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# Semilinear Evolution Equations in Banach Spaces

Tesi di Laurea Magistrale in
Mathematical Engineering - Ingegneria Matematica

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# Abstract

In this thesis we focus on the problem

$$u'(t) = Au(t) + J(u(t)) \text{ when } t > 0 \quad \text{and } u(0) = u_0, \tag{1}$$

where $u : [0, T] \to E$ is a curve in a Banach space $E$, $A$ is the infinitesimal generator of a $C_0$ $\omega$-contractive semigroup $e^{tA}$ on $E$, $J : E_J \to E$ is a nonlinear function, $E_J$ being a Banach space dense in $E$ and continuously embedded into $E$.
In chapter I, we introduce the basic notions in semigroup theory.
In chapter 1 we address problem (1).
A useful way to study problem (1) is through its corresponding integral equation

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} J(u(s))\, ds. \tag{2}$$

In the following $\theta$ is a real number with $\theta \geq 1$.
We give some sufficient conditions for the existence of solutions to (2), when $E = L^\theta(\Omega)$ and $E_J = L^{p\theta}(\Omega)$ with $p > 1$.

In the second and third chapters we focus on a particular case where in problem (1) we have $A = \Delta$ and $J(u(t)) = |u(t)|^{p-1}u(t)$ for some $p > 1$. More precisely, the second chapter examines existence of local solutions with $u_0$ in $L^\theta(\Omega)$ to the following problem:

$$\begin{cases} u'(t) = \Delta u(t) + |u(t)|^{p-1}u(t) & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(x, 0) = u_0(x) & x \in \Omega. \end{cases} \tag{3}$$

We discuss existence and uniqueness of solutions to problem (3) in an interval $[0, T)$ with $T > 0$.
In the third chapter we search for global non-negative solutions $u : [0, \infty) \to L^\theta(\mathbb{R}^n)$ satisfying problem (2) for all $t \geq 0$. In particular we give sufficient conditions for existence and non-existence of non-negative global solutions.

**Keywords:** Evolution equations, Nonlinear pde, Semigroup theory, Global solutions in $L^{\theta}(\Omega)$

# Abstract in lingua italiana

In questa tesi affrontiamo il problema

$$u'(t) = Au(t) + J(u(t)) \text{ when } t > 0 \quad \text{and } u(0) = u_0, \tag{4}$$

dove $u : [0, T] \to E$ è una curva in uno spazio di Banach $E$, $A$ è il generatore di un semigruppo $C_0$ $\omega$-contrattivo $e^{tA}$ definito su $E$, $J : E_J \to E$ è una funzione non lineare, con $E_J$ uno spazio di Banach denso in $E$ e con immersione continua in $E$.
Nel capitolo I, introduciamo le nozioni basilari di teoria dei semigruppi.
Nel capitolo 1 affrontiamo il problema (4).
Un modo utile per studiare (4) è attraverso la sua corrispondente equazione integrale

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} J(u(s))\, ds. \tag{5}$$

Nel prosieguo $\theta$ è un numero reale con $\theta \geq 1$.
Forniamo alcune condizioni sufficienti per l'esistenza di soluzioni a (5) quando $E = L^\theta(\Omega)$ e $E_J = L^{p\theta}(\Omega)$ con $p > 1$.

Il secondo e terzo capitolo sono dedicati allo studio di un caso particolare dove nel problema (4) poniamo $A = \Delta$ e $J(u(t)) = |u(t)|^{p-1}u(t)$ con $p > 1$. Il secondo capitolo prende in esame l'esistenza di soluzioni locali con $u_0$ in $L^\theta(\Omega)$ al seguente problema:

$$\begin{cases} u'(t) = \Delta u(t) + |u(t)|^{p-1}u(t) & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{6}$$

Discutiamo esistenza e unicità delle soluzioni al problema (6) in un intervallo $[0, T)$ con $T > 0$.
Nel terzo capitolo ricerchiamo soluzioni non negative globali $u : [0, \infty) \to L^\theta(\mathbb{R}^n)$ che soddisfino il problema (5) per ogni $t \geq 0$. In particolare vengono fornite condizioni per esistenza e non esistenza di soluzioni globali non negative.

**Parole chiave:** Equazioni di evoluzione, Edp non lineari, Teoria semigruppi, Soluzioni globali in $L^\theta(\Omega)$

# Contents

# Introduction

In this thesis we focus on the problem

$$\begin{cases} u'(t) = Au(t) + J(u(t)) \qquad (t > 0) \\ u(0) = u_0, \end{cases} \tag{7}$$

where $u : [0, T] \to E$ is a curve with values in a Banach space $E$ with norm $\| \ \|$, $A$ is the infinitesimal generator of a $C_0$ $\omega$-contractive semigroup $e^{tA}$ on $E$, moreover $A$ is assumed to be linear closed and densely-defined on $E$ with domain $D(A)$. $J : E_J \to E$ is a nonlinear function from a dense subset of $E$, let us call it $E_J$, continuously embedded into $E$. The set $E_J$ is a Banach space itself with norm $| \ |_J$.
A useful way to study this problem is through its corresponding integral equation

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} J(u(s))\, ds \tag{8}$$

The first thing to understand is the meaning of a semigroup and of its infinitesimal generator.
Referring to problem (7) we get rid of the nonlinear function $J$ to get

$$\begin{cases} u'(t) = Au(t) \qquad (t > 0) \\ u(0) = u_0. \end{cases} \tag{9}$$

If the operator $A$ is the infinitesimal generator of a $C_0$ semigroup, let us call it $e^{tA}$ then

$$u(t) = e^{tA}u_0.$$

The semigroup $e^{tA}$ applied to the initial datum gives for every $t \geq 0$ the solution of problem (9) at time $t$.
For densely-defined, linear and closed operator $A$ we have the following characterization (see Theorem I.4):

let $\omega \in \mathbb{R}$, $A$ is a generator of an $\omega$-contractive semigroup $\{e^{tA}\}_{t\geq 0}$ if and only if

$$(\omega, \infty) \subset \rho(A) \text{ and } ||R_\lambda|| \leq \frac{1}{\lambda - \omega} \text{ for } \lambda > \omega, \tag{10}$$

where $\rho(A) = \{\lambda \in \mathbb{R} : \lambda I - L : D(A) \to E \text{ is bijective}\}$ and $R_\lambda : E \to E$ is such that:

$$R_\lambda u = (\lambda I - A)^{-1} u \quad \forall u \in E.$$

We can now return to problem (7) and study it through the integral equation (8) knowing the meaning of the semigroup $e^{tA}$.
In particular in the first chapter we study existence of solutions to the integral equation (8) for $t \in [0, T]$ for $T > 0$ small enough, these are called local solutions.
In order to guarantee existence of local solutions to the equation (8) we need some hypothesis on the nonlinear function $J : E_J \to E$.
More precisely, we ask $J$ to be locally Lipschitz on bounded sets in $E_J$ in other words:

$$||J\phi - J\psi|| \leq l(r)|\phi - \psi|_J \quad \forall\ \phi, \psi \text{ with } |\phi|_J \leq r \text{ and } |\psi|_J \leq r,$$

where $l(r)$ is the Lipschitz constant restricted to the closed ball of radius $r$ in $E_J$.
The abstract theorem for existence of local solutions of the integral equation (8) requires two different conditions which $l(r)$ can satisfy:

$$\int_\tau^\infty r^{-\frac{1}{a}} l(r)\, dr < \infty \text{ for some } \tau > 0, \tag{11}$$

$$l(r) = O(r^{\frac{(1-a)}{b}}) \text{ as } r \to \infty; \tag{12}$$

with $a, b$ be such that $0 < b < a < 1$. Moreover, we suppose additional constraints on the semigroup $e^{tA}$:

- $\forall t > 0$, $e^{tA}$ is a bounded map $E \to E_J$ such that

  for any $T > 0$ there exists $N > 0$ such that $|e^{tA}\phi|_J \leq N t^{-a} ||\phi||$ for $t \in (0, T]$.

- $t \to e^{tA}\phi$ is continuous into $E_J$ for $t > 0$.

Under such conditions, Theorem 1.1 ensures existence of local solutions to (8) in the case where $l(r)$ behaves like in (11) and (12), respectively.
Then we study problem (7), with $E = L^\theta(\Omega)$,$E_J = L^{\theta p}(\Omega)$ with $p > 1$ and $1 \leq \theta < \infty$, $u_0$ is in $L^\theta(\Omega)$ with $\Omega \subset \mathbb{R}^n$ a bounded domain; furthermore,

- $e^{tA}$ is an analytic $C_0$ semigroup on all $L^\theta(\Omega)$ spaces for $1 < \theta < \infty$. $D_\theta(A)$ is the domain of its generator in $L^\theta(\Omega)$;
- there exists an integer $m > 0$ such that $\partial\Omega$ is of class $C^m$;
- for each $\theta$ $D_\theta(A)$ with its graph norm is continuosly embedded in $W^{m,\theta}(\Omega)$.

Theorem 1.2 ensures existence of local solutions with initial datum $u_0 \in L^\theta(\Omega)$, for carefully chosen $\theta$, to problem (7) in the case where $l(r) = O(r^{p-1})$ as $r \to \infty$. In particular we have two conditions for which problem (7) admits solutions with initial datum in $L^\theta(\Omega)$:

- $\theta > \frac{n(p-1)}{m}$ and $\theta > 1$ ($\theta \geq 1$ if $A = \Delta$).
- $\frac{n(p-1)}{mp} < \theta < \frac{n(p-1)}{m}$ and $\theta > 1$, ($\theta \geq 1$ if $A = \Delta$).

Notice that until now we have not given a particular expression to $A$ and $J$.
In the second chapter we study in detail problem (7) when $A = \Delta$ and $J(u(t)) = |u(t)|^{p-1}u(t)$ for some $p > 1$.
Indeed, we investigate existence of solutions to problem

$$\begin{cases} u'(t) = \Delta u(t) + |u(t)|^{p-1}u(t) & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{13}$$

The advantage of facing this particular case is that we know further properties of $e^{t\Delta}$ that is the Dirichlet heat semigroup in $\Omega$. In particular, one fundamental property is the following
if $1 \leq p < q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, then $||e^{t\Delta}\phi||_q \leq (4\pi t)^{-n/(2r)}||\phi||_p$ for all $t > 0$.
Solutions to problem (13) are meant in the following sense.

**Definition 0.1.** *Given a Banach space $X$ of functions defined on $\Omega$, $u_0 \in X$ and $T \in (0,\infty]$, we say that $u \in C((0,T],X)$ is a classical $X$-solution of (13) in $[0,T)$ if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times \{t = 0\})$, $u(0) = u_0$ and $u$ is a classical solution of (13) in $(0,T)$. If $\Omega$ is unbounded we also require $u \in L^\infty_{loc}((0,T), L^\infty(\Omega))$. If $X = L^\infty(\Omega)$ instead of requiring $u \in C((0,T],X)$ we require $u \in C((0,T),X)$ and $||u(t) - e^{tA}u_0||_\infty \to 0$ when $t \to 0$, where $e^{tA}$ is the heat semigroup.*

Theorem 2.1 gives sufficient conditions for existence of solutions with initial datum in $L^\theta(\Omega)$.
Let $p > 1$, $u_0 \in L^\theta(\Omega)$, $1 \leq \theta < \infty$, $\theta > \theta_c = \frac{n(p-1)}{2}$. Then there exists $T = T(||u_0||_\theta) > 0$ such that problem (13) possesses a unique classical $L^\theta$-solution in $[0,T)$ and the following

estimate holds:

$$||u(t)||_r \le C||u_0||_\theta t^{-\alpha_r}, \ \ \alpha_r = \frac{n}{2}\left(\frac{1}{\theta} - \frac{1}{r}\right) \tag{14}$$

for all $t \in (0,T)$ and $r \in [\theta, \infty]$, with $C = C(n,p,\theta) > 0$. Moreover $u \ge 0$ whenever $u_0 \ge 0$.

In the third chapter we focus on trying to find global solutions to problem (7), this means that the solution $u(t)$ satisfies (7) for all $T \ge 0$.
Thus we want to find global non-negative solutions to the problem:

$$\begin{cases} u'(t) = \Delta u(t) + u(t)^p & (t > 0) \\ u(0) = u_0, \end{cases} \tag{15}$$

where $p > 1$, the solution $u(t)$ will be a non-negative curve in $L^\theta(\mathbb{R}^n)$ for some $\theta \ge 1$.
We deal with problem (15) through its corresponding integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}(u(s)^p)\, ds. \tag{16}$$

Since here we are in $\mathbb{R}^n$, we have

$$(e^{t\Delta}\phi)(x) = \int_{\mathbb{R}^n} G_t(x-y)\phi(y)\, dy$$

with

$$G_t(x) \equiv G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

Theorem 3.1 gives sufficient conditions for non-existence of non-negative solutions to the integral equation (16). More precisely we have the following:
suppose $p \le 1 + \frac{2}{n}$ and $u_0 \ge 0$ in $L^\theta(\mathbb{R}^n)$ with $u_0$ non identically zero. Then there is no non-negative global solution $u : [0,\infty) \to L^\theta(\mathbb{R}^n)$ to the integral equation (16) with initial value $u_0$.
On the other hand, Theorem 3.2 gives conditions for global existence of non-negative solutions.
Let $u_0 \ge 0$ be in $L^\theta(\mathbb{R}^n)$, with $1 \le \theta < \infty$. Suppose the following holds

$$(p-1)\int_0^\infty ||e^{s\Delta}u_0||_\infty^{p-1}\, ds \le 1.$$

Then there exists a non-negative continuous curve $u : [0,\infty) \to L^\theta(\mathbb{R}^n)$ which is a global solution to (16) with initial datum $u_0$.

Suppose instead $p > 1 + \frac{2}{n}$. If $u_0 \geq 0$ and $||u_0||_{n(p-1)/2}$ is sufficiently small, then there exists a non-negative continuous curve $u : [0, \infty) \to L^{n(p-1)/2}$ which is a global solution to problem (16) with initial value $u_0$.

# I | Preliminary results

## I.1. Unbounded operators

Let us consider a linear operator $L$ defined on a domain $D(L)$ dense in the Banach space $X$:

$$L : D(L) \subset X \to X$$

We do not assume the boundedness of $L$ this implies that, in general the relation:
$||Lx|| \leq M||x||$ for some $M > 0$ $\forall x \in D(L)$ does not hold.
One could ask wether it is useful to study those kind of operators, with the next example we see how unbounded operators naturally arise in a standard environment.
Let $D(L) = \mathcal{C}^1([0,1])$ and $X = L^2([0,1])$:

$$L : D(L) \subset X \to X \quad \text{with} \quad Lu := \frac{d}{dt}u$$

$$||L|| = \sup_{u \in D(L)} \frac{||Lu||}{||u||}$$

Consider now the sequence $\{u_k\}_{k\in\mathbb{N}} \subset D(L)$ with $u_k(t) = e^{kt}$.
We have:

$$||Tu_k||^2 = \int_0^1 k^2 e^{2kt}\, dt = \frac{k}{2}[e^{2k} - 1]$$

On the other hand:

$$||u_k||^2 = \int_0^1 e^{2kt}\, dt = \frac{1}{2k}[e^{2k} - 1]$$

Summing up:

$$||Tu_k|| = k||u_k||$$

Hence:

$$||L|| \geq \frac{||Lu_k||}{||u_k||} \geq k \;\; \forall k \in \mathbb{N}$$

Hence $L$ is an unbounded operator.

### I.1.1. Extension of unbounded operators

We notice that in the case of unbounded operators the domain plays a crucial role. We can ask if there is a way to find the "natural" domain in some sense, consistent with the definition of the unbounded operator.
In other terms, given an unbounded operator $L : D(L) \subset X \to X$ we want to find a procedure to find an extension of the operator $L$ let's call it $\widehat{L}$ such that $D(L) \subset D(\widehat{L})$ and $\widehat{L}|_{D(L)} = L$.

There exists a standard procedure to find an optimal extension of an unbounded operator $L$, and the procedure consists in the closure of the operator.

**Definition I.1.** *Given an operator $L : D(L) \subset X \to X$ we say that $L$ is closable if there exists $\widehat{L}$ which extends $L$ and such that $\widehat{L}$ is a closed operator.*

We now give a characterization of closability:

**Proposition I.1.** *$L$ is closable if and only if for every $\{x_n\} \subset D(L)$ such that $x_n \to 0$ and $Lx_n \to y$ we necessarily have $y = 0$.*

## I.2. Semigroup theory

Semigroup theory is the study of first-order ordinary differential equations defined in Banach spaces, where we have linear bounded or unbounded operators acting on the system.
For the treatment of this argument we follow the scheme presented in [2].

### I.2.1. Definitions and basic properties

Let $X$ be a Banach space, consider the following ordinary differential equation:

$$\begin{cases} \boldsymbol{u}'(t) = L\boldsymbol{u}(t) \ (t \geq 0) \\ \boldsymbol{u}(0) = u. \end{cases} \tag{I.1}$$

Where $u \in X$ is given, and $L$ is a linear operator.
Let $D(L)$ be the domain of the operator $L$, we have:

$$L : D(L) \to X \tag{I.2}$$

We didn't impose any restriction on $L$ except for the linearity, $L$ can also be unbounded. Our aim is to study existence and uniqueness of a solution:

$$\boldsymbol{u} : [0, \infty) \to X$$

of ODE (I.1).
In particular we want to find reasonable conditions for the operator $L$ so that for every $u \in X$ the differential system (I.1) has a unique solution.

### I.2.2. Semigroups

Assume $\boldsymbol{u} : [0, \infty) \to X$ is the unique solution of (I.1) when the initial condition $u \in X$ has been fixed.
We need some notation:

$$\boldsymbol{u}(t) = S(t)u \ \ (t \geq 0) \tag{I.3}$$

(I.3) gives us the solution of (I.1) for each time $(t \geq 0)$ when the initial datum is $u \in X$. It is important to notice that $\forall t \geq 0$ the operator $S(t)$ has as its domain the entire Banach space $X$ since in our hypothesis we have a solution of (I.1) for each initial datum $u \in X$. In other words for any fixed $t \geq 0$ we have $S(t) : X \to X$.

We now focus on the properties of the family of operators $\{S(t)\}_{t \geq 0}$:

$$S(0)u = u \ (u \in X) \tag{I.4}$$

$$S(t+s)u = S(t)S(s)u = S(s)S(t)u \ (t, s \geq 0, u \in X) \tag{I.5}$$

We analyze (I.5), consider for example the first equality, and we fix $s \geq 0$, if we expand both sides we get:

$$\boldsymbol{u}(t+s) = \boldsymbol{v}(t) \ (t \geq 0) \tag{I.6}$$

Where:

$$\begin{cases} \boldsymbol{u}'(t+s) = L\boldsymbol{u}(t+s) \ (t \geq 0) \\ \boldsymbol{u}(0) = u. \end{cases} \tag{I.7}$$

$$\begin{cases} \boldsymbol{v}'(t) = L\boldsymbol{v}(t) \ (t \geq 0) \\ \boldsymbol{v}(0) = \boldsymbol{u}(s). \end{cases} \tag{I.8}$$

If we perform the substitution $\boldsymbol{w}(t) = \boldsymbol{u}(t+s)$ in (I.7), we clearly see that $\boldsymbol{w}(t) = \boldsymbol{v}(t)$ since they solve the same differential system, which is (I.6). We now understand that (I.5) is a property that naturally arises from the uniqueness of system (I.1).
The final assumption we make is:

$$t \to S(t)u \text{ is continuous from } [0, \infty) \text{ into } X \tag{I.9}$$

**Definition I.2.** *A family* $\{S(t)\}_{t\geq 0}$ *of bounded linear operators mapping* $X$ *into* $X$ *is called a semigroup if* (I.4),(I.5),(I.9) *hold.*
*If moreover* $\{S(t)\}_{t\geq 0}$ *is such that* $||S(t)|| \leq 1$ *we say that the family* $\{S(t)\}_{t\geq 0}$ *is a contraction semigroup.*

**Definition I.3.** *A semigroup for which property* (I.5) *and* (I.9) *holds with arbitrary sign of* $s, t$ *is called a* $C^0$ *semigroup.*

If $\{S(t)\}_{t\geq 0}$ is a contraction semigroup we have:

$$||S(t)u|| \leq ||u|| \ (t \geq 0, u \in X) \tag{I.10}$$

From now on $\{S(t)\}_{t\geq 0}$ is a contraction semigroup on $X$.

**Definition I.4.** $D(L) := \{u \in X | \lim_{t\to 0^+} \frac{S(t)u-u}{t}$ *exists in* $X\}$

**Definition I.5.** *If* $L$ *is defined as follows:* $Lu := \lim_{t\to 0^+} \frac{S(t)u-u}{t}$ $(u \in D(L))$ *we say that* $L : D(L) \to X$ *is the (infinitesimal) generator of the semigroup* $\{S(t)\}_{t\geq 0}$*. In particular* $D(L)$ *is the domain of defintion of* $L$*.*

**Theorem I.1.** *(Differential properties of semigroups).*
*Let* $u \in D(L)$ *then:*

1. $S(t)u \in D(L) \ \forall t \geq 0$
2. $LS(t)u = S(t)Lu \ \forall t > 0$

3. *The map* $t \to S(t)u$ *is differentiable* $\forall\ t > 0$

4. $\frac{d}{dt}S(t)u = LS(t)u\ \forall t > 0$

*Proof.* 1) and 2):

$S(t)u \in D(L) \iff \lim_{s\to 0^+} \dfrac{S(s)S(t)u - S(t)u}{s}$ exists in $X$

$\lim_{s\to 0^+} \dfrac{S(s)S(t)u - S(t)u}{s} = \lim_{s\to 0^+} \dfrac{S(t)S(s)u - S(t)u}{s}$ where we have used (I.5)

$= S(t) \lim_{s\to 0^+} \dfrac{S(s)u - u}{s}$ valid because of the boundedness of $S(t)$

$= S(t)Lu \in X$

So $S(t)u \in D(L)$ and 2) has been proven from the previous computations.

3) and 4):

Let u $\in D(L)$ we need to compute $\lim_{h\to 0} \dfrac{S(t+h)u - S(t)u}{h}$

We observe that:

$\lim_{h\to 0} \dfrac{S(t+h)u - S(t)u}{h}$ exists $\iff \lim_{h\to 0^+} \dfrac{S(t+h)u - S(t)u}{h}$ and $\lim_{h\to 0^-} \dfrac{S(t+h)u - S(t)u}{h}$ both exists.

In particular:

$$\lim_{h\to 0^-} \frac{S(t+h)u - S(t)u}{h} = \lim_{k\to 0^+} \frac{S(t-k)u - S(t)u}{-k}$$

where we performed the change of variables $k = -h$

$$\lim_{h\to 0^-} \frac{S(t+h)u - S(t)u}{h} = \lim_{k\to 0^+} \frac{S(t)u - S(t-k)u}{k} = \lim_{h\to 0^+} \frac{S(t)u - S(t-h)u}{h}$$

We now show that $\lim_{h\to 0^+} \dfrac{S(t)u - S(t-h)u}{h}$ exists and is equal to $S(t)Lu$. Indeed,

$$\lim_{h\to 0^+} \{\frac{S(t)u - S(t-h)u}{h} - S(t)Lu\} = \lim_{h\to 0^+} \{S(t-h)\left(\frac{S(h)u - u}{h}\right) - S(t)Lu\}.$$

We now add and subtract $S(t-h)Lu$. So

$\lim_{h\to 0^+} \{\dfrac{S(t)u - S(t-h)u}{h} - S(t)Lu\} = \lim_{h\to 0^+} \{S(t-h)\left(\dfrac{S(h)u - u}{h} - Lu\right) + (S(t-h) - S(t))Lu\}$

Now $(S(t-h) - S(t))Lu$ approaches 0 since (I.9) holds. Moreover,

$||S(t-h)(\frac{S(h)u-u}{h} - Lu)|| \leq ||S(t-h)||\ ||\frac{S(h)u-u}{h} - Lu|| \leq ||\frac{S(h)u-u}{h} - Lu|| \to 0$ as $h \to 0^+$

Hence we showed that $\lim_{h\to 0^-} \dfrac{S(t+h)u - S(t)u}{h} = S(t)Lu \; \forall t > 0.$

A similar argument proves that $\lim_{h\to 0^+} \dfrac{S(t+h)u - S(t)u}{h} = S(t)Lu \; \forall t > 0.$

Since in 2) we proved $S(t)Lu = LS(t)u$ also point 4) has been proved.

□

**Theorem I.2.** *(Properties of generators).*
*Let $u \in D(L)$ then:*

1. *the domain $D(L)$ is dense in $X$;*
2. *$L$ is a closed operator.*

*Proof.* 1)Let u $\in X$ be fixed, define:

$$u^t := \int_0^t S(s)u\,ds \tag{I.11}$$

Since we know that (I.9) holds, we can apply the fundamental theorem of calculus:

$$\lim_{t\to 0^+} \frac{u^t - u^0}{t} = S(0)u = u.$$

In particular $\forall\, u \in X$ $\frac{u^t}{t} \to u$ as $t \to 0^+$.

If we are able to prove that $u^t \in D(L)$ we have automatically the density of $D(L)$ into $X$.
We claim $u^t \in D(L)$ $(t > 0)$:
Let $h > 0$. We have

$$\frac{S(h)u^t - u^t}{h} = \frac{1}{h}\{\int_0^t S(h)S(s)u\,ds - \int_0^t S(s)u\,ds\} = \frac{1}{h}\int_0^t (S(h+s)u\,ds - S(s)u)\,ds$$

$$\begin{aligned}\frac{S(h)u^t - u^t}{h} &= \frac{1}{h}\int_0^t (S(h+s)u - S(s)u)\,ds = \frac{1}{h}\int_h^{t+h} S(s)u\,ds - \frac{1}{h}\int_0^t S(s)u\,ds \\ &= \frac{1}{h}\int_t^{t+h} S(s)u\,ds - \frac{1}{h}\int_0^h S(s)u\,ds \to S(t)u - u \text{ as } h \to 0^+.\end{aligned}$$

Hence $D(L)$ is dense in $X$.

2) Let $\{u_k\}_{k\in\mathbb{N}} \subset D(L)$. Suppose that $u_k \to u$ and $Lu_k \to v$ in $X$. We have to prove: $u \in D(L)$ and $v = Lu$.
In part 1) of the proof we saw that:

$$\forall u \in X : \quad Lu^t = S(t)u - u. \tag{I.12}$$

Changing $u$ with $u_k$ in (I.12), we get:

$$Lu_k^t = L\int_0^t S(s)u_k\,ds = \int_0^t LS(s)u_k\,ds = \int_0^t S(s)Lu_k\,ds = S(t)u_k - u_k$$

We obtain:

$$\int_0^t S(s)Lu_k\,ds = S(t)u_k - u_k. \tag{I.13}$$

Passing to the limit as $k \to \infty$ in (I.13) we get that the right hand side becomes: $S(t)u - u$.
While for the left hand side:

$$|\int_0^t S(s)Lu_k\,ds - \int_0^t S(s)v\,ds| = |\int_0^t (S(s)Lu_k - S(s)v)\,ds| \leq \int_0^t ||S(s)(Lu_k - v)||\,ds$$

$$\leq \int_0^t ||Lu_k - v||\,ds \to 0 \text{ as } \; k \to \infty.$$

Summing up from (I.13), letting $k \to \infty$ we have:

$$\int_0^t S(s)Lv\,ds = S(t)u - u. \tag{I.14}$$

From (I.14) we can deduce:

$$Lu = \lim_{t\to 0^+} \frac{S(t)u - u}{t} = \lim_{t\to 0^+} \frac{1}{t}\int_0^t S(s)Lv\,ds = v.$$

The equation above tells us that $u \in D(L)$ and $v = Lu$, this completes the proof. □

**Definition I.6.** *1) Given a real number $\lambda \in \mathbb{R}$ we say that it belongs to $\rho(L)$ if the operator:*

$$\lambda I - L : D(L) \to X$$

*is bijective.*

*2)Whenever $\lambda \in \rho(L)$ we define the resolvent operator $R_\lambda : X \to X$ in the following way:*

$$R_\lambda u = (\lambda I - L)^{-1} u \quad \forall u \in X.$$

**Remark I.1.** *$R_\lambda : X \to D(L) \subset X$ is a bounded linear operator in fact:*
*Since $R_\lambda$ is defined on the whole banach space $X$ and $R_\lambda$ is linear, by the closed graph theorem we have that:*
*$R_\lambda$ is continuous $\iff$ the graph of $R_\lambda$ is closed $\iff$ graph of $(\lambda I - L)$ is closed $\iff$ $L$ is closed.*
*We proved that $L$ is closed, hence $R_\lambda$ is bounded.*

**Remark I.2.** *Observe that whenever $u \in D(L)$ we have:*

$$LR_\lambda u = \lim_{t \to 0^+} \frac{S(t)R_\lambda u - R_\lambda u}{t} = R_\lambda L u. \tag{I.15}$$

*Notice that* (I.15) *holds for all operators $L$, not only for the specific one we are considering. In fact:*

$$(\lambda I - L)R_\lambda u = u \;\; \forall u \in D(L),$$

$$R_\lambda(\lambda I - L)u = u \;\; \forall u \in D(L).$$

*Summing up the above two equations we get:*

$$\forall u \in D(L) \;\; LR_\lambda u = R_\lambda L u. \tag{I.16}$$

**Theorem I.3.** *(Properties of resolvent operators).*

1. *If $\lambda,\mu \in \rho(L)$ then:*

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \tag{I.17}$$

   *and also:*

$$R_\lambda R_\mu = R_\mu R_\lambda \tag{I.18}$$

2. *If $\lambda > 0$ then $\lambda \in \rho(L)$ and:*

$$R_\lambda u = \int_0^\infty e^{-\lambda t} S(t) u \, dt \tag{I.19}$$

   *and $||R_\lambda|| \leq \frac{1}{\lambda}$.*

*Proof.* 1)

$$(R_\lambda - R_\mu) = R_\lambda(\mu I - L)R_\mu - R_\lambda(\lambda I - L)R_\mu = R_\lambda(\mu - \lambda)R_\mu = (\mu - \lambda)R_\lambda R_\mu$$

In order to prove (I.18) we use (I.17), we sum to this equation, the same expression inverting the role of $\lambda$ and $\mu$, we obtain

$$R_\lambda - R_\mu + (R_\mu - R_\lambda) = (\mu - \lambda)R_\lambda R_\mu + (\lambda - \mu)R_\mu R_\lambda.$$

So

$$(\mu - \lambda)R_\lambda R_\mu = (\mu - \lambda)R_\mu R_\lambda,$$

which gives us the thesis, since the case $\lambda = \mu$ is trivial.
2) Since $\lambda > 0$ and $||S(t)|| \leq 1$ the integral on the right hand side of (I.19) is well defined. Let $\widetilde{R}_\lambda u := \int_0^\infty e^{-\lambda t} S(t)u\,dt$, and let $h > 0$:

$$\frac{S(h)\widetilde{R}_\lambda u - \widetilde{R}_\lambda u}{h} = \frac{1}{h}\{\int_0^\infty e^{-\lambda t}(S(t+h)u - S(t)u)\,dt\} =$$

$$= \frac{1}{h}\{\int_h^\infty e^{-\lambda(y-h)} S(y)u\,dy - \int_0^\infty e^{-\lambda(y)} S(y)u\,dy\} =$$

$$= \frac{1}{h}\{\int_0^\infty e^{-\lambda(y-h)} S(y)u\,dy - \int_0^h e^{-\lambda(y-h)} S(y)u\,dy - \int_0^\infty e^{-\lambda(y)} S(y)u\,dy\} =$$

$$= -\frac{1}{h}\int_0^h e^{-\lambda(y-h)} S(y)u\,dy + \frac{1}{h}\int_0^\infty (e^{-\lambda(y-h)} - e^{-\lambda y})S(y)u\,dy =$$

$$= -e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda y} S(y)u\,dy + \frac{e^{\lambda h} - 1}{h}\int_0^\infty e^{-\lambda y} S(y)u\,dy$$

Taking the limt for $h \to 0^+$ we get:

$$L\widetilde{R}_\lambda u = -u + \lambda\widetilde{R}_\lambda u \ \ \forall u \in X.$$

So

$$\forall u \in X \ \ (\lambda I - L)\widetilde{R}_\lambda u = u.$$

If we prove that

$$\forall u \in D(L) \ \ \widetilde{R}_\lambda(\lambda I - L)u = u,$$

we have that $\widetilde{R}_\lambda$ is the inverse of $(\lambda I - L)$ that is the resolvent operator.

Let $u \in D(L)$ we have

$$L\widetilde{R}_\lambda u = L\int_0^\infty e^{-\lambda t}S(t)u\,dt = \int_0^\infty e^{-\lambda t}LS(t)u\,dt = \int_0^\infty e^{-\lambda t}S(t)Lu\,dt = \widetilde{R}_\lambda Lu.$$

So

$$\forall u \in D(L)\;\; \widetilde{R}_\lambda(\lambda I - L)u = \lambda\widetilde{R}_\lambda u - \widetilde{R}_\lambda Lu = (\lambda I - L)\widetilde{R}_\lambda u = u.$$

Hence $\forall\lambda \in \mathbb{R}$, $\lambda > 0$ $\widetilde{R}_\lambda$ is the resolvent operator $R_\lambda$.
We only need to compute an estimate on $||R_\lambda||$:

$$||R_\lambda|| = \sup_{u\in X}\frac{||R_\lambda u||}{||u||} \le \sup_{u\in X}\frac{||u||\int_0^\infty e^{-\lambda t}\,dt}{||u||} = \frac{1}{\lambda}.$$

□

**Definition I.7.** *Let $\omega \in \mathbb{R}$ we say that a semigroup $\{S(t)\}_{t\ge0}$ is $\omega$-contractive if:*

$$||S(t)|| \le e^{\omega t}.$$

If in theorem I.3 point 2) we have an $\omega$-contractive semigroup we obtain

$$R_\lambda u \text{ is well defined whenever } \lambda > \omega,$$

$(\omega,\infty) \subset \rho(L)$ and $||R_\lambda|| = \sup_{u\in X}\frac{||R_\lambda u||}{||u||} \le \sup_{u\in X}\frac{||u||\int_0^\infty e^{(\omega-\lambda)t}\,dt}{||u||} = \frac{1}{\lambda-\omega}$.

**Theorem I.4.** *(Hille-Yosida Theorem). Let $L$ be a closed, densely-defined linear operator on $X$ and let $\omega \in \mathbb{R}$.Then $L$ is a generator of an $\omega$-contractive semigroup $\{S(t)\}_{t\ge0}$ if and only if*

$$(\omega,\infty) \subset \rho(L) \text{ and } ||R_\lambda|| \le \frac{1}{\lambda-\omega} \text{ for } \lambda > \omega. \tag{I.20}$$

*Proof.* We already know that if $L$ is a generator of a $\omega$-contractive semigroup (I.20) holds. It remains to prove that, given an operator $L$ closed and densely-defined on $X$ and such that (I.20) holds, we have that $L$ generates an $\omega$-contractive semigroup.
To this end, we introduce

$$L_\lambda := -\lambda I + \lambda^2 R_\lambda = \lambda L R_\lambda.$$

Let $D(L)$ be the domain of $L$, we know it is dense in $X$ for hypothesis.
We claim

$$L_\lambda u \to Lu \text{ as } \lambda \to \infty \;\; \forall u \in D(L).$$

Exploiting the fact that

$$(\lambda I - L)R_\lambda u = u \ \ \forall u \in D(L).$$

We obtain the following relations

$$\lambda R_\lambda u - u = LR_\lambda u = R_\lambda Lu \ \ \forall u \in D(L).$$

In particular

$$||\lambda R_\lambda u - u|| = ||R_\lambda Lu|| \leq ||R_\lambda|| \, ||Lu|| \leq \frac{1}{\lambda - \omega}||Lu|| \to 0 \text{ as } \lambda \to \infty \ \forall u \in D(L).$$

Let now $u \in X$ and $\delta > 0$ , there exists $u_\delta \subset D(L)$ such that $||u - u_\delta|| < \delta$ in $X$.
Fix $\lambda > \omega$, we have

$$||\lambda R_\lambda u - u|| \leq ||\lambda R_\lambda(u - u_\delta)|| + ||\lambda R_\lambda u_\delta - u_\delta|| + ||u_\delta - u|| \leq$$

$$\leq \frac{\lambda}{\lambda - \omega}||u - u_\delta|| + ||\lambda R_\lambda u_\delta - u_\delta|| + \delta.$$

Letting $\lambda \to \infty$ we get

$$\lim_{\lambda \to \infty} ||\lambda R_\lambda u - u|| \leq ||u - u_\delta|| + \delta = 2\delta.$$

Choosing $\delta < \epsilon/2$ we proved the following fact

$$\lambda R_\lambda u \to u \text{ as } \lambda \to \infty \ \ \forall u \in X.$$

If $u \in D(L)$

$$L_\lambda u = \lambda L R_\lambda u = \lambda R_\lambda L u.$$

Letting $\lambda \to \infty$

$$\lim_{\lambda \to \infty} L_\lambda u = Lu \ \ \forall u \in D(L).$$

We define

$$S_\lambda(t) := e^{tL_\lambda} = e^{-\lambda t} e^{\lambda^2 t R_\lambda} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda^2 t)^k}{k!} R_\lambda^k.$$

From the properties of the exponential of an operator (I.4) , (I.5) and (I.9) follow immediately.

Since $||R_\lambda|| \leq \frac{1}{\lambda-\omega}$

$$||S_\lambda(t)|| \leq e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\lambda^{2k}t^k}{k!}||R_\lambda||^k \leq e^{-\lambda t}\sum_{k=0}^{\infty}\frac{\lambda^{2k}t^k}{(\lambda-\omega)^k k!} = e^{-\lambda t}e^{\lambda^2 t/(\lambda-\omega)} = e^{\lambda\omega t/(\lambda-\omega)}.$$

In particular $\{S_\lambda(t)\}_{t\geq 0}$ is an $\frac{\lambda\omega}{\lambda-\omega}$-contractive semigroup.
We now prove that for every $t \geq 0$ and $u \in D(L)$, $\{S_\lambda(t)u\}_{\lambda>\omega}$ has the Cauchy property as $\lambda \to \infty$.
Let $t > 0$, $u \in D(L)$ and define $\phi : [0,t] \to X$ by setting

$$\phi(s) = e^{(t-s)L_\lambda}e^{sL_\mu}u \text{ for } 0 \leq s \leq t.$$

We have

$$\phi(t) - \phi(0) = e^{tL_\mu}u - e^{tL_\lambda}u = S_\mu(t)u - S_\lambda(t)u.$$

Let $\lambda$,$\mu > \omega$

$$L_\lambda L_\mu = \lambda L R_\lambda\, \mu L R_\mu = \lambda L \mu R_\lambda\, R_\mu L = \lambda L \mu R_\mu\, R_\lambda L = L_\mu L_\lambda.$$

So

$$\phi'(s) = e^{(t-s)L_\lambda}e^{sL_\mu}(L_\mu u - L_\lambda u) \text{ for all } s \in [0,t].$$

$$||\phi'(s)|| \leq e^{\lambda\omega(t-s)/(\lambda-\omega)}e^{\mu\omega s/(\mu-\omega)}||L_\mu u - L_\lambda u||,$$

since the function $\lambda \to \lambda/(\lambda-\omega)$ is decreasing we have

$$||\phi'(s)|| \leq e^{\lambda\omega t/(\lambda-\omega)}||L_\mu u - L_\lambda u|| \text{ for all } \mu > \lambda > \omega \text{ for all } s \in [0,t].$$

Hence

$$||S_\mu(t)u - S_\lambda(t)u|| = ||\phi(t) - \phi(0)|| = ||\phi'(c)||t \leq te^{\lambda\omega t/(\lambda-\omega)}||L_\mu u - L_\lambda u||\ \forall u \in D(L) \tag{I.21}$$

where $c \in (0,t)$.
Since $L_\lambda u \to Lu$ as $\lambda \to \infty$ we proved that the family $\{S_\lambda(t)u\}_{\lambda>\omega}$ is Cauchy as $\lambda \to \infty$, in particular for every $t \geq 0$ there exists a linear map $S(t) : D(L) \to X$ with

$$S(t)u = \lim_{\lambda\to\infty} S_\lambda(t)u \text{ for all } u \in D(L).$$

Sending $\mu \to \infty$ in (I.21) keeping $\lambda$ fixed we obtain

$$||S(t)u - S_\lambda(t)u|| \leq te^{\lambda\omega t/(\lambda-\omega)}||Lu - L_\lambda u|| \text{ for all } u \in D(L).$$

And this tells us that $S_\lambda(t)u \to S(t)u$ uniformly with respect to $t \in [0,T]$ as $\lambda \to \infty$ for $u \in D(L)$, so that $t \to S(t)u$ is continuous on $[0,\infty)$ for $u \in D(L)$.
Since $||S_\mu(t)u|| \leq e^{\mu\omega t/(\mu-\omega)}||u||$ for every $u \in X$, we have $||S(t)|| \leq e^{\omega t}$. From this we deduce that $S_\lambda(t)u \to S(t)u$ for all $u \in X$ and $t \to S(t)u$ is continuous on $[0,\infty)$ for all $u \in X$. Properties (I.4) and (I.5) hold since they hold for each $S_\lambda$ with $\lambda > \omega$ and we have pointwise convergence of $S_\lambda u$ to $Su$ as $\lambda \to \infty$. It remains to show that $L$ is the generator of the semigroup $\{S(t)\}_{t\geq 0}$.
Let $B$ be the generator of $\{S(t)\}_{t\geq 0}$. Since $B$ is the generator of the semigroup $\{S(t)\}_{t\geq 0}$ $B$ is a closed operator with domain

$$D(B) = \{u \in X : Bu = \lim_{t\to 0^+} \frac{S(t)u - u}{t} \text{exists in } X\}.$$

Moreover $(\omega,\infty) \subset \rho(B)$. Consider that

$$S_\lambda(t)u - u = \int_0^t \frac{d}{ds} S_\lambda(s)u\, ds = \int_0^t S_\lambda(s)L_\lambda u\, ds \;\; \forall u \in D(L). \tag{I.22}$$

We also have the following inequality

$$||S_\lambda(s)L_\lambda u - S(s)Lu|| \leq ||S_\lambda(s)||\, ||L_\lambda u - Lu|| + ||(S_\lambda(s) - S(s))Lu|| \to 0 \text{ as } \lambda \to \infty.$$

Passing to the limit for $\lambda \to \infty$ in (I.22) we obtain

$$S(t)u - u = \int_0^t S(s)Lu\, ds \;\; \forall u \in D(L).$$

$$Bu = \lim_{t\to 0^+} \frac{S(t)u - u}{t} = Lu \;\; \forall u \in D(L).$$

In particular we have $D(L) \subset D(B)$.
Moreover $(\omega,\infty) \subset \rho(L) \cap \rho(B)$ and if $\lambda > \omega$

$$(\lambda I - B)(D(L)) = (\lambda I - L)(D(L)) = X.$$

The above equality holds since in $D(L)$ the operator $B$ and $L$ coincide and $\lambda \in \rho(L)$. But this means that $(\lambda I - B)|_{D(L)}$ is a bijection, also $(\lambda I - B)|_{D(B)}$ is a bijection, hence $D(L) = D(B)$ but this implies that $L = B$ and $L$ is the generator of $\{S(t)\}_{t\geq 0}$. □

We need an additional definition that will be used in the following chapters:

**Definition I.8.** *We say that $S(t)$ is an analytic semigroup if it is a $C^0$ semigroup and*

- *For some* $\phi \in (0, \pi/2)$, $S(t)$ *can be extended to* $\Delta_\phi$, *where*

$$\Delta_\phi = \{0\} \cup \{t \in \mathbb{C} : |arg(t)| < \phi\}.$$

- *for all* $t \in \Delta_\phi - \{0\}$ $S(t)$ *is analytic in* $t$ *in the uniform operator topology.*

### I.2.3. Applications

Consider the following parabolic problem

$$\begin{cases} u_t + Lu = 0 \text{ in } U_T \\ \qquad u = 0 \text{ on } \partial U \times [0, T] \\ \qquad u = g \text{ on } U \times \{t = 0\}. \end{cases} \tag{I.23}$$

Where $U$ is assumed to be an open bounded set in $\mathbb{R}^n$, $U_T = U \times (0, T]$ and $T \in \mathbb{R}$ with $T > 0$. We assume $L$ to have the divergence structure, moreover we ask $L$ to satisfy the strong ellipticity conditions and to have smooth coefficients not depending on t, that is

$$Lu := div(A(x)\nabla u) + \overrightarrow{b}(x) \cdot \nabla u + c(x)u,$$

with $A(x) = [a_{ij}(x)]_{i,j=1..n}$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\psi_i\psi_j \geq \theta|\psi|^2,$$

for some $\theta > 0$ and for all $x \in \mathbb{R}$ and $\psi \in \mathbb{R}^n$.
$A(x)$, $\overrightarrow{b}(x), c(x)$ are all smooth coefficients.
Our aim is to find solutions of (I.23) using the theory of semigroups.
Define the operator $A$ in the following way:

$$Au := -Lu \text{ if } u \in D(A),$$

where $D(A) = H_0^1(U) \cap H^2(U)$.
Setting $X = L^2(U)$, if we are able to show that $A$ generates a $\gamma$-contraction semigroup $\{S(t)\}_{t \geq 0}$ on $X$, we can apply theorem (I.1) point (4):

$$\forall\, u \in D(A) : \quad \frac{d}{dt}S(t)u = AS(t)u \;\forall\, t > 0.$$

In particular, substituting $u(t) = S(t)u$, we obtain

$$\forall u \in D(A): \quad \frac{d}{dt}u(t) = -Lu(t) \; \forall \, t > 0 \text{ and } u(0) = u.$$

We can extend the result above by density, consider $\{u_k\}_{k\in\mathbb{N}} \subset D(A)$ with $u_k \to u$ in $X$. So,

$$\lim_{k\to\infty} \frac{d}{dt}u_k(t) = \lim_{k\to\infty} \frac{d}{dt}S(t)u_k = \lim_{k\to\infty} AS(t)u_k = AS(t)u.$$

In other words we have the following result:

$$\forall u \in X: \quad \frac{d}{dt}u(t) = -Lu(t) \;\; \forall \, t > 0 \text{ and } u(0) = u,$$

where $u(t) = S(t)u$ as usual.
In particular for every $u \in X$ as initial condition, we have found a solution of problem (I.23).
We only have to show that the operator $A$ generates a $\gamma$-contraction semigroup.

**Theorem I.5.** *The operator $A$ generates a $\gamma$-contraction semigroup on $X = L^2(U)$.*

*Proof.* We have to verify the hypothesis of the Hille-Yosida theorem.
$D(A)$ is obviously dense in $L^2(U)$.
We now prove that A is closed.
Let $\{u_k\}_{k\in\mathbb{N}} \subset D(A)$ with $u_k \to u$ in $X$ and $Au_k \to y$ in $X$, we want to show that $u \in D(A)$ and $y = Au$.
According to the regularity estimates (see for example [2] section 6.3.2) we have:

$$||u_k - u_l||_{H^2(U)} \le C(||Au_k - Au_l||_{L^2(U)} + ||u_k - u_l||_{L^2(U)})$$

Hence, $u \in D(A)$ since $u_k$ is a Cauchy sequence in $H^2(U)$ and this also implies $Au_k \to Au$ in $L^2(U)$.
We proceed to check the resolvent conditions.
First of all we show $(\gamma, \infty) \subset \rho(A)$.In particular:

$$\forall \lambda \ge \gamma \;\; (\lambda I - A) : D(A) \subset X \to X \text{ is a bijection.}$$

Notice that

$$(\lambda I - A)u = Lu + \lambda u \;\; \forall u \in D(A).$$

From the theory (see [2] section 6.2.2) it is well known that the boundary value problem:

$$\begin{cases} Lu + \lambda u = f \text{ in } U \\ u = 0 \text{ on } \partial U \end{cases} \tag{I.24}$$

admits a unique solution $u \in H_0^1(U)$ for each $f \in L^2(U)$. So in principle the operator $(\lambda I - A)^{-1}$ associates to each function $f \in X = L^2(U)$ the unique solution $u \in H_0^1(U)$ of (I.24). Thus, without knowing anything else, we wouldn't be able to say that the range of this map is $D(A)$. From regularity theory we know that actually $u \in H^2(U) \cap H_0^1(U) = D(A)$. Hence

$$\forall \ \lambda \geq \gamma \ (\lambda I - A) : D(A) \to X. \text{ is bijective.}$$

It remains to prove that

$$||R_\lambda|| \leq \frac{1}{\lambda - \gamma} \ \ \forall \lambda > \gamma.$$

Consider now the weak form of (I.24):

$$B[u, v] + \lambda(u, v) = (f, v) \ \ \forall v \in H_0^1(U).$$

Where $B[u, v] := \int_U \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + (\overrightarrow{b} \cdot \nabla u)v + cuv \, dx$.

Recalling the energy estimate (see [2] subsection 6.2.2)

$$\beta ||u||_{H_0^1(U)}^2 \leq B[u, u] + \gamma ||u||_{L^2(U)}^2,$$

we have

$$||f||_{L^2(u)} \, ||u||_{L^2(U)} \geq |(f, u)| \geq (f, u) = B[u, u] + \lambda(u, u) \geq \beta ||u||_{H_0^1(U)}^2 - \gamma ||u||_{L^2(U)}^2 + \lambda ||u||_{L^2(U)}^2.$$

This implies

$$||f||_{L^2(u)} \, ||u||_{L^2(U)} \geq \beta ||u||_{H_0^1(U)}^2 - \gamma ||u||_{L^2(U)}^2 + \lambda ||u||_{L^2(U)}^2 \geq -\gamma ||u||_{L^2(U)}^2 + \lambda ||u||_{L^2(U)}^2.$$

Hence

$$(\lambda - \gamma) ||u||_{L^2(U)}^2 \leq ||f||_{L^2(U)} ||u||_{L^2(U)}.$$

Since $u = R_\lambda f$ we obtain

$$||R_\lambda f||_{L^2(U)} \leq \frac{1}{\lambda - \gamma} ||f||_{L^2(U)} \ \ \forall \ \lambda > \gamma, \ \ \forall f \in L^2(U).$$

Finally
$$||R_\lambda|| \leq \frac{1}{\lambda - \gamma} \; \forall \lambda > \gamma.$$
Hence A generates a $\gamma$-contractive semigroup and the proof is complete. $\square$

# 1 | Local existence

In the following chapter we focus our attention on the problem

$$\begin{cases} u'(t) = Au(t) + J(u(t)) & (t > 0) \\ u(0) = \phi, \end{cases} \tag{1.1}$$

where $u : [0, T] \to E$ is a curve with values in the Banach space $E$, $A$ is the infinitesimal generator of a $C_0$ semigroup $e^{tA}$ on $E$ and $J$ is a nonlinear function from a subset of $E$ into $E$.

We will study the existence of solutions of problem (1.1) through the corresponding integral equation

$$u(t) = e^{tA}\phi + \int_0^t e^{(t-s)A} J(u(s))\, ds \tag{1.2}$$

In particular we will follow the work of Weissler, see [5].

## 1.1. Abstract existence theorem

To further investigate problem (1.2) we need some definitions.

**Definition 1.1.** *Given a Banach space $E$ with norm $|| \ ||$, a family of transformation $W_t$ with $t \geq 0$ is called a semiflow on $E$ with domains $D(W_t)$ if:*

1. $W_0\phi = \phi \ \ \forall \phi \in E$.
2. *If one between $W_{t+s}\phi$ or $W_t W_s \phi$ is defined, then also the other is defined and they coincide.*
3. *For every $\phi \in E$ the map $t \to W_t\phi$ is continuous into $E$.*
4. *Every $\phi \in E$ is in $D(W_t)$ for some $t > 0$.*

**Definition 1.2.** *Given a semiflow $W_t$ on the Banach space $E$, the curve $u(t) = W_t\phi$ is called the trajectory of $\phi$ and is defined on $[0, T_\phi)$ where $T_\phi = sup\{t > 0 : \phi \in D(W_t)\}$. $T_\phi$ is called the existence time of the trajectory.*

In this section we will ask particular properties to the nonlinear function $J$ in (1.1); in particular

1. $J : E_J \to E$ where $E_J$ is a Banach space with norm $|\ |_J$ and $E_J$ is dense in the banach space $E$, and also $E_J$ is continuously embedded into $E$.
2. $J$ is locally Lipschitz on bounded sets in $E_J$, in other words:

   $$||J\phi - J\psi|| \leq l(r)|\phi - \psi|_J \quad \forall \phi, \psi \text{ with } |\phi|_J \leq r \text{ and } |\psi|_J \leq r,$$

   where $l(r)$ denotes the Lipschitz constant restricted to

   $$B_r(E_J) = \{\phi \in E_J : |\phi|_J \leq r\}.$$

3. $J(0) = 0$

In the next theorem we want to prove one of the fundamental components to be taken into account is the growth of the Lipschitz constant $l(r)$ associated to the function $J$ as $r \to \infty$ (notice that $l(r)$ is a nondecreasing function in $r$).
Let $0 < a < 1$ be a fixed constant.
We give two different conditions which $l(r)$ can satisfy:

$$\int_\tau^\infty r^{-\frac{1}{a}} l(r)\, dr < \infty \text{ for some } \tau > 0 \tag{1.3}$$

$$l(r) = O(r^{\frac{(1-a)}{b}}) \text{ as } r \to \infty \text{ for some } 0 < b < a \tag{1.4}$$

Notice that condition (1.3) is stronger than (1.4). In fact

$$\int_r^{2r} s^{-\frac{1}{a}} l(s)\, ds \geq l(r) \int_r^{2r} s^{-\frac{1}{a}}\, ds \geq l(r)(2r)^{-\frac{1}{a}} \int_r^{2r} 1\, ds = r\, l(r)(2r)^{-\frac{1}{a}}$$

Now

$$r\, l(r)(2r)^{-\frac{1}{a}} \leq \int_r^{2r} s^{-\frac{1}{a}} l(s)\, ds \leq \int_r^{\infty} s^{-\frac{1}{a}} l(s)\, ds < \infty.$$

But this implies

$$l(r)\, r r^{-\frac{1}{a}} \text{ is bounded as } r \to \infty,$$

which means

$$l(r)\, r^{\frac{a-1}{a}} \text{ is bounded as } r \to \infty,$$

and this implies (1.4).
Moreover, we ask further properties to the $C_0$ semigroup $e^{tA}$ on $E$:

1. $||e^{tA}\phi|| \leq Me^{\gamma t}||\phi|| \ \ \forall\phi \in E$ and $t \geq 0$, with $\gamma \geq 0$,
2. $\forall t > 0$, $e^{tA}$ is a bounded map $E \to E_J$ and, for any $T > 0$ there exists $N > 0$:

$$|e^{tA}\phi|_J \leq Nt^{-a}||\phi||, \ t \in (0, T], \tag{1.5}$$

   where a is the same number used for the Lipschitz constant growth control.
3. $t \to e^{tA}\phi$ is continuous into $E_J$ for $t > 0$.

We need an additional lemma regarding a basic integral inequality.

**Lemma 1.1.** *Let $f, g : [a, b] \to \mathbb{R}$. Assume $f$ to be increasing and $g$ to be decreasing in the entire interval $[a, b]$. The following inequality holds*

$$\int_a^b f(x)g(x)\,dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx \int_a^b g(x)\,dx.$$

*Proof.*

$$[f(x) - f(y)][g(x) - g(y)] \leq 0 \text{ for all } x, y \in [a, b] \tag{1.6}$$

Let $\mathcal{R} := [a, b]^2$. Integrating (1.6) over $\mathcal{R}$ we get

$$0 \geq \iint_{\mathcal{R}} [f(x) - f(y)][g(x) - g(y)]\,dx\,dy =$$

$$= (b-a)\int_a^b f(x)g(x)\,dx + (b-a)\int_a^b f(y)g(y)\,dy - \iint_{\mathcal{R}} f(y)g(x)\,dx\,dy - \iint_{\mathcal{R}} f(x)g(y)\,dx\,dy =$$

$$= 2\left[(b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx\right].$$

Hence

$$(b-a)\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx \leq 0,$$

which is the thesis. □

We can now state the following theorem.

**Theorem 1.1.** *Let $E, J, E_J$ and $e^{tA}$ have the properties defined above.*
*a) Suppose that condition (1.3) holds. Then there exists a unique semi-flow $W_t$ on $E$ such that:*

1. *If we set* $u(t) = W_t\phi$ *then the curve* $u : (0, T_\phi) \to E_J$ *is continuous, moreover* $\limsup_{t\to 0^+} |t^a u(t)|_J < \infty$ *and:*

$$u(t) = e^{tA}\phi + \int_0^t e^{(t-s)A} J(u(s))\, ds \tag{1.7}$$

(1.7) *holds for the entire trajectory of* $\phi$. *The integral is both an* $E$ *and* $E_J$ *valued Bochner integral.*

2. *If* $v : (0, T] \to E_J$ *is strongly measurable with* $|t^a v(t)|_J$ *bounded and satisfies* (1.7) *then* $v(t)$ *coincides with the semiflow* $W_t$ *on* $(0, T]$.

3. *Given* $\alpha, \beta > 0$ *with* $\alpha M < \beta$ *and* $\alpha N < \beta$, *there exists* $T > 0$ *such that:* $B_\alpha(E) = \{\psi \in E : ||\psi|| \le \alpha\} \subset D(W_T)$ *and the maps* $W_t : B_\alpha(E) \to B_\beta(E)$ *and* $t^a W_t : B_\alpha(E) \to B_\beta(E_J)$ *are uniformly Lipschitz for* $t \in (0, T]$.

4. *If the existence time* $T_\phi$ *is finite then:* $||W_t\phi|| \to \infty$ *as* $t \to T_\phi$ *moreover we have* $\limsup_{t\to T_\phi -} |t^a u(t)|_J = \infty$.

*b)Suppose instead that* (1.4) *holds for some* $b \in (0, a)$. *It follows that for sufficiently small* $K > 0$, *if* $\phi$ *is in* $E$ *and* $\limsup_{t\to 0^+} |t^b e^{tA}\phi|_J < K$ *then there exists* $T > 0$ *and a curve* $u : [0, T] \to E$ *satisfying:*

6. $u : [0, T] \to E$ *is continuous and* $u(0) = \phi$.

7. $u : (0, T] \to E_J$ *is continuous and* $|t^b u(t)|_J \le 2K$.

8. $u(t)$ *satisfies* (1.7) *for* $t \in [0, T]$ *with the integral be both an* $E$ *and* $E_J$ *valued integral.*

9. *If* $v : (0, T1] \to E_J$ *is strongly measurable with* $T_1 \le T$ *and* $|t^b v(t)|_J \le 2K$ *and also satisfies* (1.7) *then* $v(t) = u(t)\ \forall\ t \in (0, T_1]$.

*Proof.* We will start proving part a).
We will prove existence of a solution to (1.7) through a contraction mapping argument on a specific Banach Space.
Fix $\phi \in E$ and let $X$ be the space of curves $u : [0, T] \to E$ satisfying:

I $u : [0, T] \to E$ is continuous and $u(0) = \phi$.

II $||u(t)|| \le \beta$ for all $t \in [0, T]$.

III $u : (0, T] \to E_J$ is continuous.

IV $|t^a u(t)|_J \le \beta$ for all $t \in (0, T]$

We equip $X$ with the metric $d(u,v) := \max[\sup_{[0,T]} ||u(t)-v(t)||, \sup_{(0,T]} |u(t)-v(t)|_J]$.
In fact, suppose $\{u_k\}_{k\in\mathbb{N}} \subset X$ is Cauchy in $X$, we want to show that $u_k \to u \in X$. Exploiting the definition of the metric $d$, we see that $u_k(t) \to u(t)$ in $E$ uniformly with respect to $t \in [0,T]$, moreover $u_k(t) \to u(t)$ in $E_J$ for $t \in (0,T]$ uniformly with respect to $t$. Hence $u$ which is the limit is continuous in both $E$ and $E_J$. The other properties are obvious.
Let $u \in X$. Define

$$(\mathcal{F}u)(t) = e^{tA}\phi + \int_0^t e^{(t-s)A} J(u(s))\, ds.$$

Notice that $\mathcal{F}u$ is well defined $\forall\ u \in X$ since
$u : (0,T] \to E_J$ is strongly measurable and $|t^a u(t)|_J \le \beta$ for $t \in (0,T]$ and this implies that the function $s \to e^{(t-s)A} J(u(s))$ is strongly measurable into both $E$ and $E_J$.

If we show that $\mathcal{F}u \in X$ for all $u \in X$ and that $\mathcal{F}u$ is a contraction. We have that there exists a unique fixed point $u \in X$ such that $\mathcal{F}u = u$ which is (1.7).
Observe that $\mathcal{F}u : [0,T] \to E$ is continuous. In fact
Let $0 \le t_2 \le t_1$. We have

$$||\mathcal{F}u(t_1) - \mathcal{F}u(t_2)|| = ||\int_0^{t_2} (e^{(t_1-s)A} - e^{(t2-s)A})(J(u(s))\, ds + \int_{t_2}^{t_1} e^{(t_1-s)A} J(u(s))\, ds|| \le$$

$$\le \int_0^{t_2} ||(e^{(t_1-s)A} - e^{(t2-s)A})(J(u(s))||\, ds + \int_{t_2}^{t_1} ||e^{(t_1-s)A} J(u(s))||\, ds.$$

We will see later in the proof that the term $||(e^{(t_1-s)A} - e^{(t2-s)A})(J(u(s))||$ is bounded hence for the dominated convergence theorem we can pass to the limit inside the integral.

$$\lim_{t_2\to t_1-} ||\mathcal{F}u(t_1) - \mathcal{F}u(t_2)|| \le \int_0^{t_2} \lim_{t_2\to t_1-} ||(e^{(t_1-s)A} - e^{(t2-s)A})(J(u(s))||\, ds + \lim_{t_2\to t_1-} C(t_1-t_2) = 0.$$

Hence $\mathcal{F}u : [0,T] \to E$ is continuous and condition I) for $\mathcal{F}u$ in order to belong to $X$ holds. Similar reasoning can be applied to show condition III) in the requisites for $\mathcal{F}u$ to belong to $X$.
It remains to prove properties II) and IV) for $\mathcal{F}u$.

$$||\mathcal{F}u(t)|| \le ||e^{tA}\phi|| + \int_0^t ||e^{(t-s)A} J(u(s))||\, ds$$

We focus on the second term:

$$\int_0^t ||e^{(t-s)A} J(u(s))||\, ds \le Me^{\gamma T} \int_0^t ||J(u(s))||\, ds \le Me^{\gamma T} \int_0^t l(|u(s)|_J)|u(s)|_J\, ds$$

We can now use the fact that $u \in X$, in particular: $|t^a u(t)|_J \leq \beta$ and also that $r \to l(r)$ is nondecreasing. Thus

$$Me^{\gamma T}\int_0^t l(|u(s)|_J)|u(s)|_J\,ds \leq Me^{\gamma T}\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds \leq$$

$$\leq Me^{\gamma T}\int_{+\infty}^{\beta T^{-a}} l(y)\,y\beta^{\frac{1}{a}}(-\frac{1}{a})y^{\frac{-a-1}{a}}\,dy = Me^{\gamma T}\int_{\beta T^{-a}}^{+\infty}(\frac{1}{a})\beta^{\frac{1}{a}}l(y)\,yy^{\frac{-a-1}{a}}\,dy =$$

$$= Me^{\gamma T}(\frac{1}{a})\beta^{\frac{1}{a}}\int_{\beta T^{-a}}^{+\infty} l(y)\,y^{-\frac{1}{a}}\,dy < \infty.$$

Hence

$$||\mathcal{F}u(t)|| \leq Me^{\gamma T}[\alpha + (\frac{1}{a})\beta^{\frac{1}{a}}\int_{\beta T^{-a}}^{+\infty} l(y)\,y^{-\frac{1}{a}}\,dy]$$

Choosing $T$ sufficiently small, $\int_{\beta T^{-a}}^{+\infty} l(y)\,y^{-\frac{1}{a}}\,dy$ and $Me^{\gamma T}$ can be as small as we want. Hence

$$||\mathcal{F}u(t)|| \leq \beta \;\; \forall u \in X, t \in [0,T].$$

We repeat the same process for $|t^a\mathcal{F}u(t)|_J$:

$$|t^a\mathcal{F}u(t)|_J \leq t^a|e^{tA}\phi|_J + t^a\int_0^t |e^{(t-s)A}J(u(s))|_J\,ds$$

We focus on the second term:

$$t^a\int_0^t |e^{(t-s)A}J(u(s))|_J\,ds \leq Nt^a\int_0^t (t-s)^{-a}||J(u(s))||\,ds \leq$$

$$\leq Nt^a\int_0^t (t-s)^{-a}l(\beta s^{-a})\beta s^{-a}\,ds \leq Nt^a t^{-1}\int_0^t (t-s)^{-a}\,ds\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds,$$

where in the last step we used Lemma 1.1 with $f = (t-s)^{-a}$ and $g = l(\beta s^{-a})\beta s^{-a}$. Now

$$Nt^a t^{-1}\int_0^t (t-s)^{-a}\,ds\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds = Nt^a t^{-1}\int_0^t t^{-a}(1-\frac{s}{t})^{-a}\,ds\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds =$$

$$= Nt^a t^{-1}\int_0^1 t^{-a}(1-y)^{-a}t\,dy\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds =$$

$$= N\int_0^1 (1-y)^{-a}\,dy\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds = N(1-a)^{-1}\int_0^t l(\beta s^{-a})\beta s^{-a}\,ds \leq$$

$$\leq N(1-a)^{-1}(\frac{1}{a})\beta^{\frac{1}{a}}\int_{\beta T^{-a}}^{+\infty} l(y)\,y^{-\frac{1}{a}}\,dy.$$

Hence
$$|t^a \mathcal{F}u(t)|_J \leq N\alpha + N(1-a)^{-1}(\frac{1}{a})\beta^{\frac{1}{a}} \int_{\beta T^{-a}}^{+\infty} l(y)\, y^{-\frac{1}{a}}\, dy$$
Choosing $T$ sufficiently small, we obtain
$$|t^a \mathcal{F}u(t)|_J \leq \beta \;\; \forall t \in (0,T].$$

In other words we proved that, given $u \in X$ then $\mathcal{F}u \in X$. So, taking $T$ sufficiently small $\mathcal{F} : X \to X$ is a strict contraction. In fact

$$\sup_{[0,T]} || \int_0^t e^{(t-s)A}(J(u(s)) - J(v(s)))\, ds|| \leq M\beta^{-1}e^{\gamma T} \int_0^T l(\beta s^{-a})\beta s^{-a}\, ds \;\; \sup_{(0,T]} s^a |u(s) - v(s)|_J,$$

while

$$\sup_{(0,T]} | \int_0^t e^{(t-s)A}(J(u(s)) - J(v(s)))\, ds|_J \leq N\beta^{-1}t^a \int_0^t (t-s)^{-a} l(\beta s^{-a})\beta s^{-a}\, ds \;\; \sup_{(0,T]} s^a |u(s) - v(s)|_J.$$

For the choice we made of $T$, both $Me^{\gamma T}\int_0^T l(\beta s^{-a})\beta s^{-a}\, ds$ and $Nt^a \int_0^t (t-s)^{-a} l(\beta s^{-a})\beta s^{-a}\, ds$ are strictly less than $\beta$.
Now for $u, v \in X$, we have

$$d(\mathcal{F}u, \mathcal{F}v) = \max[\sup_{[0,T]} ||\mathcal{F}u(t) - \mathcal{F}v(t)||, \sup_{(0,T]} |\mathcal{F}u(t) - \mathcal{F}v(t)|_J] \leq$$

$$\leq \frac{1}{\beta} \max[Me^{\gamma T} \int_0^T l(\beta s^{-a})\beta s^{-a}\, ds \;, NT^a \int_0^T (t-s)^{-a} l(\beta s^{-a})\beta s^{-a}\, ds\;] \sup_{(0,T]} s^a |u(s) - v(s)|_J <$$
$$< \sup_{(0,T]} s^a |u(s) - v(s)|_J \leq d(u,v).$$

Hence, $\mathcal{F}$ is a strict contraction. This implies that there exists a unique fixed point i.e. a solution of the integral equation (1.7).Point 1) of the theorem has been proved.
We shall prove point 2).
For any $\phi \in E$, let $W_t\phi$ be the maximal continuous curve $u(t)$ in $E$ such that for positive $t$ it is also continuous in $E_J$ with $\limsup_{t\to 0+} t^a |u(t)|_J < \infty$ satisfying the integral equation (1.7), then the $W_t$ form a semiflow on $E$.Notice that we know $W_t$ defined as above exists since for hypothesis $u = \mathcal{F}u$ and both $||\mathcal{F}u||$ and $t^a|\mathcal{F}u|_J$ are bounded by $\beta$ since $t^a|u(t)|_J$ is bounded, hence $u(t) = W_t\phi \in X$ and is the unique solution to (1.7).
We proceed in proving point 3).

Let $\phi$ and $\psi \in B_\alpha(E)$, let $t \in (0,T]$. We have

$$t^a|W_t\phi - W_t\psi|_J = t^a|e^{tA}(\phi-\psi) + \int_0^t e^{(t-s)A}(J(W_s\phi) - J(W_s\psi))\,ds|_J \leq$$

$$\leq N||\phi-\psi|| + t^a \int_0^t N(t-s)^{-a} l(\beta s^{-a})|W_s\phi - W_s\psi|_J\,ds =$$

$$= N||\phi-\psi|| + t^a \int_0^t N(t-s)^{-a} l(\beta s^{-a}) s^{-a} s^a |W_s\phi - W_s\psi|_J\,ds \leq$$

$$\leq N||\phi-\psi|| + t^a \int_0^t N(t-s)^{-a} l(\beta s^{-a}) s^{-a}\,ds \sup_{(0,T]} s^a|W_s\phi - W_s\psi|_J \leq$$

$$\leq N||\phi-\psi|| + (1-\delta)\sup_{(0,T]} s^a|W_s\phi - W_s\psi|_J.$$

Where the last inequality holds for some $\delta > 0$.
In particular we have

$$t^a|W_t\phi - W_t\psi|_J \leq N||\phi-\psi|| + (1-\delta)\sup_{(0,T]} s^a|W_s\phi - W_s\psi|_J.$$

Taking on both sides the supremum on $(0,T]$ and rearranging the terms we obtain

$$\sup_{(0,T]} t^a|W_t\phi - W_t\psi|_j \leq N\delta^{-1}||\phi-\psi||$$

In other words, since $t^a|W_t\phi|_J \leq \beta$ we have that the mapping

$$t^a W_t : B_\alpha(E) \to B_\beta(E_J)$$

is uniformly Lispchitz.
With an identical argument, we get

$$||W_t\phi - W_t\psi|| \leq Me^{\gamma T}(||\phi-\psi|| + \int_0^T l(\beta s^{-a}) s^{-a}\,ds \sup_{(0,T]} s^a|W_s\phi - W_s\psi|_J \leq$$

$$\leq Me^{\gamma T}||\phi-\psi|| + (1-\delta)\sup_{(0,T]} s^a|W_s\phi - W_s\psi|_J.$$

But we already know that $\sup_{(0,T]} s^a|W_s\phi - W_s\psi|_J \leq N\delta^{-1}||\phi-\psi||$.
Hence

$$\sup_{(0,T]} ||W_t\phi - W_t\psi|| \leq (Me^{\gamma T} + (1-\delta)N\delta^{-1})||\phi-\psi||$$

Point 3) has finally been proved. We can now prove point 4).

Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence such that $t_k \to T_\phi$ suppose now by contradiction $||W_{t_k}\phi|| \leq C \ \ \forall k \in \mathbb{N}$.
$W_{t_k}\phi \in E$, let $u_k(t)$ be the unique solution, $u_k(t) : [0,T] \to E_J$ to problem (1.7) with initial datum $W_{t_k}\phi$, notice that $T$ is independent of $k$ since $||W_{t_k}\phi|| \leq C \ \ \forall k \in \mathbb{N}$.
By uniqueness of $W_t\phi$ we have: $u_k(t) = W_{t+t_k}\phi$ for t sufficiently small.
Fix now $k$ such that $t_k \in (T_\phi - T, T_\phi)$, set:

$$\begin{cases} \widetilde{u}(t) = W_t\phi & t \in [0,t_k] \\ \widetilde{u}(t) = u_k(t-t_k) & t \in [t_k, t_k+T]. \end{cases} \tag{1.8}$$

Note that $\widetilde{u}(t)$ is a solution on $[0, t_k+T]$ and $t_k + T > T_\phi$ but this is absurd since $T_\phi$ is the maximal time for the existence of a solution.
Hence $||W_{t_k}\phi|| \to \infty$ as $t_k \to T_\phi$.
With a similar argument by contradiction we also obtain that $\limsup_{t\to T_\phi} |W_t\phi|_J = \infty$.
We can now pass to part b) of the theorem.
Choose $C$ such that $l(r) \leq Cr^{\frac{1-a}{b}}$.
Since

$$\limsup_{t\to 0^+} |t^b e^{tA}\phi|_J < K,$$

there exists $T > 0$ such that

$$|t^b e^{tA}\phi|_J \leq K \ \ \forall t \in (0,T].$$

Let now $Y$ be the space of curves $u : [0,T] \to E$ satisfying I and III above and also $|t^b u(t)|_J \leq 2K \ \ \forall t \in (0,T]$. We equip $Y$ with the same metric $d$ of $X$.(Y,d) is a complete metric space (you can see the proof for the metric space $X$).
Let $u \in Y$ define $\mathcal{F}u$ as above. We have

$$||\mathcal{F}u(t)|| \leq Me^{\gamma T}(||\phi|| + \int_0^t l(|u(s)|_J)|u(s)|_J \, ds) \leq Me^{\gamma T}(||\phi|| + \int_0^t l(2Ks^{-b})2Ks^{-b} \, ds)$$

We now further assume that $T^b \leq 2K$ in such a way $l(2Ks^{-b}) \geq 1 \ \ \forall s \in (0,T]$. So

$$||\mathcal{F}u(t)|| \leq Me^{\gamma T}(||\phi|| + \int_0^t C(2Ks^{-b})^{\frac{1-a}{b}} 2Ks^{-b} \, ds) = Me^{\gamma T}(||\phi|| + C(2K)^{\frac{1-a+b}{b}} \int_0^t s^{a-1-b} \, ds).$$

Moreover,

$$|t^b \mathcal{F}u(t)|_J \leq K + t^b \int_0^t N(t-s)^{-a} l(2Ks^{-b})2Ks^{-b} \, ds \leq$$

$$\leq t^b NC(2K)^{\frac{1-a+b}{b}} \int_0^t (t-s)^{-a} s^{a-1-b} \, ds = K + NBC(2K)^{\frac{1-a+b}{b}},$$

where $B = \int_0^1 (1-s)^{-a} s^{a-1-b}\, ds$
Notice that $\mathcal{F}u$ satisfies condition I and III. We now choose $K$ in such a way that

$$NBC(2)^{\frac{1-a+b}{b}} K^{\frac{1-a}{b}} \leq 1.$$

This ensures that $|t^b \mathcal{F}u(t)|_J \leq 2K$,hence $\mathcal{F}u \in Y$. Requiring additionally that

$$Me^{\gamma T} C(2K)^{\frac{1-a}{b}} \int_0^t s^{a-1-b}\, ds < 1,$$

ensures that $\mathcal{F} : Y \to Y$ is a strict contraction. This proves Theorem 1.1, since the technicalities regarding measurability and continuity are analogous at the ones in the first part of the proof.

□

At this point we make an additional assumption. We require that $e^{tA}$ restricts to a $C_0$ semigroup on $E_J$. Then the maps $e^{tA}J : E_J \to E_J$ satisfy the hypotheses of theorem 1 in [4].This implies that there is a semi-flow $V_t$ on $E_J$ whose trajectories $v(t) = V_t\phi$ satisfy the integral equation (1.7) in $E_J$; moreover, the following corollary is true.

**Corollary 1.1.** *Suppose that $e^{tA}$ restricts to a $C_0$ semigroup on $E_J$. Let $V_t$ be the semi-flow on $E_J$ described above.*
*a)If* (1.3) *holds in theorem* (1.1) *then $V_t$ is the restriction of $W_t$ on $E_J$. In particular if $\phi \in E_J$, the existence time of the trajectory of $\phi$ is the same in $E_J$ and in $E$, and if $\phi \in E$ with $T_\phi < \infty$, then both $||W_t\phi||$ and $|W_t\phi|_J$ approach $\infty$ as $t \to T_\phi$.*
*b)If* (1.4) *holds, then then the curves $u(t)$ satisfying 6-9 in theorem* (1.1) *extend $V_t$, in the sense that $u(t) = V_{t-s}u(s) \quad \forall\ 0 < s < t \leq T$ and also for $s = 0$ if $\phi \in E_J$. In particular $u(t)$ can be continuously extended in $E_J$ as a solution to* (1.7) *until $|u(t)|_J \to \infty$.This extension is also continuous in $E$.(We denote the extension of $u(t)$ by $W_t\phi$ and the existence time by $T_\phi$.Thus $W_t$ is a densely defined semi-flow on $E$ which extends $V_t$).*
*c)If $e^{tA}$ is an analytic semigroup on both $E$ and $E_J$ then in both cases above, the trajectory $u(t) = W_t\phi$ is continuously differentiable on $(0, T_\phi)$ in $E$ and satisfies:*

$$u'(t) = Au(t) + J(u(t)), \quad t \in (0, T_\phi)$$

*i.e. $u(t)$ is in the domain of $A$ in $E$ for $t \in (0, T_\phi)$.*

*Proof.* Suppose that (1.3) holds. then if $\phi \in E_J$ the curve $v(t) = V_t\phi$ satisfies condition 2) in theorem (1.1) on every closed subinterval of the trajectory. Thus $V_t\phi = W_t\phi$ throughout the $E_J$-trajectory of $\phi$. The only thing is that the $E_J$-trajectory could be smaller than the $E$-trajectory since $E_J \subset E$,but this cannot happen since we proved in theorem (1.1) that $u : (0, T_\phi) \to E_J$ is continuous, throughout the $E$-trajectory. Thus $T_\phi$ is the same in both $E$ and $E_J$.

The same reasoning holds for point b).

For point c) it is sufficient to prove the result with $\phi$ replaced by $\psi = W_\epsilon \phi$ for every $\epsilon > 0$. $W_{\epsilon/2}\phi \in E_J$ and by proposition 1.2 in [4] we have that $u(t) = W_t(W_\epsilon \phi)$ is Höelder continuous on $[0, T]$ into $E_J$ for $T < T_\psi$. This implies that $J(u(t))$ is Höelder continuous from $[0, T]$ into $E$. The result follows from page 491, theorem 1.27 of [1]. $\square$

## 1.2. A class of examples

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $e^{tA}$ is an analytic $C_0$ semigroup on all $L^p(\Omega)$ spaces for $1 < p < \infty$.
We will denote $D_p(A)$ the domain of its generator in $L^p(\Omega)$. Moreover, assume there exists a positive integer $m$ such that $\partial\Omega$ is of class $C^m$ and that for each p, $D_p(A)$ with its graph norm is continuously embedded in $W^{m,p}(\Omega)$.

**Proposition 1.1.** *Let $1 < p < q \leq \infty$. Let $r \in \mathbb{R}$ be such that $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then for $t > 0$, $e^{tA} : L^p \to L^q$ is a bounded map. Moreover, for any $T > 0$ there is a constant $N$ (depending on $p, q, \Omega$) such that:*

$$||e^{tA}\phi||_q \leq N t^{\frac{-n}{mr}} ||\phi||_p \text{ for all } t \in (0, T],$$

*If $A = \Delta$, then $p = 1$ is also allowed with $m = 2$.*

We don't see the proof of this proposition; we will prove it in a particular case later.

**Theorem 1.2.** *Let $\nu > 1$ and suppose for some $p$ with $1 \leq p < \infty$, $J : L^{p\nu} \to L^p$ is locally Lipschitz satisfying $l(r) = O(r^{\nu-1})$ as $r \to \infty$.*
*a)*
*Suppose $p > \frac{n(\nu-1)}{m}$ and $p > 1$ ($p \geq 1$ if $A = \Delta$).Then there exists a semi-flow $W_t$ on $L^p(\Omega)$ satisfying*

$$W_t\phi = e^{tA}\phi + \int_0^t e^{(t-s)A} J(W_s\phi)\, ds, \tag{1.9}$$

*and having all the properties described in theorem (1.1) part a) and corollary (1.1) parts a) and c) with $E = L^p$ and $E_J = L^{p\nu}$. In particular, if $\phi \in L^p$ then $W_t\phi \in L^{p\nu} \cap D_p(A)$ for $t > 0$; and if $T_\phi < \infty$ then $||W_t\phi||_p \to \infty$ as $t \to T_\phi$. Moreover $W_t$ restricts to a semi-flow on $L^{p\nu}$ and the existence time of the trajectory of $\phi \in L^{p\nu}$ is the same in $L^{p\nu}$ and in $L^p$. Furthermore, for all $\phi \in L^p$ the curve $u(t) = W_t\phi$ is continuously differentiable $(0, T_\phi) \to L^p$ and satisfies*

$$u'(t) = Au(t) + J(u(t)), \quad u(0) = \phi. \tag{1.10}$$

*b)*
*Suppose $\frac{n(\nu-1)}{m\nu} < p < \frac{n(\nu-1)}{m}$ and $p > 1$,($p \geq 1$ if $A = \Delta$).*
*Let $b = \frac{1}{\nu-1} - \frac{n}{mp\nu}$. There exists $K > 0$ such that if $\phi \in L^p$ and $\limsup_{t\to 0^+} ||t^b e^{tA}\phi||_{p\nu} <$*

*$K$, then there exists a continuous curve $u : [0,T] \to L^p$ satisfying*

$$u(t) = e^{tA}\phi + \int_0^t e^{(t-s)A} J(u(s))\, ds,$$

*and having all the properties of theorem (1.1) part b) and corollary (1.1) part b) and c) with $E = L^p$ and $E_J = L^{p\nu}$. In particular, the semi-flow $W_t$ on $L^{p\nu}$ satisfying (1.9) as described in theorem 4 of [4] extends to include such $\phi$ and $u(t) = W_t\phi$ for $t \in [0,T]$. Thus, if $T_\phi < \infty$, $||W_t\phi||_{p\nu} \to \infty$ as $t \to T_\phi$. For all such $\phi$ the extended curve $u(t) = W_t\phi$ is continuously differentiable $(0,T_\phi) \to L^p$ and satisfies (1.10). In addition, if such a $\phi$ is in $L^q$ with $p < q < p\nu$, then $u(t) = W_t\phi$ is continuous $(0,T_\phi] \to L^q$. Furthermore, the set of $\phi$ satisfying $\limsup_{t\to 0^+} ||t^b e^{tA}\phi||_p < K$ includes every $\phi \in L^{\frac{n(\nu-1)}{m}}$, and so $W_t$ extends to a continuous semi-flow on all of $L^{\frac{n(\nu-1)}{m}}$.*

*Proof.* If we set $E = L^p$ and $E_J = L^{p\nu}$, in particular we have, using the notation for the Banach spaces in the previous section:

$$|e^{tA}\phi|_J = ||e^{tA}\phi||_{p\nu} \le N t^{\frac{-n}{m}(\frac{1}{p}-\frac{1}{p\nu})} ||\phi||_p = N t^{\frac{-n(\nu-1)}{mp\nu}} ||\phi||_p$$

We have that (1.5) holds with a $= \frac{n(\nu-1)}{mp\nu}$.If $p > \frac{n(\nu-1)}{m}$ then (1.3) holds.
In fact let $\tau > 1$, choose $C > 0$ such that for all $r > 1, l(r) \le C r^{\nu-1}$. Then

$$\int_\tau^\infty r^{-\frac{1}{a}} l(r)\, dr \le \int_\tau^\infty r^{-\frac{1}{a}} C r^{\nu-1}\, dr.$$

Now we consider the exponent in the integral:

$$-\frac{1}{a} + \nu - 1 = \frac{-mp\nu + n(\nu-1)^2}{n(\nu-1)} < \frac{\nu(-n(\nu-1)) + n(\nu-1)^2}{n(\nu-1)} = -1,$$

where in the next to last step we used $p > \frac{n(\nu-1)}{m}$. In this way we can apply the Theorem 1.1 and its corollary 1.1. Therefore part a) of the theorem is proved.
For part b) of the theorem, we first show that $0 < b < a$. Indeed,
$b > 0$ since

$$b = \frac{1}{\nu-1} - \frac{n}{mp\nu} > \frac{1}{\nu-1} - \frac{1}{\nu-1} = 0,$$

because $\frac{n(\nu-1)}{m\nu} < p$ implies $-\frac{n}{pm\nu} > -\frac{1}{\nu-1}$. Moreover,
$b < a$, since

$$b = \frac{1}{\nu-1} - \frac{n}{mp\nu} < \frac{n}{mp} - \frac{n}{mp\nu} = a.$$

This follows from the fact that $p < \frac{n(\nu-1)}{m}$ which implies $\frac{1}{\nu-1} < \frac{n}{mp}$. Hence $0 < b < a$.

It remains to prove that $l(r) = O(r^{\frac{1-a}{b}})$. I indeed, note that

$$\frac{1-a}{b} = \nu - 1$$

And for hypothesis $l(r) = O(r^{\nu-1}) = O(r^{\frac{1-a}{b}})$.
To complete the proof, we first claim that if $q < p\nu$ and $\phi \in L^q$, then we have

$$\lim_{t \to 0^+} ||t^c e^{tA}\phi||_{p\nu} = 0$$

with $c = \frac{n}{mq} - \frac{n}{mp\nu}$. By Proposition 1.1, the maps $t^c e^{tA} : L^q \to L^{p\nu}$ are uniformly bounded for $t \in (0, T]$. In fact,

$$||t^c e^{tA}\phi||_{p\nu} \leq N t^c t^{-\frac{n}{mq} + \frac{n}{mp\nu}} ||\phi||_q = N||\phi||_q.$$

We also have:

$$\lim_{t \to 0^+} ||t^c e^{tA}\phi||_q = \lim_{t \to 0^+} ||t^c \phi||_q = 0,$$

and converge strongly to 0 on the dense subset $L^{p\nu}$. This by the way shows that for every $\phi \in L^{\frac{n(\nu-1)}{m}}$:

$$\limsup_{t \to 0^+} ||t^b e^{tA}\phi||_{p\nu} < K.$$

In fact,

$$||t^b e^{tA}\phi||_{p\nu} \leq N t^b t^{-\frac{n}{m}(\frac{1}{q} - \frac{1}{p\nu})} ||\phi||_q.$$

Choosing $q = \frac{n(\nu-1)}{m}$, we obtain

$$||t^b e^{tA}\phi||_{p\nu} \leq N||\phi||_q.$$

Hence, there exists $K > 0$ such that

$$\limsup_{t \to 0^+} ||t^b e^{tA}\phi||_{p\nu} < K,$$

for all $\phi \in L^{\frac{n(\nu-1)}{m}}$.
Let now $\phi \in L^q$ with $p < q < p\nu$ be such that $\limsup_{t \to 0^+} ||t^b e^{tA}\phi||_{p\nu} < K$.
Let $u(t)$ be the solution to (1.7) described in part b) of theorem (1.1).
We claim that for small $T$ we have

$$\sup_{(0,T]} ||t^b u(t)||_{p\nu} \leq 2 \sup_{(0,T]} ||t^b e^{tA}\phi||_{p\nu}. \tag{1.11}$$

In fact

$$||t^b u(t)||_{p\nu} \le ||t^b e^{tA}\phi||_{p\nu} + t^b \int_0^t ||e^{(t-s)A} J(u(s))||_{p\nu}\, ds \le$$

$$\le ||t^b e^{tA}\phi||_{p\nu} + N t^b (2K)^{\frac{1-a}{b}} \int_0^t (t-s)^{-a} s^{a-1} s^{-b} s^b ||u(s)||_{p\nu}\, ds \le$$

$$\le ||t^b e^{tA}\phi||_{p\nu} + N(2K)^{\frac{1-a}{b}} CB \sup_{(0,T]} ||s^b u(s)||_{p\nu}.$$

Now, if we choose $K$ such that $NBC2^{\frac{1-a+b}{b}} K^{\frac{1-a}{b}} \le 1$, we have

$$||t^b u(t)||_{p\nu} \le ||t^b e^{tA}\phi||_{p\nu} + \frac{1}{2} \sup_{(0,T]} ||s^b u(s)||_{p\nu}.$$

Taking the supremum on both sides we get (1.11).
We already know from Theorem 1.1 that $u(t)$ is continuous into $L^{p\nu}$ for $t > 0$.
This implies that for $t > 0$, $u(t)$ is continuous into $L^q$, since

$$||u(t) - u(s)||_q \le D||u(t) - u(s)||_{p\nu} \;\; \forall \;\; t, s \in [0, T]$$

It remains to prove that $u(t)$ is continuous in $L^q$ up to $t = 0$:
We only need to show that

$$\lim_{t \to 0^+} \int_0^t e^{(t-s)A} J(u(s))\, ds = 0 \text{ in } L^q$$

$$|| \int_0^t e^{(t-s)A} J(u(s))\, ds - 0||_q \le \int_0^t ||e^{(t-s)A} J(u(s))||_q\, ds \le$$

$$\le N \int_0^t (t-s)^{\frac{-n}{mr}} l(||u(s)||_{p\nu}) ||u(s)||_{p\nu}\, ds,$$

where in the last step we applied the estimate of Proposition 1.1. Now:

$$N \int_0^t (t-s)^{\frac{-n}{mr}} l(||u(s)||_{p\nu}) ||u(s)||_{p\nu}\, ds \le NC(2K)^{\frac{1-a}{b}} \int_0^t (t-s)^{\frac{-n}{mr}} s^{-b\frac{1-a}{b}} s^{-b} s^b ||u(s)||_{p\nu}\, ds \le$$

$$\le NC(2K)^{\frac{1-a}{b}} \int_0^t (t-s)^{\frac{-n}{mr}} s^{-b\frac{1-a}{b}} s^{-b}\, ds \sup_{(0,t]} ||s^b u(s)||_{p\nu} =$$

$$= NC(2K)^{\frac{1-a}{b}} \int_0^t (t-s)^{\frac{-n}{mr}} s^{-b(\nu-1)} s^{-b}\, ds \sup_{(0,t]} ||s^b u(s)||_{p\nu} =$$

$$= NC(2K)^{\frac{1-a}{b}} \int_0^1 (1-y)^{\frac{-n}{mr}} y^{-n\nu}\, dy\, t^{\frac{-n}{mr} - b\nu + 1} \sup_{(0,t]} ||s^b u(s)||_{p\nu} =$$

$$= Dt^{\frac{-n}{mr}-b\nu+1} \sup_{(0,t]} ||s^b u(s)||_{p\nu},$$

where we put $D = NC(2K)^{\frac{1-a}{b}} \int_0^1 (1-y)^{\frac{-n}{mr}} y^{-n\nu}\, dy$.
Summing up

$$|| \int_0^t e^{(t-s)A} J(u(s))\, ds - 0||_q \leq Dt^{\frac{-n}{mr}-b\nu+1} \sup_{(0,t]} ||s^b u(s)||_{p\nu} \tag{1.12}$$

Now if $q < \frac{n(\nu-1)}{m}$, then $\frac{n}{mr} + b\nu < 1$. In fact

$$\frac{n}{mr} + b\nu = \frac{-n}{mq} + \frac{\nu}{\nu-1} < 1,$$

where last step holds since $q < \frac{n(\nu-1)}{m}$, which implies $\frac{-n}{mq} < \frac{-1}{\nu-1}$.
Hence, whenever $q < \frac{n(\nu-1)}{m}$, letting $t \to 0^+$ in (1.12) and noticing that for small $t$ $\sup_{(0,t]} ||s^b u(s)||_{p\nu}$ is bounded we obtain the continuity.
If $q \geq \frac{n(\nu-1)}{m}$ we can use the following argument:

$$|| \int_0^t e^{(t-s)A} J(u(s))\, ds - 0||_q \leq Dt^{\frac{-n}{mr}-b\nu+1} \sup_{(0,t]} ||s^b u(s)||_{p\nu} \leq Dt^{\frac{-n}{mr}-b\nu+1} 2 \sup_{(0,t]} ||s^b e^{tA}\phi||_{p\nu} =$$

$$= Dt^{\frac{-n}{mr}-b\nu+1} 2 \sup_{(0,t]} ||s^{b-c} s^c e^{tA}\phi||_{p\nu} \leq Dt^{\frac{-n}{mr}-b\nu+1} t^{b-c} 2 \sup_{(0,t]} ||s^c e^{tA}\phi||_{p\nu} = D2 \sup_{(0,t]} ||s^c e^{tA}\phi||_{p\nu}.$$

Summing up

$$|| \int_0^t e^{(t-s)A} J(u(s))\, ds - 0||_q \leq D2 \sup_{(0,t]} ||s^c e^{tA}\phi||_{p\nu}.$$

Taking the limit for t approaching 0 we have the continuity in $L^q$ at $t = 0$. □

**Corollary 1.2.** *Suppose $e^{tA}$ is positivity preserving and that $J$ takes non-negative functions into non-negative functions.Then the semi-flows constructed in theorem (1.2) are positivity preserving.*

*Proof.* Let $\phi \geq 0$ a.e. in $\Omega$, if we require that the curves in the spaces $X$ and $Y$ defined in theorem (1.1) are positive for a.a $t \in [0, T]$ this implies that $W_t\phi = u(t) = e^{tA}\phi + \int_0^t e^{(t-s)A} J(u(s))\, ds \geq 0$.

□

**Corollary 1.3.** *If for every $p$, $1 \leq p < \infty$, $J : L^{p\nu} \to L^p$ is locally Lipschitz with $l(r) = O(r^{\nu-1})$ as $r \to \infty$ then the semi-flow $W_t$ exists on all $L^p$ spaces with $\frac{n(\nu-1)}{m} < p < \infty$*

*and* ($p \geq 1$ *if* $A = \Delta$) *as well as* $p = \frac{n(\nu-1)}{m}$ *if* $\frac{n(\nu-1)}{m} > 1$. *(This happens in particular if* $J$ *is a polinomial of degree* $\nu$*).*
*Furthermore, for all* $\phi$ *in* $L^1(\Omega)$ *for which existence of a solution* $u(t) = W_t\phi$ *to the integral equation* (1.9) *has been shown above, this solution is continuously differentiable in in all* $L^p$, $1 < p < \infty$ *for* $t > 0$ *and satisfies* (1.10) *throughout the entire trajectory. The existence time of the curve is the same in all* $L^p$, *and if* $T_\phi < \infty$ *and* $p > \frac{n(\nu-1)}{m}$ *then* $||u(t)||_p \to \infty$ *as* $t \to T_\phi$. *Lastly,* $u(t)$ *is continuous in* $L^q$ *at* $t = 0$ *whenever* $\phi \in L^q$, $1 \leq q < \infty$.

# 2 | Local existence for a particular semilinear evolution equation

We will now shift our focus to one particular case where in (1.1) $A = \Delta$ and $J(u(t)) = |u(t)|^{p-1}u(t)$ for some $p > 1$.
We will follow [3] chapter II.

$$\begin{cases} u'(t) = \Delta u(t) + |u(t)|^{p-1}u(t) & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, not necessarily bounded.
For a given domain $\Omega$ we introduce:
$Q_T := \Omega \times (0,T)$,
$S_T := \partial\Omega \times (0,T)$ (lateral boundary),
$P_T := S_T \cup (\overline{\Omega} \times \{0\})$ (parabolic boundary).
We are interested in finding solutions (in some precise sense defined below) with the initial datum $u_0$ belonging to $L^q$ with $1 \leq q < \infty$.

**Definition 2.1.** *Given a Banach space $X$ of functions defined on $\Omega$, $u_0 \in X$ and $T \in (0,\infty]$, we say that $u \in C((0,T],X)$ is a classical $X$-solution of* (2.1) *in* $[0,T)$ *if $u \in C^{2,1}(\Omega \times (0,T)) \cap C(\overline{\Omega} \times \{t = 0\})$, $u(0) = u_0$ and $u$ is a classical solution of* (2.1) *in* $(0,T)$*. If $\Omega$ is unbounded we also require $u \in L^\infty_{loc}((0,T), L^\infty(\Omega))$. If $X = L^\infty(\Omega)$ instead of requiring $u \in C((0,T],X)$ we require $u \in C((0,T),X)$ and $||u(t) - e^{tA}u_0||_\infty \to 0$ when $t \to 0$, where $e^{tA}$ is the heat semigroup.*

**Definition 2.2.** *We say that problem* (2.1) *is well-posed in $X$ if given $u_0 \in X$, there exist $T > 0$ and a unique classical $X$-solution of* (2.1) *in* $[0,T]$.

We can now give other definitions of solution:

Rewriting (2.1) with a general $J(u(t))$ we obtain:

$$\begin{cases} u'(t) = \Delta u(t) + J(u(t)) & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{2.2}$$

**Definition 2.3.** *Any function $u \in C((0,T], L^q(\Omega))$ with $J(u) \in L^1_{loc}((0,T), L^1(\Omega) + L^\infty(\Omega))$ and $u(0) = u_0$ and such that:*

$$u(t) = e^{(t-\tau)A}u(\tau) + \int_\tau^t e^{(t-s)A} J(u(s))\, ds \text{ for all } 0 < \tau < t < T$$

*is called a mild $L^q$-solution of* (2.2).

**Definition 2.4.** *Consider problem* (2.2) *with $J$ nonnegative and $u_0 \geq 0$.*
*We say that $u$ is an integral solution of* (2.2) *in $(0,T]$ if $u : \Omega \times [0,T] \to [0,\infty]$ is measurable finite a.e. and:*

$$u(x,t) = \int_\Omega G(x,y,t)u_0(y)\, dy + \int_0^t \int_\Omega G(x,y,t-s)J(u(y,s))\, dy\, ds$$

*For a.e. $(x,t) \in Q_T$, where $G$ is the heat kernel in $\Omega$.*

**Definition 2.5.** *Assume $\Omega$ is bounded and $u_0 \in L^1_\delta(\Omega)$. A function $u \in C([0,T), L^1_\delta(\Omega))$ is called a weak solution of* (2.2) *in $[0,T)$ if the functions $u$,$\delta J(u)$ belong to $L^1_{loc}((0,T), L^1(\Omega))$, $u(0) = u_0$ and:*

$$\int_\tau^t \int_\Omega J(u)\phi = -\int_\tau^t \int_\Omega u(\phi_t + \Delta\phi) - \int_\Omega u(\tau)\phi(\tau)$$

*for any $0 < \tau < t < T$, for any $\phi \in C^2(\overline{\Omega} \times [\tau,t])$ such that $\phi = 0$ on $\partial\Omega \times [\tau,t]$ and $\phi(t) = 0$.*

Since we are now considering the case where we have the Laplace operator, we can shift our focus on some estimates on the heat semigroup, this estimates will be fundamental in the proof of existence of classical $L^q$-solutions of problem (2.1).

**Proposition 2.1.** *Let $e^{tA}$ be the heat semigroup in $\mathbb{R}^n$ and $G_t(x) = G(x,t)$ the Gaussian heat kernel. We have the following:*

1. $||G_t||_1 = 1$ *for all* $t > 0$.

2. *If* $\phi \geq 0$, *then* $e^{tA}\phi \geq 0$ *and* $||e^{tA}\phi||_1 = ||\phi||_1$.
3. *If* $1 \leq q \leq \infty, then ||e^{tA}\phi||_q \leq ||\phi||_q$ *for all* $t > 0$.
4. *If* $1 \leq p < q \leq \infty$ *and* $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$, *then* $||e^{tA}\phi||_q \leq (4\pi t)^{-n/(2r)}||\phi||_p$ *for all* $t > 0$.
5. *For an arbitrary domain* $\Omega \subset \mathbb{R}^n$ *points 3) and 4) remain valid if we replace* $e^{tA}$ *with the dirichlet heat semigroup in* $\Omega$ .

*Proof.* 1) We know that $G_t(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$. Since

$$\int_{\mathbb{R}^n} e^{-a|x|^2}\,dx = (\pi/a)^{n/2},$$

we have

$$||G_t||_1 = (4\pi t)^{-n/2}(\pi/(\frac{1}{4t}))^{n/2} = 1.$$

2)

$$||e^{tA}\phi||_1 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_t(x-y)\phi(y)\,dy\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_t(x-y)\,dx\,\phi(y)\,dy =$$

$$= \int_{\mathbb{R}^n} ||G_t||_1\phi(y)\,dy = |||\phi||_1.$$

3)
Since $e^{tA}\phi = G_t * \phi$ we can use Young inequality for convolutions:

$$||e^{tA}\phi||_q = ||G_t * \phi||_q \leq ||G_t||_1||\phi||_q = ||\phi||_q.$$

4)
Now $e^{tA}\phi = G_t * \phi$. We can use Young inequality for convolutions:

$$||e^{tA}\phi||_q = ||G_t * \phi||_q \leq ||G_t||_m||\phi||_p, \tag{2.3}$$

where $1 + \frac{1}{q} = \frac{1}{m} + \frac{1}{p}$,
But:

$$||G_t||_m = (4\pi t)^{-n/2}\left(\int_{\mathbb{R}^n} e^{\frac{-m|x|^2}{4t}}\,dx\right)^{1/m} =$$

$$= (4\pi t)^{-n/2}\left(\frac{4\pi t}{m}\right)^{\frac{n}{2m}} \leq (4\pi)^{-\frac{n}{2r}}t^{-\frac{n}{2}(1-\frac{1}{m})} = (4\pi)^{-\frac{n}{2r}}t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}.$$

Substituting in (2.3), we obtain:

$$||e^{tA}\phi||_q \leq (4\pi t)^{-\frac{n}{2r}}||\phi||_p.$$

5)
Denote by $e^{tA_\Omega}$ the Dirichlet heat semigroup in $\Omega$. Let $\widetilde{\phi}(x) = \phi(x)$ if $x \in \Omega$, $\widetilde{\phi}(x) = 0$ otherwise. We have:

$$|e^{tA_\Omega}\phi| \leq e^{tA_\Omega}|\phi| \leq e^{tA}|\widetilde{\phi}|$$

Now points 3) and 4) easily follow from the fact that $||\widetilde{\phi}||_{L^p(\mathbb{R}^n)} = ||\phi||_{L^p(\Omega)}$. □

In the next theorem we will see that in order to guarantee the well-posedness of (2.1) in $L^q$ the exponent:

$$q_c = \frac{n(p-1)}{2} \tag{2.4}$$

will play a crucial role.

**Theorem 2.1.** *Let $p > 1$, $u_0 \in L^q(\Omega)$, $1 \leq q < \infty$, $q > q_c$ ($q_c$ defined in (2.4)). Then there exists $T = T(||u_0||_q) > 0$ such that problem (2.1) possesses a unique classical $L^q$-solution in $[0, T)$ and the following estimate holds:*

$$||u(t)||_r \leq C||u_0||_q t^{-\alpha_r}, \quad \alpha_r = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right) \tag{2.5}$$

*for all $t \in (0, T)$ and $r \in [q, \infty]$, with $C = C(n, p, q) > 0$. Moreover $u \geq 0$ whenever $u_0 \geq 0$.*

*Proof.* ***Step 1. Fixed point argument.***
Let $T > 0$ be small and introduce the Banach space

$$Y_T = \{u \in L^\infty_{loc}((0, T), L^{pq}(\Omega)) : ||u||_{Y_T} < \infty\},$$

with

$$||u||_{Y_T} = \sup_{0<t<T} t^\alpha ||u(t)||_{pq},$$

where $\alpha = \frac{n(p-1)}{2pq}$. Since for our hypotheses $q > q_c$ we easily have that $\alpha < \frac{1}{p} < 1$.
Choose $M > ||u_0||_q$ and let

$$B_M = B_{M,T} = \{u \in Y_T : ||u||_{Y_T} \leq M\}.$$

We consider in the same way as Theorem 1.1 the mapping:

$$\Phi_{u_0}(u)(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}|u(s)|^{p-1}u(s)\, ds$$

Exactly in the same way as before we show that this mapping is a contraction in $Y_T$, therefore has a unique fixed point.
In the next steps we will use the estimates in Proposition 2.1 point 4.
For sake of clarity we report here the estimates which are repeatedly used.
For all $1 \leq p < q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ we have:

$$||e^{tA}\phi||_q \leq (4\pi t)^{-\frac{n}{2r}}||\phi||_p \quad \text{for all } t > 0. \tag{2.6}$$

Now

$$t^\alpha||\Phi_{u_0}(u)(t) - \Phi_{v_0}(v)(t)||_{pq} \leq$$
$$\leq (4\pi)^{-\alpha}||u_0 - v_0||_q + t^\alpha \int_0^t (4\pi(t-s))^{-\alpha}||(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s))||_q \, ds,$$

where we have used (2.6) with $q$ substituted with $pq$ and $p$ substituted with $q$.
Let us focus momentairily on the term

$$||(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s))||_q^q = \int_\Omega ||u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)|^q \, d\Omega \leq$$
$$\leq \int_\Omega (\, p|u(s) - v(s)|\, (|u(s)|^{p-1} + |v(s)|^{p-1})\, )^q \, d\Omega \leq$$
$$\leq p^q||u(s) - v(s)||_{pq}^q||(|u(s)|^{p-1} + |v(s)|^{p-1})^q||_{\frac{p}{p-1}} =$$

$$= p^q||u(s) - v(s)||_{pq}^q(\int_\Omega ((|u(s)|^{p-1} + |v(s)|^{p-1})^q)^{\frac{p}{p-1}} \, d\Omega)^{\frac{p-1}{p}} \leq$$
$$\leq p^q||u(s) - v(s)||_{pq}^q(\int_\Omega (|u(s)|^{p-1} + |v(s)|^{p-1})^{\frac{qp}{p-1}} \, d\Omega)^{\frac{p-1}{p}} =$$
$$= p^q||u(s) - v(s)||_{pq}^q||\, |u(s)|^{p-1} + |v(s)|^{p-1}\, ||_{\frac{qp}{p-1}}^q \leq$$
$$\leq p^q||u(s) - v(s)||_{pq}^q \left( ||u(s)^{p-1}||_{\frac{qp}{p-1}} + ||v(s)^{p-1}||_{\frac{qp}{p-1}} \right)^q =$$
$$= p^q||u(s) - v(s)||_{pq}^q \left( ||u(s)||_{qp}^{p-1} + ||v(s)||_{qp}^{p-1} \right)^q$$

Summing up

$$||(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s))||_q^q \leq p^q||u(s) - v(s)||_{pq}^q \left( ||u(s)||_{qp}^{p-1} + ||v(s)||_{qp}^{p-1} \right)^q$$

This implies

$$||(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s))||_q \leq p||u(s) - v(s)||_{pq} \left( ||u(s)||_{qp}^{p-1} + ||v(s)||_{qp}^{p-1} \right)$$

After some computations we obtain

$$t^{\alpha}||\Phi_{u_0}(u)(t)-\Phi_{v_0}(v)(t)||_{pq}\leq$$

$$\leq(4\pi)^{-\alpha}||u_0-v_0||_q+C(p)M^{p-1}t^{\alpha}\int_0^t(t-s)^{-\alpha}s^{-(p-1)\alpha}||u(s)-v(s)||_{pq}\,ds \tag{2.7}$$

Choosing $v_0=0$ and $v=0$ we can estimate

$$||\Phi_{u_0}(u)||_{Y_T}\leq(4\pi)^{-\alpha}||u_0||_q+C(p,\alpha)M^{p-1}T^{1-p\alpha}||u||_{Y_T}.$$

Choose $T_0$ such that $C(p,\alpha)M^{p-1}T_0^{1-p\alpha}<\min(1-(4\pi)^{-\alpha},\frac{1}{2})$, then

$$||\Phi_{u_0}(u)||_{Y_T}<M \text{ for all } t<T_0.$$

If we choose in (2.7) $u_0=v_0$ and $C(p)M^{p-1}T_0^{1-\alpha}<\frac{1}{2}$ we obtain

$$||\Phi_{u_0}(u)-\Phi_{v_0}(v)||_{Y_T}\leq\frac{1}{2}||u-v||_{Y_T}$$

Hence $\Phi_{u_0}$ is a strict contraction and has a unique fixed point belonging to $B_M$ let us call this fixed point $u$.
Note, for any $T\leq T_0$,
$u$ is the unique solution of $\Phi_{u_0}(u)=u$ such that $u\in Y_T$. In fact
given $u_1,u_2$ solutions they both belong to $B_{M,T_0'}$ for some large $M$ and small $T_0'$ hence they coincide for small time, let's say for all $t<t_1$. Let now $E$ be the set of points of $(t1,T)$ such that $u_1\neq u_2$.
By contradiction let $E$ be non-empty, let $t_0$ be its inferior limit. We have $t_1\leq t_0<T$ and that $u_1(t_0)=u_2(t_0)$ since the solutions are continuous and for $t<t_0$ they coincide.
The problem $\Phi_{u_1(t_0)}(w)=\Phi_{u_2(t_0)}(w)=w$ has a unique solution, but $u_1$ and $u_2$ are solutions in all of $(0,T)$ and they satisfy $\Phi_{u_i(t_0)}(u_i)=u_i$, $i=1,2$. Hence they coincide near $t_0$ from the right, but this contradicts the fact that $t_0$ is the infimum of $E$.
***Step 2. Regularity.***
We can now study the regularity of the solution. We have that
$|u(t)|^{p-1}u(t)\in L^1((0,T),L^q(\Omega))$. In fact

$$\int_0^t|||u(t)|^{p-1}u(t)||_q\,dt=\int_0^t||u(t)||_{pq}^p\,dt\leq\int_0^t M^pt^{-\alpha p}\,dt<\infty.$$

Now let $t \geq s$:

$$||\Phi_{u_0}(u)(t)-\Phi_{u_0}(u)(s)||_q \leq \int_0^s ||(e^{(t-y)}-e^{(s-y)})|u(y)|^{p-1}u(y)||_q\, dy+\int_s^t ||e^{(t-y)}|u(y)|^{p-1}u(y)||_q\, dy.$$

Now

$||(e^{(t-y)}-e^{(s-y)})|u(y)|^{p-1}u(y)||_q$ is bounded. I can pass under the integral sign as t goes to s.

The same holds for $||e^{(t-y)}|u(y)|^{p-1}u(y)||_q$. Hence we can pass to the limit as $t \to s$, proving continuity. In other words $u \in C([0,T], L^q(\Omega))$.
Choose $\epsilon > 0$ small and let $\gamma_1 = pq$. Obviously, $u \in L^\infty([\epsilon,T], L^{\gamma_1}(\Omega))$ since $u \in L^\infty_{loc}((0,T), L^{pq}(\Omega))$. We also have

$$u(t+\epsilon) = e^{tA}u(\epsilon) + \int_0^t e^{(t-s)A}|u(s+\epsilon)|^{p-1}u(s+\epsilon)\, ds.$$

Choose $\gamma_2 > \gamma_1$ such that $\beta_1 = \frac{n}{2}(\frac{p}{\gamma_1} - \frac{1}{\gamma 2}) < 1$ and set $\beta_2 = \frac{n}{2}(\frac{1}{\gamma_1} - \frac{1}{\gamma_2})$. We have, for all $t \in [\epsilon, T-\epsilon]$,

$$||u(t+\epsilon)||_{\gamma_2} \leq t^{-\beta_2}||u(\epsilon)||_{\gamma_1} + \int_0^t (t-s)^{-\beta_1}||u(s+\epsilon)||^p_{\gamma_1}\, ds \leq$$

$$\leq \epsilon^{-\beta_2}M\epsilon^{-\alpha} + \int_0^t (t-s)^{-\beta_1}(s+\epsilon)^{-\alpha p}\, dsM^p \leq$$

$$\leq \epsilon^{-\beta_2}M\epsilon^{-\alpha} + t^{-\beta_1+1}\int_0^1 (1-y)^{-\beta_1}(\epsilon)^{-\alpha p}\, dyM^p \leq C(\epsilon).$$

Hence $u \in L^\infty([2\epsilon,T], L^{\gamma_2}(\Omega))$. A bootstrap argument shows that $u \in L^\infty_{loc}((0,T], L^\infty(\Omega))$. Now standard existence and regularity results for linear parabolic equations imply that u is a classical $L^q$-solution.
Let us explain in the case where we are in bounded domains.
Fix $\delta > 0$ small and let $\psi : \mathbb{R} \to [0,1]$ be a smooth function satisfying $\psi(t) = 0$ for $t \leq \delta$ and $\psi(t) = 1$ for $t \geq 2\delta$. Since $u$ is a mild solution it is also a weak solution (see corollary 48.11 of [3]). Hence $\psi u$ is a weak solution of the following linear problem:

$$\begin{cases} u_t - \Delta u = f & x \in \Omega, t > 0 \\ u = 0 & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x) & x \in \Omega \end{cases} \tag{2.8}$$

With $f = \psi_t u + \psi|u|^{p-1}u \in L^\infty(Q)$ where $Q = Q_T$. Theorem 48.1 (iii) of [3] guarantees that this problem has a strong solution $v \in W^{2,1;q}(Q)$ for any $q \in (1,\infty)$. This strong solution is also a weak solution, and uniqueness of weak solutions implies $\psi u = v$. This means that also $u \in W^{2,1;q}(\Omega \times (2\delta, T))$. Fixing $q > n+2$ we see that $f(u)$ is Hölder continuous in $\Omega \times (2\delta, T)$. Now we use theorem 48.2 (ii) considering the function $\psi(t - 2\delta)u(t)$ of [3] to obtain that u is a classical solution for $t > 4\delta$.

***Step 3. Continuous dependence.***

Let us denote $U(t)u_0$ the solution constructed above. We know that $U(\cdot)u_0$ is defined and belongs to $B_{M,T}$ for $||u_0||_q < M$ and $T \le T_0$. Moreover starting from (2.7):

$$||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le ||u_0 - v_0||_q + \sup_{0<t<T} C(p)M^{p-1}t^{\alpha} \int_0^t (t-s)^{-\alpha} s^{-(p-1)\alpha} ||u(s) - v(s)||_{pq}\, ds \le$$

$$\le ||u_0 - v_0||_q + \sup_{0<t<T} C(p)M^{p-1}t^{\alpha} \int_0^t (t-s)^{-\alpha} s^{-p\alpha}\, ds ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le$$

$$\le ||u_0 - v_0||_q + \sup_{0<t<T} C(p)M^{p-1}t^{1-p\alpha} \int_0^1 (1-y)^{-\alpha} y^{-p\alpha}\, dy ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le$$

$$\le ||u_0 - v_0||_q + \sup_{0<t<T} C(p,\alpha)M^{p-1}t^{1-p\alpha} ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} =$$

$$= ||u_0 - v_0||_q + C(p,\alpha)M^{p-1}T^{1-p\alpha} ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T}$$

Summing up

$$||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le ||u_0 - v_0||_q + C(p,\alpha)M^{p-1}T^{1-p\alpha} ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T}$$

But $C(p,\alpha)M^{p-1}T^{1-p\alpha} < \frac{1}{2}$, hence

$$||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le 2||u_0 - v_0||_q.$$

Now

$$||U(t)u_0 - U(t)v_0||_q \le ||u_0 - v_0||_q + \int_0^t p(||U(s)U_0||^{p-1} + ||U(s)v_0||_{qp}^{p-1}) ||U(s)u_0 - U(s)v_0||_{qp}\, ds \le$$

$$\le ||u_0 - v_0||_q + C(p)M^{p-1} \int_0^t (s^{-\alpha(p-1)}) ||U(s)u_0 - U(s)v_0||_{qp}\, ds \le$$

$$\le ||u_0 - v_0||_q + C(p)M^{p-1} \int_0^t (s^{-\alpha(p)})\, ds ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le$$

$$\le ||u_0 - v_0||_q + C(p)M^{p-1}T_0^{1-\alpha} ||U(\cdot)u_0 - U(\cdot)v_0||_{Y_T} \le$$

$$\le 2||u_0 - v_0||_q.$$

Hence the map from $L^q(\Omega)$ into $L^q(\Omega)$ $v_0 \to U(t)v_0$ is Lipschitz continuous in a neighbourhood of $u_0$.

***Step 4. Uniqueness.***

We can now show uniqueness of the $L^q$-classical solution.

Let $v$ be a $L^q$-classical solution of (2.1) in an interval $[0, T_1)$, we have $v \in C([0, T_1), L^q(\Omega)) \cap L^\infty_{loc}((0, T1), L^\infty(\Omega)), v(0) = u(u)$ and $v$ is a classical solution of (2.1) in $(0, T_1)$.

It is enough to show that $v(t) = U(t)u_0$ for small t for the uniqueness property described above. We may assume $T_1 \leq T_0$ and $||v(s)||_q < M$ in $[0, T_1)$. Let $T = \frac{T_1}{2}$ and $\tau \in (0, T)$, define $v_\tau = v(\cdot + \tau)$, notice $v_\tau \in Y_T$.

Note that $v_\tau$ satisfies:

$$v_\tau(t) = e^{tA}v_\tau(0) + \int_0^t e^{(t-s)A}|v_\tau(s)|^{p-1}v_\tau(s)\, ds,$$

and it is the only solution in $Y_T$ such that the above equality holds. Hence:

$$v_\tau(t) = U(t)v_\tau(0).$$

That can also be rewritten as

$$v(t + \tau) = U(t)v(\tau) \text{ for all } t \in (0, T).$$

Passing to the limit as $\tau \to 0$ and using the Lipschitz continuity we obtain:

$$v(t) = U(t)u_0 \text{ for all } t \in (0, T).$$

Hence the solution $u$ is unique.

***Step 5. Smoothing estimates.***

We shall now prove the smoothing estimates.

Fix $M = 2||u_0||_q$ and notice that $T_0 = T_0(||u_0||_q)$, choose $r \geq q$ we have already proved the estimate in (2.5) for the case $r = q$ and $r = pq$ in fact for the case $r = q$ it is enough to choose $v_0 = 0$ in the estimate for the local Lipschitz continuity, while the case $r = pq$ follows immediately from $||\Phi_{u_0}(u)||_{Y_T} < M$.

Assume now that for some $m \geq \max(p, q)$

$$||u(t)||_m \leq C||u_0||_q t^{-\alpha_m} \tag{2.9}$$

and $\alpha_m$ is defined in (2.5). Notice that if we choose $m = pq$, we have $\max(p, q) \leq pq = m$, so we have at least one $m$ satisfying (2.9). Now, if we let $r > m$ the following estimate holds:

$$||u(t)||_r \leq ||e^{\frac{tA}{2}} u(t/2)||_r + \int_{\frac{t}{2}}^{t} ||e^{(t-s)} |u(s)|^{p-1} u(s)||_r \, ds,$$

where the inequality above holds since $u$ is also a mild-solution.
We can now proceed with the chain of inequalities:

$$||e^{\frac{tA}{2}} u(t/2)||_r + \int_{\frac{t}{2}}^{t} ||e^{(t-s)} |u(s)|^{p-1} u(s)||_r \, ds \leq$$

$$\leq t^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{r})} ||u(t/2)||_m + \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} || \, |u(s)|^{p-1} u(s) \, ||_{\frac{m}{p}} \, ds =$$

$$= t^{-\frac{n}{2}(\frac{1}{m}-\frac{1}{r})} ||u(t/2)||_m + \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} ||u(s)||_m^p \, ds \leq$$

$$\leq C t^{-\alpha_r} ||u_0||_q + C^p ||u_0||_q^p \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} s^{-\alpha_m p} \, ds$$

We focus only on the second term, we have

$$C^p ||u_0||_q^p \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} s^{-\alpha_m p} \, ds =$$

$$= C^p ||u_0||_q^p t^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} t^{-\alpha_m p} t \int_{\frac{1}{2}}^{1} (1-y)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} y^{-\alpha_m p} \, dy =$$

$$= C^p ||u_0||_q^p t^{-\alpha_r} t^{-\frac{n}{2}(\frac{p}{q}-\frac{1}{q})} t \int_{\frac{1}{2}}^{1} (1-y)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} y^{-\alpha_m p} \, dy$$

Putting all together

$$||u(t)||_r \leq C t^{-\alpha_r} ||u_0||_q + C^p ||u_0||_q^p t^{-\alpha_r} t^{-\frac{n}{2}(\frac{p}{q}-\frac{1}{q})} t \int_{\frac{1}{2}}^{1} (1-y)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} y^{-\alpha_m p} \, dy \leq$$

$$\leq C t^{-\alpha_r} ||u_0||_q \left( 1 + t^{1-\frac{n}{2}(\frac{p}{q}-\frac{1}{q})} \int_{\frac{1}{2}}^{1} (1-y)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} y^{-\alpha_m p} \, dy \right) \leq$$

$$\leq C ||u_0||_q t^{-\alpha_r},$$

where $C$ can change from line to line and depends on $||u_0||_q$ and the value of the finite integral $\int_{\frac{1}{2}}^{1} (1-y)^{-\frac{n}{2}(\frac{p}{m}-\frac{1}{r})} y^{-\alpha_m p} \, dy$, the important point is that it doesn't depend upon $r$.
Summing up we proved that for any $r > m$ (this condition is due to (2.6)) we have

$$||u(t)||_r \leq C ||u_0||_q t^{-\alpha_r} \text{ for any } r > m$$

We need to prove the result for any $r \in [q, \infty]$ but this follows immediately since: $u(t) \in L^q(\Omega) \cap L^\infty(\Omega)$ for all $t > 0$ hence

$$||u(t)||_r \leq ||u(t)||_q^{\frac{q}{r}} ||u(t)||_\infty^{1-\frac{q}{r}} \leq 2^{\frac{q}{r}} ||u_0||_q^{\frac{q}{r}} C^{1-\frac{q}{r}} ||u_0||_q^{1-\frac{q}{r}} (t^{-\frac{n}{2}(\frac{1}{q})})^{1-\frac{q}{r}} \leq C||u_0||_q \leq$$

$$\leq C||u_0||_q t^{-\alpha_r}.$$

We finally proved (2.5).

***Step 6. Positivity.***

We shall now pass to the positivity property, we know that $e^{tA}$ is positivity preserving, if $u_0 \geq 0$, we can construct a solution using the Banach fixed point theorem as the limit of the sequence:

$$u_{k+1} = \Phi_{u_0}(u_k)$$

with $u_1(t) = 0$, now $u_2(t) \geq 0$ for all t since $u_2(t) = e^{tA}u_0 \geq 0$, now, given $u_k(t) \geq 0$ for all t we have:

$$u_{k+1}(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} u_k(s)^p \, ds \geq 0$$

Hence $u$ found as the limit of $k \to \infty$ in $Y_T$ of $u_k(t)$ is still greater than or equal to 0.

□

Given a measurable function $\Phi : \Omega \to [0, \infty]$ we set:

$$(e^{tA}\Phi)(x) = \int_\Omega G(x, y, t)\Phi(y) \, dy$$

Where $G$ is the Dirichlet heat kernel in $\Omega$.

**Lemma 2.1.** *Let $u_0 : \Omega \to [0, \infty]$ and $u : \Omega \times [0, T] \to [0, \infty]$ be measurable functions and such that:*

$$u(t) \geq e^{tA}u_0 + \int_0^t e^{(t-s)A} u(s)^p \, ds \ \text{ a.e. in } Q_T \tag{2.10}$$

*Moreover assume that $u(x, t) < \infty$ for almost every $(x, t) \in Q_T$. Then the following inequality necessarily holds*

$$t^{\frac{1}{p-1}} ||e^{tA}u_0||_\infty \leq k_p = (p-1)^{-1/(p-1)} \text{ for all } t \in (0, T] \tag{2.11}$$

*Proof.* We preliminary notice that

$$e^{tA}\Phi^p \geq (e^{tA}\Phi)^p \text{ for all measurable } \Phi : \Omega \to [0, \infty] \tag{2.12}$$

In fact, letting $p'$ be the conjugate exponent of $p$ we have

$$e^{tA}\Phi = \int_\Omega G(x,y,t)^{1/p}\Phi(y)G(x,y,t)^{1/p'}\,dy \leq$$

$$\leq ||G(x,\cdot,t)^{1/p}\Phi||_p||G(x,\cdot,t)^{1/p'}||_{p'} = \left(\int_\Omega G(x,y,t)\Phi(y)^p\,dy\right)^{1/p}\left(\int_\Omega G(x,y,t)\,dy\right)^{1/p'}.$$

Using the fact that $\int_\Omega G(x,y,t)\,dy \leq 1$ we have

$$e^{tA}\Phi \leq (e^{tA}\Phi^p)^{1/p}.$$

Elevating both sides to the power $p$ implies (2.12).
We redefine $u$ on a null set in such a way that (2.10) holds in the entire $Q_T = \Omega \times (0,T)$.
We also have for a.e. $\tau \in (0,T)$, $u(\cdot,\tau) < \infty$ a.e. in $\Omega$.
Fix such $\tau$, let: $\Omega_\tau = \{x \in \Omega : u(x,\tau) < \infty\}$.
Let now $t \in [0,\tau]$, we have

$$e^{(\tau-t)A}u(t) \geq e^{\tau A}u_0 + e^{(\tau-t)A}\int_0^t e^{(t-s)A}u(s)^p\,ds \geq e^{\tau A}u_0 + \int_0^t e^{(\tau-s)A}u(s)^p\,ds \geq$$

$$\geq e^{\tau A}u_0 + \int_0^t e^{(\tau-s)A}u(s)^p\,ds \geq e^{\tau A}u_0 + \int_0^t (e^{(\tau-s)A}u(s))^p\,ds := h(\cdot,t).$$

We see from the second inequality that $h(\cdot,t) \leq u(\cdot,\tau)$ hence: $h(x,t) < \infty$ for all $(x,t) \in \Omega_\tau \times [0,\tau]$, this is true since $u(\cdot,\tau) < \infty$ in $\Omega_\tau$.
We fix now $x \in \Omega_\tau$, then the function of one variable $\phi(t) = h(x,t)$ is absolutely continuous hence

$$\phi'(t) = (e^{(\tau-t)A}u(t))^p(x) \geq \phi^p(t).\text{for a.a. } t \in [0,\tau] \tag{2.13}$$

Moreover $\phi(t) \geq e^{\tau A}\phi > 0$ since both $u$ and $u_0$ are non-negative.
We can thus rewrite (2.13) in the following way

$$(\phi^{1-p})' \leq -(p-1).$$

We can now integrate over the interval $[0,\tau]$ the inequality to obtain:

$$\phi^{1-p}(\tau) \leq \phi^{1-p}(0) - (p-1)\tau$$

Hence

$$[(e^{\tau A}u_0)(x)]^{1-p} = \phi^{1-p}(0) \geq \phi^{1-p}(\tau) + (p-1)\tau \geq (p-1)\tau.$$

In other words we obtained

$$\text{for a.e. } \tau \in (0,T) \text{ for all } x \in \Omega_\tau \text{ we have } \tau^{1/(p-1)}[e^{\tau A}u_0](x) \leq (p-1)^{-1/(p-1)} = k_p,$$

and this implies

$$\text{for a.e. } \tau \in (0,T): \quad \tau^{1/(p-1)}||e^{\tau A}u_0||_\infty \leq k_p.$$

In particular, we have that for a.e. $t \in (0,T) \quad e^{tA}u_0 \in L^\infty(\Omega)$.
The function $t \to ||e^{tA}v||_\infty$ is continuous for all $t > 0$ and $v \in L^\infty(\Omega)$, choose $s$ such that $e^{sA}u_0 \in L^\infty(\Omega)$ then: $||e^{tA}u_0||_\infty = ||e^{(t-s)A}e^{sA}u_0||_\infty$, so $t \to ||e^{tA}u_0||_\infty$ is continuous for $t > 0$, but this implies:

$$t \to t^{1/(p-1)}||e^{tA}u_0||_\infty \text{ is continuous for } t > 0.$$

Let now $t_0 \in (0,T]$ we can choose a sequence $\{t_k\}_{k\in\mathbb{N}} \subset (0,T)$ such that $t_k^{1/(p-1)}||e^{t_kA}u_0||_\infty \leq k_p$, letting $t_k \to t$ we have (2.11). And this concludes the proof. □

**Theorem 2.2.** *Let $p > 1 + \frac{2}{n}$ and $1 \leq q < q_c$:*

1. *There exists a nonnegative function $u_0 \in L^q(\Omega)$, such that (2.1) doesn't admit any nonnegative classical $L^q$-solution in $[0,T)$ for any $T > 0$.*
2. *Assume $p < p_S, \Omega = B_R$, let $u_0 \in L^\infty(\Omega)$ and $u_0 \geq 0$, be radial nonincreasing. Then there exists a time $T > 0$ such that problem (2.1) possesses infinitely many positive radial nonincreasing classical $L^q$-solutions in $[0,T)$. Here $p_S = \frac{2n}{n-2}$ if $n \geq 2$ and $\infty$ otherwise.*

*Proof.* 1) Fix $\alpha \in (0, \frac{n}{q})$, assume that $B(0,2\rho) \subset \Omega$, $\rho > 0$.
Let $u_0(y) = |y|^{-\alpha}\chi_{B(0,\rho)}(y)$, with this choice of $\alpha$ we have: $0 \leq u_0 \in L^q(\Omega)$.
using the heat kernel estimates in 49.10 of [3] we obtain for $t > 0$ small:

$$(e^{tA}u_0)(0) = \int_{|y|<\rho} G(0,y,t)|y|^{-\alpha}\,dy \geq c_1 t^{-\frac{n}{2}} \int_{\sqrt{t}/2<|y|<\sqrt{t}} |y|^{-\alpha}\,dy \geq ct^{-\alpha/2}$$

If we take $\alpha/2$ close enough to $n/q$ we have $\alpha/2 > 1/(p-1)$.
We can now see that the inequality (2.11) is not respected, hence there cannot be any integral solution to problem (2.1).
2) We skip the proof of the second point. □

# 3 | Global solutions

We have seen in chapter 1 that under proper conditions we can ensure existence of local solutions to problem (1.1). In the following chapter we want to see wether we can find appropriate conditions to guarantee existence of global solutions. The idea is the following: given that we have found a local solution up to time $T$ we want to see if the maximal existence time for carefully chosen initial datum is infinite. For this chapter we mainly refer to the work of Weissler, see [6].
We set ourselves in the following environment.
We want to find non-negative solutions to the problem:

$$\begin{cases} u'(t) = \Delta u(t) + u(t)^\gamma \qquad (t > 0) \\ u(0) = \phi, \end{cases} \tag{3.1}$$

which are global in time i.e. they exist for all $t \geq 0$. In problem (3.1) $\gamma$ is assumed to be greater than one, the solution $u(t)$ will be a non-negative curve in $L^p(\mathbb{R}^n)$ for some $p \geq 1$. As usual we treat problem (3.1) through its corresponding integral equation

$$u(t) = e^{t\Delta}\phi + \int_0^t e^{(t-s)\Delta}(u(s)^\gamma)\, ds \tag{3.2}$$

We remind that

$$(e^{t\Delta}\phi)(x) = \int_{\mathbb{R}^n} G_t(x-y)\phi(y)\, dy$$

with:

$$G_t(x) = G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}.$$

Before considering existence of global solutions to problem (3.2), it is natural to ask if there are cases in which we are sure there is not non-negative global solution to problem (3.2). The following theorem gives us a positive answer.

**Theorem 3.1.** *Suppose $\gamma \leq 1 + \frac{2}{n}$ and $\phi \geq 0$ in $L^p(\mathbb{R}^n)$ with $\phi$ non identically zero. Then there is no non-negative global solution $u : [0, \infty) \to L^p(\mathbb{R}^n)$ to the integral equation (3.2) with initial value $\phi$.*

*Proof.* In the proof of theorem 5 in [5] for the case $\gamma < 1 + \frac{2}{n}$, the crucial estimate is that if $u(t)$ is a non-negative solution to problem (3.2) on $[0, T)$ then:

$$t^{1/(\gamma-1)} e^{t\Delta}\phi \leq C \tag{3.3}$$

for all $t \in [0, T)$, with $C$ being a fixed constant depending on $\gamma$ but not on $\phi$ and $T$.
If $\phi \geq 0$ we have:

$$\lim_{t\to\infty} (4\pi t)^{n/2} (e^{t\Delta}\phi)(x) = \lim_{t\to\infty} \left( (4\pi t)^{n/2} \int_{\mathbb{R}^n} G_t(x-y)\phi(y)\, dy \right) =$$

$$= \lim_{t\to\infty} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t}\phi(y)\, dy = \int_{\mathbb{R}^n} \lim_{t\to\infty} e^{-|x-y|^2/4t}\phi(y)\, dy = ||\phi||_1.$$

Hence for all $x \in \mathbb{R}^n$ we have

$$\lim_{t\to\infty} (4\pi t)^{n/2} (e^{t\Delta}\phi)(x) = ||\phi||_1.$$

This means

$$\lim_{t\to\infty} (4\pi t)^{n/2} e^{t\Delta}\phi = ||\phi||_1 \text{ pointwise in } \mathbb{R}^n. \tag{3.4}$$

Now if $n(\gamma - 1)/2 < 1$, we have for sufficiently large $t$,

$$t^{n/2} e^{t\Delta}\phi \leq t^{1/(\gamma-1)} e^{t\Delta}\phi = t^{1/(\gamma-1)-n/2} t^{n/2} e^{t\Delta}\phi.$$

Taking the limit for $t \to \infty$:

$$\frac{||\phi||_1}{(4\pi)^{n/2}} = \lim_{t\to\infty} t^{n/2} e^{t\Delta}\phi \leq \lim_{t\to\infty} t^{1/(\gamma-1)-n/2} t^{n/2} e^{t\Delta}\phi = \lim_{t\to\infty} t^{1/(\gamma-1)-n/2} \lim_{t\to\infty} t^{n/2} e^{t\Delta}\phi = \infty.$$

Contradicting the estimate (3.3).
Let now $n(\gamma - 1)/2 = 1$ and suppose $u : [0, \infty) \to L^p$ is a global non-negative solution to problem (3.2). The estimate (3.3) becomes:

$$t^{n/2} e^{t\Delta}\phi \leq C$$

for all $t \geq 0$.
We also know that

$$\lim_{t\to\infty} t^{n/2} e^{t\Delta}\phi = ||\phi||_1/(4\pi)^{n/2}.$$

Hence

$$||\phi||_1/(4\pi)^{n/2} = \lim_{t\to\infty} t^{n/2} e^{t\Delta}\phi \leq C.$$

This implies

$$||\phi||_1 \leq C(4\pi)^{n/2} = C'.$$

Since $u(t)$ is a solution it belongs to $L^p$ for almost any $t \geq 0$. Hence $u(t)$ can be regarded for almost any $t$ as the initial value. Hence:

$$||u(t)||_1 \leq C'.$$

for (almost) all $t \geq 0$.
We now assume that the initial value $\phi$ dominates some Gaussian function i.e. $\phi \geq kG_\alpha = k(4\pi\alpha)^{-n/2}e^{-|x|^2/4\alpha}$ for some $k > 0$ and $\alpha > 0$.
Obviously from the integral equation (3.2) we have

$$u(t) \geq e^{t\Delta}\phi \geq e^{t\Delta}kG_\alpha.$$

So

$$||u(t)||_1 \geq \int_0^t ||e^{(t-s)\Delta}(u(s)^\gamma)||_1\, ds \geq \int_0^t ||e^{(t-s)\Delta}((e^{s\Delta}kG_\alpha)^\gamma)||_1\, ds =$$

$$= k^\gamma \int_0^t ||((e^{s\Delta}G_\alpha)^\gamma)||_1\, ds.$$

For the properties of the Gaussians $G_t$ we get:

$$(e^{s\Delta}G_\alpha)^\gamma = (G_{s+\alpha})^\gamma$$

So

$$(e^{s\Delta}G_\alpha)^\gamma = (G_{s+\alpha})^\gamma = (4\pi(s+\alpha))^{-n\gamma/2}e^{-(|x|^2\gamma)/4(s+\alpha)} =$$

$$= (4\pi(s+\alpha))^{-n\gamma/2}G_{(s+\alpha)/\gamma}\left(4\pi\frac{(s+\alpha)}{\gamma}\right)^{n/2} =$$

$$= \gamma^{-n/2}(4\pi(s+\alpha))^{-\frac{n}{2}(\gamma-1)}G_{(s+\alpha)/\gamma} = \gamma^{-n/2}(4\pi(s+\alpha))^{-1}G_{(s+\alpha)/\gamma}.$$

Hence for almost all $t \geq 0$

$$||u(t)||_1 \geq k^\gamma \int_0^t ||((e^{s\Delta}G_\alpha)^\gamma)||_1\, ds \geq k^\gamma\gamma^{-n/2}(4\pi)^{-1}\int_0^t ||(s+\alpha)^{-1}G_{(s+\alpha)/\gamma}||_1\, ds =$$

$$= k^\gamma\gamma^{-n/2}(4\pi)^{-1}\int_0^t (s+\alpha)^{-1}\, ds.$$

But this last term becomes arbitrarily large as $t \to \infty$, which contradicts the fact that for almost all t $||u(t)|| \leq C'$.
For now we proved that if the initial datum $\phi$ dominates a Gaussian we cannot have global

non-negative solutions to (3.2).
We can now prove the result in general, given a non-negative solution $u(t)$ to problem (3.2) with non-trivial initial value $\phi$, consider $v(t) = u(t+\epsilon)$ for some $\epsilon > 0$. Then $v(t)$ is a solution to (3.2) with initial value $u(\epsilon)$. If we show that $v(t)$ cannot be a global solution then obviously also $u(t)$ cannot be one.
In particular we show that $v(t)$ has the initial datum $\psi = u(\epsilon)$ that dominates a Gaussian.

$$\psi = u(\epsilon) \geq e^{\epsilon\Delta}\phi = G_\epsilon * \phi$$

Now

$$(G_\epsilon * \phi)(x) = (4\pi\epsilon)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4\epsilon} \phi(y)\, dy =$$

$$= (4\pi\epsilon)^{-n/2} e^{-|x|^2/2\epsilon} \int_{\mathbb{R}^n} e^{-|x+y|^2/4\epsilon} e^{-|y|^2/2\epsilon} \phi(y)\, dy \geq$$

$$\geq (4\pi\epsilon)^{-n/2} e^{-|x|^2/2\epsilon} \int_{\mathbb{R}^n} e^{-|y|^2/2\epsilon} \phi(y)\, dy = (1/2)^{n/2} G_{\epsilon/2}(x) \int_{\mathbb{R}^n} e^{-|y|^2/2\epsilon} \phi(y)\, dy =$$

$$= kG_{\epsilon/2}$$

where $k = (1/2)^{n/2} \int_{\mathbb{R}^n} e^{-|y|^2/2\epsilon} \phi(y)\, dy$.
In other terms

$$\psi = u(\epsilon) \geq kG_{\epsilon/2}.$$

Hence neither $v(t)$ nor $u(t)$ can be global solutions to problem (3.2). □

We shall now pass to the core of the chapter that is to find suitable conditions on the initial datum $\phi$ in order to guarantee global existence of solutions. We state and prove the following theorem.

**Theorem 3.2.** *a) Let $\phi \geq 0$ be in $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$. Suppose the following holds:*

$$(\gamma - 1) \int_0^\infty ||e^{s\Delta}\phi||_\infty^{\gamma-1}\, ds \leq 1 \tag{3.5}$$

*Then there exists a non-negative continuous curve $u : [0, \infty) \to L^p(\mathbb{R}^n)$ which is a global solution to (3.2) with initial datum $\phi$.*
*Furthermore, the following bound holds:*

$$u(t) \leq \frac{e^{t\Delta}\phi}{[1 - (\gamma - 1)\int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}\, ds]^{1/(\gamma-1)}} \tag{3.6}$$

*for all $t \geq 0$.*

*b) Suppose $n(\gamma-1)/2>1$. If $\phi\geq 0$ and $||\phi||_{n(\gamma-1)/2}$ is sufficiently small, then there exists a non-negative continuous curve $u:[0,\infty)\to L^{n(\gamma-1)/2}$ which is a global solution to problem (3.2) with initial value $\phi$.*

*Proof.* We focus now on part a).
Let
$$C(t)=[1-(\gamma-1)\int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}\,ds]^{-1/(\gamma-1)}.$$
So $C(0)=1$ and

$$C'(t)=-1/(\gamma-1)C(t)C(t)^{-1}(1-\gamma)||e^{t\Delta}\phi||_\infty^{\gamma-1}=$$
$$=||e^t\Delta\phi||_\infty^{\gamma-1}[1-(\gamma-1)\int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}\,ds]^{-\gamma/(\gamma-1)}=$$
$$=||e^{t\Delta}\phi||_\infty^{\gamma-1}C(t)^\gamma.$$
Hence
$$C(t)=1+\int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma\,ds.$$
Now, let $u(t):[0,\infty)\to L^p$ be a continuous curve such that $e^{t\Delta}\phi\leq u(t)\leq C(t)e^{t\Delta}\phi$ for all $t\geq 0$.
Let
$$\mathcal{F}u(t)=e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(u(s)^\gamma)\,ds.$$
Then
$$\mathcal{F}u(t)\leq e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(e^{s\Delta}\phi)^\gamma C(s)^\gamma\,ds\leq$$
$$\leq e^{t\Delta}\phi+\int_0^t e^{(t-s)\Delta}(e^{s\Delta}\phi)||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma\,ds=$$
$$=e^{t\Delta}\phi\left(1+\int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma\,ds\right)=e^{t\Delta}\phi C(t).$$
Hence
$$e^{t\Delta}\phi\leq\mathcal{F}u(t)\leq C(t)e^{t\Delta}\phi \text{ for all } t\geq 0$$
Let $u_0(t)=e^{t\Delta}\phi$ and for $m\geq 0$, $m\in\mathbb{N}$ we define $u_{m+1}(t)=\mathcal{F}u_m(t)$. We want to show that the sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ converges to the desired solution.
We first observe that
$$u_n(t)\leq u_{n+1}(t) \text{ for all } t\geq 0.$$

This follows from the fact that if we have $u(t) \leq v(t)$ for all $t \geq 0$ then

$$\mathcal{F}u(t) = e^{t\Delta}u(0) + \int_0^t e^{(t-s)\Delta}(u(s)^\gamma)\, ds \leq e^{t\Delta}v(0) + \int_0^t e^{(t-s)\Delta}(v(s)^\gamma)\, ds.$$

Now, since $u_0(t) \leq C(t)e^{t\Delta}\phi$ and the application of $\mathcal{F}$ preserves this property, then

$$u_n(t) \leq C(t)e^{t\Delta}\phi \text{ for all } t \geq 0 \text{ and all } n \in \mathbb{N}.$$

Summing up, $\{u_n(t)\}_{n\in\mathbb{N}}$ is an increasing sequence dominated in $L^p(\mathbb{R}^n)$ by $C(t)e^{t\Delta}\phi$ since

$$||C(t)e^{t\Delta}\phi||_p \leq C(t)||\phi||_p < \infty.$$

Hence by the dominated convergence theorem, $u_n(t)$ converges in $L^p(\mathbb{R}^n)$ to a function that we will call $u(t)$ as $n \to \infty$.
Obviously $u(t) \leq C(t)e^{t\Delta}\phi$ for all $t \geq 0$.
Moreover the functions $s \to e^{(t-s)\Delta}(u_m(s)^\gamma)$ are dominated by $e^{t\Delta}\phi||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma$ (it can be seen in the proof of the fact that $\mathcal{F}u(t) \leq C(t)e^{t\Delta}\phi$).
But $e^{t\Delta}\phi||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma \in L^1((0,t); L^p(\mathbb{R}^n))$, since

$$\int_0^t ||e^{t\Delta}\phi\; ||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma||_p\, ds \leq ||\phi||_p \int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}C(s)^\gamma\, ds \leq$$

$$\leq C(t)^\gamma||\phi||_p \int_0^t ||e^{s\Delta}\phi||_\infty^{\gamma-1}\, ds < \infty.$$

Moreover, for $s \in (0,t)$ $e^{(t-s)\Delta}(u_m(s)^\gamma) \to e^{(t-s)\Delta}(u(s)^\gamma)$ as $m \to \infty$.
In fact

$$||e^{(t-s)\Delta}(u(s)^\gamma) - e^{(t-s)\Delta}(u_m(s)^\gamma)||_p = ||e^{(t-s)\Delta}(u(s)^\gamma - u_m(s)^\gamma)||_p \leq ||u(s)^\gamma - u_m(s)^\gamma||_p \leq$$

$$\leq \gamma^p||u(s) - u_m(s)||_{\gamma p}(||u(s)||_{\gamma p}^{\gamma-1} + ||u_m(s)||_{\gamma p}^{\gamma-1}).$$

Now, since $u_m(s) \leq C(s)e^{s\Delta}\phi$ and $u(s) \leq C(s)e^{s\Delta}\phi$,
we have that $||u_m(s)||_{\gamma p}$ and $||u(s)||_{\gamma p}$ are both bounded:

$$||C(s)e^{s\Delta}\phi||_{\gamma p} \leq C(s)(4\pi s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{\gamma p})}||\phi||_p < \infty,$$

where we have used proposition (2.1) point 4).
Hence, by dominated convergence theorem $u_m(s)$ converges to $u(s)$ in $L^{p\gamma}(\mathbb{R}^n)$.

Reminding that

$$||e^{(t-s)\Delta}(u(s)^\gamma) - e^{(t-s)\Delta}(u_m(s)^\gamma)||_p \le \gamma^p ||u(s) - u_m(s)||_{\gamma p}(||u(s)||_{\gamma p}^{\gamma-1} + ||u_m(s)||_{\gamma p}^{\gamma-1}),$$

we obtain that for all $s \in (0,t)$ $e^{(t-s)\Delta}(u_m(s)^\gamma) \to e^{(t-s)\Delta}(u(s)^\gamma)$ as $m \to \infty$.
Thus we can apply the dominated convergence theorem for $L^p$-valued functions to get

$$\lim_{m\to\infty} \int_0^t e^{(t-s)\Delta}(u_m(s)^\gamma)\, ds = \int_0^t \lim_{m\to\infty} e^{(t-s)\Delta}(u_m(s)^\gamma)\, ds = \int_0^t e^{(t-s)\Delta}(u(s)^\gamma)\, ds,$$

where the limit is in $L^p$.
Letting $m \to \infty$ in the formula $u_{m+1}(t) = \mathcal{F}u_m(t)$ we get

$$u(t) = \mathcal{F}u(t).$$

In other words $u(t)$ is a global solution to the problem (3.2) since continuity in $L^p$ is proved as in Theorem 1.1.
We can now prove part b):
We use the fact that by theorem (1.2) part b) we know the existence of local solutions to problem (3.2) for all initial data $\phi \in L^{n(\gamma-1)/2}$. More precisely, choose $p$ such that $1 \le p < n(\gamma-1)/2 < p\gamma$, then theorem (1.2) part b) and its corollary guarantee that for every $\phi \in L^{n(\gamma-1)/2}$ with $\phi \ge 0$, there exists a non-negative continuous curve $u : [0,T) \to L^{n(\gamma-1)/2}$ satisfying (3.2) with initial value $\phi$. Moreover, $u(t)$ is continuous into $L^{p\gamma}$ for $t > 0$ and $t^b||u(t)||_{p\gamma}$ is bounded near 0 with $b = 1/(\gamma-1) - n/2p\gamma$.
Hence $u(t)$ can be continued to a solution of (3.2) as long as $||u(t)||_{p\gamma}$ doesn't blow-up.
Let $a = n(\gamma-1)/2p\gamma$ then remembering Proposition 2.1 point 4):

$$t^b||u(t)||_{p\gamma} \le t^b||e^{t\Delta}\phi||_{p\gamma} + t^b \int_0^t ||e^{(t-s)\Delta}(u(s)^\gamma)||_{p\gamma}\, ds \le$$

$$\le t^b(4\pi t)^{-\frac{n}{2}(\frac{2}{n(\gamma-1)} - \frac{1}{p\gamma})}||\phi||_{n(\gamma-1)/2} + t^b \int_0^t (4\pi(t-s))^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{p\gamma})}||u(s)^\gamma||_p\, ds =$$

$$= (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + t^b \int_0^t (4\pi(t-s))^{-a}||u(s)^\gamma||_p\, ds =$$

$$= (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a}t^b \int_0^t (t-s)^{-a}s^{-b\gamma}s^{b\gamma}||u(s)||_{p\gamma}^\gamma\, ds \le$$

$$\le (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a}t^b \int_0^t (t-s)^{-a}s^{-b\gamma}\, ds \sup_{(0,t)} ||s^b u(s)||_{p\gamma}^\gamma =$$

$$= (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a}t^b \int_0^1 t^{-a}(1-y)^{-a}t^{-b\gamma}y^{-b\gamma}t\, dy \sup_{(0,t)} ||s^b u(s)||_{p\gamma}^{\gamma} =$$

$$(4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a}t^{b-a-b\gamma+1} \int_0^1 (1-y)^{-a}y^{-b\gamma}\, dy \sup_{(0,t)} ||s^b u(s)||_{p\gamma}^{\gamma},$$

but $b - a - b\gamma + 1 = 0$. Hence

$$t^b||u(t)||_{p\gamma} \leq (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a} \int_0^1 (1-y)^{-a}y^{-b\gamma}\, dy \sup_{(0,t)} ||s^b u(s)||_{p\gamma}^{\gamma} \tag{3.7}$$

Note that $b\gamma < 1$. Let now $f(T) = \sup_{(0,T)} ||s^b u(s)||_{p\gamma}$.
We want to show that the function $f$ is continuous.
Fix $y > 0$, we want to show:

$$\forall \epsilon > 0 \;\; \exists \delta > 0 : \forall x \, |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Suppose w.l.o.g. that $x > y$ and $f(x) > f(y)$ (otherwise $f(x) = f(y)$ and we are done).

$$f(x) - f(y) = \max [\sup_{(0,y)} ||s^b u(s)||_{p\gamma}, \sup_{[y,x)} ||s^b u(s)||_{p\gamma}] - \sup_{(0,y)} ||s^b u(s)||_{p\gamma} =$$

$$= \sup_{[y,x)} ||s^b u(s)||_{p\gamma} - \sup_{(0,y)} ||s^b u(s)||_{p\gamma} \leq ||t_1^b u(t_1)||_{p\gamma} + \epsilon/2 - \sup_{(0,y)} ||s^b u(s)||_{p\gamma},$$

where $t_1 \in [y, x)$. Now

$$||t_1^b u(t_1)||_{p\gamma} + \epsilon/2 - \sup_{(0,y)} ||s^b u(s)||_{p\gamma} \leq ||t_1^b u(t_1)||_{p\gamma} + \epsilon/2 - ||t_2^b u(t_2)||_{p\gamma},$$

with $t_2 \in (y - \delta, y)$. Now $0 < \max(t_1 - t_2) \leq 2\delta$.
Since $||t^b u(t)||_{p\gamma}$ is continuous we have $f(x) - f(y) < \epsilon$ for $\delta$ small enough.
Moreover $f(0) = 0$. And taking the supremum on $(0, t)$ in inequality (3.7) we obtain:

$$f(t) \leq (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + Cf(t)^{\gamma}, \tag{3.8}$$

where $C$ is a fixed constant. Thus, if we choose $||\phi||_{n(\gamma-1)/2}$ sufficiently small, $f(t)$ must remain bounded in fact, choose $\alpha > 0$ such that $C2^{\gamma}\alpha^{\gamma-1} < 1$. If we choose $\phi$ such that $(4\pi)^{-b}||\phi||_{n(\gamma-1)/2} \leq \alpha$, then $f(t)$ can never be equal to $2\alpha$, in fact if it did we would have using inequality (3.8)

$$2\alpha \leq \alpha + C(2\alpha)^{\gamma},$$

that is

$$\alpha \leq C(2\alpha)^{\gamma}.$$

But we chose $\alpha$ such that $C2^{\gamma}\alpha^{\gamma-1} < 1$, that is $\alpha > C(2\alpha)^{\gamma}$. Hence $f$ is bounded for sufficiently small $\phi$.

We can rewrite (3.7) in the following way:

$$t^b||u(t)||_{p\gamma} \leq (4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a} \int_0^1 (1-y)^{-a} y^{-b\gamma} \, dy f(t)^{\gamma}$$

And $(4\pi)^{-b}||\phi||_{n(\gamma-1)/2} + (4\pi)^{-a} \int_0^1 (1-y)^{-a} y^{-b\gamma} \, dy f(t)^{\gamma}$ is bounded for sufficiently small $\phi$. We have now proved point b). □

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