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Network Analysis of Listed Italian Companies

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Abstract

Many company interactions can be described by the usage of network structure, as for example the Board of Directors network or the ownership network. To get a wider grasp of the looks of the environment composed by firms, these interactions have to be taken into account together. With this purpose, a multi layer network is constructed between companies listed in the italian stock exchange. Four different layers are built between those companies, each one based either on interlocks, ownership ties or a ownership similarity index. The ownership similarity index links together firms that present similar equity distribution across shareholders. The analysis first studies the different layers independently, trying to describe the topology of the graphs and identifying the most important companies in it according to several centrality measures. Afterwards, the layers are considered together in order to analyse the overall structure and to make a comparison between them. The results show all layers to complement each other and present very few overlaps. Moreover, no relevant correlation is found between interlocks and strongly concentrated ownership. The majority number of ties happens more for firms with dispersed equity. Interlocks and ownership ties are the features that lead to the most similar results. The overlaps between those two are mainly composed by firms sharing a physical person or an institutional investor as common shareholder.

Keywords: network, graph, layer, interlocking, ownership

Abstract in lingua italiana

Molte interazioni tra aziende possono essere descritte attraverso l'utilizzo di reti sociali. Un esempio possono essere le reti derivanti da connessioni all'interno delle varie strutture di Governance, piuttosto che da legami nell'ambito della composizione dell'azionariato. Per compredere come questi diversi aspetti caratterizzino un mercato, e come ed in quali modalità interagiscano e si influenzino a vicenda, è utile considerarli insieme. A questo scopo, all'interno della presente tesi viene construita una rete sociale a più strati, avente come oggetto diverse società quotate presso la borsa ialiana. Vengono introdotte quattro reti differenti, ognuna composta a partire da uno tra i due dataset a disposizione, riguardanti rispettivamente la composizione del CdA e dell'azionariato delle società quotate. Per costruire le diverse reti, vengono presi in considerazione casi in cui ci sia la presenza di un consigliere (interlocking), piuttosto che un azionista, in comune tra due aziende, e casi di somiglianza della distribuzione del capitale sociale tra le diverse società. Le reti vengono analizzate d'apprima singolarmente al fine di descriverne i principali aspetti topologici e identificare all'interno di esse i nodi (i.e. le società) più importanti. In un secondo momento, queste vengono considerate assieme, componendo una rete sociale a più strati con l'obiettivo di confrontare i singoli layer. I risultati ottenuti mostrano come le diverse reti tendano a mostrare un aspetto complementare e presentino poche sovrapposizioni. Non è stata identificata nessuna correlazione rilevante tra casi di interlocking e una particolare tipologia di struttura societaria. La maggior parte delle aziende aventi almeno un azionista in comune invece è rappresentata da società caratterizzate da una distribuzione più equa del capitale sociale tra i diversi azionisti. In generale, le reti che sovrappongono di più sono quelle derivanti dallo studio della composizione dei diversi CdA e quella che tiene conto di possibili azionisti in comune tra le società.
Parole chiave: network, reti sociali, grafi, interlocking, struttura azionariato

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Introduction

This thesis focuses on the ownership structure and presence of interlocks among several companies listed in the italian stock exchange. It aims to study the relationship between those features across firms by introducing a multi layer network representation of the market. This is done by creating and analysing different graphs, each one build from either data regarding ownership composition of firms or their Board of Directors structure. Those graphs show the different interconnetions taking place between companies in the italian market, and each of them gives a different representation of the overall market and of its players by displaying the degrees of connections tighting it up and the importance of each company in it. In particular, four different graphs are created; two of them consider interlocks for their creations, while the others focus on the presence of ownership ties or on the degree of similarity between companies'ownership structure. These graphs are first studied independently with the aim of identifying their general topology and the most relevant companies in them. For this purpose different centrality measures are introduced, as closeness, eccentricity and betweenness centrality. The graphs are then studied together, by constructing multi layer networks. These are studied throughout different representations of them, also allowing for the comparison between the layers. Measures as edge overlap, layer contribution, graph distance and structural reducibility are here used for understanding the relationships between the layers.
The thesis is structured as follows: the first chapter gives an overview of the literature about the subjects introduced, what has been already studied and the results shown. The second chapter is a description of the mathematical concepts used in this thesis for carrying on with the analysis. An introduction to graph theory and networks analysis is given here. The third chapter focuses on the dataset of interest by describing its main features. Chapter four describes the methods introduced for constructing the different layers. Chapters five and six respectively report the studies and results about the different layers by first considering them independently and then together. Finally, chapter seven highlights and comments the results.

1 | Literature review

The presence of interlocks within different italian firms' Board of Directors is a common feature [1]. The term Interlocking refers to the situation in which a certain person sits simultaneously on the board of two or more companies. There are different reasons as why this happens and why companies aspire to create those links. Indeed, in general, the main scope of a firm is to mitigate the level of uncertainty around its future performance [2]. In an uncertain environment as the one in which we live in today, it is fundamental for firms to maximise the amount of information at their disposal in order to make better decisions.

Joanna Szalacha in [2] lists four main reasons why firms should benefit from interlocking directorates. The first two motivations are to establish both horizontal and vertical coordination with other firms. Horizontal coordination refers to the situation in which a firm has the possibility to comunicate with other companies belonging to the same industry. This condition would allow companies to discuss for example the pricing of the services they provide. Vertical coordination on the other hand is the process of coordination that takes place within firms that do not necessarely belong to the same industry, but that are part of the same chain process running from the initial avaliable resources to the delivery of the final good or service. Having the possibility to communicate vertically with other companies could result in better accords for the supplies that a firm needs or delivers. A third reason for interlocking to take place comes from the need of expertise. The know how and problem solving abilities that a professional learns with his or her work by being employed in a different firm are skills that company eagerly search for. Finally, having a well known, respected person sitting on its board of directors would allow the company to benefit in terms of reputation, helping to paint a positive picture of the firm. This claim is also supported by Sapinski et al. in [3].

Both in [1] and [2] an interesting point is also made about the possible connection between interlocking and corporate control. Ghezzi and Picciau in [1] state that, for the case of Italy, interlocking is a "mechanism to secure the control of the biggest privately owned corporations across different sectors as well as a defensive tool against hostile takeovers.". On the same line of reasoning, Szlacha in [2] states that companies might engage in in-

terlocks with the aim of gaining control of a firm, through a cheaper and less structured operation with respect to a merge or acquisition.
An important consequence of interlocks is the possible rising of conflict of interest. Indeed, this may take place when a member of the board also serves as a member at another firm. In theory, the actions and decisions that he takes should aim to improve the overall firm's condition. As he sits on more that one board, this accounts for all the companies he works for. It might just happen then that one or more of those action he could undertake will not benefit, but possibly even damage, one of the other firms [1].
The presence of interlocks can also have some relevant consequences on the industry as a whole. In [1], the authors point out how the creation of interlocks, especially within the same industry, might reduce competition and lead to a more quiet life equilibrium, facilitating the establishment. These are actually among the reasons behind why Italy, in 2011, introduced an interlocking ban within the banking, insurance and financial sector. Related to this problem is also the one regarding a possible "loss of autonomy", meaning that a company would not be able to function successfully without several business links represented by interlocks [2].
In literature it is also possible to find several studies that tried to understand the consequences that interlocking directorates have on firm performance. In [4] the authors present a study that supports the claim that the presence of interlocks positively affects firms' ROE. They ascribe this relationship to the fact that interlocks reduce uncertainty and help in detecting threats and possible opportunities from the market. On the other hand, Kaczmarek et al. in [5] found that, relatively to different UK-listed firms, the presence of interlocks had a negative effect on companies perfomance (computed through Tobins'Q). They used this result to support the claim that, " when used in excess, interlocking is likely to compromise the attention of directors on the focal company board".

The other main argument that this thesis considers for the analysis of a network between listed firms regards company ownership structure. Many articles can be found in literature concerning the composition of the companies' ownership and its consequences on firms' performance. Knowing who owns the different firms and how they perform in their role is indeed a relevant issue [6]. Information of interest concerns mainly the categories to which shareholders belong to and the level of concentration of ownership, that is how the total amount of shares is split out among the different shareholders. In [6] relevant descriptive statistics are presented regarding these subjects for the 10000 largest listed companies of 54 different markets. Those firms all together account for more than 90% of global market capitalisation. In this study, shareholders are divided into four main categories: private corporations, public owners, individuals or family owners and finally institutional investors. Private corporations are represented by private companies, their subsidiaries and joint ventures. Public sector owners are on the other hand mainly represented by central and local governments. Institutional investors are mainly funds and asset management companies.

Institutional investors are the predominant class of investors worldwide, but the situation varies a lot with respect to which region of the world is considered. In the US for example they account for more than 72% of the market cap. In Europe on the other hand the situation is different, as nearly 38% of market cap is managed by institutional investors. In Italy, institutional investors represent the second largest investor class, owning more than 20% of the market cap.

When it comes to trying to understand the effect of institutional investors on corporate governance and firm performance, according to Lund in [7], institutional investors tend to support management proposals, rather than shareholders' ones when it comes to discussing decisions. In [14], Mizuno et al. found that, cosidering several listed japanese firms, the ones characterised by a higher institutional ownership percentage showed better performances. In [9], Dakhlallh et al. recovered that, in a study based on Jordanian public companies, institutional ownership have a significant positive effect on firm's Tobin's Q.

Another category of interest when it comes to the global market is public ownership. Also in this case, the percentage of shared owned by the public sector differs from country to country. In asian countries, especially in China and Saudi Arabia, public ownership are the most relevant. In Italy on the other hand, they account for a little more than 10% of the market capitalisation, and are mainly concentrated in the energy sector [6].

For what concerns the consequences of main public investors on firms' performance, [8] conducted a study in 2021 that showed how, on average, government owned firms outperformed other ownership types.

Some relevant information can also be retrieved for what concerns company ownership concentration. The dataset presented by De La Cruz et al. in [6] tells that dispersed ownership is globally a rare phenomenon; around 50% of the considered companies have their largest owners controlling more than 30% of the individual firms' equity. Furthermore, the three largest owners hold combined more than 49% of a firm's equity.

Data regarding ownership structure is also important because it provides knowledge about who is actively controlling the firm and how it is possible to gain control of the company [15]. When it comes to company control, a relevant threshold is usually set to 50%; in this sense, a shareholder has total control of a company if he owns more than 50% of its shares. In Italy, more than half of the traded companies are controlled solely by their

largest shareholders, and almost 3 out of 4 companies are controlled solely by their three largest shareholders [6].
Several studies have also been conducted in order to see if the degree of ownership concentration affects a company's performance. Neven et al. in [10] found that block holders positively impact firms in MENA countries. On the other hand, Bele et al. in [11] found no particular relationship between those variables when considering US listed firms.

Because of the impact that both interlocking and company ownership have on firm performance and on corporate control, it is matter of interest to understand if those two arguments are correlated or independent with each other. Related to this subject is the work brought on by Auvrail and Ossard in [12]. They conducted a study over french listed companies appearing in the CAC40 index across a nineteen years period between 1997 and 2006. They found out that overlapping directors and the network of shareholding linkages are highly correlated. Concerning the connections based on ownership composition of firms, they linked two firms whenever those had a common shareholder, of if one of the companies was investing in the other (ownership tie).
Van Lidth de Jeude et al. in [13] conducted another study on those matters considering german, british and irish firms. They introduced a multilayer structure in order to create different networks among the firms. They constructed four different layers based on ownership ties, directors interlocking, R&D collaboration and stock correlations. What they concluded was that they were able to construct non overlapping layers that managed to complement each other. Also in this case, shareholding network was constructed following a criterion similar to the one used by Auvral and Ossard in [12].
Despite those examples, when it comes to constructing a network that takes into account both interlocking as well as ownership composition in general (and not only focusing on ownership ties), literature is rather scarce. This thesis tries to consider both those arguments together in order to understand if those subjects are related. Does a company that present a strongly concentrated ownership structure have on average fewer interlocks with other companies, as it is already heavily controlled by its owners? Do companies that are controlled by important institutions present many interlocks, as in general the components of the BoD tend to reflect the choice of the main shareholders [12] ? Is it possible to witness a correlation between dispersed ownerships and interlocks? Does the category to which main shareholders belong to affect the composition of the Board of Directors?
This thesis will try to adress these questions by following a similar approach to the one introduced by Van Lidth de Jeude et al in [13].
Four different layers will be created regarding interlocks and ownership structure; they

will be analised in depth both individually and together. The layer taking into account interlocks will be construvted in two different ways, considering first "direct" and then also "indirect" interlocks. The layers based on ownership structure will focus one on the general characteristics of shareholders and one on possible ownership ties.

2 | Network Theory and Similarity Indices

Several real world situations can be described by a set of nodes representing entities and a collection of edges connecting them that enahance the presence of a relationship existing between them. Social networks are the easiest example. The intricated web connecting all different users can be conveniently represented by a graph were the nodes stand for the users and an edge is created between two nodes in case the relative users are friends on the social network.
But a graph can also be used to represent connection of cities through highways, rather than airports and the presence of direct flights between them. Any feature that can be decomposed in singular objects and relevant relationships between those is eligible to be modelled through a graph. In this thesis graphs will be used to represent the different companies listed in the Italian Stock Exchange and the relationship and similarites occurring among them. Four different graphs will be created in order to represent two networks. In all of them the nodes will embody the listed firms. The first two graphs will have edges connecting companies within which interlocking occurs, while in the third and last one edges will connect firms that present a similar overall ownership structureor an ownerhsip tie. Those networks is studied both separately and together.
In the following chapter a basic graph theory review is given, focusing mostly on the subjects, definitions and theorems that will be used when constructing and analising the networks. Before that, an overview of the concepts of distance and similarity are given. Those will come in hand when constructing the links among the different companies in the two layers.

2.1. Metrics and Similarity indices

For th elayer considering ownership structure similarity, a similarity index is constructed, that associates to every couple of companies a real number that quantifies the degree of affinity of their respective ownership structure. A link among those firms is created in the network if this number exeeds a certain minimal threshold. The concept of similarity index can be related to the one of *distance* among elements. Moreover, this last notion is also important when it comes to either quantify the difference among graphs or by computing the length of a path (i.e. set of edges) connecting two nodes in a layer. For those reasons, this chapter will briefly summarise those notions.

Def 1. Given a set S, a **metric** (or *distance*) on S is a function

$$d : S \times S \to \mathbb{R}$$

that satisfies the following properties $\forall x, y, z \in S$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

the couple (S, d) is called **metric space**, while the inequality present in property number 4 is referred to as *triangular inequality*.

If the set taken into consideration is $S = \mathbb{R}^n$, $n \in \mathbb{N}$, then several notable distances can be introduced. In the following, some of those are presented. Given $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$:

- Euclidian Distance: $d_E(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
- Minkowski Distances: $d_M^r(x, y) = \Big(\sum_{i=1}^{n} (x_i - y_i)^r \Big)^{\frac{1}{r}}$, r $\in \{\mathbb{N} \cup \infty\}$. For $r = \infty$ the Minkowski distance is set as

$$d_M^\infty(x, y) = \max_{i=1,\dots,n} |x_i - y_i|$$

Note that if $r = 2$ then $d_M^2 = d_E$.
Opposite to metrics which do have a strict mathematical definition, similarity indices are defined in different ways according to the authors. In [19], the authors define a similarity index on a set S as the function:

$$s : S \times S \to \mathbb{R}$$

such that, $\forall x, y \in S$:

1. $0 \leq s(x, y) \leq 1$

2. $s(x, y) = 1 \Leftrightarrow x = y$
3. $s(x, y) = s(y, x)$

Similarity indices can be easily constructed through the usage of distances. Indeed, if for example there exists a distance d on the set S that is unitary, in the sense that $d(x, y) \leq 1 \ \forall x, y \in S$, then a similarity index on S can just be constructed as $s = 1 - d$. Such a function satisfies properties $1, 2, 3$ and is thus a similarity index. Note that a similarity index does not satisfy, obviously, the triangular inequality, but more than that, to construct one throughout the method just described, it is not even needed that the operator d satisfies it.

2.2. Graph Teory Basics

Mathematically speaking, the definition of a graph is the following:

Def 2. Given a set V of $n \in \mathbb{N}$ elements and a set $\mathrm{E} \subset V \times V$, a graph G is defined as $G = (V, E)$. The elements $v \in V$ are called nodes of the graph while the elements $e = (u, w) \in E$ are called edges of the graph.

To a graph G it's possible to associate an *incidence function* ψ_G that associates to each edge of G a pair of vertices. Mathematically,

$$\begin{aligned} \psi_G : \ & E \to V \times V \\ & e \mapsto \psi(e) = (u, w) \end{aligned}$$

Given a graph $G = (V, E)$ of $n \in \mathbb{N}$ elements, the set of nodes V can be written as $V = \{v_1, \ldots, v_n\}$. The edge connecting the nodes v_i to the node v_j, $i, j \in \{1, \ldots, n\}$, if existing, will be indicated with e_{ij}, thus $\mathrm{e}_{ij} = (v_i, v_j)$.

For a graph $G = (V, E)$, a **subgraph** G_{sub} is a graph $G_{sub} = (V_{sub}, E_{sub})$ such that $V_{sub} \subset V$ and $E_{sub} \subset E$. A graph is said to be *directed* if the set E is composed by ordered pairs, while it is *undirected* if the pairs are unordered. In un undirected graph, if a link exists between nodes v_i and v_j, there is no distinction between the edges e_{ij} and e_{ji}. The separation between these categories allows to differentiate between the existence of mutual and one-way relationships.
Graphs can be graphically represented with dots embodying the nodes and lines connecting them embodying the edges. For directed graphs, arrows are used instead of arcs in order to depict the direction of the connections. The following picture shows an example

of two graphs belonging to the two classes respectively.

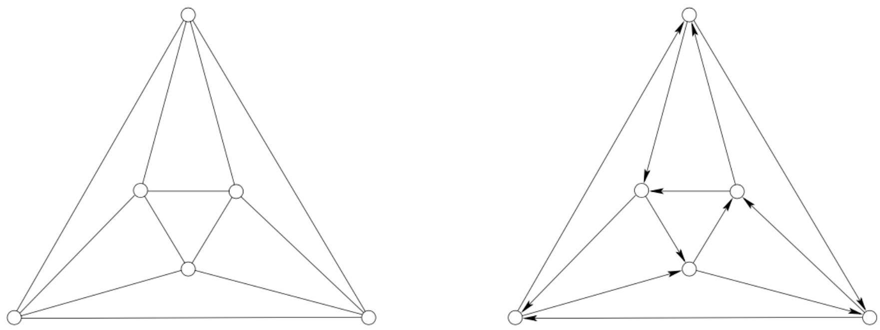


Figure 2.1: Undirected (left) and directed (right) graphs

A graph is said to be **weighted** if for every edge $e \in E$ there exists a positive number $w_e \in \mathbb{R}^+$ that represents the weight of the edge. Weighted graphs are used when the relationships between different nodes are not on the same level but can be more or less strong.

A useful way of representing a graph is through its **adjacency Matrix**:

Def 3. Given a graph $G = (V, E)$ composed by n nodes, the adjacency Matrix associated with G is the $n \times n$ matrix A such that:

$$A(i, j) = 1 \quad if \;\; e_{ij} \in E$$

$$A(i, j) = 0 \quad if \;\; e_{ij} \notin E$$

In other words, the entry $A(i, j)$ is equal to 1 if and only if the edge e_{ij} exists. For undirected graphs the adjacency matrix is symmetric.
A weighted graph can be represented in a more precise way by using the $n \times n$ **Weight Matrix** W defined as:

$$W(i, j) = w_{ij}, \quad where \begin{cases} w_{ij} = w_{e_{ij}}, \quad if \;\; A(i, j) = 1 \\ w_{ij} = 0, \quad if \;\; A(i, j) = 0 \end{cases}$$

From now on, statements will be given for undirected graphs in order to ease up the notation; moreover, graphs are going to be assumed to be unweighted unless specified otherwise.

Given a graph $G = (V, E)$, a **walk** w of length k is a sequence of edges $(e_1, \ldots, e_k)$

for which there exist a sequence of vertices $(v_1, \dots, v_{k+1})$ such that $\psi(e_i) = (v_i, v_{i+1})$ for $i = 1, \dots, k$. If the graph is weighted, the length of the walk can be computed by assigning a **cost** to each edge, representing the distance between the nodes at the end of the arc. The cost of an edge can be computed in various ways; if it is not already given, it can be computed as the inverse of the weight. The length of the walk is thus just the sum of the costs of the edges composing the walk.

A walk is *closed* if the starting node coincide with the ending one. A walk is called a **path** if all the nodes besides the first and the last one (thus also all edges) in it are distinct. If a path presents the same starting and ending node it is called a **closed path**.

A node is said to be *reachable* from another node if there exist a path connecting them, while it is said to be *isolated* if it is not reachable from any node. A network is said to be *connected* if there exist a path connecting every single node to another. A **connected component** is a maximal connected subgraph. In a connected graph $G = (V, E)$, the **distance** d_{ij} between the two nodes v_i,v_j is defined as the length of the shortest path connecting them.

The operator $d : V \times V \to \mathbb{R}$ associating to each couple of vertices the distance between them trivially defines a distance over the set V. The **diameter** D of a connected graph of n edges on the other hand is given by the maximum among all the distances. In formulas:

$$D = \max_{i,j \in \{1,\dots,n\}} \{d_{ij}\}$$

2.2.1. Single node Indices

When referring to a single node of a graph, some measures can give information about its importance in the network. Indeed, depending on the relative relationships a node has with all the others in the network of study, it bears different caratheristics. In literature, the notion of "importance" of a node is by no means unambiguous, and there is no common accepted definition for it [18]. Following each one a different specific criterion for indicating a node as important, those measures (often called indices) aim to intruduce an order of importance on the vertices or edges of a graph by assigning real values to them. One idea is that the more links a node has, the more it can be regarded as relevant. But also a node that has few links but that with its presence allows the network to be connected could be of particular interest. In order to properly quantify those features, different measures are introduced. The most basic among those are the concepts of degree and strength.

Given a graph of n nodes characterised by the adjacency matrix A, the **degree** k_i of the

$i-th$ node, for $i=1,\ldots,n$ is defined as

$$k_i = \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ij}$$

where $a_{ij} = A(i,j)$. The degree of a node gives information about how many links it has inside the graph of interest. The degree can take all integer values from 0 to n.
For a weighted graph it is also possible to define the **strength** of a node, which takes into account not only the number of edges a node is part of, but also the total weight of those. For every node $v_i \in V$ the strenght s_i is defined as:

$$s_i = \sum_{j=1}^{n} w_{ij} = \sum_{i=1}^{n} w_{ij}$$

where $w_{ij} = W(i,j)$ and W is the graph's weight matrix. A network can also have some regions of local density (clusters), meaning that some nodes are very interconnected with each other while others are not. The clustering coefficient of a node tries to give a measure of the extent to which a node belongs to such a cluster. Given an undirected network $G=(V,E)$ of n nodes, the **clustering coefficient** c_i of node v_i, for $i=1,\ldots,n$ is calculated as:

$$cl_i = \frac{\#triangles\ including\ v_i}{\#unordered\ connected\ triplets\ centered\ in\ v_i(v_j, v_i, v_l)} \quad j,l = 1,\ldots,n, \ \ j \neq l$$

Looking at the following example can help understand this definition:

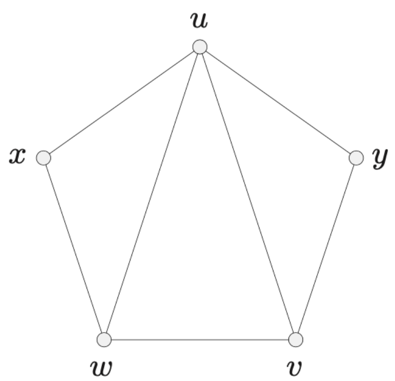


Figure 2.2: A connected graph

In the graph presented above, node w for instance belongs to 2 different triangles, the one composed by nodes (w,x,u) and the one composed by (w,v,u). The number of unordered triplets it belongs to is equal to 3 (triples (x,w,u), (x,w,v) and (u,w,v)). The clustering

coefficient for node w is thus $\frac{2}{3}$.
The clustering coefficient can take values in [0,1], where 1 indicates that all nodes that are connected to a given node also present an edge linking them, while 0 represents the situation in which a node does not belong to any triangle. For a graph $G = (V, E)$ composed by n nodes, the clustering coefficient can be written as

$$cl(v_i) = \frac{\#triangles\ including\ v_i}{k_i(k_i - 1)/2} \quad i = 1, \ldots, n$$

as one can choose the first node belonging to a triplet including v_i in k_i different ways and choose the last one among $k_i - 1$ other nodes. As the triplets are unordered, this quantity has to be divided by 2. In case the denumerator were to be 0, that is if the node considered is isolated, then the respective clustering coefficient is set to 0 as well. A global clustering coefficient can also be defined for the overall graph G, as simply the average of all clustering coefficients over all the nodes. In formulas:

$$C = \frac{1}{n} \sum_{i=1}^{n} c_i$$

Other measures try to grasp the visual concept of a node being somehow "central" in the network of study. In one sense of this word, the node regaded as most central is the one for which the distance to the furthest vertex is minimal.
Given a graph $G = (V, E)$ of n nodes, the **eccentricity centrality** index for a node $v_i \in V$ is computed as:

$$c_E(v_i) = \frac{1}{e(v_i)} = \frac{1}{\max_{j=1,\ldots,n}\{d_{ij}\}}$$

the denumerator $e(v_i)$ is defined as the *eccentricity* of node v_i.
Another centrality measure that on the other hand regards as more central the nodes that are overall closest to all others is **closeness centrality**. For node v_i this defined as:

$$c_C(v_i) = \frac{1}{\sum_{j=1}^{n} d_{ij}}$$

So while eccentricity centrality assignes bigger values to nodes with smaller maximum distance, closeness centrality takes into account the total distance of all vertices from the node considered. These measures can not be computed if the node taken into consideration is not connected to every other node. As this might not be always the case, it is possible to adjust these indices in order for them to not lose sense in these situations. One way is to only consider the reachable nodes and to adjust the indices by a dumping factor taking into account the fact that some nodes are not reachable. The adjusted indices are the following:

$$c_E(v_i) = \left(\frac{\#r_{v_i}}{n-1}\right)^2 \cdot \frac{1}{\max_{v_j \in r_{v_i}}\{d_{ij}\}} \qquad c_C(v_i) = \left(\frac{\#r_{v_i}}{n-1}\right)^2 \cdot \frac{1}{\sum_{v_j \in r_{v_i}} d_{ij}}$$

where r_{v_i} is the set of reachable nodes from v_i (not counting v_i).
A node's importance can also depend on the fraction of times it belongs to a shotest path connecting all the other nodes to eachother. For this purpose, the betweenness centrality index is defined. Always considering a graph of n nodes $G = (V, E)$, for a node $v_i \in V$ it is possible to define the following ratio:

$$\delta_{j,l}(v_i) = \frac{\sigma_{j,l}(v_i)}{\sigma_{j,l}} \quad j, l \in \{1, \ldots, n\}, \quad j, l \neq i, \quad j \neq l$$

Here $\sigma_{j,l}(v_i)$ represents the number of shortest paths connecting nodes v_j and v_l passing through v_i, while $\sigma_{j,l}$ is the number of shortest paths connecting nodes v_j and v_l. If this were to be 0 then $\delta_{j,l}(v_i)$ is set to 0 as well. The **betweenness centrality** index for node v_i is calulated as:

$$c_B(v_i) = \left(\sum_{j=1, j\neq i}^{n} \sum_{l=1, j\neq i, l\neq j}^{n} \delta_{j,l}(v_i)\right)/2$$

The maximum betweenness a node can reach is in case all shortest paths between all other nodes pass through the vertex being considered. In case of a network composed by n nodes, this is equal to $(n-1) \cdot (n-2)/2$.

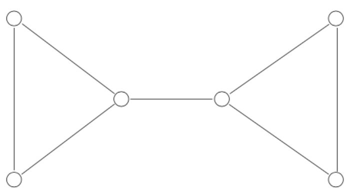


Figure 2.3

This index is useful to describe a graph as the one presented in 2.3. In this situation, the vertices in the center of the graph will present a higher betweenness centrality index with respect to the remaining ones.

2.3. Network Analysis

Since this thesis will construct different layers across the companies listed in the Italian Stock Exchange, it is useful to introduce an approach and a coherent notation for the study of multiple layers.

2.3.1. Multilayer Representation

Formally, a *multilayer network* is defined as follows:

Def 4. A **multilayer network** is a pair $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ where $\mathcal{G} = \{G_\alpha, \alpha \in \{1, \ldots, M\}\}$ is a family of (directed or indirected, weighted or unweighted) graphs $G_\alpha = (V_\alpha, E_\alpha)$ called *layers* of $\mathcal{M}$ and

$$\mathcal{C} = \{E_{\alpha\beta} \subset V_\alpha \times V_\beta | \alpha, \beta \in \{1, \ldots, M\}\}$$

is a set of interconnections between nodes of different layers G_α and G_β. The elements of $\mathcal{C}$ are called *crossed layers*, while the elements of each E_α are called *interlayer* connections.

Note that in this definition the different layers do not need to be composed by the same nodes. For the sake of this thesis, the set $\mathcal{C}$, when considered, will simply be represented by the crossed edges connecting the same nodes over the different layers. That is,

$$E_{\alpha\beta} = \{(u, v) \in V_\alpha \times V_\beta : u = v\}$$

For each layer $\alpha \in \{1, \ldots, M\}$, the set of nodes V_α is written as $V_\alpha = \{v_1^\alpha, \ldots, v_{N_\alpha}^\alpha\}$. The $N_\alpha \times N_\alpha$ adjacency matrix $A_\alpha$representing layer α (i.e. graph G_α) is

$$A^{[\alpha]}(i, j) = a_{ij}^\alpha = \begin{cases} 1 \ \ if \ \ (v_i^\alpha, v_j^\alpha) \in E_\alpha \\ 0 \ \ otherwise \end{cases}$$

This represents the interlayer adjacency matrix. To practically write the adjacency matrix given by intralayer connections between two layers α and $\beta \in \{1, \ldots, M\}$ one writes:

$$A^{[\alpha][\beta]}(i, j) = a_{ij}^{\alpha\beta} = \begin{cases} 1 \ \ if \ \ (v_i^\alpha, v_j^\alpha) \in E_{\alpha\beta} \\ 0 \ \ otherwise \end{cases}$$

$A^{[\alpha][\beta]}$ is a $N_\alpha \times N_\beta$-matrix.
This definition can be extended to include weighted multilayers. In this case, to each edge and crossed layer edge it is assigned a (positive) real number representing the weight of the edge. In formulas:

$$W^{[\alpha]}(i, j) = w_{ij}^\alpha = \begin{cases} > 0 \ \ if \ \ (v_i^\alpha, v_j^\alpha) \in E_\alpha \\ 0 \ \ otherwise \end{cases}$$

For intralayer connections

$$W^{[\alpha][\beta]}(i,j) = w_{ij}^{\alpha\beta} = \begin{cases} > 0 \;\; if \;\; (v_i^{\alpha}, v_j^{\alpha}) \in E_{\alpha\beta} \\ 0 \;\; otherwise \end{cases}$$

When it comes to studying the caractheristics of the nodes composing the multilayer, it is useful to simplify the notation and develop another representation of the network on which each layer is augmented in its nodes. That is, in order to simplify the notation, $V = \bigcup_{\alpha=1}^{M} V_\alpha$ is substituted to each V_α. We write the elements of V as $V = \{v_1, \ldots, v_N\}$ where $N = \#V$. As a consequence, also the different adjacency matrices and intralayer connection matrices change in the following:

$$A^{[\alpha]}(i,j) = a_{ij}^{\alpha} = \begin{cases} 1 \;\; if \;\; (v_i, v_j) \in E_{\alpha} \\ 0 \;\; otherwise \end{cases}$$

and

$$A^{[\alpha][\beta]}(i,j) = a_{ij}^{[\alpha][\beta]} = \begin{cases} 1 \;\; if \;\; (v_i, v_j) \in E_{\alpha\beta} \\ 0 \;\; otherwise \end{cases}$$

Both $A^{[\alpha]}$ and $A^{[\alpha][\beta]}$ are $N \times N$ matrices. The difference from the notation above is only in the labelling of the nodes. By merging the set of nodes and considering each graph on the overall set V, the relevant adjacency matrices mantain the same structure, in the sense that they represent the same edges linking the same nodes as before, but modify their representation because of the set on which they are defined.
The three dimentional matrix $\mathbf{A} = \{A^{[1]}, \ldots, A^{[M]}\}$ completely defines the multilayer structure. For a given layer α, the degree of node $i \in \{1, \ldots, N\}$ is

$$k_i^{[\alpha]} = \sum_{j=1}^{N} a_{ij}^{[\alpha]}$$

where $a_{ij}^{\alpha} = A^{[\alpha]}(i,j)$. The vector $\mathbf{k_i} = \{k_i^{[1]}, \ldots, k_i^{[M]}\}$ keeps track of the degrees of node i across all layers. The sum of this vector represents the total number of edges to which vertex v_i belongs to. The tensor $\mathbf{A}$ and the vector $\mathbf{k_i}$, $i = 1, \ldots, N$ are useful to store the total information about the nodes, separating edges stemming from different networks. To consider the number of times two nodes are directly linked in the network, the matrix $\mathcal{O}$ is defined, whose entries are:

$$o_{ij} = \sum_{\alpha=1}^{M} a_{ij}^{[\alpha]}$$

This matrix can be interpreted as the weight matrix of the aggretate network that rises when merging the different network layers. Summing over all indices $i, j \in \{1, \ldots, N\}$ bears the following

$$\sum_{i=1}^{N}\sum_{j=1}^{N} o_{ij} = 2O$$

where O represents the sum of all the edges of the aggretate network.
For weighted networks, given a set of M different layers on N different vertices, the notation can be exteded by defining the matrices $W^{[\alpha]}$ representing each one the weight matrix of layer α. One gets

$$W^{[\alpha]}(i,j) = \begin{cases} w_{ij}^{[\alpha]} \;\; if(v_i, v_j) \in E_{\alpha} \\ 0 \;\; otherwise \end{cases}$$

as for the interlayer weight matrices

$$W^{[\alpha][\beta]}(i,j) = \begin{cases} w_{ij}^{[\alpha]} \;\; if(v_i, v_j) \in E_{\alpha\beta} \\ 0 \;\; otherwise \end{cases}$$

The three dimentional matrix $\mathbf{W} = (W^{[1]}, \ldots, W^{[M]})$ stores all the information regarding the weights in all layers. The strength of node i in layer α is written as $s_i^{[\alpha]} = \sum_{j=1}^{N} w_{ij} = \sum_{j=1}^{N} w_{ji}$ and this can be defined for each layer, thus allowing to compose the following variable:

$$\mathbf{s}_i = (s_i^{[1]}, \ldots, s_i^{[M]})$$

For what concerns the aggregate weight matrix $\mathcal{O}$, similarly to the unweighted case, its entries are defined as

$$o_{ij} = \sum_{\alpha}^{M} w_{ij}^{\alpha}$$

The sum over all entries of this matrix $\sum_{i,j=1}^{N} o_{ij} = 2O$ computes O that is the overall size of the weighted multiplex.

Besides those first node and mean network measures, some interesting indices can also be introduced in order to measure the similarity of the different layers composing the network. Indeed, one of the aims of the thesis is to understand the extend to which the layers stemming from different data, either governance or ownership structure, bear common features. In this direction it is possible to introduce several notions that quantify those information: the **edge overlap**, the **graph distance** and the **mean entropy**.

Given a multilayer network $\mathcal{M} = (\mathcal{G}, \mathcal{C})$ composed by M layers such that each set of nodes $V_1, \dots, V_M = V$ is the same, the *edge overlap index* of layers $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_{n_\alpha}\}$ on layers $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_{n_\beta}\}$ ($\boldsymbol{\beta} \subset \boldsymbol{\alpha}$) is given by the fraction:

$$EO(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\sum_{i,j=1}^{N} a_{ij}^{\alpha_1} \cdot \ldots \cdot a_{ij}^{\alpha_{n_\alpha}}}{\sum_{i,j=1}^{N} a_{ij}^{\beta_1} \cdot \ldots \cdot a_{ij}^{\beta_{n_\beta}}}$$

The edge overlap between two set of layers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ takes value in $[0, 1]$, where 0 means that there is no edge belonging simultaneously to all layers in $\boldsymbol{\alpha}$, while 1 represents the situation in which all links belonging to all layers in $\boldsymbol{\beta}$ are also present in all layers in $\boldsymbol{\alpha}$.

Other indices similar to this one aim to quantify the degree to which a layer (or set of layers) contributes to the whole structure. These are for example the *contribution* and *structural reducibility* indices. These will be presented in the following paragraphs, as other relevant mathematical definition must be given before them in order to define them.

The other topic that can be introduced is the concept of a *distance* between graphs. This can be used to quantify how similar the structures of different layers in a network are. The authors describe in [22] the construction of a specific distance between connected graphs that is based on the distances between their respective nodes. This can be easily extended to disconnected layers as well.

Theorem 2.1. *Let $\mathcal{G}$ be the set of all undirected, connected graphs insisting on a set of N nodes V. Then the function $\tilde{\mathbf{d}} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ defined as*

$$\tilde{\mathbf{d}}(G_1, G_2) = \sum_{\{u,v\} \in G_1 \times G_2} |d_{G_1}(u, v) - d_{G_2}(u, v)|$$

where $d_{G_1}, d_{G_2} : V \times V \to \mathbb{R}$ are the usual distances over the set V on the graphs G_1 and G_2, is a distance over the set $\mathcal{G}$.

The definition of the distance $\tilde{\mathbf{d}}$ can be easily extended to disconnected graphs in case un unweighted edges. It is possible to define the operator $\mathbf{d} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ over the cartesian product of the set of undirected, possibly disconnected graphs of N nodes $\mathcal{G}$ with itself as:

$$\mathbf{d}(G_1, G_2) = \sum_{\{u,v\} \in E_1 \cap E_2} |d_{G_1}(u, v) - d_{G_2}(u, v)| + \sum_{\{u,v\} \in E_1 \setminus E_2} |N - d_{G_1}(u, v)| +$$

$$+ \sum_{\{u,v\} \in E_2 \setminus E_1} |N - d_{G_2}(u,v)|$$

The extension from $\tilde{\mathbf{d}}$ to $\mathbf{d}$ follows the intuitive idea of imagining to consider disconnected nodes as if they were connected by a path of length N. Transforming the graphs into connected ones, it is possible to recover the old distance definition.
The proof of that $\mathbf{d}$ is a distance can be found in the appendix.

Another useful tool to analyse the different layers composing a network is throughout what is referred to as *mean entropy*. This is a measure of how the overall degree (strength) of the nodes composing the network is spread across the different layers. Considering a multilayer structure $\mathcal{M}$ of M layers, each one insisting on the same set of nodes, in order to describe for each node the distribution of its degree among all the M layers, it is possible to introduce the following quantity:

$$H_i = -\sum_{\alpha=1}^{M} \frac{k_i^{[\alpha]}}{o_i} ln\Big(\frac{k_i^{[\alpha]}}{o_i}\Big)$$

where $o_i = \sum_{j=1}^{N} o_{ij}$ and N is the number of nodes, and regular algebra is extended to infinite values according to the conventions that $0 \cdot \infty = 0$ and $\frac{0}{0} = 0$. This quantity is called **Entropy**[1] and takes value zero if all links for the $i - th$ node are in one layer, while it takes higher values the more the links are equally distributed across the different layers. This quantity helps to understand the extent to which a nodes importance in the multiplex structure is distributed among the different layers.
The mean of these values over all nodes bears the final result

$$H = \frac{1}{N} \sum_{i=1}^{N} H_i$$

The same definition can be given for weighted networks, substituting the vectors k with s and updating the definition of the vectors o accordingly. In formulas

$$H_i = -\sum_{\alpha=1}^{M} \frac{s_i^{[\alpha]}}{o_i} ln\Big(\frac{s_i^{[\alpha]}}{o_i}\Big)$$

and

$$H = \frac{1}{N} \sum_{i=1}^{N} H_i$$

[1] see appendix for details.

2.3.2. Network Projection

Another way of studying a multiplex is by assigning a matrix to it, by either *projecting* or *flattening* the network. The projected network is important when there is no need to discern between different layers. Sometimes, it is useful indeed to aggretate these information in order to study the graph that would appear if all networks were to be merged in one. In this direction, indicate with $MN(N,M)$ the set of (possibly weighted) multi networks composed by N nodes and M layers. Then, the *projection* of order $m \in 1,\dots,M$ of layers $\boldsymbol{\alpha} = \{\alpha_1,\dots,\alpha_m\}$ is the operator $\pi_{\boldsymbol{\alpha}}$ such that:

$$\pi_{\boldsymbol{\alpha}} : \begin{array}{c} MN(N,M) \to MN(N,1) \\ \mathcal{M} \mapsto \pi_{\boldsymbol{\alpha}}(\mathcal{M}) \end{array}$$

where the network $\pi_{\boldsymbol{\alpha}}(\mathcal{M})$ is completely identified by the adjacency matrix $\mathcal{A}_{\pi_{\boldsymbol{\alpha}}}$ whose entries are

$$a_{ij}^{\pi_{\boldsymbol{\alpha}}} = \begin{cases} 1 \; if \;\; \exists \alpha \in \{\alpha_1,\dots,\alpha_m\} \, : \, a_{ij}^{[\alpha]} = 1 \\ 0 \;\; otherwise \end{cases} \qquad i,j = 1,\dots,N$$

and, in case of a weighted network, by the matrix $\mathcal{W}_{\pi_{\boldsymbol{\alpha}}}$ whose entries are:

$$\mathcal{W}_{\pi_{\boldsymbol{\alpha}}}(i,j) = w_{ij}^{\pi_{\boldsymbol{\alpha}}} = \sum_{\alpha \in \{\alpha_1,\dots,\alpha_m\}} w_{ij}^{[\alpha]}$$

Given this notation, considering a network $\mathcal{M}$ of N nodes and M layers, the aggregate *projection matrix* $\mathcal{A}_p = (a_{ij}^p)_{i,j=1}^N$ is defined as the adjacency matrix of the 1 layer network $\pi_p(\mathcal{M}) := \pi_{\{\alpha_1,\dots,\alpha_M\}}(\mathcal{M})$. The degree of node v_i in the aggregate network is

$$k_i^p = \sum_{j=1}^N a_{ij}^p = \sum_{j=1}^N a_{ji}^p$$

summing over all nodes leads to

$$\sum_{i=1}^N k_i^p = 2K_p$$

where K_p represents the total number of links present in the final structure. Note that this construction does not consider if different nodes are linked multiple times in the initial networks. In this sense, some information is lost when considering only the aggregate structures $\mathcal{A}_p$ and K^p.

If the multinetwork $\mathcal{M}$ is weighted, then the projected weight matrix $\mathcal{W}_p$ is the weight matrix of the 1 layer projected network $\pi_p(\mathcal{M})$. Once the matrices $\mathcal{A}_p$ and $\mathcal{W}_p$ are constructed, it is possible to carry one with the single layer analysis presented in the previous chapters in order to understand which are the most important nodes in the projected layer.

This notation is helpful when introducing a new index that aims to quantify the degree to which a layer affects the overall structure. This one is the *contribution* index. Considering a multilayer $\mathcal{M}$ of M different layers insisting on the same set of N nodes V, the **contribution** of a set of m layers $\boldsymbol{\alpha}$ to a set of n layers $\boldsymbol{\beta}$, with $\boldsymbol{\alpha} \subset \boldsymbol{\beta}$ and $m \leq n$, is given by the following index:

$$CO_A(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\sum_{i,j=1}^{N} a_{ij}^{\pi_{\boldsymbol{\beta}}(\mathcal{M})} - a_{ij}^{\pi_{\boldsymbol{\gamma}}(\mathcal{M})}}{\sum_{i,j=1}^{N} a_{ij}^{\pi_{\boldsymbol{\beta}}(\mathcal{M})}}$$

where $\pi_{\boldsymbol{\gamma}}(\mathcal{M})$ represents the projection of the layers $\boldsymbol{\gamma} = \boldsymbol{\beta} \backslash \boldsymbol{\alpha}$ onto one single graph. This quantity ranges in the set $[0, 1]$, indicating the fraction of the number of links to witch the layers $\boldsymbol{\alpha}$ contribute uniquely to build. It is equal to 0 when taking also into account layers $\boldsymbol{\alpha}$ does not lead to new edges in the projected layer $\pi_{\boldsymbol{\beta}}(\mathcal{M})$ with respect to $\pi_{\boldsymbol{\gamma}}(\mathcal{M})$. In case of a weighted network $\mathcal{M}$, the same definition can be given:

$$CO_B(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{\sum_{i,j=1}^{N} w_{ij}^{\pi_{\boldsymbol{\beta}}(\mathcal{M})} - w_{ij}^{\pi_{\boldsymbol{\gamma}}(\mathcal{M})}}{\sum_{i,j=1}^{N} w_{ij}^{\pi_{\boldsymbol{\beta}}(\mathcal{M})}}$$

Again, $CO_B(\boldsymbol{\alpha}, \boldsymbol{\beta})$ takes values in $[0, 1]$.

Another useful usage that can be made of this network representation is for computing the importance of a layer in the overall structure. The index quantifying this feature is linked to what is reffered to as **structural reducibility** measure. This index helps in understanding if different layers can be aggregated in case they bear similar information. It is computed by means of the **Von Neuman entropy**. This concept must not be confused with the one of *entropy* presented in the previous pharagraph. This quantity, rather than computing the distribution of the different node's degrees (or strengths) across layers, calculates the distribution of the degrees (or strengths) of all nodes in a sigle layer. Given a graph $G = (V, E)$ of N nodes, the Von Neuman entropy h_{VN} of the graph is given by:

$$h_{VN} = -Tr(\mathcal{L}_G \cdot log_2(\mathcal{L}_G))$$

where the operator Tr is the trace operator, that is $Tr(B) = \sum_{i=1}^{N} b_{ii}$, and $\mathcal{L}_G$ is the rescaled laplacian matrix associated with the graph G. Recall that $\mathcal{L}_G = c \cdot (D - A)$

where D is a diagonal matrix, having as diagonal entrances the degrees of the nodes, A is the adjacency matrix and $c = 1/\sum_{i,j=1}^{N} a_{ij}$ is a rescaling factor.
Note that, while H_i is a quantity related to a single node, computed over a network, h_{VN} is a measure associated to a layer, computed over all its nodes.
Assuming now to consider a network $\mathcal{M}$ of M layers across N nodes, it is possible to extend the concept of Von Neumann entropy to the whole network. This is given by the following quantity:

$$H_{VN}(\mathcal{M}) = \frac{1}{M}\sum_{\alpha=1}^{M} h_{VN}^{[\alpha]}$$

where $h_{VN}^{[\alpha]}$ is the Von Neumann entropy for layer α. This quantity can now be used in order to calculate structural reducibility. Structural reducibility calculates the VN entropy between a given network and its aggregate, projected representation. Structural reducibility aims to find the best representation $\mathcal{R}$ of the network $\mathcal{M}$ that maximises the quantity:

$$q(\mathcal{R}) = 1 - \frac{H_{VN}(\mathcal{R})}{H_{VN}(\pi_p(\mathcal{M}))}$$

In this notation, $\mathcal{R}$ is a multilayer network composed by m layers, $m \in \{1, \ldots, M\}$, each one being either an original layer of $\mathcal{M}$ or the result of a projection of some of those onto one layer. That is, each layer $\mathcal{R}_\beta$ of $\mathcal{R}$, $\beta = 1, \ldots, m$ is given by one of the two following statements:

$$\exists \alpha \in \{1, \ldots, M\}\ st\ \mathcal{C}_\beta = \mathcal{M}_\alpha$$

or

$$\exists n \in \{1, \ldots, M - m\}, \boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_n\} \subset \{1, \ldots, M\}\ st\ \mathcal{C}_\beta = \pi_{\boldsymbol{\alpha}}(\mathcal{M})$$

where $\mathcal{M}_\alpha$ represents layer α of the network $\mathcal{M}$. It is important to notice that each original layer of $\mathcal{M}$ can be used to construct *one and only one* layer of $\mathcal{R}$.
The index $q(\mathcal{R})$ is called *structural reducibility quantity*. It calculates the relative entropy between an aggregate representation $\mathcal{R}$ of $\mathcal{M}$ and the projected network $\pi_M(\mathcal{M})$. The larger q, the more distinguishable the layer is from the projected one. For example, supposing that the multilayer $\mathcal{R}$ is composed by n identical layers, then $q(\mathcal{R}) = 0$ as the projected graph $\pi_M(\mathcal{M})$ and the network $\mathcal{R}$ bear the same entropy. The best representation of $\mathcal{M}$ is the one maximising q, and this is because an increase in q usually corresponds to the merge of layers having very similar structure [23]. Thus, by maximising this quantity, one tends to avoid using a representation that might contain redundant layers. In case then $argmax(q) = \mathcal{M}$, this would mean that all layers bring relevant different information. In other cases, the merging of different layers implies that those are similar to each other and contribute likewise to the final information provided.

2.3.3. Network Flattening

Another way of representing the same network throughout a single layer is by constructing the *flattened matrix* $\mathcal{A}_f$. This description of the networks differs from the one just described, because it does not neglect the difference of the layers by projecting them on the same adjacency matrix. Given a network $\mathcal{M}$ of M layers, each one insisting on the same set of N nodes $V = \{v_1, \ldots, v_N\}$, the flatten adjacency matrix $\mathcal{A}_f$ is an $NM \times NM$ block matrix. The diagonal entries, i.e. M different $N \times N$ matrices, are the adjacency matrices of the M different layers. The off diagonal entries are on the other hand represented by the $N \times N$ Identity matrices. These represent the interlayer connection between nodes. Those interlayer edges connect the same nodes over the different layers. The block matrix $\mathcal{A}_f$ is given thus by:

$$\mathcal{A}_f = \begin{bmatrix} A_1 & I_d & \cdots & \cdots & I_d \\ I_d & A_2 & I_d & \cdots & I_d \\ \vdots & I_d & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & I_d \\ I_d & I_d & \cdots & I_d & A_M \end{bmatrix}$$

whre I_d represents the $N \times N$ identity matrix. The construction of $\mathcal{A}_f$ is perfectly in line with the first original network definition, were the interlayer edges are used to match the same nodes over the different layers. It is also possible to extend this definition to weighted graphs, however the flattened representation is more useful when thought in terms of *cost* of an edge, rather than weight of it. Recall that the cost of an edge is a positive real number that is assigned to that edge, representing the "price" to pay, the distance to travel in order to move from one node to another. Given the same network structure as before, but assigning moreover to each layer an $N \times N$ cost matrix C_i, $i = 1, \ldots, M$, the overall flattened cost block $NM \times NM$ matrix is defined as:

$$\mathcal{C}_f = \begin{bmatrix} C_1 & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & C_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C_M \end{bmatrix}$$

where $\mathbf{0}$ represent the null $N \times N$ matrix. Once such a matrix has been introduced then it is again possible to complute the classical single layer centrality measures. It is important to notice that in this case, as the off diagonal entries of the block matrix $\mathcal{C}_f$ are the null matrices, there is no cost in moving from one layer to the other. In other words, there is no difference in thinking of a node as belonging to one layer rather than to another. One point has to be highlighted when considering both the projected and flattened representations of a network. The matrices $\mathcal{A}_p$ and $\mathcal{A}_f$ represent the same multi layer structure, but bear different features. The first one projects all layers on a single one; moreover one node is associated with only one column (row) of the matrix. In the second representation on the other hand, to one original node correspond M different columns (rows). For these reasons, also the analysis that are going to be brought on them might lead to different results, for example in identifying the most important node according to a specific centrality measure.

3 | Dataset Review

The dataset considered regards companies listed in the italian stock exchange, the Borsa Italiana. Borsa Italiana was founded in 1998 after the privatization of stock exchanges in Italy; its role is to attend to the organisation and management of the financial market in Italy. Since the 29th of April 2021 it has been part of the Euronext Group that regulates markets in Milan as well as in Amsterdam, Bruxelles, Dublin, Lisbon, Oslo and Paris.
According to the official website of Borsa Italiana, there were 326 listed companies in 2020 at the Italian Stock Exchange, considering MTA and the companies admitted to Global Equity Market [16]. In 2020 the overall market capitalization reached more than 600000 million Euros and accounted for around 37% of the italian GDP [16].
Out of the 326 listed firms, the dataset collects information for 234 companies for which data is avaliable regarding either ownership structure or BoD composition. Information regarding BoD composition was collected either from the private websites of the different companies or directly from the Borsa Italiana web page. The dataset concerning ownership structure was retreived from the official website of CONSOB, the italian authority for the vigilance of financial markets. All information are to be considered as of end of 2020. It is worth noticing that, considering the major listed companies in terms of market capitalization, the dataset does not include the holding *Exor N.V.* and associated companies like *Stellantis N.V*, *Ferrari S.p.A.*, *CNH Industrial N.V.* nor the companies *Campari Group* and *Tenaris S.A.* for which no data was found online.

3.1. Board composition

The dataset regarding the composition of the Board of Directors was collected by looking through the reports of corporate governance and the reports on remuneration and compensation paid. Out of the 234 companies for which information was retrieved, it was possible to find data regarding BoD for 201 different companies.
For each company, the dataset shows the composition of the Board of Directors as of end of 2020 or latest information avaliable; for every director basic personal data are reported as name, gender and age. It was also possible to retrieve information about the profes-

sional position that they represent as for example their role in the BoD, if they classify as independent, the number of Committees they are member of and their yearly remuneration. For companies that adopt a two-tier system[1], rather than a one-tier system[2], the members of the Management Board were considered as directors. For the aim of this thesis and for the creation of a network among the considered companies based on the presence of possible interlocking between them, the solely information of interest is the name of the different directors. In the following lines, a small descriptive analysis of the dataset is given in order to grasp the main features about it.
The average number of directors sitting on a board is 9,8. The company that presents the smallest BoD, composed by only 2 members, is *S. S. Lazio*, which is also the only company adopting a two tier system. The maximum number of directors sitting on a board is 19. The company presenting the most numerous BoD is *Unipol Group S.p.A.*. The following bar plot shows the number of directors composing the boards across the different firms.

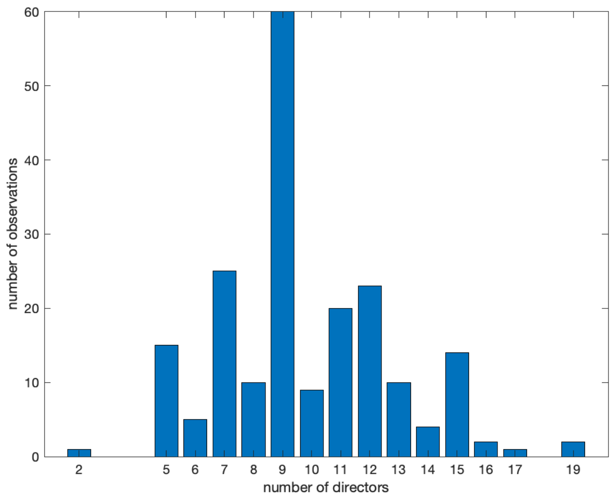


Figure 3.1: Board of Director composition across firms

For what concerns the people involved, 1780 different persons span over 1971 different director positions. That means that, on average, a director sits on 1,1 different boards. Out of the 1780 different persons, 163 occupy more than one position across the considered companies and thus account for creating an interlock between firms; in percentage this means that a little more than 9% of the directors hold more than one position. The maximum amount of roles taken by the same person is 5. The following pie chart helps in visualizing these informations.

[1] a two-tier system is a type of company governance structure in which the management and supervisory tasks are separated across two different boards.
[2] a one-tier system is a type of company governance structure composed by only one board of directors.

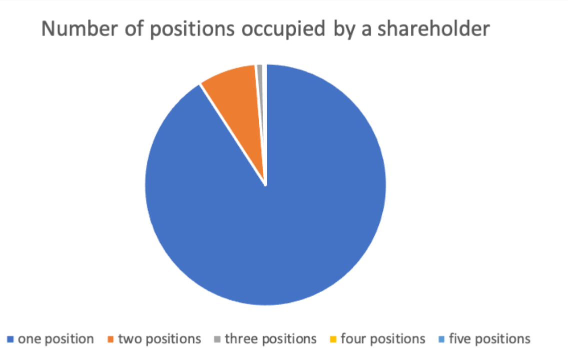
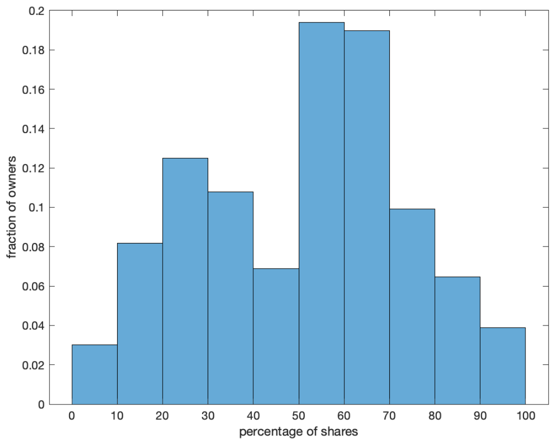


Figure 3.2: Shareholders' occupied positions

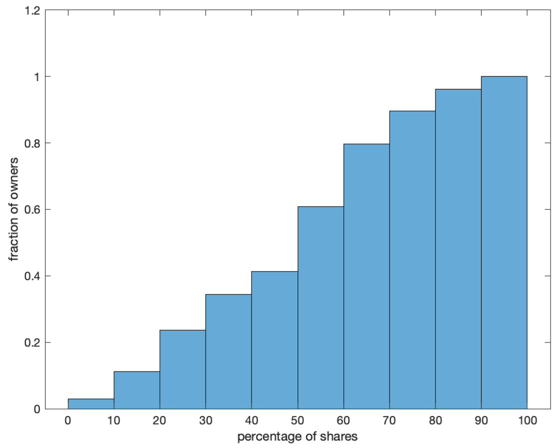
As it can be seen, the majority of directors only sit on one board among the ones considered in the dataset.

3.2. Ownership structure

For what concerns the ownership structure of the different companies, it was possible to retrieve information for all the 234 companies but *D'amico S.p.A.* and *IVS Group*, two companies working in the maritime transportation and food service sectors respectively. For both firms though, information about BoD composition was avaliable. For every company, the dataset shows, among other info, the name of the different relevant shareholders along with the percentage of shares they hold. A relevant shareholder is one whose percentage of shares is above 2%. It also shows the shareholders' societary form (for example if it is a fund, a public traded company, a bank etc).
By looking through the dataset, a few features pop up that are worth noticing: first of all, it turns out that the average percentage of shares hold by a relevant shareholder is around 50%. At the same time, 134 companies, close to 58% of the sample, present their most relevant shareholder to hold more than 50% of the total shares. This information is in line with the one presented in [6] that showed how, in Italy, around half of the companies are fully controlled by their major shareholder. The following histograms show how the percentages of shares hold by relevant shareholders are distributed across the firms considered in the dataset.



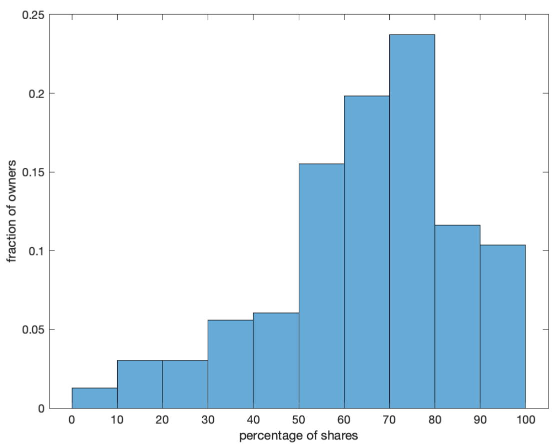
(a) Ownership distribution



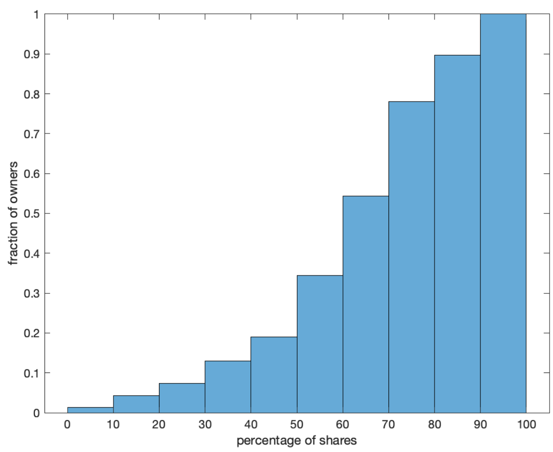
(b) Cumulative ownership distribution

Figure 3.3: Main sharelholders' ownership distribution

In (b) it appears clear how over 50% of the relevant shareholders hold more than half of the tradable shares, while from (a) we can retrieve how overall relevant shareholders tend to own a considerate amount of tradable shares. Moreover, relevant shareholders hold a combined overall percentage of shares for each company above 50% in 187 companies, that is around 80% of the sample. Also this index is in line with what was found in [6]. The average combined percentage of shares hold by relevant shareholders is little more than 65%, and the average number of relevant shareholders for each company is 2,6, with a standard deviation of 1,8. The following histograms visualise those information while showing once again the marginal and cumulative distribution of the sum of all relevant shareholders owned equity for single firms.



(a) Ownership distribution



(b) Cumulative ownership distribution

Figure 3.4: Combined relevant shareholders' ownership distribution

For what concerns the number of relevant investors, the company with the highest number of them (9) is *Hera S.p.A.*, a multi-utility company founded in Bologna. For this company, the relevant shareholders are mainly represented by public investors. The graph below shows the distribution of the number of relevant shareholders for each firm.

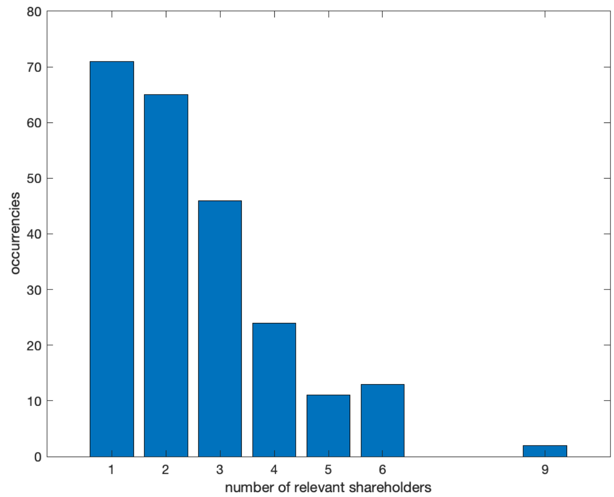


Figure 3.5: Number of relevant shareholders' distribution

Considering the category to which different shareholders can belong to, again in line with what [6] found, it turns out that the majority of shareholders are private investors. The dataset separates between 14 different categories to which a shareholder can belong to.

Categories	Shortening
Bank	BAN
Cooperative	COP
Fundation	FOND
Fund	FP/GR
Insurance	ASS
Other entity	EV
Private investor	PF
Public entity	EP
Società in Accomandita per Azioni	SAPA
Società in Accomandita Semplice	SAS
Società in Nome Collettivo	SNC
Società per Azioni	SPA
Società Semplice	SS
Trust company	TRUST

Table 3.1: Different Shareholders categories

The table above summarises all the possible categories. In this dataset, around 84% of the firms have as their first owner either a single person, a public traded company, an entity belonging to the public sector or an SRL firm. Below the number of companies divided with respect to the category of their first shareholder.

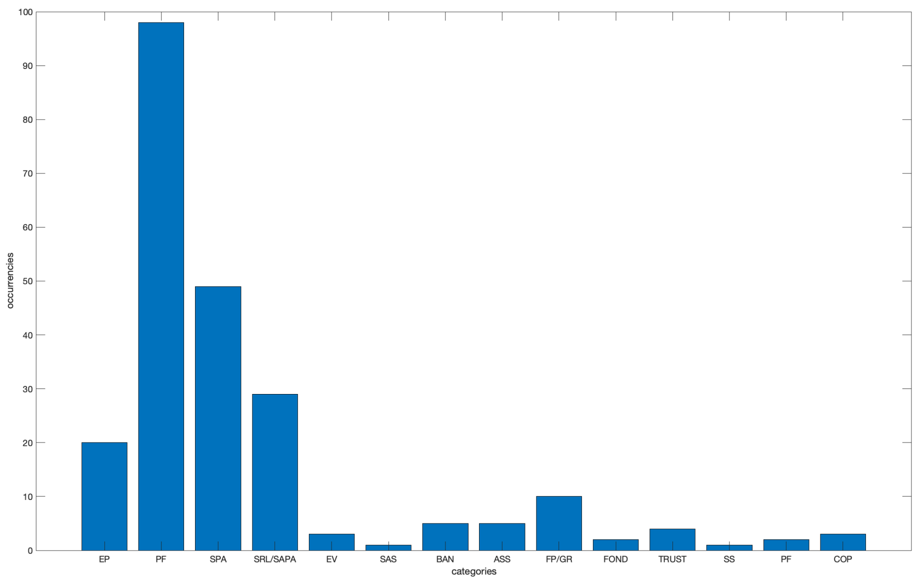


Figure 3.6: First shareholder category distribution

4 | Network construction

This chapter will present the methodology followed for constructing the different layers. When considering interlocking directorates, the construction of links among companies does not require a similarity index. The construction of the layer based on ownership similarity on the other hand is based on one. For each dataset, two types of layers will be constructed. Concerning the BoD composition, we consider two typers of interlokcing: a direct and an indirect one. The latter relaxes the definition of interlock, and allows for more links to be born in the final layer. As for what concerns the dataset on companies' ownership structure, the first layer aggregates those companies that present a similar shareholder composition. This is done by introducing a similarity index. That means that firms whose equity is distributed across shareholders in a similar way are regarded linked. The relevant information does not consider who is controlling firms, but how the shareholders structure looks like. In this, the similarity index tries to differentiate f.e. among heavily controlled companies rather than ones that do not have a big major owner. Finally, the last layer takes into account what are referred to as *ownership ties*; in other words, it links firms that have a shareholder in common. The following pages present the construction of the different adjacency and weight matrices representing the layers.

4.1. Interlocking network construction

For the construction of the first layers based on possible interlocking among companies, the dataset regarding the different Boards of Directors composition is considered. In this chapter, direct and indirect interlocks are presented. For the sake of notation, let V be the set of all companies present in this dataset and $n = \#V$ the number of all firms taken into account.

4.1.1. Direct Interlocking

Indicate with $v_i \in V$, $i \in \{1, \dots, n\}$ a general company, and with $d_i = \{d_{i,1}, \dots, d_{i,m_i}\}$ the m_i directors sitting on its board. When considering only direct interlocking, the adjacency matrix A is constructed as follows:

$$A_{di}(i,j) = \begin{cases} 1 \;\; if \;\; \#\{d_i \cap d_j\} \geq 1 \\ 0 \;\; if \;\; \#\{d_i \cap d_j\} = 0 \end{cases} \qquad i,j = 1,\ldots,n \;\; i \neq j$$

In this case interlocking occurs only if two different companies share a director. The correspondent weight matrix W is the following:

$$W_{di}(i,j) = \frac{\#\{d_i \cap d_j\}}{M} \quad i,j = 1,\ldots,n \;\; i \neq j$$

where $M = \max_{i \neq j}\{\#\{d_i \cap d_j\}\}$. By definition, $W(i,j) \leq 1$ for every i,j.

Indirect Interlocking

In literature [21], another more broad definition rather than the one just presented is given for identifying an interlock. Under this conception, for an interlock to occur it just needs to happen that directors of different firms also sit, together, on a board. For example, imagine that a is a member of the BoD of company A and b is a member of the BoD of company B. Suppose that both a and b also sit on the BoD of a third company, C. In this situation, a link will be created between A and B (and, of course, also between A and C and B and C). Mathematically speaking, this can be modelled as follows:

$$A_{ii}(i,j) = \begin{cases} 1 \;\; if \;\; \exists k \in \{1,\ldots,n\}, l \in \{1,\ldots,m_i\}, h \in \{1,\ldots,m_j\} : d_{i,l}, d_{j,h} \in d_k \\ 0 \;\; otherwise \end{cases}$$

$$i,j = 1,\ldots,n \;\; i \neq j$$

As for the direct interlocking case, the weight matrix is constructed as:

$$W_{ii}(i,j) = \frac{\chi_{ij}}{\max_{ij} \chi_{ij}}$$

where:

$$\chi_{ij} = (1 - \delta_{ij}) \cdot \#\{k \in \{1,\ldots,n\} : \exists l \in \{1,\ldots,m_i\}, h \in \{1,\ldots,m_j\} : d_{i,l}, d_{j,h} \in d_k\}$$

and δ_{ij} is the Kronecker's delta, i.e.

$$\delta_{ij} = \begin{cases} 1 \;\; if \;\; i = j \\ 0 \;\; otherwise \end{cases}$$

4.2. Ownership similarity index

The construction of the first ownership layer, as said before, is based on the definition of a similarity index, which is deduced from a distance measure. According to how this index is constructed, the final layer presents different features. The second layer takes into account possible ownership ties. The following paragraphs highlight the guidelines followed for leading this procedure. Consider again the notation of the previous paragraphs for the set of n companies V.

4.2.1. Ownership Similarity

From the dataset regarding the ownership structure, the information taken into account for constructing this layer are: the percentage of equity owned by the biggest shareholder (p_{s1}), the percentage of combined equity hold by relevant shareholders (p_{rs}), the number of relevant shareholders (n_{rs}) and finally the category to which a relevant shareholder can belong to (c). Recall that a shareholder is regarded to as "relevant" if the percentage of equity he/she owns is greater than or equal to 2%. The final distance between two companies will be constructed as a weigthed average between the marginal distances of the two in the aforementioned aspects. As the numbers p_{s1} and p_{rs}, indicated as fractions, are real numbers belonging to the interval [0,02 1] it is possible to compute the distance of two firms in terms of those information using simply the d_∞ distance. If the set of n firms is $V = \{V_1, \dots, V_n\}$, then the distances in terms of equity percentage of main shareholder and relevant shareholders between the firms $v_i, v_j \in$ are respectively:

$$\tilde{d_{s1}}(v_i, v_j) = |p^i_{s1} - p^j_{s1}| \qquad i, j \in \{1, \dots, n\}$$

and

$$\tilde{d_{rs}}(v_i, v_j) = |p^i_{rs} - p^j_{rs}| \qquad i, j \in \{1, \dots, n\}$$

where the apices in this and the following notations represent the indices identifying the companies considered. When considering percentage of shares, an important threshold is usually set to 50%[15]. Indeed, this is the percentage of equity that is usually needed to gain full control of a company. That means that there is a relevant difference between holding an amount that is more rather than less than the half of traded shares. This is why both those distances are adjusted by a factor that takes into account this consideration.

The distance measured between two companies thus results in:

$$d_{s1}(v_i, v_j) = \mathbb{1}_{\{A_{ij}\}}(p^i_{s1}, p^j_{s1}) + \tilde{d_{s1}}(c_i, c_j) \cdot \mathbb{1}_{\{A^c_{ij}\}}(p^i_{s1}, p^j_{s1})$$

and

$$d_{rs}(v_i, v_j) = \mathbb{1}_{\{B_{ij}\}}(p^i_{rs}, p^j_{rs}) + \tilde{d_{rs}}(c_i, c_j) \cdot \mathbb{1}_{\{B^c_{ij}\}}(p^i_{rs}, p^j_{rs})$$

where A_{ij} and B_{ij} are respectively given by

$$A_{ij} = \{(p^i_{s1}, p^j_{s1}) \in [0,1] \times [0,1] \quad | \quad (p^i_{s1} > 0,5 \wedge p^j_{s1} \leq 0,5) \vee (p^i_{s1} \geq 0,5) \wedge p^j_{s1} < 0,5)\}$$

$$B_{ij} = \{(p^i_{rs}, p^j_{rs}) \in [0,1] \times [0,1] \quad | \quad (p^i_{rs} > 0,5 \wedge p^j_{rs} \leq 0,5) \vee (p^i_{rs} \geq 0,5) \wedge p^j_{rs} < 0,5)\}$$

and the complementary is taken with respect to the set $[0,1] \times [0,1]$. This basically means that the distance d_{s1} (respectively d_{rs}) among two companies is 1 if one is completely controlled by its first main shareholder (respectively relevant shareholders) and the other one is not. If this is not the case, then d_{s1} (respectively d_{rs}) is just given by the absolute value of the difference of the amount of shares owned by the first shareholder (respectively relevant shareholders).
When considering the number of relevant shareholders, the distance between two firms v_i and v_j according to this criterion can be computed as $|n^i_{rs} - n^j_{rs}|$ normalised by a scaling factor. In formulas

$$d_{n_{rs}}(v_i, v_j) = \frac{|n^i_{rs} - n^j_{rs}|}{\max_i n^i_{rs} - \min_i n^i_{rs}} \qquad i, j \in \{1, \ldots, n\}$$

As for what concerns the category to which the most relevant shareholder belongs to, in this sense the distance considered has a binary form as defined below:

$$d_c(v_i, v_j) = \mathbb{1}_{\{v^i = v^j\}}(v_i, v_j) \qquad i, j \in \{1, \ldots, n\}$$

To compute the overall distance among companies while taking into account all these information, one can sum up the squares of those quantites and taking the root of the final score.

$$d(v_i, v_j) = \sqrt{d_{s1}(v_i, v_j)^2 + d_{rs}(v_i, v_j)^2 + d_{n_{rs}}(v_i, v_j)^2 + d_c(v_i, v_j)^2} \qquad i, j \in \{1, \ldots, n\}$$

Because of the normalizations imposed, $d(v_i, v_j) \in [0,1]$. Finally, the similarity index s is thus defined as

$$s(v_i, v_j) = 1 - d(v_i, v_j) \qquad i, j \in \{1, \ldots, n\}$$

Once such an index is introduced, it is rather straightforward to define the weight and adjacency matrices for the set of companies V. In order to avoid to build a fully connected network, what is missing to be identified is the minimum threshold t that the similarity value must reach in order for a link to be born. The weight and adjacency matrices will thus be defined as

$$W(i,j) = \begin{cases} \frac{s(v_i,v_j)-t}{1-t} \; if \; i \neq j \;\; \wedge \; s(v_i,v_j) > t \\ 0 \;\; otherwise \end{cases}$$

and

$$A(i,j) = \begin{cases} 1 \;\; if \; s(v_i,v_j) > t \;\; \wedge \; i \neq j \\ 0 \;\; otherwise \end{cases}$$

The choice of the threshold has a great impact on the network. Indeed, choosing low levels of t would result in a fully connected network, while values for t close to 1 would lead to a graph with very few (if any) edges.

4.2.2. Ownership Ties

The next layer concerns again the ownership structure of listed firms and it regards what are usually referred to as *ownership ties* [15]. Those arise between two firms when either one of them is holding an amount of shares in the other that is greater than 2%, or if the two companies present a shareholder in common. In this freamework, the links are undirected, thus there is made no distinction between which company is investing in which. Because of this, the resulting adjacency and weight matrices are of course symmetric. Given a set of n companies $V = \{v_1, \ldots, v_n\}$, the $n \times n$ matrix N_{ot} is calculated as follows:

$$N_{ot}(i,j) = \#\{S_i \cap S_j\} + \mathbb{1}_{\{S_i\}}(v_j) + \mathbb{1}_{\{S_j\}}(v_i)$$

where S_i and S_j represents the set of main shareholders of companies v_i and v_j respectively. N_{ot} represents the number of links that bound all companies. From this matrix, the weight and adjacency matrices W_{ot} and A_{ot} are computed as:

$$W_{ot}(i,j) = \frac{N_{ot}(i,j)}{\max_{ij} N_{ot}(i,j)}$$

$$A_{ot}(i,j) = \begin{cases} 1 \;\; if \;\; W_{ot}(i,j) > 0 \\ 0 \;\; otherwise \end{cases}$$

5 | Layers Analysis

This chapter presents the analysis of the 4 introduced layers. Those are here studied singularly and independently. For each layer, few main features about its structure are presented, as the average values of the most relevant indices and centrality measures. Afterwards, a more in depth analysis is brought on by trying to identify which are the main components of the layer and the most relevant nodes according to different measures. The graphs representing the different layers have been plotted using the software `Gephi`, while all other computations, as for example the creation of the adjacency matrices or the centrality measures have been implemented in `MATLAB` and `Excel`.

5.1. Interlocking Layers

201 companies out of the 234 that compose the dataset of study present information regarding the composition of their Board of Directors. To each of those, a unique number is assigned in order to properly identify it. Writing this in terms of the mathematical notation previously introduced, the set of companies is the set $V = \{v_1, \ldots, v_{201}\}$.

5.1.1. Direct Interlocking

The 201×201 weight and adjacency matrices W_{di},A_{di} built following the procedure exposed in the previous chapter considering direct interlocking completely identify the final layer. The graph is presented in the following picture:

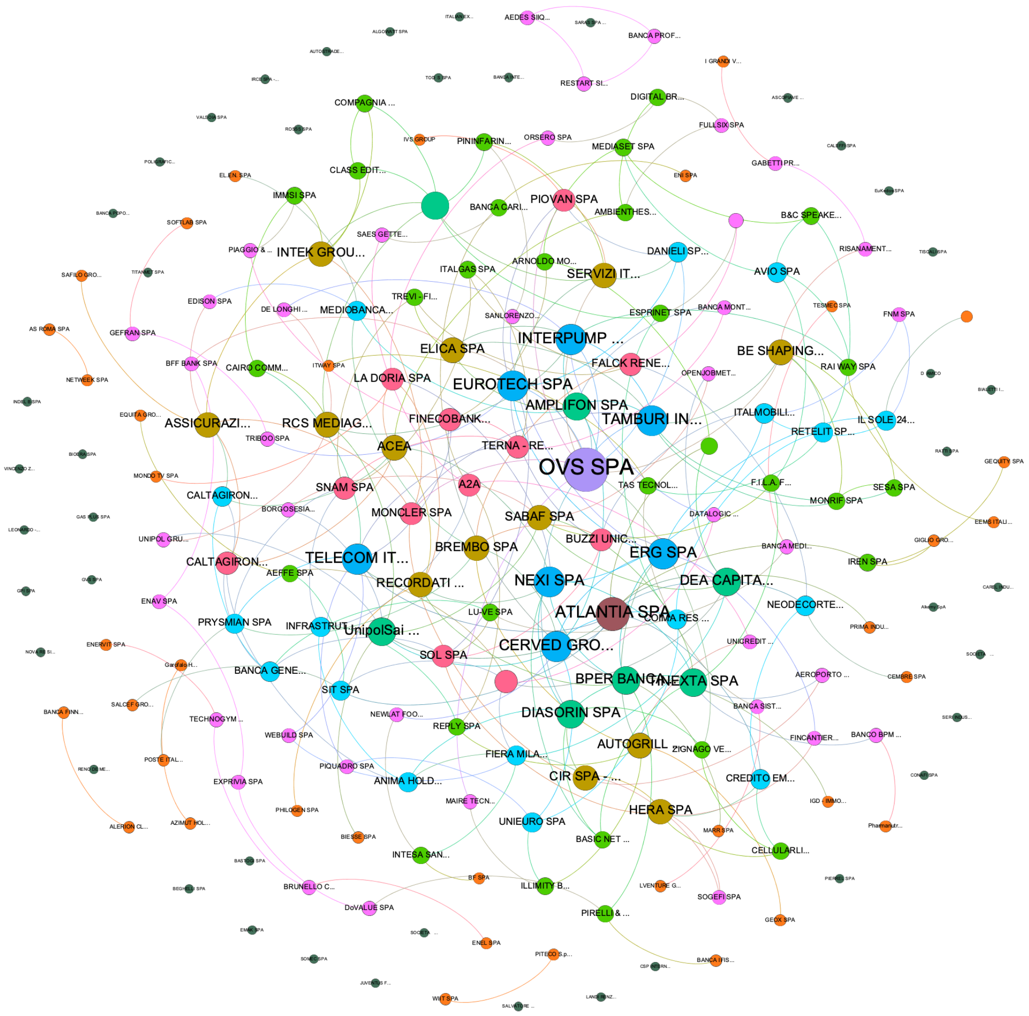


Figure 5.1: Direct Interlocking layer

The layer is composed by 201 nodes and 269 edges. The nodes' color and size reflect the degrees of the correspondent companies. Bigger nodes are the ones that present more connection in this layer. Companies depicted in pink are the ones that do not ingage in interlocks. Out of 201, 38 firms are the ones that are not connected to any other, that is around 19% of the total sample. First thing to notice is that the layer presents one giant component, a few disconnected components and then, as said before, 38 isolated nodes. 148 firms, around 73% of the ones considered belong to the giant component. In order to capture other basic features of this first layer, the average values of the main node measures are reported in the following table.

Measure	Average Value
Degree	2,67
Strength	0,33
Clustering	0,26
Closeness	$6{,}2{\cdot}10^{-4}$
Eccentricity	$5{,}7 \cdot 10^{-3}$
Betweenness	$1{,}89{\cdot}10^{2}$
Diameter	99

Table 5.1: Average layer measures

While betweenness is difficult to interpret when it comes to average layer values, other quantities are useful. The average degree and strength show how the layer is far from being highly connected. The clustering coefficient, which by definition ranges in the interval [0 1], shows that not many local clusters arise. As for closeness and eccentricity centrality, a node present the highest scores in these measures when it is connected with every other node and the edges present the less possible cost. As the cost is here computed as the inverse of the weight, and given the fact that the maximum weight is 1, the smallest distance between two nodes is 1. Thus, in such a layer composed by n nodes, the maximum closeness and eccentricity scores are respectively $\frac{1}{n-1}$ and 1. Again, the average scores presented in table 5.1 shows how the layer generally presents few and light edges. The company presenting the most edges connecting it to the rest of the layer is *OVS S.p.A.*, with 13 different interlocks. The degree distribution across all firms is shown below:

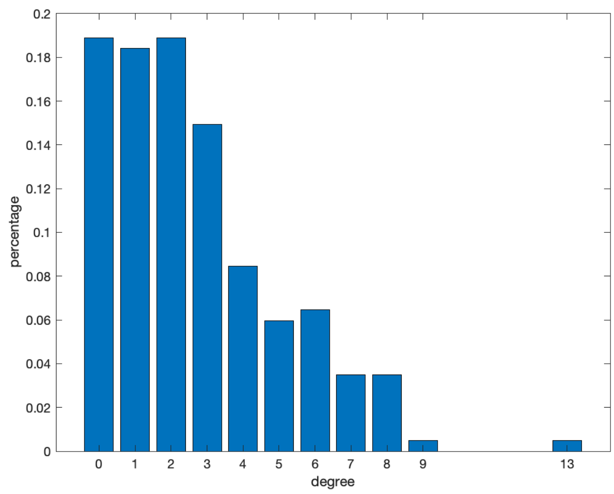


Figure 5.2: Degree distribution

The majority of the companies present less then 4 links. As for the degree distribution, it is shown in the following bar plot:

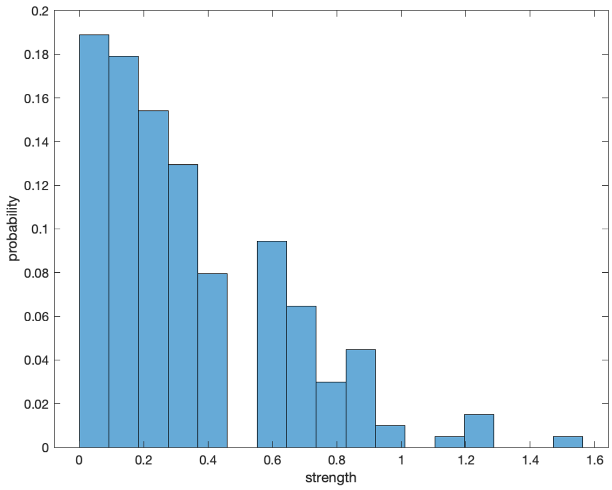
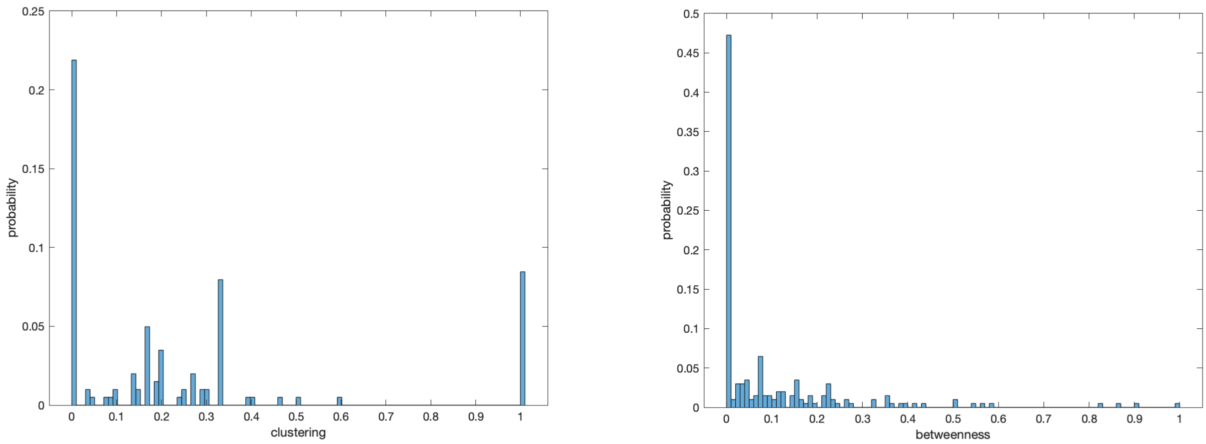


Figure 5.3: Strength distribution

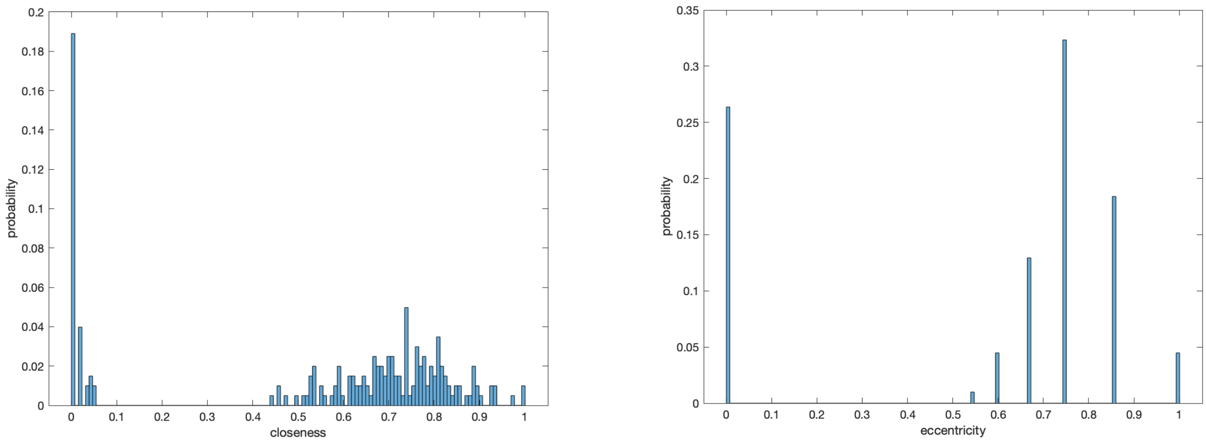
The histogram plots the different strength that a node can take in this layer against the number of occurrencies. The maximum strength is less than 1,6 as nodes tend to belong to few edges. Again, the maximum strength is hold by *OVS S.p.A.*
The clustering coefficient and the betweenness centrality measure for all nodes are presented in the following histograms.



(a) Clustering distribution (b) Betweenness distribution

Figure 5.4: Centrality distributions

Histogram (*a*) shows how the nodes belonging to local clusters are not the majority of them all. Beside 17 companies that present the maximum reachable clustering coefficient, other companies tend to present very low levels for this index. Picture (*b*) plotting the betweenness centrality measure has been normalised for the greatest value reached by a node for this quantity. It shows that the most relevant firms according to this index is again *OVS Group*.



(a) Closeness distribution

(b) Eccentricity distribution

Figure 5.5: Centrality distributions

Figure 5.5 shows the distribution of closeness and eccentricity centrality over the layer, again normalised so to fit in the interval [0 1]. The two measures present a similar trend, although eccentricity is more concentrated in fewer values. This is a consequence of its definition that takes into account the maximum of the distances of node instead of the overall sum. According to closeness centrality *OVS Group* is once again the most relevant node. This is also true when considering eccentricity, but in this case along with other 8 companies.

5.1.2. Indirect Interlocking

The layer structure stemming from the construction of indirect interlocks is the following:

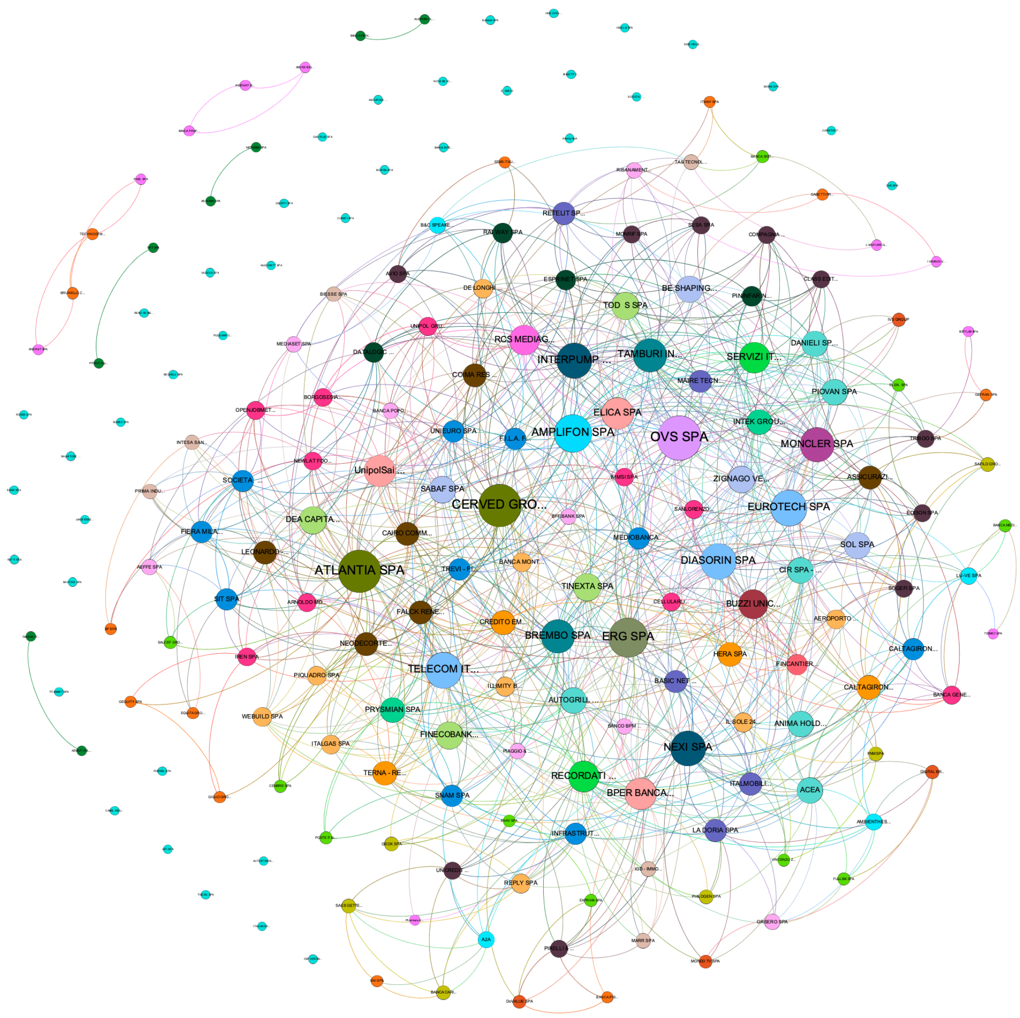


Figure 5.6: Indirect Interlocking

Exactly as in the prior case, a giant component is present to which the majority of the firms belong to. This is composed by the same 163 firms that composed the giant component in the prior layer. Moreover, the same 38 firms that were isolated in the prior layer are isolated in this structure as well. The number of total edges though changes drastically; in fact, from 268 edges for the direct interlocking layer, the number of links among companies raises to 1020. This difference is underlined by the following average measures:

Measure	**Average Value**
Degree	10,15
Strength	2,19
Clustering	0,62
Closeness	$1{,}1\ \times 10^{-3}$
Eccentricity	$1{,}8\ \times 10^{-2}$
Betweenness	81,00
Diameter	29,75

Table 5.2: Average layer measures

Indeed, mean degree and strength take higher values due to the higher number of connections. Also closeness and eccentricity tend to show smaller values on average. Clustering coefficient raises to almost the double of its prior value, showing how more local clusters arise. The following bar plots show again the distribution of those measures across all nodes.

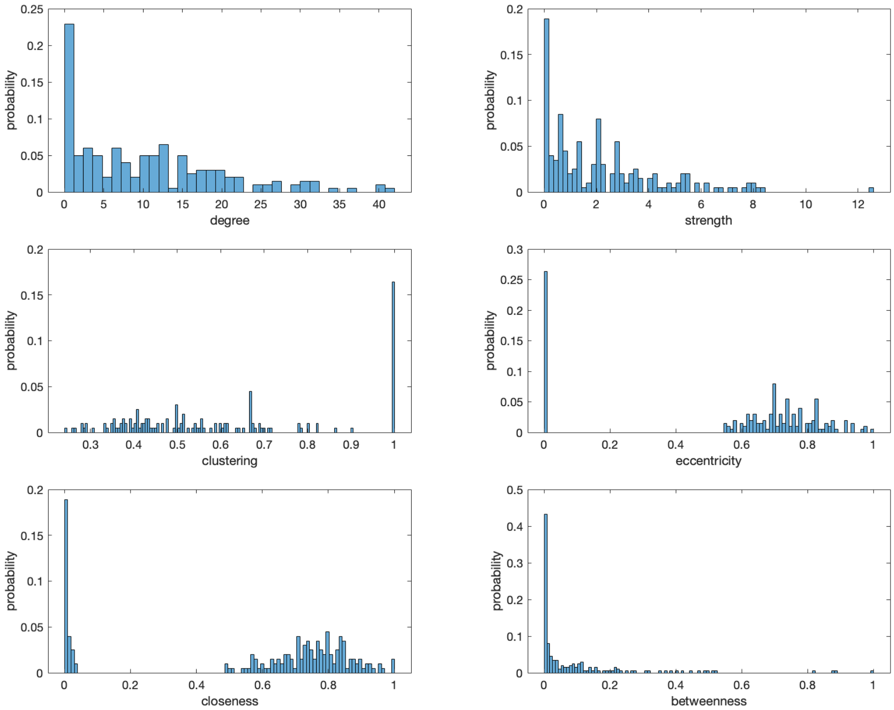


Figure 5.7: Main layer measures' distribution

The node regarded as the most important according to degree, strength, closeness and eccentricity measures is again *OVS S.p.A.*. On the other hand, betweeneess centrality indicates *Atlantia S.p.A.* as the most relevant node in the layer. Both companies present a clustering coefficient smaller than 0,3 indicating that for both firms less than 30% of companies to which they are linked are directly joined to each other.

5.2. Ownership Layer

The study of the layer based on ownership data regards 232 companies. 33 new companies are introduced with respect to the prior layer; firms *D'Amico S.p.A.* and *IVS Group* do not belong to the dataset anymore.

5.2.1. Ownership Similarity

The construction and analysis of the ownership layer is subordinated to the choice of the threshold t to indicate when it comes to building links among companies. The threshold t is important because without it the layer would result in being totally connected, in the sense that each company would be linked to every other. For this layer the choices on this parameters is $t = 0.875$. The reason behind this value is driven by the outcome of the similarity index construction. Indeed, the distribution of the similarity index s over all firms is the following:

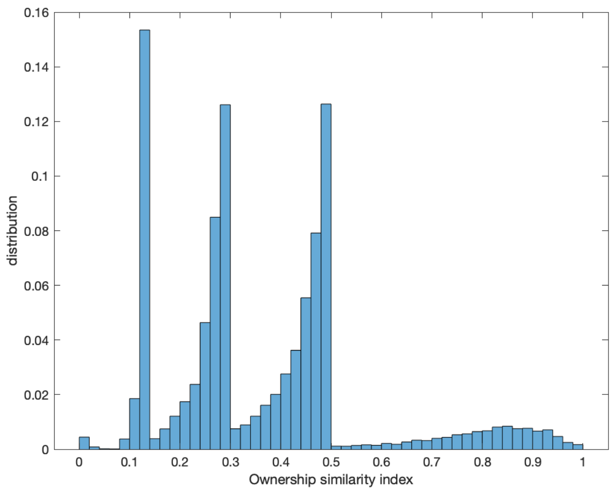


Figure 5.8: Similarity index distribution

As one can clearly see, the distribution of s follows piecewise increasing patterns alterned with drastic decreases. The values of s in correspondance with which the spikes occurs are not random, but depend on the choice of the distance introduced. This distance depends on 4 different quantities. Assuming two companies to present the exact same features in only one of those and to be completely different in all the others, the similarity index in this case becomes equal to around $0,134$. If they match in only two of those, the similarity rises to around $0,293$, and to $0,5$ when 3 of their discriminants perfectly matches. The spikes in the distribution of the similarity index are in correspondance with those values. For the dataset taken into consideration, no companies present the same exact features for all four attributes. Setting a threshold equal to $0,875$ means that companies are regarded as similar in case their similarity index equals the one of two companies that are the same in at least 3 of their main attributes, and present the distance in the last attribute to be less than 0,25 (on a scale from 0 to 1). The final structure of the graph is shown here below:

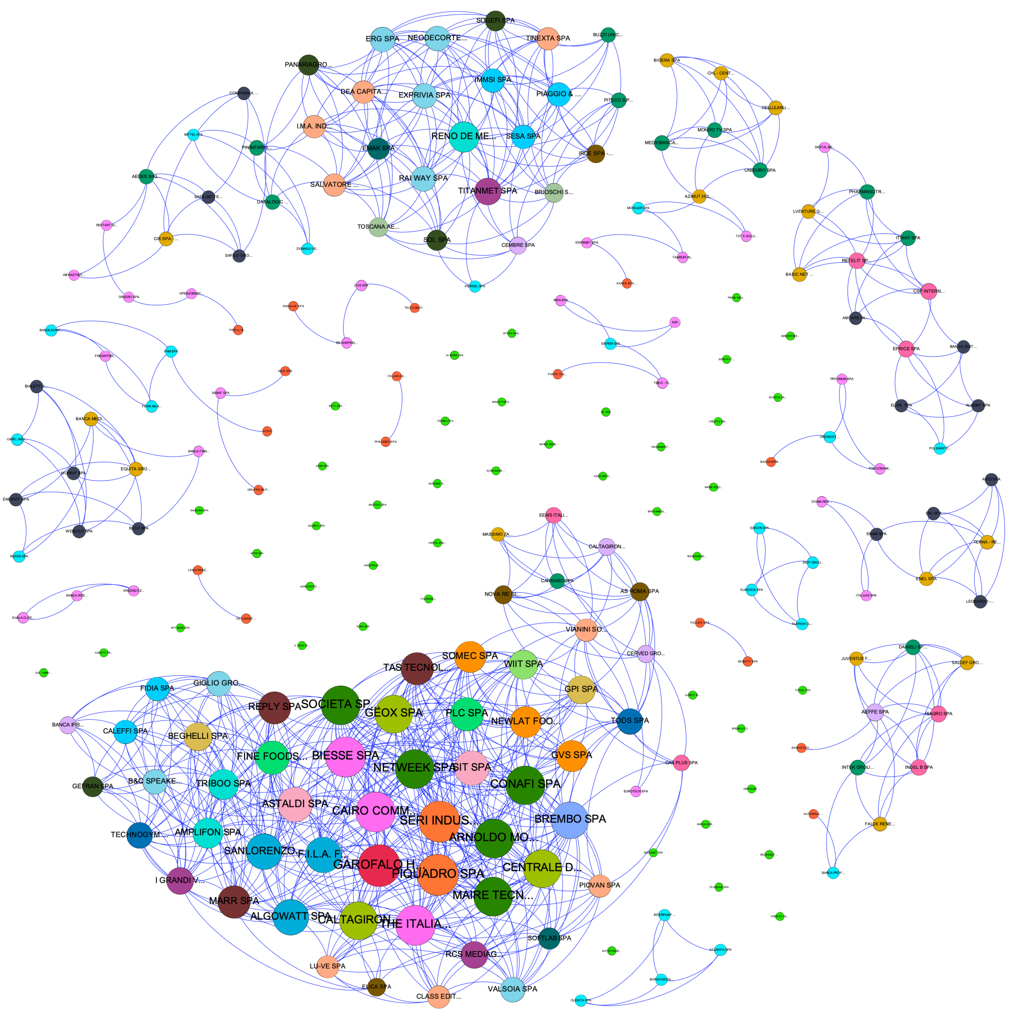


Figure 5.9: Ownership layer 1

The graph results in not being connected and being composed by 879 edges. It decomposes in two main giant components and other smaller disconnected ones. Overall, the number of connected components is 62, when including also isolated nodes. The number of those nodes adds up to 43. The main structure of the graph is represented by a giant component, formed by 59 nodes. This means that more than 22% of the firms belong to the giant connected component. It is interesting to observe how the ownership structure looks like for the companies belonging to the two main giant components. As the lower threshold t has been set to $0,875$, looking at a couple of companies for each component gives an idea on how the mean ownership structure for it looks like. It turns out that firms belonging to the main giant component are heavily controlled by their first shareholder who owns more than 50% of total shares. Others relevant shareholders do not bear a significant importance in those companies. The number of relevant shareholders is low, usually less than 4, and the main one is characterized to be a physical person. A similar structure identifies the firms belonging to the second largest connected component. In this case, companies still present a low number of relevant investors, are still controlled by their first shareholder, but this is represented by a *S.p.A.*, not a physical person. In general though, companies belonging to the main components present a heavily centralised ownership structure and few relevant shareholders. The distribution of the companies across the connected components is the following:

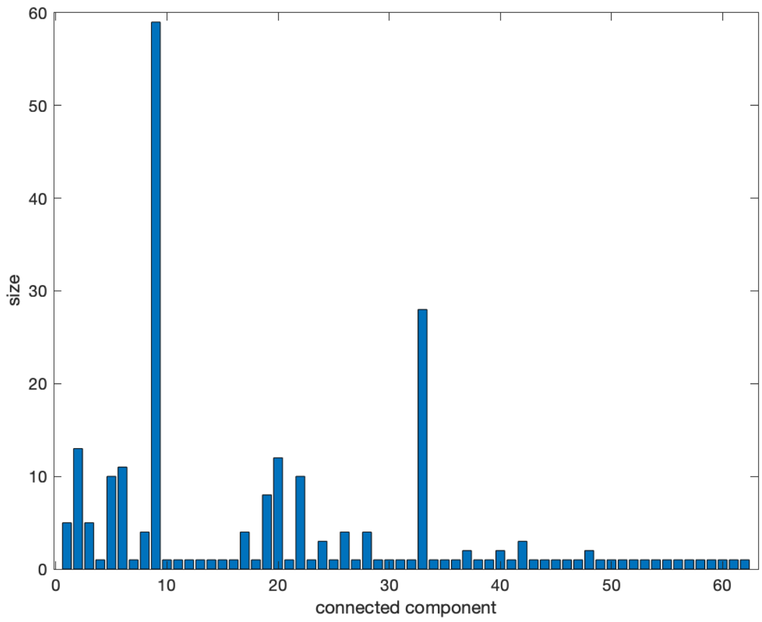


Figure 5.10: Connected components' size

The biggest connected component is composed by 59 companies and 555 link. The second biggest consists of 28 companies and 153 links between them. Since the total number of links in the layer is 879, this means that the account for more than 63% and 17% of the total edges singularly and around 80% of them combined. Looking on the other hand on the graphs mean index values, the average scores for the different node measures, together with the size of the graph's diameter, are the following:

Measure	**Average Value**
Degree	7,58
Strength	2,81
Clustering	0,73
Closeness	$2{,}00 \cdot 10^{-4}$
Eccentricity	$8{,}53 \cdot 10^{-4}$
Betweenness	11,69
Diameter	510,62

Table 5.3

The average clustering coefficient tells that, on average, if a certain company presents a connection with two other firms, the probability that those two are also directed connected is a little more than 70%. The diameter increases with respect to the prior values, as the longest shortest path between two connected nodes equals 510,62. Again, the cost of moving from one node to another is computed as the inverse of the weight of the edge connecting them. For what concerns the distribution of the degree, strength and the main centrality measures over all nodes, note the following bar plots:

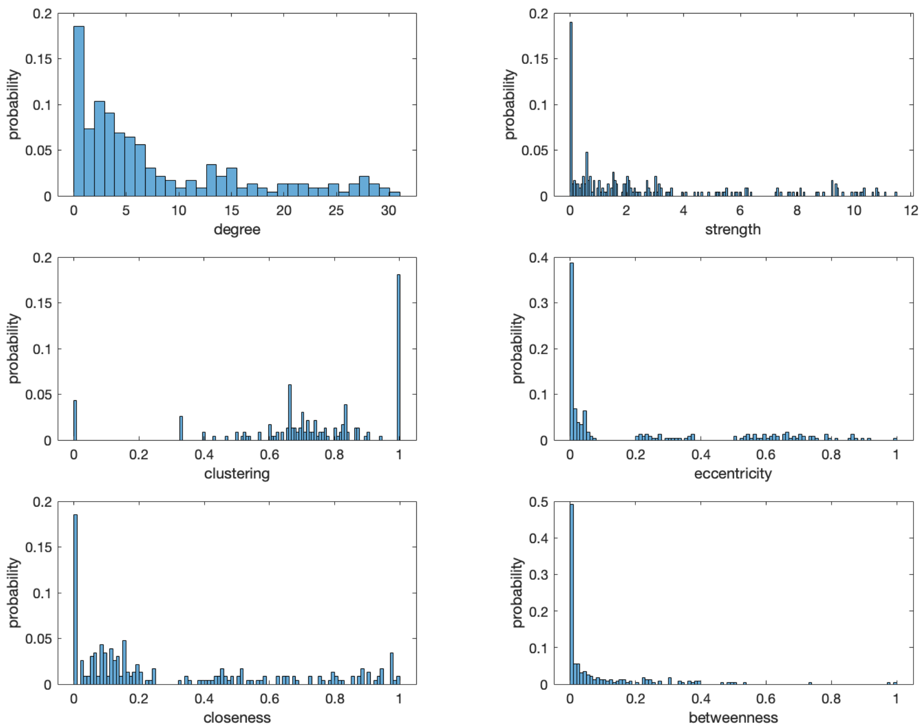


Figure 5.11: Main layer measures' distribution

For what concerns the degree distribution, this spans from a minimum of 0 to a maximum of 31, presenting a decreasing path with fewer companies ingaging in more than 7 links. Indeed, the majority of the firms tend to have less than 7 connections. The company that presents more links is *Garofalo Health Care S.p.A.*, a firm working in the private health care sector. *Piquadro S.p.A.* and *Seri Industrial S.p.A.* follow with 30 connections each. Interestingly enough though, none of them is directly linked to *Garofalo Health Care S.p.A.*.

The representation depicted by the distribution of the strength index across companies is very similar, showing a general decreasing trend with many companies having null or close to null strength value. In this case it is *Piquadro S.p.A.* that presents the highest value, equal to around 11,5. The clustering coefficient is very high, more than 0,7, meaning that many local clusters arise. Looking at the distribution it is interesting to see how around 17% of the companies present a clustering coefficient equal to 1, meaning that all possible triplets to which they could belong to actually take place. The centrality measures are once again normalized to so fit in the interval [0 1]. Betweenness and eccentricty present a similar trend, identifying fewer very important companies. They both indicate *Tod's S.p.A.* as the most relevant firm. On the other hand, according to closeness centrality, the two most central companies are *Netweek S.p.A.* and again *Garofalo S.p.A.*. It it also important to notice that all those companies mentioned above belong to the main giant component. All of them are characterized by the fact that their main shareholder is a physical person and that the number of relevant shareholder is less than 3. Moreover, their relevant owhership structure present the main shareholder to completely control the company, that is by owning more than 50% of the shares, and the other relevant shareholders to hold a marginal amount of shares. This means that these are companies that are strongly controlled by one person that is the main shareholder.

5.2.2. Ownership Ties

The next layer is again composed by the same 232 companies that were part of the prior one. It considers ownership ties to create links among companies, in a similar way to how interlocking directorates defined links in the first two layers. A picture showing out the final result is represented here:

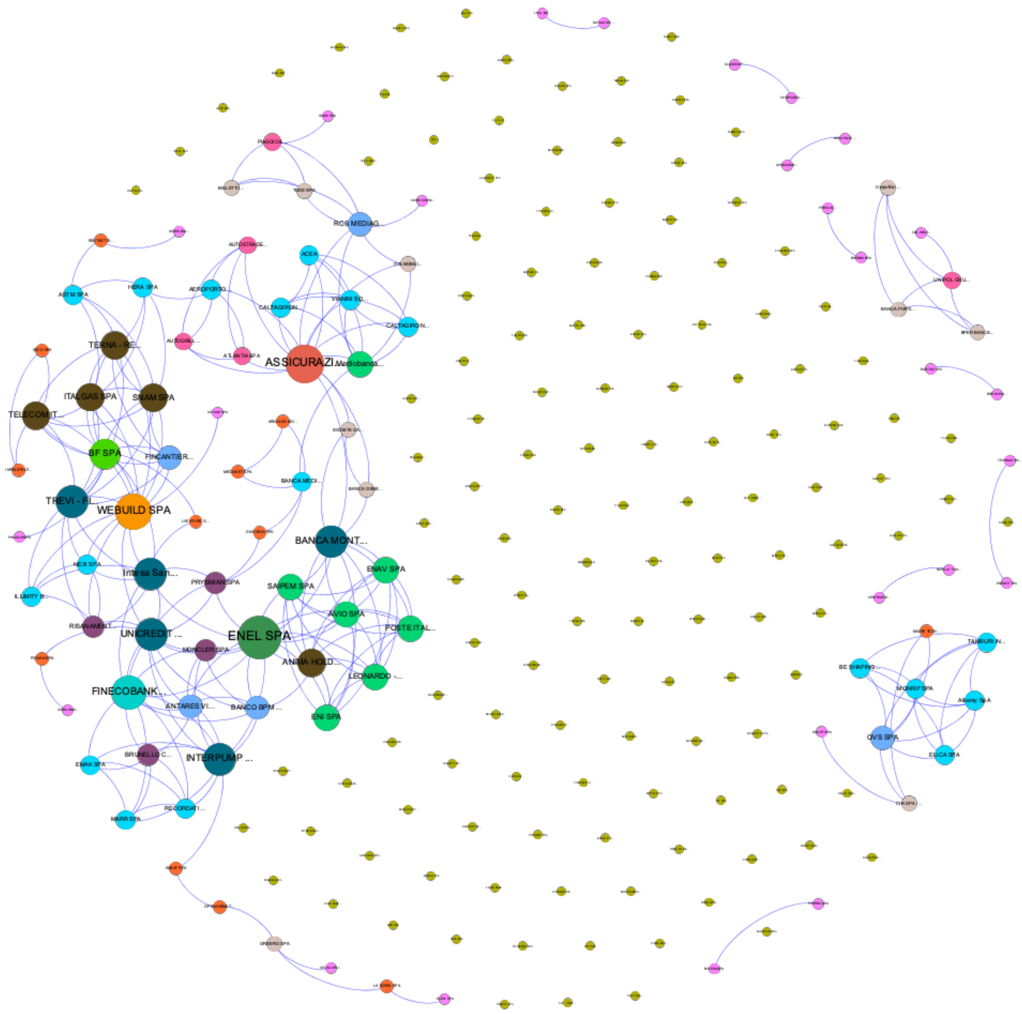


Figure 5.12: Ownership ties graph

In this framework, companies are roughly divided in three categories: isolated nodes, firms belonging to the main giant component and companies ingaging in few links outside of the main component. The total number of links is 239. The giant component is composed by 72 firms, that is a little more than 31% of the sample, while the number of isolated nodes is 130. The average values for the different node measures are:

Measure	Average Value
Degree	2,06
Strength	0,74
Clustering	0,74
Closeness	$1,17 \cdot 10^{-04}$
Eccentricity	$1,3 \cdot 10^{-03}$
Betweenness	32,01
Diameter	33

Table 5.4: Mean values

The average degree of a node is little more than 2 (as one would expect as there are around the same number of nodes and edges). Average clustering coefficient is again pretty high, around 0,74. The company presenting the most links (16) is *Enel S.p.A.*, followed by *Assicurazioni Generali S.p.A.* (14). Also for the strength index those are the most important companies. Betweenness and closeness centrality identify respectively *Assicurazioni Generali S.p.A.* and *Enel S.p.A.* as the most relevant companies, while the highest value of eccentricity is reached by *Finecobank S.p.A.*. Below the distribution of those indices across all companies:

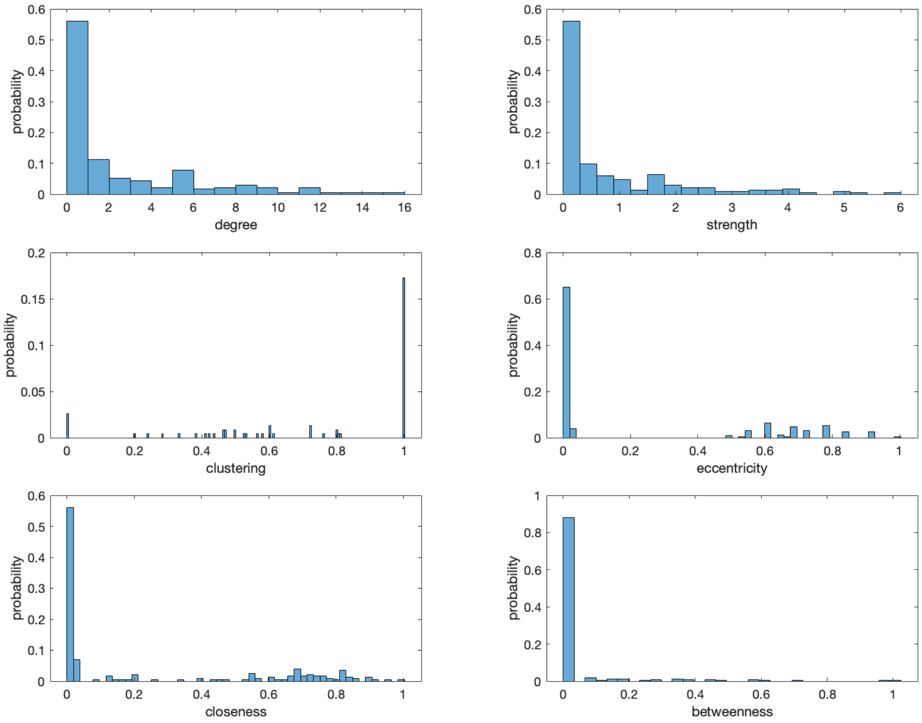


Figure 5.13: Centralities distribution

6 | Network Analysis

Now that all the different layers have been presented independently, it is possible to start analysing them together while constructing the network. As already mentioned in the previous chapters, as the analysis are based on two datasets, the set of nodes, i.e. the company considered, are not all the same across the different layers. Following the notation introduced in the mathematical review chapter, in order to reconduct ourselves to a situation in which all layers insist on the same nodes, the union of the set of companies presented in the two different datasets is considered. Two different networks, each one composed by three layers, are considered. The first one is composed by the direct interlocking, ownership ties and ownership similarity layers, while the second one considers indirect interlocking instead of direct ones.

6.1. Network 1: Direct interlocking, Ownership ties and Ownership similarity

The first multi network is composed by $N = 234$ companies, represented by the nodes of the network. These are the companies that belong to at least one of the two datasets on ownership and governance structure. Let V represent the set of all companies. In the mathematical notation introduced in the previous chapters, the multiplex being analised is $\mathcal{M} = (\mathcal{G}, \mathcal{C})$. $\mathcal{G} = (G_1, G_2, G_3)$ is composed by three different layers, $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ and $G_3 = (V, E_3)$. G_1 is the graph whose edges are composed by interlocking directorates (DI layer), G_2 is the one stemming from ownership ties (OT layer), while G_3 is the one that links companies having a similar ownership structure (OS layer). Each of those layers is weighted. The links across companies and the weights attached to the edges are the one described in the single layer analysis. The set $\mathcal{C}$ is composed only by 234×234 identity matrices representing the inter layer links connecting the same nodes. Indeed, no other inter layer links have been constructed. A helpful visual representation of the multi layer structure is presented below:

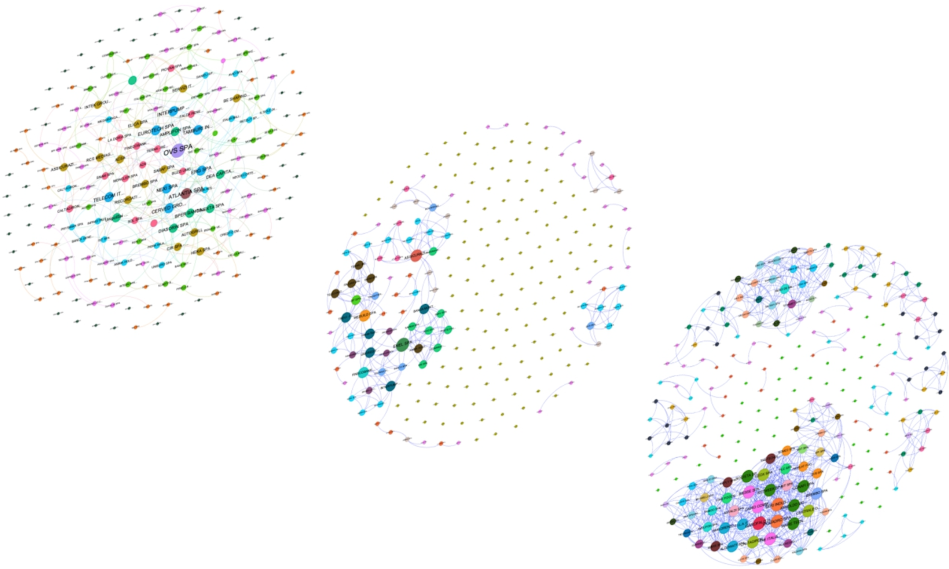


Figure 6.1: Network representation

Note that, for simplicity, the inter layer edges have not been drawed.
The networks (from now on, always not considering inter layer edges) is composed by a total of $O_A = 1387$ edges. The overall weight of the network on the other hand is $O_W = 445, 91$. The number of isolated nodes is only equal to 6. The companies that are not linked to any other firm in any of the layers are *Aquafil S.p.A*, *Biancamano S.p.A.*, *CreditoValtellinese S.p.A.*, *Ratti S.p.A.*, *Techedge S.p.A.* and *Tiscali S.p.A.*.
The distribution of the sum over the three layers of the degrees of the nodes is presented below:

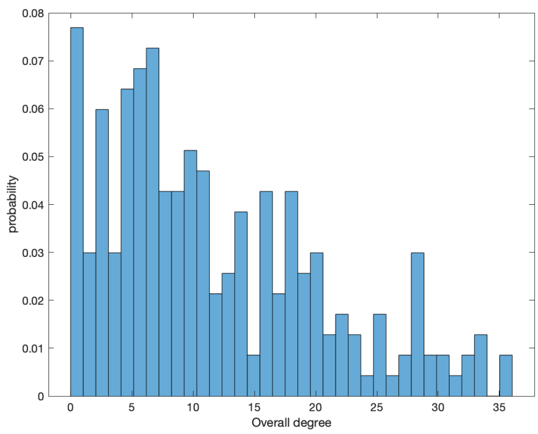


Figure 6.2: Overall degree distribution

Comparing this histogram plot with the single layer degree distributions, one can clearly see a different, but still decreasing, pattern. The maximum number of links to which a firm belongs to is 36. The two companies realising this maximum are *Caltagirone Editore S.p.A.* and *Piquadro S.p.A.*. Both of them were not the companies presented as the most important ones according to the degree index in the three single layers. Below the representation of the overall degree for the companies presenting the most links in the single layers, together with *Caltagirone Editore S.p.A.* and *Piquadro S.p.A.*.

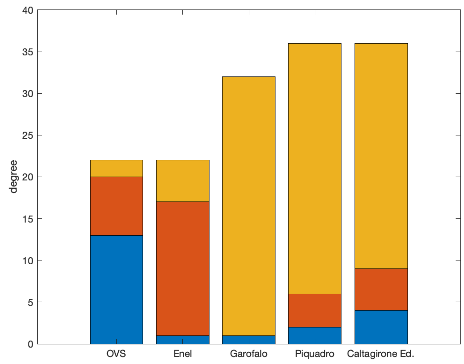


Figure 6.3: Companies' degree composition

The blue, yellow and red histograms represent the number of links to which the companies belong to in the DI, OS and OT respectively. *OVS S.p.A.*, *Enel S.p.A.* and *Garofalo S.p.A.* are the companies presenting the highest number of edges in those layers.
The same reasoning can be made for the overall strength. Of course, only the same 6 companies presented before have a total network weight equal to 0. The distribution of the overall weight across the nodes is the following:

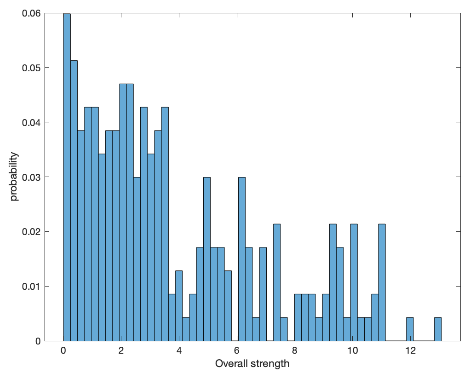


Figure 6.4: Companies strength composition

The pattern is similar to the one pictured by the probability distribution of the degree, but in this case more companies seem to present low strength values. Indeed, a little less than 25% of the firms have an overall network strength higher than 6. This has to be considered together with the fact that the company realising the maximum strength, *Piquadro S.p.A.*, has a strength of around 13,06.

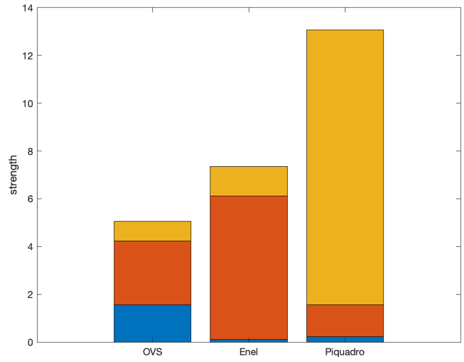


Figure 6.5: Companies' strength composition

The stacked bar plot shows again the distribution across layers of the most relevant companies in the single layers (which happens to be the same ones as before, with the exeption of *Garofalo S.p.A.*) and *Piquadro S.p.A.*.
The *entropy* of a node indicates how its overall degree or strength is distributed across layers. Low levels of entropy represent the situation in which the relative node presents the majority of its edges to belong to a single layer, while larger values for this measure indicate that links are more or less equally distributed across all layers. The mean entropy value for the network is around $0,43$. Because $\mathcal{M}$ is composed by three layers, the maximum value that entropy can take is around $1,1$. This means that the normalised average entropy value is little less than 40%. The distribution of the entropy across all nodes is the following:

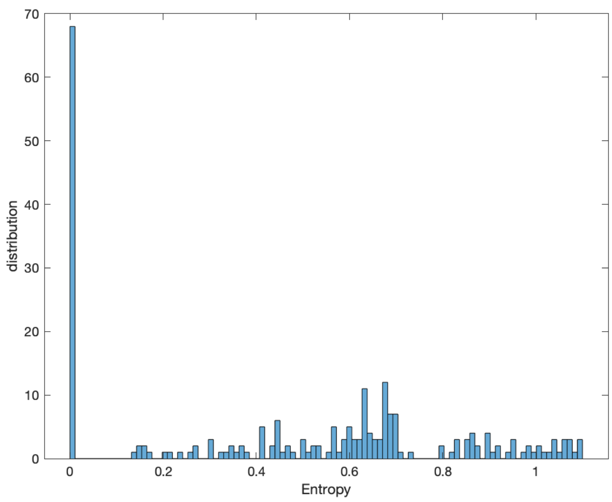


Figure 6.6: Entropy distribution

68 companies bear a null entropy. This means that more than 29% of the companies either are completely disconnected or are only linked to other firms in one layer, i.e. they would be disconnected if not for one specific layer. The rest of the companies on the other hand seem to have a uniform distribution across possible entropy values. For what concerns mean weighted entropy, this is around $0,37$, which normalised gives a result of 33%. On avergage, nodes tend to present very different features regarding degree and strength across the different layers. The distribution of weighted entropy is the follwoing:

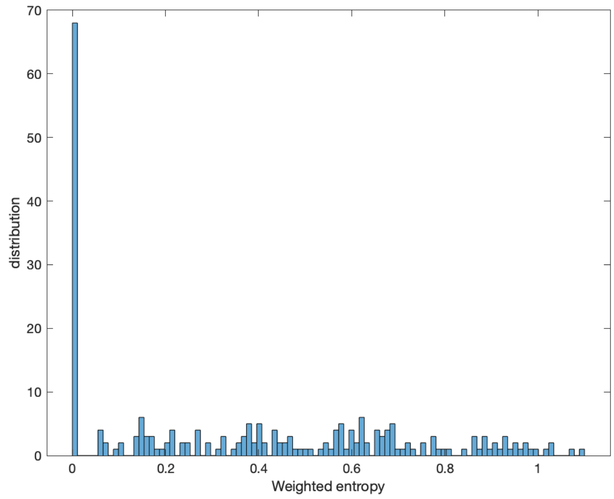


Figure 6.7: Weighted entropy distribution

The final result is similar to the one depicted by the unweighted entropy, with many nodes showing null entropy and around 70% of the sample being kind of uniformly distributed across all possible values in $(0, we_{max}]$.

To further analyse the network, let's now focus on the layers, trying understanding which are the most relevant ones and to what degree they affect the overall network structure. As said before, the network presents 1387 different edges. These are distributed across the three layers as follows:

Layer	**DI**	**OT**	**OS**	**Total**
Number of Edeges	269	239	879	1387

Table 6.1: Number of links

More than 65% of the edges belong to the ownership similarity layer. For what concerns the total weight of the network, this turns out to be equal to around 445,91. Again, its distribution across layers is the following:

Layer	DI	OT	OS	Total
Edges' overall weight	33,67	86,67	325,57	445,91

Table 6.2: Layers' weight

Also here the most relevant layer turns out to be the third one. In this case though, its overall weight is even more relevant in the final structure as it accounts for more than 73% of the overall structure's weight.
The edge overlap $EO_A(\boldsymbol{\alpha}, \boldsymbol{\beta})$ helps in understanding how many nodes are linked over different layers. This quantity has been computed aggregating the layers in all possible ways. The edge overlap values are the following:

EO$_A$	(DI, OT)	(DI, OS)	(OT, OS)	(DI, OT, OS)
DI	0,1152	0,0260		0,0186
OT	0,1297		0,0837	0,0209
OS		0,0080	0,0228	0,0057
(DI, OT)				0,1613
(DI, OS)				0,7143
(OT, OS)				0,2500

Table 6.3: Edge overlap

In this table, the first row stores all relevant aggregation of layers $\boldsymbol{\alpha}$, while the first column stores all relevant aggregation of layers $\boldsymbol{\beta}$. Notice that the DI and OT layers overlap for around 11% and 12% of their respective number of edges. On the other hand, both layers present a very now number of overlaps when they are each one considered together with the OS layer. Recall that around 80% of the links in the OS layer are between companies that are heavily controlled by their first shareholder. This leads to thinking that heavily controlled companies, in general, do not engage in ownership ties or interlocks. As stated by several authors (for example in [1] and [2]), ownership ties are also introduced with the aim of gaining control of a firm. In a company in which the majority of shares are in hand of only one shareholder, it might be more difficult to introduce interlocks as the main shareholder has an important impact on the choice of the directors. Also when considering the three layers combined over a single layer the edge overlap index is almost negligible. An increase in this index is seen when considering the overlap of all three layer combined against two of them. In particular considering $\boldsymbol{\beta} = (DI, OT, OS)$ and $\boldsymbol{\alpha} = (DI, OS)$ leads to an overlap in more than 71% of the cases. This value holds

relevant information in terms of the OT layer. Indeed, it means that the addition of this third layer to the structure impacts the least on the total number of new edges. In other words, if two nodes are linked both in the DI graph and in the OS one, then 71% of the times those two vertices are also linked in the OT layer.
The *contribution* of a multiplex to another one bears similar information in this direction. The following two matrices store the *contribution* CO_A and *weighted contribution* CO_W of all different set of layers $\boldsymbol{\alpha}$ to all possible sets of layers $\boldsymbol{\beta}$.

$\mathbf{CO}_A$	(DI, OT)	(DI, OS)	(OT, OS)	(DI, OT, OS)
DI	0,4990	0,2296		0,1769
OT	0,4361		0,1995	0,1447
OS		0,7642	0,7823	0,6424
(DI, OT)				0,3411
(DI, OS)				0,8208
(OT, OS)				0,7984

Table 6.4: Layers' Contribution

Many observations can be made about these results. Recall that, given a multi layer $\mathcal{M}$, the contribution of a set of m layers $\boldsymbol{\alpha}$ to a set of n layers $\boldsymbol{\beta}$ represents the fraction of links present in the projected network $\pi_{\boldsymbol{\beta}}(\mathcal{M})$ that are *only* contributed by the layers $\boldsymbol{\alpha}$. High levels of this index thus indicate that the addition of $\boldsymbol{\alpha}$ is significant.
Both DI and OT layers have around the same impact against the set (DI, OT). This is also due to the fact that they present a similar total number of edges. On the other hand, their respective impact on the sets (DI, OS) and (OT, OS) is less relevant. This is due to the fact that OS present many more links. Moreover, OS has a relevant impact on any set of layers $\boldsymbol{\beta}$.
As for what concerns the different *weighted contribution* index CO_W values, these are stored below:

$\mathbf{CO}_W$	(DI, OT)	(DI, OS)	(OT, OS)	(DI, OT, OS)
DI	0,2798	0,0937		0,0766
OT	0,7202		0,2102	0,1944
OS		0,9063	0,7898	0,7311
(DI, OT)				0,2699
(DI, OS)				0,8056
(OT, OS)				0,9245

Table 6.5: Weighted layers' contribution

This index quantifies the contribution of a set of layers to the overall weight of a bigger structure. The difference in the values taken by the CO_A and CO_W indices gives a hint on how the weights are distributed and which layers l are "heavier", meaning that their overall weight $W_l = \sum_{i,j=1}^{N} w_{ij}^l$is greater. In case of CO_W, the OT layer independently contributes to more than 72% of the weight of the network given by layers (DI, OT). Comparing now $CO_W(OT, (DI, OT))$ with $CO_A(OT, (DI, OT))$ we deduce that the links in OT are on average heavier than the ones in DI. Overall, it can be seen that layer OS contributes the most to all the weight matrices of the projected network it is a part of.

Another relevant index that compares the layers composing the network is the *distance* among them. For this multiplex, the distances between its layers are the following:

distances	DI	OT	OS
DI	0	0,3679	0,4288
OT	0,3679	0	0,1774
OS	0,4288	0,1774	0

Table 6.6: Distances across graphs

These distances have been normalised so to fit in the interval $[0, 1]$. This has been done by dividing the real distance for the one between the two furthest types of graphs of N nodes: a completely connected one and one with no edges. In a mathematical notation, the maximum distance between graphs is realised between G_1 and G_0, where $G_1 = (V, \mathbb{1}_N)$ and $G_0 = (V, \mathbb{0}_N)$. Here, $\mathbb{1}_N$ and $\mathbb{0}_N$ represent respectively the $N \times N$ matrices of ones in the off diagonal elements and the null matrix. It can be seen that, according to this choice for defining the distance, the layers that are more similar to each other are the ones build from the ownership dataset, that is OT and OS.

Another important index that is used for comparing layers and understanding which could be the best representation of a network is the *VN entropy*. The VN entropy of a network is the average of the VN entropies of its single layers. These ones are computed by means of the Laplace formulations of their adjacency or weight matrices. The VN entropy of a layer represents how the overall degree (or strength) of the graph is spread across its nodes. Comparing the VN entropies of different representations of a network with its projected layer's one gives information on which representation deviates the most from the projected layer portrayal. The VN entropies for the different projected networks $\mathcal{R}$, together with the respective structural reducibility quantities $q(\mathcal{R})$ are the following:

Network representations	**VN entropy**	**Structural reducibility**
(DI,OT,OS)	6,79	0,0928
((DI,OT),OS)	7,05	0,0581
((DI,OS),OT)	6,82	0,0881
(DI,(OT,OS))	7,18	0,0410
((DI,OT,OS))	7,48	0

Table 6.7: VN entropy

In this table, the network representations column stores the different multilayer that can be born after projecting some of the layers of $\mathcal{M}$. Each $\mathcal{R}_n$ of size n is written by means of its n layers, that is $\mathcal{R}_n = (R_1, \ldots, R_n)$. In this case, $n \leq 3$. The value of the VN entropy and the respective quantity q show that the representation that deviates the most from the projected layer is the original 3 layer network. On the contrary, weighted VN entropy shows a different result; indeed, it turns out that the lowest value for this index, and thus the highest score for the respective structural reducibility, is reached by the representation in which DI and OS layers are projected in one single layer.

Network representations	**VN entropy**	**Structural reducibility**
(DI,OT,OS)	6,7462	0,08563
((DI,OT),OS)	6,8789	0,0673
((DI,OS),OT)	6,7021	0,0913
(DI,(OT,OS))	7,1634	0,0288
((DI,OT,OS))	7,3754	0

Table 6.8: Weighted VN entropy

This means that the distribution of weights across all nodes in the network $((DI, OS), OT)$ differs the most, over all possible representations, from the same distribution in the projected layer. A possible explanation for this can be found in how the layer contribution changes when considering the weights in the network. Indeed, the contribution of DI to the set of layers (DI, OS) decreases when considering CO_W rather than CO_A, while the contribution of OS to the same set of layers increases. This means in other words that the intake of DI becomes less relevant and might not be worth being differentiated with a layer for itself.

6.1.1. Projected Layer

The projected layer $\pi_p(\mathcal{M})$ have been introduced many times in this analysis. This is a one layer network composed by the union of all the three graphs DI, OT and OS. Its graphical representation is presented below:

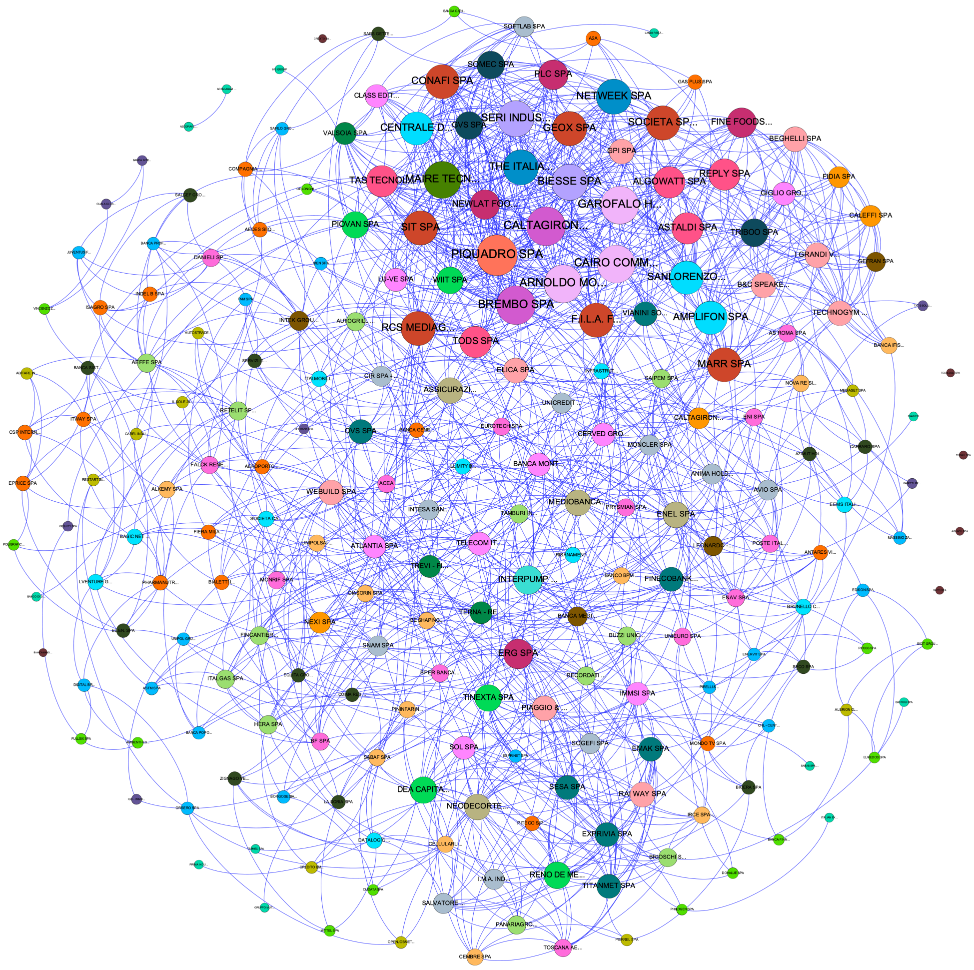


Figure 6.8: Projected graph

It is composed by 234 nodes and presents, as network $\mathcal{M}$, only 6 isolated nodes. The number of different connected components is 8. Besides the 6 isolate companies and two others that are only linked to each other, $ACSM - AGAM$ *S.p.A.* and *Ascopiave S.p.A.*, the rest of the firms belong to a giant component composed by 226 firms, that is more than 96% of the sample.

Measure	Average Value
Degree	11,40
Strength	3,81
Clustering	0,47
Closeness	$1{,}4{\cdot}10^{-3}$
Eccentricity	$2{,}59{\cdot}10^{-2}$
Betweenness	214,98
Diameter	52,70

Table 6.9: Mean values

The mean degree is higher than each of the ones for the single layer, and same goes for the mean strength. The clustering coefficient is around $0,5$, higher than the one for DI layer but significantly lower than the one reached by OT and OS layers. Mean closeness, mean eccentricity and mean betweenness observe an increase in their values with respect to the single layers values.
The distribution of those indices and centrality measures across firms are shown below:

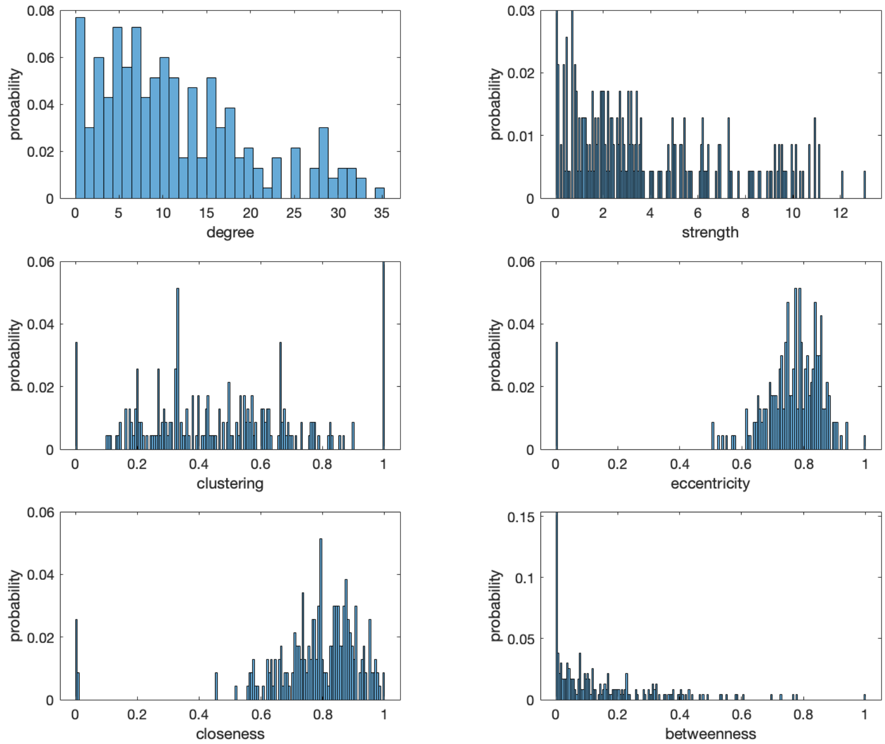


Figure 6.9: Centralities distribution

With respect to the prior cases, more companies tend to present a high level for degree and strength. *Piquadro S.p.A.* is again the company realising the maximum in these measures, respectively 35 and 13,06. Almost 6% of the companies present a clustering coefficient equal to 1. The distribution of closeness and eccentricity change a lot with respect to the prior cases, as more companies present higher relative (i.e with respect to their normalization) values in those measures. Each centrality measure indicates a different most central company; according to closeness, the most important firm in the layer is *RCS Mediagroup S.p.A.*, a multimedia publishing group. Eccentricity on the other hand indicates *Tod's S.p.A.* as the most central company. As for betweeneess centrality, *Mediobanca S.p.A.* is the one presenting the maximum value for it.

6.1.2. Flattened Layer

The flattened representation of a network also bears interesting features about it. This is a one layer graph were to each node in the starting network $\mathcal{M}$, M different nodes correspond in the flattened representation G_F, where M is the number of original layers. This representation is well suited when a cost, rather than a weight, is given to each edge. This because the cost of going from two nodes representing the same firm in the original network $\mathcal{M}$ is 0, i.e. there is no difference in thinking of a company as belonging to a layer rather than to another. The cost matrix is thus constructed as an $NM \times NM$ block matrix whose off diagonal matrices are null entries, while the diagonal matrices are the $N \times N$ cost matrices of the different layers. In this analysis, these have been computed as follows:

$$c_{ij} = \frac{1}{w_{ij}} \; \forall i, j \in \{1, \dots, N\}$$

with the convention that $\frac{1}{0} = \infty$.

The nodes stemming from the same companies in $\mathcal{M}$ are linked by edges within G_F. This is composed by 702 different nodes, and each company in the original dataset corresponds to three vertices. Having this in mind, it is possible to run the usual centrality measures (besides calculating the strength of each node). The average values for those are presented here:

Measure	Average Value
Degree	5,95
Clustering	0,58
Closeness	$2,6 \cdot 10^{-4}$
Eccentricity	$2,55 \cdot 10^{-2}$
Betweenness	1388,9
Diameter	53,90

Table 6.10: Mean measures

The average degree is of course lower than the one for the projected layer. Diameter, mean closeness and mean eccentricity present very similar features with their respective values in $\pi_p(\mathcal{M})$; this is due to the fact that the distances between nodes across the different structure representations remain similar (no cost in jumping from two nodes representing the same firm). Betweenness on the other hand heavily increases while clusters are less likely to happen, again as a consequence of the way in which the network is represented. The distribution of the values for those measures across all nodes are plotted below:

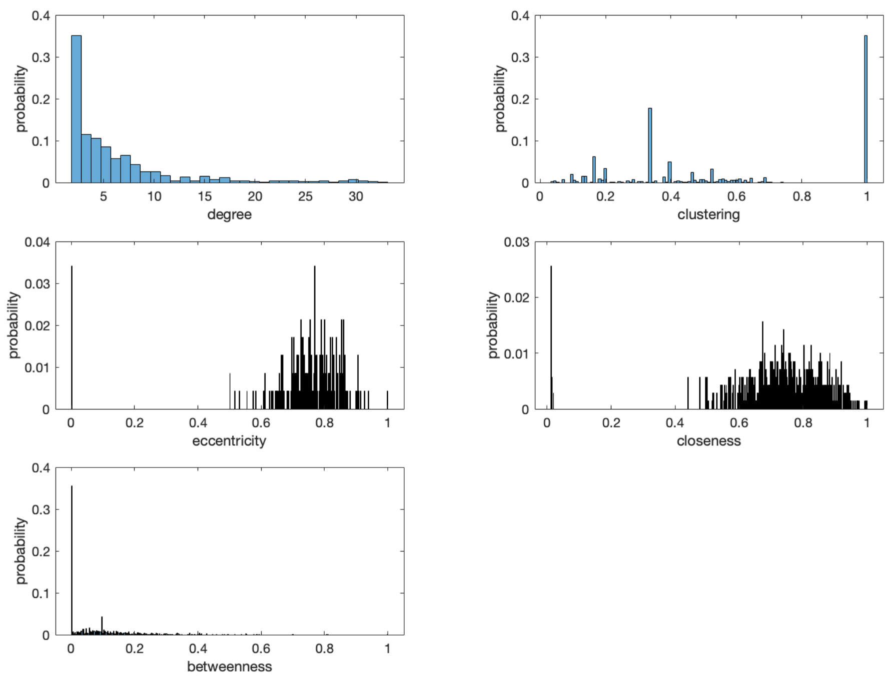


Figure 6.10: Centrality measures distribution

The degree distribution for this layer presents a steeper pattern than the one that characterises the projected network. This is no surprise as, besides the trivial edges connecting same nodes, no other inter layer edges are taken into account. For this reason no new links are added to the structure, but nodes of different layers are still kept separated, and thus there is no summation of any sort between the different adjacency matrices. $Garofalo\ S.p.A.$ is the most relevant company according to degree. Eccentricity and closeness present a pattern very similar to the one held by the projected layer, and they indicate respectively $Tod's\ S.p.A.$ and $OVS\ S.p.A.$ as the most central companies. Betweenness centrality is almost always zero for the majority of the nodes and realises its maximum in a node representing the firm $ERG\ S.p.A.$.

6.2. Network 2: Indirect interlocking, Ownership ties and Ownership similarity

The next network to be analysed is similar to the prior one, but changes in the way in which interlocks are defined. Indeed, not only direct interlocks are considered now, but also indirect ones. In other words, the only condition for an interlock to be born is that directors of different companies sit together in a board of directors of a listed company present in the dataset, but it $doesn't\ matter$ which one. This graph have already been introduced and studied independently in the prior chapter. It is characterised to have a number of edges (1029) greater than the OS graph (879) and much higher than the ones of DI and OT layers. For this reason, it is interesting to see how the general network feature changes when substituting the DI layer with the new II (Indirect Interlocking) one. The multiplex $\mathcal{M}$ of study is thus composed by the same set of nodes V as before and by three layers: $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ and $G_3 = (V, E_3)$. E_2 and E_3 are the same ones as in the prior network, while E_1 represents edges linking companies that present at least one indirect interlock. Below the representation of the three layers.

Figure 6.11: Network representation

The network is composed by a total of $O_A = 2147$ edges, and the overall weight of the network is around $633,38$. Again there are 6 isolated nodes, corresponding to the same companies that were isolated in the prior network. The distribution of the total degree of each firm is the following:

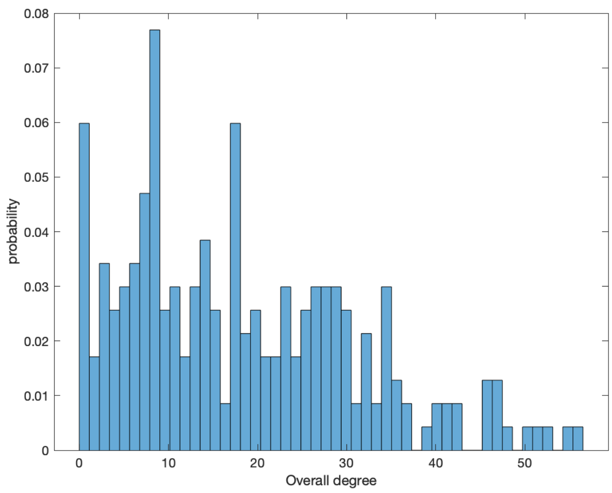


Figure 6.12: Overall degree distribution

The degree distribution differs slightly form the ones presented before, as it is not always decreasing but peaks at around the value 10. The company with the greatest degree (56) is *Brembo S.p.A.*, a manufacturing company specialised in producing automotive brake systems. It is again intersting to see how this compares to the company that, in the single layers, presented the most links.

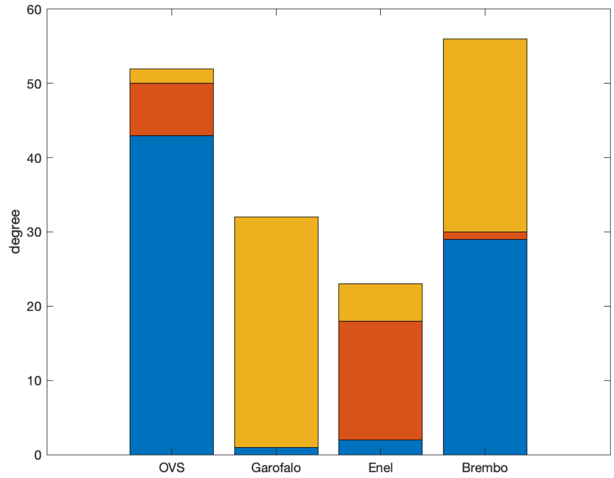


Figure 6.13: Companies' degree composition

The blue, yellow and red histograms represent the number of links to which the companies belong to in the II, OS and OT respectively. Contrary to the other companies, *Brembo S.p.A.* presents many links both in the II and in the OS layers.
The same reasoning can be made for the overall strength.

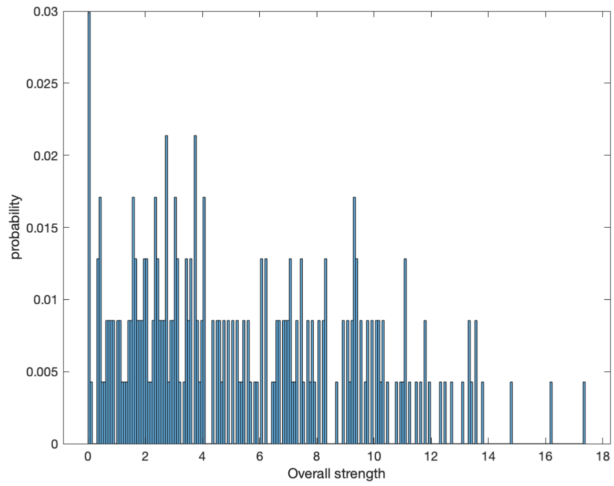


Figure 6.14: Overall strength distribution

The most important company in this case is *Amplifon S.p.A.*. Comparing its strength to the one of the single layer most important companies lead to the following result:

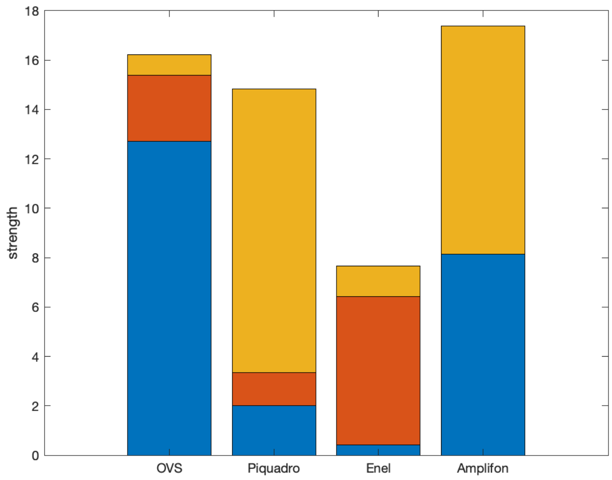


Figure 6.15: Companies' degree composition

Average entropy and weighted entropy of the network are both around 0,45. While there is a slight decrease in the value of entropy with respect to the prior network, weighted entropy increases of around 0,08, that is around 17% of its new value. This implies that in the new network, weights are on average more equally distributed across layers. The nodes distribute across all possible entropy values as follows:

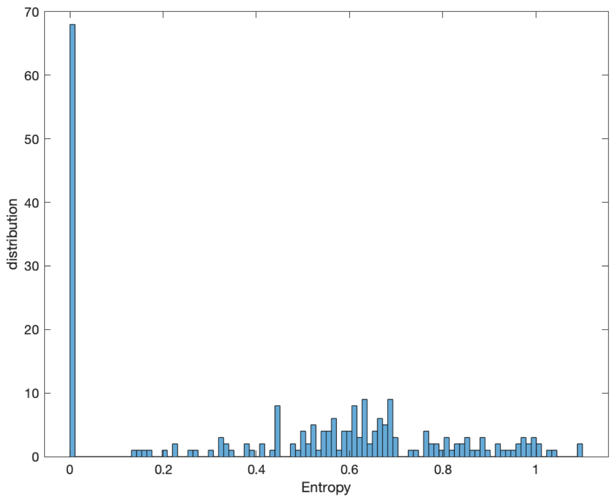


Figure 6.16: Entropy distribution

As for what concerns weighted entropy, this is the distribution:

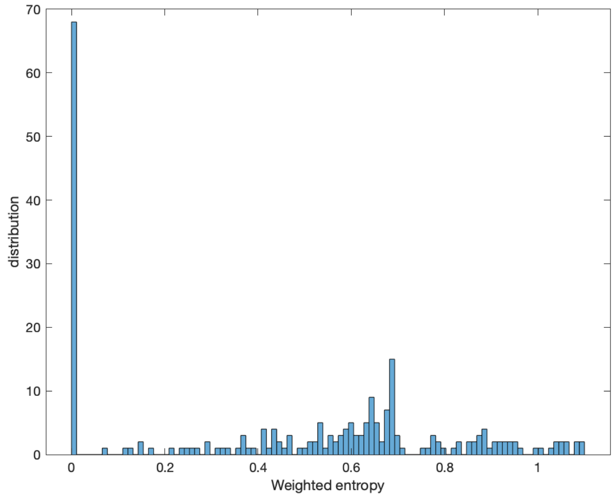


Figure 6.17: Weighted entropy distribution

Going on to analysing the different layers and their impact on the overall structure, the first thing to look at is how the total number of edges is distributed across layers:

Layer	II	OT	OS	Total
Number of Edeges	1029	239	879	2147

Table 6.11: Number of links

More than 88% of the edges is concentrated in the II and OS layers (respectively around 48% and 41%). This leads to the belief that layer OT could impact less on the structure of $\mathcal{M}$. The overall weight distribution is similar:

Layer	II	OT	OS	Total
Edges' overall weight	221,14	86,66	325,57	633,38

Table 6.12: Layers' weight

For what concerns the edge overlap indices $EO_A(\boldsymbol{\alpha}, \boldsymbol{\beta})$, this obviously does not change for the sets of layers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ that do not include II. The table storing the results is the following:

EO	*(II,OT)*	*(II,OS)*	*(OT,OS)*	*(II,OT,OS)*
II	0,0486	0,0233		0,0058
OT	0,2092		0,0837	0,0251
OS		0,0273	0,0228	0,0068
(II,OT)				0,12
(II,OS)				0,25
(OT,OS)				0,3

Table 6.13: Edge overlap

In general, edge overlap values are similar to the ones for the prior network. A relevant decrease in this value is seen though for the overlapping of the whole network over layers (II, OS). This suggests that the addition of indirect interlocks introduces links that were not present in the *OT* graph. Indeed, also the index $EO_A(OT, (II, OT))$ decreases.
Let's now go on and focus on the different values contribution for layers to the structure in both the unweighted and weighted case, CO_A and CO_W respectively.

CO$_A$	*(II,OT)*	*(II,OS)*	*(OT,OS)*	*(II,OT,OS)*
II	0,8038	0,5334		0,4667
OT	0,1552		0,1995	0,0850
OS		0,4538	0,7823	0,4085
(II,OT)				0,5731
(II,OS)				0,8839
(OT,OS)				0,5502

Table 6.14: Layers' contribution

With respect to the prior network, as one would expect, the contribution of the interlocking dataset is much more relevant. It adds around 80% of the edges when considered together with the *OT* layer and turns out to contribute even more than the *OS* one. Moreover, the projected layer $\pi_{(II,OS)}(\mathcal{M})$ accounts for more than 88% of total edges of $\pi_p(\mathcal{M})$. The situation described by CO_W is slightly different.

CO$_W$	*(II,OT)*	*(II,OS)*	*(OT,OS)*	*(II,OT,OS)*
II	0,7184	0,4045		0,3499
OT	0,2816		0,2102	0,1368
OS		0,5955	0,7898	0,5182
(II,OT)				0,4680
(II,OS)				0,8632
(OT,OS)				0,6509

Table 6.15: Weighted layers' contribution

When considering the intake of layers taking into account also edges' weights, the contribution index portrays a different picture. Indeed, the contribution of II layers lowers while the one of OT and OS grows. This can be ascribed to the fact that the mean weight of OT and OS are respectively 0,36 and 0,37, while the average weight for the II layer is just around 0,21. For those reasons, the layer contributing the most to the different projected representations is the OS one.
As for the distance between graphs, the following table stores all the results:

distance	*II*	*OT*	*OS*
II	0	0,3709	0,4319
OT	0,3709	0	0,1774
OS	0,4319	0,1774	0

Table 6.16: Distances between graphs

The situation is very similar to the one given by the prior multiplex. Layer II departs from the ownership layers with around the same magnitude as layer DI does. It is thus interesting to see what the distance is between the two governance structure based layers, and this turns out to be 0,0055. It means that the two graphs are very similar to each other (recall that the values presented in the table have been normalised so to fit in the interval $[0, 1]$). On the one hand, this should be of no surprise as the II layer is an extention of the DI one. On the other hand though, the number of links present in the two is very different, as II presents almost 4 times the number of links that DI does. Moreover, one has to recall that the distance $\mathbf{d}$ is defined here only on unweighted layers. This means that the weights of the edges are not taken into account, simplifying slightly the representation. In case one were to extend the definition of distance also for weighted graphs, the new distance between II and DI would probably result in a higher (normalised) score.

The VN entropy and the relative structural reducibility for each possible representation of the multilayer $\mathcal{M}$ are presented here:

Network representations	VN entropy	Structural reducibility
(DI,OT,OS)	6,706	0,0987
((DI,OT),OS)	7,0518	0,0598
((DI,OS),OT)	6,8682	0,0843
(DI,(OT,OS))	7,1760	0,0433
((DI,OT,OS))	7,5000	0

Table 6.17: VN entropy

The VN entropy presents the lowest value for the original $\mathcal{M}$ network and the highest for the projected layer $\pi_p(\mathcal{M})$. This means that the representation that differs the most from $\pi_p(\mathcal{M})$ is indeed the original 3 layer network $\mathcal{M}$. According thus to these indices, no layers must be projected onto each other. Weighted VN entropy bears the same conclusions, as the scores it holds are very similar. Below the exact results.

Network representations	VN entropy	Structural reducibility
(DI,OT,OS)	6,7121	0,1032
((DI,OT),OS)	6,9969	0,0651
((DI,OS),OT)	6,8335	0,0869
(DI,(OT,OS))	7,1122	0,0497
((DI,OT,OS))	7,4842	0

Table 6.18: Weighted VN entropy

6.2.1. Projected layer

The projected graph $\pi_p(\mathcal{M})$ is shown in the following picture:

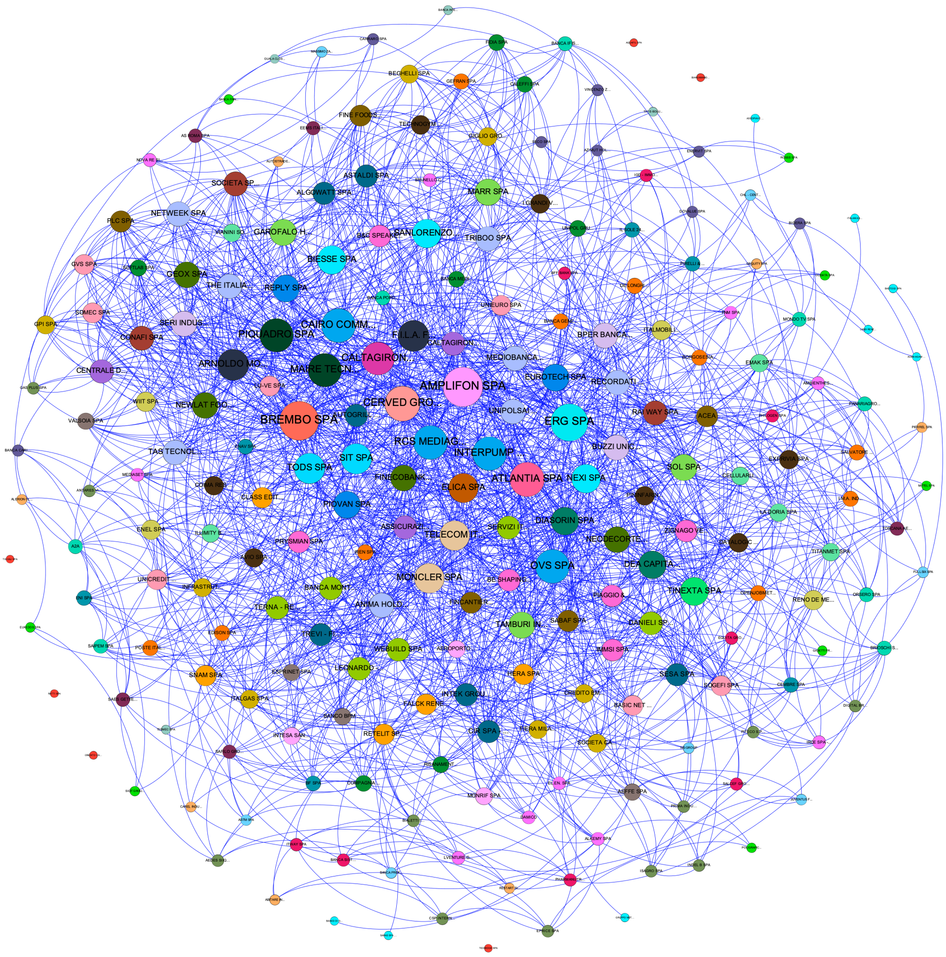


Figure 6.18: Projected network

$\pi_p(\mathcal{M})$ is composed by the usual 234 nodes and 2059 edges. It decomposes in 8 connected components, 6 of which are isolated nodes. These are again the same nodes that were isolated in the prior network. Again, two companies, *ACSM-AGAM S.p.A.* and *AscopiaveS.p.A.*, are only linked to each other, while the remaining 226 firms belong to the giant component. The average centrality values are

Measure	Average Value
degree	17,60
strength	5,41
clustering	0,51
closeness	$1,710^{-3}$
eccentricity	$3,3410^{-2}$
betweenness	150,80
diameter	34,84

Table 6.19: Mean centrality values

With respect to the prior projected layer, both mean degree and strength increase as a consequence of the higher number of links. This goes also for the clustering coefficient, closeness and eccentricity. Betweenness centrality on the other hand decreases, as well as the diameter of the graph. The distribution of those indices and centrality measures across firms are shown below:

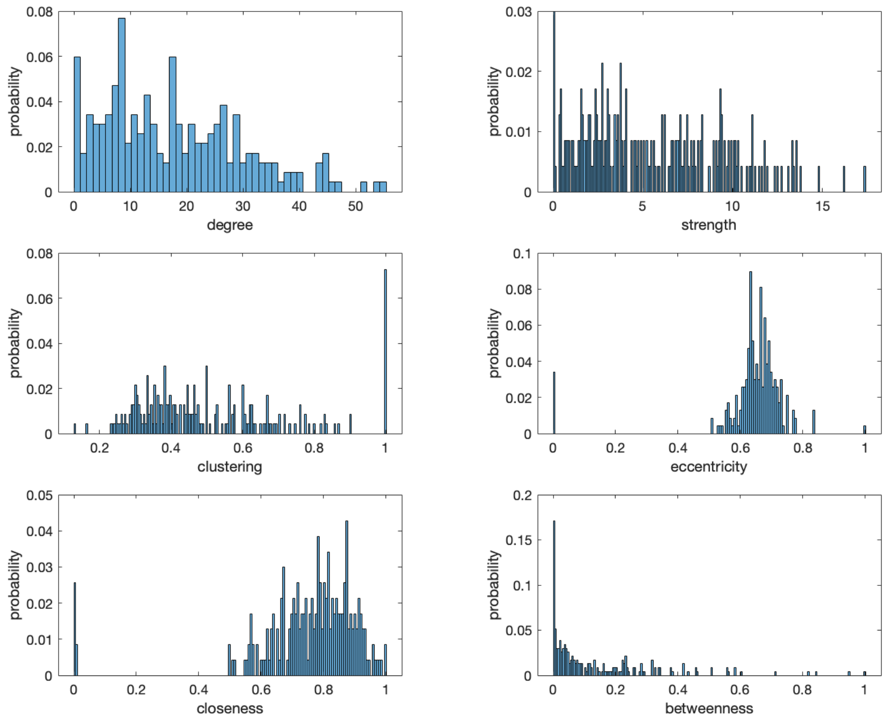


Figure 6.19: Centrality distributions

Amplifon S.p.A. is the company for which degree and strength reach their highest values. The clustering coefficient distribution looks similar to the one describing the prior projected network, while eccentricity identifies *Esprinet S.p.A* as the most central node and *Tod′s S.p.A.* becomes the second most central firms in this case. *Brembo S.p.A.* and *ERG S.p.A.* presents the highest closeness and betweenness values respectively.

6.2.2. Flattened layer

The flattened representation G_F of multilayer $\mathcal{M}$ is composed by 702 nodes, and is completely identified by its two block matrices $\mathcal{A}_f$ and $\mathcal{C}_f$. The average node measures are the following:

Measure	**Average Value**
degree	8,12
clustering	0,62
closeness	$3,1110^{-4}$
eccentricity	$3,3010_{-2}$
betweenness	1118,7
diameter	36,23

Table 6.20: Mean values

As expected mean degree increases because of the substitution of DI with II layer. Overall, there is an increase in every single measure; in line with this result, the diameter of the network on the other hand decreases. For what concerns finally the distribution of those measures across nodes:

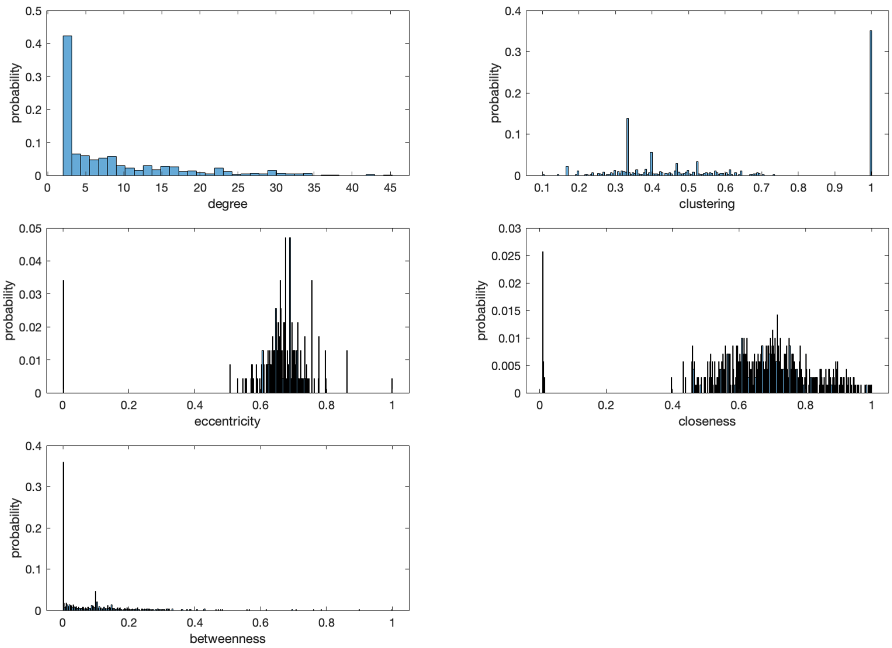


Figure 6.20: Centrality measures distribution

These distribution present again a pattern similar to the one describing the prior network. Again, betweenness shows very low values, while closeness and eccentricity seem to be negatively skewed. The most relevant companies are *OVS S.p.A.* according to degree and closeness, *Esprinet S.p.A.* for what concerns eccentricity and again *ERG S.p.A.* when considering betweenness.

7 | Results

The scores for the different nodes, layers and network measures are now to be interpreted in a more profound and broader view, in order to grasp possible interesting patterns or economical trends at their foundation. First thing is to understand if the different layers are similar and if they present analogous features. Because of the way the different layers have been constructed, comparing graphs can deliver relevant information about the relationships between Board of Director composition, specifically interlocks, ownership ties and ownership structure. Comparing the direct interlocks and the ownership ties layer is relevant, especially because the overall number of edges in the two cases is very similar. The two layers present indeed 269 and 239 links respectively, therefore it is interesting to see how those are spread across the nodes. Edge overlap is small, but not negligible as around 12% of the edges of both layers are present in both. These include companies that both have a shareholder and a director in common. As it has been stated before, italian public listed companies' ownership tend to be concentrated in few owners, and these are usually physical persons. Therefore, companies that are linked both by ownership ties and interlocks are usually the ones that are owned by private persons who also have a strong influence on the choice of the BoD composition. In the dataset of interest, links are present between firms in which one or few persons play a relevant role. It is the case of *Caltagirone S.p.A.* and *Caltagirone Editore S.p.A.* for example. Those firms, besides being strongly connected, are also linked with *Acea* and *Assicurazioni Generali*. All those companies are strongly tied and connected by the Caltagirone family members, centered around the businness man Francesco Gaetano. *Caltagirone S.p.A.*, the main holding, is a company whose interest are in real estate, manifacturing and publishing. The same goes for the companies orbiting around *Tamburi investment Partners S.p.A.* (*TIP*). *Monrif*, *Elica*, *OVS*, *BE Shaping the Future* and *Alkemy* all have *TIP* as relevant investor. *TIP* is a company engaged in the financial sector, providing services of investment banking and advisory in corporate finance transactions. Then also state owned companies present both links in the *DI* and *OT* layer. *Terna*, *Italgas Snam*, *Telecom* and *Webuild* are all linked by the *CDP*, Cassa Depositi e Prestiti investor, a state owned investment bank. When considering also indirect interlocks the result is very

similar. In this case though, there is a strong unbalance between the two layers given by the different number of links present in each graph. OT presents less than one fourth of the overall number of links created by indirect interlocks. For this reason, the overlapping index between the two drops when considering it in terms of indirect interlocks. On the other hand, the proportion of overlapping edges with respect to ownership ties grows to more than 20%, indicating that the new links stemming from indirect interlocks do overlap to some extent with the ownership ties. Even if this is the case though, the cluster of companies that ingage in both links remain the same. Indeed, the main groups of double linked companies are again the ones orbiting around either the *Caltagirone S.p.A.* group, the TIP one or present a relevant amount of shares that belong to public investors like CDP.
The contribution of each layer to each other is another measure of how well distributed edges are across the DI and OT layers. Table 6.4 stores the data of interest. Summing up the elements of the first column, that are $CO_A(DI,(DI,OT))$ and $CO_A(OT,(DI,OT))$ gives a hint at what extent the two graphs complement each other. Indeed if the two contributions were to sum up to one, it would mean that no overlap is present and the layers are perfectly complementary with respect to their projection. In this case $CO_A(DI,(DI,OT)) = 0,50$ and $CO_A(DI,(DI,OT)) = 0,44$. This implies that they only overlap for around 6% of the edges and they almost equally split over the projected network. Also when considering indirect interlocks as well, the number of links that are contributed by both layers accounts for less that 5% of the total links in the projected layer. This to further strengthen the idea that interlocks and ownership ties complement over the set of listed italian companies. The thing that changes when considering indirect interlocks is the importance of the layers in the final structure. Because indirect interlocks are much more common than ownership ties, this leads to a scenario in which firms are more likely to be connected by an interlock in the BoD rather than having a common shareholder.
For what concerns the overlap between interlocks and ties, it is also interesting to see how those distribute across the different sectors to which companies can belong to. The following bar plot depicts the situation for DI and OT layers.

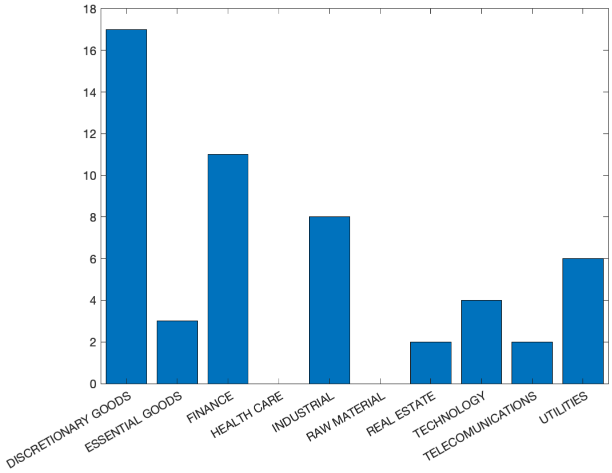


Figure 7.1: DI an OT overlaps

17 of the overlaps happen including a company working in the discretionary goods producing sector. When also considering financial and industrial companies, then more than half of the overlapping edges are about firms belonging to those sectors. In percentage with respect to the number of firms that belong to each sector, again the discretionary good sector is the most represented, followed by the utilities one. Besides 5 different double links connecting companies both from the discretionary goods sector, edge overlapping happens mostly between companies of different sectors. Indeed, only around 30% of overlaps are between same industry firms. The sectors presenting most overlaps are the discretionary goods one, together with the finance and industrial one.
When also considering indirect interlocks, the number of double links between companies belonging to different industry increases. The distribution of the overlapping edges across the sectors is the following:

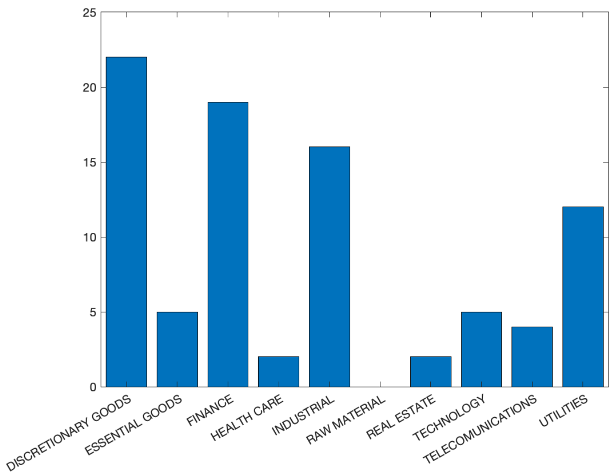


Figure 7.2: II and OT overlaps

The number of overlapping edges is 50, and only 13, the 26% of them, is between same sector companies. Overlaps, i.e. situations in which firms are tightly connected as they share both at least a director and a shareholder, happen more often again in the directionary goods, finance, industrial and utility sectors. The most common inter sectors overlap happens between the discretionary goods and industry sectors, and the utility and industry sectors. In proportion, the sectors for which it is most likely to find firms belonging to overlapping edges are the telecomunication and utility ones.

Comparing the two layers stemming from the ownership dataset is also a matter of interest. Any conclusion that can be deduced has to take into consideration the fact that the number of edges in the OS graph are more than 3 times as many as the one connecting companies because of a tie. This even if the threshold used for constructing links in the OS layer has been chosen to only connect companies that present a very similar ownership structure. Indeed, recall that two firms, in order to be linked in this layer, must present at least the same type of first shareholder, must either be completely controlled or not completely controlled both by the first shareholder and by the relevant ones, and finally have a similar number of relevant shareholders. This means in particular that companies with equally distributed shares and many shareholders are not linked with heavily controlled companies. Recall moreover that the dataset considered regarding ownership structure is composed mostly by companies whose first shareholder on average holds more than 50% of companies shares, and that more than 57% of firms are completely controlled by their first shareholder and 80% of them are controlled by the relevant shareholders. This means that, when comparing the OS layer with other ones, we are usually analysing how heavily controlled companies either engage with interlocks or ownership ties. With this consideration in mind, it is possible to start and interpret the network composed by OT and OS layers. The first thing that comes to attention is the extremely high number number that pops out when summing the contribution of OS and OT layers to the (OT, OS) network. $CO_A(OT, (OT, OS)) = 0,2$ and $CO_A(OS, (OT, OS)) = 0,78$, leading to a total contribution to the their projected layer of around 0,98. This means that less than 2% of all edges are present in both layers. Indeed, only 20 double links exist. The distribution of double links over all companies' sectors is shown below:

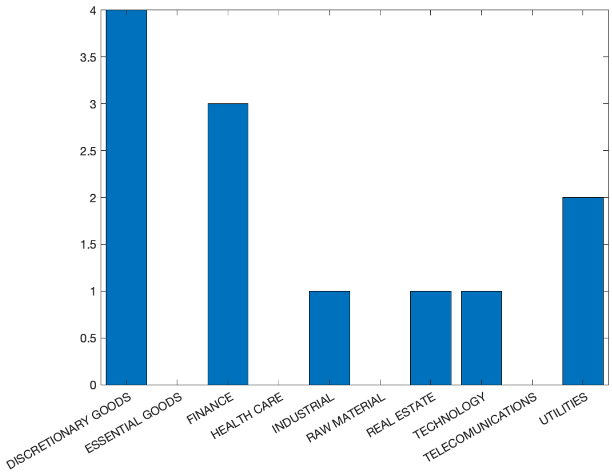


Figure 7.3: OT and OS overlaps

With respect to the previous cases, now the sectors in which more overlaps occurs are the industrial one, followed by the utilities and finance one. Two main clusters of heavily linked companies arise. The first one is composed by public utility and industrial companies. These are *ENI S.p.A.*, *Enel S.p.A.*, *Leonardo S.p.A.* and *Avio S.p.A.*. The italian *Ministero delle Economie e delle Finanze* is the public invertor linking of those. The second main cluster involves banks, respectively *Unicredit S.p.A*, *Banco BPM S.p.A.*, *Prysmian S.p.A.* and *Finecobank S.p.A.*. These companies are linked by at least one of the following common owners, *BlackRock Inc.* and *Capital Group*. Those two are financial services companies, among the largest investment management firms. The first one, *BlackRock*, is the world's largest company for AUM, and also invests in *Eni* and *Intesa Sanpaolo* among other important italian listed firms. These asset management companies specialise in active and passive management whose main aim is to profit from their investment. They usually do not heavily invest in companies so not to be involved in the management decisions. All the just mentioned companies are indeed characterised by having a very disperse ownership. None of them has its main shareholder owning more than 30% of outstanding shares, and moreover none of them is controlled by the relevant shareholders. This means that equity is mainly distributed across very small investors, who usually do not interfere in the decision making process of the company. A (maybe trivial) takeaway from this is that ownership ties tend to happen among companies whose equity is more widespread. But this happens also for heavily controlled companies. Indeed, even in the case of *Piquadro S.p.A.* and *RCS Mediagroup S.p.A.*, which are linked both in the *OS* and in the *OT* layers, the tie between them is due to the investment in both companies made by *Mediobanca*, a minority shareholder. Both *Piquadro* and *Mediagroup* have their first shareholder owning more than half of total eq-

uity. The previous claim could thus be substituted with the statement that ownership ties do not tend to happen more often for widespread equity companies, but rather they occur because of smaller investors. Still, there are also fewer examples of heavily controlled companies who present ownership ties that are due to the fact that both companies are completely controlled by the same owner. This is usually a physical person, as for the case of *Centrale del latte d'Italia S.p.A* and *Newlat Food S.p.A*, or *Vianini S.p.A* and *Caltagirone S.p.A.*.
Besides the clusters and links just mentioned though, very few companies are double linked. In general the final picture is the one of two very different layers which almost completely complement each other over their projected network. The fact that the equity distribution across shareholders in the italian stock exchange is usually not very widespread, together with the fact that OS and OT hardly ever overlap, and ,when they do, this happens for companies with distributed equity, leads to think that strongly controlled companies do not usually engage in ownership ties.
This complementary feature between the layers seems to be in contrast with what the distance between them turns out to be. Indeed $\mathbf{d}(OT, OS) = 0,18$. This is an incredible low number when considering the fact that very few edges overlap. The reason behind why those two features can coexist is hidden behind the overall number of reachable nodes from each vertex. When considering OT and OS, the number of nodes that do not present a path linking them in both layers is much higher than the one in DI or II. This is because considering interlocks leads to a layer structure in which a main connected component includes the majority of companies. Even if the number of links is relatively low, as in the case of DI, the number of couple of disconnected companies is not high. On the other hand, OS partitions the dataset of companies in many clusters. This is a consequence of the choice of the similarity index and the respective threshold. As soon as firms' most relevant shareholders do not belong to the same category, or if the two companies present very different equity distribution, they will not be linked by any path. The OT layer on the other hand links companies building a main connected component, but also leaves many disconnected nodes. For this reasons, OT and OS are regarded as completely equivalent when they are compared over a couple of disconnected firms. The distance measure $\mathbf{d}$ does not care for the topology of the layers, as it extends them by building their "most natural" connected version, without having to eliminate disconnected nodes.

The last considerations are now to be made considering the interlock layers together with OS. Taking into account DI, also in this case the situation is very similar to the one described previously, where overlaps are almost negligible. The overall marginal contribution of DI and OS to their projection is more than 99%. There are only seven double links and they are equally distributed across companies presenting different equity distribution. This means that interlocks are very rare between companies with same ownership characteristics; in other words, for the dataset of interest, direct interlocks almost strictly take place between companies with very different structure in terms of company control. The situation is similar also when including indirect interlocks into the picture. In this case, the number of overlapping edges is 24. This is a surprisingly low number when considering the fact that II and OS are the layers with the highest number number of links. The distribution of overlaps over all companies sectors is the following:

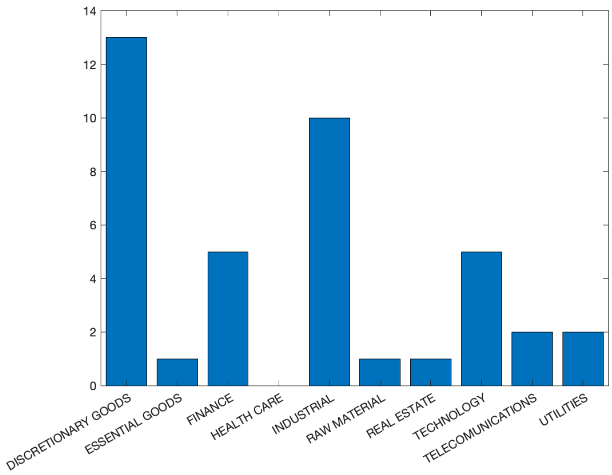


Figure 7.4: II and OS overlaps

The discretionary goods, industrial and technology sectors are the ones that engage the most in those overlaps. There doesn't seem to be any relevant pattern relating interlocks, both direct and indirect, and the degree of despersiveness of companies. Indeed, the average percentage of shares held by the first shareholder of firms engaging in double links is around 50%. This means that there is no strong evidence of interlocks to happen more often between heavily controlled companies, rather than in disperse ownership ones. In general, the main information of interest is the very low number of edge overlaps, that seem to clearly state the separation between the two layers across the dataset.

8 | Conclusions and future developments

This thesis had as subject the ownership and governance composition of the main firms listed in the italian stock exchange. It introduced a multilayer network analysis in order to study if the presence of interlocks, ownership ties or similarity in the ownership structure between companies influence each other and if those features are related. The analysis carried on in the thesis focused on the creation of different layers, linking firms according to several criterion stemming from the subjects just introduced. The aim was to understand the relationship between those firms' characteristics. This has been done by analysing the different networks, their degree of overlap and the reasons behind those. What emerged from the study was that there is a strong level of separation between the layers, meaning that considering different data leads to very different final structures. All layers complement each other, and there are very few companies engaging in more than one link. No relevant correlation is found between strongly concentrated ownership and interlocks. As the dataset considers companies that, on average, are controlled by few investors, this could lead to the conclusion that interlocks are less likely to happen for those firms. On the other hand, the majority of ties between companies happens across firms with a more dispersed ownership structure. Interlocks and ownership ties are the layers that tend to overlap the most, but still present a very complementary feature. The overlaps between those are driven by small clusters of firms, all mainly linked by either a single person or a common public investor. Considering indirect, rather than only directs interlocks leads to more connected networks, but does not change meaningfully the relationships between the layers.

A matter of interest for further future development could be to understand the impact of those features on company performance. The question that could arise, to which an answer might lead to interesting findings, is how interlocks, ownership ties and the structure of ownership of a company impact on its financial results. Carrying on an analysis of this sort could help in understanding if those features are relevant issues on this matter.

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A | Appendix A: Proof of theorem 1

Proof. The proof below demonstrates that $\mathbf{d}$ is a distance. It is thus given only for unweigthed networks. The proof that $\tilde{\mathbf{d}}$ is a distance (over connected networks) is not given, as it follows a same and simpler reasoning.

Consider two different graphs $G_1 = (V, E_1)$, $G_2 = (V, E_2)$. It is trivial to see that the operator $\mathbf{d}$ takes only positive values, that it is symmetric and that it is equal to 0 if and only if $E_1 = E_2$. What is left to be proven is thus only the triangular inequality. Consider now a third graph $G = (V, E_3)$. For simplicity, in the following mathematical reasoning, the notation $d_i = d_{G_i}(u, v)$, for $i = 1, 2, 3$ will be used. Then it has

$$\mathbf{d}(G_1, G_2) = \sum_{\{u,v\}\in E_1\cap E_2} |d_{G_1}(u,v) - d_{G_2}(u,v)| + \sum_{\{u,v\}\in E_1\setminus E_2} |N - d_{G_1}(u,v)| +$$

$$+ \sum_{\{u,v\}\in E_2\setminus E_1} |N - d_{G_2}(u,v)| =$$

$$= \sum_{E_1\cap E_2} |d_1 - d_2| + \sum_{E_1\setminus E_2} |N - d_1| + \sum_{E_2\setminus E_1} |N - d_2| \leq$$

$$\leq \sum_{E_1\cap E_2\cap E_3} |d_1 - d_3| + |d_3 - d_2| + \sum_{E_1\cap E_2\setminus E_3} |d_1 - N| + |N - d_2| +$$

$$+ \sum_{(E_1\setminus E_2)\cap E_3} |d_1 - d_3| + |d_3 - N| + \sum_{(E_1\setminus E_2)\setminus E_3} |d_1 - N| + |N - N| +$$

$$+ \sum_{(E_2\setminus E_1)\cap E_3} |N - d_3| + |d_3 - d_2| + \sum_{(E_2\setminus E_1)\setminus E_3} |N - N| + |N - d_2| = (*)$$

The inequalities in the prior passages stemps simply from the modular inequality that states that $\forall a, b, c \in \mathbb{R}$, then $|a - b| \leq |a - c| + |c - b|$. Decomposing even further the result by splitting the sums yield the result that

$$(*) = \sum_{E_1\cap E_2\cap E_3} |d_1 - d_3| + \sum_{E_1\cap E_2\cap E_3} |d_3 - d_2| + \sum_{E_1\cap E_2\setminus E_3} |d_1 - N| + \sum_{E_1\cap E_2\setminus E_3} |N - d_2| +$$

$$+\sum_{(E_1\setminus E_2)\cap E_3}|d_1-d_3|+\sum_{(E_1\setminus E_2)\cap E_3}|d_3-N|+\sum_{(E_1\setminus E_2)\setminus E_3}|d_1-N|+\sum_{(E_1\setminus E_2)\setminus E_3}|N-N|+$$

$$+\sum_{(E_2\setminus E_1)\cap E_3}|N-d_3|+\sum_{(E_2\setminus E_1)\cap E_3}|d_3-d_2|+\sum_{(E_2\setminus E_1)\setminus E_3}|N-N|+\sum_{(E_2\setminus E_1)\setminus E_3}|N-d_2|=$$

That leads to

$$\leq\sum_{E_1\cap E_3}|d_1-d_3|+\sum_{E_2\cap E_3}|d_3-d_2|+\sum_{E_1\setminus E_3}|N-d_1|+$$

$$+\sum_{E_2\setminus E_3}|N-d_2|+\sum_{(E_1\cup E_2)\cap E_3}|N-d_3|\leq$$

$$\leq\sum_{E_1\cap E_3}|d_1-d_3|+\sum_{E_2\cap E_3}|d_3-d_2|+\sum_{E_1\setminus E_3}|N-d_1|+$$

$$+\sum_{E_2\setminus E_3}|N-d_2|+\sum_{E_1\setminus E_3}|N-d_3|+\sum_{E_2\setminus E_3}|N-d_3|=$$

Adding together the different terms we get to the thesis

$$=\mathbf{d}(G_1,G_3)+\mathbf{d}(G_3,G_2)$$

□

B | Appendix B: Entropy derivation formula

The notion of "Entropy" present in this thesis has to be considered in the information theory sense of the concept. Information theory is a branch of probability theory designed to model how $information$ is conveyed from a sender to a receiver [20]. One of its purposes is to properly measure the information acquired when observing an event occurring with probability p. Defined as $I(p)$ the information gathered, this problem breaks down to how to model the function $I(\cdot)$, which is supposed continuous. This function has to satisfy some initial hypothesis; indeed, if an event is certain, then no relevant information about the state of the world can be retreived from it, thus $I(1) = 0$. Secondly, as the aim is to quantify information, $I(\cdot)$ should not hand over negative values, i.e. $I(p) \geq 0$. Finally, if two independent events with probability of happening p_1 and p_2 are observed together, then the overall information gathered should just be the sum of the one that would be retreived if those events were to be observed by themselves. Mathematically, $I(p_1 \cdot p_2) = I(p_1) + I(p_2)$. This reasoning can be generalized for every number n of events, thus yielding the equality

$$I\left(\prod_{i=1}^{n} p_i\right) = \sum_{i=1}^{n} I(p_i) = \quad \forall n \in \mathbb{N}$$

From those characteristics that the infomation function is supposed to satisfy, it follows that

$$I(p) = I((p^{\frac{1}{m}})^m) = \sum_{i=1}^{m} I(p^{\frac{1}{m}}) = m \cdot p^{\frac{1}{m}} \Rightarrow I(p^{\frac{1}{m}}) = \frac{1}{m} \cdot I(p)$$

Similarly, $I(p^{\frac{n}{m}}) = \frac{n}{m} \cdot I(p)$, $n, m \in \mathbb{N}$. This concept can be generalised and proven to hold for each real number $\alpha > 0$, i.e:

$$I(p^{\alpha}) = \alpha \cdot I(p) \quad 0 \leq p \leq 1$$

It can also be proven that the only function $I(\cdot)$ satisfying all those properties is the logarithm, thus yielding the result that

$$I(p) = -log_b(p) = log_b\left(\frac{1}{p}\right) \quad b > 0$$

Now, suppose that there are k different events that can be observed and to label them $(a_1, \ldots, a_k)$. Each event occurs with a different probability p_i. The vector $(p_1, \ldots, p_k)$ stores those probabilities. When observing event a_i the information $I(p_i) = -log(p_i)$ is gathered. Suppose N different observations are conducted. In this situation, the event $a_i,\ i \in \{1, \ldots, k\}$ will occur on average $N \cdot p_i$ times. The total information obtained throughout all observations is

$$I_N = \sum_{i=1}^{k} (N \cdot p_i) log\left(\frac{1}{p_i}\right)$$

Averaging out the information received over all observations yields the final result that

$$\frac{I_N}{N} = \sum_{i=1}^{k} p_i \cdot log\left(\frac{1}{p_i}\right)$$

If the vector $(a_1, \ldots, a_k)$ represents the support of a random variable X whose law is completely determined by the vector of probabilities $(p_1, \ldots, p_k)$, the last quantity is defined as the **Entropy** of X, and it measures the uncertainty around the outcomes of X. In formulas

$$H(X) = \sum_{i=1}^{k} p_i \cdot log\left(\frac{1}{p_i}\right)$$

For example, if $k = 2$ and X is the certain event a_1, then $H(X) = 0$ (in this case, we suppose $0 \cdot \infty = 0$). The more density function of X maximising its entropy is given by $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$.

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