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Executive Summary of the Thesis

Waveguide Quantum Electrodynamics with Two-photon interactions

Laurea Magistrale in Physics Engineering - Ingegneria Fisica

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1. Introduction

Controlling and manipulating light-matter coupling is one of the most important branches of research of quantum sciences and technologies. The greatest achievement in this field is undoubtedly the study of the interaction between a two-level quantum emitter and a single confined bosonic mode [1]. In the cavity Quantum Electrodynamics (QED) context one of the most studied models for single photon interactions (SPI) is the so called Jaynes-Cummings (JC) model. It is based on two fundamental approximations: the dipolar approximation and the rotating wave approximation (RWA). The former implies that the interaction between the electromagnetic field and the qubit is dominated only by the electric field of the wave; moreover it is seen as uniform in space by the qubit. The RWA implies that all the terms of the Hamiltonian operator rotating at high frequency are neglected, since they tend to average to zero. In this framework the number of excitations is constant and the problem is analytically solvable. The JC model is suitable for describing linear coupling between the qubit and the mode in the weak (coupling strength smaller than the system losses) and in the strong (coupling strength greater than the losses but smaller than the mode frequency) regimes. Recent researches had revealed the possibility of implementing two-photon coupling by engineering superconducting atom-resonator systems [2] or by applying analog quantum simulation schemes in trapped-ions [3] or ultracold atoms. In order to properly describe this phenomenon of two-photon interaction (TPI) it is needed to go beyond the JC model. This is done in the two-photon quantum Rabi model (QRM). It allows one to study the nonlinear coupling between confined bosonic modes and a two-level quantum emitter. Recent studies in this direction has been done in the context of cavity QED showing new and interesting phenomena such as the appearance of distinct selection rules and a two-photon blockade as a first-order process [2].

In this thesis work, for the first time, the TPI is studied in the context of waveguide QED. With the term waveguide QED we refer to a very recent research field whose aim is to study the interaction between quantum emitters and a 1D-continuum of modes. Waveguide QED experiments can be implemented for example with superconducting artificial atoms coupled to transmission-line resonators or with quantum dots coupled to photonic-crystal waveguides. . Different studies in waveguide QED context involving SPI have been done showing phenomena that is impossible to observe in other contexts such as the total reflection effect [4]. These new possibilities have motivated us to develop a general theory for TPI in the waveguide framework. In this thesis work we have re-obtained the results relative to the SPI in waveguide QED context. Furthermore, we proceeded to study the TPI in the same context, with particular focus on input-output theory. Finally we applied the fundamental results achieved to two cases of interest: spontaneous emission and two-photon scattering. This brief executive summary will be organized as follows: in section 2 the general theory of TPI in waveguide QED will be presented; in section 3 we will analyze the results related to the two example situations mentioned above; finally, in section 4 we summarize our results and discuss the research directions opened by the present work.

2. General theory

The system that we considered is constituted by a single two-level quantum emitter coupled to the modes of a superconducting waveguide. The Hamiltonian operator of such a system is given by:

$$\begin{aligned}\hat{\mathbf{H}} &= \omega_0\hat{\sigma}^+\hat{\sigma}^- + \sum_{\mu=\pm}\int \omega(\hat{a}^{\mu}_{\omega})^{\dagger}\hat{a}^{\mu}_{\omega}\, d\omega + \\ &+ \sum_{\mu,\mu'=\pm}\iint (g^{\mu\mu'}_{\omega\omega'})^{*}\hat{\sigma}^{+}\frac{\hat{a}^{\mu}_{\omega}\hat{a}^{\mu'}_{\omega'}}{\sqrt{2}}\, d\omega d\omega' + h.c\end{aligned} \tag{1}$$

where the first and the second terms refer to the energy of the emitter and of the supported modes, while the others constitute the nonlinear interaction operator. It is indeed proportional to the square of the electric field operator and, since the interaction is nonlinear, the coupling strength depends on the frequencies of the two interacting photons. The indexes μ and μ' define the direction of propagation of each mode along the waveguide: $+$ means from left to right;- means from right to left. We point out here that to obtain the expression in Eq.1 the RWA has been applied and the number of excitations is conserved. In particular, to each atomic transition correspond two field transitions (absorption/emission of two photons). Working in the RWA allows us also to define the general state of the system with the Wigner-Weisskopf ansatz. Its expression is the following:

$$\begin{aligned}|\Phi(t)\rangle &= C_e(t)\hat{\sigma}^{+}|\mathbf{0}\rangle + \\ &+ \sum_{\mu,\mu'=\pm}\iint C^{\mu\mu'}_{\omega\omega'}(t)\frac{(\hat{a}^{\mu}_{\omega})^{\dagger}(\hat{a}^{\mu'}_{\omega'})^{\dagger}}{\sqrt{2}}\, d\omega d\omega'\,|\mathbf{0}\rangle\end{aligned} \tag{2}$$

where $|\mathbf{0}\rangle$ is the system ground state. The general state of the system is expressed as the linear superposition of the possible output states ($|e\rangle|0_{\omega},0_{\omega'}\rangle$, $|g\rangle|1_{\omega},1_{\omega'}\rangle$), each one weighted by its own time-dependent amplitude probability coefficient. In order to study the dynamics of the system considered, it is necessary to find out the explicit expressions of these amplitude probabilities. The most direct way to solve this problem is to insert Eq.1 and Eq.2 inside the time-dependent Scrhödinger equation (TDSE) and, by exploiting the properties of hortonormality of the possible output states, to obtain a system of linear coupled differential equations for the two unknowns seeked. This is indeed what it has been done in this thesis work. The system obtained is the following:

$$\begin{aligned}i\dot{C}_e &= \omega_0 C_e(t) + \sum_{\mu,\mu'=\pm}\iint d\omega d\omega'(g^{\mu\mu'}_{\omega\omega'})^{*}C^{\mu\mu'}_{\omega\omega'}(t) \\ i\dot{C}^{\mu\mu'}_{\omega\omega'}(t) &= (\omega+\omega')C^{\mu\mu'}_{\omega\omega'}(t) + g^{\mu\mu'}_{\omega\omega'}C_e(t)\end{aligned} \tag{3}$$

We notice here that the system of differential equations is bidimensional in the frequency domain. This particular dependence adds more complexity to the problem, making it even more difficult to solve. Before analyzing the solutions obtained in this work, it is useful to introduce the definitions of input and output fields of the system. Their definitions are given by:

$$\Psi^{\mu\mu'\Delta}_{in}(t) = \frac{1}{\sqrt{2\pi}}\int C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0)e^{-i\bar{\omega}(t-t_0)}\, d\bar{\omega} \tag{4}$$

$$\Psi^{\mu\mu'\Delta}_{out}(t) = \frac{1}{\sqrt{2\pi}}\int C^{\mu\mu'}_{\bar{\omega},\Delta}(t_1)e^{-i\bar{\omega}(t-t_1)}\, d\bar{\omega} \tag{5}$$

where $\bar{\omega} = \omega' + \omega$ and $\Delta = \omega' - \omega$ are a new pair of variables obtained from the linear combination of the original frequencies of the modes. The input and the output fields are defined as the Fourier Transform of the fields amplitude probabilities evaluated in t_0 (far before the interaction event) and t_1 (far after the interaction event) respectively. Notice that they both depend on time and, differently from the one-photon case, they also depend on the difference of the frequencies. From the system in Eq.3 it is possible to obtain a linear differential equation in the only unknown $C_e(t)$, together with the input-output relation for the TPI, which links the output field to the input one through the nonlinear coupling with the quantum emitter. The expressions of these two fundamental equations are the following:

$$\dot{C}_e(t) = -i(\frac{\gamma}{2} + \omega_0)C_e(t) + \\ - \frac{i}{2} \sum_{\mu,\mu'=\pm} \int_{-\infty}^{+\infty} \sqrt{\gamma_\Delta^{\mu\mu'}} \Psi_{in}^{\mu\mu'\Delta}(t)\, d\Delta \tag{6}$$

$$\Psi_{out}^{\mu\mu'\Delta}(t) = \Psi_{in}^{\mu\mu'\Delta}(t) - i\sqrt{2\gamma_\Delta^{\mu\mu'}}C_e(t) \tag{7}$$

where γ is the qubit total spontaneous emission rate defined as the integral in all the possible values of Δ of the coupling strength and $\gamma_\Delta^{\mu\mu'} = \pi\left|g_\Delta^{\mu\mu'}\right|^2$. In obtaining the fundamental equations above we have assumed that in a band of frequencies around resonance the coupling parameter does not depend on the sum of the frequencies but only on the difference. The TPI problem is self-consistent thanks to Eq.6 and Eq.7: once the input field expression is known, it is possible to solve the differential equation to find $C_e(t)$; by inserting its expression in the input-output relation it is possible to obtain the output field and, with an inverse Fourier Transform operation, one can compute the output field amplitude probability, thus solving the problem.

3. Phenomenology

In this thesis work, after defining the general theory of TPI, we applied it to the two cases of interest of spontaneous emission and two-photon scattering. In this section we will show the results related to those two situations.

3.1. Spontaneous emission

For the particular system considered, the situation could be schematized as follows: the artificial atom initially in the ground state $|g\rangle$ has absorbed two incoming photons at certain frequencies ω_{in1} and ω_{in2} in the waveguide; the qubit now in the excited state $|e\rangle$, emits two photons at certain frequencies ω and ω' after a certain time t, usually of the order of the excited level lifetime. The first step that has to be done for finding the two amplitude probability coefficients is defining the initial conditions of the system, i.e. the value of the qubit and of the field coefficients at t_0, with t_0 the instant of time when the emitter is in the state $|e\rangle$. The initial conditions related to this situation are simply:

$$C_e(t_0) = 1 \quad ; \quad C_{\bar{\omega},\Delta}^{\mu\mu'}(t_0) = 0 \tag{8}$$

Since the field amplitude probability is null, it is easy to verify that also the input field $\Psi_{in}^{\mu\mu'\Delta}(t)$ is equal to zero. In this way the differential equation for the atom amplitude probability simplifies and it is directly integrable. Inserting the solution of Eq.6 in the input-output relation allows us to compute the output field $\Psi_{out}^{\mu\mu'\Delta}(t)$ and then the relative amplitude probability function. The modulus square of the two time-dependent coefficients are:

$$|C_e(t)|^2 = e^{-\gamma(t-t_0)}\mathcal{H}(t-t_0) \tag{9}$$

$$\left|C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1)\right|^2 = \frac{1}{\pi}\frac{\gamma_\Delta^{\mu\mu'}}{\frac{\gamma^2}{4} + (\omega_0 - \bar{\omega})^2} \tag{10}$$

where $\mathcal{H}(t-t_0)$ is the Haeviside function centered in $t = t_0$. The physical meaning of the above expressions is the following: they give us information about the probability of finding the emitter or the field in excited state respectively. For what concerns the atom probability distribution represented by $|C_e(t)|$, it is exactly what one could expect. Indeed, it is defined by a decreasing exponential in time with slope determined by the qubit total spontaneous emission rate. This means that the probability of finding the qubit in the state $|e\rangle$ after a certain instant t, grater than the initial one t_0, decreases exponentially as soon as we let the system evolve in time. On the other hand, the probability distribution regarding the output field does not depend on time. This derives from the fact that through the input-output relation what is retrieved is the field amplitude probability coefficient evaluated in $t = t_1$, with t_1 much larger than t_0. In addition to that, as it can be seen from Eq.10, the field probability distribution depends on the frequencies sum $\bar{\omega}$ and on the frequencies difference Δ via the coupling parameter. This is indeed a direct consequence of the bidimensionality of the system in Eq.3 in the frequency domain and it will result in a much more complicated frequency distribution for the coefficient considered. Before going on with the analyzes of the output probability distribution, it is necessary at this point to define properly the parameter $\gamma_{\Delta}^{\mu\mu'}$. Indeed, the expression of the modulus square of $C_{\bar{\omega},\Delta}^{\mu\mu'}(t_1)$ computed above, is generic, i.e. it holds for every possible profile of the coupling parameter. However, to obtain the specific value for the coupling parameter we must first study in details how the nonlinear superconducting waveguide couples with the emitter. In the following we will consider an arbitrary plausible shape for the coupling parameter. Of course, once its exact value will be computed, it would be possible to repeat the following analyzes. In the continue of this thesis work, we have chosen for the coupling parameter in the TPI case the following expression:

$$\gamma_{\Delta}^{\mu\mu'} = \frac{\Gamma_0^{\mu\mu'}}{\sqrt{2\pi}\sigma} e^{-\frac{\Delta^2}{2\sigma^2}} \tag{11}$$

It is defined as a normalized Gaussian function in the variable Δ with a full width at half maximum (FWHM) proportional to σ. Its maximum value, reached when $\Delta = \omega - \omega' = 0$ is given by the constant $\Gamma_0^{\mu\mu'}$, which in principle depends on the particular direction of propagation of the two photons. However, we will consider a symmetric waveguide, i.e. there will not be any preferential direction of emission along it. This means that it is possible to eliminate the dependence on the indexes μ and μ' from the coupling parameter. The specific expression for the modulus square of the output field amplitude probability is then:

$$\left|C_{\bar{\omega},\Delta}(t_1)\right|^2 = \frac{1}{\sqrt{2\pi^3}} \frac{\Gamma_0 e^{-\frac{\Delta^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - \bar{\omega})^2} \tag{12}$$

where $\gamma = 3\Gamma_0$. To be more precise, the expression in Eq.12 is related to only one possible combination of the indexes μ and μ'. To retrieve the total probability distribution for the output field it is necessary to add another multiplicative factor of 3, but that would not change in any case the frequencies dependence and it is omitted here. As it could be seen from Eq.12, the resonance condition in this TPI picture is given by $\omega = \omega' = \omega_0/2$. Indeed, the probability of observing at the output of the waveguide a pair of emitted frequencies is maximum when these frequencies match, or are close to, this resonance condition. This is coherent with the fact that the atom is more likely to emit photons close to its own transition frequency ω_0 and with the fact that the spontaneous emission of a pair of photons at a completely different sum of frequencies with respect to that of the atom is practically impossible. The novelty related to considering TPI with respect to the single photon ones, is that, since the problem is bidimensional, we could have in principle different bandwidths of emission probability for each direction. While it is always true that the shape of the modulus square along the direction $\bar{\omega}$ will be Lorentzian with FWHM proportional to γ, the shape on the opposite direction Δ is determined by the particular expression of $\gamma_{\Delta}^{\mu\mu'}$. In this thesis work it has been chosen with a Gaussian profile with FWHM proportional to σ. However, by a properly design of the implementation of the system, it is possible to tune it.

3.2. Two-photon scattering

The other case of interest investigated in this thesis work is the two-photon scattering event. With this term we refer to a particular situation in which the qubit, that is prepared in its ground state $|g\rangle$, interacts with an incoming electromagnetic field in input in the state $|1_{\omega_1}, 1_{\omega_2}\rangle$. As it has been done in the previous section for the spontaneous emission case, in order to study the TPI in case of scattering, it is necessary to define the initial conditions. Regarding the qubit amplitude probability coefficient, we must impose that in the instant of time in which the scattering event happens, i.e. t_0, the qubit is in its ground state $|g\rangle$. On the other hand, we must define the input field amplitude probability coefficient. For simplicity, in this thesis work, the input fields will be taken as monochromatic plane waves. These two incoming monochromatic fields will be centered in two generic input frequencies labeled as ω_1 and ω_2. However, since to derive Eq.7 and Eq.6 we have performed a change of variables, these input fields will be expressed as functions of the frequency sum $\bar{\omega}$ and the frequency difference Δ. The above considerations correspond to the following initial conditions:

$$C^{\mu\mu'}_{\bar{\omega},\Delta}(t_0) = \frac{\delta^{\mu}_{+}\delta^{\mu'}_{+}}{\sigma\sqrt{\pi}} e^{-\frac{[\bar{\omega}-(\omega_1+\omega_2)]^2}{4\sigma^2}} e^{-\frac{[\Delta-(\omega_1-\omega_2)]^2}{4\sigma^2}}$$

$$C_e(t_0) = 0 \tag{13}$$

where the Kronecker delta functions simply define the direction of propagation along the waveguide of the input, which in this case has been chosen from left to right for both the photons. The condition of monochromatic plane wave is reached when the fields have a really narrow spectral profile (limit of small σ). At this point there are three different possible situations for what concerns the relative width between the input fields in the waveguide and the coupling parameter (which also for this two-photon scattering case will be taken equal to that in Eq.11): *i*) the width of the input fields matches the width of the coupling parameter; *ii*) the width of the input fields is narrower than that of the coupling parameter; *iii*) the width of the coupling parameter is narrower than that of the input fields. In this executive summary will be presented the results in the case of spectral width matching. However, in the thesis work the situation *ii*) is also briefly analyzed. Unfortunately, due to lack of time, the last case has to be studied yet. From now on we will focus on the situation in which the width of the input fields matches the one of the coupling parameter. With the initial conditions in Eq.13 together with Eq.7 and Eq.6, we could compute the exact expression for the scattering coefficients in this TPI picture. It is important to notice that there are three different scattering situations that could be observable after the interaction with the quantum emitter, each one defined by a specific choice of the direction indexes: Reflection of two photons ($\mu = \mu' = -$), Transmission of two photons ($\mu = \mu' = +$) and Splitting ($\mu \neq \mu'$). With the term Splitting we refer to the possibility of observing one photon reflected back and the other transmitted after the interaction. Since we are working with bosonic modes, it does not matter where each photon goes, because the two alternatives represent the same situation.
The explicit expression of the scattering coefficients are the following:

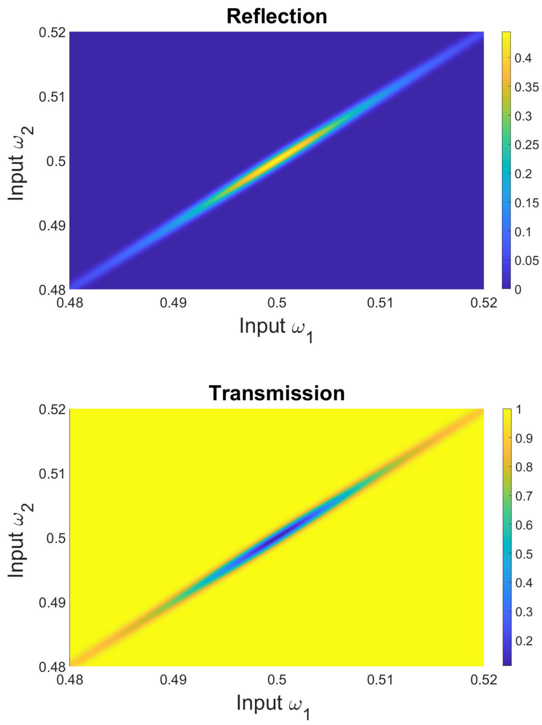

Figure 1: 2D-colormap of the Reflection and Transmission scattering coefficients as a function of the two input frequencies expressed in units of the atom characteristic frequency ω_0. The coupling parameter Γ_0 is $0.01\omega_0$, while $\sigma = 0.001\omega_0$.

$$R = S = \frac{\Gamma_0^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - (\omega_1+\omega_2))^2} \tag{14}$$

$$T = 1 - \frac{2\Gamma_0^2 e^{-\frac{(\omega_1-\omega_2)^2}{2\sigma^2}}}{\frac{9\Gamma_0^2}{4} + (\omega_0 - (\omega_1+\omega_2))^2} \tag{15}$$

As it can be seen from the previous equations, the Reflection and the Splitting coefficients share the same expression, which means that the probability of having a reflection or a splitting event after the interaction is the same. In fig.1 are showed the 2D-colormap for the Reflection (equal to the Splitting one) and for the Transmission coefficient, function of the two input frequencies. In the sum of frequencies direction $\bar{\omega}'$, the scattering coefficients have a Lorentzian linewidth with FWHM proportional to Γ_0, while in the frequency difference direction Δ', they have the same profile as that of the coupling parameter (Gaussian linewidth with FWHM proportional to σ in this case). For the particular choices that we have made through this thesis work, the parameter Γ_0 results to be greater than the parameter . Due to this reason in the direction Δ' the coefficients have a really narrower profile with respect to the one in the direction $\bar{\omega}'$. This is also coherent with the fact that in the $\bar{\omega}'$ direction, what we observe in this TPI case is the same profile that we would have observed in the SPI situation if we had considered the qubit interacting with a single photon with frequency $\Omega = \omega_1 + \omega_2$. Regarding the profile in the Δ' direction, it is so narrow in agreement with the fact that the qubit is not capable of interacting with two photons with very different frequencies. Ultimately, from Eq.14 and Eq.15, at resonance ($\omega_1 = \omega_2 = \omega_0/2$), the maximum value achievable for the Reflection coefficient is $\frac{4}{9}$, while for the Transmission one the minimum is $\frac{1}{9}$. This means that, at least in this framework, it is not possible to observe the total reflection phenomenon (achieved in SPI). Indeed, the probability of transmission is always greater than zero.

4. Conclusions

Let us now briefly summarize our results before commenting on future perspectives. We were able to obtain important results regarding the general theory of TPI in waveguide QED systems. In particular, we succeeded in defining a set of self-consistent equations (Eq.7 and Eq.6) that could allow us to compute the amplitude probability coefficients after the initial state of the system has been defined. We have analyzed two specific examples, spontaneous emission and two-photon scattering, to show explicitly how those results could be applied to cases of interest. This work represents a first analysis of two-photon couplings in the context of waveguide QED. Our result paves the way towards the exploration of a novel quantum phenomenology and to possible applications in quantum technologies. Regarding the future perspectives of this work, it could have in quantum computation a possible field of application. In the past few decades quantum computing has been object to the interest of many researches around the world, which has lead to a rapid evolution of its architectures and techniques of implementation. The origin of this incredible success has to be found in the idea of the so called quantum supremacy. There are many competing platforms, which are in principle able to implement quantum computers. Quantum information processing with propagating photons is particularly interesting also for quantum communication and cryptography tasks. Atoms and artificial atoms can be used to mediate the interaction between photons. However, standard light-matter couplings have some intrinsic limitation. For example, achieving perfect fidelity in the implementation of controlled-phase gates is not possible [5] using a single artificial atom. This fundamental limitation does not hold for TPI and so our theory could lead to alternative solutions for quantum information processing with propagating photons.

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