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# GAUSSIAN QUANTUM MARKOV SEMIGROUPS AND TRANSPORT PROPERTIES IN THE WEAK COUPLING LIMIT

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# Abstract

THE study of open quantum systems is often associated with the study of Quantum Markov Semigroups (QMSs), namely weak* continuous semigroups of completely positive, identity preserving, normal maps on the set of bounded operators over an Hilbert space. They are associated with a generator which has a normal form called the GKLS form. This depends on an operator $H$, called the Hamiltonian, that describes the evolution of the system of interest as a closed system, and the set of operators $\{L_\ell\}$, called the Kraus' operators, that describe how the environment interferes with the closed evolution of the system. This thesis deals essentially with two types of QMSs.

The first one is the class of Gaussian QMSs that models several physical systems involving Bosons. In particular this forces us to deal with unbounded operators and address their domain problems. The Hamiltonian in this case is a quadratic expression of the so called creation and annihilation operators, while the Kraus' operators are a linear expression of the same operators. In this thesis we develop a thorough study of this class of semigroups setting the foundation for future development. As a starter we unify the different definitions of Gaussian QMSs that have emerged in the literature. In particular this involves showing the equivalence between the GKLS form of the generator that we introduced, an explicit formula for the action of the semigroup itself and a qualitative definition of Gaussian QMSs involving preservation of the set of so called Gaussian states. We also studied in depth the one dimensional case, completely classifying irreducibility of the semigroup and existence of an invariant state, based only on the parameters of the GKLS generator. Eventually we started the study also in the multidimensional case, addressing some problems regarding the Decoherence-Free subalgebra.

The second model deals with a very specific physical system, namely a chain of Fermions that is linked on both ends to an independent reservoir at fixed temperature. We successfully evaluated the energy transfer through the chain between the two ends which turns out to be approximately proportional to the temperature difference of the reservoirs, as expected from the classical counterpart.

# Introduction

GAUSSIAN Quantum Markov Semigroups (QMSs) have appeared in the literature many times also under the name of quasi-free semigroups [20,21,44,72]. They are first of all Quantum Markov Semigroups, i.e. a weakly* continuous semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of weakly* continuous, completely positive, identity preserving maps, acting on the set $\mathcal{B}(\Gamma_s(\mathsf{h}))$ of bounded operators over the Bosonic Fock space of the complex separable Hilbert space $\mathsf{h} = \mathbb{C}^d$. They arise naturally in many physical models for open quantum systems (see [6] or the end of Chapters 4, 5). In this thesis we use this models as examples but focus instead on the properties of this class of semigroups in its entirety, which has traditionally been introduced either via the generator or the explicit action on the Weyl operators of the Fock space. In the first case one considers a generator in the GKLS form

$$\mathcal{L}(x) = \mathrm{i}\,[H, x] - \frac{1}{2}\sum_{\ell=1}^{m} \left(L_\ell^* L_\ell x - 2L_\ell^* x L_\ell + x L_\ell^* L_\ell\right), \quad x \in \mathcal{B}(\Gamma_s(\mathcal{H})),$$

where $H$ is a self-adjoint, second order polynomial in creation and annihilation operators, while $(L_\ell)_{\ell=1,\dots,m}$ are a set of linearly independent, first order polynomial in creation and annihilation operators (cf. (3.17), (3.18)). This definition needs some justification due to the unbounded nature of creation and annihilation operators but it can be shown that is well posed (cf. Section 3.3). In the latter case instead one considers the Weyl operators $W(z)$, that generate the whole of $\mathcal{B}(\Gamma_s(\mathsf{h}))$ (cf. Proposition 1.51), and uses them to introduce gaussian QMSs via the explicit formula

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2}\int_0^t \mathrm{Re}\left\langle \mathrm{e}^{sZ}z, C\mathrm{e}^{sZ}z\right\rangle \mathrm{d}s - \mathrm{i}\int_0^t \mathrm{Re}\left\langle \zeta, \mathrm{e}^{sZ}z\right\rangle \mathrm{d}s\right) W(\mathrm{e}^{tZ}z),$$

for some $\zeta \in \mathcal{H}$ and some real linear operators $Z, C$ satisfying a positivity condition (cf. (3.10))

$$\mathbf{C} + \mathrm{i}\,(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}) \geq 0.$$

It can be shown that definition of a gaussian QMS through the GKLS generator implies the explicit formula on Weyl operators (cf. Theorem 3.26) and this gives some intuition on why they actually identify the same class of semigroups.

An even stronger characterization for this class of semigroups (part of the original content of this thesis) involves the set of gaussian states (cf. Chapter 2). They are identified, in analogy with classical gaussian random variables, as those whose (quantum) characteristic function satisfies

$$\hat{\rho}(z) = \mathrm{tr}(\rho W(z)) = \exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \omega, S\rangle - \frac{1}{2}\mathrm{Re}\,\langle z, Sz\rangle\right), \quad z \in \mathcal{H}$$

with a mean vector $\omega \in \mathcal{H}$ and a covariance operator $S$. Gaussian QMSs, as introduced in the literature, preserve the set of gaussian states in its entirety in the sense that the time evolution of a gaussian state through the semigroup yields another gaussian state at all times (cf. Proposition 3.29). In particular they induce a time evolution of the parameters of the state and this property can actually be used to define implicitly gaussian QMSs. Indeed, if a QMS preserves the set of gaussian states in its entirety, together with some differentiability assumptions, it must also satisfy the explicit formula on Weyl operators for some $\zeta, Z, C$ and it must have a GKLS generator of the form we used to introduce gaussian QMSs (cf. Theorem 3.28). Therefore the definition of gaussian QMSs as those that preserve the set of gaussian state, would still be identifying the usual class of semigroup used in the literature.

The strong link between gaussian QMSs and gaussian states is highlighted also when delving deeper in the one-mode case, namely when $\mathsf{h} = \mathbb{C}$. The low dimensionality of the space allows one to study in detail many problems such as the invariant states and obtain very explicit answers (cf. Chapter 4). In particular we can relate the parameters used in the definition of a gaussian QMS with the existence of an invariant state and irreducibility of the semigroup itself. Both these problems are completely solved (cf. Theorem 4.27 and Figure 4.1) and strongly benefit for the low amount of parameters of the one-mode case. Considering the invariant state problem one finds that for a gaussian QMS it not always exists and that a necessary property is that in the operators $(L_\ell)$ the annihilation operators "prevail" over the creation ones (cf. Remark 4.23). However, a remarkable property is that whenever an invariant state exists it must be gaussian (cf. Theorem 4.22). Moreover the study of the irreducibility property for gaussian semigroups allows us to also state that an invariant state is always unique for this class.

When dealing instead with the general multi-mode case the treatment of the previous problems becomes much more convoluted. However another interesting result comes from the analysis of the decoherence-free subalgebra. This is a well-known object introduced as the biggest von Neumann subalgebra of $\mathcal{B}(\Gamma_s(\mathsf{h}))$ where every $\mathcal{T}_t$ is a *-homomorphism. In particular on this set the semigroup acts as if it was a closed quantum system (cf. Proposition 5.2). For gaussian QMSs the decoherence-free subalgebra is a factor (cf. Theorem 5.15) and it is determined by the spectral properties of $Z, C$ (cf. Corollary 5.16).

This thesis has the scope to present the original results obtained on this topic, which are mostly contained in Chapters 3, 4, 5 and 6, but also to be as self-contained as possible in order to provide all the necessary results in a unified notation. Whenever an ancillary but well-known result has an easily accessible proof in the literature we will just report its statement providing a reference to the source.

The content of the thesis is organized as follows.

Chapter 1 contains the introduction of the algebra of the Canonical Commutation Relations (CCR algebra) and the Bosonic Fock space. This sets the ground for the definition of gaussian QMSs acting on $\mathcal{B}(\Gamma_s(\mathsf{h}))$ as well as providing all the results and notations needed in the proofs and the rest of the thesis. In particular in this chapter it is also presented a short introduction to symplectic spaces. This is needed both for the definition of CCR algebra itself but also when dealing dealing with gaussian states and the decoherence-free subalgebra (cf. Theorem 5.15).

Chapter 2 contains the results regarding gaussian states. They are introduced both as states on the CCR algebra and as density matrices on the Fock space through some very similar definitions involving quantum analogues to the characteristic function, as previously noted. We provide a characterization property for covariance operator (cf. Theorem 2.24, Theorem 2.40), mimicking the classical requirement of positive definiteness of the covariance matrix, and we construct transformations that change the parameters of gaussian states. Eventually we provide a density result of these states in the set of all density matrices. It is of note that the first part of the chapter is devoted to recalling some relevant properties of real linear operators, such as Williamson's normal form. These kind of operators are necessary since the covariance operator itself is real linear and their understanding is needed in order to obtain all the previous cited results.

Chapter 3 is devoted to presenting the results for the equivalence of the different definitions of gaussian QMSs. It starts with the analysis of gaussian semigroups only on the CCR algebra, as partially contained in [37, 72] and then recovers QMSs on $\mathcal{B}(\Gamma_S(\mathcal{H}))$ providing the anticipated result of equivalence of definitions (cf. Theorem 3.30). Part of this chapter is also devoted to showing that we can indeed construct a unique QMS starting from an unbounded GKLS generator of the form (3.17), (3.18).

Chapter 4 deals with gaussian QMSs in the case $\mathsf{h} = \mathbb{C}$. The first section shows the complete characterization of irreducibility of the semigroup in terms of the parameters involved in the generator. The analysis is quite convoluted and long but it is fully solved in the end (cf. Figure 4.1). This allows to understand when the evolution of the semigroup could be reduced on a proper subspace and when, instead, this reduction is not possible, corresponding to the irreducible case. The second section deals instead with the study of invariant states and presents the result anticipated of existence and uniqueness (cf. Theorem 4.22). The complete characterization of irreducibility plays an important role in the proof of the uniqueness of the invariant state. In particular it provides, in the same Theorem, a result of convergence towards the invariant state starting from any other state.

Chapter 5 goes back to the general case $\mathsf{h} = \mathbb{C}^d$ and studies the decoherence-free subalgebra. It contains the anticipated (cf. Theorem 5.15) result along with the main tool that is used in order to achieve the characterization, namely that we can write the decoherence-free subalgebra as the generalized commutant of some operators (cf. Theorem 5.4).

Chapter 6 deals with a completely different topic than the rest of the thesis. Here we also concerned with QMSs defined through a GKLS generator but in this case all the operators involved are bounded. The aim is to model a system which is weakly coupled with two reservoirs at fixed temperatures and then study the energy currents through the system that arise in this way. An explicit formula for the current is given under some assumptions (cf. Theorem 6.7). Under very similar conditions we can also provide a lower and an upper bound for the current, emphasizing the explicit dependence on the difference of bath's temperatures and recovering a sort of Fourier law for heat conduction (cf. Theorem 6.12). An explicit example of these formulas applied to the one dimensional Ising chain is presented. Possible developments in the theory are to study similar quantities but in the case of gaussian QMSs.

Last but not least, I would like to thank my supervisor Franco Fagnola for all the help he gave me. Not only for the many ideas and suggestions he continuously had and that shared with, or the amount of time he dedicated to me. But also for the great example he was which is arguably even more valuable and difficult to find than everything else.

# Contents

CHAPTER 1

# CCR Algebra and the Fock Representation

In this chapter we introduce the basic spaces that will serve as foundations for the work on the rest of the thesis. Most of the results of this part are very well known in the literature and therefore their proof will be omitted, with some indication on where to find them. The first section deals with the introduction of symplectic spaces, both in the real and complex cases. We recall some of their basic properties and provide some useful examples that will be used throughout the thesis. The results presented are very well known in the literature and some basic references are [18, 23]. We then use symplectic spaces to construct, in the second section, the algebra of the Canonical Commutain Relations (CCR algebra). We provide an abstract definition and present the result of its uniqueness. The third section is instead devoted to the introduction of the Fock Space, and the Fock representation of the CCR algebra which will be essential in the definition of gaussian Quantum Markov Semigroups. The basic reference for these last two sections is [12].

## 1.1 Symplectic Spaces

**Definition 1.1.** Let $V$ be a vector space over the field $\mathbb{K}$. A *symplectic form* on $V$ is a bilinear form $\sigma : V \times V \to \mathbb{R}$ that is

(antisymmetric) $\sigma(u, v) = -\sigma(v, u)$ for every $u, v \in V$;

(non-degenerate) $\sigma(u, v) = 0$ for every $v \in V$ implies $u = 0$.

The pair $(V, \sigma)$ is called a *symplectic space*.

*Remark* 1.2. The antisymmetry condition implies $\sigma(v, v) = 0$ for every $v \in V$.

**Definition 1.3.** Let $(V, \sigma)$ be a symplectic space. We say $W$ is a *symplectic subspace* of $V$ if $W$ is a linear subspace of $V$ and if the restriction $\sigma|_{W\times W}$ is non-degenerate.

**Definition 1.4.** Let $(V, \sigma)$ be a symplectic space and let $W \subset V$ be a linear subspace of $V$. The *symplectic complement* $W^{\perp_\sigma}$ of $W$ is the set

$$W^{\perp_\sigma} = \{u \in V \mid \sigma(w, u), \text{ for every } w \in W\} .$$

*Remark* 1.5. Suppose $U, W$ are vector subspaces of $V$. Then

$$U^{\perp_\sigma} \cap W^{\perp_\sigma} = (U + W)^{\perp_\sigma} .$$

Indeed if $v \in U^{\perp_\sigma} \cap W^{\perp_\sigma}$ then for every $u \in U, w \in W$ we have

$$\sigma(u + w, v) = \sigma(u, v) + \sigma(w, v) = 0.$$

Vice versa if $v \in (U + W)^{\perp_\sigma}$, then for every $u \in U$

$$\sigma(u, v) = \sigma(u + 0, v) = 0,$$

and analogously for $W$.

*Remark* 1.6. Let $(V, \sigma)$ a symplectic space and suppose $W \subset V$ is a linear subspace of $V$. Then $W$ is a symplectic subspace if and only if $W \cap W^{\perp_\sigma} = \{0\}$.

From now on we will focus on a finite dimensional vector space $V$ since it will be the case of interest in future chapters. Suppose $n$ is the dimension of $V$ and fix $(f_k)_{k=1,\dots,n}$ a basis for the real vector space $V$. There is the natural identification of $V$ with $\mathbb{K}^n$ exploiting the decomposition of $v \in V$ as $v = \sum_{k=1}^{n} v_k f_k$ to the vector $(v_k)_k \in \mathbb{K}^n$. Under this identification the symplectic form $\sigma$ is naturally identified with the matrix $(\sigma(f_j, f_k))_{j,k}$.

*Notation* 1.7. For simplicity in this case we identify

$$\sigma = (\sigma_{jk})_{jk=1}^{n} = (\sigma(f_j, f_k))_{j,k=1}^{n} .$$

*Remark* 1.8. The matrix $\sigma$ has zeroes on the main diagonal and is antisymmetric, since the symplectic form is antisymmetric. Moreover $\sigma$ is invertible, since the symplectic form is non-degenerate.

**Proposition 1.9.** *Let $(V, \sigma)$ be a finite dimensional symplectic space. Then the dimension of $V$ is even and there exists $d \in \mathbb{N}$ such that $n = 2d$. Moreover there exists $(f_j)_{j=1}^{2d}$ a basis for $V$ such that*

$$\sigma = \begin{pmatrix} 0_d & \mathbb{1}_d \\ -\mathbb{1}_d & 0_d \end{pmatrix}, \tag{1.1}$$

*where $0_d, \mathbb{1}_d$ denote respectively the zero and identity matrices of dimension $d \times d$.*

*Proof.* See [18, Theorem 1.1] □

**Definition 1.10.** Let $(V, \sigma)$ be a symplectic space of dimension $2d$. We say $(f_k)_{k=1}^{2d}$ is a *symplectic basis* for $V$ if $(f_k)_{k=1}^{2d}$ is a basis for $V$ as a vector space and the symplectic form $\sigma$ has the matrix expression (1.1) through the identification induced by $(f_k)_{k=1}^{2d}$.

**Definition 1.11.** Let $(V_1, \sigma_1), (V_2, \sigma_2)$ be symplectic spaces and let $T : V_1 \to V_2$ be a linear map. We say $T$ is a *symplectic map* if $T$ preserves the symplectic forms, i.e.

$$\sigma_2(T(v), T(w)) = \sigma_1(v, w), \quad \forall v, w \in V_1.$$

Moreover we say $T$ is a *symplectomorphism* or *isomorphism of symplectic spaces* or *Bogoliubov transformation* if $T$ is a symplectic map and an isomorphism of vector spaces.

In order to clarify the definitions we just gave we now introduce three examples of symplectic spaces that will moreover be used in the rest of the thesis.

**Example 1.12.** Let $V = \mathbb{R}^{2d}$ with the usual scalar product $\langle \cdot, \cdot \rangle_{\mathbb{R}^{2d}}$ and define the bilinear form

$$\sigma_{\mathbb{R}}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} 0_d & \mathbb{1}_d \\ -\mathbb{1}_d & 0_d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_{\mathbb{R}^{2d}}, \quad \forall x_1, x_2, y_1, y_2 \in \mathbb{R}^d.$$

It is easy to prove that $\sigma_{\mathbb{R}}$ is a symplectic form and it is called the *canonical symplectic form* of $\mathbb{R}^{2d}$. the space $(V, \sigma_{\mathbb{R}})$ is symplectic and the canonical basis for $\mathbb{R}^{2d}$ is a symplectic base for $V$ for which

$$\sigma_{\mathbb{R}} = \begin{pmatrix} 0_d & \mathbb{1}_d \\ -\mathbb{1}_d & 0_d \end{pmatrix}.$$

**Example 1.13.** Let $\mathsf{h} = \mathbb{C}^d$ with the usual scalar product $\langle \cdot, \cdot \rangle$, antilinear in the first component, as a complex vector space. It induces a real vector space which is given by the set $\mathbb{C}^d$ with scalar product $\mathrm{Re} \langle \cdot, \cdot \rangle$. We use $\mathsf{h}_{\mathbb{R}}$ when referring to this real vector space and $\mathsf{h}$ to the complex one. Consider the bilinear form on $\mathsf{h}_{\mathbb{R}} \times \mathsf{h}_{\mathbb{R}}$ given by

$$\sigma(z_1, z_2) = \mathrm{Im} \langle z_1, z_2 \rangle, \quad \forall z_1, z_2 \in \mathsf{h}_{\mathbb{R}}.$$

$\sigma$ is a symplectic form and $(\mathsf{h}_{\mathbb{R}}, \sigma)$ is a real symplectic space. Suppose $(e_k)_{k=1}^d$ is the canonical basis of $\mathbb{C}^{2d}$ and define

$$f_k = e_k, \quad f_{d+k} = \mathrm{i} e_k, \quad \forall k = 1, \dots, d.$$

Then $(f_k)_{k=1}^{2d}$ is a symplectic basis for $(\mathsf{h}_{\mathbb{R}}, \sigma)$.

**Example 1.14.** Consider $(\mathbb{C}^{2d}, \mathrm{Re} \langle \cdot, \cdot \rangle /2)$ which is a real vector space, where $\langle \cdot, \cdot \rangle$ represents the usual scalar product on $\mathbb{C}^{2d}$, antilinear in the first component. We introduce the real subspace

$$\mathfrak{h} := \left\{ \begin{pmatrix} z \\ \overline{z} \end{pmatrix} : z \in \mathsf{h}_{\mathbb{R}} \right\}$$

which is a symplectic subspace when equipped with the symplectic form

$$\sigma_{\mathfrak{h}}\left(\begin{pmatrix} z_1 \\ \overline{z_1} \end{pmatrix}, \begin{pmatrix} z_2 \\ \overline{z_2} \end{pmatrix}\right) = \frac{1}{2} \mathrm{Im} \left\langle \begin{pmatrix} z_1 \\ \overline{z_1} \end{pmatrix}, \begin{pmatrix} \mathbb{1}_d & 0_d \\ 0_d & -\mathbb{1}_d \end{pmatrix} \begin{pmatrix} z_2 \\ \overline{z_2} \end{pmatrix} \right\rangle = \mathrm{Im} \langle z_1, z_2 \rangle,$$

for all $z_1, z_2 \in \mathsf{h}_{\mathbb{R}}$. Suppose $(e_k)_{k=1}^d$ is the canonical basis of $\mathbb{C}^d$ as a complex vector space and define

$$f_k = \begin{pmatrix} e_k \\ e_k \end{pmatrix}, \quad f_{d+k} = \begin{pmatrix} \mathrm{i} e_k \\ -\mathrm{i} e_k \end{pmatrix}, \quad \forall k = 1, \dots, d. \tag{1.2}$$

Then $(f_k)_{k=1}^{2d}$ is a symplectic basis for $(\mathfrak{h}, \sigma_{\mathfrak{h}})$.

**Example 1.15.** Consider $(\mathbb{C}^{2d}, \langle \cdot, \cdot \rangle /2)$, where $\langle \cdot, \cdot \rangle$ represents the usual scalar product on $\mathbb{C}^{2d}$, antilinear in the first component. Consider the bilinear form

$$\sigma_{\mathbb{C}^{2d}}\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}\right) = \frac{\mathrm{i}}{2} \left\langle \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}, \begin{pmatrix} 0_d & \mathbb{1}_d \\ -\mathbb{1}_d & 0_d \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right\rangle, \quad z_1, z_2, z_3, z_4 \in \mathbb{C}^d.$$

The space $(\mathbb{C}^{2d}, \sigma_{\mathbb{C}^{2d}})$ is complex symplectic and a symplectic basis is given by $(f_k)_{k=1}^{2d}$ defined in (1.2).

*Remark* 1.16. Every symplectomorphism maps a symplectic basis of $V_1$ into one of $V_2$. Vice versa if $T : V_1 \to V_2$ is a linear map that transforms a symplectic basis of $V_1$ into one of $V_2$ then it is a symplectomorphism.

**Proposition 1.17.** *Any two symplectic spaces over the same field $\mathbb{K}$ and with the same dimension are symplectomorphic.*

*Proof.* By proposition 1.9 both spaces have a symplectic basis. Since they have the same dimension we can define a linear map by transforming each element of the first basis in the corresponding one of the second basis and then extending it by linearity. Using remark 1.16 we have that this is also a symplectomorphism. □

**Example 1.18.** The spaces given by Examples 1.12, 1.13 and 1.14 are symplectomorphic. Moreover we can find the symplectomorphisms explicitly. Let $z \in \mathsf{h}_{\mathbb{R}}$ and suppose $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}^{2d}$ then

$$T_{\mathbb{R}^{2d}} : \mathsf{h}_{\mathbb{R}} \ni z \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2d},$$

and

$$T_{\mathfrak{h}} : \mathsf{h}_{\mathbb{R}} \ni z \mapsto \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \in \mathfrak{h},$$

are symplectomorphisms.

We often need to consider complexifications of symplectic spaces. So let us recall the notion of complexification.

**Definition 1.19.** Let $(V, \langle \cdot, \cdot \rangle_{\mathbb{R}})$ be a real vector space. The *complexification* of $V$ denoted by $V_{\mathbb{C}}$ is the complex vector space $V \oplus V$. An element $w \in V_{\mathbb{C}}$ will be denoted by its unique decomposition $w = u + \mathrm{i}v$ with $u, v \in V$ and the scalar product on $V_{\mathbb{C}}$ is defined by

$$\langle u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2 \rangle_{\mathbb{C}} = \langle u_1, u_2 \rangle_{\mathbb{R}} + \langle v_1, v_2 \rangle_{\mathbb{R}} - \mathrm{i} \langle v_1, u_2 \rangle_{\mathbb{R}} + \mathrm{i} \langle u_1, v_2 \rangle_{\mathbb{R}},$$

for every $u_1, u_2, v_1, v_2 \in V$.
If $\sigma$ is a real symplectic form over $V$, the *complexification* of the real symplectic space $(V, \sigma)$ is the complex symplectic space $(V_{\mathbb{C}}, \sigma_{\mathbb{C}})$ where

$$\sigma_{\mathbb{C}}(u_1 + \mathrm{i}v_1, u_2 + \mathrm{i}v_2) = \sigma(u_1, u_2) - \sigma(v_1, v_2) + \mathrm{i}\sigma(u_1, v_2) + \mathrm{i}\sigma(v_1, u_2),$$

for every $u_1, u_2, v_1, v_2 \in V$.

**Proposition 1.20.** *Suppose $V$ is a real vector space of dimension $d$. Then the complexification $V_{\mathbb{C}}$ has dimension $d$ and they share the same basis.*

*Proof.* Suppose $(f_k)_{k=1}^d$ is a basis for $V$. We can identify each $f_k$ with an element of $V_{\mathbb{C}}$ and we will prove $(f_k)_{k=1}^d$ is a basis of $V_{\mathbb{C}}$. Suppose $u, v \in V$ such that $u = \sum_{k=1}^d u_k f_k$, $v = \sum_{k=1}^d v_k f_k$ with $u_k, v_k \in \mathbb{R}$. Then

$$u + \mathrm{i}v = \sum_{k=1}^{d} (u_k + \mathrm{i}v_k) f_k \in V_{\mathbb{C}},$$

and $(f_k)_{k=1}^d$ generates $V_{\mathbb{C}}$. Suppose now there exist $u_k, v_k \in \mathbb{R}$ such that

$$0 = \sum_{k=1}^{d} (u_k + \mathrm{i} v_k) f_k = \left( \sum_{k=1}^{d} u_k f_k \right) + \mathrm{i} \left( \sum_{k=1}^{d} v_k f_k \right).$$

Then

$$\left( \sum_{k=1}^{d} u_k f_k \right) = \left( \sum_{k=1}^{d} v_k f_k \right) = 0,$$

and by linear independence of the basis $(f_k)_{k=1}^d$ of the real vector space $V$ $u_k = v_k = 0$ for all $k = 1, \dots, d$. □

**Proposition 1.21.** *Let $V, W$ be real vector spaces and let $T : V \to W$ be a linear map. There exists a unique map $T_{\mathbb{C}} : V_{\mathbb{C}} \to W_{\mathbb{C}}$ such that the diagram below commutes.*

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \big\downarrow & & \big\downarrow \\ V_{\mathbb{C}} & \xrightarrow{T_{\mathbb{C}}} & W_{\mathbb{C}} \end{array}$$

*Moreover in this case $T_{\mathbb{C}}(u + \mathrm{i}v) = T(u) + \mathrm{i}T(v)$, for every $u, v \in V$ and, if $T$ is an isomorphism, $T_{\mathbb{C}}$ is as well.*

*Proof.* Clearly $T_{\mathbb{C}}$ defined in the Proposition makes the diagram commute. Suppose $T'_{\mathbb{C}}$ is another complex linear map that does so. For every $v \in V$

$$T'_{\mathbb{C}}(v) = T(v) = T_{\mathbb{C}}(v).$$

If $u \in V$ we have

$$T'_{\mathbb{C}}(u + \mathrm{i}v) = T'_{\mathbb{C}}(u) + \mathrm{i}T'_{\mathbb{C}}(v) = T_{\mathbb{C}}(u) + \mathrm{i}T_{\mathbb{C}}(v) = T_{\mathbb{C}}(u + \mathrm{i}v),$$

where we used complex linearity of both $T_{\mathbb{C}}$ and $T'_{\mathbb{C}}$. Therefore $T_{\mathbb{C}} = T'_{\mathbb{C}}$.

Suppose now $T$ is an isomorphism and let $w_1 + \mathrm{i}w_2, w_3 + \mathrm{i}w_4 \in W_{\mathbb{C}}$. We have

$$T_{\mathbb{C}}(w_1 + \mathrm{i}w_2) = T_{\mathbb{C}}(w_3 + \mathrm{i}w_4) \iff T(w_1 - w_3) + \mathrm{i}T(w_2 - w_4) = 0,$$

which implies $w_1 = w_3$ and $w_2 = w_4$, since $T$ is an isomorphism. On the other hand there exist $u_1, u_2 \in V$ such that $T(u_1) = w_1, T(u_2) = w_2$ therefore

$$T_{\mathbb{C}}(u_1 + \mathrm{i}u_2) = w_1 + \mathrm{i}w_2.$$

Hence, $T_{\mathbb{C}}$ is an isomorphism. □

**Definition 1.22.** We say the map $T_{\mathbb{C}}$ given by Proposition 1.21 is the *complexification* of $T$.

**Example 1.23.** Consider the real symplectic space $(\mathbb{R}^{2d}, \sigma_{\mathbb{R}})$ of Example 1.12. The complexification of $\mathbb{R}^{2d}$ is the complex vector space $\mathbb{C}^{2d}$ with the usual scalar product $\langle \cdot, \cdot \rangle$. The complexified symplectic form is given by

$$\sigma_{\mathbb{C}} \left( \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right) = \left\langle \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix}, \begin{pmatrix} 0_d & \mathbb{1}_d \\ -\mathbb{1}_d & 0_d \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} \right\rangle,$$

for every $z_1, z_2, z_3, z_4 \in \mathbb{C}^d$.

**Example 1.24.** Consider the real symplectic space $(\mathsf{h}_{\mathbb{R}}, \sigma)$ of Example 1.13. The complexification of $\mathsf{h}_{\mathbb{R}}$ is not $\mathbb{C}^d$ because the complexification preserves the dimension and $\mathsf{h}_{\mathbb{R}}$ has dimension $2d$, while $\mathbb{C}^d$ ha dimension $d$ as a complex vector space. The explicit expression of the complexification of $\mathsf{h}_{\mathbb{R}}$ is not easy to use in practice but we will exploit the results of Proposition 1.21 and Example 1.18 and work with an isomorphic space which is easier to use in computations.

**Example 1.25.** Consider the real symplectic space $(\mathfrak{h}, \sigma_{\mathfrak{h}})$ of Example 1.14. The complexification of this symplectic space is given by $(\mathbb{C}^{2d}, \sigma_{\mathbb{C}^{2d}})$, introduced in Example 1.15.

## 1.2 CCR algebra

In this section we will introduce the CCR algebra as an abstract C* algebra and then recover the ususal definition using operators on a Hilbert spcae. An abstract formulation is convenient since we want to have a definition depending only on a symplectic space, without requiring any additional structure. We follow the approach of [54, 67].

Consider $(V, \sigma)$ a real symplectic space and let $\Delta(V, \sigma)$ defined as

$$\Delta(V,\sigma) = \{F : V \to \mathbb{C} \mid F(v) \neq 0 \text{ for a finite number of } v \in V\}.$$

The following proposition holds

**Proposition 1.26.** *Let $(V, \sigma)$ a symplectic space. The set $\Delta(V, \sigma)$ equipped with pointwise summation and pointwise multiplication by scalars, the multiplication rule and involution given by*

$$\begin{aligned}(FG)(v) &= \sum_{u\in V} \mathrm{e}^{-\mathrm{i}\sigma(u,v)} F(u)G(v-u) = \sum_{u\in V} \mathrm{e}^{\mathrm{i}\sigma(u,v)} F(v-u)G(u),\\ F(v)^* &= \overline{F(-v)},\end{aligned}$$

*and the norm*

$$\|F\|_1 = \sum_{u\in V} |F(u)|,$$

*is a normed $*$-algebra.*

*Proof.* Let us note that the sums in the definitions all make sense since just a finite number of summands are non-zero. The proof of the proposition just requires us to show submultiplicativity of the norm and the product rule for the involution. Let $F, G \in \Delta(V, \sigma)$, for every $v \in V$

$$\begin{aligned}(FG)^*(v) = \overline{(FG)(-v)} &= \sum_{u\in V} \mathrm{e}^{\mathrm{i}\sigma(u,-v)} \overline{F(u)G(-v-u)}\\ &= \sum_{u\in V} \mathrm{e}^{\mathrm{i}\sigma(u,v)} \overline{G(u-v)F(-u)} = (G^*F^*)(v),\end{aligned}$$

while

$$\begin{aligned}\|FG\|_1 = \sum_{u\in V} |(FG)(u)| &= \sum_{u\in V} \left| \sum_{w\in V} \mathrm{e}^{-\mathrm{i}\sigma(w,u)} F(w)G(u-w) \right|\\ &\leq \sum_{u,w\in V} |F(w)|\,|G(u-w)| = \|F\|_1 \|G\|_1,\end{aligned}$$

which concludes the proof. □

We consider a notable system of elements in $\Delta(V,\sigma)$.

**Proposition 1.27.** *Let $(V,\sigma)$ a symplectic space and consider the functions*

$$\delta_v(u) = \begin{cases} 1 & u = v \\ 0 & u \neq v \end{cases}, \quad u, v \in V.$$

*The set $\{\delta_v : v \in V\}$ is a basis for $\Delta(v,\sigma)$, $\delta_0$ is the neutral element and*

$$\delta_u \delta_v = \mathrm{e}^{-\mathrm{i}\sigma(u,v)} \delta_{u+v}, \quad u, v \in V \tag{1.3}$$

$$\delta_v^* = \delta_{-v} = (\delta_v)^{-1}, \quad v \in V. \tag{1.4}$$

*Proof.* Let us start by showing $\{\delta_v : v \in V\}$ is a basis for $\Delta(v,\sigma)$. Consider $F \in \Delta(v,\sigma)$ and observe we can write

$$F = \sum_{u \in V} F(u)\delta_u,$$

where the sum is on a finite number of summands, therefore $\{\delta_v : v \in V\}$ is a system of generators. Suppose now there exist $n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $v_1, \ldots v_n \in V$ such that $\sum_{k=1}^n \lambda_k \delta_{v_k} = 0$. Then we have

$$0 = \left\| \sum_{k=1}^n \lambda_k \delta_{v_k} \right\| = \sum_{k=1}^n |\lambda_k| \iff \lambda_k = 0 \ \forall k = 1, \ldots, n.$$

The rest of the proof follows by a straightforward application of the definitions in Proposition 1.26. □

This normed $*$-algebra is not closed, therefore we consider $\overline{\Delta(V,\sigma)}$ the closure of $\Delta(V,\sigma)$ with respect to the $\|\cdot\|_1$ norm. We have thus obtained a Banach $*$-algebra which does not need to be a C*-algebra and infact is not one. In order to define a C*-norm on $\overline{\Delta(V,\sigma)}$ we shall use representations. Recall the following definitions from [13]

**Definition 1.28.** A *representation of a normed $*$-algebra* $\mathfrak{A}$ is a pair $(\mathcal{H},\pi)$ where $\mathcal{H}$ is a complex Hilbert space and $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is $* - homomorphism$, i.e. a linear map satisfying

1. $\pi(AB) = \pi(A)\pi(B)$ for every $A, B \in \mathfrak{A}$,
2. $\pi(A^*) = \pi(A)^*$ for every $A \in \mathfrak{A}$.

Moreover we say $(\mathcal{H},\pi)$ is *non degenerate* if $\{\pi(a)h : a \in \mathfrak{A}, h \in \mathcal{H}\}$ is dense in $\mathcal{H}$ and *faithful* if $\pi$ is an isomorphism between $\mathfrak{A}$ and $\pi(\mathfrak{A})$.

**Definition 1.29.** Let $(V,\sigma)$ a real symplectic space. A representation $(\mathcal{H},\pi)$ of $\overline{\Delta(V,\sigma)}$ is *regular* if for every $v \in V$ the function

$$\mathbb{R} \ni t \to \pi(\delta_{tv}) \in \overline{\Delta(V,\sigma)},$$

is strongly continuous.

*Notation* 1.30. Let us denote with $\mathcal{R}(V,\sigma)$ the set of regular, non-degenerate representations of $\overline{\Delta(V,\sigma)}$.

We have now the following result from [54]

**Proposition 1.31.** *Let $(V,\sigma)$ a real symplectic space. For every $F \in \Delta(V,\sigma)$ we have*

$$\|F\| := \sup_{\pi \in \mathcal{R}(V,\sigma)} \|\pi(F)\|\,,$$

*is finite and thus $\|\cdot\|$ defines a norm on $\Delta(V,\sigma)$. Moreover for every $F, G \in \Delta(V,\sigma)$ it satisfies*

$$\|FG\| \leq \|F\|\,\|G\|\,,\quad \|F^*\| = \|F\|\,,\quad \|F^*F\| = \|F\|^2\,.$$

**Definition 1.32.** Let $(V,\sigma)$ a real symplectic space. The completion of $\Delta(V,\sigma)$ with respect to the norm $\|\cdot\|$ introduced in Proposition 1.31 is the C$^*$ algebra of the *Canonical Commutation Relations* or *CCR algebra* and will be denoted with $CCR\,(V,\sigma)$.

We constructed explicitly the CCR algebra but we can show that actually there is only one possibility for its definition, up to isometric *-isomorphism. Namely we have the following result, which is standard in the literature and whose proof can be found for example in [12, 61].

**Theorem 1.33.** *Let $(V,\sigma)$ a real symplectic space. Suppose $\mathfrak{A}$ is a C$^*$-algebra generated by elements $\{\delta'_u : u \in V\}$ that satisfy* (1.3)*,* (1.4)*. Then $\mathfrak{A}$ and $CCR\,(V,\sigma)$ are isometrically $*$-isomorphic.*

**Corollary 1.34.** *Let $(V,\sigma)$ be a real symplectic space and $T$ a Bogoliubov transformation on $V$, i.e. invertible and such that*

$$\sigma(Tu,Tv) = \sigma(u,v), \quad \forall u,v \in V.$$

*Then there exists $\Gamma(T)$ a unique isometric $*$-isomorphism of $CCR\,(V,\sigma)$ such that*

$$\Gamma(T)(\delta^u) = \delta_{Tu}, \quad \forall u \in V.$$

*Proof.* The proof follows since the set $\{\delta'_u = \delta_{Tu} : u \in V\}$ satisfies (1.3), (1.4) and therefore we can use the uniqueness result of Theorem 1.33. □

**Corollary 1.35.** *Let $(V_1,\sigma_1),(V_2,\sigma_2)$ two real symplectic spaces and consider $(V,\sigma)$ the symplectic space obtained through the direct sum of the previous two, namely*

$$(V_1 \oplus V_2, \sigma_1 \oplus \sigma_2).$$

*Then $CCR\,(V,\sigma)$ and $CCR\,(V_1,\sigma_1) \otimes CCR\,(V_2,\sigma_2)$ are isometrically *-isomorphic.*

*Proof.* Let $\{\delta_v : v \in V_1\}, \{\delta'_v : v \in V_2\}$ be the basis of $CCR\,(V_1,\sigma_1)\,, CCR\,(V_2,\sigma_2)$ respectively. Then, if $v = v_1 + v_2$ where $v \in V$, $v_1 \in V_1, v_2 \in V_2$, we can define

$$\delta''_v = \delta_{v_1} \otimes \delta'_{v_2}.$$

The element of the set $\{\delta''_v : v \in V\}$ satisfy (1.3), (1.4), therefore we conclude the proof via Theorem 1.33. □

*Remark* 1.36. By definition of the norm $\|\cdot\|$ of Proposition 1.31, for every $F \in \Delta(V, \sigma)$ and every representation $(\mathcal{H}, \pi) \in \mathcal{R}(V, \sigma)$

$$\|\pi(F)\| \leq \|F\|.$$

Therefore there is a unique representation $\pi'$ of $CCR\,(V, \sigma)$ such that

$$\pi' \mid_{\Delta(V,\sigma)} = \pi.$$

*Notation* 1.37. By virtue of Remark 1.36 we use the same symbol $\mathcal{R}(V, \sigma)$ to denote the set of regular, non degenerate representations of $CCR\,(V, \sigma)$. We will specify which one we are using if it is not clear from the context.

**Definition 1.38.** Let $(V, \sigma)$ be a real symplectic space. A *Weyl system* is a pair $(\mathcal{H}, W)$ where $\mathcal{H}$ is a complex Hilbert space and $W : H \to \mathcal{B}(\mathcal{H})$ is a function such that

$$W(u)^* = W(-u) = W(u)^{-1}, \forall u \in V$$

and that satisfies the *Weyl form of the CCR*:

$$W(u)W(v) = \mathrm{e}^{-\mathrm{i}\sigma(u,v)}W(u+v), \quad \forall u, v \in V,$$

We say a Weyl system is *regular* if

$$\mathbb{R} \ni t \mapsto W(tv)$$

is strongly continuous for every $v \in V$. The operators $W(v)$, $v \in V$ are called *Weyl operators*. We denote with $\mathcal{W}(V, \sigma)$ the set of regular Weyl systems over $(V, \sigma)$,

We prove now the following result from [54]

**Proposition 1.39.** *Let $(V, \sigma)$ a real symplectic space. There is a bijection between $\mathcal{R}(V, \sigma)$ and $\mathcal{W}(V, \sigma)$ which is explicitly given by*

$$(\mathcal{H}, \pi) \mapsto (\mathcal{H}, v \mapsto \pi(\delta_v)). \tag{1.5}$$

*Proof.* The map (1.5) is well posed since for every representation $(\mathcal{H}, \pi) \in \mathcal{R}(V, \sigma)$, $\pi$ is a *-homomorphism and the elements $\delta_v$ satisfy the properties of Proposition 1.27.

Consider now a Weyl system $(\mathcal{H}, W) \in \mathcal{W}(V, \sigma)$. We will show there is one and only one function $\pi : CCR\,(V, \sigma) \to \mathcal{B}(\mathcal{H})$ such that $\pi(\delta_v) = W(v)$ for every $v \in V$ and $(\mathcal{H}, \pi) \in \mathcal{R}(V, \sigma)$. Uniqueness follow from the fact that if $\pi(\delta_v) = W(v)$ for every $v \in V$, then for $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $v_1, \dots, v_n \in V$ we have

$$\pi\left(\sum_{k=1}^{n} \lambda_k \delta_{v_k}\right) = \sum_{k=1}^{n} \lambda_k W(v_k).$$

Therefore $\pi$ is uniquely defined by a density argument and Remark 1.36.

It remains to be shown the existence of a representation where $\pi$ satisfies $\pi(\delta_v) = W(v)$ for every $v \in V$. But this is trivial by the *-homomorphism properties we require for $\pi$ and strong continuity of $t \mapsto W(tv)$ by definition of a Weyl system. Eventually the representation is non degenerate because $\pi(\delta_0) = W(0) = \mathbb{1}$. □

*Notation* 1.40. We denote via $CCR\,(V, \sigma)_W$ the C*-algebra generated by the Weyl operators of the Weyl system $(\mathcal{H}, W)$. Since to each Weyl system corresponds a representation of the CCR algebra $CCR\,(V, \sigma)$, we say $CCR\,(V, \sigma)_W$ is the representation of the CCR algebra on $(\mathcal{H}, W)$ or simply on $W$. Theorem 1.33 shows that all these CCR algebras are isometrically $*$-isomorphic.

## 1.3 The Symmetric Fock Space and the Fock Representation

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, antilinear in the first component. Let $\mathcal{H}^{\otimes n} = \mathcal{H}\otimes\cdots\otimes\mathcal{H}$ denote the $n$-fold tensor product of $\mathcal{H}$ with itself. The Fock space is defined as

$$\Gamma(\mathcal{H}) = \bigoplus_{n\in\mathbb{N}} \mathcal{H}^{\otimes n},$$

where $\mathcal{H}^0 = \mathbb{C}$. A generic element $\psi$ of $\Gamma(\mathcal{H})$ is identified with a sequence $(\psi^{(n)})_{n\in\mathbb{N}}$ of vectors $\psi^{(n)} \in \mathcal{H}^{\otimes n}$. In this way we can identify $\mathcal{H}^{\otimes n}$ as the subspace of $\Gamma(\mathcal{H})$ where the elements $\psi$ have $\psi^{(m)} = 0$ except for $m = n$. We are interested in the symmetric Fock space that is a particular subspace of $\Gamma(\mathcal{H})$. Consider the operator $P_s$ on $\Gamma(\mathcal{H})$ acting on $\mathcal{H}^{\otimes n}$ as

$$P_s(f_1\otimes\cdots\otimes f_n) = \frac{1}{\sqrt{n!}}\sum_{\xi\in S_n} f_{\xi(1)}\otimes\cdots\otimes f_{\xi(n)}, \quad \forall f_1,\dots,f_n\in\mathcal{H},$$

where $S_n$ is the set of permutations of $\{1,\dots,n\}$. Extension by linearity of $P_s$ gives a densely defined operator with $\|P_s\| = 1$, therefore it can be extended to a bounded operator of norm one. We still use $P_s$ to denote this extensions.

**Definition 1.41.** Let $\mathcal{H}$ be a complex separable Hilbert space. The *Symmetric Fock Space*, denoted with $\Gamma_s(\mathcal{H})$, is

$$\Gamma_s(\mathcal{H}) = P_s\Gamma(\mathcal{H}).$$

*Remark* 1.42. It is easy to see that if the Hilbert space $\mathcal{H}$ is the direct sum of two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ $\mathcal{H} = \mathcal{H}_1\oplus\mathcal{H}_2$ then $\Gamma_s(\mathcal{H})$ is isometrically isomorphic to $\Gamma_s(\mathcal{H}_1)\otimes\Gamma_s(\mathcal{H}_2)$.

If $U$ is a unitary operator on $\mathcal{H}$ we can define $U_n$ for every $n\in\mathbb{N}$ by $U_0 = \mathbb{1}$ and for every $f_1,\dots,f_n\in\mathcal{H}$

$$U_n(P_s(f_1\otimes\cdots\otimes f_n)) = P_s(Uf_1\otimes\cdots\otimes Uf_n).$$

**Definition 1.43.** We define the *second quantization* of a unitary operator $U$ on $\mathcal{H}$ the unitary operator obtained as

$$\Gamma(U) = \bigoplus_{n\in\mathbb{N}} U_n.$$

**Definition 1.44.** Let $\mathcal{H}$ be a complex separable Hilbert space. An *exponential vector* $e(f)\in\Gamma_s(\mathcal{H})$, for $f\in\mathcal{H}$ is

$$e(f) = \sum_{n\in\mathbb{N}} \frac{f^{\otimes n}}{\sqrt{n!}}.$$

We denote by $E = \mathrm{span}\{e(f) : f\in\mathcal{H}\}$ the linear span of exponential vectors.

The following proposition from [57] gives some properties of the newly introduced set.

**Proposition 1.45.** *Let $\mathcal{H}$ be a complex separable Hilbert space. The set $\{e(f) : f \in \mathcal{H}\}$ of all the exponential vectors is linearly independent and total in $\Gamma_s(\mathcal{H})$. Moreover,*

$$\langle e(f), e(g)\rangle = \mathrm{e}^{\langle f,g\rangle}, \quad \forall f, g \in \mathcal{H}. \tag{1.6}$$

Exponential vectors can also be used to evaluate traces according to the formula of the following Lemma whose proof can be found for example in [58].

**Lemma 1.46.** *Let $z = x + \mathrm{i}y$, with $x, y \in \mathbb{R}^d$, we have*

$$\frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-|z|^2} |e(z)\rangle\langle e(z)| \,\mathrm{d}x\mathrm{d}y = \mathbb{1}, \tag{1.7}$$

*where $|a\rangle\langle a|\, b = \langle a, b\rangle\, a$, for every $a, b \in \mathcal{H}$ and the integral has to be interpreted in the weak sense. In particular we have*

$$\mathrm{tr}(\omega) = \frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-|z|^2} \langle e(z), \omega(e(z))\rangle \,\mathrm{d}x\mathrm{d}y, \tag{1.8}$$

*for every trace class operator $\omega$.*

*Proof.* We will prove this result with $d = 1$, the proof in higher dimension follows along the same lines. Let $z = x+\mathrm{i}y$, $z_1 = x_1+\mathrm{i}y_1$, $z_2 = x_2+\mathrm{i}y_2$ where $x, x_1, x_2, y, y_1, y_2 \in \mathbb{R}$. We have

$$\begin{aligned}\left\langle e(z_1), \frac{1}{\pi}\int_{\mathbb{R}^d} \mathrm{e}^{-|z|^2} |e(z)\rangle\langle e(z)| \,\mathrm{d}x\mathrm{d}y\; e(z_2)\right\rangle &= \frac{1}{\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-|z|^2+\overline{z}z_2+\overline{z_1}z}\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-x^2-y^2+x(\overline{z_1}+z_2)+\mathrm{i}y(\overline{z_1}-z_2)}\mathrm{d}x\mathrm{d}y \\ &= \exp\left(\left(\frac{\overline{z_1}+z_2}{2}\right)^2 - \left(\frac{\overline{z_1}-z_2}{2}\right)^2\right) \\ &= \mathrm{e}^{\overline{z_1}z_2} = \langle e(z_1), e(z_2)\rangle\,.\end{aligned}$$

By Proposition 1.45 the set of exponential vectors $E$ is total in $\Gamma_s(\mathcal{H})$ therefore (1.7) is proven. In order to prove (1.8) now we have

$$\begin{aligned}\mathrm{tr}(\omega) &= \mathrm{tr}\left(\omega\frac{1}{\pi}\int_{\mathbb{R}^d} \mathrm{e}^{-|z|^2} |e(z)\rangle\langle e(z)| \,\mathrm{d}x\mathrm{d}y\right) \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-|z|^2} \langle e(z), \omega e(z)\rangle \,\mathrm{d}x\mathrm{d}y\end{aligned}$$

□

Now let us fix the symplectic space $(\mathsf{h}_{\mathbb{R}}, \sigma)$ introduced in Example 1.13 and $\mathsf{h} = \mathbb{C}^d$ as a complex Hilbert space. Let us denote with $\mathcal{H} = \Gamma_s(\mathsf{h})$ and define the operators $W(z) \in \mathcal{B}(\mathcal{H})$ for $z \in \mathsf{h}_{\mathbb{R}}$ that act on the set of exponential vectors as

$$W(z)e(f) = \exp\left(-\frac{|z|^2}{2} - \langle z, f\rangle\right) e(z+f).$$

**Proposition 1.47.** *The pair $(\mathcal{H}, W)$ is a Weyl system on $(\mathsf{h}_{\mathbb{R}}, \sigma)$.*

*Proof.* It is easy to see that, for $f \in \mathcal{H}, u, v \in \mathsf{h}_{\mathbb{R}}$

$$W(u)W(v)e(f) = \mathrm{e}^{-\mathrm{i}\sigma(u,v)}W(u+v)e(f), \quad \|W(u)e(f)\|^2 = \|e(f)\|^2 .$$

By totality of exponential vectors, from Proposition 1.45 the above equalities hold on the whole $\mathcal{H}$. Therefore $W(u)$ satisfies the second requirement of Definition 1.38, and $W(u)$ are unitaries, so

$$W(u)W(u)^* = \mathbb{1} = W(0) = W(u)W(-u),$$

which concludes the first requirement. It remains to be proven just strong continuity of the maps $t \mapsto W(tu)$, for $t \in \mathbb{R}$. Note at first that

$$\|e(f+tu) - e(f)\|^2 = \mathrm{e}^{|f+tu|^2} - 2\,\mathrm{Re}\left(\mathrm{e}^{\langle f+tu, f\rangle}\right) + \mathrm{e}^{|f|^2} \xrightarrow{t\to 0} 0,$$

therefore

$$\begin{aligned}\|W(u)e(f) - e(f)\|^2 \le{}& \mathrm{e}^{-t^2|u|^2 - 2t\,\mathrm{Re}\langle u,f\rangle}\,\|e(f+tu) - e(f)\|^2 \\ &+ \left|\mathrm{e}^{-\frac{t^2|u|^2}{2} - t\langle u,f\rangle} - 1\right|^2 \|e(f)\|^2\end{aligned}$$

which converges to $0$ as $t$ goes to $0$. This yields strong continuity at $t = 0$ and we can extend to the whole real line by noticing that

$$W((t+s)u) = W(tu)W(su),$$

and recalling that $W(u)$ is unitary for every $u \in \mathsf{h}_{\mathbb{R}}$ therefore

$$\begin{aligned}\|W((t+s)u)e(f) - W(tu)e(f)\|^2 &= \|W(tu)\left(W(s)e(f) - e(f)\right)\|^2 \\ &= \|W(s)e(f) - e(f)\|^2 \xrightarrow{s\to 0} 0.\end{aligned}$$

□

**Definition 1.48.** We will call the representation $(\mathcal{H}, \pi)$, induced by the Weyl system $(\mathcal{H}, W)$, the *Fock representation*.

**Definition 1.49.** A set $\mathfrak{M}$ of bounded operators on a Hilbert Space $\mathcal{H}$ is said to be *irreducible* if the only closed subspaces of $\mathcal{H}$ which are invariant under the action of $\mathfrak{M}$ are the trivial ones, i.e. $\{0\}$ and $\mathcal{H}$. A representation $(\mathcal{H}, \pi)$ of C$^*$-algebra $\mathfrak{A}$ is said to be *irreducible* if $\pi(\mathfrak{A})$ is irreducible on $\mathcal{H}$.

The following Lemma comes from [57]

**Lemma 1.50.** *Let $T$ any bounded operator on $\Gamma_s(\mathcal{H})$ such that $TW(z) = W(z)T$ for every $z \in \mathsf{h}_{\mathbb{R}}$. Then $T$ is a scalar multiple of the identity.*

*Proof.* See [57, Proposition 20.9]. □

**Proposition 1.51.** *The Fock representation of the CCR algebra is irreducible and the von Neumann algebra generated by the Weyl operators is $\mathcal{B}(\Gamma_s(\mathsf{h}))$.*

*Proof.* Let us start by showing irreducibility. Suppose there exists a subspace $\mathcal{K} \subset \mathcal{H}$ which is invariant for $W(z)$ for every $z \in \mathsf{h}_{\mathbb{R}}$. Since, by definition, $W(z)^* = W(-z)$ we have $\mathcal{K}$ is a reducing subspace for $W(z)$ for every $z \in \mathsf{h}_{\mathbb{R}}$. Therefore, if $p$ is the projection operator onto $\mathcal{K}$ we have $pW(z) = W(z)p$ for all $z \in \mathsf{h}_{\mathbb{R}}$. By Lemma 1.50 this implies $p$ is a multiple of the identity and therefore $\mathcal{K} = \mathcal{H}$.

Consider now $\mathrm{span}\{W(z) : z \in \mathsf{h}\}$, which is a unital *-subalgebra of $\mathcal{B}(\Gamma_s(\mathsf{h}))$. Using Von Neumann's bicommutant theorem $\mathcal{K} = \{W(z) : z \in \mathsf{h}\}''$ is a *-subalgebra of $\mathcal{B}(\Gamma_s(\mathsf{h}))$ closed in the strong topology. Irreducibility of the Fock Representation now implies $\mathcal{K}' = \mathbb{C}$ and therefore

$$\mathcal{K} = \mathcal{K}'' = \mathcal{B}(\Gamma_s(\mathsf{h})).$$

□

There exists a generalization of the previous two results which has been proved by Araki (see [7, Theorem 4]) and reformulated with a new simplified proof in [52, Theorem 1.3.2].

**Definition 1.52.** For all subsets $\mathcal{M}$ of $\mathsf{h}$ we denote by $\mathcal{W}(\mathcal{M})$ the von Neumann algebra generated by Weyl operators $W(z)$ with $z \in \mathcal{M}$.

**Theorem 1.53.** *Let $M \subset \mathsf{h}_{\mathbb{R}}$ be a real vector subspace. Then*

$$\mathcal{W}(M)' = \mathcal{W}(M^{\perp_\sigma}),$$

*where $M^{\perp_\sigma}$ is the symplectic complement of $M$ (cf. Definition 1.4).*

*Proof.* See [52, Theorem 1.3.2]. □

*Remark* 1.54. By Proposition 1.51

$$\mathcal{W}(\mathsf{h}) = \mathcal{B}(\Gamma_s(\mathsf{h})).$$

Using instead Theorem 1.53 we have $\mathsf{h}^{\perp_\sigma} = \{0\}$ and

$$\mathcal{W}(\mathsf{h})' = \mathcal{W}(\mathsf{h}^{\perp_\sigma}) = \mathcal{W}(\{0\}) = \mathbb{C}\mathbb{1},$$

that implies, as in the proof of Proposition 1.51,

$$\mathcal{W}(\mathsf{h}) = \mathcal{W}(\mathsf{h})'' = (\mathbb{C}\mathbb{1})' = \mathcal{B}(\Gamma_s(\mathsf{h})).$$

We introduce now some operators that are frequently considered in the Fock representation of the CCR algebra. We will use them in future chapters when introducing Gaussian QMSs.

*Notation* 1.55. The Fock representation is regular, therefore, for every $u \in \Gamma_s(\mathsf{h})$ we denote with $p(u)$ the Stone generator such that

$$W(tu) = \mathrm{e}^{-\mathrm{i}tp(u)}, \quad \forall t \in \mathbb{R}. \tag{1.9}$$

These operators satisfy the following properties whose proof can be found in [57, Proposition 20.4]

**Proposition 1.56.** *Let $E$ be the linear span of exponential vectors on $\Gamma(\mathsf{h})$. It holds* *(i)* $E \subset D(p(u_1)\dots p(u_m))$*, for all* $m \in \mathbb{M}$ *and* $u_1, \dots, u_n \in \Gamma(\mathsf{h})$*;*

*(ii)* $E$ *is a core for* $p(u)$ *for every* $u \in \Gamma(\mathsf{h})$*;*

*(iii)* $[p(u), p(v)]\, e(w) = (2\mathrm{i}\,\mathrm{Im}\,\langle u, v\rangle)\, e(w)$*, for every* $u, v, w \in \Gamma(\mathsf{h})$.

Consider now the following definitions for $u \in \mathsf{h}$,

$$q(u) = -p(\mathrm{i}u), \quad a(u) = \frac{q(u) + \mathrm{i}p(u)}{2}, \quad a^\dagger(u) = \frac{q(u) - \mathrm{i}p(u)}{2}. \tag{1.10}$$

**Proposition 1.57.** *Let* $a(u), a^\dagger(u)$ *be defined by* (1.10) *for* $u \in \mathsf{h}$*. For any operator of the form* $T = T_1 \dots T_n$ *where* $T_j$ *is one of the operators* $a(u_j), a^\dagger(u_j)$ *with* $u_j \in \mathsf{h}$*,* $n \in \mathbb{N}$*, it holds* $E \subset D(T)$*. Furthermore it holds:*

*(i)* $a(u)e(v) = \langle u, v\rangle\, e(v)$*,* $a^\dagger(u)e(v) = \frac{d}{dt}e(v + tu)\mid_{t=0}$ *for* $u, v \in \mathsf{h}$*;*

*(ii)* $\left\langle a^\dagger(u)\psi_1, \psi_2\right\rangle = \langle \psi_1, a(u)\psi_2\rangle$*, for every* $\psi_1, \psi_2 \in E$*;*

*(iii)* $[a(u), a(v)]\, \psi = \left[a^\dagger(u), a^\dagger(v)\right] \psi = 0$*, for every* $u, v \in \mathsf{h}$ *and* $\psi \in E$

*(iv)* $\left[a(u), a^\dagger(v)\right] \psi = \langle u, v\rangle\, \psi$*, for every* $u, v \in \mathsf{h}$ *and* $\psi \in E$.

*Notation* 1.58. Let $(e_n)_{n=1}^d$ be an orthonormal basis for $\mathsf{h}$ and define

$$a_j = a(e_j), \quad a_j^\dagger = a^\dagger(e_j), \quad p_j = \frac{1}{\sqrt{2}}p(e_j), \quad q_j = \frac{1}{\sqrt{2}}q(e_j).$$

They satisfy

$$[a_j, a_k] = \left[a_j^\dagger, a_k^\dagger\right] = 0, \quad \left[a_j, a_k^\dagger\right] = \delta_{jk} \tag{1.11}$$

$$[q_j, q_k] = [p_j, p_k] = 0, \quad [q_j, p_k] = \mathrm{i}\delta_{jk}. \tag{1.12}$$

Moreover

$$a_j = \frac{q_j + \mathrm{i}p_j}{\sqrt{2}}, \quad a_j^\dagger = \frac{q_j - \mathrm{i}p_j}{\sqrt{2}}, \quad p_j = \frac{a_j - a_j^\dagger}{\mathrm{i}\sqrt{2}}, \quad q_j = \frac{a_j + a_j^\dagger}{\sqrt{2}}.$$

We also introduce $N = \sum_{j=1}^d a_j^\dagger a_j$.

Whit these notations we can now prove the following result.

**Proposition 1.59.** *For every* $z = x + \mathrm{i}y \in \mathsf{h}$ *with* $x, y \in \mathbb{R}^d$*, we have*

$$W(z) = \exp\left(-\mathrm{i}\sqrt{2}\sum_{j=1}^d (x_j p_j - y_j q_j)\right) = \exp\left(\sum_{j=1}^d \left(z_j a_j^\dagger - \overline{z_j} a_j\right)\right). \tag{1.13}$$

*Moreover, for every* $v, z \in \mathsf{h}$ *and* $\psi \in E$

$$[a(v), W(z)]\, \psi = \langle v, z\rangle\, W(z)\psi, \quad \left[a^\dagger(v), W(z)\right] \psi = \langle z, v\rangle\, W(z)\psi. \tag{1.14}$$

*Proof.* Equation (1.13) follows immediately from the notations 1.58 and (1.9).
In order to prove (1.14) we compute at first

$$[p(v), W(z)] = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} [p(v), p(z)^n] = \sum_{n=1}^{\infty} \frac{(-\mathrm{i})^n 2n\mathrm{i} \operatorname{Im} \langle v, z\rangle}{n!} p(z)^{n-1}$$
$$= 2 \operatorname{Im} \langle v, z\rangle \, W(z),$$

where all the computations are performed on $\psi \in E$ and we used the commutation rules of Proposition 1.56. Recalling now the definition of $a(v), a^\dagger(v)$ in (1.10) we complete the proof. □

**Example 1.60.** Suppose $\mathcal{H} = \mathbb{C}^d$ then there exists a unitary isomorphism $U : \Gamma_s(\mathcal{H}) \to L^2(\mathbb{R}^d)$ with the identification

$$Ue(z) = \frac{1}{\sqrt[4]{(2\pi)^d}} \exp\left(\sum_{j=1}^{d}\left(-\frac{x_j^2}{4} + x_j z_j - \frac{z_j^2}{2}\right)\right),$$

where $z_j$ represents the j-th component of $z$. Indeed the set $\{Ue(z) : z \in \mathsf{h}_\mathbb{R}\}$ is total in $L^2(\mathbb{R}^d)$ and

$$\langle Ue(z_1), Ue(z_2)\rangle_{L^2} = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^{2d}} \exp\left(\sum_{j=1}^{d}\left(-\frac{x_j}{2} + x_j z_{2j} + x_j \overline{z_{1j}} - \frac{z_{1j}^2}{2} - \frac{z_{2j}^2}{2}\right)\right) \mathrm{d}x$$
$$= \exp\left(\langle z_1, z_2\rangle\right) = \langle e(z_1), e(z_2)\rangle .$$

By totality of exponential vectors, given by Proposition 1.45, we can extend $U$ to a unitary isomorphism on the entire $\Gamma_s(\mathcal{H})$. Moreover if $x, y \in \mathbb{R}^d$ and $z \in \mathsf{h}_\mathbb{R}$

$$(UW(y)e(z))(x) = \mathrm{e}^{\sum_{j=1}^d \left(-\frac{y_j^2}{2} - y_j z_j\right)} \left(Ue(y+z)\right)(x)$$
$$= \frac{\mathrm{e}^{\sum_{j=1}^d \left(-\frac{y_j^2}{2} - y_j z_j\right)}}{\sqrt[4]{(2\pi)^d}} \exp\left(\sum_{j=1}^{d}\left(-\frac{x_j^2}{4} + x_j(y_j + z_j) - \frac{(y_j+z_j)^2}{2}\right)\right)$$
$$= \frac{1}{\sqrt[4]{(2\pi)^d}} \exp\left(\sum_{j=1}^{d}\left(-\frac{(x_y - 2y_j)^2}{4} + (x_j - 2y_j)z_j - \frac{z_j^2}{2}\right)\right)$$
$$= (Ue(z))(x - 2y)$$

In particular therefore if $x, y \in \mathbb{R}^d$ and $f = Ue(x)$

$$\left(UW(x)U^{-1}f\right)(y) = f(y - 2x). \tag{1.15}$$

Analogously one can obtain

$$\left(UW(\mathrm{i}x)U^{-1}f\right)(y) = \exp\left(\mathrm{i}\sum_{j=1}^{d} x_j y_j\right) f(y) \tag{1.16}$$

and both equalities can be extended on the entire $L^2(\mathbb{R}^d)$ by totality of the set $\{Ue(z) : z \in \mathsf{h}_{\mathbb{R}}\}$. Therefore we have

$$Up_jU^{-1}f(y) = -\mathrm{i}\sqrt{2}\frac{\partial}{\partial y_j}f(y), \quad Uq_jU^{-1}f(y) = \frac{y_j}{\sqrt{2}}f(y).$$

*Notation* 1.61. Consider $(e_j)_{j=1}^d$ an orthonormal basis of $\mathsf{h}$. We can then construct the *occupancy number* basis for $\Gamma_s(\mathsf{h})$ considering $(e_{(n_1,\dots,n_d)})_{n_1,\dots,n_d \in \mathbb{N}}$ where

$$e_{(n_1,\dots,n_d)} = P_s\left(e_1^{\otimes n_1} \otimes \cdots \otimes e_d^{\otimes n_d}\right). \tag{1.17}$$

We have the correspondence, for every $z \in \mathsf{h}$

$$e(z) \leftrightarrow \sum_{n_1,\dots,n_d=0}^{+\infty} \frac{z_1^{n_1}\dots z_d^{n_d}}{\sqrt{n_1!\dots n_d!}} e_{(n_1,\dots,n_d)}.$$

In particular using Proposition 1.57 and comparing the series expansions we obtain, the action of creation and annihilation operators on the new basis is

$$a_j e_{(n_1,\dots,n_d)} = \sqrt{n_j} e_{(n_1,\dots,n_{j-1},n_j-1,\dots,n_d)}, \tag{1.18}$$

$$a_j^\dagger e_{(n_1,\dots,n_d)} = \sqrt{n_j+1} e_{(n_1,\dots,n_{j-1},n_j+1,\dots,n_d)}, \; a_j^\dagger a_j e_{(n_1,\dots,n_d)} = n_j e_{(n_1,\dots,n_d)}. \tag{1.19}$$

In particular introducing the subspaces

$$D_n := \mathrm{span}\{e_{(n_1,\dots,n_d)} : n_1 + \dots n_d \leq n\}, \quad D = \bigcup_{n\in\mathbb{N}} D_n. \tag{1.20}$$

one has $a_j(D_n) \subset D_{n-1}$ and $a_j^\dagger(D_n) \subset D_{n+1}$, which will be a useful insight for the proofs in Chapter 4.

# CHAPTER 2

# Gaussian States

This chapter is devoted to the introduction of gaussian states. They are essential to understanding the motivation and the properties of gaussian Quantum Markov Semigroups. In the first section however we present some mathematical properties of real linear operators. They arise naturally when considering operators on a symplectic space and when approaching the definition of gaussian states. Moreover they will be essential in the solution of some of the problems presented in the thesis. The second section introduces gaussian states on the CCR algebra $CCR(\mathsf{h}_{\mathbb{R}}, \sigma)$. We study different equivalent condition for a state to be gaussian and then present a partial order relation that can be defined for those states. In the final section we define gaussian states as trace class operators on the Fock space and then recover most of the properties we had in the CCR algebra.

## 2.1 Real Linear Operators

In the following we will often deal with a complex finite dimensional Hilbert space $\mathsf{h}$ with a complex scalar product $\langle \cdot, \cdot \rangle$ antilinear in the first component and the associated real Hilbert space $\mathsf{h}_{\mathbb{R}}$ with scalar product $\mathrm{Re}\langle \cdot, \cdot \rangle$, as in the Example 1.13. On the same set $\mathsf{h}$ we can then have complex linear bounded operators, that will be denoted as usual $\mathcal{B}(\mathsf{h})$ or $\mathcal{B}(\mathsf{h}_{\mathbb{R}})$, and real linear operators, that will be denoted as $\mathcal{B}_{\mathbb{R}}(\mathsf{h})$ or $\mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$. We will denote the adjoint of an operator $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ as $A^{\sharp}$. Some of the results we will show are valid also in this general context with some minor adjustments. Although, since we will use them in the case $\mathsf{h} = \mathbb{C}^d$, we deal just with this finite dimensional case. The notations for the rest of the thesis will be the one of Example 1.13.

**Proposition 2.1.** *Let $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$. There exists two unique operators $A_1, A_2 \in \mathcal{B}(\mathsf{h}_{\mathbb{R}})$*

*such that*

$$Az = A_1 z + A_2 \overline{z}, \quad \forall z \in \mathsf{h}_{\mathbb{R}} \tag{2.1}$$

*that are given by*

$$A_1 z = \frac{A - \mathrm{i}A\mathrm{i}}{2} z, \quad A_2 z = \frac{A + \mathrm{i}A\mathrm{i}}{2} \overline{z}, \quad \forall z \in \mathsf{h}_{\mathbb{R}}. \tag{2.2}$$

*Moreover in this case we have*

$$A^\sharp z = A_1^* z + A_2^T \overline{z}, \tag{2.3}$$

*where* $A_2^T = \overline{A_2^*}$.

*Proof.* We immediately see the decomposition (2.1) holds by simply plugging in the expressions (2.2). To prove the first part it remains to be shown that $A_1, A_2$ are complex linear operators. We do this for $A_2$, consider $\lambda = x + \mathrm{i}y \in \mathbb{C}$ with $x, y \in \mathbb{R}$, we have

$$\begin{aligned} A_2(\lambda z) &= \frac{A + \mathrm{i}A\mathrm{i}}{2} \left( (x - \mathrm{i}y)\, \overline{z} \right) = x \frac{A + \mathrm{i}A\mathrm{i}}{2} \overline{z} - y \frac{A + \mathrm{i}A\mathrm{i}}{2} (\mathrm{i}\overline{z}) \\ &= x \frac{A + \mathrm{i}A\mathrm{i}}{2} \overline{z} + y \frac{\mathrm{i}A - A\mathrm{i}}{2} \overline{z} = x \frac{A + \mathrm{i}A\mathrm{i}}{2} \overline{z} + \mathrm{i}y \frac{A + \mathrm{i}A\mathrm{i}}{2} \overline{z} \\ &= \lambda A_2 \overline{z}. \end{aligned}$$

The proof for $A_1$ follows in a similar way.

We prove now (2.3). Let $z_1, z_2 \in \mathsf{h}_{\mathbb{R}}$ and observe that

$$\begin{aligned} \operatorname{Re} \left\langle A^\sharp z_1, z_2 \right\rangle &= \operatorname{Re} \left\langle z_1, A z_2 \right\rangle = \operatorname{Re} \left\langle z_1, A_1 z_2 \right\rangle + \operatorname{Re} \left\langle z_1, A_2 \overline{z_2} \right\rangle \\ &= \operatorname{Re} \left\langle A_1^* z_1, z_2 \right\rangle + \operatorname{Re} \left\langle A_2^* z_1, \overline{z_2} \right\rangle = \operatorname{Re} \left\langle A_1^* z_1, z_2 \right\rangle + \operatorname{Re} \left\langle \overline{A_2^* \overline{z_1}}, z_2 \right\rangle. \end{aligned}$$

which concludes the proof. □

The next Proposition gives some useful insight on how to interpret real linear operators. Indeed it is not always easy to force oneself to think of $\mathbb{C}^d$ as a real space and detach from all the built up automatisms one may have. For this reason it is useful to think of it through some symplectomorphism with some other space which is naturally $2d$-dimensional.

**Proposition 2.2.** *Consider the real Hilbert spaces* $\mathsf{h}_{\mathbb{R}}, \mathbb{R}^{2d}, \mathfrak{h}$ *given by Examples 1.12, 1.14 that are isomorphic by Example 1.18 via* $T_{\mathbb{R}^{2d}}, T_{\mathfrak{h}}$*, respectively. There exists a one to one correspondence between operators* $A \in \mathcal{B}_{\mathbb{R}} \mathsf{h}_{\mathbb{R}}$ *and operators* $A_{\mathbb{R}^{2d}} \in \mathcal{B}_{\mathbb{R}}(\mathbb{R}^{2d}), A_{\mathfrak{h}} \in \mathcal{B}_{\mathbb{R}}(\mathfrak{h})$ *such that*

$$T_{\mathbb{R}^{2d}} A = A_{\mathbb{R}^{2d}} T_{\mathbb{R}^{2d}}, \quad T_{\mathfrak{h}} A = A_{\mathfrak{h}} T_{\mathfrak{h}}. \tag{2.4}$$

*They are explicitly given by*

$$A_{\mathbb{R}^{2d}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \operatorname{Re}(A_1 + A_2) & \operatorname{Im}(A_2 - A_1) \\ \operatorname{Im}(A_1 + A_2) & \operatorname{Re}(A_1 - A_2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad x, y \in \mathbb{R}^d \tag{2.5}$$

*and*

$$A_{\mathfrak{h}} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ \overline{A_2} & \overline{A_1} \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}, \quad z \in \mathsf{h}. \tag{2.6}$$

*Moreover it holds*

$$T_{\mathbb{R}^{2d}} A^\sharp = A^T_{\mathbb{R}^{2d}} T_{\mathbb{R}^{2d}}, \quad T_{\mathfrak{h}} A^\sharp = A^*_{\mathfrak{h}} T_{\mathfrak{h}},$$

*where $A^T_{\mathbb{R}^{2d}}, A^*_{\mathfrak{h}}$ are adjoints with respect to the real scalar product of $\mathbb{R}^{2d}$ and the usual complex scalar product of $\mathbb{C}^{2d}$.*

*Proof.* It is easy to see through explicit calculations that (2.6), (2.5) satisfy (2.4). This relation is one-to-one and onto since $T_{\mathbb{R}^{2d}}, T_{\mathfrak{h}}$ are invertible. In the case $\mathbb{R}^{2d}$ we can find an explicit inversion formula. Indeed let $S \in \mathcal{B}_{\mathbb{R}}(\mathbb{R}^{2d})$ given by

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

and define

$$Az = \left( \frac{S_{11} + S_{22}}{2} + \mathrm{i}\frac{S_{21} - S_{12}}{2} \right) z + \left( \frac{S_{11} - S_{22}}{2} + \mathrm{i}\frac{S_{12} + S_{21}}{2} \right) \overline{z},$$

it holds $A_{\mathbb{R}^{2d}} = S$. □

**Example 2.3.** A notable example of a real linear operator is $Jz = -\mathrm{i}z$, for $z \in \mathsf{h}_{\mathbb{R}}$. In particular we have $J_1 = -\mathrm{i}\mathbb{1}$, $J_2 = 0$, moreover

$$J_{\mathbb{R}^{2d}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad J_{\mathfrak{h}} = \begin{pmatrix} -\mathrm{i}\mathbb{1} & 0 \\ 0 & \mathrm{i}\mathbb{1} \end{pmatrix}.$$

We have $J^\sharp = -J = J^{-1} = J^*$. We can use $J$ to rewrite the symplectic forms of Examples 1.13, 1.12, 1.14 as

$$\sigma(u, v) = \operatorname{Re} \langle u, Jv \rangle, \quad \sigma_{\mathbb{R}}(x, y) = \langle x, J_{\mathbb{R}^{2d}} y \rangle_{\mathbb{R}^{2d}}, \quad \sigma_{\mathfrak{h}}(h_1, h_2) = \frac{1}{2} \operatorname{Re} \langle h_1, J_{\mathfrak{h}} h_2 \rangle,$$

for all $u, v \in V$, $x, y \in \mathbb{R}^{2d}$ and $h_1, h_2 \in \mathfrak{h}$.

*Remark* 2.4. It is often useful to consider spectra of real linear operators, which can of course have also complex eigenvalues. For this reason it is often useful to consider complexification of such operators, as one does when working in $\mathbb{R}^n$. As noted in Example 1.24, it is not easy to work with the complexification of $\mathsf{h}_{\mathbb{R}}$ and, by transitivity, even with the complexification $A_{\mathbb{C}}$ of an operator $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$. It is much simpler to first identify $A$ with $A_{\mathfrak{h}}$ and then consider the complexification $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{h}$ and obtain the complexified operator $A_{\mathfrak{h}_{\mathbb{C}}}$ which will be denoted $\mathbf{A}$ in boldface character. By Proposition 1.21, $\mathfrak{h}_{\mathbb{C}}$ and the complexification of $\mathsf{h}_{\mathbb{R}}$ are still isomorphic and

$$T_{\mathfrak{h}_{\mathbb{C}}} A_{\mathbb{C}} = \mathbf{A} T_{\mathfrak{h}_{\mathbb{C}}}.$$

Moreover $\mathbf{A}$ has the same matrix expression than $A_{\mathfrak{h}}$.

Of course it would be possible to use the complexification of $A_{\mathbb{R}^{2d}}$ instead, and in fact it has been done (see [16, 58]). We prefer to use this approach since, from an application point of view, it is easier to calculate $A_{\mathfrak{h}}$ starting from the expression for $A$.

*Notation* 2.5. For $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ we denote with $\mathbf{A}$, in boldface character, the complexification of $A_{\mathfrak{h}}$. Moreover in the following whenever we speak of the complexification of an operator $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ we are always referring to $\mathbf{A}$ implying this identification.

The following result characterizes real linear operators that are Bogoliubov transformations (cf. Definition 1.10.

**Proposition 2.6.** *Let $T \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ be an invertible operator, the following are equivalent:*

*(i) $T$ is a Bogoliubov transformation;*

*(ii) $T^\sharp JT = J$;*

*(iii) $T^T_{\mathbb{R}^{2d}} J_{\mathbb{R}^{2d}} T_{\mathbb{R}^{2d}} = J_{\mathbb{R}^{2d}}$;*

*(iv) $T^*_{\mathfrak{h}} J_{\mathfrak{h}} T_{\mathfrak{h}} = J_{\mathfrak{h}}$.*

*In particular in that case $T_{\mathbb{R}^{2d}}$ and $T_{\mathfrak{h}}$ are Bogoliubov transformations on $\mathbb{R}^{2d}$ and $\mathfrak{h}$ respectively.*

*Proof.* Using the expression for $\sigma$ in Example 2.3 we obtain

$$\sigma(u,v) = \sigma(Tu,Tv) \iff \mathrm{Re}\left\langle u, \left(J - T^\sharp JT\right) v\right\rangle = 0,$$

for every $u, v \in \mathsf{h}_{\mathbb{R}}$. Therefore we have the equivalence between (i) and (ii). The equivalence among (ii), (iii) and (iv) follows from Proposition 2.2. □

*Remark* 2.7. If $T \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ is a Bogoliubov transformation then $T^\sharp$ and $T^{-1}$ are as well. Indeed by Proposition 2.6 we have $T^\sharp JT = J$ and clearly $TT^{-1} = \mathbb{1}$, therefore

$$J = (QQ^{-1})^\sharp JQQ^{-1} = (Q^{-1})^\sharp JQ^{-1}.$$

Again by Proposition 2.6 we have $Q^{-1}$ is a Bogoliubov transformation. Moreover the product of Bogoliubov transformations is still a Bogoliubov transformation, since if $S$ is another Bogoliubov transformation

$$(TS)^\sharp J(TS) = S^\sharp T^\sharp JTS = S^\sharp JS = J.$$

Therefore $T^\sharp = JT^{-1}J^{-1}$ is a Bogoliubov transformation.

We consider a useful normal form for Bogoliubov transformations.

**Proposition 2.8.** *Every symplectic automorphism $T$ of $\mathsf{h}_{\mathbb{R}}$ admits a factorization $T = U_1 B U_2$, where $U_1, U_2$ are orthogonal operators on $\mathsf{h}_{\mathbb{R}}$ and $B$ is a symplectic automorphism such that*

$$B_{\mathbb{R}^{2d}} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix},$$

*where $A$ is a positive operator on $\mathbb{R}^d$.*

*Proof.* In this proof we will always consider real linear operators as acting on $\mathbb{R}^{2d}$ through the identification of Proposition 2.2, therefore we will drop the subscript to avoid cluttering notation. Consider the polar decomposition $S = UH$ with $U$ an orthogonal operator and $H$ positive. The equivalent conditions of Proposition 2.6 for the Bogoliubov transformation $S$ yield

$$JUH = UH^{-1}J = UJJ^{-1}H^{-1}J$$

where $JU$ and $UJ$ are orthogonal operators and $H$, $J^{-1}H^{-1}J$ are positive operators. By the uniqueness of the polar decomposition we obtain

$$UJ = JU, \quad H = J^{-1}H^{-1}J. \tag{2.7}$$

Denote now with $R_0, R_1, R_2$ the eigenspaces of $H$ associated with eigenvalues smaller than 1, equal to 1 and greater than 1, respectively. The second equation of 2.7 implies that $J$ maps $R_0$ onto $R_2$ and vice versa, while $R_1$ is mapped onto itself, moreover $H$ is symplectic. Therefore $R_1$ is a symplectic space. Since $\langle x, Jx\rangle_{\mathbb{R}^{2d}} = 0$ for all $x \in \mathbb{R}^{2d}$, similar to the proof of Proposition 1.9 we can find a basis of $R_1$ given by

$$\{x_j, Jx_j : j = 1, \dots, m\}$$

where $2m$ is the dimension of $R_1$. We can then define $R_{11}$, $R_{12}$ as the subspaces spanned by $\{x_j : j = 1, \dots, m\}$ and $\{Jx_j : j = 1, \dots, m\}$, respectively, and introduce $M_1 = R_0 \oplus R_{11}$, $M_2 = R_{12} \oplus R_2$. Therefore $\mathbb{R}^{2d} = M_1 \oplus M_2$ and $J$ maps $M_1$ onto $M_2$ and vice versa. Moreover $H$ leaves both $M_1$ and $M_2$ invariant. Fix a basis $\{y_1, \dots, y_d\}$ of $M_1$ and the canonical orthonormal basis $(e_j)_{j=1}^{2d}$ of $\mathbb{R}^{2d}$. Define $V$ the orthogonal operator on $\mathbb{R}^{2d}$ that acts as

$$Vy_j = -e_j, \quad VJy_j = e_{j+d}, \quad j = 1, \dots, d.$$

We have that $V$ commutes with $J$ and by construction

$$VHV^{-1} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} =: B$$

for some positive operator $A$. Therefore by setting $U_1 = UV^{-1}, U_2 = V$ we have

$$S = UH = UV^{-1}VHV^{-1}V = U_1BU_2.$$

□

The last result we need for real linear operators is a diagonalization result called Williamson's normal form. This results does not hold for general real linear operators but just for positive ones. Recall that we say $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ is positive if $\operatorname{Re}\langle v, Av\rangle \geq 0$ for every $v \in \mathsf{h}_{\mathbb{R}}$ and is strictly positive if $A$ is also invertible. We start by stating the classical result, due to Williamson [74] on $\mathbb{R}^{2d}$ which requires a preliminary Lemma.

**Lemma 2.9.** *Let $A \in \mathcal{B}_{\mathbb{R}}(\mathbb{R}^{2d})$ an invertible operator such that $A^T = -A$. Then there exist unique $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $U \in \mathcal{B}_{\mathbb{R}}(\mathbb{R}^{2d})$ an orthogonal operator such that*

$$A = U^T \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} U, \tag{2.8}$$

*where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$ is the diagonal matrix whose diagonal entries are given in order by $\lambda_1, \dots, \lambda_d$.*

*Proof.* Since $A^T = -A$ we have that $B := A^2$ satisfies $B^T = B$. Let $\mu_1, \dots, \mu_{2d} \in \mathbb{R}$ and $f_1, \dots, f_{2d}$ corresponding eigenvectors. Notice that all the eigenvalues must be strictly negative, since $A$ is invertible and

$$\mu_k = \langle f_k, Bf_k\rangle_{\mathbb{R}^{2d}} = -\langle Af_k, Af_k\rangle_{\mathbb{R}^{2d}} < 0, \quad \forall k = 1, \dots, 2d.$$

Now notice that $f_k, Af_k$ are linearly independent eigenvectors associated with the same eigenvalue $\mu_k$, indeed

$$BAf_k = A^3 f_k = \mu_k A f_k, \quad \langle f_k, Af_k\rangle_{\mathbb{R}^{2d}} = -\langle f_k, Af_k\rangle, \quad \forall k = 1,\dots,2d.$$

In particular every eigenspace has even dimension and we can find a basis (by induction on the dimension of an eigenspace, similarly to the proof of Proposition 1.9) which is composed of pairs $(g_k, Ag_k)$ where $(g_k)$ is a sequence of mutually orthogonal, normalized eigenvectors. Therefore we can find an orthonormal basis for $\mathbb{R}^{2d}$ defined by

$$u_k = g_k, \quad u_{d+k} = \frac{\lambda_k}{A} g_k, \quad \lambda_k = \sqrt{-\mu_k}, \quad k = 1,\dots,d.$$

Let $U$ be the matrix whose rows are the vectors $u_k$, by explicit computations of the quantities $\langle u_j, Au_k\rangle_{\mathbb{R}^{2d}}$ one obtains (2.8).

We now prove uniqueness. Suppose there exists $\lambda_1' \geq \cdots \geq \lambda_d' > 0$ and $U'$ an orthogonal operator such that

$$A = U'^T \begin{pmatrix} 0 & D' \\ -D' & 0 \end{pmatrix} U',$$

where $D' = \operatorname{diag}(\lambda_1', \dots, \lambda_d')$. If $V = UU'^T$ we have $V$ is still orthogonal and

$$V \begin{pmatrix} 0 & D' \\ -D' & 0 \end{pmatrix} V^T = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

Let $(e_k)_{k=1}^{2d}$ the canonical orthonormal basis of $\mathbb{R}^{2d}$, we have

$$\begin{aligned}\lambda_k V^T e_{k+d} = V^T \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} e_k &= \sum_{j=1}^{2d} \begin{pmatrix} 0 & D' \\ -D' & 0 \end{pmatrix} \langle e_j, Ve_k\rangle e_j \\ &= \sum_{j=1}^{d} \left( \langle e_j, Ve_k\rangle \lambda_j' e_{j+d} - \langle e_{j+d}, Ve_k\rangle \lambda_j' e_j \right)\end{aligned}$$

In particular this implies $\lambda_k = \lambda_k'$ for every $k = 1,\dots,d$. ☐

We can now state the result for Williamson's normal form in $\mathbb{R}^{2d}$.

**Proposition 2.10** (Williamson's normal form)**.** *Let $A \in \mathcal{B}_{\mathbb{R}}(\mathbb{R}^{2d})$ be a strictly positive operator. Then there exist unique $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $T \in \mathcal{B}_{\mathbb{R}}(\mathbb{R}^{2d})$ a Bogoliubov transformation such that*

$$A = T^T \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} T, \tag{2.9}$$

*where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$, the diagonal matrix with entries given by $\lambda_1, \dots, \lambda_d$.*

*Proof.* Let $A^{1/2}$ stand for the unique positive square root of $A$ and define the real linear operator

$$B = A^{1/2} J_{\mathbb{R}^{2d}} A^{1/2},$$

where $J_{\mathbb{R}^{2d}}$ is the operator given in Example 2.3. We have $B^T = -B$ and $B$ is invertible, therefore by Lemma 2.9 there exist unique $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $U$ an orthogonal matrix such that

$$U^T B U = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$ . Consider then the matrix

$$L = \sqrt{A} U \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix}.$$

One has

$$\begin{aligned} L^T J_{\mathbb{R}^{2d}} L &= \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix} U^T \sqrt{A} J_{\mathbb{R}^{2d}} \sqrt{A} U \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix} \\ &= J_{\mathbb{R}^{2d}} \end{aligned}$$

and

$$\begin{aligned} L \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} L^T &= A^{1/2} U \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} D^{-1/2} & 0 \\ 0 & D^{-1/2} \end{pmatrix} U^T A^{1/2} \\ &= A. \end{aligned}$$

We can then set $T = L^T$, which is still a Bogoliubov transformation as pointed out in Remark 2.7, and complete the proof. □

Exploiting the symplectomorphisms given in Example 1.18 we can obtain the corresponding result to Williamson's normal form also for real linear operators on $\mathsf{h}_{\mathbb{R}}$ and $\mathfrak{h}$.

**Corollary 2.11.** *Let $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ be a strictly positive operator. Then there exist $\lambda_1 \geq \cdots \geq \lambda_d > 0$ and $T \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ a Bogoliubov transformation such that*

$$A = T^{\sharp} D T, \tag{2.10}$$

*where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_d) \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$, the diagonal matrix with entries given by $\lambda_1, \ldots, \lambda_d$. Similarly*

$$A_{\mathfrak{h}} = T_{\mathfrak{h}}^* \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} T_{\mathfrak{h}},$$

*and $T_{\mathfrak{h}}$ is a Bogoliubov transformation.*

*Proof.* The proof follows from Proposition 2.10 noticing that

$$D_{\mathbb{R}^{2d}} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}.$$

Eventually consider that by Proposition 2.6 if $T_{\mathbb{R}^{2d}}$ is a Bogoliubov transformation, $T$ is one as well. A similar proof can be used to prove the Corollary also on $\mathfrak{h}$. □

## 2.2 Gaussian States on the CCR algebra

In this section we define gaussian states on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ as a starting point to then move towards their definition on $\mathcal{B}(\Gamma_s(\mathsf{h}))$. When it doesn't require additional work we provide definitions and results on the general case of the CCR algebra on a generic symplectic space.

We first recall some basic definitions for states on a C* algebra.

**Definition 2.12.** Let $\mathcal{A}$ a C*-algebra and $\mathcal{S} \subset \mathcal{A}$. We set $\mathcal{S}^* = \{a^* : a \in \mathcal{S}\}$, we say $\mathcal{S}$ is *self-adjoint* when $\mathcal{S} = \mathcal{S}^*$. If $\mathcal{A}$ has a unit $\mathbb{1}$ and $\mathcal{S}$ is a self-adjoint subspace of $\mathcal{A}$ containing $\mathbb{1}$ we say $\mathcal{S}$ is an *operator system*

**Definition 2.13.** Let $\mathcal{A}, \mathcal{B}$ be unital C*-algebras and let $\mathcal{S} \subset \mathcal{A}$ be an operator system. A function linear function $\phi : \mathcal{S} \to \mathcal{B}$ is *positive* if $\phi(a) \geq 0$ whenever $a \geq 0$. We say a functional $\omega$ on $\mathcal{A}$ is a *state* if $\omega(\mathbb{1}) = 1$ and $\omega$ is positive, i.e. $\omega(x^*x) \geq 0$ for every $x \in \mathfrak{A}$.

We recall the following properties.

**Proposition 2.14.** *Let $\mathcal{A}$ a unital C*-algebra, $\mathcal{S} \subset \mathcal{A}$ an operator system and $\phi : \mathcal{S} \to \mathbb{C}$ a positive map then*

1. $\phi(a^*) = \overline{\phi(a)}$*, for every* $a \in \mathcal{S}$*;*
2. $\phi$ *is bounded and* $\|\phi\| = \phi(\mathbb{1})$*;*
3. $\phi$ *can be uniquely estended to a positive map* $\overline{\phi}$ *on the closure* $\overline{\mathcal{S}}$ *of* $\mathcal{S}$*.*

We are now ready to give the definition of a gaussian state. They have been introduced in many articles and in different equivalent ways (see [16, 45, 46, 55, 58].

**Definition 2.15.** Let $(\mathsf{h}_{\mathbb{R}}, \sigma)$ the symplectic space introduced in Example 1.13. We say $\omega$ a state on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ is *gaussian* or *quasi-free* if there exists $\mu \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ a positive, invertible operator such that

$$\omega(\delta_u) = \exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \mu, u\rangle - \frac{1}{2}\,\mathrm{Re}\,\langle u, Su\rangle\right), \quad u \in V. \tag{2.11}$$

We say $\mu$ and $S$ are the *mean vector* and the *covariance operator*, respectively, and we denote the state $\omega$ as $\omega_{(\mu,S)}$.

We denote with $\mathcal{Q}$ the set of covariance operators of gaussian states.

This definition assumes a priori that $\omega$ is a state. However we will show that there is an explicit condition on the operator $S$ such that a functional that satisfies (2.11) defines a state on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$. In particular this will show that we can characterize operators in $\mathcal{Q}$ and that it will be a proper subset of $\mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$. A fundamental tool in proving this result is Proposition 2.18, but first let us recall the following definition.

**Definition 2.16.** Let $\mathcal{X}$ be any set and $K$ a complex-valued function on $\mathcal{X} \times \mathcal{X}$ is called a *positive definite kernel* on $\mathcal{X}$ if for any $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $x_1, \ldots, x_m \in \mathcal{X}$ it holds

$$\sum_{j,k=1}^{m} \overline{c_j} c_k K(x_j, x_k) \geq 0.$$

*Remark* 2.17. Note that the kernel $K$ is positive definite if and only if the matrix $(K(x_j, x_k))_{j,k=1}^m$ is positive definite for every $m \in \mathbb{N}, c_1, \dots, c_m \in \mathbb{C}$ and $x_1, \dots, x_m \in \mathcal{X}$.

**Proposition 2.18.** *Let $(V, \sigma)$ a real symplectic space and $G : V \to \mathbb{C}$ a function. Then there exists a positive functional $\omega$ on $CCR\,(V, \sigma)$ such that*

$$\omega(\delta_v) = G(v), \quad \forall v \in V, \tag{2.12}$$

*if and only if the kernel*

$$V \times V \ni (u, v) \mapsto G(u - v)\mathrm{e}^{\mathrm{i}\sigma(u,v)} \tag{2.13}$$

*is positive definite. Moreover, in that case, $\omega$ is a state if and only if $G(\mathbb{1}) = 1$.*

*Proof.* Let us start by proving the characterization of positivity for $\omega$. Suppose $\omega$ is positive and let $n \in \mathbb{N}, \lambda_1, \dots, \lambda_n \in \mathbb{C}, v_1, \dots, v_n \in V$. Setting $x = \sum_{k=1}^n \lambda_k \delta_{v_k}$ we have

$$xx^* = \sum_{j,k=1}^n \lambda_j \overline{\lambda_k} \mathrm{e}^{\mathrm{i}\sigma(v_j, v_k)} \delta_{v_j - v_k} \geq 0.$$

Exploiting positivity of $\omega$ we obtain

$$\sum_{j,k=1}^n \lambda_j \overline{\lambda_k} \mathrm{e}^{\mathrm{i}\sigma(v_j, v_k)} G(v_j - v_k),$$

which is precisely positive definiteness of the kernel (2.13). On the other hand, by the same argument, the positive definiteness of the kernel allows us to define a positive functional $\omega$ on the operator system of finite linear combinations of $\{\delta_v : v \in V\}$ that satisfies (2.12). By Proposition 2.14 we can uniquely extend $\omega$ to a positive functional (still denoted $\omega$) that acts on the closure of this set which is the whole $CCR\,(V, \sigma)$.

Eventually the condition $G(\mathbb{1}) = 1$ follows since by definition of a state

$$1 = \omega(\mathbb{1}) = G(\mathbb{1}).$$

□

As anticipated, the previous Proposition will be the fundamental tool to characterize gaussian states based solely on their parameters. Before proving the actual Theorem however we need some preliminary Lemmas. The first one comes from [16, 61]

**Lemma 2.19.** *Let $(V, \sigma)$ a real symplectic space. If $\alpha(\cdot, \cdot)$ is a positive symmetric bilinear form on $V$ then the following conditions are equivalent:*

*(i) The kernel $(u, v) \mapsto \alpha(u, v) - \mathrm{i}\sigma(u, v)$ is positive definite;*

*(ii) $\sigma(u, v)^2 \leq \alpha(u, u)\alpha(v, v)$, for every $u, v \in V$.*

The second one states a well-known property of the so called *Hadamard* product.

**Lemma 2.20.** *Let $A = (a_{jk})_{jk=1}^d, B = (b_{jk})_{jk=1}^d$ be real positive semidefinite matrices. Then $(a_{jk}b_{jk})_{jk=1}^d$, the entry wise multiplication of $A$ and $B$, and $(\mathrm{e}^{a_{jk}})_{jk=1}^d$, the entry wise exponentiation of $A$, are still positive semidefinite matrices.*

Eventually we have the following result.

**Lemma 2.21.** *Let $A \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ is a self-adjoint operator and consider the kernel*

$$K : \mathfrak{h} \times \mathfrak{h} \ni (u, v) \mapsto \langle u, \mathbf{A} v \rangle .$$

*$K$ is positive definite if and only if $\mathbf{A}$ is positive as operator on $\mathbb{C}^{2d}$.*

*Proof.* Suppose at first that $\mathbf{A}$ is positive definite. For every $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n \in \mathfrak{h}$ we have

$$\sum_{j,k=1}^{n} \overline{c_j} c_k \langle u_j, \mathbf{A} u_k \rangle = \left\langle \sum_{j=1}^{n} c_j u_J, \mathbf{A} \left( \sum_{k=1}^{n} c_k u_k \right) \right\rangle \geq 0.$$

On the other hand if the kernel $K$ is positive definite we have for all $u \in \mathfrak{h}$

$$\operatorname{Re} \langle u, Au \rangle = \frac{1}{2} \langle u, \mathbf{A} u \rangle \geq 0,$$

where we implied the identification of $\mathfrak{h}$ with $\mathsf{h}_{\mathbb{R}}$. Similarly, for $u, v \in \mathfrak{h}$

$$\begin{aligned} \langle u + \mathrm{i} v, \mathbf{A} (u + \mathrm{i} v) \rangle &= \langle u, \mathbf{A} u \rangle + \langle v, \mathbf{A} v \rangle + \mathrm{i} \langle u, \mathbf{A} v \rangle - \mathrm{i} \langle v, \mathbf{A} u \rangle \\ &= 2 \operatorname{Re} \langle u, Au \rangle + 2 \operatorname{Re} \langle v, Av \rangle + 2\mathrm{i} \left( \operatorname{Re} \langle u, Av \rangle - \operatorname{Re} \langle v, Au \rangle \right) \\ &= 2 \operatorname{Re} \langle u, Au \rangle + 2 \operatorname{Re} \langle v, Av \rangle \geq 0 \end{aligned}$$

Therefore $\mathbf{A}$ is positive. □

We now start proving the characterization theorem for gaussian states. The proof will be split into different results coming from from [16, 37, 58, 61]. In particular it justifies different conditions that have been used to introduce gaussian states and proves they are all equivalent. We start with the following proposition that gives two equivalent conditions for $S \in \mathcal{Q}$. It also shows that we can assess whether a a functional defined by (2.11) is a gaussian states looking solely at $S$ and checking if it belongs to $\mathcal{Q}$. In particular the mean vector doesn't pose any problem in the definition of a gaussian state.

**Proposition 2.22.** *Let $\omega$ be a linear functional on $CCR(\mathsf{h}_{\mathbb{R}}, \sigma)$ such that*

$$\omega(\delta_u) = \exp\left( -\mathrm{i} \operatorname{Re} \langle \mu, u \rangle - \frac{1}{2} \operatorname{Re} \langle u, Su \rangle \right), \quad u \in \mathsf{h}_{\mathbb{R}}, \tag{2.14}$$

*where $\mu \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ is an invertible self-adjoint operator. Then $\omega$ is a (gaussian) state if and only if one of the following equivalent conditions holds:*

*(i)* $S \in \mathcal{Q}$;

*(ii)* $\mathbf{S} - \mathrm{i}\mathbf{J} \geq 0$ *on* $\mathbb{C}^{2d}$;

*(iii)* $\mathbf{S} + \mathrm{i}\mathbf{J} \geq 0$ *on* $\mathbb{C}^{2d}$;

*Proof.* We start the proof by showing equivalence between $\omega$ being a state and (i). If $\omega$ is a state then $S$ is positive, since otherwise there would be $z \in \mathsf{h}_{\mathbb{R}}$ such that $|z| = 1$ and $\operatorname{Re} \langle z, Sz \rangle < 0$. Hence $|\omega(\delta_z)| > 1$, contradicting (2) of Proposition 2.14. Therefore $\omega$

is a gaussian state and by definition $S \in \mathcal{Q}$. On the other hand, if $S \in \mathcal{Q}$ there exists $\mu_0 \in \mathsf{h}_{\mathbb{R}}$ and a gaussian state $\omega'$ such that

$$\omega'(\delta_u) = G(u) = \exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \mu_0, u\rangle - \frac{1}{2}\,\mathrm{Re}\,\langle u, Su\rangle\right).$$

Now observe that $\omega(\delta_u) = G(u)\exp\left(\mathrm{i}\,\mathrm{Re}\,\langle \mu_0 - \mu, u\rangle\right)$ and $\omega(\mathbb{1}) = 1$. By proposition 2.18 $\omega$ is positive if and only if the kernel

$$(u, v) \mapsto G(u - v)\exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \mu_0 - \mu, u - v\rangle\right)\mathrm{e}^{\mathrm{i}\sigma(u,v)},$$

is positive definite. But $G(u-v)\mathrm{e}^{\mathrm{i}\sigma(u,v)}$ is positive definite since $\omega'$ is a state and the kernel $\exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \mu_0 - \mu, u - v\rangle\right)$ is trivially positive definite. Therefore their product is still positive definite, by Remark 2.17 and Lemma 2.20.

This shows the equivalence between $\omega$ being a state and $S \in \mathcal{Q}$. In particular then we can always suppose that $\mu = 0$ for the rest of the proof since it doesn't play any relevant role.

Suppose then $\mu = 0$, we prove (ii), (iii) are equivalent with $S \in \mathcal{Q}$. Using Proposition 2.18, $S \in \mathcal{Q}$ if and only if the kernel

$$\begin{aligned} K(u,v) &= \exp\left(-\frac{1}{2}\,\mathrm{Re}\,\langle v - u, S(v-u)\rangle + \mathrm{i}\sigma(u,v)\right) \\ &= \exp\left(-\frac{1}{2}\,\mathrm{Re}\,\langle v - u, S(v-u)\rangle + \mathrm{i}\,\mathrm{Re}\,\langle u, Jv\rangle\right). \end{aligned}$$

Now denote the kernel at the exponent of $K$ by

$$N(u,v) - \frac{1}{2}\,\mathrm{Re}\,\langle v - u, S(v-u)\rangle + \mathrm{i}\,\mathrm{Re}\,\langle u, Jv\rangle\,.$$

Positive definiteness of $K$ is equivalent to that of

$$M_t(u,v) := K(\sqrt{t}u, \sqrt{t}v) = \mathrm{e}^{tN}$$

for every $t > 0$. By a known result (see [60, Lemma 1.7]) this is equivalent to conditional positivity of $N$ or positive definiteness of

$$\begin{aligned} N(u,v) - N(u,0) - N(0,v) + N(0,0) &= \mathrm{Re}\,\langle v, Su\rangle - \mathrm{i}\,\mathrm{Re}\,\langle v, Ju\rangle && (2.15) \\ &= \mathrm{Re}\,\langle u, Sv\rangle + \mathrm{i}\,\mathrm{Re}\,\langle u, Jv\rangle\,. && (2.16) \end{aligned}$$

By identification of $\mathsf{h}_{\mathbb{R}}$ into $\mathbb{C}^{2d}$ through complexification we have equation (2.15), (2.16) become

$$\frac{1}{2}\left\langle \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, (\mathbf{S} \pm \mathrm{i}\mathbf{J})\right\rangle \begin{pmatrix} v \\ \overline{v} \end{pmatrix}.$$

Using Lemma 2.21 this is equivalent to $\mathbf{S} \pm \mathrm{i}\mathbf{J} \geq 0$ on $\mathbb{C}^{2d}$. □

We present now a result with some sufficient conditions for $S \in \mathcal{Q}$.

**Proposition 2.23.** *Let $\omega$ be a linear functional on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ that satisfies* (2.14) *for some $\mu \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ invertible and self-adjoint. The following are equivalent sufficient conditions for $\omega$ to be a (gaussian) state:*

*(i)* $J^*SJ \geq S^{-1}$*;*

*(ii) the map* $(u,v) \mapsto \mathrm{Re}\,\langle u, Sv\rangle - \mathrm{i}\,\mathrm{Re}\,\langle u, Jv\rangle$ *is positive definite on* $\mathsf{h}_{\mathbb{R}}$*.*

*(iii)* $\sigma(u,v)^2 \leq \mathrm{Re}\,\langle u, Su\rangle\,\mathrm{Re}\,\langle v, Sv\rangle$*, for every* $u, v \in \mathsf{h}_{\mathbb{R}}$*.*

*Proof.* We assume as in the proof of Theorem 2.24 that $\mu = 0$. Consider $G(u) = \exp\left(-\frac{1}{2}\,\mathrm{Re}\,\langle u, Su\rangle\right)$. By Proposition 2.18 we have $\omega$ defined via (2.14) is a state if and only if the kernel (2.13) is positive definite. This means that for every $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ and $u_1, \ldots, u_n \in \mathsf{h}_{\mathbb{R}}$

$$\begin{aligned} 0 &\leq \sum_{j,k=1}^{n} \overline{c_j} c_k \exp\left(-\frac{1}{2}\,\mathrm{Re}\,\langle u_j - u_k, S\,(u_j - u_k)\rangle + \mathrm{i}\sigma(u_j, u_k)\right) \\ &= \sum_{j,k=1}^{n} \left(\overline{c_j}\mathrm{e}^{-\frac{1}{2}\mathrm{Re}\langle u_j, Su_j\rangle}\right)\left(c_k\mathrm{e}^{-\frac{1}{2}\mathrm{Re}\langle u_k, Su_k\rangle}\right)\exp\left(\mathrm{Re}\,\langle u_k, Su_j\rangle - \mathrm{i}\sigma(u_k, u_j)\right) \\ &= \sum_{j,k=1}^{n} \overline{c_j'} c_k' \exp\left(\mathrm{Re}\,\langle u_k, Su_j\rangle - \mathrm{i}\sigma(u_k, u_j)\right). \end{aligned}$$

This is true, by Lemma 2.20, if (ii) holds. Now Lemma 2.19 shows equivalence between (ii) and (iii).

In order to show equivalence with (i) suppose at first that (iii) holds. By choosing $u = JSv$ one obtains

$$\mathrm{Re}\,\langle v, Sv\rangle^2 = \mathrm{Re}\,\langle JSv, Jv\rangle^2 \geq \mathrm{Re}\,\langle JSv, SJSv\rangle\,\mathrm{Re}\,\langle v, Sv\rangle\,,$$

which implies

$$\mathrm{Re}\,\langle v, Sv\rangle \leq \mathrm{Re}\,\langle Sv, J^*SJSv\rangle\,, \quad \forall v \in \mathsf{h}_{\mathbb{R}}.$$

Letting $v = S^{-1}w$ we obtain

$$\mathrm{Re}\left\langle w, \left(J^*SJ - S^{-1}\right)w\right\rangle \geq 0, \quad \forall w \in \mathsf{h}_{\mathbb{R}},$$

which is (i). Conversely for every $u, v \in \mathsf{h}_{\mathbb{R}}$ there exists $w \in \mathsf{h}_{\mathbb{R}}$ such that $w = S^{-1}Jv$, in that case

$$\begin{aligned} \mathrm{Re}\,\langle u, Jv\rangle^2 &= \mathrm{Re}\,\langle u, Sw\rangle^2 = \mathrm{Re}\left\langle \sqrt{S}u, \sqrt{S}w\right\rangle^2 \\ &\leq \mathrm{Re}\,\langle u, Su\rangle\,\mathrm{Re}\,\langle w, Sw\rangle = \mathrm{Re}\,\langle u, Su\rangle\,\mathrm{Re}\left\langle S^{-1}Jv, Jv\right\rangle \\ &\leq \mathrm{Re}\,\langle u, Su\rangle\,\mathrm{Re}\,\langle J^*SJJv, Jv\rangle = \mathrm{Re}\,\langle u, Su\rangle\,\mathrm{Re}\,\langle v, Sv\rangle\,, \end{aligned}$$

where we used Cauchy-Schwartz inequality and (i). □

Eventually we state the result for the characterization of a gaussian state based on its parameters. This joins partial results contained in Propositions 2.22 and 2.23.

**Theorem 2.24.** *Let* $\omega$ *be a linear functional on* $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ *that satisfies* (2.14) *for some* $\mu \in \mathsf{h}_{\mathbb{R}}$ *and* $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ *invertible and self adjoint. Then* $\omega$ *is a gaussian state if and only if one of the following equivalent conditions hold:*

*(i)* $S \in \mathcal{Q}$*;*

*(ii)* $\mathbf{S} - \mathrm{i}\mathbf{J} \geq 0$ *on* $\mathbb{C}^{2d}$*;*

*(iii)* $\mathbf{S} + \mathrm{i}\mathbf{J} \geq 0$ *on* $\mathbb{C}^{2d}$*;*

*(iv)* $J^*SJ \geq S^{-1}$*;*

*(v) the map* $(u, v) \mapsto \mathrm{Re}\,\langle u, Sv\rangle - \mathrm{i}\,\mathrm{Re}\,\langle u, Jv\rangle$ *is positive definite on* $\mathsf{h}_\mathbb{R}$.

*(vi)* $\sigma(u, v)^2 \leq \mathrm{Re}\,\langle u, Su\rangle\, \mathrm{Re}\,\langle v, Sv\rangle$*, for every* $u, v \in \mathsf{h}_\mathbb{R}$.

*Proof.* Proposition 2.22 shows already the statement for (i), (ii), (iii), while Proposition 2.23 shows sufficiency and equivalence of (iv), (v), (vi). The only missing step is to prove that one of the latter conditions is also necessary. Suppose then $\omega = \omega_{(\mu,S)}$ is a gaussian state and consider $(\pi_\omega, \mathcal{H}_\omega, \Phi)$ the GNS triple associated with it.

As a first step we show that for every $n \in \mathbb{N}$, $v, v_1, \ldots, v_n \in \mathsf{h}_\mathbb{R}$ the vector $\pi_\omega(\delta_v)\Phi$ is in the domain of the operator

$$p(v_n) \ldots p(v_1).$$

We prove this by induction and to simplify the notation we will omit $\pi_\omega$. Suppose that

$$\eta = p(v_{n-1}) \ldots p(v_1)\delta_v\Phi,$$

to prove the assertion we need to show that the function $t \to \delta_{tv}\eta$ is weakly differentiable at $t = 0$. Consider now $\xi = \delta_u\Phi$ with $u \in \mathsf{h}_\mathbb{R}$, we have

$$\begin{aligned}\lim_{t\to 0} \frac{1}{t} \langle \xi, (\delta_{tv_n} - \mathbb{1})\, \eta\rangle &= (-\mathrm{i})^n \frac{\partial}{\partial t} \frac{\partial^{n-1}}{\partial t_{n-1} \ldots \partial t_1} \langle \Phi, \delta_{-u}\delta_{t_n v_n} \ldots \delta_{t_1 v_1}\delta_{tv}\Phi\rangle \\ &= (-\mathrm{i})^n \frac{\partial}{\partial t} \frac{\partial^{n-1}}{\partial t_{n-1} \ldots \partial t_1} \omega\left(\delta_{-u}\delta_{t_n v_n} \ldots \delta_{t_1 v_1}\delta_{tv}\right),\end{aligned}$$

where all the derivatives are evaluated at the point $t = t_{n-1} = \cdots = t_1 = 0$. This quantity is finite since the expression (2.11) for a gaussian state is analytic. Therefore the weak derivative of $t \to \delta_{tv}\eta$ exists on the dense set $\mathcal{D}$ defined by

$$\mathcal{D} = \mathrm{span}\{\delta_v\Phi : v \in \mathsf{h}_\mathbb{R}\}.$$

Using a similar computation one can also see that the quantity

$$C = \lim_{t\to 0} \frac{1}{t} \left\|(\delta t v_n - \mathbb{1})\, \eta\right\|^2$$

exists and is finite. Therefore

$$\lim_{t\to 0} \frac{1}{t} \langle \xi, (\delta_{tv_n} - \mathbb{1})\, \eta\rangle \leq \sqrt{C}\, \|\eta\|$$

and the weak derivative exists also on the closure of $\mathcal{D}$ which is the whole $\mathcal{H}_\omega$.

We can use this property to evaluate the following quantity

$$\begin{aligned}\langle \Phi, p(u)p(v)\Phi\rangle &= (-\mathrm{i})^2 \frac{\partial^2}{\partial t \partial s} \langle \Phi, \delta_{su})\delta_{tv})\Phi\rangle = -\frac{\partial^2}{\partial t\partial s} \mathrm{e}^{-\mathrm{i}ts\sigma(u,v)}\omega(\delta_{su+tv}) \\ &= \mathrm{Re}\,\langle v, Su\rangle - \mathrm{i}\sigma(v, u),\end{aligned}$$

for every $u, v \in \mathsf{h}_{\mathbb{R}}$. Therefore, for every $n \in \mathbb{N}$, $c_1, \dots, c_n \in \mathbb{C}$ and $v_1, \dots, v_n \in \mathsf{h}_{\mathbb{R}}$ we have

$$0 \leq \left\langle \sum_{j=1}^{n} \overline{c_j} p(v_j)\Phi, \sum_{k=1}^{n} \overline{c_k} p(v_k)\phi \right\rangle = \sum_{j,k=1}^{n} \overline{c_k} c_j \left( \mathrm{Re} \langle v_k, S v_j \rangle - \mathrm{i}\sigma(v_k, v_j) \right),$$

which is (v). □

*Remark* 2.25. The conditions of Theorem 2.24 imply that the covariance operator of a gaussian state $S$ is strictly positive. Indeed by definition $S$ is self adjoint and for $h \in \mathfrak{h}$

$$0 \leq \langle h + \mathrm{i}h, (\mathbf{S} - \mathrm{i}\mathbf{J})(h + \mathrm{i}h) \rangle = \langle h + \mathrm{i}h, S_{\mathfrak{h}} h + \mathrm{i} S_{\mathfrak{h}} h \rangle - \mathrm{i} \langle h + \mathrm{i}h, J_{\mathfrak{h}} h + \mathrm{i} J_{\mathfrak{h}} h \rangle$$
$$= 2 \langle h, S_{\mathfrak{h}} h \rangle$$

since the scalar product on the right hand side is real, hence symmetric, and $\langle h, J_{\mathfrak{h}} h \rangle = 0$, since $J_{\mathfrak{h}}^* = -J_{\mathfrak{h}}$. Therefore $S_{\mathfrak{h}} \geq 0$ and this is equivalent to $S \geq 0$, via Example 1.18. Eventually strict positivity follows from invertibility of $S$.

The following result comes from [59] and allows one to understand the condition $S \in \mathcal{Q}$ from its Williamson's normal form.

**Proposition 2.26.** *Let $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ a strictly positive operator and let*

$$S = T^{\sharp} D T$$

*be its Williamson's normal form, given by Corollary 2.11 for some Bogoliubov transformation $T \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ and $D = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$ where $\lambda_1 \geq \cdots \geq \lambda_d > 0$. Then $S \in \mathcal{Q}$ if and only if*

$$\lambda_1 \geq \cdots \geq \lambda_n \geq 1.$$

*Proof.* Let us start by assuming $S \in \mathcal{Q}$. Using the equivalent conditions of Theorem 2.24 and the characterization of Bogoliubov transformations in Proposition 2.6 we have

$$\mathbf{D} - \mathrm{i}\mathbf{J} = \mathbf{T^{-1}}^* (\mathbf{S} - \mathrm{i}\mathbf{J}) \mathbf{T^{-1}} \geq 0.$$

We can write this condition explicitly as

$$\begin{pmatrix} D & -\mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & D \end{pmatrix} \geq 0,$$

whose meaning is positivity of a complex matrix. Evaluating the minor of the second order arising considering the rows and columns of indexes $k, k+d$, one obtains

$$d_k^2 - 1 \geq 0,$$

for every $k = 1, \dots, d$, which proves the first implication.

The converse implication follows from this last consideration since we have therefore that

$$\mathbf{D} - \mathrm{i}\mathbf{J} = \begin{pmatrix} D & -\mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & D \end{pmatrix} \geq 0.$$

Therefore also

$$\mathbf{S} - \mathrm{i}\mathbf{J} = \mathbf{T}^* (\mathbf{D} - \mathrm{i}\mathbf{J}) \mathbf{T} \geq 0$$

and the proof is concluded via Theorem 2.24. □

Similar to the case with normal random variables we can transform the parameters of a gaussian state that acts on $CCR(\mathsf{h}_\mathbb{R}, \sigma)$.

*Notation* 2.27. For every $z_0 \in \mathsf{h}_\mathbb{R}$ consider the automorphism of the $CCR(\mathsf{h}_\mathbb{R}, \sigma)$ given by

$$\tau_{z_0}(x) = \delta_{-\mathrm{i}\frac{z_0}{2}} x \delta_{\mathrm{i}\frac{z_0}{2}}, \quad \forall x \in CCR(\mathsf{h}_\mathbb{R}, \sigma). \tag{2.17}$$

**Proposition 2.28.** *Let $\mu \in \mathsf{h}_\mathbb{R}$, $S \in \mathcal{B}_\mathbb{R}(\mathsf{h}_\mathbb{R})$ an invertible, positive operator and $\omega_{(\mu,S)}$ a gaussian state on $CCR(\mathsf{h}_\mathbb{R}, \sigma)$. For every $z_0 \in \mathsf{h}_\mathbb{R}$ and $T \in \mathcal{B}_\mathbb{R}(\mathsf{h}_\mathbb{R})$ a Bogoliubov transformation*

$$\omega_{(\mu,S)} \circ \tau_{z_0} = \omega_{(\mu+z_0,S)}, \quad \omega_{(\mu,S)} \circ \Gamma(T) = \omega_{\left(T^\sharp\mu, T^\sharp S T\right)}, \tag{2.18}$$

*with $\tau_{z_0}$ given by* (2.17) *and $\Gamma(B)$ introduced in Corollary 1.34.*

*In particular, if $T^\sharp S T = D$ is the Williamson's normal form of $S$ given by Corollary 2.11, then*

$$\omega_{(\mu,S)} \circ \tau_{-\mu} \circ \Gamma(T) = \omega_{(0,D)}.$$

*Proof.* Let $z \in \mathsf{h}_\mathbb{R}$. Using equation (1.3) we have

$$\tau_{z_0}(\delta_z) = \delta_{-\mathrm{i}\frac{z_0}{2}} \delta_z \delta_{\mathrm{i}\frac{z_0}{2}} = \exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle z_0, z\rangle\right)\delta_z,$$

while $\Gamma(T)\delta_z = \delta_{Tz}$. Therefore by explicit calculation one gets (2.18). We just need to show that $T^\sharp S T \in \mathcal{Q}$. By Proposition 2.6 we have $J_\mathfrak{h} = T_\mathfrak{h}^* J_\mathfrak{h} T_\mathfrak{h}$ and using its complexified counterpart we obtain

$$\mathbf{T}^*\mathbf{S}\mathbf{T} - \mathrm{i}\mathbf{J} = \mathbf{T}^*\mathbf{S}\mathbf{T} - \mathrm{i}\mathbf{T}^*\mathbf{J}\mathbf{T} = \mathbf{T}^*\left(\mathbf{S} - \mathrm{i}\mathbf{J}\right)\mathbf{T} \geq 0$$

since $\mathbf{S} - \mathrm{i}\mathbf{J} \geq 0$. Eventually, since $S$ is strictly positive from Remark 2.25, we can apply Corollary 2.11 and equation (2.18) to conclude the proof. ☐

We will further investigate maps that transform the parameters of a gaussian state similar to (2.18) in Chapter 3.

### 2.2.1 Partial Order Relation on Gaussian States

In the final part of this section we define a partial order relation on the set of gaussian states which was introduced by Fannes in Ref. [37]. It will be useful for the characterization of gaussian maps in Chapter 3.

**Definition 2.29.** Let $S_1, S_2 \in \mathcal{Q}$ and $\mu_1, \mu_2 \in \mathsf{h}_\mathbb{R}$, we say

$$\omega_{(\mu_1,S_1)} \preceq \omega_{(\mu_2,S_2)},$$

if there exists $\gamma \in \mathbb{R}$ such that

$$\omega_{(\mu_1,S_1)} \leq \gamma\omega_{(\mu_2,S_2)}.$$

We report here a short proposition from [37] showing that this is indeed the definition of a partial order relation and that gives some necessary condition for it.

**Proposition 2.30.** *Let $S_1, S_2 \in \mathcal{Q}$ and $\mu_1, \mu_2 \in \mathsf{h}_\mathbb{R}$.*

1. *If* $\omega_{(\mu_1,S_1)} \preceq \omega_{(\mu_2,S_2)}$ *then* $S_1 \leq S_2$ *and* $\mu_1 - \mu_2 \in \ker(S_2 - S_1)^\perp$.
2. $\preceq$ *is a partial order relation.*

*Proof.* We start by proving Item 1. Let $\gamma$ be given as in Definition 2.29. Since $\tau_{-\mu_2}$ is a positive transformation, via (2.18) we have

$$\omega_{(\mu_1-\mu_2,S_1)} = \omega_{(\mu_1,S_1)} \circ \tau_{-\mu_2} \leq \gamma\omega_{(\mu_2,S_2)} \circ \tau_{-\mu_2} = \gamma\omega_{(0,S_2)}.$$

Therefore $\omega_{(\mu_1-\mu_2,S_1)} \preceq \omega_{(0,S_2)}$. Consider now $z \in \mathsf{h}_\mathbb{R}$ and for $n \in \mathbb{N}$ let $t_1, \dots t_n \in \mathbb{R}$, $c_1, \dots c_n \in \mathbb{C}$. Define $x = \sum_{j=1}^n c_j \delta_{t_j z}$ and observe that

$$x^*x = \sum_{j,k=1}^n \overline{c_j} c_k \delta_{(t_k-t_j)z}.$$

Exploiting the fact that

$$\gamma\omega_{(0,S_2)}(x^*x) - \omega_{(\mu_1-\mu_2,S_1)}(x^*x) \geq 0$$

for all choices of $t_j, c_j$ we have positive definition of the function

$$t \mapsto \gamma \exp\left(-\frac{t^2}{2}\operatorname{Re}\langle z, S_2 z\rangle\right) - \exp\left(-\frac{t^2}{2}\operatorname{Re}\langle z, S_1 z\rangle - \mathrm{i}\operatorname{Re}\langle \mu_1 - \mu_2, z\rangle\right).$$

Via Bochner's theorem its Fourier transform should therefore be positive, namely the quantity

$$\frac{\gamma}{\sqrt{\operatorname{Re}\langle z, S_2 z\rangle}} \mathrm{e}^{-\frac{k^2}{2\operatorname{Re}\langle z, S_2 z\rangle}} - \frac{1}{\sqrt{\operatorname{Re}\langle z, S_1 z\rangle}} \mathrm{e}^{-\frac{|k+\operatorname{Re}\langle \mu_1-\mu_2, z\rangle|^2}{2\operatorname{Re}\langle z, S_1 z\rangle}}, \tag{2.19}$$

is positive for all $k \in \mathbb{R}$. What we obtained is the weighted difference of two one-dimensional classical guassian kernels. The existence of a $\gamma$ such that (2.19) is positive for every $k \in \mathbb{R}$ implies, taking limits at $\pm\infty$ that

$$\operatorname{Re}\langle z, S_2 z\rangle \geq \operatorname{Re}\langle z, S_1 z\rangle,$$

for every $z \in \mathsf{h}_\mathbb{R}$. Moreover in the case where the equality holds we have the further restriction that the two densities must have the same mean, hence $\operatorname{Re}\langle \mu_1 - \mu_2, z\rangle = 0$. These two conditions must hold for every $z \in \mathsf{h}_\mathbb{R}$ and this concludes the proof of Item 1.

To show 2 observe that reflexivity and transitivity follow easily from Definition (2.29). Antisymmetry comes instead from what we have just proven. □

In general it is not easy to translate $\preceq$ in terms of the parameters $\mu$ and $S$ of the gaussian states. However this can be done if we restrict ourselves to comparing states whose covariance operators can be simultaneously diagonalized via Williamson's Normal Form. The result and ideas will be similar to the one obtained in [37] but the approach taken with the use of Williamson's normal form is, in our opinion, easier to understand and follow along.

**Lemma 2.31.** *Let* $\mu \in \mathbb{C}$ *and* $Sz = \sigma z$ *for some* $\sigma \geq 1$. *Then* $S \in \mathcal{Q}$ *and*

$$\omega_{(\mu,\mathbb{1})} \preceq \omega_{(0,S)}$$

*if and only if either* $\sigma > 1$ *or* $\sigma = 1$ *and* $\mu = 0$.

*Proof.* From Proposition 2.26 we immediately obtain $S \in \mathcal{Q}$, since it is already in Williamson's normal form.

Let us proceed now assuming $\omega_{(\mu,\mathbb{1})} \preceq \omega_{(0,S)}$. Via Proposition 2.30 we have $S \geq \mathbb{1}$ and $\mu \in \ker(S-\mathbb{1})^{\perp}$. Therefore $\sigma \geq 1$ and, if $\sigma = 1$, $S - \mathbb{1} = 0$ hence

$$\mu \in \ker(0)^{\perp} = \{0\}.$$

We will now prove the converse implication of the Lemma. Clearly if $\sigma = 1$ and $\mu = 0$ we have $\omega_{(\mu,\mathbb{1})} = \omega_{(0,S)}$ and the inequality is trivial. Let us assume $\sigma > 1$. In order to prove the inequality we look at the Fock representation of the one dimensional CCR algebra. The density matrices of the gaussian states are given by

$$\rho_{(\mu,\mathbb{1})} = W\left(-\mu_0\right)|e(0)\rangle\langle e(0)|\, W\left(\mu_0\right), \quad \rho_{(0,S)} = (1-\mathrm{e}^{-s})\mathrm{e}^{-sa^\dagger a},$$

with $\mu_0 := -\mathrm{i}\frac{\mu}{2}$ and $\sigma = \coth\frac{s}{2}$. We postpone the proof of this to Remark 2.47 and Example 2.46 respectively.

In order to conlcude the proof, now let $x \in \Gamma_s(\mathbf{h})$, we have

$$\begin{aligned}\left\langle \rho_{(\mu,\mathbb{1})}x, x\right\rangle &= \left|\langle x, W\left(\mu_0\right)e(0)\rangle\right|^2 = \left|\left\langle \mathrm{e}^{-\frac{s}{2}a^\dagger a}x, \mathrm{e}^{\frac{s}{2}a^\dagger a}W\left(\mu_0\right)e(0)\right\rangle\right|^2 \\ &\leq \left\|\mathrm{e}^{\frac{s}{2}a^\dagger a}W\left(\mu_0\right)e(0)\right\|^2 \left\langle \mathrm{e}^{-sa^\dagger a}x, x\right\rangle \\ &= \frac{\left\|\mathrm{e}^{\frac{s}{2}a^\dagger a}W\left(\mu_0\right)e(0)\right\|^2}{1-\mathrm{e}^{-s}} \left\langle \rho_{(0,S)}x, x\right\rangle.\end{aligned}$$

Note that $W(\mu_0)e(0) \in \mathrm{Dom}(\mathrm{e}^{\frac{s}{2}a^\dagger a})$ and therefore it is sufficient to choose

$$\gamma = \left\|\mathrm{e}^{\frac{s}{2}a^\dagger a}W(\mu_0)e(0)\right\|^2 / (1-\mathrm{e}^{-s})$$

to conclude the proof. □

This Lemma can be easily extended to states with non zero means and slightly more general covariance operators.

**Corollary 2.32.** *Let $\sigma_1, \sigma_2 \geq 1$, $S_j z = \sigma_j z$ for $j = 1, 2$, $z \in \mathbb{C}$ and $\mu_1, \mu_2 \in \mathbb{C}$. Then*

$$\omega_{(\mu_1,S_1)} \preceq \omega_{(\mu_2,S_2)}$$

*if and only if either $\sigma_2 > \sigma_1$ or $\sigma_1 = \sigma = 2$ and $\mu_1 = \mu_2$.*

*Proof.* Note at first that $\omega_{(\mu_1,S_1)} \preceq \omega_{(\mu_2,S_2)}$ is equivalent to $\omega_{(\mu_1-\mu_2,\mathbb{1})} \preceq \omega_{(0,S_1)}$. Indeed, using (2.18)

$$\omega_{(0,S_2)} - \omega_{(\mu_1-\mu_2,S_1)} = \left(\omega_{(\mu_2,S_2)} - \omega_{(\mu_1,S_1)}\right) \circ \tau_{-\mu_2}$$

and $\tau_\mu$ is a *-automorphism. Eventually $\omega_{(\mu_1-\mu_2,\mathbb{1})} \preceq \omega_{(0,S_2-S_1+\mathbb{1})}$ is equivalent to $\omega_{(\mu_1-\mu_2,\mathbb{1})} \preceq \omega_{(0,S_2-S_1+\mathbb{1})}$ via multiplication for the positive definite function

$$\mathbb{C} \ni z \mapsto \exp\left(-\frac{1}{2}\left(\sigma_1 - 1\right)|z|^2\right),$$

and the use of Lemma 2.20. The Corollary then simply follows by applying Lemma 2.31. □

We are now ready to characterize the partial relation $\preceq$ in the multidimensional CCR algebra if we restrict ourselves to considering covariance operators that can be diagonalized with the same Bogoliubov transformation in Williamson's Normal form.

**Proposition 2.33.** *Let $S_1, S_2 \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$, $\mu_1, \mu_2 \in \mathsf{h}_{\mathbb{R}}$ and suppose there exists a symplectic transformation $M$ such that*

$$M^\sharp S_j M = D_j,$$

*for $j = 1, 2$ where $D_j$ are diagonal matrices. Then $\omega_{(\mu_1, S_1)} \preceq \omega_{(\mu_2, S_2)}$ if and only if $S_1 \leq S_2$ and $\mu_1 - \mu_2 \in \ker(S_2 - S_1)^\perp$.*

*Proof.* Let $\Gamma(M)$ be the *-automorphism of $CCR(\mathsf{h}_{\mathbb{R}}, \sigma)$ given by Corollary 1.34. Via (2.18) we have

$$\omega_{(\mu_1, S_1)}\Gamma(M) = \omega_{\left(M^\sharp \mu_1, D_1\right)}, \quad \omega_{(\mu_2, S_2)}\Gamma(M) = \omega_{\left(M^\sharp \mu_2, D_2\right)}.$$

This means that $\omega_{(\mu_1, S_1)} \preceq \omega_{(\mu_2, S_2)}$ if and only if $\omega_{\left(M^\sharp \mu_1, D_1\right)} \preceq \omega_{\left(M^\sharp \mu_2, D_2\right)}$. Let $(D_1)_j, (D_2)_j$ represent the diagonal entries of $D_1, D_2$ respectively for $j = 1, \ldots, d$ and observe that

$$\omega_{\left(M^\sharp \mu_k, D_k\right)} = \omega_{\left(\left(M^\sharp \mu_k\right)_1, (D_k)_1\right)} \otimes \cdots \otimes \omega_{\left(\left(M^\sharp \mu_k\right)_d, (D_k)_d\right)},$$

for $k = 1, 2$, where we used the *-isomorphism introduced in Corollary 1.35:

$$CCR(\mathsf{h}_{\mathbb{R}}, \sigma) \ni W(z) \mapsto W(z_1) \otimes \cdots \otimes W(z_d) \in (CCR(\mathbb{C}, \operatorname{Im}\langle\cdot,\cdot\rangle))^{\otimes d}.$$

In particular $\omega_{\left(M^\sharp \mu_1, D_1\right)} \preceq \omega_{\left(M^\sharp \mu_2, D_2\right)}$ if and only if

$$\omega_{\left(\left(M^\sharp \mu_1\right)_j, (D_1)_j\right)} \preceq \omega_{\left(\left(M^\sharp \mu_2\right)_j, (D_2)_j\right)}, \qquad \text{for } j = 1, \ldots d.$$

Using Corollary 2.32 this is equivalent to $(D_1)_j \leq (D_2)_j$ for every $j = 1, \ldots, d$ and if $(D_1)_{j_0} = (D_2)_{j_0}$ for some $j_0$ then $\left(M^\sharp \mu_1\right)_{j_0} = \left(M^\sharp \mu_2\right)_{j_0}$. The first inequality is exactly the condition $S_1 \leq S_2$, while the second condition is equivalent to $M^\sharp(\mu_1 - \mu_2) \in \ker(D_2 - D_1)^\perp$. But

$$\ker(D_2 - D_1) = M(\ker(S_2 - S_1)),$$

therefore $M^\sharp(\mu_1 - \mu_2) \in \ker(D_2 - D_1)^\perp$ if and only if $(\mu_1 - \mu_2) \in \ker(S_2 - S_1)^\perp$, which concludes the proof. □

## 2.3 Gaussian States on the Fock Space

We introduce now gaussian states on $\mathcal{B}(\Gamma_s(\mathsf{h}))$ trying to replicate most of the results obtained for gaussian states on $CCR(\mathsf{h}_{\mathbb{R}}, \sigma)$. Recall the following definitions

**Definition 2.34.** A *state* on $\mathcal{B}(\Gamma_s(\mathsf{h}))$ is a positive, normalized functional.

A state $\omega$ is *normal* if for every increasing net $(x_\alpha)_\alpha$ of positive elements of $\mathcal{B}(\Gamma_s(\mathsf{h}))$ which has an upper bound, one has

$$\omega(\sup_\alpha x_\alpha) = \sup \omega(x_\alpha).$$

We denote with $\mathcal{B}(\Gamma_s(\mathsf{h}))_*$ the set of normal states on $\mathcal{B}(\Gamma_s(\mathsf{h}))$.

One has the following classical result, whose proof can be found for example in [13, Theorem 2.4.21].

**Proposition 2.35.** *Let $\omega$ we a state on $\mathcal{B}(\Gamma_s(\mathsf{h}))$. The following conditions are equivalent:*

*(i) $\omega$ is normal;*

*(ii) $\omega$ is continuous in the $\sigma$-weak topology;*

*(iii) there exists a* density matrix *$\rho$, i.e. a positive trace class operator $\rho$ on $\Gamma_s(\mathsf{h})$, with $\rho(\mathbb{1}) = 1$ such that*

$$\omega(x) = \mathrm{tr}(\rho x), \quad x \in \mathcal{B}(\Gamma_s(\mathsf{h})).$$

*Notation* 2.36. We will denote with $L^1(\Gamma_s(\mathsf{h}))$ the set of trace-class operators on $\Gamma_s(\mathsf{h})$.

We introduce now the characteristic function of a normal state.

**Definition 2.37.** Let $\rho \in L^1(\Gamma_s(\mathsf{h}))$ be a density matrix. The *quantum characteristic function* $\hat{\rho}$ of $\rho$ is the function

$$\hat{\rho}(z) = \mathrm{tr}(\rho W(z)), \quad \forall z \in \mathsf{h}.$$

We give now the definition if a Gaussian State on $\mathcal{B}(\Gamma_s(\mathsf{h}))$.

**Definition 2.38.** We say that a normal state $\omega$ on $\mathcal{B}(\Gamma_s(\mathsf{h}))$ is *gaussian* if the quantum characteristic function of its density matrix $\rho$ satisfies

$$\hat{\rho}(z) = \exp\left(-\mathrm{i}\,\mathrm{Re}\,\langle \mu, z\rangle - \frac{1}{2}\mathrm{Re}\,\langle z, Sz\rangle\right), \quad z \in \mathsf{h}, \tag{2.20}$$

for some $\mu \in \mathsf{h}$ and an invertible, positive operator $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$. Similarly to Definition 2.15 we say $\mu$ is the *mean vector* and $S$ is the *covariance operator*. To avoid confusion we identify $\omega$ with its density matrix $\rho$ and denote it via $\rho = \rho_{(\mu,S)}$.

We now want to establish some conditions for an operator $S$ to be a covariance operator of a gaussian state. For this reason we present an analogue result to Bochner's Theorem and Proposition 2.18.

**Theorem 2.39.** *Let $G : \mathsf{h} \to \mathbb{C}$ be a function. $G$ is the quantum characteristic function of the density matrix $\rho$ if and only if:*

*(i) $G(0) = 1, F(z)$ is continuous at $z = 0$;*

*(ii) the kernel*

$$\mathsf{h} \times \mathsf{h} \ni (z_1, z_2) \mapsto G(z_1 - z_2)\mathrm{e}^{\mathrm{i}\sigma(z_1,z_2)},$$

*is positive definite.*

*Proof.* Let us start by proving necessity, therefore suppose $\rho$ is a density matrix. Clearly $1 = \mathrm{tr}(\rho W(0)) = G(0)$. Moreover, regularity of the Fock representation implies $\hat{\rho}(z)$ is continuous at $z = 0$ and (i) is proved. Condition (ii) follows in a similar way to the proof of Proposition 2.18 considering $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathsf{h}$ and

$$x = \sum_{k=1}^{n} c_k W(z_k).$$

the proof follows evaluating $\operatorname{tr}(\rho x x^*) \geq 0$.

We prove now sufficiency of the Theorem. At first we will prove that continuity at $z = 0$ and (ii) imply uniform continuity of $G$. To show this consider $n = 3$ and $z_1 = 0, z_2, z_3 \in \mathsf{h}$ then (ii), along with Remark 2.17, implies positive definiteness of the matrix

$$\begin{pmatrix} 1 & G(-z_2) & G(-z_3) \\ G(z_2) & 1 & G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)} \\ G(z_3) & G(z_3 - z_2)\mathrm{e}^{\mathrm{i}\sigma(z_3,z_2)} & 1 \end{pmatrix}.$$

Self-adjointness of the matrix implies that $G(-z) = \overline{G(z)}$ for every $z \in \mathsf{h}$. Now, using Sylvester's criterion, positive definiteness translates in these requirements

$$\begin{cases} 1 \geq |G(z_2)|^2 \\ 1 + 2\operatorname{Re}\overline{G(z_2)}G(z_3)G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)} \geq |G(z_2)|^2 + |G(z_3)|^2 + |G(z_2 - z_3)|^2. \end{cases}$$

Rearranging the second inequality and using the first one we obtain

$$\begin{aligned} |G(z_2) - G(z_3)|^2 &\leq 1 - |G(z_2 - z_3)|^2 - 2\operatorname{Re}\overline{G(z_2)}G(z_3)\left(1 - G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)}\right) \\ &\leq \left|1 - G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)}\right|^2 + 2\left|1 - G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)}\right| \\ &\leq 2\left|1 - G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)}\right| + 2\left|1 - G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)}\right| \\ &\leq 4\left|1 - G(z_2 - z_3)\mathrm{e}^{\mathrm{i}\sigma(z_2,z_3)}\right| \end{aligned}$$

which proves uniform continuity of $G$ by continuity of $G$ at $z = 0$.

We will now construct a representation of the CCR on another Hilbert space. Consider $\psi$ a function on $\mathsf{h}$ and for every $z \in \mathsf{h}$ consider the operator $\hat{W}(z)$ that acts as

$$\hat{W}(z)\psi(z_1) = \mathrm{e}^{-\mathrm{i}\sigma(z,z_1)}\psi(z + z_1).$$

It is easy to check that the operators $\hat{W}(z)$ satisfy the CCR (1.3). Consider the linear space $\mathcal{H}_0$ of the functions on $\mathsf{h}$ of the form

$$\psi(z) = \left(\sum_{k=1}^{n} c_k \hat{W}(z_k)\right) 1(z) = \sum_{k=1}^{n} c_k \mathrm{e}^{-\mathrm{i}\sigma(z_k,z)}, \quad z \in \mathsf{z},$$

for some $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$, $z_1, \ldots, z_n \in \mathsf{h}$ and $1(z)$ is the function which is identically equal to 1. On $\mathcal{H}_0$ we can consider the sesquilinear form

$$\langle \psi_1, \psi_2 \rangle_0 = \sum_{j,k} \overline{c_{1,j}} c_{2,k} G(z_{1,j} - z_{2,k})\mathrm{e}^{\mathrm{i}\sigma(z_{1,j},z_{2,k})},$$

where $\psi_k = \sum_j c_{k,j}\hat{W}(z_{k,j}j)1$ for $k = 1, 2$. By (ii) we have

$$\langle \psi, \psi \rangle_0 \geq 0, \quad \forall \psi \in \mathcal{H}_0$$

and $\langle \cdot, \cdot \rangle_0$ is a pre-scalar product, moreover

$$\left\langle \hat{W}(z)\psi_1, \hat{W}(z)\psi_2 \right\rangle_0 = \langle \psi_1, \psi_2 \rangle_0, \quad \forall \psi_1, \psi_2 \in \mathcal{H}_0, z \in \mathsf{h}.$$

We can therefore denote with $\hat{\mathcal{H}}_0$ the completion of $\mathcal{H}_0$ with respect to this scalar product, which is now an Hilbert space. Of course $\hat{W}(z)$ extends uniquely to a unitary operator on $\hat{\mathcal{H}}_0$ for every $z \in \mathsf{h}$. In particular we just need to show the continuity property to have that the pair $(\hat{\mathcal{H}}_0, \hat{W})$ is a Weyl system. Observe that for $\psi_1, \psi_2 \in \mathcal{H}_0$ we have

$$\left\langle \psi_1, \hat{W}(z)\psi_2 \right\rangle = \sum_{j,k} \overline{c_j} c_k \exp\left(\mathrm{i}\sigma(z_{1,j} - z, z_{2,k} + z)\right) G(z_{1,j} - z - z_{2,k}).$$

For uniform continuity of $G$ we have that for $t \in \mathbb{R}$

$$\left\langle \psi_1, \hat{W}(tz)\psi_2 \right\rangle \xrightarrow{t\to 0} \langle \psi_1, \psi_2 \rangle .$$

Therefore

$$\left\| \left( \hat{W}(tz) - \mathbb{1} \right) \psi \right\|^2 = \left\langle \left( \hat{W}(tz) - \mathbb{1} \right) \psi, \left( \hat{W}(tz) - \mathbb{1} \right) \psi \right\rangle \xrightarrow{t\to 0} 0,$$

for every $\psi \in \mathcal{H}_0$, by sesquilinearity this holds on the whole $\hat{\mathcal{H}}_0$. Note in particular that we can define $\rho_0 \in L^1(\hat{\mathcal{H}}_0)$ as $\rho_0 = |1\rangle\langle 1|$ which is a density matrix, moreover

$$\mathrm{tr}(\rho_0 \hat{W}(z)) = \left\langle 1, \hat{W}(z) 1 \right\rangle = G(z).$$

This is the density matrix that satisfies the Theorem, we just need to show that there exists one in $L^1(\mathcal{H})$. Now by Theorem 1.33 the CCR algebras induced by the representations $(\mathcal{H}, W)$ and $(\hat{\mathcal{H}}_0, \hat{W})$ are isometrically $*$-isomorphic. In particular the two Hilbert spaces are unitarily equivalent and therefore we can define the density matrix $\rho$ by conjugating with the unitary that realizes said equivalence. □

Since this result holds we can easily recover a result similar to Theorem 2.24, more precisely:

**Theorem 2.40.** *Let $\mu \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ an invertible positive operator. Let $\rho \in L^1(\Gamma_s(\mathsf{h}))$ that satisfies* (2.20) *and $\omega$ a functional on $CCR(\mathsf{h}_{\mathbb{R}}, \sigma)$ that satisfies* (2.11). *$\rho$ is a gaussian state if and only if $\omega$ is a gaussian state.*

*In particular $\rho$ is a gaussian state if and only if $S$ satisfies one of the equivalent conditions of Theorem 2.24.*

Analogously we can prove a similar result to the one of Proposition 2.28. Fist we state the following Lemma, which is the specialization to finite dimension of a classic result due to Shale [57, 66]

**Lemma 2.41** (Shale)**.** *For every Bogoliubov transformation $T$ on $\mathsf{h}_{\mathbb{R}}$ there exists a unitary operator $\Gamma(T)$, unique up to a scalar of modulus unity , on $\Gamma_s(\mathcal{H})$ such that*

$$\Gamma(T) W(z) \Gamma(T)^* = W(Tz)$$

**Proposition 2.42.** *Let $\rho = \rho_{(\mu,S)}$ be a gaussian state. For every Bogoliubov transformation $T \in \mathsf{h}_{\mathbb{R}}$ consider $\Gamma(T)$ the unitary operator on $\Gamma_s(\mathsf{h})$ given by Lemma 2.41, it holds*

$$W(z)^* \rho W(z) = \rho_{(\mu - 2\mathrm{i}z, S)}, \quad \Gamma(T)^* \rho \Gamma(T) = \rho_{\left(T^\sharp \mu, T^\sharp S T\right)}.$$

*Proof.* The proof is exactly the one of Proposition 2.28, indeed

$$W(z)W(z_1)W(z)^* = \exp\left(2\mathrm{i}\,\mathrm{Re}\left\langle \mathrm{i}z, z_1\right\rangle\right) W(z_1), \quad \Gamma(T)W(z)\Gamma(T)^* = W(Tz),$$

and we conclude by the ciclicity property of the trace. Moreover $T^\sharp ST$ is a suitable covariance operator by the same proof of Proposition 2.28 since Theorem 2.40 holds. □

As a final result for this chapter we consider a connection between the two kinds of gaussian states, recalling the following definition.

**Definition 2.43.** We say a functional $\omega$ on $CCR(\mathsf{h}_\mathbb{R}, \sigma)$ is normal if there exists a trace class operator, positive with tr$\rho = 1$, such that

$$\mathrm{tr}(\rho\pi(x)) = \omega(x) \ ,$$

for every $x \in CCR(\mathsf{h}_\mathbb{R}, \sigma)$, where $\pi$ is the Fock representation. We will denote with $CCR(\mathsf{h}_\mathbb{R}, \sigma)_*$ the set of normal functionals and with $\mathcal{S}(CCR(\mathsf{h}_\mathbb{R}, \sigma))$ the set of normal states.

*Remark* 2.44. Gaussian states on $CCR(\mathsf{h}_\mathbb{R}, \sigma)$ are normal. Indeed by Theorem 2.40 to every gaussian state on $CCR(\mathsf{h}_\mathbb{R}, \sigma)$ $\omega_{(\mu,S)}$ for $\mu \in \mathsf{h}_\mathbb{R}$, $S \in \mathcal{Q}$ there corresponds a gaussian state on $\mathcal{B}(\Gamma_s(\mathsf{h}))$ $\rho_{(\mu,S)}$. Clearly

$$\omega_{(\mu,S)}(\delta_z) = \mathrm{tr}(\rho W(z)), \quad \forall z \in \mathsf{h}_\mathbb{R}$$

and by continuity of the trace and of the representation the equality holds for $x \in CCR(\mathsf{h}_\mathbb{R}, \sigma)$.

The following two examples provide the intuition for possible gaussian density matrices.

**Example 2.45.** Using Lemma 1.46, we can show that

$$\rho_{(0,\mathbb{1})} = |e(0)\rangle\langle e(0)| \,.$$

Indeed $\mathbb{1}$ is an invertible positive operator and

$$\begin{aligned}\mathrm{tr}(|e(0)\rangle\langle e(0)|\, W(z)) &= \frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} \mathrm{e}^{-x^2-y^2}\left\langle e(x+\mathrm{i}y), |e(0)\rangle\langle e(0)|\, W(z)e(x+\mathrm{i}y)\right\rangle \mathrm{d}x\mathrm{d}y \\ &= \frac{1}{\pi^d}\int_{\mathbb{R}^{2d}} \exp\left(-x^2 - \langle z, x\rangle - y^2 - \mathrm{i}\,\langle z, y\rangle - \frac{|z|^2}{2}\right) \\ &= \exp\left(-\frac{1}{2}\,\mathrm{Re}\,\langle z, z\rangle\right),\end{aligned}$$

which is precisely the quantum characteristic function of $\rho_{(0,\mathbb{1})}$.

**Example 2.46.** In this example let $d = 1$, $\sigma > 1$ and $s \in \mathbb{R}$ such that $\sigma = \coth(s/2)$. We will show that

$$\rho_{(0,\sigma\mathbb{1})} = (1 - \mathrm{e}^{-s})\mathrm{e}^{-sa^\dagger a}$$

Recall that

$$\mathrm{e}^{-sa^\dagger a}e(z) = \sum_{m,n=0}^{\infty} \frac{(-s)^m}{m!}\frac{z^n}{\sqrt{n!}}(a^\dagger a)^m e_n = \sum_{n=0}^{\infty}\frac{(\mathrm{e}^{-s}z)^n}{\sqrt{n!}}e_n = e(\mathrm{e}^{-s}z).$$

Via the use of Lemma 1.46, we have

$$\begin{aligned}
\mathrm{tr}((1-\mathrm{e}^{-s})\mathrm{e}^{-sa^\dagger a}W(f)) &= \frac{1-\mathrm{e}^{-s}}{\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-|z|^2}\left\langle e(\mathrm{e}^{-s}z), e(z+f)\right\rangle \mathrm{e}^{-\frac{|f|^2}{2}-\langle f,z\rangle}\mathrm{d}x\mathrm{d}y \\
&= \frac{1-\mathrm{e}^{-s}}{\pi}\int_{\mathbb{R}^2}\exp\left(-(1-\mathrm{e}^{-s})|z|^2 + \left\langle \mathrm{e}^{-s}z, f\right\rangle - \frac{|f|^2}{2} - \langle f, z\rangle\right)\mathrm{d}x\mathrm{d}y \\
&= \frac{1-\mathrm{e}^{-s}}{\pi}\mathrm{e}^{-\frac{|f|^2}{2}}\int_{\mathbb{R}}\exp\left(-\left(1-\mathrm{e}^{-s}\right)x^2 - \left(\overline{f}-\mathrm{e}^{-s}f\right)x\right)\mathrm{d}x \\
&\times \int_{\mathbb{R}}\exp\left(-\left(1-\mathrm{e}^{-s}\right)y^2 - \mathrm{i}\left(\overline{f}+\mathrm{e}^{-s}f\right)y\right)\mathrm{d}y \\
&= \exp\left(-\frac{|f|^2}{2} + \frac{\left(\overline{f}-\mathrm{e}^{-s}f\right)^2}{4\left(1-\mathrm{e}^{-s}\right)} - \frac{\left(\overline{f}+\mathrm{e}^{-s}f\right)^2}{4\left(1-\mathrm{e}^{-s}\right)}\right) \\
&= \exp\left(-\frac{1}{2}\coth\left(\frac{s}{2}\right)|f|^2\right) = \exp\left(-\frac{1}{2}\sigma|f|^2\right).
\end{aligned}$$

which is precisely the quantum Fourier transform of $\rho_{(0,\sigma\mathbb{1})}$.

*Remark* 2.47. From Example 2.45 and Proposition 2.42 we have that

$$\begin{aligned}
\rho_{(z,\mathbb{1})} &= W\left(-\mathrm{i}\frac{z}{2}\right)\rho_{(0,\mathbb{1})}W\left(\mathrm{i}\frac{z}{2}\right) = W\left(-\mathrm{i}\frac{z}{2}\right)|e(0)\rangle\langle e(0)|\, W\left(\mathrm{i}\frac{z}{2}\right) \\
&= \mathrm{e}^{-\frac{|z|^2}{4}}\left|e\left(-\mathrm{i}\frac{z}{2}\right)\right\rangle\left\langle e\left(-\mathrm{i}\frac{z}{2}\right)\right|.
\end{aligned}$$

In particular all normal states given by

$$\mathrm{e}^{-|z|^2}|e(z)\rangle\langle e(z)|, \quad z \in \mathsf{h},$$

are gaussian.

We conclude this section by stating two results on gaussian states that will be useful in the rest of the thesis. We recall the following definition

**Definition 2.48.** Let $\rho \in L_1(\Gamma_s(\mathsf{h}))$ a density matrix. We say $\rho$ is *faithful* if

$$\mathrm{tr}(\rho x^* x) = 0 \Leftarrow x = 0, \quad x \in \mathcal{B}(\Gamma_s(\mathsf{h})).$$

Instead we say $\rho$ is *pure* if it is a projection, namely $\rho^2 = \rho$,

For $\mathsf{h} = \mathbb{C}$ we obtain the following result.

**Proposition 2.49.** *A gaussian state on* $\Gamma_s(\mathbb{C})$ *is ether faithful or pure. In particular it is faithful if and only if*

$$\mathbf{S} - \mathrm{i}\mathbf{J} > 0.$$

*Proof.* By Corollary 2.11 and Proposition 2.42 we can always assume that we are dealing with a gaussian state of the form $\rho_{(0,\sigma\mathbb{1})}$. Moreover, by Proposition 2.26, $\sigma \geq 1$. Therefore we are in either one of the cases of Examples 2.45, 2.46. In the first case the gaussian state is obviously pure, while in the latter it is faithful.

In particular the case of a faithful gaussian state corresponds, by Example 2.46, to the case $\sigma > 1$ which is precisely the condition for

$$\mathbf{S} - \mathrm{i}\mathbf{J} = \begin{pmatrix} \sigma - 1 & 0 \\ 0 & \sigma + 1 \end{pmatrix} \geq 0.$$

□

Eventually we state this density Theorem for gaussian states whose proof can be found in [38, 65].

**Theorem 2.50.** *The linear span of gaussian states on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ and $\mathcal{B}(\Gamma_s(\mathsf{h})))$ are norm dense in the set of normal states on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ and $L_1(\Gamma_s(\mathsf{h}))$, respectively.*

CHAPTER 3

# Characterization of Gaussian QMSs

This chapter is based mainly on the two articles [62, 63]. We introduce here one of the main objects of this thesis, namely gaussian Quantum Markov Semigroups. We first recall the following definition.

**Definition 3.1.** A *Quantum Dynamical Semigroup* on $\mathcal{B}(\mathcal{H})$ is a family $\mathcal{T} = (\mathcal{T})_{t\geq 0}$ with the following properties:

(i) $\mathcal{T}_0 = \mathbb{1}$, $\mathcal{T}_{t+s} = \mathcal{T}_t\mathcal{T}_s = \mathcal{T}_s\mathcal{T}_t$ for every $t, s \geq 0$;

(ii) $\mathcal{T}_t$ is completely positive for all $t \geq 0$;

(iii) $\mathcal{T}_t$ is a $\sigma$-weakly continuous (equivalently weak$^*$ continuous) operator for all $t \geq 0$;

(iv) the map $t \mapsto \mathcal{T}_(a)$ is $\sigma$-weakly continuous for all $a \in \mathcal{B}(\mathcal{H})$.

If moreover it holds $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$ we say $\mathcal{T}$ is a *Quantum Markov Semigroup*. If $\mathcal{T}$ satisfies

$$\lim_{t\to 0} \|\mathcal{T}_t - \mathbb{1}\| = 0,$$

we say $\mathcal{T}$ is *uniformly continuous*.

Recall also

**Definition 3.2.** The *predual* semigroup of a quantum dynamical semigroup $\mathcal{T}$ on $\mathcal{B}(\mathcal{H})$ is the semigroup $\mathcal{T}_*$ of operators on the trace set of trace class operators over $\mathcal{H}$ such that

$$\text{tr}\left(\mathcal{T}_{*t}(\rho)a\right) = \text{tr}\left(\rho\mathcal{T}_t a\right), \quad \forall a \in \mathcal{B}(\mathcal{H}), \rho \in L_1(\mathcal{H}).$$

We do not dwell here on the properties of these semigroups but instead we immediately provide the definition of gaussian QMSs.

**Definition 3.3.** Let $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ be a QMS on $\mathcal{B}(\mathcal{H})$. We say it is gaussian if its predual $\mathcal{T}_*$ leaves the set of gaussian states invariant.

The main results of this chapter will be Theorem 3.30, that shows equivalence of different definitions of gaussian QMSs that have appeared in the literature. The one given in Definition 3.3 is a qualitative definition which motivates clearly the naming choice of these QMSs. However it is not very practical to use in calculations and in addressing properties of this class of semigroups. On the other hand, the remaining definitions of gaussian QMSs involved in Theorem 3.30 are more explicit on the action of the semigroup. For this reason, Theorem 3.30 ties in qualitative and quantitative definitions for gaussian QMSs, making them a well behaved class to work with.

In the first section we focus on maps acting on $CCR\left(\mathsf{h}_{\mathbb{R}}, \sigma\right)$ that preserve the set of gaussian states. Their characterization will be essential for the main result in gaussian QMSs, since a semigroup that preserves the set of gaussian states at all times is composed of that kind of maps. In the second section we state this characterization theorem while in the third we prove the anticipated main result of equivalence of different definitions.

## 3.1 Gaussian Maps

In this section we will introduce one of the main objects needed to achieve the characterization, namely we consider maps that preserve the set of gaussian states on the algebra $CCR\left(\mathsf{h}_{\mathbb{R}}, \sigma\right)$. In particular we focus on the transformations that these maps induce on the parameters of the gaussian states. We will obtaining a characterization of all the possible transformations they can induce. This is an intermediate but necessary step in order to consider semigroups of that preserve the set of gaussian states and that are therefore ocmposed of maps preserving the set of gaussian states. Most of the results in this section follow the ideas of [37], with a notable difference. We do not restrict ourselves to considering just automorphism of the CCR algebra that preserve gaussian states, but we deal with all maps that do this, whether they are automorphisms or not. In particular we will recover their action on the CCR algebra only at a later stage.

**Definition 3.4.** Let $\alpha$ be a norm continuous, linear transformation on the set of normal functionals $CCR\left(\mathsf{h}_{\mathbb{R}}, \sigma\right)_*$. We say $\alpha$ is a *gaussian map* if

1. $\alpha \circ \omega_{(\mu,S)} = \omega_{(\alpha_1(\mu,S),\alpha_2(\mu,S))}$, for every $\mu \in \mathsf{h}_{\mathbb{R}}, S \in Q$.
2. the map $(\mu, S) \mapsto \left(\alpha_1\left(\mu, S\right), \alpha_2\left(\mu, S\right)\right)$ is continuous with respect to the product topology.

As anticipated we will show in Theorem 3.9 that the maps $\alpha_1, \alpha_2$ in definition 3.4 cannot be arbitrary but instead should have a specific expression. Thanks to Theorem 2.50 we can show that $\alpha_1, \alpha_2$ chracterize the map $\alpha$.

**Lemma 3.5.** *Let $\alpha$ be a gaussian map, then $\alpha$ is also positive. Suppose now $\alpha'$ is another gaussian map such that*

$$\alpha_1(\mu, S) = \alpha_1'(\mu, S) \ , \quad \alpha_2(\mu, S) = \alpha_2'(\mu, S) \ ,$$

*for all $\mu \in \mathsf{h}_{\mathbb{R}}, S \in Q$. Then $\alpha = \alpha'$.*

*Proof.* In order to prove positivity of $\alpha$ let $\eta \in CCR(\mathsf{h}_{\mathbb{R}}, \sigma)_*$ be positive. Then $\eta$ is also bounded and therefore $\eta_0 := \eta / \|\eta\| \in \mathcal{S}(CCR(\mathsf{h}_{\mathbb{R}}, \sigma))$. Via Theorem 2.50 we can then find a sequence $(\omega_n)_n$ of linear combinations of gaussian states that converge to $\eta_0$ in norm. Eventually, via norm continuity of $\alpha$ we have

$$\alpha(\eta) = \|\eta\| \, \alpha(\eta_0) = \|\eta\| \lim_{n\to\infty} \alpha(\omega_n) \geq 0 \ .$$

This proves positivity of $\alpha$. Now let $\omega$ be a gaussian state. By hypothesis we have

$$\alpha(\omega) = \omega_{(\alpha_1(\mu,S),\alpha_2(\mu,S))} = \omega_{\left(\alpha_1'(\mu,S),\alpha_2'(\mu,S)\right)} = \alpha'(\omega) \ .$$

In particular we have

$$(\alpha - \alpha')(\omega) = 0 \ ,$$

for every $\omega$ gaussian state. Using density of gaussian states, given by Theorem 2.50 we obtain $\alpha = \alpha'$. □

**Example 3.6.** Two notable examples of gaussian maps are those induced by $\tau_{z_0}$ with $z_0 \in \mathsf{h}_{\mathbb{R}}$ and $\Gamma(B)$ with $B$ a Bogoliubov transformation as in Proposition 2.28. They are *-automorphisms of $CCR(\mathsf{h}_{\mathbb{R}}, \sigma)$. Their adjoint maps $\alpha_{z_0}, \alpha_B$ defined via

$$\alpha_{z_0}(\omega) = \omega \circ \tau_{z_0}, \quad \alpha_B(\omega) = \omega \circ \Gamma(B),$$

are still norm continuous and are therefore gaussian maps. In particular

$$\alpha_{z_0,1}(\mu, S) = \mu + z_0, \quad \alpha_{z_0,2}(\mu, S) = S,$$
$$\alpha_{B,1}(\mu, S) = B^\sharp \mu, \quad \alpha_{B,2}(\mu, S) = B^\sharp S B.$$

The gaussian maps introduced in the previous example cover almost all possible gaussian maps. In fact we will show in Theorem 3.9 that the general action of a gaussian map will be similar to the ones of the example. In order to prove this result we need some preliminary Lemmas that come from [37]. We show the proof only of the first one, since it uses a different method at the beginning, due to our characterization of the ordering $\preceq$ in Proposition 2.33.

**Lemma 3.7.** *Let $\alpha$ be a Gaussian map. Then for every $S \in \mathcal{Q}$ and every $\mu \in \mathsf{h}_{\mathbb{R}}$*

$$\alpha_2(\mu, S) = \alpha_2(0, S) = \alpha_2(S) \ .$$

*Proof.* Since $S \in \mathcal{Q}$, as noted in Remark 2.25, we have $S > 0$ and let $T$ be the Bogoliubov transformation that diagonalizes $S$ according to Corollary 2.11. For every $\lambda \geq 0$ we have $S_\lambda := S + \lambda T^\sharp \mathbb{1} T \in \mathcal{Q}$ and $S_\lambda$ is clearly simultaneously diagonalizable with $S$. Therefore we can use Proposition 2.33 and obtain, for $0 < \lambda_1 < \lambda_2$

$$\omega_{(\mu,S)} \preceq \omega_{\left(0,S_{\lambda_1}\right)} \preceq \omega_{\left(\mu,S_{\lambda_2}\right)} \ .$$

Composition with $\alpha$ preserves the relation $\preceq$, since it is a positive transformation by Lemma 3.5. Therefore by composition and application of Proposition 2.33 we get

$$\alpha_2(\mu, S) \leq \alpha_2(0, S_{\lambda_1}) \leq \alpha_2(\mu, S_{\lambda_2}) \ .$$

By the continuity property in Definition 3.4, taking the limit as $\lambda_1, \lambda_2 \to 0$ we obtain

$$\alpha_2(\mu, S) \leq \alpha_2(0, S) \leq \alpha_2(\mu, S) \ ,$$

which concludes the proof. □

**Lemma 3.8.** *Suppose $s \in \mathbb{R} \mapsto f(s) \in \mathbb{R}$ is continuous and*

$$\pi^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-s^2} \mathrm{e}^{itf(s)} \mathrm{d}s = \mathrm{e}^{-\frac{1}{4}t^2}, \quad \forall t \in \mathbb{R}.$$

*Then either $f(s) = s$ or $f(s) = -s$ for every $s \in \mathbb{R}$.*

We can now state the characterization theorem for Gaussian maps. It extends the result of [37, Theorem 4.5] to maps that are not necessarily automporhisms of the CCR algebra. The proof is exactly the one of [37] except in its final part. We report the entire proof for clarity's sake, since this will be an important result in the following.

**Theorem 3.9.** *Let $\alpha$ be a Gaussian map. Then there exist $T_0, C_0 \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ and $\mu_0 \in \mathsf{h}_{\mathbb{R}}$ such that*

$$\alpha_1(z, S) = \alpha_1(z) = T_0^* z + \zeta_0 \ , \quad \alpha_2(S) = T_0^* S T_0 + C_0 \ . \tag{3.1}$$

*for every $z \in \mathsf{h}_{\mathbb{R}}$ and*

$$\mathbf{C_0} - \mathrm{i}(\mathbf{J} - \mathbf{T_0}^* \mathbf{J} \mathbf{T_0}) \geq 0 \ , \tag{3.2}$$

*where positivity is intended on $\mathfrak{h}_{\mathbb{C}}$.*

*Proof.* We use the notation $|x\rangle\langle y| \,, x, y \in \mathsf{h}_{\mathbb{R}}$ to denote the operator acting on $z \in \mathsf{h}_{\mathbb{R}}$ as $|x\rangle\langle y| \, z = \mathrm{Re} \langle y, z\rangle \, x$. For every $r > 0, z, z_0 \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{Q}$ one has

$$(\pi r)^{-\frac{1}{2}} \int_{\mathbb{R}} \mathrm{e}^{-\frac{s^2}{r}} \omega_{(sz+z_0,S)} ds = \omega_{(z_0, S+r|z\rangle\langle z|)} \ .$$

Let $x \in \mathsf{h}_{\mathbb{R}}, t \in \mathbb{R}$, by composition with $\alpha$ and evaluation on $W(tx)$ we obtain

$$\begin{aligned}(\pi r)^{-\frac{1}{2}} & \int_{\mathbb{R}} \mathrm{e}^{-\frac{s^2}{r}} \mathrm{e}^{itf(s)} \mathrm{d}s \\ &= \mathrm{e}^{it \,\mathrm{Re}\langle x, \alpha_1(S+r|z\rangle\langle z|, z_0)\rangle} \exp\left(-\frac{1}{4} t^2 \,\mathrm{Re} \left\langle x, \left(\alpha_2(S + r \,|z\rangle\langle z|) - \alpha_2(S)\right) x\right\rangle\right) ,\end{aligned}$$

where $f(s) := \mathrm{Re} \langle \alpha_1(s\mu + \mu_0, S), x\rangle$ and we used Lemma 3.7. By Lemma 3.8 we have then that $f$ satisfies $f(s) = f(0) + s(f(1) - f(0))$. Equivalently we get

$$\alpha_1(sz + z_0, S) = \alpha_1(z_0, S) + s \left(\alpha_1(z + z_0, S) - \alpha_1(z_0, S)\right) .$$

Substituting it into the integral equation and performing integration we obtain the following conditions

$$\alpha_1(z_0, S) = \alpha_1(z_0, S + r \,|z\rangle\langle z|) \ , \tag{3.3}$$

$$\begin{aligned}\alpha_2(S) + r \,|\alpha_1(z + z_0, S) - \alpha_1(z_0, S)\rangle\langle\alpha_1(z + z_0, S) - \alpha_1(z_0, S)| \\ = \alpha_2(S + r \,|z\rangle\langle z|) \ .\end{aligned} \tag{3.4}$$

Since the right hand side of (3.4) does not depend on $z_0$ it must hold

$$\alpha_1(z + z_0, S) - \alpha_1(z_0, S) = \alpha_1(z, S) - \alpha_1(0, S) \ . \tag{3.5}$$

subtracting $\alpha_1(0, S)$ on both sides we have the map

$$T_S : \mathsf{h}_{\mathbb{R}} \ni z \mapsto \alpha_1(z, S) - \alpha_1(0, S) \in \mathsf{h}_{\mathbb{R}}$$

is linear. Moreover $T_S$ is also bounded since by definition of $\alpha$ we have $\alpha_1$ is continuous. Now recall that we can write any $S \in \mathcal{Q}$ via its spectral decomposition and, using (3.3) and continuity of $\alpha_1$, we get

$$\alpha_1(0, S_1) = \alpha_1(0, S_1 + S_2) = \alpha_1(0, S_2) =: \mu_0 \in \mathsf{h}_{\mathbb{R}}.$$

In particular we have $T_{S_1} = T_{S_2} =: T_0^*$, since equality between $T_{S_1} = T_{S_2}$ holds for any $S_1, S_2 \in Q$. We can then rewrite (3.4) using $T_0$ and (3.5), obtaining

$$\alpha_2(S) + rT_0 \left|z\right\rangle\!\left\langle z\right| T_0^* = \alpha_2(S + r \left|z\right\rangle\!\left\langle z\right|) \ .$$

Let us define the operator $C$ as

$$C(S) := \alpha_2(S) - T_0^* S T_0 \ ,$$

we then have $C(S) = C(S + r \left|z\right\rangle\!\left\langle z\right|)$. By a similar argument than before we obtain $C(S_1) = C(S_2) =: C_0$, for every $S_1, S_2 \in \mathcal{Q}$. This completes the proof of (3.1). In order to prove (3.2) suppose there exists $z \in \mathsf{h}_{\mathbb{R}}$ such that

$$\langle z, (\mathbf{C_0} - \mathrm{i}\,(\mathbf{J} - \mathbf{T_0^* J T_0}))\, z\rangle < 0 \ .$$

Then we can choose $S \in \mathcal{Q}$ such that

$$\langle \mathbf{T_0} z, (\mathbf{S} - \mathrm{i}\mathbf{J})\, \mathbf{T_0} z\rangle < -\langle z, (\mathbf{C_0} - \mathrm{i}\,(\mathbf{J} - \mathbf{T_0^* J T_0}))\, z\rangle \ .$$

But this is a contradiction since it would imply $\alpha_2(S) \notin \mathcal{Q}$. □

## 3.2 Gaussian Quantum Markov Semigroups

In this section we will extend the definitions and the results obtained in the previous sections to QMSs on $\mathcal{B}(\mathcal{H})$. The Definition 3.3 of a gaussian QMS implicitly says that each map in the predual semigroup $\mathcal{T}_{*t}$ transforms gaussian states into other gaussian states exactly as gaussian maps did. This suggests we can use their characterization theorem and obtain a similar result for $\mathcal{T}_{*t}$. Moreover we can then induce a result on $\mathcal{T}_t$ by exploiting the duality of $\mathcal{T}$ and $\mathcal{T}_*$. We precisely state this in the following proposition.

**Proposition 3.10.** *Let $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ be a gaussian QMS. Then for every $t \geq 0$ there exist $T_t, C_t \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ and $\mu_t \in \mathsf{h}_{\mathbb{R}}$ satisfying*

$$\mathbf{C_t} - \mathrm{i}\,(\mathbf{J} - \mathbf{T_t^* J T_t}) \geq 0 \ , \tag{3.6}$$

*as operators on $\mathfrak{h}_{\mathbb{C}}$, such that*

$$\mathcal{T}_{*t}(\omega_{(\mu,S)}) = \omega_{\left(T_t^\sharp \mu + \zeta_t, T_t^\sharp S T_t + C_t\right)} \ , \tag{3.7}$$

*for every $\mu \in \mathsf{h}_{\mathbb{R}}$ and $S \in Q$. Moreover we have*

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2}\operatorname{Re}\langle z, C_t z\rangle - \mathrm{i}\operatorname{Re}\langle \zeta_t, z\rangle\right) W(T_t z) \ , \tag{3.8}$$

*for every $z \in \mathsf{h}_{\mathbb{R}}$.*

*Proof.* We start by noticing that each $\mathcal{T}_{*t}$ can be extended to all normal functionals on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$. Indeed, as noted in Remark 2.44 gaussian states on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$ are normal and identified with guassian states on $\mathcal{B}(\Gamma_s(\mathsf{h}))$. By norm continuity of $\mathcal{T}_{*t}$ (inherited from the one of $\mathcal{T}$) and Theorem 2.50, $\mathcal{T}_{*t}$ can be extended to $\mathcal{S}(CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma))$ and from it to the whole set of normal functionals on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)$. Let us denote with $\bar{\mathcal{T}}_{*t}$ this extension and note that it is still norm continuous and preserves the set of gaussian states. Therefore each $\bar{\mathcal{T}}_{*t}$ is a gaussian map on $CCR\,(\mathsf{h}_{\mathbb{R}}, \sigma)_*$. We can now apply Theorem 3.9 to show existence of $T_t, C_t \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ and $\zeta_t \in \mathsf{h}_{\mathbb{R}}$ satisfying (3.7) and such that

$$\bar{\mathcal{T}}_{*t}(\omega_{(\mu,S)}) = \omega_{\left(T_t^\sharp \mu + \zeta_t, T_t^\sharp S T_t + C_t\right)} \,,$$

for every $t \geq 0$, $\mu \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{Q}$. But $\bar{\mathcal{T}}_*$ coincides with $\mathcal{T}_*$ on gaussian states and (3.7) follows. Eventually one can see that

$$\begin{aligned}\mathrm{tr}\left(\omega_{(\mu,S)}\mathcal{T}_t(W(z))\right) &= \mathrm{tr}\left(\mathcal{T}_{*t}(\omega_{(\mu,S)})W(z)\right) = \mathrm{tr}\left(\omega_{\left(T_t^\sharp \mu + \mu_t, T_t^\sharp S T_t + C_t\right)} W(z)\right) \\ &= \mathrm{tr}\left(\omega_{(\mu,S)} \exp\left(-\frac{1}{2}\,\mathrm{Re}\,\langle z, C_t z\rangle - \mathrm{i}\,\mathrm{Re}\,\langle \zeta_t, z\rangle\right) W(T_t z)\right),\end{aligned}$$

for every $z, \zeta \in \mathsf{h}_{\mathbb{R}}$ and $S \in \mathcal{Q}$. By Theorem 2.50 the equality can be extended to every normal state and thus to every normal functional. Therefore equation (3.8) holds by duality. ☐

We have therefore restricted the possible transformations that a gaussian QMS can induce on the parameters of a gaussian state to the one of the kind (3.7). This fact also yields an explicit expression for the action of the QMS on Weyl operators. By assuming some regularity conditions and exploiting the fact that $(\mathcal{T}_t)_{t\geq 0}$ is a semigroup we can further specify the action (3.8) and obtain what is usually seen in the literature (see [3, 4, 72]).

**Theorem 3.11.** *Let $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ be a gaussian QMS and let $T_t, C_t \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ and $\zeta_t \in \mathsf{h}_{\mathbb{R}}$ be as given by Proposition 3.10. Suppose $t \mapsto T_t$ is continuous and $t \mapsto C_t$, $t \mapsto \zeta_t$ are differentiable. Then there exists $Z \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ such that*

$$T_t = \mathrm{e}^{tZ} \,, \quad \zeta_t = \int_0^t \mathrm{e}^{sZ^\sharp}\zeta \mathrm{d}s \,, \quad C_t = \int_0^t \mathrm{e}^{sZ^\sharp} C e^{sZ} \mathrm{d}s \,, \tag{3.9}$$

*where*

$$\zeta := \left.\frac{\mathrm{d}}{\mathrm{d}t}\zeta_t\right|_{t=0}, \quad C := \left.\frac{\mathrm{d}}{\mathrm{d}t}C_t\right|_{t=0}.$$

*Moreover, in this case,*

$$\mathbf{C} + \mathrm{i}\,(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}) \geq 0 \,, \tag{3.10}$$

*as operator on $\mathfrak{h}_{\mathbb{C}}$, and, for every $z \in \mathsf{h}_{\mathbb{R}}$, we can write $\mathcal{T}_t(W(z)) = f_t(z)W(\mathrm{e}^{tZ}z)$ where*

$$f_t(z) = \exp\left(-\frac{1}{2}\int_0^t \mathrm{Re}\left\langle z, \mathrm{e}^{sZ^\sharp} C \mathrm{e}^{sZ} z\right\rangle \mathrm{d}s - \mathrm{i}\int_0^t \mathrm{Re}\left\langle \mathrm{e}^{sZ^\sharp}\zeta, z\right\rangle \mathrm{d}s\right). \tag{3.11}$$

*Proof.* Proposition 3.10 allows us to write $\mathcal{T}_t(W(z)) = f_t(z)W(T_t z)$ where

$$f_t(z) = \exp\left(-\frac{1}{2}\operatorname{Re}\langle z, C_t z\rangle - \mathrm{i}\operatorname{Re}\langle \zeta_t, z\rangle\right).$$

Differentiability of $t \mapsto C_t$ and $t \mapsto \zeta_t$ implies differentiability of $f_t(z)$ for every $z \in \mathsf{h}_{\mathbb{R}}$. Since $\mathcal{T}$ is a semigroup it must hold

$$W(T_{t_1+t_2}z)f_{t_1+t_2}(z) = f_{t_2}(z)f_{t_1}(T_{t_2}z)W(T_{t_1}T_{t_2}z)\ , \tag{3.12}$$

for every $t_1, t_2 \geq 0$ and $z \in \mathsf{h}_{\mathbb{R}}$. This is equivalent to the requirements

$$T_{t_1+t_2}z = T_{t_1}T_{t_2}z, \quad f_{t_1+t_2}(z) = f_{t_2}(z)f_{t_1}(T_{t_2}z)\ , \tag{3.13}$$

for every $z \in \mathsf{h}_{\mathbb{R}}, t_1, t_2 \geq 0$. We also have $T_0 = \mathbb{1}$, therefore $(T_t)_{t\geq 0}$ is a uniformly continuous semigroup and $T_t = \exp(tZ)$ where $Z \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ is its generator. Using the second equation of (3.13) we can find the derivative of $f_t(z)$

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t(z) = f_t(z)\ \frac{\mathrm{d}}{\mathrm{d}s}f_s\left(\mathrm{e}^{tZ}z\right)\bigg|_{s=0}. \tag{3.14}$$

We can explicitly evaluate the derivative at $s = 0$ obtaining

$$\frac{\mathrm{d}}{\mathrm{d}s}f_s\left(\mathrm{e}^{tZ}z\right)\bigg|_{s=0} = -\mathrm{i}\operatorname{Re}\left\langle \zeta, \mathrm{e}^{tZ}z\right\rangle - \frac{1}{2}\operatorname{Re}\left\langle \mathrm{e}^{tZ}z, C\mathrm{e}^{tZ}z\right\rangle,$$

since $f_0(\exp(tZ)\,z) = 1$. Solving the differential equation (3.14) for $f_t(z)$ with initial condition $f_0(z) = 1$ yields

$$f_t(z) = \exp\left(-\mathrm{i}\operatorname{Re}\left\langle \int_0^t \mathrm{e}^{sZ^\sharp}\zeta\mathrm{d}s, z\right\rangle - \frac{1}{2}\operatorname{Re}\left\langle z, \int_0^t \mathrm{e}^{sZ^\sharp}C\mathrm{e}^{sZ}z\mathrm{d}s\right\rangle\right). \tag{3.15}$$

This proves (3.9) and (3.11). In order to show (3.10) we use the condition (3.6) for the operators. The quantity involved in (3.6) with the new expressions for $C_t, T_t$ becomes

$$\int_0^t \mathrm{e}^{s\mathbf{Z}^*}\mathbf{C}\mathrm{e}^{s\mathbf{Z}}\mathrm{d}s + \mathrm{i}\left(\mathrm{e}^{t\mathbf{Z}^*}\mathbf{J}\mathrm{e}^{t\mathbf{Z}} - \mathbf{J}\right) = \int_0^t \mathrm{e}^{s\mathbf{Z}^*}\left(\mathbf{C} + \mathrm{i}\left(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}\right)\right)\mathrm{e}^{s\mathbf{Z}}\mathrm{d}s\ .$$

Its positivity for every $t \geq 0$ is equivalent to (3.10). □

This result will be of great importance for the study of many related problems since it gives an explicit action for the semigroup on Weyl operators and differentiability assumptions are very natural. These problems include invariant states for the semigroups, which turn out to be gaussian states (see [4] or Theorem 4.22), and the study of the decoherence free subalgebra (see [3] or Theorem 5.15). We proceed now to showing the converse implication to the one of Theorem 3.11.

## 3.3 Construction of a Gaussian QMS via the generator

In this section we show that, whenever we fix $Z, C \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ and $\zeta \in \mathsf{h}_{\mathbb{R}}$ satisfying (3.10), we can define a QMS via the explicit formula (3.11) which is gaussian. In

[20, 72] it is shown that such a semigroup exists proving at first the maps $\mathcal{T}_t$, defined on the CCR algebra, are completely positive and then extending them to $\mathcal{B}(\mathcal{H})$. Here we follow a different approach using a generalized GKLS generator for the QMS and showing that it generates a QMS that satisfies (3.11).

Recall at first the following definition.

**Definition 3.12.** The *infinitesimal generator* of the QMS $\mathcal{T}$ is the operator $\mathcal{L}$ whose domain $D(\mathcal{L})$ is the vector space of elements $a \in \mathcal{B}(\mathcal{H})$ for which the limit

$$\mathcal{L}(a) = \lim_{t\to 0} \frac{\mathcal{T}_t(a) - a}{t}$$

exists in the $\sigma$-weak topology.

For a uniformly continuous QMS its infinitesimal generator has the so-called *Gorini-Kossakowski-Lindblad-Sudarshan (GKLS)* form:

$$\mathcal{L}(a) = \mathrm{i}\,[H, a] - \frac{1}{2}\left(\sum_{\ell\geq 1} L_\ell^* L_\ell a - 2\sum_{\ell\geq 1} L_\ell^* a L_\ell - \sum_{\ell\geq 1} a L_\ell^* L_\ell\right). \tag{3.16}$$

for some $H, (L_\ell)_{\ell\geq 1} \in \mathcal{B}(\mathcal{H})$.

Let $\Omega, \kappa \in \mathcal{B}(\mathbb{C}^d), \zeta \in \mathbb{C}^d$ and $U, V \in M_{m\times d}(\mathbb{C})$, with $m \leq 2d$, $\Omega = \Omega^*$, $\kappa = \kappa^T$ and $\ker(V^*) \cap \ker(U^T) = \{0\}$. Let us define

$$H = \sum_{j,k=1}^{d} \left(\Omega_{jk} a_j^\dagger a_k + \frac{\kappa_{jk}}{2} a_j^\dagger a_k^\dagger + \frac{\overline{\kappa_{jk}}}{2} a_j a_k\right) + \sum_{j=1}^{d} \left(\frac{\zeta}{2} a_j^\dagger + \frac{\overline{\zeta}}{2} a_j\right), \tag{3.17}$$

$$L_\ell = \sum_{j=1}^{d} \left(u_{\ell j} a_j^\dagger + \overline{v_{\ell j}} a_j\right), \quad \ell = 1, \ldots, m\,. \tag{3.18}$$

We will use these operators to define a generator in the GKLS form. The problem we are facing is that neither $H$ nor $L_\ell$ are bounded operators and we will cope with it in the next subsection. For now we address the motivation for the requirement $\ker(V^*) \cap \ker(U^T) = \{0\}$, which is there to avoid possible redundancies in the set $\{L_\ell : \ell \geq 1\}$.

**Definition 3.13.** A GKLS representation of $\mathcal{L}$ is *mimimal* if the number $m$ in (3.16) is minimal.

A GKLS representation is minimal if and only if the following condition on $V$ and $U$.

**Proposition 3.14.** *The pre-generator $\mathcal{L}$ has a minimal GKLS representation if and only if*

$$\ker\left(V^*\right) \cap \ker\left(U^T\right) = \{0\}. \tag{3.19}$$

*Proof.* The pre-generator (3.16) has a minimal GKLS representation if and only if $\{\mathbb{1}, L_1, \ldots, L_m\}$ is a linearly independent set (see [57], Theorem 30.16), namely, $\alpha_0\mathbb{1} + \sum_{\ell=1}^m \alpha_\ell L_\ell = 0$ for $\alpha_0, \alpha_\ell \in \mathbb{C}$ implies $\alpha_\ell = 0$ for $\ell = 0, 1, \ldots, m$. This identity is equivalent to

$$\alpha_0\mathbb{1} + \sum_{j=1}^{d} (V^*\alpha)_j a_j + \sum_{j=1}^{d} (U^T\alpha)_j a_j^\dagger = 0.$$

Since $\left\{\mathbb{1}, a_1, a_1^\dagger, \dots, a_d, a_d^\dagger\right\}$ is a linearly independent set, the last equation is equivalent to $\alpha \in \ker(V^*), \alpha \in \ker(U^T), \alpha_0 = 0$ and the proof is complete. □

We will assume this condition holds throughout the rest of the thesis.

### 3.3.1 Existence of a QDS

We now construct a QMS using the GKLS generator (3.16) with unbounded operators defined by (3.17), (3.18). We use the minimal semigroup method, which is treated in full details in [31], that starts considering a generalized form of the GKLS generator (3.16). Namely, for every $x \in \mathcal{B}(\mathcal{H})$, and $u, v$ in a suitable dense domain of $\mathcal{H}$ we define the sesquilinear form $\pounds(x)$ via

$$\langle u, \pounds(x)v\rangle = \mathrm{i}\,\langle u, [H,x]\,v\rangle - \frac{1}{2}\sum_{\ell=1}^{m}\left(\langle L_\ell u, L_\ell x v\rangle - 2\,\langle L_\ell u, x L_\ell v\rangle + \langle L_\ell x u, L_\ell v\rangle\right). \tag{3.20}$$

Moreover consider the operator

$$G := -\frac{1}{2}\sum_{\ell\geq 1} L_\ell^* L_\ell + \mathrm{i}H, \tag{3.21}$$

which can be taken as closed (see the proof of Proposition 3.17), we have the following result, whose proof can be find in [31].

**Theorem 3.15.** *Suppose that:*

- *The operator $G$ is the infinitesimal generator of a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ over $\mathcal{H}$,*
- *The domain of the operators $(L_\ell)_{\ell\geq 1}$ contains the domain of $G$ and, for every $u \in D(G)$ we have*

$$\langle u, Gu\rangle + \langle Gu, u\rangle + \sum_{\ell\geq 1}\langle L_\ell u, L_\ell u\rangle \leq 0. \tag{3.22}$$

*Then there exists at least one QDS $\mathcal{T}$ that satisfies*

$$\langle v, \mathcal{T}_t(x)u\rangle = \langle v, xu\rangle + \int_0^t \langle v, \pounds(\mathcal{T}_s(x)u\rangle\,\mathrm{d}s, \quad \forall t\geq 0, u, v \in D(G). \tag{3.23}$$

In order to apply this result in our case we just need to show the two hypotheses. Let us start by recalling the following result due to Palle E.T. Jorgensen

**Theorem 3.16.** *Let $G$ be a dissipative linear operator on a Hilbert space $\mathcal{H}$. Let $(D_n)_{n\geq 1}$ be an increasing family of closed subspaces of $\mathcal{H}$ whose union is dense in $\mathcal{H}$ and contained in the domain of $G$ and let $P_{D_n}$ be the orthogonal projection of $\mathcal{H}$ onto $D_n$. Suppose there exists an integer $n_0$ such that $GD_n \subset D_{n+n_0}$ for all $n \geq 1$.*
*Then the closure $\overline{G}$ generates a strongly continuous contraction semigroup on $\mathcal{H}$ and $\cup_{n\geq 1} D_n$ is a core for $\overline{G}$, if there exists a sequence $(c_n)_{n\geq 1}$ in $\mathbb{R}_+$ such that, for all $n \in \mathbb{N}$*

$$\|GP_{D_n} - P_{D_n}GP_{D_n}\| \leq c_n, \quad \sum_{n=1}^{\infty} c_n^{-1} = \infty.$$

*Proof.* See [48, Theorem 2]. □

Before stating the result of interest recall the definition of the set $D$ (1.20) as introduced in Notation 1.61. This set will be the core for $G$ as stated in Theorem 3.16. We can now prove the following proposition

**Proposition 3.17.** *The operator $\overline{G}$ is the infinitesimal generator of a strongly continuous contraction semigroup on $\mathcal{H}$ and $D$ is a core for this operator.*

*Proof.* We will use Theorem 3.16 with $D_n$ given by (1.20). The operator $G$ is obviously densely defined and dissipative, since $H$ is symmetric. Therefore it is closable (see [13, Lemma 3.1.14]) and its closure, denoted $\overline{G}$ is dissipative. Clearly, by the explicit form of the action of creation and annihilation operators on vectors $e_{(n_1,\dots,n_d)}$, the operator $G$ maps $D_n$ into $D_{n+2}$ for all $n \geq 0$.

A straightforward computation using (3.21)) yields

$$(GP_{D_n} - P_{D_n}GP_{D_n}) = -\frac{1}{2}\sum_{k,j=1}^{d}\left(\sum_{\ell} v_{\ell k}u_{\ell j} + \mathrm{i}\kappa\right) a_k^\dagger a_j^\dagger - \frac{\mathrm{i}}{2}\sum_{k=1}^{d}\zeta_k a_k^\dagger,$$

namely the non-zero part is the one involving only $a^\dagger$s. Let us fix $u = \sum_{|\alpha|\leq n} r_\alpha e_\alpha$ a vector in $D_n$, where $\alpha = (\alpha(1),\dots,\alpha(d))$ is a multi-index, $|\alpha| = \alpha(1) + \ldots + \alpha(d)$, and the vector $e_\alpha^T = (e_{\alpha(1)},\dots,e_{\alpha(d)})$. Clearly $a_j^\dagger u \in D_{n+1}$ and

$$\left\|a_j^\dagger u\right\|^2 \leq \sum_{|\alpha|\leq n} |r_\alpha|^2 (\alpha(j)+1)\left\|e_{\alpha+1_j}\right\|^2 \leq (n+1)\left\|u\right\|^2,$$

where $1_j$ corresponds to the multi-index with all entries 0s but for the $j$-th one which is 1. Therefore also

$$\left\|a_j^\dagger a_k^\dagger u\right\|^2 \leq (n+2)\left\|a_k^\dagger u\right\|^2 \leq (n+2)(n+1)\left\|u\right\|^2 \leq (n+2)^2\left\|u\right\|^2.$$

This means

$$\begin{aligned}\left\|(GP_{D_n} - P_{D_n}GP_{D_n})u\right\| &\leq \frac{1}{2}\sum_{jk}\left\|\left[\sum_{\ell}\left(\overline{V_{\ell\cdot}}\right)^* U_{\ell\cdot} + \mathrm{i}\kappa\right]_{jk} a_j^\dagger a_k^\dagger u\right\| + \sum_j \left\|\zeta_j a^\dagger j u\right\| \\ &\leq c\frac{n+2}{2}\left\|u\right\|\end{aligned}$$

with $c > 0$ a constant that does not depend on $n$. Since $\sum_n (n+2)^{-1}$ diverges we can use Theorem 3.16 and the proposition is proved. □

Eventually we can state the existence result.

**Corollary 3.18.** *There exists a quantum dynamical semigroup $\mathcal{T}$ that satisfied* (3.23) *where $\pounds$ and $G$ are defined by* (3.20) *and* (3.21) *with $H, L_\ell$ given by* (3.17), (3.18).

*Proof.* Proposition 3.17 shows that $G$ satisfies the first hypothesis of Theorem 3.15. Moreover looking at the definition of G (3.21) it easy to see that we can define $L_\ell$ on the domain of $G$ and, again by definition of $G$,

$$\langle u, Gu\rangle + \langle Gu, u\rangle + \sum_{\ell\geq 1}\langle L_\ell u, L_\ell u\rangle = 0.$$

Therefore we can apply Theorem 3.15 and we can construct a quantum dynamical semigroup that satisfies (3.23). □

### 3.3.2 Existence and uniqueness of a QMS

In order to show that $\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$, which is often referred to as the *conservativity* property, we will use again a result from [31].

**Theorem 3.19.** *Suppose that*

- *the operator $G$ is the infinitesimal generator of a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on $\mathcal{H}$,*
- *the domain of the operators $(L_\ell)_{\ell\geq 1}$ contains the domain of $G$ and for every $u \in D(G)$ we have*

$$\langle u, Gu\rangle + \langle Gu, u\rangle + \sum_{\ell\geq 1} \langle L_\ell u, L_\ell u\rangle = 0. \tag{3.24}$$

*and moreover that there exist a self-adjoint operator $C$ with domain $D(G)$ and a core $D$ for $C$ with the following properties*

*a) $L_\ell(D) \subset D(C^{\frac{1}{2}})$ for all $\ell \geq 1$,*

*b) there exists a self-adjoint operator $\phi$ such that*

$$-2\,\mathrm{Re}\,\langle u, Gu\rangle = \langle u, \phi u\rangle \leq \langle u, Cu\rangle\,, \quad \forall u \in D,$$

*c) there exists $b > 0$ such that*

$$2\,\mathrm{Re}\,\langle Cu, Gu\rangle + \sum_{\ell\geq 1} \left\langle C^{\frac{1}{2}} L_\ell u, C^{\frac{1}{2}} L_\ell u\right\rangle \leq b\,\langle u, Cu\rangle\,, \quad \forall u \in D. \tag{3.25}$$

*Then the quantum dynamical semigroup obtained by Theorem 3.15 is a QMS and is the unique QMS satisfying* (3.23).

As noted in the proof of Corollary 3.18 we have already proved the first two hypotheses. It remains to be shown that we can satisfy the other requirements. In order to do this one should proceed with computations on quadratic forms. However, these are equivalent to algebraic computations of the action of the formal generator $\mathcal{L}$ on first and second order polynomials in $a_j, a_j^\dagger$ therefore we will go on with algebraic computations in order to avoid cluttering notation.

**Lemma 3.20.** *It holds*

$$2\mathcal{L}(a_k) = a^\dagger\left(\left(U^T V - V^T U - 2\mathrm{i}\kappa\right)_{k\bullet}\right) + a\left(\left(U^* U - V^* V + 2\mathrm{i}\overline{\Omega}\right)_{k\bullet}\right) - \mathrm{i}\zeta_k \mathbb{1}, \tag{3.26}$$

$$2\mathcal{L}(a_k^\dagger) = a\left(\left(U^T V - V^T U - 2\mathrm{i}\kappa\right)_{k\bullet}\right) + a^\dagger\left(\left(U^* U - V^* V + 2\mathrm{i}\overline{\Omega}\right)_{k\bullet}\right) + \mathrm{i}\overline{\zeta_k} \mathbb{1}. \tag{3.27}$$

*where the notation $A_{k\bullet}$ stands for the vector of entries $(A_{kj})_{j=1}^d$.*

*Proof.* Consider at first

$$\mathcal{L}_0(X) := \sum_{\ell} \left( -\frac{1}{2} L_\ell^* L_\ell X + L_\ell^* X L_\ell - \frac{1}{2} X L_\ell^* L_\ell \right) \tag{3.28}$$

that can easily be rewritten as

$$\mathcal{L}_0(X) = \frac{1}{2} \sum_{\ell} \left( L_\ell^* [X, L_\ell] + [L_\ell^*, X] L_\ell \right). \tag{3.29}$$

By the CCR (1.11) one can evaluate

$$[a_k, L_\ell] = \sum_j [a_k, u_{\ell j} a_j^\dagger] = u_{\ell k} \mathbb{1}, \quad [L_\ell^*, a_k] = [a_k^\dagger, L_\ell]^* = \sum_j [a_k^\dagger, \overline{v}_{\ell j} a_j]^* = -v_{\ell k} \mathbb{1}$$

and obtain

$$\mathcal{L}_0(a_k) = \frac{1}{2} \sum_{\ell} \left\{ \left( \sum_j v_{\ell j} a_j^\dagger + \overline{u}_{\ell j} a_j \right) u_{\ell k} - v_{\ell k} \left( \sum_j \overline{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \right\}. \tag{3.30}$$

Using again the CCR (1.11) one can calculate

$$[H, a_k] = -\frac{\mathrm{i}\zeta_k}{2} \mathbb{1} - \sum_{j=1}^{d} \left( \Omega_{kj} a_j + \kappa_{kj} a_j^\dagger \right). \tag{3.31}$$

that together with (3.30) leads to

$$\begin{aligned}
\mathcal{L}(a_k) &= \mathrm{i}[H, a_k] + \mathcal{L}_0(a_k) \\
&= -\frac{\zeta_k}{2} \mathbb{1} - \mathrm{i} \sum_{j=1}^{d} \left( \Omega_{kj} a_j + \kappa_{kj} a_j^\dagger \right) \\
&+ \frac{1}{2} \sum_{\ell=1}^{m} \left\{ \left( \sum_{j=1}^{d} v_{\ell j} a_j^\dagger + \overline{u}_{\ell j} a_j \right) u_{\ell k} - v_{\ell k} \left( \sum_{j=1}^{d} \overline{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \right\}
\end{aligned}$$

Using the last equality and $\mathcal{L}(a_k^\dagger) = \mathcal{L}(a_k)^*$ we conclude the proof. □

We recall now the following Lemma which holds for any generator in the GKLS form (3.16).

**Lemma 3.21.** *Let $\mathcal{L}$ be given as* (3.20)*, for some $H$, $L_\ell$. Then, for $x, y \in \mathcal{B}(\mathcal{H})$,*

$$\mathcal{L}(xy) = x\mathcal{L}(y) + \mathcal{L}(x)y + \sum_{\ell \geq 1} [L_\ell, x^*]^* [L_\ell, y]. \tag{3.32}$$

*Proof.* Let be $\mathcal{L}_0$ as per (3.28). For $x, y \in \mathcal{B}(\mathcal{H})$ we can have

$$\begin{aligned}
\mathcal{L}_0(xy) - x\mathcal{L}_0(y) - \mathcal{L}(x)y &= \sum_{\ell \geq 1} \left( L_\ell^* xy L_\ell + x L_\ell^* L_\ell y - x L_\ell^* y L_\ell - L_\ell^* x L_\ell y \right) \\
&= \sum_{\ell \geq 1} \left( [L_\ell^*, x]\, y L_\ell - [L_\ell^*, x]\, L_\ell y \right) \\
&= -\sum_{\ell \geq 1} [L_\ell^*, x]\, [L_\ell, y] = \sum_{\ell \geq 1} [L_\ell, x^*]^*\, [L_\ell, y]\,.
\end{aligned}$$

Moreover

$$[H, xy] = Hxy - xyH + xHy - xHy = [H, x]\, y + x\, [H, y]$$

and so

$$\mathcal{L}(xy) - x\mathcal{L}(y) - \mathcal{L}(x)y = \mathcal{L}_0(xy) - x\mathcal{L}_0(y) - \mathcal{L}_0(x)y = \sum_{\ell} \left( [L_\ell, x^*]^* \, [L_\ell, y] \right).$$

This completes the proof. □

The following proposition is a first step towards proving conservativity via Theorem 3.19.

**Proposition 3.22.** *Let* $C = \sum_{k=1}^{d} a_k a_k^\dagger$. *There exist a constant* $b > 0$ *such that*

$$\mathcal{L}(C) \leq bC.$$

*Proof.* By Lemma 3.20 we have that

$$\mathcal{L}(a_k) = \sum_{j=1}^{d} \left( w_{kj} a_j^\dagger + z_{kj} a_j \right) - \frac{\mathrm{i}\zeta_k}{2}\mathbb{1}, \quad \mathcal{L}(a_k^\dagger) = \sum_{j=1}^{d} \left( \overline{w}_{kj} a_j + \overline{z}_{kj} a_j^\dagger \right) + \frac{\mathrm{i}\overline{\zeta}_k}{2}\mathbb{1}$$

for some complex numbers $w_{kj}, z_{kj}, \zeta_j$. While, by Lemma 3.21, we get

$$\begin{aligned}
\mathcal{L}(a_k a_k^\dagger) &= -\mathrm{i}\overline{\zeta}_k a_k/2 + \sum_{j=1}^{d} \left( \overline{w}_{kj} a_k a_j + \overline{z}_{kj} a_k a_j^\dagger \right) \\
&\quad + \mathrm{i}\zeta_k a_k^\dagger/2 + \sum_{j=1}^{d} \left( w_{kj} a_j^\dagger a_k^\dagger + z_{kj} a_j a_k^\dagger \right) + \|v_{\bullet k}\|^2 \mathbb{1} \\
&= \sum_{j=1}^{d} \left( \overline{z}_{kj} a_k a_j^\dagger + z_{kj} a_j a_k^\dagger + \overline{w}_{kj} a_k a_j + w_{kj} a_j^\dagger a_k^\dagger \right) \\
&\quad + \frac{\mathrm{i}}{2} \left( \zeta_k a_k^\dagger - \overline{\zeta}_k a_k \right) + \|v_{\bullet k}\|^2 \mathbb{1},
\end{aligned}$$

where $v_{\bullet k}$ stand for the vector of entries $(v_{\ell k})_{\ell=1}^{m}$. Note that for each $k, j$ we have

$$\begin{aligned}
\left| a_j^\dagger - z_{kj} a_k^\dagger \right|^2 &= a_j a_j^\dagger + |z_{jk}|^2 a_k a_k^\dagger - \overline{z}_{kj} a_k a_j^\dagger - z_{kj} a_j a_k^\dagger \geq 0, \\
\left| w_{kj} a_k^\dagger - a_j \right|^2 &= |w_{kj}|^2 a_k a_k^\dagger + a_j^\dagger a_j - \overline{w}_{kj} a_k a_j - w_{kj} a_k^\dagger a_j^\dagger \geq 0, \\
\left| a_k^\dagger + \mathrm{i}\zeta_k \mathbb{1} \right|^2 &= a_k a_k^\dagger + |\zeta_k|^2 \mathbb{1} - \mathrm{i}\zeta_k a_k^\dagger + \mathrm{i}\overline{\zeta}_k a_k \geq 0.
\end{aligned}$$

That respectively lead to

$$\overline{z}_{kj} a_k a_j^\dagger + z_{kj} a_j a_k^\dagger \leq a_j a_j^\dagger + |z_{jk}|^2 a_k a_k^\dagger, \tag{3.33}$$

$$\overline{w}_{kj} a_k a_j + w_{kj} a_k^\dagger a_j^\dagger \leq |w_{kj}|^2 a_k a_k^\dagger + a_j a_j^\dagger - \mathbb{1}, \tag{3.34}$$

$$\mathrm{i}\zeta_k a_k^\dagger - \mathrm{i}\overline{\zeta}_k a_k \leq a_k a_k^\dagger + |\zeta_k|^2 \mathbb{1}. \tag{3.35}$$

Using (3.33),(3.34), and (3.35) in the expression for $\pounds(a_j a_j^\dagger)$ we get

$$\pounds(C) \le \left(3d \max_{1\le k\le d}\left(1+\sum_{j=1}^{d}(|z_{kj}|^2+|w_{kj}|^2)\right)C+\sum_{j=1}^{d}\left(|\zeta_j|^2+\|v_{\bullet j}\|^2\right)\mathbb{1}\right)$$

since $C \ge d\mathbb{1}$ then $\pounds(C) \le bC$ with

$$b=\max\left\{3d\max_{1\le k\le d}\left(\sum\sum_{j=1}^{d}(|z_{kj}|^2+|w_{kj}|^2)\right),\sum_{j=1}^{d}\left(|\zeta_j|^2+\|v_{\bullet j}\|^2\right)\right\}.$$

□

Now consider the following result.

**Proposition 3.23.** *The closure $\Phi$ of the operator on $\mathcal{H}$ with domain $D$ defined by*

$$\Phi=\sum_{\ell=1}^{m}L_\ell^* L_\ell, \tag{3.36}$$

*is essentially self-djoint. Moreover, if $C$ is the operator introduced in Proposition 3.22, there exists a constant $\lambda > 1$ such that*

$$\langle u,\Phi u\rangle \le \lambda\langle u, Cu\rangle, \quad \forall u\in D, \tag{3.37}$$

*where $D$ is the set given by* (1.20).

*Proof.* We will prove essential self-adjointness of $\Phi$ by using Nelson's analytic vector theorem [64, Theorem X.39]. Indeed $\Phi$ is clearly symmetric and it is a second order polynomial in $a, a^\dagger$, in particular we have

$$\Phi=\sum_{\ell=1}^{m}\sum_{j,k=1}^{d}\left(v_{\ell j}\overline{v_{\ell k}}a_j^\dagger a_k+\overline{u_{\ell j}}u_{\ell k}a_j a_k^\dagger+v_{\ell j}u_{\ell k}a_j^\dagger a_k^\dagger+\overline{u_{\ell j}}\overline{v_{\ell k}}a_j a_k\right).$$

Note that it holds

$$\begin{aligned}\sum_{j,k=1}^{d}u_{\ell j\overline{\ell k}}a_j a_k^\dagger &= \frac{1}{2}\sum_{j,k=1}^{d}\left(v_{\ell j}\overline{v_{\ell k}}a_j a_k^\dagger+v_{\ell j}\overline{v_{\ell k}}a_j a_k^\dagger\right)\\ &=\frac{1}{2}\sum_{j,k=1}^{d}\left(v_{\ell j}\overline{v_{\ell k}}a_j a_k^\dagger+\overline{v_{\ell j}}v_{\ell k}a_k a_j^\dagger\right).\end{aligned}$$

Applying this to the expression for $\Phi$ and switching the order of summation we can therefore find $w_{jk}, z_{jk}, \in \mathbb{C}$ such that

$$\Phi=\sum_{j,k=1}^{d}\left(\overline{z_{kj}}a_k a_j^\dagger+z_{kj}a_j a_k^\dagger+\overline{w_{kj}}a_k a_j+w_{kj}a_j^\dagger a_k^\dagger\right)-\sum_{\ell=1}^{d}\|v_{\ell\cdot}\|^2\,\mathbb{1}.$$

Using again inequalities (3.33), (3.34) and following a similar reasoning to the proof of Proposition 3.22 we obtain (3.37).

For every $\alpha$ multi-index and $k \in \mathbb{N}$ we have

$$\left\langle e_\alpha, C^k e_\alpha \right\rangle = |\alpha| + d,$$

therefore

$$\sum_{k\in\mathbb{N}} \frac{\left\|\Phi^k e_\alpha\right\|^2}{k!} \leq \sum_{k\in\mathbb{N}} \frac{\lambda^k \left\|C^k e_\alpha\right\|}{k!} < \infty.$$

This means we can use Nelson's theorem and conclude the proof. □

Now we can eventually state the result on conservativity.

**Corollary 3.24.** *The QDS given by Corollary 3.18 is Markov and is the unique QDS satisfying* (3.23).

*Proof.* As stated at the beginning of the section we will use Theorem 3.19. We have already shown in the proof of Corollary 3.18 that the first two hypothesis hold. In order to prove the remaining conditions we choose the operator $C$ given by

$$D(C) = \left\{ u = \sum_\alpha u_\alpha e_\alpha \mid \sum_\alpha |\alpha|^2 |u_\alpha|^2 < \infty \right\}, \qquad Cu = \lambda \sum_{j=1}^d a_j a_j^\dagger u.$$

and the core $D$ as in the (1.20). In this way (a) is easily satisfied, while (c) follows from Proposition 3.22. Eventually consider $\phi$ as the closure of the operator given in(3.36), which is self-adjoint by Proposition 3.23. By the same proposition we also have (b). Therefore the proof is complete. □

**Definition 3.25.** We say the QMS given by Theorem 3.24 is the *Gaussian QMS associated with* $H$ and $(L_\ell)$.

We use this temporary definition to distinguish a gaussian QMS introduced via the property of preservation of gaussian states, as in Definition 3.4, and those we just constructed with the minimal QDS method. We will show that these two definitions actually are the same.

**Theorem 3.26.** *Let* $(\mathcal{T}_t)_{t\geq 0}$ *be the gaussian QMS associated with* $H, L_\ell$ *of* (3.17), (3.18) *. For all* $z \in \mathsf{h}_\mathbb{R}$ *we have*

$$\mathcal{T}_t(W(z)) = \exp\left(-\frac{1}{2}\int_0^t \mathrm{Re}\left\langle \mathrm{e}^{sZ}z, C\mathrm{e}^{sZ}z\right\rangle \mathrm{d}s + \mathrm{i}\int_0^t \mathrm{Re}\left\langle \zeta, \mathrm{e}^{sZ}z\right\rangle \mathrm{d}s\right) W\left(\mathrm{e}^{tZ}z\right) \tag{3.38}$$

*where the* $Z, C \in \mathcal{B}_\mathbb{R}(\mathsf{h}_\mathbb{R})$ *are given by*

$$Zz = \left[\left(\overline{U^*U - V^*V}\right)/2 + \mathrm{i}\Omega\right] z + \left[\left(U^T V - V^T U\right)/2 + \mathrm{i}\kappa\right] \overline{z} \tag{3.39}$$

$$Cz = \left(\overline{U^*U + V^*V}\right) z + \left(U^T V + V^T U\right) \overline{z} \tag{3.40}$$

*Proof.* As before, all the calculations are rigorous when considering the operators involved as quadratic form on exponential vectors. Here however we perform purely algebraic calculations as to avoid clutter of the notation. Let's rewrite the generator as

$$\mathcal{L}(W(z)) = \mathrm{i}\left[H, W(z)\right] + \frac{1}{2}\sum_{l=1}^m \left(L_l^* \left[W(z), L_l\right] + \left[L_l^*, W(z)\right] L_l\right).$$

Recalling that $\Omega = \Omega^*, \kappa = \kappa^T$ and through some straightforward calculations using (1.14) one gets

$$[W(z), L_l] = -W(z)\left(\overline{V}z + U\overline{z}\right)_l, \quad [L_l^*, W(z)] = W(z)\left(V\overline{z} + \overline{U}z\right)_l$$

Moreover we have

$$\begin{aligned}
[H, W(z)] &= W(z)\left[a\left(\Omega z + \kappa\overline{z}\right) + a^\dagger\left(\Omega z + \kappa\overline{z}\right) + \mathrm{Re}\left\langle\zeta, z\right\rangle\right] \\
&\quad + W(z)\left[\frac{1}{2}\left\langle z, \Omega z + \kappa\overline{z}\right\rangle + \frac{1}{2}\overline{\left\langle z, \Omega z + \kappa\overline{z}\right\rangle}\right], \\
\sum_{l=1}^m [L_l^*, [W(z), L_l]] &= -\sum_{l=1}^m W(z)\left(\overline{V}z + U\overline{z}\right)_l\left(V\overline{z} + \overline{U}z\right)_l \\
&= -W(z)\left(\left\langle z, \overline{V^*V}z + V^TU\overline{z}\right\rangle + \overline{\left\langle z, \overline{U^*U}z + U^TV\overline{z}\right\rangle}\right).
\end{aligned}$$

Using the previous results one finds that $\mathcal{L}(W(z)) = W(z)X(z)$ for some operator $X(z)$ which is explicitly given by

$$\begin{aligned}
X(z) =& a^\dagger\left(\left(\frac{\overline{U^*U - V^*V}}{2} + \mathrm{i}\Omega\right)z + \left(\frac{U^TV - V^TU}{2} + \mathrm{i}\kappa\right)\overline{z}\right) \\
&- a\left(\left(\frac{\overline{U^*U - V^*V}}{2} + \mathrm{i}\Omega\right)z + \left(\frac{U^TV - V^TU}{2} + \mathrm{i}\kappa\right)\overline{z}\right) \\
&+ \frac{1}{2}\left\langle z, \mathrm{i}\Omega z + \mathrm{i}\kappa\overline{z}\right\rangle - \frac{1}{2}\overline{\left\langle z, \mathrm{i}\Omega z + \mathrm{i}\kappa\overline{z}\right\rangle} + \mathrm{i}\,\mathrm{Re}\left\langle\zeta, z\right\rangle \\
&- \frac{1}{2}\left(\left\langle z, \overline{V^*V}z + V^TU\overline{z}\right\rangle + \overline{\left\langle z, \overline{U^*U}z + U^TV\overline{z}\right\rangle}\right).
\end{aligned}$$

Let's evaluate the derivative of (3.38) at $t = 0$. Using

$$\frac{\mathrm{d}}{\mathrm{dt}}W(e^{tZ}) = W(z)\sum_j\left((Zz)_j a_j^\dagger - (\overline{Zz})_j a_j + \frac{1}{2}\left(\overline{z_j}(Zz)_j - (\overline{Zz})_j z_j\right)\right)$$

one again has $\mathcal{L}(W(z)) = W(z)Y(z)$, where

$$\begin{aligned}
Y(z) = &\sum_j\left((Zz)_j a_j^\dagger - (\overline{Zz})_j a_j + \frac{1}{2}\left(\overline{z_j}(Zz)_j - (\overline{Zz})_j z_j\right)\right) \\
&- \frac{1}{2}\mathrm{Re}\left\langle z, Cz\right\rangle + \mathrm{i}\,\mathrm{Re}\left\langle\zeta, z\right\rangle.
\end{aligned}$$

Since $X$ and $Y$ should coincide for every $z \in \mathbb{C}^d$ the Theorem follows. □

Corollary 3.24 and Theorem 3.26 show that we can construct a QMS satisfying a relationship similar to (3.11), starting from some unbounded generator. The remaining step for the converse result to Theorem 3.11 is to prove that via the generator approach we can recover all possible gaussian QMS that satisfy the hypothesis of that Theorem.

We start with the following Lemma

**Lemma 3.27.** *Let $Z, C$ be the operator given by Theorem 3.26.*

1. *We can write*

$$C = \sqrt{C}^{\sharp}\sqrt{C} \geq 0,$$

*where* $\sqrt{C}z := \overline{U}z + V\overline{z}$ *for* $z \in \mathbb{C}^d$.

2. *It holds*

$$\mathbf{C} + \mathrm{i}\,(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}) \geq 0, \tag{3.41}$$

*and the inequality holds strictly if and only if we have exactly* $m = 2d$ *Kraus' operators.*

*Proof.* The decomposition of item 1 holds by direct computation and the inequality holds consequentially. In order now to prove 2 note that

$$\mathbf{C} + \mathrm{i}\,(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}) = 2\begin{pmatrix} \overline{U^*U} & \overline{U^*\overline{V}} \\ V^*\overline{U} & V^*V \end{pmatrix} = \begin{pmatrix}\overline{U} & V\end{pmatrix}^* \begin{pmatrix}\overline{U} & V\end{pmatrix} \geq 0, \tag{3.42}$$

where $(\overline{U} \quad V)$ is a linear operator from $\mathbb{C}^{2d}$ to $\mathbb{C}^m$. From condition (3.19) we have

$$\mathrm{Ran}\begin{pmatrix}\overline{U} & V\end{pmatrix}^{\perp} = \ker\left(\begin{pmatrix}\overline{U} & V\end{pmatrix}^*\right) = \{0\},$$

therefore $\dim(\mathrm{Ran}\begin{pmatrix}\overline{U} & V\end{pmatrix}) = m$. This leads to $\dim(\ker\begin{pmatrix}\overline{U} & V\end{pmatrix}) = 2d - m$, which implies the inequality of item 2 holds strictly if and only if $m = 2d$. □

Eventually we can show the anticipated converse result.

**Theorem 3.28.** *Let* $Z, C \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ *satisfy* (3.41). *Then there exist* $m \leq 2d$, $U, V \in M_{m\times d}(\mathbb{C})$, $\Omega, \kappa \in M_d(\mathbb{C})$ *such that* $Z, C$ *coincide with the operators given by* (3.39) *and* (3.40).
*In particular there exists a gaussian QMS* $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ *associated with some* $H, (L_\ell)$ *such that* $\mathcal{T}_t$ *satisfies* (3.11) *for every* $t \geq 0$.

*Proof.* As noted before the only missing step for the proof is to show that there is a suitable choice of $U, V, \Omega, \kappa$ such that $Z, C$ are given by formulae (3.39), (3.40). Every real linear operator, as a linear operator on the Hilbert space $\mathsf{h}_{\mathbb{R}}$, can be uniquely decomposed into the sum of a self-adjoint and anti-self-adjoint part. In particular there exist $A, B \in \mathcal{B}_{\mathbb{R}}(\mathsf{h}_{\mathbb{R}})$ such that, for every $z \in \mathsf{h}_{\mathbb{R}}$,

$$Zz = Az + Bz = (A_1 z + A_2\overline{z}) + (B_1 z + B_2\overline{z})\ , \quad C = C_1 z + C_2\overline{z}\ , \tag{3.43}$$

where $A = A^{\sharp}, B = -B^{\sharp}$ and therefore $A_1 = A_1^*, A_2 = A_2^T$, $B_1 = -B_1^*, B_2 = -B_2^T$, while $C_1 = C_1^*, C_2 = C_2^T$. Comparing (3.43) with (3.39), (3.40), by the uniqueness of the decomposition we get $\Omega = -\mathrm{i}B_1, \kappa = -\mathrm{i}A_2$ and

$$U^*U = \overline{A_1} + \frac{\overline{C_1}}{2}\ , \quad V^*V = \frac{\overline{C_1}}{2} - \overline{A_1}\ , \quad U^TV = \frac{C_2}{2} + B_2\ . \tag{3.44}$$

To conclude the proof we will now show there is a suitable choice of $U, V$ that satisfies (3.44). Since $Z, C$ satisfy (3.41) there is $\mathbf{x_0} \in \mathcal{B}(\mathfrak{h}_{\mathbb{C}})$ such that

$$\mathbf{C} + \mathrm{i}\,(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}) = \mathbf{x_0}^*\mathbf{x_0} \geq 0\ . \tag{3.45}$$

Let $m := \dim \mathbf{x_0}(\mathfrak{h}_{\mathbb{C}})$, by restricting $\mathbf{x_0}$ to its range we can assume $\mathbf{x_0} \in M_{m\times 2d}(\mathbb{C})$ and it still satisfies (3.45). Via direct computation, using (3.43), we get

$$\mathbf{x_0}^*\mathbf{x_0} = \mathbf{C} + \mathrm{i}\left(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}\right) = \begin{pmatrix} C_1 + 2A_1 & C_2 + 2B_2 \\ \overline{C_2 - 2B_2} & \overline{C_1 - 2A_1} \end{pmatrix}.$$

On the other hand for a Gaussian QMS, from (3.42), we have

$$\mathbf{C} + \mathrm{i}\left(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}\right) = 2\left(\overline{U}, \quad V\right)^*\left(\overline{U}, \quad V\right),$$

where $(\overline{U}, V) \in M_{m\times 2d}(\mathbb{C})$. Therefore it is sufficient to choose the matrices $U, V$ such that $(\overline{U}, V) = \mathbf{x_0}/\sqrt{2}$ in order to satisfy (3.44) and conclude the proof of the theorem. □

In particular we can explicitly obtain the evolution for the parameters of a gaussian state induced from a gaussian QMS associated with some $H, (L_\ell)$.

**Proposition 3.29.** *Let $(\mathcal{T}_t)_{t\geq 0}$ be the quantum Markov semigroup with GKSL generator associated with $H, L_\ell$ as in* (3.17), (3.18). *If $\rho = \rho_{(z_0,S_0)}$ is a gaussian state then $\rho_t := \mathcal{T}_{*t}(\rho)$ is still a Gaussian state for every $t \geq 0$ and*

$$\rho_t = \rho_{(z_t,S_t)} = \rho_{(T_t^* z_0 + \zeta_t, T_t^* S_0 T_t + C_t)}$$

*where*

$$T_t = \mathrm{e}^{tZ}, \quad \zeta_t = \int_0^t \mathrm{e}^{sZ^\sharp}\zeta \mathrm{d}s, \quad C_t = \int_0^t \mathrm{e}^{sZ^\sharp} C \mathrm{e}^{sZ},$$

*and $Z, C$ are given by Theorem 3.38. In particular*

$$z_t = \mathrm{e}^{tZ^\sharp} z_0 - \int_0^t \mathrm{e}^{sZ^T}\zeta \mathrm{d}s \tag{3.46}$$

$$S_t = \mathrm{e}^{tZ^\sharp} S_0 \mathrm{e}^{tZ} + \int_0^t \mathrm{e}^{sZ^\sharp} C \mathrm{e}^{sZ} \mathrm{d}s. \tag{3.47}$$

*Proof.* Applying the explicit formula (3.38) of Theorem 3.26 we can write, for $z \in \mathbb{C}^d$,

$$\begin{aligned}\hat{\rho}_t(z) &= \mathrm{tr}(\rho \mathcal{T}_t(W(z))) \\ &= \exp\left(-\frac{1}{2}\mathrm{Re}\left\langle z, \int_0^t \mathrm{e}^{sZ^T} C \mathrm{e}^{sZ} z \mathrm{d}s\right\rangle - \frac{1}{2}\mathrm{Re}\left\langle z, \mathrm{e}^{tZ^T} S_0 \mathrm{e}^{tZ} z\right\rangle\right) \\ &\cdot \exp\left(\mathrm{i}\,\mathrm{Re}\left\langle \int_0^t \mathrm{e}^{sZ^T}\zeta \mathrm{d}s, z\right\rangle - \mathrm{i}\,\mathrm{Re}\left\langle \mathrm{e}^{tZ^T}\omega_0, z\right\rangle\right).\end{aligned}$$

Comparing the previous equation with (2.20) we find (3.47) and (3.46). Now for $S_t$ to be a suitable covariance matrix it should hold $\mathbf{S}_t - \mathrm{i}\mathbf{J} \geq 0$. Indeed, using $\mathbf{S}_0 - \mathrm{i}\mathbf{J} \geq 0$ and Lemma 3.27, one gets

$$\begin{aligned}\mathbf{S}_t - \mathrm{i}\mathbf{J} &\geq \int_0^t \mathrm{e}^{s\mathbf{Z}^T}\mathbf{C} e^{s\mathbf{Z}} ds + \mathrm{e}^{t\mathbf{Z}^T}\mathrm{i}\mathbf{J}\mathrm{e}^{t\mathbf{Z}} - \mathrm{i}\mathbf{J} \\ &= \int_0^t \mathrm{e}^{s\mathbf{Z}^T}\left(\mathbf{C} + \mathrm{i}\left(\mathbf{Z}^T\mathbf{J} + \mathbf{J}\mathbf{Z}\right)\right)\mathrm{e}^{s\mathbf{Z}} \mathrm{d}s \geq 0.\end{aligned}$$

Note that all the operators in the previous inequality were considered as complex linear and therefore commutation with i was legit. □

We provide now a final result that summarizes the equivalent definitions introduced in this chapter.

**Theorem 3.30.** *Let $\mathcal{T}$ be a QMS, the following conditions are equivalent*

*(i) $\mathcal{T}$ is a gaussian QMS that satisfies the regularity condition of Theorem 3.11;*

*(ii) $\mathcal{T}$ is a gaussian QMS associated with some $H, (L_\ell)$;*

*(iii) $\mathcal{T}$ satisfies* (3.11) *for some $Z, C \in \mathcal{B}_{\mathbb{R}}\mathsf{h}_{\mathbb{R}}$ that satisfy*

$$\mathbf{C} + \mathrm{i}\,(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}) \geq 0.$$

*Proof.* From Theorem 3.11 we immediately obtain that (i) implies (iii).

From Theorem 3.28 we have instead that (iii) implies (ii).

Eventually Proposition 3.29 shows that (ii) implies (i). □

# CHAPTER 4

# Gaussian QMSs on the one-mode Fock Space

In this chapter we deal with the study of irreducibility and invariant states for a gaussian QMS that acts on $\mathcal{B}(\Gamma_s(\mathbb{C}))$, that corresponds to $d = 1$ in the general case. The low dimensionality allows for simplification of the mathematical calculations and therefore for the obtainment of explicit results. In this case the operators $H, (L_\ell)_{\ell\geq 1}$ of the GKLS generator can be rewritten as

$$H = \Omega a^\dagger a + \frac{\kappa}{2}(a^\dagger)^2 + \frac{\overline{\kappa}}{2}a^2 + \frac{\zeta}{2}a^\dagger + \frac{\overline{\zeta}}{2}a, \quad L_1 = u_1 a^\dagger + \overline{v_1}a, \quad L_2 = u_2 a^\dagger + \overline{v_2}a,$$

where $\Omega \in \mathbb{R}$ and we dropped the subscript for creation and annihilation operators, since there is only one mode. Even though we introduced to different Kraus' operators we still allow for the case $u_2 = v_2 = 0$ or the case in which there is only one Kraus' operator. Indeed in the first section we deal with irreducibility in the case of exactly two Kraus' operators, while in the second section we study irreducibility with just one of them. In the third section we consider instead the problems of finding invariant states for a gaussian QMS. In the final section we apply these results to two explicit models. The entire chapter is based upon [4]

## 4.1 Irreducibility: the case of two noise operators

We start the section by recalling two definitions

**Definition 4.1.** We recall that an element $p \in \mathcal{B}(\mathcal{H})$ is a *projection* if it is self-adjoint and $p^2 = p$. Moreover, if $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ is a QMS, we say $p$ is *subharmonic*(resp. *superharmonic*) if $\mathcal{T}_t(p) \geq p$ (resp. $\mathcal{T}_t(p) \leq p$) for every $t \geq 0$.

**Definition 4.2.** A QMS $\mathcal{T}$ on $\mathcal{B}(\mathcal{H})$ is called *irreducible* if there exists no non-trivial subharmonic projection $p$.

In the study of the evolution of an open quantum system irreducibility plays a key role because it guarantees that there is no proper subsystem which is invariant under the evolution. Therefore the system has to be regarded as a whole and reduction to subsystems is not possible. In addition, irreducibility is a key assumption of many results on the asymptotic behaviour of QMS (see [40]) and irreducible subsystems constitute the building blocks in the analysis of the structure of normal invariant states of a QMS (see [26]).

In this section we show that the Gaussian QMS with two linearly independent noise operators $L_1, L_2$ is irreducible. Gaussian QMS with only one operator $L$ will be considered in Section 4.2.

In both cases we will use this useful characterization of subharmonic projection, whose proof can be found in [33, Theorem III.1].

**Theorem 4.3.** *Suppose Hypothesis* **AA** *hold and suppose the minimal quantum dynamical semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ is Markov. Then a projection $p$ is subharmonic for $\mathcal{T}$ if and only if the range $\mathrm{Rg}(p)$ of $p$ is invariant for the operators $(P_t)_{t\geq 0}$ and*

$$L_\ell u = pL_\ell u, \tag{4.1}$$

*for all $u \in D(G) \cap \mathrm{Rg}(p)$, and all $\ell \geq 1$.*

In view of this characterization of subharmonic projections, it is now intuitively clear that, if there are two linearly independent Kraus operators, the range of a subharmonic projection should be an invariant subspace for $a$ and $a^\dagger$ and so it will be trivial by irreducibility of the Fock representation of the CCR. However, the necessary clarifications on operator domains are now in order.

Let $G_0$ be the operator defined on the domain $D$ (1.20) by

$$G_0 = -\frac{1}{2}\sum_{\ell=1}^{2} L_\ell^* L_\ell = -\frac{\Phi}{2},$$

where $\Phi$ is the operator introduced in Proposition 3.23. By the same result the closure of $G_0$ is a self-adjoint operator. The following is the key result on the domain of the operator $G$ that we need for proving irreducibility. Recall that the number operator is defined as

$$N = a^\dagger a.$$

**Theorem 4.4.** *If there are two linearly independent noise operator $L_1, L_2$ the domains of the operators $G$ and $G_0$ coincide with the domain of the number operator $N$.*

We defer the proof to the next subsection and proceed to the main result of this section. Note that the property $D(G) = D(G_0) = D(N)$ plays a key role in the proof.

**Theorem 4.5.** *The gaussian QMS with generalized GKSL generator associated with $H$ as in (3.17) and two linearly independent noise operator $L_1, L_2$ as in (3.18) is irreducible.*

*Proof.* Let $\mathcal{V}$ be a non-zero closed subspace of $\mathcal{H}$ which is invariant for the contraction operators $P_t$ of the semigroup generated by $G$ and $L_\ell\,(D(G) \cap \mathcal{V}) \subset \mathcal{V}$ for $\ell = 1, 2$.

By the linear independence of $L_1, L_2$, since $D(G) = D(N)$ we have also

$$a\,(D(N) \cap \mathcal{V}) \subset D(N^{1/2}) \cap \mathcal{V} \qquad a^\dagger\,((N) \cap \mathcal{V}) \subset D(N^{1/2}) \cap \mathcal{V}$$
$$a^\dagger a\,(D(N) \cap \mathcal{V}) \subset \mathcal{V} \qquad aa^\dagger\,(D(N) \cap \mathcal{V}) \subset \mathcal{V}$$

hence, denoting by $p$ the orthogonal projection onto $\mathcal{V}$,

$$p^\perp a p = 0 = p a p^\perp \qquad p^\perp a^\dagger p = 0 = p a^\dagger p^\perp$$

on $D(N)$ and, left multiplying by $a^\dagger$ the first identity,

$$p^\perp a^\dagger a p = 0 = p a^\dagger a p^\perp.$$

It follows that, for all $\lambda > 0$, we have the commutation $(\lambda \mathbb{1} + N)\,p = p\,(\lambda \mathbb{1} + N)$ and, left and right multiplication by the resolvent $(\lambda \mathbb{1} + N)^{-1}$ yields

$$p\,(\lambda \mathbb{1} + N)^{-1} = (\lambda \mathbb{1} + N)^{-1}\,p.$$

In particular, for all $k > 0$, considering the bounded Yosida approximations of $N$ $N_k = kN\,(k\mathbb{1} + N)^{-1}$, that converge strongly to $N$ on $D(N)$ (see [29, Lemma II.3.4]) we have

$$p\,kN\,(k\mathbb{1} + N)^{-1} = kN\,(k\mathbb{1} + N)^{-1}\,p$$

and so $p\,\mathrm{e}^{-tN_k} = \mathrm{e}^{-tN_k}p$ for all $t, k > 0$. Taking the limit as $k \to +\infty$, by the Trotter-Kato theorem ( [29, Theorem III.4.8]) we find

$$p\,\mathrm{e}^{-tN} = \mathrm{e}^{-tN}p \qquad \forall t \geq 0. \tag{4.2}$$

Let $v \in \mathcal{V}$, $v \neq 0$ with expansion in the canonical basis (1.17)

$$v = \sum_{k \geq k_0} v_k e_k$$

where $k_0$ is the minimum $k$ for which $v_k \neq 0$. Clearly, by (4.2), $\mathrm{e}^{-tN}v \in \mathcal{V}$ for all $t \geq 0$ and so

$$\mathrm{e}^{k_0 t}\mathrm{e}^{-tN}v = \sum_{k \geq k_0} \mathrm{e}^{-(k-k_0)t} v_k e_k = v_{k_0} e_{k_0} + \sum_{k > k_0} \mathrm{e}^{-(k-k_0)t} v_k e_k \in \mathcal{V}$$

for all $t \geq 0$. Taking the limit at $t \to +\infty$, we find $e_{k_0} \in \mathcal{V}$. Acting on $e_{k_0}$ with operators $a$ and $a^\dagger$ as per (1.18), (1.19) we can immediately show that every vector $e_k$ of the basis belongs to $\mathcal{V}$ and the proof is complete. Since, by Theorem 4.3, the only possible subharmonic projections would then be trivial. $\square$

### 4.1.1 Proof of Theorem 4.4

Before proving the Theorem we need some preliminary lemmas

**Lemma 4.6.** *For all $\xi \in \mathrm{Dom}(N^2)$ and all $\theta \in \mathbb{R}$ we have*

$$\left\| \left(\mathrm{e}^{\mathrm{i}\theta} a^\dagger + \mathrm{e}^{-\mathrm{i}\theta} a\right) \xi \right\|^2 \leq 2 \left\| \left(aa^\dagger + a^\dagger a\right)^{1/2} \xi \right\|^2$$
$$\left\| \left(\mathrm{e}^{\mathrm{i}\theta} a^{\dagger 2} + \mathrm{e}^{-\mathrm{i}\theta} a^2\right) \xi \right\|^2 \leq \left\| \left(aa^\dagger + a^\dagger a\right) \xi \right\|^2 + 3\,\|\xi\|^2$$

*Proof.* Computations below should be done on quadratic forms defined on the domain $D\times D$. However, we do only the algebraic computations to avoid clutter of the notation.

To prove the first inequality we begin by expanding

$$0 \leq \left|\mathrm{e}^{\mathrm{i}\theta}a^\dagger - \mathrm{e}^{-\mathrm{i}\theta}a\right|^2 = a^\dagger a - \mathrm{e}^{2\mathrm{i}\theta}a^{\dagger 2} - \mathrm{e}^{-2\mathrm{i}\theta}a^2 + aa^\dagger$$

which by (3.34) implies

$$\left|\mathrm{e}^{\mathrm{i}\theta}a^\dagger + \mathrm{e}^{-\mathrm{i}\theta}a\right|^2 \leq 2\left(a^\dagger a + aa^\dagger\right)$$

and the first inequality is proved. To prove the second inequality, first note that

$$0 \leq \left|\mathrm{e}^{\mathrm{i}\theta}a^{\dagger 2} - \mathrm{e}^{-\mathrm{i}\theta}a^2\right|^2 = a^2a^{\dagger 2} - \mathrm{e}^{2\mathrm{i}\theta}a^{\dagger 4} - \mathrm{e}^{-2\mathrm{i}\theta}a^4 + a^{\dagger 2}a^2$$

and so

$$\mathrm{e}^{2\mathrm{i}\theta}a^{\dagger 4} + \mathrm{e}^{-2\mathrm{i}\theta}a^4 \leq a^2a^{\dagger 2} + a^{\dagger 2}a^2$$

Now

$$\begin{aligned}\left(\mathrm{e}^{\mathrm{i}\theta}a^{\dagger 2} + \mathrm{e}^{-\mathrm{i}\theta}a^2\right)^2 - \left(aa^\dagger + a^\dagger a\right)^2 &= \mathrm{e}^{2\mathrm{i}\theta}a^{\dagger 4} + a^{\dagger 2}a^2 + a^2a^{\dagger 2} + \mathrm{e}^{-2\mathrm{i}\theta}a^4\\ &\quad - (aa^\dagger)^2 - (a^\dagger a)^2 - aa^{\dagger 2}a - a^\dagger a^2 a^\dagger\\ &\leq 2a^{\dagger 2}a^2 + 2a^2a^{\dagger 2} - (aa^\dagger)^2 - (a^\dagger a)^2 - aa^{\dagger 2}a - a^\dagger a^2 a^\dagger\end{aligned}$$

The right hand side is equal to

$$2N(N-1) + 2(N+1)(N+2) - (N+1)^2 - N^2 - (N+1)N - N(N+1) = 3$$

and so

$$\left(\mathrm{e}^{\mathrm{i}\theta}a^{\dagger 2} + \mathrm{e}^{-\mathrm{i}\theta}a^2\right)^2 \leq \left(aa^\dagger + a^\dagger a\right)^2 + 3$$

The claimed inequality readily follows. □

We will show that the graph norms of $G, G_0$ and $N$ are equivalent. We recall that the graph norm $\|\cdot\|_A$ of an operator $A$ is defined on its domain as

$$\|x\|_A^2 = \|Ax\|^2 + \|x\|^2. \tag{4.3}$$

We present first two preliminary Lemmas.

**Lemma 4.7.** *Let $\lambda_0$ be the smallest eigenvalue of the $2\times 2$ matrix*

$$\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix} \cdot \begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}$$

*which is strictly positive by the linear independence of $L_1, L_2$. There exists a constant $c_1 > 0$ depending on $v_1, u_1, v_2, u_2$ and uniformly bounded for $v_1, u_1, v_2, u_2$ in a bounded subset of $\mathbb{C}^4$ such that*

$$(-2G_0)^2 \geq \lambda_0^2\left(a^\dagger a + a\,a^\dagger\right)^2 - c_1\left(a^\dagger a + a\,a^\dagger\right).$$

*Proof.* Since $-2G_0 = L_1^* L_1 + L_2^* L_2$,

$$G_0 = -\frac{1}{2}\sum_{\ell=1}^{2}\left(\left(|v_\ell|^2 a^\dagger a + |u_\ell|^2 a a^\dagger\right) + v_\ell u_\ell a^{\dagger 2} + \overline{v}_\ell \overline{u}_\ell a^2\right)$$

for all $\xi \in D$, thinking of $(a\,\xi, a^\dagger \xi)$ as a vector in $\mathcal{H}\oplus\mathcal{H}$ and of product of a row vector with a column vector as the natural scalar product in $\mathcal{H}\oplus\mathcal{H}$, we can write $\langle \xi, G_0\xi\rangle$ as follows

$$\langle \xi, G_0\xi\rangle = -\frac{1}{2}\left[a^\dagger\xi, a\,\xi\right]\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix}\begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}\begin{bmatrix} a\,\xi \\ a^\dagger\xi \end{bmatrix}.$$

This notation is typical in the study of quadratic Hamiltonians (see, for instance, [22, 69–71]). Recall that, by linear independence of $L_1, L_2$, the above matrices have non-zero determinant. Therefore their product is strictly positive definite and, calling $\lambda_1$ its biggest eigenvalue, we have

$$\lambda_1 \left\langle \xi, \left(a^\dagger a + a\,a^\dagger\right)\xi\right\rangle \geq \langle \xi, -2G_0\xi\rangle \geq \lambda_0 \left\langle \xi, \left(a^\dagger a + a\,a^\dagger\right)\xi\right\rangle \tag{4.4}$$

In a similar way, dropping the vector $\xi$ and denoting by l.o.t. monomials of order 2 or less in creation and annihilation operators we have the inequalities

$$\begin{aligned}
(-2G_0)^2 &= \sum_\ell L_\ell^* \left(-2G_0\right) L_\ell + \text{l.o.t.} \\
&= [a^\dagger, a]\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix}\begin{bmatrix} -2G_0 & 0 \\ 0 & -2G_0 \end{bmatrix}\begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}\begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \text{l.o.t.} \\
&\geq \lambda_0\,[a^\dagger, a]\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix}\begin{bmatrix} a^\dagger a + aa^\dagger & 0 \\ 0 & a^\dagger a + aa^\dagger \end{bmatrix}\begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}\begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \text{l.o.t.} \\
&= \lambda_0\, a\,[a^\dagger, a]\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix}\begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}\begin{bmatrix} a \\ a^\dagger \end{bmatrix} a^\dagger \\
&\quad + \lambda_0\, a^\dagger\,[a^\dagger, a]\begin{bmatrix} v_1 & v_2 \\ \overline{u}_1 & \overline{u}_2 \end{bmatrix}\begin{bmatrix} \overline{v}_1 & u_1 \\ \overline{v}_2 & u_2 \end{bmatrix}\begin{bmatrix} a \\ a^\dagger \end{bmatrix} a + \text{l.o.t.} \\
&= \lambda_0\, a(-2G_0)a^\dagger + \lambda_0\, a^\dagger(-2G_0)a + \text{l.o.t.} \\
&\geq \lambda_0^2\, a(a^\dagger a + aa^\dagger)a^\dagger + \lambda_0^2\, a^\dagger(a^\dagger a + aa^\dagger)a + \text{l.o.t.} \\
&= \lambda_0^2 (a^\dagger a + aa^\dagger)^2 + \text{l.o.t.}
\end{aligned}$$

Since, by Lemma 4.6, we can control lower order terms with $(2N+1) = \left(a^\dagger a + a\,a^\dagger\right)$ the proof is complete. □

**Lemma 4.8.** *The commutator $[H, G_0]$ is a second order degree polynomial in $a, a^\dagger$ and*

$$|\langle \xi, [H, G_0]\xi\rangle| \leq c_2 \left\langle \xi, \left(a^\dagger a + a\,a^\dagger\right)^{1/2}\xi\right\rangle$$

*for some constant $c_2 > 0$ depending on all parameters in the model.*

*Proof.* A long but straightforward computation yields (summation on $\ell = 1, 2$ is implicit)

$$[H, G_0] = \mathrm{i}\Im\left(\kappa\left(\overline{v}_\ell \overline{u}_\ell\right)\right)\left(a^\dagger a + a a^\dagger\right) + \left(\Omega\left(\overline{v}_\ell \overline{u}_\ell\right) - \frac{\overline{\kappa}}{2}\left(|v_\ell|^2 + |u_\ell|^2\right)\right) a^2$$
$$+ \left(-\Omega\left(v_\ell u_\ell\right) + \frac{\kappa}{2}\left(|v_\ell|^2 + |u_\ell|^2\right)\right) a^{\dagger 2}$$
$$+ \left(\frac{\zeta}{2}\left(\overline{v}_\ell \overline{u}_\ell\right) - \frac{\overline{\zeta}}{2}\left(|v_\ell|^2 + |u_\ell|^2\right)\right) a + \left(-\frac{\overline{\zeta}}{2}\left(v_\ell u_\ell\right) + \frac{\zeta}{2}\left(|v_\ell|^2 + |u_\ell|^2\right)\right) a^\dagger$$

The claimed inequality follows from Lemma 4.6 and the Schwarz inequality. □

We are now ready to present the proof of Theorem 4.4

*Proof.* Clearly $D(N)$ is contained in $D(G_0)$ and $D(G)$.

In order to prove the opposite inclusion we show that there exist constants $c_3, c_4$ such that $\|N\xi\|^2 \le c_3 \|G_0\xi\|^2 + c_4 \|\xi\|^2$ for all $\xi \in D$. The conclusion follows because $D$ is a core for $G_0$ and $G$.

For all $\xi \in D$, $\epsilon > 0$ by Lemma 4.7 and Young's inequality, we have the following inequalities

$$\begin{aligned}
\|G_0\xi\|^2 &= \left\langle \xi, G_0^2\xi \right\rangle \\
&\ge \frac{\lambda_0^2}{4}\left\langle \xi, \left(a^\dagger a + a\, a^\dagger\right)^2 \xi \right\rangle - \frac{c_1}{4}\left\langle \xi, \left(a^\dagger a + a\, a^\dagger\right)\xi \right\rangle \\
&\ge \frac{\lambda_0^2}{4}\left\langle \xi, \left(a^\dagger a + a\, a^\dagger\right)^2 \xi \right\rangle - \frac{c_1}{4}\|\xi\| \cdot \left\|\left(a^\dagger a + a\, a^\dagger\right)\xi\right\| \\
&\ge \frac{\lambda_0^2}{4}\left\langle \xi, \left(a^\dagger a + a\, a^\dagger\right)^2 \xi \right\rangle - \frac{\lambda_0^2}{8}\left\|\left(a^\dagger a + a\, a^\dagger\right)\xi\right\|^2 - \frac{c_1^2}{8\lambda_0^2}\|\xi\|^2 \\
&= \frac{\lambda_0^2}{8}\left\|\left(a^\dagger a + a\, a^\dagger\right)\xi\right\|^2 - \frac{c_1^2}{8\lambda_0^2}\|\xi\|^2 .
\end{aligned}$$

Since $\left\|\left(a^\dagger a + a\, a^\dagger\right)\xi\right\|^2 \ge 4\|N\xi\|^2$ we find the inequality

$$\|N\xi\|^2 \le \frac{2}{\lambda_0^2}\|G_0\xi\|^2 + \frac{c_1^2}{4\lambda_0^4}\|\xi\|^2 \tag{4.5}$$

for all $\xi \in D$ implying that $D(G_0) \subset D(N)$.

In order to prove that the domain of $G$ is also contained in the domain of $N$ note that $G = G_0 - \mathrm{i}H$ on $D$ and write

$$\|G\xi\|^2 = \langle \xi, (G_0 + \mathrm{i}H)(G_0 - \mathrm{i}H)\xi \rangle = \left\langle \xi, (G_0^2 + H^2)\xi \right\rangle + \mathrm{i}\left\langle \xi, [H, G_0]\xi \right\rangle . \tag{4.6}$$

Now by Lemma 4.8, $\langle \xi, H^2\xi \rangle \ge 0$ and the previous inequality (4.5) we find

$$\begin{aligned}
\|G\xi\|^2 &\ge \left\langle \xi, G_0^2\xi \right\rangle - c_2\left\langle \xi, \left(a^\dagger a + a\, a^\dagger\right)^{1/2} \xi \right\rangle \\
&\ge \frac{\lambda_0^2}{2}\|N\xi\|^2 - c_2\sqrt{2}\,\|\xi\| \cdot \left\|N^{1/2}\xi\right\| - \frac{c_1^2}{4}\|\xi\|^2 .
\end{aligned}$$

We can now proceed as in the final part of the proof of (4.5) with an application of Young's inequality to show that $D(G) \subset D(N)$. □

## 4.2 Irreducibility: the case of a single noise operator

In this section we study the case where there is a single operator

$$L = \overline{v}a + ua^\dagger \qquad \text{with} \quad v \neq 0 \quad \text{or} \quad u \neq 0.$$

This case is much more convoluted. We begin by considering the algebraic aspect of the problem disregarding, for the moment, domain issues that will be considered later.

By Theorem 4.3 we are looking for common invariant subspaces for the operators $G$ and $L$ and so also for the commutator $[L, G]$. A straightforward computation yields

$$\begin{aligned} -2\,[L,G] &= [L, L^*L + 2\mathrm{i}H] \\ &= [L, L^*]\,L + 2\mathrm{i}\,(\overline{v}\Omega - u\overline{\kappa})\,a - 2\mathrm{i}\,(u\Omega - \overline{v}\kappa)\,a^\dagger + 2\mathrm{i}\,\big(\overline{v}\zeta - u\overline{\zeta}\big) \end{aligned} \tag{4.7}$$

Thus the candidate subspace must be invariant for the operators

$$G = -\frac{1}{2}L^*L - \mathrm{i}H, \qquad L = \overline{v}a + ua^\dagger, \qquad \widetilde{L} = (\overline{v}\Omega - u\overline{\kappa})\,a + (\overline{v}\kappa - u\Omega)\,a^\dagger.$$

If the operators $L$ and $\widetilde{L}$ are linearly independent, corresponding ti the condition

$$\det\begin{bmatrix} \overline{v}\Omega - u\overline{\kappa} & \overline{v}\kappa - u\Omega \\ \overline{v} & u \end{bmatrix} \neq 0, \tag{4.8}$$

then the candidate subspace must be invariant for $a$ and $a^\dagger$ and so it should be trivial as in the case of two Kraus operators $L$.

In the sequel, we prove that under condition (4.8), which is clearly a Hörmander-type iterated commutator condition the QMS is irreducible. Otherwise, we will see that irreducibility does not hold.

It is worth noticing here that a similar condition appears also in bilinear control (see [28], Definition 3.6 (ii) p. 102, *weak ad-condition*) As a matter of fact, if, starting from any initial non-zero vector $\xi_0 \in \mathcal{H}$ with time evolution one can reach a total set of vectors in $\mathcal{H}$ varying the control parameter $z \in \mathbb{C}$ in the differential equation $\dot{\xi}_t = G\xi_t + zL\xi_t$, then irreducibility holds.

**Lemma 4.9.** *Suppose* $|v| \neq |u|$. *Then* $\mathrm{Dom}(G_0) = \mathrm{Dom}(N) = \mathrm{Dom}(G)$.

*Proof.* We begin by noting that $D(N) \subset D(G_0)$ and $D(N) \subset D(G)$.

Conversely, note that for all $r \in \mathbb{R}$, on the domain $\mathrm{Dom}(N)$ of the number operator, in the same matrix notation of the proof of Lemma 4.7, we have

$$\begin{aligned} L^*L &= |v|^2a^\dagger a + vu\,a^{\dagger 2} + \overline{vu}\,a^2 + |u|^2aa^\dagger \\ &= \big(|v|^2 + r\big)\,a^\dagger a + vua^{\dagger 2} + \overline{vu}a^2 + \big(|u|^2 - r\big)\,aa^\dagger + r\mathbb{1} \\ &= \big[a^\dagger\ a\big]\begin{bmatrix} |v|^2 + r & vu \\ \overline{vu} & |u|^2 - r \end{bmatrix}\begin{bmatrix} a \\ a^\dagger \end{bmatrix} + r\mathbb{1} \end{aligned}$$

The trace of the above $2 \times 2$ matrix is strictly positive and the determinant

$$r\left(|u|^2 - |v|^2\right) - r^2$$

if we choose $r = \left(|u|^2 - |v|^2\right)/2$, it is equal to $\left(|u|^2 - |v|^2\right)^2/4 > 0$ and the lowest eigenvalue is $\left(|v| - |u|\right)^2/2$. It follows that

$$L^*L \geq \frac{\left(|v| - |u|\right)^2}{2}\left(aa^\dagger + a^\dagger a\right) + \frac{|v|^2 - |u|^2}{2}\mathbb{1}$$

and, denoting by l.o.t. monomials of order 2 or less in creation and annihilation operators,

$$\begin{aligned}
(L^*L)^2 &= L^*(LL^*)L = L^*(L^*L)L + \text{l.o.t.} \\
&\geq \frac{1}{2}\left(|v| - |u|\right)^2 L^*\left(aa^\dagger + a^\dagger a\right)L + \text{l.o.t.} \\
&= \frac{1}{2}\left(|v| - |u|\right)^2\left(aL^*La^\dagger + a^\dagger L^*La\right) + \text{l.o.t.} \\
&\geq \frac{1}{4}\left(|v| - |u|\right)^4\left(a\left(aa^\dagger + a^\dagger a\right)a^\dagger + a^\dagger\left(aa^\dagger + a^\dagger a\right)a\right) + \text{l.o.t.} \\
&= \left(|v| - |u|\right)^4\left(a^\dagger a\right)^2 + \text{l.o.t.}
\end{aligned}$$

Therefore there exists a constant $c > 0$ such that

$$\left(|v| - |u|\right)^4\left\|a^\dagger a\,\xi\right\|^2 \leq \left\|L^*L\,\xi\right\|^2 + c\left\|\xi\right\|^2 \tag{4.9}$$

for all $\xi \in D$ and $D(L^*L) \subset D(N)$. This shows the identity $D(G_0) = D(N)$.

In order to prove the other identity, for all $\xi \in D$, compute

$$\|G\xi\|^2 = \|G_0\xi\|^2 + \|H\xi\|^2 + \langle \xi, \mathrm{i}[H, G_0]\xi\rangle\,.$$

Since the commutator $[H, G_0]$ is a second order polynomial in $a, a^\dagger$ there exists a constant $c' > 0$ such that $\langle \xi, \mathrm{i}[H, G_0]\xi\rangle \geq -c'\|N^{1/2}\xi\|^2$. Recalling (4.9), by Young's inequality, we have

$$\begin{aligned}
\|G\xi\|^2 &\geq \|G_0\xi\|^2 - c'\|N^{1/2}\xi\|^2 \\
&\geq \frac{\left(|v| - |u|\right)^4}{4}\|a^\dagger a\xi\|^2 - c\|\xi\|^2 - \frac{\left(|v| - |u|\right)^4}{8}\|a^\dagger a\xi\|^2 - \frac{4}{c'^2\left(|v| - |u|\right)^4}\|\xi\|^2 \\
&= \frac{\left(|v| - |u|\right)^4}{8}\|a^\dagger a\xi\|^2 - c''\|\xi\|^2
\end{aligned}$$

where $c''$ is another constant. Thus $D(G) \subset D(N)$ and the proof is complete. □

We can now prove irreducibility when condition (4.8) holds.

**Proposition 4.10.** *Suppose that condition (4.8) holds and, moreover, $|v| \neq |u|$. Then the Gaussian QMS with $m = 1$,*

$$L = \overline{v}a + ua^\dagger,$$

*and $H$ as in* (3.17) *is irreducible.*

*Proof.* Using the result of Lemma 4.9 $D(G) = D(N)$ the proof essentially follows the line of that of Theorem 4.5.

Let $\mathcal{V}$ ($\mathcal{V} \neq \{0\}$) be a subspace of $\mathcal{H}$ which is invariant for the operators $P_t$ and $L\left(D(G) \cap \mathcal{V}\right) = L\left(D(N) \cap \mathcal{V}\right) \subset \mathcal{V}$ for $\ell = 1, 2$. Moreover, since

$$L\left(D(N^m)\right) \subset D(N^{m-1/2}), \quad G\left((N^m)\right) \subset D(N^{m-1}),$$

for all $m \geq 1$, we have also

$$[G, L]\left(D(N^{3/2}) \cap \mathcal{V}\right) \subset \mathcal{V}, \quad \widetilde{L}\left(D(N^{3/2}) \cap \mathcal{V}\right) \subset \mathcal{V}.$$

However the commutator $[G, L]$ is a first order polynomial in $a, a^\dagger$, therefore the previous inclusions can be extended to $D(N^{1/2}) \cap \mathcal{V}$.

By the linear independence of $L$ and $\widetilde{L}$, we can now follow the argument of the proof of Theorem 4.5, with $L_2 = \widetilde{L}$. □

We study separately situations in which (4.8) does not hold distinguishing three cases.

### 4.2.1 Kraus operator of annihilation type

We first consider the case where (4.8) does not hold and $|v| > |u|$. We recall the definition of the squeeze operator

**Proposition 4.11.** *For every $z \in \mathsf{h}$ let*

$$S(z) = \mathrm{e}^{(za^{\dagger 2} - \overline{z}a^2)/2}. \tag{4.10}$$

*Then $S$ is a unitary operator and if $z = \mathrm{e}^{\mathrm{i}\varphi} s$ with $s = |z|$ then*

$$S^* a S = \cosh(s)\, a + \mathrm{e}^{\mathrm{i}\varphi} \sinh(s)\, a^\dagger \qquad S^* a^\dagger S = \cosh(s)\, a^\dagger + \mathrm{e}^{-\mathrm{i}\varphi} \sinh(s)\, a \tag{4.11}$$

.

*Proof.* See for example [56] □

When can then prove the following result, answering irreducibility for the case of only one Kraus operator and $|v| > |u|$.

**Proposition 4.12.** *The Gaussian QMS with GKSL generator with only one Kraus operator $L = \overline{v}a + ua^\dagger$, $|v| > |u|$ and Hamiltonian $H$ as in (3.17) is irreducible if and only if condition (4.8) holds. If it is not irreducible, it has a unique invariant state $\mathrm{e}^{-|\nu|^2} |e(\nu)\rangle\langle e(\nu)|$ (pure) and all initial state converges to it in trace norm.*

*Proof.* Using the squeeze operator (4.10), by (4.11) with $z = s\mathrm{e}^{\mathrm{i}\varphi}$, we obtain

$$L' = S^* L S = \left(\overline{v}\cosh(s) + \mathrm{e}^{-\mathrm{i}\varphi} u \sinh(s)\right) a + \left(u \cosh(s) + \mathrm{e}^{\mathrm{i}\varphi} \overline{v} \sinh(s)\right) a^\dagger \tag{4.12}$$

and, by first choosing a $\varphi$ such that $u$ and $\mathrm{e}^{\mathrm{i}\varphi}\overline{v}$ have the same phase, and an $s$ such that

$$|u|\cosh(s) + |v|\sinh(s) = 0 \quad \Leftrightarrow \quad \tanh(s) = -|u|/|v|$$

we can assume that $L'$ is a strictly positive multiple (multiplying a Kraus operator by a phase does not change the GKLS representation) of the annihilation operator, i.e. $u = 0$. Of course also $\Omega, \kappa, \zeta$ change to $\Omega' = S^*\Omega S, \kappa' = S^*\kappa S, \zeta' = S^*\zeta S$

$$\begin{aligned}
\Omega' &= \Omega\left(\cosh^2(s) + \sinh^2(s)\right) + 2\sinh(s)\cosh(s)\Re(\mathrm{e}^{-\mathrm{i}\varphi}\kappa) \\
\kappa' &= \kappa\cosh^2(s) + \overline{\kappa}\,\mathrm{e}^{2\mathrm{i}\varphi}\sinh^2(s) + 2\Omega\,\mathrm{e}^{\mathrm{i}\varphi}\cosh(s)\sinh(s) \\
\zeta' &= \zeta\cosh(s) + \overline{\zeta}\mathrm{e}^{\mathrm{i}\varphi}\sinh(s)
\end{aligned}$$

and condition (4.8) does not hold if and only if $v\kappa' = 0$ i.e., by $v \neq 0$, $\kappa' = 0$ and (up to an irrelevant multiple of the identity operator in $H$)

$$L' = v'a, \quad H' = \Omega'\, a^\dagger a + \frac{\zeta'}{2}a^\dagger + \frac{\overline{\zeta}'}{2}a, \quad G' = -\left(\frac{|v'|^2}{2} + \mathrm{i}\Omega'\right)a^\dagger a - \frac{\mathrm{i}}{2}\left(\zeta' a^\dagger + \overline{\zeta}' a\right)$$

where $v' = (|v|^2 - |u|^2)\cosh(s)/|v|$, up to a phase factor. Eventually we have

$$\langle u, S^*\mathcal{L}(SxS^*)Sv\rangle = \langle u, \mathcal{L}'(x)v\rangle\,, \quad \forall x \in \mathcal{B}(\mathcal{H}), u, v \in \mathcal{H}$$

therefore irreducibility of $\mathcal{T}$ is equivalent to irreducibility of the semigroup generated by $\mathcal{L}'$.

Dropping the $'$ to simplify the notation, now we apply formula (3.38) with

$$Zz = -(|v|^2/2 + \mathrm{i}\Omega)z, \qquad Cz = |v|^2 z.$$

Computing $\mathrm{e}^{sZ}z = \mathrm{e}^{-(|v|^2/2 - \mathrm{i}\Omega)s}z$ and

$$\begin{aligned}
\int_0^t \Re\left(\overline{\mathrm{e}^{sZ}z}\, C\mathrm{e}^{sZ}z\right)\mathrm{d}s &= |z|^2\int_0^t |v|^2\mathrm{e}^{-s|v|^2}\mathrm{d}s = |z|^2\left(1 - \mathrm{e}^{-t|v|^2}\right) \\
\int_0^t \Re\left(\overline{\zeta}\mathrm{e}^{sZ}z\right)\mathrm{d}s &= \Re\left(\frac{\overline{\zeta}z}{|v|^2/2 - \mathrm{i}\Omega}\left(1 - \mathrm{e}^{-t(|v|^2/2 - \mathrm{i}\Omega)}\right)\right)
\end{aligned}$$

It follows that, for all $g, f \in \mathbb{C}$,

$$\begin{aligned}
\lim_{t\to+\infty}\mathrm{tr}(|e(f)\rangle\langle e(g)|\mathcal{T}_t(W(z)) &= \mathrm{e}^{-|z|^2/2 + \mathrm{i}\Re\left(\overline{\zeta}z/(|v|^2/2 - \mathrm{i}\Omega)\right)}\lim_{t\to+\infty}\left\langle e(g), W(\mathrm{e}^{tZ}z)e(f)\right\rangle \\
&= \mathrm{e}^{-|z|^2/2 + 2\mathrm{i}\Im\left(\overline{\mathrm{i}\zeta}z/(|v|^2 - 2\mathrm{i}\Omega)\right)}\,\mathrm{e}^{\overline{g}f}
\end{aligned}$$

Noting that, for all $\nu \in \mathbb{C}$

$$\mathrm{e}^{-|\nu|^2}\,\langle e(\nu), W(z)e(\nu)\rangle = \mathrm{e}^{-|z|^2/2 + 2\mathrm{i}\Im(\overline{\nu}z)},$$

defining $\nu = \mathrm{i}\zeta/(|v|^2 + 2\mathrm{i}\Omega)$ we find

$$\lim_{t\to+\infty}\mathrm{tr}(|e(f)\rangle\langle e(g)|\mathcal{T}_t(W(z)) = \mathrm{e}^{\overline{g}f}\mathrm{e}^{-|\nu|^2}\,\langle e(\nu), W(z)e(\nu)\rangle$$

In particular, $\mathrm{e}^{-|\nu|^2}|e(\nu)\rangle\langle e(\mu)|$ is a pure invariant state and the QMS is not irreducible. Moreover, since linear combinations of linear functionals $|e(f)\rangle\langle e(g)|$ are dense in the Banach space of trace class operators by totality of exponential vectors, the above identity also proves that any initial state converges in trace norm to this pure invariant state. □

### 4.2.2 Kraus operator of creation type

We consider the case where (4.8) does not hold and $|v| < |u|$. Using similar technques to what we have already used we can prove the following.

**Proposition 4.13.** *The Gaussian QMS with GKSL generator with only one Kraus operator $L = \overline{v}a + ua^\dagger$, $|v| < |u|$ and Hamiltonian $H$ as in (3.17) is irreducible if and only if condition (4.8) holds.*

*Proof.* Clearly if condition (4.8) holds we can use Proposition 4.10 and obtain irreducibility of the semigroup.

We suppose now that condition (4.8) does not hold and show that the semigroup is not irreducible. Similar to the proof of Proposition 4.12 we start by considering a new generator $\mathcal{L}'$ through a squeezing operation. This time if $z = s\mathrm{e}^{\mathrm{i}\varphi}$ we choose $\varphi$ such that $u$ and $\mathrm{e}^{\mathrm{i}\phi}\overline{v}$ have the same phase, and then $s$ such that $\tanh(s) = -|v|/|u|$. In this way we obtain $v' = 0$ and $L'$ defined by (4.12) is a multiple of the creation operator. Parameters $\Omega, \kappa, \zeta$ are transformed to $\Omega', \kappa', \zeta'$ accordingly and (4.8) does not hold if and only if $\kappa' = 0$, as in the proof of Proposition (4.12). In this way the given QMS is transformed to the unitarily equivalent QMS generated by $\mathcal{L}'$ with

$$L' = u'a^\dagger, \quad H' = \Omega'\, a^\dagger a + \frac{\zeta'}{2} a^\dagger + \frac{\overline{\zeta}'}{2} a, \quad G' = -\left(\frac{|u'|^2}{2} + \mathrm{i}\Omega'\right) a\, a^\dagger - \frac{\mathrm{i}}{2}\left(\zeta' a^\dagger + \overline{\zeta}' a\right)$$

where $u' = (|u|^2 - |v|^2)\cosh(\theta)/|u|$ up to a phase factor. In the sequel of the proof we work with the semigroup generated by $\mathcal{L}'$ but drop the $'$ to simplify the notation.

Let $\mathcal{V}$ be the range of a non-zero subharmonic projection $p$. Since, by Lemma 4.9 the operators $G$ and $N$ have the same domain, by Theorem 4.3 we have

$$G(D(N) \cap \mathcal{V}) \subseteq \mathcal{V}, \quad L(D(N) \cap \mathcal{V}) \subseteq \mathcal{V}. \tag{4.13}$$

We can add to $G$ a suitable multiple of $L$ to obtain the operator

$$\widetilde{G} = -\left(\frac{|u|^2}{2} + \mathrm{i}\Omega\right)\left(a\, a^\dagger + \overline{\eta}a + \eta a^\dagger + |\eta|^2 \mathbb{1}\right) = -\left(\frac{|u|^2}{2} + \mathrm{i}\Omega\right) W(\eta)^* a\, a^\dagger W(\eta),$$

where $\eta = \mathrm{i}\zeta/(|u|^2 - 2\mathrm{i}\Omega)$. By (4.13) we have

$$\widetilde{G}(D(N) \cap \mathcal{V}) \subseteq \mathcal{V}, \quad L(D(N) \cap \mathcal{V}) \subseteq \mathcal{V},$$

which is equivalent to $G, L$ invariance, by definition of $\widetilde{G}$.

Let $w \in \mathcal{V}$ with expansion $w = \sum_{k \geq k_0} w_k\, W(-\eta) e_k$ where $k_0$ is the minimum $k$ for which $w_k \neq 0$. Since $\widetilde{G}$ is a multiple of the number operator with strictly negative real part, arguing as in the last part of the proof of Theorem 4.5, we can show that $W(-\eta)e_{k_0} \in \mathcal{V}$. As a consequence, by the commutation $a^\dagger W(-\eta) = W(-\eta)(a^\dagger - \overline{\eta}\mathbb{1})$,

$$LW(-\eta)e_{k_0} = u\, W(-\eta)(a^\dagger - \overline{\eta}\mathbb{1})e_{k_0} = u\sqrt{k_0 + 1}\, W(-\eta)e_{k_0+1} - u\,\overline{\eta} W(-\eta)e_{k_0} \in \mathcal{V}$$

Applying $L$ we can show inductively that, for all $k_0 \geq 0$, the linear space generated by vectors $W(-\eta)e_k$ with $k \geq k_0$ is an invariant subspace determining a subharmonic projection and, in this case, the QMS associated with $G, L$ is not irreducible. □

### 4.2.3 Self-adjoint Kraus operator

We consider here the case where $|v| = |u|$. We consider at first a transformation similar to the previous subsections, here however there is no need for the squeeze operator.

**Lemma 4.14.** *Irreducibility of the Gaussian QMS with GKSL generator with only one Kraus operator $L = \overline{v}a + ua^\dagger$, $|v| = |u|$ and Hamiltonian $H$ as in (3.17) is equivalent to irreducibility of the semigroup generated by $\mathcal{L}'$ with*

$$L' = r\left(\mathrm{e}^{-\mathrm{i}\theta}a + \mathrm{e}^{\mathrm{i}\theta}a^\dagger\right), \quad H' = \Omega a^\dagger a + \frac{|\kappa|}{2}\left(\mathrm{e}^{2\mathrm{i}\phi}a^{\dagger 2} + \mathrm{e}^{-2\mathrm{i}\phi}a^2\right) + \left(\overline{\zeta}a + \zeta a^\dagger\right),$$

*for some $r > 0$, $\theta, \phi \in \mathbb{R}$. In this case condition (4.8) does not hold if and only if*

$$\Omega = |\kappa|\cos(2(\phi - \theta)). \tag{4.14}$$

*Proof.* Since $|v| = |u|$ consider $v = r\mathrm{e}^{\mathrm{i}\alpha}$, $u = r\mathrm{e}^{\mathrm{i}\alpha'}$, for $v > 0$ and $\alpha, \alpha' \in \mathbb{R}$. We can therefore multiply $L$ by the phase $\mathrm{e}^{-\mathrm{i}(\alpha'-\alpha)/2}$ and define $2\theta = \alpha + \alpha'$, to obtain the expression for $L'$.

Similarly we can write $\kappa = |\kappa|\,\mathrm{e}^{2\mathrm{i}\phi}$ to obtain the expression for $H'$. A simple computation now yields (4.14). □

*Remark* 4.15. Using the expressions of Lemma 4.14 and defining the self-adjoint operator

$$q_\theta := \frac{\mathrm{e}^{-\mathrm{i}\theta}a + \mathrm{e}^{\mathrm{i}\theta}a^\dagger}{\sqrt{2}},$$

we have

$$a^\dagger = \mathrm{e}^{-\mathrm{i}\theta}\left(q_\theta - \mathrm{i}q_{\theta+\pi/2}\right)/\sqrt{2} \qquad a = \mathrm{e}^{\mathrm{i}\theta}\left(q_\theta + \mathrm{i}q_{\theta+\pi/2}\right)/\sqrt{2} \tag{4.15}$$

$$a^\dagger a + aa^\dagger = q_\theta^2 + q_{\theta+\pi/2}^2. \tag{4.16}$$

Therefore $L' = r\sqrt{2}q_\theta$ and

$$\begin{aligned} H' &= \frac{\Omega + |\kappa|\cos(2(\phi-\theta))}{2}\,q_\theta^2 + \frac{\Omega - |\kappa|\cos(2(\phi-\theta))}{2}\,q_{\theta+\pi/2}^2 \\ &+ \frac{|\kappa|}{2}\sin(2(\phi-\theta))\left(q_\theta q_{\theta+\pi/2} + q_{\theta+\pi/2}q_\theta\right) + \left(\overline{\zeta}a + \zeta a^\dagger\right) - \frac{\Omega}{2}\mathbb{1}. \end{aligned} \tag{4.17}$$

**Proposition 4.16.** *If condition (4.8) does not hold, th Gaussian QMS with GKSL generator with only one Kraus operator $L = \overline{v}a + ua^\dagger$, $|v| = |u|$ and Hamiltonian $H$ as in (3.17) is not irreducible.*

*Proof.* Using Lemma 4.14 and Remark 4.15 we consider irreducibility of the semigroup with $L' = r\sqrt{2}q_\theta$ and $H'$ given by (4.17). We will drop the $'$ to avoid clutter of the notation.

We can now compute

$$\left[q_{\theta+\pi/2}, q_\theta\right] = -\mathrm{i}\mathbb{1}.$$

For every smooth function $f : \mathbb{R} \to \mathbb{C}$, considering for example its Taylor expansion, we have

$$[L, f(q_\theta)] = [L^*, f(q_\theta)] = 0, \quad \left[q_{\theta+\pi/2}, f(q_\theta)\right] = -\mathrm{i}f'(q_\theta).$$

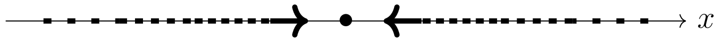
Since condition (4.8) does not hold, we have (4.14) and therefore the quadratic term $q^2_{\theta+\pi/2}$ in $H$ vanishes. In particular using

$$\begin{aligned} [a, f(q_\theta)] &= \frac{\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{2}}\left[q_\theta - \mathrm{i}q_{\theta+\pi/2}, f(q_\theta)\right] = -\frac{\mathrm{e}^{\mathrm{i}\theta}}{\sqrt{2}} f'(q_\theta), \\ \left[a^\dagger, f(q_\theta)\right] &= \frac{\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{2}}\left[q_\theta + \mathrm{i}q_{\theta+\pi/2}, f(q_\theta)\right] = \frac{\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{2}} f'(q_\theta), \end{aligned}$$

we obtain

$$\mathcal{L}(f(q_\theta)) = \mathrm{i}\,[H, f(q_\theta)] = \Big(\mathrm{Im}(\zeta \mathrm{e}^{-\mathrm{i}\theta})/\sqrt{2}) + |\kappa|\sin(2(\theta-\phi))q_\theta\Big)\, f'(q_\theta).$$

Reading again this generator as acting on functions over $\mathbb{R}$ we have obtained the generator of a deterministic translation process with drift (considering the generic case where $|\kappa|\sin(2(\theta-\phi)) \neq 0$) towards the point $x_\infty := \mathrm{Im}(\zeta\mathrm{e}^{-\mathrm{i}\theta})/(\sqrt{2}|\kappa|\sin(2(\theta-\phi)))$ (Fig. 1 below).



$x_\infty = -\mathrm{Im}(\zeta\mathrm{e}^{-\mathrm{i}\theta})/(\sqrt{2}|\kappa|\sin(2(\theta-\phi)))$

Fig. 1: deterministic translation process on the algebra generated by $q_\theta$.

The invariant density of the classical process is clearly $\delta_{x_\infty}$ which does not induce a faithful normal state on $\mathcal{B}(\mathsf{h})$. However this gives us some insight on the proof of non irreducibility.

Indeed cfor all $c > 0$ consider the projection

$$x \to 1_{[x_\infty - c, x_\infty + c]}(x)$$

which is a candidate subharmonic projection because the classical process, starting from a point in the interval $[x_\infty - c, x_\infty + c]$ does not exit for all positive times. To prove that this projection is indeed subharmonic, consider mollifier $\varphi$, namely a $C^\infty$ function $\varphi : \mathbb{R} \to \mathbb{R}_+$ with support in the interval $[-1,1]$, $\int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1$ and $\lim_{\epsilon\to 0}\varphi_\epsilon(x) = \lim_{\epsilon \to 0}\epsilon^{-1}\varphi(x/\epsilon) = \delta_0$ and, for all $\epsilon < c$ define

$$f_\epsilon(x) = \int_{-\infty}^{x} \left(\varphi_\epsilon(y - (x_\infty - c)) - \varphi_\epsilon(y - (x_\infty + c))\right)\mathrm{d}y.$$

Note that, since $\int_{\mathbb{R}} \varphi_\epsilon(x)\mathrm{d}x = 1$ for all $\epsilon > 0$ we have $f_\epsilon(x) = 0$ for $|x - x_\infty| > c + \epsilon$, $f(x) = 1$ for $|x - x_\infty| \leq c - \epsilon$ and $f'_\epsilon(x) \geq 0$ for $x_\infty - c - \epsilon < x < x_\infty - c + \epsilon$, $f'_\epsilon(x) \leq 0$ for $x_\infty + c - \epsilon < x < x_\infty + c + \epsilon$. It follows that the multiplication operator by $f_\epsilon(q_\theta)$, which belongs to the domain of the Lindbladian $\mathcal{L}$ because $\mathcal{L}(f_\epsilon(q_\theta))$ is bounded satisfies $\mathcal{L}(f_\epsilon(q_\theta)) \geq 0$ and so

$$\mathcal{T}_t(f_\epsilon(q_\theta)) \geq f_\epsilon(q_\theta)$$

for all $t \geq 0$. Taking the limit as $\epsilon$ goes to 0, $f_\epsilon$ converges to the projection $1_{[x_\infty - c, x_\infty + c]}$ in $L^2$ and almost surely, therefore $\mathcal{T}_t(1_{[x_\infty - c, x_\infty + c]}) \geq 1_{[x_\infty - c, x_\infty + c]}$ for all $t \geq 0$ and the QMS is not irreducible. A similar argument applies in the case where $\sin(2(\theta - \phi)) = 0$ and $x_\infty = +\infty$ (resp. $x_\infty = -\infty$) if $\mathrm{Im}(\zeta\mathrm{e}^{-\mathrm{i}\theta}) > 0$ (resp. $iIm(\zeta\mathrm{e}^{-\mathrm{i}\theta}) < 0$) with projections of the form $1_{[c,+\infty[}$ (resp. $1_{]-\infty,c]}$). $\square$

We now consider the case where $|u| = |v|$ and condition (4.8) holds. Again we will make use of Lemma 4.14 and Remark 4.15 , where condition (4.8) becomes $\Omega \neq |\kappa|\cos(2(\theta-\phi))$, and show that the QMS generated by $\mathcal{L}'$ is irreducible. To this end we will show coercivity of $G_0^2 + H^2 + g_l^2 \mathbb{1}$, for some constant $g_l^2$, with respect to the graph norm of the number operator $N = (q_\theta^2 + q_{\theta+\pi/2}^2 - 1)/2$. Before proving this result we need the following lemma.

**Lemma 4.17.** *Let $\mu, \lambda, x, y \in \mathbb{R}$ with $\lambda \neq 0$. For all $r > 0$ and $w > 0$ such that $w < \min\{1, (2x^2)^{-1}\}$ there exists $\epsilon > 0$ such that*

$$\begin{bmatrix} \mu^2 + r^4 & \mu x & \lambda\mu \\ \mu x & x^2 & \lambda x \\ \lambda\mu & \lambda x & \lambda^2 \end{bmatrix} \geq \epsilon \begin{bmatrix} r^4 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & \lambda^2 w \end{bmatrix}$$

*Proof.* The difference of the above matrices is

$$\begin{bmatrix} \mu^2 + r^4(1-\epsilon) & \mu x & \lambda\mu - \epsilon \\ \mu x & x^2 + \epsilon/2 & \lambda x \\ \lambda\mu - \epsilon & \lambda x & \lambda^2(1 - w\epsilon) \end{bmatrix}$$

which is positive, by the Sylvester's criterion, if and only if all principal minors are positive. For all $\epsilon > 0$, the principal minor obtained by removing the first row and column is positive if and only if $w\epsilon < 1$ and its determinant

$$\lambda^2((1-2wx^2)\epsilon - w\epsilon^2)/2 = \lambda^2\epsilon((1-2wx^2) - w\epsilon)/2$$

is positive. This is clearly the case if $\epsilon < \min\{1, w^{-1}, (1-2wx^2)/w\} := \epsilon_1$.

The principal minor obtained by removing the second row and column, namely

$$\begin{bmatrix} \mu^2 + r^4(1-\epsilon) & \lambda\mu - \epsilon \\ \lambda\mu - \epsilon & \lambda^2(1-w\epsilon) \end{bmatrix}$$

has positive diagonal elements for $0 < \epsilon < \epsilon_1$ and determinant

$$\lambda^2 r^4 + (2\lambda\mu - \lambda^2\mu^2 w - \lambda^2 r^4(1+w))\epsilon + \lambda^2 r^4 w\epsilon^2$$

which is clearly strictly positive for all $0 < \epsilon < \epsilon_2$ for some $\epsilon_2 < \epsilon_1$. Finally, the principal minor obtained by removing the third row and column, namely

$$\begin{bmatrix} \mu^2 + r^4(1-\epsilon) & \mu x \\ \mu x & x^2 + \epsilon/2 \end{bmatrix}$$

which has positive diagonal elements for $\epsilon < 1$, has determinant

$$r^4x^2 + (\mu^2 + r^4 - 2r^4x^2)\epsilon/2 - r^4\epsilon^2/2.$$

This is clearly positive for all $\epsilon$ small enough if $x \neq 0$ because it tends to $r^4x^2 \neq 0$ but also for $x = 0$ since, in this case it is equal to $\epsilon(\mu^2 + r^4 - r^4\epsilon/2)$. This completes the proof. □

**Theorem 4.18.** *If condition (4.8) holds there exist constants $g^2 > 0$, $g_l^2 \geq 0$ such that*

$$G_0^2 + H^2 \geq g^2(q_\theta^2 + q_{\theta+\pi/2}^2)^2 - g_l^2 \mathbb{1}. \tag{4.18}$$

*In particular $D(G) = D(N)$.*

*Proof.* In this proof only, to reduce the clutter of the notation, we denote $q_\theta$ by $q$, $q_{\theta+\pi/2}$ by $p$, $c := \cos(2(\phi - \theta))$, $s := \sin(2(\phi - \theta))$ and by $\{\cdot, \cdot\}$ the anticommutator.

As a first step note that, once we show that $G_0^2 + H^2 \geq g_0^2(q_\theta^2 + q_{\theta+\pi/2}^2)^2 +$ l.o.t. for some constant $g_0^2 > 0$ then, reducing the constant $g_0$ if necessary, we can get the conclusion. Indeed, if the lower order term is, for instance, $\{a, q^2 + p^2\}$ for all $\xi \in D(N^2)$ by the Schwartz and Young inequalities, we have

$$\begin{aligned}
\left\langle \xi, \{a, q^2 + p^2\}\xi \right\rangle &= \left\langle a^\dagger \xi, (q^2 + p^2)\xi \right\rangle + \left\langle (q^2 + p^2)\xi, a\,\xi \right\rangle \\
&\geq -\left\| a^\dagger \xi \right\| \cdot \left\| (q^2 + p^2)\xi \right\| - \|a\,\xi\| \cdot \left\| (q^2 + p^2)\xi \right\| \\
&\geq -\epsilon \left\| (q^2 + p^2)\xi \right\|^2 - \epsilon^{-1} \left( \|a\,\xi\|^2 + \left\| a^\dagger \xi \right\|^2 \right) \\
&= -\epsilon \left\langle \xi, (q^2 + p^2)^2 \xi \right\rangle - \epsilon^{-1} \left\langle \xi, (q^2 + p^2)\xi \right\rangle
\end{aligned}$$

for all $\epsilon > 0$. Now, again by the Schwartz and Young inequalities we have also

$$\begin{aligned}
-\epsilon^{-1} \left\langle \xi, (q^2 + p^2)\xi \right\rangle &\geq -\epsilon^{-1} \|\xi\| \cdot \left\| (q^2 + p^2)\xi \right\| \\
&\geq -\epsilon \left\| (q^2 + p^2)\xi \right\|^2 - \epsilon^{-3} \|\xi\|^2
\end{aligned}$$

Therefore we find the inequality

$$\left\langle \xi, \{a, q^2 + p^2\}\xi \right\rangle \geq -2\epsilon \left\langle \xi, (q^2 + p^2)^2 \xi \right\rangle - \epsilon^{-3} \|\xi\|^2$$

and, choosing $\epsilon$ small enough, we can reduce the constant $g^2$ in (4.18), increase $g_l^2$ and get the claimed inequality. We can proceed in a similar way if the are more lower order terms.

It is now clear that we can assume that $G_0^2 + H^2$ is a fourth order homogenous polynomial in $p, q$, or, in an equivalent way, we can proceed as if $H$ had no terms of order 1 or 0. In this case the square of $2H$ is

$$\begin{aligned}
(2H)^2 = {} & (\Omega + |\kappa| c)^2 q^4 + (\Omega - |\kappa| c)^2 p^4 + (\Omega + |\kappa| c)(\Omega - |\kappa| c)\{q^2, p^2\} \\
& + |\kappa|^2 s^2 \{q, p\}^2 + (\Omega + |\kappa| c)|\kappa| s \left\{q^2, \{q, p\}\right\} + (\Omega - |\kappa| c)|\kappa| s \left\{p^2, \{q, p\}\right\}
\end{aligned}$$

and write $(2H)^2$ as

$$\begin{bmatrix} q^2 \\ \{q, p\} \\ p^2 \end{bmatrix}^T \begin{bmatrix} (\Omega + |\kappa| c)^2 & (\Omega + |\kappa| c)|\kappa| s & (\Omega + |\kappa| c)(\Omega - |\kappa| c) \\ (\Omega + |\kappa| c)|\kappa| s & |\kappa|^2 s^2 & (\Omega - |\kappa| c)|\kappa| s \\ (\Omega + |\kappa| c)(\Omega - |\kappa| c) & (\Omega - |\kappa| c)|\kappa| s & (\Omega - |\kappa| c)^2 \end{bmatrix} \begin{bmatrix} q^2 \\ \{q, p\} \\ p^2 \end{bmatrix}$$

We now apply Lemma 4.17 Appendix C on a $3 \times 3$ matrix as above with $\lambda = \Omega - |\kappa| c$, $\mu = \Omega + |\kappa| c$, $x = |\kappa| s$. Since $L = \sqrt{2} r q$ and $G_0 = -r^2 q^2$, the operator $(2G_0)^2 + (2H)^2$ is associated with a $3 \times 3$ matrix as in Lemma 4.17 therefore is bigger than ($r^4$ becomes $4r^4$)

$$\epsilon \left( 4r^4 q^4 - \{\, p, q \,\}^2/2 + \{\, p^2, q^2 \,\} + \lambda^2 w\, q^4 \right) + \text{l.o.t.}$$

Note that $\{ p^2, q^2 \} - \{ p, q \}^2/2 = -(3/2)\mathbb{1}$ and $\{ p^2, q^2 \} \leq p^4 + q^4$ which implies

$$\begin{aligned} 4(G_0^2 + H^2) &\geq \epsilon \left(4r^4 q^4 + \lambda^2 w\, q^4\right) + \text{l.o.t.} \\ &\geq \epsilon \min\{2r^4, \lambda^2 w/2\} \left(q^4 + \{ p^2, q^2 \} + p^4\right) + \text{l.o.t.} \\ &= \epsilon \min\{2r^4, \lambda^2 w/2\} \left(q^2 + p^2\right)^2 + \text{l.o.t..} \end{aligned}$$

The above inequality together with (4.6) implies existence of constants, $g, g' > 0$ such that

$$\|N\xi\|^2 \leq g\|G\xi\|^2 + g'\|\xi\|^2$$

for all $\xi$ finite linear combination of vectors $e_n$ of the orthonormal basis (1.17). Therefore $D(G) \subseteq D(N)$. The other inclusion is trivial and the proof is complete. □

**Theorem 4.19.** *Let $\mathcal{T}$ be the QMS with generator in a generalized GKSL form associated with a single Kraus operator $L = \overline{v}a + ua^\dagger$ and $H$ as in (3.17). The following are equivalent:*

*(i) Operators $L$ and $[H, L]$ are linearly independent i.e. $2\Omega\, \overline{v}u \neq \overline{v}^2\kappa + u^2\overline{\kappa}$,*

*(ii) $\mathcal{T}$ is irreducibile.*

*Proof.* Let us start by showing (i) ⇒ (ii). If $|u| \neq |v|$, then Proposition (4.10) shows irreducibility. If $|u| = |v|$ then Theorem 4.18 shows $D(G) = D(N)$ and with the same proof as Theorem 4.5 we can prove irreducibility also in this case.

The implication (ii) ⇒ (i) follows from Propositions 4.12, 4.13, 4.16. □

Solution to the irreducibility problem is summarized by the following decision tree.

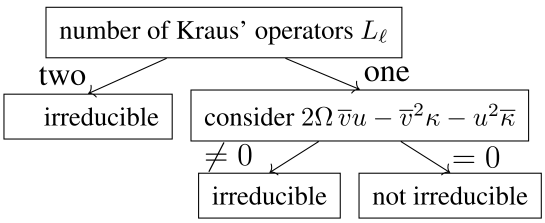



**Figure 4.1:** *Decision tree for irreducibility of a gaussian QMS on $\mathcal{B}(\Gamma_s(\mathbb{C}))$*

## 4.3 Invariant states

In this section we characterize Gaussian QMS with normal invariant states in terms of the parameters in the model. We start recalling the definition of an invariant state.

**Definition 4.20.** We say a density matrix $\rho$ is invariant for the QMS $\mathcal{T}$ (equiv. for the predual $\mathcal{T}_*$) if

$$\mathcal{T}_{t*}(\rho) = \rho, \quad \forall t \geq 0.$$

In virtue of the result of Chapter 3 we have that this semigroup preserves gaussianity of the stated. Indeed we can also specialize Proposition 3.29 to obtain explicit formulas for the mean vectors and the covariance operators of such states. A natural starting point for the study of invariant states is therefore to consider just gaussian states, where the problems can be transferred on the parameters.

To this scope we need this preliminary Lemma that further specifies the results of Lemma 3.27.

**Lemma 4.21.** *Consider the quantity*

$$\gamma = \frac{1}{2}\sum_{\ell=1}^{2}\left(|v_\ell|^2 - |u_\ell|^2\right). \tag{4.19}$$

*If $\gamma \neq 0$ then the operator $C$ given by* (3.40) *is strictly positive*

*Proof.* In virtue of Lemma 3.27 we have

$$C = \sqrt{C}^\sharp \sqrt{C} \geq 0$$

where $\sqrt{C}z = \overline{U}z + V\overline{z}$, for every $z \in \mathsf{h}_\mathbb{R}$. In order to prove strict positivity one only needs to show that $\ker\sqrt{C} = \{0\}$. By direct computation we have

$$\sqrt{C}_\mathfrak{h} = \begin{pmatrix} \overline{U} & V \\ \overline{V} & U \end{pmatrix}.$$

The minor corresponding to the first and third row (resp. the second and the fourth) is $|u_1|^2 - |v_1|^2$ (resp. $|u_2|^2 - |v_2|$). Therefore if $\gamma \neq 0$ at least one of the two minors is non-zero and $\ker\sqrt{C} = \{0\}$. □

The quantity $\gamma$ introduced by (4.19) will be relevant in addressing existence of an invariant state. It has been introduced in this way since it will appear in this form when evaluating the explicit expression of $Z_\mathfrak{h}$ (cf. (4.21)).

We now state the theorem of existence and uniqueness of invariant states.

**Theorem 4.22.** *Let $(\mathcal{T}_t)_{t\geq 0}$ be the QMS with GKSL generator associated with $H, L_1, L_2$ as in (3.18), (3.17) or with $H$ and a single Kraus operator. If $\gamma > 0$ and $\gamma^2+\Omega^2-|\kappa|^2 > 0$ the Gaussian state $\rho = \rho_{(\omega,\mathbf{S})}$ with*

$$\omega = (Z^\sharp)^{-1}\zeta = \frac{(-\gamma + \mathrm{i}\Omega)\zeta - \mathrm{i}\kappa\overline{\zeta}}{\gamma^2 + \Omega^2 - |\kappa|^2}, \qquad S = \int_0^\infty \mathrm{e}^{sZ^\sharp} C \mathrm{e}^{sZ}\mathrm{d}s. \tag{4.20}$$

*is the unique normal invariant state for the semigroup. Moreover, for all initial state $\rho_0$*

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \mathcal{T}_{*s}(\rho_0)\mathrm{d}s = \rho$$

*in trace norm.*

*Proof.* First note that, since $\gamma^2 + \Omega^2 - |\kappa|^2 > 0$, the matrix

$$Z_\mathfrak{h} = \begin{pmatrix} -\gamma + \mathrm{i}\Omega & \mathrm{i}\kappa \\ -\mathrm{i}\overline{\kappa} & -\gamma - \mathrm{i}\Omega \end{pmatrix} \tag{4.21}$$

has only eigenvalues with strictly negative real part and the integral in (4.20) is well-defined.

We now check that $\rho$ is an invariant state. Proposition 3.29 implies that $\rho_t = \mathcal{T}_{*t}(\rho)$ is still a Gaussian state with mean vector and covariance matrix given by equations (3.46) and (3.47). The state $\rho$ is invariant if and only if $\omega_t = \omega$ and $S_t = S$ for every $t \geq 0$ that means

$$\int_0^t \mathrm{e}^{sZ^\sharp}\left(Z^\sharp\omega - \zeta\right)\mathrm{d}s = 0, \qquad \int_0^t \mathrm{e}^{sZ^\sharp}\left(C + Z^\sharp S + SZ\right)\mathrm{e}^{sZ}\mathrm{d}s = 0$$

for all $t \geq 0$. Since both $e^{sZ^\sharp}$ and $e^{sZ}$ are invertible, the invariance of $\rho$ is equivalent to

$$\zeta = Z^\sharp\omega, \quad Z^\sharp S + SZ = -C. \tag{4.22}$$

Conditions on the parameters of the semigroup imply the existence of a pair $(\omega, S)$ satisfying (4.22). Indeed $\gamma^2 + \Omega^2 - |\kappa|^2 \neq 0$ implies invertibility of $Z^\sharp$, which leads to $\omega = (Z^\sharp)^{-1}\zeta$. Furthermore

$$\begin{aligned} Z^\sharp S + SZ &= \int_0^\infty \left(Z^\sharp \mathrm{e}^{s\zeta^\sharp} C \mathrm{e}^{sZ} + \mathrm{e}^{sZ^\sharp} C \mathrm{e}^{sZ} Z\right)\mathrm{d}s \\ &= \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{eZ^\sharp} C \mathrm{e}^{sZ}\right)\mathrm{d}s = \left[\mathrm{e}^{sZ^\sharp} C \mathrm{e}^{sZ}\right] = -C, \end{aligned}$$

and $S \in \mathcal{Q}$ since, as in the proof of Proposition 3.29,

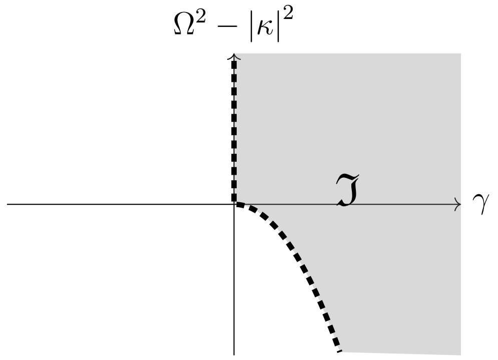
$$\mathbf{S} - \mathrm{i}\mathbf{J} = \int_0^\infty \mathrm{e}^{s\mathbf{Z}^T}\left(\mathbf{C} + \mathrm{i}\left(\mathbf{Z}^T\mathbf{J} + \mathbf{J}\mathbf{Z}\right)\right)\mathrm{e}^{s\mathbf{Z}}\mathrm{d}s \geq 0,$$

where the integral exists since $\mathbf{Z}$ has only eigenvalues with negative real part.

we consider now the uniqueness part of the proof. Whenever the semigroup considered has two linearly independent Kraus' operators it is irreducible, by Theorem 4.5. Moreover, since we have already proven it has an invariant state, this must be faithful and its uniqueness follows from standard results on irreducible QMSs with a faithful normal invariant state (see e.g. [40] Theorem 1 and Lemma 1). Convergence towards the invariant state follows again from known results on QMS with faithful normal invariant states (see e.g. [39] Theorem 2.1). If the semigroup instead has only one Kraus' operator, since $\gamma > 0$ we are in the case of Subsection 4.2.1 and uniqueness and convergence follow from Proposition 4.12. □

*Remark* 4.23. Condition $\gamma > 0$ indicates an overall higher rate of transitions to lower-level states. In order to interpret the other condition we begin by recalling that the Hamiltonian $H$ has discrete spectrum and the QMS generated by $\mathrm{i}[H, \cdot]$ has normal invariant states if and only if $|\kappa|^2 < \Omega^2$. In the case where $\Omega^2 - |\kappa|^2 < 0$ the Hamiltonian $H$ has only continuous spectrum and the additional condition $\gamma^2 > |\kappa|^2 - \Omega^2$ appears. This means that transitions to lower-level states must be stronger to compensate the effect of transitions induced by the Hamiltonian without eigenstates.

Theorem 4.22 shows that a faithful normal Gaussian invariant state exists and is unique for all parameters $(\gamma, \Omega^2 - |\kappa|^2)$ lying in the open shaded region denoted by $\mathfrak{I}$ (Fig. 2 below).




$$\mathfrak{I} = \left\{ \, (\gamma, \Omega^2 - |\kappa|^2) \,\mid\, \gamma > 0, \; \gamma^2 + \Omega^2 - |\kappa|^2 > 0 \, \right\}$$

Fig.2: parameter region $\mathfrak{I}$ (shaded) of QMS with Gaussian invariant states

We will now show that a normal invariant state, Gaussian or not, does not exist for any choice of parameters $(\gamma, \Omega^2 - |\kappa|^2)$ lying outside of the region $\mathfrak{I}$.

Equations (3.38) and (4.20) suggest that the quantity $\Re\left(\overline{\mathrm{e}^{sZ} z}\, C \mathrm{e}^{sZ} z\right)$ plays an important role in the existence of invariant states. Therefore we begin by the following two Lemmas, investigating the asymptotic behaviour of $\mathrm{e}^{tZ} z$ and the convergence of the integral (4.20).

**Lemma 4.24.** *For all choices of parameters $\gamma, \Omega, \kappa$ such that $(\gamma, \Omega^2 - |\kappa|^2)$ falls outside the region $\overline{\mathfrak{I}} \setminus \{(0,0)\}$ there exist $V_+$, a vector subspace of $\mathsf{h}_{\mathbb{R}}$, such that $\left|\mathrm{e}^{tZ} z\right|$ diverges as $t \to \infty$ for every $z \in V_+ \setminus \{0\}$.*

*Proof.* Recall that the matrix representation of the operator $Z_{\mathfrak{h}}$ is given by (4.21). We can divide the region $\overline{\mathfrak{I}} \setminus \{(0,0)\}$ in 4 subsets:

1. $\gamma < 0$ and $\Omega^2 \geq |\kappa|^2$ : eigenvalues of $Z_{\mathfrak{h}}$ are $-\gamma \pm \mathrm{i}\sqrt{\Omega^2 - |\kappa|^2}$ both with strictly positive real part,
2. $\gamma \leq 0$ and $\Omega^2 < |\kappa|^2$: eigenvalues of $Z_{\mathfrak{h}}$ are $-\gamma - \sqrt{|\kappa|^2 - \Omega^2} < -\gamma + \sqrt{|\kappa|^2 - \Omega^2}$. At least $-\gamma + \sqrt{|\kappa|^2 - \Omega^2}$ is strictly positive,
3. $\gamma > 0$ and $\gamma^2 + \Omega^2 - |\kappa|^2 < 0$ so that $\Omega^2 - |\kappa|^2 < 0$ : eigenvalues of $Z_{\mathfrak{h}}$ are $-\gamma \pm \sqrt{|\kappa|^2 - \Omega^2}$. Only the biggest eigenvalue $-\gamma + \sqrt{|\kappa|^2 - \Omega^2}$ is strictly positive,
4. $\gamma = 0$ and $\Omega = \pm\,|\kappa|$ : the only eigenvalue of $Z_{\mathfrak{h}}$ is $0$.

In each of the first three cases there is an eigenvalue $\lambda_+$ and let $z_0 \in \mathfrak{h}$ be an eigenvector associated with $\lambda_+$. We have $\left|\mathrm{e}^{tZ_{\mathfrak{h}}} z_0\right| = \mathrm{e}^{t\Re\lambda_+} |z_0|$. Setting $V_+$ as the real vector subspace generated by $z \in \mathsf{h}_{\mathbb{R}}$ corresponding to $z_0 \in \mathfrak{h}$ via Example 1.18.
In the fourth case, we see from (3.39) that $Z \neq 0$ but $Z^2 = 0$. Hence $\mathrm{e}^{tZ} = 1 + tZ$ and there exists $z_0 \in \mathsf{h}_{\mathbb{R}}$ such that $\mathbf{Zz}_0 \neq 0$. Therefore

$$\left|\mathrm{e}^{tZ} z_0\right| = |z_0 + tZz_0| \geq t\,|Zz_0| - |z_0|\,.$$

and $\left|\mathrm{e}^{tZ} z_0\right|$ diverges as $t \to \infty$. It is then sufficient to choose $V_+$ generated by $z_0$ and the proof is completed also in the last case. $\square$

**Lemma 4.25.** *For all choices of parameters $\gamma, \Omega, \kappa$ such that $(\gamma, \Omega^2 - |\kappa|^2)$ belongs to the boundary of $\mathfrak{I}$ except for the origin $(0,0)$ there exists a vector subspace $V_+$ of $\mathsf{h}_{\mathbb{R}}$ such that for every $z \in V_+ \setminus \{0\}$ the integral*

$$\int_0^t \mathrm{Re}\left\langle \mathrm{e}^{sZ} z, C\mathrm{e}^{sZ} z\right\rangle \mathrm{d}s \tag{4.23}$$

*diverges as $t \to \infty$.*

*Proof.* Consider first the case $\gamma > 0, \gamma = \sqrt{|\kappa|^2 - \Omega^2}$. Since $\gamma^2 + \Omega^2 - |\kappa|^2 = 0$, $Z_{\mathfrak{h}}$ has $0$ as an eigenvalue. Let $z_0 \in \mathfrak{h}$ be an associated eigenvector and fix $V_+ \subset \mathsf{h}_{\mathbb{R}}$ as the vector subspace generated by the element $v_0 \in \mathsf{h}_{\mathbb{R}}$ associated with $z_0$ through Example 1.18. For every $z \in V_+ \setminus \{0\}$ we have

$$\mathrm{Re}\left\langle \mathrm{e}^{tZ} z, C\mathrm{e}^{tZ} z\right\rangle = \mathrm{Re}\left\langle z, Cz\right\rangle .$$

This quantity does not depend on $t$ and is also strictly positive, since $\mathbf{C}$ is invertible thanks to Lemma 4.21. Therefore its integral (4.23) diverges as $t \to \infty$.

Consider now the case $\gamma = 0$, $\Omega^2 > |\kappa|^2$. For every such choice of the parameters $Z_{\mathfrak{h}}$ has two distinct eigenvalues, namely

$$\lambda_\pm = \pm \mathrm{i}\sqrt{\Omega^2 - |\kappa|^2} = \pm \mathrm{i}\delta$$

and thus it can be diagonalized. Let $v_+, v_- \in \mathfrak{h}$ be two eigenvectors corresponding to $\lambda_\pm$ respectively. and let $w_-, w_+ \in \mathsf{h}_{\mathbb{R}}$ be the corresponding element to $v_-, v_+$ respectively, given by Example 1.18. If for $c_+, c_- \in \mathbb{R}$ we define $z = c_- w_- + c_+ w_+$ we have

$$\mathrm{e}^{tZ} z = c_+ \mathrm{e}^{\mathrm{i}t\delta} w_+ + c_- \mathrm{e}^{-\mathrm{i}t\delta} w_- .$$

This implies in particular that the continuous function

$$f_z(t) := \mathrm{Re}\left\langle \mathrm{e}^{tZ} z, C\mathrm{e}^{tZ} z\right\rangle$$

is non negative and periodic. Now, since $C$ is not identically $0$, it exists $z_0 \in \mathsf{h}_{\mathbb{R}}$ such that $f_{z_0}(0) = \mathrm{Re}\langle z_0, Cz_0\rangle \neq 0$. In particular therefore $f_{z_0}$ is not identically zero. Setting $V_+$ as the vector subspace generated by $\mathbf{z}_0$, the integral (4.23) diverges, since its argument coincides with $f_{z_0}$ which is a non-negative, continuous, periodic function which is not identically zero. □

The previous two Lemmas can now be applied to prove the non-existence of invariant states, for some choices of parameters $\gamma, \Omega, \kappa$.

**Proposition 4.26.** *Let $W \subset \mathsf{h}_{\mathbb{R}}$ be a vector subspace. If*

$$w^* - \lim_{t\to\infty} \mathcal{T}_t(W(z)) = 0, \quad \forall z \in W \setminus \{0\} \tag{4.24}$$

*( in the weak* operator topology) then $\mathcal{T}$ has no normal invariant state.*

*In particular, for any choice of the parameters $\gamma, \Omega, \kappa$ such that $(\gamma, \Omega^2 - |\kappa|^2)$ falls outside of the region $\mathfrak{I}$ a normal invariant states for the QMS $\mathcal{T}$ does not exist.*

*Proof.* Suppose $\rho$ is a normal invariant state for $\mathcal{T}$. Then for every $z \in W \setminus \{0\}$

$$\mathrm{tr}(\rho W(z)) = \mathrm{tr}(\rho \mathcal{T}_t(W(z)))$$

for all $t \geq 0$. Taking the limit as $t \to \infty$, by Lemma 4.24, Lemma 4.25 and the explicit formula 3.38, we get $\mathrm{tr}(\rho W(z)) = 0$. But this is not possible due to (i) of Theorem 2.39.

Observe now that if we are also outside of the region $\overline{\mathfrak{I}} \setminus \{(0,0)\}$ we can use Lemma 4.24 and fix $z_0 \in V_+ \setminus \{0\}$. Let $f, g \in \mathsf{h}$, thanks to equation (3.38) we have

$$\left|\langle e(g), W(\mathrm{e}^{tZ} z_0) e(f) \rangle\right| = \exp\left\{ -\frac{\left|\mathrm{e}^{tZ} z_0\right|^2}{2} - \left\langle \mathrm{e}^{tZ} z_0, f \right\rangle - \left\langle g, \mathrm{e}^{tZ} z_0 \right\rangle + \langle g, f \rangle \right\}.$$

Since $\left|\mathrm{e}^{tZ} z_0\right|$ diverges as $t \to \infty$, we have that $W(\mathrm{e}^{tZ} z_0)$ converges weakly to 0. Moreover the Weyl operators are unitary, hence the set $\{W(\mathrm{e}^{tZ} z_0) : t \in \mathbb{R}\}$ is bounded and the weak topology coincides with the weak* one. Therefore $W(\mathrm{e}^{tZ} z_0)$ converges weakly* to 0 and

$$|\mathrm{tr}(\rho \mathcal{T}_t(W(z_0)))| = |c_t(z_0)| \left|\mathrm{tr}(\rho W(\mathrm{e}^{tZ} z_0))\right| \leq \left|\mathrm{tr}(\rho W(\mathrm{e}^{tZ} z_0))\right|,$$

where $c_t(z_0)$ is the constant multiplying the Weyl operator in equation (3.38). This means $\mathcal{T}_t(W(z_0))$ converges to 0 in the weak* topology for every $z_0 \in V_+ \setminus \{0\}$. Hence, for what we have proven at the beginning, there can be no normal invariant states.

Suppose we are now on the boundary of $\mathfrak{I}$ except for the origin $(0,0)$. Let $V_+ \subset \mathsf{h}_{\mathbb{R}}$ be the subspace given by Lemma 4.25 and fix $z_0 \in V_+ \setminus \{0\}$. Thanks to equation (3.38) one has

$$|\mathrm{tr}(\rho \mathcal{T}_t(W(z_0)))| \leq \exp\Big\{ -\frac{1}{2} \int_0^t \mathbf{z_0}^T \mathrm{e}^{s\mathbf{Z}^T} \mathbf{C} \mathrm{e}^{s\mathbf{Z}} \mathbf{z}_0 ds \Big\},$$

since $|\mathrm{tr}(\rho W(z))| \leq 1$ for every $z \in \mathsf{h}$. Letting $t \to \infty$ one has $\mathcal{T}_t(W(z_0)) \to 0$ in the weak* topology for every $z_0 \in V_+ \setminus \{0\}$. Hence also in this case there are no normal invariant states. □

Summarizing we proved the following complement to Theorem 4.22

**Theorem 4.27.** *Let $(\mathcal{T}_t)_{t \geq 0}$ be the QMS with GKSL generator associated with $H, L_1, L_2$ as in (3.18), (3.17) or with $H$ and a single Kraus operator. The QMS $\mathcal{T}$ has a normal invariant state if and only if $\gamma > 0$ and $\gamma^2 + \Omega^2 - |\kappa|^2 > 0$. The normal invariant state is also unique.*

*Proof.* Existence and uniqueness is given by Theorem 4.22. Non-existence is given by Proposition 4.26. □

Note in particular that whenever it exists a normal invariant state it is a gaussian one, justifying once more the naming choice of these semigroups. As a final result we can given some conditions for the invariant state to be pure.

**Proposition 4.28.** *The invariant state given by Theorem 4.22 is pure if and only if there is a single Kraus' operator $L = \bar{v}a + ua^\dagger$ and*

- $\kappa = 0$ *and* $u = 0$

- $\kappa \neq 0$ *both* $u \neq 0$, $\overline{v} \neq 0$ *and*

$$|v|\,|c_{\pm}| = |k|\,|u|\,, \quad uv = -\,|u|\,|v|\,\frac{\kappa}{|\kappa|}\frac{|c_{\pm}|}{c_{\pm}}, \tag{4.25}$$

*where* $c_{\pm} = -\Omega \pm \sqrt{\Omega^2 - |\kappa|^2}$.

*In all the other cases it is faithful.*

*Proof.* Thanks to Proposition 2.49 a Gaussian state is faithful if and only if $\mathbf{S} - \mathrm{i}\mathbf{J} > 0$ otherwise it is pure. As in the proof of Theorem 4.22 we can use

$$\mathbf{S} - \mathrm{i}\mathbf{J} = \int_0^\infty \mathrm{e}^{s\mathbf{Z}^*}\left(\mathbf{C} + \mathrm{i}\left(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}\right)\right)\mathrm{e}^{s\mathbf{Z}}\mathrm{d}s, \tag{4.26}$$

and study its kernel. Clearly, if $\mathbf{C} + \mathrm{i}\left(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}\right) > 0$, also $\mathbf{S} - \mathrm{i}\mathbf{J} > 0$, since $\mathrm{e}^{s\mathbf{Z}}$ is invertible. This happens whenever there are two Kraus operators, thanks to Lemma 3.27 . So the state can be pure only if there is a single Kraus' operator. We restrict ourselves to this case. Now the kernel of $\mathbf{S} - \mathrm{i}\mathbf{J}$ is non-trivial if and only if the argument of the integral (4.26). has a non-trivial kernel. Note that this can happen if and only if at least one of the eigenvectors of $\mathbf{Z}$ belongs to $\ker(\mathbf{C} + \mathrm{i}(\mathbf{Z}^T\mathbf{J} + \mathbf{J}\mathbf{Z}))$. Indeed suppose there is $\mathbf{z}_0 \in \mathsf{h}_{\mathbb{C}} \setminus \{0\}$ such that $(\mathbf{C} + \mathrm{i}(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}))\mathrm{e}^{t\mathbf{Z}}z_0 = 0$ for all $t \geq 0$. We have $\ker(\mathbf{C} + \mathrm{i}(\mathbf{Z}^T\mathbf{J} + \mathbf{J}\mathbf{Z}))$ is one dimensional since it is not identically zero and it is non trivial, by Lemma 3.27. Suppose it is generated by $v_0 \in \mathfrak{h}_{\mathbb{C}}$, we have then $\mathrm{e}^{t\mathbf{Z}}z_0 = \lambda_t v_0$ for some $\lambda_t \in \mathbb{R} \setminus \{0\}$. In particular

$$\lambda_{s+t}v_0 = \mathrm{e}^{(t+s)\mathbf{Z}}z_0 = \lambda_s \mathrm{e}^{t\mathbf{Z}}v_0$$

which means $\mathbf{v}_0$ is an eigenvector for $e^{t\mathbf{Z}}$ and therefore it is also an eigenvector for $\mathbf{Z}$. The converse implication is trivial.

Using again expression (4.21) we recall that eigenvalues of $\mathbf{Z}$ are $\lambda_{\pm} = -\gamma \pm \sqrt{|\kappa|^2 - \Omega^2}$ that are different whenever the parameters are in the region $\mathfrak{I}$, negative and their associated eigenspaces are generated by

$$v_{\pm} = \begin{pmatrix} \kappa \\ c_{\pm} \end{pmatrix}, \text{ if } \kappa \neq 0, \quad v_+ = e_1, v_- = e_2, \text{ if } \kappa = 0,$$

where $c_{\pm} = -\Omega \pm \sqrt{\Omega^2 - |\kappa|^2}$. Let us consider the case $\kappa \neq 0$ first. We have

$$(\mathbf{C} + \mathrm{i}(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}))v_{\pm} = \begin{pmatrix} |u|^2\kappa + uv\left(-\Omega \pm \sqrt{\Omega^2 - |\kappa|^2}\right) \\ \overline{uv}\kappa + |v|^2\left(-\Omega \pm \sqrt{\Omega^2 - |\kappa|^2}\right) \end{pmatrix}. \tag{4.27}$$

Since under the condition $\kappa \neq 0$ we have $c_{\pm} \neq 0$ this forces $u \neq 0$ or $v \neq 0$ because otherwise by setting the entries of (4.27) equal to zero if either $u = 0$ or $v = 0$ also the other should vanish. In this way, setting the first entry of (4.27) equal to zero and taking in consideration the modules we obtain

$$|v|\,|c_{\pm}| = |k|\,|u|\,. \tag{4.28}$$

Plugging it back into the first entry and setting it again equal to zero we obtain the further requirement

$$uv = -|u|\,|v|\,\frac{\kappa}{|\kappa|}\frac{|c_\pm|}{c_\pm}. \tag{4.29}$$

Note that the requirements (4.28), (4.29) solve also the equation obtained by setting the second entry of (4.27) equal to zero.

In the case $\kappa = 0$ we have

$$(\mathbf{C} + \mathrm{i}(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}))e_1 = \begin{pmatrix} |u|^2 \\ \overline{uv} \end{pmatrix}, \quad (\mathbf{C} + \mathrm{i}(\mathbf{Z}^*\mathbf{J} + \mathbf{J}\mathbf{Z}))e_1 = \begin{pmatrix} uv \\ |v|^2 \end{pmatrix}.$$

For them to be equal to zero we should have $u = 0$ for $v_+$ or $v = 0$ for $v_-$. In particular they satisfy (4.29) but since $\gamma > 0$ the case $v = 0$ is not possible.

In the case instead $\Omega^2 = |\kappa|^2$ the eigenvectors $v_\pm$ coincide but the calculations go through nonetheless. □

## 4.4 Examples

In this section we present the application of our results in two remarkable cases. These also serve to illustrate the relationships we have found between the parameters that determine the behaviour of the dynamics.

### 4.4.1 Open Quantum Harmonic Oscillator

Let $\mathcal{T}$ be the QMS with generator in a generalized GKSL form with

$$L_1 = \mu\, a, \qquad L_2 = \lambda\, a^\dagger, \qquad H = \Omega\, a^\dagger a + \frac{\kappa}{2}a^{\dagger 2} + \frac{\overline{\kappa}}{2}a^2 + \frac{\zeta}{2}a^\dagger + \frac{\overline{\zeta}}{2}a \tag{4.30}$$

with $\lambda, \mu \geq 0$, $\Omega \in \mathbb{R}$, $\kappa, \zeta \in \mathbb{C}$. The special case where $\kappa = \zeta = 0$ has been analyzed in [17] providing the full spectral analysis of the generator $\mathcal{L}$ in the $L^2$ space of the invariant state for $\lambda < \mu$.

In this model $\gamma = (\mu^2 - \lambda^2)/2$. Moreover, in the case where both $\lambda, \mu$ are strictly positive, the QMS is irreducible (Theorem 4.5) and admits a unique faithful normal invariant state if and only if $\lambda^2 < \mu^2$ and $(\mu^2 - \lambda^2)/4 + \Omega^2 - |\kappa|^2 > 0$ (Theorem 4.22) with the explicit mean vector $\omega$ and covariance operator $S$ as in (4.20).

If $\mu = 0$ and $\lambda > 0$ we obtain a QMS which is irreducible if and only if $\overline{\kappa}\lambda^2 \neq 0$, namely $\kappa \neq 0$ and has no normal invariant state (subsection 4.2.2).
Finally, in the case where $\lambda = 0$ and $\mu > 0$ we find a QMS which is irreducible if and only if $\kappa\mu^2 \neq 0$ i.e. $\kappa \neq 0$. It admits invariant states if and only if $|\kappa|^2 < \Omega^2 + \lambda^4/4$; these will be faithful if $\Re(\kappa) \neq 0$ and pure otherwise (subsection 4.2.1). For any initial state $\rho_0$, in both cases, $t^{-1}\int_0^t \mathcal{T}_{*s}(\rho_0)\mathrm{d}s$ converges towards the unique invariant state by Theorem 4.22.
It is worth noticing here that the Hamiltonian $H$ is bounded from below or above if and only if $\Omega^2 - |\kappa|^2 \geq 0$, in which case it has discrete spectrum. Therefore condition $|\kappa|^2 < \Omega^2 + \lambda^4/4$ appears as a relaxation of discreteness of spectrum that allows existence of normal invariant states.

### 4.4.2 Quantum Fokker-Planck model

The quantum Fokker-Planck (QFP) model is an open quantum system introduced to describe the quantum mechanical charge-transport including diffusive effects (see [8] and the references therein). In this subsection we show that a simple application of our results allows one to study the dynamics.

The formal generator

$$\begin{aligned}\mathcal{L}(x) &= \frac{\mathrm{i}}{2}\left[p^2+\omega^2q^2,x\right]+\mathrm{i}g\left\{p,[q,x]\right\}\\ &- D_{qq}[p,[p,x]]-D_{pp}[q,[q,x]]+2D_{pq}[q,[p,x]]\,,\end{aligned}$$

can be written in generalised GKLS form (3.16) with

$$H=\frac{1}{2}\left(p^2+\omega^2q^2+g(pq+qp)\right),$$

and $L_1$, $L_2$ are the operators

$$L_1=\frac{-2D_{pq}+\mathrm{i}g}{\sqrt{2D_{pp}}}p+\sqrt{2D_{pp}}\,q\,,\qquad L_2=\frac{2\sqrt{\Delta}}{\sqrt{2D_{pp}}}\,p\,.$$

where $\omega^2>0$, $D_{pp}>0, D_{qq}\geq 0$, $D_{pq}\in\mathbb{R}$ and $\Delta=D_{pp}D_{qq}-D_{pq}^2-g^2/4\geq 0$. Clearly, $L_1,L_2$ are linearly independent if and only if $\Delta>0$. Moreover

$$\begin{aligned}L_1&=\frac{-2\mathrm{i}D_{pq}-g}{2\sqrt{D_{pp}}}\left(a^\dagger-a\right)+\sqrt{D_{pp}}\left(a^\dagger+a\right),L_2=\frac{\mathrm{i}\sqrt{\Delta}}{\sqrt{D_{pp}}}\left(a^\dagger-a\right)\\ H&=\frac{\omega^2+1}{2}aa^\dagger+\frac{\omega^2-1+2\mathrm{i}g}{4}\,a^{\dagger 2}+\frac{\omega^2-1-2\mathrm{i}g}{4}\,a^2+\frac{\omega^2+1}{4}\end{aligned}$$

so that

$$\begin{aligned}\overline{v}_1&=\frac{2\mathrm{i}D_{pq}+g}{2\sqrt{D_{pp}}}+\sqrt{D_{pp}},\quad u_1=-\frac{2\mathrm{i}D_{pq}+g}{2\sqrt{D_{pp}}}+\sqrt{D_{pp}},\\ \overline{v}_2&=-\frac{\mathrm{i}\sqrt{\Delta}}{\sqrt{D_{pp}}},\quad u_2=\frac{\mathrm{i}\sqrt{\Delta}}{\sqrt{D_{pp}}},\\ \Omega&=\frac{\omega^2+1}{2},\qquad \kappa=\frac{\omega^2-1}{2}+\mathrm{i}g\end{aligned}$$

Compute

$$\gamma=\frac{1}{2}\sum_{\ell=1}^{2}\left(|v_\ell|^2-|u_\ell|^2\right)=g,\qquad \Omega^2-|\kappa|^2=\omega^2-g^2,\qquad \gamma^2+\Omega^2-|\kappa|^2=\omega^2$$

Therefore, in the case where $\Delta>0$ Kraus operators $L_1,L_2$ are linearly independent, the QFP semigroup is irreducible and a Gaussian invariant state exists if and only if $g=\gamma>0$. This is given explicitly in Theorem 4.22. Moreover, it is also faithful and it is the unique normal invariant state by irreducibility.

The case $\Delta=0$ has to be considered separately (see [8]). By Theorem (4.19) the QFP semigroup is irreducible if and only if $2\Omega\overline{v}_1u_1=\kappa\overline{v}_1^2+\overline{\kappa}u_1^2$. Taking the imaginary parts of this identity we find

$$gD_{pp}=-\omega^2D_{pq}. \tag{4.31}$$

Taking real parts we find $\omega^2\left(4D_{pq}^2 - g^2\right) + 4D_{pp}^2 = -8gD_{pq}D_{pp}$ and, from (4.31), we find the identity

$$\omega^2\left(4D_{pq}^2 - g^2\right) + 4D_{pp}^2 = -8gD_{pq}D_{pp} = 8\omega^2 D_{pq}^2$$

namely $4D_{pp}^2 = \omega^2\left(4D_{pq}^2 + g^2\right)$ and, by $\Delta = 0$ together with $D_{pp} > 0$

$$D_{pp} = \omega^2 D_{qq}. \tag{4.32}$$

Note that $\Delta = 0$ together with (4.31) and (4.32) are equivalent to conditions under which, for $\gamma > 0$, the Gaussian normal invariant state of the QFP model is pure (see [8] Lemma 9.1).

Clearly, they are equivalent to (4.25). Indeed, a straightforward computation shows that the first identity is equivalent to $D_{pq} = -gD_{qq}$ and the second one to $D_{qq} = g/(2\delta)$ which follows from $\Delta = 0$, (4.31) and (4.32) (see [8] Lemma 9.1 for details).

Convergence towards the unique invariant state of $t^{-1}\int_0^t \mathcal{T}_{*t}(\rho_0)\mathrm{d}s$ holds for any initial state $\rho_0$ by Theorem 4.22.

CHAPTER 5

# The Decoherence-free subalgebra of Gaussian QMSs

The decoherence-free subalgebra is an important object in the study of an open quantum system, in particular when trying to address long term properties of the evolution (see [5, 26, 27, 36] and references therein). Here we try to compute it for the case of gaussian QMSs. In the first section we recall the definition of the decoherence-free subalgebra along with some of its properties and a general characterization theorem. In the second section, we apply this theorem to the case of gaussian QMSs and obtain a characterization of the decoherence-free subalgebra in this case. In the third section instead we provide some examples of application to some specific models. This chapter is base on [3].

## 5.1 The Decoherence-Free subalgebra

We work in the setting of gaussian QMSs, namely we set $\mathcal{H} = \Gamma_s(\mathsf{h})$. We recall the definition of the decoherence-free subalgebra.

**Definition 5.1.** Let $\mathcal{T}$ be a QMS. The *decoherence-free subalgebra* of $\mathcal{T}$, denoted $\mathcal{N}(\mathcal{T})$ is

$$\mathcal{N}(\mathcal{T}) = \{x \in \mathcal{B}(\mathcal{H}) \mid \mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x),\ \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x^*),\ \forall t \geq 0\}\,. \tag{5.1}$$

We consider semigroups arising from a generator in a GKLS , either proper (as in (3.16)) or weak (as in (3.20)). We recall here some of the known properties of this subalgebra, whose proofs can be found in [30, Theorem 3.1].

**Proposition 5.2.** *Let $\mathcal{T}$ be a quantum Markov semigroup on $\mathcal{B}(\mathcal{H})$ and let $\mathcal{N}(\mathcal{T})$ be the set defined by (5.1). Then*

1. *$\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ on which $\mathcal{T}_t$ acts as a *-homomorphism for every $t \geq 0$.*
2. *For every $x \in \mathcal{N}(\mathcal{T})$ we have*
$$\mathcal{T}_t(x) = \mathrm{e}^{-\mathrm{i}tH} x \mathrm{e}^{\mathrm{i}tH}, \quad \forall t \geq 0,$$
*where $H$ is the Hamiltonian of the GKLS generator.*

The decoherence-free subalgebra of a QMS with a bounded generator, i.e. written in a GKLS form (3.16) with bounded operators $H, L_\ell$ is the commutator of the set of operators $\delta_H^n(L_\ell), \delta_H^n(L_\ell^*)$ with $\ell = 1, \ldots, m$, $n \geq 0$ where $\delta_H(x) = [H, x]$.

However, since we are interested in a situation where the operators of the generator are unbounded we need a different result. It turns out that $\mathcal{N}(\mathcal{T})$ can be characterized in a similar way (see [27]) considering generalized commutators of a set of unbounded operators.

**Definition 5.3.** For $Y \subset \mathcal{B}(\mathcal{H})$ we define the *generalized commutant* of $Y$, denoted $Y'$, as
$$Y' = \{x \in \mathcal{B}(\mathcal{H}) \mid xy \subset yx, \forall y \in Y\}.$$

The characterization result for $\mathcal{N}(\mathcal{T})$ is the following

**Theorem 5.4.** *The decoherence-free subalgebra of a Gaussian QMS with generator in a generalized GKLS form associated with operators $L_\ell, H$ as in (3.18),(3.17) is the generalized commutant of linear combinations of creation and annihilation operators*
$$\delta_H^n(L_\ell), \quad \delta_H^n(L_\ell^*) \qquad \ell = 1, \ldots, m,\ 0 \leq n \leq 2d-1 \tag{5.2}$$
*where $\delta_H(x) = [H, x]$ denotes the generalized commutator and $\delta_H^n$ denotes the $n$-th iterate.*

A first proof of this result appeared in [27]. Here we present a different version of it which uses a more straightforward approach than the one in [27]. Before proceeding to the proof, however, we provide a short version of it in the case where the operators $L_\ell$ and $H$ are bounded and the problem is mostly algebraic. Indeed most of the difficulties of the general case appear due to domain problems of the operators

*Proof.* (in case $H$,$L_\ell$ are bounded) We will use formula (3.32) in the case of a bounded GKLS generator. Note that, if $x \in \mathcal{N}(\mathcal{T})$ and $y$ is arbitrary then, since $\mathcal{T}_t(y^*x) = \mathcal{T}_t(y^*)\mathcal{T}_t(x)$ by Proposition 5.2, taking the derivative at $t = 0$ we get
$$\mathcal{L}(y^*x) = \mathcal{L}(y^*)x + y^*\mathcal{L}(x)$$
therefore
$$\sum_\ell [L_\ell, y]^*[L_\ell, x] = 0. \tag{5.3}$$
If the operators $L_\ell$ are bounded, we are allowed to take $x = y$ then $[L_\ell, x] = 0$ for all $\ell$. Moreover, since $x^*$ also belongs to $\mathcal{N}(\mathcal{T})$, taking the adjoint $[L_\ell^*, x^*] = 0$, $x$ also

commutes with all the operators $L_\ell^*$ and $\mathcal{L}(x) = \mathrm{i}[H,x]$. Clearly, since $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant via Proposition 5.2, $\mathcal{L}(x) = \lim_{t\to 0}(\mathcal{T}_t(x) - t)/t$ belongs to $\mathcal{N}(\mathcal{T})$. Therefore $[L_\ell, [H,x]] = 0$ for all $\ell$ and, by the Jacobi identity

$$[\,x,[H,L_\ell]\,] = -\,[\,H,[L_\ell,x]\,] - [\,L_\ell,[H,x]\,] = 0.$$

In this way, one can show inductively that $x$ commutes with the iterated commutators (5.2). $\square$

If the operators $L_\ell, H$ are unbounded, one has to cope with several problems. The operator $\mathcal{L}$ is unbounded and, even if we choose $x, y$ in the domain of $\mathcal{L}$, it is not clear whether $y^*x$ belongs to the domain of $\mathcal{L}$. Multiplication of generalized commutators $[L_\ell, y]\,[L_\ell, x]$ may not be defined. If we choose a "nice" $y \in D(\mathcal{L})$ then it is not clear whether we can take $x = y$ because we do not know a priori if our "nice" $y$ belongs to $\mathcal{N}(\mathcal{T})$.

Eventually, before moving on to the proof of Theorem 5.4 we give an example to show that inequality the $n \le 2d-1$ is sharp.

**Example 5.5.** Consider the gaussian QMS with only one Kraus' operator $L_\ell$, i.e. $m = 1$ Using Notation 1.58 we define,

$$L_1 = p_1, \qquad H = q_d^2 + \sum_{j=1}^{d-1} p_{j+1}q_j$$

It is easy to see, using (1.12) that

$$\delta_H(L_1) = 2\mathrm{i}p_2, \quad \delta_H^2(L_1) = -4p_3,\ \ldots,\ \delta_H^{d-1}(L_1) = (2\mathrm{i})^{d-1}p_d, \quad \delta_H^d(L_1) = -(2\mathrm{i})^d q_d$$

and consequentially

$$\delta_H^{d+1}(L_1) = (2\mathrm{i})^{d+1}q_{d-1}\ldots$$

Iterating $\delta_H^{d+k}(L_1)$ is proportional to $q_{d-k}$ so that, for $k = d-1$ one gets $q_1$. Clearly, for all $k$ with $0 \le k \le d-1$

$$\left\{\,\delta_H^j(L_1),\,\delta_H^j(L_1^*)\ \mid\ j \le d+k\,\right\}' = L^\infty(\mathbb{R}^{d-1-k}) \quad \text{as functions of } p_1,\ldots,p_{d-1+k}$$

Summing up, if we consider $2d-1$ iterated commutators we get all $p_j, q_j$ and $\mathcal{N}(\mathcal{T})$ is trivial.

### 5.1.1 Proof of Theorem 5.4

We derive the characterization of $\mathcal{N}(\mathcal{T})$ in terms of iterated commutators. We begin by illustrating the idea of the proof in the case where the operators $L_\ell$ and $H$ are bounded.

*Proof.* (case $H$,$L_\ell$ bounded) For all $x, y \in \mathcal{B}(\mathsf{h})$, recall the formula (3.32) from Lemma 3.21. Note that, if $x \in \mathcal{N}(\mathcal{T})$ and $y$ is arbitrary then, since $\mathcal{T}_t(y^*x) = \mathcal{T}_t(y^*)\mathcal{T}_t(x)$ by Proposition 5.2 1, taking the derivative at $t = 0$ we get $\mathcal{L}(y^*x) = \mathcal{L}(y^*)x + y^*\mathcal{L}(x)$ therefore

$$\sum_{\ell=1}^{m}[L_\ell, y]^*[L_\ell, x] = 0. \tag{5.4}$$

If the operators $L_\ell$ are bounded, we are allowed to take $x = y$, then $[L_\ell, x] = 0$ for all $\ell$. Moreover, since $x^*$ also belongs to $\mathcal{N}(\mathcal{T})$, taking the adjoint of $[L_\ell, x^*] = 0$, $x$ also commutes with all the operators $L_\ell^*$ and $\mathcal{L}(x) = \mathrm{i}[H, x]$. Clearly, since $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant, $\mathcal{L}(x) = \lim_{t\to 0}(\mathcal{T}_t(x) - x)/t$ belongs to $\mathcal{N}(\mathcal{T})$. Therefore $[L_\ell, [H, x]] = 0$ for all $\ell$ and, by the Jacobi identity

$$[\, x, [H, L_\ell]\,] = -\,[\, H, [L_\ell, x]\,] - [\, L_\ell, [H, x]\,] = 0.$$

In this way, one can show inductively that $x$ commutes with the iterated commutators (5.2). $\square$

If the operators $L_\ell, H$ are unbounded, one has to cope with several problems. The operator $\mathcal{L}$ is unbounded and, even if we choose $x, y$ in the domain of $\mathcal{L}$, it is not clear whether $y^*x$ belongs to the domain of $\mathcal{L}$ (see [32]). Multiplication of generalized commutators $[L_\ell, y]$ $[L_\ell, x]$ may not be defined. If we choose a "nice" $y \in \mathrm{Dom}(\mathcal{L})$ then it is not clear whether we can take $x = y$ because we do not know a priori if our "nice" $y$ belongs to $\mathcal{N}(\mathcal{T})$.

We begin the analysis of $\mathcal{N}(\mathcal{T})$ by a few preliminary lemmas.

**Lemma 5.6.** *The following derivative exists with respect norm topology for all $z \in \mathbb{C}$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{T}_t(W(z))e_g\Big|_{t=0} = G^*W(z)e_g + \sum_{\ell=1}^{m} L_\ell^* W(z) L_\ell e_g + W(z)Ge_g$$

*Proof.* Note at first that the right-hand side operator $G^*W(z) + \sum_{\ell=1}^m L_\ell^* W(z) L_\ell + W(z)G$ is unbounded (for $z \neq 0$) therefore $W(z)$ does not belong to the domain of $\mathcal{L}$ but we can consider the quadratic form $\pounds(W(z))$ on $D \times D$. Differentiability of functions $t \mapsto \langle \xi', \mathcal{T}_t(x)\xi\rangle$ also holds for $\xi, \xi'$ in the linear span of exponential vectors $E$. Therefore, for all such $\xi$, from (3.23)

$$\langle \xi, (\mathcal{T}_t(W(z)) - W(z) - t\pounds(W(z)))\, e_g\rangle = \int_0^t \langle \xi, \pounds(\mathcal{T}_s(W(z)) - W(z))e_g\rangle\, \mathrm{d}s.$$

Recalling that $\mathcal{T}_s(W(z)) = \varphi_z(s)W(\mathrm{e}^{sZ}z)$ as in (3.38) for a complex valued function $\varphi$ such that $\lim_{s\to 0}\varphi_z(s) = 1$, the right-hand side integrand can be written as

$$(\varphi_z(s) - 1)\left\langle \xi, \pounds(W(\mathrm{e}^{sZ}z))e_g\right\rangle + \left\langle \xi, \pounds(W(\mathrm{e}^{sZ}z) - W(z))e_g\right\rangle$$

A long but straightforward computation shows the function

$$s \mapsto \pounds(W(\mathrm{e}^{sZ}z) - W(z))e_g$$

is continuous vanishing at $s = 0$ and the function $s \mapsto \pounds(W(\mathrm{e}^{sZ}z))e_g$ is bounded with respect to the Fock space norm. Therefore, taking suprema for $\xi \in \Gamma(\mathbb{C}^d)$, $\|\xi\| = 1$, we find the inequalities

$$\begin{aligned}\|(\mathcal{T}_t(W(z)) - W(z) - t\pounds(W(z)))\, e_g\| \le &\int_0^t |\varphi_z(s) - 1|\, \left\|\pounds(W(\mathrm{e}^{sZ}z))e_g\right\| \mathrm{d}s \\ &+ \int_0^t \left\|\pounds(W(\mathrm{e}^{sZ}z) - W(z))e_g\right\| \mathrm{d}s\end{aligned}$$

The conclusion follows dividing by $t$ and taking the limit as $t \to 0^+$. $\square$

**Lemma 5.7.** *Let $x \in \mathcal{N}(\mathcal{T})$. For all exponential vectors $e_g, e_f$ and all Weyl operators $W(z)$ we have*

$$\sum_{\ell=1}^{m} \langle [L_\ell, W(-z)]\, e_g, xL_\ell\, e_f\rangle = \sum_{\ell=1}^{m} \langle L_\ell^*\, [L_\ell, W(-z)]\, e_g, x\, e_f\rangle\,. \tag{5.5}$$

*Proof.* If $x \in \mathcal{N}(\mathcal{T})$, then, for all $g, f, z \in \mathbb{C}^d$ and $t \geq 0$ we have

$$\begin{aligned}\langle e_g, (\mathcal{T}_t(W(z)x) - W(z)x e_f\rangle &= \langle e_g, (\mathcal{T}_t(W(z))\mathcal{T}_t(x) - W(z)x)\, e_f\rangle \\ &= \langle e_g, (\mathcal{T}_t(W(z)) - W(z))\, x\, e_f\rangle + \langle e_g, W(z)\, (\mathcal{T}_t(x) - x)\, e_f\rangle \\ &+ \langle(\mathcal{T}_t(W(-z)) - W(-z))\, e_g, (\mathcal{T}_t(x) - x)\, e_f\rangle\end{aligned}$$

By Lemma 5.6 means the norm limit

$$\lim_{t\to 0^+} \frac{(\mathcal{T}_t(W(-z)) - W(-z))e_g}{t}$$

exists, therefore $\sup_{t>0} t^{-1}\|\mathcal{T}_t(W(-z)) - W(-z))e_g\| < +\infty$. Moreover

$$\begin{aligned}\|(\mathcal{T}_t(x) - x)\, e_f\|^2 &= \langle e_f, (\mathcal{T}_t(x^*) - x^*)\, (\mathcal{T}_t(x) - x)\, e_f\rangle \\ &\leq \langle e_f, (\mathcal{T}_t(x^*x) - x^*\mathcal{T}_t(x) - \mathcal{T}_t(x^*)x + x^*x)\, e_f\rangle\end{aligned}$$

which tends to $0$ as $t \to 0^+$ by weak$^*$ continuity of $\mathcal{T}_t$. As a result

$$\lim_{t\to 0^+} t^{-1}\, \langle e_g, (\mathcal{T}_t(W(z)) - W(z))\, (\mathcal{T}_t(x) - x)\, e_f\rangle = 0,$$

therefore

$$\begin{aligned}\lim_{t\to 0^+} t^{-1}\langle e_g, (\mathcal{T}_t(W(z)x) - W(z)x)e_f\rangle &= \lim_{t\to 0^+} t^{-1}\, \langle e_g, W(z)\, (\mathcal{T}_t(x) - x)\, e_f\rangle \\ &+ \lim_{t\to 0^+} t^{-1}\, \langle(\mathcal{T}_t(W(-z)) - W(-z))\, e_g, x\, e_f\rangle\end{aligned}$$

and we get

$$\begin{aligned}&\langle Ge_g, W(z)x\, e_f\rangle + \sum_{\ell=1}^{m} \langle L_\ell e_g, W(z)xL_\ell e_f\rangle + \langle e_g, W(z)x\, Ge_f\rangle \\ &\quad= \left\langle \left( G^*W(-z) + \sum_\ell L_\ell^* W(-z)L_\ell + W(-z)G\right) e_g, x\, e_f\right\rangle \\ &\quad+ \langle GW(-z)e_g, xe_f\rangle + \sum_{\ell=1}^{m} \langle L_\ell W(-z)e_g, xL_\ell e_f\rangle + \langle W(-z)e_g, xGe_f\rangle\,.\end{aligned}$$

The first term in the left-hand side cancels with the third term in right-hand side and last terms in both sides cancel as well. Noting that

$$G^*W(-z)e_g + GW(-z)e_g = -\sum_\ell L_\ell^* L_\ell W(-z)e_g$$

adding the first and fourth terms in the right-hand side, we find

$$\sum_{\ell=1}^{m}\langle L_\ell e_g, W(z)xL_\ell e_f\rangle = \sum_{\ell=1}^{m}\left[\langle L_\ell^* W(-z)L_\ell e_g, x\, e_f\rangle + \langle L_\ell W(-z)e_g, xL_\ell e_f\rangle\right] - \sum_{\ell=1}^{m}\langle L_\ell^* L_\ell W(-z)e_g, xe_f\rangle .$$

Rearranging terms we get (5.5) which is a weak form of identity (5.4). □

The following lemma serves to get (5.5) for each $\ell$ fixed without summation, taking advantage of the arbitrariness of $z$.

**Lemma 5.8.** *For all $\ell_\bullet \in \{1, 2, \ldots, d\}$ fixed there exists $z \in \mathbb{C}^d$ such that*

$$\sum_{j=1}^{d}(\overline{v}_{ij}z_j + u_{ij}\overline{z}_j) = \delta_{i,\ell_\bullet} = \begin{cases} 1 & \text{if } i = \ell_\bullet \\ 0 & \text{if } i \neq \ell_\bullet \end{cases}$$

*Proof.* Note that $\overline{V}z + U\overline{z}$ arises from the map composition

$$z \;\overset{J_c}{\rightarrow}\; \begin{bmatrix} z \\ \overline{z} \end{bmatrix} \;\overset{[\,\overline{V}\,|\,U\,]}{\longrightarrow}\; \overline{V}z + U\overline{z}$$

Let $(\phi_\ell)_{1\leq\ell\leq m}$ be an orthonormal basis of $\mathbb{C}^m$. We look for a $z \in \mathbb{C}^m$ solving the real linear system

$$[\,\overline{V}\,|\,U\,]\, J_c\, z = \phi_{\ell_\bullet}$$

Since

$$\mathrm{Ran}\left([\,\overline{V}\,|\,U\,]\, J_c\right) = \ker\left(([\,\overline{V}\,|\,U\,]\, J_c)^T\right)^\perp = \ker\left(J_c^T\, [\,\overline{V}\,|\,U\,]^T\right)^\perp,$$

$J_c$ is one-to-one and, by the minimality assumption (3.19)

$$\ker\left([\,\overline{V}\,|\,U\,]^T\right) = \ker\left(\begin{bmatrix} V^* \\ U^T \end{bmatrix}\right) = \ker(V^*) \cap \ker\left(U^T\right) = \{0\},$$

we find $\mathrm{Ran}\left([\,\overline{V}\,|\,U\,]\, J_c\right) = \mathbb{C}^m$ and the proof is complete. □

**Proposition 5.9.** *The decoherence-free subalgebra $x \in \mathcal{N}(\mathcal{T})$ is contained in the generalized commutant of the operators $L_\ell, L_\ell^*$ $1 \leq \ell \leq m$.*

*Proof.* For a Weyl operator $W(z)$ we have

$$[L_\ell, W(z)] = \sum_{j=1}^{d}\left[\overline{v}_{\ell j}a_j + u_{\ell j}a_j^\dagger, W(z)\right] = \sum_{j=1}^{d}(\overline{v}_{\ell j}z_j + u_{\ell j}\overline{z}_j)\, W(z).$$

and (5.5) becomes

$$\sum_{\ell=1}^{m}\sum_{j=1}^{d}\left(\overline{v}_{\ell j}z_j+u_{\ell j}\overline{z}_j\right)\langle W(-z)e_g, xL_\ell e_f\rangle$$
$$=\sum_{\ell=1}^{m}\sum_{j=1}^{d}\left(\overline{v}_{\ell j}z_j+u_{\ell j}\overline{z}_j\right)\langle L_\ell^* W(-z)e_g, xe_f\rangle\,.$$

By Lemma 5.8, choosing some special $z_\ell \in \mathbb{C}^d$, we find

$$\langle W(-z_\ell)e_g, xL_\ell e_f\rangle = \langle L_\ell^* W(-z_\ell)e_g, xe_f\rangle$$

for all $g, f \in \mathbb{C}^d$ and all $\ell$. Therefore, by the arbitrariness of $g$ and the explicit action of Weyl operators on exponential vectors

$$\langle e_w, xL_\ell e_f\rangle = \langle L_\ell^* e_w, xe_f\rangle$$

for all $w, f \in \mathbb{C}^d$ and all $\ell$. Since exponential vectors form a core for $L_\ell^*$ and $L_\ell$ is closed, this implies that $xe_f$ belongs to the domain of $L_\ell$ and $L_\ell xe_f = xL_\ell e_f$, namely $xL_\ell \subseteq L_\ell x$.

Replacing $x$ with $x^*$ we find $x^*L_\ell \subseteq L_\ell x^*$ and standard results on the adjoint of products of operators (see e.g. [50] 5.26 p. 168) lead to the inclusions

$$xL_\ell^* \subseteq (L_\ell x^*)^* \subseteq (x^*L_\ell)^* = L_\ell x^*.$$

It follows that $x$ belongs to the generalised commutant of the set of Kraus operators, namely $\{ L_\ell, L_\ell^* \mid 1 \le \ell \le m \}$. □

**Lemma 5.10.** *The domain* $\mathrm{Dom}(N^{n/2})$ *is* $\mathrm{e}^{\mathrm{i}tH}$*-invariant for all* $t \in \mathbb{R}$ *and there exists a constant* $c_n > 0$ *such that*

$$\left\|N^{n/2}\mathrm{e}^{\mathrm{i}tH}\xi\right\|^2 \le \mathrm{e}^{c_n|t|}\left\|N^{n/2}\xi\right\|^2 \tag{5.6}$$

*for all* $\xi \in \mathrm{Dom}(N^{n/2})$.

*Proof.* Consider Yosida approximations of the identity operator $(\mathbb{1} + \epsilon N)^{-1}$ for all $\epsilon > 0$ and bounded approximations $N_\epsilon^n = N^n(\mathbb{1} + \epsilon N)^{-n}$ of the $n$-the power of the number operator $N$. Note that, the domain $D$ is invariant for these operators and also $H$ invariant. For all $u \in D$, setting $v_\epsilon = (\mathbb{1} + \epsilon N)^{-n}u$ we have

$$\langle u, (N_\epsilon^n H - HN_\epsilon^n)u\rangle = \langle v_\epsilon, (N^n H(\mathbb{1} + \epsilon N)^n - (\mathbb{1} + \epsilon N)^n HN^n)v_\epsilon\rangle$$

Computing

$$(N^n H(\mathbb{1} + \epsilon N)^n - (\mathbb{1} + \epsilon N)^n HN^n) = \sum_{k=0}^{n}\binom{n}{k}\epsilon^k(N^nHN^k - N^kHN^n)$$
$$= \sum_{k=0}^{n}\binom{n}{k}\epsilon^k N^k[N^{n-k}, H]N^k$$

and noting that the commutator $[N^{n-k}, H]$ is a polynomial in $a_j, a_k^\dagger$ of order $2(n-k)$ so that there exist a constant $c_n$ such that $\left|\langle u', [N^{n-k}, H]u'\rangle\right| \leq c_n \|N^{(n-k)/2}u'\|^2$ (for $u' \in D$) we find the inequality

$$
\begin{aligned}
|\langle u, N_\epsilon^n H u\rangle - \langle Hu, N_\epsilon^n u\rangle| &\leq c_n \sum_{k=0}^{n} \binom{n}{k} \epsilon^k \langle v_\epsilon, N^{n+k} v_\epsilon\rangle \\
&= c_n \langle v_\epsilon, N^n(\mathbb{1} + \epsilon N)^n v_\epsilon\rangle = c_n \langle u, N_\epsilon^n u\rangle .
\end{aligned}
$$

The above inequality extends to $u \in \mathrm{Dom}(H)$ by density.
Now, for all $u \in \mathrm{Dom}(H)$ and $t \geq 0$, we have

$$
\begin{aligned}
\left\|N_\epsilon^{n/2} \mathrm{e}^{\mathrm{i}tH} u\right\|^2 - \left\|N_\epsilon^{n/2} u\right\|^2 &= \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \left\|N_\epsilon^{n/2} \mathrm{e}^{\mathrm{i}sH} u\right\|^2 \mathrm{d}s \\
&= \mathrm{i} \int_0^t \left(\left\langle \mathrm{e}^{\mathrm{i}sH} u, N_\epsilon^n H \mathrm{e}^{\mathrm{i}sH} u\right\rangle - \left\langle H \mathrm{e}^{\mathrm{i}sH} u, N_\epsilon^n \mathrm{e}^{\mathrm{i}sH} u\right\rangle\right) \mathrm{d}s \\
&\leq c_n \int_0^t \left\|N_\epsilon^{n/2} \mathrm{e}^{\mathrm{i}sH} u\right\|^2 \mathrm{d}s.
\end{aligned}
$$

Gronwall's lemma implies and a similar argument for $t < 0$ yield

$$
\left\|N_\epsilon^{n/2} \mathrm{e}^{\mathrm{i}tH} u\right\|^2 \leq \mathrm{e}^{c_n|t|} \left\|N_\epsilon^{n/2} u\right\|^2
$$

for all $t \in \mathbb{R}$. Considering $u \in D$ and taking the limit as $\epsilon$ goes to zero we get (5.6) for $\xi \in D$ and, finally for $\xi \in \mathrm{Dom}(N^{n/2})$ because $D$ is a core for $N^{n/2}$. ☐

**Lemma 5.11.** *For all $j$ there exists $M_d(\mathbb{C})$ valued analytic functions $\mathsf{H}^-, \mathsf{H}^+$ such that*

$$
\mathrm{e}^{-\mathrm{i}tH} a_j \mathrm{e}^{\mathrm{i}tH} \xi = \sum_{k=1}^{d} \left(\mathsf{H}_{jk}^-(t) \mathrm{e}^{-\mathrm{i}tH} a_k \mathrm{e}^{\mathrm{i}tH} \xi + \mathsf{H}_{jk}^+(t) \mathrm{e}^{-\mathrm{i}tH} a_k^\dagger \mathrm{e}^{\mathrm{i}tH} \xi\right)
$$

*for all $t \in \mathbb{R}$, $\xi \in \mathrm{Dom}(N)$.*

*Proof.* For all $\xi', \xi \in \mathrm{Dom}(N)$ we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \xi', \mathrm{e}^{-\mathrm{i}tH} a_j \mathrm{e}^{\mathrm{i}tH} \xi\right\rangle &= \lim_{s\to 0} \frac{1}{s} \left\langle \xi', \left(\mathrm{e}^{-\mathrm{i}(t+s)H} a_j \mathrm{e}^{\mathrm{i}(t+s)H} - \mathrm{e}^{-\mathrm{i}tH} a_j \mathrm{e}^{\mathrm{i}tH}\right) \xi\right\rangle \\
&= \lim_{s\to 0} s^{-1} \left\langle \left(\mathrm{e}^{\mathrm{i}(t+s)H} - \mathrm{e}^{\mathrm{i}tH}\right) \xi', a_j \mathrm{e}^{\mathrm{i}tH} \xi\right\rangle \\
&+ \lim_{s\to 0} s^{-1} \left\langle a_j^\dagger \mathrm{e}^{\mathrm{i}tH} \xi', \left(\mathrm{e}^{\mathrm{i}(t+s)H} - \mathrm{e}^{\mathrm{i}tH}\right) \xi\right\rangle \\
&= \left\langle \mathrm{i}H \mathrm{e}^{\mathrm{i}tH} \xi', a_j \mathrm{e}^{\mathrm{i}tH} \xi\right\rangle + \left\langle a_j^\dagger \mathrm{e}^{\mathrm{i}tH} \xi', \mathrm{i}H \mathrm{e}^{\mathrm{i}tH} \xi\right\rangle .
\end{aligned}
$$

Now, for all $u, v \in D$ we have

$$
\langle \mathrm{i}Hv, a_j u\rangle + \left\langle a_j^\dagger v, \mathrm{i}Hu\right\rangle = -\mathrm{i} \langle v, [H, a_j]u\rangle = \sum_{k=1}^{d} \left(c_{jk}^- \langle v, a_k u\rangle + c_{jk}^+ \left\langle v, a_k^\dagger u\right\rangle\right)
$$

for some complex constants $c^-_{jk}, c^+_{jk}$. The left-hand and right-hand side make sense for $u, v \in \mathrm{Dom}(N)$, therefore they can be extended by density and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \xi', \mathrm{e}^{-\mathrm{i}tH}\, a_j\, \mathrm{e}^{\mathrm{i}tH}\xi\right\rangle$$

$$= \sum_{k=1}^{d}\left( c^-_{jk}\left\langle \xi', \mathrm{e}^{-\mathrm{i}tH}\, a_k\, \mathrm{e}^{\mathrm{i}tH}\xi\right\rangle + c^+_{jk}\left\langle \xi', \mathrm{e}^{-\mathrm{i}tH}\, a^\dagger_k \mathrm{e}^{\mathrm{i}tH}\xi\right\rangle\right)$$

for all $\xi', \xi \in \mathrm{Dom}(N)$. Considering the conjugate we find a differential equation for $\left\langle \xi', \mathrm{e}^{-\mathrm{i}tH}\, a^\dagger_j\, \mathrm{e}^{\mathrm{i}tH}\xi\right\rangle$ an so we get a linear system of $2d$ differential equations with constant coefficients. The solution of the system yields analytic functions $\mathsf{H}^-, \mathsf{H}^+$ as blocks of the exponential of a $2d \times 2d$ matrix. □

*Proof.* (of Theorem 5.4) Let $G_0$ be the self-adjoint extension of $-\sum_{\ell=1}^{d} L^*_\ell L_\ell/2$. By Proposition 5.9, for all $y \in \mathcal{N}(\mathcal{T})$ and all $v, u \in \mathrm{Dom}(N)$, we have

$$\begin{aligned}
&\langle G_0 v, yu\rangle + \sum_{\ell=1}^{m}\langle L_\ell v, yL_\ell u\rangle + \langle v, yG_0 u\rangle \\
&= -\frac{1}{2}\sum_{\ell=1}^{m}\left(\langle L^*_\ell L_\ell v, yu\rangle - 2\langle L_\ell v, yL_\ell u\rangle + \langle v, yL^*_\ell L_\ell u\rangle\right) \\
&= -\frac{1}{2}\sum_{\ell=1}^{m}\left(\langle L^*_\ell y^* L_\ell v, u\rangle - 2\langle L_\ell v, yL_\ell u\rangle + \langle v, L^*_\ell y L_\ell u\rangle\right) = 0
\end{aligned}$$

because $L^*_\ell y^*$ and $L^*_\ell y$ are extensions of $y^* L^*_\ell$ and $yL^*_\ell$ respectively, namely $\mathcal{L}(x) = \mathrm{i}[H, x]$ (as a quadratic form).

Now, recalling that $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_s$-invariant by Proposition 5.2 1. for all $v, u \in \mathrm{Dom}(N)$ also $\mathrm{e}^{-\mathrm{i}(t-s)H}v$ and $\mathrm{e}^{-\mathrm{i}(t-s)H}u$ belong to the domain of $N$ by Lemma 5.10, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\left\langle \mathrm{e}^{-\mathrm{i}(t-s)H}v, \mathcal{T}_s(x)\mathrm{e}^{-\mathrm{i}(t-s)H}u\right\rangle = 0$$

which implies

$$\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH}\, x\, \mathrm{e}^{-\mathrm{i}tH}.$$

From $\mathcal{T}_t$-invariance of $\mathcal{N}(\mathcal{T})$ it follows that also $\mathrm{e}^{\mathrm{i}tH}\, x\, \mathrm{e}^{-\mathrm{i}tH}$ belongs to the generalized commutant of the operators $L_\ell, L^*_\ell$ ($\ell \geq 1$).

Since $\mathrm{Dom}(N)$ is $\mathrm{e}^{\mathrm{i}tH}$-invariant by Lemma 5.10, replacing $\xi \in \mathrm{Dom}(N)$ by $\mathrm{e}^{\mathrm{i}tH}\xi \in \mathrm{Dom}(N)$ and left multiplying by $\mathrm{e}^{-\mathrm{i}tH}$ the identity $\mathrm{e}^{\mathrm{i}tH}x\,\mathrm{e}^{-\mathrm{i}tH}L_\ell\xi = L_\ell\, \mathrm{e}^{\mathrm{i}tH}x\,\mathrm{e}^{-\mathrm{i}tH}\xi$ becomes

$$x\,\mathrm{e}^{-\mathrm{i}tH}L_\ell\, \mathrm{e}^{\mathrm{i}tH}\xi = \mathrm{e}^{-\mathrm{i}tH}L_\ell\, \mathrm{e}^{\mathrm{i}tH}x\,\xi$$

Taking the scalar product with two exponential vectors and differentiating $n$ times at $t = 0$ the identity

$$\left\langle v, x\,\mathrm{e}^{-\mathrm{i}tH}\, L_\ell\, \mathrm{e}^{\mathrm{i}tH}u\right\rangle = \left\langle \mathrm{e}^{-\mathrm{i}tH}\, L^*_\ell\, \mathrm{e}^{\mathrm{i}tH}v, xu\right\rangle$$

with $u, v \in \mathrm{Dom}(N)$, we get

$$\langle v, x\, \delta_H^n(L_\ell)u\rangle = \langle \delta_H^n(L_\ell^*)v, xu\rangle \,.$$

Since iterated commutators $\delta_H^n(L_\ell)$ are first order polynomials in $a_j, a_k^\dagger$, this means that $x$ belongs to the generalized commutant of $\delta_H^n(L_\ell)$. The same argument applies to generalized commutators of $\delta_H^n(L_\ell^*)$ for all $\ell \geq 1$, $n \geq 0$.

Conversely, if $x$ belongs to the generalized commutant of operators $\delta_H^n(L_\ell)$, $\delta_H^n(L_\ell^*)$ for all $\ell \geq 1$, $0 \leq n \leq 2d-1$, recall that each one of these generalized commutators is a first order polynomial in $a_j, a_k^\dagger$ and so determines two vectors (coefficients of creation and annihilation operators) $\overline{v}, u \in \mathbb{C}^d$ and, eventually, a vector $[\overline{v}, u]^T \in \mathbb{C}^{2d}$. Let $\mathcal{V}_n$ be the complex linear subspace of $\mathbb{C}^{2d}$ determined by vectors in $\mathbb{C}^{2d}$ corresponding to generalized commutators of order less or equal than $n$. Clearly, $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ for all $n$ and so the dimensions $\dim_{\mathbb{C}}(\mathcal{V}_n)$ form a non decreasing sequence of natural numbers bounded by $2d$. Moreover, if $\dim_{\mathbb{C}}(\mathcal{V}_n)$ $=\dim_{\mathbb{C}}(\mathcal{V}_{n+1})$, then $\mathcal{V}_n = \mathcal{V}_{n+1}$ and so

$$\delta_H^{n+1}(L_\ell) = \sum_{m=0}^{n} \left( z_m \delta_H^{(m)}(L_\ell) + w_m \delta_H^{(m)}(L_\ell^*)\right) + \eta_n \mathbb{1},$$

$$\delta_H^{n+1}(L_\ell^*) = \sum_{m=0}^{n} (-1)^m \left( \overline{w}_m \delta_H^{(m)}(L_\ell) + \overline{z}_m \delta_H^{(m)}(L_\ell^*)\right) + \overline{\eta}_n \mathbb{1},$$

for some $z_1, \dots, z_n, w_1, \dots, w_n, \eta_n \in \mathbb{C}$. Iterating, it turns out that the linear part in creation and annihilation operators of $\delta_H^{n+m}(L_\ell)$ and $\delta_H^{n+m}(L_\ell)$ depends on vectors in $\mathcal{V}_n$ for all $m \geq 0$. It follows that, starting from a value $n_0 \geq 1$ (corresponding to the zero order commutators $L_\ell$ and $L_\ell^*$), the sequence of dimensions has to increase at least by $1$ before reaching the maximum value. As a consequence, this is attained in at most $2d-1$ steps.

Summarizing, if $x$ belongs to the generalised commutant of the set of operators $\delta_H^n(L_\ell)$, $\delta_H^n(L_\ell^*)$ for all $\ell \geq 1$, $0 \leq n \leq 2d-1$, then it belongs to generalized commutant of these operators for all $n \geq 0$. By Lemma 5.11, we can consider the analytic function on $\mathbb{R}$

$$t \mapsto \left\langle \xi', x\, \mathrm{e}^{-\mathrm{i}tH} L_\ell\, \mathrm{e}^{\mathrm{i}tH}\xi \right\rangle - \left\langle \mathrm{e}^{-\mathrm{i}tH} L_\ell^*\, \mathrm{e}^{\mathrm{i}tH}\xi', x\, \xi \right\rangle$$

for all $\xi, \xi' \in D$. The $n$-th derivative at $t = 0$ is $(-\mathrm{i})^n$ times

$$\langle \xi', x\, \delta_H^n(L_\ell)\xi\rangle - \langle \delta_H^n(L_\ell^*)\xi', x\, \xi\rangle = 0$$

for all $n \geq 0$ because $x$ belongs to the generalized commutant of operators $\delta_H^n(L_\ell)$. The same argument shows that $x$ belongs to the generalized commutant of operators $\delta_H^n(L_\ell^*)$. Applying Theorem 4.1 of [27] (with $C = N$ and keeping in mind that $[G, C]$, $[G^*, C]$ are second order polynomials in $a_j, a_k^\dagger$, therefore relatively bounded with respect to $C$ whence with respect to $C^{3/2}$) it follows that $\mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH} x\, \mathrm{e}^{-\mathrm{i}tH}$.

The same conclusion holds for $x^*$ and $x^*x$ because they belong to the generalized commutant of operators $\delta_H^n(L_\ell)$, $\delta_H^n(L_\ell^*)$. Therefore $x \in \mathcal{N}(\mathcal{T})$ and the proof is complete. □

## 5.2 The Decoherence-Free Subalgebra for Gaussian QMSs

In the sequel we provide a simpler characterization of $\mathcal{N}(\mathcal{T})$ in terms of *real* subspaces of $\mathbb{C}^d$ and find its structure. In order to make clear the thread of the discussion, we forget, for the moment, technicalities related with unbounded operators, and we concentrate on the linear algebraic aspect of it.

*Remark* 5.12. A straightforward computation yields

$$
\begin{aligned}
[H, L_\ell] &= \sum_{i,j=1}^{d} \left[ \Omega_{ij} a_i^\dagger a_j + \frac{\kappa_{ij}}{2} a_i^\dagger a_j^\dagger + \frac{\overline{\kappa}_{ij}}{2} a_i a_j + \frac{\zeta_j}{2} a_j^\dagger + \frac{\overline{\zeta}_j}{2} a_j, \overline{v}_{\ell k} a_k + u_{\ell k} a_k^\dagger \right] \\
&= \sum_{i,j=1}^{d} \left( -\Omega_{ij} \overline{v}_{\ell i} a_j + \Omega_{ij} u_{\ell j} a_i^\dagger - \frac{\kappa_{ij}}{2} \overline{v}_{\ell j} a_i^\dagger - \frac{\kappa_{ij}}{2} \overline{v}_{\ell i} a_j^\dagger + \frac{\overline{\kappa}_{ij}}{2} u_{\ell j} a_i + \frac{\overline{\kappa}_{ij}}{2} u_{\ell i} a_j \right) \\
&+ \frac{1}{2} \langle \zeta, u_{\ell \bullet} \rangle - \frac{1}{2} \langle v_{\ell \bullet}, \zeta \rangle \\
&= \sum_{i,j=1}^{d} \left( - \left( \Omega^T \overline{v}_{\ell \bullet} \right)_j a_j + (\Omega u_{\ell \bullet})_i a_i^\dagger - \kappa_{ij} \overline{v}_{\ell j} a_i^\dagger + \overline{\kappa}_{ij} u_{\ell j} a_i \right) \\
&+ \frac{1}{2} \langle \zeta, u_{\ell \bullet} \rangle - \frac{1}{2} \langle v_{\ell \bullet}, \zeta \rangle,
\end{aligned}
$$

where $u_{\ell\bullet}, v_{\ell\bullet}$ stand for the vectors of entries $(u_{\ell j})_{j=1}^d$, $(v_{\ell j})_{j=1}^d$ respectively. Summarizing

$$
[H, L_\ell] = \sum_{i=1}^{d} \left( (\Omega u_{\ell \bullet} - K \overline{v}_{\ell \bullet})_i a_i^\dagger - \left( \Omega^T \overline{v}_{\ell \bullet} - \overline{K} u_{\ell \bullet} \right)_i a_i \right) + \frac{1}{2} \langle \zeta, u_{\ell \bullet} \rangle - \frac{1}{2} \langle v_{\ell \bullet}, \zeta \rangle
$$

which is again a linear combination of creation and annihilation operators with an added multiple of the identity. Therefore the set of operators of which we have to consider the generalized commutant, thanks to the canonical commutation relations, is particularly simple and contains only linear combinations of creation and annihilation operators together with a multiple of the identity $\mathbb{1}$ that plays no role when we consider the generalized commutant.

It is now convenient to transfer the problem of commutators in a purely algebraic one. Each linear combination of creation and annihilation operators is uniquely determined by a pair $v, u$ of vectors in $\mathbb{C}^d$ representing coefficients of annihilation and creation operators so that, for example, the operator $L_\ell$ in (3.18) and its adjoint $L_\ell^*$ are determined by

$$
L_\ell = \sum_{j=1}^{d} \left( \overline{v}_{\ell j} a_j + u_{\ell j} a_j^\dagger \right) \rightsquigarrow \begin{pmatrix} \overline{v}_{\ell \bullet} \\ u_{\ell \bullet} \end{pmatrix}, \qquad L_\ell^* = \sum_{j=1}^{d} \left( \overline{u}_{\ell j} a_j + v_{\ell j} a_j^\dagger \right) \rightsquigarrow \begin{pmatrix} \overline{u}_{\ell \bullet} \\ v_{\ell \bullet} \end{pmatrix}
$$

In a similar way, after computing commutators,

$$
[H, L_\ell] \rightsquigarrow \begin{pmatrix} -\Omega^T & \overline{K} \\ -K & \Omega \end{pmatrix} \begin{pmatrix} \overline{v}_{\ell \bullet} \\ u_{\ell \bullet} \end{pmatrix}, \qquad [H, L_\ell^*] \rightsquigarrow \begin{pmatrix} -\Omega^T & \overline{K} \\ -K & \Omega \end{pmatrix} \begin{pmatrix} \overline{u}_{\ell \bullet} \\ v_{\ell \bullet} \end{pmatrix}
$$

Denote by $\mathbb{H}$ the above $2d \times 2d$ matrix (built by four $d \times d$ matrices)

$$
\mathbb{H} = \begin{pmatrix} -\Omega^T & \overline{K} \\ -K & \Omega \end{pmatrix}
$$

then the operators involved to characterized $\mathcal{N}(\mathcal{T})$ via (5.2) correspond to

$$\mathbb{H}^{\,n}\begin{pmatrix}\overline{v}_{\ell\bullet}\\ u_{\ell\bullet}\end{pmatrix}, \qquad \mathbb{H}^{\,n}\begin{pmatrix}\overline{u}_{\ell\bullet}\\ v_{\ell\bullet}\end{pmatrix} \tag{5.7}$$

with $\ell = 1, \ldots, m$ and $0 \leq n \leq 2d-1$.

*Notation* 5.13. We denote with $\mathcal{V}$ be the real subspace of $\mathbb{C}^{2d}$ defined as

$$\mathcal{V} = \operatorname{span}_{\mathbb{R}}\left\{\mathbb{H}^{n}\begin{pmatrix}\overline{v_{\ell\bullet}}\\ u_{\ell\bullet}\end{pmatrix}, \mathbb{H}^{n}\begin{pmatrix}\overline{u_{\ell\bullet}}\\ v_{\ell\bullet}\end{pmatrix} : \ell = 1, \ldots, m, 0 \leq n \leq 2d-1\right\}. \tag{5.8}$$

The definition of $\mathcal{V}$ allows us to give the following condition.

**Lemma 5.14.** *An operator $x \in \mathcal{B}(\mathsf{h})$ belongs to $\mathcal{N}(\mathcal{T})$ if and only if it belongs to the generalized commutant of*

$$\{\, q(\mathrm{i}w) \mid w \in \mathcal{M}\,\} \tag{5.9}$$

*where*

$$\mathcal{M} = \operatorname{Lin}_{\mathbb{R}}\left\{\, \mathrm{i}(v+u), v-u \mid [\overline{v}, u]^T \in \mathcal{V}\,\right\} \subset \mathbb{C}^d. \tag{5.10}$$

*Proof.* By Remark 5.12 we know that the operators in the set (5.2) are linear combination of annihilation and creation operators up to a multiple of the identity operator and the generalized commutant of (5.2) coincides with the generalized commutant of

$$\left\{\, a(v) + a^{\dagger}(u) \mid [\overline{v}, u]^T \in \mathcal{V}\,\right\}. \tag{5.11}$$

To conclude the proof we just need to show that the commutants of (5.9) and (5.11) are the same. Notice at first that if $[\overline{v}, u]^T \in \mathcal{V}$ also $[\overline{u}, v]^T \in \mathcal{V}$, indeed if $\delta_H^n(L_\ell) = a(v) + a^{\dagger}(u)$ then $\delta_H^n(L_\ell^*) = (-1)^n \delta_H^n(L_\ell)^* = (-1)^n\left(a(u) + a^{\dagger}(v)\right)$ on the domain $D$. Now from (1.10) we obtain

$$\begin{aligned} q(\mathrm{i}(u-v)) &= \mathrm{i}\big(a(v) + a^{\dagger}(u)\big) - \mathrm{i}\left(a(u) + a^{\dagger}(v)\right), \\ 2(a(v) + a^{\dagger}(u)) &= q(v+u) - \mathrm{i}q(\mathrm{i}(u-v)). \end{aligned}$$

Therefore every element of (5.9) is a linear combination of elements of (5.11) and vice versa, concluding the proof. □

We can now prove the following.

**Theorem 5.15.** *The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ generated by Weyl operators $W(z)$ such that $z$ belongs to the symplectic complement of (5.10). Moreover, up to unitary equivalence,*

$$\mathcal{N}(\mathcal{T}) = L^{\infty}(\mathbb{R}^{d_c}; \mathbb{C}) \,\overline{\otimes}\, \mathcal{B}(\Gamma(\mathbb{C}^{d_f})) \tag{5.12}$$

*for a pair of natural numbers $d_c, d_f \leq d$.*

*Proof.* (Subscript $c$ (resp. $f$) stands for commutative (resp. full)) By Lemma 5.14 any $x \in \mathcal{N}(\mathcal{T})$ satisfies $x\, q(\mathrm{i}w) \subseteq q(\mathrm{i}w)\, x$ for all $w \in \mathcal{M}$. Therefore, for all real number $r$, $x(\mathbb{1} + \mathrm{i}rq(\mathrm{i}w)) \subseteq (\mathbb{1} + \mathrm{i}rq(\mathrm{i}w))x$ and, right and left multiplying by the resolvent $(\mathbb{1} + \mathrm{i}rq(\mathrm{i}w))^{-1}$ which is a bounded operator

$$(\mathbb{1} + \mathrm{i}rq(\mathrm{i}w))^{-1}x = x(\mathbb{1} + \mathrm{i}rq(\mathrm{i}w))^{-1}.$$

Iterating $n$ times and considering $r = 1/n$ we find

$$(\mathbb{1} + \mathrm{i}q(\mathrm{i}w)/n)^{-n}\, x = x\, (\mathbb{1} + \mathrm{i}q(\mathrm{i}w)/n)^{-n}$$

and, taking the limit as $n$ goes to $+\infty$, by the Hille-Yosida theorem [13, Theorem 3.1.10] we have

$$\begin{aligned} W(w)x = \mathrm{e}^{-\mathrm{i}q(\mathrm{i}w)}x &= \lim_{n\to\infty} (\mathbb{1} + \mathrm{i}q(\mathrm{i}w)/n)^{-n}\, x \\ &= x \lim_{n\to\infty} (\mathbb{1} + \mathrm{i}q(\mathrm{i}w)/n)^{-n} = xW(w) \end{aligned}$$

and so $x$ belongs to $\mathcal{W}(\mathcal{M})'$ which coincides with $\mathcal{W}(\mathcal{M}^{\perp_\sigma})$ by Araki's Theorem (Theorem 1.53).

Conversely, if $z$ belongs to the symplectic complement of $\mathcal{M}$, then from (1.13), (1.10) and Proposition 1.56 we have $W(z)q(\mathrm{i}w)e_g = q(\mathrm{i}w)W(z)e_g$ for all $w \in \mathcal{M}$ and $g \in \mathbb{C}^d$. Since the linear span of exponential vectors is an essential domain for $q(\mathrm{i}w)$, for all $\xi \in \mathrm{Dom}(q(\mathrm{i}w))$ there exists a sequence $(\xi_n)_{n\geq 1}$ in $E$ such that $(q(\mathrm{i}w)\xi_n)_{n\geq 1}$ converges to $q(\mathrm{i}w)\xi$. It follows that $(q(\mathrm{i}w)W(z)\xi_n)_{n\geq 1}$ converges and, since $q(\mathrm{i}w)$ is closed, $W(z)\xi$ belongs to $\mathrm{Dom}(q(\mathrm{i}w))$ and $W(z)q(\mathrm{i}w)\xi = q(\mathrm{i}w)W(z)\xi$, namely $q(\mathrm{i}w)W(z)$ is an extension of $W(z)q(\mathrm{i}w)$. Therefore $W(z)$ belongs to the generalized commutant of all $q(\mathrm{i}w)$ with $w \in \mathcal{M}$ and therefore to $\mathcal{N}(\mathcal{T})$ by Lemma 5.14.

In order to prove (5.12) consider $\mathcal{M}^c := \mathcal{M} \cap \mathcal{M}'$ which is a real linear subspace of both $\mathcal{M}$ and $\mathcal{M}'$. Consider now $\mathcal{M}^r$ and $\mathcal{M}^f$ as the real linear complement of $\mathcal{M}^c$ in $\mathcal{M}$ and $\mathcal{M}^{\perp_\sigma}$ respectively, i.e.

$$\mathcal{M} = \mathcal{M}^c \oplus \mathcal{M}^r, \qquad \mathcal{M}^{\perp_\sigma} = \mathcal{M}^c \oplus \mathcal{M}^f.$$

($\mathcal{M}^c \perp \mathcal{M}^r$ and $\mathcal{M}^c \perp \mathcal{M}^f$, more precisely, they are orthogonal with respect to the real part of the scalar product). We will show that $\mathcal{M}^f$ is a symplectic subspace of $\mathbb{C}^d$ and that it is symplectically orthogonal to both $\mathcal{M}^c$ and $\mathcal{M}^r$. Suppose $z \in \mathcal{M}^f$ is such that $\Im\langle z, z_f\rangle = 0$ for all $z_f \in \mathcal{M}^f$. By construction $\Im\langle z, z_c\rangle = 0$ for all $z_c \in \mathcal{M}^c = \mathcal{M} \cap \mathcal{M}^{\perp_\sigma}$. Therefore

$$\mathrm{Im}\langle z, m\rangle = 0, \quad \forall m \in \mathcal{M}^{\perp_\sigma},$$

since $\mathcal{M}^{\perp_\sigma} = \mathcal{M}^c \oplus \mathcal{M}^f$. Therefore $z \in \mathcal{M}^{\perp_\sigma \perp_\sigma} = \mathcal{M}$, but then

$$z \in \mathcal{M} \cap \mathcal{M}^{\perp_\sigma} \cap \mathcal{M}^f = \mathcal{M}^c \cap \mathcal{M}^f = \{0\}.$$

Hence $\mathcal{M}^f$ is a symplectic subspace. Eventually, $\mathcal{M}^f \subset \mathcal{M}^{\perp_\sigma}$ and $\mathcal{M} = \mathcal{M}^{\perp_\sigma \perp_\sigma} \subset (\mathcal{M}^f)^{\perp_\sigma}$. In particular $\mathcal{M}^f$ is symplectically orthogonal to both $\mathcal{M}^r, \mathcal{M}^c$. Let $d_c = \dim_{\mathbb{R}} \mathcal{M}^c$ and $2d_f = \dim_{\mathbb{R}} \mathcal{M}^f$ which is even by Proposition 1.9. Still by Proposition 1.9 we can find a symplectomorphism $B$ such that

$$B : \mathcal{M}^c \to \mathrm{Lin}_{\mathbb{R}}\{e_1, \ldots, e_{d_c}\} \oplus \mathrm{Lin}_{\mathbb{R}}\{e_{d_c+1}, \mathrm{i}e_{d_c+1}, \ldots, e_{d_c+d_f}, \mathrm{i}e_{d_c+d_f}\},$$

where $(e_j)_{j=1}^{d_c+d_f}$ is the canonical complex orthonormal basis of $\mathbb{C}^{d_c+d_f}$. Eventually, from Lemma 2.41, symplectic transformation in finite dimensional symplectic spaces are always implemented by unitary transformations on the Fock space. From here we obtain the final result. □

We can then specialize this result to gaussian QMSs.

**Corollary 5.16.** *Let $\mathcal{T}$ be a guassian QMS. The decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$ is generated by Weyl operators $W(z)$ with $z$ belonging to real subspaces of* $\ker(C)$ *that are $Z$-invariant.*

*Proof.* By Theorem 5.15 it suffices to show that $z$ belongs to the symplectic complement $\mathcal{M}^{\perp_\sigma}$ of (5.10) if and only if it belongs to a real subspace of $\ker(C)$ that is $Z$-invariant.
If $z$ belongs to $\mathcal{M}^{\perp_\sigma}$ then $W(z) \in \mathcal{N}(\mathcal{T})$ and $\mathcal{T}_t(W(z)) = \mathrm{e}^{\mathrm{i}tH} W(z)\,\mathrm{e}^{-\mathrm{i}tH}$ for all $t \geq 0$, from Proposition 5.2. Comparison with (3.38) yields

$$\mathrm{e}^{\mathrm{i}tH} W(z)\,\mathrm{e}^{-\mathrm{i}tH} = \exp\left(-\frac{1}{2}\int_0^t \Re\left\langle \mathrm{e}^{sZ}z, C\mathrm{e}^{sZ}z\right\rangle \mathrm{d}s + \mathrm{i}\int_0^t \Re\left\langle \zeta, \mathrm{e}^{sZ}z\right\rangle \mathrm{d}s\right) W\left(\mathrm{e}^{tZ}z\right)$$

Unitarity of both left and right operators implies $\Re\left\langle \mathrm{e}^{sZ}z, C\mathrm{e}^{sZ}z\right\rangle = 0$ for all $s \geq 0$ and $\mathrm{e}^{sZ}z$ belongs to $\ker(C)$ for all $s \geq 0$, namely, in an equivalent way, $z$ and also $Zz$ (by differentiation) belong to $\ker(C)$.
Conversely, if $z$ belongs to a real subspace of $\ker(C)$ that is $Z$-invariant, then $\mathrm{e}^{sZ}z$ also belongs to that subset for all $s \geq 0$. The explicit formula (3.38) shows that

$$\mathcal{T}_t(W(z)) = \exp\left(\mathrm{i}\int_0^t \Re\left\langle \zeta, \mathrm{e}^{sZ}z\right\rangle \mathrm{d}s\right) W\left(\mathrm{e}^{tZ}z\right)$$

therefore

$$\mathcal{T}_t(W(z)^*)\mathcal{T}_t(W(z)) = W\left(\mathrm{e}^{tZ}z\right)^* W\left(\mathrm{e}^{tZ}z\right) = \mathbb{1} = \mathcal{T}(W(z)^*W(z))$$

and, in the same way, $\mathcal{T}_t(W(z))\mathcal{T}_t(W(z)^*) = \mathcal{T}(W(z)W(z)^*)$. It follows that $W(z) \in \mathcal{N}(\mathcal{T})$ and $z$ belongs to the symplectic complement of (5.10) by Theorem 5.15. □

## 5.3 Examples

In this final section we consider two examples of gaussian QMSs in which we compute explicitly the decoherence-free subalgebra. The first one is a toy example whose purpose is to better clarify the possible situations arising in the decomposition of Theorem 5.15. The second one instead shows its application to a physical model.

### 5.3.1 Single Kraus operator with Number Hamiltonian

The operators (3.18) and (3.17) are the closure of operators defined on $D$, defined in (1.20)

$$L = \sum_{j=1}^{d}\left(\overline{v}_j a_j + u_j a_j^\dagger\right), \qquad H = \sum_{j=1}^{d} a_j^\dagger a_j \tag{5.13}$$

(either $v$ or $u$ is non-zero). We compute recursively

$$\delta_H^{2n+1}(L) = \sum_{j=1}^{d}\left(u_j a_j^\dagger - \overline{v}_j a_j\right), \qquad \delta_H^{2n}(L) = L$$

for all $n \geq 0$, and, in the same way, $\delta_H^{2n+1}(L^*) = \sum_{j=1}^d \left( v_j a_j^\dagger - \overline{u}_j a_j \right)$, $\delta_H^{2n}(L^*) = L^*$. It follows that $\mathcal{M}$ is the real linear space

$$\mathrm{Lin}_{\mathbb{R}} \{ v-u, v+u, \mathrm{i}(v+u), \mathrm{i}(v-u) \} = \mathrm{Lin}_{\mathbb{R}} \{ v, u, \mathrm{i}v, \mathrm{i}u \} = \mathrm{Lin}_{\mathbb{C}} \{ v, u \} .$$

Thus $\mathcal{M}^{\perp_\sigma}$ is the orthogonal (for the complex scalar product) of the complex linear subspace generated by $v$ and $u$. Hence it is a complex subspace of $\mathbb{C}^d$ and

$$\mathcal{N}(\mathcal{T}) = \left\{ W(z) \mid z \in \mathcal{M}^{\perp_\sigma} \right\} = \mathcal{B}(\Gamma(\mathcal{M}^{\perp_\sigma})).$$

If $v, u$ are linearly independent, then the complex dimension of $\mathcal{M}^{\perp_\sigma}$ is $d-2$, and $\mathcal{N}(\mathcal{T})$ is isomorphic to $\mathcal{B}(\Gamma(\mathbb{C}^{d-2}))$.

### 5.3.2 Single Kraus operator with no Hamiltonian

Let $L$ be as in (5.13). If $H = 0$, then $\delta_H = 0$. In particular

$$\mathcal{M} = \mathrm{Lin}_{\mathbb{R}}\{v-u, u-v, \mathrm{i}(v+u), \mathrm{i}(u+v)\} = \mathrm{Lin}_{\mathbb{R}}\{v-u, \mathrm{i}(v+u)\}$$

and, since both $v$ and $u$ cannot be zero in our framework, $\dim_{\mathbb{R}} \mathcal{M}$ is either 1 or 2. If it is equal to 1 (first case), clearly $\mathcal{M} \cap \mathcal{M}^{\perp_\sigma} = \mathcal{M}$ and $\mathcal{M}^r = \{0\}$ therefore $d_c = 1$ and $d_r = 0$. It follows that $d_f = d-1$ and $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra unitarily equivalent to $L^\infty(\mathbb{R};\mathbb{C})\overline{\otimes}\mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$. If $\dim_{\mathbb{R}} \mathcal{M} = 2$ then, since $\mathcal{M}^r$ is a symplectic space, its real dimension must be even and so we distinguish two cases: $d_r = 0, d_c = 2$ (second case) and $d_r = 1, d_c = 0$ (third case). If $d_c = 2$, again $\mathcal{M} \cap \mathcal{M}^{\perp_\sigma} = \mathcal{M}$, and $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra $L^\infty(\mathbb{R}^2;\mathbb{C})\overline{\otimes}\mathcal{B}(\Gamma(\mathbb{C}^{d-2}))$.
If $d_c = 0, d_r = 1$, then $d_f = d-1$ and $\mathcal{N}(\mathcal{T})$ is a von Neumann algebra $\mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$. This classification is summarized by Table 5.1 in which the last column labelled "$L$" contains possible choices of the operator $L$ that realize each case.

| | $\dim_{\mathbb{R}} \mathcal{M}$ | $d_c$ | $d_r$ | $d_f$ | $\mathcal{N}(\mathcal{T})$ | $L$ |
|---|---|---|---|---|---|---|
| 1st | 1 | 1 | 0 | $d-1$ | $L^\infty(\mathbb{R};\mathbb{C})\overline{\otimes}\mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$ | $L = q_1$ |
| 2nd | 2 | 2 | 0 | $d-2$ | $L^\infty(\mathbb{R}^2;\mathbb{C})\overline{\otimes}\mathcal{B}(\Gamma(\mathbb{C}^{d-2}))$ | $L = q_1 + \mathrm{i}q_2$ |
| 3rd | 2 | 0 | 1 | $d-1$ | $\mathcal{B}(\Gamma(\mathbb{C}^{d-1}))$ | $L = a_1, L = a_1^\dagger$ |

**Table 5.1:** *$\mathcal{N}(\mathcal{T})$ that can arise with one $L$ and $H = 0$*

In the last part of the section we will characterize each case by just looking directly at the operator $L$ instead of computing $\mathcal{M}$.

Suppose $L$ is self-adjoint. In this case $\mathcal{V}$ is composed of only one vector which is of the form $[\overline{v}, v]^T$. Therefore $\mathcal{M} = \mathrm{Lin}_{\mathbb{R}}\{\mathrm{i}v\}$ and $d_c = 1$, while $d_r = 0$ (1st case).
Consider now instead the case $L$ normal but not self-adjoint. An explicit computation shows that $0 = [L, L^*] = \|v\|^2 - \|u\|^2$ on $D$. This condition shows that $\mathcal{M} = \mathcal{M} \cap \mathcal{M}^{\perp_\sigma}$ since

$$\Im\langle v-u, \mathrm{i}(v+u)\rangle = \|v\|^2 - \|u\|^2 = 0.$$

Moreover $u \neq v$ since $L$ is not self-adjoint, hence $d_c = 2$ (2nd case). If $L$ is not even normal (i.e. $\|v\|^2 \neq \|u\|^2$) then by the previous calculations $d_c = 0$ and $d_r = 1$ (3rd case).
Summing up: the 1st case arises when $L$ is self-adjoint, the case 2nd case arises when

$L$ is normal but not self-adjoint and the 3rd case arises when $L$ is not normal or, equivalently $\|v\|^2 \neq \|u\|^2$.
In the last case it can be shown that when $\|v\|^2 > \|u\|^2$ (resp. $\|v\|^2 < \|u\|^2$) there exists a Bogoliubov transformation changing $L$ to a multiple of the annihilation operator $a_1$ (resp. creation operator $a_1^\dagger$).

### 5.3.3 Two bosons in a common bath

The following model for the open quantum system of two bosons in a common environment has been considered in [15]. Here $d = 2$ and $H$ is as in equation 3.17 with $\kappa = \zeta = 0$. The completely positive part of the GKLS generator $\mathcal{L}$ is

$$\frac{1}{2}\sum_{j,k=1,2} \gamma^-_{jk}\, a_j^\dagger X a_k + \frac{1}{2}\sum_{j,k=1,2} \gamma^+_{jk}\, a_j X a_k^\dagger \tag{5.14}$$

where $(\gamma^\pm_{jk})_{j,k=1,2}$ are positive definite $2\times 2$ matrices.

Note that, by a change of phase $a_1 \to \mathrm{e}^{\mathrm{i}\theta_1}a_1$, $a_1^\dagger \to \mathrm{e}^{-\mathrm{i}\theta_1}a_1^\dagger$, $a_2 \to \mathrm{e}^{\mathrm{i}\theta_2}a_2$, $a_2^\dagger \to \mathrm{e}^{-\mathrm{i}\theta_2}a_2^\dagger$, we can always assume that $(\gamma^-_{jk})_{j,k=1,2}$ is *real* symmetric. Write the spectral decomposition

$$\gamma^\pm = \lambda_\pm|\varphi^\pm\rangle\langle\varphi^\pm| + \mu_\pm|\psi^\pm\rangle\langle\psi^\pm|$$

where the vectors $\varphi^-, \psi^-$ have *real* components. Rewrite the first term of (5.14) as

$$\begin{aligned}\sum_{j,k=1,2} \gamma^-_{jk}\, a_j^\dagger X a_k &= \lambda_-\sum_{j,k=1,2} \varphi^-_j\overline{\varphi^-_k}\, a_j^\dagger X a_k + \mu_-\sum_{j,k=1,2} \psi^-_j\overline{\psi^-_k}\, a_j^\dagger X a_k \\ &= \lambda_-\left(\sum_{j=1,2}\varphi^-_j\, a_j^\dagger\right) X \left(\sum_{k=1,2}\overline{\varphi^-_k}\, a_k\right) \\ &\quad + \mu_-\left(\sum_{j=1,2}\psi^-_j\, a_j^\dagger\right) X \left(\sum_{k=1,2}\overline{\psi^-_k}\, a_k\right)\end{aligned}$$

and write in a similar way the second term of (5.14)

$$\begin{aligned}\sum_{j,k=1,2} \gamma^+_{jk}\, a_j X a_k^\dagger &= \lambda_+\left(\sum_{j=1,2}\varphi^+_j\, a_j\right) X \left(\sum_{k=1,2}\overline{\varphi^+_k}\, a_k^\dagger\right) \\ &\quad + \mu_+\left(\sum_{j=1,2}\psi^+_j\, a_j\right) X \left(\sum_{k=1,2}\overline{\psi^+_k}\, a_k^\dagger\right)\end{aligned}$$

We can represent $\mathcal{L}$ in GKLS form with a number of Kraus operators $L_\ell$ depending on the number of strictly positive eigenvalues among $\lambda_\pm, \mu_\pm$.

$$L_1 = \lambda_-^{1/2}\sum_{k=1,2}\overline{\varphi^-_k}\, a_k \qquad L_2 = \mu_-^{1/2}\sum_{k=1,2}\overline{\psi^-_k}\, a_k$$

$$L_3 = \lambda_+^{1/2}\sum_{k=1,2}\overline{\varphi^+_k}\, a_k^\dagger \qquad L_4 = \mu_+^{1/2}\sum_{k=1,2}\overline{\psi^+_k}\, a_k^\dagger$$

Relabelling if necessary we can always assume $0 \le \lambda_- \le \mu_-$ and $0 \le \lambda_+ \le \mu_+$.

We begin our analysis by considering, for the moment, the case where $H = 0$.

If $\lambda_-$ (or $\lambda_+ > 0$) then there are four vectors $v, u$ in the defining set of $\mathcal{M}$ namely

$$\mathcal{M} = \mathrm{Lin}_{\mathbb{R}} \left\{ \varphi^-, \psi^-, \mathrm{i}\varphi^-, \mathrm{i}\psi^- \right\} \quad (\text{or } = \mathrm{Lin}_{\mathbb{R}} \left\{ \varphi^+, \psi^+, \mathrm{i}\varphi^+, \mathrm{i}\psi^+ \right\})$$

thus $\mathcal{M}^{\perp_\sigma} = \{0\}$ and $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$.

Suppose now that $\lambda_+ = \lambda_- = 0$ and $\mu_-, \mu_+ > 0$ so that there are only two Kraus operators, the above $L_2$ and $L_4$ and

$$\mathcal{M} = \mathrm{Lin}_{\mathbb{R}} \left\{ \psi^-, \psi^+, \mathrm{i}\psi^-, \mathrm{i}\psi^+ \right\}.$$

It follows that, if $\psi^-, \psi^+$ are $\mathbb{R}$-linearly independent, we have again $\mathcal{M} = \mathbb{C}^2$ whence $\mathcal{M}^{\perp_\sigma} = \{0\}$ and $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$. Otherwise, if $\psi^+$ is a *real* non-zero multiple of $\psi^-$, then, as $\psi^\pm$ and $\mathrm{i}\psi^\pm$ are $\mathbb{R}$-linearly independent, the real dimension of $\mathcal{M}$ and $\mathcal{M}^{\perp_\sigma}$ is two, $\mathcal{M} = \mathcal{M} \cap \mathcal{M}^{\perp_\sigma} = \mathcal{M}^{\perp_\sigma}$ so that $\mathcal{N}(\mathcal{T})$ is isomorphic to $\mathcal{B}(\Gamma(\mathbb{C}))$.

It is not difficult to see that, in any case, the dimension of $\mathcal{M} = \mathcal{M} \cap \mathcal{M}^{\perp_\sigma} = \mathcal{M}^{\perp_\sigma}$ cannot be $1$ or $3$ (because creation and annihilation operator always appear separately in different Kraus operators $L$, never in the same).

Summarizing: $\mathcal{N}(\mathcal{T})$ is non-trivial and isomorphic to $\mathcal{B}(\Gamma(\mathbb{C}))$ if and only if $\gamma^+$ and $\gamma^-$ are rank-one and commute.

Finally, if we consider a non-zero $H$, it is clear that $\mathcal{N}(\mathcal{T})$ is always trivial unless $\gamma^+$ and $\gamma^-$ are rank-one, commute and their one-dimensional range is an eigenvector for $\Omega$ and $\Omega^T$.

CHAPTER 6

# Energy Transfer in Open Quantum Systems Weakly Coupled with Two Reservoirs

In this chapter we depart from the analysis of gaussian QMSs or states and consider a different problem. We study models of open quantum systems rigorously deduced from the weak coupling limit. We consider a quantum system with non-degenerate Hamiltonian $H_S$ coupled with two reservoirs in equilibrium at inverse temperatures $\beta_1 \leq \beta_2$ and study variation of energy due to couplings with each reservoir. Several models have been proposed involving open quantum systems (see e.g. [10, 11, 73]), mostly phenomenological, and also numerical simulations have been done showing different behaviours. The interaction of the open quantum system with reservoirs is described through interaction operators that appear in the GKLS generator $\mathcal{L}$ of the dynamics, while the Hamiltonian part is given by the commutator with the system Hamiltonian $H_S$. However, when the GKLS generator is rigorously deduced from some scaling (weak coupling or low density limit) both the system Hamiltonian and the interaction operators appear in the GKLS generator $\mathcal{L}$ after non-trivial transformations (see [1, 2, 19, 24, 25, 47]).

It is well-known (see Lebowitz and Spohn [68] (V.28)) that, by the second law of thermodynamics, energy (heat) flows from the hotter to the cooler reservoir. The energy flow, in general, is not proportional to the difference of temperature because of the non-linear dependence of susceptibilities on temperature, namely an exact Fourier's law does not hold. However, we rigorously prove that it holds in an approximate way when the temperatures of reservoirs are not too small or, as an alternative, differences between nearest energy levels are small. More precisely, we show that the amount of energy flowing through the system, Theorem 6.12, formula (6.19), is approximately proportional to the product of the temperature differences and a constant (conductivity) which can be interpreted as the average energy needed to jump from a level to the following higher level.

## 6.1 Semigroups of weak coupling limit type

We consider an open quantum system with Hamiltonian $H_S$ acting on a complex separable Hilbert space $\mathsf{h}$ with discrete spectral decomposition

$$H_S = \sum_{m\geq 0} \varepsilon_m P_{\varepsilon_m} \tag{6.1}$$

where $\varepsilon_m$, with $\varepsilon_m < \varepsilon_n$ for $m < n$, are the eigenvalues of $H_S$ and $P_{\varepsilon_m}$ are the corresponding eigenprojectors. The system is coupled with two reservoirs each one in equilibrium with inverse temperatures $\beta_1 \leq \beta_2$ with interaction Hamiltonians

$$H_1 = D_1 \otimes a^\dagger(\phi_1) + D_1^* \otimes a^-(\phi_1), \quad H_2 = D_2 \otimes a^\dagger(\phi_2) + D_2^* \otimes a^-(\phi_2).$$

where $D_1, D_2$ are bounded operators on $\mathsf{h}$ and $A^+(\phi_j), A^-(\phi_j)$ creation and annihilation operators on the Fock space of the reservoir $j$.

It is well-known (see [2, 19, 24, 68]) that, in the weak coupling limit, the evolution of the system observables is governed by a quantum Markov semigroup (QMS) on $\mathcal{B}(\mathsf{h})$, the algebra of all bounded operators in $\mathsf{h}$, with generator of the form

$$\mathcal{L} = \sum_{j=1,2,\ \omega\in\mathsf{B}} \mathcal{L}_{j,\omega} \tag{6.2}$$

where $\mathsf{B}$ is the set of all Bohr frequencies

$$\mathsf{B} := \{\, \omega \mid \exists\, \varepsilon_n, \varepsilon_m \text{ s.t. } \omega = \varepsilon_n - \varepsilon_m > 0\}. \tag{6.3}$$

For every Bohr frequency $\omega$, $\mathcal{L}_{j,\omega}$ is a generator with the GKLS structure (3.16)

$$\begin{aligned} \mathcal{L}_{j,\omega}(x) &= \mathrm{i}[H_{j,\omega}, x] - \frac{\Gamma^-_{j,\omega}}{2}\left(D^*_{j,\omega}D_{j,\omega}x - 2D^*_{j,\omega}xD_{j,\omega} + xD_{j,\omega}D^*_{j,\omega}\right) \\ &- \frac{\Gamma^+_{j,\omega}}{2}\left(D_{j,\omega}D^*_{j,\omega}x - 2D_{j,\omega}xD^*_{j,\omega} + xD_{j,\omega}D^*_{j,\omega}\right) \end{aligned} \tag{6.4}$$

for all $x \in \mathcal{B}(\mathsf{h})$, with Kraus operators $D_{j,\omega}$ defined by

$$D_{j,\omega} = \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} P_{\varepsilon_m} D_j P_{\varepsilon_n} \tag{6.5}$$

where $\mathsf{B}_\omega = \{\, (\varepsilon_n, \varepsilon_m) \mid \varepsilon_n - \varepsilon_m = \omega\,\}$, $\Gamma^{\pm}_{j,\omega} = f_{j,\omega}\gamma^{\pm}_{j,\omega}$

$$\gamma^-_{j,\omega} = \frac{\mathrm{e}^{\beta_j\omega}}{\mathrm{e}^{\beta_j\omega}-1}, \qquad \gamma^+_{j,\omega} = \frac{1}{\mathrm{e}^{\beta_j\omega}-1}, \qquad f_{j,\omega} = \int_{\{\, y\in\mathbb{R}^3 \mid |y|=\omega\}} |\phi_j(y)|^2 \mathrm{d}_s y$$

($\mathrm{d}_s$ denotes the surface integral) and $H_{j,\omega}$ are bounded self-adjoint operators on $\mathsf{h}$ commuting with $H_S$ of the form

$$H_{j,\omega} = \kappa^-_{j,\omega} D^*_{j,\omega} D_{j,\omega} + \kappa^+_{j,\omega} D_{j,\omega} D^*_{j,\omega}$$

for some real constants $\kappa^{\pm}_{j,\omega}$.

In the sequel, following a customary convention to simplify the notation, we also denote $D^-_{j,\omega} := D_{j,\omega}$ and $D^+_{j,\omega} := D^*_{j,\omega}$ and write

$$\mathcal{Q}^{\pm}_{j,\omega}(x) = -\frac{1}{2}D^{\mp}_{j,\omega}D^{\pm}_{j,\omega}x + D^{\mp}_{j,\omega}xD^{\pm}_{j,\omega} - \frac{1}{2}xD^{\mp}_{j,\omega}D^{\pm}_{j,\omega} \tag{6.6}$$

the term of the GKLS generator arising from the interaction with the bath $j$ due the Bohr frequency $\omega$ is

$$\mathcal{L}_{j,\omega} = \Gamma^-_{j,\omega}\mathcal{Q}^-_{j,\omega} + \Gamma^+_{j,\omega}\mathcal{Q}^+_{j,\omega} + \mathrm{i}[H_{j,\omega}, \cdot\,]$$

and the term arising from the interaction with the reservoir $j$ is

$$\mathcal{L}_j = \sum_{\omega\in\mathsf{B}} \mathcal{L}_{j,\omega}.$$

We now make some assumptions on constants in such a way as to ensure boundedness of operators $\mathcal{L}_j$. First of all note that the series $\sum_\omega D^*_{j,\omega}D_{j,\omega}$ is strongly convergent. Indeed, for all vector $u = \sum_{n\geq 0} P_{\varepsilon_n}u$ in $\mathsf{h}$, we have

$$\begin{aligned}
\sum_\omega \left\langle u, D^*_{j,\omega}D_{j,\omega}u\right\rangle &= \sum_\omega \sum_{n,m\geq 0} \langle P_{\varepsilon_m-\omega}D_jP_{\varepsilon_m}u, P_{\varepsilon_n-\omega}D_jP_{\varepsilon_n}u\rangle \\
&= \sum_\omega \sum_{n\geq 0} \langle D_jP_{\varepsilon_n}u, P_{\varepsilon_n-\omega}D_jP_{\varepsilon_n}u\rangle \\
&\leq \sum_{n\geq 0} \|D_jP_{\varepsilon_n}u\|^2 \\
&= \|D_j\|^2\,\|u\|^2\,.
\end{aligned}$$

As a consequence, if we assume

$$\sup_{\omega\in\mathsf{B}} \Gamma^{\pm}_{j,\omega} < +\infty, \qquad \sup_{\omega\in\mathsf{B}} \left|\kappa^{\pm}_{j,\omega}\right| < +\infty,$$

for $j = 1, 2$ GKLS generators $\mathcal{L}_j$ turn out to be bounded. The above condition will be assumed to be in force throughout the paper.

*Remark* 6.1. Note that $\mathcal{L}_j$ depends on the inverse temperature $\beta_j$ only through the constants $\gamma^{\pm}_{j,\omega}$. The above notation follows that of [1].

*Notation* 6.2. $\mathcal{T}^j$ (resp. $\mathcal{T}^{j,\omega}$ and $\mathcal{T}^{j,\omega}_*$) stand for the QMS generated by (resp. $\mathcal{L}_{j,\omega}$ and its predual semigroup). In this paper we are concerned with normal states, therefore we shall identify them with their densities which are positive operators on $\mathsf{h}$ with unit trace.

We end this section by checking that, if reservoirs have the same temperature $\beta_1 = \beta_2 = \beta$ and $Z_\beta := \mathrm{tr}\mathrm{e}^{-\beta H_S} < +\infty$, then the Gibbs state has density

$$\rho_\beta = Z_\beta^{-1}\mathrm{e}^{-\beta H_S} \tag{6.7}$$

and is stationary.

**Proposition 6.3.** *If* $\beta_1 = \beta_2 = \beta$ *and*

$$Z_\beta := \mathrm{tr} e^{-\beta H_S} = \sum_{n \geq 0} e^{-\beta \varepsilon_n} \mathrm{dim}(P_{\varepsilon_n}) < +\infty$$

*then the Gibbs state* (6.7) *is invariant for all QMSs generated by* $\mathcal{L}$, $\mathcal{L}_1$, $\mathcal{L}_2$.

*Proof.* We begin by observing that for $(\varepsilon_n + \omega, \varepsilon_n), (\varepsilon_n, \varepsilon_n - \omega) \in \mathsf{B}_\omega$, we can compute directly

$$\begin{aligned}(\mathcal{L}_{j,\omega})_*(P_{\varepsilon_n}) =& \Gamma^-_{j,\omega}(P_{\varepsilon_n - \omega} D_j P_{\varepsilon_n} D_j^* P_{\varepsilon_n - \omega} - P_{\varepsilon_n} D_i^* P_{\varepsilon_n - \omega} D_j P_{\varepsilon_n}) + \\ & \Gamma^+_{j,\omega}(P_{\varepsilon_n + \omega} D_j^* P_{\varepsilon_n} D_j P_{\varepsilon_n + \omega} - P_{\varepsilon_n} D_j P_{\varepsilon_n + \omega} D_j^* P_{\varepsilon_n}).\end{aligned}$$

A state of the form $\rho = \sum_n \rho_{\varepsilon_n} P_{\varepsilon_n}$, which is a function of the system Hamiltonian $H_S$ (also called a *diagonal* state), satisfies

$$\begin{aligned}\mathcal{L}_{*j}(\rho) =& \sum_\omega \sum_n (\mathcal{L}_{j,\omega})_*(\rho_{\varepsilon_n} P_{\varepsilon_n}) \\ =& \sum_\omega \sum_{(\varepsilon_n + \omega, \varepsilon_n) \in \mathsf{B}_\omega} (\rho_{\varepsilon_n + \omega} \Gamma^-_{j,\omega} - \rho_{\varepsilon_n} \Gamma^+_{j,\omega}) P_{\varepsilon_n} D_j P_{\varepsilon_n + \omega} D_j^* P_{\varepsilon_n} + \\ & \sum_\omega \sum_{(\varepsilon_n, \varepsilon_n - \omega) \in \mathsf{B}_\omega} (\rho_{\varepsilon_n - \omega} \Gamma^+_{j,\omega} - \rho_{\varepsilon_n} \Gamma^-_{j,\omega}) P_{\varepsilon_n} D_j^* P_{\varepsilon_n - \omega} D_j P_{\varepsilon_n}.\end{aligned}$$

Now if $\beta_1 = \beta_2 = \beta$ and $\rho_{\varepsilon_n} = \mathrm{e}^{-\beta \varepsilon_n}$ as in (6.7), we have

$$\frac{\Gamma^+_{j,\omega}}{\Gamma^-_{j,\omega}} = \frac{\gamma^+_{j,\omega}}{\gamma^-_{j,\omega}} = e^{-\beta \omega} = \frac{\rho_{\varepsilon_n + \omega}}{\rho_{\varepsilon_n}},$$

for all $j = 1, 2$, so that $\mathcal{L}_{*j}(\rho) = 0$ and $\rho = \mathrm{e}^{-\beta H_S}/Z_\beta$ is an invariant state for the QMS generated by $\mathcal{L}_j$. Since $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ it is an invariant state also for the QMS generated by $\mathcal{L}$. □

## 6.2 Energy current

In order to compute the rate of energy transfer through the system we consider the following definition (see [68] (V.28)).

**Definition 6.4.** The rate of energy variation in the system, in a state $\rho$, due to interaction with the reservoir $j$ is

$$\mathrm{tr} \rho \mathcal{L}_j(H_S).$$

Therefore

$$\mathrm{tr} \rho \mathcal{L}_1(H_S) - \mathrm{tr} \rho \mathcal{L}_2(H_S) \tag{6.8}$$

is twice the rate at which the energy flows through the system from the hotter bath to the colder bath, namely, the energy current through the system.

Adapting a result by Lebowitz and Spohn [68] Theorem 2 and Corollary 1, it is possible to prove that the energy current is non-negative for finite dimensional systems.

**Theorem 6.5.** *Suppose that* $\mathsf{h}$ *is finite dimensional and let* $\rho$ *be a faithful invariant state, then the energy current* (6.8) *is non-negative.*

*Proof.* If a system is weakly coupled to a *single* bath $j$ at inverse temperature $\beta_j$, it is well-known that the Gibbs state $\rho_{\beta_j} = Z_{\beta_j}^{-1}\mathrm{e}^{-\beta_j H_S}$, with $Z_{\beta_j} = \mathrm{tr}\mathrm{e}^{-\beta_j H_S}$, is invariant. Consider the relative entropy of $\rho$ with respect to $\rho_{\beta_j}$ defined by

$$S(\rho|\rho_{\beta_j}) = \mathrm{tr}\rho(\log(\rho - \log\rho_{\beta_j})$$

which is a notoriously non-increasing function (see [61], Theorem 1.5), i.e.

$$S\left(\mathcal{T}^j_{*t}(\rho)|\mathcal{T}^j_{*t}(\rho_{\beta_j})\right) \leq S(\rho|\rho_{\beta_j}),$$

for all $\rho$ and $t \geq 0$. States $\mathcal{T}^j_{*t}(\rho)$, $j = 1, 2$ will still be faithful for small $t$, therefore no problem arises when considering logarithms. Since $\rho_{\beta_j}$ is invariant, denoting $\rho_t := \mathcal{T}^j_{*t}(\rho)$, and differentiating we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(\rho_t|\rho_{\beta_j}) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}\rho_t(\log\rho_t - \log\rho_{\beta_j})\\ &= \mathrm{tr}\rho_t'(\log\rho_t - \log\rho_{\beta_j}) + \mathrm{tr}\rho_t\frac{\mathrm{d}}{\mathrm{d}t}\log\rho_t.\end{aligned}$$

Since for every $x > 0$, $\log x = \int_0^{+\infty}\left(\frac{1}{1+s} - \frac{1}{x+s}\right)\mathrm{d}s$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\rho_t = \int_0^{+\infty}(s+\rho_t)^{-1}\rho_t'(s+\rho_t)^{-1}\mathrm{d}s$$

so that

$$\mathrm{tr}\rho_t\frac{\mathrm{d}}{\mathrm{d}t}\log\rho_t = \mathrm{tr}\rho_t'\int_0^{+\infty}\rho_t(s+\rho_t)^{-2}\mathrm{d}s = \mathrm{tr}\rho_t' = 0.$$

By imposing $\rho_{\beta_j} = Z_{\beta_j}^{-1}e^{-\beta_j H_S}$, and recalling that $\rho_t' = \mathcal{L}_{*j}(\rho_t)$, $\mathrm{tr}\rho_t' = 0$ by trace preservation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}S(\rho_t|\rho_{\beta_j}) &= \mathrm{tr}\rho_t'(\log\rho_t - \log\rho_{\beta_j})\\ &= \mathrm{tr}\rho_t'(\log\rho_t + \beta_j H_S - \log Z_{\beta_j}^{-1})\\ &= \mathrm{tr}\rho_t'\,\log\rho_t + \beta_j\mathrm{tr}\rho_t\mathcal{L}_j(H_S).\end{aligned}$$

In particular $\mathrm{tr}\rho_t'(\log\rho_t) + \beta_j\mathrm{tr}\rho_t\mathcal{L}(H_S) \leq 0$ by monotonicity of the relative entropy, namely

$$-\mathrm{tr}\mathcal{L}_{*j}(\rho_t)\,\log\rho_t - \beta_j\mathrm{tr}\rho_t\mathcal{L}_j(H_S) \geq 0.$$

In our context, the entropy production of the system due to interaction with the bath at inverse temperature $\beta_j$ is

$$-\,\mathrm{tr}\mathcal{L}_{*j}(\rho_t)\,\log\rho_t - \beta_j\mathrm{tr}\rho_t\mathcal{L}_j(H_S) \geq 0. \tag{6.9}$$

Now, for all $\beta, \beta_1, \beta_2$ and $\rho$ stationary state for the system $S$ interacting with both baths, By taking a sum over $j$ of the inequality before (6.9), we obtain

$$\beta_1\mathrm{tr}\rho\mathcal{L}_1(H_S) + \beta_2\mathrm{tr}\rho\mathcal{L}_2(H_S) \leq 0.$$

Moreover, $\mathrm{tr}\rho\mathcal{L}_1(H_S) = -\mathrm{tr}\rho\mathcal{L}_2(H_S)$ and so

$$(\beta_2 - \beta_1)\mathrm{tr}\rho\mathcal{L}_2(H_S) \geq 0$$

In view $\beta_1 \geq \beta_2$, we have $\mathrm{tr}\rho\mathcal{L}_1(H_S) = -\mathrm{tr}\rho\mathcal{L}_2(H_S) \geq 0$ and the proof is complete. □

In this section we prove a general explicit formula for the energy current in a stationary state $\rho$ which is a function of the system hamiltonian $H_S$. This not only confirms that it is positive also for possibly infinite dimensional systems if the eigenvalues of stationary state are a monotone system (i.e. there are no population inversions), but it allows us to establish proportionality to the difference of bath temperatures when they are not too small, namely an approximate Fourier law.

**Lemma 6.6.** *For all $\omega \in \mathsf{B}$ and $j = 1, 2$ we have*

$$\mathcal{Q}^-_{j,\omega}(H_S) = -\omega\, D^*_{j,\omega} D_{j,\omega} \qquad \mathcal{Q}^+_{j,\omega}(H_S) = \omega\, D_{j,\omega} D^*_{j,\omega} \tag{6.10}$$

*and*

$$\mathcal{L}_j(H_S) = \sum_{\omega \in \mathsf{B}} \omega \left( \Gamma^+_{j,\omega} D_{j,\omega} D^*_{j,\omega} - \Gamma^-_{j,\omega} D^*_{j,\omega} D_{j,\omega} \right). \tag{6.11}$$

*Proof.* Writing $H_S$ as in (6.1) we compute

$$\begin{aligned}
\mathcal{Q}^-_{j,\omega}(H_S) &= -\frac{1}{2} D^*_{j,\omega} D_{j,\omega} H_S + D^*_{j,\omega} H_S D_{j,\omega} - \frac{1}{2} H_S D^*_{j,\omega} D_{j,\omega} \\
&= \sum_{(\varepsilon_n,\varepsilon_m)\in \mathsf{B}_\omega} \left( \varepsilon_m\, P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j P_{\varepsilon_n} - \varepsilon_n\, P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j P_{\varepsilon_n} \right) \\
&= -\sum_{(\varepsilon_n,\varepsilon_m)\in \mathsf{B}_\omega} \omega P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j P_{\varepsilon_n} \\
&= -\omega\, D^*_{j,\omega} D_{j,\omega}.
\end{aligned}$$

The proof of the other identity (6.10) is similar. Since $[H_{j,\omega}, H_S] = 0$ for all $j, \omega$, (6.11) follows immediately. □

We can now prove our formula for the energy current in a stationary state $\rho$ which is a function of the system Hamiltonian $H_S$. We suppose that the interaction of the system with both reservoirs is similar; this property is reflected by the assumptions on $\mathrm{tr}P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j$ and $f_{1,\omega}$. In the sequel, to simplify the notation we also write $\rho_n$ instead of $\rho_{\varepsilon_n}$.

**Theorem 6.7.** *For any state $\rho$ which is a function of the system Hamiltonian $H_S$, i.e.*

$$\rho = \sum_{n\geq 0} \rho_n P_{\varepsilon_n} \tag{6.12}$$

*we have*

$$\mathrm{tr}\rho\mathcal{L}_j(H_S) = \sum_{\omega\in\mathsf{B}} \omega \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \left( \Gamma^+_{j,\omega}\rho_m - \Gamma^-_{j,\omega}\rho_n \right) \mathrm{tr}P_{\varepsilon_n} D^*_j P_{\varepsilon_m} D_j. \tag{6.13}$$

*If the state $\rho$ is also stationary and, moreover,* *1.* $\mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1 = \mathrm{tr}P_{\varepsilon_n}D_2^*P_{\varepsilon_m}D_2$ *for all* $n, m$,

*2.* $f_{1,\omega} = f_{2,\omega}$ *for all* $\omega$,

*then*

$$\mathrm{tr}\rho\mathcal{L}_1(H_S) = \frac{1}{2}\sum_{\omega\in\mathsf{B}} \omega\, f_{1,\omega}\left(\gamma_{1,\omega}^+ - \gamma_{2,\omega}^+\right) \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} (\rho_m - \rho_n)\,\mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1. \quad (6.14)$$

*Proof.* The proof of (6.13) is immediate from (6.11) and the following identities (cyclic property of the trace)

$$\mathrm{tr}P_{\varepsilon_m}D_{j,\omega}P_{\varepsilon_n}D_{j,\omega}^* = \mathrm{tr}(P_{\varepsilon_m}D_{j,\omega})P_{\varepsilon_m}D_{j,\omega}^* = \mathrm{tr}P_{\varepsilon_n}D_{j,\omega}^*P_{\varepsilon_m}D_{j,\omega}.$$

If the state $\rho$ is stationary, then $\mathrm{tr}\rho\mathcal{L}_1(H_S) = \mathrm{tr}\rho\mathcal{L}(H_S) - \mathrm{tr}\rho\mathcal{L}_2(H_S) = -\mathrm{tr}\rho\mathcal{L}_2(H_S)$, so that $\mathrm{tr}\rho\mathcal{L}_1(H_S) = \left(\mathrm{tr}\rho\mathcal{L}_1(H_S) - \mathrm{tr}\rho\mathcal{L}_2(H_S)\right)/2$. Computing the right-hand side difference by means of (6.13) with $j = 1, 2$ we can write $2\mathrm{tr}\rho\mathcal{L}_1(H_S)$ as

$$\begin{aligned}
&\sum_{\omega\in\mathsf{B}} \omega\, f_{1,\omega} \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \left(\gamma_{1,\omega}^+\rho_m - \gamma_{1,\omega}^-\rho_n - \gamma_{2,\omega}^+\rho_m + \gamma_{2,\omega}^-\rho_n\right)\mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1 \\
= &\sum_{\omega\in\mathsf{B}} \omega\, f_{1,\omega} \sum_{(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \left((\gamma_{1,\omega}^+ - \gamma_{2,\omega}^+)\rho_m - (\gamma_{1,\omega}^- - \gamma_{2,\omega}^-)\rho_n\right)\mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1.
\end{aligned}$$

Since $\gamma_{j,\omega}^- = \gamma_{j,\omega}^+ + 1$ for all $j, \omega$, then $\gamma_{1,\omega}^+ - \gamma_{2,\omega}^+ = \gamma_{1,\omega}^- - \gamma_{2,\omega}^-$ and (6.14) follows. □

*Remark* 6.8. Note that the above identity $\mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1 = \mathrm{tr}P_{\varepsilon_n}D_2^*P_{\varepsilon_m}D_2$ holds whenever there exists an isometry $R$ on $\mathsf{h}$, commuting with $H_S$, such that $D_2 = RD_1R^*$. Indeed, in this case, $R$ commutes with all spectral projections of $H_S$ and

$$\begin{aligned}
\mathrm{tr}P_{\varepsilon_n}D_2^*P_{\varepsilon_m}D_2 &= \mathrm{tr}P_{\varepsilon_n}RD_1^*R^*P_{\varepsilon_m}RD_1R^* \\
&= \mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1R^*R \\
&= \mathrm{tr}P_{\varepsilon_n}D_1^*P_{\varepsilon_m}D_1.
\end{aligned}$$

We will see later (Section 6.4) that this happens when the system interacts in the same way with the two baths.

Formula (6.14) can be applied to effectively compute the energy current in several models highlighting the dependence on the difference of temperatures. Indeed, one readily sees that, for $\beta_1, \beta_2$ very close the term $\omega\left(\gamma_{1,\omega}^+ - \gamma_{2,\omega}^+\right)$ is an infinitesimum of order $\beta_1^{-1} - \beta_2^{-1}$ while the other terms are close to some non-zero values. Moreover, it is also clear from (6.14) that the energy current is non-negative whenever the invariant state satisfies $\rho_m > \rho_n$ for all $n, m$ such that $\varepsilon_m < \varepsilon_n$ i.e. population inversion does not occur.

However, in order to find more explicit formulae we need additional information on the invariant state. This problem will be studied in the next section. We end this section by the following example

**Example 6.9.** Let $\mathsf{h} = \mathbb{C}^{n+1}$ with orthonormal basis $(e_k)_{0\le k\le n}$. Consider an $n$-level system with Hamiltonian

$$H_S = \sum_{k=0}^{n} k|e_k\rangle\langle e_k|$$

and interaction operators $D_1, D_2$ acting as

$$D_j e_k \quad = \quad e_{k-1} \quad \text{for } k = 1, \ldots, n \qquad D_j e_0 = 0.$$

Clearly $\mathsf{B} = \{1, 2, \ldots, n\}$ but the only non-zero $D_{j,\omega}$ are those corresponding to the frequency $\omega = 1$ and $D_{1,1} = D_1$, $D_{2,1} = D_2$. Moreover, since $\varepsilon_k = k$,

$$\mathrm{tr} P_{\varepsilon_k} D_1^* P_{\varepsilon_{k-1}} D_1 = \mathrm{tr} P_{\varepsilon_k} D_2^* P_{\varepsilon_{k-1}} D_2 = 1$$

for $k = 1, \ldots, n$. By Theorem 6.7 formula (6.13) we have

$$\mathrm{tr}\rho\, \mathcal{L}_j(H_S) = \sum_{k=0}^{n-1} \left( \Gamma_{j,1}^+ \rho_k - \Gamma_{j,1}^- \rho_{k+1} \right).$$

If all $\Gamma_{j,1}^{\pm}$ $(j = 1, 2)$ are nonzero, a straightforward computation shows that the unique stationary state is

$$\rho = \frac{1-\nu}{1-\nu^{n+1}} \sum_{k=0}^{n} \nu^k |e_k\rangle\langle e_k|, \qquad \nu := \frac{\Gamma_{1,1}^+ + \Gamma_{2,1}^+}{\Gamma_{1,1}^- + \Gamma_{2,1}^-}$$

and the energy current due to interaction with reservoir $j$ is

$$\begin{aligned} \mathrm{tr}\rho\, \mathcal{L}_j(H_S) &= \frac{1-\nu}{1-\nu^{n+1}} \sum_{k=0}^{n-1} \left( \Gamma_{j,1}^+ \nu^k - \Gamma_{j,1}^- \nu^{k+1} \right) \\ &= \frac{1-\nu^n}{1-\nu^{n+1}} \left( \Gamma_{j,1}^+ - \nu \Gamma_{j,1}^- \right). \end{aligned}$$

Note that, dropping the index 1 corresponding to the unique effective frequency $\omega$ to simplify the notation, we have

$$\begin{aligned} \Gamma_j^+ - \nu \Gamma_j^- &= \Gamma_j^- \left( \frac{\Gamma_j^+}{\Gamma_j^-} - \frac{\Gamma_1^+ + \Gamma_2^+}{\Gamma_1^- + \Gamma_2^-} \right) \\ &= \Gamma_j^- \left( \frac{\gamma_j^+}{\gamma_j^-} - \frac{f_1\, \gamma_1^+ + f_2\, \gamma_2^+}{f_1\, \gamma_1^- + f_2\, \gamma_2^-} \right) \\ &= \Gamma_j^- \left( \mathrm{e}^{-\beta_j} - \frac{f_1\, (\mathrm{e}^{\beta_2} - 1) + f_2\, (\mathrm{e}^{\beta_1} - 1)}{f_1\, \mathrm{e}^{\beta_1} (\mathrm{e}^{\beta_2} - 1) + f_2\, \mathrm{e}^{\beta_2} (\mathrm{e}^{\beta_1} - 1)} \right) \\ &= \Gamma_j^- \left( \mathrm{e}^{-\beta_j} - \frac{f_1\, \mathrm{e}^{-\beta_1} (1 - \mathrm{e}^{-\beta_2}) + f_2 \mathrm{e}^{-\beta_2} (1 - \mathrm{e}^{-\beta_1})}{f_1 (1 - \mathrm{e}^{-\beta_2}) + f_2 (1 - \mathrm{e}^{-\beta_1})} \right). \end{aligned}$$

For $j = 1$ we find

$$\Gamma_{j,1}^+ - \nu \Gamma_{j,1}^- \quad = \quad \Gamma_j^- f_2 (1 - \mathrm{e}^{-\beta_1}) \frac{\mathrm{e}^{-\beta_1} - \mathrm{e}^{-\beta_2}}{f_1 (1 - \mathrm{e}^{-\beta_2}) + f_2 (1 - \mathrm{e}^{-\beta_1})}$$

and so

$$\mathrm{tr}\rho\, \mathcal{L}_1(H_S) = \frac{1 - ((\Gamma_1^+ + \Gamma_2^+)/(\Gamma_1^- + \Gamma_2^-))^n}{1 - ((\Gamma_1^+ + \Gamma_2^+)/(\Gamma_1^- + \Gamma_2^-))^{n+1}} \frac{\Gamma_1^- f_2 (1 - \mathrm{e}^{-\beta_1})(\mathrm{e}^{-\beta_1} - \mathrm{e}^{-\beta_2})}{f_1 (1 - \mathrm{e}^{-\beta_2}) + f_2 (1 - \mathrm{e}^{-\beta_1})}$$

Since $\Gamma_j^+ < \Gamma_j^-$, this formula, for $n$ big and $\beta_1, \beta_2$ small becomes

$$\begin{aligned}\mathrm{tr}\rho\,\mathcal{L}_1(H_S) &\approx \frac{f_1 f_2(\mathrm{e}^{-\beta_1} - \mathrm{e}^{-\beta_2})}{f_1(1-\mathrm{e}^{-\beta_2}) + f_2(1-\mathrm{e}^{-\beta_1})} \\ &\approx \frac{f_1 f_2(\beta_2 - \beta_1)}{f_2\beta_1 + f_1\beta_2} \\ &= \frac{f_1 f_2\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)}{\frac{f_1}{\beta_1} + \frac{f_2}{\beta_2}}\end{aligned}$$

showing that, in a certain regime of high temperature a Fourier law holds for all choices $f_1, f_2$ of the interactions strength.

## 6.3 Dependence of the energy current from temperature difference and conductivity

In this section we consider systems whose Hamiltonian $H_S$ has simple spectrum, that is to say each spectral projection $P_{\varepsilon_n}$ is one-dimensional, and make explicit the dependence of the energy current on the difference of temperatures $1/\beta_1$ and $1/\beta_2$.

We begin by noting that, if spectral projections $P_{\varepsilon_n}$ are one-dimensional one can associate with the open quantum system a classical (time continuous) Markov chain with state space $V$ the spectrum $\mathrm{sp}(H_S)$ of $H_S$ in a canonical way. Indeed, for every bounded function $f$ on $V$, we have

$$\begin{aligned}\mathcal{L}(f(H_S)) &= \sum_{n\geq 0} f(\varepsilon_n)\mathcal{L}(P_{\varepsilon_n}) \\ &= \sum_{\omega\in\mathsf{B},\,(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \left(\sum_j \Gamma_{j,\omega}^- P_{\varepsilon_n} D_j^* P_{\varepsilon_m} D_j P_{\varepsilon_n}\right)(f(\varepsilon_m) - f(\varepsilon_n)) \\ &+ \sum_{\omega\in\mathsf{B},\,(\varepsilon_n,\varepsilon_m)\in\mathsf{B}_\omega} \left(\sum_j \Gamma_{j,\omega}^+ P_{\varepsilon_m} D_j P_{\varepsilon_n} D_j^* P_{\varepsilon_m}\right)(f(\varepsilon_n) - f(\varepsilon_m))\end{aligned}$$

and we find a classical Markov chain with transition rate matrix $Q = (q_{nm})$

$$q_{nm} = \begin{cases} \sum_j \Gamma_{j,\varepsilon_n-\varepsilon_m}^- \mathrm{tr} D_j^* P_{\varepsilon_m} D_j P_{\varepsilon_n}, & \text{if } \varepsilon_n > \varepsilon_m, \\ \sum_j \Gamma_{j,\varepsilon_m-\varepsilon_n}^+ \mathrm{tr} D_j P_{\varepsilon_m} D_j^* P_{\varepsilon_n}, & \text{if } \varepsilon_n < \varepsilon_m, \\ -\sum_{m\neq n} q_{nm}, & \text{if } n = m. \end{cases}$$

Now, if we consider the conditional expectation

$$\mathcal{E} : \mathcal{B}(\mathsf{h}) \to \ell^\infty(V;\mathbb{C}), \qquad \mathcal{E}(x) = \sum_{m\geq 0} P_{\varepsilon_m} x P_{\varepsilon_m},$$

where $\ell^\infty(V;\mathbb{C})$ is the abelian algebra of bounded functions on $V$, we have that

$$\mathcal{E}\circ\mathcal{L} = \mathcal{L}\circ\mathcal{E}. \tag{6.15}$$

Therefore, by defining the predual map $\mathcal{E}_*$ such that $\mathrm{tr}\mathcal{E}_*(\rho)x = \mathrm{tr}\rho\mathcal{E}(x)$, if $\rho$ is an invariant state, we have also $0 = \mathcal{E}_*(\mathcal{L}_*(\rho)) = \mathcal{L}_*(\mathcal{E}_*(\rho))$ and

$$(\pi_n) \mapsto \sum_{n\geq 0} \pi_n P_{\varepsilon_n}$$

gives a one-to-one correspondence between diagonal invariant states of the open quantum system and invariant measures of the associated Markov chain.

In the following, in order to have at hand an explicit formula for the invariant measure, we suppose, for simplicity, that the graph associated with the above Markov chain is a path graph and *jumps can occur only to nearest neighbour levels*, namely $q_{nm} = 0$ for $|n-m| \geq 2$. This assumption may hold, for instance, if the Hamiltonian $H_S$ is generic in the sense of [14], namely it is not only non-degenerate but also if $\varepsilon_n - \varepsilon_m = \varepsilon_{n'} - \varepsilon_{m'}$ then $\varepsilon_n = \varepsilon_{n'}$ and $\varepsilon_m = \varepsilon_{m'}$. Moreover, we assume that $q_{nm} \neq 0$ for $|n-m| \leq 1$. In this case the associated classical Markov chain has a simpler structure allowing one to make explicit computations and describe explicitly the structure of invariant states (see also [26] in a more general situation).

The explicit expression for the invariant state is $\rho = \sum_n \rho_n P_{\varepsilon_n}$ where

$$\rho_n = \prod_{0\leq k<n} \frac{q_{k,k+1}}{q_{k+1,k}} \rho_0 \tag{6.16}$$

with

$$q_{k,k+1} = \sum_{j=1}^{2} \Gamma^{+}_{j,\varepsilon_{k+1}-\varepsilon_k} \mathrm{tr} D_j P_{\varepsilon_{k+1}} D_j^* P_{\varepsilon_k},$$
$$q_{k+1,k} = \sum_{j=1}^{2} \Gamma^{-}_{j,\varepsilon_{k+1}-\varepsilon_k} \mathrm{tr} D_j^* P_{\varepsilon_k} D_j P_{\varepsilon_{k+1}}$$

provided that the normalization condition

$$\sum_{n\geq 1} \prod_{0\leq k<n} \frac{q_{k,k+1}}{q_{k+1,k}} < +\infty \tag{6.17}$$

holds, in which case $\rho_0$ is the inverse of the sum of the above series increased by $1$.

With the explicit formula for the invariant state we can find a Fourier's law for the energy current through the system. We begin by a technical lemma

**Lemma 6.10.** *The following inequalities hold*

$$\mathrm{e}^{-(\beta_1+\beta_2)\omega/2} \frac{\frac{1}{\beta_1} - \frac{1}{\beta_2}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} \leq \frac{\frac{1}{\mathrm{e}^{\beta_1\omega}-1} - \frac{1}{\mathrm{e}^{\beta_2\omega}-1}}{\frac{\mathrm{e}^{\beta_1\omega}}{\mathrm{e}^{\beta_1\omega}-1} + \frac{\mathrm{e}^{\beta_2\omega}}{\mathrm{e}^{\beta_2\omega}-1}} \leq \frac{\frac{1}{\beta_1} - \frac{1}{\beta_2}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}}, \tag{6.18}$$

*for all* $0 < \beta_1 \leq \beta_2$ *and* $\omega > 0$.

*Proof.* Note that $1/(\mathrm{e}^{\beta_1\omega}-1) - 1/(\mathrm{e}^{\beta_2\omega}-1) \leq 1/(\beta_1\omega) - 1/(\beta_2\omega)$ because the function $x \mapsto 1/(\mathrm{e}^{x\omega}-1) - 1/(\omega x)$ is increasing on $]0,+\infty[$ since

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\mathrm{e}^{x\omega}-1} - \frac{1}{\omega x}\right) = \frac{1}{\omega x^2} - \frac{\omega}{\left(\mathrm{e}^{\omega x/2} - \mathrm{e}^{-\omega x/2}\right)^2} \geq 0$$

by the elementary inequality $\mathrm{e}^{\omega x/2} - \mathrm{e}^{-\omega x/2} \geq \omega x$. Moreover, by another elementary inequality $1 - \mathrm{e}^{-\beta_j \omega} \leq \beta_j \omega$, we have

$$\frac{\mathrm{e}^{\beta_1\omega}}{\mathrm{e}^{\beta_1\omega}-1} + \frac{\mathrm{e}^{\beta_2\omega}}{\mathrm{e}^{\beta_2\omega}-1} = \frac{1}{1-\mathrm{e}^{-\beta_1\omega}} + \frac{1}{1-\mathrm{e}^{-\beta_2\omega}} \geq \frac{1}{\beta_1\omega} + \frac{1}{\beta_2\omega}$$

and the second inequality (6.18) follows.

In order to prove the first inequality we first write the right-hand side as

$$\begin{aligned} & \frac{(\mathrm{e}^{\beta_1\omega}-1)^{-1} - (\mathrm{e}^{\beta_2\omega}-1)^{-1}}{\mathrm{e}^{\beta_1\omega}(\mathrm{e}^{\beta_1\omega}-1)^{-1} + \mathrm{e}^{\beta_2\omega}(\mathrm{e}^{\beta_2\omega}-1)^{-1}} \\ = & \frac{\mathrm{e}^{\beta_2\omega} - \mathrm{e}^{\beta_1\omega}}{\mathrm{e}^{\beta_1\omega}\mathrm{e}^{\beta_2\omega/2}\left(\mathrm{e}^{\beta_2\omega/2} - \mathrm{e}^{-\beta_2\omega/2}\right) + \mathrm{e}^{\beta_2\omega}\mathrm{e}^{\beta_1\omega/2}\left(\mathrm{e}^{\beta_1\omega/2} - \mathrm{e}^{-\beta_1\omega/2}\right)} \\ = & \ \mathrm{e}^{-(\beta_1+\beta_2)\omega/2} \frac{\mathrm{e}^{(\beta_2-\beta_1)\omega/2} - \mathrm{e}^{-(\beta_2-\beta_1)\omega/2}}{(1-\mathrm{e}^{-\beta_2\omega}) + (1-\mathrm{e}^{-\beta_1\omega})} \end{aligned}$$

Noting that

$$\begin{aligned} \mathrm{e}^{(\beta_2-\beta_1)\omega/2} - \mathrm{e}^{-(\beta_2-\beta_1)\omega/2} & \geq 1 + \frac{(\beta_2-\beta_1)\omega}{2} - \left(1 - \frac{(\beta_2-\beta_1)\omega}{2}\right) \\ (1-\mathrm{e}^{-\beta_2\omega}) + (1-\mathrm{e}^{-\beta_1\omega}) & \leq (\beta_1+\beta_2)\,\omega \end{aligned}$$

we find

$$\frac{(\mathrm{e}^{\beta_1\omega}-1)^{-1} - (\mathrm{e}^{\beta_2\omega}-1)^{-1}}{\mathrm{e}^{\beta_1\omega}(\mathrm{e}^{\beta_1\omega}-1)^{-1} + \mathrm{e}^{\beta_2\omega}(\mathrm{e}^{\beta_2\omega}-1)^{-1}} \geq \mathrm{e}^{-(\beta_1+\beta_2)\omega/2} \frac{(\beta_2-\beta_1)\,\omega}{(\beta_1+\beta_2)\,\omega}.$$

This completes the proof. □

*Remark* 6.11. Note that the inequalities of Lemma 6.10 provide a sharp estimate in terms of the inverse temperature difference $\beta_2 - \beta_1$ for small $\beta_1, \beta_2$, i.e. when the average of temperatures $T_1, T_2$ is big. Indeed, the difference of the right-hand side and left-hand side is equal to

$$\left(1 - \mathrm{e}^{-(\beta_1+\beta_2)\omega/2}\right) \frac{\beta_2 - \beta_1}{\beta_1 + \beta_2}$$

and for temperatures $T_j > k_B \cdot 180\,\mathrm{K} = 2.49 \cdot 10^{-21}\,\mathsf{J}$ (approximately the lowest natural temperature ever recorded at ground level) we have $\beta_j < 1/(k_B \cdot 180\,\mathrm{K}) = 4.02 \cdot 10^{20}\,\mathsf{J}^{-1}$ so that the quantity that multiplies $\beta_2 - \beta_1$ is

$$\frac{1}{\beta_1 + \beta_2} < 1.24 \cdot 10^{-21}\mathsf{J}.$$

**Theorem 6.12.** *Suppose that*

1. $\mathrm{tr} P_{\varepsilon_n} D_j^* P_{\varepsilon_m} D_j = 1$ *for all* $n, m$ *and all* $j = 1, 2$*,*
2. $f_{j,\omega} = 1$ *for all* $\omega$ *and all* $j = 1, 2$*,*
3. *Jumps can occur only to nearest neighbour levels,*

4. *Formula (6.17) holds so that the state $\rho$ defined by (6.16) with $\rho_0$ determined by the normalization condition is invariant.*

*Then*

$$\kappa_{\mathrm{m}} \frac{\frac{1}{\beta_1} - \frac{1}{\beta_2}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} \kappa(\rho, H_S) \leq \mathrm{tr}\rho\mathcal{L}_1(\mathcal{H}_S) \leq \frac{\frac{1}{\beta_1} - \frac{1}{\beta_2}}{\frac{1}{\beta_1} + \frac{1}{\beta_2}} \kappa(\rho, H_S) \tag{6.19}$$

*where* $\kappa_{\mathrm{m}} = \inf_{m\geq 0} \mathrm{e}^{-(\beta_1+\beta_2)(\varepsilon_{m+1}-\varepsilon_m)/2}$ *and*

$$\widehat{H}_S = \sum_{m\geq 0} \varepsilon_{m+1} P_{\varepsilon_m}, \qquad \kappa(\rho, H_S) = \mathrm{tr}\rho(\widehat{H}_S - H_S).$$

*Proof.* By applying (6.14) in this context, we have

$$\begin{aligned}
\mathrm{tr}\rho\mathcal{L}_1(H_S) &= \frac{1}{2}\sum_{n\geq 0}(\varepsilon_{n+1} - \varepsilon_n)(\rho_n - \rho_{n+1})\left(\Gamma^+_{1,\varepsilon_{n+1}-\varepsilon_n} - \Gamma^+_{2,\varepsilon_{n+1}-\varepsilon_n}\right) \\
&= \frac{1}{2}\sum_{n\geq 0}(\varepsilon_{n+1} - \varepsilon_n)\left(1 - \frac{q_{n,n+1}}{q_{n+1,n}}\right)\rho_n\left(\Gamma^+_{1,\varepsilon_{n+1}-\varepsilon_n} - \Gamma^+_{2,\varepsilon_{n+1}-\varepsilon_n}\right) \\
&= \sum_{n\geq 0}(\varepsilon_{n+1} - \varepsilon_n)\rho_n \frac{\Gamma^+_{1,\varepsilon_{n+1}-\varepsilon_n} - \Gamma^+_{2,\varepsilon_{n+1}-\varepsilon_n}}{\Gamma^-_{1,\varepsilon_{n+1}-\varepsilon_n} + \Gamma^-_{2,\varepsilon_{n+1}-\varepsilon_n}}.
\end{aligned}$$

Now the proof follows applying Lemma 6.10 with $\omega = \varepsilon_{n+1} - \varepsilon_n$ to estimate the right-hand side ratio. □

*Remark* 6.13. Formula (6.19) shows that the energy current $\mathrm{tr}\rho\mathcal{L}_1(H_S)$ h as an explicit dependence on the difference $\beta_1^{-1} - \beta_2^{-1}$ of the reservoirs' temperatures. This dependence holds only through two inequalities, but it suggests the existence of an "approximate" Fourier law (see [9, 51]) for the current. Clearly there can be further dependencies through the term $\kappa(\rho, H_S)$, however it holds

$$\inf_k (\varepsilon_{k+1} - \varepsilon_k) \leq \kappa(\rho, H_S) \leq \sup_k (\varepsilon_{k+1} - \varepsilon_k).$$

Therefore the energy current depends on the temperature difference mainly through the explicit term and one could say that there really is an "approximate" Fourier Law. Furthermore it is worth noticing that, for $\beta_1, \beta_2$ fixed, the inequality (6.19) is better the smaller is $\sup_{m\geq 0}(\varepsilon_{m+1} - \varepsilon_m)$ so that $\kappa_{\mathrm{m}}$ is close to 1 and the inequalities are approximately equalities. However, it should also be noted that, in this case, $\kappa(\rho, H_S)$ becomes small as well. Eventually note that, due to the nature of our system, we cannot investigate spatial properties of energy flow. Therefore our discussion of the Fourier's law is concerned with proportionality to temperature difference and not with dependency on size.

*Remark* 6.14. Since the above QMS are of weak coupling limit type, one can write explicitly the entropy production (in the sense of [34, 35]).

It is tempting to study in detail what happens when $\sup_{m\geq 0}(\varepsilon_{m+1} - \varepsilon_m)$ tends to 0 so that the eigenvalues of $H_S$ increase in number and form a set more and more packed.

In a more precise way, for all $n \geq 1$ we assume that the system Hamiltonian is a self-adjoint operator $H_S^{(n)}$ on an $(n+1)$-dimensional Hilbert space $\mathsf{h}$ with simple pure point spectrum $(\varepsilon_k^{(n)})_{0\leq k\leq n}$ with $\varepsilon_0 = 0$ and, for all $a, b$ with $0 \leq a < b \leq +\infty$, we have

$$\lim_{n\to\infty} \frac{\mathrm{card}\left\{ k \mid a < \varepsilon_k^{(n)} \leq b \right\}}{n} = \mu(]a, b]) \tag{6.20}$$

for some continuous probability density $\mu$ on $[0, +\infty[$. In other words, the empirical distribution of eigenvalues of $H_S^{(n)}$ converges weakly to a probability distribution on $[0, +\infty[$ . Suppose, for simplicity, that $\mu$ has no atoms, i.e. $\mu(\{r\}) = 0$ for all $r \geq 0$.

We can now prove the following result on the distribution of eigenvalues of the stationary state and energy in stationary conditions.

**Theorem 6.15.** *Under the assumptions of Theorem 6.12, for all $n \geq 1$, let $H_S^{(n)}$ be as above and suppose that (6.20) holds. Let $\rho^{(n)}$ be the $\mathcal{L}$-invariant state (6.16) and let*

$$\widetilde{\beta} = 2\left(1/\beta_1 + 1/\beta_2\right)^{-1}$$

*be the harmonic mean of the inverse temperatures (i.e. $\widetilde{\beta}^{-1}$ is the arithmetic mean of $\beta_1^{-1}, \beta_2^{-1}$).*

*(i) Eigenvectors $\rho_k^{(n)}$ of $\rho^{(n)}$ satisfy*

$$\lim_{n\to\infty} \sum_{\{ k \mid a<\varepsilon_k\leq b\}} \rho_k^{(n)} = \frac{\int_a^b \mathrm{e}^{-\widetilde{\beta}r}\mathrm{d}\mu(r)}{\int_0^\infty \mathrm{e}^{-\widetilde{\beta}r}\mathrm{d}\mu(r)}$$

*(ii) The average energy in the system satisfies*

$$\lim_{n\to\infty} \mathrm{tr}\rho^{(n)} H_S^{(n)} = \frac{\int_0^\infty \mathrm{e}^{-\widetilde{\beta}r}\, r\mathrm{d}\mu(r)}{\int_0^\infty \mathrm{e}^{-\widetilde{\beta}r}\mathrm{d}\mu(r)}.$$

This result reminds the one in [49] where the steady state can be described by a generalized Gibbs state and the steady-state current is proportional to the difference in the reservoirs' magnetizations.

In the proof we need the following Lemma.

**Lemma 6.16.** *Let $\widetilde{\beta} = 2/\left(\beta_1^{-1} + \beta_2^{-1}\right)$ be the harmonic mean of inverse temperatures (i.e. $\widetilde{\beta}^{-1}$ is the arithmetic mean of $\beta_1^{-1}$ and $\beta_2^{-1}$). For all $1 \leq k \leq n$ and for $\sup_j \omega_j < 1/(3\beta_2)$,*

$$1 - \widetilde{\beta}\,\omega_k \leq \frac{q_{k,k+1}}{q_{k+1,k}} \leq 1 - \widetilde{\beta}\,\omega_k + \left(\widetilde{\beta}\,\omega_k\right)^2 \tag{6.21}$$

*where $\omega_k = \varepsilon_{k+1} - \varepsilon_k$ and*

$$\mathrm{e}^{-\widetilde{\beta}\varepsilon_k\left(1+\widetilde{\beta}\sup_j\omega_j\right)} \leq \prod_{j=0}^{k-1} \frac{q_{j,j+1}}{q_{j+1,j}} \leq \mathrm{e}^{-\widetilde{\beta}\varepsilon_k\left(1-\widetilde{\beta}\sup_j\omega_j\right)} \tag{6.22}$$

*Proof.* By the elementary inequality $1 - \mathrm{e}^{-\beta_j \omega_k} \le \beta_j \omega_k$ we have

$$\begin{aligned} \frac{q_{k,k+1}}{q_{k+1,k}} &= \frac{\frac{1}{\mathrm{e}^{\beta_1\omega_k-1}} + \frac{1}{\mathrm{e}^{\beta_2\omega_k-1}}}{\frac{\mathrm{e}^{\beta_1\omega_k}}{\mathrm{e}^{\beta_1\omega_k-1}} + \frac{\mathrm{e}^{\beta_2\omega_k}}{\mathrm{e}^{\beta_2\omega_k-1}}} \\ &= 1 - \frac{2}{\left(1-\mathrm{e}^{-\beta_1\omega_k}\right)^{-1} + \left(1-\mathrm{e}^{-\beta_2\omega_k}\right)^{-1}} \\ &\ge 1 - \frac{2\omega_k}{1/\beta_1 + 1/\beta_2} \end{aligned}$$

In the same way, by the elementary inequalities $1 - \mathrm{e}^{-\beta_j\omega_k} \ge \beta_j\omega_k - (\beta_j\omega_k)^2/2$ and $1/(1-\beta_j\omega_k/2) \le 1 + \beta_j\omega_k$, we find for $\beta_j\omega_k < 1$

$$\begin{aligned} \frac{q_{k,k+1}}{q_{k+1,k}} &\le 1 - \frac{2\omega_k}{1/\left(\beta_1\left(1-\beta_1\omega_k/2\right)\right) + 1/\left(\beta_2\left(1-\beta_2\omega_k/2\right)\right)} \\ &\le 1 - \frac{2\omega_k}{1/\beta_1\left(1+\beta_1\omega_k/2\right) + 1/\beta_2\left(1+\beta_2\omega_k/2\right)} \\ &\le 1 - \frac{2\omega_k}{1/\beta_1 + 1/\beta_2 + 2\omega_k} \\ &= 1 - \frac{\widetilde{\beta}\,\omega_k}{1+\widetilde{\beta}\,\omega_k} \end{aligned}$$

and so (6.21) follows.

In order to prove the upper bound in (6.22), note that, since $\log(1-x) \le -x$

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) \le \sum_{j=0}^{k-1}\log\left(1-\widetilde{\beta}\,\omega_j\left(1-\widetilde{\beta}\,\omega_j\right)\right) \le -\sum_{j=0}^{k-1}\widetilde{\beta}\omega_j\left(1-\widetilde{\beta}\omega_j\right),$$

as a consequence

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) \le -\sum_{j=0}^{k-1}\widetilde{\beta}\omega_j\left(1-\widetilde{\beta}\sup_l \omega_l\right) = -\widetilde{\beta}\varepsilon_k\left(1-\widetilde{\beta}\sup_l\omega_l\right).$$

For the lower bound, we begin by the inequality

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) = \sum_{j=0}^{k-1}\log\left(\frac{q_{j,j+1}}{q_{j+1,j}}\right) \ge \sum_{j=0}^{k-1}\log\left(1-\widetilde{\beta}\omega_j\right)$$

Note that $\log(1-x) + x + x^2 \ge 0$ for $0 \le x \le 2/3$ and, since $\widetilde{\beta}\omega_j < 2/3$ by our assumption, we have

$$\log\left(\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}\right) \ge -\sum_{j=0}^{k-1}\widetilde{\beta}\omega_j\left(1+\widetilde{\beta}\sup_l\omega_l\right) = -\widetilde{\beta}\epsilon_k\left(1+\widetilde{\beta}\sup_l\omega_l\right).$$

This completes the proof. □

*Proof of Theorem 6.15.* Let $\mu_n$ be the empirical distribution of the eigenvalues of $H_S^{(n)}$ i.e.

$$\mu_n = \frac{1}{n+1}\sum_{k=0}^{n}\delta_{\varepsilon_k}$$

and note that

$$\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\rho_k^{(n)} = \frac{\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}}{\frac{1}{n+1}\sum_{k=0}^{n}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}}}. \tag{6.23}$$

Clearly, by Lemma 6.16,

$$\begin{aligned}\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}} &\le \frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\mathrm{e}^{-\tilde\beta\varepsilon_k\left(1-\tilde\beta\sup_j\omega_j\right)}\\ &\le \mathrm{e}^{\tilde\beta^2 b\sup_j\omega_j}\int_{]a,b]}\mathrm{e}^{-\tilde\beta\varepsilon_k}\,\mathrm{d}\mu_n(r)\end{aligned}$$

and also

$$\begin{aligned}\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}} &\ge \frac{\mathrm{e}^{-\tilde\beta^2 a\sup_j\omega_j}}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\mathrm{e}^{-\tilde\beta\varepsilon_k}\\ &= \mathrm{e}^{-\tilde\beta^2 a\sup_j\omega_j}\int_{]a,b]}\mathrm{e}^{-\tilde\beta\varepsilon_k}\,\mathrm{d}\mu_n(r)\end{aligned}$$

Since $\sup_j\omega_j$ goes to 0, probability measures $\mu_n$ converge weakly to $\mu$ and the function $r\to\mathrm{e}^{-\tilde\beta r}$ is bounded continuous on $[0,+\infty[$, taking the limit as $n\to\infty$, we have

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\{\,k\,|\,a<\varepsilon_k\le b\}}\prod_{j=0}^{k-1}\frac{q_{j,j+1}}{q_{j+1,j}} = \int_{]a,b]}\mathrm{e}^{-\tilde\beta\varepsilon_k}\,\mathrm{d}\mu(r).$$

In the same way, taking $a=0$ and $b=+\infty$, we see that the denominator of (6.23) converges to

$$\int_0^{+\infty}\mathrm{e}^{-\tilde\beta r}\,\mathrm{d}\mu(r)$$

and the proof of (i) is complete. The proof of (ii) is similar. □

*Remark* 6.17. Theorem 6.15 (i) shows that, if $\mu$ has density $\mu'$, then the asymptotic distribution of eigenvalues of the stationary state is

$$\lambda\mapsto\frac{\mathrm{e}^{-\tilde\beta\lambda}\mu'(\lambda)}{\int_0^{+\infty}\mathrm{e}^{-\tilde\beta r}\mu'(r)\mathrm{d}r}.$$

The asymptotic average energy in the system can be easily computed in some remarkable cases noting that the integral of $\mathrm{e}^{-\tilde\beta r}$ with respect to $\mu$ is the moment generating function $\phi$ of $\mu$ evaluated at $-\tilde\beta$ and so the asymptotic average energy in the system is

$$-\frac{\frac{\mathrm{d}}{\mathrm{d}\tilde\beta}\phi(-\tilde\beta)}{\phi(-\tilde\beta)} = -\frac{\mathrm{d}}{\mathrm{d}\tilde\beta}\log\left(\phi(-\tilde\beta)\right).$$

We can easily find an explicit result in two cases:

$$\begin{array}{rll} \mu \text{ normal distribution } N(m,\sigma^2) & \text{average energy} & m-\tilde{\beta}\sigma \\ \mu \text{ gamma distribution } \Gamma(\alpha,\theta) & \text{average energy} & \alpha/(\tilde{\beta}+\theta) \end{array}$$

The asymptotic average energy in the system is decreasing in $\tilde{\beta}$, i.e. increasing in the average temperature as expected, for all probability measure $\mu$ because the moment generating function of a probability distribution is log-convex and the derivative of a convex function is increasing.

*Remark* 6.18. Note that, by choosing a suitable spacing of eigenvalues $\varepsilon_n$ we can control the rate of convergence to 0 of $\kappa\left(\rho^{(n)}, H_S^{(n)}\right)$ at will, as $n$ tends to $+\infty$.

## 6.4 One dimensional Ising chain

In this section we consider a one-dimensional Ising chain with nearest neighbour interaction. We will show that, in this case, if the heat baths interact locally at both ends of the chain, then the energy current is zero. Spin interaction (see 6.24) occurs only in the $z$ component. In the case where also the other components interact the derivation of the GKSL generator turns out to be really difficult (see [10]). Indeed, starting from the diagonalized $H_S$, one finds a cumbersome expression for the operators $D_\omega$.

In spite of the simple system Hamitonian $H_S$ (6.24) Theorems 6.12 and 6.15 do not apply to this model because its spectrum is degenerate.

The system space is $\mathsf{h} = \mathbb{C}^{2^{\otimes N}}$ with $N > 2$. Define Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the orthonormal basis $e_+ = [1,0]^{\mathrm{T}}$, $e_- = [0,1]^{\mathrm{T}}$ of $\mathbb{C}^2$.

Consider the one dimensional Ising chain with Hamiltonian

$$H_S = J_z \sum_{j=1}^{N-1} \sigma_j^z \sigma_{j+1}^z, \qquad J_z > 0,\ N > 2 \tag{6.24}$$

Subsequently let us define

$$e_\alpha := \otimes_{j=1}^N e_{\alpha(j)}, \quad \alpha \in \{-1,1\}^N,$$

as a basis of $\mathsf{h}$, where $e_{-1} := e_-$ and $e_{+1} := e_+$. Vectors $\{e_\alpha\}_\alpha$ form an eigenbasis for $H_S$ and the spectrum is

$$\mathrm{sp}(H_S) = \{\, J_z\,(2k-(N-1)) \mid k = 0,\dots,N-1\,\}.$$

The eigenspace associated with the eigenvalue $\varepsilon_k = J_z(2k-(N-1))$ is the linear span of the elements $e_\alpha$ such that exactly $k$ neighbouring elements in $\alpha$ have the same sign. Thus one can define the sets

$$A_k := \left\{\alpha \in \{-1,1\}^N \;\middle|\; \textstyle\sum_{j=1}^{N-1} \alpha(j)\alpha(j+1) = 2k-(N-1)\right\},$$

and the spectral projection associated with the eigenvalue $\varepsilon_k$ is given by

$$P_k := \sum_{\alpha\in A_k} |e_\alpha\rangle\langle e_\alpha| \,.$$

The system is coupled with two heat reservoirs at inverse temperature $\beta_1, \beta_2$ with $\beta_1 \leq \beta_2$ through the interaction

$$H_1 = \sigma_1^u \otimes (A^-(\phi_1) + A^+(\phi_1)), \quad H_2 = \sigma_N^v \otimes (A^-(\phi_2) + A^+(\phi_2)). \tag{6.25}$$

where $u, v \in \mathbb{R}^3$ and $\sigma_i^u$ is defined as

$$\sigma_i^u = u_1\sigma_i^x + u_2\sigma_i^y + u_3\sigma_i^z.$$

The set of positive Bohr frequencies is given by

$$\mathsf{B} := \{\, 2J_z(n-m) = \varepsilon_n - \varepsilon_m \mid n, m \in \{0, \ldots, N-1\},\, n > m \,\}.$$

while the operators $D_{j,\omega}$ are given by(6.5). Thus one has

$$D_{1,2J_z} = (u_1 - \mathrm{i}u_2) \sum_{\alpha\in C^l_{++}} \sigma_1^x \,|e_\alpha\rangle\langle e_\alpha| + (u_1 + \mathrm{i}u_2) \sum_{\alpha\in C^l_{--}} \sigma_1^x \,|e_\alpha\rangle\langle e_\alpha|$$

where $C^l_{++}$ (resp. $C^l_{--}$) denotes the set of configurations $\alpha \in \{-1, +1\}^N$ with $++$ (resp. $--$) in the first two sites ($l$ stands for left). While $D_{1,\omega} = 0$ for every $\omega \in \mathsf{B} - \{2J_z\}$ because the Pauli matrices act only on the first site and so the number of neighbouring sites with the same sign can vary of at most one after the action of $\sigma_1^u$ and for $\omega = 2J_z$ one has

$$D_{1,2J_z} = \sum_{n=1}^{N-1} \sum_{\alpha\in A_n} \sum_{\beta\in A_{n+1}} \langle e_\alpha, \sigma_1^x e_\beta\rangle \,|e_\alpha\rangle\langle e_\beta| \,.$$

With similar arguments one can see that $D_{2,\omega} = 0$ for every $\omega \in \mathsf{B} - \{2J_z\}$, while

$$D_{2,2J_z} = (v_1 - \mathrm{i}v_2) \sum_{\alpha\in C^r_{++}} \sigma_N^x \,|e_\alpha\rangle\langle e_\alpha| + (v_1 + \mathrm{i}v_2) \sum_{\alpha\in C^r_{--}} \sigma_N^x \,|e_\alpha\rangle\langle e_\alpha|$$

where $C^r_{++}$ (resp. $C^r_{--}$) denotes the set of configurations with $++$ (resp. $--$) in the last two sites ($r$ stands for right).

From now on we will drop the subscript $2J_z$ and only deal with operators related to that Bohr frequency, as the others vanish.
Recalling the definition of linear maps (6.6) and the constants

$$\gamma_i^+ = 1/(\mathrm{e}^{2J_z\beta_i} - 1), \quad \gamma_i^- = \mathrm{e}^{2J_z\beta_i}/(\mathrm{e}^{2J_z\beta_i} - 1),$$

we can write the GKLS generator of the evolution as follows

$$\mathcal{L} = \sum_{i\in\{1,N\}} \gamma_i^- \mathcal{Q}_i^- + \gamma_i^+ \mathcal{Q}_i^+.$$

A close scrutiny at the operators $D_i, D_i^*$ shows that, for each fixed configuration $\overline{\alpha} \in \{-1,+1\}^{N-2}$ of the $N-2$ inner sites of the chain the 4-dimensional projections $p_{\overline{\alpha}}$ on subspaces

$$\mathsf{h}_{\overline{\alpha}} := \operatorname{span}\{\, e_\alpha \mid \alpha(j) = \overline{\alpha}(j) \text{ for all } 2 \le j \le N-1;\ \alpha(1), \alpha(N) \in \{-1,1\}\,\}$$

commute with both $D_i$ and $D_i^*$ for $i \in \{1, N\}$, then subalgebras $p_{\alpha_1}\mathcal{B}(\mathsf{h})p_{\alpha_2}$ are invariant for the semigroup $\mathcal{T}$ generated by $\mathcal{L}$. This commutation allows us to restrict our study only to cases where the invariant state is of the form

$$\rho = \sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} p_{\overline{\alpha}}\rho p_{\overline{\alpha}} = \sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} \lambda_{\overline{\alpha}}\rho_{\overline{\alpha}}, \tag{6.26}$$

where $\rho_{\overline{\alpha}}$ is an invariant state supported only on $\mathsf{h}_{\overline{\alpha}}$ and $\lambda_{\overline{\alpha}}$ are real costants that sum up to 1. Indeed the off diagonal terms, $p_{\overline{\alpha_1}}\rho p_{\overline{\alpha_2}}$ with $\alpha_1 \neq \alpha_2$, do not contribute to current flow, since

$$\operatorname{tr} p_{\overline{\alpha_1}}\rho p_{\overline{\alpha_2}}\mathcal{L}_1(H_S) = \operatorname{tr} p_{\overline{\alpha_1}}\rho\mathcal{L}_1(H_S)p_{\overline{\alpha_2}} = 0.$$

Moreover all the conditional expectations $\mathcal{E}_{\overline{\alpha}}(x) := p_{\overline{\alpha}}xp_{\overline{\alpha}}$ commute with $\mathcal{L}$, ensuring that both $\sum_{\overline{\alpha}}\mathcal{E}_{\overline{\alpha},*}(\rho)$ and every $\mathcal{E}_{\overline{\alpha},*}(\rho)$ must also be invariant states on their own. As a further refinement we can repeat the same argument using the conditional expectation $\mathcal{E}(x) := \sum_{k=0}^{N-1} P_k x P_k$. Indeed $\mathcal{E}$ commutes with the Lindbladian $\mathcal{L}$ and

$$\operatorname{tr} P_{k_1}\rho P_{k_2}\mathcal{L}_1(H_S) = \operatorname{tr} P_{k_1}\rho\mathcal{L}_1(H_S)P_{k_2} = 0$$

for $k_1 \neq k_2$, since the spectral projections commute with $D_jD_j^*$, $D_j^*D_j$ and $\mathcal{L}_1(H_S)$ is a linear combination of these operators by Lemma 6.6, equation (6.10). In this way we can focus our study on invariant states of the form (6.26) with

$$p_{\overline{\alpha}}\rho p_{\overline{\alpha}} = \rho_{\overline{\alpha}} = \begin{pmatrix} \rho_{11}^{\overline{\alpha}} & 0 & 0 & 0 \\ 0 & \rho_{22}^{\overline{\alpha}} & \rho_{23}^{\overline{\alpha}} & 0 \\ 0 & \rho_{32}^{\overline{\alpha}} & \rho_{33}^{\overline{\alpha}} & 0 \\ 0 & 0 & 0 & \rho_{44}^{\overline{\alpha}} \end{pmatrix},$$

where we expanded the state with respect to the basis of four vectors denoted by $e_{c\,\overline{\alpha}\,c}$, $e_{d\,\overline{\alpha}\,c}$, $e_{c\overline{\alpha}d}$, $e_{d\overline{\alpha}d}$ and defined as follows: $e_{c\overline{\alpha}c}$ is the vector $e_{\overline{\alpha}(2),\overline{\alpha}(2),\dots,\overline{\alpha}(N-1),\overline{\alpha}(N-1)}$, $e_{c\overline{\alpha}d}$ corresponds to $e_{\overline{\alpha}(2),\overline{\alpha}(2),\dots,\overline{\alpha}(N-1),-\overline{\alpha}(N-1)}$ and vectors $e_{d\overline{\alpha}c}$, $e_{d\overline{\alpha}d}$ are defined in a similar way.

Now we have reduced and simplified the class of states we want to use when looking for a invariant state, without, however, losing any contribution to the current flow. In order to find the invariant state, first of all it is not too difficult to show that $\mathcal{L}_*$ leaves invariant the subspace of diagonal elements. Then compute

$$\mathcal{L}_*(\rho_{23}^{\overline{\alpha}}\,|e_{d\,\alpha\,c}\rangle\langle e_{c\,\alpha\,d}|) = -\frac{1}{2}\left[\Gamma_1^+ + \Gamma_1^- + \Gamma_N^+ + \Gamma_N^-\right]\rho_{23}^{\overline{\alpha}}\,|e_{d\,\alpha\,c}\rangle\langle e_{c\,\alpha\,d}|\,,$$

and similarly

$$\mathcal{L}_*(\rho_{32}^{\overline{\alpha}}\,|e_{c\,\alpha\,d}\rangle\langle e_{d\,\alpha\,c}|) = -\frac{1}{2}\left[\Gamma_1^+ + \Gamma_1^- + \Gamma_N^+ + \Gamma_N^-\right]\rho_{32}^{\overline{\alpha}}\,|e_{c\,\alpha\,d}\rangle\langle e_{d\,\alpha\,c}|\,,$$

where $\Gamma_1^{\pm} = \|u_1 + \mathrm{i}u_2\|^2 \gamma_1^{\pm}$ and $\Gamma_N^{\pm} = \|v_1 + \mathrm{i}v_2\|^2 \gamma_N^{\pm}$. (The above $\Gamma_i^{\pm}$ slightly differ from the constants in Section 6.1). Therefore the invariant state condition $\mathcal{L}_*(\rho) = 0$ implies $\rho_{23}^{\overline{\alpha}} = \rho_{32}^{\overline{\alpha}} = 0$. We can now just consider the reduced dynamics on diagonal elements of $p_{\overline{\alpha}}\mathcal{B}(\mathsf{h})p_{\overline{\alpha}}$, given by

$$\mathcal{L}_* = \begin{pmatrix} -(\Gamma_1^- + \Gamma_N^-) & \Gamma_1^- & \Gamma_N^- & 0 \\ \Gamma_1^+ & -(\Gamma_1^+ + \Gamma_N^-) & 0 & \Gamma_N^- \\ \Gamma_N^+ & 0 & -(\Gamma_N^+ + \Gamma_1^-) & \Gamma_1^- \\ 0 & \Gamma_N^+ & \Gamma_1^+ & -(\Gamma_N^+ + \Gamma_1^+) \end{pmatrix},$$

The unique invariant law for the time-continuous Markov chain generated by the above matrix is

$$\pi = Z^{-1}\left[1, \mathrm{e}^{2J_z\beta_1}, \mathrm{e}^{2J_z\beta_2}, \mathrm{e}^{2J_z(\beta_1+\beta_2)}\right],$$

where $Z^{-1}$ is a normalization constant that is independent of $u, v$ and is the same for all $\overline{\alpha}$. Therefore the unique $\mathcal{T}$-invariant state supported on $\mathsf{h}_\alpha$ is

$$\rho_\alpha = Z^{-1}\Big( |e_{c\overline{\alpha}c}\rangle\langle e_{c\overline{\alpha}c}| + \mathrm{e}^{2J_z\beta_1} |e_{d\,\overline{\alpha}c}\rangle\langle e_{d\,\overline{\alpha}c}| + \mathrm{e}^{2J_z\beta_2} |e_{c\overline{\alpha}\,d}\rangle\langle e_{c\overline{\alpha}\,d}| + \mathrm{e}^{2J_z(\beta_1+\beta_2)} |e_{d\,\overline{\alpha}\,d}\rangle\langle e_{d\,\overline{\alpha}\,d}| \Big).$$

Recalling (6.26) we can now write any invariant state for the semigroup $\mathcal{T}$.

We can now evaluate the energy flow $\mathrm{tr}\rho\mathcal{L}_1(H_S)$ via the expression

$$\mathcal{L}_1(H_S) = \sum_{\omega\in B^+} \omega\left(\gamma_{1,\omega}^+ D_1 D_1^* - \gamma_{1,\omega}^- D_1^* D_1\right) = 2J_z\left(\gamma_1^+ D_1 D_1^* - \gamma_1^- D_1^* D_1\right)$$

that, together with the formula for $\rho_\alpha$, yields

$$\begin{aligned} Z\,\mathrm{tr}\rho\mathcal{L}_1(H_S) &= Z\,\mathrm{tr}\sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} \lambda_{\overline{\alpha}}\rho_{\overline{\alpha}}\mathcal{L}_1(H_S) \\ &= \sum_{\overline{\alpha}\in\{-1,1\}^{N-2}} 2J_z\lambda_{\overline{\alpha}}\left(\gamma_1^+\mathrm{e}^{\beta_1\omega} + \gamma_1^+\mathrm{e}^{(\beta_1+\beta_2)\omega} - \gamma_1^-\mathrm{e}^{\beta_2\omega} - \gamma_1^-\right) \\ &= 0 \end{aligned}$$

*Remark* 6.19. For $N = 2$, it can be shown by direct computation that the energy current is strictly positive. Indeed, because of low dimensionality the ends of the chain can interact directly.

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