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Executive Summary of the Thesis

Normalized solutions for the fractional nonlinear Schrödinger equation

Laurea Magistrale in Mathematical Engineering - Ingegneria Matematica

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1. Introduction

In this thesis we deal with existence of solutions to the nonlinear fractional Schrödinger equations

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^k\Psi + \lambda\Psi - \tilde{g}(|\Psi|^2)\Psi \quad \text{in } \mathbb{R}^N, \quad (1)$$

in case when $N \geq 2$ denotes the space dimension, $k \in (0,1)$, $\lambda \in \mathbb{R}$ and with nonhomogeneous and L^2-*supercritical* nonlinearity.

More precisely, we find standing waves solutions defined by the ansatz $\Psi(x,t) = u(x)e^{-ict}$ where $c \in \mathbb{R}$ is constant and u is a time independent function. This leads to the *stationary fractional Schrödinger equation* for the density u:

$$(-\Delta)^k u - g(u) = \lambda u \quad \text{in } \mathbb{R}^N, \quad (2)$$

where $g(t) = \tilde{g}(t)t$.

It could be possible to fix $\lambda \in \mathbb{R}$ a priori and to look for solutions to (2) as critical points of the *action functional*; in this case then, the main concern regards the so called *least action solutions*, namely solutions minimizing the action functional among all non-trivial solutions.

A different choice instead, consists in searching for solutions to (2) with prescribed *mass* (namely, prescribed L^2-norm), keeping λ as part of the unknown. This alternative has a profound significance from a physical perspective. The mass indeed, constitutes both a conserved quantity for the time dependent equation (1) and a physical meaning in the fields of applications of nonlinear Schrödinger equations: for instance, it represents the power supply in the study of propagation of beams in nonlinear optics, or the total number of atoms in Bose-Einstein condensation. In addition, from a purely mathematical point of view, studying prescribed mass solutions provides a better characterization of stationary solutions to (1), e.g. in terms of their stability or instability (see [1] and [2]).

2. Main problem presentation

In this thesis we focus on the second alternative, namely we aim at proving existence of prescribed mass solutions to (2) and we formulate our problem as: to find $(u,\lambda) \in (H^k(\mathbb{R}^N) \times \mathbb{R})$ solving (2) together with the mass constraint

$$\|u\|_2^2 = \int_{\mathbb{R}^N} |u|^2 = c^2. \quad (3)$$

The benchmark case we have in mind is $g(u) = |u|^{p-2}u + |u|^{q-2}u$, with $p, q \in (0, 2_k^*)$, $p \neq q$ and

$$2_k^* = \frac{2N}{N-2k}.$$

If g were homogeneous, we could adopt a rescaling argument approach. To apply this argument, we should, at first, consider a weak solution u to equation (2) for $\lambda = 1$, whose existence is ensured by S. Dipierro in [4]; then we set $c_0^2 = \int_{\mathbb{R}^N} u^2$ and $w_{\alpha,q}(x) = \alpha^q u(\alpha x)$ and proceed calculating both $\alpha \in \mathbb{R}$ and $q > 0$, depending on c, such that $w_{\alpha,q}(x)$ solves our problem for some $\lambda_c > 0$.
However, asking for both (3) and a nonhomogeneous nonlinear term, leads us to adopt a variational approach. Solutions to our problem, indeed, can be obtained as critical points of the energy functional $F : H^k(\mathbb{R}^N) \to \mathbb{R}$

$$F(u) = \frac{C_{N,k}}{4} \lfloor u \rfloor_k^2 - \int_{\mathbb{R}^N} G(u),$$

constrained to the L^2 sphere

$$S_c := \left\{ u \in H^k(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^2 = c^2 \right\}.$$

We remark that $\lfloor u \rfloor_k$ denotes the Gagliardo seminorm related to $H^k(\mathbb{R}^N)$, $G(t) = \int_0^t g(\tau)\, d\tau$ and $C_{N,k}$, which is the is constant appearing in the fractional Laplacian definition, is defined by

$$C_{N,k} = \left(\int_{\mathbb{R}^N} \frac{1 - \cos(x_1)}{|x|^{N+2k}}\, dx \right)^{-1}, \qquad (4)$$

where $x_1 = x \cdot e_1$ and e_1 denotes the first direction in $\mathbb{R}^N$.
In particular, we continue our dissertation showing how any solution to (2) satisfying (3) corresponds to a critical point for $F|_{S_c}$, where the parameter $\lambda \in \mathbb{R}$ appears as a Lagrangian multiplier.
Our strategy aims at detecting these critical points and identifies two different scenarios. The first one is the so called L^2-*subcritical* case, namely $p, q \in (2, \frac{4k}{N} + 2)$ and we show that, under these values of p and q, the energy functional F is bounded from below. Specifically, this case can be handled with minimization techniques, such us the concentration compactness methods (see [6] and [7]). On the other side, in the L^2-*supercritical* case, namely for $p, q \in (\frac{4k}{N} + 2, 2_k^*)$, F is not bounded from below and minimization techniques cannot be applied. As a consequence, the core part of the thesis deals with solutions to our problem in the L^2-*supercritical* case, which reveals to be harder with respect to the previous one.

2.1. Existence of a bounded Palais-Smale sequence

Our first step consists in showing the existence of a bounded Palais-Smale sequence for F at some level $\gamma(c)$.
In order to achieve this point, we rely on the auxiliary functional $\tilde{F} : (H^k(\mathbb{R}^N) \times \mathbb{R}) \to \mathbb{R}$, defined by

$$\tilde{F}(u, s) = \frac{C_{N,k}\, e^{2ks}}{4} \lfloor u \rfloor_k^2 - e^{-\, sN} \int_{\mathbb{R}^N} G(e^{\frac{sN}{2}} u(x))\, dx$$

and on the continuous mapping $H : (H^k(\mathbb{R}^N) \times \mathbb{R}) \to H^k(\mathbb{R}^N)$

$$H(u, s)(x) = e^{\frac{sN}{2}} u(e^s x),$$

which, for any $u \in S_c$ is a transformation from S_c to S_c.
At first, we shall show that $\tilde{F}$ possesses a mountain pass geometrical structure on $(S_c \times \mathbb{R})$ and, since $(S_c \times \mathbb{R})$ constitutes a riemannian manifold, the existence of a Palais-Smale sequence $(u_n, s_n)_n$ for $\tilde{F}$ at some level $\tilde{\gamma}(c)$ can be proved.
In particular, two entire sections of the thesis focus on the presentation of a version of the min-max theorem valid on differential manifolds and on its adaption to our specific L^2-sphere S_c, considered as subset of the Hilbert space $\mathbf{E} := (E \times \mathbb{R})$ (where $E := H^k(\mathbb{R}^N)$), provided with the scalar product

$$\langle \cdot, \cdot \rangle_{\mathbf{E}} = \langle \cdot, \cdot \rangle_E + \langle \cdot, \cdot \rangle_{\mathbb{R}}.$$

Finally, $(v_n)_n := (H(u_n, s_n))_n$ constitutes the candidate bounded Palais-Smale sequence for F. Therefore, we just need to exploit the boundedness of both $\tilde{F}(u_n, s_n)$ and of $\partial_s \tilde{F}(u_n, s_n)$, in order to complete the first step.

2.2. Covergence of the Palais-Smale sequence

Once we have inferred the existence of such $(v_n)_n$, we are just left to deal with its convergence in $H^k(\mathbb{R}^N)$, namely ending up with the equation

$$\lim_n \|v_n - v\|_{H^k} = 0,$$

for some $v \in H^k(\mathbb{R}^N)$.
The first problem we face, regards the lack of compactness for the continuous embedding $H^k(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $2 < p < 2_k^*$, implying that the weak limit of the Palais-Smale sequence could leave the constraint. Thus, in order to prove convergence, we restrict our framework to the space $E := H_r^k(\mathbb{R}^N)$, which denotes the space of radial functions in $H^k(\mathbb{R}^N)$ and recovers compactness.
At a later stage we prove a last additional characterization of our problem, which reads as

$$(-\Delta)^k v_n - \lambda_n v_n - g(v_n) \to 0 \ \text{ in } E^*, \tag{5}$$

with $(\lambda_n)_n \subset \mathbb{R}$ such that $\lambda_n \to \lambda_c < 0$. starting from (5) we derive the convergence for $(v_n)_n$ in E.

2.3. *Ground state*

The last part of the thesis goes through a stability analysis concerning the solution we have found. In particular we prove that this solution is a *ground state*, namely a function minimizing the energy functional F among the set of all possible solutions to our problem, denoted by $\mathcal{W}(c)$ such that

$$\mathcal{W}(c) := \{u \in S_c,\ F'|_{S_c}(u) = 0\}.$$

In this aim, we define the set

$$V_c = \left\{ u \in S_c,\ C_{N,k} \lfloor u \rfloor_k^2 = \frac{N}{k} \int_{\mathbb{R}^N} \tilde{G}(u) \right\},$$

and we apply a fractional version of the Pohozaev identity in order to gather $\mathcal{W}(c) \subset V_c$. Finally, showing that $\gamma(c) = \inf_{u \in V_c} F(u)$, is the intermediate step that leads us to

$$\gamma(c) = \inf_{u \in \mathcal{W}(c)} F(u),$$

which is the definition of *ground state*.

3. Fractional Laplacian

We remark that the results we have proved constitute the fractional counterpart of the ones presented in [5], which faces the same problem in a local framework. Furthermore, in order to properly introduce the analysis reported so far, we decided to provide a self dependent first chapter introduction of the fractional Laplacian, suitably drawing from the contributions by E. Di Nezza in [3] and by E. Valdinoci in [8].
Chapter 1 of the thesis indeed, is structured on two different intents.

3.1. Theoretical characterization

In a first, purely theoretical, part, it offers a self-contained characterization of the operator: it proposes two different definitions, via the Cauchy principal value and via the Fourier transform and sheds light on its connection with the fractional Sobolev spaces. Specifically, the Fourier definition allows us to characterize this operator as a pseudo-differential operator of multiplier $|\xi|^{2k}$, $k \in (0,1)$, as we show that

$$(-\Delta)^k u = \mathcal{F}^{-1}(\mathcal{M} \cdot \mathcal{F}u),$$

with $\mathcal{M}(\xi) = |\xi|^{2k}$. Moreover, relying again on the Fourier transform, we end up with an equation linking this operator to fractional Sobolev spaces, which is

$$\lfloor u \rfloor_k^2 = C \int_{\mathbb{R}^N} |\xi|^{2k} |\mathcal{F}u(\xi)|^2 \, d\xi.$$

3.2. Nonlocality property

Then, the second part of the first chapter focuses on some of the most outstanding consequences of the fractional Laplacian's nonlocal nature.
To be specific, we enlighten the central role it plays in the Lévy flight process where the operator's nonlocality is of prime importance, since it allows long jumps along the domain, differently from the standard random walk associated to the Laplacian. In addition, if we pass to the limit from the discrete to the continuous modelling, we lead to the famous fractional heat equation

$$\partial_t u + C(-\Delta)^k u = 0,$$

which can be considered the analogue, in a fractional framework, of the classical heat equation.
An additional non negligible effect of nonlocality appears if we consider the wavefront evolution speed for the following system

$$\begin{cases} u_t + (-\Delta)^k u = au & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = \delta(0), \end{cases}$$

where $\delta(0)$ represents the Dirac delta centred in 0 and we consider both $k = 1$ and $k = \frac{1}{2}$. In the first scenario the evolution is linear in time, while, in the second one, the fractional Laplacian plays a central role and is responsible for the wavefront travel in space to be faster, specifically exponentially in time.
The last phenomenon we analyze, then, regards the maximum principles and of the Harnack inequality. To be specific we show how the classical versions of both, which are valid for sub and superharmonic functions, fail to hold in a nonlocal context. For instance, the strong maximum principle that $(-\Delta)^k$ satisfies, under regularity hypotheses on u, reads us

$$\begin{cases} (-\Delta)^k u(x) \geq 0 & x \in \Omega \\ u(x) \geq 0 & x \in \mathbb{R}^N \backslash \Omega \end{cases}$$

implies

$$u(x) > 0 \text{ in } \Omega \text{ or } u \equiv 0.$$

It is immediate that in this case, the prescription of the only boundary values on $\partial\Omega$ for u would not be enough, since our operator works globally on $\mathbb{R}^N$; as a consequence, $\partial\Omega$ is replaced by the whole $\mathbb{R}^N \backslash \Omega$.
Also the standard Harnack inequality suffers from the same issues and needs to be adopted to the fractional framework. In particular we construct, as a counter example, a nonnegative k-harmonic function u, $k \in (0, 1)$, defined in B_1 whose minimum and maximum are not comparable in B_r, for any $r \in (0, 1)$. The function u indeed, is such that

$$\begin{cases} (-\Delta)^k u(x) = 0 & \text{for } x \in B_r \\ u(x) > 0 & \text{for } x \in B_r \backslash \{0\} \\ |u(x)| \leq 1 & \text{for } x \in \mathbb{R}^N \end{cases}$$

and $u(0) = 0$. Therefore, the Harnack inequality valid for harmonic functions, fails to hold for k-harmonic functions.
Finally, for the sake of concreteness, an explicit example of a one dimensional k-harmonic function, $k \in (0, 1)$, is provided. We prove that, setting $w_k(x) = \max\{x, 0\}^k$, $k \in (0, 1)$, it holds that

$$(-\Delta)^k w_k(x) = \begin{cases} -c_k |x|^{-k} & x < 0 \\ 0 & x > 0, \end{cases}$$

which means that, on the positive real axis, w_k is a k-harmonic function. Since this proof is pretty technical, we decided to support the analytic proof for this result with a more intuitive and heuristic justification, passing through a payoff model, in terms of expected payoff received by a particle travelling on a bounded domain.

4. Conclusions

In this manuscript, we prove existence of normalized solutions for the fractional nonlinear Schrödinger equation

$$(-\Delta)^k u - g(u) = \lambda u,$$

in case when $N \geq 2$, $k \in (0, 1)$, $\lambda \in \mathbb{R}$. Even if this result is already present in literature, the originality of the method we propose is not pointless. Section 2.3, indeed, explicits how the min-max method we rely on, allows us to detect the *ground state* associated to our problem, which is the solution seeked for the most applications in numerous physical fields.
Furthermore, this work can be intended as a starting point in order to expand the research. These kind of problems in fact, constitute nowadays a booming topic and the scientific attention they attract is in great expansion. In particular, we mention the existence of *standing and travelling solitary waves* and of *ring vortices* in bounded and unbounded domains.
Moreover, these models' application are not limited to single equations in euclidean context, but have recently been studied on systems and metric graphs. Since the metric and topological properties of these structures strongly influence the existence of the solutions, their systematic study con constitute a valid expansion of this thesis.

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