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SCUOLA DI INGEGNERIA INDUSTRIALE
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Waveguide Quantum Electrodynamics with Two-photon interactions

Tesi di Laurea Magistrale in
Physics Engineering - Ingegneria Fisica

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Academic Year: 2020-21

Abstract

Controlling and manipulating light-matter coupling is one of the most important branches of research of quantum sciences and technologies. The greatest achievement in this field is undoubtedly the study of the interaction between a two-level quantum emitter and a single confined bosonic mode. In the cavity Quantum Electrodynamics (QED) context one of the most studied models for single photon interactions (SPI) is the so called Jaynes-Cummings (JC) model, which describes linear coupling between the qubit and the mode in the weak and in the strong regimes. Recent results had revealed the possibility of implementing two-photon coupling by engineering superconducting atom-resonator systems or by applying analog quantum simulation schemes in trapped-ions or ultracold atoms. In order to properly describe this phenomenon of two-photon interaction (TPI) it is needed to go beyond the JC model. The interest in TPI is motivated by the emergence of novel phenomena such as the appearance of distinct selection rules and a two-photon blockade as a first-order process.

In this thesis work, for the first time, TPI is studied in the context of waveguide QED, which is the study of the interaction between quantum emitters and a 1D-continuum of modes. Waveguide QED experiments can be implemented for example with superconducting artificial atoms coupled to transmission-line resonators or with quantum dots coupled to photonic-crystal waveguides. In waveguide QED, the 1-D confinement allows one to implement phenomena which can not be observed in free-space, such as perfect single-photon scattering. These new possibilities have motivated us to develop a general theory for TPI in the waveguide framework.

In this thesis work we have first re-obtained the results relative to SPI in waveguide QED context. Then, we generalized the photon scattering theory to TPI, finding formal solutions for input-output relations. Finally, we applied the general results achieved to two cases of interest: spontaneous emission and two-photon scattering. Our result paves the way towards the exploration of a novel quantum phenomenology and to possible applications in quantum technologies.

Keywords: Two-photon scattering, nonlinear coupling, waveguide QED, quantum circuits

Abstract in lingua italiana

Il controllo e la manipolazione dell'accoppiamento luce-materia è uno dei rami più importanti della ricerca di scienze e tecnologie quantistiche. Il più grande successo in questo campo è senza dubbio lo studio dell'interazione tra un emettitore quantistico a due livelli e un singolo modo bosonico confinato. Nel contesto di elettrodinamica quantistica (EDQ) in cavità, uno dei modelli più studiati per interazioni a un fotone (SPI) è il cosiddetto modello Jaynes-Cummings (JC). Esso descrive l'accoppiamento lineare tra qubit e modo bosonico in regime di accoppiamento debole e forte. Recenti risultati hanno rivelato la possibilità di implementare l'accoppiamento a due fotoni mediante sistemi superconduttori risonatore-atomo o applicando schemi di simulazione quantistica analogica in ioni intrappolati o atomi ultrafreddi. Per descrivere correttamente questo fenomeno di interazione a due fotoni (TPI) è necessario andare oltre il modello JC. L'interesse verso questo tipo di interazione è motivato dalla comparsa di nuovi interessanti fenomeni come regole di selezione distinte e blocco a due fotoni come processi al primo ordine.
In questo lavoro di tesi, per la prima volta, TPI viene analizzato in un contesto di EDQ in guide d'onda, ovvero lo studio dell'interazione tra emettitori quantistici e un continuo 1D di modi. In questo ambito gli esperimenti possono essere implementati ad esempio con atomi artificiali superconduttivi accoppiati a risonatori lineari o con punti quantici accoppiati a guide d'onda a cristalli fotonici. Inoltre, il confinamento in una dimensione dei modi propaganti consente di analizzare fenomeni che non possono essere osservati nello spazio libero, come la riflessione perfetta di un singolo fotone. Queste nuove possibilità ci hanno quindi motivato a sviluppare una teoria generale per interazioni a due fotoni.
In questo lavoro di tesi abbiamo innanzitutto riottenuto i risultati relativi allo SPI in guide d'onda, generalizzando poi la teoria di scattering al caso TPI e trovando le soluzioni formali per le relazioni di input-output. Infine, abbiamo applicato i risultati raggiunti a due situazioni di interesse: emissione spontanea e scattering a due fotoni. Il nostro risultato spiana la via verso l'esplorazione di una nuova fenomenologia quantistica e verso possibili applicazioni nelle tecnologie quantistiche.
Parole chiave: Scattering a due fotoni, accoppiamento non lineare, EDQ in guide d'onda, circuiti quantistici

Contents

Introduction

Before entering into the details of the main topic of this thesis work, it is useful to have a general overview of the most important scientific results that have preceded the born of the modern field of Circuit Quantum Electrodynamics (QED). With the term Circuit QED we refer to the study of the interaction between nonlinear superconducting circuits with quantized electromagnetic fields typically operating in the microwave frequency regime. This superconducting circuits indeed act like artificial atoms with a nonlinear distribution of their energy levels [1].

This new research field was mostly inspired by already existing studies on cavity QED which has in Nobel prize Serge Haroche [2] its main representative. Following the trail of cavity QED, Circuit QED recently led to advances in many areas of interest such as the fundamental study of light-matter interaction, the development of quantum information processing technologies or the exploration of novel hybrid quantum systems.

The beginning of the physics of the superconducting circuits study dates back to mid 1980s. At that time, the main question that researchers were trying to give an answer to was whether quantum phenomena, such as quantum tunneling or energy level quantization, could be observed in macroscopic systems of any kind. One example of such a macroscopic system can be found in the Josephson tunnel junction ([3]; [4]). This device is simply constituted by a thin insulating barrier at the interface between two superconductors (fig.1). For this system macroscopic quantities such as the current flowing through the junction or the voltage developed across it are governed by the dynamics of the gauge-invariant phase difference of the Cooper pair condensate across the junction. Different experiments were done on this particular system and many quantum phenomena evidences were found. The first quantum effect observed was the quantum tunneling of the phase degree of freedom of a Josephson junction soon followed by the measurement of quantized energy levels of the same degree of freedom [5]. In the 1980s it was already supposed the possibility of observing coherent quantum phenomena in superconducting circuits built using Josephson junctions, such as coherent oscillations between two quantum states of the system or quantum superposition effects [6].

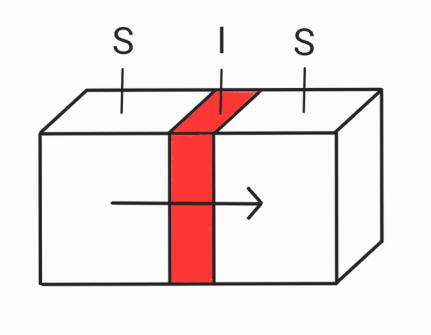

Figure 1: Schematic representation of a Josephson junction: the two blocks identified by the letter S are made of superconductors; the block labeled with I is made of an insulator.

Further studies in this direction were done in the late 1990s with the aim of realizing superconducting qubits for quantum computation. The turning point in this story is the year 1999 when for the first time time-resolved coherent oscillations with a superconducting qubit were experimentally observed [7]. Since then many progresses were done. For example coherent oscillations in coupled superconducting qubits were observed [8] and significant improvements of the coherence times of these devices were achieved.
At the same time, on the other hand, progresses have been made in the study of coherent interactions between single atoms in cavity and quantized electromagnetic fields [9] and its possible implementations for quantum computation [10]. In the early 2000s among the researchers began to spread the idea of implementing cavity QED experiments where natural atoms are replaced by non-linear superconducting circuits. These artificial atoms can be coupled with microwave radiation in open 3D cavities or in discrete LC oscillators. The main reasons for this substitution are to be found in the properties of this kind of circuits. Indeed by coupling superconducting qubits to photons stored in high-quality coplanar waveguide resonators it is possible to enter quite easily the so called strong coupling regime: a regime reached when the coupling strength overcomes dissipation rates and where coherent exchange of excitations can be observed within the excitation lifetime [11]. Moreover the fact that the circuits, that act like qubits, operate in superconducting regimes protect themselves from decohere through classical dissipation ways, i.e. by interacting with the circuit resistance. Ultimately the ease with which it is possible to read out the state of these qubits and with which it is possible to couple them to each other in a quantum computer make it clear why their use were so popular [12]. This rapid advance in experimental research culminated in the first experimental realization of a circuit QED system achieving the strong coupling regime of light-matter interaction [13].
After 15 years of development circuit QED is now the leading architecture for quantum computation. Simple quantum algorithms have been implemented, demonstrations of quantum-error correction have approached or reached the so-called break-even point

(point at which the lifetime of a qubit exceeds the lifetime of the constituents of the system, [14]), and devices with several tens of qubits have been operated with claims of quantum supremacy [15]. Furthermore circuit QED is now opening new research directions such as the development of quantum-limited amplifiers and single-microwave photon detectors, with applications ranging from quantum information processing to the search for dark matter axions.

1 | Superconducting quantum circuits

1.1. Introduction

In this section we will introduce the fundamental building blocks of circuit-QED experiments. We will start from the quantum LC oscillator and we will see how it is possible to build a nonlinear quantum circuit that can be used as a qubit from it, i.e. the transmon. Then we will describe briefly the architecture of the so called flux qubit and of the SQUID device. Finally we will discuss the implementation of the specific kind of light-matter interaction studied in this thesis. We will show how an artificial atom could be excited only by the absorption of two photons and how it is possible to realize it by using a combination of flux qubit and SQUID.

1.2. The quantum LC resonator

An LC oscillator is a resonant circuit constituted by an inductance L and a capacitance C (fig.1.1). The circuit resistance R is here neglected since all the system is working in a superconducting regime. In this way there will be no dissipations or coherence losses via scattering mechanisms inside the circuit components materials. This kind of resonator has a resonance frequency $\omega_r = \sqrt{1/LC}$ and an impedance $Z = \sqrt{L/C}$. In order to study the energy distribution of the oscillator we write the Hamiltonian function associated [16] as:

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \tag{1.1}$$

where Q is the charge of the capacitance and Φ is the magnetic flux threading the inductance. It is useful to recall here that the charge and the flux are related to the current

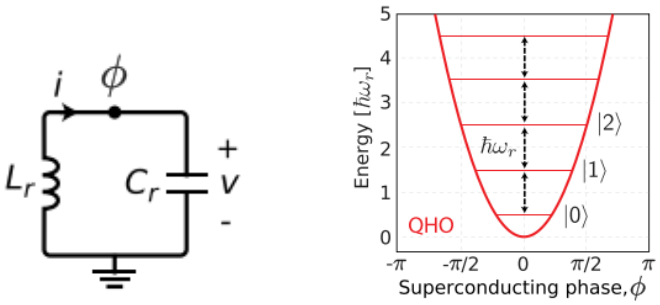

Figure 1.1: Left: LC-oscillator circuit scheme with an inductance L in parallel with a capacitance C; its superconductive phase is ϕ. Right: LC-oscillator harmonic potential energy; every energy level is equidistantly spaced by the qubit energy quantum $\hbar\omega_r$. Figure taken from [12].

and the potential by the following relations:

$$Q(t) = \int_{t_0}^{t} I(t')\,dt', \qquad \Phi(t) = \int_{t_0}^{t} V(t')\,dt' \tag{1.2}$$

that comes from the charge conservation and from the Faraday's induction laws. It is interesting to rewrite Eq.1.1 as:

$$H = \frac{Q^2}{2C} + \frac{1}{2}\omega_r^2 C\Phi^2 \tag{1.3}$$

This new form of the Hamiltonian function makes clearer the analogy between the LC oscillator and the usual mechanical oscillator. Indeed our system can be seen as a mechanical oscillator with mass equal to C and with conjugated variables Q and Φ. The second term of the Hamiltonian function proportional to the square of the magnetic flux represents the well known harmonic potential energy (fig.1.1).
From now on the steps that lead us to the quantization of the Hamiltonian function are formally the same as those of the mechanical oscillator. With this concept in mind we can promote the charge Q and the flux Φ to non-commutable quantum operators with commutator :

$$[\hat{Q}, \hat{\Phi}] = i\hbar \tag{1.4}$$

Moreover we can rerwite the expression of the operators $\hat{Q}$ and $\hat{\Phi}$ by introducing the ladder operators for this quantum LC oscillator:

$$\hat{Q} = iQ_{zpf}(\hat{a}^\dagger - \hat{a}), \qquad \hat{\Phi} = \Phi_{zpf}(\hat{a}^\dagger + \hat{a}) \tag{1.5}$$

with $Q_{zpf} = \sqrt{\hbar\omega_r C/2}$ and $\Phi_{zpf}\sqrt{\hbar/2\omega_r C}$, the characteristic magnitude of the zero point fluctuations of the charge and the flux respectively. By using these definitions for the conjugated variables ii is possible to rewrite the Hamiltonian function in the canonical way [11]:

$$\hat{H} = \hbar\omega_r(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{1.6}$$

with $\hat{a}^\dagger\hat{a} = \hat{n}$ the number operator. In proceeding we will neglect the term $\hbar\omega_r/2$ inside the Hamiltonian function since it is just a constant factor that will give at the end only a global phase. The meaning of the ladder operators here is the usual one: $\hat{a}^\dagger$ creates an excitation inside the LC circuit (equivalently we can say that it creates a photon at frequency ω_r inside the cricuit) while $\hat{a}$ has the opposite effect.
Despite the fact that all this quantization procedure seems formally correct, what we have to ask to ourselves now is whether in practice it is possible to work in a regime where all these quantum treatments are meaningful. For this to be the case two conditions must be satisfied. Firstly, the energy levels of the LC oscillator must be sufficiently separated from each other, which means that its energy levels must be less broad than their separation. This condition can be expressed equivalently by requiring an high oscillator quality factor Q=ω_r/k with k the oscillator linewidth. It is known that losses depends primarily on couplings on unwanted degrees of freedom. To avoid these losses the use of superconductive materials on low-loss dielectric substrates such as sapphire or high-resistivity silicon wafers is ideal. Secondly, we must require an energy separation between adjacent eigenstates greater than the thermal energy, which means $\hbar\omega_r \gg k_B T$. This last condition is easily satisfied by using circuits operating at microwave regime at temperatures below the superconductive materials critical temperature (usually $1 - 10$ K) [16].
Once the above two conditions are satisfied we can practically operate the LC circuit in the quantum regime and treat it like a quantum system with a certain ground state defined by $|g\rangle$ with no excitations ($\hat{n}_e\,|g\rangle = 0$, with $\hat{n}_e$ the number of excitation operator) and with energy levels separation $\hbar\omega_r$.

1.3. The transmon articial atom

Although LC oscillators behave as coherent quantum systems with their eigenstates and eigenvalues, in practice they cannot be used to implement qubits not even for quantum computing. This is made unfeasible from the fact that the LC oscillator is a linear system, which means that the energy separation between its eigenstates is always the same (fig.1.1). Indeed before using the LC oscillator as a qubit for processing information,

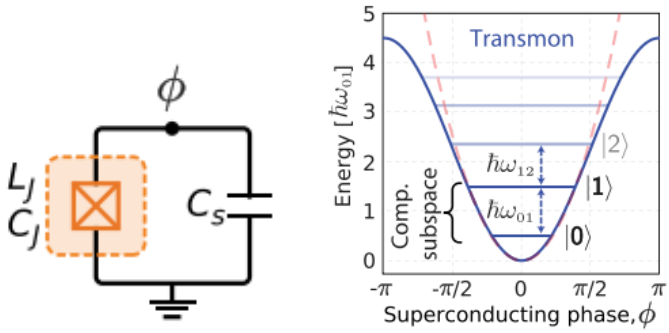

Figure 1.2: Left: Josephson qubit circuit where the orange box represents the Josephson junction subcircuit in which the nonlinear inductance L_J has substituted the linear one L. Right: Josephson circuit potential energy (in blue) where it is clear that now the energy levels are not equally spaced. Figure taken from [12].

we must be able to define a computational subspace consisting of only two energy states that we can explore without exciting other levels of the system. This is equivalent of asking a high transition frequencies difference $\omega_q^{1\to2} - \omega_q^{0\to1}$.
To solve this problem, it is necessary to add a nonlinearity in the system . In practice this is done by replacing inside the circuit the usual inductance L with a Josephson junction (fig.1.2). Indeed this kind of device is able to introduce a nonlinearity in the potential of the LC oscillator while satisfying the conditions of working at mK temperatures and providing a high quality factor. The first studies on these junctions was conducted by Brian Josephson in 1962. He managed to demonstrate that a supercurrent could flow between two superconductive layers separated by a thin insulator. It can be demonstrated that the expression of the Josephson supercurrent is [17]:

$$I = I_c \, sin(\psi) \tag{1.7}$$

where I_c is the Josephson critical current and ψ is the relative phase between the two condensates that form on both sides of the junction. The meaning of the critical current is the following: it is the maximum current that can flow across the insulator before the Cooper pairs break and it depends on materials parameters and junction size. This means that the current that flows in the junction becomes a normal current and the superconductive state is lost. For what concerns the current that flows inside the quantum circuits, it is well below this limit so there is no risk of exiting the superconductive regime. As we said before, the Josephson junction add a nonlinearity to the LC oscillator potential. This becomes possible because the new inductance added to the circuit is itself dependent on the flux Φ.

This can be demonstrated by using the following expression of the phase difference ψ time dependence [17]:

$$\frac{d\psi}{dt} = \frac{2\pi}{\Phi_0} V \tag{1.8}$$

with $\Phi_0 = \frac{h}{2e}$ the magnetic flux quantum. We note that the above equation can be integrated and by using Eq.1.2 a direct relation between the phase difference ψ and the magnetic flux Φ can be written as:

$$\psi(t) = \frac{2\pi}{\Phi_0} \Phi(t) \tag{1.9}$$

It is now evident that there is a direct link between the supercurrent flowing across the junction and the magnetic flux, which in the usual linear case we would write as $\Phi = LI$. Said so, we can define the nonlinear Josephson inductance [16] as:

$$L_J(\Phi) = \left(\frac{dI}{d\Phi}\right)^{-1} = \frac{\Phi_0}{2\pi I_c \, cos\left(\frac{2\pi\Phi}{\Phi_0}\right)} \tag{1.10}$$

Since the Josephson inductance is a nonlinear element, by replacing the usual LC oscillator inductance with it we are in practice making the circuit nonlinear. This means that the energy levels are no more equally spaced and the circuit energy spectrum is similar of that of a real atom, therefore we can call the circuit in fig.1.2 an *artificial atom*. In this case we can focus our attention only to the first two states, the ground state $|g\rangle$ and the first excited state $|e\rangle$ for which hold: $\hat{n}_e |g\rangle = 0$ and $\hat{n}_e |e\rangle = 1$. Indeed these two states constitute our qubit levels.
Now that the inductance depends on the magnetic flux, the expression of the Hamiltonian function in Eq.1.1 is no more valid. In the general case the energy stored in the inductance can be represented with the following:

$$E_L = \int V(t) I(t) \, dt \tag{1.11}$$

Using Eq.1.2 and Eq.1.7 we obtain:

$$E_L = \int I_c \, sin\left(\frac{2\pi\Phi}{\Phi_0}\right) \frac{d\Phi}{dt} \, dt = -E_J cos\left(\frac{2\pi\Phi}{\Phi_0}\right) \tag{1.12}$$

with $E_J = \Phi_0 I_c/2\pi$ the Josephson energy. Said so, we can easily write the new expression of the Hamiltonian function [18]:

$$\hat{H} = \frac{(\hat{Q} - \hat{Q}_g)^2}{2C_\Sigma^2} - E_J cos\left(\frac{2\pi\Phi}{\Phi_0}\right) = 4E_C(\hat{n} - \hat{n}_g)^2 - E_J cos(\hat{\psi}) \tag{1.13}$$

In the expression of the Hamiltonian function we define $C_\Sigma = C_S + C_J$ the total capacitance, $E_C = e^2/2C_\Sigma^2$ the charging energy, $\hat{n} = \hat{Q}/2e$ the charge number operator and $\hat{\psi} = (2\pi\Phi/\Phi_0)$ the phase operator. The term represented by the operator $\hat{n}_g$ takes into account possible bias induced by external gate potential $V_g = Q_g/C_g$. It can be seen just as an offset term that we can neglect in the general study of the Hamiltonian function. From the expression of the $\hat{H}$ operator in Eq.1.13 it is possible to see that the shape of the potential energy and the nature of the encoding of the qubit states depend on the relative strength between E_C and E_J. By tuning the circuit parameters one can achieve different trade offs between transition energy, anharmonicity and sensitivity to different noise sources.

Two are the main regimes in which this qubit can work: charge qubit and flux qubit. Being in one regime or in the other depends on the ratio E_J/E_C. Charge qubits are characterized by $E_J/E_C \ll 1$. In this case the eigenstates of the Hamiltonian operator (that we can in general label as $|j\rangle$) are given approximately by the eigenstates of the charge operator, i.e. $|j\rangle \approx |n\rangle$. Since all the characteristics of the circuit are controlled by the charge inside it, even the smallest fluctuations of the charge, that can be represented by the effect of the operator $\hat{n}_g$, have a huge impact on the system. In particular it will be much more sensitive to charge fluctuations leading to unwanted modifications in the transition energies and to dephasing. To avoid these kind of problems, or at least to reduce them, it is better to work in the opposite and most used regime: the so called *transmon regime*, for which $E_J/E_C \gg 1$. In this new situation the first energy levels become almost independent of the gate charge (fig.1.3). As a consequence in the transmon regime the device is no more sensitive to the noise due to charge fluctuations. Of course there is always a price to pay for improvements and in this case it is a reduction in the anharmonicity α defined as the energy difference of the first two energy levels $\alpha = \omega_q^{1\to2} - \omega_q^{0\to1}$. However it is possible to demonstrate that the sensitivity to the charge fluctuations decreases exponentially with the ratio E_J/E_C while on the other hand the reduction in the anharmonicity of the junction has a weaker dependence on the ratio as $(E_J/E_C)^{-\frac{1}{2}}$ [16]. This different behaviour justifies the huge implementation of the transmon qubit because the little reduction of the anharmonicity α payed for the large reduction in sensitivity to the charge fluctuations is not an impediment in controlling the system with high fidelity. To access the transmon regime the most used approach is to make the charge energy E_C

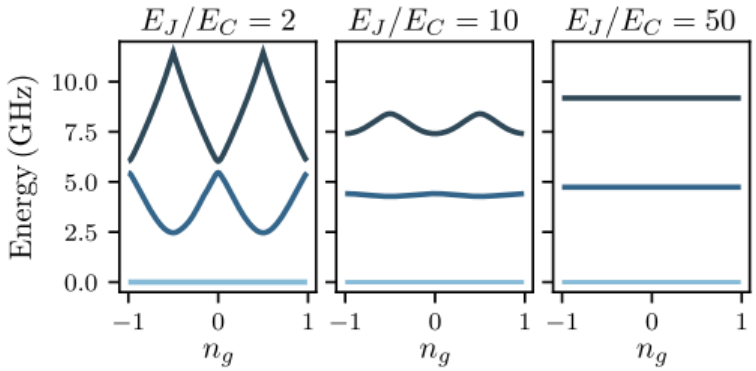

Figure 1.3: Frequency difference $\omega_j - \omega_0$ of the first three energy levels of the transmon Hamiltonian obtained from numerical diagonalization of Eq.1.13 expressed in the charge basis $\hat{n}$ for different E_J/E_C ratios. For large values of E_J/E_C the energy levels become insensitive to the offset charge $\hat{n}_g$. Figure taken from [16].

small by shunting the junction with a large capacitor $C_S \gg C_J$. In this way the superconducting phase ψ is a good quantum number and the eigenstates of the low energy levels are well localized in the potential well.
We can gain more insight by expanding the E_J term in Eq.1.13 into a power series since ψ is small [12]:

$$E_J cos(\hat{\psi}) = \frac{1}{2} E_J (\hat{\psi})^2 - \frac{1}{24} E_J (\hat{\psi})^4 + o(\hat{\psi}^6) \tag{1.14}$$

The first term in the above equation is the one that alone would give us the linear LC oscillator Hamiltonian function. The second term however being quartic breaks the harmonicity of the system and introduces a negative anharmonicity α (since the coefficient of the quartic term is negative). Moreover by using the usual relations between the ladder operators and the conjugated operators we can write the Hamiltonian function in Eq.1.13 in the standard form [12]:

$$\hat{H} = \hbar \omega_q \hat{a}^\dagger \hat{a} + \frac{\hbar \alpha}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \tag{1.15}$$

with qubit frequency $\omega_q = (\sqrt{8 E_J E_C} - E_C)/\hbar$ and anharmonicity $\alpha = -E_C$. Typical values for the α parameter are 100-300 MHz, designed to obtain $\omega_q = 3 - 6$ GHz and $E_J/E_C > 50$ [12].

1.4. Tunable qubit and SQUID

In the most recent and advanced quantum processors architectures frequency tunable qubits are needed. Indeed, in order to perform logical gate operations with high fidelity this requirement is necessary. For example, in some cases, to exchange energy two circuits must be in resonance while, on the other hand, in the idling period they must not interact with each other so to preserve their own coherence. In order to perform these kind of operations an external parameter capable of accessing the system degree of freedom is needed. The most diffused technique is to replace the single Josephson junction with a loop constituted by two different junctions (fig.1.4a). This new device is called DC-SQUID (DC-Superconducting quantum interference device). Thanks to the interference between the two arms of the loop, the total critical current I_c that flows in the system can be controlled with the application of an external magnetic flux threading the circuit ϕ_{ext}. To see why this happens, it is usefull to rewrite the inductance part of the Hamiltonian function in Eq.1.13 as [17]:

$$\hat{H}_L = -E_J cos(\hat{\psi}_1) - E_J cos(\hat{\psi}_2) \tag{1.16}$$

where the two junctions have the same Josephson energy but different phase operator. Due to the physics of the superconductors, the phase of the total wavefunction that describes the system is single valued. This implies that the total phase difference around the superconducting loop must be an integer of 2π. In addition to that, the external magnetic flux will affect the total phase of the wavefunction and the fluxoid quantization condition requires then that the algebraic sum of all the magnetic fluxes threading the loop must be a multiple of the magnetic flux quanta. This condition can be summarized in the following expression [17]:

$$\hat{\psi}_1 - \hat{\psi}_2 = 2\pi k + 2\hat{\psi}_e \tag{1.17}$$

with $\hat{\psi}_e = \pi\hat{\phi}_{ext}/\Phi_0$. Using Eq.1.17 it is possible to eliminate one degree of freedom of the system expressing the branch flux of one arm of the loop as a function of the branch flux of the other one. In this way the SQUID can be treated as a single junction with energy E_J tunable by the external flux. This last statement could be demonstrated quite easily [17] by using simple trigonometric identities and rewriting Eq.1.16 as:

$$\hat{H}_L = -2E_J cos\Big(\frac{\hat{\psi}_1 - \hat{\psi}_2}{2}\Big) cos\Big(\frac{\hat{\psi}_1 + \hat{\psi}_2}{2}\Big) \tag{1.18}$$

Inserting Eq.1.17 in Eq.1.18 and with the definition of the new phase operator $\hat{\psi} = (\hat{\psi}_1 + \hat{\psi}_2)/2$, gives the new Hamiltonian operator of the system:

$$\hat{H} = 4E_C(\hat{n} - \hat{n}_g)^2 - 2E_J\left|cos(\hat{\psi}_e)\right|cos(\hat{\psi}) \tag{1.19}$$

which is formally identical to Eq.1.13 but with the difference that now the Josephson energy is controllable. In particular it is periodically tunable in a range of values that goes from 0 to the maximum $2E_J$, with period Φ_0. The introduction of another Josephson junction in the circuit on one hand enables frequency tunability via an external magnetic flux, on the other one increases the system sensitivity to flux fluctuations (flux noise). Indeed due to the presence of a noisy environment, the qubit has now a new way through which could lose coherence. As a result, the loop size of the SQUID is often extremely small (typically of the order of 10 µm on each side of the loop) and the qubits are kept inside a shielded enclosure to minimize the sensitivity to environmental flux noise [12]. At first order of approximation the flux noise is represented by the first derivative of the qubit transition frequency with respect to the external applied flux $\partial\omega_q/\partial\phi_{ext}$. In general as it can be seen from fig.1.4b the slope of the qubit spectrum is different from zero for every possible value of the external flux except when it is equal to multiples of the magnetic flux quantum. For these particular values of ϕ_{ext} not only the noise sensitivity is minimum but also the anharmonicity is the largest possible for this particular device. In order to mitigate this unwanted effect of not negligible noise sensitivity for a very large range of values of the external flux, asymmetric Josephson junctions in the loop, each one with its own different energy E_J, could be implemented (fig.1.4c). Indeed, it is possible to demonstrate that the new Josephson energy of the asymmetric system is [12]:

$$E_J(\hat{\phi}_e) = (E_{J1} + E_{J2})\sqrt{cos^2(\hat{\phi}_e) + d^2 sin^2(\hat{\phi}_e)} \tag{1.20}$$

with $d = (\gamma - 1)/(\gamma + 1)$ and $\gamma = E_{J2}/E_{J1}$ the junction asymmetry parameter. In the cases where there is no asymmetry (d=0) or where the asymmetry is too high (d=1) the SQUID loop loses its properties and the Hamiltonian of the system becomes again the one in Eq.1.13. However as it can be seen from the qubit spectrum (fig.1.4d), for particular values of the parameter γ, the flux sensitivity could be reduced enormously with respect to the symmetric case and the shape of the energy levels becomes smoother.

1.5. Flux qubit and fluxonium

In the previous paragraph the most important characteristics of the split transmon qubit were shown. Despite the fact that this new device adds in the circuit a way with which one can tune the circuit characteristic energy, it shares with the single junction circuit the same topology and the same potential energy shape. Indeed, since the split transmon (both the symmetric and the asymmetric one) behaves like the single junction version, it is quite obvious that it will also share with it its sinusoidal potential energy and its weak anharmonicity. A residual excitation of the upper energy states cannot be eliminated unless a deeper change in the circuit topology is made. This drastic change is realized by adding in the second arm of the loop another Josephson junction. In this way the circuit is constituted by three Josephson junctions: one with energy E_{J1} and two with energy E_{J2} (fig.1.4e). The two Josephson junctions with energy E_{J2} are bigger than the other one so that the asymmetry parameter γ is indeed greater than 1. The presence of another junction might seem as a small change in the circuit. This however it is not true. The new junction not only increases the complexity of the circuit topology but also modifies intrinsically the shape of its potential energy. Moreover, to each junction is associated a phase variable and, again, by exploiting the fluxoid quantization condition:

$$\hat{\psi}_1 - \hat{\psi}_2 - \hat{\psi}_3 + \hat{\psi}_e = 2\pi k \tag{1.21}$$

it is possible to eliminate one degree of freedom from the system. Indeed, one could express the phase of one junction function of the other two. In this way the problem becomes tridimensional and this increases a lot the computational complexity with respect to the simple transmon device. Of course, this new device, called *flux qubit* will also have a tridimensional potential energy. Under the assumption that $\gamma > 1$ fortunately the system could be treated as the system of a single particle moving in a quasi one-dimensional potential. This simplifies enormously the computational part of the problem and allows one to write the Hamiltonian function of the circuit as [12]:

$$\hat{H} \approx 4E_C(\hat{n} - \hat{n}_g)^2 - E_{J1}(2\hat{\psi} + \hat{\psi}_e) - 2\gamma E_{J1} cos(\hat{\psi}) \tag{1.22}$$

where $\hat{\psi} = \frac{\hat{\psi}_2 + \hat{\psi}_3}{2}$ is the average phase of the two array junctions, calculated under the hypothesis that they are crossed by the same current in the same direction. The second term of Eq.1.22 is weighted by E_{J1} and it is due to the smaller junction. On the other end the third term of Eq. 1.22 is weighted by $2\gamma E_{J1}$, which is indeed equal to $2E_{J2}$, and

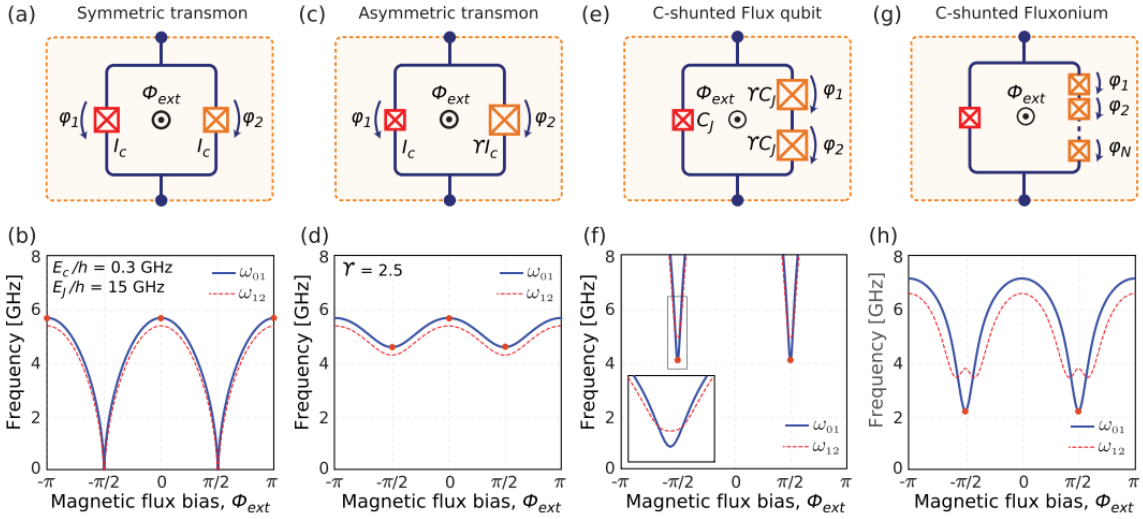

Figure 1.4: Schematic circuit representations of the symmetric and asymmetric SQUID, flux qubit and fluxonium with their corresponding transition frequencies of the first two energetic levels, function of the external magnetic flux Φ_{ext}.**a)** and **b)**: symmetric transmon qubit with Josephson energy E_J.**c)** and **d)**: asymmetric transmon qubit with asymmetry parameter $\gamma = 2.5$. **e)** and **f)**: capacitively shunted flux qubit, where a smaller junction is shunted with two larger ones. **g)** and **h)**: C-shunted fluxonium qubit, where a small junction is shunted with an array of N larger junctions. Figure taken from [12].

it is due to the two larger array junctions. It should be clear that now the eigenstates and the eigenvalues of the system depend both on the external flux and the γ parameter. Moreover now the sum of the second and the third term of the Hamiltonian operator does not give a simple cosinusoidal function and this reflects directly on the shape of the potential energy (fig1.4f).

To increase even further the anharmonicity of the flux qubit while preserving it from decohere through charge and flux environmental interactions, a new type of qubit has been developed. The *fluxonium* is indeed an extension of the flux qubit that can preserve its advantages while increasing the anharmonicity of the energy levels. The scheme of the fluxonium is quite similiar to that of the flux qubit: in one branch of the loop there is a small Josephson junction with energy E_{J_1}; on the other branch there are N different junctions with the same Josephson energy E_{J2} and bigger than the first one (fig.1.4g). In most of the cases, the number N of the junctions in the second branch of the loop could be of the order of 10^2 [12]. The Hamiltonian function of the fluxonium is similar to that of the flux qubit but with the array induction part given by the term $-N\gamma E_{J1} cos\left(\hat{\psi}/N\right)$, where the phase operator is defined as the average of the all phase contributions of all the junctions. Since the number of array junctions N can reach high values, it is possible to expand in series the cosine function to the second order. In the usual assumption of having an asymmetric parameter greater than the unity, the problem can be treated similarly to that of a particle moving in a quasi one-dimensional potential.

The fluxonium Hamiltonian function is then [12]:

$$\hat{H} \approx 4E_C(\hat{n} - \hat{n}_g)^2 - E_{J1}(\hat{\psi} + \hat{\psi}_e) + \frac{1}{2}E_L\hat{\psi}^2 \tag{1.23}$$

with $E_L = (\gamma E_{J1})/N$ the inductive energy given by the sum of all the contributions of the array junctions. It is clear that the potential of this new system could be treated as a quadratic potential modulated periodically by a cosine function weighted by the Josephson energy of the smaller junction. As it can be seen from fig.1.4h the anharmonicity of the fluxonium in the minima of the frequency levels is quite stronger with respect to each previous device analyzed. Indeed, in the ideal working point of the device a high coherent time (from hundreds of microseconds to few milliseconds) together with a higher anharmonicity is expected. Despite all the advantages that derive from the implementation of the fluxonium instead of the flux qubits, there are also some disadvantages. Indeed, it is quite difficult to combine high coherence times with fast operations such as gate operations, redout operations or others with high fidelity. This is primarly due to the large shunt inductor, composed by the hundreds of larger Josephson junctions, which is sensitive to additional dechoerence sources. A useful parameter that can be used to measure the dielectric losses is the loss tangent, defined as the ratio between the qubit decay rate and its fequency. The loss tangent of the best fluxonium qubit is approximately 10^{-6}, one order of magnitude greater than that of the state of the art transmon qubit. Therefore, fluxonium qubits need to be operated at a much lower frequency than a typical transmon qubit to reach a comparable coherence time [19].

1.6. Linear light-matter coupling

In the recent field of cavity and circuit quantum electrodynimcs, as already anticipated in the introduction part of this thesis work, the interaction between atomic systems and confined optical modes is studied. The most important model used in this studies is the so called Jaynes-Cummings (JC) model ([20];[21]). This one simply consists of a single qubit interacting with a single confined bosonic mode, for example in a cavity or in a waveguide (fig.1.5). Being it analytically solvable, the JC model has gained a large success in QED research field and it has been applied for systems made of different kind of technologies. It is based on two fundamental approximations: the dipolar approximation and the rotating wave approximation (RWA). The former implies that the interaction between the electromagnetic field and the qubit is dominated only by the electric field of the wave; moreover it is seen as uniform in space by the qubit (its wavelength is much larger than the qubit size). The RWA implies that all the terms rotating at high frequency

Figure 1.5: Left: scheme of a two-level emitter inside a resonant cavity . Right: energy levels scheme for linear light-matter coupling.

are neglected, since in normal conditions their time evolution is so rapid with respect to the other terms preserved, that they tend to average to zero [22]. This kind of model however could be used only in the weak and strong coupling regime, i.e. in a regime where the coupling strength is much smaller than the cavity losses (weak) or the cavity frequency (strong) [11]. The Hamiltonian operator related to this particular model in the context of cavity QED could be written as:

$$\hat{H}_{JC} = \frac{\omega_0}{2}\hat{\sigma}_z + \omega\hat{a}^\dagger\hat{a} + g\hat{\sigma}_x(\hat{a}^\dagger + \hat{a}) \tag{1.24}$$

where ω_0 and ω are the frequencies of the atom and the field respectively, while g is the coupling strength of the interaction. As it can be seen from Eq.1.24, the interaction between the qubit and the field in the cavity is linear, in the sense that it is directly proportional to the electric field operator. In the RWA the terms $\hat{\sigma}_+\hat{a}^\dagger$ and $\hat{\sigma}_-\hat{a}$ are neglected since they are rapidly oscillating. Thanks to this approximation the problem becomes integrable and this is one of the main reasons of its large implementation and success. Ultimately, from fig.1.5 it is possible to see that in the RWA the number of excitations is preserved, i.e. in the single photon interaction case to each atomic transition corresponds the emission or absorption of just one photon.

1.7. Nonlinear light-matter coupling

Recent progresses in the QED field had make it possible to even reach the so called ultra strong coupling regime, where the coupling strength is comparable or even higher than the cavity frequency. In the ultra strong coupling regime the dynamics of the system cannot be described anymore by the JC model since the high frequency terms, previously neglected, now becomes relevant and the rotating wave approximation ceases to be applicable. The model that better describes the interaction between the qubit and the bosonic modes in this regime is the so called Quantum Rabi model [23]. In the last years the theoretical interest for this particular regime has been growing mostly due to all the possible innovative and peculiar phenomena that happens when it is reached. One of these is the two-photon interaction process: a single atomic excitation process involving the simultaneous absorption of two photons (fig.1.6). In the ultra strong coupling regime the two-photon interaction acquires new counterintuitive properties. For example the theory predicts the collapse of the discrete energy spectrum into a continuous band for this kind of two-photon systems ([24];[25]).
For all these considerations, it is clear why huge efforts were done in the few past years to implement a physical system for which this ultra strong coupling regime is reachable and observable experimentally. To reach this kind of regime it is needed to go beyond the dipolar approximation of the JC model. So far two-photon lasing and Rabi oscillations have been observed in processes of second or higher order in resonantly driven systems. In order to obtain a two-photon interaction in an undriven system, it is necessary to have a strong nonlinear interaction between the qubit and the field. The ideal candidate of reproducing this ultrastrong nonlinear interaction is indeed a superconducting circuit. One possible circuit scheme that could be used to obtain a nondipolar interaction is proposed in fig.1.6. This superconducting circuit is constituted by a dc-SQUID with junction energy E_J, who acts like an optical resonator, coupled with a flux qubit with junction energy $E_J^{'}$ via a small inductive element L. The phase operators of each junction are showed in fig.1.6. Two different external magnetic fluxes are used to bias the SQUID ($f_s = \phi_s^{ext}/\Phi_0$) and the flux qubit ($f_q = \phi_q^{ext}/\Phi_0$). To understand why this circuit is suitable to implement the two-photon quantum Rabi model it is useful to analyze briefly the Hamiltonian function that describes it. It can be written as the sum of three contributions:

$$\hat{H}_{tp} = \hat{H}_0 + \hat{H}_{FQ} + \hat{H}_I \tag{1.25}$$

where the first term is due to the presence of the SQUID, the second is due to the flux qubit and the last is the interaction one. Without entering into the details of the derivation of

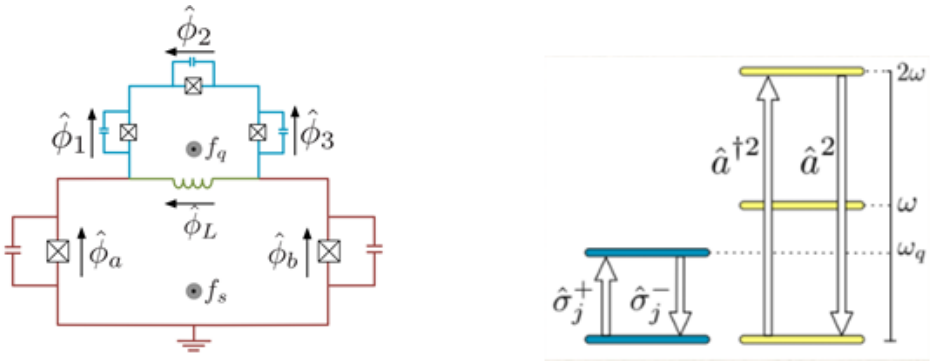

Figure 1.6: Left: scheme of the superconducting circuit used to implement the two-photon Quantum Rabi Model. It is constituted by a dc-SQUID (red) and a flux qubit (blue) coupled by a small inductance (green). Figure taken from [24]. Right: energy levels scheme for nonlinear light-matter coupling.

this Hamiltonian operator, the most important thing to point out is the interaction term $\hat{H}_I$. Its explicit expressions is the following [24]:

$$\hat{H}_I = \frac{S}{4E_L}\hat{\Sigma}_m\hat{\phi}_+^2 \tag{1.26}$$

where $S = E_J sin(f_s/2)$, E_L the coupling energy and $\hat{\Sigma}_m = \alpha E'_J sin(2\hat{\phi}_m + f_q)$. The phase operators that appear in Eq.1.26 are defined as $\hat{\phi}_+ = (\hat{\phi}_a + \hat{\phi}_b)/2$ and $\hat{\phi}_m = (\hat{\phi}_1 - \hat{\phi}_3)/2$ obtained using the flux quantization rule for the SQUID and the flux qubit respectively. The interaction Hamiltonian operator has a direct relation of proportionality with the square of the phase operator of the SQUID. It is indeed this term that is responsible for the two-photon interaction. This can be understood better if we write the explicit expressions of the conjugated variables that were used to obtain the circuit Hamiltonian operator $\hat{H}_{tp}$ [24]:

$$\hat{\phi}_+ = \sqrt{\frac{\hbar\omega_c L_{eff}}{2\Phi_0^2}}(\hat{a}^\dagger + \hat{a}) \quad ; \quad \hat{p}_+ = i\sqrt{\frac{2\Phi_0^2}{\hbar\omega_c L_{eff}}}(\hat{a}^\dagger - \hat{a}) \tag{1.27}$$

with ω_c the frequency of the bosonic mode supported by the circuit and L_{eff} the circuit effective inductance. The square of the SQUID phase operator depends on the square of the field ladders operators $\hat{a}^\dagger$ and $\hat{a}$. Since the effect of these kind of operators is the destruction or the creation of one optical excitation inside the system, the meaning of the square of one of them is therefore the creation or the destruction of a pair of excitations. Furthermore, by comparing the interaction Hamiltonian of the superconducting circuit in

Eq.1.26 with the Hamiltonian function of the quantum Rabi model [26]:

$$\hat{H}_{QRM} = \hat{H}^0_{QRM} + g_2\hat{\sigma}_x(\hat{a}^\dagger + \hat{a})^2 \tag{1.28}$$

where $\hat{\sigma}_x$ is one of the Pauli operators, it is possible to give an explicit expression of the coupling strenght of the two-photon interaction for the circuit under examination [24]:

$$g_2 = \frac{S}{4E_L}\sqrt{\frac{E_C}{K + \frac{S^2}{2E_L}}}\,\langle 0|\,\hat{\Sigma}_m\,|1\rangle \tag{1.29}$$

with E_C the energy of the Capacitors of the SQUID loop and $K = 2E_J cos(\frac{f_s}{2})$. The two-photon coupling strength depends on the circuits parameters and directly on the matrix element of the operator $\hat{\Sigma}_m$, i.e. it can be tuned via the two magnetic fluxes threading the loops f_s and f_q. The fact that the coupling strength is controllable is crucial. Indeed in the expression of the Hamiltonian function of the circuit in Eq.1.25 are present both dipolar and nondipolar terms. With the correct choice of the circuit parameters and of the external magnetic fluxes it is possible to suppress the dipolar term and enhance up to the ultra strong coupling regime the two-photon coupling coefficient. This makes this kind of superconducting circuit the ideal device with which investigate two-photon interaction phenomena.

2 | Photon scattering in waveguide QED

In the previous chapter different types of superconducting circuits were illustrated. These kind of devices are used nowadays as qubits for quantum algorithms or for QED experiments due to their tunability and reliability. In this new chapter the main topic of this thesis work, i.e. the study of photon-qubit interaction, will be presented. Starting from the simple case of the interaction between a single photon with a qubit and passing then to the more complex case of two-photon interaction in chapter 3, the general state of the considered system and its dynamics will be illustrated and analyzed. Particular attention will be put on the study of the phenomena of reflection and transmission of the input field after the scattering with the qubit. The study of this kind of effects is motivated by the perspective of making the qubit to act as a quantum switch for the coherent transport of a photon inside a waveguide [27]. Moreover the implementation of nonlinear qubits (for example the circuit in fig.1.6) makes it possible to have interesting effects that could be used to engineer quantum gates.

2.1. System dynamics

The system that it will be considered from now on is constituted by a two level quantum emitter situated in the center of a waveguide coupled linearly to electromagnetic modes. The qubit could be implemented using superconductive circuits such as the transmon or the flux qubit. Its two energy levels, the ground state and the excited state, will be labeled as $|g\rangle$ and $|e\rangle$ respectively. The electromagnetic modes that are supported by the waveguide will be labeled as $|0_\omega\rangle$ and $|1_\omega\rangle$ depending on the number of photons for each particular mode. Since one of the aims of this work is the study of single photon interactions, here the maximum number of photons in each mode will be 1. Another degree of freedom is represented by the direction of propagation μ of the photons inside the waveguide. The convention used here is to take $\mu = +$ if the photons are moving from left to right and $\mu = -$ otherwise. A simple scheme of the considered system is proposed

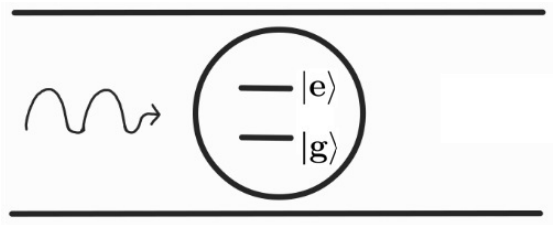

Figure 2.1: System scheme: a two level artificial atom inside a waveguide interacting with an incoming electromagnetic field.

in fig.2.1. In order to obtain the dynamics of the system it is necessary to write the explicit expression of the Hamiltonian function that better describes it. Since the qubit could be treated as a perfect two level system (in chapter 1 for example it was shown that the flux qubit has a large anharmonicity parameter) it is quite natural to use in this case the JC model adapted to the waveguide context. Then the Hamiltonian operator of the system can be written as:

$$\hat{H} = \omega_0 \hat{\sigma}^+ \hat{\sigma}^- + \sum_{\mu=\pm} \int \omega (\hat{a}^\mu_\omega)^\dagger \hat{a}^\mu_\omega d\omega + \sum_{\mu=\pm} \int g^\mu_\omega (\hat{a}^\mu_\omega)^\dagger \hat{\sigma}^- + (g^\mu_\omega)^* \hat{a}^\mu_\omega \hat{\sigma}^+ d\omega \tag{2.1}$$

In Eq.2.1 ω_0 is the qubit characteristic frequency, which means that $E_e - E_g = \omega_0$, with E the energy of the level; the operators $\hat{\sigma}^+$ and $\hat{\sigma}^-$ are the ladder operators for the artificial atom, which means that their effect on the state of the qubit is to create an excitation or to destroy it ($\hat{\sigma}^+ |g\rangle = |e\rangle$ and $\hat{\sigma}^- |e\rangle = |g\rangle$). On the other hand the operators $(\hat{a}^\mu_\omega)^\dagger$ and $\hat{a}^\mu_\omega$ are the electromagnetic field ladder operators and their effect on the field state is to create or to destroy a photon with a particular frequency ω and in a particular direction μ ($(\hat{a}^\mu_\omega)^\dagger |0\rangle = |1^\mu_\omega\rangle$ and $\hat{a}^\mu_\omega |1^\mu_\omega\rangle = |0\rangle$). The parameter g^μ_ω is the coupling strength of the interaction between the qubit and the incoming photon. It depends in general both on the particular photon frequency and the particular direction of propagation. The Hamiltonian operator in Eq.2.1 is constituted basically by three terms. The first two terms can be grouped in the operator:

$$\hat{H}_0 = \omega_0 \hat{\sigma}^+ \hat{\sigma}^- + \sum_{\mu=\pm} \int \omega (\hat{a}^\mu_\omega)^\dagger \hat{a}^\mu_\omega d\omega \tag{2.2}$$

which indeed represents the total energy of the system when the interaction is turned off. The first term of Eq.2.2 is the qubit energy where the origin of the energy reference system has been put in correspondence of the ground state energy. The second term of the non interacting Hamiltonian operator represents the total energy of the electromagnetic field expressed as the sum in all the possible directions of the energy of all the possible

supported modes. The last term in Eq.2.1 is, on the other side, the interaction operator:

$$\hat{H}_I = \sum_{\mu=\pm} \int g_\omega^\mu (\hat{a}_\omega^\mu)^\dagger \hat{\sigma}^- + g_\omega^{\mu *} \hat{a}_\omega^\mu \hat{\sigma}^+ d\omega \tag{2.3}$$

It comprehends all the energies of all the possible events that could happen, weighted by the coupling strength coefficient: the emission of a photon from an excited qubit $(\hat{a}_\omega^\mu)^\dagger \hat{\sigma}^-$ and the absorption of an incoming photon $\hat{a}_\omega^\mu \hat{\sigma}^+$. It is important to notice that in writing Eq.2.1 two important approximations were made: the dipolar approximation and the rotating wave approximation (RWA). This last one is valid only if the interaction is weak or strong, which means that the coupling strength must be much smaller than the qubit characteristic frequency [26]. In this case all the terms that would correspond to the phenomena of destruction of a photon and qubit in the ground state $\hat{a}_\omega^\mu \hat{\sigma}^-$, absorption of a photon and field in the excited state $(\hat{a}_\omega^\mu)^\dagger \hat{\sigma}^+$, are neglected. Since these phenomena that apparently seem violate the conservation of the total energy of the system are neglected, it is possible to write the total number of excitations operator as:

$$\hat{n}_1 = \sum_{\mu=\pm} \int (\hat{a}_\omega^\mu)^\dagger \hat{a}_\omega^\mu d\omega + \hat{\sigma}^+ \hat{\sigma}^- \tag{2.4}$$

It can be demonstrated that the two operators $\hat{\mathbf{H}}$ and $\hat{n}_1$ do commute, which means that, apart from the fact that they share the same eigenvectors, the expectation value of the total number of excitations operator does not change in time. This means that either the atom is excited and the field is not, either the field is excited and the atom is in the ground state. This is equivalent of saying that we cannot exit the Hilbert subspace defined by the operator $\hat{n}_1$. If we would like to write a full expression of the Hamiltonian operator in the linear algebra formalism for every possible number of excitations, this would lead to a block diagonal matrix and each block with linear dimension given by the expected value of the relative operator $\hat{n}_i$. Due to the fact that the interaction is weak, every block will not interact with the others: once the system is in a particular Hilbert subspace it cannot exit it.

After all these considerations, in order to study the system dynamics, it is possible to use as a general expression for the system state the following:

$$|\Phi(t)\rangle = \left[C_e(t)\hat{\sigma}^+ + \sum_{\mu=\pm} \int C_\omega^\mu(t) (\hat{a}_\omega^\mu)^\dagger d\omega \right] |\mathbf{0}\rangle \tag{2.5}$$

where $|\mathbf{0}\rangle = |g\rangle\,|0\rangle$ is the ground state of the two dimensional Hilbert subspace obtained as the tensor product between the possible states of the qubit and the incoming field. This kind of expression arises from the Wigner-Weisskopf model. It is a mathematical ansatz (trial state) built from a coherent superposition of an excited qubit and a propagating bosonic mode. Indeed, the two coefficients $C_e(t)$ and C_ω^μ represents the amplitude probability of the qubit and of the electromagnetic mode respectively. Their modulus square, then, represents the probability of finding the qubit or the field in the excited state for each possible instant of time. Their sum must then be equal to the unity in order to have a normalized problem:

$$|C_e(t)|^2 + \sum_{\mu=\pm} \int |C_\omega^\mu(t)|^2 \, d\omega = 1 \tag{2.6}$$

It is licit to express the general state of the system considered using Eq.2.5 thanks to the fact that the total number of excitations is constant in time, being valid the RWA hypothesis. It is clear then, that to study the dynamics of the system in fig.2.1 it is enough to find an explicit expression for the two amplitude probability coefficients and then inserting them in Eq.2.5. Indeed, once the system state is defined one could compute all the desired expectation values and analyzes.

2.2. Formal solutions for the amplitude probability coefficients

The most direct way for finding the amplitude probability coefficients $C_e(t)$ and $C_\omega^\mu(t)$ is via the time dependent Scrhödinger equation (TDSE):

$$i\frac{\partial\,|\Phi(t)\rangle}{\partial t} = \hat{H}\,|\Phi(t)\rangle \tag{2.7}$$

This is indeed a linear differential equation that links the time evolution of the state of the system with its Hamiltonian operator effect. Since we already have an explicit expression both for the Hamiltonian function and for the state $|\Phi(t)\rangle$, it is possible to solve Eq.2.7 for the unknowns $C_e(t)$ and $C_\omega^\mu(t)$.

Let us now focus our attention on the second term of Eq.2.7, dependent on the Hamiltonian operator:

$$\begin{aligned}\hat{H}\left|\Phi(t)\right\rangle &= \hat{H}\left[C_e(t)\hat{\sigma}^{+} + \sum_{\mu=\pm}\int C_{\omega}^{\mu}(t)(\hat{a}_{\omega}^{\mu})^{\dagger}d\omega\right]\left|\mathbf{0}\right\rangle = \\ &= C_e(t)\hat{H}\left|e\right\rangle\left|0\right\rangle + \sum_{\mu=\pm}\int C_{\omega}^{\mu}(t)\hat{H}(\hat{a}_{\omega}^{\mu})^{\dagger}d\omega\left|\mathbf{0}\right\rangle\end{aligned} \tag{2.8}$$

The last equation is constituted by two terms: the first is given by the action of the Hamiltonian operator onto the state $\left|0\right\rangle\left|e\right\rangle$ (corresponding to the atom in the excited state and the field in the vacuum state); the second is given by the action of the Hamiltonian operator onto the state $\left|1_{\omega}\right\rangle\left|0\right\rangle$ (field in the 1-photon state and atom in the ground state). Each term is then weighted by the corresponding amplitude coefficient. The explicit expression of the first term of Eq.2.8, is simply given by:

$$\hat{H}\left|e\right\rangle\left|0\right\rangle = \omega_0\left|e\right\rangle\left|0\right\rangle + \sum_{\mu=\pm}\int d\omega g_{\omega}^{\mu}(\hat{a}_{\omega}^{\mu})^{\dagger}\left|\mathbf{0}\right\rangle \tag{2.9}$$

This is due to fact that applying the number operator $(\hat{a}_{\omega}^{\mu})^{\dagger}\hat{a}_{\omega}^{\mu}$ to the field ground state $\left|0\right\rangle$ gives zero as result since there are no photons in that particular mode. Similarly, since the qubit can be thought of as a perfect two level system, applying the rising operator $\hat{\sigma}^{+}$ to the excited state $\left|e\right\rangle$ again gives us zero as a result. All these considerations could also be thought of as direct consequences of the impossibility of the system to leave its Hilbert subspace.
On the other hand, the second term in Eq.2.8 needs much more attention. It is useful to first analyze what is the effect of the operator $\hat{H}(\hat{a_{\omega}^{\mu}})^{\dagger}$ onto the system ground state $\left|\mathbf{0}\right\rangle$. In particular, remembering that, in principle the frequencies and the directions inside the expression of the Hamiltonian operator $\hat{H}$ could be different with respect to the ones of the field ladder operators, it is possible to write:

$$\begin{aligned}\hat{H}(\hat{a}_{\omega}^{\mu})^{\dagger}\left|\mathbf{0}\right\rangle &= \omega_0\hat{\sigma}^{+}\hat{\sigma}^{-}(\hat{a}_{\omega}^{\mu})^{\dagger}\left|\mathbf{0}\right\rangle + \sum_{\lambda=\pm}\int d\nu\nu(\hat{a}_{\nu}^{\lambda})^{\dagger}\hat{a}_{\nu}^{\lambda}(\hat{a}_{\omega}^{\mu})^{\dagger}\left|\mathbf{0}\right\rangle + \\ &+ \sum_{\lambda=\pm}\int d\nu g_{\nu}^{\lambda}(\hat{a}_{\nu}^{\lambda})^{\dagger}\hat{\sigma}^{-}(\hat{a}_{\omega}^{\mu})^{\dagger}\left|\mathbf{0}\right\rangle + \sum_{\lambda=\pm}\int d\nu(g_{\nu}^{\lambda})^{*}\hat{a}_{\nu}^{\lambda}\hat{\sigma}^{+}(\hat{a}_{\omega}^{\mu})^{\dagger}\left|\mathbf{0}\right\rangle\end{aligned} \tag{2.10}$$

The first and the third term in Eq.2.10 are equal to zero since the operator $\hat{\sigma}^-$ is acting onto the system state with the atom in the ground state $|0\rangle|g\rangle$. The other terms are non null only if $\nu = \omega$ and $\lambda = \mu$. This arises from the properties of the field ladder operators: the creation and the annihilation operators do commute if they are acting onto different field states (different frequencies or different directions of propagation), otherwise their commutator is equal to the unity. This can be summarized in the following:

$$\hat{a}^{\lambda}_{\nu}(\hat{a}^{\mu}_{\omega})^{\dagger}|0\rangle = \delta_{\nu\omega}\delta_{\lambda\mu} \quad , \quad [\hat{a}^{\lambda}_{\omega}, (\hat{a}^{\mu}_{\omega})^{\dagger}] = \delta_{\nu\omega}\delta_{\lambda\mu} \tag{2.11}$$

with $\delta_{\nu\omega}$ and $\delta_{\lambda\mu}$ the Kronecker deltas in frequency and direction respectively. Finally after combining together Eq.2.9 and the non null terms in Eq.2.10, it is possible to give the complete expression of Eq.2.8:

$$\begin{aligned}\hat{H}|\Phi\rangle = C_e(t)\omega_0|e\rangle|0\rangle + C_e(t)\sum_{\mu=\pm}\int d\omega g^{\mu}_{\omega}(\hat{a}^{\mu}_{\omega})^{\dagger}|\mathbf{0}\rangle + \\ + \sum_{\mu=\pm}\int d\omega C^{\mu}_{\omega}(t)\omega(\hat{a}^{\mu}_{\omega})^{\dagger}|\mathbf{0}\rangle + \sum_{\mu=\pm}\int d\omega C^{\mu}_{\omega}(t)(g^{\mu}_{\omega})^{*}|e\rangle|0\rangle\end{aligned} \tag{2.12}$$

Now that the second term of Eq.2.7 has been simplified and expressed explicitly, we can calculate the time derivative of the quantum state directly. This is possible since the only terms that are time dependent are the amplitude probability coefficients. The direct calculations of the time derivative lead to:

$$i\frac{\partial|\Phi\rangle}{\partial t} = i\frac{\partial C_e(t)}{\partial t}|e\rangle|0\rangle + i\sum_{\mu=\pm}\int\frac{\partial C^{\mu}_{\omega}(t)}{\partial t}(\hat{a}^{\mu}_{\omega})^{\dagger}d\omega|\mathbf{0}\rangle \tag{2.13}$$

Now that all the terms in the TDSE have been calculated explicitly, it is enough to solve it to find the time dependent coefficients. What it is necessary to do now is to obtain two different linear differential equations from Eq.2.7. This could be done by exploiting the orthonormality properties of the system state eigenvectors. Indeed, this means that the scalar product between these states is equal to the unity if they are equal, null if they are different: $\langle i|j\rangle = \delta_{ij}$, with i,j=e,0 and g,1. If Eq.2.7 is multiplied first by the bra $\langle e|\langle 0|$ and then by the bra $\langle g|\langle 0|$, a set of two coupled differential equations for the time dependent coefficients $C_e(t)$ and $C_{\omega}(t)$ can be obtained.

In particular, after all the simplifications that here are neglected for brevity, the resulting system can be written as:

$$i\dot{C}_e(t) = \omega_0 C_e(t) + \sum_{\mu=\pm} \int d\omega (g_\omega^\mu)^* C_\omega^\mu(t)$$

$$i\dot{C}_\omega^\mu(t) = \omega C_\omega^\mu(t) + g_\omega^\mu C_e(t) \tag{2.14}$$

The most direct way of solving this kind of system is to first solve the differential equation for the field amplitude coefficient $C_\omega^\mu(t)$ and, then, substitute the result in the other equation for the qubit amplitude coefficient $C_e(t)$. Indeed the second equation of the system 2.14 can be solved immediately using the usual explicit formula for the complete linear differential equations. The final result is the following:

$$C_\omega^\mu(t) = C_\omega^\mu(t_0) e^{-i\omega(t-t_0)} - i g_\omega^\mu e^{-i\omega(t-t_0)} \int_{t_0}^{t} C_e(\tau) e^{i\omega(\tau-t_0)}\, d\tau \tag{2.15}$$

where the time integration has been done from the initial instant of time t_0 and the generic final instant of time t, with $t > t_0$. The field coefficient $C_\omega^\mu(t_0)$ can be seen as the amplitude probability of having an excited mode in the waveguide at frequency ω in an instant of time t_0 far before the event of interaction with the qubit.
Inserting the result in Eq.2.15 inside the first equation of the system 2.14 allows one to find a differential equation in the only unknown $C_e(t)$:

$$\begin{aligned} i\dot{C}_e(t) = \omega_0 C_e(t) + \sum_{\mu=\pm} \int d\omega (g_\omega^\mu)^* C_\omega^\mu(t_0) e^{-i\omega(t-t_0)} + \\ - i \sum_{\mu=\pm} \int \left[(g_\omega^\mu)^* g_\omega^\mu e^{-i\omega(t-t_0)} \int_{t_0}^{t} C_e(\tau) e^{i\omega(\tau-t_0)}\, d\tau \right] d\omega \end{aligned} \tag{2.16}$$

This last equation for the qubit amplitude probability coefficient depends on three terms. While the first of them is just the coefficient $C_e(t)$ weighted by the qubit characteristic frequency, the other two require much more attention since they both depend on an integral over all the possible modes. The first integral in Eq.2.16 can be easily calculated if the initial condition of the input field is defined and an explicit expression of the coupling coefficient is given. It is possible to state that this term depends only on the input field characteristics and on the circuit model (indeed g_ω^μ is defined by the circuit components). The second integral in Eq.2.16 (called from now on **I**), on the other hand, can be simplified further.

To proceed in this simplification, we operate the following change of variable:

$$T = t - \tau \qquad dT = -d\tau \tag{2.17}$$

With this change, integral **I** can be rewritten as:

$$\mathbf{I} = -i \sum_{\mu=\pm} \int_0^{+\infty} |g_\omega^\mu|^2 \left[\int_0^{t-t_0} C_e(t-T) e^{-i\omega T}\, dT \right] d\omega \tag{2.18}$$

In addition to that, an hypothesis on the the possible final profile of the quantum emitter amplitude probability is made. Indeed we write $C_e(t)$ as the product of a term which varies slowly in time and a rapidly oscillating phase at frequency ω_0:

$$C_e(t) = C_e^{sv}(t) e^{-i\omega_0 t} \tag{2.19}$$

The slowly varying function can be then brought out from the integral over time. In this way **I** becomes simply:

$$\mathbf{I} = -i \sum_{\mu=\pm} \int_0^{+\infty} |g_\omega^\mu|^2 C_e^{sv}(t-T) e^{-i\omega_0 t} \left[\int_0^{t-t_0} e^{-i(\omega-\omega_0)T}\, dT \right] d\omega \tag{2.20}$$

If we first calculate the integral over all the possible frequencies,it is easy to recognize in the last equation the Fourier Transform of the Heaviside function, whose result is already known in the literature. It is indeed equal to a Dirac delta centered on the qubit characteristic frequency weighted by π (this weight comes out when ω is used instead of the actual real frequency $\nu = \omega/2\pi$) and a imaginary term that would give rise to the Lamb shift phenomenon. In this thesis work, this shift of the resonance frequency is neglected.
The final expression of integral **I**, after having applied all these simplifications and after having used the Dirac delta function properties, is:

$$\mathbf{I} = -iC_e(t) \sum_{\mu=\pm} \pi |g^\mu|^2 = -iC_e(t) \frac{\gamma}{2} \tag{2.21}$$

where γ is the qubit spontaneous emission rate. The value of the parameter γ gives the probability of having an atomic emission at a certain frequency ω after an absorption event. Moreover, the higher the value of γ sooner the atom will emit a photon and shorter

will be the period of time during which the atom can be found in the excited state $|e\rangle$. In general the spontaneous emission rate, as already said, depends on the particular emission frequency ω. However, for our particular system, it is taken constant in frequency. This is equivalent of saying that the coupling parameter is constant for a wide range of frequencies centered on the qubit characteristic frequency ω_0. This hypothesis is quite reasonable since the frequency dependence of g is determined by the superconducting circuit scheme and its components that could both be engineered properly. The explicit definition of the spontaneous emission rate is the following:

$$\gamma = \sum_{\mu=\pm} \gamma^{\mu} \tag{2.22}$$

with $\gamma^{\mu} = 2\pi|g^{\mu}|^2$. Finally we can write the more simplified expression of the partial differential equation for the qubit amplitude probability coefficient as:

$$i\dot{C}_e(t) = \omega_0 C_e(t) - i\frac{\gamma}{2}C_e(t) + \sum_{\mu=\pm}(g^{\mu})^* \int d\omega C^{\mu}_{\omega}(t_0)e^{-i\omega(t-t_0)} \tag{2.23}$$

In Eq.2.23 the final term is nothing but the Fourier Transform of the field amplitude probability evaluated in the instant of time t_0. We define this quantity as the input field in the time domain $\Psi^{\mu}_{in}(t)$:

$$\Psi^{\mu}_{in}(t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} d\omega C^{\mu}_{\omega}(t_0)e^{-i\omega(t-t_0)} \tag{2.24}$$

Using this new definition in Eq.2.23 allows us to rewrite the qubit coefficient differential equation as:

$$i\dot{C}_e(t) = (\omega_0 - i\frac{\gamma}{2})C_e(t) + \sum_{\mu=\pm}\sqrt{\gamma^{\mu}}\Psi^{\mu}_{in}(t) \tag{2.25}$$

If the input conditions are given, the solution of Eq.2.25 is straightforward. In order to find easily a solution for the field in output after the interaction between the qubit and the input field, it is useful to derive a direct relation that links what enters in the waveguide and what exit from it.

2.3. Input-output relation

The input-output relation is an equation that links in a linear way the field distribution in input $\Psi^{\mu}_{in}(t)$, defined in Eq.2.24, and the field distribution in output, when the interaction has already happened, $\Psi^{\mu}_{out}(t)$. This last term could be defined as follows:

$$\Psi^{\mu}_{out}(t) = \frac{1}{\sqrt{2\pi}} \int d\omega C^{\mu}_{\omega}(t_1) e^{-i\omega(t-t_1)} \tag{2.26}$$

with t_1 an instant of time much grater than t and t_0, i.e. $t_1 > t > t_0$. As it can be seen from its definition, the output field distribution is nothing but the Fourier Transform of the field amplitude probability coefficient evaluated at the instant of time t_1.
In order to obtain the input-output relation seeked, we start from the second equation in the system 2.14. In the previous section, it has been integrated forward in time, i.e. from a past instant of time t_0 to the instant of time t, to find the general solution of the field coefficient. Now what we do is to take that equation and integrate it backward in time, i.e. from the instant of time t to a future instant t_1. The result of this integration is the following:

$$C^{\mu}_{\omega}(t) = C^{\mu}_{\omega}(t_1) e^{-i\omega(t-t_1)} + ig^{\mu} e^{-i\omega(t-t_1)} \int_t^{t_1} C_e(\tau) e^{i\omega(\tau-t_1)}\, d\tau \tag{2.27}$$

With the variable substitution $T = \tau - t$ the last equation can be simplified further as:

$$C^{\mu}_{\omega}(t) = C^{\mu}_{\omega}(t_1) e^{-i\omega(t-t_1)} + ig^{\mu} \int_0^{t_1-t} C_e(T+t) e^{i\omega T}\, dT \tag{2.28}$$

Now if we take Eq.2.15 and do the following variable substitution $T = t - \tau$, we obtain:

$$C^{\mu}_{\omega}(t) = C^{\mu}_{\omega}(t_0) e^{-i\omega(t-t_0)} - ig^{\mu} \int_0^{t-t_0} C_e(T-t) e^{i\omega T}\, dT \tag{2.29}$$

The two equations for the field amplitude probability coefficient $C^{\mu}_{\omega}(t)$, i.e. Eq.2.28 and Eq.2.29, can be compared. If we multiply both of them by the constant $1/\sqrt{2\pi}$ and integrate both of them in frequency, it is possible to identify, after these operations, the exact definitions of the input and output fields in the time domain:

$$\begin{aligned} &\Psi^{\mu}_{out}(t) + \frac{ig^{\mu}}{\sqrt{2\pi}} \int_0^{+\infty} \left[\int_0^{t_1-t} C_e(T+t) e^{i\omega T}\, dT \right] d\omega = \\ &\Psi^{\mu}_{in}(t) - \frac{ig^{\mu}}{\sqrt{2\pi}} \int_0^{+\infty} \left[\int_0^{t-t_0} C_e(T-t) e^{i\omega T}\, dT \right] d\omega \end{aligned} \tag{2.30}$$

The two integrals in frequency domain that appear in the last identity are formally equal to the integral in Eq.2.18. If we proceed in the same way as it has been done in the previous section, we can simply write:

$$\Psi^{\mu}_{out}(t) + \frac{ig^{\mu}}{\sqrt{2\pi}}C_e(t) = \Psi^{\mu}_{in}(t) - \frac{ig^{\mu}}{\sqrt{2\pi}}C_e(t) \tag{2.31}$$

At this point, after some trivial mathematical simplifications, we are able to write a direct relation that give us the expression of the output field in time domain as a function of the input field in time domain, the qubit amplitude probability coefficient and the coupling strength of the interaction. Finally the input-output relation expression is:

$$\Psi^{\mu}_{out}(t) = \Psi^{\mu}_{in}(t) - i\sqrt{\gamma^{\mu}}C_e(t) \tag{2.32}$$

From this explicit relation it can be noticed that the output field in time domain is simply given by the input field in the time domain plus a correction due to the presence of the emitter, weighted by the square root of the γ^{μ} parameter (which depends directly on the coupling strength). The dependence on the direction of propagation is enclosed in the input field and in the particular value of the interaction strength. More physical insight regardin these input-output fields could be found in Appendix B.
With the input-output relation in Eq.2.32 and the Eq.2.25 for the artificial atom amplitude probability coefficient, it is possible to calculate both the unknown coefficients seeked and finally obtain the explicit expression for the system state in Eq.2.5.

2.4. Example: spontaneous emission

Now that the system of differential equations has been solved and the equations for the amplitude probability coefficients are set, different situations of interest can be investigated. The two phenomena studied in this thesis work will be the atomic spontaneous emission and the single photon scattering. In this particular section we will focus our attention on the former.
With the expression spontaneous emission we refer to the phenomenon of the emission of a radiation from an atom that had previously absorbed a photon and had undergone a transition from the ground to the excited state. For the particular system considered here, the situation could be schematized as follows: the artificial atom initially in the ground state $|g\rangle$ has absorbed an incoming photon at a certain frequency ω_{in} in the waveguide; the qubit, now in the excited state $|e\rangle$, emits a photon at a certain frequency ω_{out} after

Figure 2.2: Left: at time t_0 the qubit is in the excited state $|e\rangle$. Right: after a certain amount of time the qubit spontaneously emits a photon and relaxes in the ground state $|g\rangle$. In principle the emission could be in the right or in the left direction with a certain probability.

a certain time t, usually of the order of the excited level lifetime (fig.2.2). The first step that has to be done for finding the two amplitude probability coefficients is defining the initial conditions of the system, i.e. the value of the qubit and field coefficients at t_0, with t_0 the instant of time when the artificial atom is in the state $|e\rangle$. Said so, the initial conditions of the system can be expressed as:

$$C_e(t_0) = 1 \quad ; \quad C_\omega^\mu(t_0) = 0 \tag{2.33}$$

since there is no input field in the waveguide for $t = t_0$. With this in mind, it is easy to verify that $\Psi_{in}^\mu = 0$. The direct consequence of this condition is that now Eq.2.23 can be simplified further, becoming in this way directly integrable:

$$i\dot{C}_e(t) = (\omega_0 - i\frac{\gamma}{2})C_e(t) \tag{2.34}$$

The solution of the partial differential equation for the qubit amplitude probability coefficient is then:

$$C_e(t) = e^{-i\omega_0(t-t_0)}e^{-\frac{\gamma}{2}(t-t_0)}\mathcal{H}(t-t_0) \tag{2.35}$$

where $\mathcal{H}(t-t_0)$ is the Heaviside function centered in t_0. This function has been added in Eq.2.35 to underline the fact that the expression of the coefficient $C_e(t)$ is the one computed above only when we are looking at the system at a time t grater than t_0. If we look at the system in an instant of time before t_0, we are putting ourselves in a situation different from that described by the initial conditions in Eq. 2.33, i.e. in a situation where the input field cannot be considered null. Consequently, for $t < t_0$, the calculations that permitted us to find Eq.2.35 do not hold anymore.
Since the qubit amplitude probability coefficient has been calculated, by inserting its expression inside the input-output relation in Eq.2.32 we can find also the expression for

the output field in the time domain. Its definition, since $\Psi^{\mu}_{in}(t) = 0$, is simply:

$$\Psi^{\mu}_{out}(t) = -i\sqrt{\gamma^{\mu}}e^{-i\omega_0(t-t_0)}e^{-\frac{\gamma}{2}(t-t_0)}\mathcal{H}(t-t_0) \tag{2.36}$$

From the above expression, the field amplitude probability coefficient can be found by an inverse Fourier Transform operation of Eq.2.26:

$$C^{\mu}_{\omega}(t_1) = -\frac{i\sqrt{\gamma^{\mu}}}{\sqrt{2\pi}}\int_0^{+\infty} e^{-i\omega_0(t-t_0)}e^{-\frac{\gamma}{2}(t-t_0)}e^{i\omega(t-t_1)}\mathcal{H}(t-t_0)\,dt \tag{2.37}$$

The solution of the above integral is straightforward since the integrand is just an exponential function. The final expression of the field amplitude probability is the following:

$$C^{\mu}_{\omega}(t_1) = -\frac{i\sqrt{\gamma^{\mu}}}{\sqrt{2\pi}}\frac{e^{-i\omega(t_1-t_0)}}{\frac{\gamma}{2}+i(\omega_0-\omega)} \tag{2.38}$$

where ω is the frequency of the spontaneously emitted radiation and $\omega_0 - \omega = \Delta\omega$ the frequency detuning between the qubit characteristic frequency and the emitted photon. It is interesting now to take Eq.2.35 and Eq.2.38 and evaluate the modulus square of these coefficients. Indeed, the modulus square of the amplitude probabilities represents the probability distribution function of time, that gives us information about how probable is to find the qubit or the field in the excited state respectively. The expression of the modula squared of $C_e(t)$ and $C^{\mu}_{\omega}(t_1)$ (fig.2.3) are:

$$|C_e(t)|^2 = e^{-\gamma(t-t_0)}\mathcal{H}(t-t_0) \tag{2.39}$$

$$|C^{\mu}_{\omega}(t_1)|^2 = \frac{1}{2\pi}\frac{\gamma^{\mu}}{\frac{\gamma^2}{4}+(\omega_0-\omega)^2} \tag{2.40}$$

For what concerns the probability distribution represented by $|C_e(t)|^2$, it can be seen that it depends on time exponentially. This means that the probability of finding the qubit in the state $|e\rangle$ after a certain time t, grater than the initial one t_0, decreases exponentially as soon as we let the system evolve in time. The time constant of this decrease is given by the reciprocal of the spontaneous emission rate parameter γ. This parameter has been taken constant in frequency over a wide range around ω_0 so once the superconducting circuit defining the qubit has been set, the parameter γ is therefore defined.

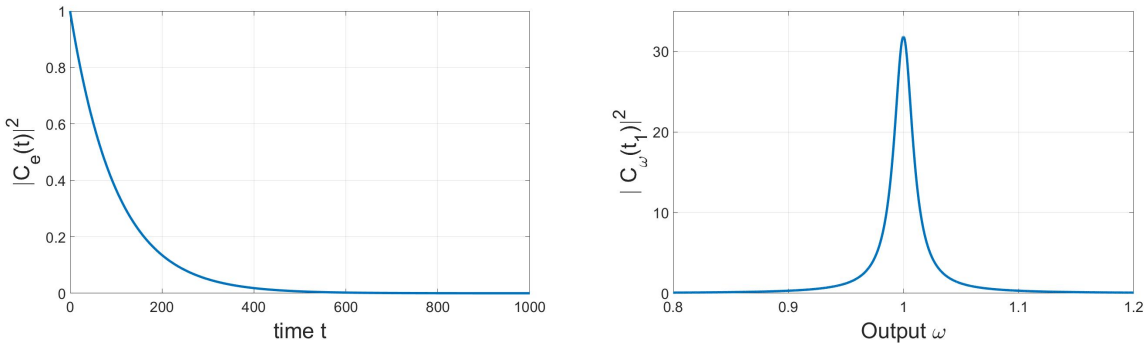

Figure 2.3: Left: plot of the modulus square of the qubit amplitude probability coefficient function of time (in units of $1/\Gamma_0$). Right: plot of the modulus square of the field amplitude probability coefficient function of the output frequency ω (in units of ω_0, which has been set equal to 1). The parameter Γ_0 has been set equal to $\omega_0/100$.

On the other hand, the probability distribution function regarding the output field, does not depend on time. This arises from the fact that through the input-output relation what is retrieved is the field amplitude probability coefficient evaluated in the instant of time $t = t_1$, with t_1 much larger than t_0. Moreover, the only time dependence in $C^\mu_\omega(t_1)$ is in the phase term that the field acquires through the propagation along the waveguide from the instant t_0 to the final one t_1, which becomes equal to 1 when the modulus square is taken. In any case, the modulus square of $C^\mu_\omega(t_1)$ depends on the output frequency ω. Its frequency distribution, as it can be seen from Eq.2.40, is a normalized Lorentzian function centered around the emitter characteristic frequency ω_0, with a full width at half maximum (FWHM) proportional to the parameter γ (fig.2.3). This means that the probability of measuring in output a photon with a frequency ω becomes lower and lower as soon as this frequency is too different from ω_0, in agreement with the fact that the spontaneous emission of a photon at a completely different frequency with respect to that of the atom is practically impossible. In addition to these considerations, the modulus square of the field coefficient depends on the direction parameter μ. In principle the two different parameters γ^+ and γ^- could be taken different from each other. However the best choice for the system we are considering is to take both of them equal to half the spontaneous emission rate, i.e. $\gamma^+ = \gamma^- = \Gamma_0$. This choice would correspond to having a symmetric coupling inside the waveguide (isotropic spontaneous emission) and a value of the total spontaneous emission rate γ equal to $2\Gamma_0$.
In conclusion of this paragraph it is useful to check whether the results found are coherent with the definition of our problem. To verify this, it is enough to check if the normalization condition in Eq.2.6 is satisfied by the two amplitude probability coefficients evaluated in the instant of time $t = t_1$. Indeed, t_1 is the time instant in which we observe our system and it is, by definition, much larger than the initial one t_0.

This leads to the following equation:

$$|C_e(t_1)|^2 + \sum_{\mu=\pm} \int |C_\omega^\mu(t_1)|^2 \, d\omega = 1 \tag{2.41}$$

For what it has been said, for $t \to t_1$ the emitter amplitude probability vanishes and, since the field probability distribution is a normalized Lorentzian function, the validity of Eq.2.6 is assured.

2.5. Example: scattering event

Another fundamental phenomenon that is worth studying related to our considered system is the opposite case with respect to what it has been analyzed before, i.e. the single photon scattering event. With this term we refer to a particular situation in which the qubit, that is prepared in its ground state $|g\rangle$, interacts with an incoming input field in the state $|1_{\omega_1}\rangle$. Usually, in the free space, this kind of interaction would bring us to observe firstly an absorption of the incoming photon by the qubit and subsequently an emission. Of course the probability of having such an interaction would depend on the detuning between the input field frequency ω_1 and the qubit characteristic frequency ω_0, with a bigger probability as soon as the detuning approximates the zero. In principle, once the scattering event has happened, it is possible to observe the emission of the absorbed photon in any space directions with equal probability. This is equivalent of saying that the scattering event in free space is isotropic. This, however, is not true for the system represented schematically in fig.2.1. Indeed, being the atom confined in the waveguide, the only possible directions in which it is possible to observe an emission are those labeled by the index μ,i.e. from left to right for $\mu = +$ and from left to right for $\mu = -$. The presence of the waveguide, thus, simplifies enormously the problem since now all the directions of emission could be neglected except for those two. A direct consequence of this simplification is that the problem studied here becomes linear in space and the only two possible output field states are represented by the kets $\left|1^+_{\omega_1}\right\rangle$ and $\left|1^-_{\omega_1}\right\rangle$.
Similarly to what has been done in the previous paragraph, in order to study the behaviour of the system in this case of single photon scattering, it is necessary to define the initial conditions. Regarding the qubit amplitude probability coefficient, we must impose that in the instant of time in which the scattering event happens, i.e. t_0, the qubit is prepared in its ground state $|g\rangle$. On the other hand, we must define a field amplitude probability coefficient in input. For simplicity, in this thesis work, the input field will be taken as a monochromatic plane wave.

This corresponds to the following definitions:

$$C_e(t_0) = 0 \quad ; \quad C^{\mu}_{\omega}(t_0) = \frac{1}{\sqrt{\sigma\sqrt{2\pi}}} e^{-\frac{(\omega-\omega_1)^2}{4\sigma^2}} \delta^{\mu}_{+} \tag{2.42}$$

The monochromatic plane wave in input has been taken as a Gaussian function centered on the generic input frequency ω_1. The condition of plane wave is reached when the width of this Gaussian field is small enough with respect to the input frequency. Formally, it is possible to write:

$$\delta(\omega - \omega_1) = \lim_{\sigma \to 0} \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(\omega-\omega_1)^2}{4\sigma^2}} \tag{2.43}$$

Indeed, to a function really narrow in the frequency domain corresponds a function that is really broad in the time domain, being the two functions linked via a Fourier Transform operation. For what concerns the direction index μ, the input field has been taken arbitrarily propagating in the direction + (from left to right). This choice is expressed by the Kronecker delta δ^{μ}_{+}. Ultimately, it is important to notice that, to be precise, in input, it has been chosen the square root of a Gaussian function. This choice is related to the normalisation condition in Eq.2.6, that has to be satisfied for every instant of time, included t_0.

Now that the initial conditions have been set, it is possible to explicitly calculate the two amplitude probability coefficients, using Eq.2.25 and Eq.2.32. Before solving the differential equation for the qubit coefficient $C_e(t)$, we need first to obtain the expression of the input field in the time domain. In order to do this, we perform a Fourier Transform operation on the coefficient $C^{\mu}_{\omega}(t_0)$, i.e. we use its definition in Eq.2.24. Substituting Eq.2.42 inside the definition of $\Psi^{\mu}_{in}(t)$ leads to the following:

$$\Psi^{\mu}_{in}(t) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} d\omega \frac{1}{\sqrt{\sigma\sqrt{2\pi}}} e^{-\frac{(\omega-\omega_1)^2}{4\sigma^2}} \delta^{\mu}_{+} e^{-i\omega(t-t_0)} \tag{2.44}$$

At this point, if the width of the input Gaussian field σ is small enough (in the extreme case $\sigma \to 0$), the function inside the integral could be replaced by a Dirac delta. With this substitution in mind, the expression of $\Psi^{\mu}_{in}(t)$ becomes:

$$\Psi^{\mu}_{in}(t) = \sqrt[4]{\frac{2}{\pi}} \sqrt{\sigma} \delta^{\mu}_{+} \int_0^{+\infty} d\omega \delta(\omega - \omega_1) e^{-i\omega(t-t_0)} \tag{2.45}$$

The integration over all the possible frequencies is now trivial, since it is enough to apply

the effect of the Dirac delta distribution on the phase function. The final expression of the input field in the time domain is:

$$\Psi_{in}^{\mu}(t) = \sqrt[4]{\frac{2}{\pi}}\sqrt{\sigma}\delta_{+}^{\mu}e^{-i\omega_1(t-t_0)} \tag{2.46}$$

As it can be seen from its expression, the time dependence of the input field is enclosed in the phase term, which represents the phase that the incoming photon acquires from the initial instant of time t_0 to the generic one t. The amplitude of $\Psi_{in}^{\mu}(t)$ depends on the square root of the width of the input Gaussian function. This is consistent with the approximation done at the beginning of the paragraph when the input monochromatic field was expressed as the limit in Eq.2.43. It is important to notice that, although the σ parameter tends to zero, the input field does not vanish since what is relevant here is its integral over all the frequencies, which, as it will be clear later, is well defined.
Now that the input field $\Psi_{in}^{\mu}(t)$ has been calculated, we can insert its expression inside Eq.2.25. The differential equation for the qubit amplitude probability coefficient reads:

$$i\dot{C}_e(t) = (\omega_0 - i\frac{\gamma}{2})C_e(t) + \sum_{\mu=\pm}\sqrt{\gamma^{\mu}}\sqrt[4]{\frac{2}{\pi}}\sqrt{\sigma}\delta_{+}^{\mu}e^{-i\omega_1(t-t_0)} \tag{2.47}$$

The Kronecker delta δ_{+}^{μ} selects inside the summation over all the possible directions the one of the input field, which, in this case, is the direction from left to right ($\mu = +$). The differential equation for the qubit amplitude probability coefficient is then linear with the addition of a driven function represented by the input field in the time domain. Using the well known closed formula for this kind of differential equations leads to the following solution for $C_e(t)$:

$$C_e(t) = -ie^{-(i\omega_0+\frac{\gamma}{2})(t-t_0)}\int_{t_0}^{t}\sqrt{\gamma^{+}}\sqrt[4]{\frac{2}{\pi}}\sqrt{\sigma}e^{-i\omega_1(\tau-t_0)}e^{(i\omega_0+\frac{\gamma}{2})(\tau-t_0)}\,d\tau \tag{2.48}$$

By grouping together all the constant terms in front of the integral, the solution for the qubit coefficient could be rewritten in a much more compact way as:

$$C_e(t) = -i\sqrt{\gamma^{+}\sigma}\sqrt[4]{\frac{2}{\pi}}e^{-(i\omega_0+\frac{\gamma}{2})(t-t_0)}\int_{t_0}^{t}e^{(i(\omega_0-\omega_1)+\frac{\gamma}{2})(\tau-t_0)}\,d\tau \tag{2.49}$$

The solution is then straightforward since inside the integral over time there are only exponential functions. At this point it is necessary to specify that the instant of time

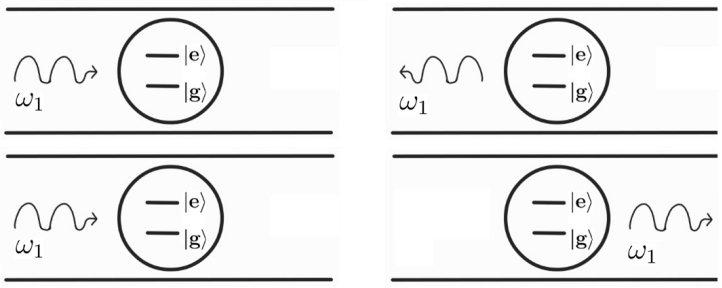

Figure 2.4: Up: schematic representation of the reflection phenomenon. Bottom: schematic representation of the transmission phenomenon.

in which the system is observed will be t_1, with t_1 much larger than the initial instant t_0. In this framework we could neglect all the terms that are directly proportional to the exponential function $e^{-\frac{\gamma}{2}(t-t_0)}$ since when evaluated in $t = t_1$, they will vanish. Said so, the final expression for the qubit amplitude probability coefficient is:

$$C_e(t) = -i\sqrt[4]{\frac{2}{\pi}}\frac{\sqrt{\sigma\gamma^+}}{(\frac{\gamma}{2} + i(\omega_0 - \omega_1))}e^{-i\omega_1(t-t_0)} \tag{2.50}$$

Inserting now the expression obtained for the coefficient $C_e(t)$ inside the input-output relation in Eq.2.32, allows us to retrieve the output field in the time domain. Its expression would be:

$$\Psi^\mu_{out}(t) = \sqrt[4]{\frac{2}{\pi}}\sqrt{\sigma}\delta^\mu_+ e^{-i\omega_1(t-t_0)} - \sqrt[4]{\frac{2}{\pi}}\frac{\sqrt{\sigma\gamma^+\gamma^\mu}}{(\frac{\gamma}{2} + i(\omega_0 - \omega_1))}e^{-i\omega_1(t-t_0)} \tag{2.51}$$

By looking at the last equation it is immediate to notice that both the terms that appear in it have as a common factor the input field in the time domain $\Psi^\mu_{in}(t)$, whose full expression is given in Eq.2.46. It is possible then to bring this common factor in front of both the terms and rewriting the expression of the output field in a simpler way as:

$$\Psi^\mu_{out}(t) = \Psi^+_{in}(t)\chi^\mu_{\omega_1} \tag{2.52}$$

where $\Psi^+_{in}(t)$ is just the input field in the time domain in the direction corresponding to $\mu = +$ and $\chi^\mu_{\omega_1}$ the function that remains after the grouping. Its explicit expression is the following:

$$\chi^\mu_{\omega_1} = \delta^\mu_+ - \frac{\sqrt{\gamma_\mu}\sqrt{\gamma_+}}{\frac{\gamma}{2} + i(\omega_0 - \omega_1)} \tag{2.53}$$

The function $\chi^\mu_{\omega_1}$ depends on the detuning $\Delta\omega = \omega_0 - \omega_1$ between the input frequency

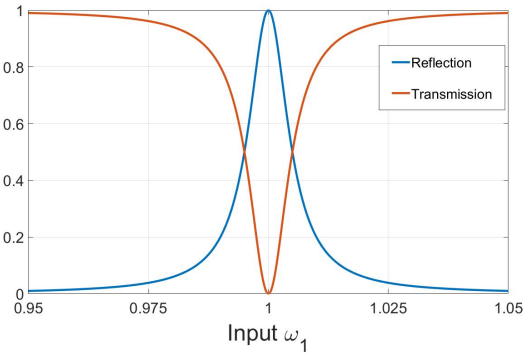

Figure 2.5: Plot of the reflection coefficient R (in blue) and of the transmission coefficient T (in orange) function of the input frequency ω_1, expressed in units of the qubit frequency ω_0. The characteristic frequency ω_0 has been set equal to 1, while the parameter Γ_0 has been set equal to $\omega_0/100$.

ω_1 and the qubit frequency ω_0. It depends also on the spontaneous emission rate γ and on the direction index μ. Indeed, this particular function could be seen as the system transfer function, i.e. it gives us information about how the input field $\Psi^+_{in}(t)$ is modified through the interaction with the qubit inside the waveguide. This interaction modifies the input field in a different way, depending on the direction of propagation of $\Psi^\mu_{out}(t)$. It is possible to define the output field $\Psi^+_{out}(t)$ as the transmitted field along the waveguide, since it propagates in the same direction of the input field, and the field $\Psi^-_{out}(t)$ as the reflected field along the waveguide, since it propagates in the opposite direction of the input field (fig.2.4). With these definitions in mind, it should be clear now that the modulus square of the transfer function $\chi^\mu_{\omega_1}$ represents the transmission coefficient T for $\mu = +$ and the reflection coefficient R for $\mu = -$. By taking the modulus square of $\chi^\mu_{\omega_1}$ for a particular value of μ allows us to calculate the reflection and transmission coefficients. Their expressions are the following:

$$T = \left|\chi^+_{\omega_1}\right|^2 = \left|1 - \frac{\sqrt{\gamma_+}\sqrt{\gamma_+}}{\frac{\gamma}{2} + i(\omega_0 - \omega_1)}\right|^2 = \frac{(\omega_0 - \omega_1)^2 + (\frac{\gamma}{2} - \gamma_+)^2}{\frac{\gamma^2}{4} + (\omega_0 - \omega_1)^2} \tag{2.54}$$

$$R = \left|\chi^-_{\omega_1}\right|^2 = \left|-\frac{\sqrt{\gamma_-}\sqrt{\gamma_+}}{\frac{\gamma}{2} + i(\omega_0 - \omega_1)}\right|^2 = \frac{\gamma_-\gamma_+}{\frac{\gamma^2}{4} + (\omega_0 - \omega_1)^2} \tag{2.55}$$

In the approximation that has been already done in the previous paragraph regarding the spontaneous emission phenomenon, it is possible to state that the coupling between the waveguide and the qubit is symmetric. This means that it is possible to state that $\gamma_- = \gamma_+ = \Gamma_0$ and, being the spontaneous emission rate the sum of the two of them, $\gamma = 2\Gamma_0$.

With this in mind, the final expression of the reflection and transmission coefficients is:

$$R = \left|\chi^{-}_{\omega_1}\right|^2 = \frac{\Gamma_0^2}{\Gamma_0^2 + (\omega_0 - \omega_1)^2} \tag{2.56}$$

$$T = \left|\chi^{+}_{\omega_1}\right|^2 = \frac{(\omega_0 - \omega_1)^2}{\Gamma_0^2 + (\omega_0 - \omega_1)^2} \tag{2.57}$$

The plot of the two coefficients function of the input field frequency is shown in fig.2.5. For what concerns the reflection coefficient R, it is clear that, with respect to the input frequency ω_1, it behaves like a Lorentzian function centered in ω_0 and FWHM proportional to Γ_0. On the other hand, the transmission coefficient T is constantly equal to the unity except for a deep well of FWHM proportional to the parameter Γ_0 around the resonance frequency. From fig.2.5 it is possible to notice that a particular phenomenon happens when the input frequency perfectly matches the qubit one. Indeed, in the resonance condition ($\omega_0 = \omega_1$) the input field is completely reflected back by the interaction with the qubit. This means that, at resonance, $R = 1$ and the transmission coefficient T is null, i.e. the probability of measuring the photon coming out from the left side of the waveguide (the one in which initially the field entered) is exactly 100%. As soon as the input frequency mismatches the emitter one, the probability of reflection soon decreases while the transmission event becomes more and more probable until, for $\omega_1 \gg \omega_0$, the incoming field does not interact with the qubit anymore. The phenomenon of total reflection is typical of the system we are considering, in which the qubit is linearly coupled with a superconducting waveguide. The interest related to this phenomenon relies in its possible applications in quantum gates controls. By driving externally the resonance frequency of the qubit, we could make the system acting like a switch where the incoming photons would totally pass or totally be reflected back [28].
The last thing that remains to do is the calculation of the output field amplitude probability coefficient $C^{\mu}_{\omega}(t_1)$. Its computation is quite straightforward, since it is just the inverse Fourier Transform of the output field $\Psi^{\mu}_{out}(t)$, which means that:

$$C^{\mu}_{\omega}(t_1) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} e^{i\omega(t-t_1)} \Psi^{+}_{in}(t) \chi^{\mu}_{\omega_1}\, dt \tag{2.58}$$

The only dependence on time is enclosed in the input field $\Psi^{+}_{in}(t)$. If we bring outside the integral all the non-time dependent terms and if we multiply and divide by the phase term $e^{i\omega t_0}$, we obtain:

$$C^{\mu}_{\omega}(t_1) = \chi^{\mu}_{\omega_1} e^{-i\omega(t_1-t_0)} \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \Psi^{+}_{in}(t) e^{i\omega(t-t_0)}\, dt \tag{2.59}$$

The integral is exactly the definition of the input field amplitude probability coefficient $C^{+}_{\omega}(t_0)$. The final expression of the output field amplitude probability is then:

$$C^{\mu}_{\omega}(t_1) = C^{+}_{\omega}(t_0) \chi^{\mu}_{\omega_1} e^{-i\omega(t_1-t_0)} \tag{2.60}$$

The physical meaning of this result is quite evident: the amplitude probability coefficient of the output field is given by the input field coefficient, defined in Eq.2.42, multiplied by the transfer function $\chi^{\mu}_{\omega_1}$ and by a phase term. The latter simply indicates that the time evolution of the electromagnetic field during the propagation along the waveguide contributes to the final result with a phase term, which corresponds to the phase accumulated by the photon from the initial instant of time t_0 to the final one t_1. The amplitude of the input coefficient, on the other hand, is modified only by the interaction with the qubit via the transfer function.
As it has been done before, to check the consistency of all the results found until now, it is necessary that the two amplitude probability coefficients $C_e(t)$ and $C^{\mu}_{\omega}(t_1)$, Eq.2.50 and Eq.2.60, satisfy the normalization condition in Eq.2.6. In the limit we have been working so far, i.e. in the limit of having a really narrow input field in the frequency domain, it is possible to neglect the contribution of the modulus square of the qubit amplitude probability coefficient inside Eq.2.6, being it multiplied by the parameter σ. Inserting the modulus square of Eq.2.60 inside the normalization condition leads to the following:

$$\sum_{\mu=\pm} \int \left|C^{\mu}_{\omega}(t)\right|^2 d\omega = \int \left|C^{+}_{\omega}(t_1)\right|^2 d\omega \left[\left|\chi^{-}_{\omega_1}\right|^2 + \left|\chi^{+}_{\omega_1}\right|^2\right] = 1 \tag{2.61}$$

It is easy to see that the normalization condition is verified. Indeed, the input field amplitude probability coefficient is normalized by definition, while the sum of the reflection coefficient $\left|\chi^{-}_{\omega_1}\right|^2$ and the transmission coefficient $\left|\chi^{+}_{\omega_1}\right|^2$ is always equal to the unity. All the results are then coherent with the formal theory developed so far for the particular system we have been considering.

3 | Photon scattering with two-photon interactions

In this chapter the analysis of the interaction between a two level quantum emitter and two-photons will be presented. This kind of light-matter interaction process has already been observed in many other cases. For example, it has been implemented in cavity QED experiments or simply in free space. However, in all those cases, two-photon interaction has been possible to implement only as a second order process. A direct consequence of this is that the coupling strength of the process is really small, requesting often high driving to be observed properly. On the other side, in the context of waveguide QED, it is possible to implement and then study this interaction as a first order process, without needing any driving. A simple way to do this is by using superconducting quantum circuits [29]. Indeed, with a proper engineerization of the circuit showed and analyzed briefly in paragraph 1.7, it is possible to implement a nonlinear coupling between the waveguide and the circuit itself. The detailed analysis of how to do this coupling in a practical way will not be presented in this work. Our main goal was to be able to define a proper model that could describe this two-photon nonlinear coupling in the context of waveguide QED. Indeed, in the following paragraphs will be presented both the theoretical model, mainly focusing on input-output theory, and its application to the two cases of interest, i.e. spontaneous emission and two-photon scattering.

3.1. System dynamics

The system that we will be considering from now on is quite similar to that already seen in chapter 2. It is constituted by a two level quantum emitter situated in the center of the waveguide and interacting with two input fields. The qubit in this particular case could be implemented using the superconducting quantum circuit in fig.1.6. Its two energy levels will be labeled as $|g\rangle$ and $|e\rangle$ for the ground and the excited state respectively. Regarding the electromagnetic fields propagating along the waveguide, they will be labeled as $|0_\omega, 0_{\omega'}\rangle$ and $|1_\omega, 1_{\omega'}\rangle$ depending on the number of photons inside them. In this case the maximum

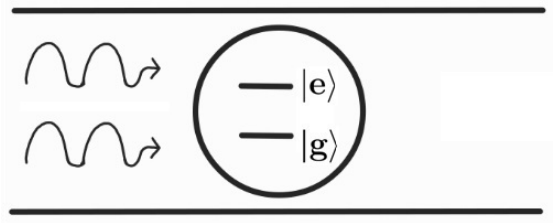

Figure 3.1: System scheme: a two level artificial atom inside a waveguide interacting with two incoming electromagnetic fields.

number of photons that we can have inside the waveguide is equal to 2. In addition to that, we have to consider the direction of propagation of each mode, defined by the two indexes μ and μ'. The convention remains the same: $+$ means propagation from left to right; - propagation from right to left. Of course, in principle the directions of propagation of the two fields could be different. A simple scheme of the system described can be seen in fig.3.1. In order to study the time evolution of the system, it is necessary to write the full expression of the Hamiltonian operator related to it. Since here the light-matter interaction involves the simultaneously absorption or emission of two photons, the Jaynes-Cummings model is no more suitable. Indeed, in the case of two-photon interaction, it is necessary to switch to the Two-photon Quantum Rabi model. Here, we have generalized its expression to adapt it to the waveguide QED context in which we are working. Said so, the Hamiltonian operator describing the system in fig.3.1 is defined as:

$$\hat{\mathbf{H}} = \omega_0 \hat{\sigma}^+ \hat{\sigma}^- + \sum_{\mu=\pm} \int \omega (\hat{a}^\mu_\omega)^\dagger \hat{a}^\mu_\omega \, d\omega + \sum_{\mu,\mu'=\pm} \iint (g^{\mu\mu'}_{\omega\omega'})^* \hat{\sigma}^+ \frac{\hat{a}^\mu_\omega \hat{a}^{\mu'}_{\omega'}}{\sqrt{2}} \, d\omega d\omega' + h.c \tag{3.1}$$

In Eq.3.1 ω_0 is, as usual, the qubit characteristic frequency defined as the difference of the energies of its two lowest eigenstates, while $\hat{\sigma}^+$ and $\hat{\sigma}^-$ are the ladder operators for the artificial atom. For this new system considered here, since there is the presence of two photons, inside the expression of the Hamiltonian operator it is necessary to have fields ladders operators for each of them. In particular with the operators $(\hat{a}^\mu_\omega)^\dagger$ and $\hat{a}^\mu_\omega$ we refer to the particular photon with frequency ω and direction of propagation μ. On the other side, with the operators $(\hat{a}^{\mu'}_{\omega'})^\dagger$ and $\hat{a}^{\mu'}_{\omega'}$ we refer to the particular photon with frequency ω' and direction of propagation μ'. Of course, the effects of these operators on the field state is simply the creation or the destruction of two photons:

$$(\hat{a}^\mu_\omega)^\dagger (\hat{a}^{\mu'}_{\omega'})^\dagger \left|0_\omega, 0_{\omega'}\right\rangle = \left|1^\mu_\omega, 1^{\mu'}_{\omega'}\right\rangle \qquad \hat{a}^\mu_\omega \hat{a}^{\mu'}_{\omega'} \left|1^\mu_\omega, 1^{\mu'}_{\omega'}\right\rangle = \left|0_\omega, 0_{\omega'}\right\rangle \tag{3.2}$$

The parameter $g^{\mu\mu'}_{\omega\omega'}$ is the coupling strength of the two-photon interaction. It depends

in principle on both the possible fields frequencies, ω and ω', and on the two possible directions of propagation, μ and μ'.
By looking at Eq.3.1, we can see that it is constituted by different terms. The first two terms can be grouped in the following expression:

$$\hat{H}_0 = \omega_0 \hat{\sigma}^+ \hat{\sigma}^- + \sum_{\mu=\pm} \int \omega (\hat{a}^\mu_\omega)^\dagger \hat{a}^\mu_\omega \, d\omega \tag{3.3}$$

This term represents the total energy of the system when the interaction between the qubit and the two photons is turned off. The first term of Eq.3.3 is the emitter energy where the origin of the energy reference system has been put in correspondence of the ground state eigenvalue. The second term of the non interacting Hamiltonian operator represents the total energy of the electromagnetic fields expressed as the sum in all the possible directions of the energy of all the possible supported modes. The last term in Eq.3.1 corresponds, on the other side, to the interaction operator:

$$\hat{H}_I = \sum_{\mu,\mu'=\pm} \iint (g^{\mu\mu'}_{\omega\omega'})^* \hat{\sigma}^+ \frac{\hat{a}^\mu_\omega \hat{a}^{\mu'}_{\omega'}}{\sqrt{2}} \, d\omega d\omega' + h.c \tag{3.4}$$

While Eq.3.3 is exactly what one would obtain in the case of single photon interaction (it is simply the total energy of the two uncoupled objects), the expression of the interaction Hamiltonian operator is fundamentally different. It is indeed dependent on a new coupling parameter $g^{\mu\mu'}_{\omega\omega'}$, which is function of two different frequencies and directions of propagation. In addition to that, it is easy to see that to each atomic transition, mediated by the atomic ladder operators $\hat{\sigma}^-$ and $\hat{\sigma}^+$, correspond two field ladder operators $(\hat{a}^\mu_\omega)^\dagger (\hat{a}^{\mu'}_{\omega'})^\dagger$ and $\hat{a}^\mu_\omega \hat{a}^{\mu'}_{\omega'}$ respectively. This means that it is possible to have only two different situations: the atom in the ground state $|g\rangle$ and the field in the excited state $\left|1^\mu_\omega, 1^{\mu'}_{\omega'}\right\rangle$ or the atom in the excited state $|e\rangle$ and the field in the ground state $|0_\omega, 0_{\omega'}\rangle$. It is important to point out that in writing down the expression of the Hamiltonian operator in Eq.3.1, we have performed the same approximations made in the single photon interaction case: the dipolar approximation and the rotating wave approximation (RWA). Indeed, here we are working in the strong coupling regime where the coupling parameter is taken greater than the losses outside the waveguide but much smaller with respect to the frequencies involved in the interaction. This translates into having the following condition for the coupling strength:

$$k < g \ll \omega \tag{3.5}$$

Due to the RWA the highly rotating terms have been discarded and only the terms in Eq.3.4 are preserved. Since the terms that apparently do not respect the law of conservation of the energy are not considered, we can write the excitation number operator as:

$$\hat{n}_2 = 2\sum_{\mu=\pm}\int (\hat{a}^{\mu}_{\omega})^{\dagger}\hat{a}^{\mu}_{\omega}d\omega + \hat{\sigma}^{+}\hat{\sigma}^{-} \tag{3.6}$$

where in this case, since to each atomic excitation correspond two field excitations, a factor 2 is needed to take into account that the weight of each field excitation is half of that of the atom. All the previous considerations make it evident that also in this two-photon interaction case the excitation number does not change in time. This means that the operators $\hat{\mathbf{H}}$ and $\hat{n}_2$ do commute and that we cannot exit from the Hilbert space defined by the operator $\hat{n}_2$. It must be clear that all what has been said until this point regarding the two-photon interaction holds only when this kind of interaction is the first order one. As it has been remarked before, this goal can be achieved, for example, by using superconducting quantum circuits coupled in a nonlinear way to the waveguide.
In addition to the Hamiltonian operator in Eq.3.1, we could write down the expression for the general state of the system considered. Indeed, it is possible to generalize the Wigner-Weisskopf ansatz to this new two-photon situation. In this way the general expression of the state of the system is given by:

$$|\Phi(t)\rangle = \left[C_e(t)\hat{\sigma}^{+} + \sum_{\mu,\mu'=\pm}\iint C^{\mu\mu'}_{\omega\omega'}(t)\frac{(\hat{a}^{\mu}_{\omega})^{\dagger}(\hat{a}^{\mu'}_{\omega'})^{\dagger}}{\sqrt{2}}\,d\omega d\omega'\right]|\mathbf{0}\rangle \tag{3.7}$$

where $|\mathbf{0}\rangle = |g\rangle\,|0_{\omega}, 0_{\omega'}\rangle$ is the ground state of the two dimensional Hilbert subspace obtained as the tensor product between the possible states of the qubit and the incoming fields. As already seen in the single photon interaction case, since the RWA holds, it is licit to express the general state of the system as the linear superposition of the possible states in which the system can be observed. Each of these states is then weighted by its own time-dependent amplitude probability coefficient. For the atom, we have again $C_e(t)$ while for the field we consider $C^{\mu\mu'}_{\omega\omega'}(t)$, which now depends in principle on two different frequencies and two different directions of propagation. For what concerns the field amplitude probability coefficient, it is necessary to add that it must be symmetric with respect to the photon swapping. With this, we mean that, being the photons propagating along the waveguide indistinguishable and being them bosonic particles, changing one with the other must not change anything in the formulation and in the results of the problem. This

leads to the following new additional property for the field weight:

$$C^{\mu\mu'}_{\omega\omega'}(t) = C^{\mu'\mu}_{\omega'\omega}(t) \tag{3.8}$$

The meaning of these weights is exactly the same that has already been discussed in chapter 2: their modulus square represents the probability distribution function of finding the atom or the fields in the excited states in a certain instant of time t. The sum of these probabilities must be equal to one to have a well posed problem, which leads to the following normalization condition for the two-photon interaction phenomenon:

$$|C_e(t)|^2 + \sum_{\mu.\mu'=\pm} \iint \left|C^{\mu\mu'}_{\omega\omega'}(t)\right|^2 d\omega = 1 \tag{3.9}$$

Again, in order to study the dynamics of the system showed in fig.3.1, we need to find out the explicit expression for the amplitude probability functions, since the time dependence of the system state in Eq.3.7 is enclosed in them. In the following paragraphs, we will try to outline the main steps that brought us to obtaining the seeked expressions for $C_e(t)$ and $C^{\mu\mu'}_{\omega\omega'}(t)$, with particular focus on the formulation of a general input-output theory for the two-photon interaction situation.

3.2. Formal solutions for the amplitude probability coefficients

At this point, in order to obtain the expressions for the two amplitude probability functions, we will follow the same method illustrated in chapter 2 when we were dealing with single photon interaction. Firstly, we start from the TDSE that we report again here for simplicity:

$$i\frac{\partial\,|\Phi(t)\rangle}{\partial t} = \hat{H}\,|\Phi(t)\rangle \tag{2.7}$$

Also in this case, we have both the expressions of the Hamiltonian operator and of the general state of the system. It is possible then to insert Eq.3.1 and Eq.3.7 inside the TDSE and solve this differential equation for the two unknowns $C_e(t)$ and $C^{\mu\mu'}_{\omega\omega'}(t)$. As one could imagine, the fact that now the coupling between the two level artificial atom and the electromagnetic fields is nonlinear (indeed $\hat{H}_I \propto [(\hat{a})^\dagger + \hat{a}]^2$) makes the solution of Eq.2.7 much more complicated than that of the linear coupling case.

Let us now try to expand the two members in the TDSE. Making explicit the right term in Eq.2.7 leads to the following:

$$\begin{aligned}\hat{H}\left|\Phi(t)\right\rangle &= \hat{H}\Big[C_e(t)\hat{\sigma}^+ + \sum_{\mu,\mu'=\pm}\iint C^{\mu\mu'}_{\omega\omega'}(t)\frac{(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger}{\sqrt{2}}\,d\omega d\omega'\Big]\left|\mathbf{0}\right\rangle = \\ &= C_e(t)\hat{H}\left|e\right\rangle\left|0_\omega, 0_{\omega'}\right\rangle + \sum_{\mu,\mu'=\pm}\iint C^{\mu\mu'}_{\omega\omega'}(t)\hat{H}\frac{(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger}{\sqrt{2}}\,d\omega d\omega'\left|\mathbf{0}\right\rangle\end{aligned} \tag{3.10}$$

The last equation is constituted by two terms: the first one is given by the action of the Hamiltonian operator onto the system state $\left|e\right\rangle\left|0,0\right\rangle$ (corresponding to the atom in the excited state and the fields in the vacuum state); the second one is given by the action of the Hamiltonian operator onto the state $\left|g\right\rangle\left|1_\omega, 1_{\omega'}\right\rangle$ (fields in the 2-photon state and atom in the ground state). Focusing on the first term in Eq.3.10, we obtain simply:

$$\hat{H}\left|e\right\rangle\left|0_\omega, 0_{\omega'}\right\rangle = \omega_0\left|e\right\rangle\left|0_\omega, 0_{\omega'}\right\rangle + \sum_{\mu,\mu'=\pm}\iint (g^{\mu\mu'}_{\omega\omega'})^*\frac{\hat{a}^{\mu}_{\omega}\hat{a}^{\mu'}_{\omega'}}{\sqrt{2}}\,d\omega d\omega'\left|\mathbf{0}\right\rangle \tag{3.11}$$

This derives from the fact that applying the number operator $(\hat{a}^{\mu}_{\omega})^\dagger\hat{a}^{\mu}_{\omega}$ to the field ground state $\left|0_\omega, 0_{\omega'}\right\rangle$ gives as result zero since there are no photons in any particular mode. Similarly, since the qubit can be thought of as a perfect two level system, applying the rising operator $\hat{\sigma}^+$ to the excited state $\left|e\right\rangle$ gives again zero as a result. Thus, the only remaining terms are those showed in Eq.3.11.

On the other hand, the second term in Eq.3.10 needs much more attention. It is useful to first analyze what is the effect of the operator $\hat{H}(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger$ onto the system ground state $\left|\mathbf{0}\right\rangle$. In particular, remembering that, in principle the frequencies and the directions of propagation inside the expression of the Hamiltonian operator could be different with respect to the ones of the field ladder operators, we could write:

$$\begin{aligned}\hat{H}(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger\left|\mathbf{0}\right\rangle &= \omega_0\hat{\sigma}^+\hat{\sigma}^-(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger\left|\mathbf{0}\right\rangle + \sum_{\lambda=\pm}\int d\nu\nu(\hat{a}^{\lambda}_{\nu})^\dagger\hat{a}^{\lambda}_{\nu}(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger\left|\mathbf{0}\right\rangle + \\ &+ \sum_{\lambda,\lambda'=\pm}\iint\frac{d\nu d\nu'}{\sqrt{2}}(g^{\lambda\lambda'}_{\nu\nu'})^*\hat{\sigma}^+\hat{a}^{\lambda}_{\nu}\hat{a}^{\lambda'}_{\nu'}(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger\left|\mathbf{0}\right\rangle + \\ &+ \sum_{\lambda,\lambda'=\pm}\iint\frac{d\nu d\nu'}{\sqrt{2}}g^{\lambda\lambda'}_{\nu\nu'}\hat{\sigma}^-(\hat{a}^{\lambda}_{\nu})^\dagger(\hat{a}^{\lambda'}_{\nu'})^\dagger(\hat{a}^{\mu}_{\omega})^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger\left|\mathbf{0}\right\rangle\end{aligned} \tag{3.12}$$

In Eq.3.12, the first and the last terms are null. This is because we are applying the atom ladder operator $\hat{\sigma}^-$ to the state $|\mathbf{0}\rangle = |g\rangle\,|0_\omega, 0_{\omega'}\rangle$. For what concerns the other two terms, the second and the third ones, they are different from zero only if the following identities are satisfied:

$$\lambda = \mu \quad \lambda' = \mu' \quad \nu = \omega \quad \nu' = \omega' \tag{3.13}$$

These constrains on the values of ν and λ comes directly from the properties of the field ladders operators explicitated in Eq.2.11.
After all the above simplifications we are able now to write the final expression for the right term in Eq.2.7 as :

$$\begin{aligned}\hat{H}\,|\Phi(t)\rangle = C_e(t)\omega_0\,|e\rangle\,|0_\omega, 0_{\omega'}\rangle + C_e(t)\sum_{\mu,\mu'=\pm}\iint (g^{\mu\mu'}_{\omega\omega'})^*\frac{\hat{a}^\mu_\omega\hat{a}^{\mu'}_{\omega'}}{\sqrt{2}}\,d\omega d\omega'\,|\mathbf{0}\rangle + \\ + \sum_{\mu,\mu'=\pm}\iint \frac{d\omega d\omega'}{\sqrt{2}}(\omega+\omega')C^{\mu\mu'}_{\omega\omega'}(t)(\hat{a}^\mu_\omega)^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger\,|\mathbf{0}\rangle + \\ + \sum_{\mu,\mu'=\pm}\iint d\omega d\omega'(g^{\mu\mu'}_{\omega\omega'})^* C^{\mu\mu'}_{\omega\omega'}(t)\,|e\rangle\,|0_\omega, 0_{\omega'}\rangle\end{aligned} \tag{3.14}$$

Here, for the sake of simplicity, the complete calculations and simplifications done to pass from Eq.3.12 and Eq.3.11 to the final expression in Eq.3.14 were omitted.
Returning back to the TDSE, since now the right term of the equation has been computed, it remains only to expand the left one. In order to do so, it is necessary to operate a time derivative onto the system state in Eq.3.7. Being the amplitude probability coefficients the only time-dependent terms, the calculation is straightforward. Indeed, we directly obtain:

$$i\frac{\partial\,|\Phi\rangle}{\partial t} = i\frac{\partial C_e(t)}{\partial t}\,|e\rangle\,|0,0\rangle + \frac{i}{\sqrt{2}}\sum_{\mu,\mu'=\pm}\iint \frac{\partial C^{\mu\mu'}_{\omega\omega'}(t)}{\partial t}(\hat{a}^\mu_\omega)^\dagger(\hat{a}^{\mu'}_{\omega'})^\dagger d\omega d\omega'\,|\mathbf{0}\rangle \tag{3.15}$$

Now that all the terms in Eq.2.7 have been calculated explicitly, it is enough to solve the partial differential equation to find the time dependent coefficients. What it is necessary to do then is to obtain two coupled linear differential equations from the TDSE. This could be done by exploiting the orthonormality properties of the state eigenvectors. Indeed, if we multiply first by the bra $\langle e|\,\langle 0_\omega, 0_{\omega'}|$ and then by the bra $\langle g|\,\langle 0_\omega, 0_{\omega'}|$ Eq.2.7, a set of two coupled differential equations for the time-dependent coefficients $C_e(t)$ and $C^{\mu\mu'}_{\omega\omega'}(t)$ can be obtained.

In particular, after all the simplifications that here are neglected for brevity, the resulting system can be written as:

$$i\dot{C}_e(t) = \omega_0 C_e(t) + \sum_{\mu,\mu'=\pm} \iint d\omega d\omega' (g^{\mu\mu'}_{\omega\omega'})^* C^{\mu\mu'}_{\omega\omega'}(t)$$

$$i\dot{C}^{\mu\mu'}_{\omega\omega'}(t) = (\omega + \omega') C^{\mu\mu'}_{\omega\omega'}(t) + g^{\mu\mu'}_{\omega\omega'} C_e(t) \tag{3.16}$$

At a first look, the system of linear coupled differential equations is quite similar to the one already seen in chapter 2. However, there is a fundamental difference between the two: while in the single photon interaction case the problem was one dimensional, here it becomes a bidimensional one due to the fact that now both the field amplitude probability and the coupling parameter depend on two different frequencies ω and ω'. This bidimensionality in the frequency dependence adds of course more complexity to the problem, making it even more difficult to solve. As we did in the previous chapter for the single photon interaction case, in order to solve the system 3.16, we will follow the most direct way. To begin with, we compute the differential equation for the field amplitude probability coefficient $C^{\mu\mu'}_{\omega\omega'}(t)$ and then, we substitute the result in the other equation for the qubit amplitude probability $C_e(t)$. Indeed, the second equation of the system 3.16 can be solved immediately using the usual explicit formula for the complete linear differential equations. The final result is the following:

$$C^{\mu\mu'}_{\omega\omega'}(t) = C^{\mu\mu'}_{\omega\omega'}(t_0) e^{-i(\omega+\omega')(t-t_0)} - i g^{\mu\mu'}_{\omega\omega'} e^{-i(\omega+\omega')(t-t_0)} \int_{t_0}^{t} C_e(\tau) e^{i(\omega+\omega')(\tau-t_0)}\, d\tau \tag{3.17}$$

where the time integration has been done from the initial instant of time t_0 to the generic final instant of time t, with $t > t_0$. The field amplitude probability function $C^{\mu\mu'}_{\omega\omega'}(t_0)$ can be seen as the amplitude probability of having two excited modes in the waveguide at frequencies ω and ω' respectively in an instant of time t_0 far before the event of interaction with the emitter.
Inserting the result obtained in Eq.3.17 inside the first equation of the system 3.16 allows us to find a linear differential equation in the only unknown $C_e(t)$:

$$\begin{aligned} i\dot{C}_e(t) = \omega_0 C_e(t) + \sum_{\mu,\mu'=\pm} \iint d\omega d\omega' (g^{\mu\mu'}_{\omega\omega'})^* C^{\mu\mu'}_{\omega\omega'}(t_0) e^{-i(\omega+\omega')(t-t_0)} + \\ - i \sum_{\mu,\mu'=\pm} \iint \left[(g^{\mu\mu'}_{\omega\omega'})^* g^{\mu\mu'}_{\omega\omega'} e^{-i(\omega+\omega')(t-t_0)} \int_{t_0}^{t} C_e(\tau) e^{i(\omega+\omega')(\tau-t_0)}\, d\tau \right] d\omega d\omega' \end{aligned} \tag{3.18}$$

This last equation for the qubit amplitude probability coefficient depends basically on three terms. While the first of them is just the coefficient $C_e(t)$ weighted by the qubit characteristic frequency (term that was present also in Eq.2.16 in the single photon interaction case), the other two require much more attention since they both depend on a double integral over all the possible modes supported by the waveguide. The first integral in Eq.3.18 could be easily calculated if the initial conditions related to the input fields are defined and an explicit expression of the two-photon interaction coupling strength is given. The second integral in Eq.3.18 (called from now on $\mathbf{I}_2$), on the other hand, can be simplified further. Firstly, we operate a change of variables inside the integral over time:

$$T = t - \tau \qquad dT = -d\tau \tag{3.19}$$

In this way $\mathbf{I}_2$ can be rewritten as:

$$\mathbf{I}_2 = -i \sum_{\mu,\mu'=\pm} \iint \left[\left| (g^{\mu\mu'}_{\omega\omega'}) \right|^2 \int_0^{t-t_0} C_e(t-T) e^{-i(\omega+\omega')T}\, dT \right] d\omega d\omega' \tag{3.20}$$

At this point another change of variables is performed but, in this case, it regards the double integral over the possible frequencies. In particular we pass from using the two frequencies ω and ω' to using as new variables the ones obtained from their linear combination. These variables can be defined as follows:

$$\bar{\omega} = \omega' + \omega \qquad \Delta = \omega' - \omega \tag{3.21}$$

Of course, since now we are working in a two dimensional space in the frequency domain, when performing a change of variables, we must also compute the Jacobian matrix associated to this change. This step is needed to properly rescale the differential area related to the function inside the integral. In our case the Jacobian matrix associated to $\bar{\omega}$ and Δ is given by:

$$J = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.22}$$

whose determinant is equal to 2. With this in mind, the differentials elements of the new variables are related to the previous ones by the following:

$$d\omega d\omega' = \frac{d\bar{\omega} d\Delta}{2} \tag{3.23}$$

Before going on with the calculations, we perform here an approximation similar to the one made for the coupling parameter in the single photon interaction case. The two-photon interaction coupling strength, which now has to be written as a function of the new variables $\bar{\omega}$ and Δ, is taken constant over a band of frequencies around the characteristic frequency of the atom ω_0 in the sum direction. This means that the coupling parameter in this case does not depend on $\bar{\omega}$ but it depends only on the frequency difference Δ. Performing this important approximation is indeed equivalent of saying that the coupling between the two-level quantum emitter and the electromagnetic fields in the waveguide does not change in a relevant way for a band of frequencies around the resonance condition. In a two-photon interaction framework it can be formally be expressed as:

$$g^{\mu\mu'}_{\bar{\omega},\Delta} \approx g^{\mu\mu'}_{\Delta} \qquad if \qquad \bar{\omega} \approx \omega \tag{3.24}$$

Indeed, in the system we are considering $\bar{\omega} = \omega + \omega'$ can be seen as the total frequency interacting with the qubit. On the other hand, it is not possible to eliminate also the dependence on the Δ variable since it gives us information about the capability of the atom to interact to very different frequencies. This concept will be analyzed better in the following paragraphs.
With these simplifications in mind, we can return back to the expression of $\mathbf{I}_2$ and rewrite it as:

$$\mathbf{I}_2 = -\frac{i}{2} \sum_{\mu,\mu'=\pm} \int_{-\infty}^{+\infty} \left[\left| (g^{\mu\mu'}_{\Delta}) \right|^2 \int_0^{t-t_0} C_e(t-T) \int_0^{+\infty} e^{-i\bar{\omega}T}\, d\bar{\omega} \right] dT d\Delta \tag{3.25}$$

If we first calculate the integral over all the possible sum of the frequencies $\bar{\omega}$, it is easy to recognize in the last equation the Fourier Transform of the Heaviside function, whose result is already known. It is indeed equal to a Dirac delta in time centered in $T = 0$ weighted by π (this weight comes out when ω is used instead of the actual real frequency $\nu = \omega/2\pi$) and an imaginary part that would lead to the Lamb shift phenomenon. However, this frequency shift will be neglected in the following.
After having applied all the above mentioned simplifications and after having used the Dirac delta distribution properties, finally the most simplified expression for $\mathbf{I}_2$ is:

$$\mathbf{I}_2 = -\frac{i}{2} C_e(t) \sum_{\mu,\mu'=\pm} \int_{-\infty}^{+\infty} \pi \left| (g^{\mu\mu'}_{\Delta}) \right|^2 d\Delta = -iC_e(t)\frac{\gamma}{2} \tag{3.26}$$

where γ is the qubit spontaneous emission rate. The value of the parameter γ gives the

probability per unit time of having an atomic emission of a pair of photons at frequencies ω and ω' after an absorption event. Moreover, the higher the value of γ sooner the atom will emit the photon and shorter will be the period of time during which the atom can be found in the excited state $|e\rangle$. The explicit expression of the qubit spontaneous emission rate in this case of two-photon interaction is:

$$\gamma = \sum_{\mu,\mu'=\pm} \int_{-\infty}^{+\infty} \gamma_{\Delta}^{\mu\mu'} \, d\Delta \tag{3.27}$$

with $\gamma_{\Delta}^{\mu\mu'} = \pi \left| g_{\Delta}^{\mu\mu'} \right|^2$. The expression of γ in Eq.3.27 can be compared directly with the qubit spontaneous emission rate in the case of single photon interaction in Eq.2.22. In the latter situation, its expression was simply given by the sum of all the possible values of the index μ of the parameter γ^{μ}, which was taken approximately constant over a band of frequencies centered around ω_0. In this new situation of two-photon interaction, the expression of γ is much more complicated. Its expression is given by the sum of all the possible pairs of the indexes μ and μ' of the integral over the frequencies difference Δ of the parameter $\gamma_{\Delta}^{\mu\mu'}$, directly linked to the coupling strength. Being the problem now bidimensional in the frequency domain, although the coupling parameter has been taken constant in the frequencies sum variable, it still has a dependence on the frequency difference. Since to retrieve the total qubit spontaneous emission rate γ we need to perform an integral on the possible values of Δ, there will be a constrain on the shape of $\gamma_{\Delta}^{\mu\mu'}$. Indeed, $\gamma_{\Delta}^{\mu\mu'}$ needs to be a rapidly decaying function as soon as the two frequencies become too different one with respect to the other so to have a convergent integral. As it was said in the previous paragraphs, the coupling parameter is directly linked to the superconducting circuit scheme, so it is possible to engineering it properly to satisfy the above constraints.

Now that all the terms in Eq.3.18 have been simplified, we are able to write a more compact expression for the atom amplitude probability differential equation:

$$i\dot{C}_e(t) = \omega_0 C_e(t) - iC_e(t)\frac{\gamma}{2} + \sum_{\mu,\mu'=\pm} \iint \frac{d\bar{\omega}d\Delta}{2} (g_{\Delta}^{\mu\mu'})^* C_{\bar{\omega}\Delta}^{\mu\mu'}(t_0) e^{-i\bar{\omega}(t-t_0)} \tag{3.28}$$

In the last term of the previous equation we can recognize the Fourier Transform of the field amplitude probability coefficient, evaluated in the instant of time t_0, in the variable $\bar{\omega}$. Analogously to what we did in the case of single photon interaction, we define here

the input field in the time domain for the two-photon interaction as:

$$\Psi_{in}^{\mu\mu'\Delta}(t) = \frac{1}{\sqrt{2\pi}} \int C_{\bar{\omega},\Delta}^{\mu\mu'}(t_0) e^{-i\bar{\omega}(t-t_0)}\, d\bar{\omega} \tag{3.29}$$

The definition of the input field in the time domain in Eq.3.29 is a straightforward generalization of the one in Eq.2.24. Even though it is defined as a Fourier Transform in the variable $\bar{\omega}$, with the change of variable $\bar{\omega} = \omega + \omega'$, it can be seen as a two dimensional Fourier Transform operation. This is indeed what we could expect from the moment in which the problem we are facing has become bidimensional in the frequency domain. Another important element that needs to be pointed out is the fact that $\Psi_{in}^{\mu\mu'\Delta}(t)$ not only depends on time but also on the frequency difference Δ. This dependence, not observable in the single photon interaction case, adds, of course, complexity to the solutions of the problem and could probably lead to new interesting aspects.
Said so, we can now write the final expression for the atom amplitude probability coefficient:

$$\dot{C}_e(t) = -i(\frac{\gamma}{2} + \omega_0)C_e(t) - \frac{i}{2} \sum_{\mu,\mu'=\pm} \int_{-\infty}^{+\infty} \sqrt{\gamma_\Delta^{\mu\mu'}}\, \Psi_{in}^{\mu\mu'\Delta}(t)\, d\Delta \tag{3.30}$$

In this case, it is not enough to define the initial conditions of the system to solve Eq.3.30. It is necessary also to define properly the coupling parameter $\gamma_\Delta^{\mu\mu'}$, now dependent on Δ. The need of having an explicit expression for the coupling opens here to new interesting possibilities, since it adds, in a certain way, a new degree of freedom to the solutions of the problem ($\gamma_\Delta^{\mu\mu'}$ depends on the circuit scheme). Ultimately, in order to obtain an explicit solution for the field amplitude probability, we need to generalize the input-output theory expressed in Eq.2.32 to this case of two-photon interaction.

3.3. Input-output theory with two-photon interaction

As we already know, the input-output relation is an equation that links in a linear way the field distribution in input, in the two-photon interaction case defined by $\Psi_{in}^{\mu\mu'\Delta}(t)$ in Eq.3.29, and the field distribution in output, when the interaction has already happened, $\Psi_{out}^{\mu\mu'\Delta}(t)$. This last term could be defined as follows:

$$\Psi_{out}^{\mu\mu'\Delta}(t) = \frac{1}{\sqrt{2\pi}} \int C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1) e^{-i\bar{\omega}(t-t_1)}\, d\bar{\omega} \tag{3.31}$$

with t_1 an instant of time much grater than t and t_0, i.e. $t_1 > t > t_0$. As we could expect, the output field distribution is nothing but the Fourier Transform of the field amplitude probability coefficient in the variable $\bar{\omega}$ evaluated at the instant of time t_1.
In order to obtain the generalized expression to the two-photon interaction case of the input-output relation, we proceed in the same way as we did in section 2.3. In the previous paragraph the second equation in system 3.16 has been integrated forward in time from the past instant t_0 to the generic instant t. Now what we do is to take that equation and integrate it backward in time, from the generic instant t to the future instant of time t_1. The result of this integration is the following:

$$C^{\mu\mu'}_{\bar{\omega},\Delta}(t) = C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)e^{-i\bar{\omega}(t-t_1)} + ig^{\mu\mu'}_{\Delta}e^{-i\bar{\omega}(t-t_1)}\int_t^{t_1} C_e(\tau)e^{i\bar{\omega}(\tau-t_1)}\,d\tau \tag{3.32}$$

With the variable substitution $T = \tau - t$ the last equation can be simplified further as:

$$C^{\mu\mu'}_{\bar{\omega},\Delta}(t) = C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)e^{-i\bar{\omega}(t-t_1)} + ig^{\mu\mu'}_{\Delta}e^{-i\bar{\omega}(t-t_1)}\int_0^{t_1-t} C_e(T+t)e^{i\bar{\omega}T}\,dT \tag{3.33}$$

Performing the same change of variables in the Eq.3.17 leads to:

$$C^{\mu\mu'}_{\bar{\omega},\Delta}(t) = C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)e^{-i\bar{\omega}(t-t_0)} - ig^{\mu\mu'}_{\Delta}e^{-i\bar{\omega}(t-t_0)}\int_0^{t-t_0} C_e(T-t)e^{i\bar{\omega}T}\,dT \tag{3.34}$$

The two equations for the field amplitude probability $C^{\mu\mu'}_{\bar{\omega},\Delta}(t)$, i.e. Eq.3.33 and Eq.3.34, can be compared. If we multiply both of them by the constant $1/\sqrt{2\pi}$ and integrate both of them in the frequency sum variable $\bar{\omega}$, it is possible to identify, after these operations, the exact definitions of the input and output fields in the time domain in the two-photon interaction case:

$$\begin{aligned}\Psi^{\mu\mu'\Delta}_{out}(t) + \frac{ig^{\mu\mu'}_{\Delta}}{\sqrt{2\pi}}\int_0^{+\infty}\left[\int_0^{t_1-t} C_e(T+t)e^{i\bar{\omega}T}\,dT\right]d\bar{\omega} = \\ \Psi^{\mu\mu'\Delta}_{in}(t) - \frac{ig^{\mu\mu'}_{\Delta}}{\sqrt{2\pi}}\int_0^{+\infty}\left[\int_0^{t-t_0} C_e(T-t)e^{i\bar{\omega}T}\,dT\right]d\bar{\omega}\end{aligned} \tag{3.35}$$

The two integrals in frequency domain that appear in the last identity are formally equal to the integral in Eq.3.25: they are simply equal to a Dirac delta centered in $T = 0$. If we proceed in the same way as it has been done in the previous section, we can simply write:

$$\Psi^{\mu\mu'\Delta}_{out}(t) + \frac{ig^{\mu\mu'}_{\Delta}}{\sqrt{2\pi}}C_e(t) = \Psi^{\mu\mu'\Delta}_{in}(t) - \frac{ig^{\mu\mu'}_{\Delta}}{\sqrt{2\pi}}C_e(t) \tag{3.36}$$

At this point, after some trivial mathematical simplifications and by keeping in mind the definition of the qubit spontaneous emission rate in Eq.3.27, we are able to write a direct relation that gives us the expression of the output field in time domain as a function of the input field distribution, the qubit amplitude probability coefficient and the coupling strength of the two-photon interaction. Finally, the input-output relation seeked is:

$$\Psi_{out}^{\mu\mu'\Delta}(t) = \Psi_{in}^{\mu\mu'\Delta}(t) - i\sqrt{2\gamma_{\Delta}^{\mu\mu'}}C_e(t) \tag{3.37}$$

From the two-photon input-output relation it is easy to see that the output field in the time domain is given by the input field in the time domain plus a correction term dependent on the interaction with the emitter. This last term is indeed the product of the qubit amplitude probability function and the coupling parameter $\gamma_{\Delta}^{\mu\mu'}$. Apart from the different multiplicative factor in the square root of the last term, the main difference of the input-output relation in Eq.3.37 with respect to the same relation for the single photon interaction case in Eq.2.32 relies in its frequency dependence. Indeed both the input-output fields and the coupling parameter depend on the frequency difference Δ. More physical insight regarding these input-output fields could be found in Appendix B. In any case, thanks to the atom amplitude probability differential equation expressed in Eq.3.18 and the input-output relation in Eq.3.37 the problem becomes self consistent. Once the input fields and the particular profile of the coupling parameter have been defined, it is possible to calculate both the emitter amplitude probability $C_e(t)$ and the output field amplitude probability $C_{\bar{\omega},\Delta}^{\mu\mu'}(t)$, thus solving the problem considered.

3.4. Example: spontaneous emission

Until now we have been able to define a general set of equations that allow us to compute quite easily the two amplitude probabilities for two-photon interaction in waveguide QED context. In the following paragraphs we will investigate two cases of interest: the spontaneous emission and two-photon scattering. In particular, after having obtained the explicit expressions for the two amplitude probability coefficients $C_e(t)$ and $C_{\bar{\omega},\Delta}^{\mu\mu'}(t)$, we will analyze and compare them with the same solutions already found in chapter 2 for single photon interaction.

Let us start in this paragraph with the example of spontaneous emission. For the particular system of two-photon interaction considered (fig.3.1), the situation could be schematized as follows: the artificial atom initially in the ground state $|g\rangle$ has absorbed two incoming photons at certain frequencies ω_{in_1} and ω_{in_2} in the waveguide; the qubit, now in the excited state $|e\rangle$, emits two photons at certain frequencies ω_{out_1} and ω_{out_2} after a

Figure 3.2: Left: at time t_0 the qubit is in the excited state $|e\rangle$. Right: after a certain amount of time the qubit spontaneously emits two photons and relaxes in the ground state $|g\rangle$. In principle the emitted photons could propagate in different directions with respect to that showed here with a certain probability.

certain time t, usually of the order of the excited level lifetime (fig.3.2). The first step that has to be done for finding the two amplitude probability coefficients is defining the initial conditions of the system, i.e. the value of the qubit and of the fields coefficients at t_0, with t_0 the instant of time when the artificial atom is in the state $|e\rangle$. Said so, the initial conditions of the system in this case can be expressed as:

$$C_e(t_0) = 1 \quad ; \quad C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0) = 0 \tag{3.38}$$

since there is no input field in the waveguide for $t = t_0$. Furthermore, being $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0) = 0$, it is easy to verify that also $\Psi^{\mu\mu'\Delta}_{in}(t)$, defined in Eq.3.29, is null. The direct consequence of these conditions is that now Eq.3.28 can be simplified further, becoming in this way directly integrable:

$$i\dot{C}_e(t) = (\omega_0 - i\frac{\gamma}{2})C_e(t) \tag{3.39}$$

The solution of the linear differential equation for the emitter amplitude probability coefficient is then:

$$C_e(t) = e^{-i\omega_0(t-t_0)}e^{-\frac{\gamma}{2}(t-t_0)}\mathcal{H}(t-t_0) \tag{3.40}$$

where $\mathcal{H}(t-t_0)$ is the Heaviside function centered in t_0. Also in this case, the Heaviside function has been added in Eq.3.40 to underline the fact that the expression of the coefficient $C_e(t)$ is the one calculated above only when we are looking at the system at an instant of time t grater than t_0. If we observe the system in an instant of time before t_0, we are putting ourselves in a situation different from that described by the initial conditions in Eq.3.38, i.e. in a situation where the input fields cannot be considered null. Consequently, for $t < t_0$, the calculations that permitted us to find Eq.3.40 do not hold anymore. If we compare the result in Eq.3.40 with the one obtained in the single photon interaction case in Eq.2.35, we will notice that they are basically identical. However, there is an important difference between the two, i.e. a different spontaneous emission

rate. This different definition for the two γ parameters results of course in a different slope of the decaying exponential function. Despite this consideration, the general shape of $C_e(t)$ remains unaltered.
Now that the qubit amplitude probability coefficient has been computed, it is possible to insert its expression inside the input output relation in Eq.3.37 to obtain the output field distribution. Its definition, since $\Phi_{in}^{\mu\mu'\Delta}(t) = 0$, is simply:

$$\Phi_{out}^{\mu\mu'\Delta}(t) = -i\sqrt{2\gamma_{\Delta}^{\mu\mu'}}e^{-i\omega_0(t-t_0)}e^{-\frac{\gamma}{2}(t-t_0)}\mathcal{H}(t-t_0) \tag{3.41}$$

From the previous expression, the field amplitude probability coefficient can be computed by an inverse Fourier Transform operation of Eq.3.31:

$$C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1) = -\frac{i\sqrt{2\gamma_{\Delta}^{\mu\mu'}}}{\sqrt{2\pi}}\int_0^{+\infty} e^{-i\omega_0(t-t_0)}e^{-\frac{\gamma}{2}(t-t_0)}e^{i\bar{\omega}(t-t_1)}\mathcal{H}(t-t_0)\,dt \tag{3.42}$$

The solution of the above integral is straightforward since the integrand is just an exponential function. The final expression of the field amplitude probability is given by:

$$C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1) = -\frac{i\sqrt{2\gamma_{\Delta}^{\mu\mu'}}}{\sqrt{2\pi}}\frac{e^{-i\bar{\omega}(t_1-t_0)}}{\frac{\gamma}{2}+i(\omega_0-\bar{\omega})} \tag{3.43}$$

where $\bar{\omega} = \omega + \omega'$ is the frequency sum of the spontaneously emitted radiation and $\omega_0 - \bar{\omega} = \Delta\omega$ the frequency detuning between the qubit characteristic frequency and the sum of the emitted photons frequencies.
As usual, it is interesting to take Eq.3.40 and Eq.3.43 and evaluate the modulus square of these coefficients. Indeed, the modulus square of the amplitude probabilities represents the probability distribution, function of time, which gives us information about how probable is to find the qubit or the field in the excited state respectively. The expression of the modula squared of $C_e(t)$ and $C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1)$ are:

$$|C_e(t)|^2 = e^{-\gamma(t-t_0)}\mathcal{H}(t-t_0) \tag{3.44}$$

$$\left|C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1)\right|^2 = \frac{1}{\pi}\frac{\gamma_{\Delta}^{\mu\mu'}}{\frac{\gamma^2}{4}+(\omega_0-\bar{\omega})^2} \tag{3.45}$$

For what concerns the atom probability distribution represented by $|C_e(t)|^2$, it can be seen that it is really similar to the modulus square in Eq.2.39, obtained in the case of single

photon interaction as it is defined by a decreasing exponential in time. This means that the probability of finding the qubit in the state $|e\rangle$ after a certain instant of time t grater than the initial one t_0 decreases exponentially as soon as we let the system evolve in time. The time constant related to this decrease is given by the reciprocal of the spontaneous emission rate parameter γ, which now, as we pointed out above, is defined in a different way (Eq.3.27). Furthermore, this parameter depends directly on the particular coupling between the superconducting circuit in fig.1.6 and the waveguide.
On the other hand, also in this two-photon interaction case, the probability distribution regarding the output field, does not depend on time. This derives again from the fact that through the input-output relation, what it is retrieved is the field amplitude probability coefficient evaluated in $t = t_1$, with t_1 much larger than t_0. Moreover, the only time dependence in $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$ is in the phase term that the photons acquire during the propagation along the waveguide from the initial instant of time t_0 to the final one t_1. The amplitude of the phase term of course becomes equal to the unity when the modulus square is taken. In addition to that, as it can be seen from Eq.3.45, $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$ depends on the frequency sum $\bar{\omega}$ and on the frequency difference Δ. This is indeed a direct consequence of the bidimensionality of the system 3.16 in the frequency domain and it will result in a much more complicated frequency distribution for the coefficient considered.
Before going on with the analyzes of the expression in Eq.3.45, it is necessary at this point to define properly the parameter $\gamma^{\mu\mu'}_{\Delta}$. Indeed, the expression of the modulus square of $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$ computed above, is generic, i.e. it holds for every possible profile of the coupling parameter that would satisfy the convergence constrain imposed by the integral in Eq.3.27. However, to obtain its specific value we must first study in details how the nonlinear superconducting circuit in fig.1.6 would couple to a waveguide. This, due to lack of time, has to be done yet. In the following, we will consider an arbitrary plausible shape for the coupling parameter. Of course, once its exact value will be computed, it would be possible to repeat the following analyzes regarding the examples considered. In the continue of this thesis work, we have chosen for the coupling parameter in the two-photon interaction case the following expression:

$$\gamma^{\mu\mu'}_{\Delta} = \frac{\Gamma^{\mu\mu'}_{0}}{\sqrt{2\pi}\sigma} e^{-\frac{\Delta^2}{2\sigma^2}} \tag{3.46}$$

It is defined as a normalized Gaussian function in the variable Δ, with a full width at half maximum (FWHM) proportional to σ. Its maximum value, reached when $\Delta = \omega - \omega' = 0$ is given by the constant $\Gamma^{\mu\mu'}_{0}$, which, at least in principle, depends on the particular direction of propagation of the two photons along the waveguide. However, as we did for

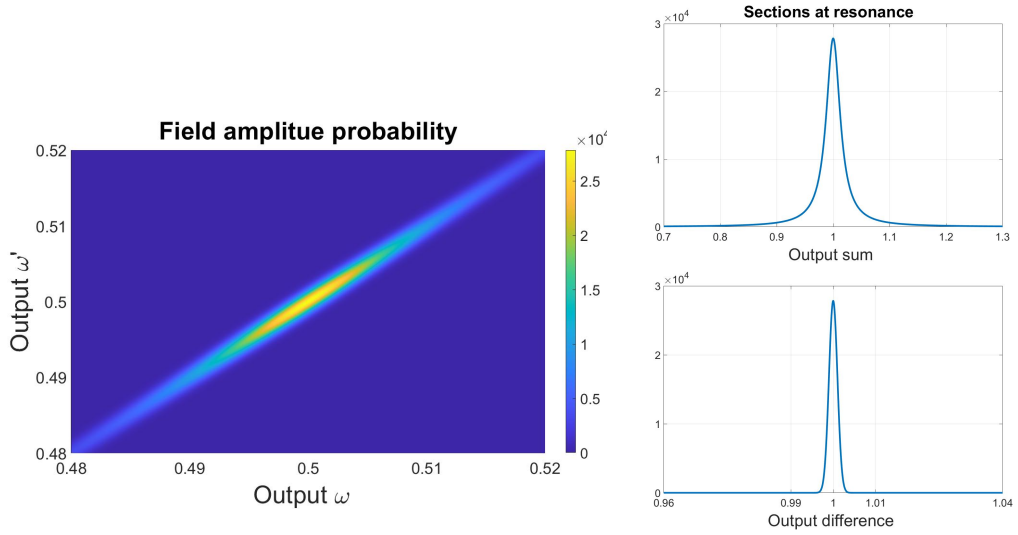

Figure 3.3: Left: 2D-colormap of the modulus square of the output field amplitude probability as a function of the two emitted frequencies ω and ω'. Right: sections of the 2D-colormap at resonance along the frequency sum $\bar{\omega}$ and frequency difference Δ directions. All the frequencies are expressed in units of ω_0, which has been set equal to 1. The coupling parameter Γ_0 has been set equal to $\omega_0/100$.

the single photon interaction case, we will consider a symmetric waveguide, i.e. there will not be any preferential direction of emission. This means that it is possible to eliminate the dependence on the indexes μ and μ' from the coupling parameter. With all the above considerations in mind, it is clear that the specific value for the qubit total spontaneous emission rate will be simply given by:

$$\gamma = 3\Gamma_0 \tag{3.47}$$

where the factor 3 derives from the fact that there are only three possible combinations of the direction indexes (remember that having $\mu\mu'$ or $\mu'\mu$ is the same). At this point, we can rewrite the expression of the modulus square of the field amplitude probability taking into account all the above assumptions. Its new expression (fig.3.3) is the following:

$$\left|C_{\bar{\omega},\Delta}(t_1)\right|^2 = \frac{1}{\sqrt{2\pi^3}} \frac{\Gamma_0 e^{-\frac{\Delta^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - \bar{\omega})^2} \tag{3.48}$$

To be more precise, the expression in Eq.3.48 is related to only one possible combination of the indexes μ and μ'. To retrieve the total probability distribution for the output field it is necessary to add another multiplicative factor of 3, but that would not change in any

case the frequency dependence and it is omitted here.
Now that the final expression for the modulus square of the field amplitude probability has been obtained, it is possible to continue with the analyzes of its frequency distribution. In fig.3.3 is showed a 2D-colormap of $C_{\bar{\omega},\Delta}(t_1)$ as a function of the two possible emitted frequencies in output, ω and ω', side by side to its sections along the directions identified by $\bar{\omega}$ and Δ at resonance. As it could be easily seen from the plots, the resonance condition in this two-photon interaction picture is given by having $\omega = \omega' = \omega_0/2$. Indeed, the probability of observing at the output of the waveguide a pair of emitted frequencies is maximum when these frequencies match, or are close to, this resonance condition. This is coherent with the fact that the atom is more likely to emit photons close to its own transition frequency ω_0 and with the fact that the spontaneous emission of a pair of photons at a completely different sum of frequencies with respect to that of the atom is practically impossible. The novelty related to considering only two-photon interactions with respect to the single photon ones, is that, since the problem is bidimensional, we could have in principle different bandwidths of emission probability for each direction. While it is always true that the shape of the modulus square along the frequency sum direction $\bar{\omega}$ will be Lorentzian with FWHM proportional to γ, the shape on the opposite direction Δ will be determined by the particular expression of $\gamma_{\Delta}^{\mu\mu'}$. In this thesis work it has been chosen to have a Gaussian profile with a FWHM proportional to σ. However, as it has already been mentioned, by a properly design of the superconducting circuit, it is possible to tune it.
In conclusion of this paragraph it is useful to check whether the results found are coherent with the definition of our problem. To verify this, it is enough to check if the normalization condition in Eq.3.9 is satisfied by the two amplitude probability coefficients evaluated in $t = t_1$. Indeed, t_1 is the time in which we observe our system, much larger than the initial instant of time t_0. Since we are working now with the two variables $\bar{\omega}$ and Δ, the normalization condition can be written as:

$$|C_e(t_1)|^2 + \sum_{\mu.\mu'=\pm} \int_0^{+\infty} \int_{-\infty}^{+\infty} \left| C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1) \right|^2 \frac{d\bar{\omega}d\Delta}{2} = 1 \tag{3.49}$$

For what it has been said, for $t = t_1$ and $t_1 \to \infty$, the qubit amplitude probability vanishes. Regarding the modulus square of the field amplitude probability, it is a normalized Gaussian function along Δ and a normalized Lorentzian function along $\bar{\omega}$. Thus, Eq.3.9 is satisfied and the solutions are coherent with the problem definition.

3.5. Example: scattering event

The other case of interest investigated in this thesis work also for the two-photon interaction case, is the opposite situation with respect to what it has been analyzed in the previous paragraph, i.e. the two-photon scattering phenomenon. With this term we refer here to a particular situation in which the qubit, that is prepared in its ground state $|g\rangle$, interacts with an incoming electromagnetic field in input in the state $|1_{\omega_1}, 1_{\omega_2}\rangle$. As we have already remarked in chapter 2, in free space, this kind of interaction would bring us to observe firstly an absorption of the two incoming photons by the qubit and subsequently an emission. Of course the probability of having such an interaction would depend on the frequency detuning between the sum of the input fields frequencies $\omega_1 + \omega_2$ and the qubit characteristic frequency ω_0, with a bigger probability as soon as the detuning approximates the zero. In principle, once the scattering event has occurred, it would be possible to observe the emission of two photons in any possible space directions with equal probability. This is equivalent of saying that the scattering event in free space is isotropic. This, however, is not true for the system represented schematically in fig.3.1. Indeed, being the atom confined in the waveguide, the only possible directions in which it is possible to observe an emission are those labeled by the parameters μ and μ', i.e. right when they are equal to $+$ and left when they are equal to $-$ respectively, for each photon. The presence of the waveguide, thus, simplifies enormously the problem since now all the directions of emission could be neglected except for those two. A direct consequence of this simplification is that the problem studied becomes linear in space.

As it has been done in the previous paragraph for the spontaneous emission example, in order to study the behaviour of the system considered in this case of two-photon scattering, it is necessary to define its initial conditions. Regarding the emitter amplitude probability coefficient we must impose that in the instant of time in which the scattering event happens, i.e. t_0, the qubit is prepared in its ground state $|g\rangle$. On the other hand, we must define the field amplitude probability coefficient in input. For simplicity and to be coherent with the choice made while studying the single photon scattering, in this thesis work, the input fields will be taken as monochromatic plane waves. These two incoming monochromatic fields will be centered in two generic input frequencies labeled as ω_1 and ω_2. However, since to derive Eq.3.37 and Eq.3.30 we have performed a change of variables, these input fields will be expressed as functions of the frequency sum $\bar{\omega}$ and the frequency difference Δ.

The above considerations correspond to the following initial conditions:

$$C_e(t_0) = 0 \qquad C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0) = \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{[\bar{\omega}-(\omega_1+\omega_2)]^2}{4\sigma^2}} e^{-\frac{[\Delta-(\omega_1-\omega_2)]^2}{4\sigma^2}} \delta^{\mu}_{+}\delta^{\mu'}_{+} \tag{3.50}$$

While the atom amplitude probability is null for $t = t_0$, the field one is expressed as the product of two normalized Gaussian functions dependent on $\bar{\omega}$ and Δ, centered in $\omega_1+\omega_2$ and $\omega_1 - \omega_2$ respectively. The condition of having two monochromatic plane waves in input is reached when the width of this Gaussian fields is small enough with respect to the input frequencies. Formally we could write:

$$\delta(\bar{\omega} - (\omega_1 + \omega_2))\delta(\Delta - (\omega_1 - \omega_2)) = \lim_{\sigma\to 0} \frac{1}{4\sigma^2\pi} e^{-\frac{[\bar{\omega}-(\omega_1+\omega_2)]^2}{4\sigma^2}} e^{-\frac{[\Delta-(\omega_1-\omega_2)]^2}{4\sigma^2}} \tag{3.51}$$

For what concerns the direction indexes μ and μ', the input fields have been taken arbitrarily propagating in the direction + (from left to right). This choice is expressed by the Kronecker deltas δ^{μ}_{+} and $\delta^{\mu'}_{+}$. Ultimately, it is important to notice that, to be precise, in input it has been chosen the square root of two Gaussian functions. This choice is related to the normalisation condition in Eq.3.9, that has to be satisfied for every instant of time, included t_0.

At this point, since the initial conditions have been set, it is possible to compute the atom and field amplitude probabilities by means of Eq.3.37 and Eq,3.30. However, before directly solving the differential equation for the qubit coefficient $C_e(t)$ it is necessary to first obtain the expression of the input field in the time domain. In order to do this, we perform a Fourier Transform operation on the field coefficient $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)$ in the variable $\bar{\omega}$, using the definition in Eq.3.29. Substituting Eq.3.50 inside the definition of $\Phi^{\mu\mu'\Delta}_{in}(t)$ leads to the following:

$$\Psi^{\mu\mu'\Delta}_{in}(t) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{[\bar{\omega}-(\omega_1+\omega_2)]^2}{4\sigma^2}} e^{-\frac{[\Delta-(\omega_1-\omega_2)]^2}{4\sigma^2}} \delta^{\mu}_{+}\delta^{\mu'}_{+} e^{-i\bar{\omega}(t-t_0)}\, d\bar{\omega} \tag{3.52}$$

If we are considering a Gaussian function with a really narrow width (ideally we can think of having $\sigma \to 0$), we can use the definition of Dirac delta in Eq.3.51. After this formal substitution and after noticing that, since the integral is performed only on the variable $\bar{\omega}$, we could bring outside the integral the Dirac delta distribution in Δ, the expression for the input field in the time domain becomes:

$$\Psi^{\mu\mu'\Delta}_{in}(t) = 2\sqrt{2}\sigma\delta^{\mu}_{+}\delta^{\mu'}_{+}\delta(\Delta - (\omega_1 - \omega_2)) \int \delta(\bar{\omega} - (\omega_1 + \omega_2)) e^{-i\bar{\omega}(t-t_0)}\, d\bar{\omega} \tag{3.53}$$

The integration over all the possible sum of frequencies is now trivial, since it is enough to apply the effect of the Dirac delta distribution on the phase function. The final expression of the input field in the time domain is then:

$$\Psi_{in}^{\mu\mu'\Delta}(t) = 2\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)}\delta(\Delta-(\omega_1-\omega_2))\delta_+^{\mu}\delta_+^{\mu'} \tag{3.54}$$

A direct comparison between the input field in the time domain just computed with the one in Eq.2.46 in the case of single photon interaction allows us to understand how different they are. Apart from the different constant terms that depend on the Gaussian functions normalization condition, it is possible to observe that now, as it might be expected, the time dependence in the phase term is weighted by the sum of the two input frequencies ω_1 and ω_2. In addition to that, the input field continues to depend on the output variable Δ. It is not, for this case of two-photon interaction, independent on frequencies. Finally, we can notice that it is directly proportional to the parameter σ, but, although it tends to zero, the output field does not vanishes. This is because what is relevant is the integral over all the possible frequencies, which is well defined.
Since the input field $\Psi_{in}^{\mu\mu'\Delta}(t)$ has been calculated, we can insert its expression inside Eq.3.30 to find $C_e(t)$. The Kronecker deltas δ_+^{μ} and $\delta_+^{\mu'}$ select inside the summation over all the possible directions the ones of the input fields, which, in this case, are the directions from left to right ($\mu = \mu' = +$). The differential equation for the qubit amplitude probability coefficient reads:

$$\dot{C}_e(t) = -i(\frac{\gamma}{2}+\omega_0)C_e(t) - i\int_{-\infty}^{+\infty}\sqrt{\gamma_{\Delta}^{\mu\mu'}}\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)}\delta(\Delta-(\omega_1-\omega_2))\,d\Delta \tag{3.55}$$

In the last equation the parameter $\gamma_{\Delta}^{\mu\mu'}$ was not explicitated on purpose. This is because there are three possible situations that could happen regarding the frequency distributions of the profiles of the input fields and of the coupling parameter. These possibilities are listed below:
i): the width of the input fields matches the width of the coupling parameter;
ii): the width of the input fields is narrower than that of the coupling parameter;
iii): the width of the coupling parameter is narrower than that of the input fields.
Depending on the particular relation between the fields and the coupling parameter, the solution of the integral in Eq.3.55 and the results will be different. In this thesis work we will present the results regarding the first situation of width-matching. For what concerns the second one (fields narrower than the coupling parameter) more details could be found in Appendix A. Unfortunately, due to lack of time the last situation (coupling parameter

narrower than the fields) has not been investigated yet. This of course opens to new future possibilities and researches.
From now on let us focus on the situation in which the width of the input fields matches the one of the coupling parameters. As it has been said previously, the exact shape of $\gamma_{\Delta}^{\mu\mu'}$ has to be determined yet. That is why, in the following analyzes, we will keep considering as its prototype shape the one in Eq.3.46. Using the formal definition of the Dirac delta as the limit of a really narrow Gaussian function (Eq.2.43), we will express the square root of the coupling parameter as:

$$\sqrt{\gamma_{\Delta}^{\mu\mu'}} = \sqrt[4]{8\pi}\sqrt{\Gamma_0\sigma}\delta(\Delta) \tag{3.56}$$

By inserting the last expression in Eq.3.55, we are able to compute the integral over the frequency difference:

$$\begin{aligned}\dot{C}_e(t) = &-i(\frac{\gamma}{2}+\omega_0)C_e(t)+ \\ &- i2\sqrt[4]{2\pi}\sqrt{\Gamma_0}\sigma^{\frac{3}{2}}e^{-i(\omega_1+\omega_2)(t-t_0)}\int_{-\infty}^{+\infty}\delta(\Delta)\delta(\Delta-(\omega_1-\omega_2))\,d\Delta\end{aligned} \tag{3.57}$$

The integral over the frequency difference in the last equation can be computed quite straightforward from the moment in which we consider the first Dirac delta as an actual mathematical distribution acting on the second Dirac delta, which in this case behaves like a normal function. This kind of operation, although might seem not so formal, does hold. A direct way of demonstrating its validity would be that of rewriting the Dirac deltas as Gaussian functions using Eq.2.43 and then apply the well known formula for the Gaussian integral. In both cases, the differential equation for the atom amplitude probability will simplify as follows:

$$\dot{C}_e(t) = -i(\frac{\gamma}{2}+\omega_0)C_e(t) - i2\sqrt[4]{2\pi}\sqrt{\Gamma_0}\sigma^{\frac{3}{2}}e^{-i(\omega_1+\omega_2)(t-t_0)}\delta(\omega_1-\omega_2) \tag{3.58}$$

The last equation for the coefficient $C_e(t)$ is simply a linear differential equation with a driving term determined by the sum and the difference of the input frequencies ω_1 and ω_2. In order to solve it, we can apply the standard resolutive formula that will lead to :

$$C_e(t) = -i2\sqrt[4]{2\pi}\sqrt{\Gamma_0}\sigma^{\frac{3}{2}}\delta(\omega_1-\omega_2)e^{-(i\omega_0+\frac{\gamma}{2})(t-t_0)}\int_{t_0}^{t}e^{(i(\omega_0-\omega_1-\omega_2)+\frac{\gamma}{2})(\tau-t_0)}\,d\tau \tag{3.59}$$

where we have taken into account the fact that $C_e(t_0) = 0$. As for the case of single photon interaction, at this point it is necessary to specify that the instant of time in which the system is observed will be t_1, with t_1 much larger than the initial instant t_0. In this framework it is possible to neglect all the terms that are directly proportional to the exponential function $e^{-\frac{\gamma}{2}(t-t_0)}$, since, when evaluated in $t = t_1$ with $t_1 \to \infty$, they will vanish. Said so, the final expression for the qubit amplitude probability coefficient is:

$$C_e(t) = -\frac{i2\sqrt[4]{2\pi}\sigma^{\frac{3}{2}}\sqrt{\Gamma_0}\delta(\omega_1-\omega_2)e^{-i(\omega_1+\omega_2)(t-t_0)}}{\frac{\gamma}{2}+i[\omega_0-(\omega_1+\omega_2)]} \tag{3.60}$$

The qubit amplitude probability function obtained is quite different from what has been obtained in the same framework for the single photon interaction in Eq.2.50. In this case $C_e(t)$ depends on time via the phase term which takes into account both the phases accumulated by the two input photons from the initial instant of time t_0 to the final generic time t. In addition to that, it is clear that now its shape is determined by the detuning between the atom characteristic frequency and the two input fields frequencies $\Delta\omega = \omega_0 - (\omega_1 + \omega_2)$, and by the relative difference between ω_1 and ω_2. It is important to notice here that in the limit of having $\sigma \to 0$, the amplitude of $C_e(t)$ will tend to zero. This is because even if at the numerator there is a Dirac delta function, it is not enough to compensate for the term $\sigma^{\frac{3}{2}}$ (Eq.2.43). However, we can use the generic expression in Eq.3.60 to compute the output field in the time domain $\Psi_{out}^{\mu\mu'\Delta}(t)$. This is done by inserting Eq.3.60, Eq.3.54 and 3.56 inside the input-output relation for the two-photon interaction in Eq.3.37. The general expression obtained in this way is given by the following:

$$\begin{aligned}\Psi_{out}^{\mu\mu'\Delta}(t) = {} & 2\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)}\delta(\Delta-(\omega_1-\omega_2))\delta_+^{\mu}\delta_+^{\mu'} + \\ & - \sqrt[4]{8\pi}\sqrt{2\Gamma_0\sigma}\delta(\Delta)\frac{2\sqrt[4]{2\pi}\sigma^{\frac{3}{2}}\sqrt{\Gamma_0}\delta(\omega_1-\omega_2)e^{-i(\omega_1+\omega_2)(t-t_0)}}{\frac{\gamma}{2}+i[\omega_0-(\omega_1+\omega_2)]}\end{aligned} \tag{3.61}$$

The above equation for the output field in the time domain, although it might seem hard to interpret, can be simplified a lot after several trivial mathematical operations and by grouping together the common factors. This results in the following simpler and clearer equation for $\Psi_{out}^{\mu\mu'\Delta}(t)$:

$$\Psi_{out}^{\mu\mu'\Delta}(t) = 2\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)}\chi_{\omega_1,\omega_2}^{\mu\mu'\Delta} \tag{3.62}$$

with $\chi_{\omega_1,\omega_2}^{\mu\mu'\Delta}$a function dependent on the input frequencies ω_1, ω_2 and the output frequency

difference Δ, whose explicit expression is given by:

$$\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2} = \delta(\Delta - (\omega_1 - \omega_2))\delta^{\mu}_{+}\delta^{\mu'}_{+} - \frac{\Gamma_0 2\sqrt{\pi}\sigma\delta(\Delta)\delta(\omega_1 - \omega_2)}{\frac{\gamma}{2} + i[\omega_0 - (\omega_1 + \omega_2)]} \tag{3.63}$$

At this point it is possible to compare directly the two expressions of the output field in time domain after an event of scattering in the cases of single photon interaction (see Eq.2.52) and two-photon interaction (see Eq.3.62). In the first case we were able to express the output field as the product of the input field in the time domain and the function $\chi^{\mu}_{\omega_1}$, which assumed the role of a transfer function for the system we were considering. Moreover that transfer function did not depend on the output frequency but only on the input ω_1. Thanks to this, we managed also to retrieve immediately the scattering coefficients, i.e. R and T (Eq.2.56 and Eq.2.57), simply by selecting the particular direction index μ and taking the modulus square of the relative transfer function (χ^{-} for the reflection event and χ^{+} for the transmission one). All the above considerations do not hold anymore in this two-photon interaction framework. This is because, as it can be seen directly by comparing Eq.3.54 and Eq.3.62, the output field in the time domain is not expressed as the product between $\Psi^{\mu\mu'\Delta}_{in}(t)$ and a transfer function. Indeed, although the common term in the expression for the output field in Eq.3.62 resembles the input field in time domain in Eq.3.54, they are not exactly equal since there is a missing dependence on the output frequency difference Δ. Furthermore, in this case the function $\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2}$ cannot be considered as the system transfer function since it depends on Δ and not only on the input frequencies ω_1 and ω_2. It should be clear now that we cannot use the same procedure implemented in the single photon interaction case to obtain the scattering coefficients. We must follow another path.

Since it seems not possible to exploit the definition of $\Psi^{\mu\mu'\Delta}_{out}(t)$ to compute the scattering coefficients, we will focus on the output fields amplitude probability function $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$. The meaning of the scattering coefficients is indeed the following: they give us information about the probability of observing a particular scattering event depending on the particular input. In this case the input is indeed given by the amplitude probability function $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)$. It follows immediately that another way of defining the seeked scattering coefficients is then:

$$X^{\mu\mu'} = \frac{\iint \frac{\left|C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)\right|^2}{2} d\bar{\omega}d\Delta}{\iint \frac{\left|C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)\right|^2}{2} d\bar{\omega}d\Delta} \tag{3.64}$$

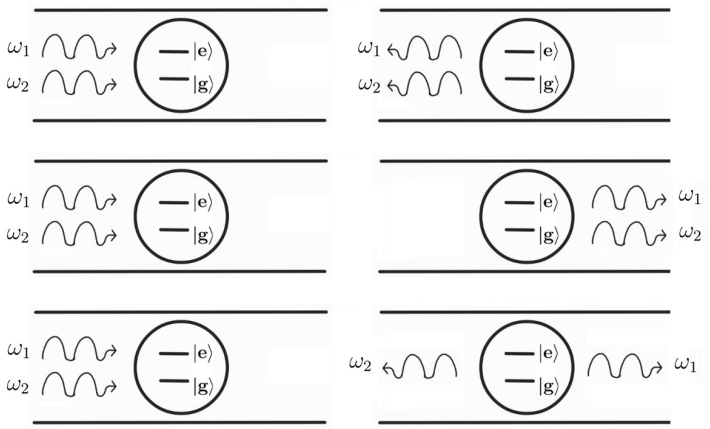

Figure 3.4: Up: schematic representation of the two-photon reflection phenomenon. Middle: schematic representation of the two-photon transmission phenomenon. Bottom: schematic representation of the two-photon splitting phenomenon.

where $X^{\mu\mu'}$ is the generic scattering coefficient. They are defined as the ratio between the integral over all the possible frequencies of the modulus square of the output fields amplitude probability function and the the integral over all the possible frequencies of the modulus square of the input fields amplitude probability function. It is clear now that once we compute $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$, we could use the definition in Eq.3.64 to find the scattering coefficients. In order to obtain the expression for the output fields amplitude probability it is enough to perform an inverse Fourier Transform operation onto the output field in Eq.3.62. Since the only time dependence of the output field in the time domain relies in its phase term, we can bring outside the integral all the other factors and write:

$$C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1) = 2\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t_1-t_0)}\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2}\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-i(\bar{\omega}-(\omega_1+\omega_2))t}\,dt \tag{3.65}$$

The integral over all the possible instants of time is indeed equal to a Dirac delta function weighted by a proper factor needed to take into account that we are working with $\omega = 2\pi\nu$, with ν the formally defined frequency. After the integration and the trivial simplifications, the output fields amplitude probability expression is:

$$C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1) = 4\sqrt{\pi}\sigma\delta(\bar{\omega}-(\omega_1+\omega_2))e^{-i(\omega_1+\omega_2)(t_1-t_0)}\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2} \tag{3.66}$$

From its expression in Eq.3.66 it is easy to see that $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$ is given by the product of two terms: the first is constituted by a phase term, which takes into account the phase accumulated by the two input photons from the initial instant of time t_0 to the final one t_1, times a Dirac delta in the frequency sum $\bar{\omega}$; the second one is exactly the function

$\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2}$, already discussed above. It is also directly proportional to the input fields width parameter σ, which in this case is entirely balanced by the presence of the two Dirac delta functions (one in $\bar{\omega}$, one in Δ). In addition to that, we can also notice that the dependence on the particular direction of propagation is enclosed in the Kronecker delta functions in the term $\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2}$. Also in this case, although the waveguide has been considered symmetric (isotropic spontaneous emission rate), the output field amplitude probability depends on μ and μ' due to the presence of the input field.

Before going on with the calculations, it is important to have in mind what kind of scattering events could be observed in this two-photon interaction situation. To individuate the particular scattering events, it is necessary to select a particular value for the two direction indexes μ and μ'. Indeed, in the single photon scattering case, to the choice $\mu = +$ corresponded the transmission of one photon, while to the choice $\mu = -$ the reflection of one photon. Now, for the system we are considering (fig.3.1), the possible combinations of the direction indexes pairs are 3. In fig.3.4 are showed the schemes of these three possible events of scattering: the pair $\mu = \mu' = -$ correspond of the reflection event of two photons; the pair $\mu = \mu' = +$ correspond to the transmission event of two photons; the pair $\mu \neq \mu'$ correspond to the reflection of one photon and the transmission of the other one. From now on we will refer to these three different events of scattering as reflection, transmission and splitting scattering event respectively. Said so, it is clear that by choosing a particular pair of direction indexes it is possible to select the amplitude probability function corresponding to that particular scattering event.

At this point we can proceed with the calculations for the seeked scattering coefficients. Let us start by selecting the pair of indexes $\mu = \mu' = -$, i.e. by considering the reflection event. Since both the direction indexes are different from those of the input fields, the Kronecker delta functions become null. From Eq.3.66, the expression of the modulus square of $C^{--}_{\bar{\omega},\Delta}(t_1)$ is:

$$\left|C^{--}_{\bar{\omega},\Delta}(t_1)\right| = \frac{16\pi\Gamma_0^2\sigma^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}\left|\delta(\bar{\omega}-(\omega_1-\omega_2))\delta(\Delta)\right|^2}{\frac{\gamma^2}{4}+(\omega_0-(\omega_1+\omega_2))^2} \tag{3.67}$$

Where we used Eq.2.43 to rewrite the term $2\sqrt{\pi}\sigma\delta(\omega_1-\omega_2)$ in the definition of $\chi^{\mu\mu'\Delta}_{\omega_1,\omega_2}$ as a Gaussian function. Now that the modulus square of the amplitude probability function of the reflection scattering event has been computed, we can insert it inside Eq.3.64 to compute R.

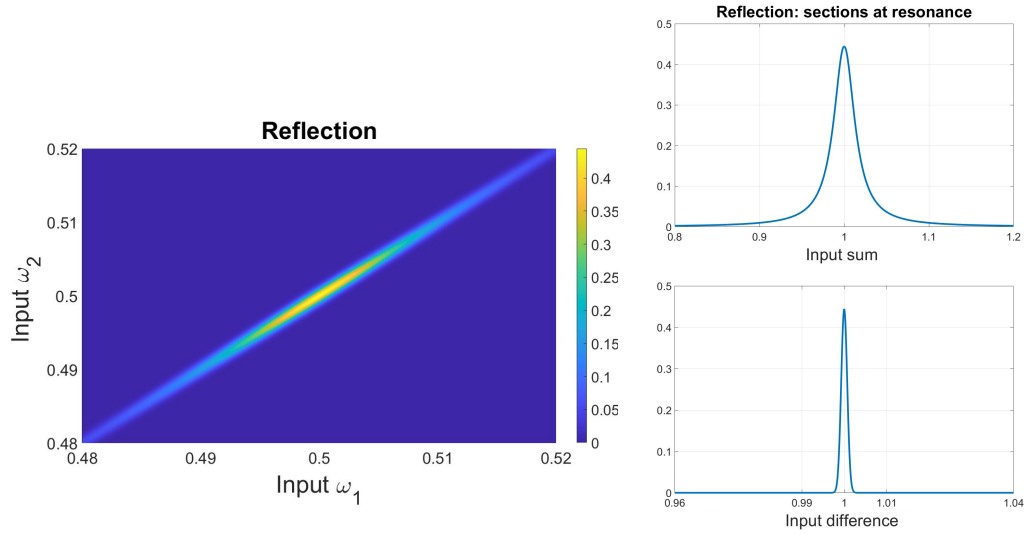

Figure 3.5: Left: 2D-colormap of the modulus square of the Reflection scattering coefficient as a function of the two input frequencies ω_1 and ω_2. Right: sections of the 2D-colormap at resonance along the input frequency sum $\bar{\omega}'$ and the input frequency difference Δ' directions. All the frequencies are expressed in units of ω_0, which has been set equal to 1. The coupling strength Γ_0 has been set equal to $\omega_0/100$.

Since, by definition, the input field amplitude probability $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)$ is normalized, the expression of the coefficient R (fig.3.5) is:

$$R = \frac{\Gamma_0^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}}{\frac{\gamma^2}{4} + (\omega_0 - (\omega_1+\omega_2))^2} \int_0^{+\infty} \int_{-\infty}^{+\infty} 16\pi\sigma^2 |\delta(\bar{\omega} - (\omega_1 - \omega_2))\delta(\Delta)|^2 \frac{d\bar{\omega}d\Delta}{2} \tag{3.68}$$

Finally, using Eq.3.51 to express the product of the two Dirac delta functions as the product of two Gaussian functions, it is possible to solve directly the integral over the frequencies. Indeed the Dirac delta functions will be substituted formally with two independent normalized Gaussian functions in $\bar{\omega}$ and Δ respectively. After this formal step, remembering that $\gamma = 3\Gamma_0$ (see Eq.3.47), we can write the explicit expression of the reflection scattering event R:

$$R = \frac{\Gamma_0^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - (\omega_1+\omega_2))^2} \tag{3.69}$$

The first thing to notice is the fact that if we had selected the pair of direction indexes $\mu \neq \mu'$ we would have obtained the same result in Eq.3.69. The proof of this statement is straightforward since also for this selected pair the Kronecker delta functions are equal to zero in Eq.3.63 and the waveguide is considered symmetric. More in general we can say

that the reflection and the splitting scattering coefficients are equal. Thus, the following identity holds:

$$R = S = \frac{\Gamma_0^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - (\omega_1+\omega_2))^2} \tag{3.70}$$

In fig.3.5 it is possible to observe the 2D-colormap of the reflection coefficient as a function of the two input frequencies ω_1 and ω_2, together with the sections performed at resonance (remember that in this two-photon interaction case it is equal to $\omega_1 = \omega_2 = \omega_0/2$) in the two directions defined by the sum of the input frequencies $\bar{\omega}' = \omega_1 + \omega_2$ and their difference $\Delta' = \omega_1 - \omega_2$. At a first look the 2D-colormap in fig.3.5 might seem the same plot as the one in fig.3.3 obtained in the spontaneous emission example. However, they are fundamentally different: in the spontaneous emission example we were looking at the modulus square of the output fields amplitude probability $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)$ function of the two possible emitted frequencies in output ω and ω'; in this situation of two-photon scattering event we are looking at the probability of observing an event of reflection (or splitting) dependent on the particular input frequencies ω_1 and ω_2.
As the reflection and the splitting scattering coefficients share the same expression, the following analyzes of Eq.3.70 do hold for both of them, despite the fact that, for the sake of simplicity, we will refer to the coefficient R only. From Eq.3.70 and from the 2D-colormap in fig.3.5 it is possible to identify directly a Lorentzian profile in the direction defined by $\bar{\omega}'$ with a FWHM directly proportional to the qubit spontaneous emission rate γ. On the other hand, in the direction defined by Δ', we can observe a Gaussian profile with FWHM proportional to the parameter σ. For the particular choices that we have made through this thesis work the parameter Γ_0 results to be greater than the parameter σ. Due to this reason in the direction Δ' the coefficient R has a really narrower profile with respect to the one in the direction $\bar{\omega}'$. This is also coherent with the fact that in the sum of the input frequencies direction what we observe in this two-photon interaction case is the same profile that we would have observed in the single photon interaction situation if we had considered the qubit interacting with a single photon with frequency $\Omega = \omega_1 + \omega_2$. Regarding the profile in the difference of the input frequencies direction, it is so narrow in agreement with the fact that the qubit is not capable of interacting with two photons with very different frequencies (this also prevent the system to interact with a single photon only). Ultimately, from the 2D-colormap in fig.3.5 it is easy to see that there is an increase of the probability of having both the input photons reflected after the interaction with the qubit as soon as those are in a band of frequencies around the resonance condition. This was quite expected since the interaction between the input fields and the two-level quantum emitter is stronger if their sum matches its characteristic frequency. It is indeed

when $\omega_1 = \omega_2 = \omega_0/2$ that the maximum probability of observing a reflection event is achieved. From Eq.3.70, at resonance we have:

$$R_{max} = S_{max} = \frac{4}{9} \tag{3.71}$$

Now that both the reflection and the splitting scattering coefficients have been computed, it remains only to calculate the transmission coefficient to have a complete overview of the two-photon scattering problem. For what concerns its amplitude probability function, it corresponds to the selection of the pair $\mu = \mu' = +$ of the direction indexes. For this particular choice the Kronecker delta functions in Eq.3.63 are equal to the unity. Said so, the modulus square of the transmission scattering event amplitude probability function is:

$$\left|C^{++}_{\bar{\omega},\Delta}(t_1)\right| = \frac{16\pi\sigma^2\left|\delta_{\bar{\omega}}\delta_{\Delta}(\frac{\gamma}{2} + i\Delta\omega') - \Gamma_0\delta_{\bar{\omega}}\delta_0 e^{-\frac{\Delta^2}{2\sigma^2}}\right|^2}{\frac{\gamma^2}{4} + (\Delta\omega')^2} \tag{3.72}$$

where $\delta_{\bar{\omega}} = \delta(\bar{\omega} - (\omega_1 + \omega_2))$, $\delta_{\Delta} = \delta(\Delta - (\omega_1 - \omega_2))$, $\delta_0 = \delta(\Delta)$ and $\Delta\omega' = \omega_0 - (\omega_1 + \omega_2)$ the detuning between the atom and the input fields. To obtain Eq.3.72 we have substituted the Dirac delta function in Eq.3.63 with a Gaussian function, using Eq.2.43, similarly to what has been done for the reflection event amplitude probability. Now that the modulus square of the amplitude probability function of the transmission scattering event has been computed, we can insert it inside Eq.3.64 to compute T. Again, since by definition the input field amplitude probability $C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)$ is normalized, the expression of the coefficient T is:

$$T = \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{16\pi\sigma^2\left|\delta_{\bar{\omega}}\delta_{\Delta}(\frac{\gamma}{2} + i\Delta\omega') - \Gamma_0\delta_{\bar{\omega}}\delta_0 e^{-\frac{\Delta^2}{2\sigma^2}}\right|^2}{\frac{\gamma^2}{4} + (\Delta\omega')^2} \frac{d\bar{\omega}d\Delta}{2} \tag{3.73}$$

The solution of the above double integral over all the possible frequencies is not trivial. To solve it in a direct and simpler way it is enough to use Eq.3.51 to express the two pairs of Dirac delta distributions as Gaussian functions. In this way we can perform in the classical sense the modulus square of the quantities inside the double integral. This will result in having a series of pairs of normalized Gaussian functions that allow us to compute directly the double integral.

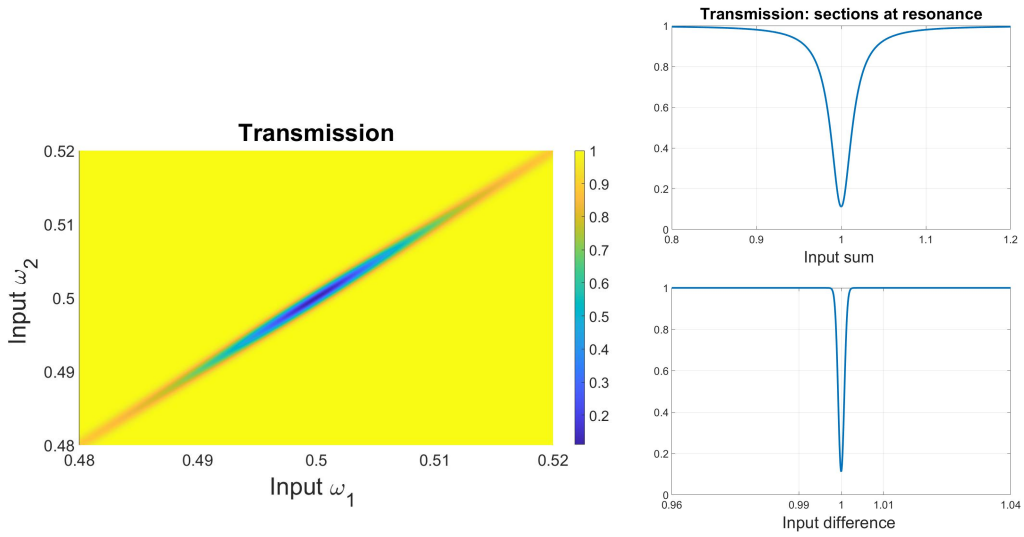

Figure 3.6: Left: 2D-colormap of the modulus square of the Transmission scattering coefficient as a function of the two input frequencies ω_1 and ω_2. Right: sections of the 2D-colormap at resonance along the input frequency sum $\bar{\omega}'$ and the input frequency difference Δ' directions. All the frequencies are expressed in units of ω_0, which has been set equal to 1. The coupling strength Γ_0 has been set equal to $\omega_0/100$.

The final result (fig.3.6), after performing all the above described mathematical steps, is:

$$T = 1 - \frac{2\Gamma_0^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - (\omega_1+\omega_2))^2} \tag{3.74}$$

To verify if the expression in Eq.3.74 of the transmission scattering coefficient is the correct one, it is enough to check if the sum of the three scattering coefficients is equal to one. Indeed, the total probability of observing a generic scattering event must be equal to %100. By making the sum of Eq.3.74 and twice the Eq.3.70, since $R = S$, it is straightforward to verify that the following expression does hold:

$$T + R + S = 1 \tag{3.75}$$

In fig.3.6 is shown the 2D-colormap of the Transmission scattering coefficient in Eq.3.74 function of the two input frequencies ω_1 and ω_2, together with with the sections performed at resonance in the two directions defined by the sum of the input frequencies $\bar{\omega}' = \omega_1 + \omega_2$ and their difference $\Delta' = \omega_1 - \omega_2$. From its definition and from the two sections, it can be seen that the frequency dependence of the coefficient T is exactly the same as that of the other two: it has a Lorentzian profile of FWHM proportional to γ in the sum of the

input frequencies direction and a Gaussian profile with FWHM proportional to σ in the difference of the input frequencies direction. However, regarding the transmission scattering event, we have the opposite situation of what has been observed for the reflection one. The probability of observing the transmission of two input photons along the waveguide increases as soon as their frequencies are much far away from the resonance condition. Of course this behaviour could be explained if we consider that, since the interaction between the two input photons and the qubit is stronger around resonance, it will be harder to observe a transmission event in that particular bandwidth around ω_0. Indeed, we have to remember that transmission could also be seen as the probability for the input fields of not interacting with the two-level quantum emitter. At perfect resonance the maximum strength of interaction is achieved and consequently we will observe a minimum in the probability of having transmission. From Eq.3.74 we can compute the minimum value of the coefficient T, which is indeed:

$$T_{min} = \frac{1}{9} \tag{3.76}$$

Since the transmission coefficient is always different from zero even at resonance, we will always have a certain probability of not observing any interaction between the input photons and the qubit.
In conclusion of this chapter it remains only to discuss if the results we have found for this two-photon interaction scattering example satisfy the normalization condition in Eq.3.9. As it can be seen from Eq.3.60, the atom amplitude probability is directly proportional to the parameter σ, which, as it has been said, ideally tends to zero. This means that the modulus square of $C_e(t_1)$ can be neglected. To see if the normalization condition is verified, it is enough then to use the result in Eq.3.66 and compute:

$$\sum_{\mu.\mu'=\pm} \iint \left| C^{\mu\mu'}_{\omega\omega'}(t_1) \right|^2 d\omega = 1 \tag{3.77}$$

Expanding explicitly the sum over all the possible pairs of the direction indexes leads to having the sum of all the scattering coefficients. Since we have already checked that the sum of all the scattering probabilities is equal to the unity, the normalization condition is always verified and the results obtained are coherent with the definition of the problem considered.

4 | Conclusions

At the end of this thesis work we were able to obtain important results regarding the general theory of two-photon interaction in waveguide QED systems. In particular, we succeeded in defining a set of self-consistent equations (Eq.3.37 and Eq.3.30) that could allow us to compute the amplitude probability coefficients after the initial state of the system has been defined. We have analyzed two specific examples, spontaneous emission and two-photon scattering, to show explicitly how those results could be applied to cases of interest. This work represents a first analysis of two-photon couplings in the context of waveguide QED. Our result paves the way towards the exploration of a novel quantum phenomenology and to possible applications in quantum technologies.
In conclusion, in the following last paragraphs we will compare the general results achieved in the case of two-photon interaction with those regarding the single photon interaction situation. In this way it would be possible to have a general overview of all the specific characteristics of both the situations of interest with particular focus on the differences between the two. Lastly, we will propose an example of possible practical application of the general theory of two-photon interaction in the field of quantum engineering.

4.1. Single photon and two-photon scattering in waveguide QED: a direct comparison

In this direct comparison between the two different situations of interest, we will first start by considering the single photon interaction case. In this particular situation the two-level quantum emitter, situated in the middle of a superconducting waveguide, interacts with a single photon. Since we have been working in the strong regime where the coupling parameter g is greater than the losses k in the directions outside the waveguide but much smaller with respect to the frequencies involved in the interaction, the RWA, together with the dipolar approximation did hold. This means that the total number of excitations does not change in time and to each atomic transition correspond only one field transition (emission or absorption of one photon, fig.1.5). With these approximations, it is possible to describe the system using the well known Jaynes-Cummings model adapted to the

context of waveguide QED. The JC Hamiltonian operator can be decomposed in two parts: the non-interacting and the interacting one. In particular, the interaction term of this operator is given by:

$$\hat{H}_I = \sum_{\mu=\pm} \int g_\omega^\mu (\hat{a}_\omega^\mu)^\dagger \hat{\sigma}^- + (g_\omega^\mu)^* \hat{a}_\omega^\mu \hat{\sigma}^+ d\omega \tag{4.1}$$

The most important thing to notice here is that $\hat{H}_I$ is directly proportional to the electric field operator (remember that in general for a single mode it is possible to state that $\hat{E} \propto [\hat{a} + \hat{a}^\dagger]$). The consequence of this proportionality is that the coupling between the qubit and the incoming field in the waveguide is linear. The linearity of the coupling in this single photon interaction case reflects also on the frequency dependence of the coupling parameter. Indeed, in this case the problem is one dimensional in the frequency domain. For this particular reason the solutions of the associated system of linear differential equations for the amplitude probability coefficients (system 3.16) were obtained without any particular computational effort. For what concerns the input-output relation in Eq.2.32, it links the output and the input fields in time domain through the interaction with the qubit. It is important to underline that those fields defined in time domain depends only on time and not on frequency. It is for this particular reason that in applying the input-output theory to the specific example of single photon scattering we were able to define the output field in the time domain as the product between the input field and the system transfer function (Eq.2.53). Thanks to this direct equation, the scattering coefficients were computed immediately by calculating the modulus square of the transfer function for a particular choice of the direction index μ. The most important result in the study of single photon scattering in the waveguide QED context is the fact that at resonance it is possible to achieve the condition of total reflection of the input field, which is not possible to observe in free space.

On the other hand, we can analyze the situation of having two-photon interaction. In this case we are considering the interaction between a two-level quantum emitter situated in the middle of the superconducting waveguide and two incoming photons. Also in this new case we have been working in the strong regime with a two-photon coupling parameter g_2 greater than the losses k in the directions outside the waveguide but much smaller than the frequencies involved in the interaction. Said so, both the RWA and the dipolar approximation were made. This means that the total number of excitations does not change in time and to each atomic transition correspond two field transitions (emission or absorption of two photons, fig.1.6). In addition to that, each atomic transition has twice the weight of the single field transition (for the single photon interaction case the ratio

was 1:1). For what concerns two-photon interaction systems, they cannot be described using simply the Jaynes-Cummings model. Indeed, in order to describe them properly it is necessary to use the two level quantum Rabi model adapted to the waveguide QED context. In the regime of validity of the above mentioned approximations, the Hamiltonian operator related to the two-level Rabi model can be decomposed in two terms: the non-interacting and the interacting one. The interaction term in this framework can be written as:

$$\hat{H}_I^{2ph} = \sum_{\mu,\mu'=\pm} \iint (g_{\omega\omega'}^{\mu\mu'})^* \hat{\sigma}^+ \frac{\hat{a}_\omega^\mu \hat{a}_{\omega'}^{\mu'}}{\sqrt{2}} \, d\omega d\omega' + h.c \tag{4.2}$$

The fundamental difference between this two-photon interaction Hamiltonian operator and the one analyzed above relies in the fact that now it is directly proportional to the square of the electric field operator ($\hat{E} \propto [\hat{a} + \hat{a}^\dagger]^2$). The terms that compare in Eq.4.2 are then the ones that after the expansion of the square of the sum of the fields ladder operators have not been neglected due to the RWA. The most important consequence of this dependence is that the coupling between the two-level quantum emitter and the incoming electromagnetic fields is nonlinear. Indeed, in this new situation of two-photon coupling, the problem is bidimensional in the frequency domain. Due to this more complicated frequency dependence the solutions for the system 3.16 were obtained with much more computational effort. Moreover, the spontaneous emission rate of the qubit could not be taken constant in the frequency domain as it has been done with the single photon interaction case. This frequency dependence introduces constrains on the particular functional shape of the coupling parameter. Although during this thesis work it has been chosen to be a Gaussian function, a complete microscopic description of the two-photon coupling between the superconducting circuit in fig.1.6 and the waveguide is still missing. Further studies in that direction would probably lead to a deeper insight of the phenomenology behind this particular two-photon interaction and would give also the exact expression for the coupling parameter. Of course, since we are working with superconducting circuits, we could expect that by engineering properly the circuit scheme it would be possible to tune in a desired way the main characteristic of the system such as the bandwidth or even the coupling parameter itself. For what concerns the two-photon interaction input-output relation in Eq.3.37, it is a really important result achieved. Indeed, this relation could allow one to study really different and interesting situations that, due to lack of time, have not been investigated yet in this thesis. Also in this case the input-output relation links the output field in time domain with the input field in the time domain through the interaction with the qubit. The main aspect to notice here is that the fields considered

depend not only on time but also on the difference of the frequencies in output. For this particular reason, in the example of application of the two-photon interaction theory to the scattering event, it was not possible to obtain an equation that could express the output field as the product between the input field and the system transfer function. In order to compute the scattering coefficients we had to first go back from the definition of the output field in the time domain to the amplitude probability function in output. After that mathematical step, using Eq.3.72, we were able to obtain the expressions for the seeked scattering coefficients. In the context of two-photon interaction in waveguide QED, the possible observable scattering events are three. Indeed, in addition to the reflection and the transmission events, already present in the single photon scattering case, it is possible to have a phenomenon of splitting. With this term we refer to the situation in which the two incoming photons are split, i.e. one is reflected back and one is transmitted. At the end, by selecting a particular pair of the direction indexes μ and μ', we managed to compute the modulus square of the amplitude probability coefficient related to each scattering phenomenon and, finally, the expression of all the scattering coefficients. The most interesting difference with respect to the results obtained in the single photon scattering case is that the phenomenon of total reflection cannot be observed anymore. Indeed the maximum probability of observing a reflection event is $4/9$. In addition to that, the probability of observing a splitting event is the same of that of observing a reflection one. This means that, with equal probability, we will see two photons interacting with the qubit and consequently being reflected back or the reflection of just one photon and the other being transmitted without any interaction with the two-level emitter. Ultimately, the probability of transmission is never equal to zero. This leads of course to having always a certain probability of not observing any interaction between the incoming photons and the qubit.

It has to be remembered that all the considerations made until now, do hold for the particular framework we have been working in, i.e. for the particular input fields shape considered and for the chosen expression of the coupling parameter. This does not mean that it is not possible to have different expressions or even to tune the probabilities of observing a certain scattering event. Working in a different framework would lead certainly to different results, which might be more suitable for possible practical implementations of the two-photon interaction phenomenon.

4.2. Perspectives: possible application in controlled quantum phase-gate

In the past few decades quantum computing has been object to the interest of many researches around the world which has lead to a rapid evolution of its architectures and techniques of implementation. The origin of this incredible success has to be found in the idea of the so called quantum supremacy. With this term we refer to the capability of quantum devices to solve problems that cannot be solved by usual classical computers in a feasible amount of time. The main ingredients for quantum computation are basically three: preparation of the input state, preparation of the unitary transformation acting on the state and measurement of the output state [30]. There are different ways of implementing the above mentioned operations such as using trapped ions, superconducting circuits or photonics. The last two possible ways of implementation are surely related to the nature of this thesis work. In particular, for what concerns photonic quantum computing architectures, what it is required is a source of single indistinguishable photons and a way in which they could interact. Since this requirements were first stated, single photon sources have been hugely improved [31]. Photon-photon interactions have been implemented using nonlinearities outside the waveguide in which they are propagating (measurement and feed forward). However, in principle this interactions could be mediated by the presence of a nonlinear few-level emitter situated in the waveguide (as sketched in fig.3.1) in a deterministic way ([32];[33]). The potentially deterministic nature of this in-line nonlinearities makes this approach particular attractive for the realization of controlled photonic phase gates. A possible implementation example of such a device is shown in fig.4.1. It is constituted by two phase shifters, two directional couplers and two two-level quantum emitters. The main idea behind this device is that it could be used to implement two-photon gate. The state of the electromagnetic field in input could be only one of this four alternatives:

$$|0_s\rangle\,|0_c\rangle \quad |0_s\rangle\,|1_c\rangle \quad |1_s\rangle\,|0_c\rangle \quad |1_s\rangle\,|1_c\rangle \tag{4.3}$$

where c denotes the control zone and s the signal one. To better understand how the showed device works, it is useful to consider in input monochromatic fields and analyze how they are modified through the propagation in the waveguides. Consider first in input two photons in the state $|0_s\rangle\,|0_c\rangle$: they will propagate and each of them would simply acquire a phase equal to ϕ. For the input states $|0_s\rangle\,|1_c\rangle$ or $|1_s\rangle\,|0_c\rangle$, the photon in the state $|0\rangle$ will simply acquire the phase ϕ while the other one will be affected by the

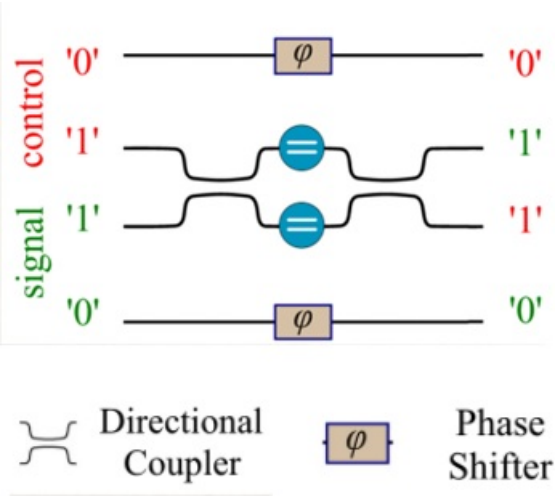

Figure 4.1: Schematic of the controlled-PHASE gate, which uses chiral waveguides, directional couplers, phase shifters, and two identical quantum emitters.

directional coupler and the two-level emitter. The combination of their effects simply modifies the $|1\rangle$ states as follows:

$$|1\rangle \rightarrow e^{i\theta} |1\rangle \tag{4.4}$$

Ultimately, regarding the input state $|1_s\rangle |1_c\rangle$, it will enter the first directional coupler whose main effect is to give rise to the Hong-Ou Mendel effect. This means that the directional coupler, acting as a 50/50 beam splitter, will create a superposition of the situations in which two photons interact with each emitter. Immediately after the first directional coupler the state is defined by:

$$[(\hat{a}_{1c}^{\dagger})^2 + (\hat{a}_{1s}^{\dagger})^2] |\psi\rangle \tag{4.5}$$

where $|\psi\rangle$ is the vacuum. At the end, the possible input states are transformed in the following schematic way:

$$\begin{aligned} |0_s\rangle |0_c\rangle &\rightarrow e^{i2\phi} |0_s\rangle |0_c\rangle \\ |1_s\rangle |0_c\rangle &\rightarrow e^{i\phi} e^{i\theta} |1_s\rangle |0\rangle \\ |0_s\rangle |1_c\rangle &\rightarrow e^{i\phi} e^{i\theta} |0\rangle |1_c\rangle \\ |1_s\rangle |1_c\rangle &\rightarrow e^{i\chi} |1_s\rangle |1_c\rangle \end{aligned} \tag{4.6}$$

where χ is the phase accumulated by the input state $|1_s\rangle |1_c\rangle$. If the photon-emitter interaction could be tuned in such a way to have $\phi = \theta$ and $\chi = 2\phi + \pi$ the control

gate implements the unitary transformation diag(1,1,1,-1). The situation becomes more complicated when we consider in input single photons with spectral profiles having FWHM proportional to the parameter σ. However, standard light-matter couplings have some intrinsic limitation. For example, achieving perfect fidelity in the implementation of controlled-phase gates is not possible [34]. Without entering too much into the details, it can be proved that the maximum fidelity of the device analyzed is around %84. This is because in order to maximize this value two different constrains must be satisfied. First let us consider the transformation of the states $|0_s\rangle\,|0_c\rangle$ and $|0\rangle\,|1\rangle$. Since there is at most one photon entering into the directional coupler, the transformation is linear (single photon interaction). On the other hand, for the state $|1_s\rangle\,|1_c\rangle$, as stated in Eq.4.5, the transformation is nonlinear (two-photon interaction). To maximize the fidelity of the linear transformation it is required that the spectral pulse width must be narrow compared with the emitter linewidth. On the other hand, to maximize the fidelity of the nonlinear interaction the pulse width and emitter linewidth must be comparable. These two constrains are clearly in contrast with each other. It is at this point that the general theory of two-photon interaction developed in this thesis work might come in help. Indeed, as it was remarked many times in the previous chapters, the coupling between the two-level quantum emitter and two-photon in input is nonlinear. The relevant aspect here is that the coupling will be nonlinear in any case, independently from the fact that the interaction between the input photons and the qubit will happen or not. In this way one of the two above mentioned constrains does not hold anymore and just the constrain related to the input state $|1_s\rangle\,|1_c\rangle$ must be satisfied. Hopefully, in the future a possible new implementation of devices similar to that in fig.4.1 with superconducting circuits exploiting two-photon interactions and a complete microscopic theory regarding the two-photon coupling of the circuit in fig.1.6 in waveguide QED context will lead to a fidelity higher than the fundamental limit of %84, making this two-photon controlled phase-gate actually useful for quantum computing.

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A | Appendix A

In this section we will analyze the two-photon scattering situation in the case in which the input fields spectral width is narrower than that of the coupling parameter. In the following we will express the coupling parameter as:

$$\gamma_\Delta = \Gamma_0 f(\Delta) \tag{A.1}$$

where $f(\Delta)$ is a function of the output frequency difference with spectral width much larger than the parameter σ. Let us now start by computing the expression for the atom amplitude probability coefficient using Eq.3.30. With the input field defined in Eq.3.54, the differential equation for the coefficient $C_e(t)$ can be written as:

$$\dot{C}_e(t) = -i(\frac{\gamma}{2} + \omega_0)C_e(t) - i\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)} \int_{-\infty}^{+\infty} \sqrt{\gamma_\Delta}\delta(\Delta - (\omega_1 - \omega_2))\, d\Delta \tag{A.2}$$

In this case the Dirac delta centered in the input frequencies difference $\omega_1 - \omega_2$ could be considered as a distribution acting on the function γ_Δ. The result of this effect is the following:

$$\dot{C}_e(t) = -i(\frac{\gamma}{2} + \omega_0)C_e(t) - i\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)} \sqrt{\gamma_{\omega_1-\omega_2}} \tag{A.3}$$

The expression in Eq.A.3 is a simple linear differential equation that can be solved for the amplitude probability $C_e(t)$ by using the well known resolutive formula. The final result is then:

$$C_e(t) = -\frac{i\sqrt{2}\sigma e^{-i(\omega_1+\omega_2)(t-t_0)}\sqrt{\gamma_{\omega_1-\omega_2}}}{\frac{\gamma}{2} + i[\omega_0 - (\omega_1 + \omega_2)]} \tag{A.4}$$

In obtaining the result in Eq.A.4, the terms proportional to $e^{-\frac{\gamma}{2}(t-t_0)}$ have been discarded since the system at the end is observed in an instant of time t_1 much larger to the initial one t_0 (ideally $t_1 \to \infty$). At this point, since the expression of the coefficient $C_e(t)$ has been computed, it is possible to use the input-output relation in Eq.3.37 to obtain the

output field in the time domain. Its expression is given by:

$$\Psi_{out}^{\mu\mu'\Delta}(t) = \Psi_{in}^{\mu\mu'\Delta}(t) - \frac{i4\sigma e^{-i(\omega_1+\omega_2)(t-t_0)}\sqrt{\gamma_\Delta \gamma_{\omega_1-\omega_2}}}{\frac{\gamma}{2} + i[\omega_0 - (\omega_1+\omega_2)]} \tag{A.5}$$

Also in this case, it is not possible to express the output field in the time domain as the product between the input field in the time domain and the system transfer function. In addition to that, the second term in Eq.A.5 is directly proportional to the input fields spectral width parameter σ. The fundamental difference here is that there is not a Dirac delta function that could compensate the fact that σ is ideally close to zero. Indeed, the coupling parameter has a really large spectral width compared to that of the input fields. In this way, analogously to what has been done with the modulus square of the atom amplitude probability function in checking the normalization condition, the second term in Eq.A.4 can be neglected. The consequence is that now the output field is simply equal to the input field in the time domain:

$$\Psi_{out}^{\mu\mu'\Delta}(t) = \Psi_{in}^{\mu\mu'\Delta}(t) \tag{A.6}$$

The system transfer function is then just equal to the unity and the only scattering event that is possible to observe is the total transmission of both the input photons:

$$T = 1 \quad ; \quad R = S = 0 \tag{A.7}$$

B | Appendix B

In this last section we will investigate the physical meaning of the input fields expressed in the time domain for both the single photon and the two-photon interaction cases. Let us begin by considering the single photon interaction situation. In order to have a better insight of the function $\Psi_{out}^{\mu\mu'\Delta}(t)$, we can start by calculating the intensity of the electromagnetic field at the output of the waveguide in an instant of time $\bar{t}$ grater than t_1. From the quantum mechanics theory, it is known that the expected value of any observable is given by the scalar product between the state of the system considered and the operator associated to that particular observable applied to the state. The intensity operator for single photon interaction is defined as follows:

$$\hat{I} = (\hat{a}_{\omega'})^{\dagger}(\bar{t})\hat{a}_{\omega}(\bar{t}) = \frac{1}{2\pi}\iint e^{-i(\omega-\omega')(\bar{t}-t_1)}(\hat{a}_{\omega'})^{\dagger}\hat{a}_{\omega}\, d\omega d\omega' \tag{B.1}$$

On the other hand, the state of the system after the scattering event could be obtained by inserting Eq.2.50 and Eq.2.60 inside the state definition in Eq.2.5. This will lead in having:

$$|\Phi(t_1)\rangle = \sum_{\mu=\pm}\int C_{\omega}^{\mu}(t_1)(\hat{a}_{\omega}^{\mu})^{\dagger}d\omega\,|\mathbf{0}\rangle \tag{B.2}$$

since at time $t = t_1$ the coefficient $C_e(t)$ can be neglected. Said so, the expected value of the intensity of the output fields can be computed as follows:

$$\begin{aligned} I^{\mu}(\bar{t}) &= \langle\Phi(t_1)|\,(\hat{a}_{\omega'}^{\mu})^{\dagger}(\bar{t})\hat{a}_{\omega}^{\mu}(\bar{t})\,|\Phi(t_1)\rangle = \\ &= \frac{1}{2\pi}\iint d\omega d\omega' e^{-i(\omega-\omega')(\bar{t}-t_1)}\,\langle\Phi(t_1)|\,(\hat{a}_{\omega'}^{\mu})^{\dagger}\hat{a}_{\omega}^{\mu}\,|\Phi(t_1)\rangle \end{aligned} \tag{B.3}$$

In order to calculate the effect of the operator $\hat{a}_{\omega}^{\mu}$ onto the system state $|\Phi(t_1)\rangle$, we must take into account the properties of the field ladder operators in Eq.2.11. Since in the definition of the system state we are applying a creation operator onto the vacuum state with a certain frequency ν and direction λ, applying consequently the destruction operator

$\hat{a}^{\mu}_{\omega}$ will give zero apart when the two frequencies and directions do match. In that case we have:

$$\hat{a}^{\mu}_{\omega}\left|\Phi(t_1)\right\rangle = \sum_{\lambda=\pm}\int C^{\lambda}_{\nu}(t_1)\hat{a}^{\mu}_{\omega}(\hat{a}^{\lambda}_{\nu})^{\dagger}d\nu\left|\mathbf{0}\right\rangle = C^{\mu}_{\omega}(t_1) \tag{B.4}$$

Finally the expected value of the intensity observable is:

$$I^{\mu}(\bar{t}) = \frac{1}{2\pi}\iint d\omega d\omega' e^{-i(\omega-\omega')(\bar{t}-t_1)}(C^{\mu}_{\omega'})^{*}(t_1)C^{\mu}_{\omega}(t_1) = (\Psi^{\mu}_{out}(\bar{t}))^{*}\Psi^{\mu}_{out}(\bar{t}) = \left|\Psi^{\mu}_{out}(\bar{t})\right|^2 \tag{B.5}$$

where in the last expression we recognized the definition of the output field in the time domain (Eq.2.26). From this derivation it is clear that the physical meaning of the field in the time domain is the following: it is the amplitude probability of measuring one single photon in a certain instant of time t in a certain direction given by μ. In particular, the modulus square of $\Psi^{\mu}_{out}(\bar{t})$ gives the probability of measuring a photon far after the scattering event.

Let us now focusing on the two-photon interaction case. Since now we are dealing with pairs of photons, the observable we are interested in is the two-photon flux outside the waveguide. This means that the two photons outside the waveguide are measured with two photodetectors in a certain instant of time, which in principle could be different for each photon. The expected value desired could be written as:

$$\begin{aligned} I(\bar{t},\bar{t}') &= \left\langle\Phi(t_1)\right|(\hat{a}^{\mu}_{\omega})^{\dagger}(\bar{t})(\hat{a}^{\mu'}_{\omega'})^{\dagger}(\bar{t}')\hat{a}^{\mu''}_{\omega''}(\bar{t}')\hat{a}^{\mu'''}_{\omega'''}(\bar{t})\left|\Phi(t_1)\right\rangle = \\ &= \frac{1}{(2\pi)^2}\int d\omega d\omega' d\omega'' d\omega''' e^{i(\omega\bar{t}+\omega'\bar{t}'-\omega''\bar{t}'-\omega'''\bar{t})}\left\langle\Phi(t_1)\right|(\hat{a}^{\mu}_{\omega})^{\dagger}(\hat{a}^{\mu'}_{\omega'})^{\dagger}\hat{a}^{\mu''}_{\omega''}\hat{a}^{\mu'''}_{\omega'''}\left|\Phi(t_1)\right\rangle \end{aligned} \tag{B.6}$$

where for sake of simplicity we have omitted the phase term accumulated until the instant of time t_1. To simplify the above expression, we can study the effect of the double field ladder operator on the system state in Eq.3.7:

$$\hat{a}^{\mu''}_{\omega''}\hat{a}^{\mu'''}_{\omega'''}\left|\Phi(t_1)\right\rangle = \sum_{\lambda,\lambda'=\pm}\iint C^{\lambda\lambda'}_{\Omega\Omega'}(t_1)\frac{\hat{a}^{\mu''}_{\omega''}\hat{a}^{\mu'''}_{\omega'''}(\hat{a}^{\lambda}_{\Omega})^{\dagger}(\hat{a}^{\lambda'}_{\Omega'})^{\dagger}}{\sqrt{2}}d\Omega d\Omega'\Big]\left|\mathbf{0}\right\rangle \tag{B.7}$$

By using the properties of the fields ladder operators one can demonstrate that finally it is possible to write:

$$\hat{a}^{\mu''}_{\omega''}\hat{a}^{\mu'''}_{\omega'''}\left|\Phi(t_1)\right\rangle = \sqrt{2}C^{\mu\mu'}_{\omega''\omega'''}(t_1)\left|\mathbf{0}\right\rangle \tag{B.8}$$

Inserting the result in Eq.B.8 in the expression for the two-photon flux in Eq.B.6 and focusing on the same arrival time $\bar{t}$ lead to:

$$\begin{aligned} I^{\mu\mu'}(\bar{t},\bar{t}) &= \iint \frac{d\omega d\omega'}{2\pi^2} e^{i(\omega+\omega')\bar{t}} (C^{\mu\mu'}_{\omega\omega'})^*(t_1) \iint d\omega'' d\omega''' e^{-i(\omega''+\omega''')\bar{t}} C^{\mu\mu'}_{\omega''\omega'''}(t_1) = \\ &= \iint \frac{d\bar{\omega}\Delta}{8\pi^2} e^{i\bar{\omega}\bar{t}} (C^{\mu\mu'}_{\bar{\omega}\Delta})^*(t_1) \iint d\bar{\omega}'\Delta' e^{-i\bar{\omega}'\bar{t}} C^{\mu\mu'}_{\bar{\omega}'\Delta'}(t_1) = \\ &= \frac{1}{4\pi}\int d\Delta (\Psi^{\mu\mu'\Delta}_{out})^*(\bar{t}) \int d\Delta' \Psi^{\mu\mu'\Delta'}_{out}(\bar{t}) = \\ &= \left|\Psi^{\mu\mu'}_{out}(\bar{t})\right|^2 \end{aligned} \tag{B.9}$$

where from the second to the third line we identified the definition of the output field in the time domain (Eq.3.31) and from the third to the fourth we used the definition:

$$\Psi^{\mu\mu'}_{out}(\bar{t}) = \frac{1}{2\sqrt{\pi}}\int d\Delta \Psi^{\mu\mu'\Delta}_{out}(\bar{t}) \tag{B.10}$$

It is now clear from Eq.B.9 that the physical meaning of the output field in the time domain is the following: it is the amplitude probability distribution in the difference of the output frequencies of measuring outside the waveguide two photons in the same instant of time. Integrating it in the frequency difference Δ and taking the modulus square give us the probability of measuring a couple of photons in the same instant of time in the directions μ and μ'. If we had computed the expected value of the two-photon flux in different instants of time, we would have obtained as result simply the product between the output field in the time domain evaluated in $\bar{t}$ and the output field in time domain evaluated in $\bar{t}'$. Its physical meaning remains however unchanged.

List of Figures

Acknowledgements

This thesis work has been developed under the supervision of Roberto Osellame (Politecnico di Milano) and Simone Felicetti (CNR-IFN), and it is part of an international collaboration involving Tomas Ramos and Juanjo-Garcia Ripoll (CSIC, Madrid) and Roberto Di Candia (Aalto University).