POLITECNICO MILANO 1863

SCUOLA DI INGEGNERIA INDUSTRIALE E DELL'INFORMAZIONE

Executive Summary of the Thesis

On Kolmogorov-Fokker-Planck operators with linear drift and time dependent measurable coefficients

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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Academic year: 2021-2022

1. Introduction

In this thesis we study a class of degenerate parabolic operators called of Kolmogorov-Fokker-Planck type. The operators we study are given by:

$$\mathcal{L} = \sum_{i,j=1}^{q} a_{ij}(t)\partial_{ij} + \sum_{i,j=1}^{N} b_{ij}x_j\partial_i - \partial_t \qquad (1)$$

where q is a positive integer strictly less than N. We assume that the coefficients $\{a_{ij}\}_{i,j=1}^{q}$ and $\{b_{ij}\}_{i,j=1}^{N}$ satisfy the following conditions:

- There exists $\nu > 0$ such that for almost every t and every $\xi \in \mathbb{R}^q$:

$$\nu|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}(t)\xi_i\xi_j \leq \frac{1}{\nu}|\xi|^2$$

- The matrix of coefficients $\mathbb{B} = \{b_{ij}\}_{i,j=1}^{N}$ assumes the following form:

$$\mathbb{B} = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{B}_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{B}_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{B}_\kappa & \mathbb{O} \end{pmatrix} \qquad (2)$$

where there exists a sequence of integers $q = m_0 \geq \cdots \geq m_j \geq \cdots \geq m_\kappa \geq 1$ with sum equal to N and such that $\mathbb{B}_j$ is an $m_j \times m_{j-1}$ matrix of maximal rank.

1.1. The operator with constant coefficient

The starting point for our research is the article by Lanconelli and Polidoro [4] in which the case of constant coefficients $\{a_{ij}\}_{i,j=1}^{q}$ has been studied. Actually, in [4] the matrix $\mathbb{B}$ satisfies the more general assumption:

$$\mathbb{B} = \begin{pmatrix} * & * & \dots & * & * \\ \mathbb{B}_1 & * & \dots & * & * \\ \mathbb{O} & \mathbb{B}_2 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{B}_\kappa & * \end{pmatrix} \qquad (3)$$

where the $*$-blocks are arbitrary while $\mathbb{B}_j$ are as before.

1.2. The homogeneous group structure

Here we list some notions concerning the homogeneous group structure related to the operator (1):

• The group law $\circ$ ([4]) on $\mathbb{R}^{N+1}$:

$$(x,t)\circ(y,s) = (y + E(s)x, t+s)$$

where $E(s) = \exp(-s\mathbb{B})$.
• The family of automorphisms $\{D(\lambda)\}_{\lambda>0}$ ([4]):

$$D(\lambda) = diag(\lambda^{q_1},\ldots,\lambda^{q_N},\lambda^2)$$

where the constants q_i are defined as

$$(q_1,\ldots,q_N) := (\underbrace{1,\ldots,1}_{m_0},\ldots, \ldots\underbrace{2i+1,\ldots,2i+1}_{m_i},\ldots, \underbrace{2\kappa+1,\ldots,2\kappa+1}_{m_\kappa}) \ .$$

• The homogeneous norm (see [1]):

$$\rho((x,t)) = \sum_{i=1}^{N} |x_i|^{\frac{1}{q_i}} + \sqrt{|t|}$$

• The homogeneous dimension:

$$Q = \sum_{i=0}^{\kappa} m_i(2i+1) \ .$$

With these definitions we can remark (following [4]) that, in the case of constant coefficients $\{a_{ij}\}_{i,j=1}^{N}$, the operator is invariant with respect to left translations in $(\mathbb{R}^{n+1},\circ)$:
$\forall\zeta\in\mathbb{R}^{N+1}$, $\forall u\in C_c^\infty(\mathbb{R}^{N+1})$, $\forall\xi\in\mathbb{R}^{N+1}$

$$\mathcal{L}_\xi u(\zeta\circ\xi) = (\mathcal{L}u)(\zeta\circ\xi)$$

($\mathcal{L}_\xi$ means that the operator is evaluated with respect to the ξ variable). Moreover, the condition (2) is equivalent to:
$\forall\lambda>0$, $\forall u\in C_c^\infty(\mathbb{R}^{N+1})$, $\forall\xi\in\mathbb{R}^{N+1}$

$$\mathcal{L}(u(D(\lambda)\xi)) = \lambda^2(\mathcal{L}u)(D(\lambda)\xi) \ .$$

1.3. The case of varying coefficients

As said at the beginning our goal is to proceed further with the case of coefficients depending on time in a nonsmooth way. Actually, there are many papers (see [2] section 5.1) dealing with Hölder continuous coefficients depending on (x,t). But, the Hölder continuity is assumed with respect to a quasi-distance d defined by:

$$d(\xi,\zeta) := \rho(\zeta^{-1}\circ\xi) \quad \xi,\zeta\in\mathbb{R}^{N+1}$$

and a global Hölder continuity with respect to d turns out to be a quite restrictive condition (as shown in [5] Example 1.3).

2. Some results for the operator

Here we mention two papers which are the basis for our results. The first paper is [3] by Bramanti and Polidoro. It contains a construction of the fundamental solution for the operator (1) together with some estimates on the constructed fundamental solution, uniqueness for the Cauchy problem and existence in the homogeneous case. We remark that these results are proved under (3) instead of the more restrictive (2). In our case the fundamental solution given in [3] reduces to:

$$\Gamma(x,t;y,s) = \\ = \frac{e^{-\frac{1}{4}(x-E(t-s)y)^T C(t,s)^{-1}(x-E(t-s)y)}}{\sqrt{(4\pi)^N det(C(t,s))}})$$

where $(x,t)\in\mathbb{R}^{N+1}$ and $(y,s)\in\mathbb{R}^{N+1}$ are such that $t>s$ while the matrix $C(s,t)$ is defined as follows:

$$C(t,s) := \int_s^t E(t-\sigma)\mathbb{A}(\sigma)E(t-\sigma)^T d\sigma \ .$$

Actually, for every fixed $(y,s)\in\mathbb{R}^{N+1}$, function $\Gamma(\cdot;y,s)$ is a solution of $\mathcal{L}u=0$ indeed, by [Theorem 4.4 [3]]), we have:
for almost every $t>s$ and every $x\in\mathbb{R}^N$

$$\mathcal{L}_{(x,t)}\Gamma(x,t;y,s) = 0 \quad .$$

Moreover, it satisfies also the following condition (see [Theorem 4.11 [3], point iii)]):
if $\phi\in C_c(\mathbb{R}^N)$ then the function

$$u(x,t) = \int_{\mathbb{R}^N}\Gamma(x,t;y,s)\phi(y)dy \quad (x,t)\in\mathbb{R}^{N+1}$$

satisfies:

$$u(\cdot,t)\to\phi \quad \text{uniformly as } t\to 0^+ \ .$$

The second paper is [1] by Bramanti and Biagi. This paper contains many results concerning the operator we study, for instance it contains some sharp estimates on the fundamental solution, representation formulas and global Schauders estimates. Actually, the Schauder estimates are proved in the more general case of coefficients which vary also in space and satisfy a Hölder condition only with respect the x variable (see Definition 2.1 below). We remark that this paper constitutes the starting point for the definition of solution given in the thesis. Actually, we exploit also the functional spaces defined there.

Definition 2.1 (See Definition 1.2 [1]). *Let* $\Omega = D \times I$ *where* I *is an open interval and* D *is an open subset of* $\mathbb{R}^N$ *moreover, let* $f : \Omega :\to \mathbb{R}$ *and* $\alpha \in (0,1)$. *We define:*

$$|f|_{C^\alpha(\Omega)} := \sup_{\substack{\xi,\eta\in\Omega \\ \xi\neq\eta}} \frac{|f(\xi) - f(\eta)|}{d(\xi,\eta)^\alpha},$$

$$||f||_{C^\alpha(\Omega)} := |f|_{C^\alpha(\Omega)} + ||f||_{L^\infty(\Omega)}$$

$$|f|_{C^\alpha_x(\Omega)} := \operatorname*{ess\,sup}_{t\in I} \sup_{x\neq y} \frac{|f(x,t) - f(y,t)|}{d((x,t),(y,t))^\alpha},$$

$$||f||_{C^\alpha_x(\Omega)} := |f|_{C^\alpha_x(\Omega)} + ||f||_{L^\infty(\Omega)}$$

and

$$C^\alpha(\Omega) := \big\{ f \in C(\Omega) : ||f||_{C^\alpha(\Omega)} < +\infty \big\}$$

$$C^\alpha_x(\Omega) := \big\{ f \in L^\infty(\Omega) : ||f||_{C^\alpha_x(\Omega)} < +\infty \big\} \ .$$

Then, given $S_T := \mathbb{R}^N \times (-\infty, T)$ we recall the definition of the spaces $S^0(S_T)$ and $S^\alpha(S_T)$:

Definition 2.2 (see Definition 1.4 [1]).

$$S^0(S_T) := \{u \in C(\overline{S_T}) \cap L^\infty(S_T) : \\ \forall i,j \in \{1,\dots,q\}\ \partial_{ij} u \in L^\infty(S_T), \\ Yu \in L^\infty(S_T)\}$$

and if $\alpha \in (0,1)$ *then*

$$S^\alpha(S_T) := \{u \in S^0(S_T) : \forall i,j \in \{1,\dots,q\} \\ \partial_{ij} u \in C^\alpha_x(S_T), Yu \in C^\alpha_x(S_T)\} \ .$$

Now we recall the Schauder estimates (in a simplified form) since they let us understand from where the definition 3.2 comes (see section 3).

Theorem 2.1 (Global Schauder Estimates (see Theorem 4.7 [1])). *Let* $T > \tau > -\infty$ *and* $\alpha \in (0,1)$. *Then, there exists* $c > 0$, *only depending on* $(T-\tau)$, α, ν, $\mathbb{B}$, *such that* $\forall u \in S^\alpha(S_T)$

$$\sum_{h,k=1}^{q} ||\partial^2_{i,j} u||_{C^\alpha_x(S_T)} + ||Yu||_{C^\alpha_x(S_T)} + \\ + \sum_{k=1}^{q} ||\partial^2_k u||_{C^\alpha(S_T)} + ||u||_{C^\alpha(S_T)} \\ \leq c\big\{ ||\mathcal{L}u||_{C^\alpha_x(S_T)} + ||u||_{L^\infty(S_T)} \big\} \ .$$

3. Original contributions

The thesis contains three main results, the first is the well-posedness of the Cauchy problem:

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (0,T) \\ u(\cdot,0) = g & \mathbb{R}^N \end{cases} .$$

The other two results consists in a regularity result for solutions and some local estimates.

3.1. Existence of a solutions - preliminary results

Our initial goal was to prove the well-posedness of the following Cauchy problem:

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (-\infty,T) \\ u(\cdot,t) = 0 & \forall t \leq 0 \end{cases} . \tag{4}$$

The first difficulty is the definition of solution since due to the low regularity of the coefficients it is not straightforward. The definition of solution exploits the spaces defined in [1].

Definition 3.1. *We say that* $u \in S^0(S_T)$ *is a solution of (4) if* $u \equiv 0$ *when* $t \leq 0$ *and for almost every* $(x,t) \in S_T$:

$$\sum_{i,j=1}^{q} a_{ij}(t)\partial_{ij} u(x,t) + Yu(x,t) = f(x,t) \ . \tag{5}$$

We stress that in (5) the derivatives are considered in a weak sense. For instance the term Yu represents the L^∞ function such that for any $\phi \in D(\mathbb{R}^{N+1})$:

$$\int_{\mathbb{R}^{N+1}} Yu\,\phi = \int_{\mathbb{R}^{N+1}} u\,Y^*\phi \ .$$

This definition seemed more natural in order to solve the nonhomogeneous Cauchy problem, however, we cannot exploit the uniqueness result of [3].
The next theorem gives the well-posedness of (4).

Theorem 3.1. *Let* $f \in C^\alpha_x(S_T)$ *be such that* $supp(f) \subset \mathbb{R}^N \times [0,T]$.
Then, there exists a unique $u \in S^0(S_T)$ *solution of (4) (in the sense of 3.1). Moreover,* $u \in S^\alpha(S_T)$ *and there exists a constant* c *depending only on* ν, T *and* α *such that the following stability estimate holds:*

$$\sum_{i,j=1}^{q} ||\partial_{ij} u||_{C^\alpha_x(S_T)} + ||Yu||_{C^\alpha_x(S_T)} + \\ + \sum_{i=1}^{q} ||\partial_i u||_{C^\alpha(S_T)} + ||u||_{C^\alpha(S_T)} \leq c||f||_{C^\alpha_x(S_T)} \ .$$

The main point in the proof of this theorem is to construct a solution since the stability and the uniqueness follow by the Schauder estimates (Theorem 4.7, [1]) and the representation formula (Theorem 3.11, [1]). The first step is to approximate the solution when the datum is $C_c^\infty(\mathbb{R}^{N+1})$, to this aim we have the following lemma:

Lemma 3.1. *Let $f \in C_c^\infty(S_T)$ and let $\varepsilon > 0$. Moreover, let $u_\varepsilon : S_T \to \mathbb{R}$ be defined by:*

$$u_\varepsilon(x,t) = -\int_{-\infty}^{t-\varepsilon} \int_{\mathbf{R}^N} \Gamma(x,t;y,s) f(y,s) dyds \ .$$

Then, for any $\alpha \in (0,1)$, $u_\varepsilon \in S^\alpha(S_T)$ and

$$\mathcal{L}u_\varepsilon(x,t) = \int_{\mathbf{R}^N} \Gamma(x,t;y,t-\varepsilon) f(y,t-\varepsilon) dy \ .$$

Thanks to the estimates on Γ [Theorem 3.5 [1]] it is proved that if f is $C_c^\infty(\mathbb{R}^{N+1})$ the function u_ε converges, in a suitable sense, to a solution u. Hence, we obtain existence for $f \in C_c^\infty(\mathbb{R}^{N+1})$. The next step is to consider $f \in C_x^\alpha$ whit compact support. Thanks to the regularizing properties of the convolution, we can approximate f with C_c^∞ functions and then, by using some compactness properties, we can obtain that the sequence of solutions, obtained for the regularized datum, converges (up to a subsequence) to a solution for our problem.

Theorem 3.2. *If $f \in C_x^\alpha(S_T)$ and is compactly supported, then, the function $u : S_T \to \mathbb{R}$ defined by*

$$u(x,t) = -\int_{-\infty}^{t} \int_{\mathbf{R}^N} \Gamma(x,t;y,s) f(y,s) dyds$$

is such that $u \in S^\alpha(S_T)$ and $\mathcal{L}u = f$.

We remark that the compactness properties are obtained thanks to the Banach Alaoglu Bourbaki Theorem and the Schauder estimates (Theorem 4.7, [1])). Finally, we obtain the existence of a solution when the datum is not compactly supported by exploiting the same machinery with a different approximation: a partition of unity.

3.2. Local estimates and Regularity

The last chapter of the thesis is devoted to the extension of the previous results. More precisely it contains some local estimates, a regularity result and the well-posedness for the nonhomogeneous Cauchy problem with nontrivial data. We shall start describing the local estimates and the regularity result, but before we need to introduce some notation. In this section the sets D, D' are open bounded sets of $\mathbb{R}^N$ while I and I' are open bounded intervals of $\mathbb{R}$.

Definition 3.2. *Let $\alpha \in (0,1)$ then define:*

$$C^\alpha_{loc}(D' \times I') := \{f \in C(D' \times I') : \\ ||f||_{C^\alpha(D\times I)} < +\infty \ \forall D \times I \subset\subset D' \times I'\}$$

$$C^\alpha_{x,loc}(D' \times I') := \{f \in L^\infty_{loc}(D' \times I') : \\ ||f||_{C^\alpha_x(D\times I)} < +\infty \ \forall D \times I \subset\subset D' \times I'\}$$

where the norms are the same of Definition 2.1.

We need also the following definition.

Definition 3.3. *Let $\alpha \in (0,1)$, then consider the following spaces:*

$$X^\alpha(D' \times I') := \{u \in C^\alpha(D' \times I') : \forall i,j \le q \\ \partial_i u \in C^\alpha(D' \times I'),\ \partial_{ij} u \in C^\alpha_x(D' \times I'), \\ Yu \in C^\alpha_x(D' \times I')\}$$

$$X^\alpha_{loc}(D' \times I') := \{u \in C^\alpha_{loc}(D' \times I') :\ \forall i,j \le q \\ \partial_i u \in C^\alpha_{loc}(D' \times I'),\ \partial_{ij} u \in C^\alpha_{x,loc}(D' \times I'), \\ Yu \in C^\alpha_{x,loc}(D' \times I')\}$$

$$S^\alpha_{loc}(D' \times I') := \{u \in C(D' \times I') :\ \forall i,j \le q \\ \partial_{ij} u \in C^\alpha_{x,loc}(D' \times I'),\ Yu \in C^\alpha_{x,loc}(D' \times I')\} \ .$$

Moreover, we define the following norm:

$$||u||_{X^\alpha(D\times I)} := \\ \sum_{i,j=1}^q ||\partial_{ij}u||_{C^\alpha_x(D\times I)} + \sum_{i,j=1}^q ||\partial_i u||_{C^\alpha(D\times I)} + \\ ||Yu||_{C^\alpha_x(D\times I)} + ||u||_{C^\alpha(D\times I)} \ .$$

We remark that the space $X^\alpha(D' \times I')$ with the norm $||\cdot||_{X^\alpha(D'\times I')}$ is a Banach space while $X^\alpha_{loc}(D' \times I')$ when endowed with the family of seminorms $\{||\cdot||_{X^\alpha(D\times I)}\}_{D\times I \subset\subset D'\times I'}$ is a Fréchet space. Moreover, it is easily seen that by the Schauder estimates (Theorem 2.1) we obtain:

$$X^\alpha(\mathbb{R}^{N+1}) = S^\alpha(\mathbb{R}^{N+1}) \ . \tag{6}$$

Now we can state the Local estimates:

Theorem 3.3. *For every $D \times I$ and $D' \times I'$ nonempty open sets satisfying $D \times I \subset\subset D' \times I' \subset\subset \mathbb{R}^{N+1}$, there exists $c > 0$, depending only on $\mathcal{L}$, $D \times I$ and $D' \times I'$, such that:*
$\forall u \in X^\alpha_{loc}(\mathbb{R}^{n+1})$

$$||u||_{X^\alpha(D\times I)} \le c(||u||_{L^\infty(D'\times I')} + ||\mathcal{L}u||_{C^\alpha_x(D'\times I')}) \ .$$

This theorem is proved through the open mapping theorem and the results contained in [1]. The other result is the following.

Theorem 3.4. *Let $a^* > 1 + \frac{Q}{2}$ and let Ω be a nonempty open set. Moreover, let u be such that:*
1) $u \in L^1_{loc}(\Omega)$;
2) $\partial_{ij}u \in L^{a^*}_{loc}(\Omega)$ for any $i,j \in \{1,\dots,q\}$;
3) $Yu \in L^{a^*}_{loc}(\Omega)$.
If $\mathcal{L}u \in C^{\alpha}_{x,loc}(\Omega)$ for some $\alpha \in (0,1)$ then $u \in X^{\alpha}_{loc}(\Omega)$.

This theorem is proved by a representation formula obtained from to the one given in [1] (Theorem 3.11) together with the Schauder estimates (Theorem 4.7 [1]). As an immediate consequence we have the following remarkable fact:

$$S^{\alpha}_{loc}(\mathbb{R}^{N+1}) = X^{\alpha}_{loc}(\mathbb{R}^{N+1}) \ .$$

Notice that thanks to this result if $u \in S^{\alpha}_{loc}(\mathbb{R}^{N+1})$ then for any $\phi \in D(\mathbb{R}^{N+1})$, $\phi u \in S^{\alpha}(\mathbb{R}^N)$. This was not clear before since the first derivatives of u enter the expression for the second order x-derivatives of ϕu.

3.3. Well posedness

Thanks to the same representation formulas used to prove Theorem 3.4 we obtain also the following theorem.

Theorem 3.5 (Uniqueness). *Let $a^* > 1 + \frac{Q}{2}$. If u is a function such that:*
1) $u \in C(\mathbb{R}^N \times (0,T))$;
2) $\partial_{ij}u \in L^{a*}_{loc}(\mathbb{R}^N \times (0,T))$ for any $i,j \leq q$;
3) $Yu \in L^{a*}_{loc}(\mathbb{R}^N \times (0,T))$;
4) $\mathcal{L}u = 0$ in $\mathbb{R}^N \times (0,T)$;
5) For any $\phi \in C_c(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} u(x,t)\phi(x)dx \xrightarrow[t\to 0^+]{} 0$$

(i.e. in the sense of zero order distributions);
5) There exists $c > 0$ such that:

$$\int_0^T \int_{\mathbb{R}^N} |u(x,t)|e^{-c|x|^2} dxdt < +\infty \ . \qquad (7)$$

Then $u \equiv 0$.

This result gives a definition of solution for the nonhomogeneous Cauchy problem.

Definition 3.4. *Let $f \in L^{\infty}_{loc}(\mathbb{R}^N \times (0,T))$, let g be a zero order distribution. Then, a function u is said to be a solution to the Cauchy Problem:*

$$\begin{cases} \mathcal{L}u = f & \mathbb{R}^N \times (0,T) \\ u(\cdot,0) = g & \mathbb{R}^N \end{cases} \qquad (8)$$

if:
1) $u \in C(\mathbb{R}^N \times (0,T))$;
2) $Yu, \partial_{ij}u \in L^{\infty}_{loc}(\mathbb{R}^N \times (0,T))$ for any $i,j \in \{1,\dots,q\}$;
3) $\mathcal{L}u = f$ in $\mathbb{R}^N \times (0,T)$;
4) $u(\cdot,t) \xrightarrow[t\to 0^+]{} g(\cdot)$ in the sense f zero order distributions.

Notice that any L^p_{loc} function is a zero order measure and if the initial datum is achieved in L^p_{loc} sense then it is achieved also in the sense of zero order distributions, actually, the convergence in the sense of zero order distributions still holds when the datum is achieved almost everywhere and the solution is locally bounded in $\mathbb{R}^N \times [0,T)$ (thanks to dominated convergence theorem). Concerning the existence of a solution, thanks to the local estimates (Theorem 3.3), in order to construct a solution we only need to approximate the datum and then proceed exploiting some compactness properties as in Theorem 3.1. We remark that the idea of approximating the datum is employed also for the initial datum g indeed, we can think (at least formally) that the solution of the homogeneous problem with initial datum g is the solution of a nonhomogeneous problem with datum which is concentrated at $t = 0$. In other words we are thinking to solve $\mathcal{L}u = -g \otimes \delta$. This approach exploits the local estimates (Theorem 3.3) and therefore it does not care about how good is g since in order to have $\mathcal{L}u = 0$ in $\mathbb{R}^N \times (0,T)$ we are only interested in the regularity for positive times. Moreover, the function $g \otimes \delta$ is approximated by a convolution hence it is considered as a whole. All these remarks let us think that it is worth paying for some little additional technicalities and assume the datum to be satisfied as in definition 3.4. Now we state the last main result:

Theorem 3.6. *Let g be a zero order distribution in $\mathbb{R}^N$ and let $f \in C^{\alpha}_{x,loc}(\mathbb{R}^N \times (0,+\infty))$. If there exists $c > 0$ such that:*

$$\sup_{\substack{\varphi \in C_c(\mathbb{R}^N) \\ ||\varphi||_\infty \leq 1}} \langle g(x), \varphi(x)e^{-c|x|^2} \rangle < +\infty$$

$$\operatorname*{ess\,sup}_{t>0} \sup_{x \in \mathbb{R}^N} |f(x,t)|e^{-c|x|^2} < +\infty$$

then, there exists $T > 0$ and a unique solution of (8) satisfying

$$\int_0^T \int_{\mathbb{R}^N} |u(x,t)| e^{-c'|x|^2} dxdt$$

for some $c' > 0$. The unique solution u belongs to $S^{\alpha}_{loc}(\mathbb{R}^N \times (0,T))$ and assumes the form:

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} \Gamma(x,t;y,s) f(y,s) dyds + \int_{\mathbb{R}^N} \Gamma(x,t;y,0) g(dy)$$

Moreover, for any $D \times I$ and $D' \times I'$ satisfying $D \times I \subset\subset D' \times I' \subset\subset \mathbb{R}^N \times (0,T)$ there exist a constant $C > 0$, depending only on c, $\mathcal{L}$, $D \times I$ and $D' \times I'$, such that:

$$\sum_{i,j=1}^{q} ||\partial_{ij} u||_{C^{\alpha}_x(D\times I)} + \sum_{i,j=1}^{q} ||\partial_i u||_{C^{\alpha}(D\times I)} + ||Yu||_{C^{\alpha}_x(D\times I)} + ||u||_{C^{\alpha}(D\times I)} \leq$$
$$\leq C\Big(||f||_{C^{\alpha}_x(D'\times I')} + \operatorname*{ess\,sup}_{t>0} \sup_{x\in\mathbb{R}^N} |f(x,t)| e^{-c|x|^2} + \sup_{\substack{\varphi\in C_c(\mathbb{R}^N)\\ ||\varphi||_\infty \leq 1}} \langle g(x), \varphi(x) e^{-c|x|^2}\rangle\Big) \,.$$

4. Conclusions

In this thesis we give a definition of solution to (8) and we prove existence, uniqueness and some stability estimates. Moreover, two additional results of independent interest are proved, namely the local estimates and the regularity result.

5. Acknowledgements

My sincere thanks go to Professor Marco Bramanti and Doctor Stefano Biagi for introducing me to these very interesting topics, for the valuable ideas and suggestions and for the time dedicated to me during the development of the thesis.

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