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EXECUTIVE SUMMARY OF THE THESIS

Syzygy Design Strategy for the Multiple Gravity Assist Problem

LAUREA MAGISTRALE IN SPACE ENGINEERING - INGEGNERIA SPAZIALE

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1. Introduction

In the past 60 years interplanetary spaceflight has shown the great scientific value of data gathered from the observation of the Solar System. A growing need for missions that enable deep interplanetary travel has brought forward the development of new technologies and strategies to overcome the limitations dictated by the current propulsion systems.
A solution to this issue has been identified in exploiting Gravity Assist Maneuvers (GAMs). A gravity assist manoeuvre is the use of the relative movement and gravity field of a planet or other massive celestial body to change the velocity of a spacecraft. This is achieved with a close proximity swing-by of the celestial body so that its gravity produces a change in the velocity vector of the spacecraft [6]. This idea has been exploited widely in the last decades, in missions such as Bepicolombo, Cassini-Huygens and many others.
However, in many cases, a single GAM is not sufficient to achieve the mission target and it is necessary to perform a definite number of GAMs to reach the objective. These kind of missions are identified as Multiple Gravity Assist missions (MGA).
In the last decades, several methods were proposed to tackle this issue, based on the keplerian map [7] or the Tisserand-Poincarè map [2] and the flyby [8].
In this work, an alternative solution to the problem is proposed: following the idea outlined by Menzio et al.[3], the trajectory is designed by means of the *syzygy* algorithm. The model presented by Menzio is further improved by removing one of its main simplifying assumptions, namely that the transfer leg must either start or end at one of it's absidal points, and it's extended to hyperbolic orbits. The initial model is then augmented by implementing the B-plane formalism introduced by Öpik, [4] and following Carusi et al. [1], the flyby problem is resolved.
The developed model is finally combined with a dynamic programming approach to address the design of an optimal solution.

2. The *Syzygy* Algorithm

The design of the interplanetary arcs is performed in a circular, planar two body problem, in a patched conics framework. Under this assumptions, once the wanted planetary sequence is defined, and removing the assumption of tangential departure and arrival, an interplanetary transfer is completely defined by parameterizing the departure time t_1, the eccentricity and the

true anomaly at departure $\theta_1(t_1)$.
Interception with the target planet occurs after travelling an aperture of [3]:

$$\Delta\theta_{21}(e_1) = acos(\frac{1}{e_1}(\frac{r_p}{r_2}(1-e_1^2)-1)) \quad (1)$$

$$\theta_2 = \Delta\theta_{21}(e_1) \quad (2)$$

Evaluating the time of flight relative to the aperture just computed, the encounter takes place if the feasible transfer condition (FTC), defined as [3]:

$$FTC = n_2 tof + \phi_{21}(t_1) - \theta_2(t_2) + \theta_1(t_1) \quad (3)$$

where n_2 is the angular velocity of the target planet, $\phi_{21}(t_1)$ is the initial phasing between the celestial bodies, tends to zero.
The FTC represents the difference, in terms of angular position, between the spacecraft and the target planet. As the FTC tends to zero, so does the difference of angular position between the spacecraft and the target planet, meaning that the two objects share the same spatial position.
The trajectory design problem is therefore reduced to a fully combinatorial matter: finding the correct combinations of shaping parameters (t_i, e_i, θ_i) that make the FTC null.

3. Close Encounters In The B-plane And Flyby Characterization

3.1. Encounter in the B-plane

Once the pre-encounter interplanetary leg has been evaluated in terms of Keplerian parameters by means of the *syzygy* algorithm, the post-encounter orbital elements can be evaluated by exploiting the B-plane and Öpik theory for close encounters.
Following the procedure outlined by Carusi et al. [1], the components of the spacecraft velocity are computed in a planetocentric reference frame (X, Y, Z), centred in the planet's center of mass such that the X-axis is directed from the Sun to the planet's position, the Y-axis is aligned with its direction of motion and the and Z-axis completes the right-handed triad. In this framework, all the orbital parameters are computed by using Jacobi normalized units. The expressions for the velocity components are:

$$\begin{cases} U_x = \pm\sqrt{2 - \frac{1}{a} - a(1-e^2)} \\ U_y = \sqrt{a(1-e^2)}cos(i) - 1 \\ U_z = \pm\sqrt{a(1-e^2)}sin(i) \end{cases} \quad (4)$$

Calling with T the Tisserand parameter of the particle's orbit, it can be shown a relation between T and the magnitude of the planetocentric velocity U:

$$U = 3 - T = \sqrt{3 - \frac{1}{a} - 2\sqrt{a(1-e^2)}cos(i)} \quad (5)$$

T is an invariant for the CR3BP, therefore, both U and T are conserved during a close encounter. By tackling the close encounter with this strategy, the flyby will automatically respect the conservation of the Tisserand parameter.
The encounter is then characterized geometrically by means of two angles, θ and ϕ, where θ is the angle between U and the y-axis, and ϕ, the angle between the y-z plane and that containing U and the x-axis, as shown in figure 1:

$$cos(\theta) = \frac{1 - \frac{1}{a} - U^2}{2U} \quad (6)$$

Being the problem bi-dimensional ϕ is null (the same holds for the post-encounter ϕ'), and the rotation of the incoming velocity is planar of magnitude γ. It's possible to derive the deflection angle γ, from:

$$tan(\frac{\gamma}{2}) = \frac{c}{b} \quad (7)$$

Where b is the impact parameter and c the characteristic length defined by:

$$c = m/U^2 \quad (8)$$

Where m is the planet's mass in terms of the Sun mass.
Computing the post-encounter angle θ' by means of a rotation γ, finally, the post-encounter orbital parameters are evaluated as:

$$\begin{bmatrix} U'_x \\ U'_y \\ U'_z \end{bmatrix} = \begin{bmatrix} Usin(\theta')sin(\phi') \\ Ucos(\theta') \\ Usin(\theta')cos(\phi') \end{bmatrix} \quad (9)$$

$$a' = \frac{1}{1 - U^2 - 2U'_y} \quad (10)$$

$$e' = \sqrt{U^4 + 4U_y^{2\prime} + U_x^{2\prime}(1 - U^2 - 2U_y^{2\prime}) + 4U^2U'_y} \quad (11)$$

Now that the close encounter effect has been evaluated, the specific point where the flyby takes place can be identified on the B-plane. The B-plane is defined as the plane orthogonal to U and containing the centre of the planet. In this context it is introduced another planetocentric reference frame (ξ, η, ζ) such that the (ξ, ζ)-axes lie on the b-plane and η is perpendicular to it. In particular, ζ is parallel to the projection of the planet's velocity V_{pl} on the b-plane but with opposite direction and ξ completes a right-handed reference system, as shown in figure 1:

$$\hat{\eta} = \frac{U}{|U|} \quad (12)$$

$$\hat{\xi} = \frac{U \times v_{pl}}{|U \times v_{pl}|} \quad (13)$$

$$\hat{\zeta} = \hat{\xi} \times \hat{\eta} \quad (14)$$

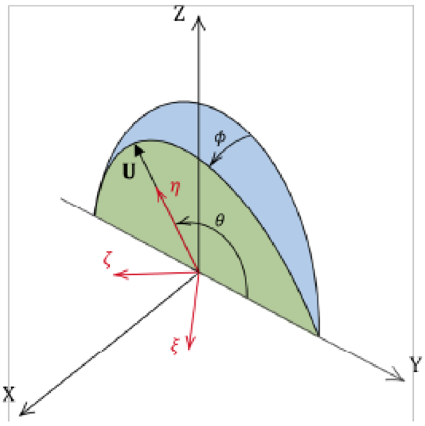

Figure 1: The figure shows the two reference frames presented: B-plane and planetocentric frame. The figure is taken from Campiti et al. work [5]

Starting from the geometry considerations made so far, it is possible to identify the point on the B-plane corresponding to the close encounter considered:

$$\zeta = \frac{(b^2 + c^2) * cos(\theta')}{2csin(\theta)} - \frac{(b^2 - c^2) * cos(\theta)}{2csin(\theta)} \quad (15)$$

$$\xi = \sqrt{b^2 - \zeta^2} \quad (16)$$

$$\eta = 0 \quad (17)$$

3.2. Flyby Characterization

It is possible to employ the classical Orbital Mechanics relations for the characterization of the flyby, in order to verify if the deflection studied by means of the B-plane is actually feasible for the unique incoming interplanetary leg designed with the *syzygy* algorithm.

By first evaluating the specific angular momentum of the planet and spacecraft, it is possible to compute the radial and orthogonal components of the spacecraft and planet's velocity. At this point, it is possible to compute the planetocentric relative velocity (velocity at infinity) of the spacecraft, first evaluating it's radial and orthogonal components:

$$v_{r\infty} = v_{rpl} - v_{rsc} \quad (18)$$

$$v_{o\infty} = v_{opl} - v_{osc} \quad (19)$$

$$\mathbf{v}_\infty = \begin{bmatrix} v_{r\infty} \\ v_{o\infty} \end{bmatrix} \quad (20)$$

By fixing the impact parameter, it is possible to completely define the close encounter hyperbolic trajectory in terms of pericenter radius r_p:

$$r_p = \frac{-\mu_{pl}}{v_\infty^2} + \sqrt{\frac{\mu_{pl}^2}{v_\infty^4} + b^2} \quad (21)$$

Where μ_{pl} is the gravitational parameter of the flyby planet, v_∞ is the magnitude of $\mathbf{v}_\infty$ and b is the impact parameter. The flyby hyperbola will feature an eccentricity given by:

$$e = 1 + \frac{r_p v_\infty^2}{\mu_{pl}} \quad (22)$$

and turn angle δ computed as:

$$\delta = 2asin(\frac{1}{e}) \quad (23)$$

The data gathered by means of these computations can be exploited in order to evaluate the feasibility of the close encounter prescribed by the B-plane deflection model, by checking if the perigee radius r_p is actually greater than the planetary radius of the flyby planet. The feasibility condition therefore reads:

$$r_p > r_{planet} \tag{24}$$

Any flyby that respects this condition is considered feasible.

4. Dynamic programming

Dynamic programming is a useful mathematical technique for making a sequence of interrelated decisions. It provides a computationally efficient method for finding optimal solutions to problems that can be formulated as multi-stage decision processes, where the decisions made at one time influence the later available choices.
The general approach requires to structure the problem as a multistage decision process, where the stages identify the points in which a policy decision is required. The problem is tackled starting from either the first or last stage and proceeding one stage at a time, such that the solution to a stage is necessary to solve the next one. At each stage, the system might be in different possible conditions called states, and a policy decision has the effect to transform the current state into a different one associated with the next stage. The problem must comply with the so called Markovian property [9], which prescribes that the chosen state must retain all the necessary information to determine the optimal policy henceforth.
In practice, the optimization is carried out by defining a recursive relationship that provides the optimal policy to any sub-problem, given the solution to all the smaller sub-problems.

5. Solution Strategy

The preliminary design problem addressed in this work can be expressed as: given a spacecraft departing from a given position at an initial time, evaluate the optimal trajectory that leads the spacecraft to a target celestial object, performing ballistic flybys on a sequence of selected planets.

The problem is tackled under the following assumptions:

- patched-conics model
- planets move on circular, co-planar orbits
- no resonant flybys
- unpowered flybys
- no orbital perturbations

Following the ideas presented in sections 2 and 3, finding the optimal sequence of GAMs therefore translates into the search of the optimal triplet (e_i, θ_{1i}, t_i) for the first arc and the optimal series of couplets (t_i, γ_i) that lead from the initial planet to the target one by performing an arbitrary number of flybys:

$$\text{Initial planet} \rightarrow (e_1, \theta_1, t_1) \rightarrow (t_2, \gamma_2) \rightarrow ... \rightarrow \text{Target planet}$$

The approach proposed in the thesis features a structure with a variable number (depending on the number of wanted GAMs) of nested *for* loops, used to analyse all possible combinations of departure date, eccentricity and departure true anomaly on the first transfer orbit, and combinations of departure date and deflection angle for the following arcs, that satisfy the chosen optimality policy, The objective function is defined as:

$$f_n(s_n, x_n) = \sum_{i=1}^{N} |FTC| \tag{25}$$

As already stated, the function FTC represents the precision of the interplanetary transfer: the closer it is to zero, the more precisely the transfer arc leads the spacecraft to the target planet. The dynamic programming approach is then exploited to evaluate the optimal solution between the ones identified by the algorithm.

6. Test Cases and Results

The proposed solution strategy performance is assessed by reproducing Voyager-like trajectories and by designing a third arbitrary mission towards the inner planets, in the framework introduced in the previous sections.
The goodness of the proposed solutions is assessed by a validation process using a Lambert solver, that showed that the relative error committed by the *syzygy* is negligible from a trajectory design point of view, and doesn't depend on the number of close encounters underwent.

In this summary only the Voyager-1 mission will be reported, for the sake of conciseness.

6.1. Voyager-1

The computed trajectory is shown in figure 2:

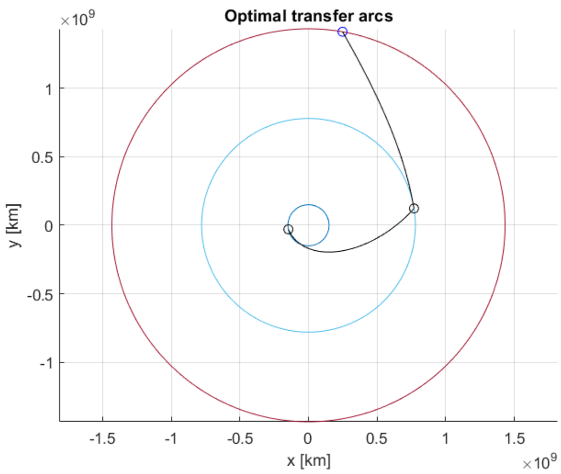

Figure 2: Optimal trajectory in terms of transfer precision (FTC). In blue Earth's orbit, in light blue Jupiter's orbit and in red Saturn's orbit. The small circles represent the planetary position at encounter.

The orbital elements of the transfer legs are reported in table 1:

a	**e**	ω	θ
$6.54 \cdot 10^8$ km	0.783	-2.45 rad	5.79 rad
$-5.29 \cdot 10^8$ km	2.306	0.19 rad	-0.0204 rad

Table 1: Orbital elements of the computed transfer legs.

The presented trajectory shows the minimum value for the *FTC* function, corresponding to the most precise transfer, that leads to a distance error with respect of the target planets shown in table (table 2):

Planet	*FTC* **value**	**Distance Error**
Jupiter	$2.28 \cdot 10^{-4}$	$1.7749 \cdot 10^5$ km
Saturn	$-7.7369 \cdot 10^{-4}$	$1.108 \cdot 10^6$ km

Table 2: *FTC* value and associated distance error of each interplanetary leg.

The departure and arrival time for each leg, corresponding to the shown planetary configuration are reported in table 3:

Departure date	02-01-82
First encounter date	05-08-83
Arrival date	04-08-85

Table 3: Departure, first encounter and arrival date for the computed solution.

Finally, the flyby is characterized more in detail. In order to obtain the prescribed variation of keplerian parameters, the close encounter must take place at a specific location in the B-plane, namely (table 4):

ξ	η	ζ
0	0	$8.2426 \cdot 10^{-4}$

Table 4: B-plane coordinates of the close encounter. Note that all the values are adimensional, due to the fact that all quantities used in the B-plane computations are normalized.

It is also interesting to evaluate the characteristic quantities that define the close encounter, shown in table 5:

b	r_p	e	δ
$-6.4167 \cdot 10^5$ km	$1.7225 \cdot 10^5$ km	1.1553	2.0926 rad

Table 5: Values of the impact parameter b, perigee radius of the flyby hyperbola r_p, eccentricity and turn angle δ.

Both the values for perigee radius and impact parameter show a value compatible with the minimum flyby altitude (the planetary radius),

making the presented close encounter indeed feasible.
The algorithm took 246.69 s for the first arc generation, 9.582 s for the second arc and only 0.085 s for the dynamic programming optimization.

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7. Conclusions and Outlook

The developed algorithm was used to simulate two real-life missions, Voyager-1 and Voyager-2. The computed solutions present differences with the baseline mission profiles, mainly due to the bi-dimensionality of the model, but nevertheless can be considered good initial guesses for more complex solution strategies.
The algorithm was finally validated by comparing the goodness of the computed solutions with the typical and fool-proof Lambert solver, showing very small discrepancies between the two solutions. The performance might be comparable in terms of solution goodness, but the *syzygy* algorithm real merit is the great quickness in the solution space search.
The main limitation of the proposed procedure is the bi-dimensionality of the model: the out of plane motion leads to non-negligible differences in the trajectory computed. The B-plane procedure to evaluate post encounter parameters is already defined in the three dimensional case, so future works could focus on the extension of the syzygy algorithm to the 3-D case, exploiting considerations on spherical geometry. Another step forward could be to include perturbations within the planet's SOI in the model, similarly to what has already been introduced in the resonant close encounter scenario.
The complexity of the problems presented lies in the size of the solution space and on the highly combinatorial nature of the latter. The computational quickness of the dynamic programming approach, together with the speed of the syzygy formulation of the problem, represent a very useful tool for the preliminary trajectory design for MGA mission scenarios.
A possible direction that could be followed in the future works to fully exploit the presented optimization strategy is looking for more complex and efficient optimality policies for the selection of the optimal trajectory.
In conclusion, the proposed algorithm provides a fast and reliable implement for the preliminary design of MGA trajectories, which could serve as reasonable starting point for numerical methods to faster converge to more complete trajectory solutions.

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