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Managing Cliff Effect in Benchmarks Reform's LIBOR transition

Tesi di Laurea Magistrale in
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Abstract

Over the past few years, an unprecedented revolution and the most challenging issues are taking place: the Benchmarks Reform.

Interest reference rates (IBORs) are widely used in the global financial system as benchmarks for a large volume and broad range of financial products. Since the 2008 Credit Crunch, the post-crisis decline in liquidity in interbank unsecured deposit markets, together with the cases of attempted market manipulation and false reporting of global reference rates by some of the most important banks, have undermined confidence in the reliability and robustness of existing interbank benchmark interest rates.

Starting from 2013, following the recommendations of the Financial Stability Board (FSB) and the International Organization of Securities Commissions (IOSCO), public authorities in many countries and jurisdictions have taken steps to reform interest rate benchmarks, encouraging market participants to ensure timely progress towards it. In this light, the whole set of all the individual initiatives aimed at revising the major interest reference rates to make them compliant with the provisions of current regulations are known as Benchmarks Reform.

Such reform also brings the need to revise the way interest rates term structures, the cornerstones for the proper evaluation of a wide range of financial instruments, are built. Managing a transition between reference rates undertaking different (intrinsic) characteristics and risks, practitioners must take into account, especially during stressed market conditions as those occurred for the Covid-19 pandemic and Russo-Ukrainian conflict, the possibly disruptive and sharp jump estimating forward rates also known as Cliff Effect.

Based on the market Best Practice and the state-of-the-art curve bootstrapping techniques, this work aims to deeply understand the Cliff Effect and present some practical solutions to manage it following the path traced by leading investment banks and software houses whose develop their own methodologies.

To reach our goals, the thesis is structured as follows. In chapter one we introduce LIBOR with its definition and calculation mechanisms, showing the main issues and scandals surrounding it, and we also provide an overview on the key pillars within the scope of the Benchmarks Reform with their own milestones.

After fixing notation and explaining the meaning of the most important quantities we will work with, in chapter two we give a comprehensive description of interest rates term structures with a focus on market instruments' selection and their pricing formula in a multi-curve framework, on calibration techniques and algorithms and, finally, on most widely used interpolation schemes comparing them and describing their strengths and weakness.

Chapter three explains in detail what Cliff Effect is and what the main challenges practitioners are led to face are: a target forward curve is defined as benchmark and reasons why an underlying term structure cannot be determined from it are explained.

In chapter four we present our contribution to addressing the Cliff Effect related problems by proposing three different solutions to manage it and describing pros and cons of each one.

The thesis closes with chapter five on which all the proposed solutions are compared against each other, both in terms of distance from the target forward curve and in terms of market instruments' pricing not used for term structure construction.

Keywords: Cliff Effect, Benchmarks Reform, LIBOR, transition curve, interest rates term structure, bootstrapping.

Sommario

In questi anni si sta assistendo a una sfida in ambito finanziario senza precedenti: la cosiddetta "Benchmarks Reform".

I tassi d'interesse di riferimento (IBOR) vengono ampiamente utilizzati nel sistema finanziario globale come benchmark di un'ampia gamma di prodotti finanziari. Con il Credit Crunch del 2008, il progressivo diminuirsi della liquidità nei mercati interbancari a depositi non garantiti e con gli eventi di manipolazione del mercato da parte di alcune delle banche più importanti del mondo, la fiducia nella solidità dei tassi di interesse interbancari di riferimento esistenti è venuta sempre meno.

A partire dal 2013, seguendo le raccomandazioni del Financial Stability Board (FSB) e dell'International Organization of Securities Commissions (IOSCO), le autorità pubbliche di molti Paesi e giurisdizioni hanno preso provvedimenti per il rinnovo dei benchmark. In quest'ottica, l'insieme di tutte le singole iniziative volte a rivedere i principali tassi di interesse di riferimento per renderli conformi alle disposizioni della normativa vigente è noto come Benchmarks Reform.

Tale riforma comporta, tra le altre cose, la necessità di dover rivedere il modo in cui vengono costruite le strutture a termine dei tassi d'interesse, che costituiscono i pilastri per una corretta valutazione di un'ampia gamma di strumenti finanziari. Nella gestione di un passaggio tra tassi di riferimento con caratteristiche (intrinseche) e rischi diversi, gli operatori devono tenere conto, soprattutto in condizioni di mercato stressate come quelle verificatesi per la pandemia Covid-19 e il conflitto russo-ucraino, dell'eventuale brusco e dirompente salto nella stima dei tassi forward, noto anche come Cliff Effect.

Basandosi sulle Best Practice di mercato e sullo stato dell'arte delle tecniche di bootstrapping, l'obiettivo di questa tesi è quello di comprendere a fondo il Cliff Effect e di presentare alcune pratiche soluzioni per riuscire a gestirlo, seguendo il percorso tracciato dalle principali banche d'investimento e dalle software house che si sono mosse a sviluppare le proprie metodologie.

La tesi è quindi strutturata come segue. Nel primo capitolo introduciamo il LIBOR dando la sua definizione e presentando i suoi meccanismi di calcolo, mostrando i principali problemi e scandali ad esso associati, fornendo anche una panoramica su quelli che sono

i pilastri chiave nell'ambito della Benchmarks Reform.

Dopo aver introdotto la notazione, nonché il significato delle quantità più importanti con cui lavoreremo, nel secondo capitolo forniamo una descrizione completa delle strutture a termine dei tassi d'interesse, con particolare attenzione alla selezione degli strumenti di mercato e alla loro formula di pricing in un contesto multicurva, alle tecniche e agli algoritmi di calibrazione e, infine, agli schemi di interpolazione più utilizzati, per poi confrontarli e descriverne i punti di forza e di debolezza di ciascuno di essi.

Il terzo capitolo spiega in dettaglio cos'è il Cliff Effect e quali sono le principali sfide che l'industry è portata ad affrontare: viene definita una curva forward che sarà il target del nuovo benchmark e viene spiegato il motivo per cui non è possibile determinare una struttura a termine partendo da tale curva.

Nel quarto capitolo presentiamo quello che è il nostro contributo nell'affrontare i problemi legati al Cliff Effect, proponendo tre diverse soluzioni e descrivendo i pro e i contro di ciascuna di esse.

La tesi si chiude con il quinto capitolo, in cui tutte le soluzioni proposte vengono confrontate tra loro, sia in termini di distanza dalla curva forward target sia in termini di pricing degli strumenti di mercato non utilizzati per la costruzione della struttura a termine.

Contents

1 | Introduction

Interest rate benchmarks, also known as reference rates or just benchmarks, are regularly updated interest rates, publicly accessible and useful basis for all kinds of financial contracts such as mortgages, bank overdrafts and other more complex financial transactions. They most often reflect the cost of borrowing money in different markets, or how much it costs for banks to borrow from each other, or even how much it costs banks to obtain funds from other sources, such as pension funds, insurance companies and money market funds.

Taking London Interbank Offered Rate (LIBOR) as an example, it is nowadays the international standard for interest rate markets and is considered the world's most important number so far as to be called the "London crown jewel". Indeed, despite the market to which it refers (the London interbank market) is characterized by more or less $500M daily transaction volumes, LIBOR helps set interest rates worldwide and affects the price of more than $200T in mortgage, loans and derivatives (see Figure 1.1 for a detailed breakdown of instruments referenced to LIBOR). Only in the 2000s derivative products indexed to LIBOR grew ten-fold from $10T up to more than $100T of notional outstanding in the USD market alone. It is probably the most important benchmark rate in the world [3].

Anyway, starting from the 2008 financial crisis (also known as Credit Crunch) and the Lehman Brothers bankruptcy, banks became increasingly wary of lending money to each other. Due to the limited activity that has generated the lack of effective transactions data and the consequent decrease in liquidity in the unsecured interbank funding markets, LIBOR has gradually become extremely based on expert Panel Bank judgment rather than actual transaction.

For these reasons, in addition to the excessive leverage and the announcement of some scandals concerning the LIBOR's manipulation by some banks (see Section 1.2 for more details), starting from 2013 both supranational and sector authorities, such as the G20 nations, the Financial Stability Board (FSB) and the International Organization of Securities Commissions (IOSCO), embark on a process of undertaking a fundamental review of major interest rate benchmarks (see Section 1.3 for more details), with the aim of making them more reliable and, where not possible, to develop more robust alternative reference rates.

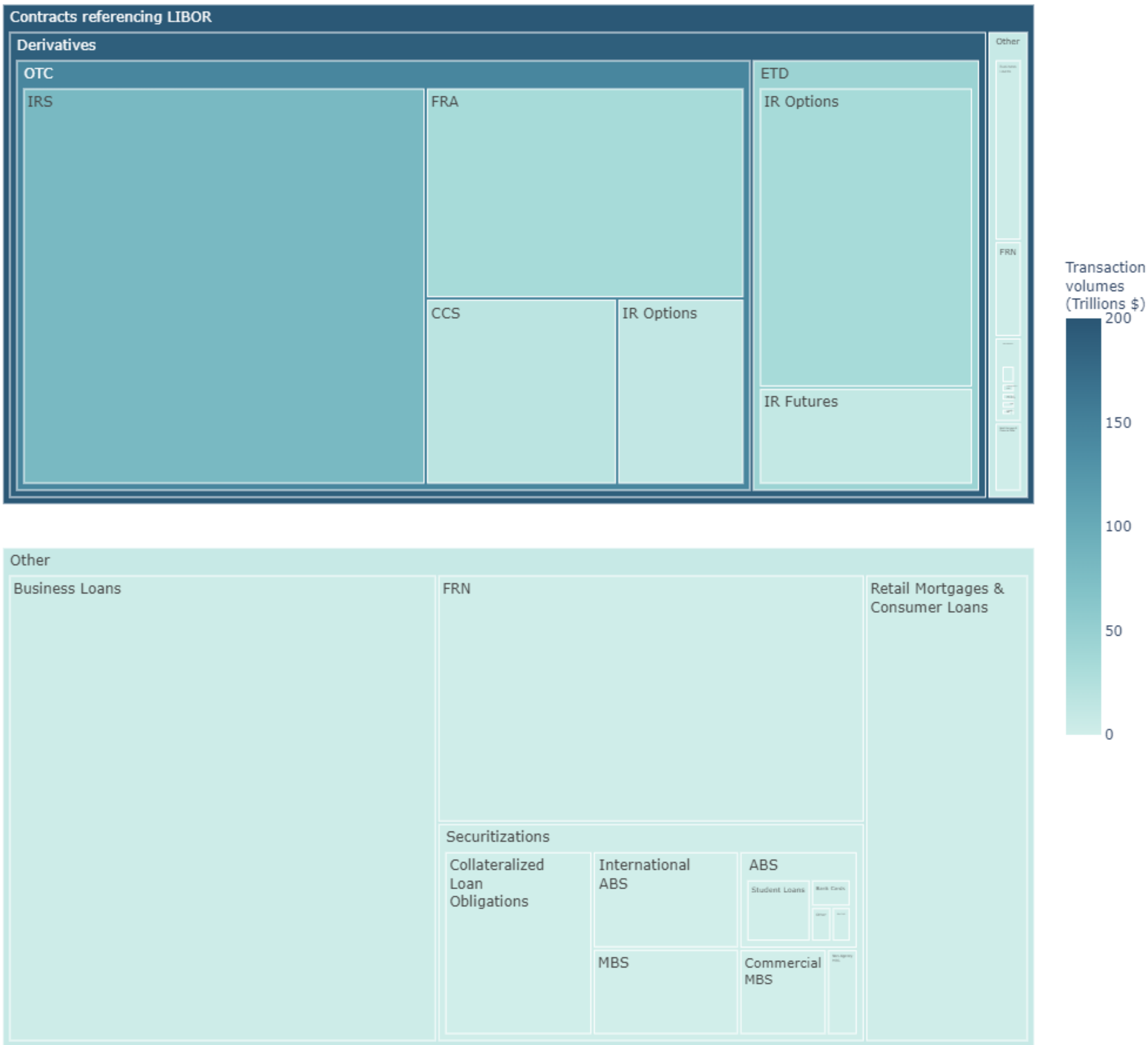

Figure 1.1: Overall (upper figure) and partial (lower figure) instruments' notional amount outstanding referenced to LIBOR

In this context the Financial Conduct Authority (FCA), the UK regulator overseeing the LIBOR benchmarks, has stated that it expects that some Panel Banks will withdraw from making submissions after end-2021, deciding to no longer make the LIBOR publication mandatory beyond that date and calling for significant efforts by regulator and financial firms in order to mitigate potential risks arising from the expected cessation of LIBOR.

The financial risks associated with migrating to alternative reference rates, typically based on overnight transactions to make them "as risk-free as possible", are indeed impactful in a undreamable way for non-experts. We have to consider that pricing models, financial systems and market infrastructure have been built around LIBOR benchmarks for several decades, which have made the size, scale and scope of the migration to new benchmark rates arguably one of the biggest challenges facing the financial industry.

From a more technical perspective, the unfixable structural differences between LIBOR and their alternative reference rates make it challenging to estimate the future value of floating rate to use when LIBOR no longer exist as expected. The problem we will dealing with regards the construction of the LIBORs' term structure and, in correspondence of the day in which LIBORs will cease to exist, the jump on the forward rates curve due to the switch from LIBORs to the alternative benchmark later [14]. This jump, that we will call Cliff Effect from now on, could create a lot of non-trivial problems in pricing financial instruments and will be the subject of this thesis.

In particular, this work aims to deeply understand the Cliff Effect and present some practical solutions to manage it following the path traced by leading investment banks and software houses whose develop their own methodologies within the position keeping and revaluation systems most widely used in the financial industry. To reach our goals, the thesis is structured as follows.

In chapter one we introduce LIBOR with its definition and calculation mechanisms, showing the main issues and scandals surrounding it, and we also provide an overview on the key pillars within the scope of the Benchmarks Reform with their own milestones.

After fixing notation and explaining the meaning of the most important quantities we will work with, in chapter two we give a comprehensive description of interest rates term structures with a focus on market instruments' selection and their pricing formula in a multi-curve framework, on calibration techniques and algorithms and, finally, on most widely used interpolation schemes comparing them and describing their strengths and weakness.

Chapter three explains in detail what Cliff Effect is and what the main challenges practitioners are led to face are: a target forward curve is defined as benchmark and reasons why an underlying term structure cannot be determined from it are explained.

In chapter four we present our contribution to addressing the Cliff Effect related problems by proposing three different solutions to manage it and describing pros and cons of each one. As we shall see, this will result in the introduction of new market instruments, carefully constructed in order to be able to obtain a comprehensive curve.

The thesis closes with chapter five on which all the proposed solutions are compared against each other, both in terms of distance from the target forward curve and in terms of market instruments' pricing not used for term structure construction. In the end, we present some further ideas to solve Cliff Effect problem.

As support of our work we used QuantLib [16], an open-source library written in C++ for (and by) financial quantitative analyst and developers. All the numerical evaluations are performed through the QuantLibXL project [18] that export an object-oriented interface on Microsoft Excel and results are shown using the Python QuantLib package [17] via a Jupyter notebook.

1.1. Understanding LIBOR

The London Interbank Offered Rate (LIBOR) is a benchmark interest rate at which major global banks lend to one another in the international interbank market for short-term loans. There are 35 LIBORs, spanning seven tenors (Overnight, 1-Week, 1-Month, 2-Months, 3-Months, 6-Months, 12-Months) and five currencies (USD, EUR, GBP, JBY, CHF).

Each day, InterContinental Exchange (ICE) asks major global banks, so-called Panel Banks, how much they would charge other banks for short-term loans.

The 25% higher and lower rates are taken out from the calculation, then the LIBOR is computed as the average from the remaining numbers. All this procedure is conducted by Thomson Reuters provider and the rate is posted each morning as the daily rate. Once the rates for each maturity and currency are calculated and finalized, they are fixed and published once a day at around 11:55 a.m. London time by the ICE Benchmark Administration (IBA) [11].

Current USD LIBOR Panel Banks
Bank of America N.A. (London Branch)
Barclays Bank plc
Citibank N.A. (London Branch)
Cooperatieve Rabobank U.A.
Crédit Agricole Corporate & Investment Bank
Credit Suisse AG (London Branch)
Deutsche Bank AG (London Branch)
HSBC Bank plc
JPMorgan Chase Bank, N.A. (London Branch)
Lloyds Bank plc
MUFG Bank, Ltd
Royal Bank of Canada
SMBC Bank International plc
The Norinchukin Bank
UBS AG

Table 1.1: List of current USD LIBOR Panel Banks [9]

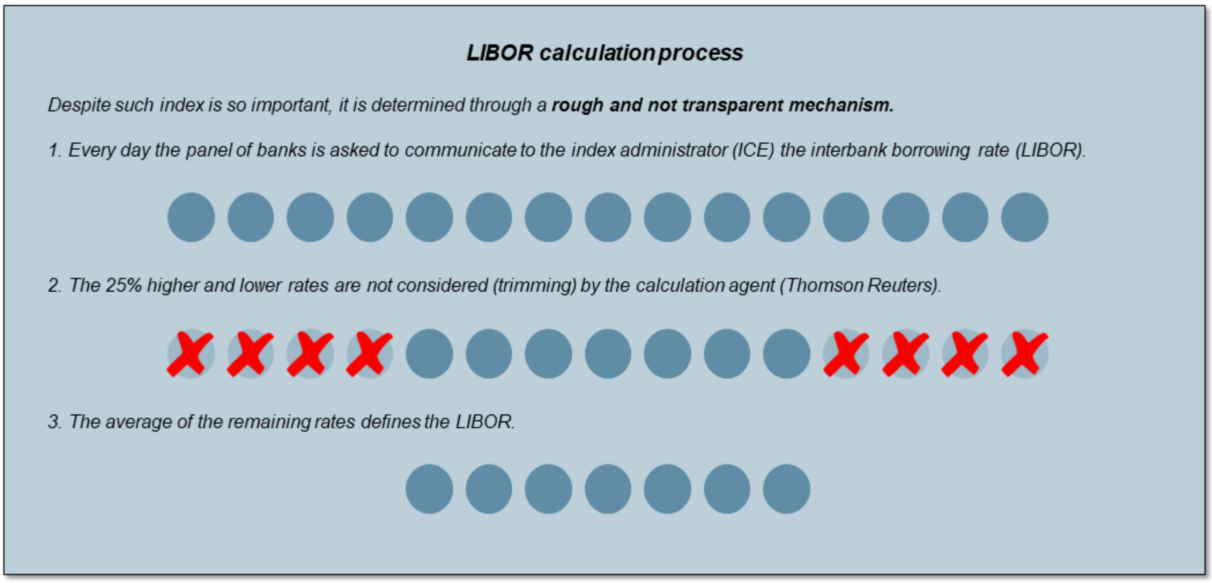

Figure 1.2: Scheme of LIBOR calculation process

LIBOR rates have an international counterpart termed (IBOR), a term used indicate interbank lending rates set in major financial centres other than London e.g. EURIBOR (Europe/Brussels), HIBOR (Hong Kong), TIBOR (Tokyo) et al. The Association for Financial Markets in Europe (AFME) report market exposure based on total notional amount outstanding for IBOR products is greater than USD 370 trillion [3].

1.2. LIBOR weakness and scandals

Despite the importance of LIBOR, it has been victim of some manipulations on the downside, leading Panel Banks traders to improper earnings in the period between 2005 and 2010. An example of these illicit is described below.

In June of 2012, Barclays plc, one of the world's largest and most important banks, admitted that, between 2005 and 2009, manipulated LIBOR to gain profits and/or limit losses from derivative trades. In addition, between 2007 and 2009 the firm had made dishonestly low LIBOR submission rates to dampen market speculation and negative media comments about the firm's viability during the financial crisis. By manipulating the LIBOR, the traders in question were indirectly causing a cascade of mispriced financial assets throughout the entire global financial system.

Indeed a month later, in July of 2012, the city of Baltimore (Maryland, USA) accused Barclays plc and the other Panel Banks to have robbed the city of millions of dollars in returns on investments such as Interest Rate Swaps, keeping LIBOR artificially low. Baltimore bought hundreds of millions of dollars worth of Interest Rate Swaps during the period when the alleged fixing took place. Swaps were used by many government agencies that funded public infrastructure such as transit systems, water-works and stadiums that allowed the agencies to exchange the floating interest rates promised to bond investors for fixed rates paid by banks. The following and inevitable increase of LIBOR led Baltimore (and other cities involved in the scandal) to massive losses [5].

At the same time, LIBOR started to show some weakness related to low volume and liquidity in transactions, geographical, sector and Panel Bank bias, lack of transparency and reliable on expert judgment [3]. For these and other reasons financial world started to think it was time to change something for real.

1.3. The Benchmarks Reform

Due to weakness and scandals associated to LIBOR, as explained above, a series of reforms started to be published in different countries and in different forms but all collectable under the name of "Benchmarks Reform": in Figure 1.3 is represented the timeline of the most relevant Benchmarks Reform events. One of these reforms is about the transition from IBOR rates to new one called Alternative Reference Rates (ARRs). The one we are interest in is the transition from USD LIBOR benchmarks to its associate ARR: the SOFR rate.

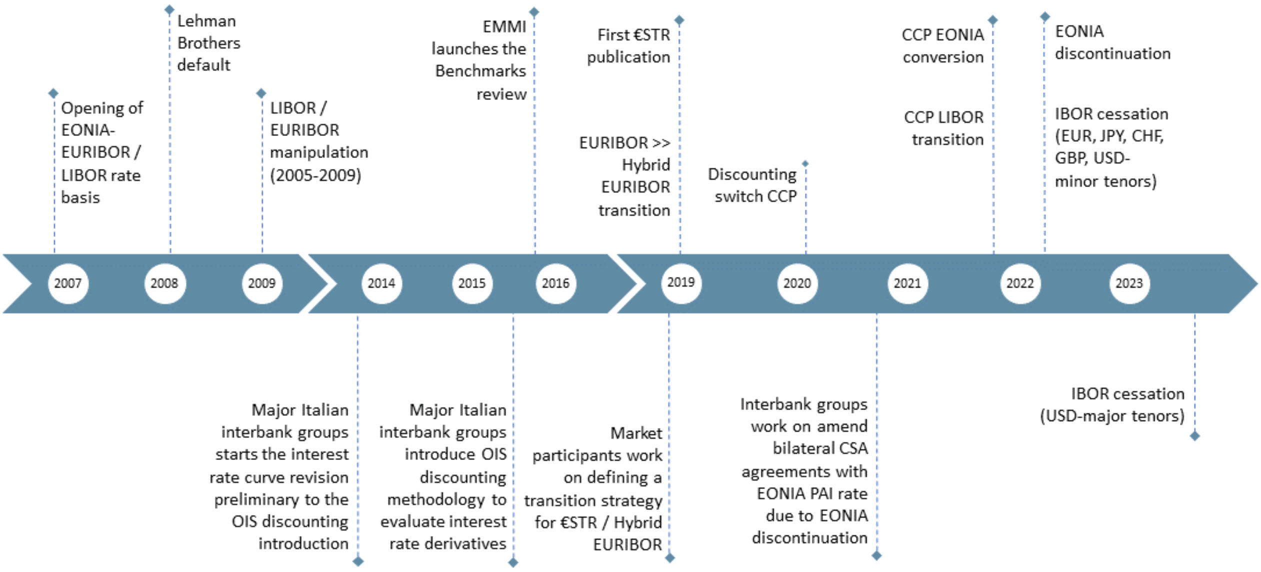

Figure 1.3: Timeline of the most relevant Benchmarks Reform events

In July of 2013, the International Organization of Securities Commissions (IOSCO) published a robust framework of principles for underlying benchmark rates in financial markets [12]. The principles, as described by IOSCO, are:

- **Governance**, Administrators must have the appropriate government arrangements in place to protect the integrity of the benchmark determination process and address conflicts of interest.
- **Quality**, The design and data used to construct the benchmark rate should be such that the benchmark provides an accurate, reliable and transaction-based representation of the underlying market.
- **Methodology**, The calculation methodology must be transparent with clear published controls in place to manage when material changes are planned. Furthermore,

administrators must establish credible contingency policies in case the benchmark becomes unavailable, ceases to exist or stakeholders need to transition to another benchmark.

- **Accountability**, The administrator must establish complaints processes, documentation standards and audit reviews and provide evidence of compliance to its quality standards.

These Principles were intended to promote the reliability of benchmark determinations, and address benchmark governance, quality and accountability mechanisms. The Principles provide a framework of standards that Administrators should implement according to the specificities of each benchmark. In particular, the application and implementation of the Principles should be proportional to the size and risks posed by each benchmark and/or Administrator and the benchmark-setting process.

On March 5, 2021 FCA announced the cessation (/loss of representativeness) of the 35 LIBORs. Each of them, except five (which asked for a postponing, also due to delays linked to Covid-19 pandemic: USD Overnight, 1-Month, 3-Months and 6-Months, 1-Year) ceased to be published on December 31, 2021. The last five will cease to exist on June 30, 2023. Our study on the Cliff Effect will involve them.

SOFR rate differ from USD LIBOR rates in several respects. USD LIBOR represents interest rates for unsecured interbank loans across various tenors. So, it incorporates unsecured bank credit risk. In contrast, SOFR is an overnight interest rate which incorporate little or no credit risk and, for this reason, is lower than USD LIBOR rates. Furthermore, the markets underpinning SOFR rate are significantly more active than the markets underpinning USD LIBORs. Hence, while USD LIBOR relies significantly on expert judgement, SOFR rate is transaction-based (see Table 1.2) [20].

Benchmark	USD LIBORs	SOFR
Tenors	Published for seven various tenors	Overnight
Secured vs. Unsecured	Unsecured	Secured
Credit risk	Incorporate a term bank credit spread	Minimal credit risk
Rate determination method	Rely in large part on expert judgement, due to low volume in underlying markets	Transaction based with significant volume in underlying markets

Table 1.2: Core differences between USD LIBORs and SOFR

As the transition for an existing contract from LIBOR to an ARR has to be economically neutral, a so-called credit adjustment spread (CAS) is used to bridge the gap to minimise the economic impact during the transition (see Figure 1.4).

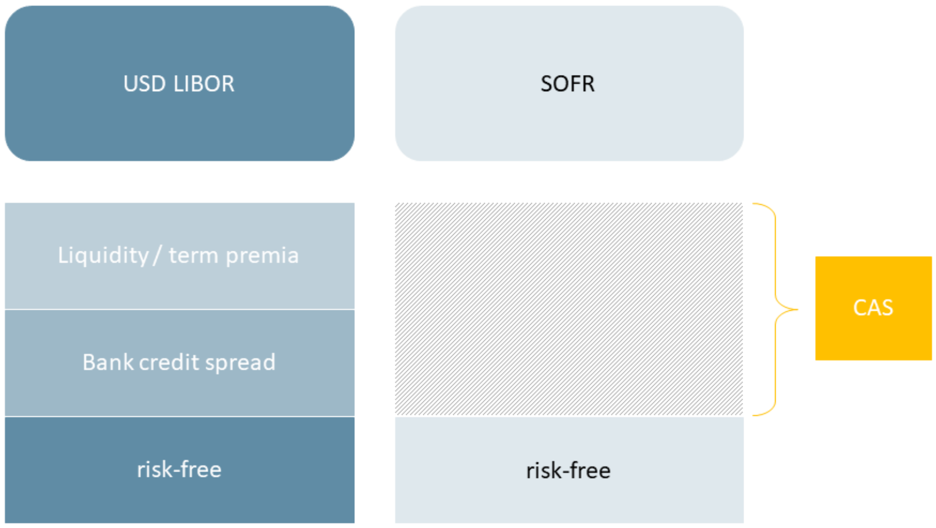

Figure 1.4: Reasons behind the credit adjustment spread

The transition to new ARRs involved also other IBOR rates. In Figure 1.5 is shown the way through which every jurisdiction has chosen a "near risk-free" rate to replace its old legacy benchmark [20].

Legacy benchmark	USD LIBOR	EURIBOR, EONIA, Euro LIBOR	CHF LIBOR	GBP LIBOR	JPY LIBOR, JPY TIBOR, EUROYEN TIBOR
Recommended Alternative Reference Rate	**SOFR** Secured Overnight Financing Rate	**€STR** Euro Short Term Rate	**SARON** Swiss Average Rate Overnight	**Reformed SONIA** Sterling Overnight Index Average	**TONA** Tokyo Overnight Average Rate
Administrator	Federal Reserve Bank of New York	European Central Bank	SIX Swiss Exchange	Bank of England	Bank of Japan
Secured vs. Unsecured	Secured	Unsecured	Secured	Unsecured	Unsecured
Underlying Market	US Treasury Repo market	Overnight unsecured fixed rate deposit transactions over €1mn	CHF Repo Market	Overnight unsecured sterling deposit transactions	Overnight unsecured call rate market
Initial ARR Publication Date	3 April 2018	2 October 2019	2009	1997; Reformed 23 April 2018	1992
Publication Time	Published at 8am Eastern Time on following business day	Published at 9am CET on following business day	Published daily every 10 minutes from 8:30AM, with fixings at 12pm, 4pm, 6pm	Published at 9am CET on following business day	Published at 10am on following business day

Figure 1.5: Recommended ARRs and their characteristics

2 | Interest Rates Yield Curves

Interest rates term structures (or yield curves) are the founding stones of all the quantitative finance. Without them it would be impossible to correctly price any security, given that there would be no other way to estimate future flows, nor to correctly assess the their expected value at the evaluation date.

Unfortunately, term structures are not directly observable as market data due to the non-existence of zero-coupon instruments over a continuum of maturity dates. As a consequence, financial institutions and software houses have been found a way to extract those data from quoted prices of a finite number a market instruments. Of course there is not a unique financially right choice of market instruments and none construction methodology to build yield curves is absolutely better than any other: as Ametrano and Bianchetti [4] say and describe exhaustively, the construction of a yield curve is more a "question of art" than a science.

Before proceeding, let's note that we can recognize two main different classes of construction procedures:

- **Best-fit algotithms**: assuming that there exists a functional form for the term structure, we calibrate its parameters using a selection of instruments quoted on market such that to minimize the repricing error. This approach is mostly used for the smoothness of the resulting curves and the intuitive financial interpretation of the functional form parameters (e.g. level, slope, curvature etc.);
- **Exact-fit algorithms**: fixing the yield curve on a time grid of N points called pillars, we want to exactly reprice N pre-selected instruments quoted on market. The implementation of this method is usually done with a recursive approach called bootstrap, which extend the yield curve step-by-step with the increasing maturity of the ordered instruments and the intermediate values are obtained interpolating on the bootstrapping grid.

As we can easly imagine, best-fit algorithms ensure smoothest curves but the fit quality is typically not good enough for pricing purpose in a liquid market; to the other hand, exact-fit algorithms exactly reprice the input bootstrapping instruments but produce nastiest curves.

Since the final goal of this thesis is to find a yield curve, to use for pricing purposes, that can best handle the Cliff Effect we will describe in detail in Chapter 3, we will adopt (and in the following give a step-by-step description of) an exact-fit approach widely used by practitioners to build interest rate term structures.

After Section 2.1 where we will fix the notation, in the following sections we will provide a description as comprehensive as possible of the main steps used to build a yield curve that can be easily summed up as:

1. financial instruments' selection: define a set of the most liquid vanilla interest rate instruments traded in real time on the OTC market and with increasing maturities (usually Deposits for short-term, Forwards / Futures for mid-term and Swaps for long-terms, see Section 2.2);
2. curve's calibration: detect an iterative process to define a term structure, in terms of discount factor or zero-coupon rates, that exactly reprice the selected instruments;
3. pillars' interpolation: choose an interpolation scheme (to use already during the calibration process)in order to get a complete term structure allowing the estimation of its values at any time between pillars.

2.1. Notation

The aim of this section is to fix the notation and to explain the meaning of all the variables that will appear in the next pages. We mainly refer to Brigo and Mercurio [2] adapting the notation to the modern market environment as made in Ametrano and Bianchetti [4].

2.1.1. Time structure

When we deal with interest rate instruments, we need to have clear in mind what are the periods on which they consist of and how the time intervals are counted. To describe the time structure of such instruments, we will use a time grid $\mathbf{T} = \{t_0, ..., t_n\}$ that collects all the relevant contract dates, which are:

- **Fixing date**: date in which contracts are stipulated;

- **Settlement date**: date in which contracts start to be valid. For USD market, is usually 2 business days after the fixing date;
- **Start date**: date in which contracts start to accrue interests. For spot contracts, it is equal to the settlement date;
- **Cash flows or payment dates**: dates, if provided by the contract, in which interests are paid;
- **End date**: date in which contracts expire.

Moreover, when it is necessary to compute cash amounts for time periods shorter than a year, it is very important to count in the right way the year fraction between the two dates. Here we will denote it with one of the equivalent notation

$$\tau_{x,y} = \tau(t_x, t_y) = \tau[t_x, t_y; \mathrm{dc}]$$

where dc represents the day count convention as widely described in OpenGamma [19].

2.1.2. Interest rates

Here below we define all the different types on interest rates we will use in the following, also giving an unambiguous representation of the main quantities and objects associated with them.

Before we start, it is fundamental to take into account that in the present market environment each interest rate (and any quantity associated with it) must relate to a specific tenor for which there exists a reference index or benchmark like, in our case, LIBOR. For this reason, we introduce the notation $\_^B$ to link each quantity to a specific LIBOR tenor with $B \in \{ON, 1W, 1M, 2M, 3M, 6M, 1Y\}$.

First of all, for any couple of times $t_x \leq t_y$ we define the discount factor $D^B_{x,y} = D^B(t_x, t_y)$ as such function that gives the value at time t_x of one monetary unit received in t_y and that satisfies

- $D^B(t,t) = 1 \quad \forall t \geq 0;$
- $\forall t_x \geq 0,\ D^B_x(\cdot) = D^B(t_x, \cdot) : t_y \mapsto D^B(t_x, t_y)$ is differentiable, hence $D^B(t_x, \cdot) \in$ $\in \mathcal{C}^1(\mathbb{R}^+)$;
- $\forall t_y \geq 0,\ D^B_y(\cdot) = D^B(\cdot, t_y) : t_x \mapsto D^B(t_x, t_y)$ is stochastic.

On the contrary, if we want to know the value accrued in t_y of one monetary unit holds in t_x we define the zero-coupon rate function $z^B_{x,y} = z^B(t_x, t_y)$ that is linked to the discount factor through the injective relation

$$D^B(t_x, t_y) = e^{-z^B(t_x,t_y)\cdot\tau_{x,y}}. \tag{2.1}$$

Once the simplest and key definitions are given, we can now denote with $L^B_{x,y} = L^B(t_x, t_y)$ the spot Libor rate, fixed on the market at time t_x and spanning the time interval $[t_x, t_y]$ where, in general, $\tau_{x,y} \approx B$. The market practice assumes that the Libor rate in expressed in a simple interest regime as follows:

$$1 + L^B(t_0, t_y) \cdot \tau_{x,y} = \frac{1}{D^B(t_0, t_y)} \tag{2.2}$$

Notice that, according to the definition of the discount factor function, $L^B_{x,y}$ is stochastic for any time t_x in the future. For this reason, we define the forward Libor rate (or simply forward rate) $F^B_{x,t}(t_0) = F^B(t_0; t_x, t_y)$ as the expectation, in an appropriate risk-neutral forward measure $\mathbb{Q}$, at time t_0 of the future value of the spot Libor rate, hence [6]

$$F^B(t_0; t_x, t_y) = \mathbb{E}^{\mathbb{Q}}_0 \left[L^B(t_x, t_y)\right] = \mathbb{E}^{\mathbb{Q}} \left[L^B(t_x, t_y)|\mathcal{F}_{t_0}\right] \tag{2.3}$$

where $\mathcal{F}_{t_0}$ is the filtration in t_0.
We do not spend additional time figuring out which is the right forward measure $\mathbb{Q}$ because this is not the scope of the thesis and, more important, forward Libor rates are given by the market and can be deduced from the interest rate term structure. Indeed, through a simple reasoning in terms of absence of arbitrage opportunity, one can prove that the following relations holds:

$$F^B(t_0; t_x, t_y) = \frac{1}{\tau_{x,y}} \left(\frac{D^B(t_0, t_x)}{D^B(t_0, t_y)} - 1 \right) \tag{2.4}$$

Finally, thinking on what a term structure means, we also define different types of yield curves $\mathcal{C}^B$ (potentially one for each LIBOR tenor) as follows:

- **discount factors** curve $\mathcal{C}^B_D(t) = \left\{T \mapsto D^B(t, T)\right\}$;
- **zero-coupon rates** curve $\mathcal{C}^B_z(t) = \left\{T \mapsto z^B(t, T)\right\}$;
- **log-discounts** curve $\mathcal{C}^B_{\ln D}(t) = \left\{T \mapsto \ln(D^B(t, T))\right\} = \left\{T \mapsto -z^B(t, T) \cdot \tau(t, T)\right\}$.

Remark 2.1. *From here on, without loss of formality, the notation* $\_^B$ *will be omitted if* *it does not create ambiguity: the implied LIBOR tenor associated to any function will be deduced from the corresponding period between* t_x *and* t_y*.*

Remark 2.2. *Always with the aim of simplifying the notation, from now on it will happen to use* D_i *and* z_i *instead of the extended version* $D(t_0, t_i)$ *and* $z(t_0, t_i)$ *for discount factors and zero-coupon rates respectively.*

2.2. Financial instruments selection

As previously anticipated, bootstrap procedure in exact-fit algorithms needs an initial selection of financial instruments traded in the market and with increasing maturities. This is done via an info-provider software, like Bloomberg or Thomson Reuters, and the way through which we choose these instruments is done in such a way the selection includes the most liquid contracts traded on the market and, so, they are a good proxy for imply a reasonable term structure for interest rates. This because of, traded a lot, they can represent better than other instruments the sentiment of financial market.

According to the market best practice, a representative selection of market instruments includes:

- Deposit contracts for maturities from today up to one year;
- Forward Rate Agreement (FRA) contracts covering the window from one month up to two years;
- Short Term Interest Rate (STIR) Futures contracts for maturities from three months up to two years and more (depending on the currency);
- Interest Rate Swap (IRS) contracts covering the window from two or three years up to sixty years.

The evolution of the financial markets after the crisis of 2008, also known as Credit Crunch crisis, has triggered a general reflection about the methodology used to select instruments and build interest rate term structures for pricing and hedging purposes.

Briefly (see [4] for an exhaustive discussion about it), the pre-crisis standard market practice was to build one single interest rate term structure because of the belief that interbank credit/liquidity issues did not matter for pricing and LIBORs were a good proxy for risk free rates. Trivially with the Lehman Brothers bankruptcy everything changed: practitioners realized that:

1. the so-called single-curve approach did not take into account the market information carried by the Basis Swap spreads, now much larger than in the past and no longer negligible (see Figure 2.1);

2. it does not take into account that the interest rate market is segmented into sub-areas corresponding to instruments with different underlying rate tenors, characterized by different dynamics;

3. there is no mention to collateral agreements and funding costs associated to the pricing of market instruments.

So now, the post-crisis and actual market best practice (the so-called multiple-curve approach) can be summarized as follows:

1. by no-arbitrage, discounting must be unique: decide the appropriate funding rate, also according to collateral agreement in place with your own counterparties, then select the corresponding market instruments (i.e. those that have the same rate as underlying) and build one single discounting curve;

2. select multiple separated sets of vanilla interest rate instruments traded on the market with increasing maturities, each set homogeneous in the underlying rate (typically with 1M, 3M, 6M, 12M tenors);

3. build multiple separated forward curves using the selected instruments and the unique discounting curve within a chosen bootstrap technique;

4. compute the relevant forward rates and the corresponding cash flows from the forward curve with the appropriate tenor;

5. compute the relevant discount factors from the discounting curve to estimate the present value of the cash flows obtained above.

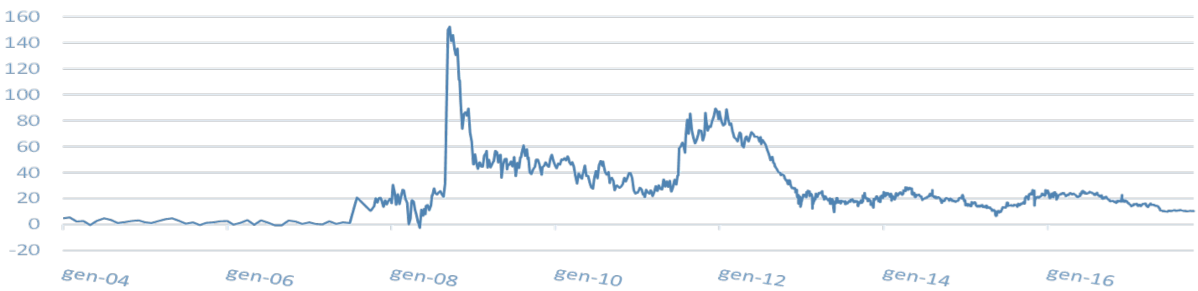

Figure 2.1: Historical Basis Swap spreads (bps) from 2004 to 2017

In the following paragraphs we examine in detail the market instruments selected, accordingly to the actual multi-curve approach, to build our own yield curves described in Chapter 3 and 4 to manage the Cliff Effect.

2.2.1. Deposit

Interbank Deposits are unsecured loans between banks. They can be viewed as zero coupon contracts, where the Lender pays a notional amount N to the Borrower and, after the period associated to the deposit, he receives back the notional amount N plus the interest earned over the period.

Schematically, we have three dates:

- the fixing date (t_0^F) when the interest rate that will be applied on our Deposit is fixed;
- the settlement Date (t_0) when the notional amount is paid by the Lender;
- the maturity Date (t_1) when the contract ends and the notional amount plus the interests earned are paid back by the Borrower.

The interest rate applied to the contract is called Deposit rate and is represented as K^{Depo}. Trivially, the payoff of this contract at maturity date t_1 (see Figure 2.2) is

$$\mathbf{Depo}(t_1) = N \cdot (1 + K^{Depo} \cdot \tau_{0,1}) \tag{2.5}$$

If we want to compute the value at settlement date of the Deposit we have to discount this unique cashflow to that date, so:

$$\mathbf{Depo} = D_1 \cdot \mathbf{Depo}(t_1) \tag{2.6}$$

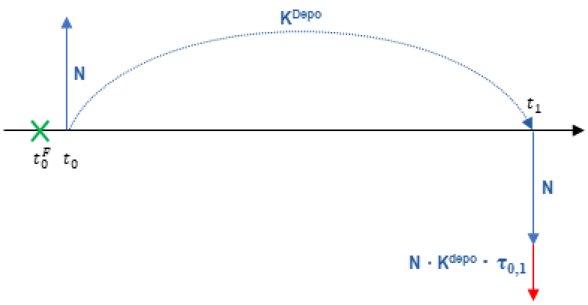

Figure 2.2: Representation of a Deposit contact

These contracts are quoted and exchanged on the OTC market for various currencies and maturities, that usually do not go beyond the year [13].

2.2.2. Forward Rate Agreement

A Forward Rate Agreement (FRA) is a forward contract traded on OTC markets in which two counterparties fix a rate K^{FRA} that will be applied on a deposit, with notional N, that starts at future date and ends at a later date.

Schematically, also in this case we have three dates:

- the fixing date (t_0) when the interest rate that will be applied on our FRA is fixed and nothing is paid;
- the settlement Date (t_1) when the contract ends (the date in which our deposit theoretically starts to accrue interest) and the payoff of the contract is paid back by the Borrower or the Lender, depending on the value of the underlying;
- the maturity Date (t_2) when the deposit expires.

In practice, the notional amount is not exchanged and the contract, neither the counterparties enter into the subsequent deposit. What actually happens is that a cash amount is exchanged based on the difference between the rate K^{FRA} established at the fixing date t_0 and the value of an interest rate index, usually LIBOR or EURIBOR, that will be fixed at settlement date.
In this way, the payoff of this contract at settlement date (see Figure 2.3) is

$$\mathbf{FRA}(t_1) = N \cdot (L(t_1, t_2) - K^{FRA}) \cdot \tau_{1,2} \cdot D_{1,2} \tag{2.7}$$

and the the net present value is given by

$$NPV(t_0) = \mathbf{FRA}(t_1) \cdot D_1 = N \cdot (F(t_0; t_1, t_2) - K^{FRA}) \cdot \tau_{1,2} \cdot D_2 \tag{2.8}$$

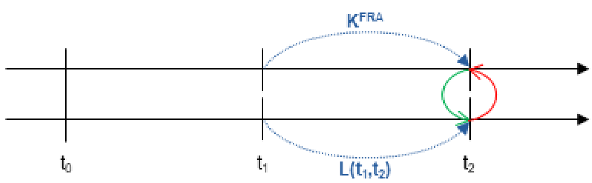

Figure 2.3: Representation of a Forward Rate Agreement contact

FRAs are quoted in terms of par rates, namely that rate K^{FRA} which sets the NPV of the contract in t_0 equal to 0. Solving 2.8 with respect to K^{FRA}, we simply have:

$$K^{FRA} = F(t_0; t_1, t_2) \tag{2.9}$$

2.2.3. Futures

Short Term Interest Rate (STIR) Futures are exchange-traded contracts equivalent to the OTC market FRAs. They are marked-to-market every day, i.e. any profit and loss is regulated through a daily settlement, reducing credit risk and transaction costs. For all of this reasons, Futures contracts are very liquid.

The most traded contracts are the Internationa Monetary Market (IMM) Futures from the Chicago Mercantile Exchange. Their peculiarity is to insist on 3-Months LIBOR (or EURIBOR) and to expire on the third Wednesday every March, June, September and December (the so-called IMM Dates). The first front contract is the most liquid interest rate instrument, with longer expiry contracts having reasonable liquidity up to about the 8-th contract (with expiry to two years from fixing date).

Futures are quoted in terms of prices instead of rates and the relationship between price and rate is given by:

$$Price = 100 - K^{FUT} \tag{2.10}$$

2.2.4. Interest Rate Swap

An Interest Rate Swap (IRS) is OTC contract in which two counterparties agree to exchange two streams of cash flows typically tied to a floating LIBOR or EURIBOR rate, $L(t_x, t_y)$, versus a fixed rate K^{IRS}. They are long-term contracts and for this reason IRS lifetime is typically measured in years. During this period, cash flow exchanges happen at fixed dates (e.g. annually, semi-annual, quarterly and so on) and payments dates can be different for each counterparty.

All payments are based on the same notional N and are generally settled at the end of each period. Schematically, we have two streams of cash flows called legs (see Figure 2.4):

- a floating leg that involves payments based on the floating rate at some dates $t_0, t_1, ..., t_n$;

- a fixed leg representing payments based on the fixed rate done at dates $s_0, s_1, ..., s_m$, with $t_0 = s_0$ and $t_n = s_m$.

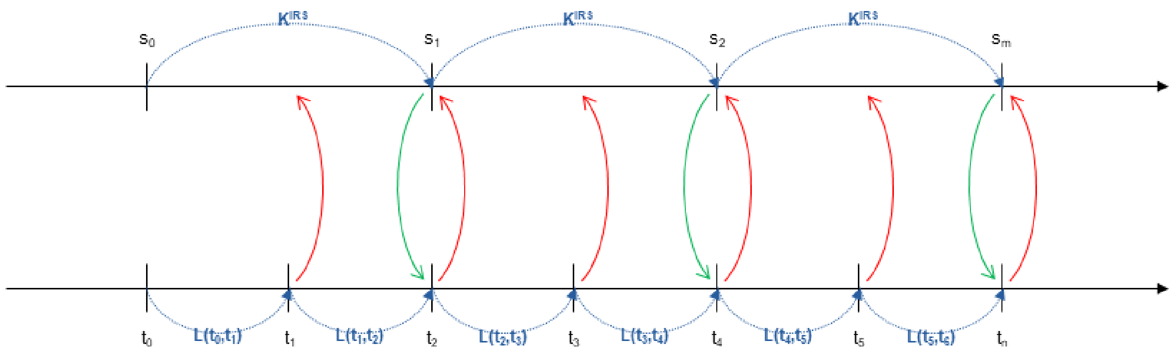

Figure 2.4: Interest Rate Swap functioning

For each time interval $[t_{i-1}, t_i]$ and $[s_{j-1}, s_j]$ related to the floating and the fixed leg respectively, the coupons' payoff called **IRSlet** is:

$$\begin{aligned} \mathbf{IRSlet}_{float}(t_0; t_{i-1}, t_i, F) =& N \cdot F(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \\ \mathbf{IRSlet}_{fixed}(s_0; s_{j-1}, s_j, K^IRS) =& N \cdot K^IRS \cdot \tau_{j-1,j} \end{aligned} \tag{2.11}$$

So, if we want to compute the NPVs of the two legs we have:

$$\begin{aligned} NPV_{float}(t_0) &= \sum_{i=1}^{n} \mathbf{IRSlet}_{float}(t_0; t_{i-1}, t_i, F) \cdot D_i \\ &= \sum_{i=1}^{n} N \cdot F(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \cdot D_i \\ NPV_{fixed}(t_0) &= \sum_{j=1}^{m} \mathbf{IRSlet}_{fixed}(s_0; s_{j-1}, s_j, K^IRS) \cdot D_j \\ &= \sum_{j=1}^{m} N \cdot K^IRS \cdot \tau_{j-1,j} \cdot D_j \end{aligned} \tag{2.12}$$

Putting all together, we deduce the NPV for payer IRS (i.e. an IRS where we pay cash flows tied to fixed rate) as:

$$\begin{aligned} NPV(t_0) =& NPV_{float}(t_0) - NPV_{fixed}(t_0) \\ =& N \cdot \left(\sum_{i=1}^{n} F(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \cdot D_i - \sum_{j=1}^{m} K^IRS \cdot \tau_{j-1,j} \cdot D_j \right) \end{aligned} \tag{2.13}$$

As for the FRAs, IRS are quoted in terms of par rates, namely that rate K^{IRS} which sets the NPV of the contract in t_0 equal to 0. Solving 2.13 with respect to K^{IRS}, we simply have:

$$K^{IRS} = \frac{\sum_{i=1}^{n} F(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \cdot D(t_0, t_i)}{\sum_{j=1}^{m} \tau_{j-1,j} \cdot D(s_0, s_j)} \tag{2.14}$$

Overnight Indexed Swap

An Overnight Indexed Swap (OIS) is a particular IRS where the floating leg is tied to an overnight rate. Payments of the floating leg are not done every day, but at specific dates that coincide with the fixed leg ones. For this reason, the floating rate is daily compounded over that period on which interest accrued. Recalling the notation used in 2.11, the floating leg is given by:

$$\mathbf{IRSlet}^{OIS}_{float}(t_0; t_{i-1}, t_i, R) = N \cdot R(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \tag{2.15}$$

where $R(t_0; t_{i-1}, t_i)$ is the daily compounded interest rate, hence:

$$R(t_0; t_{i-1}, t_i) = \frac{1}{\tau_{i-1,i}} \cdot \left(\prod_{k=1}^{M} [1 + r^{ON}(t_0; t_{i-1_{k-1}}, t_{i-1_k}) \cdot \tau^{ON}_{k-1,k}] - 1 \right) \tag{2.16}$$

with M indicating the number of business days between t_{i-1} and t_i, ($t_{i-1_0} = t_{i-1}$ and $t_{i-1_M} = t_i$)
To determine the par rate K^{OIS}, we proceed like a simple IRS contract: we calculate the NPV of the Swap and we impose that its value in t_0 is equal to 0.

Since in this case overnight rates are used at the same time for forwarding and discounting purposes because are the best proxy for risk-free rates, we can apply the telescopic property to our floating leg obtaining:

$$\begin{aligned} \sum_{i=1}^{n} (R(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \cdot D(t_0, t_i)) &= \sum_{i=1}^{n} (D(t_0, t_{i-1}) - D(t_0, t_i)) \\ &= 1 - D(t_0, t_n) \end{aligned} \tag{2.17}$$

Hence the par rate K^{OIS} is now given by:

$$K^{OIS} = \frac{1 - D(t_0, t_n)}{\sum_{i=1}^{n} \tau_{i-1,i} \cdot D(t_0, t_i)} \tag{2.18}$$

Basis Swap

Interest Rate Basis Swaps (IRBS) are contracts in which two counterparties agree to exchange two streams of cash flows tied to two floating rates with different tenors, for example USD LIBOR 3M vs USD LIBOR 6M.

Before Credit Crunch, their values were of the order of few basis points and they were usually neglected: receive USD LIBOR 3M and than reinvest the amount for another 3-Months period, or receive USD LIBOR 6M rate were substantially the same thing: the different credit risk between these two operations were neglected.

IRBS are fundamental element for long term multiple-curve bootstrapping because we can use them for imply levels of non quoted IRS on different underlying tenors.

On the OTC market we can find two different types of quotation:

1. IRBS as two IRS (typically in EUR market);
2. IRBS as single IRS (typically in other markets).

Due to we will treat in the following chapters the US market, here below we present the analytic framework behind the second approach.

Also in this case we have two streams of cash flows called legs, but unlike before there are:

- a floating leg x that involves payments on a floating rate x at dates $t_0, t_1, ..., t_n$;
- another floating leg y that instead involves payments on a floating rate y at dates $s_0, s_1, ...s_m$ with $t_0 = s_0$ and $t_n = s_m$.

With $t^x_{n_x} = t^y_{n_y}$ and $t^x_0 = t^y_0$.
The $NPVs$ of the two legs are:

$$\begin{aligned} NPV^x_{float}(t_0) &= \sum_{i=1}^{n} N \cdot F_x(t_0; t_{i-1}, t_i) \cdot \tau_{(i-1,i)} \cdot D(t_0, t_i) \\ NPV^y_{float}(t_0) &= \sum_{j=1}^{m} N \cdot F_y(t_0; s_{j-1}, s_j) \cdot \tau_{(j-1,j)} \cdot D(t_0, s_j) \end{aligned} \tag{2.19}$$

To have zero initial cost we need to insert a constant interest rate K^{IRBS}, for example in leg y (market practice requires to insert this spread in the leg the shortest tenor related to the floating rate), so the NPV for leg y becomes:

$$NPV^{y,spread}_{float}(t_0) = \sum_{j=1}^{m} N \cdot (F_y(t_0; s_{j-1}, s_j) + K^{IRBS}) \cdot \tau_{(j-1,j)} \cdot D(t_0, s_j) \tag{2.20}$$

By imposing $NPV(t_0) = NPV^y_{float}(t_0) - NPV^x_{float}(t_0) = 0$ we obtain K^{IRBS} imposing

$$K^{IRBS} = \frac{NPV^x_{float}(t_0) - NPV^y_{float}(t_0)}{\sum_{j=1}^{m} N \cdot \tau_{(j-1,j)} \cdot D(t_0, s_j)} \tag{2.21}$$

Remark 2.3. *We can outline the second approach (IRBS as two IRS) in this way: assume you have three counterparties (let's say A, B and C). Then, they exchange two IRS in the following way: A exchange an IRS (floating,x vs fixed) with B and B exchange the other IRS (fixed vs floating,y) with C.*

2.3. Curve calibration

Calibrating the curve consists in determining a zero-rate function that fits the market, i.e. zeroes the NPV of all the instruments involved in the curve building.
The underlying principle of rate curve fitting is that instruments (Swaps, FRA, Futures and so on) are always quotes at par, so the corresponding contract is fair to both parties without the need for any upfront payment.
Defining $K^{contract}$ as the market quote of the instrument, the evaluation of any linear instrument candidate to be selected in a rate curve is function of the market quote and

the zero-coupon rates:

$$NPV(K^{contract}, \mathbf{z}) = 0 \tag{2.22}$$

where NPV is the net present value of the instrument and $\mathbf{z}$ is the vector of all the zero-coupon rates used to estimate its future cash flows.

Knowing $\mathbf{z}$, it is trivial to calculate the value of an instrument ($K^{contract}$ is a scalar that can be easily calculated from the pricing formula). Curve calibration consists in solving the inverse problem of finding such $\mathbf{z}$ that recovers the quoted $K^{contract}$.

Let's define $\mathbf{t} = (t_1, \ldots, t_n)$ the schedule of an instrument and $\mathbf{z} = (z(t_1), \ldots z(t_n))$. As we can see, there are n unknown variables against only one equation. One solution would be to use n linearly independent equations with these unknowns. Anyway:

1. the number of such unknowns can be very large, leading to computational problems;
2. the market does not provide the n corresponding equations.

The solution is therefore to choose a set of dates $\mathbf{T} = (T^{(1)}, \ldots T^{(n)})$ at which we will determine the zero-coupon rates. A convenient choice is to select the dates corresponding to the maturity date of each curve instrument. Each element $T^{(i)}$ is called pillar. To obtain the intermediary zero-coupon rates, an interpolation function must be used (see Section 2.4).

Thereby, the pricing formula of the i^{th} instrument becomes $NPV^{(i)}(K^{contract(i)}, z) = 0$ with $K^{contract(i)}$ the market quote of the i^{th} instrument.

Let's define $\mathbf{NPV^T} = (NPV^{(1)}, \ldots, NPV^{(n)})$ the total NPV of the curve.
$\mathbf{Z}$ is the solution of $\mathbf{NPV^T}(\mathbf{K}, \mathbf{Z}) = \mathbf{0}$, with $\mathbf{K} = (K^{contract(1)}, \ldots, K^{contract(n)})$.

Generally, this equation is not linear, and an approximate algorithm must be used to solve the problem. One of the most used choice can be the Newton-Raphson method, which starts from an arbitrary point, approximates the function by its tangent line and then computes the x-intercept of this tangent line. The x-intercept will typically be a better approximation of the function's root than the original guess (see Figure 2.5). Reproducing recursively this step, the method will converge quadratically to the root (in the assumption that the starting point is in the convergence set).

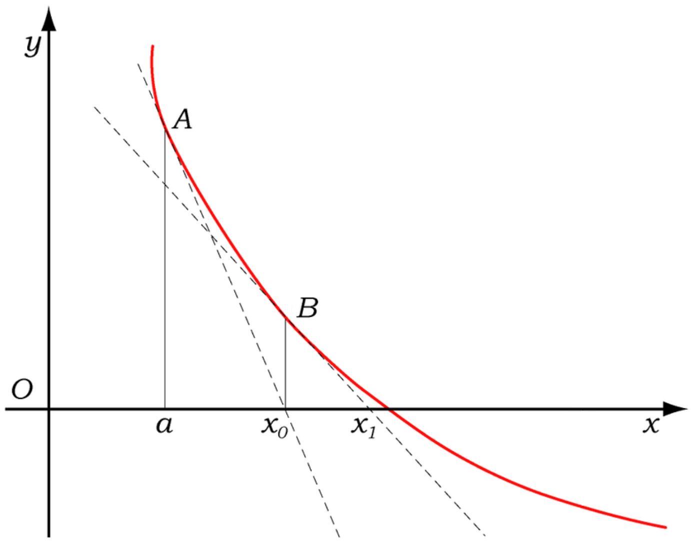

Figure 2.5: Newton-Raphson method representation

Let's $f : E \subset \mathbb{R}^m \to \mathbb{F} \subset \mathbb{R}^n$ a differentiable function and $m, n \geq 1$ and $x_0 \in E$ a point. The tangent function t of f at x_0 is

$$t(x) = f(x_0) + J_f(x_0)(x - x_0) \tag{2.23}$$

where J_f is the Jacobian of the function f. In our case $f : \mathbf{Z} \mapsto \mathbf{NPV^T}(\mathbf{K}, \mathbf{Z})$:

$$J_f = \begin{bmatrix} \dfrac{\partial NPV^{(1)}(\mathbf{Z})}{\partial Z_1} & \cdots & \dfrac{\partial NPV^{(1)}(\mathbf{Z})}{\partial Z_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial NPV^{(n)}(\mathbf{Z})}{\partial Z_1} & \cdots & \dfrac{\partial NPV^{(n)}(\mathbf{Z})}{\partial Z_n} \end{bmatrix} \tag{2.24}$$

So, the intersection with the abscisse is

$$x = x_0 - J_f(x_0)^{-1} f(x_0) \tag{2.25}$$

So, if we continue this step recursively we obtain the following formula for the n^{th} iteration

$$x_n = x_{n-1} - J_f(x_{n-1}) f(x_{n-1}) \tag{2.26}$$

In numerical computation, the iteration will stop when a certain criterion will be reached as $\|x_n - x_{n-1}\| < \epsilon$ or simply $\|f(x_n)\| < \epsilon$ with ϵ equal, for instance, to 0.1.

As the method can be unstable or have difficulty to reach the root in certain cases, the number of iteration can be also capped (to 30, for example).

Remark 2.4. *As previously said, the starting point is crucial in the Newton method and leads the its convergence. Given the nature of the instruments used in the curve, the zero-coupon rates of the curve satisfying the* $\mathbf{NPV^T}$ *equation are closed to the instruments market quotes. Thereby, the chosen starting point is* $\mathbf{K}$*. If the curve has already been calibrated and the market quotes are shifted, the last root* $\mathbf{Z}$ *will be used.*

2.4. Interpolation

As we widely have seen above, to price an generic instrument we need to know the value of discount factors related to its cash flows' payment dates. In this way a problem can arise: when for instance we price an Interest Rate Swap with monthly payments, we may need to a grid of discount factors (or zero-coupon rates) more dense that the one used to build the pricing curve (see Figure 2.6 for a schematic representation of the problem).

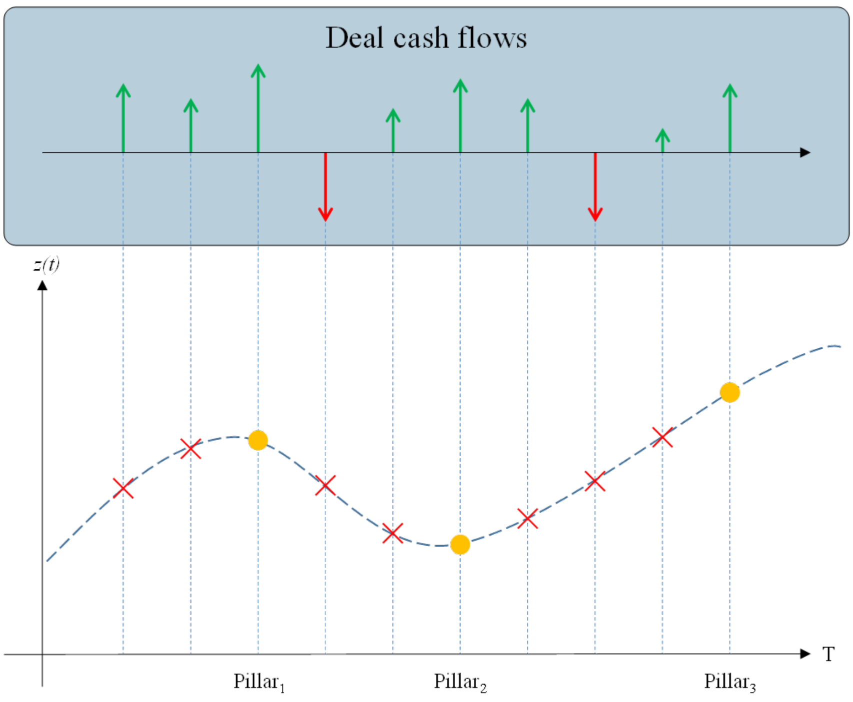

Figure 2.6: Need of having an interpolation schemes to price an instrument

Therefore, to complete the information needed for pricing purposes, it is necessary to use an interpolation scheme witch, in other words, means to find an "reasonable" function f passing through a finite set of predetermined points $\{(x_i, y_i)\}_{i\in I}$ such that $f(x_i) = y_i$.

Moreover, we also need to keep in mind that the interpolation is already used during (thus is highly correlated with) the calibration process, hence robustness and speed must be the key points for our choice.

Recalling the notation given in Section 2.1, the most common quantities on which an interpolation scheme is applied are:

- discount factors curve $\mathcal{C}^D(t) = \{T \mapsto D(t,T)\}$;
- zero-coupon rate curve $\mathcal{C}^z(t) = \{T \mapsto z(t,T)\}$;
- log-discounts curve $\mathcal{C}^{\ln D}(t) = \{T \mapsto \ln(D(t,T))\} = \{T \mapsto -z(t,T) \cdot \tau(t,T)\}$;

where all the quantities are linked each other by the relation $D(t,T) = e^{-z(t,T)\cdot\tau(t,T)}$.

Generally speaking, the interpolation schemes can be distinguished into two different categories:

- local methods, like linear interpolation, where the value of any point $y_t = f(t)$ is given only by the previous and the following grid-point:

$$y_t = f(t) = g((t_i, y_i), (t_{i+1}, y_{i+1})), \quad t \in [t_i, t_{i+1}] \tag{2.27}$$

- global methods, like polynomial or spline interpolation, where all the grid-points come into account for the evaluation of any point outside the grid.

In addition to the choice of the curve to apply the interpolation to, in the following paragraphs we will present the most widely used interpolation methods in finance comparing them and describing their strengths and weaknesses.

2.4.1. Linear interpolation

The most common interpolation method is for sure the linear one due to its high stability and ease of implementation.
Knowing a finite set of points in the plan $\{(t_0, y_0), ..., (t_n, y_n)\}$, the value y_t as function of the point $t \in [t_i, t_{i+1}]$ is given by

$$\begin{aligned} y_t =& y_i + \frac{y_{i+1} - y_i}{t_{i+1} - t_i} \cdot (t - t_i) \\ =& \frac{t - t_i}{t_{i+1} - t_i} \cdot y_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i} \cdot y_i \end{aligned} \tag{2.28}$$

Applying this formula to one of the three quantities used to define a yield curve (discount factors, log-discounts and zero-coupon rates), due to the simple relation between them and the injectivity of the exponential function we can deduce formulas for the other two remaining quantities as shown as follows.

Interpolating on discount factors ($y_t = D_t$) we have

Variable	Interpolating on discount factors
discount factor	$D_t = \frac{t - t_i}{t_{i+1} - t_i} D_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i} D_i$
log-discount	$\ln(D_t) = \ln\left(\frac{t - t_i}{t_{i+1} - t_i} D_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i} D_i\right)$
zero-coupon rate	$z_t = -\frac{1}{t}\ln\left(\frac{t - t_i}{t_{i+1} - t_i} D_{i+1} + \frac{t_{i+1} - t}{t_{i+1} - t_i} D_i\right)$

Table 2.1: Linear interpolation on discount factors

Interpolating on log-discount ($y_t = ln(D_t)$) we have

Variable	Interpolating on log-discounts
log-discount	$\ln(D_t) = \frac{t - t_i}{t_{i+1} - t_i} \ln(D_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \ln(D_i)$
discount factor	$D_t = exp\left\{\frac{t - t_i}{t_{i+1} - t_i} \ln(D_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \ln(D_i)\right\}$
zero-coupon rate	$z_t = -\frac{1}{t}\left(\frac{t - t_i}{t_{i+1} - t_i} \ln(D_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \ln(D_i)\right)$

Table 2.2: Linear interpolation on log-discounts

Interpolating on zero-coupon rates ($y_t = z_t$) we finally have

Variable	Interpolating on zero-coupon rates
zero-coupon rate	$z_t = \frac{t-t_i}{t_{i+1}-t_i} z_{i+1} + \frac{t_{i+1}-t}{t_{i+1}-t_i} z_i$
discount factor	$D_t = exp\Big\{ -t\Big(\frac{t-t_i}{t_{i+1}-t_i} z_{i+1} + \frac{t_{i+1}-t}{t_{i+1}-t_i} z_i \Big) \Big\}$
log-discount	$\ln(D_t) = -t\Big(\frac{t-t_i}{t_{i+1}-t_i} z_{i+1} + \frac{t_{i+1}-t}{t_{i+1}-t_i} z_i \Big)$

Table 2.3: Linear interpolation on zero-coupon rates

Remark 2.5. *When each instrument has a different maturity, i.e. each instrument has a maturity $T^i \neq T^j$, $\forall i \neq j$, with linear interpolation the Jacobian matrix of f, J_f, (see Section 2.3) is a triangular matrix. In these cases the matrix inversion is easier and this accelerates the computing.*

2.4.2. Natural cubic splines interpolation

To obtain a smoother behaviour for the curve, one of the most common choice in finance is to consider a (natural) cubic spline interpolation. This method belongs to the global methods cited above and consists in finding a polynomial of degree 3 (cubic) on each segment $[t_i, t_{i+1}]$ (spline) passing through each y_i.

Since a polynomial of degree k requires $k+1$ constraints to obtain an unique solution, in our case for each segment we need to find 4 equations to well-define the problem. Assuming to have a finite set of points in the plan $\{(t_i, y_i)\}_i$ with $i \in \{1, ...n\}$, what we want is to determine for each segment $[t_i, t_{i+1}]$ a separate cubic polynomial S_i each with its own coefficient:

$$S_i(t) = a_i + b_i \cdot (t-t_i) + c_i \cdot (t-t_i)^2 + d_i \cdot (t-t_i)^3, \quad t \in [t_i, t_{i+1}]. \tag{2.29}$$

such that following conditions about the whole interpolating function will be satisfied:

- it passes through all the given data y_i;
- it is continuous and twice differentiable in each points.

We can easly calculate the derivatives of equation 2.29:

$$\begin{aligned} y'_t =& b_i + 2c_i(t - t_i) + 3d_i(t - t_i)^2 \\ y''_t =& 2c_i + 6d_i(t - t_i) \end{aligned} \tag{2.30}$$

In this way we can rewrite all the condition as:

$$\begin{cases} a_i = y_i, & i = 1, ..., n-1 \\ y_n = a_{n-1} + b_{n-1} \cdot (t_n - t_{n-1}) + c_{n-1} \cdot (t_n - t_{n-1})^2 + d_{n-1} \cdot (t_n - t_{n-1})^3 & \\ a_{i+1} = a_i + b_i(t_{i+1} - t_i) + c_i(t_{i+1} - t_i)^2 + d_i(t_{i+1} - t_i)^3 & i = 1, ..., n-2 \\ b_{i+1} = b_i + 2c_i(t_{i+1} - t_i) + 3d_i(t_{i+1} - t_i)^2 & i = 1, ..., n-2 \\ c_i + 3d_i(t_{i+1} - t_i) = c_{i+1} & i = 1, ..., n-2 \end{cases}$$

We can see that there are only $(1)+(n-1)+(n-2)+(n-2)+(n-2) = 4n-6$ conditions, so we need to add two more boundary conditions to solve that system. For this reason we apply the so-called "natural boundary conditions" (hence the name of such interpolation scheme), which require that the second derivative at each endpoint is 0, i.e.

$$\begin{cases} c_1 = 0 \\ c_{n-1} + 3d_{n-1} \cdot (t_n - t_{n-1}) = 0 \end{cases} \tag{2.31}$$

As for the linear interpolation, this method can be applied to zero-coupon rates, discount factors and log-discounts and it also ensures a smoother and financially best behaviour that the previous one as we will soon see.

Remark 2.6. *Natural cubic spline interpolation suffer of well-documented problems, such as spurious inflection points, excessive convexity and lack of locality after input price perturbations. What it is done in practice is to apply the so-called Hyman filter [8] which ensures that in regions of local monotony of the input, the interpolating cubic remains monotonic.*

2.4.3. Interpolation methods comparison

Here below we want to make a comparison on the effects that the different types of interpolation discussed above have on the construction of the term structure. In particular, we will show which kind of behaviours linear and monotonic natural cubic spline interpolation produce on discount factors, zero-coupon rates and forward rates.

Depending on a interpolation method, we always get some benefits and some drawbacks. As we said before (see Remark 2.5), linear interpolation ensures a fast convergence of the calibration process, so at first glance it seems to be a good candidate to use to build an interest rate term structure. On the contrary, natural cubic spline interpolation should be ensure a smoother and financially best behaviour.

As a first step, let's see how both the interpolation schemes perform on discount factors and zero-coupon rate curves (see Figure 2.7 and Figure 2.8).

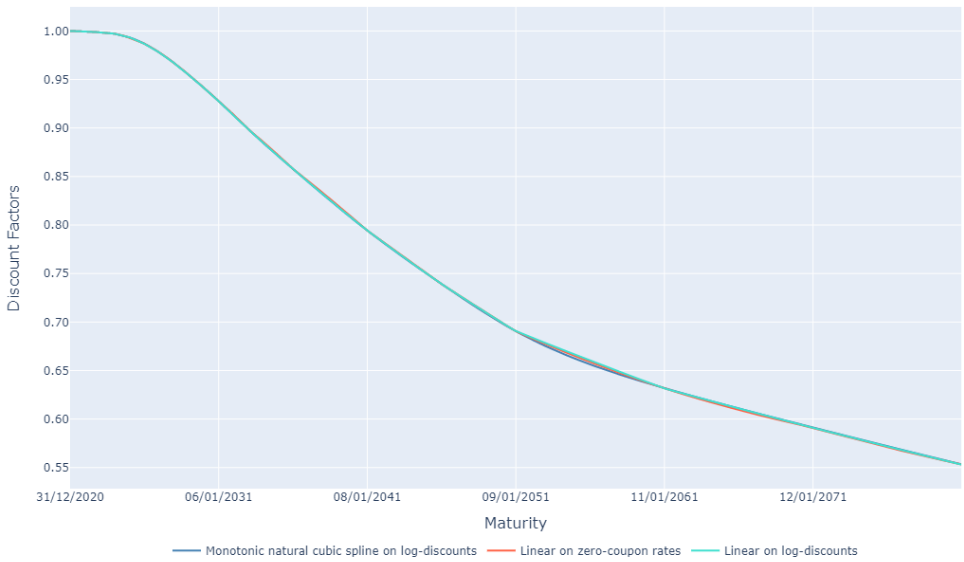

Figure 2.7: Different interpolation methods compared on discount factors curve

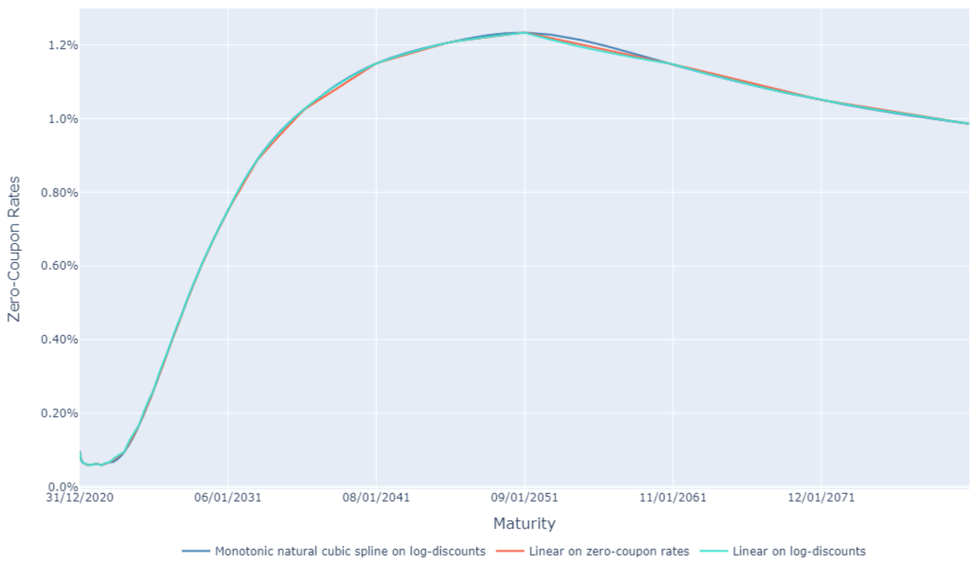

Figure 2.8: Different interpolation methods compared on zero-coupon rates curve

As we can see, in these cases all the curves appears to be smooth enough and an appreciable difference between the two interpolation schemes can be retrieved only at those (rare) points where the curves exhibit a more pronounced convexity .

Remembering that one of the key points for a term structure is to estimate forward rates, the last check to perform before taking a decision on which interpolation method is to compare them on the behaviour of the forward rates curve.

Figure 2.9: Different interpolation methods compared on forward curve

As you can see in Figure 2.9, the only interpolation with a smooth enough behavior is the one made with monotonic natural cubic spline interpolation while the other two present a nasty piecewise linear behavior (the so-called stepwise and sawtooth effect for log-discounts and zero-coupon rates respectively). In particular, the behaviour on log-discounts can be easily justified just recalling equation 2.4 for forward rates and the deduced formula for discount factors shown in Table 2.2.

Given a time interval $[t_1, t_2]$ on which the linear interpolation is applied, let us consider two dates t and $t + \Delta$ such that both are in the interval, $t, t + \Delta \in [t_1, t_2]$. Thanks to the properties of exponential function, we have:

$$
\begin{aligned}
F(t_0;t,t+\Delta) =& \frac{1}{\Delta}\left(\frac{D(t_0,t)}{D(t_0,t+\Delta)}-1\right)\\
=& \frac{1}{\Delta}\cdot\left(\frac{\exp\left\{\frac{t-t_1}{t_2-t_1}\cdot\ln(D_2)+\frac{t_2-t}{t_2-t_1}\cdot\ln(D_1)\right\}}{\exp\left\{\frac{(t+\Delta)-t_1}{t_2-t_1}\cdot\ln(D_2)+\frac{t_2-(t+\Delta)}{t_2-t_1}\cdot\ln(D_1)\right\}}-1\right)\\
=& \frac{1}{\Delta}\cdot\left(\exp\left\{\left(\frac{t-t_1}{t_2-t_1}-\frac{(t+\Delta)-t_1}{t_2-t_1}\right)\cdot\ln(D_2)+\right.\right.\\
&+\left.\left.\left(\frac{t_2-t}{t_2-t_1}-\frac{t_2-(t+\Delta)}{t_2-t_1}\right)\cdot\ln(D_1)\right\}-1\right)\\
=& \frac{1}{\Delta}\cdot\left(\exp\left\{\frac{-\Delta}{t_2-t_1}\cdot\ln(D_2)+\frac{\Delta}{t_2-t_1}\cdot\ln(D_1)\right\}-1\right)\\
=& \frac{1}{\Delta}\cdot\left(\exp\left\{\frac{\Delta}{t_2-t_1}\cdot(\ln(D_1)-\ln(D_2))\right\}-1\right)
\end{aligned}
\tag{2.32}
$$

Since we have lost any reference to the date t at which the forward rate is calculated, this implies that its value only depends on the quantities Δ, $\ln(D_1)$ and $\ln(D_2)$ and is therefore constant.

3 | Understanding Cliff Effect

As we anticipated in previous chapters, on the today's USD LIBOR 3M (and also for ON, 1M, 6M and 1Y) forward curve there is a jump discontinuity in correspondence of the June 30, 2023 due to the instantaneous change of the underlying rate: the Cliff Effect is this brutal transition from a LIBOR rate to a SOFR + CAS rate.

About the CAS, there are two general approaches for calculating it.

1. A backward-looking approach: this approach uses the median spread between LIBOR and the related risk-free rate over a five-year lookback period. The so-calculated spreads are sometimes referred to as the ISDA spreads or five-year historical median spreads.

2. A forward-looking approach: this approach uses the forward-looking basis swap spreads that are taken from publicly available market information. These spreads reflect the market's current view about implied future spreads between LIBOR and the relevant risk-free rate. [10]

Following industry consultation, the ARR Committee has recommended the use of the first approach to define the fallback rates for USD LIBOR-indexed cash products (including loans) following a cessation or pre-cessation trigger. These recommendations are also in line with the ISDA approach proposed for the derivative fallbacks.

In such a context, any announcement of indexes' discontinuation occurring on a given date would immediately trigger the five-year median spread calculation and will fix it in stone until discontinuation effectively occurs. Regarding the LIBORs, CAS was defined on March 5, 2021 when the FCA formally announced the cessation of certain LIBOR settings (see Section 3.1 for more detail). This statement locked in the CAS which has been posted by Bloomberg [1].

Between the announcement and the actual discontinuation on the indexes (a period possibly long more than one year), practitioners would need to manage LIBOR-indexed positions with a two-regime set of estimated fixings. The first regime, for all fixings before discontinuation date, would need to be estimated off the usual strip of LIBOR futures and fixing. The second one, for all fixings after the discontinuation date, would need to be properly estimated off the risk-free curve, adding the CAS to the resulting rate.

So, to get a correct valuation of trades, new curves are needed. For example, take a trade indexed on the 3-Months USD LIBOR with a maturity date greater than the discontinuation one. Theoretically, all rate estimations with a fixing date before the discontinuation date should be estimated with the actual USD LIBOR 3M curve, while all the rate estimations with a fixing date after that should be estimated quarterly compounding SOFR curve plus 3M CAS. As it is quite hard (even impossible) in practice for a trade to use two different curves depending on the fixing date, we must introduce a curve (see Section 3.3) that will:

- provide 3-Months forward rates as the same deduced from the USD LIBOR 3M cruve before the discontinuation date;
- provide 3-Months forward rates as the same deduced from the 3M daily compounding overnight forward rate (retrieved from SOFR curve) plus CAS 3M after the discontinuation date.

Moreover, the Covid-19 pandemic demonstrated that SOFR and LIBOR rates are significantly decorrelated, which is obvious from their fundamental difference in nature (see again Figure 1.5 and Figure 1.4 to remember it). Hence, depending on the respective movements of those rates, the difference between the two regimes (the Cliff Effect) could be sharp. If it is, practitioners will need to significantly amend their pricing, valuation and risk calculation practices to take the Cliff Effect into account until LIBOR definitely dies.

3.1. FCA announcement of LIBOR cessation

The Financial Conduct Authority (FCA) formally announced that, in the cessation day, the LIBOR rates will either stop being published, or be deemed ‘not representative’ in the way shown in Table 3.1.

From December 31, 2021, 30 of the 35 LIBORs have ceased to be published. To avoid a disruptive effect on legacy trades after that date, the FCA confirmed that certain LIBOR settings would still be published for the duration of 2022 under a synthetic methodology, based on term risk-free rates. This was addressed in the Critical Benchmarks (References and Administrators' Liability) Bill.

Broadly speaking, this proposed legislation has aimed to ensure that references to Sterling LIBOR in existing contracts would be interpreted as references to a synthetic LIBOR rate where a transition to an alternative rate has not been made in time. The Bill also provided that contracting parties could not argue that use of synthetic LIBOR constitutes breach of contract. Anyway, while the Bill offered some comfort to the markets, there remained significant risks for lenders in the transition process.

The theory behind the publication of synthetic LIBOR rates is that continued use of synthetic LIBOR will allow some contracts to reach their natural maturity. However, the FCA has been very clear that in most cases, it should be viewed as providing a further period to complete transition of legacy contracts, rather than an alternative. For this reason, synthetic LIBOR is not to be viewed as a permanent solution. Furthermore, the availability of synthetic LIBOR rates is subject to annual review and therefore the availability of synthetic LIBOR beyond 2022 cannot be assured [7].

Index name	USD LIBOR			EUR LIBOR	CHF LIBOR	GBP LIBOR		JPY LIBOR	
Tenor	1W, 2M	ON, 1Y	1M, 3M, 6M	All	All	ON, 1W, 2M, 1Y	1M, 3M, 6M	ON, 1W, 2M, 1Y	1M, 3M, 6M
End of Panel Bank Submissions	Dec 31, 2021	Jun 30, 2023	Jun 30, 2023	Dec 31, 2021	Dec 31, 2021	Dec 31, 2021	Dec 31, 2021	Dec 31, 2021	Dec 31, 2021
Cessation Type	Permanent Cessation	Permanent Cessation	Loss of representativ.	Permanent Cessation	Permanent Cessation	Permanent Cessation	Loss of representativ.	Permanent Cessation	Loss of representativ.
Potential Synthetic LIBOR Publication	Not Applicable	Not Applicable	Jul 1, 2023 Jun 30, 2033	Not Applicable	Not Applicable	Not Applicable	Jan 1, 2022 Dec 31, 2031	Not Applicable	Jan 1, 2022 Dec 31, 2022

Figure 3.1: FCA announcement on benchmarks cessation [15]

3.2. The twofold challenge for practitioners

Based on what has been said so far, the Cliff Effect's challenge is twofold:

1. First of all, managing a wide brutal jump in forward rates estimation is unnatural for many financial analytics' libraries. Although it is not the first time discontinuity

in forwards must be modelled in curves (e.g. in the Overnight-indexed curves for the TOY effect and the central banks meetings' announcements), processing a single, potentially large jump at a fixed date while using completely different calibration instruments on both sides of the milestone is something rather new.

2. The second challenge is the necessity to amend risk calculations: for traders, that means taking into account the double regime of the forward curve when calculating sensitivities and hedge ratios. This is yet another challenge that may not be the same as previous ones from a technical perspective and it represents a critical point of attention.

In any case, appropriate modelling of this jump requires significant adaptations of analytics libraries from pricing, valuation and risk perspectives. Given the sudden appearance of Cliff Effect in the last period, these changes would need to be implemented in a very short amount of time, and would only remain for a limited period [14].

3.3. Finding a transition curve

Here above we have tried to understand why the Benchmarks Reform, with the consequent introduction of new ARRs to replace the old LIBORs generating the so-called Cliff Effect, has to be considered as one of the most challenging reform in financial framework.

For all that has been said so far, one fundamental question arises: if the magnitude of the cliff is high (as in this period), how can we try to model the jump building the interest rate term structure without any undesirable forward oscillations around the discontinuation date?

To answer this question, we must keep in mind:

- how and where it appears in a graphic representation;
- how and whether a term structure can be deduced to justify such representation

We proceed as follows. First of all, we show the steps through which we have created the Cliff Effect, plotting it in a graph, then, we explain why it is an issue to be fixed.

The idea is that, considering that from June 30, 2023 last five LIBOR rates (Overnight, 1-Month, 3-Months, 6-Months, 1-Year) will be replaced by SOFR + CAS, we need a curve through which we can price financial instruments (such as Swap, FRA, . . .) with expiry date before discontinuation date with current USD LIBOR 3M and financial instruments with expiry date after discontinuation date with SOFR + 3M CAS. From now on, all the computations have been made using Jan 31, 2023 as instruments evaluation date. The ideal (and forward) curve we want to use will be something like this (see Figure 3.2).

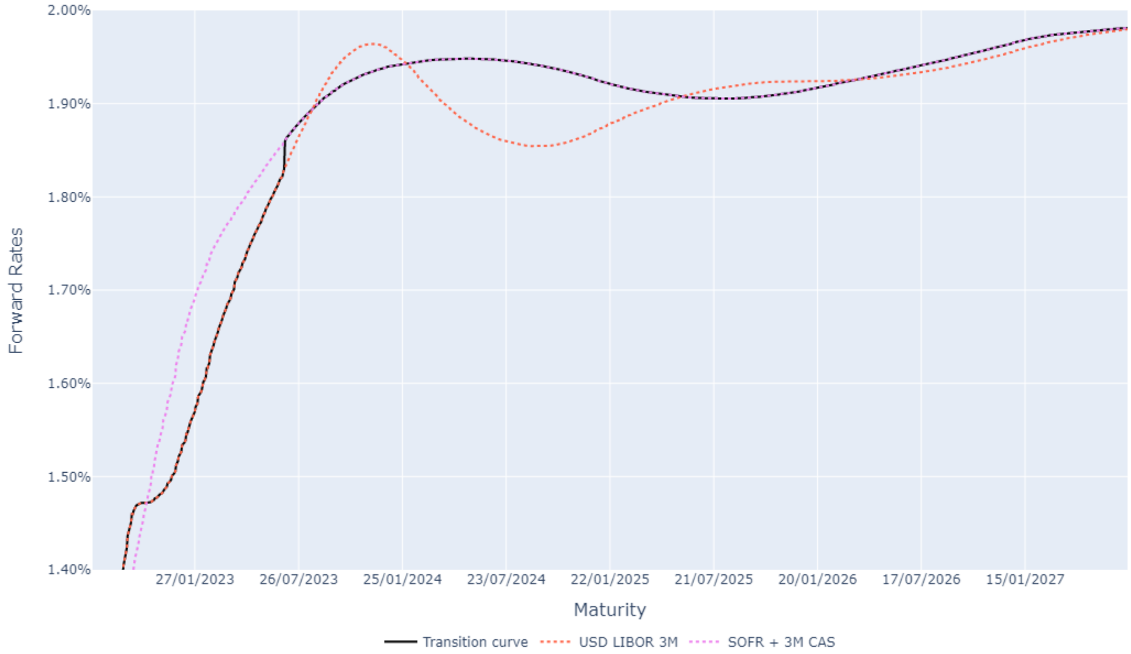

Figure 3.2: Transition curve as combination of USD LIBOR 3M and SOFR + 3M CAS

As you can notice from the graph, it is evident the presence of a jump in this ideal curve, the so-called Cliff Effect, in correspondence of the discontinuation date.
To create such a curve we operated as follows:

- build the USD LIBOR 3M curve as usual taking the most liquid market instruments;
- plot the 3M forward rates implied on it (dotted orange line in Figure 3.2);
- build the SOFR curve as usual taking the most liquid market instruments;
- plot the 3M forward rate deduced from the 3M daily composition of the overnight forward rate + 3M CAS (dotted violet line in Figure 3.2).

The curve resulting to join the combination between the two forward curves (continuous black line in Figure 3.2) is our target and, form now on, it will be called transition curve. The transition curve is a forward curve obtained through the day-by-day calculation of 3-Months forward rates. If we also would like to obtain the related term structure (through discount factors or zero-coupon rates), starting from which we could exactly replicate these forward rates, we couldn't use any interpolation method. The only possible solution would consist in defining a pillar for each day taken into account, bringing this approach to an unmanageable one.

For this reason, in the next chapter, our goal is to build new curves and in different ways but such that, through the inclusion of suitable financial instruments, they could replicate as much as possible our transition cuve.

4 | How To Manage Cliff Effect

After showing Cliff Effect and explaining why it cannot be easily used for elementary financial operations, the immediate following step is the one to understand how to bypass this inconvenient. This is a topic which was, and still is, in the spotlight of major banks from all over the world.

We remember that our proposed solutions are supported by the main Italian investment (and not) banks, as well as implemented by the main software houses which use systems employed to re-evaluate positions held by banks. All the computations that follows has been done using QuantLib open source library in the extension for Microsoft Excel.

4.1. USD LIBOR 3M curve as-is

The first approach we want to study is the trivial one, as well as, not necessarily the first thing one can think to do: leave everything as it is, by changing nothing. This means you are required to leave USD LIBOR 3M forward curve as if Benchmarks Reform never took place. This may seem that we are ignoring the reform, but it is not the case. In fact, since the Bloomberg's publication of CAS, the current USD LIBOR 3M forward curve is converging to the SOFR + 3M CAS one. The plot of the two curves is shown in Figure 4.1.

Despite it is the most immediate and easy way to bypass Cliff Effect issue, it also leads to a new, big issue. Continuing to use old USD LIBOR 3M curve after the discontinuation date leads to bad re-evaluations of financial instruments already existing in the market, and for this reason quoted, with expiry date around discontinuation date. As we already known, the goal of a good calibration is twofold. On one hand, to evaluate not quoted, less liquid or more complex instruments. On the other, to correct re-evaluate existing and already quoted market instruments. With the transition curve constructed in this way it is immediate to prove that the second request it is not satisfied. If we try to re-price these instruments using the created curve, we can find out that the new quote will have an error of different basis points respect to the market quote. This is a sign of a bad calibration.

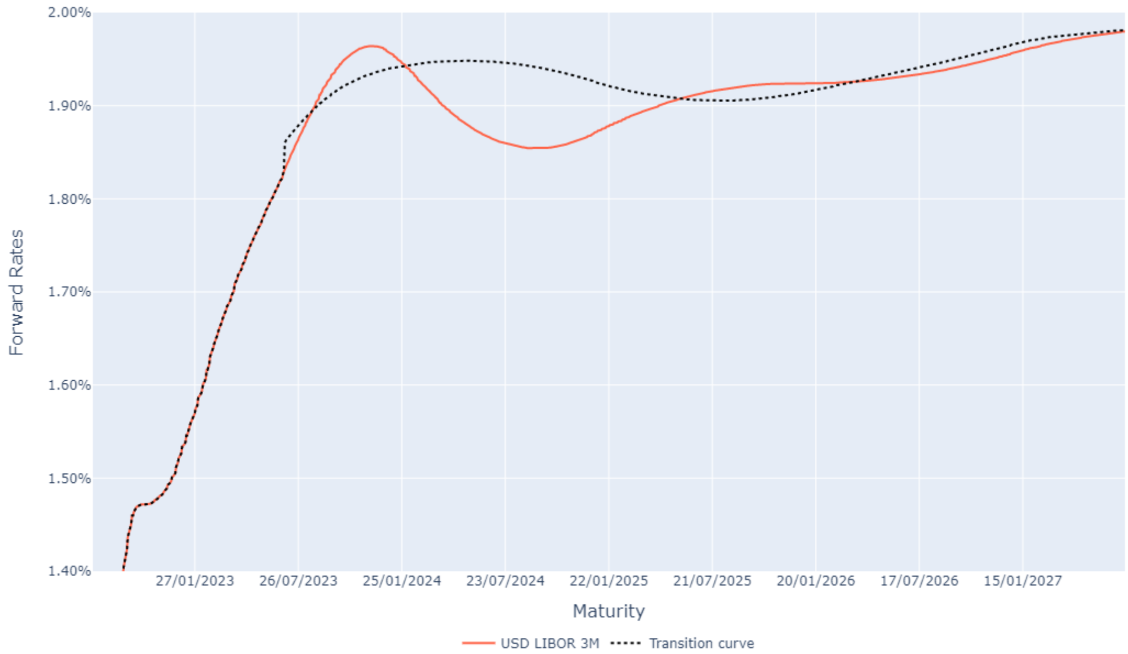

Figure 4.1: Transition curve compared with the as-is USD LIBOR 3M curve

4.2. New USD LIBOR 3M curve build with Basis Swaps

We have seen with the first approach that, leaving the USD LIBOR 3M forward curve as it is, changing nothing, we have a convergence to the SOFR + 3M CAS forward curve. The problem is that this convergence is too slow, leading to incorrect valuations of future instruments, as well as large errors in difference between market quote and repriced quote of traded instruments not used for bootstrap procedure. The idea could be the one to find out some market instruments which can help in creating a different transition curve, coherent with the requirements, but different from the previous one.

Knowing that, from June 30, 2023, it will be true that USD LIBOR 3M = SOFR + 3M CAS, we can construct some Basis Swap instruments, forward starting and with start date after discontinuation date, based on SOFR + 3M CAS vs USD LIBOR 3M from which, starting from the value of SOFR + 3M CAS, we can compute the value of the new USD LIBOR 3M. Then, we can interpolate between the forward rates of old USD LIBOR 3M instruments, with the one of the new created Basis Swap, obtaining the new USD LIBOR 3M forward curve (see Figure 4.2).

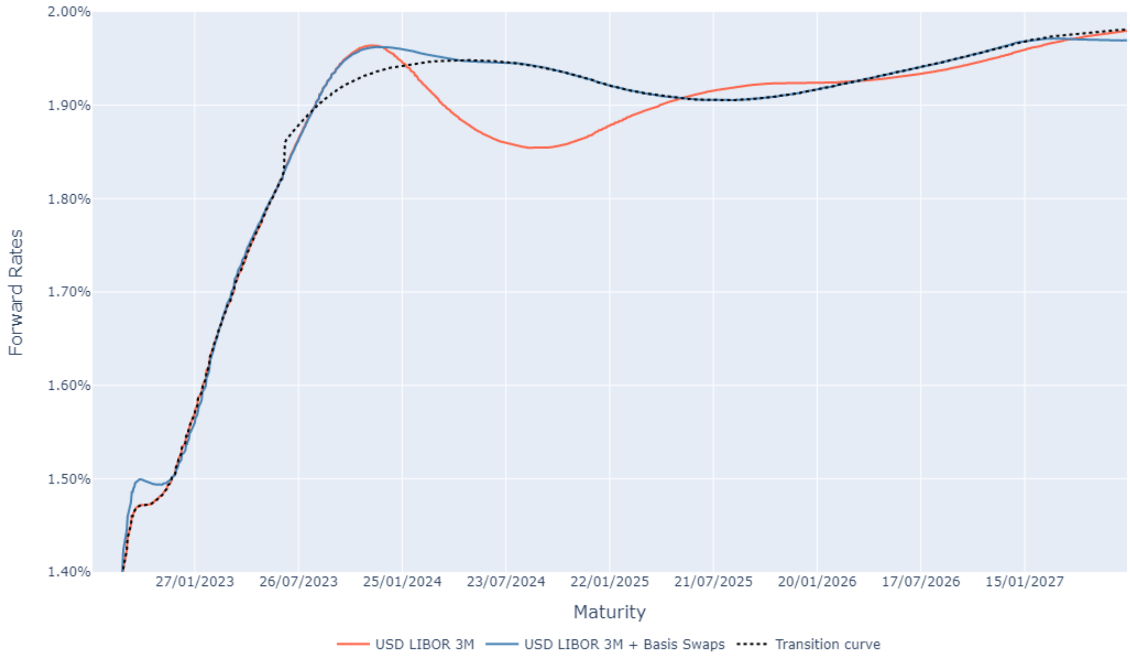

Figure 4.2: Transition curve compared with the USD LIBOR 3M curve build with Basis Swaps

In particular, we have constructed our Basis Swaps in this way:

- forward starting, with start date after the discontinuation date;
- 3-Months tenor, so that every forward rate will correspond to new USD LIBOR 3M forward rate;
- their frequency has to be the following
  - every three months from discontinuation date up to five years;
  - every year, from five up to ten years;
  - every five years, up to thirty years;
  - every ten years, up to fifty years;
- one leg based on 3-Months daily compounding SOFR + 3M CAS, where all the quantities are well known;
- one other leg based on the USD LIBOR 3M rate we want to deduce.

So, for each brand new Basis Swap, the $NPVs$ of the two legs are:

$$\begin{aligned} NPV_{USD\ 3M}(t_0) =& F_{USD\ 3M}(t_0; t_i, t_i + 3M) \cdot \tau_{(i,i+3M)} \cdot D(t_0, t_i + 3M) \\ NPV_{SOFR+CAS}(t_0) =& (R_{SOFR}(t_0; t_i, t_i + 3M) + CAS) \cdot \tau_{(i,i+3M)} \cdot D(t_0, t_i + 3M) \end{aligned} \tag{4.1}$$

We know that the zero initial cost is imposed by taking the 3M CAS equal to the value fixed on March 5, 2021 when the discontinuation was announced (3M CAS = 26.161 bps). Hence, by imposing $NPV(t_0) = NPV_{USD\ 3M}(t_0) - NPV_{SOFR+CAS}(t_0) = 0$ we obtain

$$F_{USD\ 3M}(t_0; t_i, t_i + 3M) = R_{SOFR}(t_0; t_i, t_i + 3M) + CAS \tag{4.2}$$

Then, summing up the procedure to build this new curve we have:

- took the most liquid market instruments based on USD LIBOR 3M with expiry date before discontinuation date;
- created Basis Swap instruments based on SOFR + 3M CAS with starting date after discontinuation date;
- applied a bootstrap procedure with monotonic natural spline interpolation on these instruments obtaining a interest rate term structure.

As you can see in Fig 4.2, even if we have a faster convergence to SOFR + 3M CAS curve with respect to the previous case, we still have bad oscillations in a left hand-side neighborhood of the discontinuation date. This is true for values immediately before and after the discontinuation date. Also in this case, if we try to reprice existing and already quoted market instruments with expiry date near discontinuation date using the created curve, we can find out that the new quote will have an error of different basis points respect to the market quote. Anyway, this approach seems to be a good one, especially because the curve reaches the stability really faster respect to the as-is USD LIBOR 3M curve.

4.3. New USD LIBOR 3M curve build with Multi-phase Interest Rate Swaps

We have tried firstly, to leave USD LIBOR 3M curve as it is, changing nothing. Secondly, to transit from old USD LIBOR 3M curve to SOFR + 3M CAS curve through the introduction of some Basis Swaps. In the second case, we got closer to our goal to construct a transition curve smooth and not too far from market quotations. The idea, now, is also in this case the one to search for some other financial instruments placed in a smart way into our bootstrap calibration with the aim to obtain such a curve. How to choose them is the matter of this section.

Let's think to do something like this. Take all the market instruments you took to construct old USD LIBOR 3M forward curve. Now, proceed as follows. All of these instruments (it will be all Swaps) which have an expiry date after the discontinuation date will be modified in such a way:

1. payments of the instrument will be based on old USD LIBOR 3M curve, as it already is, up to discontinuation date;
2. payments of the instrument will be based on SOFR + 3M CAS curve after the discontinuation date.

So, also for this type of IRS we have two legs:

- Floating Leg: it involves payments of the floating rate at dates $t_0, t_1, ..., t_n$;
- Fixed Leg: it involves payments of the fixed rate at dates $s_0, s_1, ..., s_m$.

With $t_0 = s_0$ and $t_n = s_m$.

The coupon payoffs in one time interval, between t_{i-1} and t_i for the floating leg and between s_{j-1} and s_j for the fixed leg, evaluated at the end date of the period, are:

$$\begin{aligned}
\mathbf{IRSlet}_{float,1}(t_0; t_{i-1}, t_i, F_{USD\ 3M}) =& N \cdot F_{USD\ 3M}(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \\
\mathbf{IRSlet}_{float,2}(t_0; t_{i-1}, t_i, F_{SOFR+3M\ CAS}) =& N \cdot F_{SOFR+CAS}(t_0; t_{i-1}, t_i) \cdot \tau_{i-1,i} \\
\mathbf{IRSlet}_{fixed}(s_0; s_{j-1}, s_j, K^{IRS}) =& N \cdot K^{IRS} \cdot \tau_{j-1,j}
\end{aligned} \tag{4.3}$$

So, the $NPVs$ of the two legs are:

$$\begin{aligned}
NPV_{float}(t_0) =& \sum_{i=1}^{h} \mathbf{IRSlet}_{float,1}(t_0; t_{i-1}, t_i, F_{USD\ 3M}) \cdot D(t_0, t_i) + \\
&+ \sum_{k=h+1}^{n} \mathbf{IRSlet}_{float,2}(t_0; t_{k-1}, t_k, F_{SOFR+3M\ CAS}) \cdot D(t_0, t_k) \\
NPV_{fixed}(t_0) =& \sum_{j=1}^{m} \mathbf{IRSlet}_{fixed}(s_0; s_{j-1}, s_j, K^{IRS}) \cdot D(s_0, s_j)
\end{aligned} \tag{4.4}$$

Where h is the number of payments before discontinuation date. The fixed interest rate of the contract, K^{IRS}, is calculated at fixing date such that the NPV of the contract at that day is equal to 0:

$$NPV(t_0) = NPV_{float}(t_0) - NPV_{fixed}(t_0) = 0 \tag{4.5}$$

Solving for K^{IRS} we can find the fair rate, called in this case the Multi-phase Swap par rate.

In this way, we are transforming existing Interest Rate Swaps in Multi-phase Swaps: Swaps indexed on a rate, up to a specific payment date, on another one later. In this way, we will have new market instruments, starting from some already existing, but with a recomputed quote different from the market one. With this new quotes, the new USD LIBOR 3M forward curve will be the one shown in Figure 4.3.

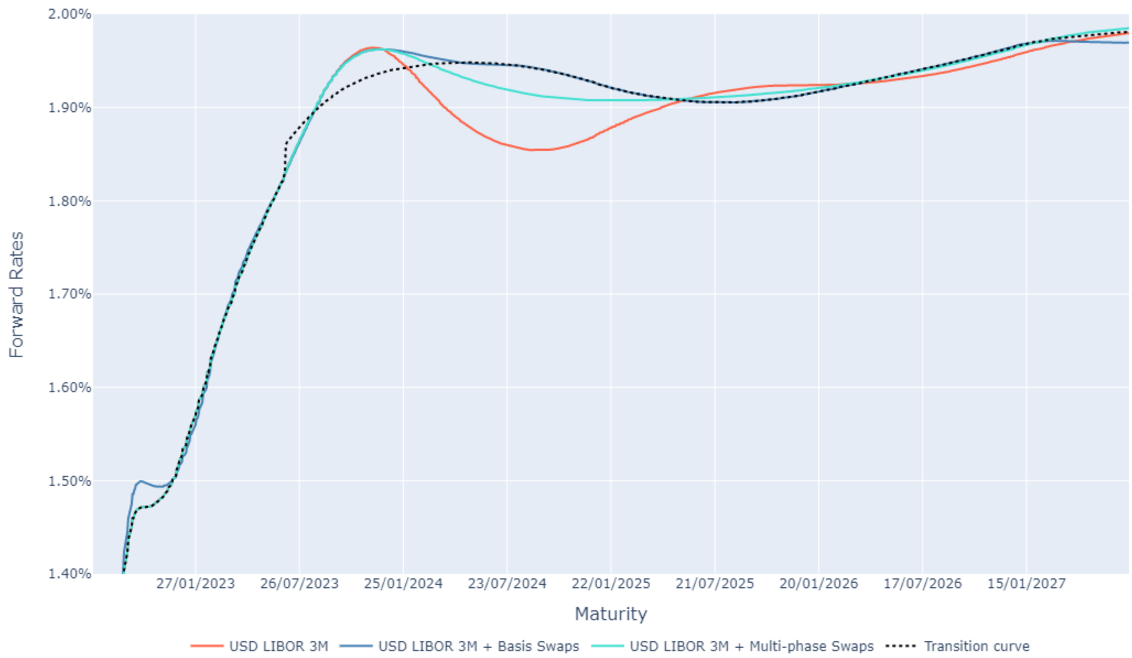

Figure 4.3: Transition curve introducing Multi-phase IRS

This is another good approach for constructing a transition curve respecting the constraints we imposed: smooth; transiting from old USD LIBOR 3M curve from SOFR + 3M CAS curve; re-evalueting in a good way existing market quotes. As you can see from the graph, this approach is a double-edged sword with respect to the previous one. If we do not have oscillations in the values before the discontinuation date, on the other hand we obtain a slower convergence to the SOFR + 3M CAS curve. Also in this case, following this procedure we do not obtain our ideal curve but one close enough, we can say.

5 | Conclusions and Future Developments

We have introduced Benchmarks Reform, we have talked about the historical motivations that brought the financial world to this reform, giving first of all the definition of the most involved benchmark, such as LIBOR, and trying to let you understand the importance of it and, consequently, of its cessation. Then, we have tried to give all the technical and analytical notions useful to understand the rest of the work.

The one of Cliff Effect is and has been a real big challenge for banks of all around the world. Each of them is trying to find out a way to bypass the problem and this make it a great theme of actuality. We have created a framework finalized to shown the Cliff Effect starting from the ideal curve we wanted to have as final result. From this one we started by listing all the possible ways we proposed to fix it.

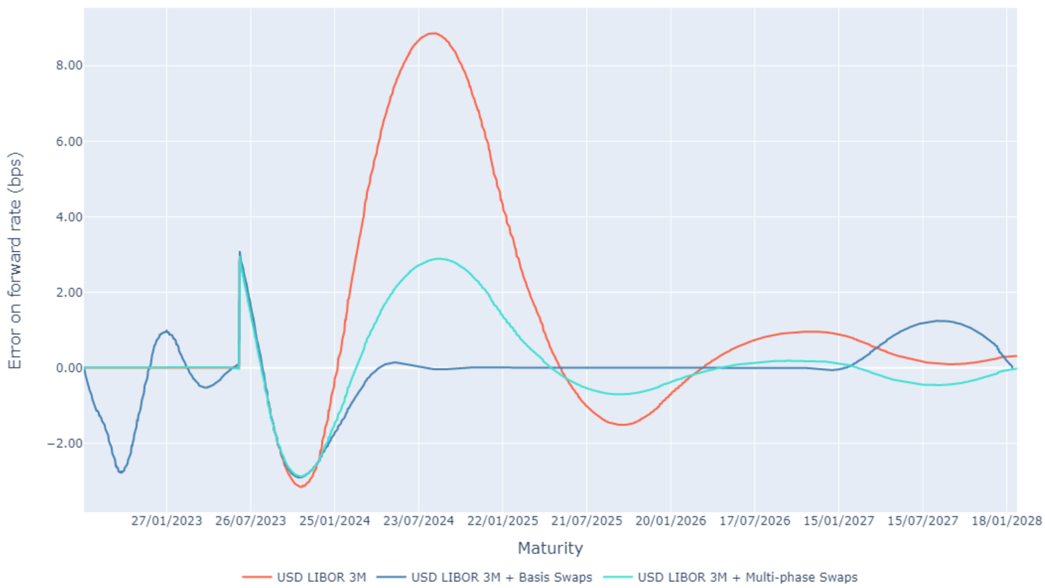

Figure 5.1: Errors (bps) comparing transition curve w.r.t. our proposed USD LIBOR 3M curves

The distance between ideal transition curve and build transition curves are in the Figure 5.1.

Curve	MSE
USD LIBOR 3M	7.4868
USD LIBOR 3M with Basis Swaps	0.8475
USD LIBOR 3M with Multi-Step IRS	1.0868

Table 5.1: Mean Square Errors (bps) comparing transition curve w.r.t. our proposed USD LIBOR 3M curves

From Figure 5.1 and Table 5.1 we can argue that:

1. the first approach was the one to continue using old USD LIBOR 3M curve, ignoring the discontinuation, seeing that, starting from March 5, 2021 (which is the FCA's cessation announcement, as well as the Bloomberg's CAS publication), old USD LIBOR 3M curve is converging to the SOFR + 3M CAS. As you can see, in this case, it turned out this is a bust. Convergence is too slow and errors in repricing financial instruments are too high;

2. the second approach, the one introducing new Basis Swaps indexed to SOFR + 3M CAS curve, is the one which lead to the smallest mean square error (see Table 5.1). In this way we have some oscillations before discontinuation date but, later, a immediate convergence. The reason why we have an almost zero error in the five years following the discontinuation date is because is the period in which we have decided to build one Basis Swap instrument every 3-Months. In this way, the curve is forced to be attached to the SOFR + 3M CAS one. Indeed, starting from the fifth year (the time when we started building Basis Swap instruments one a year), we have some small oscillations;

3. also with the last approach, the one in which we change USD LIBOR 3M Interest Rate Swaps in Multi-phase Swaps, we have good results. Contrary to the previous one, in this case we have no oscillations before transition date, conversely the convergence to the target curve is slower.

Considering that one of the goal of a good forward curve is the one to price and reprice in a good way every financial instrument, to help us in choosing the best approach between the two proposed, we have thought to evaluate some IRS with the two forward curve, comparing the results with the market quote (see Figure 5.2).

Instrument				Error (bps)		
Type	Start	Tenor	Par Rate	USD 3M	Basis Swaps	M-p Swaps
Swap	SPOT	2Y	1,3486%	0,2032	0,4358	0,1619
Swap	SPOT	3Y	1,5483%	-1,7322	0,4260	0,0000
Swap	SPOT	4Y	1,6393%	-1,4278	0,3220	0,0000
Forward Swap	1Y	2Y	1,8741%	-2,5884	0,3408	0,0293
Forward Swap	1Y	3Y	1,8899%	-1,8996	0,2289	0,0198
Forward Swap	1Y	4Y	1,9044%	-1,5262	0,1729	0,0149
Forward Swap	2Y	2Y	1,9394%	-3,1117	0,2046	-0,1768
Forward Swap	2Y	3Y	1,9427%	-2,2128	0,1373	-0,1189
Forward Swap	2Y	4Y	1,9552%	-1,8266	0,0521	-0,0900
MSE				3,9593	0,0819	0,0090

Figure 5.2: Errors between ideal transition curve and proposed transition curves

As you can see from the Figure 5.2, the approach with the smallest Mean Square Error (MSE) is the third one: change old IRS in Multi-phase IRS. It presents the best calibration, although the second approach (the one introducing Basis Swaps) presents, in MSE, the smallest distance to the ideal transition curve. This brings us to prefer it.

5.1. Next steps and further considerations

As next step could be interesting, starting from the chosen approach, to try to add two stub deposits to our basket of financial instruments chosen for the bootstrap procedure: one in correspondence of the ending date of the last USD LIBOR 3M rate (end of September, 2023), in a way to fix forward rate until discontinuation date, the second one with expiry date N days later (ideally $N = 1$), to impose the jump to SOFR + 3M CAS rates. We are expecting even better results.

In addition, a further way of dealing with the Cliff Effect could be to handle the whole period after the discontinuation (or at least the part immediately following it) by means of jump modelling as already done for the TOY effect and central banks' announcements.

Last but not least, one could actually leave things as they are in the hope that the closer we get to the discontinuation date, the closer LIBOR 3M index will get to the estimation of 3-Months daily compounding SOFR plus 3M CAS, thus making the Cliff Effect negligible.

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A | Market terminology

Here below we report a list of acronyms used in the thesis, with a short description.

Acronym	Definition
ARR	**Alternative Reference Rate** Risk-free rates used as new benchmarks due to Benchmarks Reform
bps	**basis points** One hundredth of 1 percentage point. Changes of interest rates are often stated in basis points
CAS	**Credit Adjustment Spread** Term used for the adjustment between LIBOR and Risk Free Rate to reduce or eliminate the economic value transfer between the lender and the borrower when the index changes from LIBOR to the replacement Risk Free Rate
FCA	**Financial Conduct Authority** Conduct regulator for around 50000 firms in the UK to ensure that financial markets are honest, competitive and fair
FRA	**Forward Rate Agreement** OTC contract between parties that determine the rate of interest to be paid on an agreed-upon date in the future
IBA	**ICE Benchmark Administration** One of the world's most experienced administrators of regulated benchmarks, is leading the way in Benchmarks Reform
ICE	**InterContinental Exchange** American company formed in 2000 that operates global financial exchanges and clearing houses and provides mortgage technology, data and listing services
IMM	**International Monetary Market** Division of the Chicago Mercantile Exchange (CME) that deals with the trading of currency and interest rate futures and options

IOSCO	**International Organization of Securities Commissions**	Global cooperative of securities regulatory agencies that aims to establish and maintain worldwide standards for efficient, orderly and fair markets
IRBS	**Interest Rate Basis Swap**	Swap agreement in which two parties agree to swap variable interest rates based on different money market reference rates
IRS	**Interest Rate Swap**	Forward contract in which one stream of future interest payments is exchanged for another based on a specified principal amount
LIBOR	**London Interbank Offered Rate**	Benchmark interest rate at which major global banks lend to one another in the international interbank market for short-term loans
MSE	**Mean Squared Error**	Average squared difference between the estimated values and the actual value
NPV	**Net Present Value**	Difference between the present value of cash inflows and the present value of cash outflows over a period of time
OIS	**Overnight Indexed Swap**	Interest Rate Swap involving the overnight rate being exchanged for a fixed interest rate
OTC	**Over The Counter**	Process of trading securities via a broker-dealer network as opposed to on a centralized exchange (like NYSE)
SOFR	**Secured Overnight Financing Rate**	Secured interbank overnight interest rate established as an alternative to LIBOR

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