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EXECUTIVE SUMMARY OF THE THESIS

Optimal market making with Hawkes Processes: a viscosity solution approach

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1. Introduction

Modern financial markets operate through electronic platforms built around the Limit Order Book, which is a centralized ledger where buy and sell orders are queued and matched by price-time priority. Within this structure, a market maker (dealer) continuously quote bid and ask prices, providing liquidity to the market in exchange for earning the bid-ask spread.

However, the market maker faces two types of risk operating this way. The first is inventory risk: an imbalance between incoming buy and sell orders forces the dealer to accumulate a net position in the asset, exposing them to adverse price movements. The second is asymmetric information risk: the dealer may unknowingly trade against agents who has a better, predictive information about the asset's future price. To hedge inventory risk, the dealer adjusts both quotes dynamically, lowering prices to attract buyers when holding a long position and raising them to attract sellers when holding a short position.

The mathematical formalization of the inventory risk has been addressed by Ho and Stool [3], who first modeled the behavior of a monopolistic dealer optimizing their prices in a single stock. However, this framework was adapted to

the continuous-time dynamics of modern Limit Order Books in the model of Avallaneda and Stoikov [1], in which the dealer maximizes the expected exponential utility of their terminal profit and loss over a finite horizon T , with order arrivals driven by independent inhomogeneous Poisson processes, whose intensity decay exponentially with the spread by characterizing a Constant Absolute Risk Aversion (CARA) function. This model gives an approximate closed-form solution via asymptotic expansion in the inventory variable, however it lacks the mathematical foundations needed to prove existence and uniqueness of the optimal control. Moreover, the Poisson assumption fails to capture the self-exciting clustering behavior of real order flow, where the execution of an order significantly raises the probability of further executions shortly afterwards. Therefore, a Poisson model systematically underestimates inventory risk during periods of market stress.

The model presented in this thesis addresses these limitations, replacing independent Poisson arrivals with Hawkes processes.

The main architecture of this model was introduced by Jusselin [4]. The main contribution of this thesis consists in providing a fully detailed and rigorous mathematical formalization

of this framework, reconstructing and extending all the underlying mathematical steps and providing several analytical derivations and topological proofs of [4].

This executive summary is organized in the following chapters:

- Chapter 2 presents the definitions and formalization of the stochastic framework. We introduce the canonical space and the state variables, define the Hawkes driven order flow and construct an infinite-dimensional topological space in $L^1(\mathbb{R}^+)$ along with a controlled probability measure to restore the Markov property of the system.
- Chapter 3 formulates the stochastic optimal control problem. We derive the Hamilton-Jacobi-Bellman equation associated to our problem and compute the explicit analytical form of the optimal bid and ask spreads.
- Chapter 4 analyzes the theoretical well-posedness of the problem via the viscosity framework. By adapting the Crandall-Ishii-Lions theory by the definition of a Lyapunov function to handle unbounded variables, we prove a generalized Comparison principle. Finally, through a Verification Argument based on the Dynamic programming principle, we establish that the value function is the unique continuous viscosity solution.
- Further Developments concludes the summary, outlining potential extensions of the current mathematical framework and practical implementation.

2. Stochastic framework

2.1. Canonical space and state variables

Let $T > 0$ be the trading horizon, representing the time at which the market maker either liquidates or evaluates their position.

We build the canonical sample space $\Omega = \Omega_p \times \Omega_d \times \Omega_a$, where $\Omega_p = C([0, T]; \mathbb{R})$ and Ω_d is the set of non-decreasing, right-continuous piecewise constant càd-làg functions from $[0, T]$ to $\mathbb{N}$ starting from zero with unitary jumps.

The introduced canonical processes are the Brownian motion $W = (W_t)_{t \in [0, T]}$, which generates the right-continuous and completed filtration $(\mathcal{F}_t^p)_{t \in [0, T]}$ and the counting processes

$N^a = (N_t^a)_{t \in [0, T]}$ (buy order arrivals) and $N^b = (N_t^b)_{t \in [0, T]}$ (sell order arrivals), which generate the right-continuous and completed filtration $(\mathcal{F}_t^d)_{t \in [0, T]}$.

We equip the sample set with the natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ defined by $\mathcal{F}_t = \mathcal{F}_t^p \otimes \mathcal{F}_t^d \otimes \mathcal{F}_t^a$.

We construct a probability measure $\mathbb{P}_0$, under which the process W is a standard Brownian motion, the compensated processes, defined by $M_t^j = N_t^j - \lambda_0 t$, $j \in \{a, b\}$, for a constant $\lambda_0 > 0$ are local martingales and W , N^a and N^b are mutually independent.

The fundamental price $P = (P_t)_{t \in [0, T]}$ follows the dynamics:

$$dP_t = d(t, P_t)dt + \sigma dW_t, \quad (1)$$

where $\sigma > 0$ is the price volatility and d is a Lipschitz continuous function, guaranteeing existence and uniqueness of a strong solution for the price.

Furthermore, the inventory process $i = (i_t)_{t \in [0, T]}$ describes the dealer's net position for any $t \in [0, T]$

$$i_t = dN_t^b - dN_t^a, \quad i_0 = 0.$$

2.2. Hawkes processes and Markovianization

The stochastic intensity associated to an univariate Hawkes process is the process $(\lambda_t^j)_{t \in [0, T]}$, $j \in \{a, b\}$ defined as

$$\lambda_t^j = \mu + \int_0^t K(t-s) dN_s^j, \quad j \in \{a, b\}$$

where $\mu > 0$ is a baseline exogenous rate and $K : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, belonging in $L^1(\mathbb{R}^+)$, is a monotone non-increasing continuous kernel describing how past events excite future arrivals. Each new execution boosts the arrival rate, which then decays according to K .

Since λ^j depends on the entire history of N^j , for any $t \in [0, T]$ the tuple $(P_t, i_t, \lambda_t^a, \lambda_t^b)$ is not Markovian. To restore the Markov property we introduce auxiliary random functions $(\theta_t^j)_{t \in [0, T]}$, $j \in \{a, b\}$ defined as

$$\theta_t^j(u) = \int_0^t K(u-s) dN_s^j, \quad u \geq t, j \in \{a, b\},$$

which summarize the influence of all past jumps on future intensities.

We define the topological space in which these functions take values:

$$\Theta_T = \left\{ \sum_{i=1}^n K(\cdot - T_i) \mid n \in \mathbb{N}, 0 \leq T_1 \leq \dots \leq T_n \leq T \right\}$$

which is shown in Proposition 3.1 to be a locally compact, closed subset of $L^1(\mathbb{R}^+)$.

Therefore, we formally define the state of our system. For any $t \in [0, T]$, we consider the set

$$\mathcal{X}_t = \{(p, i, \theta^a, \theta^b) \in \mathbb{R} \times \mathbb{Z} \times \Theta_t \times \Theta_t\}.$$

For any generic element $x = (p, i, \theta^a, \theta^b) \in \mathcal{X}_t$, we define the norm on $\mathcal{X}_t$ as

$$\|x\| = \sqrt{p^2 + i^2 + \|\theta^a\|_1^2 + \|\theta^b\|_1^2}$$

Furthermore, we define the space

$$\mathcal{E} = \{(t, x) \in [0, T] \times \mathcal{X}_t \mid x \in \mathcal{X}_t\}$$

For any $(t, x) \in \mathcal{E}$, we define the shifted state process

$$X_s^{t,x} = (P_s^{t,x}, i_s^{t,x}, \theta_s^{a,t,x}, \theta_s^{b,t,x}), \quad s \in [t, T],$$

which describes the dynamics of the state starting from $(t, x) \in \mathcal{E}$. In particular, the state components are initialized as follows: $P_t^{t,x} = p \in \mathbb{R}$, $i_t^{t,x} = i \in \mathbb{Z}$ and $\theta_t^{t,x,j}(\cdot) = \theta^j(\cdot)$, $\theta^j \in \Theta_t$, $j \in \{a, b\}$.

2.3. Admissible controls and controlled measure

The dealer's controls are the ask and bid spreads $\delta = (\delta^a, \delta^b)$. We denote by $\mathcal{A}$ the set of $\mathcal{F}$ -predictable processes taking values in $(\mathbb{R}^+)^2$.

The controlled intensities are given by

$$\lambda^j(u, x, r) = e^{-\frac{k}{\sigma}r} \Phi(\theta^j(u)), \quad j \in \{a, b\},$$

where $k, \sigma > 0$, $r > 0$ represents the spread and $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function with at most sub-linear growth. In particular, the factor $e^{-\frac{k}{\sigma}r}$ captures the decrease in execution probability as spread increases.

We introduce the process $Z^{t,x,\delta}$ as the stochastic integral with respect to the $\mathbb{P}_0$ -martingales M^a and M^b :

$$Z_s^{t,x,\delta} = \int_0^s \sum_{j=a,b} \frac{\lambda^j(u, X_u^{t,x}, \delta_u^j) - \lambda_0}{\lambda_0} \mathbb{1}_{u \geq t} dM_u^j,$$

where we denote by $\mathbb{1}_{\{\cdot\}}$ the indicator function.

We then define $L = (L_s^{t,x,\delta})_{s \in [t, T]}$ as the Doléans-Dade exponential of $Z^{t,x,\delta}$. Furthermore, it is a martingale under $\mathbb{P}_0$ since $\lambda^j(u, X_u^{t,x}, \delta_u^j)$, $j \in \{a, b\}$ is sub-linear with respect the state norm. By means of Girsanov's theorem for semimartingales, for any $\delta \in \mathcal{A}$, we define a new measure of probability $\mathbb{P}^{t,x,\delta}$:

$$\frac{d\mathbb{P}^{t,x,\delta}}{d\mathbb{P}_0} \Big|_{\mathcal{F}_T} = L_T^{t,x,\delta}.$$

In particular, by construction, the process $N_u^j - \int_t^u \lambda^j(s, X_s^{t,x}, \delta_s^j) ds$ is a local martingale under $\mathbb{P}^{t,x,\delta}$. Furthermore, we denote by $\mathbb{E}_{t,x}^\delta[\cdot]$ the expectation taken under this probability measure. Under this measure, the process $(s, X_s^{t,x})_{s \geq t}$ is Markovian (Proposition 3.2), which is established via Bayes' formula, the martingale property of the Doléans-Dade exponential and the Freezing Lemma.

3. The optimal control problem

The market maker maximizes the discounted expected reward

$$\sup_{\delta \in \mathcal{A}} \mathbb{E}_{0,0}^\delta \left[G(i_T, P_T) e^{-rT} + \int_0^T e^{-rs} (g(i_s, P_s) + \delta_s^a \lambda_s^{a,\delta} + \delta_s^b \lambda_s^{b,\delta}) ds \right],$$

where $r > 0$ is a discount factor, $G(i_T, P_T)$ is the terminal payoff, g is a running reward independent of the reward generated by the spread terms and $\delta^j \lambda^{j,\delta}$ represent the profit generated by the spread terms after each execution.

3.1. Hamilton-Jacobi-Bellman equation

We define the value function $U : \mathcal{E} \rightarrow \mathbb{R}$, belonging in $C^{2,2,0,0,0}(\mathcal{E}; \mathbb{R})$, which represents, for any initial condition $(t, x) \in \mathcal{E}$, the supremum of the expected utility functional over all admissible controls, discounted back to time t .

Moreover, we define the finite difference operators D^a and D^b , which capture the effect of a buy or sell execution on the state, applied to a generic function $\psi : \mathcal{E} \rightarrow \mathbb{R}$ and belonging in

$C^{2,2,0,0,0}(\mathcal{E}; \mathbb{R})$ as:

$$\begin{aligned} D^a \psi(t, x) &= \psi(t, p, i-1, \theta^a + K(\cdot - t), \theta^b) \\ &\quad - \psi(t, x), \\ D^b \psi(t, x) &= \psi(t, p, i+1, \theta^a, \theta^b + K(\cdot - t)) \\ &\quad - \psi(t, x). \end{aligned}$$

By means of Itô's formula for semimartingales applied to the auxiliary process $V(t, x, \delta) = \int_0^t e^{-rs} \tilde{g} ds + e^{-rt} U(t, x)$ and by applying the Dynamic Programming Principle, we retrieve the Hamilton-Jacobi-Bellman equation associated with our control problem:

$$F(t, x, U, (\partial_t U, \partial_p U), \partial_{pp} U, DU) = 0, \quad (2)$$

with terminal condition $U(T, x) = G(i, p)$, where

$$\begin{aligned} F &= rU - \partial_t U - d(t, p) \partial_p U - \frac{1}{2} \sigma^2 \partial_{pp} U - g(i, p) \\ &\quad - \sup_{\delta^a \in \mathbb{R}^+} [\Phi(\theta^a(t)) e^{-\frac{k}{\sigma} \delta^a} (\delta^a + D^a U)] \\ &\quad - \sup_{\delta^b \in \mathbb{R}^+} [\Phi(\theta^b(t)) e^{-\frac{k}{\sigma} \delta^b} (\delta^b + D^b U)]. \end{aligned}$$

3.2. Optimal spreads

Assuming that the value function $U \in C^{2,2,0,0,0}(\mathcal{E}; \mathbb{R})$ is a solution to the Hamilton-Jacobi-Bellman equation, we retrieve the optimal bid and ask spreads by maximizing the two suprema independently (Proposition 3.4):

$$\delta^{*,j} = \left(\frac{\sigma}{k} - D^j U(t, x) \right)_+, \quad j \in \{a, b\}, \quad (3)$$

where $(\cdot)_+ = \max(\cdot, 0)$.

This result has an intuitive interpretation: the dealer quotes a base spread $\frac{\sigma}{k}$ (determined by price volatility and sensitivity), adjusted by the marginal value $D^j U$ of executing one more order, decreasing the spread when the marginal values is high and increasing it otherwise.

4. Viscosity framework

The assumption made on the solutions to the Hamilton-Jacobi-Bellman equation (2) requires them to belong to $C^{2,2,0,0,0}(\mathcal{E}, \mathbb{R})$. However, in general, value functions for stochastic optimal control are not sufficiently smooth. Therefore we rely on the theory of viscosity solutions, allowing us to interpret the equation in a weak sense.

4.1. Adapted definitions

Since the state has both continuous (t, p) and discrete jump (i, θ^a, θ^b) components, we adapt the viscosity framework as follows:

- Derivatives with respect to (t, p) are evaluated on a smooth test function $\phi \in C^{2,2,0,0,0}(\mathcal{E}; \mathbb{R})$, which is touched by U at relevant extrema.
- The finite difference operators D^a and D^b act directly on U , requiring only point-wise evaluation, relaxing the assumption of differentiability.

We then define viscosity sub and super solutions (Definition 4.1) and provide an equivalent characterization via generalized semi-jets (the semi super-jet $\bar{\mathcal{J}}^+$ and semi sub-jet $\bar{\mathcal{J}}^-$) (Definitions 4.3-4.5 and Proposition 4.1), extending the standard Crandall-Ishii framework to our setting with Hawkes variables.

4.2. Main result

The main contribution of this work is the following well-posedness theorem.

Theorem 4.1 (Well-posedness - Theorem 4.7). *The following hold:*

1. *There exists a unique continuous viscosity solution U with polynomial growth to the Hamilton-Jacobi-Bellman equation.*
2. *This solution satisfies the stochastic control problem (2).*
3. *The problem admits an optimal control $\delta_t^* = \delta(t, X_t)$ given by (3).*

The proof proceeds in four steps.

4.3. Step 1 - Comparison principle

We characterize the uniqueness for bounded viscosity solution via the comparison principle.

Let $USC(\mathcal{E})$ and $LSC(\mathcal{E})$ be respectively the set of upper-semi-continuous functions and lower semi-continuous functions defined on $\mathcal{E}$.

Theorem 4.2 (Comparison Principle - Theorem 4.6). *Let $U \in USC(\mathcal{E})$ be a bounded from above viscosity sub-solution and $V \in LSC(\mathcal{E})$ be a bounded from below viscosity super-solution of the Hamilton-Jacobi-Bellman equation, satisfying $U(T, \cdot) \leq V(T, \cdot)$. Then $U \leq V$ on $\mathcal{E}$.*

The proof relies on a doubling-variables technique: to isolate a supremum point, we define a

penalized supremum

$$N_\epsilon^\alpha = \sup_{(t,x),(t,y) \in \mathcal{E}} U(t,x) - V(t,y) - \Psi(t,x,y),$$

where the penalization function is given by $\Psi(t,x,y) = \epsilon(\|x\|^2 + \|y\|^2) + \alpha\|x - y\|^2$. By applying an adapted version of Crandall-Ishii's lemma, we extract the elements of the semi-jets characterized by the derivatives of Ψ . Furthermore, by evaluating the non-linear operator F with these elements, we derive a mathematical contradiction as $\alpha \rightarrow +\infty$ and $\epsilon \rightarrow 0$.

Since the state variables in our framework are unbounded, we extend this result to functions with polynomial growth. In particular, we construct a Lyapunov function $w(t,x) = e^{M(T-t)}(1+\|x\|^2)^k$ that dominates the growth of the envelopes at infinity. By perturbing the sub and super solutions with $\pm\epsilon w(t,x)$, we bound them respectively from above and below, allowing us to apply the bounded comparison result and conclude $U \leq V$ on $\mathcal{E}$.

4.4. Step 2 - Continuation utility function

We formally define the continuation utility function $J(t,x,\delta)$ and the value function:

$$U(t,x) = \sup_{\delta \in \mathcal{A}_t} \mathbb{E}_{t,x}^\delta \left[G(i_T, P_T) e^{-r(T-t)} + \int_t^T e^{-r(s-t)} \tilde{g}(s,x,\delta) ds \right]$$

where $\mathcal{A}_t$ restricts the admissible controls to those independent of the past filtration $\mathcal{F}_t$, by means of the Markovianity of the system. We rely on a priori estimates (Appendix A) and we establish that U satisfies polynomial growth of degree 2, meaning that there exists $C > 0$ such that $|U(t,x)| \leq C(1 + \|x\|^2)$ for all $(t,x) \in \mathcal{E}$.

4.5. Step 3 - Dynamic Programming Principle

The Dynamic Programming Principle provides two key inequalities, evaluated at an arbitrary stopping time $\theta^\delta \in [t, T]$:

- Sub-optimality inequality: $U(t,x)$ is bounded from below by the expected reward evaluated using the lower semi-continuous envelope $U_*(\theta^\delta, X_{\theta^\delta}^{t,x})$.
- Super-optimality inequality: $U(t,x)$ is bounded from above by the expected re-

ward evaluated using the upper semi-continuous envelope $U^*(\theta^\delta, X_{\theta^\delta}^{t,x})$.

In order to prove these inequalities, we first approximate from below the lower semi-continuous envelope U_* by a continuous test function by Baire's theorem. We then construct a countable partition of the state space where specific local controls satisfy a lower bound, which are after summarized together at the stopping time θ^δ into a global control $\delta^{\epsilon,n}$. By evaluating the expected reward under this strategy and applying the Tower and Markov properties, we obtain a lower bound for $U(t,x)$, which represent the sub-optimality inequality after passing to the limit.

Conversely, to prove the super-optimality inequality we define $\bar{\delta}$ as the controlled process obtained by freezing the trajectory of δ up to time θ^δ and we derive the upper bound using the Tower Property. Finally the desired inequality follows from the definition of the value function as a global supremum over all admissible strategies.

4.6. Step 4 - Verification Argument

We finally prove that the upper envelope U^* is a viscosity sub-solution and the lower envelope U_* is a viscosity super-solution. We argue by contradiction: assuming that the Hamilton-Jacobi-Bellman equation is strictly violated at a local extremum, we define a small cylinder around it. By applying Itô's formula to the smooth test function up to the first exit time τ_n from the cylinder and evaluating the conditional expectation under an arbitrary control, we contradict the bounds established by the Dynamic Programming Principle.

Therefore, U^* is a sub-solution and U_* is a super-solution. Finally by applying the extended comparison principle from Step 1, we obtain $U^* \leq U_*$ and remarking that the inequality $U_* \leq U^*$ holds intrinsically by the definition of semi-continuous envelopes, we conclude that $U = U^* = U_*$ globally on the domain. Therefore, we conclude that the value function U is the unique continuous viscosity solution to our optimal control problem.

5. Conclusions and Further Development

This thesis provides a complete argument of the continuous-time optimal market making problem with Hawkes driven order flow.

The key contributions are:

- Markovianization, we indeed introduce an infinite-dimensional functional space $\Theta_T \subset L^1(\mathbb{R}^+)$ to summarize the excitation history and prove its topological properties, which are essential for the well-posedness arguments.
- Optimal spreads, we indeed derive the analytical form of the optimal bid-ask spreads (3), explicitly characterizing how the dealer should adjust their quotes based on the marginal value of executing an order.
- Well-posedness, we indeed prove the existence and uniqueness of a continuous viscosity solution with polynomial growth for the infinite-dimensional Hamilton-Jacobi-Bellman equation (Theorem 4.1), providing the mathematical foundations that earlier asymptotic approaches lacked.

From a practical perspective, by shifting to a Hawkes-driven framework, algorithmic dealers and high-frequency trading firms can deploy more robust market making models, accurately pricing the microstructural risk of order clustering and self-exciting market crashes, which would not be possible by using a Poisson-driven framework. Moreover, the quotes can be dynamically adjusted in real-time by means of the optimal spread formulas explicitly retrieved in (3), without relying on computationally heavy simulations.

Further developments of this model include numerical approaches for the resolution of the infinite-dimensional Hamilton-Jacobi-Bellman equation, calibration of the Hawkes kernel K to empirical order-book data, and extension to multi-asset setting or Limit Order Books with adverse selection.

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