the number of rounds. Ideally, we would like to find an algorithm that generates a sequence $\{\phi^t\}_{t\in[T]}$ with the following properties: (i) the regret is polynomial in the size of the problem instance, *i.e.*, $\mathsf{poly}(m,\varrho,d)$, and goes to zero as a polynomial of T; (ii) the per-round running time is $\mathsf{poly}(T,m,\varrho,d)$. An algorithm satisfying property (i) is usually called a *no-regret* algorithm.

When the receiver is allowed to play an ϵ-best response at any t, we would like to design a sequence $\{\phi^t\}_{t\in[T]}$ of signaling schemes which have which have small regret with respect to the best signaling scheme in hindsight with a receiver playing a best response. In this setting, we measure the performance of an algorithm with the following different notion of regret:

$$R^T_\epsilon := \max_{\phi}\left\{\frac{1}{T}\sum_{t=1}^{T}\big(f(\phi,k^t)-\mathbb{E}\big[f^\epsilon(\phi^t,k^t)\big]\big)\right\}. \tag{9.1}$$

In the case in which requiring no-regret is too limiting, we use the following relaxed notion of regret. An algorithm has *no-α-additive-regret* if there exists a constant $c>0$ such that: $R^T\le\alpha+\frac{1}{T^c}\mathsf{poly}(m,\varrho,d)$. The idea of no-$\alpha$-regret is that the regret approaches α after a sufficiently large number of rounds (polynomial in the size of the game). In the remaining of the chapter, we focus on α-additive-regret and we write no-α-regret instead of no-α-additive-regret.

9.3 Hardness of Sub-linear Regret

Our first result is negative: for any $\alpha<1$, it is unlikely (*i.e.*, technically, it is not the case unless $\mathsf{NP}\subseteq\mathsf{RP}$) that there exists a no-α-regret algorithm for the online Bayesian persuasion problem requiring a per-round running time polynomial in the size of the instance. In order to prove the result, we provide an intermediate step, showing that the problem of approximating an optimal signaling scheme is computationally intractable even in the *offline* Bayesian persuasion problem in which the sender knows the probability distribution over the receiver's types (see Theorem 9.2 below).

Definition 9.3 (OPT-SIGNAL). *Given an offline Bayesian persuasion problem in which the distribution over the receiver's types $\rho \in \Delta_{\mathcal{K}}$ is uniform, i.e., $\rho_k = \frac{1}{m}$ for all $k \in \mathcal{K}$, we call* OPT-SIGNAL *the problem of finding an optimal signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$, i.e., one maximizing the sender's expected utility $\frac{1}{m} \sum_{k \in \mathcal{K}} f(\phi, k)$.*

In order to prove the hardness of OPT-SIGNAL, we resort to a result by Guruswami and Raghavendra (2009) (see Theorem 9.1 below), which is about the following *promise problem* related to the satisfiability of a fraction of linear equations with rational coefficients and variables restricted to the hypercube.[1]

Definition 9.4 (LINEQ-MA$(1-\zeta, \delta)$). *For any two constants $\zeta, \delta \in \mathbb{R}$ satisfying $0 \leq \delta \leq 1 - \zeta \leq 1$,* LINEQ-MA$(1-\zeta, \delta)$ *is the following promise problem: Given a set of linear equations $A\mathbf{x} = \mathbf{c}$ over variables $\mathbf{x} \in \mathbb{Q}^{n_{\text{var}}}$, with coefficients $A \in \mathbb{Q}^{n_{\text{eq}} \times n_{\text{var}}}$ and $\mathbf{c} \in \mathbb{Q}^{n_{\text{eq}}}$, distinguish between the following two cases:*

- *there exists a vector $\hat{\mathbf{x}} \in \{0,1\}^{n_{\text{var}}}$ that satisfies at least a fraction $1-\zeta$ of the equations;*
- *every possible vector $\mathbf{x} \in \mathbb{Q}^{n_{\text{var}}}$ satisfies less than a fraction δ of the equations.*

Theorem 9.1 ((Guruswami and Raghavendra, 2009)). *For all valid $\zeta, \delta > 0$,* LINEQ-MA$(1-\zeta, \delta)$ *is* NP*-hard.*

Then, we can prove the following result.

Theorem 9.2. *For every $0 \leq \alpha < 1$, it is* NP*-hard to compute an α-optimal solution to* OPT-SIGNAL.

Proof. We introduce a reduction from LINEQ-MA$(1-\zeta, \delta)$ to OPT-SIGNAL, showing the following:

- *Completeness*: If an instance of LINEQ-MA$(1-\zeta, \delta)$ admits a $1-\zeta$ fraction of satisfiable equations when variables are restricted to lie the hypercube $\{0,1\}^{n_{\text{var}}}$, then an optimal solution to OPT-SIGNAL provides the sender with an expected utility at least of $1 - 2\zeta$;
- *Soundness*: If at most a δ fraction of the equations can be satisfied, then an optimal solution to OPT-SIGNAL has sender's expected utility at most δ.

[1] In the definition in (Guruswami and Raghavendra, 2009), the vector $\hat{\mathbf{x}}$ can be non-binary. However, Guruswami and Raghavendra (2009) use a binary vector $\hat{\mathbf{x}}$ in their proof and hence the hardness result holds also for our definition.

Since ζ and δ can be arbitrary (with $0 \leq \delta \leq 1-\zeta \leq 1$), the two properties above immediately prove the result. In the rest of the proof, given a vector of variables $\mathbf{x} \in \mathbb{Q}^{n_{\text{var}}}$, for $i \in [n_{\text{var}}]$, we denote with x_i the component corresponding to the i-th variable. Similarly, for $j \in [n_{\text{eq}}]$, c_j is the j-th component of the vector $\mathbf{c}$, whereas, for $i \in [n_{\text{var}}]$ and $j \in [n_{\text{eq}}]$, the (j,i)-entry of A is denoted by A_{ji}.

Reduction As a preliminary step, we normalize the coefficients by letting $\bar{A} := \frac{1}{\tau}A$ and $\bar{\mathbf{c}} := \frac{1}{\tau^2}\mathbf{c}$, where we let

$$\tau := 2\max\left\{\max_{i\in[n_{\text{var}}],j\in[n_{\text{eq}}]} A_{ji}, \max_{j\in[n_{\text{eq}}]} c_j, n_{\text{var}}^2\right\}.$$

It is easy to see that the normalization preserves the number of satisfiable equations. Formally, the number of satisfied equations of $A\mathbf{x} = \mathbf{c}$ is equal to the number of satisfied equations of $\bar{A}\bar{\mathbf{x}} = \bar{\mathbf{c}}$, where $\bar{\mathbf{x}} = \frac{1}{\tau}\mathbf{x}$. For every variable $i \in [n_{\text{var}}]$, we define a state of nature $\theta_i \in \Theta$. Moreover, we introduce an additional state $\theta_0 \in \Theta$. The prior distribution $\mu \in \text{int}(\Delta_\Theta)$ is defined in such a way that $\mu_{\theta_i} = \frac{1}{n_{\text{var}}^2}$ for every $i \in [n_{\text{var}}]$, while $\mu_{\theta_0} = \frac{n_{\text{var}}-1}{n_{\text{var}}}$ (notice that $\sum_{\theta\in\Theta}\mu_\theta = 1$). We define a receiver's type $k_j \in \mathcal{K}$ for each equation $j \in [n_{\text{eq}}]$ (recall that the distribution over receiver's types $\rho \in \Delta_{\mathcal{K}}$ is uniform by definition of OPT-SIGNAL). The receiver has three actions available, namely $\mathcal{A} := \{a_0, a_1, a_2\}$, whereas, for every $k_j \in \mathcal{K}$, the utilities of type k_j are $u_{\theta_i}^{k_j}(a_0) = \frac{1}{2}$, $u_{\theta_i}^{k_j}(a_1) = \frac{1}{2} - \bar{A}_{ji} + \bar{c}_j$, and $u_{\theta_i}^{k_j}(a_2) = \frac{1}{2} + \bar{A}_{ji} - \bar{c}_j$ for every $i \in [n_{\text{var}}]$, while $u_{\theta_0}^{k_j}(a_0) = \frac{1}{2}$, $u_{\theta_0}^{k_j}(a_1) = \frac{1}{2} + \bar{c}_j$, and $u_{\theta_0}^{k_j}(a_2) = \frac{1}{2} - \bar{c}_j$. Finally, the sender's utility is 1 when the receiver plays a_0, while it is 0 otherwise, independently of the state of nature. Formally, $f_\theta(a_0) = 1$ and $f_\theta(a_1) = f_\theta(a_2) = 0$ for every $\theta \in \Theta$.

Completeness Suppose there exists a vector $\hat{\mathbf{x}} \in \{0,1\}^{n_{\text{var}}}$ such that at least a fraction $1-\zeta$ of the equations in $A\hat{\mathbf{x}} = \mathbf{c}$ are satisfied. Let $X^1 \subseteq [n_{\text{var}}]$ be the set of variables $i \in [n_{\text{var}}]$ with $x_i = 1$, while $X^0 := [n_{\text{var}}] \setminus X^1$. Given the definition of $\bar{A}$ and $\bar{\mathbf{c}}$, there exists a vector $\bar{\mathbf{x}} \in \{0, \frac{1}{\tau}\}^{n_{\text{var}}}$ such that at least a fraction $1-\zeta$ of the equations in $\bar{A}\bar{\mathbf{x}} = \bar{\mathbf{c}}$ are satisfied, and, additionally, $\bar{x}_i = \frac{1}{\tau}$ for all the variables in $i \in X^1$, while $\bar{x}_i = 0$ whenever $i \in X^0$. Let us consider an (indirect) signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ defined for the set of signals $\mathcal{S} := \{s_1, s_2\}$. Let $q := \frac{n_{\text{var}}(n_{\text{var}}-1)}{\tau - |X^1|}$. For every $i \in [n_{\text{var}}]$, we define $\phi_{\theta_i}(s_1) = q$ and $\phi_{\theta_i}(s_2) = 1-q$ if $i \in X^1$, while $\phi_{\theta_i}(s_1) = 0$ and $\phi_{\theta_i}(s_2) = 1$ otherwise. Moreover, we let $\phi_{\theta_0}(s_1) = 1$ and $\phi_{\theta_0}(s_2) = 0$. Now, let us take the receiver's posterior $\boldsymbol{\xi}^1 \in \Delta_\Theta$ induced by signal s_1. Let $h := \frac{\frac{q}{n_{\text{var}}^2}}{\sum_{i \in X^1} \frac{q}{n_{\text{var}}^2} + \frac{n_{\text{var}}-1}{n_{\text{var}}}}$. Then, using the definition of $\boldsymbol{\xi}^1$, it is easy to check that $\xi^1_{\theta_i} = h$ for every $i \in X^1$, $\xi^1_{\theta_i} = 0$ for every $i \in X^0$, while $\xi^1_{\theta_0} = \frac{\frac{n_{\text{var}}-1}{n_{\text{var}}}}{\sum_{i \in X^1} \frac{q}{n_{\text{var}}^2} + \frac{n_{\text{var}}-1}{n_{\text{var}}}} = 1 - h\,|X^1|$. Next, we prove that given the posterior $\boldsymbol{\xi}^1$ at least a fraction $1-\zeta$ of the receiver's types has action a_0 as a best response, implying that the expected utility of the sender is equal to $\frac{1}{m}\sum_{k \in \mathcal{K}} f(\phi, k) \geq \frac{m-1}{m}(1-\zeta) \geq 1-2\zeta$, which holds for m large enough. For each satisfied equality $j \in [n_{\text{eq}}]$ in $\bar{A}\bar{\mathbf{x}} = \bar{\mathbf{c}}$, the receiver of type $k_j \in \mathcal{K}$ experiences a utility of $\sum_{\theta \in \Theta} \xi^1_\theta u^{k_j}_\theta(a_0) = \frac{1}{2}$ by playing action a_0. Instead, the utility she gets by playing a_1 is defined as follows:

$$\begin{aligned}
\sum_{\theta \in \Theta} \xi^1_\theta u^{k_j}_\theta(a_1) &= \sum_{i \in X^1} h\left(\frac{1}{2} - \bar{A}_{ji} + \bar{c}_j\right) + \xi^1_{\theta_0}\left(\frac{1}{2} + \bar{c}_j\right) \\
&= h\,|X^1|\left(\frac{1}{2} + \bar{c}_j\right) - h\sum_{i \in X^1} \bar{A}_{ji} + \left(1 - h\,|X^1|\right)\left(\frac{1}{2} + \bar{c}_j\right) \\
&= \frac{1}{2} + \bar{c}_j - h\sum_{i \in X^1} \bar{A}_{ji} = \frac{1}{2} + \bar{c}_j - \frac{1}{\tau}\sum_{i \in X^1} \bar{A}_{ji} = \frac{1}{2},
\end{aligned}$$

where the second to last equality holds since $h = \frac{1}{\tau}$ (by definition of h and q), while the last equality follows from the fact that the j-th equation is satisfied, and, thus, $\frac{1}{\tau}\sum_{i \in X^1} \bar{A}_{ji} = \bar{c}_j$ (recall that $\bar{x}_i = \frac{1}{\tau}$ for all $i \in X^1$). Using similar arguments, we can write $\sum_{\theta \in \Theta} \xi^1_\theta u^{k_j}_\theta(a_2) = \frac{1}{2}$, which concludes the completeness proof.

Soundness Suppose, by contradiction, that there exists a signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ providing the sender with an expected utility greater than δ. This implies, by an averaging argument, that there exists a signal inducing a posterior $\boldsymbol{\xi} \in \Delta_\Theta$ in which at least a fraction δ of the receiver's types best responds by playing action a_0. Let $\mathcal{K}^1 \subseteq \mathcal{K}$ be the set of such reviver's types. For every receiver's type $k_j \in \mathcal{K}$, it holds $\sum_{\theta \in \Theta} \xi_\theta u^{k_j}_\theta(a_0) = \frac{1}{2}$. Moreover, it is the case that:

$$\begin{aligned}
\sum_{\theta \in \Theta} \xi_\theta u^{k_j}_\theta(a_1) &= \sum_{i \in [n_{\text{var}}]} \xi_{\theta_i}\left(\frac{1}{2} - \bar{A}_{ji} + \bar{c}_j\right) + \xi_{\theta_0}\left(\frac{1}{2} + \bar{c}_j\right) \\
&= \frac{1}{2} + \bar{c}_j - \sum_{i \in [n_{\text{var}}]} \xi_{\theta_i}\bar{A}_{ji}.
\end{aligned}$$

Similarly, it holds:

$$\sum_{\theta \in \Theta} \xi_\theta u_\theta^{k_j}(a_2) = \frac{1}{2} - \bar{c}_j + \sum_{i \in [n_{\text{var}}]} \xi_{\theta_i} \bar{A}_{ji}.$$

By assumption, for every type $k_j \in \mathcal{K}^1$, it is the case that $\sum_{\theta \in \Theta} \xi_\theta u_\theta^{k_j}(a_0) \geq \sum_{\theta \in \Theta} \xi_\theta u_\theta^{k_j}(a_1)$, which implies that $\bar{c}_j - \sum_{i \in [n_{\text{var}}]} \xi_{\theta_i} \bar{A}_{ji} \leq 0$, whereas $\sum_{\theta \in \Theta} \xi_\theta u_\theta^{k_j}(a_0) \geq \sum_{\theta \in \Theta} \xi_\theta u_\theta^{k_j}(a_2)$, implying $-\bar{c}_j + \sum_{i \in [n_{\text{var}}]} \xi_{\theta_i} \bar{A}_{ji} \leq 0$. Thus, $\sum_{i \in [n_{\text{var}}]} \xi_{\theta_i} \bar{A}_{ji} = \bar{c}_j$ for every $j \in [n_{\text{eq}}]$ such that $k_j \in \mathcal{K}^1$ and the vector $\hat{\mathbf{x}} \in \mathbb{Q}^{n_{\text{var}}}$ with $\hat{x}_i = \xi_{\theta_i}$ for all $i \in [n_{\text{var}}]$ satisfies at least a fraction δ of the equations, reaching a contradiction. □

Now, we use the approximation-hardness of the offline Bayesian persuasion problem to provide lower bounds on the α-regret in the online setting. In order to do this, we employ a set of techniques introduced by Roughgarden and Wang (2019), which lead to the following result.

Theorem 9.3. *For every $\alpha < 1$, there is no polynomial-time algorithm for the online Bayesian persuasion problem providing no-α-regret, unless* $\mathsf{NP} \subseteq \mathsf{RP}$.

Proof. The theorem follows applying Theorem 6.2 by Roughgarden and Wang (2019) to the NP-hard problem in Theorem 9.2. Notice that we use an additive notion of α-regret while the proof of Theorem 6.2 by Roughgarden and Wang (2019) focuses on multiplicative α-regret. However, the proof can be easily extended to work with additive α-regret. □

9.4 Full Information Feedback Setting

The negative result of the previous section (Theorem 9.3) rules out the possibility of designing an algorithm which satisfies the no-regret property and requires a $\mathsf{poly}(T, m, \varrho, d)$ per-round running time. A natural question is whether it is possible to devise a no-regret algorithm for the online Bayesian persuasion problem by relaxing the running-time constraint. This is not a trivial problem because, at every round t, the sender has to choose a signaling scheme among an infinite number of alternatives and her/his utility depends on the receiver's best response, which yields a function that is not linear nor convex (or even continuous in the space of the signaling schemes). However, we show that it is possible to provide a no-regret algorithm for the full information setting by restricting the sender's action space to a finite set of posteriors.

First, we show that it is always possible to design a sender-optimal signaling scheme defined as a convex combination of a specific finite set of posteriors. For each type $k \in \mathcal{K}$ and action $a \in \mathcal{A}$, we define $\Xi_{a_k} \subseteq \Xi$ as the set of posterior beliefs in which a is a receiver's best response. Formally, $\Xi_{a_k} := \left\{ \boldsymbol{\xi} \in \Xi \mid a \in \mathcal{B}^k_{\boldsymbol{\xi}} \right\}$. Let $\mathbf{a} = (a_k)_{k \in \mathcal{K}} \in \times_{k \in \mathcal{K}} \mathcal{A}$ be a tuple specifying one action for each receiver's type k. Then, for each tuple $\mathbf{a}$, let $\Xi_{\mathbf{a}} \subseteq \Delta_\Theta$ be the (potentially empty) polytope such that each action a_k is optimal for the corresponding type k, *i.e.*, $\Xi_{\mathbf{a}} := \bigcap_{k \in \mathcal{K}} \Xi^k_{a_k}$. The polytope $\Xi_{\mathbf{a}}$ has a simple interpretation: a probability distribution over posteriors in $\Xi_{\mathbf{a}}$ yields a signaling scheme such that, for every type k, the receiver has no interest in deviating from a_k in the induced posteriors $\Xi_{\mathbf{a}}$. Then, let $\hat{\Xi} \subseteq \Xi$ be the set of posteriors defined as $\hat{\Xi} := \bigcup_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} V(\Xi_{\mathbf{a}})$. [2] Finally, we define the following set of consistent (according to Equation (3.4)) distributions over posteriors in $\hat{\Xi}$:

$$\hat{\Gamma} := \left\{ \boldsymbol{\gamma} \in \Delta_{\hat{\Xi}} \mid \sum_{\boldsymbol{\xi} \in \hat{\Xi}} \gamma_\theta \xi_\theta = \mu_\theta, \ \forall \theta \in \Theta \right\}. \tag{9.2}$$

By letting M be a suitably defined $|\Theta| \times |\hat{\Xi}|$-dimensional matrix with one column for each $\boldsymbol{\xi} \in \hat{\Xi}$, then the affine hyperplanes defined by Equation (3.4) are in the form $M \cdot \boldsymbol{\gamma} = \boldsymbol{\mu}$. Since $\boldsymbol{\gamma} \in \Delta_{\hat{\Xi}}$, we can safely rewrite the consistency constraints as $M \cdot \boldsymbol{\gamma} \geq \boldsymbol{\mu}$ (see the example below for a better intuition). Then, $\hat{\Gamma}$ can be seen as the intersection between the simplex $\Delta_{\hat{\Xi}}$ and a finite number of half-spaces. Therefore, $\hat{\Gamma}$ is a convex polytope, whose vertices compose the finite action space that will be employed by the no-regret algorithm. Specifically, let

$$\Gamma^\star := V(\hat{\Gamma}). \tag{9.3}$$

9.4.1 A More Complex Example with Types

Consider the game of Section 9.1.1 (see Figure 9.1–Left) where the receiver has a single type (*type 1*). We obtain $\hat{\Xi}$ by partitioning the space of posteriors in different best response regions and by taking the vertices of the resulting polytopes (see Figure 9.2–Left). Then, we provide a visual depiction of $\hat{\Gamma}$ and $\Gamma^\star$, which are obtained, respectively, by intersecting $\Delta_{\hat{\Xi}}$ with the hyperplanes corresponding to consistency constraints (see Equation (9.2)), and then taking the vertices of the resulting polytope (see figure

[2] $V(X)$ denotes the set of vertices of polytope X.

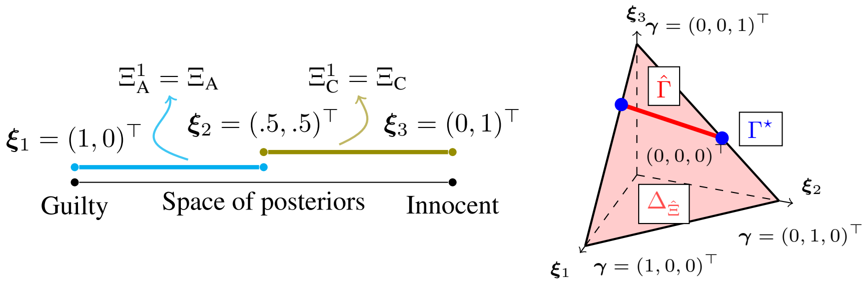


Figure 9.2: Left: *Subdivision of the space of posteriors Ξ in the two best-response regions. If $\boldsymbol{\xi} \in \Xi_A$ (resp., $\boldsymbol{\xi} \in \Xi_C$) then the judge's best response under $\boldsymbol{\xi}$ is acquitting (resp., convicting) the defendant. When $\boldsymbol{\xi} = \boldsymbol{\xi}_2$, the judge is indifferent among her/his available actions. We have $\hat{\Xi} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$.* **Right**: *Visual depiction of $\Delta_{\hat{\Xi}}$, $\hat{\Gamma} \subseteq \Delta_{\hat{\Xi}}$, and $\Gamma^\star = V(\hat{\Gamma})$. The set $\hat{W}$ comprises of the distributions over posteriors in $\hat{\Xi}$ consistent with the prior $\boldsymbol{\mu} = (.3, .7)^\top$ and it is obtained by intersecting $\Delta_{\hat{\Xi}}$ with $[\boldsymbol{\xi}_1 \mid \boldsymbol{\xi}_2 \mid \boldsymbol{\xi}_3] \cdot \boldsymbol{\gamma} \geq \boldsymbol{\mu}$. As a result, we obtain $\hat{\Gamma} = \text{conv}\{(.3,0,.7)^\top, (0,.6,.4)^\top\}$. Finally, $\Gamma^\star = V(\hat{\Gamma}) = \{(.3,0,.7)^\top, (0,.6,.4)^\top\}$.*

Type 1					
		State G ($\mu_G = .3$)		State I ($\mu_I = .7$)	
$\mathcal{A}$	A	0	0	0	1
	C	1	1	1	0

Type 2					
		State G ($\mu_G = .3$)		State I ($\mu_I = .7$)	
$\mathcal{A}$	A	0	−1	0	1
	C	1	1	1	0

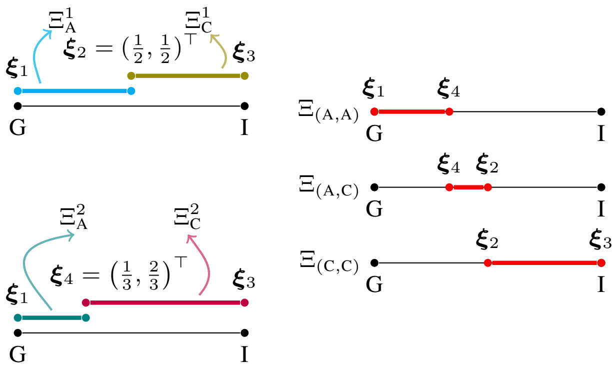


Figure 9.3: Left: *A prosecutor/judge game with two types. When the judge is of type 2 she has a worse perception of acquitting a guilty defendant.* **Center**: *A visual depiction of Ξ_A^k and Ξ_C^k for each possible type $k \in \{1,2\}$. When $k = 2$, the judge is less inclined towards acquitting and, therefore, the best-response boundary is $\boldsymbol{\xi}_4$. When $k = 1$ (resp., $k = 2$) and the posterior is $\boldsymbol{\xi}_2$ (resp., $\boldsymbol{\xi}_4$), the judge is indifferent between acquitting and convicting the defendant.* **Right**: *Best-response regions for the possible joint actions. When $\boldsymbol{a} = (\text{C}, \text{A})$ we have $\Xi_{\boldsymbol{a}} = \varnothing$ because there is no posterior for which A is a best response for a receiver of type 1, and C is a best response for a receiver of type 2. We have $\hat{\Xi} = \{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4\}$.*

9.2–Right). [3]

Figure 9.3 shows a more complex example of the classical prosecutor/-judge game. Here, the judge has two possible types. A judge of *type 1* gets payoff 1 for a just decision, and 0 otherwise. A judge of *type 2* has a worse perception of acquitting a guilty defendant, for which she gets -1. In this case, the computation of best-response regions is more involved because different judge's types yield different boundaries on the space of posteriors. Specifically, by Equation (9.2), $\hat{\Gamma}$ is the result of the intersection between the simplex $\Delta_{\hat{\Xi}}$ and the closed half-spaces specified by $[\boldsymbol{\xi}_1|\boldsymbol{\xi}_2|\boldsymbol{\xi}_3|\boldsymbol{\xi}_4] \cdot \boldsymbol{\gamma} \geq \boldsymbol{\mu}$. The vertices of the resulting polytope are $\boldsymbol{\gamma}_1 = (3/10, 0, 0, 7/10)^\top$, $\boldsymbol{\gamma}_2 = (0, 9/10, 0, 1/10)^\top$, and $\boldsymbol{\gamma}_3 = (0, 0, 3/5, 2/5)^\top$. Then, the new sender's action space can be restricted to $\Gamma^\star = \{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3\}$.

9.4.2 A No-regret Algorithm with Full Information Feedback

For an arbitrary sequence of receiver's types, we show that there exists $\boldsymbol{\gamma}^\star \in \Gamma^\star$ guaranteeing to the sender an expected utility that is equal to the best-in-hindsight signaling scheme.

Lemma 9.1. *For every sequence of receiver's types $\boldsymbol{k} = \{k^t\}_{t\in[T]}$, it holds*

$$\max_{\boldsymbol{\gamma}\in\Gamma} \sum_{t=1}^{T} f(\boldsymbol{\gamma}, k^t) = \max_{\boldsymbol{\gamma}^\star\in\Gamma^\star} \sum_{t=1}^{T} f(\boldsymbol{\gamma}^\star, k^t).$$

Proof. The idea to prove the lemma is the following: any posterior distribution $\boldsymbol{\xi}$ in supp$(\boldsymbol{\gamma})$ can be represented as the convex combination of elements of $\hat{\Xi}$. We denote such convex combination by $\boldsymbol{\gamma}^{\boldsymbol{\xi}} \in \Delta_{\hat{\Xi}}$. We define a new signaling scheme $\boldsymbol{\gamma}^\star \in \Delta_{\hat{\Xi}}$ as follows:

$$\gamma^\star_{\boldsymbol{\xi}'} := \sum_{\substack{\boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}):\\ \boldsymbol{\xi}'\in\text{supp}(\boldsymbol{\gamma}^{\boldsymbol{\xi}})}} \gamma_{\boldsymbol{\xi}} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'} \quad \text{for each} \quad \boldsymbol{\xi}' \in \hat{\Xi}. \tag{9.4}$$

Since $\boldsymbol{\gamma}$ is consistent (*i.e.,* $\boldsymbol{\gamma} \in \Gamma$) we have by construction that $\boldsymbol{\gamma}^\star$ is consistent, and therefore $\boldsymbol{\gamma}^\star \in \hat{\Gamma}$. Finally, we show that $\boldsymbol{\gamma}^\star$ guarantees to the sender an expected utility which is greater than or equal to that achieved via $\boldsymbol{\gamma}$. The crucial point here is showing that whenever the decomposition over $\hat{\Xi}$ involves a vertex (*i.e.,* a posterior) where the receiver is indifferent between two or more actions, her/his choice does not damage the sender. This happens at the boundaries of best-response regions (see, *e.g.*, what

[3] The polytopes were computed using `Polymake`, a tool for computational polyhedral geometry (Assarf et al., 2017; Gawrilow and Joswig, 2000).

happens at $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_4$ in the example of Figure 9.3). The sender's expected utility is a linear function of the signaling scheme $\boldsymbol{\gamma}^\star$. Therefore, the sender can limit her attention to $\Gamma^\star$, since her/his maximum expected utility is attained at one of the vertices of $\hat{\Gamma}$.

Consider a posterior $\boldsymbol{\xi} \in \Xi$ and let $\mathbf{a} = \{b^k_{\boldsymbol{\xi}}\}_{k\in\mathcal{K}}$ (*i.e.*, $\mathbf{a}$ is the tuple specifying the best-response action under posterior $\boldsymbol{\xi}$ for each receiver's type k). Tuple $\mathbf{a}$ defines polytope $\Xi_{\mathbf{a}} \subseteq \Xi$. By Carathéodory's theorem, any $\boldsymbol{\xi} \in \Xi_{\mathbf{a}}$ is the convex combination of a finite number of points in $\Xi_{\mathbf{a}}$. Specifically, there exists $\boldsymbol{\gamma}^{\boldsymbol{\xi}} \in \Delta_{V(\Xi_{\mathbf{a}})}$ such that, for each $\theta \in \Theta$, $\sum_{\boldsymbol{\xi}'\in V(\Xi_{\mathbf{a}})} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'}\xi'_\theta = \xi_\theta$.

Let $\boldsymbol{\gamma} \in \hat{\Gamma}$ (*i.e.,* $\boldsymbol{\gamma}$ is consistent). By following Equation (9.4), we define a distribution $\boldsymbol{\gamma}^\star$ such that, for each $\boldsymbol{\xi}' \in \hat{\Xi}$,

$$\gamma^\star_{\boldsymbol{\xi}'} := \sum_{\substack{\boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}):\\ \boldsymbol{\xi}'\in\text{supp}(\boldsymbol{\gamma}^{\boldsymbol{\xi}})}} \gamma_{\boldsymbol{\xi}}\gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'}.$$

By construction, $\boldsymbol{\gamma}^\star$ is a well-defined convex combination of elements of $\hat{\Xi}$. Moreover, since $\boldsymbol{\gamma}$ is consistent, the same holds true for $\boldsymbol{\gamma}^\star$, which implies $\boldsymbol{\gamma}^\star \in \hat{\Gamma}$.

Fix a type $k \in \mathcal{K}$ and a posterior $\boldsymbol{\xi} \in \Xi$, and let $\mathbf{a}$ be defined as the tuple specifying the best response under $\boldsymbol{\xi}$ for each k. At each posterior $\boldsymbol{\xi}' \in V(\Xi_{\mathbf{a}})$, the receiver plays $b^k_{\boldsymbol{\xi}'}$. The following holds:

$$b^k_{\boldsymbol{\xi}'} \in arg\,max_{a'\in\mathcal{B}^k_{\boldsymbol{\xi}'}} \sum_{\theta\in\Theta} \xi'_\theta f_\theta(a') \geq \sum_{\theta\in\Theta} \xi'_\theta f_\theta(b^k_{\boldsymbol{\xi}}), \tag{9.5}$$

where the inequality holds because, by construction, $b^k_{\boldsymbol{\xi}} \in \mathcal{B}^k_{\boldsymbol{\xi}'}$. Therefore, we can show that the sender's expected utility when decomposing $\boldsymbol{\xi}$ as $\boldsymbol{\gamma}^{\boldsymbol{\xi}} \in \Delta_{V(\Xi_{\mathbf{a}})}$ is guaranteed to be greater than or equal to the expected utility under $\boldsymbol{\xi}$. Specifically,

$$\begin{aligned}
\sum_{\boldsymbol{\xi}'\in V(\Xi_{\mathbf{a}})} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'} f(\boldsymbol{\xi}', k) &= \sum_{\boldsymbol{\xi}'\in V(\Xi_{\mathbf{a}})} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'} \sum_{\theta\in\Theta} \xi'_\theta f_\theta(b^k_{\boldsymbol{\xi}'}) \\
&\geq \sum_{\boldsymbol{\xi}'\in V(\Xi_{\mathbf{a}})} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'} \sum_{\theta\in\Theta} \xi'_\theta f_\theta(b^k_{\boldsymbol{\xi}}) && \text{(By Equation (9.5))} \\
&= \sum_{\theta\in\Theta} \xi_\theta f_\theta(b^k_{\boldsymbol{\xi}}) && \text{(By definition of } \boldsymbol{\gamma}^{\boldsymbol{\xi}}\text{)} \\
&= f(\boldsymbol{\xi}, k).
\end{aligned}$$

Let $\boldsymbol{\gamma} \in \Gamma$ be the best-in-hindsight signaling scheme. We show that, for any sequence of receiver's types $\boldsymbol{k} = \{k^t\}_{t\in[T]}$, the sender's expected utility achieved via $\boldsymbol{\gamma}$ is matched by the expected utility guaranteed by $\boldsymbol{\gamma}^\star \in \hat{\Gamma}$ defined as in Equation (9.4). We have

$$
\begin{aligned}
\sum_{t\in[T]} \sum_{\boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}^\star)} \gamma^\star_{\boldsymbol{\xi}} f(\boldsymbol{\xi}, k^t) &= \sum_{t\in[T]} \sum_{\boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}^\star)} \sum_{\substack{\boldsymbol{\xi}'\in\text{supp}(\boldsymbol{\gamma}):\\ \boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}^{\boldsymbol{\xi}'})}} \gamma_{\boldsymbol{\xi}'} \gamma^{\boldsymbol{\xi}'}_{\boldsymbol{\xi}} f(\boldsymbol{\xi}, k^t) \\
&= \sum_{t\in[T]} \sum_{\boldsymbol{\xi}'\in\text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}'} \sum_{\boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}^{\boldsymbol{\xi}'})} \gamma^{\boldsymbol{\xi}'}_{\boldsymbol{\xi}} f(\boldsymbol{\xi}, k^t) \\
&\geq \sum_{t\in[T]} \sum_{\boldsymbol{\xi}'\in\text{supp}(\boldsymbol{\gamma})} f(\boldsymbol{\xi}', k^t) \\
&= \sum_{t\in[T]} f(\boldsymbol{\gamma}, k^t).
\end{aligned}
$$

Finally, since $\sum_{t\in[T]} f(\boldsymbol{\gamma}^\star, k^t) = \sum_{t\in[T]} \sum_{\boldsymbol{\xi}\in\text{supp}(\boldsymbol{\gamma}^\star)} \gamma^\star_{\boldsymbol{\xi}} f(\boldsymbol{\xi}, k^t)$ is a linear function in the signaling scheme $\boldsymbol{\gamma}^\star$, its maximum is attained at a vertex of $\hat{\Gamma}$. This concludes the proof. □

The size of the sender's finite action space grows exponentially in the number of states of nature d.

Lemma 9.2. *The size of* $\Gamma^\star$ *is* $|\Gamma^\star| \in O\left((m\,\varrho^2 + d)^d\right)$.

Proof. By definition, for any $\mathbf{a} = (a_k)_{k\in\mathcal{K}}$, $\Xi_{\mathbf{a}} \subseteq \Xi$. Then, each $\boldsymbol{\gamma} \in V(\Xi_{\mathbf{a}})$ is an extreme point of a $(d-1)$-dimensional convex polytope, and therefore the point lies at the intersection of $(d-1)$ linearly independent defining half-spaces of the polytope. Now, to provide a bound for $|\hat{\Xi}|$ we first compute the number of half-spaces separating best-response regions corresponding to different actions. For each type $k \in \mathcal{K}$, there are at most $\binom{\varrho}{2}$ half-spaces each separating Ξ_{a_k} and $\Xi_{a'_k}$ for two actions $a \neq a'$. Then, in order to take all the incentive constraints into account, we have to sum over all possible reveiver's types, obtaining $O(m\,\varrho^2)$ half-spaces. The set $\hat{\Xi}$ is the result of the intersection between the region defined by such half-spaces, and the d constraints defining the simplex. Each extreme point of the polytope defined by points in $\hat{\Xi}$ lies at the intersection of $d-1$ half-spaces. Therefore, there are at most $\binom{m\varrho^2+d}{d-1} \in O\left((m\varrho^2 + d)^d\right)$ such extreme points. The convex polytope $\hat{\Gamma}$ is the result of the intersection between the simplex defined over $\hat{\Xi}$, which has $O\left((m\varrho^2 + d)^d\right)$ extreme points, and d half-spaces defining consistency constraints. Then, $\hat{\Gamma}$ has a number of extreme points which is less than or equal to $O\left((m\varrho^2 + d)^d\right)$. □

Now, by letting $\eta \in [0,1]$ be the maximum absolute payoff value, we can employ any algorithm satisfying $R^T \leq O\Big(\eta\sqrt{\log|A|/T}\Big)$ as a black box (see, *e.g.*, *Polynomial Weights* (Cesa-Bianchi and Lugosi, 2006) and *Follow the Lazy Leader* (Kalai and Vempala, 2005)), where A is the action set of the learner. By taking $\Gamma^\star$ as the sender action space, we obtain the following.

Theorem 9.4. *Given an online Bayesian persuasion problem with full information feedback, there exists an online algorithm such that, for every sequence of receiver's types* $\boldsymbol{k} = \{k^t\}_{t\in[T]}$*:*

$$R^T \leq O\left(\sqrt{\frac{d\,\log(m\,\varrho^2+d)}{T}}\right).$$

Proof. We employ an arbitrary algorithm satisfying $R^T \leq O\Big(\eta\sqrt{\log|A|/T}\Big)$ with action set $A = \Gamma^\star$. Let $\boldsymbol{\gamma}^* \in \Gamma$ be the sender-optimal signaling scheme in hindsight. Then,

$$\begin{aligned}\sum_{t\in[T]} \mathbb{E}[f(\boldsymbol{\gamma}^t,k^t)] &\geq \sum_{t\in[T]} f(\boldsymbol{\gamma}^*,k^t) - O\left(\sqrt{T\log|\Gamma^\star|}\right)\\ &\geq \sum_{t\in[T]} f(\boldsymbol{\gamma}^*,k^t) - O\left(\sqrt{T\log{(m\varrho^2+d)^d}}\right) \qquad \text{(By Lemma 9.2)}\\ &= \sum_{t\in[T]} f(\boldsymbol{\gamma}^*,k^t) - O\left(\sqrt{Td\log{(m\varrho^2+d)}}\right).\end{aligned}$$

This completes the proof. □

Notice that any no-regret algorithm working on $\Gamma^\star$ requires a per-round running time polynomial in n, m and exponential in d (see the bound in Lemma 9.2). This shows that the source of the hardness result in Theorem 9.3 is the number of states of nature d, while achieving no-regret in polynomial time is possible when the parameter d is fixed.

9.5 Partial Information Feedback Setting

In this setting, at every round t, the sender can only observe the action a^t played by the receiver. Therefore, the sender has no information on the utility $f(\boldsymbol{\gamma},k^t)$ that she/he would have obtained by choosing any signaling scheme $\boldsymbol{\gamma} \in \Gamma^\star$ other than $\boldsymbol{\gamma}^t$. We show how to design no-regret algorithms with regret bounds that depend polynomially in the size of the problem instance by exploiting a reduction from the partial information setting to the full information one.[4] The main idea is to use a full-information no-regret algorithm in combination with a mechanism to estimate the sender's utilities corresponding to signaling schemes different from the one recommended by the algorithm. In particular, the overall time horizon T is split into a given number of equally-sized blocks, each corresponding to a window of time simulating a single round of a full information setting. During this window, the strategy suggested by the full-information algorithm is played in most of the rounds (exploitation phase), while only few rounds are chosen uniformly at random and used by the mechanism that estimates the utilities provided by other signaling schemes (exploration phase).

We split the rest of this section in two parts. Subsection 9.5.1 describes in details the overall structure of the partial-information algorithm and shows its regret bound, while Subsection 9.5.2 shows the details about the utility estimates built by the algorithm.

9.5.1 Overall Structure of the Algorithm

Algorithm 9.1 provides a sketch of the overall procedure, where Z (Line 1) denotes the number of blocks, which are the intervals of consecutive rounds $\{I_\tau\}_{\tau \in [Z]}$ defined in Line 4. The FULL-INFORMATION$(\cdot)$ sub-procedure is a black box representing a no-regret algorithm for the full information setting, working on a subset $\Gamma^\circ \subseteq \Gamma^\star$ of signaling schemes. After the execution of all the rounds of each block $\tau \in [Z]$, it takes as input the utility estimates computed during I_τ and returns a recommended strategy $\mathbf{q}^{\tau+1} \in \Delta_{\Gamma^\circ}$ for the next block $I_{\tau+1}$ (see Line 16). During each block I_τ with $\tau \in [Z]$, Algorithm 9.1 alternates between two tasks: (i) *exploration* (Line 8), trying all the signaling schemes in a subset $\Gamma^{\circledcirc} \subseteq \Gamma^\star$ given as input, so as to compute the required estimates of the sender's expected utilities; and (ii) *exploitation* (Line 10), playing strategy $\mathbf{q}^\tau$ recommend by FULL-INFORMATION$(\cdot)$ for I_τ.

Our main result is the proof that Algorithm 9.1 achieves the no-regret property.

In order to prove this result, we show that Algorithm 9.1 provides a regret bound that depends on the number $|\Gamma^{\circledcirc}|$ of signaling schemes used for exploration, the logarithm of $|\Gamma^\circ|$, and the range and bias of the estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$. To do this, we extend a result shown by (Balcan et al., 2015, Lemma 6.2) to the more general case in which only *biased* utility estima-

[4] The reduction is an extension of those proposed by Balcan et al. (2015) and Awerbuch and Mansour (2003).

Algorithm 9.1 ONLINE BAYESIAN PERSUASION WITH PARTIAL INFORMATION FEEDBACK

Inputs ▷ See Subsection 9.5.2 for the definitions of subsets Γ° and $\Gamma^{\circledcirc}$

- Full-information no-regret algorithm FULL-INFORMATION$(\cdot)$ working on a subset $\Gamma^\circ \subseteq \Gamma^\star$ of signaling schemes.
- Subset $\Gamma^{\circledcirc} \subseteq \Gamma^\star$ of signaling schemes used for exploration.

Let Z be defined as in Lemma 9.3
Let $\mathbf{q}^1 \in \Delta_{\Gamma^\circ}$ be the uniform distribution over Γ°
for $\tau = 1, \dots, Z$ **do**
    $I_\tau \leftarrow \left\{(\tau-1)\frac{T}{Z}+1, \dots, \tau\frac{T}{Z}\right\}$
    Choose a random permutation $\pi : [|\Gamma^{\circledcirc}|] \to \Gamma^{\circledcirc}$ and $t_1, \dots, t_{|\Gamma^{\circledcirc}|}$ rounds at random from I_τ
    for $t = (\tau-1)\frac{T}{Z}+1, \dots, \tau\frac{T}{Z}$ **do**
        if $t = t_j$ for some $j \in [|\Gamma^{\circledcirc}|]$ **then**
            $\mathbf{q}^t \leftarrow \mathbf{q} \in \Delta_{\Gamma^\star}$ such that $q_{\boldsymbol{\gamma}} = 1$ for the signaling scheme $\boldsymbol{\gamma} = \pi(j)$ ▷ Exploration phase
        else
            $\mathbf{q}^t \leftarrow \mathbf{q}^\tau$ ▷ Exploitation phase
        end if
        Play a signaling scheme $\boldsymbol{\gamma}^t \in \Gamma^\star$ randomly drawn from $\mathbf{q}^t$
        Observe sender's utility $f(\boldsymbol{\gamma}^t, k^t)$ and receiver's action $a^t \in \mathcal{A}$
    end for
    Compute estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ of $f^{I_\tau}(\boldsymbol{\gamma}) := \frac{1}{|I_\tau|}\sum_{t\in[T]:t\in I} f(\boldsymbol{\gamma}, k^t)$ for all $\boldsymbol{\gamma} \in \Gamma^\circ$
    $\mathbf{q}^{\tau+1} \leftarrow$ FULL-INFORMATION$\left(\left\{\tilde{f}^{I_\tau}(\boldsymbol{\gamma})\right\}_{\boldsymbol{\gamma}\in\Gamma^\circ}\right)$
end for

tors are available, rather than unbiased ones. This result can be generalized to any partial information setting (beyond online Bayesian persuasion).

In any block I_τ with $\tau \in [Z]$, for every $\boldsymbol{\gamma} \in \Gamma^\circ$, we assume that Algorithm 9.1 has access to an estimator $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ of the sender's average utility $f^{I_\tau}(\boldsymbol{\gamma}) = \frac{1}{|I_\tau|}\sum_{t\in[T]:t\in I} f(\boldsymbol{\gamma}, k^t)$ obtained by committing to $\boldsymbol{\gamma}$ during the block I_τ, with the following properties:

(i) the *bias is bounded* by a given constant $\iota \in (0,1)$, *i.e.*, it holds that $\left| f^{I_\tau}(\boldsymbol{\gamma}) - \mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\gamma})\right]\right| \le \iota$;

(ii) the *range is limited*, *i.e.*, there exists a $\eta \in \mathbb{R}$ such that $\tilde{f}^{I_\tau}(\boldsymbol{\gamma}) \in [-\eta, +\eta]$.

Lemma 9.3. *Suppose that Algorithm 9.1 has access to estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ with properties (i) and (ii) for some constants $\iota \in (0,1)$ and $\eta \in \mathbb{R}$, for every signaling scheme $\boldsymbol{\gamma} \in \Gamma^\circ$ and block I_τ with $\tau \in [Z]$. Moreover, let $Z := T^{2/3}|\Gamma^{\circledcirc}|^{-2/3}\eta^{2/3}\log^{1/3}|\Gamma^\circ|$. Then, Algorithm 9.1 guarantees regret:*

$$R^T \le O\left(\frac{|\Gamma^{\circledcirc}|^{1/3}\eta^{2/3}\log^{1/3}|\Gamma^\circ|}{T^{1/3}}\right) + O(\iota).$$

Proof. In order to prove the desired regret bound for Algorithm 9.1, we rely on two crucial observations:

- during the exploration phase of each block I_τ with $\tau \in [Z]$, *i.e.*, the iterations $t_1, \dots, t_{|\Gamma^{\circledcirc}|}$, the algorithm plays a strategy $\mathbf{q}^t \neq \mathbf{q}^\tau$, where $\mathbf{q}^\tau$ is the last strategy recommended by FULL-INFORMATION$(\cdot)$, resulting in a corresponding utility loss that can be as large as -1 (since the utilities are in the range $[0,1]$);
- running the full-information no-regret algorithm (*i.e.*, the sub-procedure FULL-INFORMATION$(\cdot)$) using biased estimates of the sender's utilities (rather than their real values) results in the regret bound being worsened by only a term that is proportional to the bias ι of the adopted estimators.

In the following, we denote with R^Z_{full} the cumulative regret achieved by FULL-INFORMATION$(\cdot)$, where we remark the fact that each block I_τ simulates a single iteration of the full information setting, and, thus, the number of iterations for the full-information algorithm is Z rather than T. Formally, we have the following definition:

$$R^Z_{\text{full}} := \max_{\boldsymbol{\gamma}\in\Gamma^\circ} \sum_{\tau\in[Z]} \tilde{f}^{I_\tau}(\boldsymbol{\gamma}) - \sum_{\tau\in[Z]}\sum_{\boldsymbol{\gamma}\in\Gamma^\circ} q^\tau_{\boldsymbol{\gamma}} \tilde{f}^{I_\tau}(\boldsymbol{\gamma}),$$

where we notice that the regret is computed with respect to the estimates $\tilde{f}^{I_\tau}(\gamma)$ of the sender's average utilities $f^{I_\tau}(\gamma)$ experienced in each block I_τ, defined as $f^{I_\tau}(\gamma) = \frac{1}{|I_\tau|}\sum_{t\in I_\tau} f(\gamma, k^t)$ for every $\gamma \in \Gamma^\circ$. We also remark that the full-information algorithm is run on a subset $\Gamma^\circ \subseteq \Gamma^\star$ of signaling schemes, and, thus, the regret R^Z_{full} is defined with respect to them. Moreover, from Section 9.4, we know that there exists an algorithm satisfying the regret bound $R^Z_{\text{full}} \leq O\left(\eta\sqrt{Z \log|\Gamma^\circ|}\right)$, where η is the range of the utility values observed by the algorithm that, in our case, corresponds to the range of the estimates observed by the algorithm, which is limited thanks to property (ii) of the estimators.

In order to prove the result, we also need the following relation, which holds for every $\tau \in [Z]$ and signaling scheme $\gamma \in \Gamma^\circ$:

$$\sum_{t\in I_\tau} f(\gamma, k^t) = |I_\tau| f^{I_\tau}(\gamma) \geq |I_\tau|\Big(\mathbb{E}[\tilde{f}^{I_\tau}] - \iota\Big) = \frac{T}{Z}\Big(\mathbb{E}[\tilde{f}^{I_\tau}] - \iota\Big), \quad (9.6)$$

where the first equality holds by definition, the inequality holds thanks to property (i) of the estimators, while the last equality is given by $|I_\tau| = \frac{T}{Z}$.

Letting U be the sender's expected utility achieved by playing according to Algorithm 9.1, the following relations hold:

$$\begin{aligned}
\frac{1}{T}U &:= \frac{1}{T}\sum_{\tau\in[Z]}\sum_{t\in I_\tau}\sum_{\gamma\in\Gamma^\circ} q^t_\gamma f(\gamma, k^t) \\
&\qquad (\mathbf{q}^t \neq \mathbf{q}^\tau \text{ in } |\Gamma^{\circledcirc}| \text{ iterations and max. loss} = -1) \\
&\geq \frac{1}{T}\sum_{\tau\in[Z]}\sum_{\gamma\in W^\circ} q^\tau_\gamma \sum_{t\in I_\tau} f(\gamma, k^t) - \frac{|\Gamma^{\circledcirc}|Z}{T} \\
&\geq \frac{1}{T}\sum_{\tau\in[Z]}\sum_{\gamma\in W^\circ} q^\tau_\gamma \frac{T}{Z}\Big(\mathbb{E}\left[\tilde{f}^{I_\tau}(\gamma)\right] - \iota\Big) - \frac{|\Gamma^{\circledcirc}|Z}{T} && \text{(By Equation (9.6))} \\
&= \frac{1}{Z}\sum_{\tau\in[Z]}\sum_{\gamma\in\Gamma^\circ} q^\tau_\gamma\Big(\mathbb{E}\left[\tilde{f}^{I_\tau}(\gamma)\right] - \iota\Big) - \frac{|\Gamma^{\circledcirc}|Z}{T} \\
&\qquad \Big(\text{Since } \sum_{\tau\in[Z]}\sum_{\gamma\in\Gamma^\circ} q^\tau_\gamma = Z, \text{ being } \mathbf{q}^\tau \in \Delta_{\Gamma^\circ}\Big) \\
&= \frac{1}{Z}\sum_{\tau\in[Z]}\sum_{\gamma\in\Gamma^\circ} q^\tau_\gamma \mathbb{E}\left[\tilde{f}^{I_\tau}(\gamma)\right] - \iota - \frac{|\Gamma^{\circledcirc}|Z}{T} \\
&= \frac{1}{Z}\mathbb{E}\left[\sum_{\tau\in[Z]}\sum_{\gamma\in\Gamma^\circ} q^\tau_\gamma \tilde{f}^{I_\tau}(\gamma)\right] - \iota - \frac{|\Gamma^{\circledcirc}|Z}{T} \\
&= \frac{1}{Z}\mathbb{E}\left[\max_{\gamma\in\Gamma^\circ}\sum_{\tau\in Z}\tilde{f}^{I_\tau}(\gamma) - R^Z_{\text{full}}\right] - \iota - \frac{|\Gamma^{\circledcirc}|Z}{T} && \text{(Definition of } R^Z_{\text{full}})
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{Z} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{\tau \in [Z]} \mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\gamma})\right] - \frac{1}{Z} R^Z_{\text{full}} - \iota - \frac{|\Gamma^{\circledcirc}| Z}{T} && \text{(Jensen's inequality)} \\
&\geq \frac{1}{Z} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{\tau \in [Z]} \left(f^{I_\tau}(\boldsymbol{\gamma}) - \iota\right) - \frac{1}{Z} R^Z_{\text{full}} - \iota - \frac{|\Gamma^{\circledcirc}| Z}{T} && \text{(By property (i))} \\
&= \frac{1}{Z} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{\tau \in [Z]} f^{I_\tau}(\boldsymbol{\gamma}) - \iota - \frac{1}{Z} R^Z_{\text{full}} - \iota - \frac{|\Gamma^{\circledcirc}| Z}{T} \\
&= \frac{1}{Z} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \frac{Z}{T} \sum_{\tau \in [Z]} \sum_{t \in I_\tau} f(\boldsymbol{\gamma}, k^t) - \frac{1}{Z} R^Z_{\text{full}} - 2\iota - \frac{|\Gamma^{\circledcirc}| Z}{T} \ \text{(By def. of } f^{I_\tau}(\boldsymbol{\gamma}) \text{ and } |I_\tau| = \frac{T}{Z}) \\
&= \frac{1}{T} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{\tau \in [Z]} \sum_{t \in I_\tau} f(\boldsymbol{\gamma}, k^t) - \frac{1}{Z} R^Z_{\text{full}} - 2\iota - \frac{|\Gamma^{\circledcirc}| Z}{T} \\
&= \frac{1}{T} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{t \in [T]} f(\boldsymbol{\gamma}, k^t) - \frac{1}{Z} R^Z_{\text{full}} - 2\iota - \frac{|\Gamma^{\circledcirc}| Z}{T} = \\
&\geq \frac{1}{T} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{t \in [T]} f(\boldsymbol{\gamma}, k^t) - \frac{1}{Z} O\left(\eta \sqrt{Z \log |\Gamma^{\circ}|}\right) - 2\iota - \frac{|\Gamma^{\circledcirc}| Z}{T} \\
&\geq \frac{1}{T} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{t \in [T]} f(\boldsymbol{\gamma}, k^t) - O\left(\frac{|\Gamma^{\circledcirc}|^{1/3} \eta^{2/3} \log^{1/3} |\Gamma^{\circ}|}{T^{1/3}}\right) - 2\iota - \frac{|\Gamma^{\circledcirc}|^{1/3} \eta^{2/3} \log^{1/3} |\Gamma^{\circ}|}{T^{1/3}} \\
&\geq \frac{1}{T} \max_{\boldsymbol{\gamma} \in \Gamma^{\circ}} \sum_{t \in [T]} f(\boldsymbol{\gamma}, k^t) - O\left(\frac{|\Gamma^{\circledcirc}|^{1/3} \eta^{2/3} \log^{1/3} |\Gamma^{\circ}|}{T^{1/3}}\right) - O(\iota)
\end{aligned}
$$

By using the definition of the regret R^T of Algorithm 9.1, we get the statement. □

lemma 9.3 shows that even if utility estimators have small bias, we can still hope for a no-regret algorithm. However, we have to guarantee that $\Gamma^{\circledcirc}$ has a polynomial size, and that the estimator has a limited range. These requirements can be achieved by estimating sender's utilities indirectly by means of other related estimates, at the cost of giving up on the unbiasedness of the estimators.

The key observation that allows to get the desired estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ by only exploring a polynomially-sized set $\Gamma^{\circledcirc}$ is that the utilities $f^{I_\tau}(\boldsymbol{\gamma})$ that we wish to estimate are *not* independent, but they all depend on the frequency of each receiver's type during block I_τ. Thus, only these (polynomially many) quantities need to be estimated. In order to do so, we use the concept of *barycentric spanners* (Awerbuch and Kleinberg, 2008). A direct application of barycentric spanners to our setting would require being able to induce *any* receiver's posterior during the exploration phase. Unfortunately, this is not possible as the sender is forced to play consistent signaling schemes (see Equation (3.3)), which could prevent her from inducing certain posteriors. We achieve the goal of keeping the bias and the range of the estimators small by adopting the following two technical caveats:

(i) we focus on posteriors that can be induced by a signaling scheme with at least some ('not too small') probability, which ensures that the resulting estimators have a limited range; and

(ii) we restrict the full-information algorithm to signaling schemes $\Gamma^\circ \subseteq \Gamma^\star$ inducing a small number of posteriors, which guarantees to have estimators with a small bias.

Our complete technical results on utilities estimation are provided in the following subsection.

9.5.2 Utilities Estimation

We show in details how to compute the estimates needed by Algorithm 9.1 by using random samples from a polynomially-sized set $\Gamma^{\circledcirc} \subseteq \Gamma^\star$. Let us recall that, during each block I_τ with $\tau \in [Z]$, Algorithm 9.1 needs to compute the estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ of $f^{I_\tau}(\boldsymbol{\gamma}) = \frac{1}{|I_\tau|}\sum_{t \in I_\tau} f(\boldsymbol{\gamma}, k^t)$ for all the signaling schemes $\boldsymbol{\gamma} \in \Gamma^\circ$ (Line 15). Notice that the set $\Gamma^\circ \subseteq \Gamma^\star$ is defined (as shown in Lemma 9.6) in order to be able to build estimators with the desired properties (i) and (ii).

As discussed at the end of the preceding Subsection 9.5.1, the key insight that allows us to get the required estimates by using only a polynomial number of random samples is that the utilities to be estimated are *not* independent. This is because they depend on the frequencies of the receiver's actions during bock I_τ, which depend, in turn, on the frequencies of the receiver's types. Thus, the goal is to devise estimators for the frequencies of the receiver's types during each block I_τ. As an intuition, imagine that the sender commits to a signaling scheme such that each receiver's type best responds by playing a different action. Then, by observing the receiver's action, the sender gets to know the receiver's type with certainty. In general, for a given signaling scheme, there might be many different receiver's types that are better off playing the same action. In order to handle this problem and build the required estimates of the frequencies of the receiver's types, we use insights from the *bandit linear optimization* literature, and, in particular, we use the concept of *barycentric spanner* introduced by Awerbuch and Kleinberg (2008).

For every block I_τ with $\tau \in [Z]$, we let $g_\tau : [0,1]^m \to \mathbb{R}$ be a function that, given a vector $\mathbf{x} = [x_1, \ldots, x_m] \in [0,1]^m$, returns the sum of the number of times the receiver's types in $\mathcal{K}$ were active during block I_τ weighted by the coefficients defined by the vector $\mathbf{x}$. Formally, the following definition holds:

$$g_\tau(\mathbf{x}) \coloneqq \sum_{k \in \mathcal{K}} x_k \sum_{t \in B_\tau} \mathbb{I}[k^t = k],$$

where $\mathbb{I}[k^t = k]$ is an indicator function that is equal to 1 if and only if it is the case that $k^t = k$, while it is 0 otherwise. Notice that, for a given $\tau \in [Z]$ and $k \in \mathcal{K}$, the term $\sum_{t \in B_\tau} \mathbb{I}[k^t = k]$ is a constant, and, thus, the function g_τ is linear. Intuitively, g_τ is the key element that allows us to connect the utilities that we need to estimate with the actual quantities we can estimate through the use of barycentric spanners.

The first crucial step is to restrict the attention to posteriors that can be induced with at least some ('not too small') probability. This ensures that our estimators have a limited range. Given a probability threshold $\sigma \in (0, 1)$, we denote with $\Xi^\circledcirc \subseteq \Xi$ the set of posteriors that can be induced with probability at least σ by some signaling scheme. We can verify whether a given posterior $\boldsymbol{\xi} \in \Xi$ belongs to $\Xi^\circledcirc$ by solving an LP. Formally, $\boldsymbol{\xi} \in \Xi^\circledcirc$ if and only if the following set of linear equations admits a feasible solution $\boldsymbol{\gamma} \in \Delta_\Xi$:

$$\gamma_{\boldsymbol{\xi}} \geq \sigma \tag{9.8a}$$

$$\sum_{\boldsymbol{\xi} \in \Xi} \gamma_{\boldsymbol{\xi}} \boldsymbol{\xi}_\theta = \mu_\theta \qquad \forall \theta \in \Theta. \tag{9.8b}$$

We define $\mathcal{Q}$ as the set of all the tuples $\mathbf{a} = (a_k)_{k \in \mathcal{K}} \in \times_{k \in \mathcal{K}} \mathcal{A}$ for which there exists a posterior $\boldsymbol{\xi} \in \Xi^\circledcirc$ such that, for every receiver's type $k \in \mathcal{K}$, the action a_k specified by the tuple is a best response to $\boldsymbol{\xi}$ for type k. Formally:

$$\mathcal{Q} \coloneqq \bigcup_{\boldsymbol{\xi} \in \Xi^\circledcirc} \left(b^1_{\boldsymbol{\xi}}, \ldots, b^m_{\boldsymbol{\xi}}\right),$$

where we recall that $b^k_{\boldsymbol{\xi}}$ denotes the best response of type $k \in \mathcal{K}$ under posterior $\boldsymbol{\xi}$. Intuitively, $\mathcal{Q}$ is the set of tuples of receiver's best responses which result from the posteriors that the sender can induce with probability at least σ.[5]

Given a tuple $\mathbf{a} = (a_k)_{k \in \mathcal{K}} \in \mathcal{Q}$ and a receiver's action $a \in \mathcal{A}$, we denote with $\mathbb{I}_{(\mathbf{a}=a)} \in \{0, 1\}^m$ an indicator vector whose k-th component is equal to 1 if and only if type $k \in \mathcal{K}$ plays action a in $\mathbf{a}$, *i.e.*, it holds $a_k = a$.

[5] Let us remark that the sets $\Xi^\circledcirc$ and $\mathcal{Q}$ depend on the given threshold $\sigma \in (0, 1)$. In the following, for the ease of notation, we omit such dependence, as the actual value of σ that the two sets refer to will be clear from context.

Moreover, we define $\mathcal{X}$ as the set of all the indicators vectors; formally, $\mathcal{X} := \{\mathbb{I}_{(\mathbf{a}=a)} \mid \mathbf{a} \in \mathcal{Q}, a \in \mathcal{A}\}$.

Since the set $\mathcal{X}$ is a finite (and hence compact) subset of the Euclidean space $\mathbb{R}^m$, we can use the following proposition due to Awerbuch and Kleinberg (2008) to introduce the *barycentric spanner* of $\mathcal{X}$.

Proposition 9.1 (Awerbuch and Kleinberg (2008), Proposition 2.2)**.** *If $\mathcal{X}$ is a compact subset of an m-dimensional vector space $\mathcal{V}$, then there exists a set $\mathcal{H} = \{\mathbf{h}^1, ..., \mathbf{h}^m\} \subseteq \mathcal{X}$ such that for all $\mathbf{x} \in \mathcal{X}$, $\mathbf{x}$ may be expressed as a linear combination of elements of $\mathcal{H}$ using coefficients in $[-1, 1]$. That is, for all $\mathbf{x} \in \mathcal{X}$, there exists a vector of coefficients $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m] \in [-1, 1]^m$ such that $\mathbf{x} = \sum_{i \in [m]} \lambda_i \mathbf{h}^i$. The set $\mathcal{H}$ is called* barycentric spanner *of $\mathcal{X}$.*

In the following, we denote with $\mathcal{H} := \{\mathbf{h}^1, ..., \mathbf{h}^m\} \subseteq \mathcal{X}$ a barycentric spanner of $\mathcal{X}$. Notice that, since each element $\mathbf{h} \in \mathcal{H}$ of the barycentric spanner belongs to $\mathcal{X}$ by definition, there exist a tuple $\mathbf{a} \in \mathcal{Q}$ and a receiver's action $a \in \mathcal{A}$ such that $\mathbf{h}$ is equal to the indicator vector $\mathbb{I}_{(\mathbf{a}=a)}$. Moreover, by definition of $\mathcal{Q}$, there exists a posterior $\boldsymbol{\xi} \in \Xi^{\circledcirc}$ such that the tuple of best responses $\left(b^1_{\boldsymbol{\xi}}, \ldots, b^m_{\boldsymbol{\xi}}\right)$ coincides with $\mathbf{a}$.

Next, we describe how Algorithm 9.1 computes the required estimates. During the exploration phase of block I_τ with $\tau \in [Z]$, one iteration is devoted to each element $\mathbf{h} \in \mathcal{H}$ of the barycentric spanner, so as to get an estimate of $g_\tau(\mathbf{h})$. During such iteration, the algorithm plays a signaling scheme $\boldsymbol{\gamma} \in \Delta_\Xi$ that is feasible for the LP defined by Constraints (9.8) where the posterior $\boldsymbol{\xi} \in \Xi^{\circledcirc}$ is that associated to $\mathbf{h}$. This means that the set of all such signaling schemes is used as $\Gamma^{\circledcirc}$ in Algorithm 9.1. Moreover, when the induced receiver's posterior is $\boldsymbol{\xi}$ and the receiver responds by playing action a, the algorithm sets a variable $p_\tau(\mathbf{h})$ to the value $\frac{1}{\gamma_{\boldsymbol{\xi}}}$, otherwise $p_\tau(\mathbf{h})$ is set to 0.

The following lemma shows that the variables $p_\tau(\mathbf{h})$ computed by the algorithm during each block I_τ with $\tau \in [Z]$ are unbiased estimates of the values $g_\tau(\mathbf{h})$.

Lemma 9.4. *For any $\tau \in [Z]$ and $\mathbf{h} \in \mathcal{H}$, it holds $\mathbb{E}\left[p_\tau(\mathbf{h}) \cdot |I_\tau|\right] = g_\tau(\mathbf{h})$.*

Proof. First, recall that $p_\tau(\mathbf{h}) = \frac{1}{\gamma_{\boldsymbol{\xi}}}$ if and only if during the iteration of exploration devoted to $\mathbf{h}$, the induced receiver's posterior is $\boldsymbol{\xi}$ and she/he best responds by playing a (otherwise, $p_\tau(\mathbf{h}) = 0$). Since the iteration is selected uniformly at random over the block I_τ and the sequence of receiver's types $\mathbf{k} = \{k^t\}_{t \in [T]}$ is chosen adversarially before the beginning of the game, we can conclude that also the receiver's type for that iteration is picked uniformly at random. Thus,

$$\mathbb{E}\left[p_\tau(\mathbf{h})\right] = \frac{1}{\gamma_{\boldsymbol{\xi}}} \cdot \gamma_{\boldsymbol{\xi}} \cdot \mathbb{P}\Big\{\text{randomly chosen type from } I_\tau \text{ best responds to } \boldsymbol{\xi} \\ \text{consistently with } \mathbf{h}\Big\},$$

where by best responding consistently we mean that the type $k \in \mathcal{K}$ is such that $h_k = 1$, *i.e.*, she plays action a in $\mathbf{a}$. By using the definition of $g_\tau(\mathbf{h})$, we can write the following:

$$\mathbb{E}\left[p_\tau(\mathbf{h})\right] = \frac{\sum_{k \in \mathcal{K}: h_k = 1} g_\tau(\mathbf{e}^k)}{|I_\tau|} = \frac{g_\tau(\mathbf{h})}{|I_\tau|},$$

where $\mathbf{e}^k \in \mathbb{R}^m$ denotes an m-dimensional vector whose k-th component is 1, while others components are 0. □

For any $\mathbf{x} \in \mathcal{X}$, we let $\boldsymbol{\lambda}(\mathbf{x}) = [\lambda_1(\mathbf{x}), \ldots, \lambda_m(\mathbf{x})] \in [-1, 1]^m$ be the vector of coefficients representing $\mathbf{x}$ with respect to basis $\mathcal{H}$. Formally, we can write $\mathbf{x} = \sum_{i \in [m]} \lambda_i(\mathbf{x})\mathbf{h}^i$.

For any posterior $\boldsymbol{\xi} \in \Xi^{\circledcirc}$, let $\mathbf{a}[\boldsymbol{\xi}] \in \mathcal{Q}$ be such that $\mathbf{a}[\boldsymbol{\xi}] = \left(b^1_{\boldsymbol{\xi}}, \ldots, b^m_{\boldsymbol{\xi}}\right)$. Then, for each $\tau \in [Z]$, let us define

$$\tilde{f}^{I_\tau}(\boldsymbol{\xi}) := \sum_{a \in \mathcal{A}} \sum_{k \in \mathcal{K}} \lambda_k \left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}] = a)}\right) p_\tau\left(\mathbf{h}^k\right) \sum_{\theta \in \Theta} \xi_\theta f_\theta(a).$$

Letting $f^{I_\tau}(\boldsymbol{\xi}) := \frac{1}{|I_\tau|} \sum_{t \in \tau} f(\boldsymbol{\xi}, k^t)$ be the sender's average utility achieved by inducing the receiver's posterior $\boldsymbol{\xi} \in \Xi^{\circledcirc}$ during each iteration of block I_τ with $\tau \in [Z]$, the following lemma shows that $\tilde{f}^{I_\tau}(\boldsymbol{\xi})$ is an unbiased estimator of $f^{I_\tau}(\boldsymbol{\xi})$, and, additionally, the range in which the estimator values lie is not to large.

Lemma 9.5. *For any posterior $\boldsymbol{\xi} \in \Xi^{\circledcirc}$ and $\tau \in [Z]$, it holds $\mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\xi})\right] = f^{I_\tau}(\boldsymbol{\xi})$. Moreover, $\tilde{f}^{I_\tau}(\boldsymbol{\xi}) \in [-\frac{\varrho m}{\sigma}, \frac{\varrho m}{\sigma}]$.*

Proof. The first statement follows from the following relations:

$$\begin{aligned}\mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\xi})\right] &= \mathbb{E}\left[\sum_{a \in \mathcal{A}} \sum_{k \in \mathcal{K}} \lambda_k \left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}] = a)}\right) p_\tau\left(\mathbf{h}^k\right) \sum_{\theta \in \Theta} \xi_\theta f_\theta(a)\right] \\ &= \sum_{a \in \mathcal{A}} \sum_{k \in \mathcal{K}} \lambda_k \left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}] = a)}\right) \mathbb{E}\left[p_\tau\left(\mathbf{h}^k\right)\right] \sum_{\theta \in \Theta} \xi_\theta f_\theta(a)\end{aligned}$$

$$= \sum_{a \in \mathcal{A}} \sum_{\theta \in \Theta} \xi_\theta f^\theta(a) \sum_{k \in \mathcal{K}} \lambda_k \left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}]=a)}\right) \mathbb{E}\left[p_\tau\left(\mathbf{h}^k\right)\right]$$

$$= \sum_{a \in \mathcal{A}} \sum_{\theta \in \Theta} \xi_\theta f_\theta(a) \sum_{k \in \mathcal{K}} \lambda_k \left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}]=a)}\right) \frac{g_\tau\left(\mathbf{h}^k\right)}{|I_\tau|} \qquad \text{(By Lemma 9.4)}$$

$$= \sum_{a \in \mathcal{A}} \sum_{\theta \in \Theta} \xi_\theta f_\theta(a) \sum_{k \in \mathcal{K}} \frac{g_\tau\left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}]=a)}\right)}{|I_\tau|} \qquad \text{(By definition of } g_\tau\text{)}$$

$$= f^{I_\tau}(\boldsymbol{\xi}),$$

where the last equality holds by using again the definition of g_τ and rearranging the terms.

As for the second statement, since $\lambda_k\left(\mathbb{I}_{(\mathbf{a}[\boldsymbol{\xi}]=a)}\right) \in [-1,1]$, it holds $\sum_{\theta \in \Theta} \xi_\theta f_\theta(a) \in [0,1]$, and $p_\tau\left(\mathbf{h}^k\right) \in \left[0, \frac{1}{\sigma}\right]$, it is easy to show that $\tilde{f}^{I_\tau}(\boldsymbol{\xi}) \in [-\frac{\varrho m}{\sigma}, \frac{\varrho m}{\sigma}]$. □

In the next lemma, we show that there always exists a best-in-hindsight signaling scheme that uses (*i.e.*, induces with positive probability) only a small number of posteriors. This is the final step needed to show that the estimators $\tilde{f}^{I_\tau}(\boldsymbol{\xi})$ allow to compute slightly biased estimates of the utilities needed by the full-information algorithm.

Lemma 9.6. *Given a sequence of receiver's types* $\mathbf{k} = \{k^t\}_{t \in [T]}$*, there always exists a best-in-hindsight signaling scheme* $\boldsymbol{\gamma}^\star \in \Gamma^\star$ *such that the set of posteriors it induces with positive probability* $\left\{\boldsymbol{\xi} \in \Xi \mid \gamma^\star_{\boldsymbol{\xi}} > 0\right\}$ *has cardinality at most the number of states* d*.*

Proof. By Lemma 9.1, there always exists an optimal signaling scheme in $\Gamma^\star$. Notice that a best-in-hindsight signaling scheme $\boldsymbol{\gamma}^\star$ in $\Gamma^\star$ can be computed by solving the following LP:

$$\max_{\boldsymbol{\gamma} \in \Gamma^\star} \quad \sum_{t \in [T]} \sum_{\boldsymbol{\xi} \in \hat{\Xi}} \gamma_{\boldsymbol{\xi}} f(\boldsymbol{\gamma}, k^t)$$

$$\text{s.t.} \quad \sum_{\boldsymbol{\xi} \in \hat{\Xi}} \gamma_{\boldsymbol{\xi}} \xi_\theta = \mu_\theta \qquad \forall \theta \in \Theta.$$

Since the LP has d equalities, it always admits an optimal basic, *i.e.*, a vertex of $\hat{\Gamma}$, feasible solution in which at most d variables $\gamma_{\boldsymbol{\xi}}$ are greater than 0. This concludes the proof. □

Then, we define the Γ° used by Algorithm 9.1 as the set of signaling schemes $\boldsymbol{\gamma} \in \Gamma^\star$ whose support is at most d, *i.e.*, it is the case that

$|\{\boldsymbol{\xi} \in \Xi \mid \gamma_{\boldsymbol{\xi}} > 0\}| \leq d$. By Lemma 9.6, a best-in-hindsight signaling scheme is always guaranteed to be in the set Γ°.

Letting $\tilde{f}^{I_\tau}(\boldsymbol{\gamma}) := \sum_{\boldsymbol{\xi} \in \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} \tilde{f}^{I_\tau}(\boldsymbol{\xi})$ for every $\boldsymbol{\gamma} \in \Gamma^\circ$ and $\tau \in [Z]$, the following lemma shows that each $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ is a biased estimator of the sender's average utility $f^{I_\tau}(\boldsymbol{\gamma})$ in block I_τ, while also providing bounds on the bias and the range of the estimators. This final result allows us to effectively use the estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ defined above in Algorithm 9.1.

Lemma 9.7. *For any signaling scheme $\boldsymbol{\gamma} \in \Gamma^\circ$ and $\tau \in [Z]$, it holds $f^{I_\tau}(\boldsymbol{\gamma}) \geq \mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\gamma})\right] \geq f^{I_\tau}(\boldsymbol{\gamma}) - d\sigma$. Moreover, it is the case that $\tilde{f}^{I_\tau}(\boldsymbol{\gamma}) \in [-\frac{\varrho m}{\sigma}, \frac{\varrho m}{\sigma}]$.*

Proof. By using Lemma 9.5, it is easy to check that the left inequality in the first statement holds:

$$f^{I_\tau}(\boldsymbol{\gamma}) = \sum_{\boldsymbol{\xi} \in \Xi} \gamma_{\boldsymbol{\xi}} f^{I_\tau}(\boldsymbol{\xi}) \geq \sum_{\boldsymbol{\xi} \in \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} f^{I_\tau}(\boldsymbol{\xi}) = \sum_{\boldsymbol{\xi} \in \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} \mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\xi})\right] = \mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\gamma})\right].$$

Moreover, it is the case that:

$$\begin{aligned}
\mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\gamma})\right] &= \sum_{\boldsymbol{\xi} \in \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} \mathbb{E}\left[\tilde{f}^{I_\tau}(\boldsymbol{\xi})\right] \\
&= \sum_{\boldsymbol{\xi} \in \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} f_{I_\tau}(\boldsymbol{\xi}) && \text{(By Lemma 9.5)} \\
&= f^{I_\tau}(\boldsymbol{\gamma}) - \sum_{\boldsymbol{\xi} \in \Xi \setminus \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} f^{I_\tau}(\boldsymbol{\xi}) && \text{(By definition of } f^{I_\tau}(\boldsymbol{\gamma})\text{)} \\
&\geq f^{I_\tau}(\boldsymbol{\gamma}) - \sum_{\boldsymbol{\xi} \in \Xi \setminus \Xi^\circledcirc} \gamma_{\boldsymbol{\xi}} && \text{(Since } f^{I_\tau}(\boldsymbol{\gamma}) \leq 1\text{)} \\
&\geq f^{I_\tau}(\boldsymbol{\gamma}) - \sum_{\boldsymbol{\xi} \in \Xi \setminus \Xi^\circledcirc} \sigma && \text{(By definition of } \Xi^\circledcirc\text{, it must be } \gamma_{\boldsymbol{\xi}} < \sigma\text{)} \\
&\geq f^{I_\tau}(\boldsymbol{\gamma}) - d\sigma && \text{(Since } \boldsymbol{\gamma} \in \Gamma^\circ\text{)}
\end{aligned}$$

Finally, $\tilde{f}^{I_\tau}(\boldsymbol{\gamma}) \in [-\frac{\varrho m}{\sigma}, \frac{\varrho m}{\sigma}]$ follows from the fact that, by definition, $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ is the weighted sum of quantities within the range $[-\frac{\varrho m}{\sigma}, \frac{\varrho m}{\sigma}]$, with the weights sum being at most 1. □

Finally, we can prove the following theorem.

Theorem 9.5. *Given an online Bayesian persuasion problem with partial feedback, there exist $\Gamma^\circ \subseteq \Gamma^\star$, $\Gamma^\circledcirc \subseteq \Gamma^\star$, and estimators $\tilde{f}^{I_\tau}(\boldsymbol{\gamma})$ such that*

Algorithm 9.1 provides the following regret bound:

$$R^T \leq O\left(\frac{m\varrho^{2/3} d \log^{1/3}(\varrho m + d)}{T^{1/5}}\right).$$

Proof. By setting $\sigma := d^{-2/5}T^{-1/5}$, it is sufficient to run Algorithm 9.1 with estimators $f^{I_\tau}(\boldsymbol{\gamma})$ for every $\boldsymbol{\gamma} \in \Gamma^\circ$ computed as previously described in this section. Thus, it holds $|\Gamma^{\circledcirc}| = m$ and $\eta = \varrho m d^{2/5} T^{1/5}$. By Theorem 9.3, the following holds:

$$\begin{aligned}
R^T &\leq O\left(\frac{|\Gamma^{\circledcirc}|^{1/3}\eta^{2/3} log^{1/3}|\Gamma^\circ|}{T^{1/3}}\right) + O(\iota) \\
&= O\left(\frac{m^{1/3}\left(\varrho m d^{2/5} T^{1/5}\right)^{2/3} \log^{1/3}|\Gamma^\circ|}{T^{1/3}}\right) + O\left(\frac{d}{d^{2/5}T^{1/5}}\right) \\
&= O\left(\frac{m\varrho^{2/3} d^{4/15}\left(d\log\left(\varrho^2 m + d\right)\right)^{1/3}}{T^{1/5}}\right) + O\left(\frac{d^{3/5}}{T^{1/5}}\right) \\
&= O\left(\frac{m\varrho^{2/3} d^{3/5} \log^{1/3}(\varrho m + d)}{T^{1/5}}\right).
\end{aligned}$$

This concludes the proof.

□

9.6 A No-α-regret Algorithm for ϵ-persuasive Signaling Schemes

Theorem 9.3 shows that, for all $\alpha < 1$, there is no polynomial-time algorithm for the online Bayesian persuasion problem guaranteeing no-α-regret, unless NP $\subseteq$ RP. This implies that achieving sublinear regret with a polynomial-time algorithm is unlikely. Section 9.4 and 9.5 described no-regret algorithms for the full information and partial information feedback requiring an exponential per-iteration running time. In this section we focus on the following natural question: *is it possible to design an algorithm with a better running time, at the cost of relaxing the persuasiveness constraints?*

We consider the notion of regret defined in Equation (9.1). When computing R^T_ϵ, we compare the performance of the best-in-hindsight persuasive signaling scheme with the sequence of ϵ-persuasive signaling schemes computed via the online algorithm. Then, we are interested in online algorithms guaranteeing, for any $\alpha > 0$ and $\epsilon > 0$, the no-α-regret property.

Specifically, there must be a constant $c > 0$ such that, after T iterations, it holds: $R_\epsilon^T \leq \alpha + \frac{1}{T^c}\mathsf{poly}(m, \varrho, d)$. We show that in many settings it is possible to devise an online algorithm for the online Bayesian persuasion problem exhibiting the no-α-regret property for each constant α that works in polynomial time in the size of the problem. In particular, we provide an algorithm that works in time quasi-polynomial in the number of receiver'actions. Hence, assuming that the number of receiver'actions is fixed, the algorithm runs in polynomial time. In many applications, it is oftentimes enough to set $\varrho = 2$. This is the case, for example, in common voting problems (Alonso and Câmara, 2016) and product marketing problems (Babichenko and Barman, 2017; Candogan, 2019). Notice that Theorem 9.3 shows that even with three actions it is computationally intractable to compute a sequence of *persuasive* signaling scheme with no-α-regret for each $\alpha < 1$. The main result reads as follows:

Theorem 9.6. *Given a Bayesian persuasion problem with partial feedback, for any $\alpha > 0$ and $\epsilon > 0$, there exists a* poly $\left(Tmd^{\frac{\log(\varrho/\alpha)}{\epsilon^2}}\right)$ *time algorithm such that:*

$$R_\epsilon^T \leq O\left(\frac{d^{log(\varrho/\alpha)/\epsilon^2}}{\sqrt{T}}\right) + \alpha.$$

As a corollary, for each constant α and ϵ, we obtain a polynomial-time no- α-regret algorithm when the number of actions ϱ is fixed.

Corollary 9.1. *Given a Bayesian persuasion problem with partial feedback, for any constant number of actions $\varrho \geq 1$, constant $\alpha > 0$, and constant $\epsilon > 0$, there exists a polynomial-time algorithm such that:*

$$R_\epsilon^T \leq \frac{\text{poly}(m, d)}{\sqrt{T}} + \alpha.$$

In order to prove Theorem 9.6, we need to introduce some additional machinery and to prove two fundamentals auxiliary results. We follow a reasoning similar to that of Section 9.4. Specifically, we show that there exists a set of signaling schemes of size quasi-polynomial in the number of actions and polynomial in the instance size that for each possible sequence of receiver' types includes a signaling scheme nearly as good as the optimal one.

Let Ξ^q be the set of q-uniform posteriors and recall that $|\Xi^q| \in O(d^q)$. We show that, in order to prove Theorem 9.6, it is enough to set $q = \frac{2\log(\varrho/\alpha)}{\epsilon^2}$, and to limit the sender's action space to signaling schemes defined over posteriors in Ξ^q. The following lemma shows that for any $\epsilon > 0$

and $\alpha > 0$, any posterior $\boldsymbol{\xi}^* \in \Xi$ can be represented as a convex combination of elements of Ξ^q which guarantees a sender's expected utility (when the receiver is ϵ-best responding) at most distant by α from the expected utility at $\boldsymbol{\xi}^*$.

Lemma 9.8. *For each $\epsilon > 0$ and $\alpha > 0$, and each posterior $\boldsymbol{\xi}^* \in \Xi$, there exists a $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$, with $q = \frac{2\log(\varrho/\alpha)}{\epsilon^2}$, such that: for each $\theta \in \Theta$,*

$$\sum_{\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}} \xi_\theta = \xi^*_\theta,$$

and, for each type $k \in \mathcal{K}$,

$$\sum_{\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}} f^\epsilon(\boldsymbol{\xi}, k) \geq f(\boldsymbol{\xi}^*, k) - \alpha, \tag{9.9}$$

where $|\Gamma^*| = d^{\frac{2log(\varrho/\alpha)}{\epsilon^2}}$.

Proof. Let $\tilde{\boldsymbol{\xi}} \in \Xi^q$ be the empirical distribution of q i.i.d. samples drawn according to $\boldsymbol{\xi}^*$, where each $\theta \in \Theta$ has probability ξ^*_θ of being sampled. Therefore, the vector $\tilde{\boldsymbol{\xi}}$ is a random variable supported on q-uniform posteriors with expectation $\boldsymbol{\xi}^*$. Moreover, let $\boldsymbol{\gamma} \in \Delta_{\Xi^q}$ be a probability distribution such as, for each $\boldsymbol{\xi} \in \Xi^q$, $\gamma_{\boldsymbol{\xi}} := \Pr(\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi})$.

Given an arbitrary $k \in \mathcal{K}$, we show that $\boldsymbol{\gamma}$ satisfies Equation (9.9). Let $\Xi^{q,\epsilon}$ be the set of posteriors such that $\boldsymbol{\xi} \in \Xi^{q,\epsilon}$ iff, for each $a \in \mathcal{A}$, it holds:

$$\left| \sum_{\theta \in \Theta} \left(\xi_\theta u^k_\theta(a) - \xi^*_\theta u^k_\theta(a) \right) \right| \leq \frac{\epsilon}{2}. \tag{9.10}$$

Then, for each $\boldsymbol{\xi} \in \Xi^{q,\epsilon}$, we have that $\mathcal{B}^k_{\boldsymbol{\xi}^*} \subseteq \mathcal{B}^k_{\epsilon,\boldsymbol{\xi}}$. In particular, for any $a^* \in \mathcal{B}^k_{\boldsymbol{\xi}^*}$, $\boldsymbol{\xi} \in \Xi^{q,\epsilon}$ and $a \in \mathcal{A}$:

$$\begin{aligned}
\sum_{\theta \in \Theta} \xi_\theta u^k_\theta(a^*) &\geq \sum_{\theta \in \Theta} \xi^*_\theta u^k_\theta(a^*) - \frac{\epsilon}{2} && \text{(By Eq. (9.10) and the definition of } \mathcal{B}^k_{\boldsymbol{\xi}^*}\text{)} \\
&\geq \sum_{\theta \in \Theta} \xi^*_\theta u^k_\theta(a) - \frac{\epsilon}{2} && \text{(By Definition 9.2)} \\
&\geq \sum_{\theta \in \Theta} \xi_\theta u^k_\theta(a) - \epsilon && \text{(By Equation (9.10))}
\end{aligned}$$

which is precisely the definition of $\mathcal{B}^k_{\epsilon,\boldsymbol{\xi}}$.

For each $a \in \mathcal{A}$, let $\tilde{t}_a^k := \sum_{\theta \in \Theta} \tilde{\xi}_\theta u_\theta^k(a)$ and $t_a^k := \sum_{\theta \in \Theta} \xi_\theta^* u_\theta^k(a)$. By the Hoeffding's inequality we have that, for each $a \in \mathcal{A}$,

$$\Pr\Big(\big|\tilde{t}_a^k - \mathbb{E}[\tilde{t}_a^k]\big| \geq \frac{\epsilon}{2}\Big) \leq 2e^{-2q(\epsilon/2)^2} = 2e^{-\log(\varrho/\alpha)} \leq \frac{\alpha}{\varrho}. \tag{9.11}$$

Moreover, Equation (9.10) and the union bound yield the following:

$$\begin{aligned}
\sum_{\boldsymbol{\xi} \in \Xi^{q,\epsilon}} \gamma_{\boldsymbol{\xi}} &= \Pr(\tilde{\xi} \in \Xi^{q,\epsilon}) \\
&= \Pr\left(\bigcap_{a \in \mathcal{A}} \big|\tilde{t}_a^k - t_a^k\big| \leq \frac{\epsilon}{2}\right) \\
&\geq 1 - \sum_{a \in \mathcal{A}} \Pr\Big(\big|\tilde{t}_a^k - t_a^k\big| \geq \frac{\epsilon}{2}\Big) \\
&\geq 1 - \alpha. && \text{(By Equation (9.11))}
\end{aligned}$$

Let $\bar{\boldsymbol{\alpha}}$ be a d-dimensional vector defined as $\bar{\alpha}_\theta := \sum_{\boldsymbol{\xi} \in \Xi^q \setminus \Xi^{q,\epsilon}} \gamma_{\boldsymbol{\xi}} \xi_\theta$. By definition and for the previous result we have: $\sum_{\theta \in \Theta} \bar{\alpha}_\theta \leq \alpha$.

Finally, we can show that Equation (9.9) is satisfied:

$$\begin{aligned}
\sum_{\boldsymbol{\xi} \in \Xi_q} \gamma_{\boldsymbol{\xi}} f^\epsilon(\boldsymbol{\xi}, k) &\geq \sum_{\boldsymbol{\xi} \in \Xi^{q,\epsilon}} \gamma_{\boldsymbol{\xi}} f^\epsilon(\boldsymbol{\xi}, k) && (\Xi^{q,\epsilon} \subseteq \Xi^q) \\
&= \sum_{\boldsymbol{\xi} \in \Xi^{q,\epsilon}} \gamma_{\boldsymbol{\xi}} \left(\sum_{\theta \in \Theta} \xi_\theta f_\theta(b_{\epsilon,\boldsymbol{\xi}}^k)\right) \\
&\geq \sum_{\boldsymbol{\xi} \in \Xi^{q,\epsilon}} \gamma_{\boldsymbol{\xi}} \left(\sum_{\theta \in \Theta} \xi_\theta f_\theta(b_{\boldsymbol{\xi}^*}^k)\right) && (\mathcal{B}_{\boldsymbol{\xi}^*}^k \subseteq \mathcal{B}_{\epsilon,\boldsymbol{\xi}}^k \text{ for each } \boldsymbol{\xi} \in \Xi^{q,\epsilon}) \\
1 &= \sum_{\theta \in \Theta} f_\theta(b_{\boldsymbol{\xi}^*}^k) \left(\sum_{\boldsymbol{\xi} \in \Xi^{q,\epsilon}} \gamma_{\boldsymbol{\xi}} \xi_\theta\right) \\
&= \sum_{\theta \in \Theta} f_\theta(b_{\boldsymbol{\xi}^*}^k) \left(\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \xi_\theta - \bar{\alpha}_\theta\right) && \text{(By definition of } \bar{\boldsymbol{\alpha}}) \\
&= \sum_{\theta \in \Theta} f_\theta(b_{\boldsymbol{\xi}^*}^k) \left(\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \xi_\theta\right) - \sum_{\theta \in \Theta} u_\theta^{\mathsf{s}}(b_{\boldsymbol{\xi}^*}^k) \bar{\alpha}_\theta \\
&\geq \sum_{\theta \in \Theta} f_\theta(b_{\boldsymbol{\xi}^*}^k) \left(\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \xi_\theta\right) - \sum_{\theta \in \Theta} \bar{\alpha}_\theta && \text{(Utilities in } [0,1])
\end{aligned}$$

$$\geq \sum_{\theta \in \Theta} f_\theta(b^k_{\boldsymbol{\xi}^*}) \left(\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \xi_\theta \right) - \alpha.$$

By definition of $\boldsymbol{\gamma}$, we have that, for each $\theta \in \Theta$:

$$\sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \xi_\theta = \xi^*_\theta.$$

Then, the above implies that:

$$\sum_{\boldsymbol{\xi} \in \Xi_q} \gamma_{\boldsymbol{\xi}} f^\epsilon(\boldsymbol{\xi}, k) \geq \sum_{\theta \in \Theta} \xi^*_\theta f_\theta(b^k_{\boldsymbol{\xi}^*}) - \alpha = f(\boldsymbol{\xi}^*, k) - \alpha.$$

This concludes the proof. □

We showed that for any $\epsilon > 0$ and $\alpha > 0$, any posterior $\boldsymbol{\xi}^* \in \Xi$ can be represented as a convex combination of elements of Ξ^q which guarantees a sender's expected utility (when the receiver is ϵ-best responding) at most distant by α from the expected utility at $\boldsymbol{\xi}^*$. We exploit this result to show that there exists a set of signaling schemes with quasi-polynomial size which guarantees ϵ-persuasiveness as well as near optimal sender's expected utility. This set is defined analogously to what we did in Equations (9.2) and (9.3), with the only difference that now the set of consistent distributions is built starting from Δ_{Ξ^q}. In particular,

$$\Gamma^* := V\left(\left\{ \boldsymbol{\gamma} \in \Delta_{\Xi^q} \mid \sum_{\boldsymbol{\xi} \in \Xi^q} \gamma_{\boldsymbol{\xi}} \xi_\theta = \mu_\theta, \quad \forall \theta \in \Theta \right\} \right). \tag{9.12}$$

Therefore, the action-space Γ^* is defined as the set of extreme points of the convex polytope obtained by intersecting Δ_{Ξ^q} with d half-spaces corresponding to consistency constraints. By construction, we have that $|\Gamma^*| \in O(d^q)$. We characterize Γ^* via the following lemma:

Lemma 9.9. *For each sequence of receivers $\boldsymbol{k} = \{k^t\}_{t \in [T]}$,*

$$\max_{\boldsymbol{\gamma} \in \Gamma} \sum_{t \in [T]} f(\boldsymbol{\gamma}, k^t) - \max_{\boldsymbol{\gamma}^* \in \Gamma^*} \sum_{t \in [T]} f^\epsilon(\boldsymbol{\gamma}^*, k^t) \leq \alpha T.$$

Proof. Given an arbitrary $\boldsymbol{\gamma} \in \Gamma$, each posterior $\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})$ can be rewritten according to Lemma 9.8 as a convex combinations of q-uniform posteriors, which we denote by $\boldsymbol{\gamma}^{\boldsymbol{\xi}} \in \Delta_{\Xi^q}$. Let $\boldsymbol{\gamma}^* \in \Delta_{\Xi^q}$ be a signaling scheme such that, for each $\boldsymbol{\xi}' \in \Xi^q$, it holds:

$$\gamma^*_{\boldsymbol{\xi}'} = \sum_{\boldsymbol{\xi} \in \text{supp}(\boldsymbol{\gamma})} \gamma_{\boldsymbol{\xi}} \gamma^{\boldsymbol{\xi}}_{\boldsymbol{\xi}'}.$$

The signaling scheme γ^* is consistent by construction. Moreover, the following holds:

$$\begin{aligned}
\sum_{t\in[T]} f(\gamma, k^t) &= \sum_{k\in\mathcal{K}} \sum_{\substack{t\in[T]:\\ k^t=k}} \left(\sum_{\xi\in\Xi} w_{\xi} \left(\sum_{\theta\in\Theta} \xi_\theta f_\theta(b^k_{\xi}) \right) \right) \\
& \qquad \text{(By Lemma 9.8)} \\
&\le \sum_{k\in\mathcal{K}} \sum_{\substack{t\in[T]:\\ k^t=k}} \left(\sum_{\xi\in\Xi} w_{\xi} \left(\sum_{\xi'\in\Xi^q} w^{\xi}_{\xi'} \left(\sum_{\theta\in\Theta} \xi'_\theta f_\theta(b^k_{\xi'}) \right) + \alpha \right) \right) \\
&= \sum_{k\in\mathcal{K}} \sum_{\substack{t\in[T]:\\ k^t=k}} \left(\sum_{\xi'\in\Xi^q} w^*_{\xi'} \left(\sum_{\theta\in\Theta} \xi'_\theta f_\theta(b^k_{\xi'}) \right) + \alpha \right) \\
&= \sum_{k\in\mathcal{K}} \sum_{\substack{t\in[T]:\\ k^t=k}} \left(f(\gamma^*, k^t) + \alpha \right) \\
&= \sum_{t\in[T]} f(\gamma^*, k^t) + \alpha T.
\end{aligned}$$

Then, there exists an ϵ-persuasive signaling scheme γ^* which is consistent and belongs to Δ_{Ξ^q} while satisfying the lemma. Since $\sum_{t\in[T]} f(\gamma^*, k^t)$ is a linear function of γ^*, its maximum is attained precisely at one of the extreme points of the set of consistent signaling schemes in Δ_{Ξ^q}, *i.e.*, $\gamma^* \in \Gamma^*$. This concludes the proof. □

At this point, we can easily provide a proof of the main theorem (Theorem 9.6) by limiting the sender's strategy space to Γ^*. The last component of the proof is to have a no-regret algorithm for the sender's strategy space to Γ^*. We can obtain no-regret with respect to the optimal signaling scheme in W^* employing any algorithm for the adversarial MAB problem satisfying $R^T \le O(\sqrt{|A|T})$ where A is an arbitrary action set. This can be achieved for example via the INF (Implicitly Normalized Forecaster) algorithm by Audibert and Bubeck (2009).

Proof of Theorem 9.6. Take any no-regret algorithm satisfying

$$R^T \le O(\sqrt{|A|T}),$$

where A is the action set. Then, by taking Γ^* as the sender's action-set we obtain:

$$\max_{\boldsymbol{\gamma}\in\Gamma^*}\frac{1}{T}\sum_{t\in[T]}\big(f^\epsilon(\boldsymbol{\gamma},k^t)-\mathbb{E}[f^\epsilon(\boldsymbol{\gamma}^t,k^t)]\big)\le O\left(\sqrt{\frac{|\Gamma^*|}{T}}\right).$$

By Lemma 9.9,

$$\frac{1}{T}\left(\max_{\boldsymbol{\gamma}\in\Gamma}\sum_{t\in[T]}f(\boldsymbol{\gamma},k^t)-\max_{\boldsymbol{\gamma}^*\in\Gamma^*}\sum_{t\in[T]}f^\epsilon(\boldsymbol{\gamma}^*,k^t)\right)\le\alpha.$$

Then, we have

$$\begin{aligned}
&R^T_\epsilon=\max_{\boldsymbol{\gamma}\in\Gamma}\frac{1}{T}\sum_{t\in[T]}\big(f(\boldsymbol{\gamma},k^t)-\mathbb{E}[f^\epsilon(\boldsymbol{\gamma}^t,k^t)]\big)\le\\
&\frac{1}{T}\big(\max_{\boldsymbol{\gamma}\in\Gamma}\sum_{t\in[T]}f(\boldsymbol{\gamma},k^t)-\max_{\boldsymbol{\gamma}\in\Gamma^*}\sum_{t\in[T]}f^\epsilon(\boldsymbol{\gamma},k^t)\\
&\qquad\qquad+\max_{\boldsymbol{\gamma}\in\Gamma^*}\sum_{t\in[T]}f^\epsilon(\boldsymbol{\gamma},k^t)-\sum_{t\in[T]}\mathbb{E}[f^\epsilon(\boldsymbol{\gamma}^t,k^t)]\big)\\
&\le O\left(\sqrt{\frac{|\Gamma^*}{T}}\right)+\alpha.
\end{aligned}$$

By substituting $|\Gamma^*|=d^{\frac{2\log(\varrho/\alpha)}{\epsilon^2}}$ (see Lemma 9.8) in the above expression we have

$$R^T_\epsilon\le O\left(\frac{d^{\log(\varrho/\alpha)/\epsilon^2}}{\sqrt{T}}\right)+\alpha.$$

This concludes the proof. $\square$

CHAPTER 10

Online Multi-receiver Bayesian Persuasion

In this Chapter, we extend the online Bayesian persuasion framework introduced in Chapter 9 to multiple receivers. In Section 10.1 we extend the Bayesian persuasion framework with multiple receivers to include receivers' types, focusing on the setting in which each receiver has a constant number of types. In Section 10.2, we introduce the online learning framework with multiple receivers. Then, in Section 10.3 we show that designing no-α-multiplicative-regret algorithms with polynomial per-iteration running time is an intractable problem when the sender's utility function is supermodular or anonymous, even if each receiver has a constant number of types. In Section 10.4, we design a general online gradient descent scheme with approximate projection oracles. This algorithm can be applied to any online learning problem with a finite number of possible loss (under mild assumptions). The main result of the section is to show that the gradient descent scheme provides no-α-multiplicative-regret given access to an α-approximate projection oracle. In Section 10.5, we show that, in the multi-receiver Bayesian persuasion setting, such an oracle can be implemented in polynomial time given access to a approximate separation oracle. Finally, Section 10.6 concludes the construction of the no-$(1-1/e)$-multiplicative-regret algorithm for submodular sender's utility functions by showing how to implement in polynomial time an $(1 - 1/e)$-approximate separation oracle for settings in which the sender's utility is submodular.

10.1 Multi-receiver Bayesian Persuasion with Types

We consider a generalization to a multi-receiver setting of the online problem introduced in Chapter 9. There is a finite set $\mathcal{R} := \{r_i\}_{i=1}^{\bar{n}}$ of $\bar{n}$ receivers, and each receiver $r \in \mathcal{R}$ has a type chosen from a finite set $\mathcal{K}_r := \{k_{r,i}\}_{i=1}^{m_r}$ of m_r different types (let $m := \max_{r \in \mathcal{R}} m_r$). We introduce $\mathcal{K} := \bigtimes_{r \in \mathcal{R}} \mathcal{K}_r$ as the set of type profiles, which are tuples $\boldsymbol{k} \in \mathcal{K}$ defining a type $k_r \in \mathcal{K}_r$ for each receiver $r \in \mathcal{R}$. Each receiver $r \in \mathcal{R}$ has two actions available, defined by $\mathcal{A}_r := \{a_0, a_1\}$. We let $\mathcal{A} := \bigtimes_{r \in \mathcal{R}} \mathcal{A}_r$ be the set of action profiles specifying an action for each receiver. We assume no inter-agent externalities. Hence, the payoff of a receiver depends on the action played by her/him and the state of nature, while it does *not* depend on the actions played by the other receivers. Formally, a receiver $r \in \mathcal{R}$ of type $k \in \mathcal{K}_r$ has a utility $u^{r,k} : \Theta \times \mathcal{A}_r \to [0, 1]$. For the ease of notation, we let $u_\theta^{r,k} := u_\theta^{r,k}(a_1, \theta) - u_\theta^{r,k}(a_0)$ be the payoff difference for a receiver r of type k when the state of nature is $\theta \in \Theta$. The sender's utility in a state θ depends on the actions played by all the receivers, and it is defined by $f_\theta : \mathcal{A} \to [0, 1]$. In this chapter, we focus on *private* signaling, where each receiver has her/his own signal that is privately communicated to her/him. We remark that, given a signaling scheme ϕ, a receiver $r \in \mathcal{R}$ of type $k \in \mathcal{K}_r$ observing a private signal $s \in \mathcal{S}_r$ experiences an expected utility $\sum_{\theta \in \Theta} \mu_\theta \sum_{\mathrm{s} \in \mathcal{S}: s_r = s} \phi_\theta(\mathrm{s})\, u_\theta^{r,k}(a)$ (up to a normalization constant) when playing action $a \in \mathcal{A}_r$. Assuming the receivers' type profile is $\boldsymbol{k} \in \mathcal{K}$, the goal of the sender is to commit to an *optimal* signaling scheme ϕ, which is one maximizing her/his expected utility $f(\phi, \boldsymbol{k}) := \sum_{\theta \in \Theta} \mu_\theta \sum_{\mathrm{s} \in \mathcal{S}} \phi_\theta(\mathrm{s})\, f_\theta(R_{\mathrm{s}}^{\boldsymbol{k}})$, where we let $R_{\mathrm{s}}^{\boldsymbol{k}} \subseteq \mathcal{R}$ be the set of receivers who play a_1 after observing their private signal s_r in s, under signaling scheme ϕ. By well-known revelation-principle-style arguments (Kamenica and Gentzkow, 2011; Arieli and Babichenko, 2019), we can restrict our attention to signaling schemes that are direct and persuasive. In our setting, a direct signal sent to a receiver specifies an action recommendation for each receiver's type; thus, we let $\mathcal{S}_r := 2^{\mathcal{K}_r}$ for every $r \in \mathcal{R}$. A signal $s \in \mathcal{S}_r$ for a receiver $r \in \mathcal{R}$ is encoded by a subset of her/his types, namely $s \subseteq \mathcal{K}_r$. Given a direct and persuasive signaling scheme ϕ, for a signal profile $\mathrm{s} \in \mathcal{S}$ and a type profile $\boldsymbol{k} \in \mathcal{K}$, the set $R_{\mathrm{s}}^{\boldsymbol{k}}$ appearing in the definition of the sender's expected utility $f(\phi, \boldsymbol{k})$ can be formally expressed as $R_{\mathrm{s}}^{\boldsymbol{k}} := \{r \in \mathcal{R} \mid k_r \in s_r\}$.

In the rest of this chapter, we assume that the the sender's utility is *monotone non-decreasing* in the set of receivers playing a_1. Formally, for each state $\theta \in \Theta$, we let $f_\theta(R) \leq f_\theta(R')$ for every $R \subseteq R' \subseteq \mathcal{R}$, while $f_\theta(\varnothing) = 0$ for the ease of presentation. Moreover, we assume that the number of types m_r of each receiver $r \in \mathcal{R}$ is fixed; in other words, the value of m cannot grow arbitrarily large. [1]

10.2 Online Settings

The sender plays a repeated game in which, at each iteration $t \in [T]$, she/he commits to a signaling scheme ϕ^t, observes the realized state of nature $\theta^t \sim \boldsymbol{\mu}$, and privately sends signals determined by $\mathrm{s}^t \sim \phi^t_{\theta^t}$ to the receivers. Then, each receiver (whose type is unknown to the sender) selects an action maximizing her/his expected utility given the observed signal (in the *one-shot* interaction at iteration t).

We focus on the problem of computing a sequence $\{\phi^t\}_{t\in[T]}$ of signaling schemes maximizing the sender's expected utility when the sequence of receivers' types $\{\boldsymbol{k}^t\}_{t\in[T]}$, with $\boldsymbol{k}^t \in \mathcal{K}$, is adversarially selected beforehand. After each iteration $t \in [T]$, the sender gets payoff $f(\phi^t, \boldsymbol{k}^t)$ and receives a *full-information feedback* on her/his choice at t, which is represented by the type profile $\boldsymbol{k}^t$. Therefore, after each iteration, the sender can compute the expected utility $f(\phi, \boldsymbol{k}^t)$ guaranteed by any signaling scheme ϕ she/he could have chosen during that iteration.

We are interested in an algorithm computing ϕ^t at each iteration $t \in [T]$. We measure the performance of one such algorithm using the α-multiplicative-*regret* $R^T_{\mathrm{M},\alpha}$. Formally, for $0 < \alpha \leq 1$,

$$R^T_{\mathrm{M},\alpha} := \alpha \max_{\phi} \sum_{t\in[T]} f(\phi, \boldsymbol{k}^t) - \mathbb{E}\left[\sum_{t\in[T]} f(\phi^t, \boldsymbol{k}^t)\right],$$

where the expectation is on the randomness of the algorithm. The classical notion of regret is obtained for $\alpha = 1$. In the remaining of the chapter, we focus on the notion of α-multiplicative-regret, and we write α-regret instead of α-multiplicative-regret and R^T_α instead of $R^T_{\mathrm{M},\alpha}$.

Ideally, we would like an algorithm that returns a sequence $\{\phi^t\}_{t\in[T]}$ with the following properties:

- the α-regret is sublinear in T for some $0 < \alpha \leq 1$;

[1] The monotonicity assumption is w.l.o.g., since our positive result (Theorem 10.9) relies on it. Instead, assuming a fixed number of types is necessary, since, even in single-receiver settings, designing no-regret algorithms with running time polynomial in ϱ is intractable as proven in Theorem 9.2 of the previous chapter.

- the number of computational steps it takes to compute ϕ^t at each iteration $t \in [T]$ is $\mathsf{poly}(T, \bar{n}, d)$, that is, it is a polynomial function of the parameters T, $\bar{n}$, and d.

An algorithm satisfying the first property is called a *no-α-regret algorithm* (it is *no-regret* if it does so for $\alpha = 1$). In this work, we focus on the weaker notion of α-regret since, as we discuss next, requiring no-regret is oftentimes too limiting in our setting (from a computational perspective).

10.3 Hardness of Being No-α-Regret

We start with a negative result. We show that designing no-α-regret algorithms with polynomial per-iteration running time is an intractable problem (formally, it is impossible unless NP $\subseteq$ RP) when the sender's utility is such that functions f_θ are *supermodular* or *anonymous*. This hardness result is deeply connected with the intractability of the offline version of our multi-receiver Bayesian persuasion problem that we formally define in the following Section 10.3.1. Then, Section 10.3.2 collects all the hardness results.

10.3.1 Offline Multi-Receiver Bayesian Persuasion

We consider an offline setting where the receivers' type profile $\boldsymbol{k} \in \mathcal{K}$ is drawn from a known probability distribution (rather then being selected adversarially at each iteration). Given a subset of possible type profiles $K \subseteq \mathcal{K}$ and a distribution $\boldsymbol{\lambda} \in \text{int}(\Delta_K)$, we call BAYESIAN-OPT-SIGNAL the problem of computing a signaling scheme that maximizes the sender's expected utility. This can be achieved by solving the following LP of exponential size.[2]

$$\max_{\phi} \quad \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} \sum_{\theta \in \Theta} \mu_\theta \sum_{\mathbf{s} \in \mathcal{S}} \phi_\theta(\mathbf{s}) f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}}) \tag{10.1a}$$

$$\text{s.t.} \sum_{\theta \in \Theta} \mu_\theta \sum_{\mathbf{s} \in \mathcal{S}: s_r = s} \phi_\theta(\mathbf{s}) u^{r,k}_\theta \geq 0 \qquad \forall r \in \mathcal{R}, \forall s \in \mathcal{S}_r, \forall k \in \mathcal{K}_r : k \in s \tag{10.1b}$$

$$\sum_{\mathbf{s} \in \mathcal{S}} \phi_\theta(\mathbf{s}) = 1 \qquad \forall \theta \in \Theta \tag{10.1c}$$

[2]Constraints (10.1b) encode persuasiveness for the signals recommending to play a_1. The analogous constraints for a_0 can be omitted. Indeed, by assuming that each f_θ is non-decreasing in the set of receivers who play a_1, any signaling scheme in which the sender recommends a_0 when the state is θ and the receiver prefers a_1 over a_0 can be improved by recommending a_1 instead.

$$\phi_\theta(\mathbf{s}) \geq 0 \qquad \forall \theta \in \theta, \forall \mathbf{s} \in \mathcal{S}. \qquad (10.1d)$$

10.3.2 Hardness Results

First, we study the computational complexity of finding an approximate solution to BAYESIAN-OPT-SIGNAL. In particular, given $0 < \alpha \leq 1$, we look for an α-approximate solution in the multiplicative sense, *i.e.*, a signaling scheme providing at least a fraction α of the sender's optimal expected utility (the optimal value of LP 10.1). Theorem 7.2 provides our main hardness result, which is based on a reduction from the *promise-version* of LABEL-COVER. The following is the formal definition of an instance of the LABEL-COVER problem.

Definition 10.1 (LABEL-COVER instance)**.** *An instance of* LABEL-COVER *consists of a tuple* (G, Σ, Π)*, where:*

- $G := (U, V, E)$ *is a* bipartite graph *defined by two disjoint sets of nodes* U *and* V*, connected by the edges in* $E \subseteq U \times V$*, which are such that all the nodes in* U *have the same degree;*
- Σ *is a finite set of* labels*; and*
- $\Pi := \{\Pi_e : \Sigma \to \Sigma \mid e \in E\}$ *is a finite set of* edge constraints.

Definition 10.2 (Labeling)**.** *Given an instance* (G, Σ, Π) *of* LABEL-COVER*, a* labeling *of the graph* G *is a mapping* $\pi : U \cup V \to \Sigma$ *that assigns a label to each vertex of* G *such that all the edge constraints are satisfied. Formally, a labeling* π *satisfies the constraint for an edge* $e = (u, v) \in E$ *if* $\pi(v) = \Pi_e(\pi(u))$.

The classical LABEL-COVER problem is the search problem of finding a valid labeling for a LABEL-COVER instance given as input. In the following, we consider a different version of the problem, which is the *promise problem* associated with LABEL-COVER instances, defined as follows.

Definition 10.3 (GAP-LABEL-COVER$_{c,b}$)**.** *For any pair of numbers* $0 < b < c < 1$*, we define* GAP-LABEL-COVER$_{c,b}$ *as the following promise problem.*

- `Input`**:** *An instance* (G, Σ, Π) *of* LABEL-COVER *such that either one of the following is true:*
  - *there exists a labeling* $\pi : U \cup V \to \Sigma$ *that satisfies at least a fraction* c *of the edge constraints in* Π*;*

  - *any labeling $\pi : U \cup V \to \Sigma$ satisfies less than a fraction b of the edge constraints in Π.*

- `Output`**:** *Determine which of the above two cases hold.*

In order to prove Theorem 10.2, we make use of the following result due to Raz (1998) and Arora et al. (1998).

Theorem 10.1 ((Raz, 1998; Arora et al., 1998))**.** *For any $\epsilon > 0$, there exists a constant $k_\epsilon \in \mathbb{N}$ that depends on ϵ such that the promise problem* GAP-LABEL-COVER$_{1,\epsilon}$ *restricted to inputs (G, Σ, Π) with $|\Sigma| = k_\epsilon$ is* NP*-hard.*

Finally, we can prove the following.

Theorem 10.2. *For every $0 < \alpha \leq 1$, it is* NP*-hard to compute an α-approximate solution to* BAYESIAN-OPT-SIGNAL*, even when the sender's utility is such that, for every $\theta \in \Theta$, $f_\theta(R) = 1$ iff $|R| \geq 2$, while $f_\theta(R) = 0$ otherwise.*

Proof. We provide a reduction from GAP-LABEL-COVER$_{1,\epsilon}$. Our reduction maps an instance (G, Σ, Π) of LABEL-COVER to an instance of BAYESIAN-OPT-SIGNAL with the following properties:

- (*completeness*) if the LABEL-COVER instance admits a labeling satisfying all the edge constraints (recall $c = 1$), then the BAYESIAN-OPT-SIGNAL instance has a signaling scheme with sender's expected utility $\geq \left(1 - \frac{\epsilon}{|\Sigma|}\right) \frac{1}{|\Sigma|} \geq \frac{1}{2|\Sigma|}$;
- (*soundness*) if the LABEL-COVER instance is such that any labeling satisfies at most a fraction ϵ of the edge constraints, then an optimal signaling scheme in the BAYESIAN-OPT-SIGNAL instance has sender's expected utility at most $\frac{2\epsilon}{|\Sigma|}$.

By Theorem 10.1, for any $\epsilon > 0$ there exists a constant $k_\epsilon \in \mathbb{N}$ that depends on ϵ such that GAP-LABEL-COVER$_{1,\epsilon}$ restricted to inputs (G, Σ, Π) with $|\Sigma| = k_\epsilon$ is NP-hard. Given $0 < \alpha \leq 1$, by setting $\epsilon = \frac{\alpha}{4}$ and noticing that $\frac{2\epsilon/|\Sigma|}{1/2|\Sigma|} = 4\epsilon = \alpha$, we can conclude that it is NP-hard to compute an α-approximate solution to BAYESIAN-OPT-SIGNAL.

Construction Given an instance (G, Σ, Π) of LABEL-COVER defined over a bipartite graph $G := (U, V, E)$, we build an instance of BAYESIAN-OPT-SIGNAL as follows.

- For each label $\sigma \in \Sigma$, there is a corresponding state of nature $\theta_\sigma \in \Theta$. Moreover, there is an additional state $\theta_0 \in \Theta$. Thus, the total number of possible states is $d = |\Sigma| + 1$.
- The prior distribution is $\boldsymbol{\mu} \in \text{int}(\Delta_\Theta)$ such that $\mu_{\theta_\sigma} = \frac{\epsilon}{|\Sigma|^2}$ for every $\theta_\sigma \in \Theta$ and $\mu_{\theta_0} = 1 - \frac{\epsilon}{|\Sigma|}$.
- For every vertex $v \in U \cup V$ of the graph G, there is a receiver $r_v \in \mathcal{R}$. Thus, $n = |U \cup V|$.
- Each receiver $r_v \in \mathcal{R}$ has $m_{r_v} = |\Sigma| + 1$ possible types. The set of types of receiver r_v is $\mathcal{K}_{r_v} = \{k_\sigma \mid \sigma \in \Sigma\} \cup \{k_0\}$.
- A receiver $r_v \in \mathcal{R}$ of type $k_\sigma \in \mathcal{K}_{r_v}$ has utility such that $u_{\theta_\sigma}^{r_v,k_\sigma} = \frac{1}{2}$ and $u_{\theta_{\sigma'}}^{r_v,k_\sigma} = -1$ for all $\theta_{\sigma'} \in \Theta : \theta_{\sigma'} \neq \theta_\sigma$, while $u_{\theta_0}^{r_v,k_\sigma} = -\frac{\epsilon}{2|\Sigma|^2}$. Moreover, a receiver $r_v \in \mathcal{R}$ of type k_0 has utility such that $u_\theta^{r_v,k_0} = -1$ for all $\theta \in \Theta$.
- The sender's utility is such that, for every $\theta \in \Theta$, the function $f_\theta : 2^{\mathcal{R}} \to [0,1]$ satisfies $f_\theta(R) = 1$ if and only if $R \subseteq \mathcal{R} : |R| \geq 2$, while $f_\theta(R) = 0$ otherwise.
- The subset $K \subseteq \mathcal{K}$ of type profiles that can occur with positive probability is $K := \{\boldsymbol{k}^{uv,\sigma} \mid e = (u,v) \in E, \sigma \in \Sigma\}$, where, for every edge $e = (u,v) \in E$ and label $\sigma \in \Sigma$, the type profile $\boldsymbol{k}^{uv,\sigma} \in \mathcal{K}$ is such that $k_{r_u}^{uv,\sigma} = k_\sigma$, $k_{r_v}^{uv,\sigma} = k_{\sigma'}$ with $\sigma' = \Pi_e(u)$, and $k_{r_{v'}}^{uv,\sigma} = k_0$ for every $r_{v'} \in \mathcal{R} : r_{v'} \notin \{r_u, r_v\}$.
- The probability distribution $\boldsymbol{\lambda} \in \text{int}(\Delta_K)$ is such that $\lambda_{\boldsymbol{k}} = \frac{1}{|E||\Sigma|}$ for every $\boldsymbol{k} \in K$.

Notice that, in the BAYESIAN-OPT-SIGNAL instances used for the reduction, the sender's payoff is 1 if and only if at least two receivers play action a_1, while it is 0 otherwise. Let us also recall that direct signals for a receiver $r_v \in \mathcal{R}$ are defined by the set $\mathcal{S}_{r_v} := 2^{\mathcal{K}_{r_v}}$, with a signal being represented as the set of receiver's types that are recommended to play action a_1.

Completeness Let $\pi : U \cup V \to \Sigma$ be a labeling of the graph G that satisfies all the edge constraints. We define a corresponding direct signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ as follows. For any label $\sigma \in \Sigma$, let $\mathrm{s}^\sigma \in \mathcal{S}$ be a signal profile such that the signal sent to receiver $r_v \in \mathcal{R}$ is $s_{r_v}^\sigma = \{k_\sigma\}$, *i.e.*, only a receiver of the type k_σ is told to play a_1, while all the other types are recommended to play a_0. Moreover, let $\mathrm{s}^\pi \in \mathcal{S}$ be a signal profile in which the signal sent to receiver $r_v \in \mathcal{R}$ is $s^{\pi}_{r_v} = \{k_\sigma\}$ with $\sigma \in \Sigma : \sigma = \pi(v)$, *i.e.*, each receiver r_v is told to play action a_1 only if her/his type is k_σ for the label σ assigned to vertex v by the labeling π, otherwise she/he is recommended to play a_0. Then, we define $\phi_{\theta_\sigma}(\mathrm{s}^\sigma) = 1$ for every state of nature $\theta_\sigma \in \Theta$, while $\phi_{\theta_0}(\mathrm{s}^\pi) = 1$. Notice that the signaling scheme ϕ is deterministic, since each state of nature is mapped to only one signal profile (with probability one). As a first step, we prove that the signaling scheme ϕ is *persuasive*. Let us fix a receiver $r_v \in \mathcal{R}$. After receiving a signal $s = \{k_\sigma\} \in \mathcal{S}_{r_v}$ with $\sigma \in \Sigma : \sigma \neq \pi(v)$, by definition of ϕ, the receiver's posterior belief is such that state of nature θ_σ is assigned probability one. Thus, if the receiver has type k_σ, then she/he is incentivized to play action a_1, since $u^{r_v,k_\sigma}_{\theta_\sigma} = \frac{1}{2} > 0$ (recall that $u^{r_v,k_\sigma}_{\theta_\sigma}$ is the utility different "action a_1 minus action a_0" when the state is θ_σ). Instead, if the receiver has type $k \in \mathcal{K}_{r_v} : k \neq k_\sigma$, then she/he is incentivized to play action a_0, since either $k = k_0$ and $u^{r_v,k_0}_{\theta_\sigma} = -1 < 0$ or $k = k_{\sigma'}$ with $\sigma' \in \Sigma : \sigma' \neq \sigma$ and $u^{r_v,k_{\sigma'}}_{\theta_\sigma} = -1 < 0$. After receiving a signal $s = \{k_\sigma\} \in \mathcal{S}_{r_v}$ with $\sigma = \pi(v)$, the receiver's posterior belief is such that the states of nature θ_σ and θ_0 are assigned probabilities proportional to their corresponding prior probabilities, respectively μ_{θ_σ} and μ_{θ_0} (she/he cannot tell whether s^σ or s^π has been selected by the sender). Thus, if the receiver has type k_σ, then she/he is incentivized to play action a_1, since her expected utility difference "action a_1 minus action a_0" is the following:

$$
\begin{aligned}
&\frac{\mu_{\theta_\sigma}}{\mu_{\theta_\sigma} + \mu_{\theta_0}} u^{r_v,k_\sigma}_{\theta_\sigma} + \frac{\mu_{\theta_0}}{\mu_{\theta_\sigma} + \mu_{\theta_0}} u^{r_v,k_\sigma}_{\theta_0} \\
&\quad = \frac{1}{\mu_{\theta_\sigma} + \mu_{\theta_0}} \left[\frac{\epsilon}{|\Sigma|^2} \frac{1}{2} - \left(1 - \frac{\epsilon}{|\Sigma|}\right) \frac{\epsilon}{2|\Sigma|^2} \right] \\
&\quad > \frac{1}{\mu_{\theta_\sigma} + \mu_{\theta_0}} \left[\frac{\epsilon}{2|\Sigma|^2} - \frac{\epsilon}{2|\Sigma|^2} \right] = 0.
\end{aligned}
$$

If the receiver has a type different from k_σ, simple arguments show that the expected utility difference is negative, incentivizing action a_0. This proves that the signaling scheme ϕ is persuasive. Next, we bound the sender's expected utility in ϕ. Notice that, when the state of nature is θ_0, if the receivers' type profile is $\boldsymbol{k}^{uv,\sigma} \in K$ with $\sigma = \pi(u)$ for some edge $e = (u, v) \in E$, then both receivers r_u and r_v play action a_1. This is readily proved since $k^{uv,\sigma}_{r_u} = k_\sigma$ and $k^{uv,\sigma}_{r_v} = k_{\sigma'}$ with $\sigma = \pi(u)$ and $\sigma' = \pi(v)$ (recall that $\pi(v) = \Pi_e(u)$ as ϕ satisfies all the edge constraints), and, thus, both r_u and r_v are recommended to play a_1 when the state is θ_0. As a result, under signaling scheme ϕ, when the receivers' type profile is $\boldsymbol{k}^{uv,\sigma} \in K$, then the sender's resulting payoff is one (recall the definition of functions f_θ). By recalling that each type profile $\boldsymbol{k}^{uv,\sigma} \in K$ with $\sigma = \pi(u)$ (for each edge $e = (u, v) \in E$) occurs with probability $\lambda_{\boldsymbol{k}^{uv,\sigma}} = \frac{1}{|E||\Sigma|}$, we can lower bound the sender's expected utility (see the objective of Problem (10.1)) as follows:

$$\begin{aligned}
\sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} \sum_{\theta \in \Theta} \mu_\theta \sum_{\mathrm{s} \in \mathcal{S}} \phi_\theta(\mathrm{s}) f_\theta \left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) \\
&\geq \mu_{\theta_0} \sum_{\boldsymbol{k}^{uv,\sigma} \in K : \sigma = \pi(u)} \lambda_{\boldsymbol{k}^{uv,\sigma}} \\
&= \mu_{\theta_0} \frac{1}{|\Sigma|} = \left(1 - \frac{\epsilon}{|\Sigma|}\right) \frac{1}{|\Sigma|}.
\end{aligned}$$

Soundness By contradiction, suppose that there exists a direct and persuasive signaling scheme $\phi : \Theta \to \Delta_{\mathcal{S}}$ that provides the sender with an expected utility greater than $\frac{2\epsilon}{|\Sigma|}$. Since the sender can extract an expected utility at most of $\frac{\epsilon}{|\Sigma|}$ from states of nature $\theta \in \Theta$ with $\theta \neq \theta_0$ (as $\sum_{\theta \in \Theta : \theta \neq \theta_0} \mu_\theta = \frac{\epsilon}{|\Sigma|}$ and the maximum value of functions f_θ is one), then it must be the case that the expected utility contribution due to state θ_0 is greater than $\frac{\epsilon}{|\Sigma|}$. Let us consider the distribution over signal profiles $\phi_{\theta_0} \in \Delta_{\mathcal{S}}$ induced by state of nature θ_0. We prove that, for each signal profile $\mathrm{s} \in \mathcal{S}$ such that $\phi_{\theta_0}(\mathrm{s}) > 0$ and each receiver $r_v \in \mathcal{R}$, it must hold that $|s_r| \leq 1$, *i.e.*, at most one type of receiver r_v is recommended to play a_1. First, notice that a receiver of type k_0 cannot be incentivized to play a_1, since $u^{r_v,k_0}_\theta = -1$ for all $\theta \in \Theta$. By contradiction, suppose that there are two receiver's types $k_\sigma, k_{\sigma'} \in \mathcal{K}_{r_v}$ with $k_\sigma \neq k_{\sigma'}$ such that $k_\sigma, k_{\sigma'} \in s_r$ (*i.e.*, they are both recommended to play a_1). By letting $\boldsymbol{\xi} \in \Delta_\Theta$ be the posterior belief of receiver r_v induced by s_r, for type k_σ it must be the case that:

$$\begin{aligned}
\xi_{\theta_\sigma} u^{r_v,k_\sigma}_{\theta_\sigma} + \sum_{\theta_{\sigma''} \in \Theta : \theta_{\sigma''} \neq \theta_\sigma} \xi_{\theta_{\sigma''}} u^{r_v,k_\sigma}_{\theta_{\sigma''}} + \xi_{\theta_0} u^{r_v,k_\sigma}_{\theta_0} \\
&= \frac{1}{2} \xi_{\theta_\sigma} - \sum_{\theta_{\sigma''} \in \Theta : \theta_{\sigma''} \neq \theta_\sigma} \xi_{\theta_{\sigma''}} - \frac{\epsilon}{2|\Sigma|^2} \xi_{\theta_0} > 0,
\end{aligned}$$

since the signaling scheme is persuasive, and, thus, a receiver of type k_σ must be incentivized to play action a_1. This implies that

$$\xi_{\theta_\sigma} > 2 \sum_{\theta_{\sigma''} \in \Theta : \theta_{\sigma''} \neq \theta_\sigma} \xi_{\theta_{\sigma''}} \geq 2 \xi_{\theta_{\sigma'}}.$$

Analogous arguments for type $k_{\sigma'}$ imply that $\xi_{\theta_{\sigma'}} > 2\xi_{\theta_\sigma}$, reaching a contradiction. This shows that, for each $\mathrm{s} \in \mathcal{S}$ such that $\phi_{\theta_0}(\mathrm{s}) > 0$ and each $r_v \in \mathcal{R}$, it must be the case that $|s_r| \leq 1$. Next, we provide the last contradiction proving the result. Let us recall that, by assumption, the sender's expected utility contribution due to θ_0 is

$$\sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} \sum_{\mathrm{s} \in \mathcal{S}} \phi_{\theta_0}(\mathrm{s}) f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) \geq \frac{\epsilon}{|\Sigma|}.$$

By an averaging argument, this implies that there must exist a signal profile $\mathrm{s} \in \mathcal{S}$ such that $\phi_{\theta_0}(\mathrm{s}) > 0$ and $\sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) \geq \frac{\epsilon}{|\Sigma|}$. Let $\mathrm{s} \in \mathcal{S}$ be such signal profile. Let us define a corresponding labeling $\pi : U \cup V \to \Sigma$ of the graph G such that, for every vertex $v \in U \cup V$, it holds $\pi(v) = \sigma$, where $\sigma \in \Sigma$ is the label corresponding to the unique type k_σ of receiver r_v that is recommended to play action a_1 under s (if any, otherwise any label is fine). Since $\sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) \geq \frac{\epsilon}{|\Sigma|}$ and it holds $\lambda_{\boldsymbol{k}} = \frac{1}{|E||\Sigma|}$ and $f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) \in \{0, 1\}$ for every $\boldsymbol{k} \in K$, it must be the case that there are at least $\epsilon|E|$ type profiles $\boldsymbol{k} \in K$ such that $f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) = 1$. Since a receiver of type k_0 cannot be incentivized to play action a_1, the value of $f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right)$ can be one only if there are at least two receivers with types different from k_0 that play action a_1. Thus, it must hold that $f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right) = 0$ for all the type profiles $\boldsymbol{k}^{uv,\sigma} \in K$ such that $\sigma \neq \pi(u)$ (as $k^{uv,\sigma}_{r_u}$ would be equal to k_σ with $\sigma \neq \pi(u)$ and $k_\sigma \notin s_{r_u}$). For the type profiles $\boldsymbol{k}^{uv,\sigma} \in K$ such that $\sigma = \pi(u)$ (one per edge $e = (u, v) \in E$ of the graph G), the value of $f_{\theta_0}\left(R^{\boldsymbol{k}}_{\mathrm{s}}\right)$ is one if and only if $\pi(v) = \Pi_e(u)$, so that both receivers r_u and r_v are told to play action a_1. As a result, this implies that there must be at least $\epsilon|E|$ edges $e \in E$ for which the labeling π satisfies the corresponding edge constraint Π_e, which is a contradiction. □

Notice that Theorem 7.2 holds for problem instances in which functions f_θ are anonymous. Moreover, the reduction can be easily modified so that functions f_θ are supermodular and satisfy $f_\theta(R) = \max\{0, |R| - 1\}$ for $R \subseteq \mathcal{R}$. Thus:

Corollary 10.1. *For* $0 < \alpha \leq 1$, *it is* NP*-hard to compute an* α*-approximate solution to* BAYESIAN-OPT-SIGNAL, *even when the sender's utility is such that functions* f_θ *are* supermodular *or* anonymous *for every* $\theta \in \Theta$.

By using arguments similar to those employed in the proof of Theorem 6.2 by Roughgarden and Wang (2019), the hardness of computing an α-approximate solution to the offline problem can be extended to designing no-α-regret algorithms in the online setting. Then:

Theorem 10.3. *For every $0 < \alpha \leq 1$, there is no polynomial-time no-α-regret algorithm for the multi-receiver online Bayesian persuasion problem, unless* $\mathsf{NP} \subseteq \mathsf{RP}$*, even when functions f_θ are* supermodular *or* anonymous *for all $\theta \in \Theta$.*

In the rest of the work, we show how to design a polynomial-time no-$(1-\frac{1}{e})$-regret algorithm for the case in which the sender's utility is such that functions f_θ are submodular. This result is tight since even in the setting without types the problem is NP-hard to approximate to within a factor better that $1 - 1/e$ (Babichenko and Barman, 2017).

10.4 An Online Gradient Descent Scheme with Approximate Projection Oracles

As a first step in building our polynomial-time algorithm, we introduce our OGD scheme with an *approximate projection oracle*. Intuitively, it works by transforming the multi-receiver online Bayesian persuasion setting into an equivalent online learning problem whose decision space does not need to explicitly deal with signaling schemes (thus avoiding the burden of having an exponential number of possible signal profiles). The OGD algorithm is then applied on this new domain. In our setting, we do *not* have access to a polynomial-time (exact) projection oracle, and, thus, we design and analyze the algorithm assuming access to an approximate one only. As we show later in Sections 10.5 and 10.6, such approximate projection oracle can be implemented in polynomial time when the functions f_θ are submodular.

Let us recall that the OGD scheme that we describe in this section is general and applies to any online learning problem with a finite number of possible loss functions.

10.4.1 A General Approach

Consider an online learning problem in which the learner takes a decision $y^t \in \mathcal{Y}$ at each iteration $t \in [T]$. Then, the learner observes a feedback $e^t \in \mathcal{E}$, where $\mathcal{E}$ is a finite set of p possible feedbacks. The reward (or negative loss) of a decision $y \in \mathcal{Y}$ given feedback $e \in \mathcal{E}$ is defined by $u(y, e)$ for a given function $u : \mathcal{Y} \times \mathcal{E} \to [0, 1]$. Thus, the learner is awarded $u(y^t, e^t)$ for decision y^t at iteration t, while she/he would have achieved $u(y, e^t)$ for any other choice $y \in \mathcal{Y}$.

We transform this general online learning problem to a new one in which the learner's decision set is $\mathcal{X} \subseteq [0,1]^p$ with:

$$\mathcal{X} := \bigcup_{y \in \mathcal{Y}} \Big\{ \mathbf{x} \in [0,1]^p \mid x_e \leq u(y,e) \quad \forall e \in \mathcal{E} \Big\}. \tag{10.2}$$

Intuitively, the set $\mathcal{X}$ contains all the vectors whose components x_e (one for each feedback $e \in \mathcal{E}$) are the learner's rewards $u(y,e)$ for some decision $y \in \mathcal{Y}$ in the original problem. Moreover, the inequality "$\leq$" in the definition of $\mathcal{X}$ also includes all the reward vectors that are dominated by those corresponding to some decision in $\mathcal{Y}$. At each iteration $t \in [T]$, the learner takes a decision $\mathbf{x}^t \in \mathcal{X}$ and observes a feedback $e^t \in \mathcal{E}$. The reward of decision $\mathbf{x} \in \mathcal{X}$ at iteration t is the e^t-th component of $\mathbf{x}$, namely x_{e^t}. It is sometimes useful to write it as $1_{e^t}^\top \mathbf{x}$, where $1_{e^t} \in \{0,1\}^p$ is a vector whose e^t-th component is 1, while all the others are 0. Thus, the learner's reward at iteration t is $x_{e^t}^t$. Notice that the size of the decision set $\mathcal{X}$ of the new online learning setting does *not* depend on the dimensionality of the original decision set $\mathcal{Y}$ (which, in our setting, would be exponential), but only on the number of feedbacks p.

If $\mathcal{Y}$ and u are such that $\mathcal{X}$ is compact and convex, then we can minimize the α-regret R_α^T in the original problem by doing that in the new setting. Let us introduce the set $\alpha\mathcal{X} := \{\alpha \mathbf{x} \mid \mathbf{x} \in \mathcal{X}\}$ for any $0 < \alpha \leq 1$. Given a sequence of feedbacks $\{e^t\}_{t \in [T]}$ and a sequence of decisions $\{\mathbf{x}^t\}_{t\in[T]}$, with $e^t \in \mathcal{E}$ and $\mathbf{x}^t \in \mathcal{X}$, we have that:

$$\begin{aligned} R_\alpha^T &:= \max_{\mathbf{x} \in \alpha\mathcal{X}} \sum_{t \in [T]} 1_{e^t}^\top \Big(\mathbf{x} - \mathbf{x}^t \Big) \\ &\geq \alpha \max_{y \in \mathcal{Y}} \sum_{t \in [T]} u(y, e^t) - \sum_{t \in [T]} u(y^t, e^t), \end{aligned}$$

where $\{y^t\}_{t \in [T]}$ is a sequence of decisions $y^t \in \mathcal{Y}$ for the original problem such that $x_e^t \leq u(y^t, e)$ for $e \in \mathcal{E}$.

We assume to have access to an approximate projection oracle for $\alpha\mathcal{X}$, which we define in the following. By letting $E \subseteq \mathcal{E}$ be a subset of feedbacks, we define $\tau_E : \mathcal{X} \to [0,1]^p$ as the function mapping any vector $\mathbf{x} \in \mathcal{X}$ to another one that is equal to $\mathbf{x}$ in all the components corresponding to feedbacks $e \in E$, while it is 0 everywhere else. Moreover, we let $\mathcal{X}_E := \{\tau_E(\mathbf{x}) \mid \mathbf{x} \in \mathcal{X}\}$ be the image of $\mathcal{X}$ through τ_E, while $\alpha\mathcal{X}_E := \{\alpha\mathbf{x} \mid \mathbf{x} \in \mathcal{X}_E\}$ for $0 < \alpha \leq 1$.

Definition 10.4 (Approximate projection oracle)**.** *Consider a subset of feedbacks $E \subseteq \mathcal{E}$, a vector $\mathbf{y} \in [0,2]^p$ such that $y_e = 0$ for all $e \notin E$, and an* *approximation error* $\epsilon \in \mathbb{R}_+$. *Then, for any* $0 < \alpha \leq 1$, *an* approximate projection oracle $\varphi_\alpha(E, \boldsymbol{y}, \epsilon)$ *is an algorithm returning a vector* $\mathbf{x} \in \mathcal{X}_E$ *and a decision* $y \in \mathcal{Y}$ *with* $x_e \leq u(y, e)$ *for all* $e \in \mathcal{E}$, *such that:*

$$||\mathbf{x}' - \mathbf{x}||^2 \leq ||\mathbf{x}' - \boldsymbol{y}||^2 + \epsilon \qquad \forall \mathbf{x}' \in \alpha\mathcal{X}_E.$$

Intuitively, φ_α returns a vector $\mathbf{x} \in \mathcal{X}_E$ that is an approximate projection of **y** onto the subspace $\alpha\mathcal{X}_E$. The vector $\mathbf{x}$ can be outside of $\alpha\mathcal{X}_E$. However, it is “better” than a projection onto $\alpha\mathcal{X}_E$, since, ignoring the ϵ error, $\mathbf{x}$ is closer than **y** to any vector in $\alpha\mathcal{X}_E$. Moreover, φ_α also gives a decision $y \in \mathcal{Y}$ that corresponds to the returned vector $\mathbf{x}$. Notice that, if $\alpha = 1$ and $\epsilon = 0$, this is equivalent to find an exact projection onto the subspace $\mathcal{X}_E$.

10.4.2 A Particular Setting: Multi-Receiver Online Bayesian Persuasion

Our setting can be easily cast into the general learning framework described so far. The possible feedbacks are type profiles, namely $\mathcal{E} := \mathcal{K}$, while the receivers' type profile $\boldsymbol{k}^t \in \mathcal{K}$ is the feedback observed at iteration $t \in [T]$, namely $e^t := \boldsymbol{k}^t$. Notice that the number of possible feedbacks is $p = m^{\bar{n}}$, which is exponential in the number of receivers. The decision set of the learner (sender) $\mathcal{Y}$ is the set of all the possible signaling schemes ϕ, with $y^t := \phi^t$ being the one chosen at iteration t. The rewards observed by the sender are the utilities $f(\phi, \boldsymbol{k})$; formally, for every signaling scheme ϕ and type profile $\boldsymbol{k} \in \mathcal{K}$, which define a pair $y \in \mathcal{Y}$ and $e \in \mathcal{E}$ using the generic notation, we let $u(y, e) := f(\phi, \boldsymbol{k})$. Then, the new decision set $\mathcal{X} \subseteq [0, 1]^{|\mathcal{K}|}$ is defined as in Equation (10.2). Notice that $\mathcal{X}$ is a compact and convex set, since it can be defined by a set of linear inequalities. In the following, we overload the notation and, for any subset $K \subseteq \mathcal{K}$ of types profiles, we let $\mathcal{X}_K := \mathcal{X}_E$ for $E \subseteq \mathcal{E} : E = K$.

10.4.3 OGD with Approximate Projection Oracle

Algorithm 10.1 is an OGD scheme that operates in the $\mathcal{X}$ domain by having access to an approximate projection oracle φ_α (we call the algorithm OGD-APO).

The procedure in Algorithm 10.1 keeps track of the set $E^t \subseteq \mathcal{E}$ of different feedbacks observed up to each iteration $t \in [T]$. Moreover, it works on the subspace $\mathcal{X}_{E^t}$, whose vectors are zero in all the components corresponding to feedbacks $e \notin E^t$. Since it is the case that $|E^t| \leq t$, the procedure in Algorithm 10.1 attains a per-iteration running time that is independent of the number of possible feedbacks p.

Algorithm 10.1 OGD-APO

Input: • approximate projection oracle φ_α
• learning rate $\eta \in (0,1]$
• approximation error $\epsilon \in [0,1]$

Initialize $y^1 \in \mathcal{Y}$, $E^0 \leftarrow \varnothing$, and $\mathbf{x}^1 \leftarrow \mathbf{0} \in \mathcal{X}_{E^1}$
for $t = 1, \dots, T$ **do**
  Take decision y^t
  Observe feedback $e^t \in \mathcal{E}$ and reward $u(y^t, e^t) = x^t_{e^t}$
  $E^t \leftarrow E^{t-1} \cup \{e^t\}$
  $\mathbf{y}^{t+1} \leftarrow \mathbf{x}^t + \eta 1_{e^t}$
  $\left(\mathbf{x}^{t+1}, y^{t+1}\right) \leftarrow \varphi_\alpha\left(E^t, \mathbf{y}^{t+1}, \epsilon\right)$
end for

Next, we bound the α-regret incurred by Algorithm 10.1.

Theorem 10.4. *Given an oracle φ_α (as in Definition 10.4) for some $0 < \alpha \leq 1$, a learning rate $\eta \in (0,1]$, and an approximation error $\epsilon \in [0,1]$, Algorithm 10.1 has α-regret*

$$R^T_\alpha \leq \frac{|E^T|}{2\eta} + \frac{\eta T}{2} + \frac{\epsilon T}{2\eta},$$

with a per-iteration running time $\mathsf{poly}(t)$.

Proof. First, we bound the per-iteration running time of Algorithm 10.1. For any $t \in [T]$, we have $E^t = \bigcup_{t' \in [t]} e^{t'}$, which represents the set of feedbacks observed up to iteration t. Thus, it holds $|E^t| \leq t$. At iteration $t \in [T]$, the algorithm works with vectors $\mathbf{x}^t$ and $\mathbf{y}^{t+1}$. The first one belongs to $\mathcal{X}_{E^{t-1}}$ (as it is returned by φ_α at iteration $t-1$), and, thus, it has at most $t-1$ non-zero components. Similarly, since $\mathbf{y}^{t+1} = \mathbf{x}^t + \eta 1_{e^t}$, it holds that $\mathbf{y}^{t+1} \in [0,2]^p$ and $y^{t+1}_e = 0$ for all $e \notin E^t$, which implies that $\mathbf{y}^{t+1}$ has at most t non-zero components. As a result, we can sparsely represent vectors $\mathbf{x}^t$ and $\mathbf{y}^{t+1}$ so that Algorithm 10.1 has a per-iteration running time bounded by t for any iteration $t \in [T]$, independently of the actual size p of the vectors. Moreover, notice that $\mathbf{y}^{t+1}$ satisfies the conditions required by the inputs of the oracle φ_α.

Next, we bound the α-regret of Algorithm 10.1. For the ease of notation, in the following, for any vector $\mathbf{x} \in \mathcal{X}$ and subset $E \subseteq \mathcal{E}$, we let $\mathbf{x}_E := \tau_E(\mathbf{x})$. Moreover, for any $t \in [T]$, we let $\mathbb{I}_t := \mathbb{I}\left[e^t \notin E^{t-1}\right]$, which is the indicator function that is equal to 1 if and only if $e^t \notin E^{t-1}$, *i.e.*, when the feedback e^t at iteration t has never been observed before. Fix $\mathbf{x} \in \alpha\mathcal{X}$. Then, the following relations hold:

$$\left|\left|\mathbf{x}_{E^t} - \mathbf{x}^{t+1}\right|\right|^2 \leq \left|\left|\mathbf{x}_{E^t} - \mathbf{y}^{t+1}\right|\right|^2 + \epsilon \tag{10.3a}$$

$$
\begin{aligned}
&= \left|\left|\mathbf{x}_{E^t} - \mathbf{x}^t - \eta \mathbf{1}_{e^t}\right|\right|^2 + \epsilon && \text{(10.3b)}\\
&= \left|\left|\mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t} - \mathbf{x}^t - \eta \mathbf{1}_{e_t}\right|\right|^2 + \epsilon && \text{(10.3c)}\\
&= \left|\left|\mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t} - \mathbf{x}^t\right|\right|^2 + \eta^2 \\
&\qquad - 2\eta \mathbf{1}_{e^t}^\top \Big(\mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t} - \mathbf{x}^t\Big) + \epsilon && \text{(10.3d)}\\
&= \left|\left|\mathbf{x}_{E^{t-1}} - \mathbf{x}^t_{E^{t-1}}\right|\right|^2 + \mathbb{I}_t \left|x_{e^t} - x^t_{e^t}\right|^2 + \eta^2 \\
&\qquad - 2\eta \mathbf{1}_{e^t}^\top \Big(\mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t} - \mathbf{x}^t\Big) + \epsilon && \text{(10.3e)}\\
&\leq \left|\left|\mathbf{x}_{E^{t-1}} - \mathbf{x}^t_{E^{t-1}}\right|\right|^2 + \mathbb{I}_t + \eta^2 \\
&\qquad - 2\eta \mathbf{1}_{e^t}^\top \Big(\mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t} - \mathbf{x}^t\Big) + \epsilon. && \text{(10.3f)}
\end{aligned}
$$

Notice that Equation (10.3b) holds by definition of φ_α since $\mathbf{x}_{E^t} \in \alpha \mathcal{X}_{E^t}$, Equation (10.3d) follows from $\mathbf{x}_{E^t} = \mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t}$, while Equation (10.3e) can be derived by decomposing the first squared norm in the preceding expression. By using the last relation above, we can write the following:

$$
\begin{aligned}
&\sum_{t \in [T]} \mathbf{1}_{e^t}^\top \Big(\mathbf{x} - \mathbf{x}^t\Big) && \text{(10.4a)}\\
&= \sum_{t \in [T]} \mathbf{1}_{e^t}^\top \Big(\mathbf{x}_{E^{t-1}} + \mathbb{I}_t\, x_{e^t} \mathbf{1}_{e^t} - \mathbf{x}^t\Big) && \text{(10.4b)}\\
&\leq \frac{1}{2\eta} \sum_{t \in [T]} \left(\left|\left|\mathbf{x}_{E^{t-1}} - \mathbf{x}^t_{E^{t-1}}\right|\right|^2 - \left|\left|\mathbf{x}_{E^t} - \mathbf{x}^{t+1}\right|\right|^2 + \mathbb{I}_t + \eta^2 + \epsilon\right) && \text{(10.4c)}\\
&= \frac{1}{2\eta} \sum_{t \in [T]} \Big(\mathbb{I}_t + \eta^2 + \epsilon\Big) && \text{(10.4d)}\\
&= \frac{1}{2\eta} \Big(|E^T| + T\eta^2 + T\epsilon\Big), && \text{(10.4e)}
\end{aligned}
$$

where Equation (10.4d) is obtained by telescoping the sum. Then, the following concludes the proof:

$$
\begin{aligned}
R^T_\alpha &:= \alpha \max_{y \in \mathcal{Y}} \sum_{t \in [T]} u(y, e^t) - \sum_{t \in [T]} u(y^t, e^t) \\
&\leq \alpha \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [T]} x_{e^t} - \sum_{t \in [T]} x^t_{e^t} \\
&= \alpha \max_{\mathbf{x} \in \mathcal{X}} \sum_{t \in [T]} \mathbf{1}_{e^t}^\top \left(\mathbf{x} - \mathbf{x}^t\right)
\end{aligned}
$$

$$= \max_{\mathbf{x} \in \alpha \mathcal{X}} \sum_{t \in [T]} \mathbf{1}_{e^t}^\top \left(\mathbf{x} - \mathbf{x}^t \right)$$
$$\leq \frac{1}{2\eta} \Big(|E^T| + T\eta^2 + T\epsilon \Big).$$

□

By setting $\eta = \frac{1}{\sqrt{T}}$, $\epsilon = \frac{1}{T}$, we get $R^T_\alpha \leq \sqrt{T} \left(1 + \frac{|E^T|}{2} \right)$.

Notice that the bound only depends on the number of observed feedbacks $|E^T|$, while it is independent of the overall number of possible feedbacks p. This is crucial for the multi-receiver online Bayesian persuasion case, where p is exponential in the the number of receivers $\bar{n}$. On the other hand, as T goes to infinity, we have $|E^T| \leq p$, so that the regret bound is sublinear in T.

10.5 Constructing a Poly-Time Approximate Projection Oracle

The crux of the OGD-APO algorithm (Algorithm 10.1) is being able to perform the approximate projection step. In this section, we show that, in the multi-receiver Bayesian persuasion setting, the approximate projection oracle φ_α required by OGD-APO can be implemented in polynomial time by an appropriately-engineered ellipsoid algorithm. This calls for an *approximate separation oracle* $\mathcal{O}_\alpha$ (see Definition 10.5).

We proceed as follows. In Section 10.5.1, we define an appropriate notion of approximate separation oracle, and show how to find, in polynomial time, an α-approximate solution to the offline problem BAYESIAN-OPT-SIGNAL. This is a preparatory step towards the understanding of our main result in this section, and it may be of independent interest. Then, in Section 10.5.2, we exploit some of the techniques introduced for the offline setting in order to build φ_α starting from an approximate separation oracle $\mathcal{O}_\alpha$.

10.5.1 Warming Up: The Offline Setting

An approximate separation oracle $\mathcal{O}_\alpha$ finds a signal profile $\mathbf{s} \in \mathcal{S}$ that approximately maximizes a weighted sum of the f_θ functions, plus a weight for each receiver which depends on the signal s_r sent to that receiver. Formally:

Definition 10.5 (Approximate separation oracle)**.** *Consider a state $\theta \in \Theta$, a subset $K \subseteq \mathcal{K}$, a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^{|K|}$, weights $\mathbf{w} = (w_{r,s})_{r \in \mathcal{R}, s \in \mathcal{S}_r}$ with*

$w_{r,s} \in \mathbb{R}$ and $w_{r,\varnothing} = 0$ for all $r \in \mathcal{R}$, and an approximation error $\epsilon \in \mathbb{R}_+$. Then, for any $0 < \alpha \leq 1$, an approximation oracle $\mathcal{O}_\alpha(\theta, K, \boldsymbol{\lambda}, \mathbf{w}, \epsilon)$ is an algorithm returning an $\mathrm{s} \in \mathcal{S}$ such that:

$$\sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}}) + \sum_{r \in \mathcal{R}} w_{r,s_r} \geq \max_{\mathbf{s}^\star \in \mathcal{S}} \left\{ \alpha \sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}^\star}) + \sum_{r \in \mathcal{R}} w_{r,s^\star_r} \right\} - \epsilon, \quad (10.5)$$

in time poly$\left(n, |K|, \max_{r,s} |w_{r,s}|, \max_{\boldsymbol{k}} \lambda_{\boldsymbol{k}}, \frac{1}{\epsilon}\right)$.

As a preliminary result, we show how to use an oracle $\mathcal{O}_\alpha$ to find in polynomial time an α-approximate solution to BAYESIAN-OPT-SIGNAL (see Section 9.3). This problem is interesting in its own right, and allows us to develop a line of reasoning that will be essential to prove Theorem 10.6.

Theorem 10.5. *Given $\epsilon \in \mathbb{R}_+$ and an approximate separation oracle $\mathcal{O}_\alpha$, with $0 < \alpha \leq 1$, there exists a polynomial-time approximation algorithm for* BAYESIAN-OPT-SIGNAL *returning a signaling scheme with sender's utility at least α*OPT *$- \epsilon$, where* OPT *is the value of an optimal signaling scheme. Moreover, the algorithm works in time* poly$(\frac{1}{\epsilon})$.

Proof. The dual problem of LP 10.1 reads as follows:

$$\min_{\mathbf{z},\mathbf{d}} \quad \sum_{\theta \in \Theta} d_\theta \quad (10.6a)$$

$$\text{s.t. } \mu_\theta \sum_{r \in \mathcal{R}} \sum_{k \in s_r} u^{r,k}_\theta z_{r,s_r,k} + d_\theta \geq \mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}}) \; \forall \theta \in \Theta, \forall \mathbf{s} \in \mathcal{S} \quad (10.6b)$$

$$z_{r,s,k} \leq 0 \qquad \forall r \in \mathcal{R}, \forall s \in \mathcal{S}_r, \forall k \in \mathcal{K}_r : k \in s. \quad (10.6c)$$

where $\mathbf{d} \in \mathbb{R}^{|\Theta|}$ is the vector of dual variable corresponding to the primal Constraints (10.1c), and $\mathbf{z} \in \mathbb{R}_-^{|\mathcal{R} \times \mathcal{S}_r \times \mathcal{K}_r|}$ is the vector of dual variable corresponding to Constraints (10.1b) in the primal. We rewrite the dual LP 10.6 so as to highlight the relation between an approximate separation oracle for Constraints (10.6b) and the oracle $\mathcal{O}_\alpha$. Specifically, we have

$$\min_{\mathbf{z} \geq 0, \mathbf{d}} \quad \sum_{\theta \in \Theta} d_\theta \quad (10.7a)$$

$$\text{s.t. } d_\theta \geq \mu_\theta \left(\sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}}) + \sum_{r \in \mathcal{R}} \sum_{k \in s_r} u^{r,k}_\theta z_{r,s_r,k} \right) \quad (10.7b)$$

$$\forall \theta \in \Theta, \forall \mathbf{s} \in \mathcal{S}. \quad (10.7c)$$

Now, we show that it is possible to build a binary search scheme to find a value $\gamma^\star \in [0,1]$ such that the dual problem with objective $\gamma^\star$ is feasible, while the dual with objective $\gamma^\star - \beta$ is infeasible. The constant $\beta \geq 0$ will be specified later in the proof. The algorithm requires $\log(\beta)$ steps and works by determining, for a given value $\bar{\gamma} \in [0,1]$, whether there exists a feasible pair $(\mathbf{d}, \mathbf{z})$ for the following feasibility problem Ⓕ:

$$\text{Ⓕ} \begin{cases} \sum_{\theta \in \Theta} d_\theta \leq \bar{\gamma} \\ d_\theta \geq \mu_\theta \left(\sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}}) + \sum_{r \in \mathcal{R}} \sum_{k \in s_r} u_\theta^{r,k} z_{r,s_r,k} \right) & \forall \theta \in \Theta, \forall \mathbf{s} \in \mathcal{S} \\ \mathbf{z} \geq 0. \end{cases}$$

At each iteration of the bisection algorithm, the feasibility problem Ⓕ is solved via the ellipsoid method. The algorithm is inizialized with $l = 0$, $h = 1$, and $\bar{\gamma} = \frac{1}{2}$. If Ⓕ is infeasible for $\bar{\gamma}$, the algorithm sets $l \leftarrow (l+h)/2$ and $\bar{\gamma} \leftarrow (h + \bar{\gamma})/2$. Otherwise, if Ⓕ is (approximately) feasible, it sets $h \leftarrow (l+h)/2$ and $\bar{\gamma} \leftarrow (l+\bar{\gamma})/2$. Then, the procedure is repeated with the updated value of $\bar{\gamma}$. The bisection procedure terminates when it determines a value $\gamma^\star$ such that Ⓕ is feasible for $\bar{\gamma} = \gamma^\star$, while it is infeasible for $\bar{\gamma} = \gamma^\star - \beta$. In the following, we present the approximate separation oracle which is employed at each iteration of the ellipsoid method.

Separation Oracle Given a point $(\bar{\mathbf{d}}, \bar{\mathbf{z}})$ in the dual space, and $\bar{\gamma} \in [0,1]$, we design an approximate separation oracle to determine if the point $(\bar{\mathbf{d}}, \bar{\mathbf{z}})$ is approximately feasible, or to determine a constraint of Ⓕ that is violated by such point. For each $\theta \in \Theta$, $r \in \mathcal{R}$, and $s \in \mathcal{S}_r$, let

$$w^\theta_{r,s} := \mu_\theta \sum_{k \in s} u_\theta^{r,k} \bar{z}_{r,s,k}.$$

When the magnitude of the weights $|w^\theta_{r,s}|$ is small, we show that it is enough to employ the optimization oracle $\mathcal{O}_\alpha$ in order to find a violated constraint, or to certify that all the constraints are approximately satisfied. On the other hand, when the weights $|w^\theta_{r,s}|$ are large (in particular, when the largest weight has exponential size in the size of the problem instance), the optimization oracle $\mathcal{O}_\alpha$ loses its polynomial time guarantees (see Definition 10.5). We show how to handle those specific settings in the following case analysis:

- Equation (10.7c) implies that $d_\theta \geq 0$ for each $\theta \in \Theta$. Then, if there exists a $\theta \in \Theta$ such that $\bar{d}_\theta < 0$, we return the violated constraint $(\theta, \varnothing)$ (that is, $d_\theta \geq 0$).
- If there exists $\theta \in \Theta$ such that $\bar{d}_\theta > 1$, then the first constraint of Ⓕ must be violated as $\bar{\gamma} \in [0, 1]$.
- If there exists a receiver $r \in \mathcal{R}$ and a signal $s \in \mathcal{S}_r$ such that $w^\theta_{r,s} > 1$, then the constraint of Ⓕ corresponding to the pair (θ, s) is violated, because $d_\theta \leq 1$.
- If no violated constraint was found in the previous steps, we proceed by checking if there exists a state $\theta' \in \Theta$, a receiver $r' \in \mathcal{R}$, and a signal $s' \in \mathcal{S}_r$, such that $w^{\theta'}_{r',s'} \leq -|\mathcal{R}|$. If this is the case, we observe that for any pair (θ', s), with $\mathrm{s} \in \mathcal{S} : s_r = s'$, the corresponding constraint in Ⓕ reads

$$\mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathrm{s}}) + \sum_{r \in \mathcal{R} \setminus \{r'\}} w^{\theta'}_{r,s_r} + w^{\theta'}_{r',s'} \leq 0,$$

  since $\bar{\mathbf{d}} \geq 0$ if the current step is reached. For $w^{\theta'}_{r',s'} \leq -|\mathcal{R}|$ the above constraints are trivially satisfied, and therefore we can safely manage (for the current iteration of the ellipsoid method) any such constraint by setting $w^{\theta'}_{r',s'} = -|\mathcal{R}|$.

If none of the previous steps returned a violated constraint, we can safely assume that $0 \leq d_\theta \leq 1$ and $-|\mathcal{R}| \leq w^\theta_{r,s} \leq 1$, for each $\theta \in \Theta$, $r \in \mathcal{R}$, and $s \in \mathcal{S}_r$. Moreover, we observe that, by definition, for each $r \in \mathcal{R}$ and $\theta \in \Theta$, it holds $w^\theta_{r,\varnothing} = 0$. Since the magnitude of the weights is guaranteed to be small (that is, weights are guaranteed to be in the range $[-|\mathcal{R}|, 1]$), for each $\theta \in \Theta$ we can invoke $\mathcal{O}_\alpha(\theta, \mathcal{K}, \lambda, \mathbf{w}^\theta, \delta)$ to determine an $\mathrm{s}^\theta \in \mathcal{S}$ such that

$$\mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathrm{s}^\theta}) + \sum_{r \in \mathcal{R}} w^\theta_{r,s^\theta_r} \geq \max_{\mathrm{s} \in \mathcal{S}} \left\{ \alpha \mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R^{\boldsymbol{k}}_{\mathrm{s}}) + \sum_{r \in \mathcal{R}} w^\theta_{r,s_r} \right\} - \delta,$$

where δ is an approximation error that will be defined in the following. If at least one s^θ is such that $(\theta, \mathrm{s}^\theta)$ is violated, we output that constraint, otherwise the algorithm returns that the LP is feasible.

Putting It All Together The bisection algorithm computes a $\gamma^\star \in [0, 1]$ and a pair $(\bar{\mathbf{d}}, \bar{\mathbf{z}})$ such that the approximate separation oracle does not find a violated constraint. The following lemma defines a modified LP and shows that $(\bar{\mathbf{d}}, \bar{\mathbf{z}})$ is a feasible solution for this problem and has value at most $\gamma^\star$.

Lemma 10.1. *The pair* $(\bar{\mathbf{d}}, \bar{\mathbf{z}})$ *is a feasible solution to the following LP and has value at most* $\gamma^\star$*:*

$$\begin{aligned} \min_{\mathbf{z}\geq 0,\mathbf{d}} \quad & \sum_{\theta\in\Theta} d_\theta \\ \text{s.t. } & d_\theta \geq \alpha\mu_\theta \sum_{\boldsymbol{k}\in\mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}}) + \mu_\theta \sum_{r\in\mathcal{R}} \sum_{k\in s_r} u_\theta^{r,k} z_{r,s_r,k} - \delta \quad \forall\theta\in\Theta, \forall \mathbf{s}\in\mathcal{S}. \end{aligned}$$

Proof. The value is at most $\gamma^\star$ by assumption (that is, the separation oracle does not find a violated constraint for $(\bar{\mathbf{d}}, \bar{\mathbf{z}})$ in Ⓕ with objective $\gamma^\star$). Analogously, it holds that $\bar{d}_\theta \in [0,1]$ for each $\theta\in\Theta$, and $w_{r,s}^\theta \leq 1$ for each $r\in\mathcal{R}$, $s\in\mathcal{S}_r$, and $\theta\in\Theta$. Suppose, by contradiction, that (θ, s') is a violated constraint of the modified LP above. Then, given $\bar{\mathbf{d}}$, oracle $\mathcal{O}_\alpha$ would have found an $\mathrm{s}\in\mathcal{S}$ such that

$$\begin{aligned} & \mu_\theta \sum_{\boldsymbol{k}\in\mathcal{K}} \lambda_{\boldsymbol{k}} f(R_{\mathbf{s}}^{\boldsymbol{k}}) + \mu_\theta \sum_{r\in\mathcal{R}} \sum_{k\in s_r} u_\theta^{r,k} \bar{z}_{r,s_r,k} \\ & \qquad \geq \alpha \sum_{\theta\in\Theta} \mu_\theta \sum_{\boldsymbol{k}\in\mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R_{\mathbf{s}'}^{\boldsymbol{k}}) + \mu_\theta \sum_{r\in\mathcal{R}} \sum_{k\in s'_r} u_\theta^{r,k} \bar{z}_{r,s'_r,k} - \delta > \bar{d}_\theta, \end{aligned}$$

where the first inequality follows by Definition 10.5, and the second from the assumption that the modified dual is infeasible. Hence, $\mathcal{O}_\alpha$ would return a violated constraint, reaching a contradiction. □

The dual problem of the LP of Lemma 10.1 reads as follows:

$$\begin{aligned} \max_\phi & \sum_{\mathbf{s}\in\mathcal{S}} \sum_{\theta\in\Theta} \phi_\theta(\mathbf{s}) \left(\alpha\mu_\theta \sum_{\boldsymbol{k}\in\mathcal{K}} \lambda_{\boldsymbol{k}}\, f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}}) - \delta \right) \\ \text{s.t. } & \sum_{\theta\in\Theta} \mu_\theta \sum_{\mathbf{s}: s_r = s'} \phi_\theta(\mathbf{s})\, u_\theta^{r,k} \geq 0 \qquad \forall r\in\mathcal{R}, \forall s'\in\mathcal{S}_r, \forall k\in\mathcal{K}_r : k\in s' \\ & \sum_{\mathbf{s}\in\mathcal{S}} \phi_\theta(\mathbf{s}) = 1 \qquad \forall\theta\in\Theta \\ & \phi_\theta(\mathbf{s}) \geq 0 \qquad \forall\theta\in\Theta, \mathbf{s}\in\mathcal{S}. \end{aligned}$$

By strong duality, Lemma 10.1 implies that the value of the above problem is at most $\gamma^\star$. Then, let OPT be value of the optimal solution to LP 10.1. The same solution is feasible for the LP we just described, where it has value

$$\alpha\text{OPT} - |\Theta|\delta \leq \gamma^\star. \tag{10.10}$$

Now, we show how to find a solution to the original problem (LP 10.1) with value at least $\gamma^\star - \beta$. Let $\mathcal{H}$ be the set of constraints returned by the ellipsoid method run on the feasibility problem Ⓕ with objective $\gamma^\star - \beta$.

Lemma 10.2. *LP 10.1 with variables restricted to those corresponding to dual constraints $\mathcal{H}$ returns a signaling scheme with value at least $\gamma^\star - \beta$. Moreover, the solution can be determined in polynomial time.*

Proof. By construction of the bisection algorithm, Ⓕ is infeasible for value $\gamma^\star - \beta$. Hence, the following LP has value at least $\gamma^\star - \beta$:

$$\min_{\mathbf{z}\geq 0,\mathbf{d}} \quad \sum_{\theta\in\Theta} d_\theta$$

$$\text{s.t.} \quad d_\theta \geq \mu_\theta \left(\sum_{\boldsymbol{k}\in\mathcal{K}} \lambda_{\boldsymbol{k}} f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}}) + \sum_{r\in\mathcal{R}} \sum_{k\in s_r} u_\theta^{r,k} z_{r,s_r,k} \right) \qquad \forall(\theta,\mathbf{s})\in\mathcal{H}.$$

Notice that the primal of the above LP is exactly LP 10.1 with variables restricted to those corresponding to dual constraints in $\mathcal{H}$, and that the former (restricted) LP has value at least $\gamma^\star - \beta$ by strong duality. To conclude the proof, the ellipsoid method guarantees that $\mathcal{H}$ is of polynomial size. Hence, the LP can be solved in polynomial time. □

Let APX be the value of an optimal solution to LP 10.1 restricted to variables corresponding to dual constraints in $\mathcal{H}$. Then,

$$\begin{aligned}\text{APX} &\geq \gamma^\star - \beta \\ &\geq \alpha\text{OPT} - |\Theta|\delta - \beta \\ &\geq \alpha\text{OPT} - \epsilon,\end{aligned}$$

where the first inequality holds by Lemma 10.2, the second inequality follows from Equation (10.10), and the last inequality is obtained by setting $\delta = \frac{\epsilon}{2|\Theta|}$ and $\beta = \frac{\epsilon}{2}$. □

10.5.2 From an Approximate Separation Oracle to an Approximate Projection Oracle

Now, we show how to design a polynomial-time approximate projection oracle φ_α using an approximate separation oracle $\mathcal{O}_\alpha$. The proof employs a convex linearly-constrained quadratic program that computes the optimal projection on $\mathcal{X}$, the ellipsoid method, and a careful primal-dual analysis.

Theorem 10.6. *Given a subset $K \subseteq \mathcal{K}$, a vector $\boldsymbol{y} \in [0,2]^{|\mathcal{K}|}$ such that $y_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \notin K$, and an approximation error $\epsilon \in \mathbb{R}_+$, for any $0 < \alpha \leq 1$, the approximate projection oracle $\varphi_\alpha(K, \boldsymbol{y}, \epsilon)$ can be computed in polynomial time by querying the approximate separation oracle $\mathcal{O}_\alpha$.*

Proof. The problem of computing the projection of point $\mathbf{y}$ on $\mathcal{X}_K$ (see Equation (10.2)) can be formulated via the following convex programming problem, which we denote by Ⓟ:

$$
\text{Ⓟ}\begin{cases}
\min_{\phi,\mathbf{x}} \sum_{\boldsymbol{k}\in K}(x_{\boldsymbol{k}}-y_{\boldsymbol{k}})^2 \\
\text{s.t.} \sum_{\theta\in\Theta}\mu_\theta\left(\sum_{\substack{\mathbf{s}\in\mathcal{S}:\\ s_r=s'}}\phi_\theta(\mathbf{s})u_\theta^{r,k}\right)\geq 0 \;\forall r\in\mathcal{R},\forall s'\in\mathcal{S}_r,\forall k\in\mathcal{K}_r: k\in s' \\
\sum_{\mathbf{s}\in\mathcal{S}}\phi_\theta(\mathbf{s})=1 \qquad \forall\theta\in\Theta \\
\phi_\theta(\mathbf{s})\geq 0 \qquad \forall\theta\in\Theta,\forall \mathbf{s}\in\mathcal{S} \\
x_{\boldsymbol{k}}\leq\sum_{\theta\in\Theta}\sum_{\mathbf{s}\in\mathcal{S}}\mu_\theta\,\phi_\theta(\mathbf{s})f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}}) \qquad \forall\boldsymbol{k}\in K.
\end{cases}
$$

Then, we compute the Lagrangian of Ⓟ by introducing dual variables $z_{r,s,k}\leq 0$ for each $r\in\mathcal{R}$, $s\in\mathcal{S}_r$, and $k\in s$, $d_\theta\in\mathbb{R}$ for each $\theta\in\Theta$, $v_{\theta,\mathbf{s}}\leq 0$ for each $\theta\in\Theta$, $\mathbf{s}\in\mathcal{S}$, and $\nu_{\boldsymbol{k}}\geq 0$ for each $\boldsymbol{k}\in K$. Specifically, the Lagrangian of Ⓟ reads as follows

$$
\begin{aligned}
L(\phi,\mathbf{x},\mathbf{z},\mathbf{v},\boldsymbol{\nu},\mathbf{d}) := &\sum_{\boldsymbol{k}\in K}(x_{\boldsymbol{k}}-y_{\boldsymbol{k}})^2 \\
&+\sum_{r\in\mathcal{R}}\sum_{s'\in\mathcal{S}_r}\sum_{k\in s'} z_{r,s,k}\left(\sum_{\theta\in\Theta}\mu_\theta\sum_{\mathbf{s}:s_r=s'}\phi_\theta(\mathbf{s})\,u_\theta^{r,k}\right) \\
&+\sum_{\theta\in\Theta,\mathbf{s}\in\mathcal{S}} v_{\theta,\mathbf{s}}\phi_\theta(\mathbf{s})+\sum_{\theta\in\Theta}d_\theta\left(\sum_{\mathbf{s}\in\mathcal{S}}\phi_\theta(\mathbf{s})-1\right) \\
&+\sum_{\boldsymbol{k}\in K}\nu_{\boldsymbol{k}}\left(x_{\boldsymbol{k}}-\sum_{\theta\in\Theta,\mathbf{s}\in\mathcal{S}}\mu_\theta\phi_\theta(\mathbf{s})f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}})\right).
\end{aligned}
$$

We observe that Slater's condition holds for Ⓟ (all constraints are linear, and by setting $\mathbf{x}=0$ any signaling scheme ϕ constitutes a feasible solution). Therefore, by strong duality, an optimal dual solution must satisfy the KKT conditions. In particular, in order for stationarity to hold, it must be $\mathbf{0}\in\partial_{\phi_\theta(\mathbf{s})}(L)$ for each $\mathbf{s}$ and θ. Then, for each $\theta\in\Theta$ and $\mathbf{s}\in\mathcal{S}$, we have

$$
\partial_{\phi_\theta(\mathbf{s})}(L)=\sum_{r\in\mathcal{R}}\sum_{k\in s_r}\mu_\theta z_{r,s_r,k}u_\theta^{r,k}+v_{\theta,\mathbf{s}}+d_\theta-\sum_{\boldsymbol{k}\in K}\nu_{\boldsymbol{k}}\mu_\theta f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}})=0.
$$

Then, for each $\theta \in \Theta$ and $\mathrm{s} \in \mathcal{S}$, we obtain

$$\sum_{r\in\mathcal{R}}\sum_{k\in s_r} \mu_\theta z_{r,s_r,k} u_\theta^{r,k} + d_\theta - \sum_{\boldsymbol{k}\in K} \nu_{\boldsymbol{k}} \mu_\theta f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}})) \geq 0. \tag{10.12}$$

Moreover, stationarity has to hold with respect to variables $\mathbf{x}$. Formally, for each $\boldsymbol{k} \in K$,

$$\partial_{x_{\boldsymbol{k}}}(L) = 2(x_{\boldsymbol{k}} - y_{\boldsymbol{k}})\nu_{\boldsymbol{k}} = 0.$$

Therefore, for each $\boldsymbol{k} \in K$,

$$x_{\boldsymbol{k}} = y_{\boldsymbol{k}} - \frac{\nu_{\boldsymbol{k}}}{2}. \tag{10.13}$$

By Equations (10.12) and (10.13), we obtain the following dual quadratic program

$$\text{Ⓓ} \begin{cases} \max\limits_{\mathbf{z},\mathbf{v},\boldsymbol{\nu},\mathbf{d}} \ \sum\limits_{\boldsymbol{k}\in K}\left(\nu_{\boldsymbol{k}} y_{\boldsymbol{k}} - \dfrac{\nu_{\boldsymbol{k}}^2}{4}\right) - \sum\limits_{\theta\in\Theta} d_\theta & \\ \text{s.t.} \quad d_\theta \geq \sum\limits_{\boldsymbol{k}\in K} \nu_{\boldsymbol{k}} \mu_\theta f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}}) + \sum\limits_{r\in\mathcal{R}}\sum\limits_{k\in s_r} \mu_\theta z_{r,s_r,k} u_\theta^{r,k} & \forall \theta \in \Theta, \forall \mathrm{s} \in \mathcal{S} \\ \quad\quad z_{r,s,k} \geq 0 & \forall r \in \mathcal{R}, \forall s \in \mathcal{S}_r, \forall k \in \mathcal{K}_r : k \in s \\ \quad\quad \nu_{\boldsymbol{k}} \geq 0 & \forall \boldsymbol{k} \in K. \end{cases}$$

in which the objective function is obtained by observing that each term $\phi_\theta(\mathrm{s})$ in the definition of L is multiplied by $\partial_{\phi_\theta(\mathrm{s})}(L)$, which has to be equal to zero by stationarity. Similarly to what we did in the proof of Theorem 10.5, we repeatedly apply the ellipsoid method equipped with an approximate separation oracle to problem Ⓓ. In this case, the analysis is more involved than what happens in Theorem 10.5, because we are interested in computing an approximate projection on $\alpha\mathcal{X}_K$ rather than an approximate solution of Ⓟ. We proceed by casting Ⓓ as a feasibility problem with a certain objective (analogously to Ⓕ in Theorem 10.5). In particular, given objective $\gamma \in [0, 1]$, the objective function of Ⓓ becomes the following constraint in the feasibility problem

$$\sum_{\boldsymbol{k}\in K}\left(\nu_{\boldsymbol{k}} y_{\boldsymbol{k}} - \frac{\nu_{\boldsymbol{k}}^2}{4}\right) - \sum_{\theta\in\Theta} d_\theta \geq \gamma. \tag{10.14}$$

Then, given an approximation oracle $\mathcal{O}_\alpha$ which will be specified later, we apply to the feasibility problem the search algorithm described in Algorithm 10.2.

Algorithm 10.2 SEARCH ALGORITHM

Input: Error ϵ, $\mathbf{y} \in \mathbb{R}_+^{|\mathcal{K}|}$, subspace $K \subseteq \mathcal{K}$.

Initialization: $\beta \leftarrow \frac{\epsilon}{2}$, $\delta \leftarrow \frac{\epsilon}{2|\Theta|}$, $\gamma \leftarrow |K| + \beta$, and $\mathcal{H} \leftarrow \varnothing$.
repeat
  $\gamma \leftarrow \gamma - \beta$
  $\mathcal{H}_{\text{UNF}} \leftarrow \mathcal{H}$
  $\mathcal{H} \leftarrow \{$violated constraints returned by the ellipsoid method on $\textcircled{D}$, with objective $\gamma$and constraints $\mathcal{H}_{\text{UNF}}\}$
until $\textcircled{D}$ is feasible with objective γ (see Equation (10.14))
return $\mathcal{H}_{\text{UNF}}$

At each iteration of the main loop, given an objective value γ, Algorithm 10.2 checks whether the problem $\textcircled{D}$ is approximately feasible or unfeasible, by applying the ellipsoid algorithm with separation oracle $\mathcal{O}_\alpha$. Let $\mathcal{H}$ be the set of constraints returned by the separation oracle (the separating hyperplanes due to the linear inequalities). At each iteration, the ellipsoid method is applied on the problem with explicit constraints in the current set $\mathcal{H}_{\text{UNF}}$ (that is, each constraint in $\mathcal{H}_{\text{UNF}}$ is explicitly checked for feasibility), while the other constraints are checked through the approximate separation oracle. Algorithm 10.2 returns the set of violated constraints $\mathcal{H}_{\text{UNF}}$ corresponding to the last value of γ for which the problem was unfeasible. Now, we describe how to implement the approximate separation oracle employed in Algorithm 10.2. Then, we conclude the proof by showing how to build an approximate projection starting from the set $\mathcal{H}_{\text{UNF}}$ computed as we just described.

Approximate Separation Oracle Let $(\bar{\mathbf{z}}, \bar{\mathbf{v}}, \bar{\boldsymbol{\nu}}, \bar{\mathbf{d}})$ be a point in the space of dual variables. Then, let, for each $\theta \in \Theta$, $r \in \mathcal{R}$, and $s \in \mathcal{S}_r$,

$$w_{r,s}^{\theta} := \sum_{k \in s} \bar{z}_{r,s,k} \mu_\theta u_\theta^{r,k}.$$

First, we can check in polynomial time if one of the constraint in $\mathcal{H}$ is violated. If at least one of those constraint is violated, we output that constraint. Moreover, if the constraint corresponding to the objective is violated, we can output a separation hyperplane in polynomial time since the constraint has a polynomial number of variables. Then, by following the same rationale of the proof of Theorem 10.5 (offline setting), we proceed with a case analysis in which we ensure it is possible to output a violated constraint when $|\nu_{\boldsymbol{k}}|$ or $|w_{r,s}^{\theta}|$ are too large to guarantee polynomial-time sovability by Definition 10.5.

- First, it has to hold $d_\theta \in [0, 4|K|]$ for each $\theta \in \Theta$. Indeed, if $d_\theta < 0$, then the constraint relative to $(\theta, \varnothing)$ would be violated. Otherwise, suppose that there exists a θ with $\bar{d}_\theta > 4|K|$. Two cases are possible: (i) the constraint corresponding to the objective is violated, which allows us to output a separation hyperplane; (ii) it holds

$$\sum_{\boldsymbol{k} \in K} \left(\bar{\nu}_{\boldsymbol{k}} y_{\boldsymbol{k}} - \frac{\bar{\nu}_{\boldsymbol{k}}^2}{4} \right) > 4|K|,$$

which implies that there exists a $\boldsymbol{k} \in K$ such that $\bar{\nu}_{\boldsymbol{k}} y_{\boldsymbol{k}} - \bar{\nu}_{\boldsymbol{k}}^2/4 > 4$. However, we reach a contradiction since, by assumption, $y_{\boldsymbol{k}} \leq 2$ for each $\boldsymbol{k} \in K$, and therefore it must hold $\bar{\nu}_{\boldsymbol{k}} y_{\boldsymbol{k}} - \bar{\nu}_{\boldsymbol{k}}^2/4 \leq 2\bar{\nu}_{\boldsymbol{k}} - \bar{\nu}_{\boldsymbol{k}}^2/4 \leq 4$.

- Second, we show how to determine a violated constraint when $\bar{\nu}_{\boldsymbol{k}} \notin [0, |K| + 10]$. Specifically, if there exists a $\boldsymbol{k} \in K$ for which $\bar{\nu}_{\boldsymbol{k}} < 0$, then the objective is negative, and we can return a separation hyperplane (corresponding to Equation (10.14)). If there exists a $\nu_{\boldsymbol{k}} > |K| + 10$, then

$$\begin{aligned}
\sum_{\boldsymbol{k}' \in K} \left(\bar{\nu}_{\boldsymbol{k}'} y_{\boldsymbol{k}'} - \frac{\bar{\nu}_{\boldsymbol{k}'}^2}{4} \right) &\leq 2\nu_{\boldsymbol{k}} - \frac{\bar{\nu}_{\boldsymbol{k}}^2}{4} + \sum_{\boldsymbol{k}' \in K \setminus \{\boldsymbol{k}\}} \left(2\bar{\nu}_{\boldsymbol{k}'} - \frac{\bar{\nu}_{\boldsymbol{k}'}^2}{4} \right) \\
&\leq 2|K| + 20 - \frac{|K|^2}{4} - 5|K| - 25 + 4|K| \\
&= -\frac{|K|^2}{4} + |K| - 5 \\
&< 0,
\end{aligned}$$

where the first inequality follows by the assumption that $y_{\boldsymbol{k}} \leq 2$ for each $\boldsymbol{k} \in K$, and the second inequality follows from the fact that $2\nu_{\boldsymbol{k}} - \bar{\nu}_{\boldsymbol{k}}^2/4$ has its maximum in $\bar{\nu}_{\boldsymbol{k}} = 4$ and, when $\bar{\nu}_{\boldsymbol{k}} \geq |K| + 10$, the maximum is at $\bar{\nu}_{\boldsymbol{k}} = |K| + 10$ since the function in concave. Hence, we obtain that Constraint (10.14) is violated.

- Finally, suppose that there exists a $\theta \in \Theta$, $r \in \mathcal{R}$, $s \in \mathcal{S}_r$ such that $w_{r,s}^\theta > 4|K|$. Then, the constraint corresponding to (θ, s) is violated (because $d_\theta \leq 4|K|$, otherwise we would have already determined a violated constraint in the first case of our analysis). If, instead, there exists a $\theta \in \Theta$, $r \in \mathcal{R}$, $s \in \mathcal{S}_r$ such that $w_{r,s}^\theta < -4|K||\mathcal{R}| - 10$, then, for all the inequalities (θ, s') with $s'_r = s$, it holds $\bar{d}_\theta \geq 0$ and

$$\mu_\theta \sum_{\boldsymbol{k} \in K} \bar{\nu}_{\boldsymbol{k}} f_\theta(R_{\mathrm{s}}^{\boldsymbol{k}}) + \sum_{r' \in \mathcal{R} \setminus \{r\}} w_{r', s'_{r'}}^\theta + w_{r, s'_r}^\theta \leq 0.$$

In this last case, all the inequalities corresponding to (θ, s') with $s'_r = s$ are guaranteed to be satisfied. Then, we can safely manage all the inequalities comprising of $w^\theta_{r,s} \leq -4|K||\mathcal{R}| - 10$ by setting $w^\theta_{r,s} = -4|K||\mathcal{R}| - 10$.

After the previous steps, it is guaranteed that $|w^\theta_{r,s}| \leq 4|K||\mathcal{R}|+10$ for each θ, r, s, and $\nu_{\boldsymbol{k}} \in [0, |K| + 10]$ for each $\boldsymbol{k}$. Hence, we can employ an oracle $\mathcal{O}_\alpha$ with $|w^\theta_{r,s}|$ and $\lambda^\theta_{\boldsymbol{k}} = \nu_{\boldsymbol{k}}\mu_\theta$, which is guaranteed to be polynomial in the size of the instance by Definition 10.5. Let δ be an error parameter which will be defined in the remainder of the proof. For each $\theta \in \Theta$, we call the oracle $\mathcal{O}_\alpha(\theta, K, \{\nu_{\boldsymbol{k}}\}_{\boldsymbol{k}\in K}, \mathrm{w}^\theta, \delta)$. Each query to the oracle returns an s^θ. If at least one of the constraints corresponding to a pair $(\theta, \mathrm{s}^\theta)$ is violated, we output that constraint. Otherwise, if for each $\theta \in \Theta$ the constraint $(\theta, \mathrm{s}^\theta)$ is satisfied, we conclude that the point is in the feasible region.

Putting It All Together Algorithm 10.2 terminates at objective $\gamma^\star$. It is easy to see that the algorithm terminates in polynomial time because it must return *feasible* when $\gamma = 0$. Our proof proceeds in two steps. First, we prove that a particular problem obtained from Ⓟ has value at least $\gamma^\star$. Then, we prove that the solution of Ⓟ with only variables in $\mathcal{H}_{\text{UNF}}$ has value close to $\gamma^\star$. Finally, we show that the two solutions are, respectively, the projection and an approximate projection on a set that includes $\alpha\mathcal{X}_K$. This will complete the proof.

If the algorithm terminates at objective γ^*, the following convex optimization problem is feasible (see Theorem 10.5).[3]

$$\begin{cases} \displaystyle\sum_{\boldsymbol{k}\in K} \left(\nu_{\boldsymbol{k}} y_{\boldsymbol{k}} - \nu^2_{\boldsymbol{k}}/4\right) - \sum_{\theta\in\Theta} d_\theta \geq \gamma^\star \\ \displaystyle d_\theta \geq \sum_{\boldsymbol{k}\in K} \nu_{\boldsymbol{k}}\mu_\theta f_\theta(R^{\boldsymbol{k}}_{\mathrm{s}}) - \sum_{r\in\mathcal{R}, k\in s_r} z_{r,s_r,k}\, \mu_\theta u^{r,k}_\theta & \forall (\theta, \mathrm{s}) \in \mathcal{H}_{\text{UNF}} \\ \displaystyle d_\theta \geq \sum_{\boldsymbol{k}\in K} \alpha\nu_{\boldsymbol{k}}\mu_\theta f_\theta(R^{\boldsymbol{k}}_{\mathrm{s}}) - \sum_{r\in\mathcal{R}, k\in s_r} z_{r,s_r,k}\, \mu_\theta u^{r,k}_\theta - \delta & \forall (\theta, \mathrm{s}) \notin \mathcal{H}_{\text{UNF}}. \end{cases}$$

By strong duality, the following convex optimization problem has value at

[3] In the following, we will refer to the proof of Theorem 10.5 when the steps of the two proofs are analogous.

least $\gamma^\star$

$$
\text{(Pf)}\begin{cases}
\min\limits_{\phi,\mathbf{x}} \sum\limits_{\boldsymbol{k}\in K}(x_{\boldsymbol{k}}-y_{\boldsymbol{k}})^2+\delta\sum\limits_{(\theta,\mathbf{s})\notin\mathcal{H}_{\text{UNF}}}\phi_\theta(\mathbf{s}) \\
\text{s.t. } \sum\limits_{\theta\in\Theta}\mu_\theta\left(\sum\limits_{\mathbf{s}':s'_r=s}\phi_\theta(\mathbf{s}')u_\theta^{r,k}\right)\geq 0 \;\forall r\in\mathcal{R},\forall s\in\mathcal{S}_r,\forall k\in\mathcal{K}_r: k\in s \\
\sum\limits_{\mathbf{s}\in\mathcal{S}}\phi_\theta(\mathbf{s})=1 \qquad \forall\theta\in\Theta \\
\phi_\theta(\mathbf{s})\geq 0 \qquad \forall\theta\in\Theta,\forall\mathbf{s}\in\mathcal{S} \\
x_{\boldsymbol{k}}\leq\sum\limits_{\theta\in\Theta}\left(\sum\limits_{\mathbf{s}:(\theta,\mathbf{s})\in\mathcal{H}_{\text{UNF}}}\mu_\theta\,\phi_\theta(\mathbf{s})f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}})+\alpha\sum\limits_{\mathbf{s}:(\theta,\mathbf{s})\notin\mathcal{H}_{\text{UNF}}}\mu_\theta\,\phi_\theta(\mathbf{s})f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}})\right) \\
\qquad \forall\boldsymbol{k}\in K.
\end{cases}
$$

Moreover, since the algorithm did not terminate at value $\gamma^\star+\beta$, problem (D) with value $\gamma^\star+\beta$ is unfeasible when restricting the set of constraints to $\mathcal{H}_{\text{UNF}}$. The primal problem (P) restricted to primal variables corresponding to dual constraints in $\mathcal{H}_{\text{UNF}}$ reads as follows

$$
\begin{cases}
\min\limits_{\phi,\mathbf{x}} \sum\limits_{\boldsymbol{k}\in K}(x_{\boldsymbol{k}}-y_{\boldsymbol{k}})^2 \\
\text{s.t. } \sum\limits_{\theta\in\Theta}\mu_\theta\left(\sum\limits_{\substack{\mathbf{s}:(\theta,\mathbf{s})\in\mathcal{H}_{\text{UNF}},\\ s_r=s'}}\phi_\theta(\mathbf{s})u_\theta^{r,k}\right)\geq 0 \;\forall r\in\mathcal{R}, s'\in\mathcal{S}_r,\forall k\in\mathcal{K}_r: k\in s' \\
\sum\limits_{\mathbf{s}:(\theta,\mathbf{s})\in\mathcal{H}_{\text{UNF}}}\phi_\theta(\mathbf{s})=1 \qquad \forall\theta\in\Theta \\
\phi_\theta(\mathbf{s})\geq 0 \qquad \forall(\theta,\mathbf{s})\in\mathcal{H}_{\text{UNF}} \\
x_{\boldsymbol{k}}\leq\sum\limits_{\theta\in\Theta}\sum\limits_{\mathbf{s}:(\theta,\mathbf{s})\in\mathcal{H}_{\text{UNF}}}\mu_\theta\,\phi_\theta(\mathbf{s})f_\theta(R_{\mathbf{s}}^{\boldsymbol{k}}) \qquad \forall\boldsymbol{k}\in K.
\end{cases}
$$

By strong duality, the above problem has value at most $\gamma^\star+\beta$. Moreover, it has a polynomial number of variables and constraints because the ellipsoid method returns a set of constraints $\mathcal{H}_{\text{UNF}}$ of polynomial size. Therefore, the above problem can be solved in polynomial time.

A solution to the above problem is a feasible signaling scheme. Let

$(\mathbf{x}^\epsilon, \phi)$ be its solution. We have that $\mathbf{x}^\epsilon \in \bar{\mathcal{X}}_K$, with

$$\bar{\mathcal{X}}_K = \Big\{\mathbf{x} : x_{\boldsymbol{k}} \le \sum_{\theta\in\Theta}\Big(\sum_{\mathbf{s}:(\theta,\mathbf{s})\in\mathcal{H}_{\text{UNF}}} \mu_\theta\, \phi_\theta(\mathbf{s}) f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}}) + \alpha \sum_{\mathbf{s}:(\theta,\mathbf{s})\notin\mathcal{H}_{\text{UNF}}} \mu_\theta\, \phi_\theta(\mathbf{s}) f_\theta(R^{\boldsymbol{k}}_{\mathbf{s}})\Big) \quad \forall \boldsymbol{k}\in K, \phi\in\Phi\Big\}.$$

It holds $\alpha\mathcal{X}_K \subseteq \bar{\mathcal{X}}_K$. Now, we show that $\mathbf{x}^\epsilon$ is *close* to $\mathbf{x}^\star$, where $\mathbf{x}^\star$ is the projection of $\mathbf{y}$ on $\bar{\mathcal{X}}_K$ (that is the solution of (Pf) with $\delta = 0$). Since $x^\star$ is a feasible solution of (Pf) and the minimum of (Pf) is at least $\gamma^\star$, it holds $||\mathbf{x}^\star - \mathbf{y}||^2 + \delta|\Theta| \ge \gamma^\star$. Then,

$$\begin{aligned}
||\mathbf{x}^\star - \mathbf{y}||^2 + \delta|\Theta| + \beta &\ge \gamma^\star + \beta \\
&\ge ||\mathbf{x}^\epsilon - \mathbf{y}||^2 \\
&= ||\mathbf{x}^\epsilon - \mathbf{x}^\star + \mathbf{x}^\star - \mathbf{y}||^2 \\
&= ||\mathbf{x}^\epsilon - \mathbf{x}^\star||^2 + ||\mathbf{x}^\star - \mathbf{y}||^2 + 2\langle \mathbf{x}^\epsilon - \mathbf{x}^\star, \mathbf{x}^\star - \mathbf{y}\rangle \\
&\ge ||\mathbf{x}^\epsilon - \mathbf{x}^\star||^2 + ||\mathbf{x}^\star - \mathbf{y}||^2,
\end{aligned}$$

where the last inequality follows from $\langle \mathbf{x}^\epsilon - \mathbf{x}^\star, \mathbf{x}^\star - \mathbf{y}\rangle \ge 0$, because $\mathbf{x}^\star$ is the projection of $\mathbf{y}$ on $\bar{\mathcal{X}}_K$ and $\mathbf{x}^\epsilon \in \bar{\mathcal{X}}_K$. Hence, $||\mathbf{x}^\epsilon - \mathbf{x}^\star||^2 \le \delta|\Theta| + \beta$. Finally, let $\mathbf{x}$ be a point in $\alpha\mathcal{X}_K$. Then,

$$\begin{aligned}
||\mathbf{x}^\epsilon - \mathbf{x}||^2 &\le ||\mathbf{x}^\epsilon - \mathbf{x}^\star||^2 + ||\mathbf{x}^\star - \mathbf{x}||^2 \\
&\le ||\mathbf{x}^\epsilon - \mathbf{x}^\star||^2 + ||\mathbf{y} - \mathbf{x}||^2 \\
&\le ||\mathbf{y} - \mathbf{x}||^2 + \delta|\Theta| + \beta,
\end{aligned}$$

where the second inequality follow from the fact that $\mathbf{x}^\star$ is the projection of $\mathbf{y}$ on a superset of $\alpha\mathcal{X}_K$. Setting $\delta = \frac{\epsilon}{2|\Theta|}$ and $\beta = \frac{\epsilon}{2}$ concludes the proof. □

10.6 A Poly-Time No-α-Regret Algorithm for Submodular Sender's Utilities

In this section, we conclude the construction of our polynomial-time no-$(1-\frac{1}{e})$-regret algorithm for settings in which sender's utilities are submodular. The last component that we need to design is an approximate separation oracle $\mathcal{O}_\alpha$ (see Definition 10.5) running in polynomial time. Next, we show how to obtain this by exploiting the fact that functions f_θ are submodular in the set of receivers playing action a_1.

First, we establish a relation between direct signals $\mathcal{S}$ and matroids. We define a matroid $\mathcal{M}_\mathcal{S} := (\mathcal{G}_\mathcal{S}, \mathcal{I}_\mathcal{S})$ such that:

- the ground set is $\mathcal{G}_\mathcal{S} := \{(r,s) \mid r \in \mathcal{R}, s \in \mathcal{S}_r\}$;
- a subset $I \subseteq \mathcal{G}_\mathcal{S}$ belongs to $\mathcal{I}_\mathcal{S}$ if and only if I contains *at most one* pair for each receiver $r \in \mathcal{R}$.

The elements of the ground set $\mathcal{G}_\mathcal{S}$ represent receiver, signal pairs. However, sets $I \in \mathcal{I}_\mathcal{S}$ do *not* characterize signal profiles, as they may not define a signal for each receiver. Indeed, direct signal profiles are captured by the basis set $\mathcal{B}(\mathcal{M}_\mathcal{S})$ of the matroid $\mathcal{M}_\mathcal{S}$. Let us recall that $\mathcal{B}(\mathcal{M}_\mathcal{S})$ contains all the maximal sets in $\mathcal{I}_\mathcal{S}$, and, thus, a subset $I \subseteq \mathcal{I}_\mathcal{S}$ belongs to $\mathcal{B}(\mathcal{M}_\mathcal{S})$ if and only if I contains *exactly one* pair for each receiver $r \in \mathcal{R}$. Intuitively, a basis $I \in \mathcal{B}(\mathcal{M}_\mathcal{S})$ defines a direct signal profile $\mathrm{s} \in \mathcal{S}$ in which, for each receiver $r \in \mathcal{R}$, all the receiver's types in $s \in \mathcal{S}_r$ such that $(r,s) \in I$ are recommended to play action a_1, while the others are told to play a_0.

The following Theorem 10.7 provides a polynomial-time approximation oracle $\mathcal{O}_{1-\frac{1}{e}}$ for instances in which f_θ is submodular for each state of nature $\theta \in \Theta$. The core idea of its proof is that $\sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} f_\theta(R_{\mathrm{s}}^{\boldsymbol{k}})$ (see Equation (10.5)) can be seen as a submodular function defined for the ground set $\mathcal{G}_\mathcal{S}$ and optimizing over direct signal profiles $\mathrm{s} \in \mathcal{S}$ is equivalent to doing that over the bases $\mathcal{B}(\mathcal{M}_\mathcal{S})$ of the matroid $\mathcal{M}_\mathcal{S}$. Then, the result is readily proved by exploiting some results concerning the optimization over matroids.[4]

Theorem 10.7. *If the sender's utility is such that function f_θ is* submodular *for each $\theta \in \Theta$, then there exists a polynomial-time separation oracle $\mathcal{O}_{1-\frac{1}{e}}$.*

To prove Theorem 10.7, we need some preliminary results concerning the optimization over matroids. Given a *non-decreasing submodular* set function $f : 2^\mathcal{G} \to \mathbb{R}_+$ and a *linear* set function $\ell : 2^\mathcal{G} \ni I \mapsto \sum_{i \in I} w_i$ defined for finite ground set $\mathcal{G}$ and weights $\mathrm{w} = (w_i)_{i \in \mathcal{G}}$ with $w_i \in \mathbb{R}$ for each $i \in \mathcal{G}$, let us consider the problem of maximizing the sum $f(I) + \ell(I)$ over the bases $I \in \mathcal{B}(\mathcal{M})$ of a given matroid $\mathcal{M} := (\mathcal{G}, \mathcal{I})$. We make use of a theorem due to Sviridenko et al. (2017), which, by letting $v_f := \max_{I \in 2^\mathcal{G}} f(I)$, $v_\ell := \max_{I \in 2^\mathcal{G}} |\ell(I)|$, and $v := \max\{v_f, v_\ell\}$, reads as follows:

Theorem 10.8 (Essentially Theorem 3.1 by Sviridenko et al. (2017))**.** *For every $\epsilon > 0$, there exists an algorithm running in time* poly$\left(|\mathcal{G}|, \frac{1}{\epsilon}\right)$ *that*

[4] The separation oracle provided in Theorem 10.7 guarantees the desired approximation factor with arbitrary high probability. It is easy to see that, since the algorithm fails with arbitrary small probability, this does not modify our regret bound except for an (arbitrary small) negligible term.

produces a basis $I \in \mathcal{B}(\mathcal{M})$ *satisfying* $f(I) + \ell(I) \geq \left(1 - \frac{1}{e}\right) f(I') + \ell(I') - O(\epsilon)v$ *for every* $I' \in \mathcal{B}(\mathcal{M})$ *with high probability.*

Next, we provide the proof of Theorem 10.7.

Proof of Theorem 10.7. We show how to implement an approximation oracle $\mathcal{O}_\alpha(\theta, K, \boldsymbol{\lambda}, \mathrm{w}, \epsilon)$ (see Definition 10.5) with $\alpha = 1 - \frac{1}{e}$ that has a running time poly $\left(n, |K|, \max_{r,s} |w_{r,s}|, \max_{\boldsymbol{k}} \lambda_{\boldsymbol{k}}, \frac{1}{\epsilon}\right)$. Let $\mathcal{M}_{\mathcal{S}} := (\mathcal{G}_{\mathcal{S}}, \mathcal{I}_{\mathcal{S}})$ be a matroid defined as in Section 10.6 for direct signal profiles $\mathcal{S}$. Let us recall that, given the relation between the bases of $\mathcal{M}_{\mathcal{S}}$ and direct signals, each direct signal profiles $\mathrm{s} \in \mathcal{S}$ corresponds to a basis $I \in \mathcal{B}(\mathcal{M}_{\mathcal{S}})$, which is defined as $I := \{(r, s_r) \mid r \in \mathcal{R}\}$. In the following, given a subset $I \subseteq \mathcal{G}_{\mathcal{S}}$ and a type profile $\boldsymbol{k} \in K$, we let $R_I^{\boldsymbol{k}} \subseteq \mathcal{R}$ be the set of receivers $r \in \mathcal{R}$ such that there exits a pair $(r, s) \in I$ (for some signal $s \in \mathcal{S}_r$) with the receiver's type k_r being recommended to play a_1 under signal s; formally,

$$R_I^{\boldsymbol{k}} := \{r \in \mathcal{R} \mid \exists (r, s) \in I : k_r \in s\}.$$

First, we show that, when using matroid notation, the left-hand side of Equation (10.5) can be expressed as the sum of a non-decreasing submodular set function and a linear set function. To this end, let $f_\theta^{\boldsymbol{\lambda}} : 2^{\mathcal{G}_{\mathcal{S}}} \to \mathbb{R}_+$ be defined as $f_\theta^{\boldsymbol{\lambda}}(I) = \sum_{\boldsymbol{k} \in K} \lambda_{\boldsymbol{k}} f_\theta(R_I^{\boldsymbol{k}})$ for every subset $I \subseteq \mathcal{G}_{\mathcal{S}}$. We prove that $f_\theta^{\boldsymbol{\lambda}}$ is submodular. Since $f_\theta^{\boldsymbol{\lambda}}$ is a suitably defined weighted sum of the functions f_θ, it is sufficient to prove that, for each type profile $\boldsymbol{k} \in K$, the function $f_\theta : 2^{\mathcal{R}} \to [0, 1]$ is submodular in the sets $R_I^{\boldsymbol{k}}$. For every pair of subsets $I \subseteq I' \subseteq \mathcal{G}_{\mathcal{S}}$, and for every receiver $r \in \mathcal{R}$ and signal $s \in \mathcal{S}_r$, the marginal contribution to the value of function f_θ due to the addition of element (r, s) to the set I is:

$$\begin{aligned}
&f_\theta(R_{I \cup (r,s)}^{\boldsymbol{k}}) - f_\theta(R_I^{\boldsymbol{k}}) \\
&\quad = \mathbb{I}\left\{k_r \in s \wedge \nexists (r, s') \in I : k_r \in s'\right\} \left(f_\theta(R_I^{\boldsymbol{k}} \cup \{r\}) - f_\theta(R_I^{\boldsymbol{k}})\right) \geq \\
&\quad \geq \mathbb{I}\left\{k_r \in s \wedge \nexists (r, s') \in I' : k_r \in s'\right\} \left(f_\theta(R_I^{\boldsymbol{k}} \cup \{r\}) - f_\theta(R_I^{\boldsymbol{k}})\right) \geq \\
&\quad \geq \mathbb{I}\left\{k_r \in s \wedge \nexists (r, s') \in I' : k_r \in s'\right\} \left(f_\theta(R_{I'}^{\boldsymbol{k}} \cup \{r\}) - f_\theta(R_{I'}^{\boldsymbol{k}})\right) = \\
&\quad = f_\theta(R_{I' \cup (r,s)}^{\boldsymbol{k}}) - f_\theta(R_{I'}^{\boldsymbol{k}}),
\end{aligned}$$

where the last inequality holds since the functions f_θ are submodular by assumption. Since the last expression is the marginal contribution to the value of function f_θ due to the addition of element (r, s) to the set I', the relations above prove that the function $f_\theta^{\boldsymbol{\lambda}}$ is submodular. Let $\ell^{\mathrm{w}} : 2^{\mathcal{G}_{\mathcal{S}}} \to$

$\mathbb{R}_+$ be a linear function such that $\ell^{\mathbf{w}}(I) = \sum_{r\in\mathcal{R}} w_{r,s_r}$ for every basis $I \subseteq \mathcal{B}(\mathcal{M}_\mathcal{S})$, with each $s_r \in \mathcal{S}_r$ being the signal of receiver $r \in \mathcal{R}$ specified by the signal profile corresponding to the basis, namely $(r, s_r) \in I$. Then, we have that finding a signal profile $\mathrm{s} \in \mathcal{S}$ satisfying Equation (10.5) is equivalent to finding a basis $I \in \mathcal{B}(\mathcal{M}_\mathcal{S})$ of the matroid $\mathcal{M}_\mathcal{S}$ (representing a direct signal profile) such that:

$$f_\theta^{\boldsymbol{\lambda}}(I) + \ell^{\mathbf{w}}(I) \geq \max_{I^\star \in \mathcal{B}(\mathcal{M}_\mathcal{S})} \left\{ \alpha \sum_{\boldsymbol{k}\in K} f_\theta^{\boldsymbol{\lambda}}(I^\star) + \ell^{\mathbf{w}}(I^\star) \right\} - \epsilon.$$

Notice that, for $\epsilon' > 0$, the algorithm of Theorem 10.8 by Sviridenko et al. (2017) can be employed to find a basis $I \in \mathcal{B}(\mathcal{M}_\mathcal{S})$ such that $f_\theta^{\boldsymbol{\lambda}}(I) + \ell^{\mathbf{w}}(I) \geq \left(1 - \frac{1}{e}\right) f_\theta^{\boldsymbol{\lambda}}(I') + \ell^{\mathbf{w}}(I') - O(\epsilon')v$ for every $I' \in \mathcal{B}(\mathcal{M})$ with high probability, employing time polynomial in $|\mathcal{G}_\mathcal{S}|$ and $\frac{1}{\epsilon}$. Since $|\mathcal{G}_\mathcal{S}|$ is polynomial in n and v is polynomial in $|K|, \max_{r,s} |w_{r,s}|$ and $\max_{\boldsymbol{k}} \lambda_{\boldsymbol{k}}$, by setting $\epsilon' = O(\frac{\epsilon}{v})$ and $\alpha = 1 - \frac{1}{e}$, we get the result. □

In conclusion, by letting $\mathcal{K}^T \subseteq \mathcal{K}$ be the set of receivers' type profiles observed by the sender up to iteration T, the following Theorem 10.9 provides our polynomial-time no-$(1 - \frac{1}{e})$-regret algorithm working with submodular sender's utilities.

Theorem 10.9. *If the sender's utility is such that function f_θ is* submodular *for each $\theta \in \Theta$, then there exists a no-$(1 - \frac{1}{e})$-regret algorithm having $(1 - \frac{1}{e})$-regret*

$$R^T_{1-\frac{1}{e}} \leq O\left(\sqrt{T}\,|\mathcal{K}^T|\right),$$

with a per-iteration running time $\mathsf{poly}(T, n, d)$.

Proof. We can run Algorithm 10.1 on an instance of our multi-receiver online Bayesian persuasion problem. By Theorem 10.4, if we set $\eta = \frac{1}{\sqrt{T}}$, $\epsilon = \frac{1}{T}$, and $\alpha = 1 - \frac{1}{e}$, we get the desired regret bound (notice that the set of observed feedbacks is $E^t = \mathcal{K}^t$ in our setting). Algorithm 10.1 employs an approximate projection oracle $\varphi_{1-\frac{1}{e}}$ that we can implement in polynomial time by using the algorithm provided in Theorem 10.6. This requires access to a polynomial-time approximate separation oracle $\mathcal{O}_{1-\frac{1}{e}}$, which can be implemented by using Theorem 10.7, under the assumption that the sender's utility is such that functions f_θ are submodular. □

Notice that the regret bound only depends on the number $|\mathcal{K}^T|$ of receivers' type profiles observed up to iteration T, while it is independent of the overall number of possible type profiles $|\mathcal{K}| = m^{\bar{n}}$, which is exponential in the number of receivers. Thus, the $(1 - \frac{1}{e})$-regret is polynomial in the size of the problem instance provided that the type profiles received as feedbacks by the sender are polynomially many (though the sender does not have to know which are these type profiles in advance). This is reasonable in many practical applications, where not all the type profiles can occur, since, *e.g.*, receivers' types are highly correlated. On the other hand, let us remark that, as T goes to infinity, we have $|\mathcal{K}^T| \leq m^{\bar{n}}$, so that the regret is sublinear in T.

CHAPTER 11

Bayesian Persuasion with Type Reporting

In Chapters 9 and 10, we showed that the first crucial step towards the computation of *efficient* online algorithms is the study of the problem in which the receiver's payoffs depend on her unknown type, which is randomly determined by a known finite-support probability distribution. However, we have shown that this problem is intractable in the classical Bayesian persuasion framework. In this chapter, we circumvent this issue extending the Bayesian persuasion framework with a type reporting step. In Section 11.1, we introduce the formal model of Bayesian persuasion with type reporting. In Section 11.2, we focus on the case with a single receiver, showing that an optimal sender's strategy can be computed in polynomial time. In Section 11.3, we study the case with multiple receivers, showing that an optimal sender's strategy can be computed in polynomial time when the sender's utility function is supermodular or anonymous, while when the sender's utility function is submodular we design an algorithm that provides a tight $(1 - 1/e)$-approximation.

11.1 Model with Type Reporting

In this section, we formally introduce the Bayesian persuasion framework with type reporting that we study in the rest of this chapter. In particular, in Subsection 11.1.1, we describe the model with a single receiver, while in Subsection 11.1.2 we extend it to multi-receiver settings. A model of signaling with type reporting is introduce in Xu et al. (2016a); Gan et al. (2019) for the specific setting of Bayesian Stackelberg Games. However, they design IC menus of signaling schemes for settings in which the sender has no private information.

11.1.1 Model with a Single Receiver

The receiver has a finite set $\mathcal{A} := \{a_i\}_{i=1}^{\varrho}$ of ϱ available actions and a type chosen from a finite set $\mathcal{K} := \{k_i\}_{i=1}^{m}$ of m possible types. For each type $k \in \mathcal{K}$, the receiver's payoff function is $u^k : \mathcal{A} \times \Theta \to [0, 1]$, where $\Theta := \{\theta_i\}_{i=1}^{d}$ is a finite set of d states of nature. We denote by $u_\theta^k(a) \in [0, 1]$ the payoff obtained by the receiver of type $k \in \mathcal{K}$ when the state of nature is $\theta \in \Theta$ and they play action $a \in \mathcal{A}$. The sender's payoffs are described by the functions $f_\theta : \mathcal{A} \to [0, 1]$ for $\theta \in \Theta$.

The sender commits to a *signaling scheme* ϕ, which is a randomized mapping from states of nature to *signals* for the receiver. Formally, $\phi : \Theta \to \Delta_{\mathcal{S}}$, where $\mathcal{S}$ is a set of available signals. For convenience, we let ϕ_θ be the probability distribution employed by the sender to draw signals when the state of nature is $\theta \in \Theta$ and we denote by $\phi_\theta(s)$ the probability of sending signal $s \in \mathcal{S}$. Moreover, we slightly abuse the notation and use ϕ to also denote the probability distribution over signals induced by the signaling scheme ϕ and the prior distribution $\boldsymbol{\mu}$. We recall that we denote as $\mathcal{B}_{\boldsymbol{\xi}}^k := arg\,max_{a\in\mathcal{A}} \sum_{\theta\in\Theta} \xi_\theta u_\theta^k(a)$ the set of actions that maximize the expected utility of the receiver of type $k \in \mathcal{K}$ in any posterior $\boldsymbol{\xi} \in \Xi$, while we denote by $b_{\boldsymbol{\xi}}^k \in arg\,max_{a\in\mathcal{B}_{\boldsymbol{\xi}}^k} \sum_\theta \xi_\theta f_\theta(a)$ the action in $\mathcal{B}_{\boldsymbol{\xi}}^k$ that is actually played by the receiver of type k in posterior $\boldsymbol{\xi}$.

In our *Bayesian persuasion framework with type reporting*, the sender asks the receiver to report their type before observing the realized state of nature. This enables the sender to increase their expected utility. In particular, before the receiver reports their type, the sender proposes to the receiver a *menu* $\Phi = \{\phi^k\}_{k\in\mathcal{K}}$ of signaling schemes, committing to send signals according to the signaling scheme ϕ^k if the receiver reports their type to be $k \in \mathcal{K}$. In details, the interaction goes on as follows: (i) the sender proposes a menu $\Phi = \{\phi^k\}_{k\in\mathcal{K}}$ to the receiver; (ii) the receiver reports a type $k \in \mathcal{K}$ that maximizes their expected utility give the proposed menu; (iii) the sender observes the realized state of nature $\theta \sim \boldsymbol{\mu}$; (iv) the sender draws a signal $s \in \mathcal{S}$ according to ϕ_θ^k and communicates it to the receiver; finally, the interaction terminates with steps (iv) and (v) of the classical setting (see Section 3.1).

Notice that, in step (ii), the receiver of type $k \in \mathcal{K}$ can compute their expected utility for each signaling scheme $\phi^{k'}$ in the menu as

$$\sum_{\theta \in \Theta} \mu_\theta \mathbb{E}_{s \sim \phi_\theta^{k'}} \left[u_\theta^k \left(b_{\boldsymbol{\xi}^s}^k \right) \right],$$

and, then, they can report a type $k' \in \mathcal{K}$ whose corresponding signaling scheme $\phi^{k'}$ maximizes their expected utility.

We focus on menus of signaling schemes that are *incentive compatible* (IC), *i.e.*, in which the receiver of type k is incentivized to report their true type, for any $k \in \mathcal{K}$.[1] Formally, a menu $\Phi = \{\phi^k\}_{k \in \mathcal{K}}$ is IC if, for every type $k \in \mathcal{K}$, the following constraints are satisfied:

$$\sum_{\theta \in \Theta} \mu_\theta \mathbb{E}_{s \sim \phi_\theta^k} \left[u_\theta^k \left(b_{\boldsymbol{\xi}^s}^k \right) \right] \geq \sum_{\theta \in \Theta} \mu_\theta \mathbb{E}_{s \sim \phi_\theta^{k'}} \left[u_\theta^k \left(b_{\boldsymbol{\xi}^s}^k \right) \right] \quad \forall k' \neq k. \tag{11.1}$$

We say that a signaling scheme is *direct* if $\mathcal{S} = \mathcal{A}$, which means that signals correspond to action recommendations for the receiver. Moreover, we say that a direct signaling scheme is *persuasive* if the receiver has an incentive to follow the action recommendations that they receive as signals, when they report their true type. It is easy to check that a menu $\Phi = \{\phi^k\}_{k \in \mathcal{K}}$ of direct and persuasive signaling schemes is IC if

$$\sum_{a \in \mathcal{A}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta^k(a) u_\theta^k(a) \geq \sum_{a \in \mathcal{A}} \max_{a' \in \mathcal{A}} \sum_{\theta \in \Theta} \mu_\theta \phi_\theta^{k'}(a) u_\theta^k(a') \; \forall k' \neq k. \tag{11.2}$$

11.1.2 Model with Multiple Receivers

In a multi-receiver setting, there is a finite set $\mathcal{R} := \{r_i\}_{i=1}^{\bar{n}}$ of $\bar{n}$ receivers, and each receiver $r \in \mathcal{R}$ has a type chosen from a finite set $\mathcal{K}_r := \{k_{r,i}\}_{i=1}^{m_r}$ of m_r different types. We introduce $\mathcal{K} := \times_{r \in \mathcal{R}} \mathcal{K}_r$ as the set of type profiles, which are tuples $\boldsymbol{k} \in \mathcal{K}$ defining a type $k_r \in \mathcal{K}_r$ for each receiver $r \in \mathcal{R}$. Each receiver $r \in \mathcal{R}$ has two actions available, defined by $\mathcal{A}_r := \{a_0, a_1\}$. We let $\mathcal{A} := \times_{r \in \mathcal{R}} \mathcal{A}_r$ be the set of action profiles specifying an action for each receiver. We assume that there are *no inter-agent*

[1] Notice that, by a revelation-principle-style argument (see the book by Shoham and Leyton-Brown (2008) for some examples of these kind of arguments), focusing on IC menus of signaling schemes is w.l.o.g. when looking for a sender-optimal menu.

externalities. Formally, a receiver $r \in \mathcal{R}$ of type $k \in \mathcal{K}_r$ has a payoff function $u^{r,k} : \mathcal{A}_r \times \Theta \to [0,1]$. The sender's payoffs depend on the actions played by all the receivers, and they are defined by $f : \mathcal{A} \times \Theta \to [0,1]$. In the rest of this chapter, we assume that the sender's payoffs are *monotone non-decreasing* in the set of receivers playing a_1. As it is customary, we focus on three families of functions: *submodular*, *supermodular*, and *anonymous*. With multiple receivers, the sender must send a signal to each of them. We focus on *private* signaling, where each receiver has their own signal that is privately communicated to them.

The interaction between the sender and the receivers goes on as follows: (i) the sender proposes to each receiver $r \in \mathcal{R}$ a menu of marginal signaling schemes $\Phi^r = \{\phi_r^k\}_{k \in \mathcal{K}_r}$; (ii) each receiver $r \in \mathcal{R}$ reports a type $k_r \in \mathcal{K}_r$ such that $\phi_r^{k_r}$ is the marginal signaling scheme maximizing their expected utility; (iii) the sender commits to a signaling scheme ϕ whose resulting marginal signaling schemes ϕ_r are such that $\phi_r := \phi_r^{k_r}$ for all $r \in \mathcal{R}$; (iv) the sender observes the realized state of nature $\theta \sim \boldsymbol{\mu}$ and draws a signal profile s $\sim \phi_\theta$; (v) each receiver $r \in \mathcal{R}$ observes their signal s_r, rationally updates their prior belief over Θ according to the *Bayes* rule, and selects an action maximizing their expected utility. Notice that the sender only needs to propose marginal signaling schemes to the receivers (rather than general ones), since the expected utility of each receiver only depends on their marginal signaling scheme, and *not* on the others. Thus, the sender can delay the choice of the (general) signaling scheme after types have been reported.

Similarly to the single-receiver case, we restrict the attention to IC menu of marginal signaling schemes. Thus, in a multi-receiver setting, a sender's strategy is composed by an IC menu of marginal signaling scheme $\Phi^r = \{\phi_r^k\}_{k \in \mathcal{K}_r}$ for each receiver $r \in \mathcal{R}$, and a set of signaling schemes $\{\phi^{\boldsymbol{k}}\}_{\boldsymbol{k} \in \mathcal{K}}$ (one per type profile possibly reported by the receivers) such that the resulting marginal signaling scheme satisfy $\phi^{\boldsymbol{k},r} = \phi_r^{k_r}$ for all $\boldsymbol{k} \in \mathcal{K}$ and $r \in \mathcal{R}$.

11.1.3 Sender's Computational Problems

We consider the computational problem in which, given the probability distribution over the receivers' types, the sender wants to maximize their expected utility. In the single-receiver case, the receiver's type $k \in \mathcal{K}$ is drawn from a known distribution $\boldsymbol{\lambda} \in \Delta_{\mathcal{K}}$. We call MENU-SINGLE the problem of computing an IC menu of signaling schemes $\Phi = \{\phi^k\}_{k \in \mathcal{K}}$ that maximizes the sender's expected utility, given a probability distribution

$\boldsymbol{\lambda} \in \Delta_{\mathcal{K}}$ as input. In the multi-receiver case, the types profiles $\boldsymbol{k} \in \mathcal{K}$ are drawn from a known distribution $\boldsymbol{\lambda} \in \Delta_{\bar{\mathcal{K}}}$, where $\bar{\mathcal{K}} \subseteq \mathcal{K}$ is a subset of possible types vectors, *i.e.*, the support of $\boldsymbol{\lambda}$. We call MENU-MULTI the problem of computing a sender's strategy—made by an IC menu of marginal signaling schemes $\Phi^r = \{\phi_r^k\}_{k \in \mathcal{K}_r}$ for each receiver $r \in \mathcal{R}$ and a set of signaling schemes $\{\phi^{\boldsymbol{k}}\}_{\boldsymbol{k} \in \bar{\mathcal{K}}}$—that maximizes the sender's expected utility, given a probability distribution $\boldsymbol{\lambda} \in \Delta_{\bar{\mathcal{K}}}$ as input.[2]

11.2 Single-receiver Problem

We show how to solve MENU-SINGLE in polynomial time. By using the well-known equivalence between signaling schemes and distributions over posteriors, it is easy to check that an optimal menu of signaling schemes can be computed by the following LP 11.3 with an *infinite* number of variables, namely $\boldsymbol{\gamma}^k \in \Delta_\Xi$ for $k \in \mathcal{K}$. In LP 11.3, the objective is the sender's expected utility assuming the receiver reports their true type, the first set of constraints encodes IC conditions, while the last one ensures that the distributions over posteriors correctly represent signaling schemes.

$$\begin{aligned}
&\max_{\boldsymbol{\gamma}} \sum_{k \in \mathcal{K}} \lambda_k \mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^k} \sum_{\theta \in \Theta} \xi_\theta f_\theta \left(b_{\boldsymbol{\xi}}^k\right) \quad \text{s.t.} && (11.3)\\
&\mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^k} \left[\sum_{\theta \in \Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right)\right] \geq \mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^k} \left[\sum_{\theta \in \Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right)\right] && \forall k \neq k' \in \mathcal{K}\\
&\mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^k} \left[\xi_\theta\right] = \mu_\theta && \forall \theta \in \Theta, \forall k \in \mathcal{K}\\
&\boldsymbol{\gamma}^k \in \Delta_\Xi && \forall k \in \mathcal{K}.
\end{aligned}$$

As a first step, we show that there always exists an optimal solution to LP 11.3 in which the probability distributions $\boldsymbol{\gamma}^k \in \Delta_\Xi$ have finite support. This allows us to compute an optimal menu of signaling schemes by solving an LP with a *finite* number of variables. In the following, for every $k \in \mathcal{K}$ and $a \in \mathcal{A}$, let $\Xi^{k,a} := \left\{\boldsymbol{\xi} \in \Xi : a \in \mathcal{B}_{\boldsymbol{\xi}}^k\right\}$ and $\hat{\Xi}^{k,a} := \left\{\boldsymbol{\xi} \in \Xi : a = b_{\boldsymbol{\xi}}^k\right\}$. Moreover, for every $\mathbf{a} \in \times_{k \in \mathcal{K}} A$, let $\Xi^{\mathbf{a}} := \bigcap_{k \in \mathcal{K}} \Xi^{k,a_k}$ and $\hat{\Xi}^{\mathbf{a}} := \bigcap_{k \in \mathcal{K}} \hat{\Xi}^{k,a_k}$, where a_k is the k-th component of $\mathbf{a}$. Finally, let Ξ^* be such that $\Xi^* := \bigcup_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} V(\Xi^{\mathbf{a}})$, where $V(\Xi^{\mathbf{a}})$ denotes the set of vertices of the polytope $\Xi^{\mathbf{a}}$. The following Lemma 11.1 shows that there always exists an optimal menu of signaling schemes that can be encoded as

[2]A polynomial-time algorithm for MENU-MULTI must run in time polynomial in the size of the instance and in the size of the support of the distribution $\boldsymbol{\lambda}$. Notice that, in general, the latter may be exponential in the number of receivers $\bar{n}$.

probability distributions over Ξ^*. Formally, the lemma is proved by showing that the following LP 11.4 is equivalent to LP 11.3.

$$\max_{\boldsymbol{\gamma}} \sum_{k \in \mathcal{K}} \lambda_k \sum_{\boldsymbol{\xi} \in \Xi^*} \gamma^k_{\boldsymbol{\xi}} \sum_{\theta \in \Theta} \xi_\theta f_\theta \left(b^k_{\boldsymbol{\xi}} \right) \quad \text{s.t.} \tag{11.4a}$$

$$\sum_{\boldsymbol{\xi} \in \Xi^*} \gamma^k_{\boldsymbol{\xi}} \sum_{\theta \in \Theta} \xi_\theta u^k_\theta \left(b^k_{\boldsymbol{\xi}} \right) \geq \sum_{\boldsymbol{\xi} \in \Xi^*} \gamma^{k'}_{\boldsymbol{\xi}} \sum_{\theta \in \Theta} \xi_\theta u^k_\theta \left(b^k_{\boldsymbol{\xi}} \right) \qquad \forall k \neq k' \in \mathcal{K} \tag{11.4b}$$

$$\sum_{\boldsymbol{\xi} \in \Xi^*} \gamma^k_{\boldsymbol{\xi}} \xi_\theta = \mu_\theta \qquad \forall k \in \mathcal{K}, \forall \theta \in \Theta \tag{11.4c}$$

$$\sum_{\boldsymbol{\xi} \in \Xi^*} \gamma^k_{\boldsymbol{\xi}} = 1 \qquad \forall k \in \mathcal{K}. \tag{11.4d}$$

Intuitively, the result is shown by noticing that, once fixed the receiver's best responses to $\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}$, the sums over Θ in the objective and the constraints of LP 11.3 are linear in the posterior $\boldsymbol{\xi}$, which allows to apply Carathéodory theorem to replace each posterior with a probability distributions over the vertices of $\Xi^{\mathbf{a}}$.

Lemma 11.1. *In single-receiver instances, there always exists a sender-optimal menu of signaling schemes that can be encoded as probability distributions over the finite set of posteriors* Ξ^*.

Proof. We show that, given a menu of signaling schemes $\Phi = \{\phi^k\}_{k \in \mathcal{K}}$ with each ϕ^k encoded as a probability distribution $\boldsymbol{\gamma}^k \in \Delta_\Xi$, we can construct a new menu of signaling schemes $\bar{\Phi} = \{\bar{\phi}^k\}_{k \in \mathcal{K}}$ with each $\bar{\phi}^k$ encoded as a finite-supported probability distribution $\bar{\boldsymbol{\gamma}}^k \in \Delta_{\Xi^*}$ and such that the sender's expected utility for $\bar{\Phi}$ is greater than or equal to that for Φ. This immediately proves the statement.

In order to do so, we split the posteriors in Ξ into the sets $\hat{\Xi}^{\mathbf{a}}$ for $\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}$. Notice that $\Xi = \bigcup_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \hat{\Xi}^{\mathbf{a}}$. Then, we replace the distributions γ^k with other probability distributions supported on sets $V(\Xi^{\mathbf{a}}) \subseteq \Xi^*$. For every action profile $\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}$ and type $k \in \mathcal{K}$, we let $\boldsymbol{\xi}^{k,\mathbf{a}} := \mathbb{E}_{\boldsymbol{\xi} \sim \gamma^k} \left[\boldsymbol{\xi} \mid \boldsymbol{\xi} \in \hat{\Xi}^{\mathbf{a}} \right]$. Since $\hat{\Xi}^{\mathbf{a}} \subseteq \Xi^{\mathbf{a}}$ and $\Xi^{\mathbf{a}}$ is a bounded convex polytope, by Carathéodory theorem there exists a probability distribution $\boldsymbol{\gamma}^{k,\mathbf{a}} \in \Delta_{\Xi^*}$ such that its support is a subset of the set of vertices $V(\Xi^{\mathbf{a}})$ and it holds $\mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^{k,\mathbf{a}}} [\boldsymbol{\xi}] = \boldsymbol{\xi}^{k,\mathbf{a}}$. Then, let us define the probability distributions $\bar{\boldsymbol{\gamma}}^k \in \Delta_{\Xi^*}$ for $k \in \mathcal{K}$ so that, for every posterior $\boldsymbol{\xi} \in \Xi^*$, it holds

$$\bar{\gamma}^k_{\boldsymbol{\xi}} = \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \gamma^{k,\mathbf{a}}_{\boldsymbol{\xi}} \Pr_{\boldsymbol{\xi}' \sim \gamma^k} \left\{ \boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}} \right\}.$$

Next, we show that the distributions $\bar{\gamma}^k \in \Delta_{\Xi^*}$ for $k \in \mathcal{K}$ defined above constitute a feasible solution to LP 11.4 and the sender's expected utility in the resulting menu of signaling schemes $\bar{\Phi}$ is at least as large as the sender's expected utility for the menu of signaling schemes Φ. First, let us notice that, for every $\mathbf{a} \in \times_{k\in\mathcal{K}} \mathcal{A}$, $k \in K$, and $k' \in K$, it holds

$$\sum_{\boldsymbol{\xi}\in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}} \left[\sum_{\theta\in\Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right)\right] = \sum_{\boldsymbol{\xi}\in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}} \left[\sum_{\theta\in\Theta} \xi_\theta u_\theta^k(a_k)\right] \tag{11.5a}$$

$$= \sum_{\theta\in\Theta} \xi_\theta^{k',\mathbf{a}} u_\theta^k(a_k) \tag{11.5b}$$

$$= \mathbb{E}_{\boldsymbol{\xi}\sim\gamma^{k'}} \left[\sum_{\theta\in\Theta} \xi_\theta u_\theta^k(a_k) \mid \boldsymbol{\xi} \in \hat{\Xi}^{\mathbf{a}}\right], \tag{11.5c}$$

where the second equality comes from the fact that action a_k is the best response of the receiver of type k in each posterior $\boldsymbol{\xi} \in \Xi^{\mathbf{a}}$. Similarly, we can prove that, for every $\mathbf{a} \in \times_{k\in\mathcal{K}} \mathcal{A}$ and $k \in K$, it holds

$$\sum_{\boldsymbol{\xi}\in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k,\mathbf{a}} \left[\sum_{\theta\in\Theta} \xi_\theta f_\theta \left(b_{\boldsymbol{\xi}}^k\right)\right] \geq \sum_{\boldsymbol{\xi}\in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k,\mathbf{a}} \left[\sum_{\theta\in\Theta} \xi_\theta f_\theta(a_k)\right] \tag{11.6a}$$

$$= \sum_{\theta\in\Theta} \xi_\theta^{k,\mathbf{a}} f_\theta(a_k) \tag{11.6b}$$

$$= \mathbb{E}_{\boldsymbol{\xi}\sim\gamma^{k}} \left[\sum_{\theta\in\Theta} \xi_\theta f_\theta(a_k) \mid \boldsymbol{\xi} \in \hat{\Xi}^{\mathbf{a}}\right]. \tag{11.6c}$$

Then, we can show that the IC constraints, namely Constraints (11.4b), are satisfied. Formally, for every $k \in \mathcal{K}$ and $k' \in \mathcal{K} : k \neq k'$, we have:

$$\sum_{\boldsymbol{\xi}\in\Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^{k'} \sum_{\theta\in\Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right) = \sum_{\boldsymbol{\xi}\in\Xi^*} \sum_{\mathbf{a}\in\times_{k\in\mathcal{K}}\mathcal{A}} \gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}} \Pr_{\boldsymbol{\xi}'\sim\gamma^{k'}} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \sum_{\theta\in\Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right)$$

$$= \sum_{\mathbf{a}\in\times_{k\in\mathcal{K}}\mathcal{A}} \Pr_{\boldsymbol{\xi}'\sim\gamma^{k'}} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \sum_{\boldsymbol{\xi}\in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}} \left[\sum_{\theta\in\Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right)\right]$$

$$= \sum_{\mathbf{a}\in\times_{k\in\mathcal{K}}\mathcal{A}} \Pr_{\boldsymbol{\xi}'\sim\gamma^{k}} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \mathbb{E}_{\boldsymbol{\xi}\sim\gamma^{k'}} \left[\sum_{\theta\in\Theta} \xi_\theta u_\theta^k(a_k) \mid \boldsymbol{\xi} \in \hat{\Xi}^{\mathbf{a}}\right]$$

$$= \mathbb{E}_{\boldsymbol{\xi}\sim\gamma^{k'}} \left[\sum_{\theta\in\Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right)\right],$$

where the second equality comes from the fact that $\gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}}$ is non-zero only for posteriors $\boldsymbol{\xi} \in V(\Xi^{\mathbf{a}})$ and in third equality we use Equation (11.5). Hence, for every $k \in \mathcal{K}$ and $k' \in \mathcal{K} : k \neq k'$, we have

$$\begin{aligned}\sum_{\boldsymbol{\xi} \in \Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^k \sum_{\theta \in \Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right) &= \mathbb{E}_{\boldsymbol{\xi} \sim \gamma^k} \left[\sum_{\theta \in \Theta} \xi_\theta u^k \left(b_{\boldsymbol{\xi}}^k\right)\right] \\ &\geq \mathbb{E}_{\boldsymbol{\xi} \sim \gamma^{k'}} \left[\sum_{\theta \in \Theta} \xi_\theta u^k \left(b_{\boldsymbol{\xi}}^k\right)\right] \\ &= \sum_{\boldsymbol{\xi} \in \Xi^*} \gamma_{\boldsymbol{\xi}}^{k'} \sum_{\theta \in \Theta} \xi_\theta u_\theta^k \left(b_{\boldsymbol{\xi}}^k\right),\end{aligned}$$

where the inequality comes from the w.l.o.g. assumption that the menu Φ is IC. This proves that Constraints (11.4b) hold. Similarly, we can prove that the sender's expected utility does *not* decrease when using $\bar{\Phi}$ rather than Φ. Formally,

$$\begin{aligned}&\sum_{k \in \mathcal{K}} \lambda_k \sum_{\boldsymbol{\xi} \in \Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^k \sum_{\theta \in \Theta} \xi_\theta f_\theta \left(b_{\boldsymbol{\xi}}^k\right) \\ &\quad= \sum_{k \in \mathcal{K}} \lambda_k \sum_{\boldsymbol{\xi} \in \Xi^*} \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}} \operatorname{Pr}_{\boldsymbol{\xi}' \sim \gamma^k} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \sum_{\theta \in \Theta} \xi_\theta f_\theta \left(b_{\boldsymbol{\xi}}^k\right) \\ &\quad= \sum_{k \in \mathcal{K}} \lambda_k \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \operatorname{Pr}_{\boldsymbol{\xi}' \sim \gamma^k} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{a}\right\} \sum_{\boldsymbol{\xi} \in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k',\mathbf{a}} \sum_{\theta \in \Theta} \xi_\theta f_\theta \left(b_{\boldsymbol{\xi}}^k\right) \\ &\quad\geq \sum_{k \in K} \lambda_k \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \operatorname{Pr}_{\boldsymbol{\xi}' \sim \boldsymbol{\gamma}^k} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \mathbb{E}_{\boldsymbol{\xi} \sim \gamma^k} \left[\sum_{\theta \in \Theta} \xi_\theta f_\theta(a_k) \mid \boldsymbol{\xi} \in \hat{\Xi}^{\mathbf{a}}\right] \\ &\quad= \sum_{k \in \mathcal{K}} \lambda_k \mathbb{E}_{\boldsymbol{\xi} \sim \gamma^k} \left[\sum_{\theta \in \Theta} \xi_\theta f_\theta \left(b_{\boldsymbol{\xi}}^k\right)\right],\end{aligned}$$

where the inequality comes from Equation (11.6). Moreover, Constraints (11.4c) are satisfied, since

$$\begin{aligned}\sum_{\boldsymbol{\xi} \in \Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^k \xi_\theta &= \sum_{\boldsymbol{\xi} \in \Xi^*} \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \gamma_{\boldsymbol{\xi}}^{k,\mathbf{a}} \operatorname{Pr}_{\boldsymbol{\xi}' \sim \gamma^k} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \xi_\theta \\ &= \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \operatorname{Pr}_{\boldsymbol{\xi}' \sim \boldsymbol{\gamma}^k} \left\{\boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}}\right\} \sum_{\boldsymbol{\xi} \in V(\Xi^{\mathbf{a}})} \gamma_{\boldsymbol{\xi}}^{k,\mathbf{a}} \xi_\theta\end{aligned}$$

$$= \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \Pr_{\boldsymbol{\xi}' \sim \gamma^k} \left\{ \boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}} \right\} \mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^k} \left[\xi_\theta \mid \boldsymbol{\xi} \in \hat{\Xi}^{\mathbf{a}} \right]$$
$$= \mathbb{E}_{\boldsymbol{\xi} \sim \boldsymbol{\gamma}^k} \left[\xi_\theta \right] = \mu_\theta.$$

Finally, it is easy to see that the $\bar{\gamma}^k$ are valid probability distributions. Indeed, for every $k \in \mathcal{K}$, it holds

$$\sum_{\boldsymbol{\xi} \in \Xi^*} \bar{\gamma}^k_{\boldsymbol{\xi}} = \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \Pr_{\boldsymbol{\xi}' \sim \gamma^k} \left\{ \boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}} \right\} \sum_{\boldsymbol{\xi} \in \Xi^*} \gamma^{k,\mathbf{a}}_{\boldsymbol{\xi}}$$
$$= \sum_{\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}} \Pr_{\boldsymbol{\xi}' \sim \gamma^k} \left\{ \boldsymbol{\xi}' \in \hat{\Xi}^{\mathbf{a}} \right\} = 1.$$

This concludes the proof. □

Next, we show that there always exists an optimal menu of direct and persuasive signaling schemes, and that it can be computed in polynomial time by solving a polynomially-sized LP obtained by further simplifying LP 11.4 (Theorem 11.1). Notice that, in a Bayesian persuasion problem without type reporting, an optimal signaling scheme must employ a signal for each action profile $\mathbf{a} \in \times_{k \in \mathcal{K}} \mathcal{A}$. Since these profiles are exponentially many, an optimal direct and persuasive signaling scheme cannot be computed in polynomial time by linear programming. Indeed, in Chapter 9 we show that without typer reporting the problem is NP-hard.

An intuition behind the proof of Theorem 11.1 is provided the following. Fix type $k \in \mathcal{K}$ and action $a \in \mathcal{A}$. Suppose that an optimal menu of signaling schemes employs $\boldsymbol{\gamma}^k \in \Delta_{\Xi^*}$ for the type k, and that $\boldsymbol{\gamma}^k$ has in the support two posteriors $\boldsymbol{\xi}^1, \boldsymbol{\xi}^2 \in \hat{\Xi}^{k,a}$ with probabilities $\gamma^k_{\boldsymbol{\xi}^1}$ and $\gamma^k_{\boldsymbol{\xi}^2}$. Consider a new signaling scheme that replaces the two posteriors $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$ with their convex combination $\boldsymbol{\xi}^* \in \Delta_{\Xi^*}$, so that

$$\xi^*_\theta = \frac{\gamma^k_{\boldsymbol{\xi}^1} \xi^1_\theta + \gamma^k_{\boldsymbol{\xi}^2} \xi^2_\theta}{\gamma^k_{\boldsymbol{\xi}^1} + \gamma^k_{\boldsymbol{\xi}^2}} \text{ for every } \theta \in \Theta$$

and

$$\gamma^k_{\boldsymbol{\xi}^*} = \gamma^k_{\boldsymbol{\xi}^1} + \gamma^k_{\boldsymbol{\xi}^2}.$$

Both $\boldsymbol{\xi}^1$ and $\boldsymbol{\xi}^2$ induce the same best response of the receiver of type k, and Objective (11.4a) and Constraints (11.4c) are linear in $\boldsymbol{\xi}$. Hence, replacing the two posteriors with their convex combination $\boldsymbol{\xi}^*$ preserves the value of the objective, while maintaining the constraints satisfied. The same does *not* hold for Constraints (11.4b), which are linear in the posterior only if we fix the best responses of all the receiver's types. For Constraints (11.4b), if we consider an inequality in which $\boldsymbol{\gamma}^k$ appears in the left hand side, the sum over Θ is linear in $\boldsymbol{\xi}$ and

$$\gamma^k_{\boldsymbol{\xi}^1} \sum_{\theta \in \Theta} \xi^1_\theta u^k_\theta(a) + \gamma^k_{\boldsymbol{\xi}^2} \sum_{\theta \in \Theta} \xi^2_\theta u^k_\theta(a) = \gamma^k_{\boldsymbol{\xi}^*} \sum_{\theta \in \Theta} \xi^*_\theta u^k_\theta(a).$$

Instead, if $\boldsymbol{\gamma}^k$ appears in the right hand side, by the convexity of the max operator it hods:

$$\begin{aligned}
&\gamma^k_{\boldsymbol{\xi}^1} \max_{a' \in \mathcal{A}} \sum_{\theta \in \Theta} \xi^1_\theta u^k_\theta(a') + \gamma^k_{\boldsymbol{\xi}^2} \max_{a' \in \mathcal{A}} \sum_{\theta \in \Theta} \xi^1_\theta u^k_\theta(a') \\
&\quad \geq \max_{a' \in \mathcal{A}} \left[\gamma^k_{\boldsymbol{\xi}^1} \sum_{\theta \in \Theta} \xi^1_\theta u^k_\theta(a') + \gamma^k_{\boldsymbol{\xi}^2} \sum_{\theta \in \Theta} \xi^1_\theta u^k_\theta(a') \right] \\
&\quad = \max_{a' \in \mathcal{A}} \sum_{\theta \in \Theta} \xi^*_\theta u^k_\theta(a').
\end{aligned}$$

Therefore, if we replace two posteriors that induce the same receiver's best responses with their convex combination, the left hand side of Constraints (11.4b) is preserved, while the value of the right hand side can only decrease, guaranteeing that Constraints (11.4b) remain satisfied. By using this idea, we can join all the posteriors that induce the same best responses. Finally, by resorting to the equivalence between signaling schemes and distributions over, we obtain the following LP 11.7 of polynomial size. Hence, an optimal menu of signaling schemes can be computed in polynomial time.

$$\max_{\phi, l} \sum_{k \in \mathcal{K}} \lambda_k \sum_{\theta \in \Theta} \mu_\theta \sum_{a \in \mathcal{A}} \phi^k_\theta(a) f_\theta(a) \quad \text{s.t.} \tag{11.7a}$$

$$\sum_{a \in \mathcal{A}} \sum_{\theta \in \Theta} \mu_\theta \phi^k_\theta(a) u^k_\theta(a) \geq \sum_{a \in \mathcal{A}} l^{k,k'}_a \qquad \forall k \neq k' \in \mathcal{K} \tag{11.7b}$$

$$l^{k,k'}_a \geq \sum_{\theta \in \Theta} \mu_\theta \phi^k_\theta(a) u^k_\theta(a') \qquad \forall k \neq k' \in \mathcal{K}, \forall a, a' \in \mathcal{A} \tag{11.7c}$$

$$\sum_{\theta \in \Theta} \mu_\theta \phi^{k'}_\theta(a) u^k_\theta(a) \geq \sum_{\theta \in \Theta} \mu_\theta \phi^{k'}_\theta(a) u^k_\theta(a') \; \forall k \in \mathcal{K}, \forall a, a' \in \mathcal{A} \tag{11.7d}$$

$$\sum_{a \in \mathcal{A}} \phi^k_\theta(a) = 1 \qquad \forall k \in \mathcal{K}, \forall \theta \in \Theta. \tag{11.7e}$$

Notice that Constraints (11.7b) and (11.7c) are equivalent to the IC constraints for direct and persuasive signaling schemes, which are those specified in Equation (11.2), where $\max_{a' \in A} \sum_{\theta \in \Theta} \mu_\theta \phi^{k'}_\theta(a) u^k_\theta(a')$ is the best response of the receiver of type $k \in \mathcal{K}$ to the direct signal a for the receiver of type $k' \in \mathcal{K}$. Moreover, Constraints (11.7d) force the signaling schemes to be persuasive.

Theorem 11.1. *In single-receiver instances, there always exists an optimal menu of direct and persuasive signaling schemes. Moreover, it can be computed in polynomial time.*

Proof. Since LP 11.7 has polynomially-many variables and constraints, an optimal menu of direct and persuasive signaling schemes can be computed in polynomial time by solving the LP. Thus, we only need to show that, in any single-receiver instance, there always exists an optimal menu of direct and persuasive signaling schemes. In particular, we show that, given an optimal solution $\{\gamma^k\}_{k\in\mathcal{K}}$ to LP 11.4, there exists a solution to LP 11.7 with the same value. The menu $\Phi = \{\phi^k\}_{k\in\mathcal{K}}$ of signaling schemes defined by the solution to LP 11.7 is the desired optimal menu of direct and persuasive signaling schemes. We define the solution to LP 11.7 as follows. For every $k \in \mathcal{K}$, $a \in \mathcal{A}$, and $\theta \in \Theta$, we let $\phi^k_\theta(a) = \frac{\sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \gamma^k_{\boldsymbol{\xi}}\xi_\theta}{\mu_\theta}$. First, we prove that the two solutions have the same objective value. Formally,

$$\begin{aligned}\sum_{k\in\mathcal{K}}\lambda_k\sum_{\theta\in\Theta}\mu_\theta\sum_{a\in\mathcal{A}}\phi^k_\theta(a)f_\theta(a) &= \sum_{k\in\mathcal{K}}\lambda_k\sum_{\theta\in\Theta}\sum_{a\in\mathcal{A}}\sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*}\gamma^k_{\boldsymbol{\xi}}\xi_\theta f_\theta(a)\\ &= \sum_{k\in\mathcal{K}}\lambda_k\sum_{\boldsymbol{\xi}\in\Xi^*}\gamma^k_{\boldsymbol{\xi}}\sum_{\theta\in\Theta}\xi_\theta f_\theta\left(b^k_{\boldsymbol{\xi}}\right),\end{aligned}$$

where the last equality follows from the fact that $b^k_{\boldsymbol{\xi}} = a$ for all the posteriors in $\boldsymbol{\xi} \in \hat{\Xi}^{k,a}$. Thus, we are left to check that the solution is feasible. Recall that Constraints (11.7b) and (11.7c) are equivalent to the constraints in Equation (11.2). The latter are satisfied since, for every $k \neq k' \in \mathcal{K}$, it holds

$$\begin{aligned}\sum_{a\in\mathcal{A}}\sum_{\theta\in\Theta}\mu_\theta\phi^k_\theta(a)u^k_\theta(a) &= \sum_{a\in\ A}\sum_{\theta\in\Theta}\sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*}\gamma^k_{\boldsymbol{\xi}}\xi_\theta u^k_\theta(a)\\ &= \sum_{\boldsymbol{\xi}\in\Xi^*}\gamma^k_{\boldsymbol{\xi}}\sum_{\theta\in\Theta}\xi_\theta u^k_\theta\left(b^k_{\boldsymbol{\xi}}\right)\\ &\geq \sum_{\boldsymbol{\xi}\in\Xi^*}\gamma^{k'}_{\boldsymbol{\xi}}\sum_{\theta\in\Theta}\xi_\theta u^k_\theta\left(b^k_{\boldsymbol{\xi}}\right)\\ &= \sum_{a\in\mathcal{A}}\sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*}\gamma^{k'}_{\boldsymbol{\xi}}\sum_{\theta\in\Theta}\xi_\theta u^k_\theta\left(b^k_{\boldsymbol{\xi}}\right)\end{aligned}$$

$$= \sum_{a\in\mathcal{A}} \sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}} \gamma_{\boldsymbol{\xi}}^{k'} \max_{a'\in\mathcal{A}} \sum_{\theta\in\Theta} \xi_\theta u_\theta^k(a')$$
$$\geq \sum_{a\in\mathcal{A}} \max_{a'\in\mathcal{A}} \sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \gamma_{\boldsymbol{\xi}}^{k'} \sum_{\theta\in\Theta} \xi_\theta u_\theta^k(a')$$
$$= \sum_{a\in\mathcal{A}} \max_{a'\in\mathcal{A}} \sum_{\theta\in\Theta} \mu_\theta \phi_\theta^{k'}(a) u_\theta^k(a').$$

Moreover, each signaling scheme ϕ^k is persuasive, since, for every $k \in \mathcal{K}$, and $a \neq a' \in \mathcal{A}$, it holds

$$\sum_{\theta\in\Theta} \mu_\theta \phi_\theta^k(a) u_\theta^k(a) = \sum_{\theta\in\Theta} \sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \gamma_{\boldsymbol{\xi}}^k \xi_\theta u_\theta^k(a)$$
$$= \sum_{\theta\in\Theta} \sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \gamma_{\boldsymbol{\xi}}^k \xi_\theta u_\theta^k\left(b_{\boldsymbol{\xi}}^k\right)$$
$$\geq \sum_{\theta\in\Theta} \sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \gamma_{\boldsymbol{\xi}}^k \xi_\theta u_\theta^k(a')$$
$$= \sum_{\theta\in\Theta} \mu_\theta \phi_\theta^k(a) u_\theta^k(a'),$$

and it is well defined since, for every $k \in \mathcal{K}$ and $\theta \in \Theta$, it holds

$$\sum_{a\in A} \phi_\theta^k(a) = \sum_{a\in\mathcal{A}} \sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \frac{\gamma_{\boldsymbol{\xi}}^k \xi_\theta}{\mu_\theta} = \sum_{\boldsymbol{\xi}\in\Xi^*} \frac{\gamma_{\boldsymbol{\xi}}^k \xi_\theta}{\mu_\theta} = \frac{\mu_\theta}{\mu_\theta} = 1.$$

This concludes the proof.

□

11.3 Multi-receiver Problem

In this section, we switch the attention to MENU-MULTI. As we will show in the following (Theorem 11.2), given any multi-receiver instance, there always exists an optimal sender's strategy that uses menus of direct and persuasive marginal signaling schemes. This allows us to formulate the sender's problem as the following LP 11.8, which will be crucial for the results in the rest of this section.

Since $\phi_{r,\theta}^k(a_0) = 1 - \phi_{r,\theta}^k(a_1)$ for every $r \in \mathcal{R}$, $k \in \mathcal{K}_r$, and $\theta \in \Theta$, by letting $x_\theta^{r,k} = \phi_{r,\theta}^k(a_1)$ we can formulate the following LP:

$$\max_{\phi\geq 0,\mathbf{x}\geq 0,1} \sum_{\theta\in\Theta} \mu_\theta \sum_{\boldsymbol{k}\in\bar{\mathcal{K}}} \lambda_{\boldsymbol{k}} \sum_{R\subseteq\mathcal{R}} \phi_\theta^{\boldsymbol{k}}(R) f_\theta(R) \quad \text{s.t.} \tag{11.8a}$$

$$\sum_{R \subseteq \mathcal{R}: r \in R} \phi_\theta^{\boldsymbol{k}}(R) = x_\theta^{r,k_r} \qquad \forall \boldsymbol{k} \in \bar{\mathcal{K}}, \forall r \in \mathcal{R}, \forall \theta \in \Theta \quad (11.8b)$$

$$\sum_{\theta \in \Theta} \mu_\theta x_\theta^{r,k} u_\theta^{r,k}(a_1) + \sum_{\theta \in \Theta} \mu_\theta \left(1 - x_\theta^{r,k}\right) u_\theta^{r,k}(a_0)$$
$$\geq l_{a_1}^{r,k,k'} + l_{a_0}^{r,k,k'} \qquad \forall r \in \mathcal{R}, \forall k \neq k' \in \mathcal{K}_r \quad (11.8c)$$

$$l_{a_1}^{r,k,k'} \geq \sum_{\theta \in \Theta} \mu_\theta x_\theta^{r,k'} u_\theta^{r,k}(a)$$
$$\forall r \in \mathcal{R}, \forall a \in \mathcal{A}_r, \forall k \neq k' \in \mathcal{K}_r \quad (11.8d)$$

$$l_{a_0}^{r,k,k'} \geq \sum_{\theta \in \Theta} \mu_\theta \left(1 - x_\theta^{r,k'}\right) u_\theta^{r,k}(a)$$
$$\forall r \in \mathcal{R}, \forall a \in \mathcal{A}_r, \forall k \neq k' \in \mathcal{K}_r \quad (11.8e)$$

$$\sum_{\theta \in \Theta} \mu_\theta x_\theta^{r,k} \left[u_\theta^{r,k}(a_1) - u_\theta^{r,k}(a_0)\right] \geq 0 \qquad \forall r \in \mathcal{R}, \forall k \in \mathcal{K}_r \quad (11.8f)$$

$$\sum_{\theta \in \Theta} \mu_\theta \left(1 - x_\theta^{r,k}\right) \left[u_\theta^{r,k}(a_0) - u_\theta^{r,k}(a_1)\right] \geq 0 \; \forall r \in \mathcal{R}, \forall k \in \mathcal{K}_r \; (11.8g)$$

$$\sum_{R \subseteq \mathcal{R}} \phi_\theta^{\boldsymbol{k}}(R) = 1 \qquad \forall \boldsymbol{k} \in \bar{\mathcal{K}}, \forall \theta \in \Theta. \quad (11.8h)$$

In the LP, Constraints (11.8b) represent consistency conditions ensuring that the general signaling scheme $\phi^{\boldsymbol{k}}$ results in the marginal signaling schemes $\phi^{k,r}$, which are defined by means of variables $x_\theta^{r,k}$. Constraints (11.8c), (11.8d), and (11.8e) represent IC constraints for the menus of marginal signaling schemes, where, as in LP 11.7, we use Constraints (11.8d) and (11.8e) with variables $l_{a_0}^{r,k,k'}$, $l_{a_1}^{r,k,k'}$ to compute receivers' expected utilities of playing a best response. Finally, Constraints (11.8f) and (11.8g) encode the persuasiveness conditions, while Constraints (11.8h) require the signaling scheme be well defined.

Next, we prove our main existence result supporting LP 11.8.

Theorem 11.2. *In multi-receiver instances, there always exists an optimal sender's strategy that uses menus of direct and persuasive marginal signaling schemes.*

Proof. The key insight of the proof is that, in a multi-receiver instance, the sender's expected utility only depends on the marginal probabilities with which the receivers play actions a_1 and a_0 given each state of nature. In order to see that, observe that, once the marginal probabilities $x_\theta^{r,k}$ are fixed, an optimal (general) signaling scheme $\phi^{\boldsymbol{k}}$ can be computed by solving LP 11.8 with Constraints (11.8b) and (11.8h) only. Thus, we only need to show that, given a receiver r and an arbitrary menu of marginal signaling schemes $\{\phi_r^k\}_{k\in\mathcal{K}_r}$, we can always build a menu of *direct* marginal signaling scheme $\{\bar{\phi}_r^k\}_{k\in\mathcal{K}_r}$ such that $\bar{\phi}_{r,\theta}^k(a_1) \geq \phi_{r,\theta}^k(a_1)$ for each $\theta \in \Theta$ and $k \in \mathcal{K}_r$. By the monotonicity assumption on f_θ the optimal sender's strategy with marginal signaling scheme $\{\bar{\phi}_r^k\}_{r\in\mathcal{R},k\in\mathcal{K}_r}$ has an utility greater or equal to the one with $\{\phi_r^k\}_{r\in\mathcal{R},k\in\mathcal{K}_r}$.

This can be proved by following steps similar to those of Lemma 11.1 and Theorem 11.1 for each menu of marginal signaling schemes. In particular, let r be a receiver and $\Phi^r = \{\phi_r^k\}_{k\in\mathcal{K}_r}$ be a menu of marginal signaling schemes that induces probability distribution γ_r^k over the posteriors when the reported type is k. Notice that the probability that the receiver of type $k \in \mathcal{K}_r$ plays an action $a \in \mathcal{A}_r$ is given by $\sum_{\xi\in\hat{\Xi}^{k,a_1}} \gamma_{\boldsymbol{\xi}}^{r,k}\xi_\theta$.[3] We can obtain a menu of probability distribution $\{\bar{\gamma}^{r,k}\}_{k\in\mathcal{K}_r}$ over Ξ^* such that the probability that the receiver plays a_1 increases for each k and θ, *i.e.*, $\sum_{\xi\in\hat{\Xi}^{k,a_1}\cap\Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^{r,k}\xi_\theta \geq \sum_{\xi\in\hat{\Xi}^{k,a_1}} \gamma_{\boldsymbol{\xi}}^{r,k}\xi_\theta$. To see this, it is sufficient to follow the proof of Lemma 11.1 and notice that the receiver always breaks ties in favor of a_1 by the monotonicity assumption on f_θ. Finally, setting $\bar{\phi}_{r,\theta}^k(a) = \frac{\sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a}\cap\Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^k\xi_\theta}{\mu_\theta}$ for each $k \in \mathcal{K}_r$, $\theta \in \Theta$ and $a \in \mathcal{A}_r$, we obtain a menu of signaling schemes such that

$$\bar{\phi}_{r,\theta}^k(a_1) = \frac{\sum_{\boldsymbol{\xi}\in\hat{\Xi}^{k,a_1}\cap\Xi^*} \bar{\gamma}_{\boldsymbol{\xi}}^k\xi_\theta}{\mu_\theta} \geq \frac{\sum_{\xi\in\hat{\Xi}^{k,a_1}} \gamma_{\boldsymbol{\xi}}^{r,k}\xi_\theta}{\mu_\theta} \quad \forall\theta \in \Theta, \forall k \in \mathcal{K}_r$$

To conclude, following the proof of Theorem 11.1 we can show that the menu of marginal signaling schemes $\{\bar{\phi}_r^k\}_{k\in\mathcal{K}_r}$ is IC and persuasive. □

11.3.1 Supermodular/Anonymous Sender's Utility

LP 11.8 has an exponential number of variables and a polynomial number of constraints. Nevertheless, we show that it is possible to apply the ellipsoid algorithm to its dual formulation in polynomial time, provided access to a suitably-defined separation oracle.

Theorem 11.3. *Given access to an oracle that solves* $\max_{R\subseteq\mathcal{R}} f_\theta(R) + \sum_{r\in R} w_r$ *for any* $\mathbf{w} \in \mathbb{R}^{\bar{n}}$*, there exists a polynomial-time algorithm that finds an optimal sender's strategy in any multi-receiver instance.*

[3] For the ease of presentation, we assume that $\gamma^{r,k}$ has finite support. Formally, we should replace $\sum_{\xi\in\hat{\Xi}^{k,a}} \gamma_{\boldsymbol{\xi}}^{r,k}\xi_\theta$ with $Pr_{\boldsymbol{\xi}\sim\gamma^{r,k}}\left\{\boldsymbol{\xi} \in \hat{\Xi}^{k,a}\right\}\mathbb{E}\left[\xi_\theta \mid \boldsymbol{\xi} \in \hat{\Xi}^{k,a}\right]$.

Proof. Since 11.8 has an exponential number of constraints, we work on the dual formulation.

$$\min_{\boldsymbol{q},\mathbf{t}\le 0,\mathbf{z}\le 0,\mathbf{y}\le 0,\boldsymbol{p}} -\sum_{r\in\mathcal{R},k\neq k'\in\mathcal{K}_r}\sum_{\theta}\mu_\theta u_\theta^{r,k}(a_0)t_{r,k,k'}$$
$$+\sum_{r\in\mathcal{R},a\in\mathcal{A}_r,k\neq k'\in\mathcal{K}_r}\sum_{\theta\in\Theta}\mu_\theta u_\theta^{r,k}(a)z_{a_0,r,a,k,k'}$$
$$-\sum_{r\in\mathcal{R},k\in\mathcal{K}_r}\sum_{\theta}\mu_\theta[u_\theta^{r,k}(a_0)-u_\theta^{r,k}(a_1)]y_{a_0,r,k}+\sum_{\boldsymbol{k}\in\bar{\mathcal{K}},\theta\in\Theta}p_{\boldsymbol{k},\theta} \quad (11.9a)$$
$$\text{s.t.} -\sum_{\boldsymbol{k}'\in\bar{\mathcal{K}}:k'_r=k} q_{\boldsymbol{k}',r,\theta}+\sum_{k'\neq k}t_{r,k,k'}[\mu_\theta u_\theta^{r,k}(a_1)-\mu_\theta u_\theta^{r,k}(a_0)]$$
$$-\sum_{a\in\mathcal{A}_r,k'\neq k}\mu_\theta u_\theta^{k'}(a)z_{a_1,r,a',k',k}+\sum_{a\in\mathcal{A}_r,k'\neq k}\mu_\theta u_\theta^{k'}(a)z_{a_0,r,a',k',k}$$
$$+\mu_\theta[u_\theta^{r,k}(a_1)-u_\theta^{r,k}(a_0)]y_{a_1,r,k}$$
$$-\mu_\theta[u_\theta^{r,k}(a_0)-u_\theta^{r,k}(a_0)]y_{a_0,r,k}\ge 0 \forall r\in\mathcal{R},k\in\mathcal{K}_r,\forall\theta\in\Theta \quad (11.9b)$$
$$-t_{r,k,k'}+\sum_{a'\in\mathcal{A}_r}z_{a_1,r,a',k,k'}\ge 0 \qquad \forall r\in\mathcal{R},k\neq k'\in\mathcal{K}_r \quad (11.9c)$$
$$-t_{r,k,k'}+\sum_{a'\in\mathcal{A}_r}z_{a_0,r,a',k,k'}\ge 0 \qquad \forall r\in\mathcal{R},k\neq k'\in\mathcal{K}_r \quad (11.9d)$$
$$\sum_{r\in R}q_{\boldsymbol{k},r,\theta}+p_{\boldsymbol{k},\theta}\ge\mu_\theta\lambda_{\boldsymbol{k}}f_\theta(R) \qquad \forall\boldsymbol{k}\in\bar{\mathcal{K}},\forall\theta\in\Theta,\forall R\subseteq\mathcal{R} \quad (11.9e)$$

where variables $q_{\boldsymbol{k},r,\theta}$ relative to constraints (11.8b), $t_{r,k,k'}$ to (11.8c), $z_{a,r,a',k,k'}$ to (11.8d) and (11.8e), $y_{a,r,k}$ to (11.8f) and (11.8g), $p_{\boldsymbol{k},\theta}$ to (11.8h).

To solve the problem with the ellipsoid method it is sufficient to design a polynomial time separation oracle. We focus on the separation oracle that returns a violated constraint. Given an assignment to the variables, there are a polynomial number of Constraints (11.9b), (11.9c), and 11.9d (with polynomially many variables) and we can check if one of these constraints is violated in polynomial time. Moreover, for each $\bar{\theta}$, $\bar{\boldsymbol{k}}$, we can find if there exists a violated constraint $(\bar{\boldsymbol{k}},\bar{\theta},R)$. We can use the oracle to find $\max_{R\subseteq\mathcal{R}}\lambda_k f_\theta(R)-\sum_{r:r\in R}q_{\bar{\boldsymbol{k}},r,\bar{\theta}}$. If it is greater than $p_{\bar{\boldsymbol{k}},\bar{\theta}}$, we can return a violated constraint, while if it is smaller or equal to $p_{\bar{\boldsymbol{k}},\bar{\theta}}$, all the constraints $\{(\bar{\boldsymbol{k}},\bar{\theta},R)\}_{R\subseteq\mathcal{R}}$ are satisfied. □

An oracle that solves $\max_{R\subseteq\mathcal{R}}f_\theta(R)+\sum_{r\in R}w_r$ can be implemented in polynomial time for supermodular and anonymous functions, as shown by Dughmi and Xu (2017). As a consequence, we obtain the following corollary.

Corollary 11.1. *In multi-receiver instances with supermodular or anonymous sender's utility functions, there exists a polynomial-time algorithm that computes an optimal sender's strategy.*

Proof. By Theorem 11.3, we only need to design a polynomial time oracle. Since the sum of a supermodular and a modular function is supermodular, and unconstrained supermodular maximization can be solved in polynomial time, an oracle can be designed in polynomial time for supermodular functions. For anonymous functions we can construct a polynomial time oracle as follows. We can enumerate over all $n \in \{0, \dots, \bar{n}\}$. Once we fix the size of the set to n, the optimal set includes the n receiver with higher values of weights $\mathbf{w}$. □

11.3.2 Submodular Sender's Utility

In this section, we show how to obtain in polynomial time a $\left(1 - \frac{1}{e}\right)$-approximation to an optimal sender's strategy in instances with submodular utility functions, modulo an additive loss $\epsilon > 0$. This is the best approximation result that can be achieved in polynomial time, since, as it follows from results in the literature, it is NP-hard to obtain an approximation factor better than $\left(1 - \frac{1}{e}\right)$. Indeed, if we consider settings without types, *i.e.*, in which $|\mathcal{K}_r| = 1$ for all $r \in \mathcal{R}$, the problem reduces to computing an optimal signaling scheme when the sender knows receivers' utilities. Then, in the restricted case in which there are only two states of nature, Babichenko and Barman (2017) show that, for each $\epsilon > 0$, it is NP-hard to provide a $\left(1 - \frac{1}{e} + \epsilon\right)$-approximation of an optimal signaling scheme.

Then, the following theorem provides a tight approximation algorithm that runs in polynomial time.

Theorem 11.4. *For each $\epsilon > 0$, there exists an algorithm with running time polynomial in the instance size and $\frac{1}{\epsilon}$ that returns a sender's strategy with utility at least $\left(1 - \frac{1}{e}\right) OPT - \epsilon$ in expectation, where OPT is the sender's expected utility in an optimal strategy.*

In order to prove the result, we reduce the problem of computing the desired (approximate) sender's strategy to solving the following linearly-constrained mathematical program (Program 11.10). The program exploits the fact that, as we will show next, there always exists an "almost" optimal sender's strategy in which the sender employs signaling schemes $\phi^{\boldsymbol{k}}$

(for $\boldsymbol{k} \in \bar{\mathcal{K}}$) such that the distributions $\phi_\theta^{\boldsymbol{k}}$ are q-uniform over the set $2^{\mathcal{R}}$. Recall that we say that a distribution is q-uniform if it follows a uniform distribution on a multiset of size q. Then, the mathematical program reads as follows.

$$\max_{\mathbf{x}} \sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{k} \in \bar{\mathcal{K}}} \lambda_{\boldsymbol{k}} \frac{1}{q} \sum_{j \in [q]} F_\theta \left(x^{j,\boldsymbol{k},\theta} \right) \quad \text{s.t.} \tag{11.10a}$$

$$\sum_{j \in [q]} \frac{1}{q} x_r^{j,\boldsymbol{k},\theta} \leq x_\theta^{r,k_r} \qquad \forall r \in \mathcal{R}, \forall \boldsymbol{k} \in \bar{\mathcal{K}}, \forall \theta \in \Theta \tag{11.10b}$$

$$\sum_{\theta \in \Theta} \mu_\theta x_\theta^{r,k} u_\theta^{r,k}(a_1) + \sum_{\theta \in \Theta} \mu_\theta \left(1 - x_\theta^{r,k} \right) u_\theta^{r,k}(a_0)$$
$$\geq l_{a_1}^{k,k'} + l_{a_0}^{k,k'} \qquad \forall r \in \mathcal{R}, \forall k \neq k' \in \mathcal{K}_r \tag{11.10c}$$

$$l_{a_1}^{k,k'} \geq \sum_{\theta \in \Theta} \mu_\theta x_\theta^{r,k'} u_\theta^{r,k}(a)$$
$$\forall r \in \mathcal{R}, \forall a \in \mathcal{A}_r, \forall k \neq k' \in \mathcal{K}_r \tag{11.10d}$$

$$l_{a_0}^{k,k'} \geq \sum_{\theta \in \Theta} \mu_\theta \left(1 - x_\theta^{r,k'} \right) u_\theta^{r,k}(a)$$
$$\forall r \in \mathcal{R}, \forall a \in \mathcal{A}_r, \forall k \neq k' \in \mathcal{K}_r \tag{11.10e}$$

$$\sum_{\theta \in \Theta} \mu_\theta x_\theta^{r,k} \left[u_\theta^{r,k}(a_1) - u_\theta^{r,k}(a_0) \right] \geq 0 \qquad \forall r \in \mathcal{R}, k \in \mathcal{K}_r \tag{11.10f}$$

$$\sum_{\theta \in \Theta} \mu_\theta \left(1 - x_\theta^{r,k} \right) \left[u_\theta^{r,k}(a_0) - u_\theta^{r,k}(a_1) \right] \geq 0 \forall r \in \mathcal{R}, \forall k \in \mathcal{K}_r \tag{11.10g}$$

$$0 \leq x_\theta^{r,k} \leq 1 \qquad \forall r \in \mathcal{R}, \forall k \in \mathcal{K}_r, \forall \theta \in \Theta \tag{11.10h}$$

$$0 \leq x_r^{j,\boldsymbol{k},\theta} \leq 1 \qquad \forall j \in [q], \forall r \in \mathcal{R}, \forall k \in \mathcal{K}_r, \forall \theta \in \Theta. \tag{11.10i}$$

In Program 11.10, each variable $x_\theta^{r,k}$ represents the probability $\phi_{r,,\theta}^k(a_1)$ that the sender recommends action a_1 to receiver $r \in \mathcal{R}$ of type $k \in \mathcal{K}_r$ in state $\theta \in \Theta$. Constraints (11.10c)–(11.10h) force the marginal signaling schemes to be well defined, where Constraints (11.10c), (11.10d), and (11.10e) encode the IC conditions, Constraints (11.10f) and (11.10g) ensure the persuasiveness property, and Constraints (11.10h) require the marginal signaling schemes to be feasible, *i.e.*, $\phi_\theta^{r,k}(a_1) + \phi_\theta^{r,k}(a_0) = 1$ and $\phi_\theta^{r,k}(a) \geq 0$ for every $a \in \{a_0, a_1\}$. Moreover, the program uses variables $x_r^{j,\boldsymbol{k},\theta} \in \{0, 1\}$ to represent whether the recommended action to receiver $r \in \mathcal{R}$ is a_1 or a_0 in the j-th action profile in the support of $\phi_\theta^{\boldsymbol{k}}$. Notice that we relaxed these variables to $x_r^{j,\boldsymbol{k},\theta} \in [0, 1]$ and use the multi-linear extension of the sender's utility functions f_θ, which, for every $\theta \in \Theta$, reads as

$$F_\theta(x) \coloneqq \sum_{R \subseteq \mathcal{R}} f_\theta(R) \prod_{r \in R} x_r \prod_{r \notin R} (1 - x_r).$$

Moreover, we also relax the constraints ensuring the consistency of the marginal signaling schemes, namely Constraints (11.10b), by replacing the condition $\sum_{j \in [q]} \frac{1}{q} x_r^{j,\boldsymbol{k},\theta} = x_\theta^{r,k_r}$ for all $r \in \mathcal{R}, \boldsymbol{k} \in \bar{\mathcal{K}}, \theta \in \Theta$ with $\sum_{j \in [q]} \frac{1}{q} x_r^{j,\boldsymbol{k},\theta} \leq x_\theta^{r,k_r}$ for all $r \in \mathcal{R}, \boldsymbol{k} \in \bar{\mathcal{K}}, \theta \in \Theta$.

In order to reduce the problem of computing the desired sender's strategy to solving Program 11.10, we need the following two lemmas (Lemma 11.2 and Lemma 11.3). We show that the value of Program 11.10 for a suitably-defined q approximates the value of an optimal sender's strategy (*i.e.*, an optimal solution to LP 11.8) and that, given a solution to Program (11.10), we can recover a sender's strategy with approximately the same expected utility for the sender. Our result is related to those in (Dughmi and Xu, 2017) for the case without types. However, Dughmi and Xu (2017) use a probabilistic method to show the existence of an "almost" optimal signaling scheme that uses q-uniform distributions over the signals. This approach cannot be applied to our problem since it slightly modifies the receivers' utilities. In the case of persuasiveness constraints, they show how to maintain feasibility. However, this approach does *not* work for the IC constraints. We propose a different technique based on the fact that LP 11.8 has a polynomial number of constraints. Let β be the number of constraint of LP 11.10. Notice that β is polynomial in the size of the LP. We show that, for each $\epsilon > 0$, there exist a q such that LP 11.10 has value at least $OPT - \epsilon$, where OPT is the value of an optimal sender' strategy.

Lemma 11.2. *For each $\epsilon > 0$, the optimal value of Program 11.10 with $q = \lceil \beta/\epsilon \rceil$ is at least $OPT - \epsilon$, where OPT is the value of an optimal sender's strategy and β is the number of constraints of LP 11.8.*

Proof. Given an optimal solution (ϕ, x) to LP 11.8, we show how to build a solution to LP 11.10 with almost the same value. Since LP (11.8) has β constraints, there exists an optimal solution (ϕ, x) to LP 11.8 with support at most β. We construct a solution to Program 11.10 with the same values of variables $x_\theta^{r,k}$ (representing marginalsignalign schemes). Then, we show how to obtain a q-uniform distribution for every $\boldsymbol{k} \in \mathcal{K}$ and $\theta \in \Theta$. Fix $\boldsymbol{k} \in \mathcal{K}$ and $\theta \in \Theta$. Let $G^{\boldsymbol{k},\theta} \subseteq 2^{\mathcal{R}}$ be the subsets of $R \subseteq \mathcal{R}$ that are in the support of distribution $\phi_\theta^{\boldsymbol{k}}$, namely $\phi_\theta^{\boldsymbol{k}}(R) > 0$. Notice that $|G^{\boldsymbol{k},\theta}| \leq \beta$, since the solution has support at most β. For every $R \in G^{\boldsymbol{k},\theta}$, we define $N^{\boldsymbol{k},\theta}(R)$ as the greatest integer i such that $\phi_\theta^{\boldsymbol{k}}(R) \geq \frac{i}{q}$. Finally, for every $R \in G^{\boldsymbol{k},\theta}$, we choose $N^{\boldsymbol{k},\theta}(R)$ indexes $j \in [q]$ (with each index being selected at most one time) for which we set $x_r^{j,\boldsymbol{k},\theta} = 1$ for every $r \in R$, and $x_r^{j,\boldsymbol{k},\theta} = 0$ for every $r \notin R$. Since $\sum_{R \in G^{\boldsymbol{k},\theta}} |N^{\boldsymbol{k},\theta}(R)| \leq \sum_{R \in G^{\boldsymbol{k},\theta}} q\phi_\theta^{\boldsymbol{k}}(R) = q$, we have defined values for at most q indexes. For all the remaining indexes $j \in [q]$, we set $x_r^{j,\boldsymbol{k},\theta} = 0$ for $r \in \mathcal{R}$.

It is easy to see that the defined solution is feasible since, for every $\boldsymbol{k} \in \mathcal{K}$, $\theta \in \Theta$, and $r \in \mathcal{R}$, it holds that

$$\sum_{j \in [q]} \frac{1}{q} x_r^{j,\boldsymbol{k},\theta} = \frac{1}{q} \sum_{R \in G^{\boldsymbol{k},\theta} : r \in R} N^{\boldsymbol{k},\theta}(R) \leq \sum_{R \in G^{\boldsymbol{k},\theta} : r \in R} \phi_\theta^{\boldsymbol{k}}(R) = x_\theta^{r,k_r}.$$

Moreover, for every $\boldsymbol{k} \in \mathcal{K}$ and $\theta \in \Theta$, the sender's expected utility in a state of nature $\theta \in \Theta$ is at least

$$\begin{aligned}\frac{1}{q} \sum_{j \in [q]} F_\theta\left(x^{j,\boldsymbol{k},\theta}\right) &= \frac{1}{q} \sum_{R \in G^{\boldsymbol{k},\theta}} N^{\boldsymbol{k},\theta}(R) f_\theta(R) \\ &\geq \sum_{R \in G^{\boldsymbol{k},\theta}} \left(\phi_\theta^{\boldsymbol{k}}(R) f_\theta(R) - \frac{1}{q}\right) \\ &\geq \sum_{R \in G^{\boldsymbol{k},\theta}} \phi_\theta^{\boldsymbol{k}}(R) f_\theta(R) - \frac{\beta}{q} \\ &\geq \sum_{R \subseteq \mathcal{R}} \phi_\theta^{\boldsymbol{k}}(R) f_\theta(R) - \epsilon,\end{aligned}$$

where the equality follows from $x_r^{j,\boldsymbol{k},\theta} \in \{0,1\}$, the first inequality by $\frac{1}{q} N^{\boldsymbol{k},\theta}(R) \geq \phi_\theta^{\boldsymbol{k}}(R) - \frac{1}{q}$, the second one from the fact that $|G^{\boldsymbol{k},\theta}| \leq \beta$, and the last one by the definitions of q and $G^{\boldsymbol{k},\theta}$. Hence, the sender's expected utility is at least

$$\begin{aligned}&\sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} \frac{1}{q} \sum_{j \in [q]} F_\theta(x^{j,\boldsymbol{k},\theta}) \\ &\qquad \geq \sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} \left(\sum_{R \subseteq \mathcal{R}} \phi_\theta^k(R) f_\theta(R) - \epsilon \right) \\ &\qquad = \sum_{\theta \in \Theta} \mu_\theta \sum_{\boldsymbol{k} \in \mathcal{K}} \lambda_{\boldsymbol{k}} \sum_{R \subseteq \mathcal{R}} \phi_\theta^k(R) f_\theta(R) - \epsilon\end{aligned}$$

This concludes the proof. □

Then, we show how to obtain a signaling scheme given a solution of Program 11.10. Dughmi and Xu (2017) build a signaling scheme by using a technique whose generalization to our setting works as follows. Given a state of nature $\theta \in \Theta$ and a vector of types $\boldsymbol{k} \in \bar{\mathcal{K}}$, it selects a $j \in [q]$ uniformly at random and recommends signal a_1 to receiver $r \in \mathcal{R}$ with probability $x_r^{j,\boldsymbol{k},\theta}$, while it recommends a_0 otherwise. By definition of multi-linear extension, using this technique the sender achieves expected utility equal to the value of the given solution to Program 11.10. However, this signaling scheme uses an exponential number of signal profiles, and, thus, it cannot be represented explicitly. In the following lemma, we show how to obtain a sender's strategy in which signaling schemes use a polynomial number of signal profiles.

Lemma 11.3. *Given a solution to Program 11.10 with value APX, for each $\iota > 0$, there exists an algorithm with running time polynomial in the instance size and $1/\iota$ that returns a sender's strategy with utility at least $APX - \bar{n}/q - \iota$ in expectation. Moreover, such sender's strategy employs signaling schemes using polynomially-many signal profiles.*

Proof. Let $\mathbf{x}$ be a solution to Program 11.10 with value APX. Next, we show how to obtain the desired sender's strategy.

First, we build a new solution to Program 11.10 such that Constraints (11.10b) are satisfied with equality. Since the functions F_θ are monotonic, we can simply obtain such solution by increasing the values of variables $x_r^{j,\boldsymbol{k},\theta}$. It is easy to see that, by the monotonicity of F_θ, the objective function does *not* decrease.

Then, we obtain an "almost binary" solution by applying, for every $\theta \in \Theta$ and $\boldsymbol{k} \in \bar{\mathcal{K}}$, the procedure outlined in Algorithm 11.1. An a first operation, the algorithm iterates over the receivers, doing the operations described in the following for each receiver $r \in \mathcal{R}$.

For each $j \in [q]$, the algorithm computes an estimate of the following partial derivative

$$\frac{\partial F_\theta\left(x^{j,\boldsymbol{k},\theta}\right)}{\partial x_r^{j,\boldsymbol{k},\theta}} = \sum_{R \subseteq \mathcal{R} \setminus \{r\}} \left[f_\theta(R \cup \{r\}) - f_\theta(R)\right] \prod_{r' \in R} x_r^{j,\boldsymbol{k},\theta} \prod_{r' \notin R, r' \neq r} \left(1 - x_r^{j,\boldsymbol{k},\theta}\right),$$

This is accomplished by drawing $\sigma = \frac{-8}{\iota^2} \bar{n}^2 \log \frac{p}{2}$ samples of the random variable $f_\theta(\tilde{R} \cup \{r\}) - f_\theta(\tilde{R})$ (with $p = \frac{\iota}{2|\bar{\mathcal{K}}| d q \bar{n}}$), where $\tilde{R} \subseteq \mathcal{R}$ is obtained by randomly picking each receiver $r' \in \mathcal{R} : r' \neq r$ independently with probability $x_{r'}^{j,\boldsymbol{k},\theta}$. It is easy to see that the expected value of the random variable is exactly equal to value of the partial derivative above. Letting $\tilde{e}_r^{j,\boldsymbol{k},\theta}$ be the empirical mean of the samples, by an Hoeffding bound, we get

$$\Pr\left\{\left|\tilde{e}_r^{j,\boldsymbol{k},\theta} - \frac{\partial F_\theta\big(x^{j,\boldsymbol{k},\theta}\big)}{\partial x_r^{j,\boldsymbol{k},\theta}}\right| \geq \frac{\iota}{4\bar{n}}\right\} \leq p.$$

Moreover, consider the event $\mathcal{E}$ in which $\left|\tilde{e}_r^{j,\boldsymbol{k},\theta} - \frac{\partial F_\theta\left(x^{j,\boldsymbol{k},\theta}\right)}{\partial x_r^{j,\boldsymbol{k},\theta}}\right| \leq \frac{\iota}{4\bar{n}}$ for all $j \in [q], \boldsymbol{k} \in \bar{\mathcal{K}}, \theta \in \Theta$, and $r \in \mathcal{R}$. By a union bound, the event $\mathcal{E}$ holds with probability at least $1 - p|\bar{\mathcal{K}}|dq\bar{n}$.

As a second step, the algorithm re-labels the indexes so that, if $j < j'$, then $\tilde{e}_r^{j,\boldsymbol{k},\theta} \geq \tilde{e}_r^{j',\boldsymbol{k},\theta}$. Notice that the value of the partial derivative with respect to $x_r^{j,\boldsymbol{k},\theta}$ does *not* depend on its value. Hence, given two indexes $j < j'$, by "moving" a value t from $x_r^{j',\boldsymbol{k},\theta}$ to $x_r^{j,\boldsymbol{k},\theta}$, the sum $\sum_{j\in[q]} F_\theta\big(x_r^{j,\boldsymbol{k},\theta}\big)$ decreases at most of

$$t\left(\frac{\partial F_\theta(x^{j',\boldsymbol{k},\theta})}{\partial x_r^{j',\boldsymbol{k},\theta}} - \frac{\partial F_\theta(x^{j,\boldsymbol{k},\theta})}{\partial x_r^{j,\boldsymbol{k},\theta}}\right) \leq t\left(\frac{\iota}{2n} + \tilde{e}_r^{j',\boldsymbol{k},\theta} - \tilde{e}_r^{j,\boldsymbol{k},\theta}\right) \leq t\frac{\iota}{2n}.$$

Let $Q^{\boldsymbol{k},\theta,r} = \{1,\ldots,j^*\}$ be the set of the $j^* = \left\lfloor \sum_{j\in[q]} x_r^{j,\boldsymbol{k},\theta} \right\rfloor$ smallest indexes in $[q]$. Then, the algorithm updates the solution x by setting $x_r^{j,\boldsymbol{k},\theta} = 1$ for all indexes $j \in Q^{\boldsymbol{k},\theta,r}$ and setting

$$x_r^{j^*,\boldsymbol{k},\theta} = \sum_{j'\in Q^{\boldsymbol{k},\theta,r}} x_r^{j',\boldsymbol{k},\theta} - \left\lfloor \sum_{j'\in Q^{\boldsymbol{k},\theta,r}} x_r^{j',\boldsymbol{k},\theta} \right\rfloor.$$

After having iterated over all the receivers, the algorithm has built a new feasible solution $\bar{x}$ to Program 11.10 such that

$$\sum_{j\in[q]} [F_\theta(\bar{x}_r^{j,\boldsymbol{k},\theta}) - F_\theta(x_r^{j,\boldsymbol{k},\theta})] \geq -q\iota/2,$$

since the algorithm moved at most a value q from variables indexed by j' to variables indexed by$j < j'$. Moreover, each receiver $r \in \mathcal{R}$ has at most a non-binary element among variables $\bar{x}_r^{j,\boldsymbol{k},\theta}$.

As a final step, the algorithm first builds a set $Q^{\boldsymbol{k},\theta}$ of indexes $j \in [q]$ such that $\bar{x}^{j,\boldsymbol{k},\theta}$ is a binary vector. Notice that there always exists one such set $Q^{\boldsymbol{k},\theta}$ of size at least $q - \bar{n}$. Then, the algorithm constructs a signaling scheme such that

$$\phi_\theta^{\boldsymbol{k}}(R) = \frac{1}{q}\left|\left\{j \in Q^{\boldsymbol{k},\theta} : x_r^{j,\boldsymbol{k},\theta} = 1 \forall r \in R, x_r^{j,\boldsymbol{k},\theta} = 0 \forall r \notin R\right\}\right|.$$

Notice that $\sum_{R\in\mathcal{R}:r\in R}\phi_\theta^{\boldsymbol{k}}(R)\le x^{r,k_r}$ and, by the monotonicity assumption on f_θ, it is easy to build a signaling scheme such that $\sum_{R\in\mathcal{R}:r\in R}\phi_\theta^{\boldsymbol{k}}(R)=x^{r,k_r}$ with greater sender's expected utility. Finally, the algorithm outputs the sender's strategy made by $\{\phi^{\boldsymbol{k}}\}_{\boldsymbol{k}\in\bar{\mathcal{K}}}$ and $\{x^{r,k}\}_{r\in\mathcal{R},k\in\mathcal{K}_r}$, where the menus of marginal signaling schemes are those given as input.

To conclude the proof, we show that the utility of the sender's strategy described above is at least $APX-\bar{n}/q-\iota$ in expectation. If the event $\mathcal{E}$ holds, the utility of the solution is at least

$$\begin{aligned}
&\sum_{\theta\in\Theta}\mu_\theta\sum_{\boldsymbol{k}\in\bar{\mathcal{K}}}\lambda_{\boldsymbol{k}}\phi_\theta^{\boldsymbol{k}}(R)f_\theta(R)\\
&\qquad\ge\sum_{\theta\in\Theta}\mu_\theta\sum_{\boldsymbol{k}\in\bar{\mathcal{K}}}\lambda_{\boldsymbol{k}}\frac{1}{q}\sum_{j\in Q^{\boldsymbol{k},\theta}}F_\theta\big(\bar{x}^{j,\boldsymbol{k},\theta}\big)\\
&\qquad\ge\sum_{\theta\in\Theta}\mu_\theta\sum_{\boldsymbol{k}\in\bar{\mathcal{K}}}\lambda_{\boldsymbol{k}}\frac{1}{q}\left[\sum_{j\in[q]}F_\theta\big(\bar{x}^{j,\boldsymbol{k},\theta}\big)-\bar{n}\right]\\
&\qquad\ge\sum_{\theta\in\Theta}\mu_\theta\sum_{\boldsymbol{k}\in\bar{\mathcal{K}}}\lambda_{\boldsymbol{k}}\frac{1}{q}\left[\sum_{j\in[q]}F_\theta\big(x^{j,\boldsymbol{k},\theta}\big)-\bar{n}-\frac{\iota q}{2}\right]\\
&\qquad=\sum_{\theta\in\Theta}\mu_\theta\sum_{\boldsymbol{k}\in\bar{\mathcal{K}}}\lambda_{\boldsymbol{k}}\left[\frac{1}{q}\sum_{j\in[q]}F_\theta\big(x^{j,\boldsymbol{k},\theta}\big)-\bar{n}/q-\iota/2\right]\\
&\qquad\ge\sum_{\theta\in\Theta}\mu_\theta\sum_{\boldsymbol{k}\in\bar{\mathcal{K}}}\lambda_{\boldsymbol{k}}\frac{1}{q}\sum_{j\in[q]}F_\theta\big(x^{j,\boldsymbol{k},\theta}\big)-\bar{n}/q-\iota/2\\
&\qquad=APX-\bar{n}/q-\iota/2.
\end{aligned}$$

Hence, the sender's expected utility is at least

$$\begin{aligned}
\Pr\{\mathcal{E}\}\left(APX-\bar{n}/q-\iota/2\right)&\ge(1-p|\bar{\mathcal{K}}|dq\bar{n})(APX-\bar{n}/q-\iota/2)\\
&\ge APX-\bar{n}/q-\iota/2-p|\bar{\mathcal{K}}|dq\bar{n}\\
&\ge APX-\bar{n}/q-\iota.
\end{aligned}$$

Since we the marginal signaling schemes do not change, all the persuasiveness and IC constraints are satisfied. Moreover, for every $\boldsymbol{k}\in\bar{\mathcal{K}},\theta\in\Theta$, and $r\in\mathcal{R}$, it holds

$$\sum_{R\subseteq\mathcal{R}:r\in R}\phi_\theta^{\boldsymbol{k}}(R)=\frac{1}{q}\sum_{j\in[q]}\bar{x}_r^{j,\boldsymbol{k},\theta}=\frac{1}{q}\sum_{j\in[q]}x_r^{j,\boldsymbol{k},\theta}=x_\theta^{r,k_r},$$

while it is easy to see that $\sum_{R \subseteq \mathcal{R}} \phi_\theta^{\boldsymbol{k}}(R) = 1$ for every $\boldsymbol{k} \in \bar{\mathcal{K}}$ and $\theta \in \Theta$. This concludes the proof of the lemma. $\square$

Algorithm 11.1 Algorithm in Lemma 11.3

Input: Number of samples $\sigma > 0$; A solution $\mathbf{x}$ to Program 11.10; $\boldsymbol{k} \in \bar{\mathcal{K}}$; $\theta \in \Theta$

for $r \in \mathcal{R}$ **do**
  Compute $\tilde{e}_r^{j,\boldsymbol{k},\theta}$ that estimate $\frac{\partial F_\theta(x^{j,\boldsymbol{k},\theta})}{\partial x_r^{j,\boldsymbol{k},\theta}}$ with σ samples
  Re-label indexes $j \in [q]$ in decreasing order of $\tilde{e}_r^{j,\boldsymbol{k},\theta}$
  $j^* \leftarrow \left\lfloor \sum_{j \in [q]} x_r^{j,\boldsymbol{k},\theta} \right\rfloor$
  $x_r^{j^*+1,\boldsymbol{k},\theta} \leftarrow \sum_{j \in [q]} x_r^{j,\boldsymbol{k},\theta} - j^*$
  $Q^{\boldsymbol{k},\theta,r} \leftarrow \{1, \dots, j^*\}$
  for $j \in Q^{\boldsymbol{k},\theta,r}$ **do**
    $x_r^{j,\boldsymbol{k},\theta} \leftarrow 1$
  end for
  for $j \geq j^* + 2$ **do**
    $x_r^{j,\boldsymbol{k},\theta} \leftarrow 0$
  end for
end for
Construct $\phi_\theta^{\boldsymbol{k}}$ such that:
$\phi_\theta^{\boldsymbol{k}}(R) = \frac{1}{q} \left| \left\{ j \in [q] : x_r^{j,\boldsymbol{k},\theta} = 1 \forall r \in R, x_r^{j,\boldsymbol{k},\theta} = 0 \forall r \notin R \right\} \right| \forall R \subseteq \mathcal{R}$
Update $\phi_\theta^{\boldsymbol{k}}$ to make it consistent with the menus of marginal signaling schemes $\{x_\theta^{r,k_r}\}_{r \in \mathcal{R}}$
return $\phi_\theta^{\boldsymbol{k}}$

Now, we can prove Theorem 11.4.

proof of Theorem 11.4. By Lemmas 11.2 and 11.3, we only need to provide an algorithm that approximates the optimal solution of LP 11.10. The objective is a linear combination with non-negative coefficients of the multi-linear extension of monotone submodular functions. Hence, it is smooth, monotone and submodular. Moreover, since we relaxed Constraints (11.10b), the feasible region is a down-monotone polytope[4] and it is defined by a set of polynomially-many constraints. For each $\delta > 0$, this problem admits a $\left(1 - \frac{1}{e}\right) OPT - \delta$-approximation in time polynomial in the instance size and δ, see the continuous greedy algorithm in (Calinescu et al., 2011) and (Dughmi and Xu, 2017) for a formulation in a similar problem.[5] Finally, we can obtain an arbitrary good approximation choosing an arbitrary large value for q and an arbitrary small value for δ and ι. $\square$

[4] A polytope $\mathcal{P} \in \mathbb{R}_+^n$ is down-monotone if $\mathbf{0} \leq \mathbf{x} \leq \mathbf{y}$ coordinate-wise and $\mathbf{y} \in \mathcal{P}$ imply $\mathbf{x} \in \mathcal{P}$.

[5] The bound holds only for arbitrary large probability. This reduces the total expected utility by an arbitrary small factor.

CHAPTER 12

Conclusions and Future Research

In this thesis, we significantly advance the state of the art on algorithmic Bayesian persuasion along two different directions. First, we study the algorithmic problem of designing optimal information disclosure policies in real-world scenarios. In particular, we study several voting problems, including majority voting, plurality voting and district-based elections characterizing the computational complexity of each problem under private and public signaling. In doing so, we provide some insights on the complexity of general persuasion problems, such as the characterization of bi-criteria approximations in public signaling problems. Moreover, we show how the partial disclosure of information can be used to reduce the social cost in routing games and to increase the revenue in posted price auctions. Then, we relax the assumptions that the sender knows the receiver's utility function, initiating the study of online Bayesian persuasion. This is the first step in designing adaptive information disclosure policies that deals with the uncertainty intrinsic in all real-world applications.

We conclude the chapter proposing some future research directions. Despite the great attention received by the economics and artificial intelligence communities and the large class of potential real-world applications, the use of Bayesian persuasion in the real world is still limited. We believe that one of the main obstacle to the design of information disclosure policies in practice is the perfect knowledge assumption. An interesting direction is to study how the general online Bayesian persuasion framework introduced in the second part of the thesis can be applied to structured games like the one studied in the first part of the thesis. This posses various challenges. First, despite the design of no-regret algorithms is computationally intractable in general, it would be interesting to find some structured games for which it is possible to design *efficient* no-regret algorithms. As a second point, while for the single-receiver online Bayesian persuasion problem we provide no-regret algorithms with both full information and partial information feedback, our analysis of settings with multiple-receiver is limited to the case with full feedback and no externalities. While this assumptions are reasonable in some settings, they do not fit with some applications. For instance, routing games requires to take in account externalities among the players.

Another interesting direction is to deal with the computational challenges introduced by the online learning framework. In particular, we showed that the computation of no-regret algorithms in the online Bayesian persuasion problem is often computational intractable, making it difficult to apply in practice. We concluded the thesis proposing a way to solve this problem, showing that the intractability of an offline version of the problem can be circumvented with a type reporting step. It remains an open question if a type reporting step can be used to design *efficient* online learning algorithms.

Moreover, in our online learning framework we assume that the receivers have a *finite* number of known possible types. Despite this is a significant improvement over the perfect knowledge of the receivers' utilities, this approach assumes some prior knowledge of the receivers. It would be interesting to extend our results to the case in which the receivers can have arbitrary utilities and hence an *infinite* number of possible types.

In this thesis, we show how to deal with uncertainly over the receivers' utility functions. However, this is not the only unreasonable assumption of the classical Bayesian persuasion framework. For instance, another important assumption is that the sender and receivers share the same prior belief. In practice, these beliefs come from past observations, and thus are uncertain and approximated. Zu et al. (2021) study a game between a sender and a receiver that do not know the prior distribution. It would be interesting to consider uncertainly on the receiver's payoffs and the prior belief simultaneously.

Bibliography

H. Allcott and M. Gentzkow. Social media and fake news in the 2016 election. *J ECON PERSPECT*, 31(2):211–36, 2017.

R. Alonso and O. Câmara. Persuading voters. *American Economic Review*, 106(11):3590–3605, 2016.

G. Antioch et al. Persuasion is now 30 per cent of US GDP: Revisiting McCloskey and Klamer after a quarter of a century. *Economic Round-up*, (1):1, 2013.

I. Arieli and Y. Babichenko. Private bayesian persuasion. *Journal of Economic Theory*, 182:185–217, 2019.

S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and the hardness of approximation problems. *J. ACM*, 45(3):501–555, May 1998. ISSN 0004-5411. doi: 10.1145/278298.278306. URL `https://doi.org/10.1145/278298.278306`.

B. Assarf, E. Gawrilow, K. Herr, M. Joswig, B. Lorenz, A. Paffenholz, and T. Rehn. Computing convex hulls and counting integer points with `polymake`. *Mathematical Programming Computation*, 9(1):1–38, 2017. doi: 10.1007/s12532-016-0104-z. URL `http://dx.doi.org/10.1007/s12532-016-0104-z`.

J.-Y. Audibert and S. Bubeck. Minimax policies for adversarial and stochastic bandits. In *Proceedings of the 22nd Annual Conference on Learning Theory (COLT)*, January 2009. URL `https://www.microsoft.com/en-us/research/publication/minimax-policies-adversarial-stochastic-bandits/`. Best Student Paper Award.

R. Aumann. Subjectivity and correlation in randomized strategies. *Journal of Mathematical Economics*, 1(1):67–96, 1974. URL `https://EconPapers.repec.org/RePEc:eee:mateco:v:1:y:1974:i:1:p:67-96`.

B. Awerbuch and R. Kleinberg. Online linear optimization and adaptive routing. *Journal of Computer and System Sciences*, 74(1):97–114, 2008.

B. Awerbuch and Y. Mansour. Adapting to a reliable network path. In *Proceedings of the twenty-second annual symposium on Principles of distributed computing*, pages 360–367, 2003.

Y. Babichenko and S. Barman. Algorithmic aspects of private Bayesian persuasion. In *Innovations in Theoretical Computer Science Conference*, 2017.

Y. Babichenko, C. Papadimitriou, and A. Rubinstein. Can almost everybody be almost happy? pcp for ppad and the inapproximability of nash. *arXiv preprint arXiv:1504.02411*, 2015.

Y. Babichenko, I. Talgam-Cohen, H. Xu, and K. Zabarnyi. Regret-minimizing bayesian persuasion. In P. Biró, S. Chawla, and F. Echenique, editors, *EC '21: The 22nd ACM Conference on Economics and Computation, Budapest, Hungary, July 18-23, 2021*, page 128. ACM, 2021. doi: 10.1145/3465456.3467574. URL `https://doi.org/10.1145/3465456.3467574`.

A. Badanidiyuru, K. Bhawalkar, and H. Xu. Targeting and signaling in ad auctions. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '18, pages 2545–2563, USA, 2018. Society for Industrial and Applied Mathematics. ISBN 9781611975031.

M.-F. Balcan, A. Blum, N. Haghtalab, and A. D. Procaccia. Commitment without regrets: Online learning in stackelberg security games. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, pages 61–78, 2015.

S. Barman and K. Ligett. Finding any nontrivial coarse correlated equilibrium is hard. *SIGecom Exch.*, 14(1):76–79, 2015.

D. Bergemann and S. Morris. Bayes correlated equilibrium and the comparison of information structures in games. *THEOR ECON*, 11(2):487–522, 2016a.

D. Bergemann and S. Morris. Information design, Bayesian persuasion, and Bayes correlated equilibrium. *AM ECON REV*, 106(5):586–91, 2016b.

U. Bhaskar, Y. Cheng, Y. K. Ko, and C. Swamy. Hardness results for signaling in bayesian zero-sum and network routing games. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 479–496, 2016.

A. Blais, L. Massicotte, and A. Dobrzynska. Direct presidential elections: a world summary. *ELECT STUD*, 16(4):441–455, 1997.

F. Brandt, V. Conitzer, U. Endriss, J. Lang, and A. D. Procaccia. *Handbook of computational social choice*. Cambridge University Press, 2016.

P. Bro Miltersen and O. Sheffet. Send mixed signals: earn more, work less. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, pages 234–247, 2012.

G. Calinescu, C. Chekuri, M. Pal, and J. Vondrak. Maximizing a monotone submodular function subject to a matroid constraint. *SIAM Journal on Computing*, 40(6):1740–1766, 2011. doi: 10.1137/080733991. URL `https://doi.org/10.1137/080733991`.

O. Candogan. Persuasion in networks: Public signals and k-cores. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, pages 133–134, 2019.

M. Castiglioni and N. Gatti. Persuading voters in district-based elections. In *The AAAI Conference on Artificial Intelligence, AAAI*. AAAI Press, 2021.

M. Castiglioni, A. Celli, and N. Gatti. Persuading voters: It's easy to whisper, it's hard to speak loud. In *The Thirty-Fourth AAAI Conference on Artificial Intelligence*, pages 1870–1877, 2020a.

M. Castiglioni, A. Celli, and N. Gatti. Public bayesian persuasion: Being almost optimal and almost persuasive. *CoRR*, abs/2002.05156, 2020b. URL https://arxiv.org/abs/2002.05156.

M. Castiglioni, A. Celli, A. Marchesi, and N. Gatti. Online bayesian persuasion. In H. Larochelle, M. Ranzato, R. Hadsell, M. F. Balcan, and H. Lin, editors, *Advances in Neural Information Processing Systems*, volume 33, pages 16188–16198. Curran Associates, Inc., 2020c. URL https://proceedings.neurips.cc/paper/2020/file/ba5451d3c91a0f982f103cdbe249bc78-Paper.pdf.

M. Castiglioni, A. Celli, A. Marchesi, and N. Gatti. Signaling in bayesian network congestion games: the subtle power of symmetry. In *The Thirty-Fifth AAAI Conference on Artificial Intelligence*, 2021a.

M. Castiglioni, A. Marchesi, A. Celli, and N. Gatti. Multi-receiver online bayesian persuasion. In M. Meila and T. Zhang, editors, *Proceedings of the 38th International Conference on Machine Learning*, volume 139 of *Proceedings of Machine Learning Research*, pages 1314–1323. PMLR, 18–24 Jul 2021b. URL https://proceedings.mlr.press/v139/castiglioni21a.html.

M. Castiglioni, A. Marchesi, and N. Gatti. Bayesian persuasion meets mechanism design: Going beyond intractability with type reporting. In *The 21st International Conference on Autonomous Agents and MultiAgent Systems*, 2022a.

M. Castiglioni, G. Romano, A. Marchesi, and N. Gatti. Signaling in posted price auctions. In *The Thirty-Sixth AAAI Conference on Artificial Intelligence*, 2022b.

A. Celli, S. Coniglio, and N. Gatti. Private bayesian persuasion with sequential games. In *The Thirty-Fourth AAAI Conference on Artificial Intelligence*, pages 1886–1893, 2020.

N. Cesa-Bianchi and G. Lugosi. *Prediction, learning, and games*. Cambridge university press, 2006.

Y. Cheng, H. Y. Cheung, S. Dughmi, E. Emamjomeh-Zadeh, L. Han, and S.-H. Teng. Mixture selection, mechanism design, and signaling. In *56th Annual Symposium on Foundations of Computer Science*, pages 1426–1445, 2015.

R. Clifford and A. Popa. Maximum subset intersection. *Information Processing Letters*, 111(7): 323–325, 2011.

V. Conitzer and D. Korzhyk. Commitment to correlated strategies. In *Proceedings of the Twenty-Fifth AAAI Conference on Artificial Intelligence*, pages 632–637, 2011.

V. Conitzer and T. Sandholm. Computing the optimal strategy to commit to. In *Proceedings of the 7th ACM Conference on Electronic Commerce*, pages 82–90, 2006.

A. Deligkas, J. Fearnley, and R. Savani. Inapproximability results for approximate nash equilibria. *CoRR*, abs/1608.03574, 2016.

I. Dinur. The pcp theorem by gap amplification. *Journal of the ACM*, 54(3):12, 2007.

S. Dughmi and H. Xu. Algorithmic bayesian persuasion. In *ACM STOC*, pages 412–425, 2016.

S. Dughmi and H. Xu. Algorithmic persuasion with no externalities. In *ACM EC*, pages 351–368, 2017.

E. Elkind, E. Markakis, S. Obraztsova, and P. Skowron. Equilibria of plurality voting: Lazy and truth-biased voters. In *SAGT*, pages 110–122. Springer, 2015.

Y. Emek, M. Feldman, I. Gamzu, R. PaesLeme, and M. Tennenholtz. Signaling schemes for revenue maximization. *ACM Transactions on Economics and Computation*, 2(2):1–19, 2014.

S. Even, A. Selman, and Y. Yacobi. The complexity of promise problems with applications to public-key cryptography. *Information and control*, 61(2):159–173, 1984.

A. Fabrikant, C. Papadimitriou, and K. Talwar. The complexity of pure nash equilibria. In *Proceedings of the thirty-sixth annual ACM symposium on Theory of computing*, pages 604–612, 2004.

J. Gan, H. Xu, Q. Guo, L. Tran-Thanh, Z. Rabinovich, and M. Wooldridge. Imitative follower deception in stackelberg games. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, EC '19, pages 639–657, New York, NY, USA, 2019. Association for Computing Machinery. ISBN 9781450367929. doi: 10.1145/3328526.3329629. URL `https://doi.org/10.1145/3328526.3329629`.

M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. WH Freeman and Company, 1979.

E. Gawrilow and M. Joswig. `polymake`: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

E. N. Gilbert. A comparison of signalling alphabets. *The Bell System Technical Journal*, 31(3): 504–522, 1952.

O. Goldreich. On promise problems: A survey. In *Theoretical computer science*, pages 254–290. 2006.

M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

V. Guruswami and P. Raghavendra. Hardness of learning halfspaces with noise. *SIAM Journal on Computing*, 39(2):742–765, 2009. doi: 10.1137/070685798. URL `https://doi.org/10.1137/070685798`.

S. Ieong, R. McGrew, E. Nudelman, Y. Shoham, and Q. Sun. Fast and compact: A simple class of congestion games. In M. M. Veloso and S. Kambhampati, editors, *Proceedings, The Twentieth National Conference on Artificial Intelligence and the Seventeenth Innovative Applications of Artificial Intelligence Conference, July 9-13, 2005, Pittsburgh, Pennsylvania, USA*, pages 489–494. AAAI Press / The MIT Press, 2005. URL `http://www.aaai.org/Library/AAAI/2005/aaai05-077.php`.

A. Kalai and S. Vempala. Efficient algorithms for online decision problems. *Journal of Computer and System Sciences*, 71(3):291–307, 2005.

E. Kamenica. Bayesian persuasion and information design. *Annual Review of Economics*, 11:249–272, 2019.

E. Kamenica and M. Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

L. G. Khachiyan. Polynomial algorithms in linear programming. *USSR COMP MATH*, 20(1):53–72, 1980.

S. Khot and R. Saket. Hardness of finding independent sets in almost q-colorable graphs. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 380–389, Los Alamitos, CA, USA, oct 2012. IEEE Computer Society. doi: 10.1109/FOCS.2012.75. URL `https://doi.ieeecomputersociety.org/10.1109/FOCS.2012.75`.

Y. Liu and A. B. Whinston. Efficient real-time routing for autonomous vehicles through bayes correlated equilibrium: An information design framework. *Information Economics and Policy*, 47:14–26, 2019.

Y. Mansour, A. Slivkins, V. Syrgkanis, and Z. S. Wu. Bayesian exploration: Incentivizing exploration in bayesian games. In *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 661–661, 2016.

C. A. Meyers and A. S. Schulz. The complexity of welfare maximization in congestion games. *Networks*, 59(2):252–260, 2012.

H. Moulin and J.-P. Vial. Strategically zero-sum games: the class of games whose completely mixed equilibria cannot be improved upon. *International Journal of Game Theory*, 7(3-4):201–221, 1978.

J. Nachbar and H. Xu. The power of signaling and its intrinsic connection to the price of anarchy. *CoRR*, abs/2009.12903, 2020. URL `https://arxiv.org/abs/2009.12903`.

J. Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36:48–49, 1950.

J. B. Orlin. A polynomial time primal network simplex algorithm for minimum cost flows. *MATH PROGRAM*, 78(2):109–129, 1997.

P. Paruchuri, J. P. Pearce, J. Marecki, M. Tambe, F. Ordonez, and S. Kraus. Playing games for security: An efficient exact algorithm for solving bayesian stackelberg games. In *Proceedings of the 7th International Joint Conference on Autonomous Agents and Multiagent Systems*, pages 895–902, 2008.

Z. Rabinovich, A. X. Jiang, M. Jain, and H. Xu. Information disclosure as a means to security. In *Proceedings of the 2015 International Conference on Autonomous Agents and Multiagent Systems*, pages 645–653, 2015.

R. Raz. A parallel repetition theorem. *SIAM Journal on Computing*, 27(3):763–803, 1998.

T. Richardson and R. Urbanke. *Modern coding theory*. 2008.

R. W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

T. Roughgarden and J. R. Wang. Minimizing regret with multiple reserves. *ACM Transactions on Economics and Computation (TEAC)*, 7(3):1–18, 2019.

A. Rubinstein. Honest signaling in zero-sum games is hard, and lying is even harder. *arXiv preprint arXiv:1510.04991*, 2015.

A. Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2003.

Y. Shoham and K. Leyton-Brown. *Multiagent systems: Algorithmic, game-theoretic, and logical foundations*. Cambridge University Press, 2008.

R. P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge University Press, 2nd edition, 2011.

M. Sviridenko, J. Vondrák, and J. Ward. Optimal approximation for submodular and supermodular optimization with bounded curvature. *Mathematics of Operations Research*, 42(4):1197–1218, 2017.

S. Vasserman, M. Feldman, and A. Hassidim. Implementing the wisdom of waze. In *Twenty-Fourth International Joint Conference on Artificial Intelligence*, pages 660–666, 2015.

E. C. Xavier. A note on a maximum k-subset intersection problem. *Information Processing Letters*, 112(12):471–472, 2012.

T. Xiao, Z. Liu, and W. Huang. On the complexity of sequential posted pricing. In *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*, AAMAS '20, pages 1521–1529, Richland, SC, 2020. International Foundation for Autonomous Agents and Multiagent Systems. ISBN 9781450375184.

H. Xu. On the tractability of public persuasion with no externalities. In *Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2708–2727. SIAM, 2020.

H. Xu, R. Freeman, V. Conitzer, S. Dughmi, and M. Tambe. Signaling in bayesian stackelberg games. In *Proceedings of the 2016 International Conference on Autonomous Agents and Multiagent Systems*, AAMAS '16, pages 150–158, Richland, SC, 2016a. International Foundation for Autonomous Agents and Multiagent Systems. ISBN 9781450342391.

H. Xu, R. Freeman, V. Conitzer, S. Dughmi, and M. Tambe. Signaling in bayesian stackelberg games. In *Proceedings of the 2016 International Conference on Autonomous Agents and Multiagent Systems*, pages 150–158, 2016b.

C. Zhou, T. H. Nguyen, and H. Xu. Algorithmic information design in multi-player games: Possibility and limits in singleton congestion. *CoRR*, abs/2109.12445, 2021. URL `https://arxiv.org/abs/2109.12445`.

Y. Zu, K. Iyer, and H. Xu. Learning to persuade on the fly: Robustness against ignorance. In P. Biró, S. Chawla, and F. Echenique, editors, *EC '21: The 22nd ACM Conference on Economics and Computation, Budapest, Hungary, July 18-23, 2021*, pages 927–928. ACM, 2021. doi: 10.1145/3465456.3467593. URL `https://doi.org/10.1145/3465456.3467593`.