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EXECUTIVE SUMMARY OF THE THESIS

Variational Methods for Viscous Ergodic Mean Field Games

LAUREA MAGISTRALE IN MATHEMATICAL ENGINEERING - INGEGNERIA MATEMATICA

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Introduction

This master thesis work concentrates on the study of the Mean Field Games (MFGs) theory, which models differential games involving infinitely many interacting players. MFGs systems were introduced around 2005 by the seminal works of Lasry and Lions and independently by Caines, Huang and Malhamé. The theory has developed rapidly in the years and different approaches can be considered for the problem, both analytic and probabilistic. For an introduction to the subject, we refer to the notes of the lectures given by Pierre-Louis Lions in the years 2007-2012 at the C ollege de France and the CIME lecture notes by Cardaliaguet and Porretta [1]. We follow the analytic PDE approach and in particular we focus on Variational Mean Field Games, a specific class of MFGs that admits a variational structure which enables to find solutions to the system as critical points of a given energy under a specific constraint (see [5]), relying on techniques of convex analysis, properties of the Legendre-Fenchel transform and Gagliardo-Nirenberg Inequality. In particular we follow [2] and deal with Viscous Ergodic Mean Field Games with local coupling.

The thesis is divided into three chapters. The first chapter introduces MFGs, giving a heuristic derivation and presenting some standard results. The second chapter presents some results on Hamilton-Jacobi and Fokker-Planck equations, which are propaedeutic for the work. The third chapter is the core of the thesis, in which we prove some original result, extending the results of [2] to problems with a coupling term with stronger non-linearities (up to the "Sobolev critical" exponent included) and including Neumann boundary conditions.

Mean Field Games

The first chapter is an introduction to the theory of MFGs and follows [1]. We begin by giving an heuristic derivation of MFGs, introducing the equations involved in the problem. Then, we give some standard existence results. Moreover, we show in which sense the MFGs can be seen as an approximation of N players differential games. Finally, we introduce Viscous Ergodic Mean Field Games and derive the system that we study in Chapter 3.

The Mean Field Games Theory was developed in recent years to study differential games involving a large number of rational agents, each controlling their path according to some common strategy which is affected by the state of other agents. As the number N of agents goes to infinity, the evolution of the system is described by a Nash equilibrium of a game with a continuum number of players. Each agent controls the SDE

$$dX_s = b(X_s, \alpha_s, \bar{m}(s))ds + \sqrt{2\nu}dB_s \quad (1)$$

with a control α taking values in a set A , where $\bar{m}(s)$ is the distribution of all agents, which for the moment we consider as fixed, and B_t is a standard brownian motion. We consider the problem in an interval $[0, T]$ and for simplicity we assume the state space to be the whole \mathbb{R}^N . Each agent aims at minimizing the quantity

$$\mathbb{E} \left[\int_0^T L(X_s, \alpha_s, \bar{m}(s)) ds + G(X_T, \bar{m}(T)) \right] \quad (2)$$

where L is the running cost and G is the final cost, both possibly depending on the distribution of all agents. As it is usual in control problems we can introduce the value function u

$$u(x, t) = \inf_{\alpha} \mathbb{E} \left[\int_t^T L(X_s^{x,t}, \alpha_s, \bar{m}(s)) ds + G(X_T^{x,t}, \bar{m}(T)) \right] \quad (3)$$

where $X_s^{x,t}$ is the solution of (1) with $X_t = x$. Classical results of control theory imply that u satisfies

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u, \bar{m}) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ u(x, T) = G(x, \bar{m}(T)) & \text{in } \mathbb{R}^N \end{cases} \quad (4)$$

where

$$H(x, p, \bar{m}) = \sup_{\alpha \in A} [-b(x, \alpha, \bar{m}) \cdot p - L(x, \alpha, \bar{m})]. \quad (5)$$

It is known that an agent plays optimally if he/she chooses the feedback $\alpha^*(x, t)$ such that $b(x, \alpha^*(t, x), \bar{m}(t)) = -\nabla_p H(x, \nabla u(x, t), \bar{m}(t))$. Now if all agents argue this way with independent brownian motions it can be derived that their distribution m must satisfy the equation

$$\begin{cases} \partial_t m - \nu \Delta m - \operatorname{div}(\nabla_p H(x, \nabla u, m)m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ m(0) = m_0 & \text{in } \mathbb{R}^N \end{cases} \quad (6)$$

We can now see the interpretation of this system as a Nash equilibrium: since an agent is "small" compared to totality of the agents, his/her deviation does not affect the population dynamics and thus we can take the distribution of all agents $\bar{m}(t)$ as given in the individual optimization problem. At the same time, we can suppose that at equilibria $m(t) = \bar{m}(t)$, hence we end up with the coupled system of PDEs

$$\begin{cases} -\partial_t u - \nu \Delta u + H(x, \nabla u, m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ \partial_t m - \nu \Delta m - \operatorname{div}(\nabla_p H(x, \nabla u, m)m) = 0 & \text{in } (0, T) \times \mathbb{R}^N \\ m(0) = m_0 \quad u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^N. \end{cases} \quad (7)$$

The two equations involved in this system, the Hamilton-Jacobi equation and the Fokker-Planck equation, are well known and studied. For our thesis we make the assumption that the term H decouples into two distinct contributions, $H(x, \nabla u, m) = H(\nabla u) - f(x, m)$.

After the derivation of the MFG system, a section is dedicated to present standard existence results. In particular we show the differences between the case of smoothing coupling, in which f depends on m as a function on the space of measures, and local coupling, in which f depends only on the local behavior of m .

In this chapter we also provide some results on the approximation of N players differential games. We deal with systems of SDEs of the form

$$dX_s^i = b(X_s^i, \alpha_s^i, \mu_X^{N,i}(s)) ds + \sqrt{2\nu} dB_s \quad (8)$$

for $i = 1 \dots N$. This means that each agent controls his/her dynamics evaluating the behavior of the other agents through the empirical measure $\mu_X^{N,i}(t) = \sum_{j \neq i} \delta_{X_t^j}$. Each agent minimizes the quantity

$$J_i^N(\alpha_i, (\alpha_j)_{j \neq i}) = \mathbb{E} \left[\int_0^T L(X_s, \alpha_s^i, \mu_X^{N,i}(s)) ds + G(X_T, \mu_X^{N,i}(s)) \right] \quad (9)$$

through his/her control α_i . We are interested in finding Nash equilibria for this system, that is a collection $(\bar{\alpha}_i)_i$ of controls such that

$$J_i^N(\bar{\alpha}_i, (\bar{\alpha}_j)_{j \neq i}) \leq J_i^N(\alpha_i, (\bar{\alpha}_j)_{j \neq i})$$

for all admissible controls α_i and for all $i = 1 \dots N$. Using the solution of the MFG system, it is possible to deduce a collection of controls $(\alpha_i)_i$ which approximates a Nash equilibrium for (8) in a suitable sense. Finally in this chapter we introduce Ergodic Mean Field Games and derive the system that we analyze in Chapter 3. Ergodic MFG are a special case of MFGs derived from optimal control problems with cost in the form of a long time average

$$J(x, \alpha_t) = \liminf_{T \rightarrow \infty} \frac{1}{T} \mathbb{E} \left[\int_0^T [L(\alpha_t) + f(X_t)] dt \right]. \quad (10)$$

This results in a stationary system with an additional unknown Lagrange multiplier λ . Moreover in this thesis work we consider a bounded state space $\Omega \subset \mathbb{R}^N$ with reflection at the boundary, which leads to Neumann boundary conditions. The resulting system is

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial \Omega \\ \int_{\Omega} m = 1, \quad \int_{\Omega} u = 0 \end{cases} \quad (11)$$

where H in this case is the Legendre-Fenchel transform of L defined in (10).

Hamilton-Jacobi and Fokker Planck equations

In the second chapter we give some standard results on existence and uniqueness for Hamilton-Jacobi and Fokker-Planck equations, and in particular we present some a priori bounds on the solutions.

For the Hamilton-Jacobi equation, usually the most fitting notion of solution to be considered is the notion of viscosity solution. However, in the case studied in this thesis work the notion of classical solution is sufficient, so we give existence results in this case. Moreover, we show that $\|\nabla u\|_{\infty} \leq C$ where C depends only on the L^{∞} norm of the data.

The Fokker-Planck equation is interpreted in general in the distributional sense, with m that could be a measure of probability. In our case, we limit ourselves to the case of absolutely continuous measures, with m being the L^1 density of a probability. In this case we can interpret the equation in a weak $W^{1,2}$ sense. By regularity, this solution is also in $W^{1,p}$ for all $p > 1$ and we show that the $W^{1,p}$ norm is controlled by a constant depending only on p and the L^{∞} norm of the data.

Variational methods for Viscous Ergodic Mean Field Games

In the third chapter we focus on existence results for Viscous Ergodic Mean Field Games with local coupling using Variational methods. The system that we consider, derived in Chapter 1, is

$$\begin{cases} -\Delta u + H(\nabla u) + \lambda = f(x, m(x)) & \text{on } \Omega \\ -\Delta m - \operatorname{div}(m \nabla H(\nabla u)) = 0 & \text{on } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \\ \frac{\partial m}{\partial n} + m \nabla H(\nabla u) \cdot n = 0 & \text{on } \partial \Omega \\ \int_{\Omega} m = 1. \end{cases} \quad (12)$$

We can see that in our model the dependence of H on ∇u and on m is decoupled. For this system finding a solution means finding a triple $(u, \lambda, m) \in C^{2,\theta}(\bar{\Omega}) \times \mathbb{R} \times W^{1,p}(\Omega)$ for all $p > 1$ and all $\theta \in (0, 1)$ such that the first equation is solved in a classical sense and the second equation is solved in a weak sense. We assume that H is convex and satisfies some structure conditions, in particular $H(p) \leq C(|p|^{\gamma} + 1)$. We also assume that f is locally Lipschitz in x and

$$|f(x, m(x))| \leq C_f m(x)^{q-1} + K_f \quad (13)$$

Difficulties in the problem arise since f is unbounded and depends only on the local values of m . In particular, two values for q delineate different regimes, the "mass critical" exponent $\bar{q} = 1 + \frac{\gamma'}{N}$ (γ' here means the Hölder conjugate exponent of γ) and the "Sobolev critical" exponent $q^* = 1 + \frac{\gamma'}{N-\gamma'}$ if $N > \gamma'$ or $q^* = +\infty$ otherwise, which is the highest exponent that allows for the use of Sobolev embedding theorems. The main original result of this thesis work is the existence of a variational solution for this system in the case of $\bar{q} \leq q \leq q^*$, extending the work of Cesaroni and Cirant [2], in which they consider the case $q < \bar{q}$. This requires nontrivial modifications

of the variational structure used in [2], that does not work for $q \geq \bar{q}$. The procedure to find a solution consists of several steps. To exploit the variational structure of the problem, we introduce the energy

$$\mathcal{E}(m, w) = \int_{\Omega} mL\left(-\frac{w}{m}\right) + F[m] dx \quad (14)$$

where L is the Legendre transform of H and $F = \int_0^m f(x, n) dn$. Both functions are extended in an opportune way to take into account the singular case $m = 0$. We also introduce the constraint

$$\begin{aligned} \mathcal{K} := \{ & (m, w) \in L^q(\Omega) \cap W^{1,r} \times L^1(\Omega) \text{ s.t.} \\ & \int_{\Omega} \nabla m \cdot \nabla \phi \, dx = \int_{\Omega} w \cdot \nabla \phi \, dx \, \forall \phi \in C^\infty(\bar{\Omega}), \\ & \int_{\bar{\Omega}} m \, dx = 1, m \geq 0 \text{ a.e.} \}. \end{aligned} \quad (15)$$

(here r is fixed and depends on q and γ). We begin by proving that functions m satisfying the constraint are bounded in norm by the quantity

$$E := \int_{\Omega} \frac{|w|^{\gamma'}}{m^{\gamma'-1}} dx \quad (16)$$

which is a lower bound for the kinetic part of \mathcal{E} . In particular, we can distinguish two cases. If $q < \bar{q}$, we can say that

$$\|m\|_q^{q(1+\delta)} \leq C(E+1) \quad \|m\|_{1,r} \leq C(E+1) \quad (17)$$

for some $\delta > 0$. If $q \geq \bar{q}$, the first result fails, but supposing $q \leq q^*$ we can say that $\|m\|_q \leq C(E+1)$. These a priori bounds are crucial in the minimization of \mathcal{E} and make use of the Gagliardo-Nirenberg inequality.

Since the coupling term f is local, to proceed we need to perform an approximation. We consider a sequence of approximating f_ε , smooth and with global coupling, defined as

$$f_\varepsilon[m](x) := f(\cdot, m * \chi_\varepsilon(\cdot)) * \chi_\varepsilon(x) = \int_{\mathbb{R}^N} \chi_\varepsilon(x-y) f\left(y, \int_{\mathbb{R}^N} m(z) \chi_\varepsilon(y-z) dz\right) dy. \quad (18)$$

Moreover, we consider for each f_ε the corresponding MFG and the corresponding energy \mathcal{E}_ε defined as

$$\mathcal{E}_\varepsilon(m, w) = \int_{\Omega} mL\left(-\frac{w}{m}\right) dx + F_\varepsilon[m] \quad (19)$$

where F_ε is the L^2 antiderivative of f . For this energy, we can find the lower bound

$$\mathcal{E}_\varepsilon(m, w) \geq E - \|m\|_q^q - K. \quad (20)$$

We can now see the key role played by the mass critical exponent \bar{q} : if we are below \bar{q} and we have access to the estimate in (17), we can deduce that \mathcal{E}_ε is bounded by below. Hence it is possible using standard variational techniques to prove the existence of a global minimum. If instead we are above \bar{q} , \mathcal{E}_ε may be unbounded by below and we would have no hope of finding a global minimum. However, a local minimum may be present.

To find it, the idea, coming from [4], is to restrict our minimization on a set $B_\alpha = \{\|m\|_q^q \leq \alpha\}$ for any $\alpha \geq 1$. Once we have a minimum on any α , if we find a specific $\bar{\alpha}$ such that the minimum in the interior of $B_{\bar{\alpha}}$ is strictly less than the infimum on the boundary, then we can conclude that there exists a local minimum in the interior of $B_{\bar{\alpha}}$. This is possible assuming some restriction on C_f , the coefficient controlling the polynomial growth of f (see (13)).

In [4] this procedure is done in the case of the Schrödinger equation. However, a link between the MFG system and the equation can be found in the case $\gamma = 2$ via the so-called Hopf-Cole Trasformation. This motivates the idea to carry out the analysis also for a general γ , where the Hopf-Cole transformation is not available.

Using this procedure, for each of the approximating systems it is possible to use the variational structure to deduce the existence of a minimum $(m_\varepsilon, w_\varepsilon)$. Moreover we can find uniform bounds for the collection of minima, namely

$$\|m_\varepsilon\|_q \leq C, \quad \|m_\varepsilon\|_{1,r} \leq C \quad (21)$$

and

$$\|w_\varepsilon\|_{\frac{\gamma'q}{\gamma'+q-1}} \leq C \quad (22)$$

The following step is to perform a convex duality argument. The energy functional \mathcal{E}_ε is in general not convex, hence we perform a linearization in a neighborhood of m_ε . We introduce the functional

$$J_\varepsilon(m, w) = \int_{\Omega} mL\left(-\frac{w}{m}\right) + f_\varepsilon[m_\varepsilon](x)m \, dx. \quad (23)$$

We notice that this functional is convex. It is possible to prove that the minima $(m_\varepsilon, w_\varepsilon)$ found before, both in the global case and in the local case, are global minima of this functional. We can then proceed with the duality argument. We introduce

$$\mathcal{A}(m, w, u, c) = \int_{\Omega} [mL\left(-\frac{w}{m}\right) + f_\varepsilon[m_\varepsilon](x)m + m\Delta u + \nabla u \cdot w - cm] dx + c. \quad (24)$$

Here, the role of u and c is to act as a Lagrange multiplier for the constraints in \mathcal{K} . Indeed we can say

$$J_\varepsilon(m_\varepsilon, w_\varepsilon) = \inf_{(m, w) \in W^{1,r}(\Omega) \times L^{\frac{\gamma'q}{\gamma'+q-1}}(\Omega)} \sup_{(u, c) \in C^2(\Omega) \times \mathbb{R}, \frac{\partial u}{\partial n} = 0} \mathcal{A}(m, w, u, c).$$

Then it is possible to use some standard exchange theorems (in particular the Min-Max exchange theorem and the Rockafellar Integral exchange theorem) and the existence theorem for the Hamilton-Jacobi equation to deduce that there exists a solution $(\lambda_\varepsilon, u_\varepsilon)$ and moreover that $\lambda_\varepsilon = J_\varepsilon(m_\varepsilon, w_\varepsilon)$. Finally, using the Legendre-Fenchel transform properties we can deduce that $w_\varepsilon = -\nabla H(\nabla u_\varepsilon)m_\varepsilon$ and $(u_\varepsilon, \lambda_\varepsilon, m_\varepsilon)$ is a solution of the approximated problem.

Before passing to the limit we need to obtain stronger a priori bounds. Here, the case of $q = q^*$ is critical for the Sobolev embedding. If $q < q^*$, it is possible to use some blow-up arguments to deduce uniform L^∞ bounds for the sequence of minima m_ε . If $q = q^*$ the rescaling arguments that are necessary to find an uniform bound for m_ε fail, due to the critical scaling of the system. However, under stricter hypothesis for C_f , it is possible to use finer regularity results for the Hamilton-Jacobi equation and obtain uniform $L^{N(\gamma-1)}$ bounds for ∇u_ε , provided that we assume some additional smallness on the coefficient C_f .

As a last step, we perform a bootstrap argument to deduce the regularity necessary to pass to the limit. If $q < q^*$, we can use the Hamilton-Jacobi estimates and the L^∞ bounds to deduce uniform bounds in L^∞ for ∇u_ε . Then using the assumptions on H and the Fokker-Planck bound we can deduce uniform bounds for m_ε in C^θ for all $\theta \in (0, 1)$. Iterating, we obtain sufficient regularity to pass to the limit in the equations, hence using Ascoli-Arzelà we can finally deduce the existence of a solution (u, λ, m) of (12). If $q = q^*$, we can start with the uniform bound for ∇u_ε and use finer regularity results in the Fokker-Planck equation to deduce uniform boundedness in $W^{1,p}$ for all $p > 1$. From this, we can adopt the same argument as above and conclude.

Conclusions

In this thesis work we proved an existence result for Viscous Ergodic Mean Field Games using Variational Methods. We extended the results of [2] to stronger nonlinearities up to the "Sobolev critical" exponent q^* included. We need to remark that solutions for $q < q^*$ were already found using fixed point methods in [3], however using variational methods we obtain more insight on the nature of such solutions. The case $q = q^*$, as far as we know, has never been considered in the literature. Let us talk about possible further developments. A first possible generalization is to consider evolutive systems, where an analogous variational structure is present but we have to deal with possible finite time blow up. A further step would be to include common noise for the system in the derivation of the Mean Field Game. In this case the PDE system is not a suitable model and in general SPDEs have to be considered. A parallel path to be explored would be to consider systems of Mean Field Games, modelling games where different populations are present. Finally, we need to remark that Mean Field Games find various applications in Finance and can be solved using Neural Networks. Thus, it could be interesting, once we have developed the existence theory for these new models, to test their performance in these fields of study.

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