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# Jump-Diffusion Processes Application in Finance

**Supervisor:**
Prof. Carlo Sgarra

**Student:**
Guilherme Matiussi Ramalho (Student Id. Number:720718)

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**Abstract**

When we talk about applications of Mathematics in finance, a reasonable question is "Do the results fit with the market being?". This question can be answered in many ways, but the first point is, if you are looking for a model that predicts the price of a stock, a tax, or something in a such way, maybe Mathematical Finance will disappoint you. In another way, if you are looking for the most honest price for that underlying, i.e. one more information to ground you in order to take an attitude, the Mathematical Finance can start to be useful. In this work we try to evaluate this link between the real world, i.e. an European Call from a well known company immersed into the Brazilian market, and four models that evaluates this European Call. Using the historical data of this asset, observed for four years, we replicate and compare these four asset pricing models to the real price from the market. The aim of this work is to explore the pricing of an European Call in a Jump model, where the asset can have jumps in its paths. We start with the mathematical basis to evaluate an European Call when the asset is modeled like a *Jump-Diffusion* process. After that, we use the hypothesis of "normal" distribution to the jumps and evaluate the Call using its analytic equation. Thereafter, we detail the construction of two numerical Monte-Carlo models , differentiated by the jumps distribution, the "Normally distributed" *Jump-Diffusion* model and the "Double Exponentially distributed" *Jump-Diffusion* model. Afterwards, It's priced using the well known Black-Scholes model, used as a reference for all the results. Finally, all the prices are compared, with special attention to the numerical *Jump-Diffusion* models. At the end we hope to give enough basis to justify the use of an algorithm in order to price this specific European Call.

**Abstract**

Quando parliamo di applicazioni della matematica in finanza, una ragionevole domanda è la seguente "I risultati ottenuti aderiscono alla realtà del mercato"? Le risposte alla domanda possono essere molteplici ma comunque il punto focale della questione è stabilire se si stia cercando un modello che predica il prezzo di un titolo azionario, un tasso o qualunque altra cosa in modo tale che la finanza matematica possa risultare insoddisfacente. D'altra parte, se si sta cercando di determinare il miglior prezzo per il sottostante, i.e. una ulteriore informazione che permetta di prendere delle scelte fondate, la finanza matematica può iniziare ad essere utile. In questo lavoro vogliamo analizzare la connessione tra il mondo reale, rappresentato da una opzione di tipo Call Europea di una ben nota emittente che opera nel mercato brasiliano e quattro modelli atti a valutare il suddetto strumento derivato. Utilizzando i dati storici relativi ad una osservazione quadriennale di questo asset, replichiamo e compariamo il prezzo fornito dai menzionati modelli a quello fornito dal mercato reale. Il principale obiettivo del lavoro è quello di esplorare la procedura di pricing di una opzione Call Europea in un modello con salti, ovvero dove l'asset può avere salti nelle traiettorie. Innanzi tutto, i prerequisiti matematici necessari alla valutazione di una Call Europea, quando l'asset è modellato come un processo *Jump-Diffusion*, sono presentati. Quindi ipotizziamo che i salti siano distribuiti in modo "normale" e valutiamo la Call per mezzo della sua equazione analitica. Dopodiché, diamo i dettagli della costruzione di un metodo di tipo Monte Carlo che differiscono per la tipologia di distribuzione ipotizzata per i salti: distribuzione "normale" e distribuzione "doble exponential". Dunque il prezzo è ottenuto utilizzando il ben noto modello di Black-Scholes, utilizzato come riferimento per tutti i risultati. In conclusione, tutti i prezzi sono raffrontati dando particolare attenzione ai modelli numerici per *Jump-Diffusion.* Alla fine speriamo di dare una base sufficiente a giustificare l'uso di un algoritmo per il prezzaggio di questa specifica opzione.

# Contents

# List of Figures

# Introduction

The study on *Jump-Diffusion* models applied to finance has started with R.C.Merton in his paper of 1976 [9] proposing that the returns process consists of three components, a linear drift, a Brownian motion repesenting "normal" price variation, and a compound Poisson process that set jumps to its paths generated by unexpected new informations. Since this, the use of such models has been incrising in real markets and theoretical studies in applied mathematics and quantitative finance. Its use, as the pure diffusion models did, came from the pricing evaluation theory and in a second stage has spread its application to headging, replication and riskless arbitrage strategies of portfolios.

By the way, a good question should be: "Why does some body choose the use of a *Jump-Diffusion* model, if the Diffusion models, like the Black-Scholes one and its derivations, are so generalized and in many cases fit so well with the Market being?". This is not a simple question and we will try to summarize here in few lines the main ideas of more than trirty years of study. For futher reading consult R.Cont and P. Tankov [7].

The first point are the large and sudden movements in prices. Although diffusion models have a good "shape" for long-term asset models, for example more than one month of observations, It's very difficult to model the intraday or short-term movements of an asset where the continue property of the asset price can be doubtful, needing a very large volatility to simulate it. For these short periods the assets seem to be better modelled by a jump models when compared to empirical observations, where these movements are a generic property that can be calibrated depending on the asset. The jump models also permit concentrated "instantenous" large losses, this way It can be modeled the large downward moves, the most common manner of loosing in a stock investment.

Another difficulty of diffusion models are to deal with heavy tail asset returns. This can be reduced by choosing non linear volatility structures, as a diffusion-based stochastic volatility models, or simulated by technics of volatility "smile" where the dependence of the volatility with respect to the strike "K" and the maturity "T" are taken into account. This is also a generic property in the simple *Jump-Diffusion* process models, and can be even better well characterized in *Jump-Diffusion*-based stochastic volatility models.

The completeness of the market is another point that must be discussed. In diffusion models the market is complete and so options can be hedged in a risk-free manner. As we will see in the chapter 5, *Jump-Diffusion* models are based in incomplete markets because of the large number of variables in order to construct a risk-free measure, therefore option is a risky investment as we can see in empirical observations. This means that some strategies can not be hedged in all risks, but some sorts of hedges are still available. In pure diffusion methods all the headging strategies as Delta, Vega, Theta hedgings lead you to the zero residual risk by the right choice of assets, options and free-risk investments in the portfolio. In jump models the hedging is more realistic because its done by solving a portfolio stochastic optmization problem, and so the risk still exists in a probabilistic way. For further information consult B.Oksendal and A.Sulem in [2].

These are the main principles in order to justify the use of Jump Models. In this work we will give an empirical assessment on the use of *Jump-Diffusion* processes, by choosing an especific well known European Call Option from the Brazilian Market with maturity of four months and evaluate it in four manners on the hope of finding a good method to model it. This work is pointed on the study of this especific Call Option, and can be used as inspiration in order to evaluate the price of other assets of the same origin, i.e. an option with high liquidity, this is the second most traded asset in the Brazilian market, and of middle Maturity, something between three months and six months.

These last remarks should be taken into account because the stimation of the parameters to short-term and long-term assets fallow different procedures, by the fact the model is too much affected by its choices. Therefore this work can not be taken as complete study of parametrization, and for more information on the generic study of estimation and system calibration for jump models, mainly to the "Double-Exponentially Distributed" *Jump-Diffusion* processes consult C.A. Ramezani and Y. Zeng on [11], where It's used Maximum Likelihood Estimation technics take in generic procedures in order to evaluate parameters.

The work is so split in five chapters and a conclusion, where all the models are put together and compared. The aim of the first chapter is to give the basic mathematical concepts in order to introduce the Jump models. First of all, fallowing S. E. Shreve [1], from the very start It's quoted the Exponential Distribution, Poisson Process and Compound Poisson Process on which is supported all *Jump-Diffusion* theory. Is given also a topological introdution to the broader class of the Lévy Process, fallowing P.Protter [3], where the *Jump-Diffusion* Process takes part. In this topological view is presented the Lévy measure $\nu$ necessary to set up the Jump-Diffusion process into the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. In this chapter is also take into considetarion the essential properties of *memorylessness* and *stationarity* of the Poisson Process, which permits ourselves, taking them as hypothesis, use the independence in time ot these processes, enabling us to use Martingale considerations in the fallowing chapters.

The second chapter is started defining ideas of continuity, a very important issue when one is dealing with process that can *jump* as the *Jump-Diffusion* processes do. After that, fallowing the approach of S. E. Shreve in [1], B.Oksendal and A.Sulem in [2], the *Jump-Diffusion* process "$X(t)$" are finally defined into the *Levy decomposition* perspective, i.e. a process composed by four parts: an initial deterministic condition "$X(0)$", an *Itô Integral* "$I(t)$", a *Riemann integral* "$R(t)$", and a *pure jump process*, like a Poisson or a compound Poisson process, "$J(t)$". It is also presented two examples of applications of how can this process be used to model a strategy with one stock. The chapter ends introducing the Quadratic Variation to jump process, that will be necessary to expose the Itô-Doeblin formula for the jump process.

The third chapter, fallowing R.Cont and P.Tankov in [7], is introduced the Stochastic Calculus for Jump Processes. It's started with a introduction for the Stochastic Calculus for continuous-path process, and from that by the inclusion of the right-continuous pure jump term to the process, is then presented the Itô-Doeblin formula for one jump process. This way we are able to set the jump process into a free risk mesure by the change of measure theory, in the same approach of the Girsanov's Theorem does for Brownian process. In the end is presented the Change of Measure for a compound Poisson Process and Brownian Motion, the generic case of *Jump-Diffusion* Process. This way the mathematical basis necessary to the development of the Vanilla European Call price into a *jump-diffusion* viewpoint is offered.

Using the achievement of the last chapters, the fourth chapter is all centred to the development of the analytic formula to the European Call price with the underlying asset driven by a Brownian motion and a compound Poisson process. Fallowing the procedure of R.C. Merton in [9] and S.E. Shreve in [1], the chapter starts with the Itô's product rule for jump process, hence is calculated the the differential equation to the stock price "$S(t)$" and its solution, given an initial condition. After that the intention is the construction of a risk-neutral measure which actually changes the compound poisson process intensity "$\lambda$" to "$\tilde{\lambda}$" and the jumps distribution "$Y$" to "$\tilde{Y}$".

Using this measure we finally perform the Black-Scholes-Merton analytic formula to price this European Call, using the hypothesis of normal distribution to the jumps. By the end of chapter, is also introduced some concepts to the precification of the European Call when the jumps "$Y$" are "Double-Exponentially" distibuted i.e. the up and down jumps fallow different and independent exponential distributions, fallowing the approach of S.G. Kou in [10]. This last part will not be used to construct an analytic formula to the Call, but will be used into the numerical model of the fallowing chapter.

The main chapter of this work is the fifth chapter, where the models are presented and the Call is priced in different ways, depending on the model. The models that are experimented are the analytic formula to the Jump-Diffusion Model, the Monte-Carlo numerical method in two perspectives, with jumps "Normally" distributed and "Double-Exponentially" distributed, and the Black-Scholes classic method. In the first part of the chapter the Call is presented as It can be found in the Market, where is offered its original price to the current date, and the data set of the stock in a period of four years is presented. This data set is used to calirate all the models on their own specific characteristics.

Then in this chapter It's explained how the systems are modelled, being the first one the analytic *Jump-diffusion* formula, which is nothing more than a formula truncated on the summatory function, and the European Call is priced on that perspective. After that is presented the algorithm, whose the main ideas where get from R.Cont and P.Tankov [7], and the calibration to both Monte-Carlo numerical methods, differing between them by the jumps distribution "normal" or "double exponential", at the end once again the European Call is priced. Afterwards the Call is priced using the Black-Scholes formula, this price is given as a benchmark.

In the conclusion chapter the prices for all the models are compared between them, and It's exposed how these prices are different from the Market one. The two prices that more fit with the Market are the two *Jump-Diffusion* Monte-Carlo numerical methods, with less than 10% of difference from the market price. This way, these two models are explored in their own strengths and weaknesses side by side taking into account th rate of convergence of each model, the range of the confidence interval of the final price and observations about the calibration, all of them observed during the algorithm contruction and simulations. In the last part of the work is also made other relevant considerations of the models, as suggestions of why the analytic *Jump-Diffusion* method doesn't fit so well with the Market being for this product, and why the Black-Scholes price does so well.

# Chapter 1

# Stochatic Calculus with Jump diffusions

## 1.1 Introduction to Jump diffusions

This work points to develop an introduction to *jump-diffusion* processes, and perform some examples of how They can be applied in the finance field. The name "diffusion" cames from the well-known Brownian motion component of its differential equation, and will be essential to our work show up some concepts of these kind of processes. In addition, with that basic class of processes, we will introduce jumps on their paths. In order to give us a more applicable approach, in this project we will work with only finitely many jumps in each finite time interval.

To model the jumps in their paths we will need to introduce the *Poisson process*, the fundamental pure jump process. This kind of process is, in fact, a different way to face the model of an asset on the stochastic calculus applied to finance. How it was born, the Poisson process jumps are of size one. To generalize the concept in a more general case we will present the *compound Poisson process*, which has the same origin of a Poisson process, despite of the jumps with a defined probability distribuition size. A *pure jump* process begins at zero, has finitely many jumps in each finite time interval, and has a constant value,a stair, between jumps.

Defining a jump process as a specific case of a Lévy Process, we will split it in some components, i.e. in a deterministic initial condition, in a differential Brownian motion $dW(t)$ component, in a differential deterministic component respect to time $t$, and in a *pure jump* process. Afterwards, We will define, in a differential and integral way, the jump processes. Once introduced the stochastic integral, we will be able to develop the stochastic calculus for jump processes, as a result of an extension of the Itô-Doeblin equation in a *pure diffusion* case.

Like in the *pure diffusion* case, we must perform a change of probability measure for the Poisson process and for the compound Poisson process. Subsequently, we will find out the measure which handle simultaneously with a change of measure to both process in the stochastic differential equation, the Brownian motion and the compound Poisson process. Using a Jump-Diffusion model and its Backward equation we will be able to price an European Call based in that asset.

The last part of the work consists on simulations on a real data set from the market. First we will take a real Option from the market to compare all the results. Then we will propose four algorithms: an analytic formula to price an European Call Option with underlying based on Jump-Diffusion process with the jumps distributed in a normal density function, a numeric method based on Monte Carlo simulation in a "Normally Distributed" Jump-Diffusion process, a numeric method based on Monte Carlo simulation in a "Double Exponentially Distributed" Jump-Diffusion process, and finally the Black-Scholes model. At the end we will compare these four results between them and the official price from the market. We will take with special attention the comparision between the two numeric methods.

## 1.2 Basic definitions and results on Poisson Process

The Poisson process is the building block for jump process. In this section we will expose basic results and properties of these processes.

### 1.2.1 Exponential Random Variables

**Definition 1.1** ([1]). *We will say the random variable $\tau$ has a exponential distribution if it has the following density :*

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, t \geq 0, \\ 0, t < 0, \end{cases} \tag{1.1}$$

*where the constant $\lambda \in \Re^+$.*

As a result we present the expected value of $\tau$:

$$\mathbb{E}[\tau] = \int_0^\infty t f(t) dt = \lambda \int_0^\infty t e^{-\lambda t} dt = -te^{\lambda t}|_{t=0}^{t=\infty} + \int_0^\infty e^{-\lambda t} dt = 0 - \frac{1}{\lambda} e^{-\lambda t}|_{t=0}^{t=\infty} = \frac{1}{\lambda}.$$

And for the cumulative distribution function, we stand :

$$F(t) = \mathbb{P}\{\tau \leq t\} = \int_0^t \lambda e^{-\lambda u} du = -e^{-\lambda u}|_{u=0}^{u=t} = 1 - e^{-\lambda t} \quad , t \geq 0,$$

And so, by the complementary probability,

$$\mathbb{P}\{\tau > t\} = e^{-\lambda t} \quad , t \geq 0, \tag{1.2}$$

The conditional probability of an event which can occur at the time $t+s$, knowing it didn't occured up to the time $s$, is given by :

$$\mathbb{P}\{\tau > t+s|\tau > s\} = \frac{\mathbb{P}\{\tau > t+s \text{ and } \tau > s\}}{\mathbb{P}\{\tau > s\}} = \frac{\mathbb{P}\{\tau > t+s\}}{\mathbb{P}\{\tau > s\}} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t}.$$

Therefore, the probability of occur an event when we have to wait an additional time $t$, when we have already waited until $s$, is the same of the probability of waiting from the start time $t = 0$. That result is a property of the exponencial distribuition, i.e. the indifference in the probability distribuition of our position in time. That property called *memorylessness*.

### 1.2.2 Construction of a Poisson Process

Suppose a sequence of i.i.d exponential random variables $\tau_1$, $\tau_2$, $\tau_3$..., all of them with mean $\frac{1}{\lambda}$. We will define the "jump" model in the following way: The sign of a random variable will be replaced by its first occurrence, each of them called *interarrival times*. So, the first jump occurs at time $\tau_1$, the second occurs at $\tau_2$ time units after the first one, the third occurs $\tau_3$ after the second, and so on.

**Definition 1.2** ([1]). *We will call the arrival time the unit of time of the nth jump, in that way:*

$$S_n = \sum_{k=1}^{n} \tau_k \tag{1.3}$$

**Definition 1.3** ([1]). *The Poisson process $N(t)$ counts the number of jumps that occured at or before time t. In that way:*

$$N(t) = \begin{cases} 0 \text{ if } 0 \leq t < S_1, \\ 1 \text{ if } S_1 \leq t < S_2, \\ . \\ . \\ n \text{ if } S_n \leq t < S_{n+1}, \end{cases}$$

*And we say the Poisson process N(t) has the an intensity $\lambda$, which measure the avarage rate of jumps in a unit time.*

Defined the Poisson process we can present some basic topological concepts of it. $N(t)$ is *right-continous*, i.e. $N(t) = \lim_{s \downarrow t} N(s)$. We can define $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ the filtered probability space , with $\mathcal{F}(t) = \mathcal{F}_t$ the $\sigma$-algebra generated by the observation of the Poisson process $N(s)$ at the time $0 \leq s \leq t$, it means that we know the paths, what occured, up to the time t.

And so we can define, in a topological sense, a more generic process:

### 1.2.3 Basic definitions and results on Lévy Process

Now we will present some basic concepts needed to apply the calculus of jump diffusions.

**Definition 1.4** ([3])**.** *Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space. A $\mathcal{F}_t$-adapted process $\{\eta_t\}_{t \geq 0} \subset \Re$ is called a Lévy process if :*

1. *$\eta_0 = 0$ a.s.;*

2. *$\eta_t$ is continous in probability;*

3. *has stationary, independent increments;*

We say convergence in probability when there is a succession $(X_n)_n$, signed by $X_n \xrightarrow{P} X$, if for each $\delta > 0$, $\lim_{n \to \infty} P(d(X_n, X) > \delta) = 0$. [4] A result from any Lévy process $\{\eta_t\}$ is that we can perform a *cadlag* version (right continous with left limits), which is also a Lévy process.[3] Hence, hereafter we will assume all the Lévy processes as cadlag processes.

**Definition 1.5.** *We will define a jump of $\{\eta_t\}$ at $t \geq 0$ by*

$$\Delta \eta_t = \eta_t - \eta_{t^-}. \tag{1.4}$$

**Definition 1.6** ([3])**.** *Given $\mathcal{B}_0$ the family of Borel sets $U \subset \Re$, whose $0 \notin \bar{U}$ (closure of U). For $\forall U \in \mathcal{B}_0$ we define the Poisson random measure of $\eta(.)$ as :*

$$N(t, U) = N(t, U, \omega) = \sum_{s:0<s \leq t} \chi_U(\Delta \eta_t) \tag{1.5}$$

*In differential notation is written $N(dt, dz)$*

Where $\chi_U(.)$ is the indicator function of the set $U$, i.e., $\chi_U(x) = 1$ if $x \in U$ or $\chi_U(x) = 0$ if $x \notin U$. By the right continuity, we can conclude $N(t, U)$ is finite $\forall U \in \mathcal{B}_0$.

**Definition 1.7** ([4]). *A process* $B = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, (B_t)_{t\geq 0}, P)$ *with real values is a Brownian motion if:*

1. $B_0 = 0$ *a.s;*

2. *for each* $0 \leq s \leq t$ *the random variable* $B_t - B_s$ *is independent of* $\mathcal{F}_t$*;*

3. *for each* $0 \leq s \leq t$ *the random variable* $B_t - B_s$ *has the following distribution* $N(0, t-s)$*;*

Hence, by the definition, The Brownian motion $\{B_t\}_{t\geq 0}$ is a particular case of Lévy processes.

Another Lévy process is the *Poisson process* $\pi(t)$ of intensity $\lambda > 0$, taking values in natural numbers. This process has the following distribution of probability:

$$P[\pi(t) = n] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \qquad n = 0, 1, 2, 3...$$

### 1.2.4 Distribution of Poisson Process Increments

First of all, we must determine the distribution of the arrival jump times $S_n$ for $n = 1, 2, 3...$ :

**Lemma 1.8** ([1]). *The random variable* $S_n$*, such that* $n \in \mathbb{N}$ *and* $n \geq 1$*, defined by the definition 1.2 has following the gamma density:*

$$g_n(s) = \tfrac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s} \quad , s \geq 0. \tag{1.6}$$

*where the constant, said density,* $\lambda \in \Re^+$*.*

*Proof.* We will prove 1.6 by induction on $n$.

For $n = 1$, we have that $S_1 = \tau_1$ is given by the density 1.1, and so,

$$g_1(s) = \lambda e^{-\lambda s} \quad , s \geq 0.$$

Now, by induction, let us assume that 1.6 holds to $n$ and let it expand to $n+1$. Hense, accepting $S_n$ let us find $S_{n+1} = S_n + \tau_{n+1}$ by the independence of $S_n$ and $\tau_{n+1}$, and the convolution, we reach the density :

$$\begin{aligned}&\textstyle\int_0^s g_n(\upsilon) f(s-\upsilon) d\upsilon = \int_0^s \frac{(\lambda \upsilon)^{n-1}}{(n-1)!} \lambda e^{-\lambda \upsilon} \cdot \lambda e^{-\lambda(s-\upsilon)} d\upsilon \\ &= \tfrac{\lambda^{n+1} e^{-\lambda s}}{(n-1)!} \textstyle\int_0^s \upsilon^{n-1} d\upsilon = \frac{\lambda^{n+1} e^{-\lambda s}}{n!} \upsilon^n \big|_{\upsilon=0}^{\upsilon=s} \\ &= \tfrac{(\lambda s)^n}{n!} \lambda e^{-\lambda s} = g_{n+1}(s)\end{aligned}$$

, and so, 1.6 is proved by induction. □

And, in the same way, we can find the distribution of the *Poisson process* *N(t) :*

**Lemma 1.9** ([1]). *The Poisson process $N(t)$ of intensity $\lambda$ has the following distribution*

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, .. \tag{1.7}$$

*,and this result is well known.*

By the memorylessness proper from Poisson process, we know the probability distribution of the process $N(t-s) - N(s)$, and so the probability distribution of the number of jumps in the interval $(s, t+s]$.They are independent of the number of jumps before the time $s$, and so,it is independent of $\mathcal{F}_s$. In the same way, we conclude that the distribution of $N(t-s) - N(s)$ has the same distribution of $N(t)$ alone, an exponential distribution with intensity $\lambda$. Thus, the important parameters for its distribution is the intensity $\lambda$ and the time interval of the increment. That property of a process increment depending only on a time interval, independent of which one in the time progress, is called *stationarity.* As we saw, also the Brownian motion has that property.

We finally can develop the distribution of an increment.

**Theorem 1.10.** *Let $N(t)$ be a Poisson process, with the distribution given by 1.7, and the given times $0 = t_0 < t_1 < t_2 < .. < t_n$. The sequential increments $N(t_1) - N(t_0), N(t_2) - N(t_1), ..., N(t_n) - N(t_{n-1})$ are stationary and independent, with the distribution,*

$$\mathbb{P}\{N(t_{j+1}) - N(t_j) = k\} = \frac{\lambda^k (t_{j+1} - t_j)^k}{k!} e^{-\lambda(t_{j+1} - t_j)}, k = 0, 1, .... \tag{1.8}$$

*Proof.* The proof is given by the independence and stationarity of Poisson increments, explained at the last point, and is a consequence of the lemma 1.9. □

In a topological way, with those instruments we can propose the following theorem, which will offer us a measure whose we will be required to build up the Lévy decomposition and all the *Jump difusion* theory.

**Theorem 1.11** ([3]).

1. *The set function $U \rightarrow N(t, U, \omega)$ defines a $\sigma - finite$ measure on $\mathcal{B}_0$ for each fixed t,w;*

2. *The set function $\nu(U) = E[N(1, U)]$ defines a $\sigma - finite$ measure on $\mathcal{B}_0$, called the Lévy measure of $\{\eta_t\}$, where $E = E_P$ denotes expectation respect to the probability measure $P$;*

*3. Fix $U \in \mathcal{B}_0$. Then the process*

$$\pi_U(t) := \pi_U(t,\omega) := N(t,U,\omega)$$

*is a Poisson process of intensity $\lambda = \nu(U)$;*

As a result we can perform the mean and the variance of Poisson Increments:

By the theorem 1.10, the Poisson Increment $N(t)-N(s)$ has distribution

$$\mathbb{P}\{N(t)-N(s)=k\} = \tfrac{\lambda^k(t-s)^k}{k!}e^{-\lambda(t-s)}, k=0,1,.... \qquad (1.9)$$

Is interesting to see that the distribution density follows the basic cumulative probability property, in that way,

$$\sum_{k=0}^{\infty} \mathbb{P}\{N(t)-N(s)=k\} = e^{-\lambda(t-s)} \sum_{k=0}^{\infty} \frac{\lambda^k(t-s)^k}{k!} = e^{-\lambda(t-s)}e^{\lambda(t-s)} = 1.$$

and this is an outcome of the exponential time series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

We next compute the expected value of a increment,

$$\begin{aligned} \mathbb{E}[N(t)-N(s)] &= \textstyle\sum_{k=0}^{\infty} k\frac{\lambda^k(t-s)^k}{k!}e^{-\lambda(t-s)} \\ &= \lambda(t-s)e^{-\lambda(t-s)} \textstyle\sum_{k=0}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!} \\ &= \lambda(t-s)e^{-\lambda(t-s)}e^{\lambda(t-s)} \\ &= \lambda(t-s). \end{aligned} \qquad (1.10)$$

Then, the average of jumps in the time interval $[s,t]$, for $s \leq t$, is given by $\mathbb{E}[N(t)-N(s)] = \lambda(t-s)$.

And, by the second moment of the increment,

$$\begin{aligned} \mathbb{E}[(N(t)-N(s))^2] &= \textstyle\sum_{k=0}^{\infty} k^2\frac{\lambda^k(t-s)^k}{k!}e^{-\lambda(t-s)} \\ &= e^{-\lambda(t-s)} \textstyle\sum_{k=1}^{\infty} \frac{\lambda^k(t-s)^k}{(k-1)!}(k-1+1) \\ &= e^{-\lambda(t-s)} \textstyle\sum_{k=2}^{\infty} \frac{\lambda^k(t-s)^k}{(k-2)!} + e^{-\lambda(t-s)} \textstyle\sum_{k=1}^{\infty} \frac{\lambda^k(t-s)^k}{(k-1)!} \\ &= \lambda^2(t-s)^2e^{-\lambda(t-s)} \textstyle\sum_{k=2}^{\infty} \frac{\lambda^{k-2}(t-s)^{k-2}}{(k-2)!} + \lambda(t-s)e^{-\lambda(t-s)} \textstyle\sum_{k=1}^{\infty} \frac{\lambda^{k-1}(t-s)^{k-1}}{(k-1)!} \\ &= \lambda^2(t-s)^2 + \lambda(t-s). \end{aligned}$$

We can find the increment's variance,

$$\begin{aligned} Var[N(t)-N(s)] &= \mathbb{E}[(N(t)-N(s))^2] - \mathbb{E}[N(t)-N(s)]^2 \\ &= \lambda^2(t-s)^2 + \lambda(t-s) - \lambda^2(t-s)^2 \\ &= \lambda(t-s). \end{aligned} \qquad (1.11)$$

A special case of the Poisson process is the following definition

**Definition 1.12** ([1]). *Let $N(t)$ be a Poisson process with intensity $\lambda$. We define a compensated Poisson process as*

$$M(t) = N(t) - \lambda t$$

As a result we find out a compensated Poisson process $M(t)$ is a *martingale*. Then, for $0 \leq s < t$,

$$\mathbb{E}[M(t)|\mathcal{F}_s] = M(s)$$

## 1.3 Compound Poisson Process

To be more generic we need a kind of jump which performs more than a process that jumps up only one unit, as the Poisson process and the compensated Poisson Process, for instance. Furthermore, we need to build up a process which the jump's size is given by a random distribution.

### 1.3.1 Construction of a Compound Poisson Process

**Definition 1.13** ([1]). *Let $N(t)$ be a Poisson process with intensity $\lambda$, and let $Y_1, Y_2, ...$ be a sequence of i.i.d. random variables with mean $\beta = \mathbb{E}[Y_i]$, and also independent of $N(t)$. We define a Compound Poisson process as*

$$Q(t) = \textstyle\sum_{i=1}^{N(t)} Y_i \quad t \geq 0. \tag{1.12}$$

The jumps of $Q(t)$ occur at the same time of the pure Poisson process of size 1, whereas the compound Poisson Process has jumps of random size, following the $Y_i$ random distribution. As occurred with the Poisson process, the increments of a compound Poisson process $Q(t)$ are independent. For instance, for $0 \leq s < t$, we have

$$Q(t) - Q(s) = \sum_{i=N(s)+1}^{N(t)} Y_i,$$

It sums up the jumps between the time interval $(s, t]$.In addition, the compound Poisson process has the stationarity and memorylessness properties, i.e. $N(t) - N(s)$ is distributed as $N(t - s)$, depending only on the time interval $(s, t]$.

The expected value of a compound Poisson process is

$$\begin{aligned}\mathbb{E}[Q(t)] &= \textstyle\sum_{k=0}^{\infty} \mathbb{E}[\sum_{i=1}^{k} Y_i | N(t) = k] \mathbb{P}\{N(t) = k\} \\ &= \textstyle\sum_{k=0}^{\infty} \beta k \frac{(\lambda t)^k}{k!} e^{-\lambda t} = \beta \lambda t e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!} = \beta \lambda t\end{aligned}$$

We read that result in the following way: The average of a process $Q(t)$ is given by $\lambda t$ ($\frac{\text{jumps}}{\text{time}} \times \text{time}$) jumps in the time interval $[0, t]$, with the average size $\beta$. As hypothesis the number of jumps is independent of theirs sizes.

In the same way of the result for the compensated Poisson process, we can give the following theorem,

**Theorem 1.14.** *With $Q(t)$ given as in 1.13 the compensated compound Poisson process*

$$Q(t) - \beta\lambda t$$

*is a martingale.*

*Proof.* For $0 \leq s < t$, the increment $Q(t) - Q(s)$ is independent of $\mathcal{F}_s$ and mean $\beta\lambda(t-s)$. Then we have,

$$\begin{aligned} \mathbb{E}[Q(t) - \beta\lambda t|\mathcal{F}(s)] &= \mathbb{E}[Q(t) - Q(s)|\mathcal{F}(s)] + Q(s) - \beta\lambda t \\ &= \beta\lambda(t-s) + Q(s) - \beta\lambda t = Q(s) - \beta\lambda s \end{aligned}$$

□

Then, as already named, the compound Poisson process has stationary independent increments. The general case is

**Proposition 1.15** ([1]). *With $Q(t)$ given as in 1.13 and the given times $0 = t_0 < t_1 < t_2 ... < t_n$. The increments*

$$Q(t_1) - Q(t_0), Q(t_2) - Q(t_1), ..., Q(t_n) - Q(t_{n-1})$$

*are independent and stationary. The process $Q(t_j) - Q(t_{j-1})$ has the same distribution of $Q(t_j - t_{j-1})$.*

We can also propose the compound Poisson Process in a topological more generic way,

**Proposition 1.16** (The compound Poisson process[2]). *Let $Y_n$, with $n \in \mathbb{N}$, be a sequence of i.i.d. random variables taking vulues in $\Re$ with common distribution $\mu_{Y_1} = \mu_Y$ and let $N(t)$ be a Poisson process of intensity $\lambda$, independent of all the $Y_n$'s. The compound Poisson process is given by 1.13 and its increments are*

$$Q(t) - Q(s) = \sum_{i=N(s)+1}^{N(t)} Y_i,$$

*This is independent of $\mathcal{F}_{N(s)+1}$ and depends only on the difference $(s-t]$. Thus, $Q(t)$ is also a Lévy process.*

*To find the Lévy measure $\nu$ of $Q(t)$ note that if $U \in \mathcal{B}_0$ then, by the independence*

$$\nu(U) = \mathbb{E}[N(1,U)] = \mathbb{E}[\sum_{s;0\leq s\leq 1} \chi_U(\Delta Q(s))] =$$
$$\mathbb{E}[(\textit{number of jumps} \cdot \chi_U((\textit{jump})] = \mathbb{E}[N(1)\chi_U(Y)] = \lambda\mu_Y(U),$$

*And we conclude that $\nu = \lambda\mu_Y$*

### 1.3.2 Moment-Generating Function

For this section we will follow [1]. As the formula for the density of a compound Poisson process increment $Q(t_j - t_{j-1})$ is quite complicated , we will present its moment-generating function formula. Let $Q(t)$ defined as in 1.13. Denote, in a generic way, the moment-generating function of a Random variable $Y_i$ by

$$\varphi_Y(u) = \mathbb{E}[e^{uY_i}] \tag{1.13}$$

Because of all $Y_i$'s are i.i.d the distribution does not depend on the index $i$.

**Proposition 1.17** (Moment-Generating function for the Compound Poisson Process)**.** *The Moment-Generating function for the Compound Poisson Process Q(t) is, by the independence of $Y_i$ and $N(t)$,*

$$\begin{aligned}\varphi_{Q(t)}(u) &= \mathbb{E}[e^{uQ(t)}] \\ &= \mathbb{E}[exp\{u \textstyle\sum_{i=1}^{N(t)} Y_i\}] \\ &= \mathbb{P}\{N(t)=0\} + \textstyle\sum_{k=1}^{\infty} \mathbb{E}[exp\{u \sum_{i=1}^{k} Y_i | N(t)=k\}]\mathbb{P}\{N(t)=k\} \\ &= \mathbb{P}\{N(t)=0\} + \textstyle\sum_{k=1}^{\infty} \mathbb{E}[exp\{u \sum_{i=1}^{k} Y_i\}]\mathbb{P}\{N(t)=k\} \\ &= e^{-\lambda t} + \textstyle\sum_{k=1}^{\infty} \mathbb{E}e^{uY_1}\mathbb{E}e^{uY_2}...\mathbb{E}e^{uY_k} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \\ &= e^{-\lambda t} + e^{-\lambda t} \textstyle\sum_{k=1}^{\infty} \frac{(\varphi_Y(u)\lambda t)^k}{k!} \\ &= e^{-\lambda t} \textstyle\sum_{k=0}^{\infty} \frac{(\varphi_Y(u)\lambda t)^k}{k!} \\ &= exp\{\lambda t(\varphi_Y(u) - 1)\}\end{aligned} \tag{1.14}$$

In the case of $Y_i$ follows a deterministic distribution, taking the constant value $y$, then the compound Poisson process $Q(t)$ is actually give by $yN(t)$ and 1.13 assumes the deterministic value $\varphi_Y(u) = e^{uy}$. By the way, if we take $y$ times a Poisson process it holds the following moment-generating function

$$\varphi_{yN(t)}(u) = \mathbb{E}[e^{uyN(t)}] = exp\{\lambda t(e^{uy} - 1)\} \tag{1.15}$$

Thus, for $y = 1$ we have the moment-generating function of a Poisson process,

$$\varphi_{N(t)}(u) = \mathbb{E}[e^{uN(t)}] = exp\{\lambda t(e^{u} - 1)\} \tag{1.16}$$

Another interesting case is when $Y_i$ takes one of the finitely possible deterministic values $y_1, y_2, .., y_M$ with the distribution $p(y_m) = P\{Y_i = y_m\}$, with $p(y_m) > 0$ for every $m$ and the cummulative property $\sum_{m=1}^{M} p(y_m) = 1$. And so, from 1.13 we have $\varphi_Y(u) = \sum_{m=1}^{M} p(y_m)e^{uy_m}$, and from 1.17 it follows

$$\begin{aligned}\varphi_{Q(t)}(u) &= exp\{\lambda t(\textstyle\sum_{m=1}^{M} p(y_m)e^{uy_m} - 1)\} \\ &= exp\{\lambda t \textstyle\sum_{m=1}^{M} p(y_m)(e^{uy_m} - 1)\} \\ &= exp\{\lambda p(y_1)t(e^{uy_1} - 1)\}...exp\{\lambda p(y_M)t(e^{uy_M} - 1)\}.\end{aligned} \tag{1.17}$$

This expression is a moment-generating function of a product of the moment-generating functions of kind 1.15, with the $m$th process with an intensity $\lambda p(y_m)$ and jump size $y_m$. If it holds we have the following theorem, essential to the theory of Jump-process.

**Theorem 1.18** (Decomposition)**.** *[1] Let $y_1, y_2, .., y_M$ be a finite set of deterministic numbers, with associated probability $p(y_i)$, for $i = 1, .., M$, and $\sum_{M}^{m=1} p(y_m) = 1$. Assume the intensity $\lambda > 0$, and let us define $\bar{N}_1(t), ..., \bar{N}_M(t)$ be independent Poisson processes, with $\bar{N}_m(t)$ having the intensity $\lambda p(y_m)$. We define a new process*

$$\bar{Q}(t) = \textstyle\sum_{m=1}^{M} y_m \bar{N}_m(t), t \geq 0. \tag{1.18}$$

*Thus, $\bar{Q}(t)$ is a compound Poisson process. Let us assume, for instance, $\bar{Y}_1$ is the size of the first jump of $\bar{Q}(t)$, $\bar{Y}_2$ is the size of the second jump, etc., and*

$$\bar{N}(t) = \textstyle\sum_{m=1}^{M} \bar{N}_m(t), t \geq 0. \tag{1.19}$$

*where $\bar{N}(t)$ is a cumulative function which measures the total number of jumps on the time interval $(0, t]$, It holds :*

1. *$\bar{N}(t)$ is a Poisson process with intensity $\lambda$;*
2. *the random variables $\bar{Y}_1, \bar{Y}_2, .., \bar{Y}_M$ are independent with probability distribution $\mathbb{P}\{\bar{Y}_i = y_m\} = p(y_m)$, for $m = 1, ..., M$;*
3. *the random variables $\bar{Y}_1, \bar{Y}_2, .., \bar{Y}_M$ are independent of $\bar{N}(t)$;*
4. *$\bar{Q}(t) = \sum_{i=0}^{\bar{N}(t)} \bar{Y}_i, t \geq 0$;*

So, we can conclude from Theorem 1.18 that there are two ways to face a compound Poisson process that has finitely many possible jump sizes. It can be thought as a Single Poisson process in which we use jumps of random size, instead size-one jumps. It can also be regarded as a sum of independent Poisson processes in each of which the size one jumps are replaced by jumps of a pre-fixed size. To reassume is given the following corollary.

**Corollary 1.19.** *Let $y_1, y_2, .., y_M$ be a finite set of deterministic numbers, with associated probability $p(y_i)$, for $i = 1, .., M$, and $\sum_M^{m=1} p(y_m) = 1$. The random variables $\bar{Y}_1, \bar{Y}_2, .., \bar{Y}_M$ are independent with probability distribution $\mathbb{P}\{\bar{Y}_i = y_m\} = p(y_m)$, for $m = 1, ..., M$. Let $N(t)$ be a Poisson process and define the compound Poisson process*

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

*At the another way of regarding, for $m = 1, ..., M$, let $N_m(t)$ denote the number of jumps in $Q(t)$ of size $y_m$ in the time interval $[0, t]$. Then*

$$N(t) = \sum_{m=1}^{M} N_m(t) \text{ and } Q(t) = \sum_{m=1}^{M} y_m N_m(t).$$

*The processes $N_m$'s are independent Poisson process, with intensity $\lambda p(y_m)$.*

We are able to understand the fallowing theorem, that will be regarded with more considerations in the next chapter.

**Theorem 1.20** (Lévy decomposition ([5]))**.** *Let $\{\eta_t\}$ be a Lévy process. Then $\eta_t$ has the following decomposition*

$$\eta_t = \alpha t + \beta B(t) + \int_{|z|<R} z\tilde{N}(t, dz) + \int_{|z|\geq R} zN(t, dz), \tag{1.20}$$

*for some constants $\alpha, \beta \in \Re$ , $R \in \Re^+$.*

# Chapter 2

# Integrals of Jump Process

In this chapter will be introduced the stochastic integral when the integrator, i.e. the $dX$ part is a process with jumps. We will also discuss some properties of these process. In our case the process of our integrator has a Brownian motion and a Poisson or a Compound Poisson parts. First of all, let's give some main ideas of continuity:

## 2.1 Directional continuity

A function may happen to be continuous in only one direction, either from the "left" or from the "right". A right-continuous function is a function which is continuous at all points when approached from the right. Technically, the formal definition is similar to the definition above for a continuous function but modified as follows:

**Definition 2.1.** *The function $f(t)$ is said to be right-continuous at the point $c$ if $f(t) = lim_{s\downarrow t} f(s)$,and so the following holds: For any number $\epsilon > 0$ however small, there exists some number $\delta > 0$ such that for all $x$ in the domain with $c < x < c + \delta$, the value of $f(x)$ will satisfy:*

$$|f(x) - f(c)| < \epsilon.$$

Notice that $x$ must be larger than $c$, that is on the right of $c$. If $x$ were also allowed to take values less than $c$, this would be the definition of continuity. This restriction makes possible the function to have a discontinuity at $c$, but still be right continuous at $c$, as pictured.

Likewise a left-continuous function is a function which is continuous at all points when approached from the left, that is, $c - \delta < x < c$.

Denoting a generic process with jumps $J(t)$ , a right-continuous version of such process. The left-continuous version will be denoted by $J(t-)$. We should interpret this in the following manner: If there is a jump on the process $J$,a right-continuous process, at the time $t$, then $J(t)$ is the value of $J$ at the very time immediately after the jump, and J(t-) is its value immediately before the jump.

A function is continuous if and only if It has both properties, the right-continuity and the left-continuity.

## 2.2 Defining a Jump Processes

We will define the terminogy to make a connection between a process and $\sigma$-algebra or filtration.

**Definition 2.2** ([1]). *Let the usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, with $\{\mathcal{F}_t\}_{t\geq 0} = \mathcal{F}(t)$, $t \geq 0$, being a filtration of this space. We define a Brownian motion $W(t)$ to be a process relative to $\mathcal{F}(t)$ if $W(t)$ is $\mathcal{F}(t)$-measurable for every $t$ and for $u > t$ the increment $W(u) - W(t)$ is independent of $\mathcal{F}(t)$. It holds for a Poisson process too. We say $N(t)$ is a Poisson process relative to the filtration $\mathcal{F}(t)$ if $N(t)$ is $\mathcal{F}(t)$-measurable for every $t$ and for $u > t$ the increment $N(u) - N(t)$ is independent of $\mathcal{F}(t)$. This definition holds in the same way for a compound Poisson process $Q(t)$.*

Let's consider the following right-continuous process, also called the *Levy decomposition*[2]:

$$X(t) = X(0) + I(t) + R(t) + J(t) \tag{2.1}$$

Where $X(t)$ is the integrator of the following stochastic integral,

$$\int_0^t \Phi(s)dX(s) \tag{2.2}$$

Let $(\Omega, \mathcal{F}, P)$ the usual probability space, by the $\sigma$-algebra we generates the filtration $\mathcal{F}(t)$, for $t \geq 0$. By now all the processes will be adapted to this filtration. Let's better specify each element of the process 2.1

- $X(0)$ in this case will be a deterministic inicial condition.
- $I(t)$ will be stochastic integral, called an *Itô Integral*

  $$I(t) = \int_0^t \Gamma(s)dW(s) \tag{2.3}$$

  where $\Gamma(s)$ is a adapted process, with the integrator as a Brownian motion $dW(s)$ relative to the filtration $\mathcal{F}(t)$.

- $R(t)$ is a *Riemann integral* for the adapted process $\Theta(s)$

$$R(t) = \int_0^t \Theta(s)ds \tag{2.4}$$

- $J(t)$ is an adapted, right-continuous *pure jump process* with $J(0)$=0. The left-continuous version of that process will be denoted $J(t-)$. Our process in this case is a *pure jump process*, hence $J$ does not jump at time zero, has only finitely many jumps on each finite time interval (0,T], and is constant between jumps. This constance is a characteristic from Poisson and compound Poisson processes. But a compensated Poisson process does not have that characteristic because of the decreasing characteristic between jumps. So, $J(t)$ is the pure jump part of $X(t)$.

Hence, can be defined:

**Definition 2.3.** *The continous part of the process $X(t)$ will be*

$$X^c(t) = X(0) + I(t) + R(t) = X(0) + \int_0^t \Gamma(s)dW(s) + \int_0^t \Theta(s)ds \tag{2.5}$$

And we finally define a jump process

**Definition 2.4** ([1])**.** *A process X(t) define as in 2.1, will be called a jump process.*

By definition the pure jump process $J(t)$ is right-continuous and adapted. Because of the continuity of $I(t)$ and $R(t)$ the left-continuous version of $X(t)$ is given

$$X(t-) = X(0) + I(t) + R(t) + J(t-). \tag{2.6}$$

And we will define the jump size of the jump process $X(t)$ [2] as

$$\Delta X(t) = X(t) - X(t-). \tag{2.7}$$

If at a time $t$, there is no jump, by the continuity it implies $\Delta X(t) = 0$. As a result, it can be seen that the jump size cames from the pure jump process $J(t)$, hence the jump size $\Delta X(t) = X(t) - X(t-)$ and the jump size $\Delta J(t) = J(t) - J(t-)$ have both the same sizes. How it was already defined, there are no jumps at the time zero, then $\Delta X(0) = 0$.

Given the stochastic integral and the jump process, we will split the stochastic integral in its parts.

**Definition 2.5.** *[1] Let $X(t)$ be a jump process as defined above, and let $\Phi(s)$ be an adapted process to the current filtration $\mathcal{F}(t)$. The stochastic integral of $\Phi$ with respect to the integrator $X$ is such that*

- *In integral form*

$$\int_0^t \Phi(s)dX(s) = \int_0^t \Phi(s)\Gamma(s)dW(s) + \int_0^t \Phi(s)\Theta(s)ds + \sum_{0<s\leq t} \Phi(s)\Delta J(s). \tag{2.8}$$

- *Or in differential notation,*

$$\begin{aligned} \Phi(t)dX(t) &= \Phi(t)dI(t) + \Phi(t)dR(t) + \Phi(t)dJ(t) \\ &= \Phi(t)\Gamma(t)dW(t) + \Phi(t)\Theta(t)dt + \Phi(t)dJ(t) \\ &= \Phi(t)dX^c(t) + \Phi(t)dJ(t) \end{aligned} \tag{2.9}$$

Let's see in a pratice way what this implies.

**Example 1** : Holding a asset with a strategy "$\Phi(t)$"[1]

The path of a generic underlying will be modeled as $X(t) = M(t) = N(t) - \lambda t$, and so $X(t)$ will be a compensated Poisson process with a Poisson process $N(t)$ with an intensity $\lambda$. Following the notation of the equation 2.1 we split our equation in its different parts:

- Itô Integral $I(t) = 0$;
- Continuous part $X^c(t) = R(t) = -\lambda t$;
- J(t)=N(t)

With the strategy $\Phi(s) = \Delta N(s)$, i.e.

$$\Phi(s) = \begin{cases} 1, \text{if there is a jump of the process N(t) at time s,} \\ 0, \text{otherwise,} \end{cases} \tag{2.10}$$

$\Phi$ has finitely many values 1 at the time interval $[0, t]$, each of them during a time with measure zero (instantaneous). It is reflected at the continuous part of the integral,

$$\int_0^t \Phi(s)dX^c(s) = \int_0^t \Phi(s)dR(s) = -\lambda \int_0^t \Phi(s)ds = 0. \tag{2.11}$$

However, by the pure jump part,

$$\int_0^t \Phi(s)dN(s) = \sum_{0<s\leq t} (\Delta N(s))^2 = N(t). \tag{2.12}$$

And so it implies,

$$\int_0^t \Phi(s)dM(s) = -\lambda \int_0^t \Phi(s)ds + \int_0^t \Phi(s)dN(s) = N(t). \tag{2.13}$$

In our case is interesting to have the stochastic integral,

$$I(t) = \int_0^t \Gamma(s)dW(s).$$

like a martingale. To make this, we aproximate the integrand $\Gamma(s)$ by simple integrands $\Gamma_n(s)$, hence we choose the stochastic integral in that way,

$$I_n(t) = \int_0^t \Gamma_n(s)dW(s).$$

and verifying ,for each $n$, that $I_n(t)$ is a martingale. Defining :

$$\lim_{n\to\infty} I_n(t) = I(t)$$

$I(t)$ will be a martingale. It will be possible when the stochastic integral with the integrand $\Gamma(s)$ satisfies the following technical condition,

$$\mathbb{E}[\int_0^t \Gamma^2(s)ds] < \infty, \text{for every t>0.}$$

In a financial view, the stochastic integral will be the gain, if we replace $\Gamma(s)$ by the position in an asset and we replace the pure Brownian motion $W(t)$ by the price of that asset. If this price is a martingale, i.e. with a pure volatility path which can goes up and down, and the position $\Gamma(s)$ respects the condition above, the gain we make will be also a martingale.

In our example, an agent who invests in the compensated Poisson process $M(t)$ by choosing his position according to the formula $\Phi(s) = \Delta N(s)$ will create an arbitrage in the market. An arbitrage portfolio is basically a deterministic money make machine, and we considerate the existence of an arbitrage portfolio as equivalent to a serious case of mispricing on the market [6].

The agent holds a zero position at all times except when there is a jump of the process $N(s)$, when he has a one position. At the same time we have the compensated Poisson process $M(s)$ jumping positively, and so he will reap the upside gain from all these jumps and have no possibility of loss, i.e. the gain is assured (arbitrage).

Actually, the position $\Phi(s) = \Delta N(s)$ cannot be implemented because the investidor must take his positions before the jump, the other fact is the impossibility to him to take a instantaneous position. If an agent don't have an inside information, and it is rightfully forbidden, is impossible to him to know when the asset will jump, and so is impossible to him to hold this asset at only exatly that time. This is true by the fact that our position $\Phi(s)$ depends only on the path of the asset $M(s)$, for $t \leq s$, do not depending on the future path of the asset. This is axatly the definition of adapted process when we have contructed a stochastic integral respect to a Brownian motion.

To assure the condition of a martingale and no arbitrage processes holding for our strategy, is not enought to require only the integrand like a adapted process to the underlying, we will need also to put the extra condition of our strategy is a *left-continuous* process. With this, during the jumps of the underlying $M(s)$ path, our strategy $\Phi(s)$ will have a zero position between and during the jump times. All these facts are assured by the following theorem.

**Theorem 2.6** ([1]). *Assume that the jump process $X(s)$ defined as in 2.1, is a martingale, the integrand $\Phi(s)$ is left-continuous and adapted, and*

$$\mathbb{E}[\int_0^t \Gamma^2(s)\Phi^2(s)ds] < \infty \text{ ,for all } t \geq 0 \ .$$

*If all those conditions hold, the stochatic integral $\int_0^t \Phi(s)dX(s)$ is a martingale.*

It is important to verify that, despite the fact that we require the integrand process $\Phi(s)$ to be left-contiuous, we have the integrator process $X(t)$ always taken as a right-continuous process and so the integral $\int_0^t \Phi(s)dX(s)$ will be right-continuous in the upper limit of integration $t$. The integral jumps whenever $X$ jumps and $\Phi$ is simultneously not zero, and so the exatly time of the jump is "protected" avoiding the arbitrage. By the stochastic integral 2.8 we have included the value of integral at the final time $t$ the possible jump. To better represent the idea, we will give another example:

**Example 2** : Holding a asset with a strategy "$\Phi(t)$"[1]

Let $M(t)$ be the process of an underlying, given by a compensated Poisson process, with $N(t)$ the Poisson process with intensity $\lambda$. The strategy will be given by the process,

$$\Phi(s) = \mathbb{I}_{[0,S_1]}(s)$$

This process is given by an left-continuous process with value 1 up to and included the time of the first jump of the Poisson process $N(t)$ with which the process is adapted, and zero thereafter. Therefore we have the gain:

$$\int_0^t \Phi(s)dM(s) = \begin{cases} -\lambda t, 0 \le t < S_1, \\ 1 - \lambda S_1, t \ge S_1, \end{cases}$$

It follows,

$$\int_0^t \Phi(s)dM(s) = \mathbb{I}_{[S_1,\infty)}(t) - \lambda(t \wedge S_1). \tag{2.14}$$

To verify the martingale property of the sthocastic integral, we perform by the definition. For $0 \le s < t$, we have

$$\mathbb{E}[\mathbb{I}_{[S_1,\infty)}(t) - \lambda(t \wedge S_1)|\mathcal{F}(s)] = \mathbb{P}\{S_1 \le t|\mathcal{F}(s)\} - \lambda\mathbb{E}[t \wedge S_1)|\mathcal{F}(s)]. \tag{2.15}$$

Now we split this equation in two cases, differing in time location:

• For $S_1 \le s$:

In that case as we are at the time $s$, by the filtration $\mathcal{F}(s)$ we know exatly the time of first jump $S_1$ and the conditional expectations give us the correct value of the random variables being estimated. In particular, we have

$$\begin{aligned}\mathbb{E}[\mathbb{I}_{[S_1,\infty)}(t) - \lambda(t \wedge S_1)|\mathcal{F}(s)] = \; & = \mathbb{P}\{S_1 \le t|\mathcal{F}(s)\} - \lambda\mathbb{E}[t \wedge S_1)|\mathcal{F}(s)] \\ & = 1 - \lambda S_1 \\ & = \mathbb{I}_{[S_1,\infty)}(s) - \lambda(s \wedge S_1).\end{aligned} \tag{2.16}$$

Hence the martingale property is satisfied.

• For $S_1 > s$: By the probability property, and the memorylessness of the exponential random variables, we develop the first part of the equation 2.15:

$$\begin{aligned}\mathbb{P}\{S_1 \le t|\mathcal{F}(s)\} & = 1 - \mathbb{P}\{S_1 > t|S_1 > s\} \\ & = 1 - \tfrac{\mathbb{P}\{S_1 \ge t\}}{\mathbb{P}\{S_1 \ge s\}} \\ & = 1 - \tfrac{e^{-\lambda t}}{e^{-\lambda s}} \\ & = 1 - e^{-\lambda(t-s)}\end{aligned} \tag{2.17}$$

The second part of the equation 2.15 will be given by:

$$\begin{aligned}\lambda\mathbb{E}[t \wedge S_1|\mathcal{F}(s)] & = \lambda\mathbb{E}[t \wedge S_1|S_1 > s] \\ & = \lambda^2 \textstyle\int_s^\infty (t \wedge u)e^{-\lambda(u-s)}du \\ & = \lambda^2 \textstyle\int_s^t ue^{-\lambda(u-s)}du + \lambda^2 \int_t^\infty ue^{-\lambda(u-s)}du \\ & = -\lambda ue^{-\lambda(u-s)}|_{u=s}^{u=t} + \lambda \textstyle\int_s^t e^{-\lambda(u-s)}du - \lambda te^{-\lambda(u-s)}|_{u=t}^{u=\infty} \\ & = \lambda s - \lambda te^{-\lambda(t-s)} - e^{-\lambda(u-s)}|_{u=s}^{u=t} + \lambda te^{-\lambda(t-s)} \\ & = \lambda s - e^{-\lambda(t-s)} + 1.\end{aligned} \tag{2.18}$$

And joining together the equations 2.17 and 2.18, we obtain :

$$\begin{aligned}\mathbb{E}[\mathbb{I}_{[S_1,\infty)}(t) - \lambda(t \wedge S_1)|\mathcal{F}(s)] &= \mathbb{P}\{S_1 \leq t|\mathcal{F}(s)\} - \lambda\mathbb{E}[t \wedge S_1)|\mathcal{F}(s)] \\ &= 1 - e^{-\lambda(t-s)} - (\lambda s - e^{-\lambda(t-s)} + 1) \\ &= -\lambda s \\ &= \mathbb{I}_{[S_1,\infty)}(s) - \lambda(s \wedge S_1).\end{aligned} \quad (2.19)$$

For both cases $S_1 \leq s$ and $S_1 > s$ we have the stochastic integral 2.14 as a martingale, and so the maringale property is verified.

## 2.3 Quadratic Variation

In this section we will follow the approach [1][3]. It is essential to the continuity of the study to give some principles of *quadratic variation*, mainly to write down the Itô-Doeblin formula for the process with jumps. Let $X(t)$ be a jump process. To compute its quadratic variation on $[0, T]$, we choose the following time points $0 = t_0 < t_1 < t_2 < ... < t_n = T$. Let's denote the set of these times by $\Pi = t_0, t_1, ..., t_n$, we also define the length of the longest subinterval as $\|\Pi\| = max_j(t_{j+1} - t_j)$ and finally will be defined,

$$Q_\Pi = \sum_{j=0}^{n-1}(X(t_{j+1}) - X(t_j))^2$$

**Definition 2.7.** *The quadratic variation of X on [0,T] is defined as*

$$[X, X](T) = lim_{\|\Pi\|\to 0}Q_\Pi(X),$$

By this definition, when the time interval measure $\|\Pi\|$ goes to 0, $\Pi$ has infinity time intervals. The quadratic variation is random number in general, depending on its path. But for the Brownian motion, for instance, we know that [W,W](T)=T has a deterministic value T, not depending on its path. In the case of an Itô Integral $I(T) = \int_0^T \Gamma(s)dW(s)$, in respect to the pure Brownian motion "W", $[I, I](T) = \int_0^T \Gamma^2(s)ds$ will depend on the path $\Gamma$.

Another concept will be needed:

**Definition 2.8.** *Given $X_1(t)$ and $X_2(t)$ two jump process. We have*

$$C_\Pi(X_1, X_2) = \sum_{j=0}^{n-1}(X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j))$$

*The cross variation between $X_1(t)$ and $X_2(t)$ on [0,T] is defined to be,*

$$[X_1, X_2](T) = lim_{\|\Pi\|\to 0}C_\Pi(X_1, X_2).$$

With these two conceps we can stand the following theorem, for a proof consult [3],

**Theorem 2.9.** *Let the usual jump process* $X_1(t) = X_1(0) + I_1(t) + R_1(t) + J_1(t)$*, where* $I_1(t) = \int_0^t \Gamma_1(s)dW(s)$*,* $R_1(t) = \int_0^t \Theta_1(s)ds$*, and* $J_1(t)$ *is the usual pure right-continuous jump process, remembering* $X_1^c(t) = X_1(0) + I_1(t) + R_1(t)$*. The quadratic variation of the process* $X_1(t)$ *will be,*

$$[X_1, X_1](T) = [X_1^c, X_1^c](T) + [J_1, J_1](T) = \int_0^T \Gamma_1^2(s)ds + \sum_{0<s\leq T} (\Delta J_1(s))^2 \tag{2.20}$$

*Let the jump process* $X_2(t) = X_2(0) + I_2(t) + R_2(t) + J_2(t)$*, where* $I_2(t) = \int_0^t \Gamma_2(s)dW(s)$*,* $R_2(t) = \int_0^t \Theta_2(s)ds$*, and* $J_2(t)$ *is the usual pure right-continuous jump process, and* $X_2^c(t) = X_2(0) + I_2(t) + R_2(t)$*. The cross variation between the processes* $X_1(t)$ *e* $X_2(t)$ *will be,*

$$\begin{aligned} [X_1, X_2](T) &= [X_1^c, X_2^c](T) + [J_1, J_2](T) \\ &= \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s). \end{aligned} \tag{2.21}$$

The proof is an application of the definition 2.8. We can use the theorem above in the differential notation, for $X_1(t) = X_1(0) + X_1^c(t) + J_1(t)$ and $X_2(t) = X_2(0) + X_2^c(t) + J_2(t)$ by the definition 2.7 and the theorem 2.9 we have:

$$dX_1(t)dX_2(t) = dX_1^c(t)dX_2^c(t) + dJ_1(t)dJ_2(t).$$

In particular, it follows

$$dX_1^c(t)dJ_2(t) = dX_2^c(t)dJ_1(t) = 0;$$

It can be concluded that the cross variation between a continuous process, a Brownian motion for instance, and a pure jump process is zero. It holds for any cross variation between a continuous process and a process without a Itô integral part. So, in order to have a non zero cross variation both process must have a Brownian motion term or the process must have simultaneous jumps, furthermore a cross variation between a Brownian motion and a compensated Poisson process will be zero. The general case is given in the following corollary:

**Corollary 2.10.** *Let* $W(t)$ *be a Brownian motion and* $M(t) = N(t) - \lambda t$ *be a compesated Poisson process relative to the same filtration* $\mathcal{F}(s)$*. Then, for* $t \geq 0$

$$[W, M](t) = 0$$

*Proof.* To prove that is enought to use the theorem 2.9, with $I_1(t) = W(t)$, $R_1(t) = J_1(t) = 0$ and take $I_2(t) = 0$, $R_2(t) = -\lambda t$, and $J_2(t) = N(t)$. □

An implication of this corollary is the fact that a Brownian motion $W$ and a compensated Poisson process $M$ are independent, and also $W$ and Poisson process $N$ relative to the same filtration $\mathcal{F}(s)$ are independent. Now, will be shown how the cross variation holds for the stochastic integral relative to a jump process, in the following corollary:

**Corollary 2.11.** *For i=1,2, let $X_i(t)$ be the usual adapted, right-continuous jump process. Therefore, in our case it means $X_i(t) = X_i(0) + I_i(t) + R_i(t) + J_i(t)$, where $I_i(t) = \int_0^t \Gamma_i(s)dW(s)$, $R_i(t) = \int_0^t \Theta_i(s)ds$, and $J_i(t)$ is a pure jump process. Let $\tilde{X}_i(0)$ be a constant, and let $\Phi_i(s)$ be and adpted process, and assume*

$$\tilde{X}_i(t) = \tilde{X}_i(0) + \int_0^t \Phi_i(s)dX_i(s)$$

*And so, it holds*

$$\tilde{X}_i(t) = \tilde{X}_i(0) + \tilde{I}_i(t) + \tilde{R}_i(t) + \tilde{J}_i(t),$$

*Where, obviously, the notation is given by $\tilde{I}_i(t) = \int_0^t \Phi_i(s)\Gamma_i(s)dW(s)$, $\tilde{R}_i(t) = \int_0^t \Phi_i(s)\Theta_i(s)ds$, $\tilde{J}_i(t) = \sum_{0<s\leq t} \Phi_i(s)\Delta J_i(s)$.*

*Look that, at the same way of the process $X_i(t)$, the process $\tilde{X}_i(t)$ has a continuous part and a pure jump part. We have*

$$\begin{aligned}[\tilde{X}_1(t), \tilde{X}_2(t)] &= [\tilde{X}_1^c, \tilde{X}_2^c] + [\tilde{J}_1, \tilde{J}_2] \\ &= \int_0^t \Phi_1(s)\Phi_2(s)\Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq t} \Phi_1(s)\Phi_2(s)\Delta J_1(s)\Delta J_2(s) \\ &= \int_0^t \Phi_1(s)\Phi_2(s)d[X_1, X_2](s).\end{aligned}$$

In differential notation we have, for $i = 1, 2$ we have $d\tilde{X}_i(t) = \Phi_i(t)dX_i(t)$, and it holds

$$d\tilde{X}_1(t)d\tilde{X}_2(t) = \Phi_1(t)\Phi_2(t)dX_1(t)dX_2(t)$$

# Chapter 3

# Stochastic Calculus for Jump Processes

In this chapter will be presented the one dimensional Stochastic Calculus for Jump Processes. In this chapter will be followed, for the main ideas of Stochastic Calculus for Jump Process, the procedure of [1] and [7].

## 3.1 Itô-Doeblin formula for One Jump Process

Remembering, the Itô-Doeblin formula for a continuous-path process is [6],

$$X^c(t) = X^c(0) + \int_0^t \Gamma(s)dW(s) + \int_0^t \Theta(s)ds \tag{3.1}$$

Or in differential notation

$$dX^c(t) = \Gamma(s)dW(s) + \Theta(s)ds \tag{3.2}$$

With initial condition $X^c(0)$.

The quadratic variation is the usual:

$$dX^c(s)dX^c(s) = \Gamma^2(s)ds$$

,where $\Gamma(s)$ and $\Theta(s)$ are $\mathcal{F}(s)$ adapted process.

Let $f(x) \in \mathcal{C}^1 \cap \mathcal{C}^2$, remembering the Itô-Doeblin formula for continuous process, is given by:

$$\begin{aligned} df(X^c(s)) &= f'(X^c(s))dX^c(s) + \tfrac{1}{2}f''(X^c(s))dX^c(s)dX^c(s) \\ &= f'(X^c(s))\Gamma(s)dW(s) + f'(X^c(s))\Theta(s)ds + \tfrac{1}{2}f''(X^c(s))\Gamma^2(s)ds. \end{aligned} \tag{3.3}$$

If we write this in the integral form, we also must include the initial contidion:

$$\begin{aligned} f(X^c(t)) = f(X^c(0)) + \textstyle\int_0^t f'(X^c(s))\Gamma(s)dW(s) + \int_0^t f'(X^c(s))\Theta(s)ds \\ +\tfrac{1}{2}\textstyle\int_0^t f''(X^c(s))\Gamma^2(s)ds. \end{aligned}$$

That are the well known Itô formula for the continuous diffusion case.

In the jump diffusion case we must add a right-continuous pure jump term, and then [1]

$$X(t) = X^c(0) + \int_0^t \Gamma(s)dW(s) + \int_0^t \Theta(s)ds + J(t)$$

Between jumps we have the same differential equation for the continuous diffusion case,

$$\begin{aligned} df(X(s)) &= f'(X(s))dX(s) + \tfrac{1}{2}f''(X(s))dX(s)dX(s) \\ &= f'(X(s))\Gamma(s)dW(s) + f'(X(s))\Theta(s)ds + \tfrac{1}{2}f''(X(s))\Gamma^2(s)ds. \end{aligned} \tag{3.4}$$

In the jump of the X process, from $X(s-)$ to $X(s)$, for instance, there will be also a jump in the $f(X)$ process from $f(X(s-))$ to $f(X(s))$. The procedure to develop a jump diffusion case is to hold the equation 3.4 and integrate its both sides form 0 to $t$, and add all the jumps occurred up to the final time. This important result is given by the following theorem.

**Theorem 3.1** (The Itô-Doeblin formula for one jump process[1])**.** *Let $X(t)$ be a jump process with $f(x) \in \mathcal{C}^1 \cap \mathcal{C}^2$ Then*

$$\begin{aligned} f(X(t)) = f(X(0)) + \textstyle\int_0^t f'(X(s))dX^c(s) + \tfrac{1}{2}\int_0^t f''(X(s))dX^c(s)dX^c(s) \\ \textstyle\sum_{0<s\leq t}[f(X(s)) - f(X(s-))] \end{aligned}$$

*Proof.* We start the proof fixing a path $w \in \Omega$ on the probability space, and fixing the jump times $0 < \tau_1 < \tau_2 < ... < \tau_{n-1} < t$, all of them in the time period $[0, t)$, in the path of the process $X$. Given the start point like a non jump time $\tau_0 = 0$, and the final time $\tau_n = t$, which may be or not a jump time. We choose two time points $u < v$ are both in the same interval

$(\tau_j, \tau_{j+1})$. By this choice there is no jumps between u and v. The equation 3.4 for the continuous case for the Itô-Doeblin formula, in the integral form, will be:

$$f(X(v)) - f(X(u)) = \int_u^v f'(X(s))dX^c(s) + \frac{1}{2}\int_u^v f''(X(s))dX^c(s)dX^c(s).$$

Choosing v and u by its limits, $u \downarrow \tau_j$ and $v \uparrow \tau_{j+1}$ and by the right-continuity of X, we conlclude that,

$$f(X(\tau_{j+1-})) - f(X(\tau j)) = \int_{\tau_j}^{\tau_{j+1}} f'(X(s))dX^c(s) + \frac{1}{2}\int_{\tau_j}^{\tau_{j+1}} f''(X(s))dX^c(s)dX^c(s).$$

Here is important to integrate respect to the integrator $X^c(s)$ and not the $X(s)$, remembering that the process $X(s)$ is right-continuous and so the jump will be at the next stair. Choosing the process $X^c(s)$ as an integrator we can handle with the left-continuous version at the jump $X(\tau_{j+1-})$ and so we can aprossimate in that way

$$lim_{v\uparrow\tau+1} \int_u^v f'(X(s))dX^c(s) = \int_u^{\tau+1} f'(X(s))dX^c(s)$$

Now we sum at the continuous parts the jump at both hands of the last equation, as following

$$\begin{aligned} f(X(\tau_{j+1})) - f(X(\tau_j)) = \textstyle\int_{\tau_j}^{\tau_{j+1}} f'(X(s))dX^c(s) \\ + \tfrac{1}{2}\textstyle\int_{\tau_j}^{\tau_{j+1}} f''(X(s))dX^c(s)dX^c(s) + f(X(\tau_{j+1})) - f(X(\tau_{j+1-})) \end{aligned} \tag{3.5}$$

And summing all the jumps over j=0,...,n-1, we will have

$$\begin{aligned} f(X(t)) - f(X(0)) = \textstyle\int_0^{\tau_t} f'(X(s))dX^c(s) \\ + \tfrac{1}{2}\textstyle\int_0^t f''(X(s))dX^c(s)dX^c(s) + \sum_{j=0}^{n-1} f[(X(\tau_{j+1})) - f(X(\tau_{j+1-})) \end{aligned} \tag{3.6}$$

And the theorem is prooved. □

In many cases is not possible to find a differential equation for the sum of jumps, and because of that characteristic is not always easy to find a differential form for the Itô-Doeblin formula 3.1. Let's see a specific case where It is possible to find a differential form for the equation.

**Example 3 :** Geometric Poisson process [1]

Let the geometric Poisson process

$$S(t) = S(0)e^{N(t)log(\sigma+1)-\lambda\sigma t} = S(0)e^{-\lambda\sigma t}(\sigma+1)^{N(t)}, \tag{3.7}$$

,where we choose the constant $\sigma > -1$ in that way: If $\sigma > 0$, this process jumps up and by the diffusion part moves down between jumps. If $-1 < \sigma < 0$, it jumps down and moves up between jumps. By its move characteristics, i.e. the process moving down and up, it can be a candidate to be a martingale process.

For $f(x) = e^x$, we can write the equation in that way $S(t) = S(0)f(X(t))$ and so,

$$X(t) = N(t)log(\sigma + 1) - \lambda\sigma t$$

Writing the equation in like in 2.1 it follows:

- $X^c(t) = -\lambda\sigma t$
- $J(t) = N(t)log(\sigma + 1)$

By the Itô-Doeblin formula 3.1,

$$\begin{aligned} S(t) &= f(X(t)) \\ &= f(X(0)) - \lambda\sigma \textstyle\int_0^t f'(X(u))du + \sum_{0<u\leq t}[f(X(u)) - f(X(u-))] \\ &= S(0) - \lambda\sigma \textstyle\int_0^t S(u)du + \sum_{0<u\leq t}[S(u) - S(u-)] \end{aligned} \tag{3.8}$$

If there is a jump at time $u$, then $S(u) = (\sigma + 1)S(u-)$. And the jump for the $S$ process will be,

$$S(u) - S(u-) = \sigma S(u-) = \sigma S(u-)\Delta N(u). \tag{3.9}$$

Look that if there is no jump at the time u $\Delta N(u) = 0$. Be able to write the jump in terms of $S(u-)$ is the characterisc which permits us to write the equation in its differential form, as a final result. This result is the basis for writing the sum of jumps of the equation 3.8 in the differential way,

$$\sum_{0<u\leq t}[S(u) - S(u-)] = \sum_{0<u\leq t}\sigma S(u-)\Delta N(u) = \sigma\int_0^t S(u-)dN(u).$$

As the two Riemann integral $\int_0^t S(u)du$ and $\int_0^t S(u-)du$ differs only in finite many points with measure zero, the two integrals are equivalent a.s. and so, we can rewrite the equation

$$\begin{aligned} S(t) &= S(0) - \lambda\sigma \textstyle\int_0^t S(u)du + \sigma\int_0^t S(u-)dN(u) \\ &= S(0) - \lambda\sigma \textstyle\int_0^t S(u-)du + \sigma\int_0^t S(u-)dN(u) \\ &= S(0) - \sigma \textstyle\int_0^t S(u-)dM(u) \end{aligned}$$

Where M is the well-known compensated process $M(u) = N(u) - \lambda u$, which , as was already proved, is a martingale. By the theorem 2.6, and knowing that we were working with limited and adpated processes, we have $S(t)$ as a martingale process.

Is also known how to write the equation in the differetial form,

$$dS(t) = \sigma S(t-)dM(t) = -\lambda\sigma S(t)dt + \sigma S(t-)dN(t) \tag{3.10}$$

And obviously, the equation 3.7 is the solution for the differential equation above, for an initial condition $S(0)$.
It implies an important corollary,

**Corollary 3.2.** *[1] Let the probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ *with* $\{\mathcal{F}_t\}_{t\geq 0}$ *the filtration and the adapted process* $W(t)$ *be a Brownian motion and let* $N(t)$ *be a Poisson process with intensity* $\lambda > 0$*. Then* $W(t)$ *and* $N(t)$ *are independent.*

For a key step in proof consult [1].

## 3.2 Change of Measure

Given a process $X_t, t \in [O, T]$ it can be considered a random variable on a space $\Omega$ of cadlag processes adapted to the filtration $\mathcal{F}_t$. The distribution of $X$ defines a probability measure $\mathbb{P}$ on the space of paths. Now we consider another process $Y_t, t \in [0, T]$ and $\tilde{\mathbb{P}}$ its distribution on the same space of paths $\Omega$. We will see in this section the conditions, in a jump scenario, for $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent probability measures, i.e. the stochastic processes models $X$ and $Y$ give us the same set of possible evolutions for the paths. The procedure to construct a new process on the same set of paths by assigning news equivalents probabilities to events is the so called *change of measure*[7].

As we can use the Girsanov's Theorem to change the measure of a Brownian motion with drift for a Brownian motion without drift, we can change the measure for Poisson processes and compound Poisson processes. As we will see, the change of measure affects the intensity for a Poisson process, and for a compound Poisson process it can interfere also in the distribution of the jumps sizes. In this chapter will be treated these three situations: a process with only a Poisson process, a process with a compound Poisson process and at the end we will treat the case with also a Brownian motion in the path.

### 3.2.1 Change of Measure for a Poisson Process

Initially will be presented how to perform a change of measure in stochastic calculus without jumps. By the Girsanov's Theorem the change of measure using the Radon-Nikodym derivative process [4].

Suppose $\int_0^T |\Gamma|^2 ds \leq K$, where $K$ is a constant.

$$Z(t) = exp\{-\int_0^t \Gamma(s)dW(s) - \frac{1}{2}\int_0^t \Gamma^2(s)ds\}$$

We can also prove that this process satisfies the following stochastic equation

$$dZ(t) = -\Gamma(t)Z(t)dW(t) = Z(t)dX^c(t),$$

Taking $X^c(t) = -\int_0^t \Gamma(s)dW(s)$ and $[X^c, X^c](t) = \int_0^t \Gamma^2(s)ds$, we can rewrite $Z(t)$ as,

$$Z(t) = exp\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\} \tag{3.11}$$

For processes with jumps we have an analogous stochastic differential equation, but now the path of the process $X(t)$ may have jumps.

$$dZ^X(t) = Z^X(s-)dX(t) \tag{3.12}$$

The solution is similar to the continuous case but with the jumps added.

$$Z^X(s) = Z^X(s-) + \Delta Z^X(s) = Z^X(s-)(1 + \Delta X(s)) \tag{3.13}$$

The final result will be given by the following corollary, in the same way of the Girsanov thorem, but for jump diffusions processes.

**Corollary 3.3.** *[1] Let $X(t)$ be a jump process. We call the Doleans-Dale exponential of $X$, the process:*

$$Z^X(t) = exp\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\} \prod_{0<s\leq t}(1 + \Delta X(s)).$$

*This process is the solution for the process 3.12 with initial condition $Z^X(0) = 1$. The integral form is the fallowing equation:*

$$Z^X(t) = 1 + \int_0^t Z^X(s-)dX(s).$$

Let the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration. We denote the poisson process $N(t)$ with intensity $\lambda > 0$. As already showed the compesated Poisson process $M(t) = N(t) - \lambda t$ is a martingale under the probability measure $\mathbb{P}$. Let fix $\tilde{\lambda}$ as a positive number. We define a new process,

$$Z(t) = e^{(\lambda - \tilde{\lambda})t} \left( \frac{\tilde{\lambda}}{\lambda} \right)^{N(t)}. \tag{3.14}$$

The idea of change the probability measure is similar to the way of changing the measure of the Radon-Nikodym theorema for a continuous process. We fix a time $T > 0$ and we will use that fixed value $Z(T)$ to change to a new measure $\tilde{\mathbb{P}}$. In that new measure the process $N(t)$ has an intensity $\tilde{\lambda}$ rather than $\lambda$. In order to use $Z(T)$ to change the measure, we must guarantee $\mathbb{E}[Z(T)] = 1$

**Lemma 3.4.** *[1] The process $Z(t)$, for all $t > 0$, satisfies the differential equation:*

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM(t). \tag{3.15}$$

*And $Z(t)$ is a martingale under $\mathbb{P}$, and $\mathbb{E}Z(T) = 1 \forall t$*

*Proof.* We define $X(t) = \frac{\tilde{\lambda}-\lambda}{\lambda} M(t)$. By the property of the compesated Poisson process this process is a martingale with continuous part $X^c(t) = (\lambda - \tilde{\lambda})t$ and pure jump part $\frac{\tilde{\lambda}-\lambda}{\lambda} N(t)$. How it was already showed $[X^c, X^c](t) = 0$, and if there is a jump at time t, then $\Delta X(t) = \frac{\tilde{\lambda}-\lambda}{\lambda}$,and obviously $1 + \Delta X(t) = \frac{\tilde{\lambda}}{\lambda}$.

The process 3.14 by the corollary 3.3, can be rewritten as

$$Z(t) = exp\{X^c(t) - \frac{1}{2}[X^c, X^c](t)\} \prod_{0<s\leq t} (1 + \Delta X(s))$$

Or in the integral way,

$$Z(t) = 1 + \int_0^t Z(s-) dX(s)$$

To prove the martingale property we have to see the fact that $X(t)$ being a martingale, $Z(s-)$ is left-continuous, and so $Z(t)$ is also a martingale. With this in hands we know that the integral is a martingale. This fact enduces,

$$\mathbb{E}[Z(t)] = \mathbb{E}[1 + \int_0^t Z(s-) dX(s)] = 1 + 0$$

and we conclude $\mathbb{E}[Z(t)] = 1 \forall t \geq 0$

□

Using the Radon-Nikodym theorem, for a fixed time $T$ we will use the constant value $Z(T)$ to change the measure. Therefore we define, for $\forall A \in \mathcal{F}$

$$\tilde{\mathbb{P}}(A) = \int_A Z(T) d\mathbb{P} \tag{3.16}$$

Now we will see how the change of measure of a Poisson process leads to a change of intensity.

**Theorem 3.5** (Change of Poisson intensity[1]). *Under the probability measure $\tilde{\mathbb{P}}$, the process $N(t)$, $0 < t \leq T$, is a Poisson process with intensity $\tilde{\lambda}$.*

*Proof.* We will see the change of measure by computing the moment-generating function of $N(t)$ under the measure $\tilde{P}$. Using the Radon-Nikodým theorem, we can change the expectation $\tilde{\mathbb{E}}$ of the moment-generating $e^{uN(t)}$ to the $\mathbb{E}$ expectation by using, for a fixed t under $0 \leq t \leq T$, $Z(t)$. Therefore using the formula of $Z(t)$, the moment-generating function and the formula 1.16:

$$\begin{aligned}
\mathbb{E}[e^{uN(t)}Z(t)] &= e^{\lambda-\tilde{\lambda}}\mathbb{E}[e^{uN(t)}\tfrac{\tilde{\lambda}}{\lambda}] \\
&= e^{\lambda-\tilde{\lambda}}\mathbb{E}[exp\{(u + log\tfrac{\tilde{\lambda}}{\lambda}N(t)\}] \\
&= e^{\lambda-\tilde{\lambda}}exp\{\lambda t(e^{u+log\frac{\tilde{\lambda}}{\lambda}} - 1)\} \\
&= e^{\tilde{\lambda}t(e^u-1)}
\end{aligned}$$

As we can see this is the moment-generating function for a Poisson process with intensity $\tilde{\lambda}$ □

We can better see by an example[1]:
**Example 3** : Change of measure

In the same way of the classic geometric Poisson process, we will model a stock with the following geometric Poisson process,

$$S(t) = S(0)e^{\alpha t+N(t)log(\sigma+1)-\lambda\sigma t} = S(0)e^{(\alpha-\lambda\sigma)t}(\sigma + 1)^{N(t)}$$

It is assumed $\sigma > -1$, $\sigma \neq 0$, and the process $N(t)$ is a Poisson process with intensity $\lambda$ under the usual probility measure $\mathbb{P}$. As we have seen the process $e^{\alpha t}S(t)$ is a martingale under $\mathbb{P}$, it means that $S(t)$ has a mean rate of return $\alpha$. Changing the Itô-Doeblin differential equation form 3.10, we introduce one,

$$dS(t) = \alpha S(t)dt + \sigma S(t-)dM(t), \tag{3.17}$$

where $M(t)$ is the usual compensated Poisson process $M(t) = N(t) - \lambda t$. The ideia is to change to a probability measure $\tilde{\mathbb{P}}$, for which the differential equation becomes:

$$dS(t) = rS(t)dt + \sigma S(t-)d\tilde{M}(t), \tag{3.18}$$

, where $r$ is the interest rate, $N(t)$ is a Poisson process with, under a probability measure $\tilde{\mathbb{P}}$, has intensity $\tilde{\lambda}$, and has a compensated Poisson process $\tilde{M}(t) = N(t) - \tilde{\lambda}t$. With this probability measure we have the classical situation where the geometric Poisson process would have mean rate of return equal to the free risk interest rate, and $\tilde{\mathbb{P}}$ would be a risk neutral measure. The continuous "$dt$" part of both equations above can be developed by the following relation, remembering that for the continuous part of these equations, $S(-t)dt$ and $S(t)dt$ have the same integral.

$$\tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}.$$

So using the Radon-Nykodym formula 3.16 to change the probability measure, by the choice of the measure we are working with a risk neutral process. We have to hold some hypothesis to change the a risk-neutral measure . We must have $\tilde{\lambda} > 0$, and so, $\lambda > \frac{\alpha - r}{\sigma}$. If it doesn't hold we must have an arbitrage. We can have these cases of arbitrage :

- If $\sigma > 0$ then,

  $$S(t) \geq S(0)e^{rt}(\sigma + 1)^N(t) \geq S(0)e^{rt}$$

  And the arbitrage consists in borrowing at the interest rate "r" and invest the amount in the stock, the value of the stock will be always greater than the loan, and we can be sure that we are winning money when we sell the stock for pay the loan (arbitrage).

- If $-1 < \sigma < 0$ then, the arbitrage consists of short the stock and invest the amount in a market account. The value of the money in the account will be always greater than the value of the stock (arbitrage)

### 3.2.2 Change of Measure for a Compound Poisson Process

We will now develop the change of measure for a more complex process, the Compound Poisson Process, i.e. a Poisson process with the jumps not only sized one, but with their amplitude given by a probability distribution[1].

Let the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, $N(t)$ be a Poisson process with intensity $\lambda$, and let $Y_1, Y_2, Y_3...$ be a sequence of i.i.d random variables, also independent form the Poisson process, all of them defined in the probability space. Remembering the compound Poisson process will be given by

$$Q(t) = \sum_{i=1}^{N(t)} Y_i$$

For notation we have the jump size for a compound Poisson process,

$$\Delta Q(t) = Y_{N(t)}. \tag{3.19}$$

In this section we intend to change the measure for a given compound Poisson process where the $N(t)$'s intensity and the probability distribuition of the jump sizes $Y_1, Y_2...$ both will change. To introduce some concepts we will first, in the most simple case, consider when the jump sizes distribuition is discrete, and then each $Y_i$ takes its value in a finitely many non zero values $y_1, y_2, ..., y_M$. We will denote the probability of occurrence a jump of size $y_m$ by $p(y_m)$, and so,

$$p(y_m) = \mathbb{P}\{Y_i = y_m\}, \text{ m = 1,...,M}$$

We will assume that $p(y_m) > 0$ for every $m$ and, $\sum_{m=1}^{M} p(y_m) = 1$. Let define some new variables too. $N_m(t)$ is the number of jumps in $Q(t)$ of size $y_m$ up to and including time t, so that

$$N(t) = \sum_{m=1}^{M} N_m(t)$$

Like a result we also have this,

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

How was already proved $N_1, N_2, ..., N_M$ are independent Poisson processes and each $N_m$ has intensity $\lambda_m = \lambda p(y_m)$. In the same way of the Poisson process case, It's defined:

$$Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t} (\frac{\tilde{\lambda}_m}{\lambda_m})^{N_m(t)}. \tag{3.20}$$

$$Z(t) = \prod_{m=1}^{M} Z_{m(t)}. \tag{3.21}$$

Where $\tilde{\lambda}_m$, for $m = 1, ..., M$, are given positive numbers. In the same way of the change of probability measure for a Poisson process, It holds the following lemma, in order to prove this Lemma fallow the steps of the Poisson process case:

**Lemma 3.6.** *[1] The process $Z(t)$ of 3.21 is a martingale. In particular we have $\mathbb{E}[Z(t)] = 1$ for all $t$.*

For a fixed $T > 0$, $Z(T) > 0$ and $\mathbb{E}[Z(T)] = 1$, $Z(T)$ is used to change the measure, the new measure is defined by

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)dP \text{ for all Z} \in \mathcal{F}. \tag{3.22}$$

It suggests the theorem,

**Theorem 3.7** (Change of compound Poisson intensity and jump size distribution for finitely many jumps sizes[1])**.** *Let $Q(t)$ be a compound Poisson process with intensity $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$, the random variables $Y_i$ are i.i.d with*

$$\tilde{\mathbb{P}}Y_i = y_m = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\lambda} \tag{3.23}$$

*Proof.* We will give some hints for the proof, the whole proof is to much technical. By the independence of the process $N_1, N_2, .., N_M$ under $\mathbb{P}$ to compute the moment-generating function of Q(t) under $\tilde{\mathbb{P}}$. Using the formula 1.16, for $0 \leq t \leq T$ it holds,

$$\begin{aligned}
\tilde{\mathbb{E}}[e^{uQ(t)}] &= \mathbb{E}[e^{uQ(t)}Z(t)] \\
&= \mathbb{E}[exp\{u\textstyle\sum_{m=1}^{M} y_m N_m(t)\}\prod_{m=1}^{M} exp\{(\lambda_m - \tilde{\lambda}_m)t\}(\frac{\tilde{\lambda}_m}{\lambda_m})^{N_m(t)}] \\
&= \textstyle\prod_{m=1}^{M} exp\{(\lambda_m - \tilde{\lambda}_m)t\} \cdot \mathbb{E}[exp\{(uy_m + log\frac{\tilde{\lambda}_m}{\lambda_m})N_m(t)\}] \\
&= \textstyle\prod_{m=1}^{M} exp\{(\lambda_m - \tilde{\lambda}_m)t\}exp\{\lambda_m t(e^{uy_m + log(\frac{\tilde{\lambda}_m}{\lambda_m})} - 1)\} \\
&= \textstyle\prod_{m=1}^{M} exp\{(\lambda_m - \tilde{\lambda}_m)t + \tilde{\lambda}_m t e^{uy_m} - \lambda_m t\} \\
&= \textstyle\prod_{m=1}^{M} exp\{\tilde{\lambda}_m t(e^{uy_m - 1})\} \\
&= \textstyle\prod_{m=1}^{M} exp\{\tilde{\lambda} t \tilde{p}(y_m)e^{uy_m} - \tilde{\lambda}_m t\} \\
&= exp\{\tilde{\lambda} t(\textstyle\sum_{m=1}^{M} \tilde{p}(y_m)e^{uy_m} - 1)\}
\end{aligned}$$

And this last equation by the equation 1.17 is the moment-generating function for a compound Poisson process with intensity $\tilde{\lambda}$ and a jump size distibution given by $\tilde{\mathbb{P}}Y_i = y_m = \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\lambda}$ □

In the more general case, when we have instead of discrete values of "$y_m$" for the possible values of "$Y_i$", a density function of probability values, i.e.

each "$Y_i$" takes its values in a continuous density we will perform this way: The equation $Z(T)$ of 3.21 could be rewritten in that way,

$$Z(t) = exp\{\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)t\} \cdot \prod_{m=1}^{M}(\frac{\tilde{\lambda}\tilde{p}(y_m)}{\lambda p(y_m)})^{N_m(t)} = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}$$

We can suppose that we don't have anymore discrete values for "$Y_i$" but they take their values in a commom probability density $f(y)$, and so we can use "$Z(t)$" to change the probability measure of the process $Q(t)$ and then changing its intensity to $\tilde{\lambda}$ and its probility density function to $\tilde{f}(y)$. The $Z(t)$ to use in the Radon-Nicodým derivative process of changing measure will be

$$Z(t) = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}. \tag{3.24}$$

In order to avoid the division by zero, we will suppose that if a size of jump has probability zero to occur under the measure $\mathbb{P}$, so it will be also probability zero under the probability $\tilde{\mathbb{P}}$. We will indicate how to lead with that situtation in the following theorem. We won't prove this theorem, but its principles will be in the same way of the discrete case. To their prove consult [1] and [7].

**Lemma 3.8.** *[1] The process $Z(t)$ given by the formula 3.24 is a martingale with mean $\mathbb{E}[Z(t) = 1]$ for all $t \geq 0$.*

In integral form, the differential equation where $Z(t)$ will take its value is given by,

$$Z(t) = 1 + \int_0^t Z(s-)d(H(s) - \tilde{\lambda}s) - \int_0^t Z(s-)d(N(s) - \lambda s) \tag{3.25}$$

And in the differential form,

$$dZ(t) = Z(t-)d(H(t) - \tilde{\lambda}t) - Z(t-)d(N(t) - \lambda t) \tag{3.26}$$

And the jump of this process is,

$$\Delta Z(t) = Z(t-)\Delta H(t) - Z(t-)\Delta N(t) \tag{3.27}$$

As was already done, fixing a positive time T, we define

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P} \text{ for all } A \in \mathcal{F} \tag{3.28}$$

**Theorem 3.9** (Change of compound Poisson intensity and jump distribution for a continuum of jump sizes[1]). *Under the probability measure $\tilde{\mathbb{P}}$, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$, and its jumps are i.i.d with density $\tilde{f}(y)$.*

### 3.2.3 Change of Measure for a Compound Poisson Process and a Brownian Motion

Now we will study the case that more interests us, when besides the Compound Poisson process we have also a Brownian Motion. In this section we will follow the approach of [1] and [7]. Let the usual probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with $\{\mathcal{F}_t\}_{t\geq 0}$ a single Filtration which a Brownian motion $W(t)$ and a compound Poisson process $Q(t)$ will be adapted in, and independent between them. The compound Poisson process will have a density function $f(y)$ and an intensity $\lambda$. Given a different positive intensity $\tilde{\lambda}$, and a another density function $\tilde{f}(t)$, with the property that $f(y) = 0 \Leftrightarrow \tilde{f}(y) = 0 \forall y$, and let $\Theta$ be an adapted process. We define the three process:

$$Z_1(t) = exp\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\} \tag{3.29}$$

$$Z_2(t) = exp\{(\lambda - \tilde{\lambda})t\} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)} \tag{3.30}$$

$$Z(t) = Z_1(t)Z_2(t) \tag{3.31}$$

**Lemma 3.10.** *[1] The process $Z(t)$ 3.31 is a martingale, in particular $\mathbb{E}[Z(t)] = 1$ for all $t \geq 0$.*

For a fixed positive T, we will have

$$\tilde{\mathbb{P}}(A) = \int_A Z(T)d\mathbb{P} \text{ for all } A \in \mathcal{F} \tag{3.32}$$

And so we propose, the following theorem,

**Theorem 3.11.** *[1] Under the probability measure $\tilde{\mathbb{P}}$, the process*

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s)ds \tag{3.33}$$

*Is a Brownian motion, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$ and i.i.d random distribution for the jump sizes, all of them with intensity $\tilde{f}(y)$, and the processes $\tilde{W}(t)$ and $Q(t)$ are independent.*

For the discrete jumps sizes, the more simple case, we have the same result, hence in the case of $Z_2$ in 3.30 replaced by Z of the equation 3.21 we can sustain the following theorem,

**Theorem 3.12.** *[1] Under the probability measure $\tilde{\mathbb{P}}$, the process*

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(s)ds \tag{3.34}$$

*Is a Brownian motion, $Q(t)$ is a compound Poisson process with intensity $\tilde{\lambda}$ and i.i.d random distribution for the jump sizes, all of them satistying $\tilde{\mathbb{P}}\{Y_i = y_m\} = \tilde{p}(y_m)$ for any $i$ and $\forall m = m_1, .., M$, and the processes $\tilde{W}(t)$ and $Q(t)$ are independent.*

# Chapter 4

# Pricing an European Call in a Jump Model

## 4.1 The analytic equation to evaluate an European Call

In this chapter we will price an European Call with the underlying asset driven by a Brownian motion and a compound Poisson process. Let's before redifine some variables and the filtrated probability space $(\Omega, \mathcal{F}, \mathcal{F}(t), \mathbb{P})$. On this probability space we define the Brownian motion $W(t)$, $0 \le t \le T$, and $M$ independent Poisson processes $N_1(t), N_2(t), ..., N_M(t)$, for $0 \le t \le T$. This filtration $\mathcal{F}(t)$ is generated by the Brownian motion and the $M$ Poisson processes. First of all, let's propose the following corollary,

**Corollary 4.1** (Itô's product rule for jump processes[1]). *Let $X_1(t)$ and $X_2(t)$ be jump processes. Then*

$$\begin{aligned} X_1(t)X_2(t) &= X_1(0)X_2(0) + \textstyle\int_0^t X_2(s)dX_1^c(s) + \int_0^t X_1(s)dX_2^c(s) \\ &+ [X_1^c, X_2^c](t) + \textstyle\sum_{0<s\le t}[X_1(s)X_2(s) - X_1(s-)X_2(s-)] \\ &= X_1(0)X_2(0) + \textstyle\int_0^t X_2(s-)dX_1(s) + \int_0^t X_1(s-)dX_2(s) + [X_1, X_2](t) \end{aligned} \tag{4.1}$$

Let's also fix $Q(t)$ the compound Poisson process,

$$N(t) = \sum_{m=1}^{M} N_m(t)$$

$$Q(t) = \sum_{m=1}^{M} y_m N_m(t)$$

Where $\lambda_m > 0$ is the intensity of $N_m(t)$th Poisson process, and a serie of possible size jumps given by $-1 < y_1 < y_2 < .... < y_M$. In our case the intensity of Poisson process $N(t)$ will be,

$$\lambda = \sum_{m=1}^{M} \lambda_m$$

Is defided also $Y_i$ denoting the size of the $i$th jump of $Q$ taking their values in the set $y_1, .., y_M$, as already explained, we redefine,

$$Q(t) = \sum_{i=1}^{N(t)} Y_i.$$

And defining the probability of the jump size $y_m$ will be,

$$p(y_m) = \frac{\lambda_m}{\lambda}$$

Defining also the random variable $Y_i$, they are i.i.d. with distribution $\mathbb{P}\{Y_i = y_m\} = p(y_m)$. It holds,

$$\beta = \mathbb{E}[Y_i] = \sum_{m=1}^{M} y_m p(y_m) = \frac{1}{\lambda} \sum_{m=1}^{M} \lambda_m y_m \tag{4.2}$$

In order to define our underlying asset we will also need the compensated compound Poisson process

$$Q(t) - \beta\lambda t = Q(t) - t \sum_{m=1}^{M} \lambda_m y_m$$

This process is a martingale process. The stock price will be modeled in the following way, on the stochastic differential equation version,

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t) + S(t-)d(Q(t) - \beta\lambda t) \\ &= (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW(t) + S(t-)dQ(t) \end{aligned} \tag{4.3}$$

In the original probability measure $\mathbb{P}$, the mean rate of return is given by the coefficient of drift,then $\alpha$. The stock price is limited for negative values by limiting the down jumps $y_i > -1$ for $i = 1, ..., M$. The initial price for the stock must be positive and different from zero otherwise it will be zero for all t.

We can reach the solution for the 4.3 differential equation by the following theorem,

**Theorem 4.2.** *[1] The solution to the diffential equation 4.3, given an initial value $S(0)$ is*

$$S(t) = S(0)exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\} \prod_{i=1}^{N(t)} (Y_i + 1) \tag{4.4}$$

*Proof.* For the proof, we will follow [1]. We must show the process $S(t)$ of 4.4 solve the differential equation 4.3. Hence, we split that solution in that way,

$$X(t) = S(0)exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\}$$

the continuous stochastic process, and

$$J(t) = \prod_{i=1}^{N(t)} (Y_i + 1).$$

the pure jump process. Then $S(t) = X(t)J(t)$, and we will show the equation is a solution for the process 4.3. By the Itô equation for continuous processes,

$$dX(t) = (\alpha - \beta\lambda)X(t)dt + \sigma X(t)dW(t). \tag{4.5}$$

At the time of the $i$th jump, $J(t) = J(t-)(Y_i + 1)$ and then

$$\Delta J(t) = J(t) - J(t-) = J(t-)Y_i = J(t-)\Delta Q(t)$$

This equation holds also for non jump times, when both sides are equal to zero. So, in a differential form

$$dJ(t) = J(t-)dQ(t) \tag{4.6}$$

By the Itô product rule, we have

$$S(t) = X(t)J(t) = S(0) + \int_0^t X(s-)dJ(s) + \int_0^t J(s)dX(s) + [X, J](t) \tag{4.7}$$

Since J is a pure jump process and $X$ is continuous, [X,J](t)=0. Replacing the last equation, in integral form,

$$\begin{aligned} S(t) &= X(t)J(t) \\ &= S(0) + \int_0^t X(s-)J(s-)dQ(s) + (\alpha - \beta\lambda) \int_0^t J(s)X(s)ds \\ &+\sigma \int_0^t J(s)X(s)dW(s). \end{aligned} \tag{4.8}$$

and in differential form,

$$\begin{aligned} dS(t) &= d(X(t)J(t)) \\ &= X(t-)J(t-)dQ(t) + (\alpha - \beta\lambda)J(t)X(t)dt + \sigma J(t)X(t)dW(t) \\ &= S(t-)dQ(t) + (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW(t) \end{aligned} \tag{4.9}$$

□

Thereafter can be performed the construction of a risk-neutral measure. Let $\Theta$ be a constant and the intensities $\tilde{\lambda}_1, .., \tilde{\lambda}_M$ be positive constants. It's defined,

$$\begin{aligned} &Z_0(t) = exp\{-\Theta W(t) - \tfrac{1}{2}\Theta^2 t\}, \\ &Z_m(t) = exp\{(\lambda_m - \tilde{\lambda}_m)t \left(\tfrac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}\} \text{ , m=1,...,M,} \\ &Z(t) = Z_0(t)\textstyle\prod_{m=1}^{M} Z_m(t), \\ &\tilde{\mathbb{P}}(A) = \textstyle\int_A Z(T)d\mathbb{P} \text{ for all A} \in \mathcal{F} \end{aligned}$$

The next statements follow the Theorem 3.12, and the independece between the Poisson processes $N_1, .., N_M$ and the Brownian motion is proved by the corollary 3.2. Under the probability measure $\tilde{\mathbb{P}}$, we redefine some variables

- the process $\tilde{W}(t) = W(t) + \Theta t$
- each $N_m$ is a Poisson process with intensity $\tilde{\lambda}_m$
- The Brownian $\tilde{W}$ process and the Poisson processes $N_1, ..., N_m$ are independent of one another.

In the same way of the measure $\mathbb{P}$,

$$\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$$

$$\tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}$$

We also have $N(t) = \sum_{m=1}^{M} N_m(t)$ is Poisson with intensity $\tilde{\lambda}$, the jump-size random variables $Y_1, ..., Y_N(t)$ are i.i.d. with distribution $\tilde{\mathbb{P}}\{Y_i = y_m\} = \tilde{p}(y_m)$, and we have also the compensated compound Poisson process $Q(t) - \tilde{\beta}\tilde{\lambda}t$ as a martingale process. On this measure holds,

$$\tilde{\beta} = \tilde{\mathbb{E}}[Y_i] = \sum_{m=1}^{M} y_m \tilde{p}(y_m) = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m y_m.$$

As is expected the probability measure $\tilde{\mathbb{P}}$ is risk-neutral if, and only if, the mean rate of return of the stock under $\tilde{\mathbb{P}}$ is the interest rate $r$. Therefore, $\tilde{\mathbb{P}}$ is risk-neutral if

$$\begin{aligned} dS(t) &= (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW(t) + S(t-)dQ(t) \\ &= rS(t)dt + \sigma S(t)d\tilde{W}(t) + S(t-)d(Q(t) - \tilde{\beta}\tilde{\lambda}t). \end{aligned} \tag{4.10}$$

By this equation we can develop, remembering the definition of the $\beta$ and $\tilde{\beta}$.

$$\alpha - \beta\lambda = r + \sigma\Theta - \tilde{\beta}\tilde{\lambda}, \tag{4.11}$$

$$\begin{aligned} \alpha - r &= \sigma\Theta + \beta\lambda + \tilde{\beta}\tilde{\lambda} \\ &= \sigma\Theta + \textstyle\sum_{m=1}^{M}(\lambda_m - \tilde{\lambda}_m)y_m. \end{aligned} \tag{4.12}$$

In this case, we have the variables $\Theta, \lambda_1, ..., \lambda_M$ and just one equation. As this equation is indeterminate, i.e. more unknown variables than equations, we have many choices of risk-neutral measures one for each combination of the variables that satisfies 4.12 . That is a characteristic of an incomplete market. In order to solve this problem an option is to add more stocks to determine a unique risk-neutral measure. In this work we will use a model with a single stock given by 4.3 and 4.4. Choosing one combination of variables $\Theta, \tilde{\lambda}_1, ..., \tilde{\lambda}_M$ that satisfies the market price of risk equations 4.12. Then, in the notation pattern of the process $S(t)$, under the $\tilde{\mathbb{P}}$ probability measure we have,

$$\begin{aligned} dS(t) &= rS(t) + \sigma S(t)d\tilde{W}(t) + S(t-)d(Q(t) - \tilde{\beta}\tilde{\lambda}t) \\ &= (r - \tilde{\beta}\tilde{\lambda})dt + \sigma S(t)d\tilde{W}(t) + S(t-)dQ(t) \end{aligned} \tag{4.13}$$

And the solution is given by,

$$S(t) = S(0)exp\{\sigma\tilde{W}(t) + (r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2)t\}\prod_{i=1}^{N(t)}(Y_i + 1) \tag{4.14}$$

It's important to understand that by the change of probability measure, we don't change the stock price process, but only change its distribution. We will compute the risk-neutral price of a Call on the stock price process given by 4.14. For this we must choose the variables $\Theta, \tilde{\lambda}_1, ..., \tilde{\lambda}_M$. The $\Theta$ will not appear in our pricing formula. However,

$$\tilde{\beta}\tilde{\lambda} = \sum_{m=1}^{M}\tilde{\lambda}y_m$$

will appear in the formula, so we can choose any risk-neutral intensities $\tilde{\lambda}_1, ..., \tilde{\lambda}_M$, all of them positive, and choose $\Theta$ to satisfy the equation 4.12. With that variables fixed in the such way. In the practice way is common to use these free parameters to calibrate the model to market data.

**Definition 4.3.** *Let $k(\tau, x)$ be the standard Black-Scholes-Merton call price on a geometric Browian motion. We define*

$$k(\tau, x) = xN(d_+(\tau, x)) - Ke^{-r\tau}N(d_-(\tau, x)), \tag{4.15}$$

*where*

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[log\frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]$$

*and $N(y)$ is the cumulative normal distribution,*

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_\infty^y e^{-\frac{1}{2}z^2}dz$$

With volatility $\sigma$, the current stock price is $x$, the maturity is $\tau$ time units in the future, the interest rate is $r$, and the strike price is K. For these conditions, the stock price is given by

$$k(\tau, x) = \tilde{\mathbb{E}}\left[e^{-r\tau}\left(xe^{\{-\sigma\sqrt{\tau}Y + (r-\frac{1}{2}\sigma^2)\tau\}} - K\right)^+\right]$$

Where Y is a standard normal random variable under the measure $\tilde{\mathbb{P}}$.

We finally propose the theorem,

**Theorem 4.4.** *[1] For $0 \leq t < T$, let*

$$V(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}(t)]$$

*where V(t)=c(t,S(t)), is the risk-neutral price of a call, and finally*

$$c(t, x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)}\frac{\tilde{\lambda}^j(T-t)^j}{j!}\tilde{\mathbb{E}}k\left(T - t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)}\prod_{i=1}^{j}(Y_i + 1)\right) \tag{4.16}$$

To see the prove consult [1] and also [7]. In the previous measure $\mathbb{P}$ we have,

$$c(t, x) = \mathbb{E}k\left(\tau, xe^{-\tilde{\beta}\tilde{\lambda}\tau}\prod_{i=N(t)+1}^{N(T)}(Y_i + 1)\right). \tag{4.17}$$

That theorem holds also for a continuous distribution for the jump size. Suppose the jump sizes $Y_i$ with a density $f(y)$, and this function is strictly positive on a set $B \subset (-1, \infty)$ and zero elsewhere. In this case, we replace the formulation of $\beta$ by the formula,

$$\beta = \mathbb{E}Yi = \int_{-1}^{\infty} yf(y)dy$$

And to assure a risk-neutral measure, we can choose $\Theta, \tilde{\lambda} > 0$ and a density $\tilde{f}(y)$ that is strictly positive on B and zero elsewhere, hence the market price of risk equation must satisfy

$$\alpha - r = \sigma\Theta + \beta\sigma - \tilde{\beta}\tilde{\lambda}$$

And we finally have for the risk-free measure $\tilde{\mathbb{P}}$

$$\tilde{\beta} = \tilde{\mathbb{E}}Y_i = \int_{-1}^{\infty} y\tilde{f}(y)dy.$$

On the discrete jump size model, It is presented the differential equation of that process by the fallowing theorem :

**Theorem 4.5.** *[1] The call price $c(t, x)$ of the theorem 4.4 satisfies the equation*

$$\begin{aligned} &-rc(t,x) + c_t(t,x) + (r - \tilde{\beta}\tilde{\lambda})xc_x(t,x) + \tfrac{1}{2}\sigma^2 x^2 c_{xx}(t,x) \\ &+\tilde{\lambda}\left[\textstyle\sum_{m=1}^{M} \tilde{p}(y_m)c(t,(y_m+1)x) - c(t,x)\right] = 0 \ ,0 \leq t < T, x \geq 0 \end{aligned} \tag{4.18}$$

*With the Call characteristic boundary condition,*

$$c(T, x) = (x - K)^+, x \geq 0.$$

## 4.2 General concepts of the "Double-Exponentially Distributed" Jump-Diffusion model

In this section will be presented some concepts for the asset pricing model proposed by Kou [10], based on the "Double-Exponentially Distributed" Jump-Diffusion process. In the next chapter between the simulations we will provide this alternative model to the more used "Normally Distributed" Jump-Diffusion model for the asset. It will be fallowed the same steps in order to construct and calibrate the model under both kinds of Jumps, the "Normally Distributed" and the "Double-Exponentially Distributed" ones. Here will be not presented the analytic formula for the option price, only offering the equations which will be useful for the model construction. For a general study of its explicit deduction consult [10] and [11].

The generic differential equation for an asset guided by a "Double-Exponentially Distributed" Jump-Diffusion process can be represented by the fallowing [11] :

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t-)\sum_{j=u,d}(V^j_{N^j(\lambda^j t)} - 1)dN^j(\lambda^j t) \tag{4.19}$$

Where $\alpha$ is the drift, $\sigma$ is the volatility, $W(t)$ is the standard Wiener process, $N^j(\lambda^j)$ are the independent Poisson process with intensities up and down parameters $\lambda^u$ and $\lambda^d$, and $V^j$ is the jump magnitude. This last partition of the equation is the central point that differs from the original "Normally distributed" Jump-Diffusion model.

The up-jump magnitudes ($V^u$) has the fallowing density function :

$$f_{V^u}(x) = \left(\frac{\eta_u}{x^{\eta_u+1}}\right)$$

This distribution is called Pareto($\eta_u$) with parameter $\eta_u$, and providing $V^u \geq 1$. The mean and variance are fallowing :

$$E(V^u) = \frac{\eta_u}{\eta_u - 1}$$
$$var(V^u) = \frac{\eta_u}{(\eta_u - 2)(\eta_u - 1)^2}$$

These two equations will be used in the fallowing steps, at the calibration of the model. An useful characteristic of the Pareto distribution is its relation with the exponential distribution (also this characteristic is used in the model). The relation is, if $Y$ is exponetially distributed with intensity $\eta_u$, then $e^Y$ is Pareto-distributed with parameter $\eta_u$.

On the other hand the down-jump magnitudes($V^d$) are distributed Beta($\eta_d$,1) with density function:

$$f_{V^d}(x) = \eta_d x^{\eta_d - 1}$$

Where the jump sizes are nested in $0 < V^d < 1$. The mean and variance are:

$$E(V^d) = \frac{\eta_d}{\eta_d + 1}$$
$$var(V^d) = \frac{\eta_d}{(\eta_d + 2)(\eta_d + 1)^2}$$

All the jumps are assumed to be independent, the path has stationary and independent increments and is continuous in probability. As the "Normally distributed" case, the Doléans-Dade formula [3] provides an explicit solution for 4.19:

$$S(t) = S(0)exp\left\{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\} \prod_{j=u,d} V^j(N(\lambda^j t)) \tag{4.20}$$

We can set togheter the both up and down jumps in a unique density distribution. Let $\lambda = \lambda_u + \lambda_d$ the intensity of the jumps. Setting $p = \frac{\lambda_u}{\lambda}$, and $q = 1 - p$. The density function of the mixture of Pareto and Beta distribution is fallowing:

$$f_V(x) = p\frac{\eta_u}{x^{\eta_u+1}} I_{x>1}(x) + q\eta_d x^{\eta_d - 1} I_{0<x<1} \tag{4.21}$$

Where $\eta_u > 1$ , $\eta_d > 0$.

If we set $Y = ln(V)$, as have Ramezani and Zeng propose in [11], It can be noticed that the distribution of the logarithm of Pareto and Beta is exponential. Then,

$$f_Y(x) = p\eta_u e^{-\eta_u y} I_{y\geq 0} + q\eta_d e^{\eta_d y} I_{y<0} \tag{4.22}$$

Under a model with the probability distribution of the logarithm of the jumps given by 4.22 we can say we are dealing with a "Double Exponentially Distributed Jump-Diffusion process. This model has intensity $\lambda$ and $Y$ has an i.i.d. mixture distribution of Exponential($\eta_u$) with probability $p$ and Exponential($\eta_d$)with probability $q$.

# Chapter 5

# Simulations and Results

In this chapter we will perform some results comparing four methods to verify which one is more suitable to evaluate a specific European Call Option. On our study, we mark as reference the market price of an European Call Option VALEC38, traded on 17/11/2009, with strike price $K = 38$ and maturity on 15/03/2010.

$$C_{or} = R\$6,82.$$

Where "$R\$$" is the Brazilian currency "Real" ($2,40$ Reais $\approx 1,00$ Euro). The difference in time between the trade date and the maturity date is about $T = 4$ months.

The procedure that we will follow to reach the results will be:

- Present how the parameters were estimated from the historical data set.
- Chosen a distribution for the jumps, we will perform a formal analytic formula for the European Call, with underlying asset modeled by a jump-diffusion process.
- By a Monte Carlo simulation evaluate the same Call Option, with the underlying driven by a jump-diffusion process, using two methods.
- Using a Lognormal Brownian process model for the underlying asset, get the analytic price for the Call Option by Black-Scholes formula.

## 5.1 Parameters and the historical data

In this section, we will see firstly how was the procedure to calibrate the parameters for the jump-diffusion model, after we will see the case for a Lognormal Brownian model.

### 5.1.1 Jump-diffusion model for the underlying asset

On our system there are two main random processes which add stochasticity to the trajectory of the stock. The first one is the diffusion process, or Lognormal Brownian motion, and the second one is the jump process driven by a compensated compound Poisson process. Therefore, our model implies the stock guided by the equation 4.4 and so,

$$S(t) = S_0 exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\} \prod_{i=1}^{N(t)} (Y_i + 1) \tag{5.1}$$

Which is solution to the differential equation,

$$\begin{aligned} &dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t-)d(Q(t) - \beta\lambda t) \\ &S(0) = S_0 \end{aligned} \tag{5.2}$$

Where, in units per year:

1. $S(t)$ is the asset price at time $t$,
2. $\alpha$ is the drift of the diffusion process,
3. $\sigma$ is the standard deviation of the diffusion process,
4. $\beta$ is the mean of the jumps at the compensated compound Poisson process,
5. $\lambda$ is the intensity for the jumps.
6. $\tau$ is the period of time in years.

This generic model is independent of distributions of the jumps. On our model we work with a risk free probability and so the drift parameter will be the interest rate of a risk free asset. The Brasilian risk free interest rate can be given by Brazil's Selic Interest rate, which is the Interest rate paid by securities backed by the national treasury. Nowadays this interest rate is about $8,75\%$ per year, but our market is also influenced by foreign markets which have a lower one. The Black-Scholes model give us good results using a lower interest rate, so we standardize our stock to have a drift of $6,00\%$ per year.

As we will explain at the next section, Let us assume the jumps $Y_i$ have a normal distribution with mean $\beta$ and standard deviation $\delta$. All these parameters were evaluated using a daily data set of VALE5 stock price, traded on Brazilian stock market. The gotten data set is between the period from 2005 to 2009, totalizing 1091 work days, about $\tau = 4,3$ years, of daily returns.

The asset's price is observed in daily intervals, the price on the close of market. In finance we usually work with the daily returns, so if $S_i$ is the price at the end of $i$th interval, the return is,

$$u_i = ln\left(\frac{S_i}{S_{i-1}}\right).$$

The criterion to define when the asset jumps is generic and depends on each case and of the asset itself. In this study we calibrate the algorithm to understand like a jump when $u_i = ln\left(\frac{S_i}{S_{i-1}}\right) > 0,04$ therefore when the daily return goes more than 4% approximately. This choice gives us an intensity $\lambda = 30,95$ jumps per year, giving a good result for the Monte Carlo convergence as we will see at the next section.

Given the criterion to define a jump, we split the returns of the data set in two groups, the group one when the process is ruled by the lognormal diffusion process, and the group two where the process is ruled by the compound Poisson process, and so when it jumps.

Taking the group one data set we can estimate the standard deviation by the following formula,

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_{1i} - \bar{u_1})^2}$$

In the literature, see [8], we have many examples of choices for the size of the time series to get good numbers for stimation. Those suggested numbers are always from 180 to 360 returns. In our case we get a bigger sample than it, about 957 returns, with this we try to compensate the last year financial crisis, because it can influence the results in a wrong way. Therefore as we are nowadays in a time of recuperation of the economy, we prefer to enlarge the sample trying to compensate the instability of the market.

The stimation $s$ is, in fact, the stimation for the standard deviation in a given period. Hence, $s \approx \sigma\sqrt{\tau}$, where $\tau$ is the series period of time. It results,

$$\sigma \approx \frac{s}{\sqrt{\tau}} = 0,0176 \text{ per year}$$

For the group two, the group of jumps, we use the same procedure of stimation, with a sample size of 134 returns assumed to be normal distributed jumps with mean $\beta$ and standard deviation $\delta$. The mean $\beta$ was estimated by $\beta \approx -0,001$, in the algorithm it is assumed as $\beta \approx 0$. The standard deviation will be stimated by:

$$\delta_s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_{2i} - \bar{u_2})^2}$$

It results

$$\delta \approx \frac{\delta_s}{\sqrt{\tau}} = 0,064 \text{ per year}$$

The calibration of the system when the jumps are distributed in a "Double Exponential" density function will be explained togheter to the model itself, because we will need some specific equations.

### 5.1.2 Lognormal Brownian model for the underlying asset

We will use also the Black and Scholes model as a reference, and for this model, all the process is given by a lognormal Brownian trajectory. On that case the process will be distributed as :

$$lnS_T \cong \phi[lnS_0 + \left(\mu - \frac{\sigma_{BS}^2}{2}\right)T, \sigma_{BS}\sqrt{T}]$$

Where , $\phi[X, Y]$ is the normal distribution with mean $X$ and standard deviation $Y$. The first parameter $\mu$ is the drift of the lognormal process and will be, on the risk free probability, equal to $0,06$ per year as already signed. The standard deviation will be estimated, for period $\tau$:

$$\delta_{bs} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2}$$

It results

$$\sigma_{BS} \approx \frac{\delta_{bs}}{\sqrt{\tau}} = 0,028 \text{ per year}$$

These parameters will be used in the simulations.

## 5.2 Analytic formula for an European Call with the underlying asset driven by a jump-diffusion process

By the theorem 4.4 the European Call Option price is given by :

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^j (T-t)^j}{j!} \tilde{\mathbb{E}} k \left( T-t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j} (Y_i + 1) \right) \tag{5.3}$$

Merton [9] has proposed a model where the size of the jumps of the jump-diffusion process has a normal distribution. On this model the jumps are included on the formula as a correction to the standard deviation and to the drift of the original lognormal process, as we will see at the final formula.

As already signed the process $S(t)$ will be,

$$S(t) = S_0 exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\} \prod_{i=1}^{N(t)} (Y_i + 1) \tag{5.4}$$

and Call Option Price will be

$$C_{JD} = \sum_{n=0}^{\infty} \frac{e^{\breve{\lambda}\tau} (\breve{\lambda}\tau)^n}{n!} C_{BS}(S_0, K, r_n, \sigma_n^2, \tau) \tag{5.5}$$

Where:

- $C_{BS}$ is the usual price on the Black-Scholes model. $S_0$ is the initial price, or the today price of the asset at the market. We evaluated the expression in 24/11/2009 when the market value of the asset was $S_0 = R\$43.10$. $K = R\$38.00$ will be the strike price.

- $\tau$ is the period of time, on our case we will evaluate an Option in a period of 4 months, and so $\tau = \frac{1}{3}$ year.

- $\breve{\lambda} = \lambda(1+\beta)$ where $\lambda = 30.95$ per year is the intensity of the compensated compound Poisson process and $\beta = 0$ is its mean.

- $\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$ is the corrected standard deviation of merton's jump-diffusion model. The diffusion standard deviation is $\sigma = 0.017$ per year and the jump's size standard deviation is $\delta = 0.064$ per year.

- $r_n = r - \lambda\beta + \frac{nln(1+\beta)}{\tau}$ is the corrected drift on a risk free probability, where $r = 0.06$ per year.

On the implemention of the formula 5.5, we need to teminate the infinite sum at some point. The factorial term $n!$ grows in a much higher rate than any other, so on the algorithm we terminate the process when $n = 50$ because the term $\frac{e^{\check{\lambda}\tau}(\check{\lambda}\tau)^n}{n!} = 7.3983 \cdot e^{-77}$. The sum over this point can be ignored.

Finally the price for an European Call Option, for the analytic jump-diffusion model implemented by an algorithm with those parameters will be:

$$C_{JD} = R\$3.63$$

## 5.3 Monte Carlo simulation for Jump-Diffusion process

The main result of this work is the construction of Monte Carlo simulations to implement the jump-diffusion process properly. In order to construct that we will follow these steps:

1. Constructing the algorithm for the compensated compound Poisson process.
2. Constructing the algorithm for the lognormal Brownian diffusion process.
3. Constructing the jump-diffusion model, apply the Monte Carlo Method and calibrate the system. It will be done using the hypothesis of "normally distributed" jumps, and "double exponentially distributed" jumps.
4. Evaluate the price to the European Call Option to a given time interval.

The final models will be based on the original formula of a jump-diffusion process:

For the "Normally distributed" Jump-Diffusion model:

$$S(t) = S_0 exp\{\sigma W(t) + (\alpha - \beta\lambda - \frac{1}{2}\sigma^2)t\} \prod_{i=1}^{N(t)} (Y_i + 1)$$

And, for the "Double Exponentially Distributed" Jump-Diffusion model:

$$S(t) = S_0 exp\left\{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)\right\} \prod_{j=u,d} V^j(N(\lambda^j t)) \tag{5.6}$$

Actually, these models are the same, differing only in the jump size distribution.

### 5.3.1 The algorithm for the compensated compound Poisson process

At this point we will present an algorithm to evaluate the compensated compound Poisson process,

$$CM(t) = Q(t) - \beta\lambda t.$$

The basis for the algorithm is given by R.Cont and P.Tankov on [7] and the steps are:

1. Simulate a random variable N from a Poisson distribution with parameter $\lambda \cdot T$, where $\lambda = 30.95$ per year. $N$ gives the total number of jumps on the interval $[0, T]$. At the calibration time we have used a grid of 252 points and $T = 1$.

2. Simulate N independent random values, $U_i$, sorted uniformly distributed on the interval $[0, T]$. These variables correspond to the jumps' times.

3. Simulate the jumps' sizes: N independent random values $Y_i$ normally distributed with mean $\beta = 0$ and standard deviation $\delta = 0.064$. Each $Y_i$ will be the jump size at time $U_i$.

4. The trajectory is given by,

$$X(t) = \beta t + \sum_{i=1}^{N} 1_{U_i < t} Y_i.$$

An example of the path of a compensated compound Poisson process is in the picture 5.1.

### 5.3.2 The algorithm for the lognormal pure diffusion process

The lognormal Brownian process has the following formula,

$$S(t) = S_0 exp\{\sigma W(t) + (\mu - \frac{1}{2}\sigma^2)t\}.$$

Where W(t) is a Brownian motion with mean 0 and standard deviation 1. The basic algorithm to construct a lognormal pure diffusion process is a discretization of the lognormal process in many small times $\Delta t$, constructing the path step by step. It holds,

$$S(t+1) = S(t) \cdot exp\{\sigma\sqrt{\Delta t}\epsilon + (\mu - \frac{1}{2}\sigma^2)\Delta t\} \tag{5.7}$$

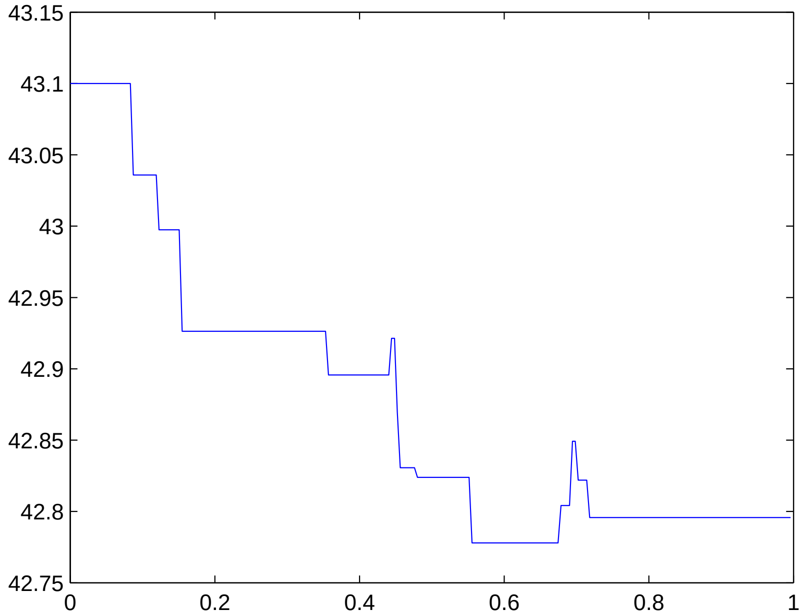


Figure 5.1: Typical trajectory of a compound Poisson process.

Where $\epsilon$ is a Normally distributed random variable with mean 0 and standard deviation 1. A typical trajectory of a simulated lognormal process is given by 5.2

### 5.3.3 The "Normally distributed" Jump-Diffusion model contruction: Monte Carlo simulation and calibration of the sytem

On this algorithm the Jump-Diffusion model is constructed with its two components simulated separately, i.e. the lognormal component, based in a diffusion process guided by a Brownian motion process, and the jump component of a compound Poisson process with the jumps fallowing a Normal density function. Utilizing the independence hypothesis, the algorithm contructs them separately and locates them togheter step by step on the simulation.

Remembering the basic differential equation 4.3 for a Jump-Diffusion process is:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t-)d(Q(t) - \beta\lambda t) \tag{5.8}$$

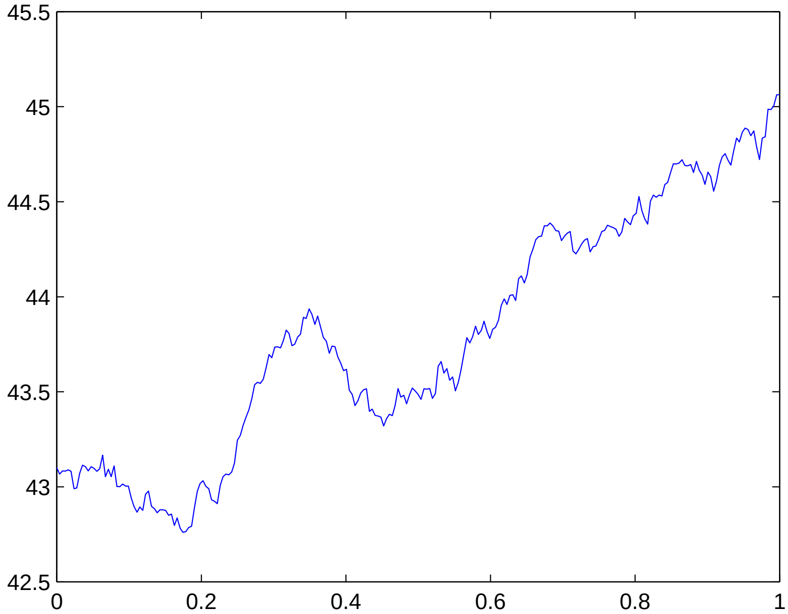


Figure 5.2: Typical trajectory of a simulated lognormal process

Here it holds some assumptions:

1. We work with the values in a fixed time grid, the number of points in the grid will be given by the number N.

2. The compound Poisson process and the Brownian motion are independent.

3. In each step of time can exist only one jump.

4. The jump influence the process proportionally, hence if at the time $t$ there is jump of size +0.2 the process grows 20%.

Initially, the algorithm is such that, with initial value $S_0$ :

1. If at the time interval $\Delta t$ there is a jump of size $Y_i$, with "normal" distribution:

$$S(t+1) = S(t)exp\left\{\sigma w(t) + (\mu - \beta\lambda - \frac{1}{2}\sigma^2)\Delta t\right\}(Y_i + 1) \tag{5.9}$$

If there is no jump at that time interval:

$$S(t+1) = S(t)exp\left\{\sigma w(t) + (\mu - \beta\lambda - \frac{1}{2}\sigma^2)\Delta t\right\} \tag{5.10}$$

2. We work with that algorithm until the evalution of the final value of the asset $S(N)$. At this point It's priced the payoff for an European Call Option of the asset, given by:

   if $(S(N) - K < 0)$ payoff= 0.

   else payoff= $S(N) - K$.

   Where $K$ is the strike price.

3. Finally we evaluate the price of the call by the risk free argument, and so we put that payoff at the present time, in this way :

   $$C_{JD} = exp(-\mu * T) * \text{payoff}.$$

   Where $\mu$ is the risk free interest rate. On our algorithm $\mu = 0.06$ per year.

An example of a simulation of a "Normally Distributed" Jump-Diffusion process is given on the figure 5.3.

Now, we have the price for one simulation for an European Call in a Jump-Diffusion model. The Monte Carlo method is nothing else than simulate many times a process and get the mean of the prices as the value of the official price for the European Call. In our case we achieve good results to the system with 100 simulations. An example of a Jump-Diffusion Monte Carlo simulation is on the figure 5.4.

Some observations about Monte Carlo convergence were observed during the simulations:

- The error decreases with increasing number of paths.
- The error is smaller if the variance of the jump-size is smaller.
- If the jump intensity ($\lambda$) is high, then the error decreases at slower rate with increasing number of paths.

To calibrate the system we used the simulation with $T = 1$, comparing the price result with the Black-Scholes reference formula, and then such that $R\$7.21$ and the parameters given by the data set.

At our model, the parameter that offer a more generic calibration is the intensity $\lambda$ of the compound Poisson process. The question is, "when the

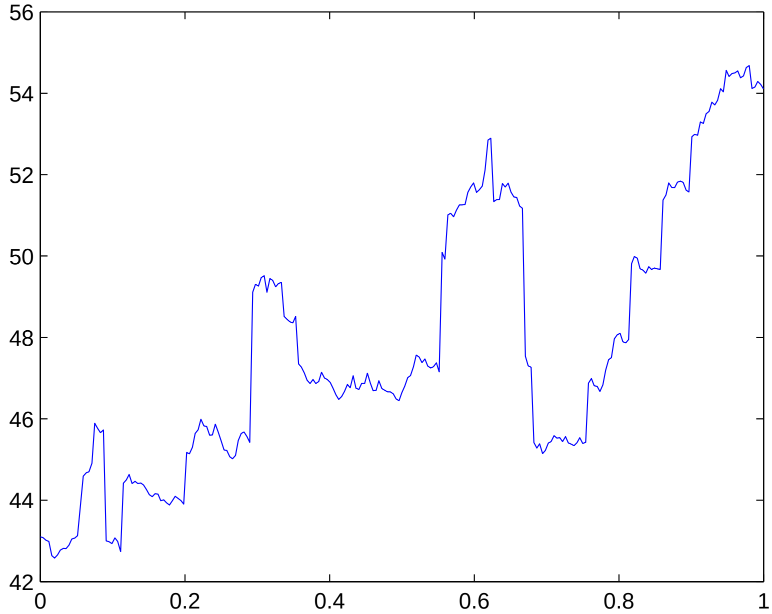


Figure 5.3: Typical trajectory of a "Normally Distributed" Jump-Diffusion process

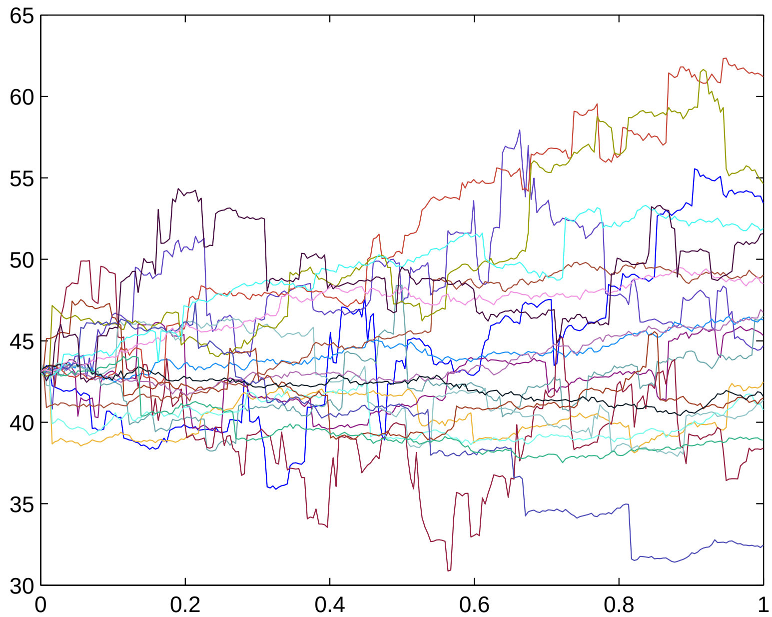


Figure 5.4: Typical trajectory of a simulated jump-diffusion with the Monte Carlo method

process jumps?". In order to answer this question we made a plenty of simulations, changing the intensity by changing the criterion to define when the process jumps from the data set, remembering:

$$u_i = ln\left(\frac{S_i}{S_{i-1}}\right)$$

So when $u_i$ is greater than a value, $V_1 = 0.032$ for example, i.e the process jumps more than 3% of its value, we staked that It jumped, and for this value on our data set we have the intensity $\lambda_1 = 43.3$ jumps per year. Changing the value $V_i$ we have others intensities $\lambda_i$. The calibration of the system for the period of one year give us the result that for $V_i = 0.04$ and the intensity $\lambda_i = 30.95$ jumps per year are good choices when we compare to the Black-Scholes reference price.

Finally, we simulate the Jump-Diffusion process with Monte Carlo method to evaluate an European Call, with strike price $K = 38$, initial price $S_0 = 43,1$ , $T = \frac{1}{3}$ (4 months), number of points at the grid $N = 252/3 = 84$, with 100 Monte Carlo simulations, and parameters gotten from the data set of the stock, as specified on section 5.1. We got the mean price:

$$C_{NJDMT} = R\$6.52$$

Assuming a normal distribution for the Call final price, we can also give a confidence interval with 95% of significance to the mean price:

$$IC_{NJDMT} = [5.67 \ 7.38]$$

The simulation of the process is given on the figure 5.5.

### 5.3.4 The "Double Exponentially Distributed" Jump-Diffusion model contruction: Monte Carlo simulation and calibration of the sytem

In the same way of the last section, here we will evaluate the European Call Option in the hyphotesis that the asset fallows a "Double Exponentially Distributed" Jump-Diffusion Process. Here, the model will be based on two building blocks.

Firstly the lognormal component, based in a diffusion process guided by a Brownian motion process, and after the jump component with intensity given by a compound Poisson process with jumps fallowing a "Double Exponential" density function. The algorithm constructs the components separately.

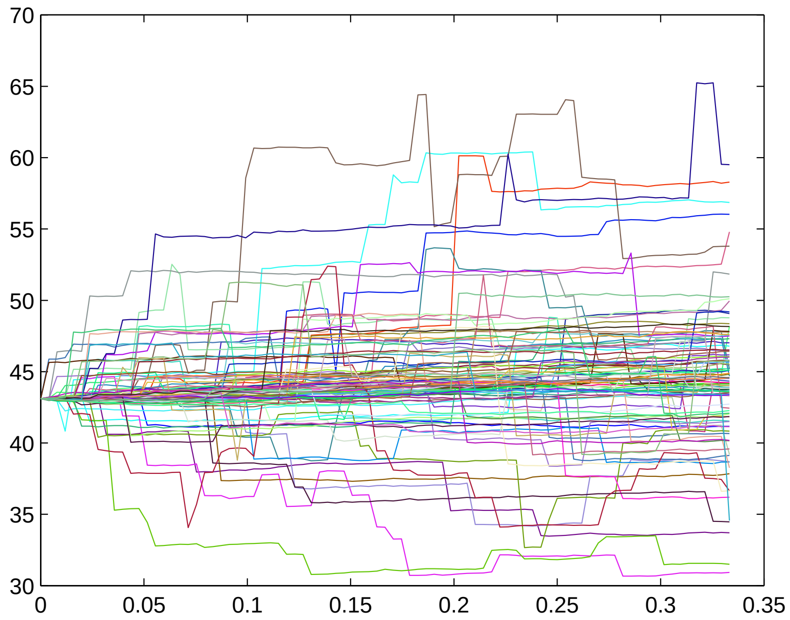


Figure 5.5: Final Monte Carlo simulation on a "Normally Distributed" Jump-Diffusion model for the asset

An asset modeled as a "Double Exponentially Distributed" Jump-Diffusion model is given by the equation 4.19:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) + S(t-)\sum_{j=u,d}(V^j_{N^j(\lambda^j t)} - 1)dN^j(\lambda^j t) \tag{5.11}$$

Which is quite similar to the "normally" distributed version of the equation. The assumptions to the grid of time, independence of the components, rate and influence of jumps in the path of the asset still holds for this model.

In the same idea of the last algorithm we construct the path of the asset step by step with the characteristic function given by 4.20, with initial value $S_0$ in this way:

1. If at the time interval $\Delta t$ there is a jump of size $V^j$, fallowing the "Double-Exponential" distribuition equation 4.21:

$$S(t+1) = S(t)exp\left\{\sigma w(t) + (\alpha - \frac{1}{2}\sigma^2)\Delta t\right\}(V^j) \tag{5.12}$$

   Here something particular is the kind of jumping we are dealing:

   - The asset has a $p$ probability of jumping up, and in this case it fallows the density function Pareto($\eta_u$) of the equation $f_{V^u}(x) = \left(\frac{\eta_u}{x^{\eta_u+1}}\right)$
   - Else, the asset has a $q = 1-p$ probability of jumping down, and in this case it fallows the density function Beta($\eta_d$,1) of the equation $f_{V^d}(x) = \eta_d x^{\eta_d - 1}$.

   Hence, if there is a jump in this time interval, the algorithm chooses randomly if it goes up or down, and apply the correct formula. This way we are simulating the mixture density function 4.21.

   If there is no jump at that time interval:

$$S(t+1) = S(t)exp\left\{\sigma w(t) + (\alpha - \frac{1}{2}\sigma^2)\Delta t\right\} \tag{5.13}$$

2. We work with that algorithm until the evalution of the final value of the asset $S(N)$. At this point It's priced the payoff of the European Call Option of the asset for this model, given by:

   if $(S(N) - K < 0)$ payoff$= 0$.

   else payoff$= S(N) - K$.

   Where $K$ is the strike price.

3. Finally we evaluate the price of the Call with the risk-free argument, and so we put that payoff at the present time, in this way :

$$C_{JD} = exp(-\alpha * T) * \text{payoff}.$$

Where $\alpha$ is the risk-free interest rate. On our algorithm, as already signed $\alpha = 0.06$ per year.

An example of a simulation of a "Double Exponentially Distributed" Jump-Diffusion process is given on the figure 5.3.

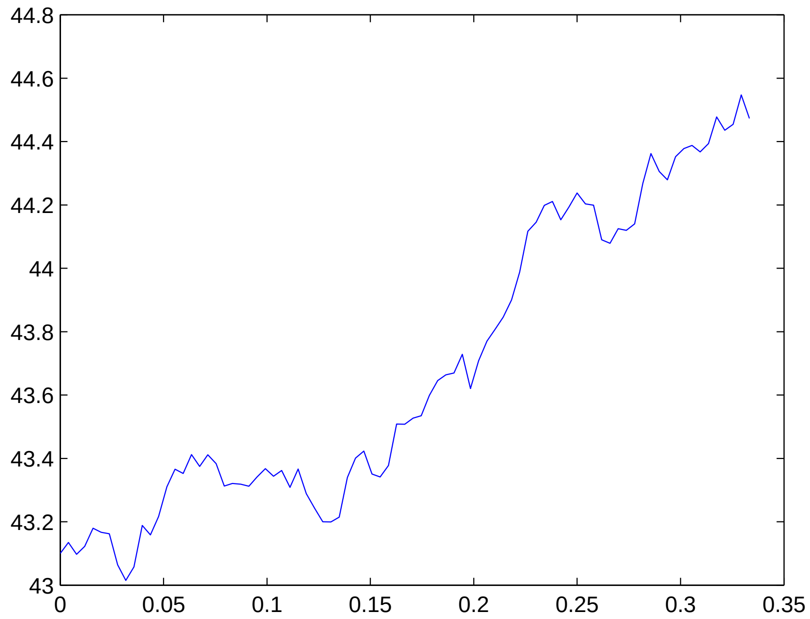


Figure 5.6: Typical trajectory of a simulated "Double Exponentially Distributed" Jump-Diffusion process

At this figure we can notice that the path of the asset in the "double exponential" simulation case has a smoother pattern than the "normal" case. This is a characteristic of the "Double-Exponential" distribution, and can be controled by the parameters $\eta_u$ and $\eta_d$.

Up to now, we presented the algorithm that provides the price for an European Call in the Jump-Diffusion model. Then we can start with the process of the Monte Carlo simulation, which ,as already was explained, is nothing else than simulating many times a process and get the mean of the prices as the value of the official price for the European Call. In our case we calibrate the system to do 100 simulations.

Some observations about Monte Carlo convergence for these kind of process were observed during the simulations:

- The rate of convergence increase with the namber of simulations.

- It's quite sensible the influence of the change in the intensity of the ups ($\lambda_u$) and downs ($\lambda_d$) jumps, in both the path and the final value.

The criteria of calibration of the system for the "Double Exponentially Distributed" Jump-Diffusion process goes in the same way of the "normal" case. In order to calibrate the system we have this set of parameters: $\theta = (\alpha, \sigma, \lambda_u, \lambda_d, \eta_u, \eta_d)$. Remembering that the probability of the up jumps is given by $p = \frac{\lambda_u}{\lambda_u + \lambda_d}$ and so for the down jumps is $q = 1 - p$.

Using the Black-Scholes model as reference in the period of one year, and the same data set already presented, the calibration of the system is set using the simulation with $T = 1$, comparing the price result with the Black-Scholes reference formula, and then such that $R\$7.21$. Again for this model the problem is to define when we can consider, in the data set, that between two days the asset jumped. Defining the intensities of the jumps, $\lambda_u$ and $\lambda_d$, becomes easy to define the other parameters.

In order to set up these two parameters, It's used the same procedure of the last section, therefore we reach to the intensities by changing the criterion that defines when the process jumps from the data set, always regarding the reference price given by the Black-Sholes model. The procedure is to looking for returns up to $V_i = 0.04$ in absolute value. With this criterion we are able to separate between these returns the up jumps from the down jumps. With these jumps, we can finally stimate the intensities of the jumps, i.e. $\lambda_u = 15.23$ and $\lambda_d = 15.69$.

The other parameters that we need to estimate are the probability densities $\eta_u$ from the Pareto($\eta_u$)distribution and the $\eta_d$ from the Beta($\eta_d$,1)distribution. As we have already set the jumps up and jumps down, we can use their own equations for the mean and variance on the historical data:

$$E(V^u) = \frac{\eta_u}{\eta_u - 1}$$
$$var(V^u) = \frac{\eta_u}{(\eta_u - 2)(\eta_u - 1)^2}$$

and,

$$E(V^d) = \frac{\eta_d}{\eta_d + 1}$$
$$var(V^d) = \frac{\eta_d}{(\eta_d + 2)(\eta_d + 1)^2}$$

And solving these systems the parameters are stimated, i.e. $\eta_u = 17.35$ and $\eta_d = 16.35$.

So we can present our calibrated system :

$$\theta = (\alpha = 0.06, \sigma = 0.018, \lambda_u = 15.23, \lambda_d = 15.69, \eta_u = 17.35, \eta_d = 16.35)$$

Where the diffusion parameters are stimated from the data set as we have already explained.

Finally, we simulate the process by Monte Carlo method to evaluate an European Call in a "Double Exponentially Distributed" Jump-Diffusion model, with strike price $K = 38$, initial price $S_0 = 43,1$ , $T = \frac{1}{3}$ (4 months), number of points at the grid $N = 252/3 = 84$, with 100 simulations. We got the mean price:

$$C_{DEJDMT} = R\$6.40$$

Assuming a normal distribution for the Call Price, we can also give a confidence interval with 95% of significance for the mean price:

$$IC_{DEJDMT} = [5.02 \ 7.79]$$

As we can see, the "Double-Exponential" model offer a larger confidence interval for the mean price than the "normal" Jump-Diffusion model. This characteristic will be retake on the conclusion.

The simulation of the process is given on the figure 5.7

## 5.4 The Black-Scholes model as reference

In this section we will use the classical model for price stock options, the Black-Scholes model. It assumes the percentage changes in the stock price for short periods, in our case one day, are lognormally distributed. The stock price process will be given by

$$dS = \mu S dt + \sigma S dz \tag{5.14}$$

Where $z$ is a Brownian Motion, $\mu$ is the Expected return on stock per year, and $\sigma$ is the volatility of the stock price per year. Both of them were stimated by a stock historical series. If $f$ is the price of an European Call Option, it has the differential equation, called Black-Scholes-Merton differential equation,

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \tag{5.15}$$

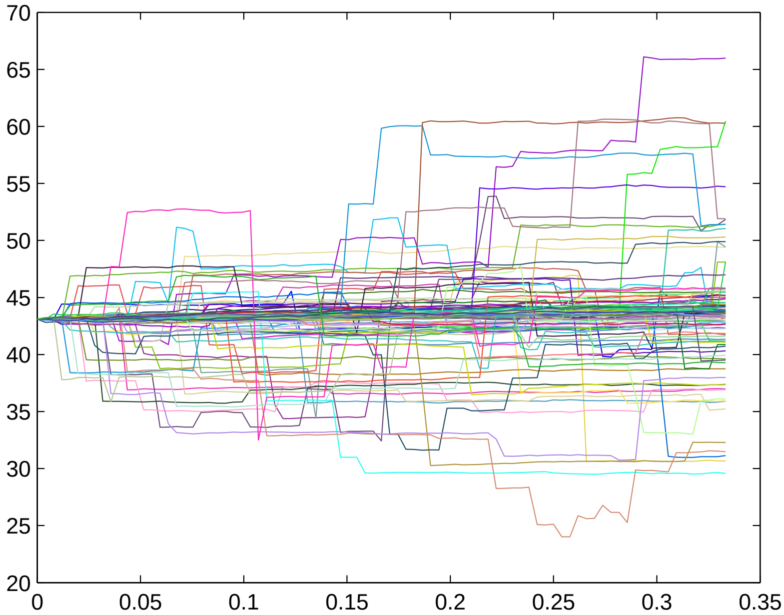


Figure 5.7: Final Monte Carlo simulation on a "Double Exponentially Distributed" Jump-Diffusion model for the asset

Where $r$ is the risk-free expected return stock.For the European Call Option, the differential equation has the boundary condition:

$$f = max(S - K, 0) \text{ when t=T} \tag{5.16}$$

As was already done for pricing a stock modeled by a Jump-Diffusion process, we will use the risk-free measure to price our option. So in practice we will do the following procedure [8]:

1. Assume that the expected return from the underlying asset is the risk-free interest rate, r (i.e. assume $\mu$ = r)
2. Calculate the expected payoff from the derivative
3. Discount the expected payoff at the risk-free interest rate

Finally we can perform the well known Black-Scholes formula for price at time 0 an European Call Option,

$$C_{BS} = S_0 N(d_1) - K e^{-rT} N(d_2) \tag{5.17}$$

where,

$$\begin{aligned} d_1 &= \tfrac{ln(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= \tfrac{ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{aligned} \tag{5.18}$$

The function $N(x)$ is the cumulative probability distribution function for a standardized normal distribution. On our simulation, we use a stock price with the parameters: $S_0 = 43$, $K = 38$, $r = 0.06$ per year, $\sigma = 0.03$ per year, and $T = 1/3$ of a year. Applying these parameter to the equation 5.17 we have,

$$C_{BS} = 43N(d_1) - 38e^{-0.06*\frac{1}{3}} N(d_2)$$

where,

$$\begin{aligned} d_1 &= \tfrac{ln(43/38)+(0.06+(0.03)^2/2)*\frac{1}{3}}{0.03\sqrt{\frac{1}{3}}} \\ d_2 &= \tfrac{ln(43/38)+(0.06-0.03^2/2)*\frac{1}{3}}{0.03\sqrt{\frac{1}{3}}} \end{aligned}$$

And It results,

$$C_{BS} = R\$5.75.$$

# Chapter 6

# Conclusions

When one evaluate the price of an European Call by a model, or any other stock product, the first question that comes in our mind is how close this solution is to the market prices. In fact, do the traders really use these models to determining a price for an option?

The answer is yes. Besides the fact that the Black-Scholes model is the most used model, traders use it in a different way in respect to the manner that Black and Scholes orinally presented it. Actually, the Black-Scholes model have some problems with its lognormal Brownian hypothesis model for drivig the trajectory of an asset. There are many ways to solve the peakness problems, and the heavy tails problems of a lognormal distribution, the most aplied are the volatility smile, and stochastic volatility, for this see [8] and [1].

This work tried to evaluate another Levy process model, the Jump-Diffusion model, which has a continuous lognormal characteristic with some sporadic jumps on its path. Introducing these jumps, some problems of the lognormal Brownian motion are solved [7].

So, this work, first of all brought into focus the problem of finding an analytic formula of an European Call Option price. Once arrived to it, we implemented some algorithms to evaluate this Call option in a Jump-Diffusion view, using a real data set to parameterize them. In this work We used four different models to price the option, futher on the market price itself, and it enables us to compare them. Here It's summarized the results,

| Model | Price | Market difference |
|---|---|---|
| Market price | $R\$6,82$ | 0% |
| Analytic Jump-Diffusion | $R\$3,63$ | 47% |
| Monte Carlo "Normally Distributed" Jump-Diffusion | $R\$6,52$ | 4% |
| Monte Carlo "Double-Exponentially Distributed" Jump-Diffusion | $R\$6,40$ | 6.1% |
| Black-Scholes | $R\$5,75$ | 16% |

These results come from the same data set of parameters, differening between them only on the proper model characteristics. The data set was taken from real values from the market, the stock VALE5, an ordinary stock from one of the biggest companies in Brazil.

Talking about the results we have seen that between the models, the Monte Carlo simulations for Jump-Diffusion process are which best suit with the market value, with a difference of 4% for the "Normally Distributed" Jump-Diffusion process from the market value, and 6.1% for the "Double Exponentially Distributed" version. Here we can compare the results for the both models, observed under the model contruction and simulation:

| **Subject** | **"Normal" jumps** | **"Double Exponential" jumps** |
|---|---|---|
| Final price | $R\$6,52$ | $6,40\%$ |
| Market difference | 4% | 6.1% |
| Confidence Interval | [5.67 7.38] | [5.02 7.79] |
| Range of confidence Interval | Strait | Large |
| Converge of the model | Faster | Slower |
| Control of simulations | The same probability to up and down jumps | The distribution of the up and down jumps can be set |
| Number of parameters to calibrate | Less parameters(5) | More parameters(6) |

Comparing the results we can see that if we assume a normal distribution for the Monte Carlo prices, we have a confidence interval of $IC_{NJDMT} =$ [5.67 7.38] for the "Normal" version and $IC_{DEJDMT} =$ [5.02 7.79] for the "Double Exponential" , with 95% of significance. For both models It's observed that the market price is in the confidences intervals of the means, despite the fact that the "normal version" has a straiter interval. This is caused mainly because of the rate of convergence characteristic of the model, faster to the "normal version" as observed into the simulations.

An advantage of the "Double Exponential" version is the control of the ups and downs jumps, selecting in the right way the parameters we are free to calibrate an asset of a company as we expect. For example, if the company is passing for a good financial situation, we can set the model to have more up jumps than down jumps, of course always based in the historical data. This kind of control is impossible in a "normal" version of the jump-diffusion model. By the way, this advantage turns to a disadvantage if we think in the number of parameters, more in the "double exponential" version. This greater number of parameters brings to the model more parameters that can be set mistakenly, and so the precision of the model could be more doubtful.

The both Jump-Diffusion methods are also good because they can be easily calibrated, and here differing totally from the analytic methods as the Black-Sholes and the analytic formula for Jump-Diffusion process. It means that when you change some parameter value, It can be seen directly the change on the simulated path, or in the final value. On analytic methods the calibration is more generic because when there is a change on the parameters, the only value that can be evaluated in this choice is the final value,i.e the formula works like a Black-Box where one cannot evaluate how the change affects the trajectory of the underlying.

The analytic formula for the European Call Option with jump-diffusion underlying works with many parameters, and there is a natural difficult in order to calibrate all of them at the same time. We have saw during the simulations that It influences too much the final value of the product. Hence, the formula is too much sensitive to the set of parameters, and it's hard to see how much each one of the parameters interfere to the final price.

Another problem of the final formula, and this one was stated during the simulations, is the fact that the exponential term penalizes too strongly the final value, and maybe this term needs some kind of correction. So, we have seen that analytic formula for jump-diffusion process didn't sufficiently suit with the market value, with a difference price of 47%.

The Black-Scholes formula was used like a reference model in all the work, and as we saw it evalutes very well the market price of this European Call Option, with just 16% of difference. Maybe it occurred because, with some corrections, the Black-Scholes model is the most used model to evaluate options in the market, therefore maybe it suits with the market value actually because is the benchmark model for all the market.

Finally, we can conclude that the models which best suit with the market value for this specific European Call Option are the Monte Carlo Jump-Diffusion models, after the second best one is the Black-Scholes model, and the last is the analytic "Jump-Diffusion" model which isn't a good manner to evaluate the price of this product.

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# Appendix A

# Algorithms

Here It's present the algorithms used to model the European Call. They are sorted this way:

1. The Black-Sholes Formula
2. The analytic *Jump-Diffusion* function
3. The Monte-Carlo "Normally Distributed" *Jump-Diffusion* process model
4. The "Normally Distributed" compound Poisson process model
5. The Monte-Carlo "Double Exponentially Distributed" *Jump-Diffusion* process model
6. The "Double Exponentially Distributed" compound Poisson process model

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%(1)Black-Scholes formula %%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function price=callPrice(S0,K,miu,sig,T)

    d1=(log(S0/K)+(miu+0.5*(sig^2))*T)/(sig*sqrt(T));
    d2= d1 - sig*sqrt(T);
    price=S0*cdf('Normal',d1,0,1) - K*exp(-miu*T)*(cdf('Normal',d2,0,1));
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%(2)Analytic Jump-Diffusion Function %%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%



function pricejdp=jdpanal()

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%Compound Poisson Data%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%intensity
lamb = 30.95;

%average
k = 0;

%standard deviation
stdjump = 0.064;

%%%%%%%%%%%%%%%%%%%
%%%Diffusion Data%%%%%%%%%
%%%%%%%%%%%%%%%%%%%

% Diffusion variance a.a.(stddev)
sig = 0.0176;

%mi a.a.
miu = 0.06;

%Initial value
S0 = 43.1;

%strike
K = 38;

%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%Generic Data%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
T = 1/3;

%%%%%%%%%%%%%%%%%%
%%%%%%%%PRICING%%%%%%
%%%%%%%%%%%%%%%%%%

lambl= lamb*(1+k);
price1 = 0;
for i=1:50
    sign = sqrt(sig^2 + (i*stdjump^2)/T);
    rn = miu - lamb*k + i*log(1 + k)/T;
    soma = 0;
    for j=1:i
        soma = soma + log(i);
    end
    expo = exp(-lambl*T + i*log(lambl*T)-soma)
    pricecall=callPrice(S0,K,rn,sign,T)
    expo*pricecall
    price1 = price1 + expo*pricecall;
end

pricejdp = price1;



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(3)Monte Carlo normal Jump-Diffusion Function %%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function price=jdp()


%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%Generic Data%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%


simnum = 100;
K = 38;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%Compound Poisson Data%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%


%So
S0=43.1;

%T
T=1/3;


%lambda
lamb = 30.95;


% Grid
N = round(252/3);


%average
beta = 0;

%standard deviation
stjump = 0.064;


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%Diffusion Data%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%



%stddev da difusão a.a.
sig = 0.0176;


%mi a.a. miu = 0.06;
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%Simulating%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

dt = T/N;
soma2=0;
x=1;

%Constructing the grid
for j=1:(N + 1)
    temp(j)= soma2;
    soma2 = soma2 + dt;
end
for j=1:simnum
    x=1;
    z=0;
    sumtime = 0;
    S(1)=S0;
    [caro]=cpp2(S0,T,lamb,stjump,N)
    Tk=caro(:,1);
    Yk=caro(:,2);
    NTk = length(Tk);
    for i=1:N
        sumtime= sumtime + dt;
        eps=randn(1);
        v=miu-(sig^2)/2;
        if (sumtime > Tk(x)) & (x ~= NTk)
            z = z + 1;
            S(i+1)=S(i)*exp(v*dt+sig*(dt^(1/2))*eps)*(Yk(x)+1);
            x=x+1;
        else
            S(i+1)=S(i)*exp(v*dt+sig*(dt^(1/2))*eps);
        end
    end
    if(S(N+1)-K<0)
        payoff(j)=0;
    else
        payoff(j)=S(N+1)-K;
    end
    if (j==1)
        figure('Name','JUMP DIFFUSION PROCESS');
```

```
        plot(temp',S');
    else
        line(temp',S','color',[rand rand rand]);
    end
end
trag=[temp',S'];
cp =[Tk,Yk'];

% Confidence Interval 95%
price2=exp(-miu*T)*payoff;
media = mean(price2)
stdev = std(price2)
Icinf= media -1.96*stdev/sqrt(simnum);
Icsup= media +1.96*stdev/sqrt(simnum);
priceIC = [Icinf Icsup]
price=mean(exp(-miu*T)*payoff);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%(4)Compound Poisson Process normal model%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [caro]=cpp2(X0,T,lamb,stddev,pontos)
N= round(exprnd(lamb*T));
if (N==0) N=1; end
Tk = sort(rand(N,1));
Yk = randn(N,1)*stddev;
i=0;
st = T / pontos ;
soma = 0;
for j=1:pontos
    temp(j)= soma;
    soma = soma + st;
end
sumtime = 0;
Xk(1)=X0;X=X0;
x = 1;
for i = 2:pontos
    sumtime = sumtime + st;
    if (sumtime > Tk(x)) & (x ~= N)
        X = X + Yk(x);
```

```
        x = x + 1;
    end
    Xk(i) = X;
end
caro = [Tk,Yk];


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%(5)Double Exponential Jump-Diffusion Function %%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%


function price=dbexpfinal()


%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%Generic Data%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%

simnum = 100;
K = 38;


%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%Compound Poisson Data%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%So
S0=43.1;
%T
T=1/3;
%eta
etaup = 17.35;
etadow = 16.35;
%lambda
lambup = 15.23;
lambdow = 15.69;
% grid
N = round(252/3);
%probabilities
probpos = lambup/(lambup+lambdow);
```

```
probneg = lambdow/(lambup+lambdow);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%Diffusion Data%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%stddev da difusão a.a.
sig = 0.0176;
%mi a.a.
miu = 0.06;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%Simulating%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

dt = T/N;
soma2=0;
x=1;

%%%%%%%Constructing the grid
for j=1:(N + 1)
    temp(j)= soma2;
    soma2 = soma2 + dt;
end
for j=1:simnum
    x=1;
    z=0;
    sumtime = 0;
    S(1)=S0;
    [caro]=cdexp(S0,T,lambdow,lambup,etaup,etadow,N,probpos);
    Tk=caro(:,1);
    Yk=caro(:,2);
    NTk = length(Tk);
    for i=1:N
        sumtime= sumtime + dt;
        eps=randn(1);
        v=miu-(sig^2)/2;
        if (sumtime > Tk(x)) & (x ~= NTk)
            z = z + 1;
        S(i+1)=S(i)*exp(v*dt+sig*(dt^(1/2))*eps)*(Yk(x));
        x=x+1;
```

```
        else
        S(i+1)=S(i)*exp(v*dt+sig*(dt^(1/2))*eps);
        end
    end
    if(S(N+1)-K<0)
        payoff(j)=0;
    else
        payoff(j)=S(N+1)-K;
    end
    if (j==1)
        figure('Name','DOUBLE EXPONENTIAL JUMP DIFFUSION PRO-
CESS');
        plot(temp',S');
    else
        line(temp',S','color',[rand rand rand]);
    end
end
trag=[temp',S'];
cp =[Tk,Yk'];
% Confidence Interval of 95%
price2=exp(-miu*T)*payoff;
media = mean(price2)
stdev = std(price2)
Icinf= media -1.96*stdev/sqrt(simnum);
Icsup= media +1.96*stdev/sqrt(simnum);
priceIC = [Icinf Icsup]
price=mean(exp(-miu*T)*payoff);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%(6)Compound Poisson Process double exponential model%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function [caro]=cdexp(X0,T,lambdo,lambup,etaup,etadow,pontos,probp)
lamb=lambdo+lambup;
N= round(exprnd(lamb*T));
if (N==0) N=1; end
Tk = sort(rand(N,1));

for j=1:N
    if (rand(1)<probp)
        Yk(j)=exp(exprnd(1/etaup));
```

```
    else
        Yk(j)=betarnd(etadow,1);
    end
end

i=0;
st = T / pontos ;
soma = 0;
for j=1:pontos
    temp(j)= soma;
    soma = soma + st;
end
sumtime = 0;
Xk(1)=X0; X=X0;
x = 1;
for i = 2:pontos
    sumtime = sumtime + st;
    if (sumtime > Tk(x)) & (x ~= N)
        X = X + Yk(x);
        x = x + 1;
    end
    Xk(i) = X;
end

[Tk,Yk']
caro = [Tk,Yk'];


%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%END%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```