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Equilibrium Configurations of a 3D Fluid-Airfoil Interaction Problem

TESI DI LAUREA MAGISTRALE IN
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Abstract

The aim of the thesis is to study a stationary fluid-structure interaction problem between a viscous incompressible fluid, namely the wind blowing into a wind tunnel, and an elastic structure representing an airfoil with one endpoint fixed to a wall of the wind tunnel and the other one left free to move. The 3D fluid domain is bounded, simply connected and, because of the presence of the edges and vertices of the wind tunnel itself, Lipschitz. The fluid is modelled by the stationary Navier-Stokes equations subject to inflow/outflow conditions. The solid structure instead is modelled by a 1D beam equation, with a load density involving the force exerted by the fluid on the structure and, thereby, may vary its position. In a regime governed by small boundary data, we prove the existence and uniqueness of the strong solution to the PDE-ODE coupled system, together with its Lipschitz continuity with respect to the boundary datum.

Keywords: Fluid-structure interaction · Navier-Stokes equations · Beam equation · PDE-ODE system · Existence and uniqueness of equilibria

Abstract in lingua italiana

Lo scopo della tesi è di studiare un problema di interazione fluido-struttura stazionario tra un fluido viscoso e incompressibile, nel caso in questione il vento che soffia all'interno di una galleria del vento, e una struttura elastica che rappresenta un profilo alare, avente un'estremità fissata a un muro della galleria del vento e l'altra libera di muoversi. Il dominio 3D del fluido è limitato, semplicemente connesso e, per la presenza degli spigoli e vertici della galleria del vento, Lipschitz. Il fluido è modellato dalle equazioni di Navier-Stokes dotate di condizioni di inflow/outflow. La struttura solida è invece modellata dall'equazione della trave in 1D, con una densità di carico che comprende la forza esercitata dal fluido sulla struttura che, quindi, può variarne la posizione. In un regime di dato al bordo piccolo, dimostriamo l'esistenza e l'unicità della soluzione forte per il problema accoppiato EDP-EDO, oltre che la sua Lipschitzianità rispetto al dato di bordo.

Parole chiave: Interazioni fluido-struttura · Equazioni di Navier-Stokes · Equazione della trave · Sistema EDP-EDO · Esistenza e unicità degli equilibri

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Introduction

The interaction between fluid flows and elastic structures is a fundamental phenomenon that arises in a wide variety of natural and engineered systems. From blood circulation in arteries to offshore platforms subjected to ocean currents, from wind-induced vibrations of civil infrastructures to the aerodynamic behaviour of aircraft components, or to the interactions between air and humidity with the Earth's surface, whose not well known complexity still represents a challenge for the validity of weather forecast, the mutual influence between fluids and solids plays a crucial role in determining the overall response of the system. The mathematical description of these phenomena has become an active field of research over the last decades, attracting the interest of engineers, physicists and mathematicians.

Among the many applications of fluid–structure interaction, aerospace engineering provides some of the most challenging and technologically relevant examples. Airfoils are designed to be sufficiently flexible to reduce weight and improve aerodynamic efficiency, yet sufficiently stiff to withstand the loads generated during flight. The airflow surrounding the wing produces pressure forces that induce structural deformations of the solid structure, while the resulting change in the geometry modifies the fluid motion itself, see Figure 1 from a realistic situation where this phenomenon occurs. The performance and safety of an aircraft therefore depend on a delicate balance between aerodynamic and structural effects.



Figure 1: Boeing 787-8 Dreamliner, whose wings are flexed upward by pressure forces. Images taken from [6]

Historically, the importance of these interactions became evident with the development of modern aviation, which was highly triggered also by the necessity for rapid engineering responses during First World War. The study of aeroelastic phenomena such as divergence, control reversal, and flutter revealed that neglecting structural deformations may lead to inaccurate predictions and, in extreme situations, to catastrophic failures. In Figure 2, one of the flight tests for the heavy bomber Handley-Page O/400 (1916) is shown, which represents the first documented case of flutter. These observations motivated the development of increasingly sophisticated mathematical models capable of capturing the coupling between fluid mechanics and elasticity. Today, aeroelastic analysis constitutes a fundamental component of aircraft design, contributing to the optimization of structural performances and the assessment of operational safety.

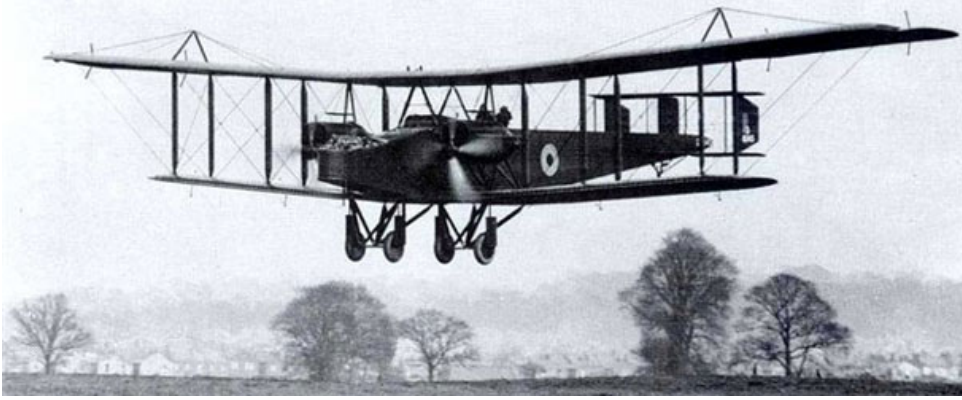


Figure 2: Heavy bomber Handley-Page O/400 (1916). Image taken from [17]

The mathematical modelling of fluid–structure interaction requires the combination of equations governing different physical processes. On the one hand, the fluid behaviour is commonly described by the Navier–Stokes equations, which represent one of the cornerstones of continuum mechanics. On the other hand, structural deformations are modelled through elasticity theories whose complexity depends on the geometry and mechanical properties of the solid. Even when simplified structural models are employed, the resulting coupled systems remain highly nonlinear due to the mutual dependence between the fluid forces and the structural displacement.

The present thesis focuses on a stationary fluid–structure interaction problem motivated by the analysis of aeroelastic equilibrium configurations. More precisely, we consider a viscous incompressible fluid governed by the stationary Navier–Stokes equations interacting with a flexible wing modelled as a one-dimensional elastic beam. The deformation of the beam determines the shape of the fluid domain, while the stresses exerted by the fluid act as external loads in the structural equation. The equilibrium configuration is

therefore characterized by a nonlinear coupling between the fluid velocity, the pressure field, and the structural displacement.

From a mathematical viewpoint, this problem presents several analytical difficulties. Since the fluid domain depends on the unknown deformation of the structure, the governing equations must be studied in a framework involving moving geometries. Furthermore, the coupling conditions generate nonlinear interactions that prevent a direct application of classical existence theories. As a consequence, establishing the well-posedness of the model requires a careful analysis of both the fluid and structural components and of their interplay.

The objective of this thesis is to investigate the stationary coupled system and to provide a rigorous analysis of its solutions. In particular, we study the existence, uniqueness and stability of strong solutions describing equilibrium configurations of the fluid–beam system. These properties are of primary importance, as they guarantee that the mathematical model is consistent and that its predictions are robust with respect to perturbations of the data. The analysis presented here is highly based on the work performed in [7], where the same model was formulated and studied, but for the interaction between the fluid and the solid structure of a bridge fixed at both endpoints, instead of an airfoil free to move at one endpoint. Building upon those results, we present the main ideas and techniques underlying the proof of well-posedness and highlight the mathematical mechanisms governing the fluid–structure coupling.

In the following of the Introduction we settle the mathematical basis of the problem. We consider the fluid-structure interaction (FSI) problem in the rectangular parallelepiped

$$P = (-R, R) \times (-1, 1)^2, \quad R \gg 1$$

representing a wind tunnel in which artificial airflows are set at the inlet and outlet. Inside the wind tunnel the airfoil occupies a compact connected set $\mathcal{B}$ such that $\Omega = P \setminus \mathcal{B}$ is still simply connected, differently from the bridge in [7]. The fluid is viscous, homogeneous, incompressible and governed by the stationary Navier-Stokes equations

$$-\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1)$$

where $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$ is the velocity vector field, p is the scalar pressure and $\eta > 0$ is the kinematic viscosity. The fluid is moving due to inflows/outflows located at the faces $\{x = \pm R\}$. We are going to set the wing $\mathcal{B}$ as *clamped* to ∂P at $y = 1$, considering two different study cases:

1. in the first study case, the clamped condition fixes the function describing the wing displacement and its derivatives, up to third order, at $y = 1$;
2. in the second study case, the clamped condition fixes only the function and its first derivative at $y = 1$, while it fixes second and third derivative at the free boundary of the wing $y = 0$.

The formulation of the second case, in particular, was also studied in [32] for modelling the *horizontal flagpole* [32, p. 12], in a review in which other models for the beam were also presented. From a mathematical point of view, the difference between the two cases lies in how we analyse the problem that involves a beam-type equation, which is a fourth order ODE (see (2) below):

1. in the first study case, being all the boundary conditions set at $y = 1$, we treat the model as a Cauchy problem centred at $y = 1$;
2. in the second study case, we see the model as a boundary value problem, since the boundary conditions are set in both the endpoints of the beam, $y = 0$ and $y = 1$.

We then suppose the cross-sections of $\mathcal{B}$, denoted with $\mathcal{B}_y$ for $y \in [0, 1]$, having a *relatively smooth* shape ($C^{1,1}$ regularity is enough). The wind hits the wing horizontally, which modifies the flow that goes around it: in the (downwind) leeward, the flow creates asymmetric vortices that generate an orthogonal force $L = L(y, \mathbf{u}, p)$ named *lift* [1, 30], which triggers the vertical displacements of the wing; see the experiment in [22]. In Figure 3 a possible section of the wing is represented, around which we can see the velocity and pressure distribution producing the lift force.

Therefore, $\mathcal{B}$ moves vertically and reaches a new position $\mathcal{B}^h$, where $h = h(y)$ is a smooth function describing its vertical displacement. This also changes the fluid domain and (1) is satisfied in $\Omega_h = P \setminus \mathcal{B}^h$. Further forces (e.g., the drag and torque) may also appear, see again [1, 30], but here we do not consider them. The resulting movements of the wing create more perturbations of the flow and generate a full FSI.

In absence of a flow, the wing $\mathcal{B}$ is maintained in its equilibrium position ($h \equiv 0$) by gravity combined with some elastic forces due to the attachment of the wing to the fuselage of the airplane, so that the overall forces $f = f(h)$ have the same sign as h . In the static setting inside a wind tunnel, the vertical position $z = h(y)$ of the barycenter of the cross-section $\mathcal{B}_y$ obeys to the beam-type equation

$$h''''(y) + f(h(y)) = L(y, \mathbf{u}, p) = L(y, h(y)), \quad (2)$$

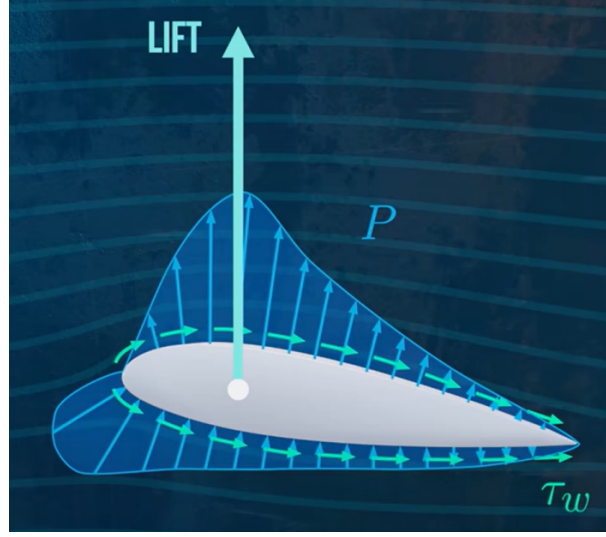


Figure 3: Velocity and pressure distributions around the airfoil, generating the lift. Source: CFD LAND [14]

which must be complemented with the two different types of boundary conditions that we discussed earlier, which give rise to two distinguished problems (see Chapter 1 for their definition). In (2) the lift L also depends on h since $(\mathbf{u}, p)$ solves (1) in Ω_h .

The equilibrium configurations of the FSI problem are obtained by solving the *system* (1)-(2). It is our purpose to analyse existence, uniqueness, stability of the equilibrium configurations of (1)-(2) as the inflows/outflows vary. In [5, 8, 9, 19] the 2D version of this problem was investigated, where the structure represents the cross-section of the deck of a bridge. The associated dynamical setting was studied in [12, 20, 31]. In 3D, the only known result [18] is that, when $\mathcal{B}$ is a rectangular plate and under *symmetry* assumptions on both Ω and the inflow/outflow (Poiseuille type), for small flows the unique equilibrium position remains Ω ; see also [13, 15] for related problems. These contributions are connected to one of the classical Leray's problems [2, 3]. Other interactions between fluids and elastic structures were extensively studied in the last decade, both mathematically [4, 11, 21, 28] and from an engineering point of view [24], see also references therein.

In Chapter 1 we describe in full detail the geometric and physical frameworks of the model. For this, defining suitable functional spaces and relations is required, as it will also give the precise assumptions used in the sequel.

In Chapter 2 we discuss existence, uniqueness, and a priori bounds for the solutions to (1), *assuming that h is given*, see Theorem 2.2. The difficult task here is the choice of the functional spaces for dealing with *strong solutions*, see Definition 2.1. We need enough regularity to have embedding into boundary spaces where the (1D) lift $L = L(y, \mathbf{u}, p)$ is

well-defined, see (1.12) below. Since all the Ω_h are Lipschitz (and nonconvex), to gain regularity we perform a reflection with respect to some faces of P . Differently from [7], we consider the wing as *smoothly* attached to the fuselage of the airplane, so that we do not treat the subtleties of having angular points of $\partial\Omega_h$, but only the ones from ∂P .

In Chapter 3 we take advantage of this functional framework and we tackle the FSI problem (1)-(2) in Ω_h ; we prove that in the uniqueness regime for (1) (small flows) and in a neighbourhood of $h = 0$ there exists a unique equilibrium configuration that varies with Lipschitz continuity with respect to the flows, see Theorem 3.1. The proof is obtained with a fixed-point procedure that requires the introduction of suitable functional spaces also for the beam displacement h . By exploiting Theorem 2.2, we first show that for a given $\bar{h}$ there exists a unique solution $(\mathbf{u}(\bar{h}), p(\bar{h}))$ to (2.2) satisfying some bounds. Then we prove that, for such $(\mathbf{u}(\bar{h}), p(\bar{h}))$, there exists a unique (strong) solution h to (2) with $L(y, \bar{h})$ under the “physical” boundary conditions described in Chapter 1. Finally, through a contraction principle we find a unique fixed point $(h, \mathbf{u}, p)$, solving (1)-(2). For further results in a symmetric framework, and consequent physical phenomenon, see again [7].

1 | Functional, Geometric and Physical Frameworks

1.1. Functional analysis setting

We dedicate this first section of Chapter 1 to setting the necessary functional relations and spaces, in which we look for the solutions of the beam-type equation (2). In particular, we distinguish between the functions satisfying the Cauchy problem rising from (2) and the ones satisfying the associated boundary value problem, which we called respectively first and second study case, in the Introduction. Firstly, we introduce the functional space

$$\mathbb{H} = H_{\{1\}}^2(0, 1) = \{v \in H^2(0, 1) : v(1) = v'(1) = 0\},$$

together with the quantities

$$(h_1, h_2)_{\mathbb{H}} = \int_0^1 h_1'' h_2'', \quad \|h\|_{\mathbb{H}} = \|h''\|_{L^2(0,1)} \sim \|h\|_{H^2(0,1)}, \quad (1.1)$$

for any $h, h_1, h_2 \in \mathbb{H}$. Moreover, we have the embedding inequality

$$\|h\|_{L^\infty(0,1)} \leq S_{\mathbb{H}} \|h\|_{\mathbb{H}}, \quad \frac{1}{S_{\mathbb{H}}} = \min_{h \in \mathbb{H}} \frac{\|h\|_{\mathbb{H}}}{\|h\|_{L^\infty(0,1)}}. \quad (1.2)$$

Analogously, we define the functional spaces

$$\begin{aligned} \mathbb{H}_C^4 &= \{v \in H^4(0, 1) : v(1) = v'(1) = v''(1) = v'''(1) = 0\} \\ \mathbb{H}_{BV}^4 &= \{v \in H^4(0, 1) : v(1) = v'(1) = 0, v''(0) = v'''(0) = 0\}, \end{aligned}$$

where C stands for *Cauchy* and BV stands for *Boundary Value*, making explicit the distinction between the two types of functions solving the two different study cases: in $\mathbb{H}_C^4$ we set to zero the function and all its derivatives, up to the third order, in $y = 1$; in $\mathbb{H}_{BV}^4$ we set to zero the function and its first derivative in $y = 1$ and its second and third derivative in $y = 0$. Notice that, since we are dealing with 1D functions, we have

the embedding $H^4(0,1) \subset C^3([0,1])$, which justifies the punctual setting of functions up to their third derivatives (this, in general, is false in higher dimensions; thus, these spaces are not appropriate for modelling structures like 2D plates). For both spaces, we prescribe the following quantities

$$(h_1, h_2)_{\mathbb{H}^4} = \int_0^1 h_1'''' h_2'''' , \quad \|h\|_{\mathbb{H}^4} = \|h''''\|_{L^2(0,1)} \sim \|h\|_{H^4(0,1)}. \quad (1.3)$$

Then, we have the following

Proposition 1.1. *Given the functional spaces $\mathbb{H}$, $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$ defined above, we have that*

1. $(\cdot, \cdot)_{\mathbb{H}}$ and $\|\cdot\|_{\mathbb{H}}$ are respectively an inner product and a norm in $\mathbb{H}$, which is a Hilbert space if endowed with them;
2. $(\cdot, \cdot)_{\mathbb{H}^4}$ and $\|\cdot\|_{\mathbb{H}^4}$ are respectively an inner product and a norm in $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$, which are both Hilbert spaces if endowed with them;

Furthermore, if we consider the embedding inequality (1.2), we can find the following upper bound for the constant $S_{\mathbb{H}}$: $S_{\mathbb{H}} \leq \frac{1}{\sqrt{3}}$

Proof. Being $(\cdot, \cdot)_{\mathbb{H}}$ and $(\cdot, \cdot)_{\mathbb{H}^4}$ trivially bilinear and symmetric, to prove that they are inner products, we need to show their positive definiteness:

$$(h, h)_{\mathbb{H}} = \int_0^1 (h'')^2 \geq 0 \quad \forall h \in \mathbb{H},$$

$$(h, h)_{\mathbb{H}} = \int_0^1 (h'')^2 = 0 \iff \begin{cases} h'' = 0 \\ h(1) = 0 \\ h'(1) = 0 \end{cases} \iff h \equiv 0$$

for any $h \in \mathbb{H}$,

$$(h, h)_{\mathbb{H}^4} = \int_0^1 (h'''')^2 \geq 0 \quad \forall h \in \mathbb{H}_C^4 \quad \text{and} \quad \forall h \in \mathbb{H}_{BV}^4,$$

$$(h, h)_{\mathbb{H}^4} = \int_0^1 (h'''')^2 = 0 \iff \begin{cases} h'''' = 0 \\ h(1) = h'(1) = 0 \\ h''(1) = h'''(1) = 0 \end{cases} \iff h \equiv 0$$

for any $h \in \mathbb{H}_C^4$, and

$$(h, h)_{\mathbb{H}^4} = \int_0^1 (h''')^2 = 0 \iff \begin{cases} h'''' = 0 \\ h(1) = h'(1) = 0 \\ h''(0) = h'''(0) = 0 \end{cases} \iff h \equiv 0$$

for any $h \in \mathbb{H}_{BV}^4$. Hence $(\cdot, \cdot)_{\mathbb{H}}$ and $(\cdot, \cdot)_{\mathbb{H}^4}$ are inner products, which induce the norms $\|h\|_{\mathbb{H}} = \sqrt{(h, h)_{\mathbb{H}}} = \|h''\|_{L^2(0,1)}$ and $\|h\|_{\mathbb{H}^4} = \sqrt{(h, h)_{\mathbb{H}^4}} = \|h''''\|_{L^2(0,1)}$ respectively.

We now prove that $\|\cdot\|_{\mathbb{H}}$ is equivalent to $\|\cdot\|_{H^2(0,1)}$ in $\mathbb{H}$ and $\|\cdot\|_{\mathbb{H}^4}$ is equivalent to $\|\cdot\|_{H^4(0,1)}$ in both $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$, from which it follows that $\mathbb{H}$, $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$ are complete, since $\mathbb{H} \subset H^2(0,1)$, $\mathbb{H}_C^4$, $\mathbb{H}_{BV}^4 \subset H^4(0,1)$ and $H^2(0,1)$ and $H^4(0,1)$ are complete themselves.

By using Fundamental Theorem of Calculus twice with $h'(1) = 0$ and $h(1) = 0$ we have that

$$h(y) = - \int_y^1 h'(t) dt = \int_y^1 \int_t^1 h''(s) ds dt,$$

thanks to Cauchy-Schwarz

$$|h'(y)| \leq \int_y^1 |h''(s)| ds \leq \sqrt{1-y} \|h''\|_{L^2(0,1)},$$

then squaring and integrating

$$\|h'\|_{L^2(0,1)}^2 \leq \int_0^1 (1-y) dy \|h''\|_{L^2(0,1)}^2 = \frac{1}{2} \|h''\|_{L^2(0,1)}^2,$$

thus

$$\|h'\|_{L^2(0,1)} \leq \frac{1}{\sqrt{2}} \|h''\|_{L^2(0,1)}.$$

We can proceed analogously to obtain

$$\|h\|_{L^2(0,1)} \leq \frac{1}{\sqrt{2}} \|h'\|_{L^2(0,1)} \leq \frac{1}{2} \|h''\|_{L^2(0,1)},$$

thus the inequality for equivalence follows:

$$\|h\|_{H^2(0,1)}^2 = \|h\|_{L^2(0,1)}^2 + \|h'\|_{L^2(0,1)}^2 + \|h''\|_{L^2(0,1)}^2 \leq \left(\frac{1}{4} + \frac{1}{2} + 1 \right) \|h''\|_{L^2(0,1)}^2$$

$$\|h\|_{H^2(0,1)} \leq \frac{\sqrt{7}}{2} \|h''\|_{L^2(0,1)}.$$

For $\mathbb{H}_C^4$, recall that $h(1) = h'(1) = h''(1) = h'''(1) = 0$. Then, by using Fundamental Theorem of Calculus twice with $h'''(1) = 0$ and $h''(1) = 0$ we have that

$$h''(y) = - \int_y^1 h'''(t) dt = \int_y^1 \int_t^1 h''''(s) ds dt,$$

thanks to Cauchy-Schwarz

$$|h'''(y)| \leq \int_y^1 |h''''(s)| ds \leq \sqrt{1-y} \|h''''\|_{L^2(0,1)},$$

then squaring and integrating

$$\|h'''\|_{L^2(0,1)}^2 \leq \int_0^1 (1-y) dy \|h''''\|_{L^2(0,1)}^2 = \frac{1}{2} \|h''''\|_{L^2(0,1)}^2,$$

thus

$$\|h'''\|_{L^2(0,1)} \leq \frac{1}{\sqrt{2}} \|h''''\|_{L^2(0,1)}.$$

We can proceed analogously to obtain

$$\|h''\|_{L^2(0,1)} \leq \frac{1}{\sqrt{2}} \|h'''\|_{L^2(0,1)} \leq \frac{1}{2} \|h''''\|_{L^2(0,1)},$$

thus the inequality for equivalence follows:

$$\begin{aligned} \|h\|_{H^4(0,1)}^2 &= \|h\|_{L^2(0,1)}^2 + \|h'\|_{L^2(0,1)}^2 + \|h''\|_{L^2(0,1)}^2 + \|h'''\|_{L^2(0,1)}^2 + \|h''''\|_{L^2(0,1)}^2 \\ &\leq \frac{7}{4} \|h''\|_{L^2(0,1)}^2 + \|h'''\|_{L^2(0,1)}^2 + \|h''''\|_{L^2(0,1)}^2 \\ &\leq \left(\frac{7}{16} + \frac{1}{2} + 1 \right) \|h''''\|_{L^2(0,1)}^2 \end{aligned}$$

$$\|h\|_{H^4(0,1)} \leq \frac{\sqrt{31}}{4} \|h''''\|_{L^2(0,1)}$$

For $\mathbb{H}_{BV}^4$, recalling that $h(1) = h'(1) = h''(0) = h'''(0) = 0$, the computations are the same, but we now use Fundamental Theorem of Calculus with $h'''(0) = 0$ and $h''(0) = 0$, which gives

$$h''(y) = \int_0^y h'''(t) dt = \int_0^y \int_0^t h''''(s) ds dt.$$

Since the functions y and $1-y$ provide the same value when integrated in the interval $(0,1)$, we prove the same equivalence inequality as before, thus completing the proof.

We now prove the upper bound for the constant $S_{\mathbb{H}}$ in (1.2): from the Fundamental

Theorem, having $h'(1) = 0$, for any $y \in (0, 1)$ we have

$$h'(y) = - \int_y^1 h''(t) dt,$$

$$h(y) = \int_y^1 \int_s^1 h''(t) dt ds = \int_y^1 \int_y^t h''(t) ds dt = \int_y^1 (t - y) h''(t) dt,$$

hence, by using Cauchy-Schwarz

$$|h(y)| \leq \int_y^1 |t - y| |h''(t)| dt \leq \left(\int_y^1 (t - y)^2 dt \right)^{1/2} \|h''\|_{L^2(0,1)} = \frac{(1 - y)^{3/2}}{\sqrt{3}} \|h''\|_{L^2(0,1)}.$$

By taking $\|\cdot\|_{L^\infty(0,1)}$, we arrive at

$$\|h\|_{L^\infty(0,1)} \leq \frac{1}{\sqrt{3}} \|h''\|_{L^2(0,1)}.$$

and the inequality is then proved. $\square$

1.2. Geometric and physical settings

We denote the faces of the parallelepiped P by

$$\begin{aligned} \Gamma_x^- &= \{-R\} \times (-1, 1) \times (-1, 1), & \Gamma_x^+ &= \{R\} \times (-1, 1) \times (-1, 1), \\ \Gamma_y^- &= (-R, R) \times \{-1\} \times (-1, 1), & \Gamma_y^+ &= (-R, R) \times \{1\} \times (-1, 1), \\ \Gamma_z^- &= (-R, R) \times (-1, 1) \times \{-1\}, & \Gamma_z^+ &= (-R, R) \times (-1, 1) \times \{1\}. \end{aligned}$$

Consider then a $C^{1,1}$ compact, connected and simply connected set $\mathcal{B} \subset \overline{P}$ such that

$$\begin{aligned} X^+ &= \max_{(x,y,z) \in \mathcal{B}} x < R, & X^- &= - \min_{(x,y,z) \in \mathcal{B}} x < R, \\ Z^+ &= \max_{(x,y,z) \in \mathcal{B}} z < 1, & Z^- &= - \min_{(x,y,z) \in \mathcal{B}} z < 1, \\ \max_{(x,y,z) \in \mathcal{B}} y &= 1, & \min_{(x,y,z) \in \mathcal{B}} y &= 0, & \mathcal{H}^2(\mathcal{B}_y) &> 0 \text{ for } y \in (0, 1] \end{aligned} \tag{1.4}$$

where $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure and $\mathcal{B}_y = \mathcal{B} \cap (\mathbb{R} \times \{y\} \times \mathbb{R})$ is the cross-section of $\mathcal{B}$, namely the intersection with a plane parallel to both the x and z axes. Its boundary $\partial\mathcal{B}$ consists then of two parts

$$\partial\mathcal{B} = \mathcal{B}_1 \cup \mathcal{S} \tag{1.5}$$

where $\mathcal{S}$ is a $C^{1,1}$ -surface that does not intersect the face $\{y = 1\}$ and such that the set $\mathcal{S} \cup \overline{\Gamma_y^+ \setminus \mathcal{B}_1}$ is a $C^{1,1}$ -surface as well. For any $h \in \mathbb{H}$, the simply connected domain

$$\Omega_h = P \setminus \mathcal{B}^h = P \setminus \{(x, y, z + h(y)) \text{ with } (x, y, z) \in \mathcal{B}\}, \quad (1.6)$$

is the region occupied by the fluid; in particular, $\mathcal{B}^0 = \mathcal{B}$ and $\Omega_0 = \Omega$. Hence, $\partial\Omega_h$ consists of the two closed squares $\overline{\Gamma_x^\pm}$, the three closed rectangles $\overline{\Gamma_z^\pm}$ and $\overline{\Gamma_y^-}$, the closed non-simply connected planar region $\overline{\Gamma_y^+ \setminus \mathcal{B}_1}$ and $\mathcal{S}$. We assume that $\mathcal{B}_y \subset (-R, R) \times (-1, 1)$ is a connected planar compact set with barycenter at $(x, z) = (0, 0)$ for all $y \in [0, 1]$. Then, in view of (1.6), the barycenter of the cross-section $\mathcal{B}_y^h$ is located at $(x, z) = (0, h(y))$. We want to stress that the airfoil is still treated as a 3D object, whose cross-sections translate vertically following the movement of their barycenters described by $h = h(y)$. After defining $\varepsilon^\pm(h) = \text{dist}(\mathcal{B}^h, \Gamma_z^\pm)$, it follows from (1.4) that:

$$\mathcal{B}^h \subset [-X^-, X^+] \times [0, 1] \times [-1 + \varepsilon^-(h), 1 - \varepsilon^+(h)].$$

Differently from [7], since we have asked $\mathcal{S} \cup \overline{\Gamma_y^+ \setminus \mathcal{B}_1}$ to be $C^{1,1}$, the domain $\Omega_h = P \setminus \mathcal{B}^h$ is *Lipschitz* only because of the vertices and edges of the parallelepiped P , not because of the intersection between $\mathcal{B}^h$ and the face Γ_y^+ of P . A vertical displacement $h = h(y)$ could in general affect the regularity of $\mathcal{S}$, and so of $\mathcal{S} \cup \overline{\Gamma_y^+ \setminus \mathcal{B}_1}$, but we shall see a posteriori that h will be sufficiently smooth. $C^{1,1}$ regularity for the boundary of the fluid domain Ω_h , away from edges and vertices, represents the minimal geometric regularity requested by classical elliptic regularity theory to prove that the solutions are strong (see Definition 2.1 and the proof of Theorem 2.2), given an inflow/outflow datum as in (1.9). To better visualize the domain Ω (and so its transformation Ω_h), in Figure 1.1, a sketch resembling the idea of the domain Ω is presented. Note, though, that in our setting the airfoil and its junction to the wind tunnel are smoother. Furthermore the wing protrudes only up to the center of the wind tunnel, which leaves the domain simply connected, differently from what is sketched in the image.

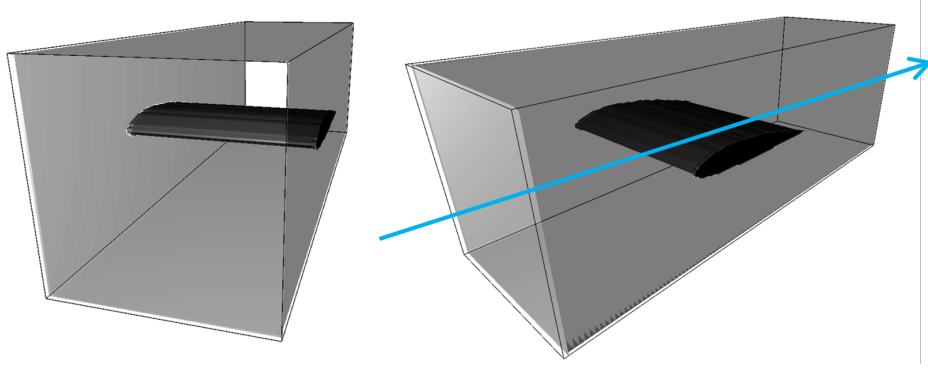


Figure 1.1: Example of fluid domain Ω . Source: Flowsquare+ Wind Tunnel [29]

Since $\mathbb{H}$ compactly embeds into $C([0, 1])$, we have the lower bounds

$$\begin{aligned} \varepsilon^+(h) &\geq 1 - \max_{[0,1]} h - Z^+ \geq 1 - \|h\|_{L^\infty(0,1)} - Z^+ \\ \varepsilon^-(h) &\geq 1 + \min_{[0,1]} h - Z^- \geq 1 - \|h\|_{L^\infty(0,1)} - Z^- \end{aligned} \quad \forall h \in \mathbb{H}. \quad (1.7)$$

A sufficient condition to prevent collisions (intersections) between $\mathcal{B}^h$ and the top/bottom boundaries of P , namely $\varepsilon^\pm(h) = 0$, is then that there exists

$$1 \leq \omega < \frac{1}{\max\{Z^+, Z^-\}}$$

such that

$$\|h\|_{L^\infty(0,1)} < 1 - \omega \max\{Z^+, Z^-\} \quad (1.8)$$

We now complement (1) with inflow/outflow conditions at the inlet/outlet $\Gamma_x^\pm$: for any $\sigma > 0$, we define the function

$$\begin{aligned} V &\in W^{2-\frac{1}{2+\sigma}, 2+\sigma}(-1, 1)^2 \subset C([-1, 1]^2) \\ \text{such that } V(y, -1) &= 0 \text{ for } |y| \leq 1, \quad V(\pm 1, z) = 0 \text{ for } |z| \leq 1, \end{aligned} \quad (1.9)$$

where we stress the fact that we are not imposing the value of V on the upper side of $[-1, 1]^2$, i.e. $V(y, 1)$ for $|y| \leq 1$ (we only impose it continuous). Then, we take

$$V_i, V_o \in W^{2-\frac{1}{2+\sigma}, 2+\sigma}(\Gamma_x^\pm)$$

verifying

$$V_i = V_o = V \text{ on } \overline{\Gamma_y^+ \setminus \mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm, \quad \int_{\Gamma_x^-} V_i = \int_{\Gamma_x^+} V_o = 1. \quad (1.10)$$

Differently from [7], introducing non Hilbertian spaces for the solution of (1), and so for the inflow/outflow datum, is not needed because of the junction between the structure and the wind tunnel, which is now smooth enough not to potentially compromise the regularity of the solution (see the proof of Theorem 2.2), but because the definition of the lift requires one-dimensional traces in the fluid domain, which would not be defined starting from an H^2 velocity, see (1.12). Thus we still need $\sigma > 0$, but without any upper constraint.

For $\gamma > 0$ we consider the boundary value problem

$$\begin{aligned} -\eta\Delta\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p &= 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_h \\ \mathbf{u} &= 0 \quad \text{on } \partial\mathcal{B}^h, \quad \mathbf{u} = \gamma V \mathbf{e}_1 \quad \text{on } \overline{\Gamma_y^+ \setminus \mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm, \\ \mathbf{u} &= \gamma V_i \mathbf{e}_1 \quad \text{on } \Gamma_x^-, \quad \mathbf{u} = \gamma V_o \mathbf{e}_1 \quad \text{on } \Gamma_x^+, \end{aligned} \quad (1.11)$$

where the boundary condition $\mathbf{u} = \gamma V \mathbf{e}_1$ on $\overline{\Gamma_y^+ \setminus \mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm$, taking also into account the definition of V in (1.9), imposes zero velocity at the floor and side walls of the wind tunnel, and continuous, but not necessarily zero, velocity on the roof of the wind tunnel. Since the fluid is incompressible, the flux $\gamma > 0$ is constant through any cross-section $\{x = x_0\}$ (for $|x_0| \leq R$) of Ω_h : this is why (1.10) is a *necessary condition* for the existence of solutions to (1.11).

For a solution $(\mathbf{u}, p)$ to (1.11), the lift force acting vertically (in the z -direction) on a cross-section $\mathcal{B}_y^h$ for a given $y \in [0, 1]$ is computed through the formula

$$L(y, \mathbf{u}, p) = -\mathbf{e}_3 \cdot \int_{\partial\mathcal{B}_y^h} \mathbb{T}(\mathbf{u}, p) \mathbf{n} = -\mathbf{e}_3 \cdot \int_{\partial\mathcal{B}_y^h} [\eta(\nabla\mathbf{u} + \nabla^T\mathbf{u}) - p\mathbb{I}] \mathbf{n}, \quad (1.12)$$

where $\mathbb{I}$ is the 3×3 identity matrix and $\mathbf{n}$ is the outward unit normal vector to $\partial\mathcal{B}^h$, pointing inside $\mathcal{B}^h$. Since $\mathcal{B}$ is of class $C^{1,1}$, the normal vector $\mathbf{n}$ is well-defined and, as we shall see, the solution $(\mathbf{u}, p)$ is smooth enough to ensure that the integrand in (1.12) is bounded.

To model the behaviour of $\mathcal{B}$, we set up a one-dimensional horizontal thin beam covering the interval $(0, 1)$. If $h(y)$ denotes the vertical displacement of the beam (in the z -direction) from the equilibrium position ($\equiv 0$) at the point $y \in (0, 1)$, we know that the energy density E_b stored by *bending* the beam is proportional to the square of the curvature:

$$E_b(h) = \frac{EI}{2} \int_0^1 \frac{h''(y)^2}{(1 + h'(y)^2)^3} \sqrt{1 + h'(y)^2} dy = \frac{EI}{2} \int_0^1 \frac{h''(y)^2}{(1 + h'(y)^2)^{5/2}} dy, \quad (1.13)$$

where $EI > 0$ is the flexural rigidity. Formula (1.13) highlights the curvature and the arclength. The functional E_b *does not include the stretching* energy, a term measuring the length variation of the beam, since we assume the airfoil to have a fixed length, as we neglect possible thinning phenomena. For small displacements of the beam, in particular for small h' , the approximated bending energy reads

$$E_b(h) \approx \int_0^1 \frac{h''(y)^2}{2} dy, \quad (1.14)$$

in which we have normalised the flexural rigidity.

The beam is subject to restoring forces $f = f(h)$ that satisfy

$$f \in \text{Lip}(\mathbb{R}), \quad f(0) = 0, \quad h \mapsto f(h) \text{ is non-decreasing} \quad (1.15)$$

As mentioned in the Introduction, these forces are due to gravity and the attachment of the wing to the fuselage of the airplane. Due to the fluid flow, the beam is also subject to the lift force $L = L(y, h(y))$; in fact, $L = L(y, \mathbf{u}, p)$, see (1.12), where $(\mathbf{u}, p)$ solves (1.11) in Ω_h and, hence, depends on h . As we shall see in Lemma 3.2 and Proposition 3.8, L is bounded with respect to y and Lipschitz continuous with respect to h . Hence, after introducing the *potentials* $F : \mathbb{R} \rightarrow \mathbb{R}$ and $G : (0, 1) \times \mathbb{R} \rightarrow \mathbb{R}$ such that

$$F'(h) = f(h), \quad G_h(y, h) = L(y, h),$$

the overall (approximated) energy of the beam is

$$\int_0^1 \frac{h''(y)^2}{2} dy + \int_0^1 F(h(y)) dy - \int_0^1 G(y, h(y)) dy \quad \forall h \in \mathbb{H}. \quad (1.16)$$

Under suitable assumptions on G (satisfied in our context), this functional is well-defined over $\mathbb{H}$ and its critical points satisfy the Euler-Lagrange equation

$$\int_0^1 \left[h''(y)\phi''(y) + f(h(y))\phi(y) - L(y, h(y))\phi(y) \right] dy = 0 \quad \forall \phi \in \mathbb{H}$$

and, in strong form,

$$h''''(y) + f(h(y)) = L(y, h(y)) \quad \text{for a.e. } y \in (0, 1), \quad (1.17)$$

that coincides with (2). In Chapter 3, we will prove that $h \in \mathbb{H}_C^4$ or $h \in \mathbb{H}_{BV}^4$ and that (1.17) is satisfied in strong form, see Proposition 3.4 below. However, for the regularity

of solutions to (1.11), it is sufficient to consider the larger space

$$C_{\{1\}}^{1,1}([0, 1]) = \{h \in C^{1,1}([0, 1]) \text{ s.t. } h(1) = 0\}.$$

Finally, the differential equation (1.17) has to be complemented with some boundary conditions, depending on how the beam is fixed at the endpoints. As said in the Introduction, for a beam representing the wing of an airplane, we have two different study cases:

1. $h(1) = h'(1) = h''(1) = h'''(1) = 0$,
2. $h(1) = h'(1) = 0$ and $h''(0) = h'''(0) = 0$.

In both cases we require $h \in \mathbb{H}$. In the first case, since we are facing a Cauchy problem centred at $y = 1$, we will make use of the theory of ODEs for the existence, uniqueness and regularity of the solution, while for the second case, having a boundary value problem, we will proceed analogously to what done in [7, Proposition 4.4]. h'' and h''' on $y = 0$ may not be defined, but we will see in Chapter 3 that $h \in \mathbb{H}_{BV}^4$, which implies the aforementioned conditions on $y = 0$ for h'' and h''' .

2 | Fluid Boundary Value Problem

To simplify notations, we group the boundary conditions in (1.11) by defining $u_* : \partial\Omega_h \rightarrow \mathbb{R}^3$ as

$$\mathbf{u}_* = \begin{cases} 0 & \text{on } \partial\mathcal{B}^h \\ \gamma V \mathbf{e}_1 & \text{on } \overline{\Gamma_y^+ \setminus \mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm \\ \gamma V_i \mathbf{e}_1 & \text{on } \Gamma_x^- \\ \gamma V_o \mathbf{e}_1 & \text{on } \Gamma_x^+. \end{cases} \quad (2.1)$$

Let us explain what is meant by solution to (1.11), that is,

$$\begin{cases} -\eta \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0, & \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega_h \\ \mathbf{u} = \mathbf{u}_* & & \text{on } \partial\Omega_h. \end{cases} \quad (2.2)$$

Definition 2.1. *We say that*

- A pair $(\mathbf{u}, p) \in H^1(\Omega_h) \times L^2(\Omega_h)$ is a **weak solution** to (2.2) if $\mathbf{u}$ satisfies the boundary condition (2.2)₂ in the trace sense and for every $\boldsymbol{\psi} \in H_0^1(\Omega_h)$ we have

$$\eta \int_{\Omega_h} \nabla \mathbf{u} : \nabla \boldsymbol{\psi} - \int_{\Omega_h} p \nabla \cdot \boldsymbol{\psi} + \int_{\Omega_h} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\psi} = 0; \quad (2.3)$$

- A pair $(\mathbf{u}, p) \in W^{2,2+\sigma}(\Omega_h) \times W^{1,2+\sigma}(\Omega_h)$, for $\sigma > 0$, is a **strong solution** to (2.2) if $\mathbf{u}$ satisfies the boundary condition (2.2)₂ pointwise and (2.2)₁ a.e. in Ω_h .

As mentioned after the definition of the inflow/outflow datum (1.9), such a regularity in non Hilbertian spaces is essential in order for the lift to be well-defined, since its definition (1.12) requires one-dimensional traces in the fluid domain, see also the proof of Lemma 3.2 below, where the application of the trace theory, used to define the gradient of the velocity and the pressure on the boundary of the cross-sections of $\mathcal{B}^h$, makes this argument clearer. Thus, from now on, starting from the following statement, we always assume $\sigma > 0$. We show that, under suitable assumptions, (2.2) admits a strong solution.

Theorem 2.2. *Let $h \in C_{\{1\}}^{1,1}([0, 1])$ be small enough to ensure (1.8). For any $\gamma \geq 0$, (2.2) admits a weak solution $(\mathbf{u}, p)$ in Ω_h ; there exists $\bar{\gamma} > 0$ such that the weak solution is unique for $\gamma \in [0, \bar{\gamma})$. Moreover, any weak solution is a strong solution and there exists $C^* > 0$, depending on σ but not on h , such that*

$$\|\mathbf{u}\|_{W^{2,2+\sigma}(\Omega_h)} + \|p\|_{W^{1,2+\sigma}(\Omega_h)} \leq C^* \gamma \quad \forall \gamma \in [0, \bar{\gamma}). \quad (2.4)$$

Proof. We preliminarily observe that $\partial\mathcal{B}^h = \mathcal{B}_1 \cup \mathcal{S}^h$, where $\mathcal{S}^h$ is the vertical translation of $\mathcal{S}$ through the function h and, hence, it is still of class $C^{1,1}$.

Existence and uniqueness of weak solutions. Since Ω_h is a Lipschitz domain with $\partial\Omega_h$ having only one connected component and since

$$\int_{\partial\Omega_h} \mathbf{u}_* \cdot \mathbf{n} = 0,$$

by [23, 25] for every $\varepsilon > 0$ there exists a Leray-Hopf solenoidal extension $\mathbf{s} \in H^1(\Omega_h)$ of the boundary datum $\mathbf{u}_*$ satisfying

$$-\int_{\partial\Omega_h} (\mathbf{v} \cdot \nabla \mathbf{s}) \mathbf{v} \leq \varepsilon \|\nabla \mathbf{v}\|_{L^2(\Omega_h)}^2 \quad \forall \mathbf{v} \in H_0^1(\Omega_h).$$

Letting $\mathbf{v} = \mathbf{u} - \mathbf{s}$, we recast (2.2) into

$$\begin{cases} -\eta \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p = \mathbf{f}, & \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega_h \\ \mathbf{v} = 0 & & \text{on } \partial\Omega_h \end{cases} \quad (2.5)$$

where $\mathbf{f} = \eta \Delta \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{s} - \mathbf{v} \cdot \nabla \mathbf{s} - \mathbf{s} \cdot \nabla \mathbf{s}$. A definition of weak and strong solution to (2.5) follows from Definition 2.1, the only difference being that the right-hand side is now given by the L^2 -inner product between $\mathbf{f}$ and every $\boldsymbol{\psi} \in H_0^1(\Omega_h)$. Let $V(\Omega_h) = \{\boldsymbol{\varphi} \in H_0^1(\Omega_h) : \nabla \cdot \boldsymbol{\varphi} = 0 \text{ a.e. in } \Omega_h\}$. From [16, Lemma IX.1.2] it follows that $\mathbf{v}$ is a weak solution to (2.5) if and only if for every $\boldsymbol{\varphi} \in V(\Omega_h)$ we have

$$\eta \int_{\Omega_h} \nabla \mathbf{v} : \nabla \boldsymbol{\varphi} + \int_{\Omega_h} \mathbf{v} \cdot \nabla \mathbf{v} \cdot \boldsymbol{\varphi} = \langle \mathbf{f}, \boldsymbol{\varphi} \rangle, \quad (2.6)$$

where $\langle \cdot, \cdot \rangle$ represents the duality pairing in $H^{-1}(\Omega_h)$. From [16, Theorem IX.4.1] we know that there exists $\mathbf{v} \in V(\Omega_h)$ satisfying (2.6) which is equivalent to say that $\mathbf{u} = \mathbf{v} + \mathbf{s} \in H^1(\Omega_h)$, together with the associated pressure $p \in L^2(\Omega_h)$ (which is chosen with

zero mean in Ω_h), is a weak solution to (2.2). In addition, from [16, Theorem IX.4.1] it follows that

$$\|\mathbf{u}\|_{H^1(\Omega_h)} \leq C \left[\left(1 + \frac{1}{\eta}\right) \|\mathbf{s}\|_{H^1(\Omega_h)} + \frac{1}{\eta} \|\mathbf{s}\|_{H^1(\Omega_h)}^2 \right] \leq C (\gamma + \gamma^2), \quad (2.7)$$

where $C > 0$ does not depend on h due to the smallness assumption on its $C^{1,1}$ -norm.

Next, we prove that the weak solution $\mathbf{u} \in H^1(\Omega_h)$ constructed above is unique for sufficiently small $\gamma \geq 0$. Let $\mathbf{u}_1, \mathbf{u}_2 \in H^1(\Omega_h)$ be any two weak solutions to (2.2), and define $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$. Then $\mathbf{w}$ solves

$$\eta \int_{\Omega_h} \nabla \mathbf{w} : \nabla \boldsymbol{\varphi} + \int_{\Omega_h} \mathbf{w} \cdot \nabla \mathbf{w} \cdot \boldsymbol{\varphi} = - \int_{\Omega_h} \mathbf{w} \cdot \nabla \mathbf{u}_2 \cdot \boldsymbol{\varphi} + \mathbf{u}_2 \cdot \nabla \mathbf{w} \cdot \boldsymbol{\varphi} \quad (2.8)$$

for every $\boldsymbol{\varphi} \in V(\Omega_h)$. Hence, choosing $\boldsymbol{\varphi} = \mathbf{w}$ in (2.8), and using Hölder and Sobolev inequality, we get

$$\begin{aligned} \eta \|\nabla \mathbf{w}\|_{L^2(\Omega_h)}^2 &= - \int_{\Omega_h} \mathbf{w} \cdot \nabla \mathbf{u}_2 \cdot \mathbf{w} \leq \|\nabla \mathbf{u}_2\|_{L^2(\Omega_h)} \|\mathbf{w}\|_{L^4(\Omega_h)}^2 \\ &\leq C \|\nabla \mathbf{u}_2\|_{L^2(\Omega_h)} \|\nabla \mathbf{w}\|_{L^2(\Omega_h)}^2. \end{aligned} \quad (2.9)$$

Since $\mathbf{u}_2$ is a weak solution to (2.2), injecting (2.7) into (2.9), we get

$$\eta \|\nabla \mathbf{w}\|_{L^2(\Omega_h)}^2 \leq C (\gamma + \gamma^2) \|\nabla \mathbf{w}\|_{L^2(\Omega_h)}^2. \quad (2.10)$$

Taking $\gamma > 0$ small enough such that

$$C(\gamma + \gamma^2) < \eta$$

from (2.10) we arrive at $\|\nabla \mathbf{w}\|_{L^2(\Omega_h)} = 0$, and since $\mathbf{w} = 0$ on $\partial\Omega_h$ we conclude that $\mathbf{w} \equiv 0$, i.e., $\mathbf{u}_1 = \mathbf{u}_2$ in Ω_h .

Regularity. The boundary $\partial\Omega_h$ is locally of class $C^{1,1}$ except at the vertices and the edges of the wind tunnel (differently from [7], we have modelled the junction between wing and wind tunnel in such a way that $\partial\Omega_h$ is locally $C^{1,1}$ also close to $\partial\mathcal{B}_1^h$, thus we do not need to introduce a theory for polyhedral domains, as presented for instance in [26]). Therefore, we need to analyse only the behaviour of a solution $(\mathbf{u}, p)$ close to the points belonging to the closed edges of P , thereby including the eight vertices.

Consider the parallelepiped $P^l = (-R, R) \times [1, 3) \times (-1, 1)$ and the reflection $\mathcal{B}^{h,l}$ of $\mathcal{B}^h$

with respect to the plane $y = 1$, and let $\Omega_h^l = P^l \setminus \mathcal{B}^{h,l}$. If we define $\mathbf{u}^l : \Omega_h^l \rightarrow \mathbb{R}^3$ and $p^l : \Omega_h^l \rightarrow \mathbb{R}$ as

$$\begin{aligned}\mathbf{u}^l(x, y, z) &= (u_1^l(x, y, z), u_2^l(x, y, z), u_3^l(x, y, z)) \\ &= (u_1(x, 2 - y, z), -u_2(x, 2 - y, z), u_3(x, 2 - y, z)) \\ p^l(x, y, z) &= p(x, 2 - y, z),\end{aligned}$$

then $(\mathbf{u}^l, p^l)$ solves

$$-\eta \Delta \mathbf{u}^l + \mathbf{u}^l \cdot \nabla \mathbf{u}^l + \nabla p^l = 0, \quad \nabla \cdot \mathbf{u}^l = 0 \quad \text{in } \Omega_h^l.$$

Hence, the pair $(\bar{\mathbf{u}}, \bar{p})$ defined as

$$(\bar{\mathbf{u}}, \bar{p}) = \begin{cases} (\mathbf{u}, p) & \text{in } \Omega_h \\ (\mathbf{u}^l, p^l) & \text{in } \Omega_h^l \end{cases}$$

is smooth across $\Gamma_y^+ \setminus \partial \mathcal{B}_1^h$ and solves

$$-\eta \Delta \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{p} = 0, \quad \nabla \cdot \bar{\mathbf{u}} = 0 \quad \text{in } \Omega_h \cup \Omega_h^l.$$

Thanks to this reflection, all the points of $\Gamma_y^+ \setminus \partial \mathcal{B}_1^h$ are interior points of $\Omega_h \cup \Omega_h^l$. In a similar manner, we introduce reflections with respect to the planes $y = -1$ and $z = -1$ over which homogeneous Dirichlet boundary conditions for u are satisfied. We then find a pair $(\tilde{\mathbf{u}}, \tilde{p})$, defined in $\tilde{\Omega}_h = (-R, R) \times (-3, 3) \times (-3, 1) \setminus \{\mathcal{B}^h \text{ and its reflections}\}$, solving

$$-\eta \Delta \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} + \nabla \tilde{p} = 0, \quad \nabla \cdot \tilde{\mathbf{u}} = 0 \quad \text{in } \tilde{\Omega}_h$$

complemented with reflected boundary conditions that we do not need to make fully explicit. We then reflect $\tilde{\Omega}_h$ with respect to the plane $z = 1$, which is slightly more delicate because of the inhomogeneous Dirichlet condition. However, we have $u_3 = 0$ on Γ_z^+ which allows an odd reflection for this (normal) component; the tangential components u_1 and u_2 may be evenly reflected without altering their boundary regularity. Defining the so-obtained domain $\hat{\Omega}_h \supsetneq \tilde{\Omega}_h$, the points in the edges of ∂P belong either to the interior of $\hat{\Omega}_h$ or to flat regions of $\partial \hat{\Omega}_h$, where standard elliptic regularity theory applies (interior or up to the boundary, see [16]). Hence, $(\mathbf{u}, p) \in W^{2,2+\sigma}(U) \times W^{1,2+\sigma}(U)$ in a neighborhood U of any edge of ∂P and, in view of (2.1), the following estimate holds

$$\|\mathbf{u}\|_{W^{2,2+\sigma}(U)} + \|p\|_{W^{1,2+\sigma}(U)} \leq C \|\mathbf{u}_*\|_{W^{2-\frac{1}{2+\sigma}, 2+\sigma}(\partial \Omega_h)} \leq C\gamma \quad \forall \gamma \in [0, \bar{\gamma})$$

for some $C > 0$ independent of h . Finally we infer that there exists $C^* > 0$, independent of h (due to the smallness assumption on its $C^{1,1}$ -norm), such that

$$\|\mathbf{u}\|_{W^{2,2+\sigma}(\Omega_h)} + \|p\|_{W^{1,2+\sigma}(\Omega_h)} \leq C^* \gamma \quad \forall \gamma \in [0, \bar{\gamma}).$$

The proof is so complete. □

3 | Fluid-Structure Interaction Problem

Aiming to study the equilibrium configurations for the FSI problem (1.11)-(1.17), we let the domain $\mathcal{B}^h$, representing the region occupied by the airfoil, vary. For any $y \in (0, 1)$, the cross-section $\mathcal{B}_y^h$ of the airfoil has a barycenter with vertical displacement $h(y)$, which is now unknown.

The two FSI problems we are led to study consist in the PDE system

$$\begin{aligned} -\eta\Delta\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p &= 0, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega_h \\ \mathbf{u} &= 0 \quad \text{on } \partial\mathcal{B}^h, \quad \mathbf{u} = \gamma V \mathbf{e}_1 \quad \text{on } \overline{\Gamma_y^+} \setminus \overline{\mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm, \\ \mathbf{u} &= \gamma V_i \mathbf{e}_1 \quad \text{on } \Gamma_x^-, \quad \mathbf{u} = \gamma V_o \mathbf{e}_1 \quad \text{on } \Gamma_x^+, \end{aligned} \quad (3.1)$$

coupled with one of the two ODE systems

$$\begin{aligned} h''''(y) + f(h(y)) &= L(y, h) \quad \text{in } (0, 1), \\ h(1) &= h'(1) = h''(1) = h'''(1) = 0, \end{aligned} \quad (3.2)$$

or

$$\begin{aligned} h''''(y) + f(h(y)) &= L(y, h) \quad \text{in } (0, 1), \\ h(1) &= h'(1) = h''(0) = h'''(0) = 0, \end{aligned} \quad (3.3)$$

where

$$L(y, h) = L(y, \mathbf{u}(h), p(h)) = -\mathbf{e}_3 \cdot \int_{\partial\mathcal{B}_y^h} \mathbb{T}(\mathbf{u}(h), p(h)) \mathbf{n} \quad (3.4)$$

with $\mathbb{T}(\cdot, \cdot)$ as in (1.12). Analysing the two different coupled problems (3.1)-(3.2) and (3.1)-(3.3) involves two distinguishes issues: since in the first case we face a Cauchy problem, we need to guarantee the extendability of the solution from $y = 1$ up to $y = 0$, thus relying on the classical theory for ODE systems; for the second case, in which we cannot see the problem as a Cauchy system, the procedure is more similar to what discussed in [7], as we treat (3.3) as the Euler-Lagrange equation of a suitable energy functional, thus relying on the minimization techniques.

Let us state the main result on the equilibrium configurations of the fluid-beam interaction problems (3.1)-(3.2) and (3.1)-(3.3). To this end, we introduce the open balls

$$\mathbb{B}_\delta^C = \{h \in \mathbb{H}_C^4 \text{ s.t. } \|h\|_{\mathbb{H}^4} < \delta\} \quad (3.5)$$

and

$$\mathbb{B}_\delta^{BV} = \{h \in \mathbb{H}_{BV}^4 \text{ s.t. } \|h\|_{\mathbb{H}^4} < \delta\} \quad (3.6)$$

and we prove the following:

Theorem 3.1. *Given $\sigma > 0$, there exists $\beta_0 > 0$ and $\gamma_0 \in (0, \bar{\gamma}]$ such that for any $\gamma \in [0, \gamma_0]$, the FSI problem (3.1)-(3.2) admits a unique solution*

$$(h, \mathbf{u}, p) \in \mathbb{B}_{\beta_0}^C \times W^{2,2+\sigma}(\Omega_h) \times W^{1,2+\sigma}(\Omega_h).$$

Analogously, the FSI problem (3.1)-(3.3) admits a unique solution

$$(h, \mathbf{u}, p) \in \mathbb{B}_{\beta_0}^{BV} \times W^{2,2+\sigma}(\Omega_h) \times W^{1,2+\sigma}(\Omega_h).$$

In addition, there exists $\gamma^* \in (0, \gamma_0]$ such that the map $\gamma \mapsto (h(\gamma), \mathbf{u}(\gamma), p(\gamma))$ is Lipschitz continuous on the interval $[0, \gamma^*)$, for both FSI problems (3.1)-(3.2) and (3.1)-(3.3). More precisely, for any $\gamma_1, \gamma_2 \in [0, \gamma^*)$,

$$\begin{aligned} & \|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4} + \|\mathbf{u}(\gamma_1) - \mathbf{u}(\gamma_2) \circ \varphi_{h(\gamma_2)-h(\gamma_1)}\|_{W^{2,2+\sigma}(\Omega_{h(\gamma_1)})} \\ & + \|p(\gamma_1) - p(\gamma_2) \circ \varphi_{h(\gamma_2)-h(\gamma_1)}\|_{W^{1,2+\sigma}(\Omega_{h(\gamma_1)})} \leq C(\beta_0)|\gamma_1 - \gamma_2|, \end{aligned}$$

where the diffeomorphism φ_h is defined in Lemma 3.6 below.

The proof of Theorem 3.1, given in Section 3.2, is based on a fixed-point procedure that involves several steps:

- we derive technical results about the lift L ;
- for any $\bar{h} \in C_{\{1\}}^{1,1}([0, 1])$, we apply Theorem 2.2 to obtain $(\mathbf{u}(\bar{h}), p(\bar{h}))$ solving (1.11) in $\Omega_{\bar{h}}$;
- we reduce (3.2) and (3.3) to two related beam equations with a given source depending on $\bar{h}$, (3.2) being a Cauchy problem centred at $y = 1$ and (3.3) being a boundary value problem with mixed-type boundary conditions;
- we prove existence and uniqueness of this new equation thereby providing a solution

$\mathfrak{h}(\bar{h}) \in C_{\{1\}}^{1,1}([0, 1])$: for this, we rely on ODE theory for the Cauchy problem (3.2) and on Calculus of Variations for the boundary value problem (3.3);

- we show that $\bar{h} \mapsto \mathfrak{h}(\bar{h})$ admits a unique fixed point in a suitable space.

3.1. Preliminary results

We first prove two properties of the lift force, which concern the boundedness and continuity of the lift L in $(0, 1)$ for a fixed $h = h(y)$. From the proof of the following lemma, in particular, will be finally evident why we have decided to work in non Hilbertian spaces for the strong solution pair $(\mathbf{u}, p)$ of the problem (2.2), having fixed $\sigma > 0$.

Lemma 3.2. *Let $h \in C_{\{1\}}^{1,1}([0, 1])$ satisfy the smallness assumption in Theorem 2.2. Let $\gamma \in [0, \bar{\gamma})$ and $(\mathbf{u}, p) \in W^{2,2+\sigma}(\Omega_h) \times W^{1,2+\sigma}(\Omega_h)$ be the unique solution to (2.2) from Theorem 2.2. Then, there exists $C > 0$, independent of h , such that*

$$\|L(\cdot, h)\|_{L^\infty(0,1)} \leq C\gamma. \quad (3.7)$$

Moreover, $L(\cdot, h) \in C(0, 1)$.

Proof. Since $\mathbf{u} \in W^{2,2+\sigma}(\Omega_h)$, we have that $\nabla \mathbf{u} \in W^{1,2+\sigma}(\Omega_h)$ and it follows from the Trace Theorem [27, Theorem 3.37] that $\nabla \mathbf{u} \in W^{1-\frac{1}{2+\sigma}, 2+\sigma}(\partial \mathcal{B}^h)$. The boundedness of $\partial \mathcal{B}^h$ and Hölder inequality then imply that $\nabla \mathbf{u} \in H^{1-\frac{1}{2+\sigma}}(\partial \mathcal{B}^h)$. Since $\sigma > 0$, we can apply the Trace Theorem once again to get $\nabla \mathbf{u} \in H^{\frac{1}{2}-\frac{1}{2+\sigma}}(\partial \mathcal{B}_y^h)$ and, in particular, $\nabla \mathbf{u} \in L^2(\partial \mathcal{B}_y^h)$ for a.e. $y \in (0, 1)$. Moreover, the following chain of inequalities holds:

$$\begin{aligned} \|\nabla \mathbf{u}\|_{L^1(\partial \mathcal{B}_y^h)} &\leq C \|\nabla \mathbf{u}\|_{L^2(\partial \mathcal{B}_y^h)} \leq C \|\nabla \mathbf{u}\|_{H^{\frac{1}{2}-\frac{1}{2+\sigma}}(\partial \mathcal{B}_y^h)} \\ &\leq C \|\nabla \mathbf{u}\|_{W^{1-\frac{1}{2+\sigma}, 2+\sigma}(\partial \mathcal{B}^h)} \leq C \|\mathbf{u}\|_{W^{2,2+\sigma}(\Omega_h)}, \end{aligned}$$

where the constants $C > 0$ may differ but are all independent of y . Similarly, since $p \in W^{1,2+\sigma}(\Omega_h)$ we obtain that $p \in H^{\frac{1}{2}-\frac{1}{2+\sigma}}(\partial \mathcal{B}_y^h)$ and, in particular, $p \in L^2(\partial \mathcal{B}_y^h)$. Thus, we end up with the following inequality

$$\|p\|_{L^1(\partial \mathcal{B}_y^h)} \leq C \|p\|_{W^{1,2+\sigma}(\Omega_h)}.$$

Therefore, we infer from the definition (3.4) that

$$|L(y, h)| \leq C \left(\|\mathbf{u}\|_{W^{2,2+\sigma}(\Omega_h)} + \|p\|_{W^{1,2+\sigma}(\Omega_h)} \right)$$

for a.e. $y \in (0, 1)$ and, using the uniform regularity estimate (2.4), we find (3.7).

As already noticed in the proof of Theorem 2.2, $\mathcal{S}^h$ is of class $C^{1,1}$. Since $\mathbf{u} = 0$ on $\mathcal{S}^h$, by elliptic regularity, we then infer that $\mathbf{u} \in W^{2,q}(\mathcal{N})$ in any (closed) neighbourhood $\mathcal{N} \subset \overline{\Omega}_h$ of any point $\xi \in \mathcal{S}^h$. By Embedding Theorems, this implies $\mathbf{u} \in C^{1,\alpha}(\mathcal{N})$ for any $\alpha < 1$ and, hence, $\nabla \mathbf{u} \in C^{0,\alpha}(\mathcal{N})$. By using (2.2), we then infer that $p \in C^{0,\alpha}(\mathcal{N})$ for any $\alpha < 1$. Therefore, the integrand in (3.4) is continuous for any $y \in (0, 1)$ and, by letting y vary, $L(\cdot, h) \in C(0, 1)$. $\square$

Remark 3.3. *The fact that $L(\cdot, h) \in C(0, 1)$ validates the pointwise definition (3.4) of the lift.*

Thanks to Theorem 2.2, we next discuss the well-posedness of (3.2) and (3.3) for the fluid velocity and pressure $(\mathbf{u}(\bar{h}), p(\bar{h}))$ determined by a given profile $\bar{h}$. We consider the following modified ODE problems: the Cauchy problem

$$\begin{aligned} h''''(y) + f(h(y)) &= L(y, \bar{h}) \quad \text{in } (0, 1), \\ h(1) &= h'(1) = h''(1) = h'''(1) = 0, \end{aligned} \tag{3.8}$$

and the boundary value problem

$$\begin{aligned} h''''(y) + f(h(y)) &= L(y, \bar{h}) \quad \text{in } (0, 1), \\ h(1) &= h'(1) = h''(0) = h'''(0) = 0, \end{aligned} \tag{3.9}$$

We want to point out that this splitting of the original fluid-structure interaction problem into two diverse types of problem is the true novelty of this work, with respect to [7], especially for the presence of the Cauchy problem. Indeed, differently from [7], in which the structure of the bridge made it obligatory to set two symmetric boundary conditions at the two endpoints, in our discussion, having to do with a structure fixed at only one endpoint forces us to break such a symmetry: while the two conditions $h(1) = h'(1) = 0$ are still present to ensure the clamping of the airfoil to the wind tunnel, exactly as a bridge may be clamped at its endpoints, we still lack two more conditions to close the ODE system. On one hand, choosing to set $h'(0) = h''(0) = 0$ seems to be natural, as it is already a known model (see [32]) and allows us to proceed with the same tools used in [7]; on the other hand, choosing to set $h''(1) = h'''(1) = 0$, thus discharging all the conditions onto $y = 1$, allows us to make use of another well established theory of differential equations, that of Cauchy problems.

We introduce the nonlinear map that will have a fixed point.

Proposition 3.4. *Let $\bar{h} \in C_{\{1\}}^{1,1}([0, 1])$ satisfy the smallness assumption in Theorem 2.2 and let $\gamma \in [0, \bar{\gamma})$. Then, (3.8) admits a unique solution $h \in \mathbb{H}_C^4 \subset C_{\{1\}}^{1,1}([0, 1])$.*

Analogously, (3.9) admits a unique solution $h \in \mathbb{H}_{BV}^4 \subset C_{\{1\}}^{1,1}([0, 1])$.

In both cases, there exists $C > 0$, independent of $\bar{h}$, such that

$$\|h\|_{\mathbb{H}^4} \leq C\gamma \quad (3.10)$$

Proof. Since $\bar{h}$ is given, we may rewrite the common ODE in (3.8) and (3.9) as

$$h''''(y) + f(h(y)) = g(y) = L(y, \bar{h}(y)) \quad \text{for a.e. } y \in (0, 1), \quad (3.11)$$

where $g \in L^\infty(0, 1)$ is a given function of y .

a. Existence and uniqueness for (3.8). In this first case, since we are facing a Cauchy problem centred at $y = 1$, we proceed with the standard theory of ODEs. After introducing $x_1 = h$, $x_2 = h'$, $x_3 = h''$, $x_4 = h'''$, let

$$\begin{aligned} X &= (x_1, x_2, x_3, x_4), \\ \Phi(y, X) &= (x_2, x_3, x_4, g(y) - f(x_1)), \end{aligned}$$

thus (3.11), together with boundary conditions $h(1) = h'(1) = h''(1) = h'''(1) = 0$, reads:

$$\begin{cases} X'(y) = \Phi(y, X(y)), & y \in (0, 1) \\ X(1) = 0 \end{cases}$$

From Picard-Lindelöf theorem, being Φ measurable in y , since $g = L(\cdot, \bar{h}) \in L^\infty(0, 1)$ from Lemma 3.2, and globally Lipschitz, since $f \in \text{Lip}(\mathbb{R})$ from (1.15), we obtain the existence and uniqueness of a global solution $X \in W^{1,\infty}(0, 1; \mathbb{R}^4)$, from which we get that $h \in W^{4,\infty}(0, 1) \subset H^4(0, 1) \subset C^3([0, 1])$. Hence, since the last embedding validates the pointwise conditions in $y = 1$, we have $h \in \mathbb{H}_C^4$.

b. Existence and uniqueness for (3.9). In this second case, we proceed through Calculus of Variations arguments, thus analogously to what done in [7, Proposition 4.4], seeing weak solutions to (3.11), together with boundary conditions $h(1) = h'(1) = h''(0) =$

$h'''(0) = 0$, as critical points of the related energy functional

$$h \mapsto E(h) = \int_0^1 \left(\frac{h''(y)^2}{2} + F(h(y)) - g(y)h(y) \right) dy, \quad (3.12)$$

defined in $\mathbb{H}$. Indeed, the ODE in (3.9) represents the associated Euler-Lagrange equation of (3.12), while the boundary conditions in (3.9) are:

- $h(1) = h'(1) = 0$, which are the boundary conditions already prescribed in the weak space $\mathbb{H}$ (notice that, since we are treating 1D functions, $\mathbb{H} \subset C^1([0, 1])$, so the pointwise conditions make sense)
- $h''(0) = h'''(0) = 0$, which are natural boundary conditions not prescribed in $\mathbb{H}$, since, in general, second and third derivatives of H^2 functions may not be defined pointwisely. So, by proving the H^4 regularity for the solution of (3.9), we also succeed in giving these natural boundary conditions a pointwise sense.

Due to (1.15), E is strictly convex with respect to h in $\mathbb{H}$, so that it admits a unique minimizer in $\mathbb{H}$ that weakly solves (3.11). Moreover, being $f(h) \in L^\infty(0, 1)$ due to (1.15) and $g \in L^\infty(0, 1)$ from Lemma 3.2, by exploiting the weak formulation of (3.11) and elliptic regularity, it follows that $h \in W^{4,\infty}(0, 1) \subset H^4(0, 1) \subset C^3([0, 1])$. Hence, since the last embedding validates the pointwise conditions in $y = 0$, we have $h \in \mathbb{H}_{BV}^4$.

Finally, in both cases, using the weak formulation of (3.11) together with (1.15) and (1.2), we get the bound

$$\|h\|_{\mathbb{H}} \leq 2S_{\mathbb{H}}\|g\|_{L^\infty(0,1)}. \quad (3.13)$$

Hence, there exist constants $C > 0$ (which may vary in each inequality) such that

$$\begin{aligned} \|h\|_{H^4(0,1)} &\leq C (\|f(h)\|_{L^2(0,1)} + \|g\|_{L^2(0,1)}) \\ &\leq C (\|h\|_{L^\infty(0,1)} + \|g\|_{L^\infty(0,1)}) \stackrel{(3.13)}{\leq} C\|g\|_{L^\infty(0,1)} \leq C\gamma \end{aligned}$$

where the last inequality is due to Lemma 3.2 applied to $L(\cdot, \bar{h}) = g(\cdot)$. $\square$

The proof of Proposition 3.4 shows that the solution to (3.11) belongs to $W^{4,\infty}(0, 1)$, while Lemma 3.2 implies that it also belongs to $C^4(0, 1)$. However, this additional regularity will not be used in the sequel.

Remark 3.5. *The second study case, which is represented by (3.9) and is a natural model in 1D, being the formulation for the horizontal flagpole presented in [32, p. 12], may not work well in higher dimensions with fourth-order elliptic PDEs, as the use of mixed*

boundary conditions on parts of the boundary that share some junction points may make elliptic regularity fail in general.

In order to study the dependence of the lift on the beam profile h , it is convenient to introduce a setting where the fluid domain is independent of h . To this end, we define a suitable map φ_h .

Lemma 3.6. *Let $\mathcal{D} \subset \Gamma_y^+$ be an open set such that the cylinder $\mathcal{C} = \mathcal{D} \times [0, 1]$ strictly contains any $\mathcal{B}^h$ for h sufficiently small. Consider a smooth cut-off function $\xi : [-R, R] \times [-1, 1] \rightarrow [0, 1]$ such that*

$$\xi = 1 \quad \text{in} \quad \overline{\mathcal{D}} \quad \text{and} \quad \xi = 0 \quad \text{in} \quad ([-R \times R] \times [-1, 1]) \setminus \lambda \mathcal{D} \quad (3.14)$$

with $\lambda > 1$ ensuring that the homothetic $\lambda \mathcal{C} \subset \Gamma_y^+$. For $h \in \mathbb{H}$, consider the map

$$\varphi_h(x, y, z) = (x, y, z + \xi(x, z)h(y)). \quad (3.15)$$

Then, there exists $\beta > 0$ such that, for $\|h\|_{\mathbb{H}} < \beta$, φ_h is a C^1 -diffeomorphism from $\overline{\Omega}_0$ to $\overline{\Omega}_h$ satisfying

$$\begin{aligned} \varphi_h(\partial \mathcal{B}) &= \partial \mathcal{B}_h, \quad \varphi_h(\overline{\Gamma_x^\pm}) = \overline{\Gamma_x^\pm}, \quad \varphi_h(\overline{\Gamma_y^-}) = \overline{\Gamma_y^-}, \\ \varphi_h(\overline{\Gamma_y^+ \setminus \mathcal{B}_1}) &= \overline{\Gamma_y^+ \setminus \mathcal{B}_1} \quad \text{and} \quad \varphi_h(\overline{\Gamma_z^\pm}) = \overline{\Gamma_z^\pm}. \end{aligned} \quad (3.16)$$

Proof. First, we take β small enough such that (1.8) holds, which is possible in view of (1.2). The embedding $\mathbb{H} \subset C^1([0, 1])$ and the smoothness of ξ yield C^1 -regularity of φ_h . Next, we want to show that the determinant of its Jacobian

$$J_h(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \partial_x \xi(x, z)h(y) & \xi(x, z)h'(y) & 1 + \partial_z \xi(x, z)h(y) \end{pmatrix}$$

is positive and uniformly bounded in Ω_0 , that is, there exists $C > 0$ such that

$$C \leq \det(J_h) = 1 + \partial_z \xi(x, z)h(y) \leq \frac{1}{C}, \quad (x, y, z) \in \Omega_0 \quad (3.17)$$

It follows from (1.2) that

$$1 - C_1 \|h\|_{\mathbb{H}} \leq \det(J_h) \leq 1 + C_2 \|h\|_{\mathbb{H}} \quad (3.18)$$

for some $C_1, C_2 > 0$ depending on ξ and $S_{\mathbb{H}}$, so that there exists (a possibly smaller) $\beta > 0$ such that

$$0 < C(\beta) < \det(J_h) \leq \frac{1}{C(\beta)}$$

whenever $\|h\|_{\mathbb{H}} < \beta$. Moreover, the properties of the cut-off ξ and the boundary condition $h(1) = 0$ yield

$$\begin{aligned} \varphi_h(x, y, z) &= (x, y, z + h(y)) \quad \text{for } (x, y, z) \in \partial\mathcal{B}, \\ \varphi_h(x, 1, z) &= (x, 1, z + \xi(x, z)h(1)) = (x, 1, z) \quad \text{for } (x, z) \in [-R, R] \times [-1, 1], \end{aligned} \quad (3.19)$$

Moreover, φ_h is the identity map on each of the five faces $\bar{\Gamma}_x^\pm$, $\bar{\Gamma}_z^\pm$ and $\bar{\Gamma}_y^-$. Hence, the C^1 -regularity, (3.17) and (3.19) prove that φ_h is a C^1 -diffeomorphism from $\bar{\Omega}_0$ to $\bar{\Omega}_h$ that satisfies (3.16). $\square$

Remark 3.7. *The diffeomorphism (3.15) is the most direct that can be introduced to transform Ω_h into Ω_0 while ensuring (3.16). Its regularity is the same as that of h , but this does not represent the optimal choice. To improve regularity, it is necessary to introduce a so-called regularizing diffeomorphism that permits to gain half derivative compared to the regularity of h , see [10] for an example. However, we will show later that $h \in \mathbb{H}_C^4$ or $h \in \mathbb{H}_{BV}^4$ (according to the study case), which in both cases provides us with enough regularity to carry out the analysis working with (3.15).*

Assuming that $\|h\|_{\mathbb{H}} < \beta$ with $\beta > 0$ as in Lemma 3.6, by means of the diffeomorphism φ_h in (3.15), we transform (2.2) in Ω_h as an elliptic problem in Ω_0 . After introducing the matrices

$$A_h = \det(J_h)M_h^T M_h \quad \text{and} \quad B_h = \det(J_h)M_h \quad \text{with} \quad M_h = (J_h^{-1})^T, \quad (3.20)$$

we obtain that $(\mathbf{U}, P) = (\mathbf{u}, p) \circ \varphi_h$ solves the elliptic problem

$$\begin{aligned} -\eta \nabla \cdot (A_h \nabla \mathbf{U}) + \mathbf{U} \cdot B_h \nabla \mathbf{U} + B_h \nabla P &= 0, \quad \nabla \cdot (B_h^T \mathbf{U}) = 0 \quad \text{in } \Omega_0 \\ \mathbf{U} &= 0 \quad \text{on } \partial\mathcal{B}, \quad \mathbf{U} = \gamma V \mathbf{e}_1 \quad \text{on } \bar{\Gamma}_y^+ \setminus \bar{\mathcal{B}}_1 \cup \bar{\Gamma}_y^- \cup \Gamma_z^\pm, \\ \mathbf{U} &= \gamma V_i \mathbf{e}_1 \quad \text{on } \Gamma_x^-, \quad \mathbf{U} = \gamma V_o \mathbf{e}_1 \quad \text{on } \Gamma_x^+. \end{aligned} \quad (3.21)$$

We are now ready to study the dependence of $L(y, h)$, defined in (3.4), on h while keeping y fixed. In the next proposition, we prove its Lipschitz continuity.

Proposition 3.8. *There exists $\beta_0 \in (0, \beta]$ with β as in Lemma 3.6 such that $h \mapsto L(y, h)$*

is Lipschitz continuous in both $\mathbb{B}_{\beta_0}^C$ and $\mathbb{B}_{\beta_0}^{BV}$.

More precisely, for any $h_1, h_2 \in \mathbb{B}_{\beta_0}^C$ or for any $h_1, h_2 \in \mathbb{B}_{\beta_0}^{BV}$,

$$\|L(\cdot, h_1) - L(\cdot, h_2)\|_{L^\infty(0,1)} \leq \gamma C(\beta_0) \|h_1 - h_2\|_{\mathbb{H}^4} \quad (3.22)$$

with $\gamma \in [0, \bar{\gamma})$ as in Theorem 2.2.

Proof. Since we have proved that both (3.8) and (3.9) admit a unique solution, respectively in $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$, and since the distinction between such spaces is not relevant for the present proof (recall that the two spaces share the same inner product and norm (1.3)), we are going to use the symbol $\mathbb{B}_\beta$ to indicate both $\mathbb{B}_\beta^C$ and $\mathbb{B}_\beta^{BV}$.

With φ_h at hand, the lift in (3.4) can be written in terms of the h -independent fluid domain Ω_0 , namely,

$$\begin{aligned} L(y, h) &= -\mathbf{e}_3 \cdot \int_{\partial \mathcal{B}_y} \tilde{\mathbb{T}}_h(\mathbf{U}, P) M_h \mathbf{n}_0 \\ &= -\mathbf{e}_3 \cdot \int_{\partial \mathcal{B}_y} [\eta(M_h \nabla \mathbf{U} + (M_h \nabla \mathbf{U})^T) - P \mathbb{I}] M_h \mathbf{n}_0 \end{aligned} \quad (3.23)$$

where $(\mathbf{U}, P)$ solves (3.21) and $\mathbf{n}_0$ is the unit normal vector to $\partial \mathcal{B}$ pointing inside $\mathcal{B}$. In (3.23), we have used the fact $\det(J_h) = 1$ on $\partial \mathcal{B}_y$. We now show that, there exists $\beta_0 > 0$ such that

$$h \mapsto (\mathbf{U}(h), P(h)) \in W^{2,2+\sigma}(\Omega_0) \times W^{1,2+\sigma}(\Omega_0) \quad \text{belongs to } C^1(\mathbb{B}_{\beta_0}). \quad (3.24)$$

To this end, with β as in Lemma 3.6 we express (3.21) as

$$\mathcal{H}(h, \mathbf{U}, P) = 0, \quad (3.25)$$

where $\mathcal{H} : \mathbb{B}_\beta \times X \rightarrow Y$ with

$$\begin{aligned} X &= W^{2,2+\sigma}(\Omega_0) \times W^{1,2+\sigma}(\Omega_0), \\ Y &= \left\{ (\chi_1, \chi_2, \chi_3) \in L^{2+\sigma}(\Omega_0) \times W^{1,2+\sigma}(\Omega_0) \times W^{2-\frac{1}{2+\sigma}, 2+\sigma}(\partial \Omega_0) \right. \\ &\quad \left. \text{such that } \int_{\Omega_0} \chi_2 = \int_{\partial \Omega_0} B_h^T \chi_3 \cdot \mathbf{n} \right\}, \end{aligned}$$

and the components of $\mathcal{H}$ read

$$\begin{aligned}\mathcal{H}_1(h, \mathbf{U}, P) &= -\eta \nabla \cdot (A_h \nabla \mathbf{U}) + \mathbf{U} \cdot B_h \nabla \mathbf{U} + B_h \nabla P, \\ \mathcal{H}_2(h, \mathbf{U}, P) &= \nabla \cdot (B_h^T \mathbf{U}), \\ \mathcal{H}_3(h, \mathbf{U}, P) &= \mathbf{U}|_{\partial\Omega_0} - \mathbf{u}_*,\end{aligned}\tag{3.26}$$

where $\mathbf{u}_*$ is as in (2.1). We explicitly compute the matrices in (3.20), which read

$$A_h = \mathbb{I} + \begin{pmatrix} \partial_z \xi h & 0 & -\partial_x \xi h \\ 0 & \partial_z \xi h & -\xi h' \\ -\partial_x \xi h & -\xi h' & \frac{-\partial_z \xi h + (\partial_x \xi h)^2 + (\xi h')^2}{1 + \partial_z \xi h} \end{pmatrix}, \quad B_h = \mathbb{I} + \begin{pmatrix} \partial_z \xi h & 0 & -\partial_x \xi h \\ 0 & \partial_z \xi h & -\xi h' \\ 0 & 0 & 0 \end{pmatrix}.\tag{3.27}$$

First, we have that $h \mapsto \mathcal{H}(h, \mathbf{U}, P)$ is $C^1(\mathbb{B}_\beta)$ since Lemma 3.6 yields the same regularity for the maps $h \mapsto A_h$, $h \mapsto B_h$ and $h \mapsto B_h^T$. Second, $(\mathbf{U}, P) \mapsto \mathcal{H}(h, \mathbf{U}, P)$ is $C^1(X)$ for any $h \in \mathbb{B}_\beta$; indeed, the linear differential operator $\mathcal{L} = D_{(\mathbf{U}, P)} \mathcal{H}[h, \mathbf{U}, P]$, whose components read

$$\begin{aligned}\mathcal{L}_1(\boldsymbol{\chi}, \Pi) &= -\eta \nabla \cdot (A_h \nabla \boldsymbol{\chi}) + \boldsymbol{\chi} \cdot B_h \nabla \mathbf{U} + \mathbf{U} \cdot B_h \nabla \boldsymbol{\chi} + B_h \nabla \Pi, \\ \mathcal{L}_2(\boldsymbol{\chi}, \Pi) &= \nabla \cdot (B_h^T \boldsymbol{\chi}), \\ \mathcal{L}_3(\boldsymbol{\chi}, \Pi) &= \boldsymbol{\chi}|_{\partial\Omega_0},\end{aligned}$$

is bounded from X to Y , hence it is Fréchet differentiable. Moreover, it is continuous with respect to $(h, \mathbf{U}, P) \in \mathbb{B}_\beta \times X$. Thus, $(h, \mathbf{U}, P) \mapsto \mathcal{H}(h, \mathbf{U}, P)$ belongs to $C^1(\mathbb{B}_\beta \times X)$.

Moreover, we claim that $\mathcal{L}$ is an isomorphism from X to Y . To prove this assertion, let us consider $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3) \in Y$ and the linear elliptic problem

$$\mathcal{L}(\boldsymbol{\chi}, \Pi) = (\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \boldsymbol{\phi}_3),\tag{3.28}$$

which can be written as the inhomogeneous Stokes-type system

$$\begin{aligned}
-\eta \Delta \boldsymbol{\chi} + \nabla \Pi &= -\boldsymbol{\chi} \cdot B_h \nabla \mathbf{U} - \mathbf{U} \cdot B_h \nabla \boldsymbol{\chi} \\
&\quad - \eta \nabla \cdot ((\mathbb{I} - A_h) \nabla \boldsymbol{\chi}) + (\mathbb{I} - B_h) \nabla \Pi + \boldsymbol{\phi}_1 \quad \text{in } \Omega_0, \\
\nabla \cdot \boldsymbol{\chi} &= \nabla \cdot (\mathbb{I} - B_h^T \boldsymbol{\chi}) + \phi_2 \quad \text{in } \Omega_0, \\
\boldsymbol{\chi}|_{\partial \Omega_0} &= \boldsymbol{\phi}_3.
\end{aligned} \tag{3.29}$$

By (3.27) and since $h \in \mathbb{B}_\beta$, there exist $C_1(\beta), C_2(\beta) > 0$, with $C_1(\beta), C_2(\beta) \rightarrow 0$ as $\beta \rightarrow 0$, such that

$$\|\mathbb{I} - A_h\|_{W^{1,\infty}(\Omega_0)} \leq C_1(\beta) \quad \text{and} \quad \|\mathbb{I} - B_h\|_{W^{2,\infty}(\Omega_0)} \leq C_2(\beta).$$

We hence infer that there exist $\beta_0 \in (0, \beta]$ and $r_0 > 0$ such that, by a contraction argument, (3.29) admits a unique solution $(\boldsymbol{\chi}, \Pi) \in X$ provided that

$$\|h\|_{\mathbb{H}^4} < \beta_0 \quad \text{and} \quad \|\mathbf{U}\|_{W^{2,2+\sigma}(\Omega_0)} < r_0. \tag{3.30}$$

Moreover, the properties of the diffeomorphism φ_h imply that

$$\begin{aligned}
c\|\mathbf{u}\|_{W^{2,2+\sigma}(\Omega_h)} &\leq \|\mathbf{U}\|_{W^{2,2+\sigma}(\Omega_0)} \leq C\|\mathbf{u}\|_{W^{2,2+\sigma}(\Omega_h)}, \\
c\|p\|_{W^{1,2+\sigma}(\Omega_h)} &\leq \|P\|_{W^{1,2+\sigma}(\Omega_0)} \leq C\|p\|_{W^{1,2+\sigma}(\Omega_h)},
\end{aligned} \tag{3.31}$$

with constants $0 < c \leq C$ independent of h , by taking $\gamma \in (0, \bar{\gamma})$ sufficiently small, the bound (2.4) yields the second condition in (3.30). Thus, $\mathcal{L}$ is an isomorphism and there exists $C > 0$, independent of h , such that the unique solution to (3.28) satisfies

$$\begin{aligned}
&\|\boldsymbol{\chi}\|_{W^{2,2+\sigma}(\Omega_0)} + \|\Pi\|_{W^{1,2+\sigma}(\Omega_0)} \\
&\leq C \left(\|\boldsymbol{\phi}_1\|_{L^{2+\sigma}(\Omega_0)} + \|\phi_2\|_{W^{1,2+\sigma}(\Omega_0)} + \|\boldsymbol{\phi}_3\|_{W^{2-\frac{1}{2+\sigma}, 2+\sigma}(\partial \Omega_0)} \right)
\end{aligned} \tag{3.32}$$

Therefore, applying the Implicit Function Theorem to (3.25), we conclude (3.24).

In addition, it follows from (3.24) that the differential operators

$D_h \mathbf{U}[h] : \mathbb{B}_{\beta_0} \rightarrow W^{2,2+\sigma}(\Omega_0)$ and $D_h P[h] : \mathbb{B}_{\beta_0} \rightarrow W^{1,2+\sigma}(\Omega_0)$ are bounded. We also know from (3.25) that, for any $\tilde{h} \in \mathbb{B}_{\beta_0}$,

$$\mathbf{U}_h(\tilde{h}) := D_h \mathbf{U}[h] \tilde{h} \quad \text{and} \quad P_h(\tilde{h}) := D_h P[h] \tilde{h}$$

solve the linear elliptic problem

$$\mathcal{L}(\mathbf{U}_h(\tilde{h}), P_h(\tilde{h})) = -D_h \mathcal{H}[h, \mathbf{U}, P] \tilde{h},$$

which explicitly reads

$$\begin{aligned} -\eta \nabla \cdot (A_h \nabla \mathbf{U}_h(\tilde{h})) + \mathbf{U}_h(\tilde{h}) \cdot B_h \nabla \mathbf{U} + \mathbf{U} \cdot B_h \nabla \mathbf{U}_h(\tilde{h}) + B_h \nabla P_h(\tilde{h}) &= F(h, \tilde{h}, \mathbf{U}) \quad \text{in } \Omega_0, \\ \nabla \cdot (B_h^T \mathbf{U}_h(\tilde{h})) &= G(h, \tilde{h}, \mathbf{U}) \quad \text{in } \Omega_0, \\ \mathbf{U}_h(\tilde{h})|_{\partial\Omega_0} &= 0, \end{aligned}$$

with the source terms

$$\begin{aligned} F(h, \tilde{h}, \mathbf{U}, P) &= -\eta \nabla \cdot (A_h(\tilde{h}) \nabla \mathbf{U}) + \mathbf{U} \cdot B_h(\tilde{h}) \nabla \mathbf{U} + B_h(\tilde{h}) \nabla P, \\ G(h, \tilde{h}, \mathbf{U}) &= \nabla \cdot (B_h^T(\tilde{h}) \mathbf{U}) \end{aligned}$$

and

$$A_h(\tilde{h}) = -D_h A_h[h] \tilde{h}, \quad B_h(\tilde{h}) = -D_h B_h[h] \tilde{h}, \quad B_h^T(\tilde{h}) = -D_h B_h^T[h] \tilde{h},$$

where $D_h A_h[h]$, $D_h B_h[h]$ and $D_h B_h^T[h]$ are the differential operators associated with the maps $h \mapsto A_h$, $h \mapsto B_h$ and $h \mapsto B_h^T$. Since $\mathcal{L}$ is an isomorphism, we know that $(\mathbf{U}_h(\tilde{h}), P_h(\tilde{h}))$ is uniquely determined and, by (3.32), it satisfies

$$\begin{aligned} \|\mathbf{U}_h(\tilde{h})\|_{W^{2,2+\sigma}(\Omega_0)} + \|P_h(\tilde{h})\|_{W^{1,2+\sigma}(\Omega_0)} \\ \leq C \left(\|F(h, \tilde{h}, \mathbf{U}, P)\|_{L^{2+\sigma}(\Omega_0)} + \|G(h, \tilde{h}, \mathbf{U})\|_{W^{1,2+\sigma}(\Omega_0)} \right). \end{aligned} \quad (3.33)$$

For any $h \in \mathbb{B}_{\beta_0}$, it follows from the explicit expressions (3.27) that there exists $C(\beta_0) > 0$ such that

$$\|A_h(\tilde{h})\|_{H^2(\Omega_0)} \leq C(\beta_0) \|\tilde{h}\|_{\mathbb{H}^4}, \quad \|B_h(\tilde{h})\|_{H^2(\Omega_0)} \leq C(\beta_0) \|\tilde{h}\|_{\mathbb{H}^4},$$

so that, using the continuous embedding $H^2(\Omega_0) \subset L^\infty(\Omega_0)$, the right-hand side of (3.33) can be estimated by

$$\begin{aligned} &\|F(h, \tilde{h}, \mathbf{U}, P)\|_{L^{2+\sigma}(\Omega_0)} + \|G(h, \tilde{h}, \mathbf{U})\|_{W^{1,2+\sigma}(\Omega_0)} \\ &\leq C \left(\|A_h(\tilde{h})\|_{H^2(\Omega_0)} + \|B_h(\tilde{h})\|_{H^2(\Omega_0)} \right) (\|\mathbf{U}\|_{W^{2,2+\sigma}(\Omega_0)} + \|P\|_{W^{1,2+\sigma}(\Omega_0)}) \\ &\leq C(\beta_0) \|\tilde{h}\|_{\mathbb{H}^4} (\|\mathbf{U}\|_{W^{2,2+\sigma}(\Omega_0)} + \|P\|_{W^{1,2+\sigma}(\Omega_0)}). \end{aligned} \quad (3.34)$$

Due to (3.31) and (2.4), we then obtain

$$\|\mathbf{U}_h(\tilde{h})\|_{W^{2,2+\sigma}(\Omega_0)} + \|P_h(\tilde{h})\|_{W^{1,2+\sigma}(\Omega_0)} \leq C(\beta_0)\gamma\|\tilde{h}\|_{\mathbb{H}^4},$$

which, in turn, implies the uniform bound for the norms of the differential operators $D_h\mathbf{U}[h]$ and $D_hP[h]$

$$\sup_{h \in \mathbb{B}_{\beta_0}} \|D_h\mathbf{U}[h]\| + \sup_{h \in \mathbb{B}_{\beta_0}} \|D_hP[h]\| \leq C(\beta_0)\gamma. \quad (3.35)$$

Finally, let us estimate the variation of the lift, defined in (3.23), with respect to the variation of h . Using the same argument and the Trace Theorem as in the proof of Lemma 3.2, we have that, for any $h_1, h_2 \in \mathbb{B}_{\beta_0}$,

$$\begin{aligned} & |L(y, h_1) - L(y, h_2)| \\ &= \left| \int_{\partial\mathcal{B}_y} \left(\tilde{\mathbb{T}}_{h_1}(\mathbf{U}(h_1), P(h_1)) M_{h_1} \mathbf{n}_0 - \tilde{\mathbb{T}}_{h_2}(\mathbf{U}(h_2), P(h_2)) M_{h_2} \mathbf{n}_0 \right) \right| \\ &= \left| \int_{\partial\mathcal{B}_y} \left[\left(\tilde{\mathbb{T}}_{h_1}(\mathbf{U}(h_1), P(h_1)) - \tilde{\mathbb{T}}_{h_2}(\mathbf{U}(h_2), P(h_2)) \right) M_{h_1} \mathbf{n}_0 \right. \right. \\ &\quad \left. \left. + \tilde{\mathbb{T}}_{h_2}(\mathbf{U}(h_2), P(h_2)) (M_{h_1} - M_{h_2}) \mathbf{n}_0 \right] \right| \\ &\leq C(\beta_0) \left(\|\mathbf{U}(h_1) - \mathbf{U}(h_2)\|_{W^{2,2+\sigma}(\Omega_0)} + \|P(h_1) - P(h_2)\|_{W^{1,2+\sigma}(\Omega_0)} \right. \\ &\quad \left. + (\|\mathbf{U}(h_2)\|_{W^{2,2+\sigma}(\Omega_0)} + \|P(h_2)\|_{W^{1,2+\sigma}(\Omega_0)}) \|h_1 - h_2\|_{\mathbb{H}^4} \right). \end{aligned} \quad (3.36)$$

Combining the Mean Value Inequality with (3.35) and using (3.31) together with (2.4), we obtain that

$$\begin{aligned} & \|L(\cdot, h_1) - L(\cdot, h_2)\|_{L^\infty(0,1)} \\ &\leq C(\beta_0) \left(\gamma + \|\mathbf{u}(h_2)\|_{W^{2,2+\sigma}(\Omega_{h_2})} + \|p(h_2)\|_{W^{1,2+\sigma}(\Omega_{h_2})} \right) \|h_1 - h_2\|_{\mathbb{H}^4} \\ &\leq \gamma C(\beta_0) \|h_1 - h_2\|_{\mathbb{H}^4}, \end{aligned}$$

which proves the Lipschitz continuity of $L(\cdot, h)$ with respect to h . $\square$

3.2. Proof of Theorem 3.1

We are now in position to set the fixed point method to uniquely solve the two coupled systems (3.1)-(3.2) and (3.1)-(3.3) and, hence, to prove Theorem 3.1. As argued at the beginning of the proof of Proposition 3.8, since both (3.8) and (3.9) admit a unique solution, respectively in $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$, and the distinction between such spaces is not relevant as they share same norms and inner products, we use the symbol $\mathbb{H}^4$ to indicate both $\mathbb{H}_C^4$ and $\mathbb{H}_{BV}^4$, and the symbol $\mathbb{B}_\beta$ to indicate both $\mathbb{B}_\beta^C$ and $\mathbb{B}_\beta^{BV}$.

Consider a given $\bar{h} \in \mathbb{H}^4 \subset C_{\{1\}}^{1,1}([0, 1])$. Theorem 2.2 yields a unique pair $(\mathbf{u}(\bar{h}), p(\bar{h})) \in W^{2,2+\sigma}(\Omega_{\bar{h}}) \times W^{1,2+\sigma}(\Omega_{\bar{h}})$ that solves (3.1) with inflow/outflow magnitude $\gamma \in [0, \bar{\gamma})$. The lift $L(\cdot, \bar{h})$ generated by the solution $(\mathbf{u}(\bar{h}), p(\bar{h}))$ belongs to $C(0, 1)$ due to Lemma 3.2 and we view $L(\cdot, \bar{h})$ as the source term of the beam equation common to (3.8) and (3.9). Thanks to Proposition 3.4, it admits a unique solution that depends on $\bar{h}$. This defines the map

$$\mathfrak{h} : \begin{array}{ccc} \mathbb{B}_{\beta_0} \subset \mathbb{H}^4 & \rightarrow & \mathbb{H}^4 \\ \bar{h} & \mapsto & \mathfrak{h}(\bar{h}) \end{array} \quad (3.37)$$

where $\mathfrak{h}(\bar{h})$ is the unique solution to one of the two problems (3.8) and (3.9). To continue, we show that $\mathfrak{h}$ is a contraction map on $\mathbb{B}_{\beta_0}$.

First, we show that $\mathfrak{h}(\mathbb{B}_{\beta_0}) \subset \mathbb{B}_{\beta_0}$ for γ sufficiently small. In fact, given any $\bar{h} \in \mathbb{B}_{\beta_0} \subset C_{\{1\}}^{1,1}([0, 1])$, we know from Proposition 3.4 that $\mathfrak{h}(\bar{h}) \in \mathbb{H}^4$ and, for $\gamma \in [0, \bar{\gamma})$,

$$\|\mathfrak{h}(\bar{h})\|_{\mathbb{H}^4} \leq C\gamma;$$

this implies that there exists $\gamma_0 \in (0, \bar{\gamma}]$ such that

$$\|\mathfrak{h}(\bar{h})\|_{\mathbb{H}^4} < \beta_0, \quad (3.38)$$

whenever $\gamma \in [0, \gamma_0)$. Consider $\mathfrak{h}(\bar{h}_1)$ and $\mathfrak{h}(\bar{h}_2)$ both solutions to the same problem, either (3.8) or (3.9), with source term $L(\cdot, \bar{h}_1)$ and $L(\cdot, \bar{h}_2)$, respectively. Their difference $\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)$ satisfies

$$\begin{aligned} (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))'''' + f(\mathfrak{h}(\bar{h}_1)) - f(\mathfrak{h}(\bar{h}_2)) &= L(\cdot, \bar{h}_1) - L(\cdot, \bar{h}_2) \quad \text{in } (0, 1), \\ (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))(1) &= (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))'(1) = 0, \\ (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))''(0) &= (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))'''(0) = 0 \quad \text{or} \\ (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))''(1) &= (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2))'''(1) = 0 \end{aligned} \quad (3.39)$$

Testing (3.39) with $\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)$ yields

$$\begin{aligned} \|\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)\|_{\mathbb{H}}^2 + \int_0^1 (f(\mathfrak{h}(\bar{h}_1)) - f(\mathfrak{h}(\bar{h}_2))) (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)) \\ = \int_0^1 (L(\cdot, \bar{h}_1) - L(\cdot, \bar{h}_2)) (\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)) \end{aligned}$$

which, due to (1.15), (1.2) and (3.22), implies that

$$\|\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)\|_{\mathbb{H}} \leq 2S_{\mathbb{H}} \|L(\cdot, \bar{h}_1) - L(\cdot, \bar{h}_2)\|_{L^\infty(0,1)} \leq C\gamma \|\bar{h}_1 - \bar{h}_2\|_{\mathbb{H}^4} \quad (3.40)$$

By elliptic regularity for (3.39), by (1.15) and the continuous embedding $\mathbb{H} \subset L^\infty(0,1)$, it follows that

$$\begin{aligned} \|\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)\|_{\mathbb{H}^4} &\leq C (\|f(\mathfrak{h}(\bar{h}_1)) - f(\mathfrak{h}(\bar{h}_2))\|_{L^2(0,1)} + \|L(\cdot, \bar{h}_1) - L(\cdot, \bar{h}_2)\|_{L^2(0,1)}) \\ &\leq C (\|\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)\|_{\mathbb{H}} + \|L(\cdot, \bar{h}_1) - L(\cdot, \bar{h}_2)\|_{L^\infty(0,1)}) \\ &\leq C\gamma \|\bar{h}_1 - \bar{h}_2\|_{\mathbb{H}^4}, \end{aligned}$$

(3.41)

where in the last inequality we used (3.40) and (3.22). Then, we deduce from (3.41) that there exists $\gamma_0 \in (0, \bar{\gamma}]$, possibly smaller than the one implying (3.38), such that

$$\|\mathfrak{h}(\bar{h}_1) - \mathfrak{h}(\bar{h}_2)\|_{\mathbb{H}^4} \leq M \|\bar{h}_1 - \bar{h}_2\|_{\mathbb{H}^4} \quad (3.42)$$

with $0 < M < 1$. Finally, gathering (3.38) and (3.42) together, we conclude that the map (3.37) is a contraction on $\mathbb{B}_{\beta_0}$ for $\gamma \in [0, \gamma_0)$. Banach fixed-point theorem then implies the existence and uniqueness of a fixed point, that is, there exists a unique $h \in \mathbb{B}_{\beta_0}$ such that $\mathfrak{h}(h) = h$. This is equivalent to say that there exists a unique solution $(h, \mathbf{u}, p) \in \mathbb{B}_{\beta_0} \times W^{2,2+\sigma}(\Omega_h) \times W^{1,2+\sigma}(\Omega_h)$ to one of the coupled systems (3.1)-(3.2) and (3.1)-(3.3).

We now show the Lipschitz continuity of $\gamma \mapsto (h(\gamma), \mathbf{u}(\gamma), p(\gamma))$. To this end, let us consider for $\gamma_1, \gamma_2 \in [0, \gamma_0)$, the solutions $(h(\gamma_1), \mathbf{u}(\gamma_1), p(\gamma_1))$ and $(h(\gamma_2), \mathbf{u}(\gamma_2), p(\gamma_2))$ to the same FSI problem, either (3.1)-(3.2) or (3.1)-(3.3), with flow magnitude γ_1 and γ_2 , respectively. In order to compare the two solutions to the fluid boundary value problem (3.1), which are defined in different domains $\Omega_{h(\gamma_1)}$ and $\Omega_{h(\gamma_2)}$, we transform one domain into the other through the diffeomorphism in Lemma 3.6. Introducing the transformed

velocity and pressure

$$\mathbf{U}(\gamma_2) = \mathbf{u}(\gamma_2) \circ \varphi_{h(\gamma_2)-h(\gamma_1)}, \quad P(\gamma_2) = p(\gamma_2) \circ \varphi_{h(\gamma_2)-h(\gamma_1)},$$

defined in $\Omega_{h(\gamma_1)}$, it follows that $(\mathbf{U}(\gamma_2), P(\gamma_2))$ solves the transformed problem

$$\begin{aligned} & -\eta \nabla \cdot (A_{h(\gamma_2)-h(\gamma_1)} \nabla \mathbf{U}(\gamma_2)) + \mathbf{U}(\gamma_2) \cdot B_{h(\gamma_2)-h(\gamma_1)} \nabla \mathbf{U}(\gamma_2) \\ & \quad + B_{h(\gamma_2)-h(\gamma_1)} \nabla P(\gamma_2) = 0 \quad \text{in } \Omega_{h(\gamma_1)}, \\ & \nabla \cdot (B_{h(\gamma_2)-h(\gamma_1)}^T \mathbf{U}(\gamma_2)) = 0 \quad \text{in } \Omega_{h(\gamma_1)}, \\ & \mathbf{U}(\gamma_2) = 0 \quad \text{on } \partial \mathcal{B}^{h(\gamma_1)}, \quad \mathbf{U}(\gamma_2) = \gamma_2 V \mathbf{e}_1 \quad \text{on } \overline{\Gamma_y^+} \setminus \overline{\mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm, \\ & \mathbf{U}(\gamma_2) = \gamma_2 V_i \mathbf{e}_1 \quad \text{on } \Gamma_x^-, \quad \mathbf{U}(\gamma_2) = \gamma_2 V_o \mathbf{e}_1 \quad \text{on } \Gamma_x^+. \end{aligned} \tag{3.43}$$

The differences $\mathbf{W} = \mathbf{u}(\gamma_1) - \mathbf{U}(\gamma_2)$ and $Q = p(\gamma_1) - P(\gamma_2)$ solve

$$\begin{aligned} & -\eta \Delta \mathbf{W} + \nabla Q = -\mathbf{u}(\gamma_1) \cdot \nabla \mathbf{W} - \mathbf{W} \cdot \nabla \mathbf{U}(\gamma_2) \\ & \quad + S_1(h(\gamma_2) - h(\gamma_1), \mathbf{U}(\gamma_2), P(\gamma_2)) \quad \text{in } \Omega_{h(\gamma_1)}, \\ & \nabla \cdot \mathbf{W} = S_2((h(\gamma_2) - h(\gamma_1), \mathbf{U}(\gamma_2))) \quad \text{in } \Omega_{h(\gamma_1)}, \\ & \mathbf{W} = 0 \quad \text{on } \partial \mathcal{B}^{h(\gamma_1)}, \quad \mathbf{W} = (\gamma_1 - \gamma_2) V \mathbf{e}_1 \quad \text{on } \overline{\Gamma_y^+} \setminus \overline{\mathcal{B}_1} \cup \overline{\Gamma_y^-} \cup \Gamma_z^\pm, \\ & \mathbf{W} = (\gamma_1 - \gamma_2) V_i \mathbf{e}_1 \quad \text{on } \Gamma_x^-, \quad \mathbf{W} = (\gamma_1 - \gamma_2) V_o \mathbf{e}_1 \quad \text{on } \Gamma_x^+. \end{aligned} \tag{3.44}$$

with

$$\begin{aligned} S_1(h(\gamma_2) - h(\gamma_1), \mathbf{U}(\gamma_2), P(\gamma_2)) &= -\eta \nabla \cdot ((A_{h(\gamma_2)-h(\gamma_1)} - \mathbb{I}) \nabla \mathbf{U}(\gamma_2)) \\ & \quad + \mathbf{U}(\gamma_2) \cdot (B_{h(\gamma_2)-h(\gamma_1)} - \mathbb{I}) \nabla \mathbf{U}(\gamma_2) + (B_{h(\gamma_2)-h(\gamma_1)} - \mathbb{I}) \nabla P(\gamma_2), \\ S_2(h(\gamma_2) - h(\gamma_1), \mathbf{U}(\gamma_2)) &= \nabla \cdot ((B_{h(\gamma_2)-h(\gamma_1)}^T - \mathbb{I}) \mathbf{U}(\gamma_2)). \end{aligned}$$

Applying [26, Theorem 10.5.4] to (3.44) then yields

$$\begin{aligned} & \|\mathbf{W}\|_{W^{2,2+\sigma}(\Omega_{h(\gamma_1)})} + \|Q\|_{W^{1,2+\sigma}(\Omega_{h(\gamma_1)})} \\ & \leq C(\beta_0) \left(\|\mathbf{u}(\gamma_1) \cdot \nabla \mathbf{W} + \mathbf{W} \cdot \nabla \mathbf{U}(\gamma_2)\|_{L^{2+\sigma}(\Omega_{h(\gamma_1)})} + |\gamma_1 - \gamma_2| \right. \\ & \quad + \|S_1(h(\gamma_2) - h(\gamma_1), \mathbf{U}(\gamma_2), P(\gamma_2))\|_{L^{2+\sigma}(\Omega_{h(\gamma_1)})} \\ & \quad \left. + \|S_2(h(\gamma_2) - h(\gamma_1), \mathbf{U}(\gamma_2))\|_{W^{1,2+\sigma}(\Omega_{h(\gamma_1)})} \right). \end{aligned} \tag{3.45}$$

After combining (3.31) (up to obvious changes) with (2.4) and estimating the terms S_1 , S_2 in the right-hand side of (3.45) in the same way as in (3.34), we infer that there exists

$\gamma^* \in (0, \gamma_0]$, depending on β_0 , such that for $\gamma_1, \gamma_2 \in [0, \gamma^*)$,

$$\begin{aligned} \|\mathbf{W}\|_{W^{2,2+\sigma}(\Omega_{h(\gamma_1)})} + \|Q\|_{W^{1,2+\sigma}(\Omega_{h(\gamma_1)})} \\ \leq C(\beta_0) (|\gamma_1 - \gamma_2| + \gamma_2 \|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4}). \end{aligned} \quad (3.46)$$

Fixing $y \in (0, 1)$, the variation of the lift is

$$\begin{aligned} L(y, h(\gamma_1)) - L(y, h(\gamma_2)) \\ = \int_{\partial \mathcal{B}_y^{h(\gamma_1)}} \left(\mathbb{T}(\mathbf{u}(\gamma_1), p(\gamma_1)) \mathbf{n}_{h(\gamma_1)} - \tilde{\mathbb{T}}_{h(\gamma_2)-h(\gamma_1)}(\mathbf{U}(\gamma_2), P(\gamma_2)) M_{h(\gamma_2)-h(\gamma_1)} \mathbf{n}_{h(\gamma_1)} \right), \end{aligned}$$

where $\mathbf{n}_{h(\gamma_1)}$ is the unit normal vector to $\partial \mathcal{B}^{h(\gamma_1)}$ pointing inside $\mathcal{B}^{h(\gamma_1)}$. Arguing as in (3.36) and by (3.46), we get the estimate

$$\begin{aligned} \|L(\cdot, h(\gamma_1)) - L(\cdot, h(\gamma_2))\|_{L^\infty(0,1)} \\ \leq C(\beta_0) \left(\|\mathbf{W}\|_{W^{2,2+\sigma}(\Omega_{h(\gamma_1)})} + \|Q\|_{W^{1,2+\sigma}(\Omega_{h(\gamma_1)})} + \gamma_2 \|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4} \right) \\ \leq C(\beta_0) (|\gamma_1 - \gamma_2| + \gamma_2 \|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4}). \end{aligned} \quad (3.47)$$

Since $h(\gamma_1)$ and $h(\gamma_2)$ solve the same beam problem, either (3.2) or (3.3), it follows that

$$\begin{aligned} (h(\gamma_1) - h(\gamma_2))'''' + f(h(\gamma_1)) - f(h(\gamma_2)) &= L(\cdot, h(\gamma_1)) - L(\cdot, h(\gamma_2)) \quad \text{in } (0, 1), \\ (h(\gamma_1) - h(\gamma_2))(1) &= (h(\gamma_1) - h(\gamma_2))'(1) = 0, \\ (h(\gamma_1) - h(\gamma_2))''(0) &= (h(\gamma_1) - h(\gamma_2))'''(0) = 0 \quad \text{or} \\ (h(\gamma_1) - h(\gamma_2))''(1) &= (h(\gamma_1) - h(\gamma_2))'''(1) = 0 \end{aligned} \quad (3.48)$$

Analogously to (3.41), by elliptic regularity for (3.48), exploiting (1.15) and (3.47), we derive

$$\begin{aligned} \|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4} &\leq C \|L(\cdot, h(\gamma_1)) - L(\cdot, h(\gamma_2))\|_{L^\infty(0,1)} \\ &\leq C(\beta_0) (|\gamma_1 - \gamma_2| + \gamma_2 \|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4}), \end{aligned}$$

which implies that there exists (a possibly smaller) $\gamma^* \in (0, \gamma_0)$ such that

$$\|h(\gamma_1) - h(\gamma_2)\|_{\mathbb{H}^4} \leq C(\beta_0) |\gamma_1 - \gamma_2| \quad \text{for any } \gamma_1, \gamma_2 \in [0, \gamma^*].$$

Consequently, by (3.46), we also infer that

$$\|\mathbf{u}(\gamma_1) - \mathbf{U}(\gamma_2)\|_{W^{2,2+\sigma}(\Omega_{h(\gamma_1)})} + \|p(\gamma_1) - P(\gamma_2)\|_{W^{1,2+\sigma}(\Omega_{h(\gamma_1)})} \leq C(\beta_0) |\gamma_1 - \gamma_2|.$$

The proof is so complete.

4 | Conclusions and future developments

In this thesis we have analysed a stationary fluid-structure interaction problem that describes the equilibrium configurations between a viscous incompressible fluid and an elastic one-dimensional beam modelling an airfoil. The coupling of the two subsystems is inherently nonlinear, since the shape of the fluid domain changes according to the deformation of the solid structure, which in turn is driven by the stresses generated by the fluid flow.

The main result that we have achieved is the existence and uniqueness of strong solutions for two resulting PDE-ODE systems, one with a Cauchy problem and one with a boundary value problem, under the assumptions of sufficiently small boundary data. Furthermore, we have proved a Lipschitz continuity dependence of the solution on the inflow/outflow datum, which provides a stability property for the equilibrium configuration and, thus, the well-posedness of the two coupled models. This guarantees a controlled response of the system to perturbations of possible external forcing.

More generally, this work contributes to the rigorous mathematical understanding of fluid-structure interaction systems in deformable geometries, which have strong relevance in several physical and engineering applications.

Furthermore, this stationary framework provides a first rigorous description of fluid-structure interaction equilibrium states, although, to capture dynamical phenomena like oscillations and instability mechanisms, a discussion of the evolutive framework is needed. For this reason, a natural continuation of the present work involves studying the corresponding evolutive fluid-structure interaction problem, which brings additional mathematical difficulties due to the more complex structure of the system coupling. An example is the issue of the stability of the stationary configurations, which may lead to bifurcation phenomena or the analysis of long-time behaviour, which suggest using energy estimates for weak solutions, in the spirit of Leray solutions for Navier-Stokes incompressible equations, see [25]. The analytical tools developed in this thesis represents a natural starting point for such an extension, whose next step could be to apply the tools developed in [20].

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