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Mathematical analysis of a simplified Ericksen-Leslie model for nematic liquid crystal flows

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Sintesi del lavoro

Oggetto di questo lavoro di tesi è lo studio di un modello di Ericksen-Leslie semplificato che descrive il flusso di un cristallo liquido nematico. Più precisamente il sistema che è stato analizzato è il seguente:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = -\nabla\mathbf{d}^t\Delta\mathbf{d} + \mathbf{g}(t) & \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty); \\ \partial_t\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d} = \Delta\mathbf{d} - \mathbf{f}(\mathbf{d}) & \\ |\mathbf{d}| \le 1 & \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0, \quad \mathbf{d}(\mathbf{x},0) = \mathbf{d}_0 & \text{per } \mathbf{x}\in\Omega; \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0}, \quad \mathbf{d}(\mathbf{x},t) = \mathbf{h}(\mathbf{x},t) & \text{su } \partial\Omega\times(0,\infty). \end{cases} \tag{0.1}$$

dove $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, $\mathbf{f}(\mathbf{d}) = \frac{1}{\epsilon^2}(|\mathbf{d}|^2 - 1)\mathbf{d}$. Il vettore $\mathbf{u}$ rappresenta il campo di velocità incognito, p il campo di pressione mentre $\mathbf{d}$ è il parametro di ordine che descrive l'orientamento locale delle molecole del fluido. I cristalli liquidi nematici, infatti, possono essere considerati, almeno in prima approssimazione, come costituiti da molecole che presentano una forma "a bastoncino" e il vettore $\mathbf{d}$ rappresenta esattamente il loro asse di simmetria.

Notiamo che il sistema (0.1) può essere dedotto dalle equazioni di bilancio della massa, della quantità di moto e del momento angolare applicate ad un continuo dotato di microstruttura e descritto dalla variabile interna aggiuntiva $\mathbf{d}$ (vd. capitolo 1). Le equazioni che danno luogo al sistema considerato in questa tesi sono dedotte in modo da essere oggettive (cioè compatibili con cambi inerziali del sistema di riferimento) e compatibili con principi di dissipatività consistenti con l'esperienza fisica. Osserviamo che, dal punto di vista fisico, il parametro d'ordine $\mathbf{d}$ dovrebbe essere un versore, ma per l'analisi matematica viene spesso preso in considerazione un rilassamento di questo vincolo imponendo solo $|\mathbf{d}| \le 1$ e penalizzando eventuali parametri d'ordine più piccoli dell'unità tramite un potenziale di tipo Ginzburg-Landau $\mathcal{F}(\mathbf{d}) \doteq \frac{1}{4\epsilon^2}(|\mathbf{d}|^2 - 1)^2$.

L'analisi di buona positura Si può dimostrare che il problema (0.1) è ben posto nel senso qui di seguito descritto (vd. capitolo 2). Siano $\mathbf{L}^2$ e $\mathbf{H}^1$ gli usuali spazi funzionali costituiti da funzioni a valori vettoriali in L^2 e H^1 rispettivamente e siano $\mathbf{H}$ e $\mathbf{V}$ gli spazi vettoriali comunemente utilizzati per l'analisi delle equazioni di Navier-Stokes e definiti come segue:

$$\mathbf{H} = \overline{\left\{\mathbf{u} \in \mathbf{C}_0^\infty \mid \nabla \cdot \mathbf{u} = 0\right\}}^{\mathbf{L}^2}$$

e

$$\mathbf{V} = \overline{\left\{\mathbf{u} \in \mathbf{C}_0^\infty \mid \nabla \cdot \mathbf{u} = 0\right\}}^{\mathbf{H}_0^1}$$

La buona positura del problema può essere espressa in termini delle seguenti nozioni di soluzione.

Definizione. Sia $T > 0$. Una coppia $(\mathbf{u}, \mathbf{d})$ è una *soluzione debole* per il problema (0.1) se $(\mathbf{u}, \mathbf{d}) \in L^2(0, T; \mathbf{V} \times \mathbf{H}^2)$, $(\partial_t \mathbf{u}, \partial_t \mathbf{d}) \in L^p(0, T; \mathbf{V}^*) \times L^2(0, T; \mathbf{L}^2)$ (con $p = 2$ se $n = 2$ e $p = 4/3$ se $n = 3$), , $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ in $\mathbf{L}^2$ e $\mathbf{d}(\mathbf{x}, 0) = \mathbf{d}_0(\mathbf{x})$ in $\mathbf{H}^1$, se $\mathbf{d}(\mathbf{x}, t) = \mathbf{h}(\mathbf{x}, t)$ su $\partial\Omega \times (0, T)$ nel senso delle tracce e se:

$$\begin{aligned}\langle \partial_t \mathbf{u}(t), \mathbf{v}\rangle + \langle (\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t), \mathbf{v}\rangle + \nu\,(\nabla \mathbf{u}(t), \nabla \mathbf{v})& \\ + (\Delta \mathbf{d}(t), \nabla \mathbf{d}(t)\mathbf{v}) &= \langle \mathbf{g}(t), \mathbf{v}\rangle\end{aligned}$$

è verificata per ogni $\mathbf{v} \in \mathbf{V}$, q.o. $t \in (0, T)$ e

$$\partial_t \mathbf{d}(t) + (\mathbf{u}(t) \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}(t)) \qquad \text{e} \qquad |\mathbf{d}(\mathbf{x}, t)| \leq 1$$

è soddisfatta quasi ovunque in $\Omega \times (0, T)$.

Definizione. Una coppia $(\mathbf{u}, \mathbf{d})$ è una *soluzione forte* del problema (0.1) se è una soluzione debole, se, inoltre, $(\mathbf{u}, \mathbf{d}) \in L^2(0, T; (\mathbf{H} \cap \mathbf{H}^2) \times \mathbf{H}^3)$, $(\partial_t \mathbf{u}, \partial_t \mathbf{d}) \in L^2(0, T; \mathbf{H} \times \mathbf{H}^1)$ e se:

$$\begin{cases} \partial_t \mathbf{u}(t) + (\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t) - \nu \Delta \mathbf{u}(t) + \nabla p(t) = -(\nabla \mathbf{d}(t))^T \Delta \mathbf{d}(t) \mathbf{g}(t) \\ \nabla \cdot \mathbf{u}(t) = 0 \\ \partial_t \mathbf{d}(t) + (\mathbf{u}(t) \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}(t)) \\ |\mathbf{d}(\mathbf{x}, t)| \leq 1 \end{cases}$$

sono soddisfatte quasi ovunque in $\Omega \times (0, T)$.

Dimostriamo i seguenti risultati:

Teorema (Esistenza debole)**.** *Sia $\Omega \subset \mathbb{R}^n$ con $n = 2, 3$ un dominio limitato e regolare*[1]*, sia $\mathbf{g} \in L^2(0, T; \mathbf{V}^*)$ e $\mathbf{h} \in L^2(0, T; \mathbf{H}^{3/2}(\partial\Omega))$, $\partial_t \mathbf{h} \in L^2(0, T; \mathbf{H}^{-1/2}(\partial\Omega))$ tale che $|\mathbf{h}| \leq 1$ q.o. su $\partial\Omega \times (0, T)$. Sia, inoltre,*

[1] Per esempio $\Omega \in \mathbf{C}^{1,1}$.

$\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{d}_0 \in \mathbf{H}^1$ tale che $|\mathbf{d}_0| \leq 1$ q.o. in Ω, allora esiste una soluzione debole $(\mathbf{u}, \mathbf{d})$ di (0.1). Inoltre, se $n = 2$, tale soluzione è anche unica e dipende con continuità dal dato iniziale, dai termini forzanti e dal dato al bordo.

Teorema (esistenza forte)**.** *Sia $\Omega \subset \mathbb{R}^2$ un dominio limitato e regolare*[2]*, sia $\mathbf{g} \in L^2_{\text{loc}}(0, T; \mathbf{H})$ e $\mathbf{h} \in L^2(0, T; \mathbf{H}^{5/2}(\partial\Omega))$, $\partial_t \mathbf{h} \in L^2(0, T; \mathbf{H}^{1/2}(\partial\Omega))$ tale che $|\mathbf{h}| \leq 1$ q.o. su $\partial\Omega \times (0, T)$. Sia, inoltre, $\mathbf{u}_0 \in \mathbf{V}$, $\mathbf{d}_0 \in \mathbf{H}^2$ tale che $|\mathbf{d}_0| \leq 1$ q.o. in Ω, allora esiste una (unica) soluzione forte $(\mathbf{u}, \mathbf{d})$ di (0.1).*

Osserviamo che la dimostrazione dell'esistenza di soluzioni deboli per il sistema in esame sfrutta un risultato di punto fisso applicato al seguente splitting del sistema stesso. In primo luogo deve essere studiato il problema per il parametro d'ordine supponendo dato il campo di velocità $\overline{\mathbf{u}}$:

$$\begin{cases} \partial_t \mathbf{d} + (\overline{\mathbf{u}} \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}) & \text{in } \Omega \times (0, T); \\ \mathbf{d}(\mathbf{x}, 0) = \mathbf{d}_0 & \text{in } \Omega; \\ \mathbf{d}(\mathbf{x}, t) = \mathbf{h} & \text{su } \partial\Omega \times (0, T). \end{cases}$$

Quindi si analizza il problema per $\mathbf{u}$ con $\mathbf{d}$ fissato:

$$\begin{cases} (\partial_t \mathbf{u}(t), \mathbf{v}) + ((\overline{\mathbf{u}}(t) \cdot \nabla)\mathbf{u}(t), \mathbf{v}) + \nu\, (\nabla \mathbf{u}(t), \nabla \mathbf{v}) & \\ \qquad = -\,(\Delta \mathbf{d}(t), \nabla \mathbf{d}(t)\mathbf{v}) + \langle \mathbf{g}(t), \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{V}; \\ \mathbf{u}(0) = \mathbf{u}_0 & \text{in } \Omega. \end{cases}$$

Attrattori globali Stabilita la buona positura del modello considerato, il passo successivo è lo studio del comportamento asintotico delle soluzioni. Più precisamente viene affrontato il problema dell'esistenza di attrattori globali ed esponenziali. Per quanto concerne l'esistenza di un attrattore globale per un sistema dinamico non-autonomo, ricordiamo i seguenti risultati generali.

Definizione. Una famiglia di applicazioni a due parametri $\{U(t, \tau)\}$, $t > \tau$, $U(t, \tau) : X \to X$ (dove X è uno spazio di Banach) è detta *processo* in X se:

- la seguente identità è verificata:

$$U(t, s)U(s, \tau) = U(t, \tau) \qquad \forall t, s \geq 0, \forall \tau \in \mathbb{R}$$

- soddisfa per ogni $\tau \in \mathbb{R}$ la condizione $U(\tau, \tau) = \text{Id}$.

Solitamente si considerano famiglie di processi indicizzate da un parametro, detto *simbolo* dell'equazione, che rappresenta tutti i termini non-autonomi presenti nel problema di interesse. Nel caso in esame il simbolo è dato dalla coppia $(\mathbf{g}, \mathbf{h})$. Nel seguito indicheremo con Σ l'insieme dei simboli considerati.

[2] Per esempio $\Omega \in \mathbf{C}^{2,1}$.

Definizione. Un insieme $K \subset X$ è *uniformemente (rispetto a $\sigma \in \Sigma$) attraente* per la famiglia di processi $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ se soddisfa, per ogni $\tau \in \mathbb{R}$ fissato e per ogni $B \in \mathcal{B}(X)$, la relazione seguente:

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}_X(U_\sigma(t,\tau)B, K) = 0$$

Definizione. Un insieme chiuso $\mathcal{A}_\Sigma \subset X$ è l'*attrattore uniforme (rispetto a $\sigma \in \Sigma$)* della famiglia di processi $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ se:

- $\mathcal{A}_\Sigma$ è uniformemente (rispetto a $\sigma \in \Sigma$) attraente;
- $\mathcal{A}_\Sigma$ è contenuto in ogni altro insieme chiuso uniformemente attraente.

Nel nostro caso, l'esistenza dell'attrattore (globale) è ottenuta sotto ipotesi generali per lo spazio dei simboli. In particolare si utilizza il seguente risultato astratto, al quale premettiamo due definizioni.

Definizione. Sia B un insieme limitato in uno spazio metrico E. La sua *misura di non-compattezza secondo Kuratowski* è data da:

$$\alpha(B) \doteq \inf\{\delta > 0 \mid B \text{ può essere ricoperto da un numero finito di insiemi di diametro } \leq \delta\}.$$

Definizione. Una famiglia di processi $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ ha *ω-limite uniformemente (rispetto a $\sigma \in \Sigma$) compatto* se, per ogni $\tau \in \mathbb{R}$ e ogni insieme $B \in \mathcal{B}(X)$, l'insieme:

$$B_t = \bigcup_{\sigma\in\Sigma} \bigcup_{s\geq t} U_\sigma(s,\tau)B$$

è limitato per ogni t e, inoltre, $\lim_{t\to\infty} \alpha(B_t) = 0$.

Teorema. *Sia $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ una famiglia di processi in X $(X \times \Sigma, X)$-continua debolmente che abbia ω-limite uniformemente (rispetto a $\sigma \in \Sigma$) compatto. Sia B_0 un insieme debolmente compatto (cioè limitato) e uniformemente (rispetto a $\sigma \in \Sigma$) debolmente attraente per $\{U_\sigma(t,\tau)\}$ e sia Σ un sottoinsieme debolmente compatto di uno spazio di Banach. Sia, inoltre, $\{T(t)\}$ un semigruppo debolmente continuo e invariante ($T(t)\Sigma = \Sigma$) agente su Σ che soddisfi l'identità di traslazione:*

$$U_\sigma(t+s, \tau+s) = U_{T(s)\sigma}(t,\tau) \qquad \forall \sigma \in \Sigma, t, s, \tau \in \mathbb{R}, t \geq \tau, s \geq 0.$$

Allora il semigruppo esteso $\{S(t)\}$ definito da:

$$S(t) : X \times \Sigma \to X \times \Sigma, \qquad S(t)(u,\sigma) = (U_\sigma(t,0), T(t)\sigma), \qquad \forall t \geq 0.$$

possiede l'attrattore $\mathfrak{A} = \omega(B_0 \times \Sigma)$ compatto (nella topologia debole) che è strettamente invariante rispetto a $\{S(t)\}$: $S(t)\mathfrak{A} = \mathfrak{A}$. Inoltre:

- $\Pi_X \mathfrak{A} = \mathcal{A}_\Sigma$ *è l'attrattore uniforme (rispetto a* $\sigma \in \Sigma$*) della famiglia di processi* $\{U_\sigma(t, \tau)\}$, $\sigma \in \Sigma$ *(nella topologia forte!);*
- $\Pi_\Sigma \mathfrak{A} = \Sigma$*;*
- *l'attrattore globale soddisfa:*
$$\mathfrak{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\};$$
- *l'attrattore uniforme soddisfa:*
$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) = \omega_{0,\Sigma}(B_0)$$
dove $\mathcal{K}_\sigma(0)$ *è la sezione al tempo* $t = 0$ *del nucleo* $\mathcal{K}_\sigma$ *del processo* $\{U_\sigma(t, \tau)\}$ *(cioè dell'insieme di tutte le traiettorie complete e limitate del processo).*

Nel nostro caso è possibile considerare la seguente classe di funzioni come spazio dei simboli.

Definizione. Sia E uno spazio di Banach. Una funzione $f \in L^2_{loc}(\mathbb{R}; E)$ è *normale* se per ogni $\epsilon > 0$ esiste $\eta > 0$ tale che:

$$\sup_{t \in \mathbb{R}} \int_t^{t+\eta} |\varphi(s)|_E^2 \, ds \leq \epsilon.$$

Indicheremo con $L^2_n(\mathbb{R}; E)$ lo spazio di tutte le funzioni normali a valori in E.

Introduciamo inoltre la seguente definizione.

Definizione. L'insieme:

$$\mathcal{H}_\mathcal{T}(f) \doteq \overline{\{T(h)f \mid h \in \mathbb{R}\}}^{\mathcal{T}}$$

è l'*inviluppo* di f nella topologia $\mathcal{T}$.

Dimostriamo i seguenti risultati.

Teorema. *Siano* $\mathbf{g} \in L^2_n(\mathbb{R}, \mathbf{H})$, $\mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{5/2}(\partial\Omega))$, *tale che* $\partial_t \mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{1/2}(\partial\Omega))$*. Il processo* $\{U_{(\mathbf{g},\mathbf{h})}(t, \tau)\}$ *generato dall'operatore soluzione del problema* (0.1) *ha un attrattore* $\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})}$ *compatto e uniforme (rispetto a* $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*) in* $\mathbf{V} \times \mathbf{H^2}$ *che attrae uniformemente (rispetto a* $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*) i sottoinsiemi limitati di* $\mathbf{H} \times \mathbf{H}^1$ *nella norma di* $\mathbf{H} \times \mathbf{H}^1$*. Inoltre si ha:*

$$\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})} = \bigcup_{(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})} \mathcal{K}_{(\mathbf{g},\mathbf{h})}(0)$$

dove $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ *è il nucleo del processo* $\{U_{(\mathbf{g},\mathbf{h})}(t, \tau)\}$ *e dove* $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ *non è vuoto per ogni* $(\mathbf{g}, \mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$.

Teorema. *Siano* $\mathbf{g} \in L^2_n(\mathbb{R}, \mathbf{V}^*)$, $\mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{3/2}(\partial\Omega))$, *tale che* $\partial_t \mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{-1/2}(\partial\Omega))$. *Il processo* $\{U_{(\mathbf{g},\mathbf{h})}(t,\tau)\}$ *generato dal problema* (2.1) *possiede un attrattore* $\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})}$ *compatto e uniforme (rispetto a* $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*) in* $\mathbf{H} \times \mathbf{H}^1$ *che attrae uniformemente (rispetto a* $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*) i sottoinsiemi limitati di* $\mathbf{H} \times \mathbf{H}^1$ *nella norma di* $\mathbf{H} \times \mathbf{H}^1$. *Inoltre si ha:*

$$\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})} = \bigcup_{(\mathbf{g},\mathbf{h})\in\mathcal{H}_w(\mathbf{g})\times\mathcal{H}_w(\mathbf{h})} \mathcal{K}_{(\mathbf{g},\mathbf{h})}(0)$$

dove $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ *è il nucleo del processo* $\{U_{(\mathbf{g},\mathbf{h})}(t,\tau)\}$ *e dove* $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ *non è vuoto per ogni* $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$.

Attrattori esponenziali Gli attrattori globali non rappresentano l'unico oggetto che caratterizza l'evoluzione di un sistema dinamico. Si possono, per esempio, introdurre gli attrattori esponenziali, che, benché non unici, attraggono esponenzialmente le traiettorie del sistema (vd. il capitolo 4). In particolare riportiamo le seguenti definizioni.

Definizione. Sia E uno spazio di Banach. Un sottoinsieme compatto $\mathcal{M} \subset E$ è un *attrattore esponenziale* per il semigruppo $\{S(t)\}$ se:

- ha dimensione frattale finita;
- è positivamente invariante, cioè se $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$;
- attrae esponenzialmente le immagini degli insiemi limitati di E sotto l'azione di S:

  $$\forall B \subset E \text{ limitato }, \mathrm{dist}_E(S(t)B, \mathcal{M}) \leq Q(|B|_E)e^{-\alpha t}, t \geq 0,$$

  dove α è un numero positivo e Q è una funzione monotona entrambi indipendenti da B.

Definizione. Siano E, E_1 spazi di Banach tali che l'immersione di E_1 in E sia compatta. Sia inoltre X un sottoinsieme limitato di E_1 e sia $S : E \to E$. Allora S è *regolarizzante* su X se esiste $C = C(X) > 0$ tale che

$$|Su - Sv|_{E_1} \leq C|u - v|_E \qquad \forall u, v \in X.$$

Definizione. Siano E e E_1 spazi di Banach tali che l'immersione di E_1 in E sia compatta. Sia inoltre X un sottoinsieme limitato di E_1. Date due costanti positive δ e K, un operatore (nonlineare) $S : E \to E$ appartiene alla classe degli *operatori regolarizzanti* $\mathbb{S}_{\delta,K}(X)$ se:

- $S\mathcal{O}_\delta(X) \subset X$ dove $\mathcal{O}_\delta(X)$ è un intorno di X di raggio δ nella topologia di E_1;

- S è regolarizzante su $\mathcal{O}_\delta(X)$, cioè:

$$|Su - Sv|_{E_1} \leq C|u - v|_E \qquad \forall u, v \in \mathcal{O}_\delta(X).$$

Utilizziamo il seguente risultato astratto.

Teorema. *Per ogni $S \in \mathbb{S}_{\delta,K}(X)$, esiste un attrattore esponenziale $\mathcal{M}_S$ nella topologia di E_1, cioè*

1. $\dim_F(\mathcal{M}_S) \leq C$;
2. $S\mathcal{M}_S \subset \mathcal{M}_S$;
3. $\mathrm{dist}_{E_1}(S(n)X, \mathcal{M}_S) \leq Ce^{-\alpha n}, n \in \mathbb{N}$.

Inoltre l'applicazione $S \mapsto \mathcal{M}_S$ può essere scelta in modo da essere Hölder-continua nel senso che segue:

$$\mathrm{dist}_{E_1}^{symm}(\mathcal{M}_{S_1}, \mathcal{M}_{S_2}) \leq C|S_1 - S_2|_{\mathbb{S}}^{\kappa}.$$

Infine, α, κ e tutte le altre costanti che compaiono nelle stime precedenti dipendono solo da X, δ e K, e sono altresì indipendenti dal particolare semigruppo $S \in \mathbb{S}_{\delta,K}(X)$ considerato.

Nel nostro caso considereremo come spazio dei simboli quello generato dall'inviluppo di funzioni quasi-periodiche:

Definizione. Sia Ξ uno spazio di Banach e sia $(\alpha^1, \dots, \alpha^k)$ una k-upla di numeri reali incommensurabili. Sia inoltre $\phi : \mathbb{R}^k \to \Xi$ una funzione continua 2π-periodica in ogni argomento, cioè:

$$\phi(\omega^1, \dots, \omega^i + 2\pi, \dots, \omega^k) = \phi(\omega^1, \dots, \omega^i, \dots, \omega^k).$$

Allora $\sigma(s) \doteq \phi(\alpha^1 s, \alpha^2 s, \dots, \alpha^k s) \doteq \phi(\alpha s)$ è una funzione *quasi-periodica* a valori in Ξ.

Dimostriamo i seguenti risultati.

Teorema. *Sia $\Omega \subset \mathbb{R}^2$ un dominio regolare e limitato. Siano $\mathbf{g}$ e $\mathbf{h}$ funzioni quasi-periodiche a valori rispettivamente in $\mathbf{L}^2$ e $\mathbf{H}^{5/2}(\partial\Omega)$ tali che anche $\partial_t\mathbf{h}$ sia quasi-periodica a valori in $\mathbf{H}^{1/2}(\partial\Omega)$. Sia, inoltre, $\{S(t)\}$ il semigruppo esteso associato all'operatore di soluzione del problema (0.1) che agisce sullo spazio delle fasi esteso $\mathbf{H} \times \mathbf{H}^1 \times \mathbb{T}^k$ (qui k è pari alla somma del numero di periodi incommensurabili di $\mathbf{h}$ e $\mathbf{g}$)[3]. Allora esiste un tempo finito t^* per cui il semigruppo discreto generato dalla mappa $S(t^*)$ possiede un attrattore esponenziale uniforme (rispetto allo fase iniziale del simbolo $\theta \in \mathbb{T}^k$).*

[3]Osserviamo che il dominio delle funzioni quasi-periodiche può essere naturalmente identificato con il toro k-dimensionale $\mathbb{T}^k$. Inoltre vale l'isomorfismo algebrico e geometrico $\mathbb{T}^l \oplus \mathbb{T}^m = \mathbb{T}^{l+m}$.

Teorema. *Sotto le stesse ipotesi del teorema precedente esiste un attrattore esponenziale* $\mathcal{M}$ *per il semigruppo esteso* $\{S(t)\}$ *su* $\mathbf{H} \times \mathbf{H}^1 \times \mathbb{T}^k$*. Inoltre, se* Π_1 *e* Π_2 *sono le proiezioni dello spazio delle fasi esteso rispettivamente su* $\mathbf{H} \times \mathbf{H}^1$ *e* $\mathbb{T}^k$*, allora* $\Pi_1\mathcal{M}$ *è l'attrattore esponenziale uniforme (rispetto a* $\theta \in \mathbb{T}^k$*) per la famiglia di processi e* $\Pi_2\mathcal{M} = \mathbb{T}^k$*.*

L'approssimazione numerica In quest'ultimo capitolo lo studio analitico dei capitoli centrali è applicato alla formulazione di semplici metodi numerici per l'approssimazione numerica delle soluzione del sistema (0.1). Il calcolo della soluzione di equilibrio (minimo della energia libera del sistema) può essere condotto per una ampia gamma di valori di ϵ tramite una opportuna linearizzazione alla Newton dell'equazione di equilibrio. In particolare lo schema implementato è il seguente:

$$\Delta\mathbf{d}^{(n+1)} - \frac{1}{\epsilon^2}\left(|\mathbf{d}^{(n)}|^2 - 1\right)\mathbf{d}^{(n+1)} - \frac{2}{\epsilon^2}\left(\mathbf{d}^{(n)} \cdot \mathbf{d}^{(n+1)}\right)\mathbf{d}^{(n)} = -\frac{2}{\epsilon^2}|\mathbf{d}^{(n)}|^2\mathbf{d}^{(n)}.$$

Per la simulazione del sistema evolutivo completo, invece, si sono confrontate le prestazioni del metodo di punto fisso di Newton applicato a tutto il sistema con quello di uno schema iterativo basato sullo splitting del problema differenziale introdotto per provare l'esistenza di soluzioni del problema. Come spazi di elementi finiti si è scelta una coppia che soddisfi la condizione di Ladyzhenskaya-Babuška-Brezzi per l'approssimazione dei campi di velocità e di pressione (in particolare la coppia $\mathbb{P}_1$bolla/$\mathbb{P}_1$) e di uno spazio di elementi finiti di un ordine più accurato per il parametro d'ordine (nel nostro caso $\mathbb{P}_2$): ciò serve a garantire un ordine di convergenza ottimale allo schema risultante.

Gli esperimenti numerici evidenziano un notevole vantaggio nell'uso di questo secondo schema numerico che riportiamo per completezza:

$$\begin{aligned}
&\mathbf{u}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n)} \cdot \nabla\right)\mathbf{u}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n+1)} \cdot \nabla\right)\mathbf{u}_j^{(n)} - dt\Delta\mathbf{u}_j^{(n+1)} \\
&\qquad = dt\left(\mathbf{u}_j^{(n)} \cdot \nabla\right)\mathbf{u}_j^{(n)} + dt\left(\mathbf{d}_j^{(n)}\right)^T\Delta\mathbf{d}_j^{(n)} + dt\,\mathbf{g}_j + \mathbf{u}_{j-1}, \\
&\mathbf{d}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n+1)} \cdot \nabla\right)\mathbf{d}_j^{(n+1)} - dt\Delta\mathbf{d}_j^{(n+1)} \\
&\qquad\qquad + \frac{dt}{\epsilon^2}\left(|\mathbf{d}_j^{(n)}|^2 - 1\right)\mathbf{d}_j^{(n+1)} + \frac{2dt}{\epsilon^2}\left(\mathbf{d}_j^{(n)} \cdot \mathbf{d}_j^{(n+1)}\right)\mathbf{d}_j^{(n)} \\
&\qquad = \frac{2dt}{\epsilon^2}|\mathbf{d}_j^{(n)}|^2\mathbf{d}_j^{(n)} + \mathbf{d}_{j-1}.
\end{aligned}$$

Dagli esperimenti numerici riportati si nota, inoltre, come l'accoppiamento non-lineare tra le equazioni di Navier-Stokes e quelle per il parametro d'ordine abbia degli effetti qualitativi notevoli se confrontato con la soluzione delle sole equazioni dei fluidi incomprimibili purché il numero di Reynolds sia abbastanza grande.

Contents

“Nissuna umana investigazione si può dimandare vera scienza, s’essa non passa per le matematiche dimostrazioni.”

(Leonardo da Vinci — Trattato della pittura, I, 1)

Introduction

In the study of the physical world, differential equations surely represent one of the most powerful mathematical tools available to the researcher. Their unifying language allows the description of systems arising from different settings and is apt to describe a wealth of phenomena: from wave propagation to the transport and diffusion of substances, from the deformation of solids to the description of the interaction between electromagnetic fields and matter.

However powerful such an instrument could be, it cannot be free of drawbacks. Although unifying and concise, both ordinary and partial differential equations do not easily reveal their secrets. It is, indeed, often difficult to obtain thorough information on the phenomenon under consideration. Only in very few cases a full understanding of the differential problems is available and in most of them also basic results may be hard to obtain. Several difficulties can arise during the analysis: starting from existence and uniqueness of solutions, one is usually furthermore interested in understanding what the main features and characteristics of the phenomenon are and therefore wishes to understand the regularity of the solution and/or its long term behaviour.

Although such issues could seem quite abstract and academical at a first glance, their importance must not be underestimated: thanks to the amazing computing capabilities made available in the last decades, many effective numerical methods which profoundly rely on these information have been invented and applied to the simulation of potentially any known equation or system of equations. This new aspect of applied sciences has led to new ways to test the predictions of phenomenological theories and to improve technological applications.

This last aspect could not be overemphasized: however deep and complete the description and theoretical understanding of any physical system could be, it has no practical value without some insight to applications. The study of any physical system, described by PDEs, must be aware at any moment of the original setting for which it was written and of the problem from which it originated. Therefore abstract questions as the well-posedness of an evolution system are to be understood as an analysis of the physical soundness of the description adopted. Moreover, when dissipation principles are available (as it is often the case in mechanical systems subjected to friction or in fluid dynamics), the study of the asymptotic evolution and the problem of the existence of finite-dimensional attractors for a set of evolution equations have to be related to the complexity of the non-transient behaviour the system can exhibit. These aspects are then tightly bounded to the robustness of the system under changes in the external forcing terms and to the difficulty arising from the numerical simulation in view of technological applications.

In the present work we study a particular system arising from the physically and technologically important field of soft matter, trying to get as a deep insight as possible on the equations ruling the evolution of liquid crystal flows in view of numerical simulations for potential applications. This field has been object of great interest in the last years and many important and deep results are already available in the literature. For some of the most recent developments see [19], [20] and [41] and the references therein.

Our work has been organized as follows:

Chapter 1 A simplified model for nematic liquid crystals
In this first part of our work we briefly summarize the physical origin of the system of PDEs which is the object of our study in all other chapters. Our system is a simplified version of the full Ericksen-Leslie equations first proposed by Liu et al. at the beginning of the 90's and which has revealed itself to be compliant with mathematical investigations. The system consists of Navier-Stokes equations for incompressible fluids non-linearly coupled with a vector equation of Ginzburg-Landau type. This equation determines the evolution of an order parameter describing the orientation of the crystal molecules. As is often the case, even a simple understanding of the nature of the phenomena to be described will help us in the understanding of its mathematical features.

Chapter 2 Well posedness Here we study the main aspects of the mathematical well posedness of the simplified system introduced in the first chapter, namely existence and uniqueness. We consider a non-autonomous version of the system presenting a time-dependent volume force for the velocity equation and a time-dependent Dirichlet boundary condition which varies with time for the order parameter. In particular, we prove existence of weak solutions both in two and three dimensions. Moreover we are also able to show the uniqueness of weak solutions and existence of strong solutions in two dimensions. We observe that here the same difficulties as in the study of Navier-Stokes equations arise (i.e. the lack of uniqueness and existence of global strong solutions).

Chapter 3 Global attractors After having proven the well-posedness of the model which we are working with, in this part of our work we study the asymptotic evolution of its solutions. In recent years much energy has been poured in the investigation of the long time description of important equations in order to characterize the non-transient dynamics. The main focus of contemporary research has been the identification of suitable "small" (i.e. finite-dimensional) attracting sets of configurations for the system of interest. The emphasis on the smallness is justified by the following important result originally due to Hölder and Mañe (see [27] and the references cited therein).

Theorem. *Let E be a Banach space and let $X \subset E$ be compact and such that $\dim X = d$ (where $\dim$ is the fractal dimension in E) and let $N > 2d$ be an integer. Then almost every bounded linear projector $P : E \to \mathbb{R}^N$ is one-to-one on X and has a Hölder continuous inverse.*

In other words, if we can identify a suitable compact finite-dimensional set which captures the long-time behavior of the system we are studying, then (almost) any sufficiently detailed finite-dimensional reduction of the evolution equations completely captures its evolution.

In this chapter we prove the existence of a global attractor for the nonautonomous equations presented above, considering quite general forcing terms belonging to a wide class recently introduced by Lu and Wu.

Chapter 4 Exponential attractors The global attractors introduced in the previous chapter, however, are not fully satisfactory from some points of view: in particular, one cannot guarantee how fast they are able to capture the evolution of the system starting from an arbitrary initial state. In order to overcome this problem, Eden, Foias, Nicolaenko and Temam in the 90's and more recently Efendiev, Miranville and Zelik developed the notion of exponential attractors respectively for autonomous semigroups and non-autonomous processes. The main result of this section is the proof of the existence of such an invariant attracting set for our system considering quasi-periodic forcing terms. This result also proves that the global attractor in this case has finite dimension.

Chapter 5 Some numerical experiments Finally we conclude our work by discussing how some of the abstract results of the previous chapters can be useful when designing numerical schemes for the numerical approximation of the solution of our system. This section, although not exhaustive, proposes some interesting results and shows how drastic gains in the computation time can easily be obtained after an analytical study of the mathematical equations.

Finally I wish to thank some of the many persons that have been essential in writing an important (at least for me) work as the present without whom it would not probably have been written. Most of my acknowledgments go to professor Maurizio Grasselli of the Mathematical Department of the Politecnico di Milano for his continuous and careful support in identifying the main mathematical aspects of this work and through supplying much interesting material for all the other chapters. A special thank goes to Professor Paolo Biscari from the Mathematical Department of the Politecnico too, whose explanations have been enlightening for the physical meaning and derivation of the model. Among the many people that have kindly borne the last months of intense study, helping me with much discussion (often leading to nonsense or sudden ideas), I cannot forget to thank Alessandro and Andrea, incomparable mates during the five years at the Politecnico, to whom I also wish the best in the years to come. Finally a special thank goes to my parents which have stood long months of study in recent times, seeing me going to bed "early in the morning" rather than "late in the evening".

Milano, July 2010.

Chapter 1

A simplified model for nematic liquid crystals

In this chapter we will briefly review the classical Ericksen-Leslie model for the dynamics of nematic liquid crystals. In the development of the theory we will essentially follow [9] and [35]. Then we will discuss some simplifications (as in [21]) to the full model which are particularly suitable for the analysis of the following chapters.

As it is customary, the model splits naturally in two parts: some general equations are directly derived from basic conservation (or balance) laws and the closure of the system is given by some specific constitutive equations. As it will be apparent in a few pages, despite the simplicity of the assumption upon which the Ericksen-Leslie model relies, the resulting equations are far from being easy to analyze. Therefore this somehow justifies the simplifications we will introduce later on.

1.1 Balance laws

We begin by considering the usual balance equations derived in the continuum theory (see [16]). In particular, we consider conservation of mass, linear momentum and angular momentum:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\, dV &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{v}\, dV &= \int_V \rho\mathbf{F}\, dV + \int_S \mathbf{t}\, dS\\
\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{x}\wedge\mathbf{v}\, dV &= \int_V \rho(\mathbf{x}\wedge\mathbf{F}+\mathbf{K})\, dV + \int_S \mathbf{x}\wedge\mathbf{t}+\mathbf{l}\, dS
\end{aligned}$$

Here ρ is the density, $\mathbf{x}$ is the position vector, $\mathbf{F}$ is the external body force, $\mathbf{t}$ is the surface force per unit area, $\mathbf{l}$ is the surface torque per unit area and $\mathbf{K}$ is the external body moment. We observe immediately that taking into account body moments is a somehow unusual starting point. However, one of the interesting characteristics of liquid crystals is that they can transmit torques (see [9, Chapter 2]) and therefore the role of body moments can be quite important in the description of both the statics and the dynamics of such systems.

In order to transform the integral form of the just recalled conservation laws into a more analytically convenient differential form, we need some classical results. First of all we begin by recalling Reynolds transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f\,dV = \int_V (\dot{f} + f(\nabla\cdot\mathbf{v}))\,dV$$

where the dot notation is a shorthand for the total derivative with respect to time (explicitly $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}t}f = \partial_t f + (\mathbf{v}\cdot\nabla)f$) and $\mathbf{v}$ is the velocity field.

We will also use the divergence theorem in its vector and tensor forms:

$$\int_S \mathbf{f}\cdot\nu\,dS = \int_V \nabla\cdot\mathbf{f}\,dV$$
$$\int_S \mathbf{T}\nu\,dS = \int_V \nabla\cdot\mathbf{T}\,dV$$

Finally we remember that, from Cauchy stress theorem, the surface force per unit area can be written as a double tensor acting on the normal vector to the surface itself: $\mathbf{t} = \mathbf{T}\nu$. However note that, since we are also considering localized moments, the stress tensor $\mathbf{T}$ need not to be symmetric.

By using the just recalled results we can easily write the infinitesimal form of the conservation laws introduced above:

$$\dot{\rho} = -\rho(\nabla\cdot\mathbf{v})$$
$$\rho\dot{\mathbf{v}} = \rho\mathbf{F} + \nabla\cdot\mathbf{T}$$
$$\epsilon_{ijk}\rho\mathbf{x}_j\dot{\mathbf{v}}_k = \epsilon_{ijk}\rho\mathbf{x}_j\mathbf{F}_k + \epsilon_{ijk}\mathbf{x}_j\mathbf{T}_{kp,p} + \rho\mathbf{K}_i + \epsilon_{ijk}\mathbf{T}_{kj} + \mathbf{L}_{ij,j}$$

A few considerations are now necessary. Consider the equation for the conservation of mass. If the liquid crystal can be considered incompressible (i.e. if $\dot{\rho} = 0$), then we easily deduce the familiar divergence-free condition for the velocity field: $\nabla\cdot\mathbf{v} = 0$.

Moreover, we can simplify the equation for the angular momentum by using the balance of linear momentum and obtain the simpler expression:

$$\rho\mathbf{K}_i + \epsilon_{ijk}\mathbf{T}_{ij} + \mathbf{L}_{ij,j} = 0.$$

Therefore we retain the following system:

$$\nabla\cdot\mathbf{v} = 0 \tag{1.1a}$$
$$\rho\dot{\mathbf{v}} = \rho\mathbf{F} + \nabla\cdot\mathbf{T} \tag{1.1b}$$
$$\rho\mathbf{K}_i + \epsilon_{ijk}\mathbf{T}_{ij} + \mathbf{L}_{ij,j} = 0. \tag{1.1c}$$

As it is obvious at first glance, the just written equations cannot form a closed system. Actually we need some more relations which describe the phenomenological link between the dynamic variables ($\mathbf{T}$ and $\mathbf{L}$) on one side and the kinematic ones (ρ and $\mathbf{v}$) on the other.

1.2 Constitutive equations I: Frank-Oseen elastic energy

In this section we will review Frank and Oseen's theory. Our main goal is to introduce an elastic energy (or free energy) which can account (in the usual framework of elastic theory) for the static configuration of a nematic liquid crystal under assigned body forces, torques and imposed boundary anchoring. This energy will also be used later when we will be discussing on the dynamic constitutive equations.

We introduce an internal variable (or order parameter) for our continuum. We will indicate with $\mathbf{n}$ the local orientation of the molecules of the liquid crystal. Since $\mathbf{n}$ is a versor, we will suppose $\mathbf{n} \in \mathbb{S}^{n-1}$ where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$ and n is the dimension of the physical space considered ($n = 2$ or $n = 3$ for the aims of the present work). We are here considering the classical theory for which this order parameter field is a deterministic quantity. We incidentally note that also a more recent approach is possible by taking into account the natural variability (or noise) of the orientation of the molecules. We refer the interested reader to [40, Chapter 6].

In writing Frank's energy, we will care of some fundamental principles: in particular, we must check for frame indifference and the equation has to satisfy all the additional symmetries of the system we can identify. In this context, these requirements translate into the following mathematical constraints on Frank's energy σ_F:

- σ_F can only depend on $\mathbf{n}$ and $\nabla\mathbf{n}$;
- if $\mathbf{Q}$ is the matrix representing any proper rotation, then $\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = \sigma_F(\mathbf{Q}\mathbf{n}, \mathbf{Q}\nabla\mathbf{n}\mathbf{Q}^T)$ must be satisfied;
- in our context, reversing the orientation of the nematic liquid crystal molecules should not affect the overall energy of the medium. We therefore also require $\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = \sigma_F(-\mathbf{n}, -\nabla\mathbf{n})$.

Moreover, in order to fix the arbitrary constant which arises in the definition of this new energy we will require $\sigma_F(\mathbf{n}, \nabla\mathbf{n}) \geq 0$ for all order parameter fields and $\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = 0$ only for a uniform order parameter field $\mathbf{n}(\mathbf{x}) = \mathbf{n}$. With some careful considerations, the most general form of Frank's energy for a nematic liquid crystals is shown to be[1]:

$$\sigma_F(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}K_1(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_2\left(\mathbf{n}\cdot(\nabla\wedge\mathbf{n})\right)^2 + \frac{1}{2}K_3\left(\mathbf{n}\wedge(\nabla\wedge\mathbf{n})\right)^2. \quad (1.2)$$

[1]Actually, a further term should be added to Frank's energy σ_F:

$$\nabla\cdot((\mathbf{n}\cdot\nabla)\mathbf{n} - (\nabla\cdot\mathbf{n})\mathbf{v}) = \operatorname{tr}\nabla(\mathbf{n})^2 - (\nabla\cdot\mathbf{n})^2$$

However, this part of free energy can be left behind without any harm to the physical meaning of our model. From the point of view of the calculus of variations and Euler-Lagrange equilibrium equations, this term is indeed a null lagrangian and therefore, giving no contribution to the equilibrium equations, will be omitted in the following passages.

Equation (1.2) contains really a bunch of information about the elastic behaviour of nematic liquid crystals. Although we are not interested in the fine details (see, for example, [9, Chapter 3]), we will give a short insight in the physical meaning of the different terms in Frank's free energy. Actually, the dependence on $\nabla \cdot \mathbf{n}$ is related to *splay* order parameter fields, that is fields that look like a fan with molecules oriented as fan-sticks. The second term, $\mathbf{n} \cdot (\nabla \wedge \mathbf{n})$ involves *twist* fields: these can be obtained from an unperturbed situation by rigidly rotating by different angles parallel planes of the liquid crystal which contain the order parameter vector. Finally the last contribution to Frank-Oseen's energy is given by *bend* fields. These also show a fan-like alignment of the molecules of the medium, but in this case these are orthogonal to the fan-sticks. In practical situation this subdivision in splay, twist and bend contributions is usually more delicate.

We observe that, from a mathematical point of view, it is often convenient to adopt the so called "one-constant approximation" in order to simplify the analytical form of Frank's free energy. One can actually suppose that all constants K_i in (1.2) are equal to K. Frank's energy then reduces to:

$$\sigma_F = \frac{1}{2} K |\nabla \mathbf{n}|^2 = \frac{1}{2} K \mathbf{n}_{i,j} \mathbf{n}_{i,j}.$$

As we will see below, this free energy can easily be linked to the elastic (and static) part of the stress tensor by the following elastic relation:

$$\mathbf{T}^{\text{el}}_{ij} = -p\delta_{ij} - \frac{\partial\, \sigma_F}{\partial \mathbf{n}_{k,j}} \mathbf{n}_{k,i}$$

where $\mathbf{T} = \mathbf{T}^{\text{el}} + \mathbf{T}^{\text{irr}}$. If we consider the one-constant approximation, then the elastic part of the stress tensor reduces to:

$$\mathbf{T}^{\text{el}} = -p\mathbf{I} - (\nabla \mathbf{n})^T (\nabla \mathbf{n}).$$

1.3 Constitutive equations II: A dissipation principle and objectivity

We now have to consider the effect of motion on the stress configuration of a nematic liquid crystal. The basic idea of this section will be the introduction of a particular dissipation function which represents the "viscous" dissipation of our medium. We start by writing the usual energy balance equation in integral form:

$$\int_V \rho(\mathbf{F}\cdot\mathbf{v}+\mathbf{K}\cdot\mathbf{w})\, dV + \int_S (\mathbf{t}\cdot\mathbf{v}+\mathbf{l}\cdot\mathbf{w})\, dS = \frac{\mathrm{d}}{\mathrm{d}t}\int_V (\tfrac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}+\sigma_F)\, dV + \int_V D\, dV$$

where $\mathbf{w}$ is the local angular velocity, i.e. the angular velocity of the molecules of the medium (and not the usual curl pseudovector associated to the velocity field $\mathbf{v}$) and where D is the rate of viscous dissipation per unit volume. Indeed we have $\dot{\mathbf{n}} = \mathbf{w} \wedge \mathbf{n}$. In accordance to the physical meaning, we will assume that D is always positive.

By using some standard results (Reynolds' transport theorem and divergence theorem) and the mass and linear momentum balance equations (1.1a) and (1.1b), passing to the point form, we deduce:

$$\mathbf{T}_{ij}\mathbf{v}_{i,hj} + \mathbf{L}_{ij}\mathbf{w}_{i,j} - \epsilon_{ijk}\mathbf{w}_i\mathbf{T}_{kj} = \dot{\sigma}_F + D$$

We now want to evaluate $\dot{\sigma}_F$. We therefore recall the following useful identities, the second of which due to Ericksen:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{n}_{i,j} = (\dot{\mathbf{n}}_i)_{,j} - \mathbf{n}_{i,k}\mathbf{v}_{k,j}$$

$$\epsilon_{ijk}\left(\mathbf{n}_j\frac{\partial\sigma_F}{\partial\mathbf{n}_k} + \mathbf{n}_{j,p}\frac{\partial\sigma_F}{\partial\mathbf{n}_{k,p}} + \mathbf{n}_{p,j}\frac{\partial\sigma_F}{\partial\mathbf{n}_{p,k}}\right) = 0 \tag{1.3}$$

Some simple calculations then lead to the following expression for the time derivative of the free energy:

$$\dot{\sigma}_F = \epsilon_{iqp}\left(\mathbf{n}_q\frac{\partial\sigma_F}{\partial\mathbf{n}_{p,j}}\mathbf{w}_{i,j} - \mathbf{n}_{k,q}\frac{\partial\sigma_F}{\partial\mathbf{n}_{k,p}}\mathbf{w}_i\right) - \mathbf{n}_{p,i}\frac{\partial\sigma_F}{\partial\mathbf{n}_{p,j}}\mathbf{v}_{i,j}$$

The energy balance equation in local forms can then be rewritten as follows to get an expression for the dissipation D:

$$D = \left(\mathbf{T}_{ij} + \frac{\partial\sigma_F}{\partial\mathbf{n}_{p,j}}\mathbf{n}_{p,i}\right)\mathbf{v}_{i,j} + \left(\mathbf{L}_{ij} - \epsilon_{iqp}\frac{\partial\sigma_F}{\partial\mathbf{n}_{p,j}}\mathbf{n}_q\right)\mathbf{w}_{i,j} - \epsilon_{iqp}\left(\mathbf{T}_{pq} - \frac{\partial\sigma_F}{\partial\mathbf{n}_{k,p}}\mathbf{n}_{k,q}\right)\mathbf{w}_i.$$

We now use the positiveness constraint on D. Since the velocity fields can be arbitrary, we deduce:

$$\begin{aligned}\mathbf{T}_{ij} &= \overbrace{-p\delta_{ij} - \frac{\partial\sigma_F}{\partial\mathbf{n}_{p,j}}\mathbf{n}_{p,i}}^{\mathbf{T}^{\mathrm{el}}_{ij}} + \mathbf{T}^{\mathrm{irr}}_{ij} \\ \mathbf{L}_{ij} &= \epsilon_{ipq}\frac{\partial\sigma_F}{\partial\mathbf{n}_{q,j}}\mathbf{n}_p + \mathbf{L}^{\mathrm{irr}}\end{aligned} \tag{1.4}$$

where the dissipative parts of the strain tensors still have to satisfy the following dissipative relation:

$$\mathbf{T}^{\mathrm{irr}}_{ij}\mathbf{v}_{i,j} + \mathbf{L}^{\mathrm{irr}}_{ij}\mathbf{w}_{i,j} - \epsilon_{ijk}\mathbf{w}_i\mathbf{T}^{\mathrm{irr}}_{kj} = D \geq 0$$

We now suppose that $\mathbf{T}^{\text{irr}}$ and $\mathbf{L}^{\text{irr}}_{ij}$ are functions of $\mathbf{n}$, $\mathbf{w}$ and $\nabla\mathbf{v}$. More precisely, objectivity requirements impose to restrict this dependence only to the quantities $\mathbf{n}$, $\mathbf{N}$ and $\mathbf{A}$ where $\mathbf{N} \doteq \dot{\mathbf{n}} - \mathbf{W}\mathbf{n}$ and where $\mathbf{A}$ and $\mathbf{W}$ are respectively the symmetric part (strain tensor) and skew-symmetric part (vorticity tensor) of the gradient of velocity. By imposing again the positivity condition we immediately deduce:

$$\mathbf{L}^{\text{irr}} = \mathbf{0}$$

and, after some tedious calculations (see [35, Section 4.2]), imposing, accordingly to experiments, $\mathbf{T}^{\text{irr}}$ to be linear in $\mathbf{A}$ and $\mathbf{N}$, we also have:

$$\begin{aligned}\mathbf{T}^{\text{irr}}_{ij} = (\mu_1 + \mu_2 \mathbf{n}_k \mathbf{A}_{kp} \mathbf{n}_p)\delta_{ij} + (\mu_3 + \alpha_1 \mathbf{n}_k \mathbf{A}_{kp} \mathbf{n}_p)\mathbf{n}_i \mathbf{n}_j \\ + \alpha_2 \mathbf{N}_i \mathbf{n}_j + \alpha_3 \mathbf{N}_j \mathbf{n}_i + \alpha_4 \mathbf{A}_{ij} + \alpha_5 \mathbf{n}_j \mathbf{A}_{ik} \mathbf{n}_k + \alpha_6 \mathbf{n}_i \mathbf{A}_{jk} \mathbf{n}_k\end{aligned}$$

If we consider again the positivity assumption on the dissipative term D, we have $\mu_3 = 0$. Since the first term is a multiple of the identity, we can incorporate it into the pressure term giving the following expression for the dissipative part of the stress tensor:

$$\begin{aligned}\mathbf{T}^{\text{irr}}_{ij} = \alpha_1 \mathbf{n}_k \mathbf{A}_{kp} \mathbf{n}_p \mathbf{n}_i \mathbf{n}_j + \alpha_2 \mathbf{N}_i \mathbf{n}_j + \alpha_3 \mathbf{N}_j \mathbf{n}_i \\ + \alpha_4 \mathbf{A}_{ij} + \alpha_5 \mathbf{n}_j \mathbf{A}_{ik} \mathbf{n}_k + \alpha_6 \mathbf{n}_i \mathbf{A}_{jk} \mathbf{n}_k\end{aligned} \tag{1.5}$$

Moreover, the coefficients α_i must satisfy some inequalities in order to ensure the positivity of D. We skip here the details referring again the interested reader to [35, Section 4.2.3].

Before putting everything together, we also note that:

$$\epsilon_{ijk} \mathbf{T}^{\text{irr}}_{ij} = \epsilon_{ijk} \mathbf{n}_j \mathbf{g}_k \qquad \text{with} \qquad \mathbf{g}_k \doteq -\gamma_1 \mathbf{N}_k - \gamma_2 \mathbf{A}_{kp} \mathbf{n}_p$$

where we have set $\gamma_1 \doteq \alpha_3 - \alpha_2$ and $\gamma_2 \doteq \alpha_6 - \alpha_5$.

Remark. In the derivation of the constitutive relation for nematic liquid crystals, we explicitly supposed that the stress tensors depend upon $\mathbf{n}$, $\mathbf{N}$ and $\mathbf{A}$ only. Actually a more general assumption is possible involving also the dependence on $\ddot{\mathbf{n}}$ through a coefficient which represents the rotational inertial constant of the particles of the medium. Since this coefficient can generally be considered small (unless in some very quick transient flows), it is customary to simplify the constitutive equation by neglecting this contribution.

1.4 The full Ericksen-Leslie model

We can now write all the equations which form the standard Ericksen-Leslie model for nematic liquid crystal flows.

From the balance of angular momentum (1.1c), by using the constitutive relation of last section and Ericksen identity (1.3), if we suppose that bulk body moments can be written as $\rho\mathbf{K} = \mathbf{n} \wedge \mathbf{G}$, we obtain:

$$\epsilon_{ipq}\mathbf{n}_p \left(\left(\frac{\partial \sigma_F}{\partial \mathbf{n}_{q,j}} \right)_{,j} - \frac{\partial \sigma_F}{\partial \mathbf{n}_q} + \mathbf{g}_q + \mathbf{G}_q \right) = 0$$

and therefore:

$$\nabla \cdot \left(\frac{\partial \sigma_F}{\partial \nabla \mathbf{n}} \right) - \frac{\partial \sigma_F}{\partial \mathbf{n}} + \mathbf{g} + \mathbf{G} = \lambda \mathbf{n} \tag{1.6}$$

where λ is an arbitrary Lagrange multiplier which is associated with the normalization constraint $\mathbf{n} \cdot \mathbf{n} = 1$.

We can deal analogously with the linear momentum equation. Starting from (1.1b) and substituting the constitutive relation (1.4), we deduce:

$$\rho \dot{\mathbf{v}}_i = \rho \mathbf{F}_i - p_{,i} - \left(\frac{\partial \sigma_F}{\partial \mathbf{n}_{k,j}} \mathbf{n}_{k,i} \right)_{,j} + \mathbf{T}^{\text{irr}}_{ij,j}$$

or, in vector notation:

$$\rho \dot{\mathbf{v}} = \rho \mathbf{F} - \nabla p + \nabla \cdot \left((\nabla \mathbf{n})^T \frac{\partial \sigma_F}{\partial \nabla \mathbf{n}} \right) + \nabla \cdot \mathbf{T}^{\text{irr}} \tag{1.7}$$

Equations (1.6) and (1.7) together with the constitutive relation (1.5) and the definitions of $\mathbf{g}$, $\mathbf{N}$, $\mathbf{A}$ and $\mathbf{W}$ form the so called Ericksen-Leslie system for nematic liquid crystal flows. As it can be easily seen, this system has eight unknowns (namely $\mathbf{v}$, $\mathbf{n}$, p and λ) and after substituting everywhere the constitutive relations we remain with eight equations (actually (1.6), (1.7), (1.1a) and the constraint $\mathbf{n} \cdot \mathbf{n} = 1$).

Although we have obtained a closed system of equations, it is evident that this form is far too complex to be studied without any further simplification. In the following section, we will therefore try to reduce the number of unknowns and to discard some unessential terms.

1.5 Some simplifications

We start by considering the angular momentum equation (1.6). If we consider the one constant approximation introduced in section 1.2, the terms involving Frank's free energy simply reduce to $\Delta\mathbf{n}$. If we further suppose that the generalized body moments $\mathbf{G}$ are null, we obtain:

$$K\Delta\mathbf{n} - \gamma_1 \dot{\mathbf{n}} + \gamma_1 \mathbf{W}\mathbf{n} - \gamma_2 \mathbf{A}\mathbf{n} = \lambda \mathbf{n} \tag{1.8}$$

In order to make the analysis of equation (1.8) even more amenable, we will consider the following relaxation for the order parameter constraint.

Instead of imposing exactly the constraint $\mathbf{n}\cdot\mathbf{n}=1$, we will consider an additional potential term in Frank's free energy which will be penalizing anytime the constraint is not verified. In particular we will consider the potential $\mathcal{F}\doteq\frac{1}{4\epsilon^2}(|\mathbf{n}|^2-1)^2$ and set $\mathbf{f}(\mathbf{n})\doteq\nabla_{\mathbf{n}}\mathcal{F}=\frac{1}{\epsilon^2}(|\mathbf{n}|^2-1)\mathbf{n}$. We can thus write:

$$K\Delta\mathbf{n}-\gamma_1\dot{\mathbf{n}}+\gamma_1\mathbf{W}\mathbf{n}-\gamma_2\mathbf{A}\mathbf{n}=\mathbf{f}(\mathbf{n}) \tag{1.9}$$

instead of (1.8) with the evident advantage of having eliminated an unknown (namely λ) and having substituted the highly non-linear constraint $\mathbf{n}\in\mathbb{S}^{n-1}$ with the more analytically amenable polynomial nonlinearity $\mathbf{f}(\mathbf{n})$.

We now consider equation (1.7). Thanks to the angular momentum equation (1.6), to the definition of $\mathbf{g}$ and to the identity $(\nabla\mathbf{n})^T\mathbf{n}=\mathbf{0}$, after renaming the pressure, we obtain:

$$-\nabla p+\nabla\cdot\left((\nabla\mathbf{n})^T\frac{\partial\sigma_F}{\partial\nabla\mathbf{n}}\right)=-\nabla p-(\nabla\mathbf{n})^T(\gamma_1\mathbf{N}+\gamma_2\mathbf{D}\mathbf{n}).$$

Substituting in (1.7) we obtain:

$$\rho\dot{\mathbf{v}}=\rho\mathbf{F}-\nabla p+(\nabla\mathbf{n})^T(\gamma_1\mathbf{N}+\gamma_2\mathbf{D}\mathbf{n})+\nabla\cdot\mathbf{T}^{\mathrm{irr}}. \tag{1.10}$$

In order to further simplify equations (1.9) and (1.10), we will consider a perturbative expansion for the backflow terms. This is physically sound since experimentally we have that the coefficients α_i in (1.5) are all very small except α_4. Following [2], we will therefore suppose $\rho=O(\epsilon^{-1})$ and $\alpha_4=(\epsilon^{-1})$ and we will write:

$$\rho=\frac{\widetilde{\rho}}{\epsilon}\qquad\alpha_4=\rho\nu=\frac{\widetilde{\alpha}_4}{\epsilon}$$

with $\widetilde{\rho}=O(1)$ and $\widetilde{\alpha}_4=O(1)$. We define the perturbation parameter ϵ as: $\epsilon\doteq\max\{|\alpha_1|,|\alpha_2|,|\alpha_3|,|\alpha_5|,|\alpha_6|\}/\alpha_4$. We will consider the following expansion of the macroscopic fields:

$$\mathbf{u}=\epsilon\mathbf{u}_1+O(\epsilon^2),\qquad\mathbf{n}=\mathbf{n}_0+\epsilon\mathbf{n}_1+O(\epsilon^2),\qquad p=p_0+\epsilon p_1+O(\epsilon^2).$$

Simple calculations then also show that $\mathbf{N}=\dot{\mathbf{n}}_0+O(\epsilon)$. Moreover the external forcing $\mathbf{F}$ will also be considered of order $O(\epsilon)$; $\mathbf{F}=\epsilon\mathbf{F}_1+O(\epsilon^2)$.

We now consider the $O(1)$ term arising form equation (1.9). Easily we get:

$$\gamma_1\dot{\mathbf{n}}_0=\mathbf{f}(\mathbf{n}_0)+K\Delta\mathbf{n}_0 \tag{1.11}$$

Analogously we consider the $O(1)$ term of equation (1.10) (we note that the $O(\epsilon^{-1})$ contribution is automatically satisfied under our assumptions). We immediately obtain:

$$\widetilde{\rho}\dot{\mathbf{u}}_1=-\nabla p-\gamma_1(\nabla\mathbf{n}_0)^T\mathbf{N}_0+\nabla\cdot(\alpha_2\mathbf{N}_0\otimes\mathbf{n}_0+\alpha_3\mathbf{n}_0\otimes\mathbf{N}_0+\widetilde{\alpha}_4\mathbf{A})+\widetilde{\rho}\mathbf{F}_1 \tag{1.12}$$

We now substitute in (1.12) $\mathbf{N}_0 = \dot{\mathbf{n}}_0$ and use equation (1.11) to obtain:

$$\begin{aligned}\widetilde{\rho}\dot{\mathbf{u}}_1 = \widetilde{\rho}\,\mathbf{F}_1 - \nabla p_0 + K(\nabla \mathbf{n}_0)^T \Delta \mathbf{n}_0 + (\nabla \mathbf{n}_0)^T \mathbf{f}(\mathbf{n}_0) \\ + \frac{\alpha_2}{\gamma_1}\mathbf{f}(\mathbf{n}_0)\nabla \cdot \mathbf{n}_0 + \frac{\alpha_2 K}{\gamma_1}\Delta \mathbf{n}_0 \nabla \cdot \mathbf{n}_0 \\ + \frac{\alpha_3}{\gamma_1}\mathbf{n}_0 \nabla \cdot \mathbf{f}(\mathbf{n}_0) + \frac{\alpha_3 K}{\gamma_1}\mathbf{n}_0 \Delta(\nabla \cdot \mathbf{n}_0) + \frac{\widetilde{\alpha}_4}{2}\Delta \mathbf{u}_1. \quad (1.13)\end{aligned}$$

This last equation seems anything but a progress! We remark, however, that we can largely simplify it. We begin by observing that, far from defects, $|\mathbf{n}_0| \approx 1$ and therefore we can suppose that all terms on the right hand side of (1.13) containing $\mathbf{f}(\mathbf{n}_0)$ or its derivatives are very small. We also remark that we cannot use a similar argument in equation (1.11) because otherwise we would loose any reminiscence of the constraint $\mathbf{n} \cdot \mathbf{n} = 1$.

From a mathematical viewpoint, the three terms $(\nabla \mathbf{n}_0)^T \Delta \mathbf{n}_0$, $\Delta \mathbf{n}_0 \nabla \cdot \mathbf{n}_0$ and $\mathbf{n}_0 \Delta(\nabla \cdot \mathbf{n}_0)$ can be treated in a very similar way in our analysis. In order to keep the mathematical discussion of our model as simple as possible, we will therefore keep only the first of these terms. For some extension of our results to the complete system see, for example, [36].

After these considerations and hypothesis, forgetting subindexes and tildes and considering all the experimental constants except viscosity to 1, we have obtained the following simplified system:

$$\begin{cases} \rho \dot{\mathbf{u}} - \nu \Delta \mathbf{u} + \nabla p = -(\nabla \mathbf{n})^T \Delta \mathbf{n} + \rho \mathbf{F} \\ \nabla \cdot \mathbf{u} = 0 \\ \dot{\mathbf{n}} = \Delta \mathbf{n} - \mathbf{f}(\mathbf{n}) \end{cases} \quad (1.14)$$

This system was firstly obtained by Lin and Liu back in 1995 (see [21]) and since then has proved to be a very interesting benchmark for studying the long term behaviour of solutions for non-linear non-autonomous systems of partial differential equations. The goal of the following chapters is to delve into these problems under very general assumptions.

Chapter 2

Well posedness

We now want to study the well posedness of the simplified Ericksen-Leslie model (1.14) derived in last chapter. Essentially we will follow the analysis in [8] for the system without external driving force. In particular, we will prove existence of weak solutions (to be defined below) for the bi- and three-dimensional problem and we will complete the analysis in the bi-dimensional case by obtaining uniqueness of weak solutions and continuous dependence on the data. Moreover, always in the bi-dimensional case, we will be able to prove existence of strong solutions.

2.1 Existence of weak solutions

We will consider the following slightly amended version of system (1.14) where the order parameter will be denoted by $\mathbf{d}$ instead of using $\mathbf{n}$, as is common practice in the mathematical literature, and where the newly introduced fourth equation is a reminder of the relaxed constraint on the order parameter and is justified by the weak maximum principle holding for this system that will be introduced below (see lemma 2.1.7):

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = -\nabla\mathbf{d}^t\Delta\mathbf{d} + \mathbf{g}(t) & \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,\infty); \\ \partial_t\mathbf{d} + (\mathbf{u}\cdot\nabla)\mathbf{d} = \Delta\mathbf{d} - \mathbf{f}(\mathbf{d}) & \\ |\mathbf{d}| \le 1 & \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0, \quad \mathbf{d}(\mathbf{x},0) = \mathbf{d}_0 & \text{for } \mathbf{x}\in\Omega; \\ \mathbf{u}(\mathbf{x},t) = \mathbf{0}, \quad \mathbf{d}(\mathbf{x},t) = \mathbf{h}(\mathbf{x},t) & \text{on } \partial\Omega\times(0,\infty). \end{cases} \tag{2.1}$$

We remember that $\mathbf{f}(\mathbf{d}) = \frac{1}{\epsilon^2}(|\mathbf{d}|^2-1)\mathbf{d}$.

We start by introducing some functional spaces which are useful in the analysis of (2.1). With $\mathbf{L}^2$ we will denote the standard function space made up by vector valued $L^2(\Omega)$ functions. Analogously, $\mathbf{H}^1$ will be the usual vector Sobolev space constituted by componentwise $H^1(\Omega)$ functions. Moreover, let

$$\mathbf{H} = \overline{\left\{\mathbf{u} \in \mathbf{C}_0^\infty \mid \nabla \cdot \mathbf{u} = 0\right\}}^{\mathbf{L}^2}$$

and

$$\mathbf{V} = \overline{\left\{\mathbf{u} \in \mathbf{C}_0^\infty \mid \nabla \cdot \mathbf{u} = 0\right\}}^{\mathbf{H}_0^1}$$

be the usual divergenceless spaces used in the analysis of Navier-Stokes equations (see, for example, [39] or [31]). The family $\{\mathbf{w}_n\}_n$ will be the Hilbert basis of $\mathbf{V}$ given by the eigenfunctions of Stokes' problem:

$$\text{find } \mathbf{w}_i : \quad (\nabla \mathbf{w}_i, \nabla \mathbf{v}) = \lambda_i \, (\mathbf{w}_i, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \qquad |\mathbf{w}_i|_2 = 1 \tag{2.2}$$

where, thanks to the spectral theorem, the sequence of λ_i is monotonically increasing. From the well known spectral theory for compact operators (see, for example, [4]) we know that the functions $\mathbf{w}_i$ form a complete orthonormal basis in $\mathbf{H}$ which is also orthogonal in $\mathbf{H}^1$. For convenience we will write $\mathbf{V}^m = \langle \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \rangle$ for the finite dimensional subspace of $\mathbf{V}$ spanned by the first m eigenfunction of the just introduced Stokes' problem. From the regularity results for this problem (see [39]) and thanks to the finite-dimensionality of $\mathbf{V}^m$, we know that the canonical embedding $\mathbf{V}^m \hookrightarrow \mathbf{H}^2(\Omega)$ is compact. Finally, we will denote by $\mathbf{V}^*$ the dual space of $\mathbf{V}$.

Notation. We will write $|\mathbf{w}|_p$ to indicate the $\mathbf{L}^p$ norm of $\mathbf{w}$ and $|\mathbf{w}|_{\mathbf{H}^s}$ when referring to its $\mathbf{H}^s$ norm. Sometimes, as in the definition of the nonlinear potential $\mathbf{f}$ just after problem (2.1), we will write $|\mathbf{f}(\mathbf{x},t)|$ or shortly $|\mathbf{f}|$ when referring to the usual euclidean norm of vectors in $\mathbb{R}^n$. Therefore, while at fixed time $|\mathbf{d}|_2$ will be a real number, $|\mathbf{d}|$ will be a real valued function on Ω.

We now give the definition of weak solutions for system (2.1).

Definition 2.1.1. Let $T > 0$. A pair $(\mathbf{u}, \mathbf{d})$ is a *weak solution* to problem (2.1) if $(\mathbf{u}, \mathbf{d}) \in L^2(0,T;\mathbf{V}\times\mathbf{H}^2)$, $(\partial_t \mathbf{u}, \partial_t \mathbf{d}) \in L^p(0,T;\mathbf{V}^*)\times L^2(0,T;\mathbf{L}^2)$ (with $p = 2$ when $n = 2$ and $p = 4/3$ when $n = 3$), $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$ in $\mathbf{L}^2$ and $\mathbf{d}(\mathbf{x},0) = \mathbf{d}_0(\mathbf{x})$ in $\mathbf{H}^1$, if $\mathbf{d}(\mathbf{x},t) = \mathbf{h}(\mathbf{x},t)$ on $\partial\Omega \times (0,T)$ in the sense of trace spaces and if:

$$\begin{aligned} \langle \partial_t \mathbf{u}(t), \mathbf{v} \rangle + \langle (\mathbf{u}(t) \cdot \nabla)\mathbf{u}(t), \mathbf{v} \rangle + \nu \, (\nabla \mathbf{u}(t), \nabla \mathbf{v}) \\ + (\Delta \mathbf{d}(t), \nabla \mathbf{d}(t)\mathbf{v}) = \langle \mathbf{g}(t), \mathbf{v} \rangle \end{aligned}$$

holds for every $\mathbf{v} \in \mathbf{V}$, a.e. $t \in (0,T)$ and

$$\partial_t \mathbf{d}(t) + (\mathbf{u}(t) \cdot \nabla)\mathbf{d} = \Delta \mathbf{d} - \mathbf{f}(\mathbf{d}(t)) \qquad \text{and} \qquad |\mathbf{d}(\mathbf{x},t)| \leq 1$$

hold almost everywhere in $\Omega \times (0,T)$.

In this section we will prove the following result:

Theorem 2.1.1 (Weak existence). *Let $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$ be a regular bounded domain*[1]*, let $\mathbf{g} \in L^2(0,T;\mathbf{V}^*)$ and $\mathbf{h} \in L^2(0,T;\mathbf{H}^{3/2}(\partial\Omega))$, $\partial_t \mathbf{h} \in L^2(0,T;\mathbf{H}^{-1/2}(\partial\Omega))$ with $|\mathbf{h}| \leq 1$ a.e., let $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{d}_0 \in \mathbf{H}^1$ with $|\mathbf{d}_0| \leq 1$ a.e. Then there exists a weak solution $(\mathbf{u}, \mathbf{d})$ of (2.1).*

The proof of this result will occupy all the remaining part of this section. For the ease of the reader we give now a brief sketch of it before introducing all the details in the following pages. We start by studying the regularity of solution for a lifting problem for the nonautonomous boundary conditions on the order parameter and then give a semi-Galerkin formulation of problem (2.1) by considering the discretized problem for the velocity field leaving the other equations in the lifted form. We then prove local existence of solutions for the approximating problem through a fixed point argument. We remark that the lifespan of these solutions depends in a critical way on the dimension of the approximating subspace $\mathbf{V}^m$. Thus we need to extend the approximating solutions before passing to the limit. This is achieved by proving the following basic uniform energy estimate which also holds for the approximating solutions (see pages 30 and 31 below).

Lemma 2.1.2. *Under the assumptions of theorem 2.1.1, any weak solution of (2.1) satisfies for all $t > 0$ the estimate:*

$$\begin{aligned}|\mathbf{u}(t)|_2^2 + |\nabla \mathbf{d}(t)|_2^2 &\leq C_\Omega + |\mathbf{h}(t)|^2_{\mathbf{H}^{1/2}} + e^{-Ct}(|\mathbf{u}_0|_2^2 + |\nabla \mathbf{d}_0|_2^2) \\ &+ C_\Omega \int_0^t |\mathbf{h}(s)|^2_{\mathbf{H}^{3/2}}\, ds + C_\Omega \int_0^t |\partial_t \mathbf{h}(s)|^2_{\mathbf{H}^{-1/2}}\, ds + \frac{1}{\nu}\int_0^t |\mathbf{g}(s)|^2_{\mathbf{V}^*}\, ds \qquad (2.3)\end{aligned}$$

Finally we pass to the limit by means of standard arguments.

Remark. We note that lemma 2.1.2 justify us in introducing global solutions defined for all positive $t \in \mathbb{R}$. One actually only needs to observe that, when the data $\mathbf{g}$ is in $L^2_{\text{loc}}(0,\infty;\mathbf{V}^*)$ and $\mathbf{h}$ is in $L^2_{\text{loc}}(0,\infty;\mathbf{H}^{3/2}(\partial\Omega))$ and in $L^\infty(0,\infty;\mathbf{H}^{1/2}(\partial\Omega))$ such that $\partial_t \mathbf{h} \in L^2_{\text{loc}}(0,\infty;\mathbf{H}^{-1/2}(\partial\Omega))$, then the size of the time interval $[t, t+T]$ on which the local solutions obtained by theorem 2.1.1 are defined is independent of t. Moreover, estimate (2.3) is uniform in t and therefore any local solution can be extended by successive steps up to ∞.

The nonlinear potential f Before starting the proof of theorem 2.1.1 as outlined above, we summarize in this section some useful results about the nonlinear forcing term $\mathbf{f}(\mathbf{d})$ which appears in the equation for the order parameter field.

[1] Although we are not interested here in optimal regularity results for the domain, $\Omega \in \mathbf{C}^{1,1}$ should be a sufficient assumption.

Notation. In analogy with the notation for vector norms introduced above, we will write $|\mathbf{T}|$ while referring to the usual euclidean norm for tensors which is defined as $|\mathbf{T}|^2 \doteq \mathbf{T}_{ij}\mathbf{T}_{ij}$ (where we use the standard Einstein's sum convention on repeated indices).

Lemma 2.1.3. *If* $|\mathbf{d}| \leq 1$ *a.e. on* $\Omega \subset \mathbb{R}^n$*, then*

$$|\mathbf{f}(\mathbf{d})| \leq \frac{2\sqrt{3}}{9\epsilon^2} \qquad \text{and} \qquad |\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})|^2 \leq \frac{n+5}{\epsilon^4}.$$

Moreover if both $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *satisfy the above assumption, then*

$$|\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)| \leq \frac{2}{\epsilon^2} |\mathbf{d}_1 - \mathbf{d}_2| .$$

and

$$|\nabla_{\mathbf{d}}(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2))| \leq \frac{2\sqrt{n+8}}{\epsilon^2} |\mathbf{d}_1 - \mathbf{d}_2|$$

Proof. The proof of this lemma is a simple exercise in univariate and multivariate calculus. The first statement can be obtained by considering the function $|\mathbf{f}(\mathbf{d})| = \frac{1}{\epsilon^2}(1 - |\mathbf{d}|^2) |\mathbf{d}|$ as depending only on the real variable $|\mathbf{d}|$ in the interval $[0, 1]$. An easy calculation then shows that this function is maximized when $|\mathbf{d}| = \frac{1}{\sqrt{3}}$ and gives the desired result.

In order to obtain the remaining results, we need to compute explicitly the gradient $\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})$. We have:

$$\mathbf{f}(\mathbf{d})_{i,j} = \frac{1}{\epsilon^2} d_{i,j}(|\mathbf{d}|^2 - 1) + \frac{2}{\epsilon^2} d_i d_j = \frac{1}{\epsilon^2}\delta_{ij}(|\mathbf{d}|^2 - 1) + \frac{2}{\epsilon^2} d_i d_j.$$

where with $g_{,i}$ we mean the derivation with respect to d_i, that is $\frac{\partial g}{\partial d_i}$. Then the following simple estimate holds:

$$\begin{aligned}|\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})| = \mathbf{f}(\mathbf{d})_{i,j}\mathbf{f}(\mathbf{d})_{i,j} &= \frac{n}{\epsilon^4}(|\mathbf{d}|^2 - 1)^2 + \frac{4}{\epsilon^4} |\mathbf{d}|^4 + \frac{4}{\epsilon^2}(|\mathbf{d}|^2 - 1) |\mathbf{d}|^2 \\ &\leq \frac{1}{\epsilon^4}(n + 4 + 1) = \frac{n+5}{\epsilon^4}.\end{aligned}$$

The continuity constant of $\mathbf{f}$ can be obtained by evaluating the maximum eigenvalue of the gradient tensor calculated above. Due to the symmetric form of $\mathbf{f}$, we can assume that $\mathbf{d}$ is directed along the first versor of the canonical basis in $\mathbb{R}^n$. A simple substitution then gives $\lambda_1 = \frac{1}{\epsilon^2}(3|\mathbf{d}|^2 - 1)$ and $\lambda_i = \frac{1}{\epsilon^2}(|\mathbf{d}|^2 - 1)$ for $i = 2, \ldots, n$. We can now easily conclude that the maximum eigenvalue is obtained by evaluating λ_1 when $|\mathbf{d}| = 1$ and thus $|\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)| \leq \frac{2}{\epsilon^2}|\mathbf{d}_1 - \mathbf{d}_2|$ as claimed.

Finally, the last statement of the lemma can be obtained through a direct computation where we have set $\mathbf{e} \doteq \mathbf{d}_1 - \mathbf{d}_2$ as follows:

$$\begin{aligned}|\nabla_{\mathbf{d}}(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2))|^2 = \frac{1}{\epsilon^4}(d_{1\,k}e_k + e_k d_{2\,k})\delta_{ij}\delta_{ij}(d_{1\,l}e_l + e_l d_{2\,l}) \\ + \frac{4}{\epsilon^4}(e_i d_{1\,j} + d_{2\,i}e_j)(e_i d_{1\,j} + d_{2\,i}e_j) \\ + \frac{4}{\epsilon^4}(d_{1\,k}e_k + e_k d_{2\,k})\delta_{ij}(e_i d_{1\,j} + d_{2\,i}e_j).\end{aligned}$$

Remembering that $\delta_{ij}\delta_{ij} = n$ and using repeatedly that $|\mathbf{d}_i| \leq 1$, $i = 1, 2$ and the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned}|\nabla_{\mathbf{d}}(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2))|^2 &= \frac{n+4}{\epsilon^4}((\mathbf{d}_1 + \mathbf{d}_2)\cdot\mathbf{e})^2 \\ &\quad + \frac{4}{\epsilon^4}|\mathbf{e}|^2\left(|\mathbf{d}_1|^2 + |\mathbf{d}_2|^2\right) + \frac{8}{\epsilon^4}(\mathbf{e}\cdot\mathbf{d}_1)(\mathbf{e}\cdot\mathbf{d}_2) \\ &\leq \frac{n+4}{\epsilon^4}4|\mathbf{e}|^2 + \frac{8}{\epsilon^4}|\mathbf{e}|^2\,\frac{8}{\epsilon^4}|\mathbf{e}|^2 = \frac{4(n+8)}{\epsilon^4}|\mathbf{e}|^2\end{aligned}$$

from which our claim can be immediately deduced. □

Regularity of a time dependent lifting problem We start the proof of theorem 2.1.1 by considering this simple linear lifting problem:

$$\begin{cases}\partial_t\widetilde{\mathbf{d}} - \Delta\widetilde{\mathbf{d}} = 0 & \text{in } Q_T \doteq \Omega\times(0,T);\\ \widetilde{\mathbf{d}} = \mathbf{h} & \text{on } \partial\Omega\times(0,T);\\ \widetilde{\mathbf{d}}(0) = \mathbf{d}_0 & \text{in } \Omega.\end{cases} \tag{2.4}$$

Existence and uniqueness of solutions for problem (2.4) follow easily from the standard Galerkin method for linear parabolic problems (see, for instance, [22]). In particular the following lemma holds.

Lemma 2.1.4. *Let $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$ be a regular domain, let $\mathbf{d}_0 \in \mathbf{L}^2$ and let $\mathbf{h} \in L^2(0,T;\mathbf{H}^{1/2}(\partial\Omega))$. Then there exists a unique weak solution $\widetilde{\mathbf{d}} \in L^\infty(0,T;\mathbf{L}^2) \cap L^2(0,T;\mathbf{H}^1)$ $\forall T > 0$ of problem (2.4). Moreover the following estimate holds:*

$$|\widetilde{\mathbf{d}}(T)|_2^2 + \int_0^T |\nabla\widetilde{\mathbf{d}}|_2^2\,dt \leq |\mathbf{d}_0|_2^2 + \int_0^T C|\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}\,dt.$$

Proof. The lemma can be proven through the following formal estimate[2]. Choosing $\widetilde{\mathbf{d}}$ as test function in the weak formulation of problem (2.4), we

[2]We remember that all the formal estimates computed in this chapter are rigorously valid only for the approximating Galerkin problems. However, the estimate obtained for the sequence $\{\mathbf{w}_n\}$ of approximating solutions being uniform in n, we can always extract from $\{\mathbf{w}_i\}$ a weakly convergent subsequence whose limit is the solution of the

obtain:

$$\begin{aligned}&\left\langle \partial_t\widetilde{\mathbf{d}},\widetilde{\mathbf{d}}\right\rangle-\left(\Delta\widetilde{\mathbf{d}},\widetilde{\mathbf{d}}\right)=0\\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{\mathbf{d}}|_2^2+|\nabla\widetilde{\mathbf{d}}|_2^2={}_{\mathbf{H}^{-1/2}(\partial\Omega)}\left\langle\partial_\nu\widetilde{\mathbf{d}},\mathbf{h}\right\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}\\ &\qquad\le|\partial_\nu\widetilde{\mathbf{d}}|_{\mathbf{H}^{-1/2}(\partial\Omega)}|\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}\overset{T}{\le}C|\widetilde{\mathbf{d}}|_{\mathbf{H}^1}|\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}\end{aligned}$$

We remember the equivalent norm in $\mathbf{H}^1$: $|\mathbf{g}|_{\mathbf{H}^1}\le C|\nabla\mathbf{g}|_2+C|\mathbf{g}|_{\mathbf{H}^{1/2}(\partial\Omega)}$ (see, for example, [28, Theorem 2.4.20]). Therefore we have:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\widetilde{\mathbf{d}}|_2^2+|\nabla\widetilde{\mathbf{d}}|_2^2&\overset{P}{\le}C|\nabla\widetilde{\mathbf{d}}|_2|\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}+C|\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}\\ &\overset{Y}{\le}\frac{1}{2}|\nabla\widetilde{\mathbf{d}}|_2^2+C|\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}\end{aligned}$$

After reordering all terms and integrating in time between 0 and T, we get the sought a priori estimate. □

Beyond this basic existence result, we can also prove stronger regularity properties for the solution $\widetilde{\mathbf{d}}$ of the lifting problem (2.4).

Lemma 2.1.5. *Let the same assumptions of lemma 2.1.4 be verified.*

- *If* $\mathbf{d}_0\in\mathbf{H}^1$, $\mathbf{h}\in L^2(0,T;\mathbf{H}^{3/2}(\partial\Omega))$ *and* $\partial_t\mathbf{h}\in L^2(0,T;\mathbf{H}^{-1/2}(\partial\Omega))$ *then* $\widetilde{\mathbf{d}}\in L^\infty(0,T;\mathbf{H}^1)\cap L^2(0,T;\mathbf{H}^2)$ *and the following estimate holds:*

$$|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^1}+\int_0^t|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^2}\le|\mathbf{d}_0|^2_{\mathbf{H}^1}+C\int_0^t|\partial_t\mathbf{h}|^2_{\mathbf{H}^{-1/2}(\partial\Omega)}+|\mathbf{h}|^2_{\mathbf{H}^{3/2}(\partial\Omega)}\,dt. \tag{2.5}$$

- *If* $\mathbf{d}_0\in\mathbf{H}^2$, $\mathbf{h}\in L^2(0,T;\mathbf{H}^{5/2}(\partial\Omega))$ *and* $\partial_t\mathbf{h}\in L^2(0,T;\mathbf{H}^{1/2}(\partial\Omega))$ *then* $\widetilde{\mathbf{d}}\in L^\infty(0,T;\mathbf{H}^2)\cap L^2(0,T;\mathbf{H}^3)$. *Moreover the following estimate holds:*

$$|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^2}+\int_0^t|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3}\le|\mathbf{d}_0|^2_{\mathbf{H}^2}+C\int_0^t|\partial_t\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}+|\mathbf{h}|^2_{\mathbf{H}^{5/2}(\partial\Omega)}\,dt. \tag{2.6}$$

In order to prove this result we will proceed formally as usual. Take $-\Delta\widetilde{\mathbf{d}}$ as test function in the weak formulation of the lifting problem. We

---

original problem. Thanks to the weak lower semi-continuity of norms, this limiting solution obviously satisfies the same estimates proved for the approximating solutions.

We also observe that, by solving the approximate Galerkin problem associated with equation (2.4), it is necessary to consider a further lifting problem for the boundary condition $\mathbf{h}$ in $[0,T]$, by choosing a convenient function $\mathbf{H}\in\mathbf{L}^2(0,T;\mathbf{H}^2)$ which satisfies the boundary requirements. However, thanks to the theory of trace operators, all the estimates of this section remain correct. We refer the interested reader to [22] for all the results on trace spaces used here and in the sequel.

observe that:

$$\int_\Omega \partial_t \widetilde{\mathbf{d}} \cdot \Delta \widetilde{\mathbf{d}}\, d\omega = -\int_\Omega \partial_t \nabla \widetilde{\mathbf{d}} : \nabla \widetilde{\mathbf{d}}\, d\Omega + {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\left\langle \partial_t \widetilde{\mathbf{d}}, \partial_\nu \widetilde{\mathbf{d}} \right\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}$$
$$= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \widetilde{\mathbf{d}}|_2^2 + {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\left\langle \partial_t \mathbf{h}, \partial_\nu \widetilde{\mathbf{d}} \right\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}$$

and remembering the equivalent form of the $\mathbf{H}^2$ norm $|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^2} \sim |\Delta \widetilde{\mathbf{d}}|_2^2 + |\widetilde{\mathbf{d}}|^2_{\mathbf{H}^{3/2}(\partial\Omega)}$, we get the following estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \widetilde{\mathbf{d}}|_2^2 + |\Delta \widetilde{\mathbf{d}}|_2^2 = {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\left\langle \partial_t \mathbf{h}, \partial_\nu \widetilde{\mathbf{d}} \right\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}$$
$$\overset{CS-T}{\leq} C|\partial_t \mathbf{h}|_{\mathbf{H}^{-1/2}(\partial\Omega)} |\widetilde{\mathbf{d}}|_{\mathbf{H}^2}$$
$$\overset{P-Y}{\leq} C|\partial_t \mathbf{h}|^2_{\mathbf{H}^{-1/2}(\partial\Omega)} + \frac{1}{2}|\Delta \widetilde{\mathbf{d}}|_2^2 + \frac{1}{2}|\mathbf{h}|^2_{\mathbf{H}^{3/2}(\partial\Omega)}.$$

We now integrate in time and obtain:

$$|\nabla \widetilde{\mathbf{d}}(t)|_2^2 + \int_0^t |\Delta \widetilde{\mathbf{d}}|_2\, dt \leq |\nabla \widetilde{\mathbf{d}}(0)|_2^2 + C\int_0^t |\partial_t \mathbf{h}|^2_{\mathbf{H}^{-1/2}(\partial\Omega)} + |\mathbf{h}|^2_{\mathbf{H}^{3/2}(\partial\Omega)}\, dt$$

from which (2.5) easily follows. We therefore deduce that $\widetilde{\mathbf{d}} \in L^\infty(0,T;\mathbf{H}^1) \cap L^2(0,T;\mathbf{H}^2)$.

In order to prove the second part of lemma 2.1.5, we start by multiplying equation (2.4) by $-\partial_t \Delta \widetilde{\mathbf{d}}$. If we integrate by parts the time derivative term, we obtain:

$$|\partial_t \nabla \widetilde{\mathbf{d}}|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta \widetilde{\mathbf{d}}|_2^2 = {}_{\mathbf{H}^{1/2}(\partial\Omega)}\left\langle \partial_t \mathbf{h}, \partial_t \partial_\nu \widetilde{\mathbf{d}} \right\rangle_{\mathbf{H}^{-1/2}(\partial\Omega)}$$
$$\overset{CS-T}{\leq} C|\partial_t \mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)} |\partial_t \widetilde{\mathbf{d}}|_{\mathbf{H}^1}$$

However, this time a problem arises: actually Poincarè's inequality does not hold for $\partial_t \widetilde{\mathbf{d}}$. Nevertheless, from the strong form of the lifting problem (2.4) we have $|\partial_t \widetilde{\mathbf{d}}|_2 = |\Delta \widetilde{\mathbf{d}}|_2$. Therefore applying Young's inequality to the last estimate we get:

$$|\partial_t \nabla \widetilde{\mathbf{d}}|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta \widetilde{\mathbf{d}}|_2^2 \leq C|\partial_t \mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)} + \frac{1}{2}|\partial_t \nabla \widetilde{\mathbf{d}}|_2^2 + \frac{1}{2}|\Delta \widetilde{\mathbf{d}}|_2$$

Now we integrate in time and thanks the estimates of the previous paragraphs and through standard arguments we deduce $\widetilde{\mathbf{d}} \in L^\infty(0,T;\mathbf{H}^2)$ and $\partial_t \widetilde{\mathbf{d}} \in L^2(0,T;\mathbf{H}^1)$. Finally, using again directly the equation (2.1), we obtain $\widetilde{\mathbf{d}} \in L^2(0,T;\mathbf{H}^3)$, $\partial_t \widetilde{\mathbf{d}} \in L^\infty(0,T;\mathbf{L}^2)$ and estimate (2.6).

We end this summary of regularity results for problem (2.4) by noting that, under physically sound assumptions, the following maximum principle holds (see, for example, [14]).

Lemma 2.1.6. *Let the same assumptions of lemma 2.1.4 be verified. If in addition* $|\mathbf{h}| \leq 1$ *a.e. on* $\partial\Omega \times [0,T]$ *then* $|\widetilde{\mathbf{d}}| \leq 1$ *a.e.* $(\mathbf{x},t) \in \Omega \times [0,T]$.

The semi-Galerkin approximation We now go back to the weak formulation of problem (2.1) and give its announced Galerkin approximation. We will use the usual Faedo-Galerkin method only for the velocity field. In particular, we will search a solution $\mathbf{u}^m \in \mathbf{C}^1(0,T;\mathbf{V}^m)$ and $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^2) \cap L^\infty(0,T;\mathbf{H}^1)$ such that:

$$\begin{cases} (\partial_t \mathbf{u}^m(t), \mathbf{v}_m) + ((\mathbf{u}^m(t)\cdot\nabla)\mathbf{u}^m(t), \mathbf{v}_m) + \nu\,(\nabla \mathbf{u}^m(t), \nabla \mathbf{v}_m) & \\ \qquad + (\Delta \mathbf{d}^m(t), \nabla \mathbf{d}^m(t)\mathbf{v}_m) = (\mathbf{g}(t), \mathbf{v}_m) & \forall \mathbf{v}_m \in \mathbf{V}^m; \\ \partial_t \mathbf{d}^m(t) + (\mathbf{u}^m(t)\cdot\nabla)\mathbf{d}^m(t) = \Delta \mathbf{d}^m(t) - \mathbf{f}(\mathbf{d}^m(t)) & \text{a.e. in } (0,T)\times\Omega; \\ |\mathbf{d}^m| \le 1 & \\ \mathbf{u}^m(0) = \mathbf{u}_{0m} \doteq P_m \mathbf{u}_0 & \\ \mathbf{d}^m(\mathbf{x},0) = \mathbf{d}_0 \qquad \text{for } \mathbf{x}\in\Omega; & \\ \mathbf{d}^m(\mathbf{x},t) = \mathbf{h}(\mathbf{x},t) \qquad \text{on } \partial\Omega\times(0,T) & \end{cases} \tag{2.7}$$

holds, where the linear operator $P_m : \mathbf{H} \to \mathbf{V}^m$ is the orthogonal (in $\mathbf{L}^2$) projection on $\mathbf{V}^m$ (we remember that $\mathbf{u}_{0m} \to \mathbf{u}_0$ in $\mathbf{L}^2$ for the dominated convergence theorem and the completeness of the basis we are using). Actually also the following lifted problem will play an important role:

$$\begin{cases} (\partial_t \mathbf{u}^m(t), \mathbf{v}_m) + ((\mathbf{u}^m(t)\cdot\nabla)\mathbf{u}^m(t), \mathbf{v}_m) + \nu\,(\nabla \mathbf{u}^m(t), \nabla \mathbf{v}_m) & \\ \qquad + (\Delta \mathbf{d}^m(t), \nabla \mathbf{d}^m(t)\mathbf{v}_m) = (\mathbf{g}(t), \mathbf{v}_m) & \forall \mathbf{v}_m \in \mathbf{V}^m; \\ \partial_t \widehat{\mathbf{d}}^m(t) + (\mathbf{u}^m(t)\cdot\nabla)\mathbf{d}^m(t) = \Delta \widehat{\mathbf{d}}^m(t) - \mathbf{f}(\mathbf{d}^m(t)) & \text{a.e. in } (0,T)\times\Omega; \\ |\mathbf{d}^m| \le 1 & \\ \mathbf{u}^m(0) = \mathbf{u}_{0m} \doteq P_m \mathbf{u}_0 & \\ \widehat{\mathbf{d}}^m(\mathbf{x},0) = \mathbf{0} \qquad \text{for } \mathbf{x}\in\Omega; & \\ \widehat{\mathbf{d}}^m(\mathbf{x},t) = \mathbf{0} \qquad \text{on } \partial\Omega\times(0,T) & \end{cases} \tag{2.8}$$

where we write $\widehat{\mathbf{d}}^m \doteq \mathbf{d}^m - \widetilde{\mathbf{d}}$ and $\widetilde{\mathbf{d}}$ is the solution of the lifting problem (2.4).

Local time existence of solutions We will now apply a fixed point argument to prove existence of (at least) a solution on the time interval $[0,T_m]$ for the approximating problem (2.7). We start by introducing the following splitting.

1. Let $\overline{\mathbf{u}}^m \in \mathbf{C}(0,T;\mathbf{V}^m)$ be a given velocity field. We look after the order parameter field $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^2) \cap L^\infty(0,T;\mathbf{H}^1)$ which solves:

$$\begin{cases} \partial_t \mathbf{d}^m + (\overline{\mathbf{u}}^m\cdot\nabla)\mathbf{d}^m = \Delta \mathbf{d}^m - \mathbf{f}(\mathbf{d}^m) & \text{in } \Omega\times(0,T); \\ \mathbf{d}^m(\mathbf{x},0) = \mathbf{d}_0 & \text{in } \Omega; \\ \mathbf{d}^m(\mathbf{x},t) = \mathbf{h} & \text{on } \partial\Omega\times(0,T). \end{cases} \tag{2.9a}$$

2. Let $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^2) \cap L^\infty(0,T;\mathbf{H}^1)$ be the order parameter field just determined. The second part of the splitting consists in finding a velocity field $\mathbf{u}^m \in H^1(0,T;\mathbf{V}^m)$ such that the following equation is satisfied.

$$\begin{cases} (\partial_t \mathbf{u}^m(t), \mathbf{v}) + ((\overline{\mathbf{u}}^m(t) \cdot \nabla)\mathbf{u}^m(t), \mathbf{v}) + \nu\,(\nabla \mathbf{u}^m(t), \nabla \mathbf{v}) & \\ \qquad = -\,(\Delta \mathbf{d}^m(t), \nabla \mathbf{d}^m(t)\mathbf{v}) + \langle \mathbf{g}(t), \mathbf{v}\rangle & \forall \mathbf{v} \in \mathbf{V}^m; \\ \mathbf{u}^m(0) = \mathbf{u}_0^m & \text{in } \Omega. \end{cases} \tag{2.9b}$$

We stress that in this problem the order parameter field $\mathbf{d}^m$ is given.

Remark. The just introduced splitting and the fixed point argument of the following pages can be considered as possible starting points in the design of an efficient numerical scheme to solve problem (2.1). One possible advantage of this numerical strategy is that one can use only existing programs that already efficiently solve Navier-Stokes equations and simple transport-diffusion equations without the need of implementing from scratch a whole new numerical algorithm.

Existence and uniqueness for problem (2.9a) In order to obtain the existence of an order parameter field which satisfies equation (2.9a) we will use again a fixed point argument. Consider the following linearization of problem (2.9a):

$$\begin{cases} \partial_t \mathbf{d}^m + (\overline{\mathbf{u}}^m \cdot \nabla)\mathbf{d}^m = \Delta \mathbf{d}^m - \mathbf{f}(\mathbf{e}) & \text{in } \Omega \times (0,T); \\ \mathbf{d}^m(\mathbf{x},0) = \mathbf{d}_0 & \text{in } \Omega; \\ \mathbf{d}^m(\mathbf{x},t) = \mathbf{h} & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{2.10}$$

and let $\mathbf{e} \in L^2(0,T;\mathbf{H}^1)$. Noting that $|\mathbf{f}(\mathbf{e})| \leq 1 + |\mathbf{e}|^3$ and using Sobolev embedding theorems, we easily deduce that $\mathbf{f}(\mathbf{e}) \in L^2(0,T;\mathbf{L}^2)$. Since by hypothesis we have $\overline{\mathbf{u}}^m \in \mathbf{C}(0,T;\mathbf{H}^1)$, the standard theory for linear parabolic equations immediately tells us that the solution of the problem obtained using the lifting (2.4) satisfies $\widehat{\mathbf{d}}^m \in L^2(0,T;\mathbf{H}^1)$ and $\partial_t \widehat{\mathbf{d}}^m \in L^2(0,T;\mathbf{H}^{-1})$ (see [14]). Thanks to the regularity of the lifting $\widetilde{\mathbf{d}}$ proved above (see lemmas 2.1.4 and 2.1.6), we conclude that also the solution $\mathbf{d}^m$ of problem (2.10) satisfies the same regularity properties.

We now chose $\partial_t \widehat{\mathbf{d}}^m$ as test function in the weak formulation of the linearized lifted version of problem (2.9a). Remembering that $\mathbf{V}^m \subset \mathbf{L}^\infty$, we obtain the following estimate:

$$\begin{aligned} |\partial_t \widehat{\mathbf{d}}^m|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \widehat{\mathbf{d}}^m|_2^2 &= -\int_\Omega (\overline{\mathbf{u}}^m \cdot \nabla)\mathbf{d}^m \partial_t \widehat{\mathbf{d}}^m \, d\Omega - \int_\Omega \mathbf{f}(\mathbf{e})\partial_t \widehat{\mathbf{d}}^m \, d\Omega \\ &\overset{H}{\leq} |\overline{\mathbf{u}}^m|_\infty |\nabla \mathbf{d}^m|_2 |\partial_t \widehat{\mathbf{d}}^m|_2 + |\mathbf{f}(\mathbf{e})|_2 |\partial_t \widehat{\mathbf{d}}^m|_2 \\ &\overset{Y}{\leq} C|\nabla \overline{\mathbf{u}}^m|_\infty^2 |\nabla \mathbf{d}^m|_2^2 + \frac{1}{4}|\partial_t \widehat{\mathbf{d}}^m|_2^2 + |\mathbf{f}(\mathbf{e})|_2^2 + \frac{1}{4}|\partial_t \widehat{\mathbf{d}}^m|_2^2. \end{aligned}$$

Simple algebra and Gronwall's inequality then give $\widehat{\mathbf{d}}^m \in L^\infty(0,T;\mathbf{H}^1)$ and $\partial_t\widehat{\mathbf{d}}^m \in L^2(0,T;\mathbf{L}^2)$.

From well-known elliptic-regularity results applied to problem (2.10) (see, for example, [15] for a comprehensive exposition), we actually have $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^2)$. We therefore deduce that the solution operator $\mathcal{S} : \mathbf{e} \mapsto \mathbf{d}^m$ of problem (2.10) maps bounded subsets of $L^2(0,T;\mathbf{H}^1)$ into bounded subsets of $L^2(0,T;\mathbf{H}^2)\cap W^{1,2}(0,T;\mathbf{L}^2)$ and henceforth into precompact subsets of $L^2(0,T;\mathbf{H}^1)$. This solution operator is also continuous as a function of $\mathbf{e}$ as was shown in lemma 2.1.3. Therefore we have proven that $\mathcal{S}$ is compact.

In order to apply the Leray-Schauder fixed point argument (see, for example, [10]) we still need an "a priori" estimate for the solutions of problem $\mathbf{d}^m_s = s\mathcal{S}(\mathbf{d}^m_s)$ with $s \in [0,1]$. This time we take $\mathbf{d}^m$ as test function in the weak formulation of problem (2.10) and obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{d}^m_s|_2^2 + |\nabla\mathbf{d}^m_s|_2^2 + s\int_\Omega \mathbf{f}(\mathbf{d}^m_s)\cdot\mathbf{d}^m_s\,d\Omega = {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\langle\partial_\nu\mathbf{d}^m_s,\mathbf{h}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}\,.$$

However, due to the particular structure of the nonlinear forcing term $\mathbf{f}$ we have:

$$\int_\Omega \mathbf{f}(\mathbf{d}^m_s)\cdot\mathbf{d}^m_s\,d\Omega = \frac{1}{\epsilon^2}\int_\Omega(|\mathbf{d}^m_s|^2-1)|\mathbf{d}^m_s|^2\,d\Omega \geq -\frac{1}{4\epsilon^2}|\Omega|$$

and therefore, remembering $|\mathbf{g}|_{\mathbf{H}^1} \leq C|\nabla\mathbf{g}|_2 + C|\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}$, we deduce:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{d}^m_s|_2^2 + |\nabla\mathbf{d}^m_s|_2^2 \leq C|\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)} + \frac{1}{2}|\nabla\mathbf{d}^m_s|_2^2 + \frac{1}{4\epsilon^2}|\Omega|$$

that is $|\mathbf{d}^m_s|_{L^2(0,T;\mathbf{H}^1)} \leq C$ where C does not depend on s. Using Leray-Schauder theorem there exists a fixed point for the operator $\mathcal{S}$ i.e. a solution in $[0,T]$ of problem (2.9a).

Before continuing the well-posedness analysis of the problem for the order parameter field, we observe that the following weak maximum principle holds for solutions of problem (2.9a) (see [7] for a proof).

Lemma 2.1.7 (Weak maximum principle)**.** *Let* $\mathbf{d}_0 \in \mathbf{H}^1$ *such that* $|\mathbf{d}_0(\mathbf{x})| \leq 1$ *a.e.* $\mathbf{x} \in \Omega$ *and let* $\mathbf{h} \in L^\infty(0,T;\mathbf{H}^{1/2}(\partial\Omega))$ *such that* $|\mathbf{h}(\mathbf{x},t)| \leq 1$ *a.e.* $(\mathbf{x},t) \in \partial\Omega\times[0,T]$*. Then every weak solution* $\mathbf{d}$ *of problem* (2.9a) *verifies* $|\mathbf{d}(\mathbf{x},t)| \leq 1$ *a.e.* $\Omega\times[0,T]$

Consider now the lifted problem obtained by subtracting system (2.4) from (2.9a):

$$\begin{cases}\partial_t\widehat{\mathbf{d}}^m + (\overline{\mathbf{u}}^m\cdot\nabla)\mathbf{d}^m = \Delta\widehat{\mathbf{d}}^m - \mathbf{f}(\mathbf{d}^m) & \text{in } \Omega\times(0,T);\\ \widehat{\mathbf{d}}^m(\mathbf{x},0) = \mathbf{0} & \text{in } \Omega;\\ \widehat{\mathbf{d}}^m(\mathbf{x},t) = \mathbf{0} & \text{on } \partial\Omega\times(0,T)\end{cases} \tag{2.9a$'$}$$

As usual, let $-\Delta\widehat{\mathbf{d}}^m$ be the test function in the weak form of (2.9a′). We obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\widehat{\mathbf{d}}^m|_2^2 + |\Delta\widehat{\mathbf{d}}^m|_2^2 = \Big(\mathbf{f}(\mathbf{d}^m), \Delta\widehat{\mathbf{d}}^m\Big) + \Big((\overline{\mathbf{u}}^m\cdot\nabla)\mathbf{d}^m, \Delta\widehat{\mathbf{d}}^m\Big)$$
$$\overset{H}{\leq} |\mathbf{f}(\mathbf{d}^m)|_2|\Delta\widehat{\mathbf{d}}^m|_2 + |\overline{\mathbf{u}}^m|_6|\nabla\mathbf{d}^m|_3|\Delta\widehat{\mathbf{d}}^m|_2$$
$$\overset{Y-S}{\leq} \frac{1}{2}|\mathbf{f}(\mathbf{d}^m)|_2^2 + \frac{1}{2}|\Delta\widehat{\mathbf{d}}^m|_2^2 + C|\nabla\overline{\mathbf{u}}^m|_2|\nabla\mathbf{d}^m|_3|\Delta\widehat{\mathbf{d}}^m|_2$$

that is:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\widehat{\mathbf{d}}^m|_2^2 + |\Delta\widehat{\mathbf{d}}^m|_2^2 \leq |\mathbf{f}(\mathbf{d}^m)|_2^2 + C|\nabla\overline{\mathbf{u}}^m|_2|\nabla\mathbf{d}^m|_3|\Delta\widehat{\mathbf{d}}^m|_2. \tag{2.11}$$

We now consider the right hand side of this last inequality and notice that both contributions are easily bounded. First of all, since $\mathbf{f}$ is continuous and since the weak maximum principle of lemma 2.1.7 holds, we deduce from lemma 2.1.3 that $\sup_{t\in(0,\infty)}|\mathbf{f}(\mathbf{d}^m)|_2 \leq \frac{2}{3\sqrt{3}}\frac{1}{\epsilon^2}|\Omega|^{1/2}$. Moreover, we have $\overline{\mathbf{u}}^m \in \mathbf{C}(0,T;\mathbf{V}^m)$ which implies $\sup_{t\in[0,T]}|\nabla\overline{\mathbf{u}}^m|_2 \leq M$. Thanks to Sobolev embedding theorems (see [37]), we then have $|\nabla\mathbf{d}^m|_3 \leq C|\nabla\mathbf{d}^m|_2^{1/2}|\nabla\mathbf{d}^m|_{\mathbf{H}^1}^{1/2}$ so that the last term (2.11) becomes:

$$|\nabla\overline{\mathbf{u}}^m|_2|\nabla\mathbf{d}^m|_3|\Delta\widehat{\mathbf{d}}^m|_2 \leq CM(|\nabla\mathbf{d}^m|_2^{1/2} + |\mathbf{d}^m|_{\mathbf{H}^2}^{1/2})|\nabla\mathbf{d}^m|_2^{1/2}|\Delta\widehat{\mathbf{d}}^m|_2$$
$$\leq CM|\nabla\mathbf{d}^m|_2|\Delta\widehat{\mathbf{d}}^m|_2 + CM|\nabla\mathbf{d}^m|_2^{1/2}|\Delta\widehat{\mathbf{d}}^m|_2^{3/2}$$
$$+ CM|\widetilde{\mathbf{d}}^m|_{\mathbf{H}^2}^{1/2}|\nabla\mathbf{d}^m|_2^{1/2}|\Delta\widehat{\mathbf{d}}^m|_2$$
$$\overset{Y}{\leq} CM^2|\nabla\mathbf{d}^m|_2^2 + \frac{1}{4}|\Delta\widehat{\mathbf{d}}^m|_2^2 + CM^4|\nabla\mathbf{d}^m|_2^2$$
$$+ \frac{1}{4}|\Delta\widehat{\mathbf{d}}^m|_2^2 + CM^2|\nabla\mathbf{d}^m|_2|\widetilde{\mathbf{d}}^m|_{\mathbf{H}^2} + \frac{1}{4}|\Delta\widehat{\mathbf{d}}^m|_2^2.$$

Substituting back this estimate in (2.11) we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\widehat{\mathbf{d}}^m|_2^2 + \frac{1}{4}|\Delta\widehat{\mathbf{d}}^m|_2^2 \leq \frac{2}{3\sqrt{3}}\frac{1}{\epsilon^2}|\Omega|^{1/2} + C(M^4+M^2)|\nabla\mathbf{d}^m|_2^2 + CM^2|\widetilde{\mathbf{d}}^m|_{\mathbf{H}^2}^2.$$

Integrating in time from 0 to T and remembering from our previous discussion that $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^1)$, we eventually get:

$$|\nabla\widehat{\mathbf{d}}^m(t)|_2^2 + \frac{1}{4}\int_0^t|\Delta\widehat{\mathbf{d}}^m(s)|_2^2\,ds \leq T\frac{2}{3\sqrt{3}}\frac{1}{\epsilon^2}|\Omega|^{1/2}$$
$$+ C(M^4+M^2)|\nabla\mathbf{d}|_{L^2(0,T;\mathbf{L}^2)}^2 + CM^2|\widetilde{\mathbf{d}}|_{L^2(0,T;\mathbf{H}^2)}^2 \doteq K(T,M) \tag{2.12}$$

for almost every $t \in [0,T]$. We remind that, from the above results, we have $\widetilde{\mathbf{d}} \in L^2(0,T;\mathbf{H}^2)$ and therefore the right hand side of last estimate is bounded. We have thus obtained that $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^2)\cap L^\infty(0,T;\mathbf{H}^1)$. Observe that we also have $\lim_{T\to 0}K(T,M) = 0$.

Before ending this part of the proof, we still have to show uniqueness of solutions for problem (2.9a) and the continuity of the solution operator $\mathcal{S}_{\mathbf{d}}^m : \mathbf{C}(0,T;\mathbf{V}^m) \to L^\infty(0,T;\mathbf{H}^1) \cap L^2(0,T;\mathbf{H}^2)$, $\mathcal{S}_{\mathbf{d}}^m : \overline{\mathbf{u}}^m \mapsto \mathbf{d}^m$. Consider now a pair of velocity fields $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{C}(0,T;\mathbf{V}^m)$ and let $\mathbf{d}_1$, $\mathbf{d}_2$ be the respective solutions of problem (2.9a). Choosing $\delta\mathbf{d} \doteq \mathbf{d}_1 - \mathbf{d}_2$ as test function for the difference between the equations satisfied by $\mathbf{d}_1$ and $\mathbf{d}_2$ and using lemma 2.1.3, we get:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\mathbf{d}|_2^2 + |\nabla\delta\mathbf{d}|_2^2 = \int_\Omega (\delta\mathbf{u}\cdot\nabla)\mathbf{d}_1 \cdot \delta\mathbf{d}\, d\Omega + \int_\Omega (\mathbf{u}_2\cdot\nabla)\delta\mathbf{d}\cdot\delta\mathbf{d}\, d\Omega \\
+ \int_\Omega (\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2))\cdot\delta\mathbf{d}\, d\Omega \\
\overset{H}{\le} |\delta\mathbf{u}|_\infty |\nabla\mathbf{d}_1|_2 |\delta\mathbf{d}|_2 + \frac{2}{\epsilon^2}|\delta\mathbf{d}|_2^2 \\
\overset{Y}{\le} \frac{1}{2}|\delta\mathbf{d}|_2^2 + \frac{1}{2}|\delta\mathbf{u}|_\infty^2 |\nabla\mathbf{d}_1|_2^2 + \frac{2}{\epsilon^2}|\delta\mathbf{d}|_2^2
\end{aligned}
$$

where we have set $\delta\mathbf{u} \doteq \mathbf{u}_1 - \mathbf{u}_2$. Upon reordering the terms of last estimate we get:

$$
\frac{\mathrm{d}}{\mathrm{d}t}|\delta\mathbf{d}|_2^2 + 2|\nabla\delta\mathbf{d}|_2^2 \le |\delta\mathbf{u}|_\infty^2 |\nabla\mathbf{d}_1|_2^2 + \left(\frac{4}{\epsilon^2} + 1\right)|\delta\mathbf{d}|_2^2
$$

If $\delta\mathbf{u} = 0$, we have thus obtained uniqueness of solutions for the first part of our split problem. Otherwise, in the general case, we have the continuity of the solution operator $\mathcal{S}_{\mathbf{d}}^m$ in $L^2(0,T;\mathbf{H}^1) \cap L^\infty(0,T;\mathbf{L}^2)$. We conclude this part of our proof by observing that further regularity of the solving order parameter field can be obtained by completely analogous estimates if we start by taking $-\Delta\delta\mathbf{d}$ as test function.

Existence and uniqueness for problem (2.9b) We now turn our attention to problem (2.9b). As before, we want to prove uniqueness of solutions (this time in $H^1(0,T;\mathbf{H}^1)$) and continuity for the solution operator. We immediately observe that, since $\nabla\mathbf{d}^m(t) \in \mathbf{L}^6$, $\Delta\mathbf{d}^m(t) \in \mathbf{L}^2$ a.e. $t \in [0,T]$, the $\mathbf{d}^m$-dependent forcing term can indeed be read as a scalar product in $\mathbf{L}^2$ instead of being a duality.

Since $\mathbf{u}^m(t) \in \mathbf{V}^m$ we can write the solution to our problem as $\mathbf{u}^m = \sum_{i=0}^n \eta_i^m(t)\mathbf{w}_i$ where $\mathbf{w}_i$ is the orthonormal basis made up by the eigenfunctions of Stokes' problem introduced above. The approximating Galerkin solution $\mathbf{u}^m$ we are looking for is therefore determined by the solution of the following system of ordinary differential equations for the coefficients $\eta_j^m(t)$ which can be obtained by choosing $\mathbf{w}_j$ as test functions in (2.9b):

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}\eta_j^m(t) + \sum_{i=1}^m ((\overline{\mathbf{u}}^m(t)\cdot\nabla)\mathbf{w}_i, \mathbf{w}_j)\,\eta_i^m(t) + \lambda_j\nu\eta_j^m(t) & \\
\qquad = -(\Delta\mathbf{d}^m(t), \nabla\mathbf{d}^m(t)\mathbf{w}_j) + \langle\mathbf{g}(t), \mathbf{w}_j\rangle & j = 1,\dots,m \\
\eta_j(0) = (\mathbf{u}_0, \mathbf{w}_j) & j = 1,\dots,m
\end{cases}
\tag{2.13}
$$

From Cauchy-Lipschitz classical result (see [29]), this system has a unique solution in the time interval $[0, T_m)$ with $T_m \le T$.

If we take $\mathbf{u}^m(t)$ (with $t \in [0, T_m)$) as test function in (2.9b) and using the usual orthogonality relation for the trilinear convective term, we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}^m|_2^2 + \nu|\nabla\mathbf{u}^m|_2^2 = -\Big((\nabla\mathbf{d}^m)^t\Delta\mathbf{d}^m, \mathbf{u}^m\Big) + \langle\mathbf{g}, \mathbf{u}^m\rangle\,.$$

We also observe that the following vector identity holds:[3]

$$\nabla\cdot(\nabla\mathbf{d}^m \odot \nabla\mathbf{d}^m) = (\nabla\mathbf{d}^m)^t\Delta\mathbf{d}^m + \frac{1}{2}\nabla|\nabla\mathbf{d}^m|^2 \tag{2.14}$$

Using this last relation, remembering that $\nabla\cdot\mathbf{u}^m = 0$ and thanks to the embedding $\mathbf{V}^m \subset \mathbf{L}^\infty$, which holds for all $m \in \mathbb{N}$, after an integration by parts, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}^m|_2^2 + \nu|\nabla\mathbf{u}^m|_2^2 &= \Big((\nabla\mathbf{d}^m)^t\nabla\mathbf{d}^m, \nabla\mathbf{u}^m\Big) + \langle\mathbf{g}, \mathbf{u}^m\rangle \\
&\overset{H}{\le} |\nabla\mathbf{d}^m|_2^2|\nabla\mathbf{u}^m|_\infty + |\nabla\mathbf{u}^m|_2|\mathbf{g}|_{\mathbf{V}^*} \\
&\le C_m|\nabla\mathbf{d}^m|_2^2|\nabla\mathbf{u}^m|_2 + |\nabla\mathbf{u}^m|_2|\mathbf{g}|_{\mathbf{V}^*} \\
&\overset{Y}{\le} \frac{C_m}{\nu}|\nabla\mathbf{d}^m|_2^4 + \frac{\nu}{4}|\nabla\mathbf{u}^m|_2 + \frac{1}{\nu}|\mathbf{g}|_{\mathbf{V}^*}^2 + \frac{\nu}{4}|\nabla\mathbf{u}^m|_2^2
\end{aligned}$$

Standard algebra then gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{u}^m|_2^2 + \nu|\nabla\mathbf{u}^m|_2^2 \le \frac{C_m}{\nu}|\nabla\mathbf{d}^m|_2^4 + \frac{2}{\nu}|\mathbf{g}|_{\mathbf{V}^*}^2$$

and integrating from 0 to T_m we obtain the desired estimate:

$$\begin{aligned}
&|\mathbf{u}^m(t)|_2^2 + \nu|\nabla\mathbf{u}^m|_{L^2(0,t;\mathbf{L}^2)}^2 \\
&\qquad\le |\mathbf{u}_0|_2^2 + \frac{2}{\nu}|\mathbf{g}|_{L^2(0,T_m;\mathbf{V}^*)}^2 + \frac{C_m}{\nu}T_m|\nabla\mathbf{d}^m|_{L^\infty(0,T_m;\mathbf{L}^2)}^4
\end{aligned} \tag{2.15}$$

which holds for a.e. t in $[0, T_m]$. From estimate (2.12) we have $\nabla\mathbf{d}^m \in L^\infty(0, T; \mathbf{L}^2)$ and we therefore obtain $\mathbf{u}_m \in L^\infty(0, T; \mathbf{H}) \cap L^\infty(0, T; \mathbf{V})$.

In addition to $\eta_i^m \in L^2(0, T)$, we also observe that we can easily prove $\frac{\mathrm{d}}{\mathrm{d}t}\eta_i^m \in L^2(0, T)$. Actually, thanks to the finite dimension of $\mathbf{V}^m$, we have $\overline{\mathbf{u}}^m \in L^\infty(0, T; \mathbf{V}^m) \subset L^\infty(0, T; \mathbf{L}^\infty)$ and from the previous discussion we

[3] We remember that for two second-rank tensors we have by definition $(\mathbf{f}\odot\mathbf{g})_{ij} \doteq f_{ki}g_{kj}$, where we use the common Einstein sum convention on repeated indices.

The vector identity we are interested in follows easily then:

$$\begin{aligned}
[\nabla\cdot(\nabla\mathbf{d}^m \odot \nabla\mathbf{d}^m)]_i &= (\nabla\mathbf{d}^m \odot \nabla\mathbf{d}^m)_{ij,j} = (d_{k,i}^m d_{k,j}^m)_{,j} = d_{k,ij}^m d_{k,j}^m + d_{k,i}^m d_{k,jj}^m \\
&= \frac{1}{2}(d_{k,j}^m d_{k,j}^m)_{,i} + (\nabla\mathbf{d}^m)_{ik}^t\Delta\mathbf{d}_k^m = \frac{1}{2}|\nabla\mathbf{d}^m|_{,i}^2 + \big[(\nabla\mathbf{d}^m)^t\Delta\mathbf{d}^m\big]_i
\end{aligned}$$

recall that $\mathbf{d}^m \in L^2(0,T;\mathbf{H}^2)\cap L^\infty(0,T;\mathbf{H}^1)$. Moreover, the standard eigenfunction theory for elliptic equations tells us that the $\mathbf{w}_i$ are regular. Directly from equation (2.13) we then obtain:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta_j^m(t)\right)^2 &\overset{Y}{\leq} 4m\sum_{i=1}^{m}\left((\overline{\mathbf{u}}^m(t)\cdot\nabla)\mathbf{w}_i,\mathbf{w}_j\right)^2(\eta_i^m(t))^2 + 4\lambda_j^2\nu^2(\eta_j^m(t))^2\\ &\quad + 4\left((\nabla\mathbf{d}^m(t))^t\Delta\mathbf{d}^m(t),\mathbf{w}_j\right)^2 + 4\left\langle\mathbf{g}(t),\mathbf{w}_j\right\rangle^2\\ &\overset{H}{\leq} 4m\sum_{i=1}^{m}|\overline{\mathbf{u}}^m(t)|_\infty^2|\nabla\mathbf{w}_i|_2^2|\mathbf{w}_j|_2^2(\eta_i^m(t))^2 + 4\lambda_j^2\nu^2(\eta_j^m(t))^2\\ &\quad + 4|\nabla\mathbf{d}^m(t)|_2^2|\Delta\mathbf{d}^m(t)|_2^2|\mathbf{w}_j|_\infty^2 + 4|\mathbf{g}(t)|_{\mathbf{V}^*}^2|\mathbf{w}_j|_2^2\end{aligned}$$

Finally, integrating on $[0,T_m]$ and observing that all terms on the right hand side of the resulting estimate are bounded, we obtain the desired result. The time regularity just proved for the coefficients η_i^m gives $\mathbf{u}^m \in H^1(0,T_m;\mathbf{H}^1)$.

With this regularity of solutions for problem (2.9b) we can easily prove uniqueness and, as we did before for the order parameter, we can straightforwardly obtain continuity for the solution operator: $\mathcal{S}_{\mathbf{u}}^m : L^2(0,T;\mathbf{H}^2)\cap L^\infty(0,T;\mathbf{H}^1) \to H^1(0,T;\mathbf{V}^m)$, $\mathcal{S}_{\mathbf{u}}^m(\mathbf{d}^m) = \mathbf{u}^m$. As before, these results can be obtained by considering the difference between the problems solved by $\mathbf{u}_1^m$ and $\mathbf{u}_2^m$ having respectively $\mathbf{d}_1^m$, $\overline{\mathbf{u}}_1$ and $\mathbf{d}_2^m$, $\overline{\mathbf{u}}_2$ as forcing terms and by choosing $\delta\mathbf{u}^m \doteq \mathbf{u}_1^m - \mathbf{u}_2^m$ as test function in the corresponding weak formulation.

A fixed point result We now have everything which is needed in order to prove existence and uniqueness of solutions for problem (2.7) for sufficiently short time intervals. Note that the composition of the solution operator $\mathcal{S}_{\mathbf{u}}^m$ e $\mathcal{S}_{\mathbf{d}}^m$ introduced before is compact for all $m \in \mathbb{N}$. Actually, the continuity follows from the continuous dependence on data which has been proved for both operators whereas pre-compactness is a direct consequence of the regularity estimates of the previous sections. In particular we have $\mathcal{S}_{\mathbf{u}}^m\circ\mathcal{S}_{\mathbf{d}}^m : \mathbf{C}(0,T_m;\mathbf{V}^m) \to H^1(0,T_m;\mathbf{V}^m)$, $\mathcal{S}_{\mathbf{u}}^m\circ\mathcal{S}_{\mathbf{d}}^m : \overline{\mathbf{u}}^m \mapsto \mathbf{u}^m(\mathbf{d}^m(\overline{\mathbf{u}}^m))$ where the immersion $H^1(0,T_m;\mathbf{V}^m) \hookrightarrow \mathbf{C}(0,T_m;\mathbf{V}^m)$ is actually compact due to Rellich theorem (see [4]) and to the finite-dimensionality of $\mathbf{V}^m$.

Let now $M>0$ such that $|\mathbf{u}_0|_2^2 + \frac{2}{\nu}|\mathbf{g}|_{L^2(0,T;\mathbf{V}^*)}^2 \leq \frac{M}{2}$. Thanks to estimates (2.12) and (2.15), if $|\overline{\mathbf{u}}^m(t)|_2^2 \leq M$ for all $t\in[0,T_m]$, we have:

$$|\mathbf{u}^m(t)|_2^2 \leq \frac{M}{2} + \frac{C_m}{\nu}T|\nabla\mathbf{d}^m|_{L^\infty(0,T;\mathbf{L}^2)}^4$$

that is $|\mathbf{u}^m(t)|_2^2 \leq M$ for all $t\in[0,\widetilde{T}_m]$ where $0<\widetilde{T}_m\leq T$ is sufficiently small so that the norms of $\mathbf{d}^m$ are suitably bounded. We can therefore apply Schauder's Theorem (see, for example, [10]) to the composition operator $\mathcal{S}_{\mathbf{u}}^m\circ\mathcal{S}_{\mathbf{d}}^m$ acting on the closed ball of radius M of $\mathbf{C}(0,T_m;\mathbf{L}^2)$. Thus we have proved that problem (2.7) has at least a solution $\mathbf{u}^m \in H^1(0, T_m; \mathbf{V}^m)$, $\mathbf{d}^m \in L^\infty(0, T_m; \mathbf{H}^1) \cap L^2(0, T_m; \mathbf{H}^2)$.

Uniqueness for the just constructed solution can be proven in a standard way by following the same strategy we used to prove it in the split problems. Therefore we leave out the straightforward details.

Extending approximating solutions Ideally, we could now use the sequence of approximating solutions just obtained from the semi-Galerkin approximation (2.7) to pass to the limit in the original equation and deduce the existence of solutions for our model. However, this is not possible since the time interval on which these approximating solutions are defined depends on the dimension m of the linear space considered: actually a careful analysis of the estimates of the previous paragraphs reveals that $T_m \to 0$ when $m \to \infty$.

In order to find a turnaround for this problem we need to extend the time interval of existence of the approximating solutions. This can be achieved through some a priori estimates obtained by considering a different lifting problem for system (2.1).

We will consider the following lifting problem:

$$\begin{cases} -\Delta \mathring{\mathbf{d}} = 0 & \text{in } \Omega; \\ \mathring{\mathbf{d}} = \mathbf{h} & \text{su } \partial\Omega. \end{cases} \tag{2.16}$$

From the standard theory for elliptic partial differential equations we know the following existence and regularity result.

Lemma 2.1.8. *Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ be a regular bounded domain.*

- *If $\mathbf{h} \in H^1(0, \infty; \mathbf{H}^{-1/2}(\partial\Omega)) \cap L^2(0, \infty; \mathbf{H}^{3/2}(\partial\Omega))$ then the lifting problem (2.16) has a unique solution $\mathring{\mathbf{d}}$ in $H^1(0, T; \mathbf{L}^2) \cap L^\infty(0, T; \mathbf{H}^1) \cap L^2(0, T; \mathbf{H}^2)$ and the following estimates hold for a.e. $t > 0$:*

$$\int_0^T |\mathring{\mathbf{d}}|_{\mathbf{H}^1}\, dt \le C \int_0^T |\mathbf{h}|_{\mathbf{H}^{3/2}(\partial\Omega)}\, dt$$

and

$$\int_0^T |\partial_t \mathring{\mathbf{d}}|_{\mathbf{L}^2}\, dt \le C \int_0^T |\partial_t \mathbf{h}|_{\mathbf{H}^{-1/2}(\partial\Omega)}\, dt.$$

- *If $\mathbf{h} \in H^1(0, \infty; \mathbf{H}^{1/2}(\partial\Omega)) \cap L^2(0, \infty; \mathbf{H}^{5/2}(\partial\Omega))$ then the lifting problem (2.16) has a unique solution $\mathring{\mathbf{d}}$ in $H^1(0, T; \mathbf{H}^1) \cap L^\infty(0, T; \mathbf{H}^2) \cap L^2(0, T; \mathbf{H}^3)$. Moreover, the following estimates hold for a.e. $t > 0$:*

$$\int_0^T |\mathring{\mathbf{d}}|_{\mathbf{H}^2}\, dt \le C \int_0^T |\mathbf{h}|_{\mathbf{H}^{5/2}(\partial\Omega)}\, dt$$

and

$$\int_0^T |\partial_t \mathring{\mathbf{d}}|_{\mathbf{H}^1}\, dt \le C \int_0^T |\partial_t \mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}\, dt.$$

In the following, we will write $\check{\mathbf{d}}^m \doteq \mathbf{d}^m - \mathring{\mathbf{d}}$ to refer to this differently lifted function. We immediately observe that $\check{\mathbf{d}}_m$ is a solution of the following problem:

$$\begin{cases} (\partial_t \mathbf{u}^m(t), \mathbf{v}_m) + ((\mathbf{u}^m(t)\cdot\nabla)\mathbf{u}^m(t), \mathbf{v}_m) + \nu\,(\nabla\mathbf{u}^m(t), \nabla\mathbf{v}_m) & \\ \qquad + \Big((\nabla\mathbf{d}^m)^t(t)\Delta\check{\mathbf{d}}^m(t), \mathbf{v}_m\Big) = (\mathbf{g}(t), \mathbf{v}_m) & \forall \mathbf{v}_m \in \mathbf{V}^m; \\ \partial_t \check{\mathbf{d}}^m(t) + (\mathbf{u}^m(t)\cdot\nabla)\mathbf{d}^m = \Delta\check{\mathbf{d}}^m - \mathbf{f}(\mathbf{d}^m(t)) - \partial_t\mathring{\mathbf{d}}, & \text{a.e. in } (0,T)\times\Omega; \\ \mathbf{u}^m(0) = \mathbf{u}_{0m} = P_m\mathbf{u}_0 & \\ \check{\mathbf{d}}^m(\mathbf{x},0) = \mathbf{d}_0 \qquad \text{for } \mathbf{x}\in\Omega; & \\ \check{\mathbf{d}}^m(\mathbf{x},t) = \mathbf{0} \qquad \text{on } \partial\Omega\times(0,T). & \end{cases} \tag{2.17}$$

Analogously we will write $\mathring{\mathbf{d}}$ and $\check{\mathbf{d}}$ for the lifting of the original equation (2.1).

If we choose $\mathbf{u}^m$ and $-\Delta\check{\mathbf{d}}^m$ as test functions in (2.17), summing the resulting two equations we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\mathbf{u}^m|_2^2 + |\nabla\check{\mathbf{d}}^m|_2^2) + \nu|\nabla\mathbf{u}^m|_2^2 + |\Delta\check{\mathbf{d}}^m|_2^2 + \Big((\nabla\mathbf{d}^m)^t\Delta\check{\mathbf{d}}^m, \mathbf{u}^m\Big)$$
$$= \Big((\mathbf{u}^m\cdot\nabla)\mathbf{d}^m, \Delta\check{\mathbf{d}}^m\Big) + \Big(\mathbf{f}(\mathbf{d}^m), \Delta\check{\mathbf{d}}^m\Big) + \Big(\partial_t\mathring{\mathbf{d}}, \Delta\check{\mathbf{d}}^m\Big) + (\mathbf{g}(t), \mathbf{u}^m)\,.$$

We remember the following useful vector identity[4]:

$$\Big((\nabla\mathbf{d}^m)^t\Delta\check{\mathbf{d}}^m, \mathbf{u}^m\Big) = \Big((\mathbf{u}^m\cdot\nabla)\mathbf{d}^m, \Delta\check{\mathbf{d}}^m\Big) \tag{2.18}$$

Using Young's inequality we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\mathbf{u}^m|_2^2 + |\nabla\check{\mathbf{d}}^m|_2^2) + \nu|\nabla\mathbf{u}^m|_2^2 + |\Delta\check{\mathbf{d}}^m|_2^2 \le 2(|\mathbf{f}(\mathbf{d}^m)|_2^2 + |\partial_t\mathring{\mathbf{d}}|_2^2) + \frac{1}{\nu}|\mathbf{g}(t)|_{\mathbf{V}^*}^2.$$

Thanks to the weak maximum principle introduced before we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\mathbf{u}^m|_2^2 + |\nabla\check{\mathbf{d}}^m|_2^2) + \nu|\nabla\mathbf{u}^m|_2^2 + |\Delta\check{\mathbf{d}}^m|_2^2 \le \frac{8}{27\epsilon^4} + C_\Omega|\partial_t\mathbf{h}|_{\mathbf{H}^{-1/2}}^2 + \frac{1}{\nu}|\mathbf{g}(t)|_{\mathbf{V}^*}^2. \tag{2.19}$$

[4]This relation follows easily using the index notation. We actually have:

$$\Big((\nabla\mathbf{d}^m)^t\Delta\check{\mathbf{d}}^m, \mathbf{u}^m\Big) = ((\nabla\mathbf{d}^m)^t\Delta\check{\mathbf{d}}^m)_j u_j^m = d_{i,j}^m \Delta\check{d}_i^m u_j^m$$

and, analogously:

$$\Big((\mathbf{u}^m\cdot\nabla)\mathbf{d}^m, \Delta\check{\mathbf{d}}^m\Big) = ((\mathbf{u}^m\cdot\nabla)\mathbf{d}^m)_i\Delta\check{d}_i^m = u_j^m d_{i,j}^m \Delta\check{d}_i^m.$$

Using now Poincarè's inequality in the left hand side, we can write:

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\mathbf{u}^m|_2^2+|\nabla\check{\mathbf{d}}^m|_2^2)+C_0(|\mathbf{u}^m|_2^2+|\nabla\check{\mathbf{d}}^m|_2^2) \leq \frac{8}{27\epsilon^4}+C_\Omega|\partial_t\mathbf{h}|^2_{\mathbf{H}^{-1/2}}+\frac{1}{\nu}|\mathbf{g}(t)|^2_{\mathbf{V}^*}.$$

where C_Ω does not depend on m and $C_0 = \min\{\frac{\nu}{C_P}, \frac{1}{C_P}\} = \min\{\nu\mu_1, \mu_1\}$ with C_P Poincarè constant for the domain Ω and with μ_1 first eigenvalue of the homogeneous Laplace-Dirichlet operator on Ω. Multiplying by $e^{C_0 t}$ and integrating from 0 to t we obtain:

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} e^{C_0 s}(|\mathbf{u}^m|_2^2+|\nabla\check{\mathbf{d}}^m|_2^2)\,ds \\ \leq \frac{8}{27C_0\epsilon^4}+C_\Omega\int_0^t e^{C_0 s}|\partial_t\mathbf{h}(s)|^2_{\mathbf{H}^{-1/2}}\,ds+\frac{1}{\nu}\int_0^t e^{C_0 s}|\mathbf{g}(s)|^2_{\mathbf{V}^*}\,ds.$$

which can easily be written as;

$$|\mathbf{u}^m|_2^2+|\nabla\check{\mathbf{d}}^m|_2^2 \leq e^{-C_0 t}(|\mathbf{u}_0|_2^2+|\nabla\mathbf{d}_0|_2^2)+\frac{8}{27C_0\epsilon^4} \\ +C_\Omega e^{-C_0 t}\int_0^t e^{C_0 s}|\partial_t\mathbf{h}(s)|^2_{\mathbf{H}^{-1/2}}\,ds+\frac{1}{\nu}e^{-C_0 t}\int_0^t e^{C_0 s}|\mathbf{g}(s)|^2_{\mathbf{V}^*}\,ds \quad (2.20)$$

We note that, since $|\nabla\mathring{\mathbf{d}}|_2 \leq C|\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)} \leq C|\mathbf{h}|_{L^2(\mathbf{H}^{3/2})}+C|\partial_t\mathbf{h}|_{L^2(\mathbf{H}^{-1/2})}$, this last inequality implies lemma 2.1.2 for the approximating solutions $(\mathbf{u}^m, \mathbf{d}^m)$.

With this estimate we have shown that the $\mathbf{L}^2$ norm of $\mathbf{u}^m$ and $\nabla\mathbf{d}^m$ are uniformly bounded in m. Therefore we can extend all approximating solutions beyond T_m up to any fixed time T.

Passing to the limit We can now pass to the limit in (2.7). For the reader convenience we recall all the results we have obtained in the above discussion:

- the sequence $\mathbf{d}^m$ is bounded in $L^\infty(0,T;\mathbf{H}^1)$, in $L^2(0,T;\mathbf{H}^2)$ and in $L^\infty(0,T;\mathbf{L}^\infty)$;
- the sequence $\mathbf{u}^m$ is bounded in $L^\infty(0,T;\mathbf{H})$ and in $L^2(0,T;\mathbf{V})$.

With these results at hand, using identity (2.14), we can deduce directly from equation (2.7) that $\partial_t\mathbf{u}^m$ is bounded in $L^p(0,T;\mathbf{V}^*)$, with $p=2$ when $n=2$ and $p=4/3$ when $n=3$, and that $\partial_t\check{\mathbf{d}}^m$ is bounded in $L^2(0,T;\mathbf{L}^2)$[5].

[5] We remember the estimates used in this passage (see [31, Proposition 9.2]):

$$|((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})| \leq C\begin{cases}|\mathbf{u}|_2^{1/2}|\mathbf{u}|_{\mathbf{H}^1}^{1/2}|\mathbf{v}|_{\mathbf{H}^1}|\mathbf{w}|_2^{1/2}|\mathbf{w}|_{\mathbf{H}^1}^{1/2} & \text{for } n=2\\ |\mathbf{u}|_2^{1/4}|\mathbf{u}|_{\mathbf{H}^1}^{3/4}|\mathbf{v}|_{\mathbf{H}^1}|\mathbf{w}|_2^{1/4}|\mathbf{w}|_{\mathbf{H}^1}^{3/4} & \text{for } n=3\end{cases}$$

and

$$|(\Delta\mathbf{e},\nabla\mathbf{d}\,\mathbf{v})| \leq C\begin{cases}|\mathbf{e}|_{\mathbf{H}^2}|\mathbf{d}|_{\mathbf{H}^1}^{1/2}|\mathbf{d}|_{\mathbf{H}^2}^{1/2}|\mathbf{v}|_2^{1/2}|\mathbf{v}|_{\mathbf{H}^1}^{1/2} & \text{for } n=2\\ |\mathbf{e}|_{\mathbf{H}^2}|\mathbf{d}|_{\mathbf{H}^1}^{1/2}|\mathbf{d}|_{\mathbf{H}^2}^{1/2}|\mathbf{v}|_{\mathbf{H}^1} & \text{for } n=3\end{cases}$$

Using Banach-Alaoglu theorem it is thus possible to extract subsequences which converge weakly(-$*$) in each one of the just enumerated functional spaces. From the standard theory we deduce that this limit is the same in all spaces and that $\lim \partial_t \mathbf{u}^m = \partial_t \lim \mathbf{u}^m$.

We now briefly review and justify this limit passage in all the nonlinear terms of (2.7).

- Since from the weak maximum principle we have $|\mathbf{d}^m| \leq 1$, the potential term $\mathbf{f}(\mathbf{d}^m)$ is dominated by a constant. Thus we can use the dominated convergence theorem to pass to the limit.
- We can control $((\mathbf{u}^m \cdot \nabla)\mathbf{d}^m, \mathbf{w}_j)$ by observing that the sequence $\mathbf{d}^m$ admits a subsequence which converges strongly in $L^2(0,T;\mathbf{H}^1)$. We then have:

$$\int_{\Omega\times[0,T]} (\mathbf{u}^m \cdot \nabla)\mathbf{d}^m \cdot \mathbf{w}_j \; d\Omega dt$$
$$= \int_{\Omega\times[0,T]} (\mathbf{u}^m \cdot \nabla)(\mathbf{d}^m - \mathbf{d}) \cdot \mathbf{w}_j \; d\Omega dt + \int_{\Omega\times[0,T]} (\mathbf{u}^m \cdot \nabla)\mathbf{d} \cdot \mathbf{w}_j \; d\Omega dt.$$

  Since $|\nabla \mathbf{d}^m - \nabla \mathbf{d}|_{L^2(\mathbf{H}^1)} \to 0$ when $m \to \infty$, the first term in this expression converges to zero. The second term converges to $((\mathbf{u} \cdot \nabla)\mathbf{d}, \mathbf{w}_j)$ thanks to the weak-$*$ convergence in $\mathbf{H}$ of $\mathbf{u}^m$.
- We then have to consider the usual convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ in the velocity field equation. This can be dealt with by the same techniques used for the Navier-Stokes equation. We shortly remember the existing results (see [31, Chapter 9] for a comprehensive discussion of the details): if $n = 2$ when we pass to the limit in equation (2.7), the (sub-)sequence converges in $L^2(0,T;\mathbf{V}^*)$, whereas for $n = 3$, we only have convergence in $L^{4/3}(0,T;\mathbf{V}^*)$.
- Finally, we can deal with the nonlinear term $\left((\nabla \mathbf{d}^m)^t \Delta \mathbf{d}^m, \mathbf{v}\right)$ in an equivalent manner using the strong convergence (up to subsequences) of $\mathbf{d}^m$ in $L^2(0,T;\mathbf{H}^1)$ and its boundedness $L^\infty(0,T;\mathbf{H}^2)$. In particular this term convergences in $L^2(0,T;\mathbf{V}^*)$ when $n = 2$ and in $L^{4/3}(0,T;\mathbf{V}^*)$ when $n = 3$.

This concludes the proof of theorem 2.1.1 both for $n = 2$ and for $n = 3$. In the next sections we will prove uniqueness and existence of strong solutions in the two dimensional case. However, before continuing, we note that, passing to the limit in (2.20), we can easily prove lemma 2.1.2. This result let us extend local weak solutions up to arbitrarily large times T under very mild assumptions. These global solutions can indeed be obtained just supposing $\mathbf{g} \in L^2_{\text{loc}}(0,\infty;\mathbf{V}^*)$ and $\mathbf{h} \in L^2(0,\infty;\mathbf{H}^{3/2}(\partial\Omega)) \cap L^\infty(0,\infty;\mathbf{H}^{1/2}(\partial\Omega))$, $\partial_t \mathbf{h} \in L^\infty(0,\infty;\mathbf{H}^{-1/2}(\partial\Omega))$.

Corollary 2.1.9 (Global weak existence)**.** *Let* $\Omega \subset \mathbb{R}^n$ *with* $n = 2, 3$ *be a regular domain, let* $\mathbf{g} \in L^2_{\text{loc}}(0, \infty; \mathbf{V}^*)$ *and* $\mathbf{h} \in L^2(0, \infty; \mathbf{H}^{3/2}(\partial\Omega)) \cap L^\infty(0, \infty; \mathbf{H}^{1/2}(\partial\Omega))$*,* $\partial_t \mathbf{h} \in L^2(0, \infty; \mathbf{H}^{-1/2}(\partial\Omega))$ *with* $|\mathbf{h}| \leq 1$ *a.e., let* $\mathbf{u}_0 \in \mathbf{H}$*,* $\mathbf{d}_0 \in \mathbf{H}^1$ *with* $|\mathbf{d}_0| \leq 1$ *a.e. Then there exists a global weak solution* $(\mathbf{u}, \mathbf{d})$ *satisfying* (2.1) *for all* $T > 0$*.*

2.2 Uniqueness and continuous dependence on initial conditions in the 2D case

We now want to prove uniqueness of solutions and continuous dependence on initial conditions in the 2D case. In particular, this section is dedicated to the proof of the following basic result.

Theorem 2.2.1 (Uniqueness and continuous dependence)**.** *Under the same assumptions of theorem 2.1.1, if* $n = 2$*, the weak solution of problem* (2.1) *is unique. Moreover, it continuously depends on the initial conditions* $\mathbf{d}_0$*,* $\mathbf{u}_0$ *and on the forcing terms* $\mathbf{g}$ *and* $\mathbf{h}$*, and the following estimate holds:*

$$|\delta\mathbf{u}(t)|_2^2 + |\nabla\delta\mathbf{d}(t)|_2^2 + \int_0^t \left(\nu|\nabla\delta\mathbf{u}|_2^2 + |\Delta\delta\mathbf{d}|_2^2\right) ds \leq \Psi(t)\left(1 + \Phi(t)\right) e^{\Phi(t)} \tag{2.21}$$

where:

$$\Phi(t) = C \int_{t_0}^t \left(\frac{1}{\nu}|\nabla\mathbf{u}_1|_2^2 + |\Delta\mathbf{d}_1|_2^2 + |\mathbf{u}_1|_2^2|\nabla\mathbf{u}_1|_2^2 + \frac{1}{\epsilon^2}\right) ds$$

$$\Psi(t) = |\delta\mathbf{u}(t_0)|_2^2 + |\nabla\delta\mathbf{d}(t_0)|_2^2 + \int_{t_0}^t \left(\tfrac{3}{\nu}|\delta\mathbf{g}|_{\mathbf{V}^*}^2 + C|\partial_t\delta\mathbf{h}|_{\mathbf{H}^{-1/2}}^2 + C\left(1 + \tfrac{1}{\epsilon^2}\right)|\delta\mathbf{h}|_{\mathbf{H}^{3/2}}^2\right) ds$$

Remark. We observe also, as a simple corollary of estimate (2.21), that $(\mathbf{u}, \mathbf{d}) \in \mathbf{C}(\mathbf{L}^2 \times \mathbf{H}^1)$.

We begin the proof of theorem 2.2.1 by summarizing briefly the results we have obtained up to now:

- any velocity field $\mathbf{u}$ solution of system (2.1) is bounded in $L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V})$. This follows from estimates for the system (2.1) similar to inequalities (2.19) and (2.20) which can be easily obtained also for solutions of the original problem;

- analogously, any order parameter field $\mathbf{d}$ solution of (2.1) is bounded in $L^\infty(0, T; \mathbf{H}^1(\Omega))$ and in $L^2(0, T; \mathbf{H}^2(\Omega))$ as it can be verified using the maximum principle previously stated and again estimates similar to (2.19) and (2.20) for the original system.

Actually, from (2.19) and (2.20), we have the estimate:

$$|\mathbf{u}(t)|_2^2 + |\mathbf{d}(t)|_{\mathbf{H}^1}^2 + \int_0^t \nu|\mathbf{u}|_{\mathbf{H}}^2 + |\mathbf{d}|_{\mathbf{H}^2}^2\, ds$$
$$\le |\mathbf{u}_0|_2^2 + |\mathbf{d}_0|_{\mathbf{H}^1}^2 + \frac{C_\Omega}{\epsilon^4}t + C|\mathbf{h}(t)|_{\mathbf{H}^{1/2}(\partial\Omega)}^2 + \frac{1}{\nu}\int_0^t |\mathbf{g}(s)|_{\mathbf{V}^*}^2\, ds$$
$$+ C\int_0^t \Big(|\mathbf{h}(s)|_{\mathbf{H}^{3/2}(\partial\Omega)}^2 + |\partial_t \mathbf{h}(s)|_{\mathbf{H}^{-1/2}(\partial\Omega)}^2\Big)\, ds.$$

We remember that the embedding $H^1(0,t;\mathbf{H}^{-1/2}(\partial\Omega)) \cap L^2(0,t;\mathbf{H}^{3/2}(\partial\Omega)) \to \mathbf{C}(0,t;\mathbf{H}^{1/2}(\partial\Omega))$ is continuous (and compact) for all $t > 0$.

As usual, let $(\mathbf{u}_1, \mathbf{d}_1)$ and $(\mathbf{u}_2, \mathbf{d}_2)$ be two solutions of system (2.1) respectively with forcing terms $\mathbf{g}_1$ and $\mathbf{g}_2$ and boundary conditions $\mathbf{h}_1$ and $\mathbf{h}_2$. We will use $\delta\mathbf{u} \doteq \mathbf{u}_1 - \mathbf{u}_2$ and $\delta\mathbf{d} \doteq \mathbf{d}_1 - \mathbf{d}_2$ to denote the difference between these two solutions and $\delta\mathbf{g} \doteq \mathbf{g}_1 - \mathbf{g}_2$, $\delta\mathbf{h} \doteq \mathbf{h}_1 - \mathbf{h}_2$ for the difference between the non-autonomous terms. By considering the difference of the equations solved by $(\mathbf{u}_1, \mathbf{d}_1)$ and $(\mathbf{u}_2, \mathbf{d}_2)$ we have:

$$\partial_t \delta\mathbf{u} + (\delta\mathbf{u}\cdot\nabla)\mathbf{u}_1 + (\mathbf{u}_2\cdot\nabla)\delta\mathbf{u} - \nu\Delta\delta\mathbf{u}$$
$$= -(\nabla\delta\mathbf{d})^t\Delta\mathbf{d}_1 - (\nabla\mathbf{d}_2)^t\Delta\delta\mathbf{d} + \mathbf{g}_1 - \mathbf{g}_2$$

and

$$\partial_t \delta\mathbf{d} + (\mathbf{u}_1\cdot\nabla)\delta\mathbf{d} + (\delta\mathbf{u}\cdot\nabla)\mathbf{d}_2 - \Delta\delta\mathbf{d} = -(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2))$$

where $\delta\mathbf{u}$ and $\delta\mathbf{d}$ satisfy suitable Dirichlet boundary conditions and have given initial value.

We now recall that the first equality holds in $L^2(0,T;\mathbf{V}^*)$ and the second in $L^2(0,T;\mathbf{L}^2(\Omega))$. We can therefore evaluate the first equation against the test function $\delta\mathbf{u}$ and take the inner product (in $\mathbf{L}^2$) of the second with $-\Delta\delta\mathbf{d}$. We obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\delta\mathbf{u}|_2^2 + \langle(\delta\mathbf{u}\cdot\nabla)\mathbf{u}_1, \delta\mathbf{u}\rangle + \nu|\nabla\delta\mathbf{u}|_2^2 =$$
$$- \Big\langle(\nabla\delta\mathbf{d})^t\Delta\mathbf{d}_1, \delta\mathbf{u}\Big\rangle - \Big\langle(\nabla\mathbf{d}_2)^t\Delta\delta\mathbf{d}, \delta\mathbf{u}\Big\rangle + \langle\delta\mathbf{g}, \delta\mathbf{u}\rangle$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\delta\mathbf{d}|_2^2 - ((\mathbf{u}_1\cdot\nabla)\delta\mathbf{d}, \Delta\delta\mathbf{d}) - ((\delta\mathbf{u}\cdot\nabla)\mathbf{d}_2, \Delta\delta\mathbf{d}) + |\Delta\delta\mathbf{d}|_2^2 =$$
$$(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2), \Delta\delta\mathbf{d}) + {}_{\mathbf{H}^{1/2}(\partial\Omega)}\langle\partial_\nu\delta\mathbf{d}, \partial_t\delta\mathbf{h}\rangle_{\mathbf{H}^{-1/2}(\partial\Omega)}\,.$$

Summing up, recalling identity (2.18), the estimate of lemma 2.1.3, and finally using Hölder's and Sobolev's inequalities we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\delta\mathbf{u}|_2^2+|\nabla\delta\mathbf{d}|_2^2\right)+\nu|\nabla\delta\mathbf{u}|_2^2+|\Delta\delta\mathbf{d}|_2^2$$
$$\overset{H}{\le}|\delta\mathbf{u}|_4^2|\nabla\mathbf{u}_1|_2+|\nabla\delta\mathbf{d}|_4|\mathbf{\Delta d}_1|_2|\delta\mathbf{u}|_4$$
$$+|\mathbf{u}_1|_4|\nabla\delta\mathbf{d}|_4|\Delta\delta\mathbf{d}|_2+\frac{2}{\epsilon^2}|\delta\mathbf{d}|_2|\Delta\delta\mathbf{d}|_2$$
$$+|\delta\mathbf{g}|_{\mathbf{V}^*}|\nabla\delta\mathbf{u}|_2+|\partial_\nu\delta\mathbf{d}|_{\mathbf{H}^{1/2}(\partial\Omega)}|\partial_t\delta\mathbf{h}|_{\mathbf{H}^{-1/2}(\partial\Omega)}$$
$$\overset{S}{\le}C|\delta\mathbf{u}|_2|\nabla\delta\mathbf{u}|_2|\nabla\mathbf{u}_1|_2+C|\nabla\delta\mathbf{d}|_2^{1/2}|\delta\mathbf{d}|_{\mathbf{H}^2}^{1/2}|\mathbf{\Delta d}_1|_2|\delta\mathbf{u}|_2^{1/2}|\nabla\delta\mathbf{u}|_2^{1/2}$$
$$+C|\mathbf{u}_1|_2^{1/2}|\nabla\mathbf{u}_1|_2^{1/2}|\nabla\delta\mathbf{d}|_2^{1/2}|\delta\mathbf{d}|_{\mathbf{H}^2}^{3/2}+\frac{C}{\epsilon^2}|\delta\mathbf{d}|_2|\Delta\delta\mathbf{d}|_2$$
$$+|\delta\mathbf{g}|_{\mathbf{V}^*}|\nabla\delta\mathbf{u}|_2+C|\delta\mathbf{d}|_{\mathbf{H}^2}|\partial_t\delta\mathbf{h}|_{\mathbf{H}^{-1/2}(\partial\Omega)}.$$

We observe that, when uniqueness estimates are of concern, thanks to the Poincarè inequality, the $\mathbf{H}^2$ norm of $\delta\mathbf{d}$ can be replaced by the $\mathbf{L}^2$ norm of $\Delta\delta\mathbf{d}$. In the general case we are treating now, the $\mathbf{H}^2$ norm can be easily estimated as follows:

$$|\delta\mathbf{d}|_{\mathbf{H}^2}\le|\delta\check{\mathbf{d}}|_{\mathbf{H}^2}+|\delta\mathring{\mathbf{d}}|_{\mathbf{H}^2}$$
$$\le C|\delta\mathbf{\Delta}\check{\mathbf{d}}|_2+C|\delta\mathbf{h}|_{\mathbf{H}^{3/2}(\partial\Omega)}\le C|\delta\mathbf{\Delta d}|_2+C'|\delta\mathbf{h}|_{\mathbf{H}^{3/2}(\partial\Omega)}.$$

We now repeatedly use Young's inequality and deduce the following estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\delta\mathbf{u}|_2^2+|\nabla\delta\mathbf{d}|_2^2\right)+\nu|\nabla\delta\mathbf{u}|_2^2+|\Delta\delta\mathbf{d}|_2^2$$
$$\overset{Y}{\le}\frac{\nu}{6}|\nabla\delta\mathbf{u}|_2^2+\frac{C}{\nu}|\nabla\mathbf{u}_1|_2^2|\delta\mathbf{u}|_2^2$$
$$+\frac{\nu}{6}|\nabla\delta\mathbf{u}|_2^2+\frac{1}{8}|\Delta\delta\mathbf{d}|_2^2+C|\mathbf{\Delta d}_1|_2^2|\delta\mathbf{u}|_2^2+C|\mathbf{\Delta d}_1|_2^2|\nabla\delta\mathbf{d}|_2^2$$
$$+\frac{1}{8}|\Delta\delta\mathbf{d}|_2^2+C|\mathbf{u}_1|_2^2|\nabla\mathbf{u}_1|_2^2|\nabla\delta\mathbf{d}|_2^2$$
$$+\frac{1}{8}|\Delta\delta\mathbf{d}|_2^2+\frac{C}{\epsilon^2}|\nabla\delta\mathbf{d}|_2^2+\frac{C}{\epsilon^2}|\delta\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}^2$$
$$+\frac{\nu}{6}|\nabla\delta\mathbf{u}|_2^2+\frac{3}{2\nu}|\delta\mathbf{g}|_{\mathbf{V}^*}^2+\frac{1}{8}|\Delta\delta\mathbf{d}|_2^2+C|\partial_t\delta\mathbf{h}|_{\mathbf{H}^{-1/2}(\partial\Omega)}^2+C|\delta\mathbf{h}|_{\mathbf{H}^{3/2}(\partial\Omega)}^2.$$

After reordering we obtain the inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\delta\mathbf{u}|_2^2+|\nabla\delta\mathbf{d}|_2^2\right)+\nu|\nabla\delta\mathbf{u}|_2^2+|\Delta\delta\mathbf{d}|_2^2$$
$$\le C\left(\frac{1}{\nu}|\nabla\mathbf{u}_1|_2^2+|\mathbf{\Delta d}_1^2|_2\right)|\delta\mathbf{u}|_2^2$$
$$C\left(|\mathbf{\Delta d}_1|_2^2+|\mathbf{u}_1|_2^2|\nabla\mathbf{u}_1|_2^2+\frac{1}{\epsilon^2}\right)|\nabla\delta\mathbf{d}|_2^2$$
$$\frac{3}{\nu}|\delta\mathbf{g}|_{\mathbf{V}^*}^2+C|\partial_t\delta\mathbf{h}|_{\mathbf{H}^{-1/2}(\partial\Omega)}^2+C\left(1+\tfrac{1}{\epsilon^2}\right)|\delta\mathbf{h}|_{\mathbf{H}^{3/2}(\partial\Omega)}^2.$$

Neglecting the positive terms on the left hand side, we finally deduce:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\delta\mathbf{u}|_2^2+|\nabla\delta\mathbf{d}|_2^2\right)+\nu|\nabla\delta\mathbf{u}|_2^2+|\Delta\delta\mathbf{d}|_2^2$$
$$\leq C\left(\frac{1}{\nu}|\nabla\mathbf{u}_1|_2^2+|\Delta\mathbf{d}_1^2|_2+|\mathbf{u}_1|_2^2|\nabla\mathbf{u}_1|_2^2+\frac{1}{\epsilon^2}\right)\left(|\delta\mathbf{u}|_2^2+|\nabla\delta\mathbf{d}|_2^2\right)$$
$$+\frac{3}{\nu}|\delta\mathbf{g}|_{\mathbf{V}^*}^2+C|\partial_t\delta\mathbf{h}|^2_{\mathbf{H}^{-1/2}(\partial\Omega)}+C\left(1+\tfrac{1}{\epsilon^2}\right)|\delta\mathbf{h}|^2_{\mathbf{H}^{3/2}(\partial\Omega)}.$$

If we apply Gronwall's inequality to this last estimate we eventually obtain (2.21).

Considering now the estimate just obtained, we notice that we have proved Lipschitz continuous dependence on initial conditions. Moreover, if $\delta\mathbf{u}(t_0)=0$, $\nabla\delta\mathbf{d}(t_0)=0$, $\delta\mathbf{g}=0$ and $\delta\mathbf{h}=0$, we obtain $\delta\mathbf{u}=0$ and $\nabla\delta\mathbf{d}=0$ a.e. $t\geq 0$, $\mathbf{x}\in\Omega$. Since when studying uniqueness of solutions we have $\delta\mathbf{d}|_{\partial\Omega}=0$ for the "difference" problem, we have finally proved that $\delta\mathbf{d}=0$ a.e. $t\geq 0$, $\mathbf{x}\in\Omega$ that means uniqueness of solution for problem (2.1) if $\Omega\subset\mathbb{R}^2$.

2.3 Strong solution in the 2D case

Having proved existence and uniqueness of weak solutions for system (2.1), we are now ready to investigate existence and regularity of strong solutions. We start by introducing the notion of strong solution for system (2.1).

Definition 2.3.1. A pair $(\mathbf{u},\mathbf{d})$ is a *strong solution* for problem (2.1) if it is a weak solution and moreover $(\mathbf{u},\mathbf{d})\in L^2(0,T;(\mathbf{H}\cap\mathbf{H}^2)\times\mathbf{H}^3)$, $(\partial_t\mathbf{u},\partial_t\mathbf{d})\in L^2(0,T;\mathbf{H}\times\mathbf{H}^1)$, $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$ in $\mathbf{H}^1$ and $\mathbf{d}(\mathbf{x},0)=\mathbf{d}_0(\mathbf{x})$ in $\mathbf{H}^2$ and if:

$$\begin{cases}\partial_t\mathbf{u}(t)+(\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t)-\nu\Delta\mathbf{u}(t)+\nabla p(t)=-(\nabla\mathbf{d}(t))^T\Delta\mathbf{d}(t)\mathbf{g}(t)\\ \nabla\cdot\mathbf{u}(t)=0\\ \partial_t\mathbf{d}(t)+(\mathbf{u}(t)\cdot\nabla)\mathbf{d}=\Delta\mathbf{d}-\mathbf{f}(\mathbf{d}(t))\\ |\mathbf{d}(\mathbf{x},t)|\leq 1\end{cases}$$

hold almost everywhere in $\Omega\times(0,T)$.

Our main objective is now to prove the following existence result whose proof will occupy the remaining part of this chapter.

Theorem 2.3.1 (Strong existence)**.** *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded regular domain*[6]*, let* $\mathbf{g}\in L^2_{\mathrm{loc}}(0,T;\mathbf{H})$ *and* $\mathbf{h}\in L^2(0,T;\mathbf{H}^{5/2}(\partial\Omega))$ *such that* $\partial_t\mathbf{h}\in L^2(0,T;\mathbf{H}^{1/2}(\partial\Omega))$ *with* $|\mathbf{h}|\leq 1$ *a.e., let* $\mathbf{u}_0\in\mathbf{V}$, $\mathbf{d}_0\in\mathbf{H}^2$ *with* $|\mathbf{d}_0|\leq 1$ *a.e. Then there exists a strong solution* $(\mathbf{u},\mathbf{d})$ *of* (2.1)*.*

[6]Here $\Omega\in\mathbf{C}^{2,1}$ is sufficient.

We remark that, in the proof of this result, we should consider again the approximate solutions obtained by the semi-Galerkin formulation (2.7) and derive bounds (independent of m) on $\mathbf{u}^m$ and $\mathbf{d}^m$ strong enough to justify passing to the limit in (2.7). For simplicity, however, we will state only the formal estimates leaving out other inessential technicalities.

We start by considering again lifting (2.4) and the lifted problem (2.8) (without ms!). By using $-\Delta\mathbf{u}$ as a test function in the equation for the velocity field $\mathbf{u}$ and remembering that in the 2D case $\langle(\mathbf{u}\cdot\nabla)\mathbf{u},\Delta\mathbf{u}\rangle = 0$, we get:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}|_2^2+\nu|\Delta\mathbf{u}|_2^2 &\le |\nabla\mathbf{d}|_\infty|\Delta\mathbf{d}|_2|\Delta\mathbf{u}|_2+|\mathbf{g}|_2|\Delta\mathbf{u}|_2\\ &\overset{S}{\le} C|\nabla\mathbf{d}|_2^{1/2}|\nabla\mathbf{d}|_{\mathbf{H}^2}^{1/2}|\Delta\mathbf{d}|_2|\Delta\mathbf{u}|_2+|\mathbf{g}|_2|\Delta\mathbf{u}|_2\\ &\overset{Y}{\le}\frac{\nu}{4}|\Delta\mathbf{u}|_2^2+\delta|\mathbf{d}|_{\mathbf{H}^3}^2+\frac{C}{\delta\nu^2}|\nabla\mathbf{d}|_2^2|\Delta\mathbf{d}|_2^4+\frac{2}{\nu}|\mathbf{g}|_2^2. \quad (2.22)\end{aligned}$$

where δ will be determined later.

To get regularity estimates for the order parameter $\mathbf{d}$ we can take the duality of the second equation in (2.8) with $\Delta(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))$. We note that, since $\mathbf{u}|_{\partial\Omega}=0$ and $\widehat{\mathbf{d}}|_{\partial\Omega}=0$, from the lifted equation for the order parameter we have $(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_{\partial\Omega}=0$ and therefore Poincarè's inequality holds for the test function now chosen. Integrating by parts and observing that boundary terms vanish, after a few calculations we have:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2^2+|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2^2\\ &\qquad=\Big(\partial_t\mathbf{f}(\mathbf{d}),\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\Big)+\Big\langle(\nabla\mathbf{u}^t\cdot\nabla)\mathbf{d},\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\Big\rangle\\ &\qquad\qquad+\Big\langle\nabla\nabla\mathbf{d}\cdot\mathbf{u},\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\Big\rangle \quad (2.23)\end{aligned}$$

We now have to find bounds for every term on the right hand side of this last equation. We start by observing that $\partial_t\mathbf{f}(\mathbf{d})=\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})\cdot\partial_t\mathbf{d}$. Remembering that $|\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})|_\infty\le C$ because $|\mathbf{d}|<1$ by the maximum principle (see lemma 2.1.3) and using again the lifted equation, we obtain:

$$\begin{aligned}&-\Big(\partial_t\mathbf{f}(\mathbf{d}),\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\Big)\\ &\quad=-\Big(\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})),\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\Big)+\Big(\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})\Delta\widetilde{\mathbf{d}},\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\Big)\\ &\qquad+\Big(\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d})(\mathbf{u}\cdot\nabla)\mathbf{d},\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\Big)\end{aligned}$$

Remember that under the regularity assumptions of this section we have $\widetilde{\mathbf{d}}\in L^\infty(0,T;\mathbf{H}^2)$ and that we can use the weak regularity estimates of the previous sections $\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2)$ and $\mathbf{d}\in L^\infty(0,T;\mathbf{H}^1)$. Therefore, by proceeding as usual, we get:

$$
\begin{aligned}
-\Big(\partial_t \mathbf{f}(\mathbf{d}), \Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})\Big) &\overset{H}{\leq} C|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2^2 + C|\Delta\widetilde{\mathbf{d}}|_2|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2 \\
&\qquad + C|\mathbf{u}|_\infty|\nabla\mathbf{d}|_2|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2 \\
&\overset{S}{\leq} C(1+|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2^2 + |\mathbf{u}|_2^{1/2}|\Delta\mathbf{u}|_2^{1/2}|\nabla\mathbf{d}|_2|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2) \\
&\overset{Y}{\leq} \frac{\nu}{4}|\Delta\mathbf{u}|_2^2 + C|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2^2 + C\left(1+\frac{1}{\nu}\right)
\end{aligned}
$$

Likewise we can bound the second term in the right hand side of equation (2.23):

$$
\begin{aligned}
\Big\langle(\nabla\mathbf{u}^t\cdot\nabla)\mathbf{d}, \nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\Big\rangle &\overset{H}{\leq} |\nabla\mathbf{u}|_4|\nabla\mathbf{d}|_4|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2 \\
&\overset{S}{\leq} C|\nabla\mathbf{u}|_2^{1/2}|\Delta\mathbf{u}|_2^{1/2}|\nabla\mathbf{d}|_2^{1/2}|\nabla\mathbf{d}|_{\mathbf{H}^1}^{1/2}|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2 \\
&\overset{Y}{\leq} \frac{1}{4}|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2^2 + \frac{\nu}{4}|\Delta\mathbf{u}|_2^2 + \frac{C}{\nu}|\mathbf{d}|_{\mathbf{H}^2}^4 + \frac{C}{\nu}|\nabla\mathbf{u}|_2^4.
\end{aligned}
$$

Proceeding similarly for the third and last term, we have:

$$
\begin{aligned}
\Big\langle\nabla\nabla\mathbf{d}\cdot\mathbf{u}, \nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))\Big\rangle &\overset{H}{\leq} |\mathbf{d}|_{\mathbf{W}^{2,4}}|\mathbf{u}|_4|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2 \\
&\overset{S}{\leq} C|\mathbf{d}|_{\mathbf{H}^2}^{1/2}|\mathbf{d}|_{\mathbf{H}^3}^{1/2}|\mathbf{u}|_2^{1/2}|\nabla\mathbf{u}|_2^{1/2}|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2 \\
&\overset{Y}{\leq} \frac{1}{4}|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2^2 + \delta|\mathbf{d}|_{\mathbf{H}^3}^2 + \frac{C}{\delta}|\mathbf{d}|_{\mathbf{H}^2}^4 + \frac{C}{\delta}|\nabla\mathbf{u}|_2^4.
\end{aligned}
$$

where $\delta > 0$ will be determined in a few passages.

We now gather all the results obtained in this section. Summing up estimates (2.22) and (2.23) and using the last three inequalities, after reordering all terms we get:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(|\nabla\mathbf{u}|_2^2 + |\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2^2\Big) &+ \frac{\nu}{4}|\Delta\mathbf{u}|_2^2 + \frac{1}{2}|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2^2 \\
&\leq 2\delta|\mathbf{d}|_{\mathbf{H}^3}^2 + \frac{C}{\delta\nu^2}|\Delta\mathbf{d}|_2^4 + C|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d})|_2^2 \\
&\qquad + K|\mathbf{d}|_{\mathbf{H}^2}^4 + \frac{2}{\nu}|\mathbf{g}|_2^2 + C\left(1+\frac{1}{\nu}\right) + K|\nabla\mathbf{u}|_2^4
\end{aligned} \tag{2.24}
$$

where we have set $K = C(1+1/\nu+1/\delta)$ for simplicity reasons.

Recalling the triangle inequality, we can easily bound the norms $|\mathbf{d}|_{\mathbf{H}^i}$, with $i = 2, 3$. In particular we have:

$$
\begin{aligned}
|\mathbf{d}|_{\mathbf{H}^2}^2 &\leq 2|\widehat{\mathbf{d}}|_{\mathbf{H}^2}^2 + 2|\widetilde{\mathbf{d}}|_{\mathbf{H}^2}^2 \leq C|\Delta\widehat{\mathbf{d}}|_2^2 + 2|\widetilde{\mathbf{d}}|_{\mathbf{H}^2}^2 \\
&\leq C|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2^2 + C|\mathbf{f}(\mathbf{d})|_2^2 + 2|\widetilde{\mathbf{d}}|_{\mathbf{H}^2}^2 \\
&\leq C|\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|_2^2 + C
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathbf{d}|^2_{\mathbf{H}^3} &\leq 2|\widehat{\mathbf{d}}|^2_{\mathbf{H}^3} + 2|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3} \leq C|\nabla\Delta\widehat{\mathbf{d}}|^2_2 + C|\Delta\widehat{\mathbf{d}}|^2_2 + 2|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3} \\
&\leq C|\nabla(\Delta\widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}))|^2_2 + C|\nabla\mathbf{f}(\mathbf{d})|^2_2 + C|\Delta\widehat{\mathbf{d}}|^2_2 + 2|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3} \\
&\leq C|\nabla(\Delta\widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}))|^2_2 + 2|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3} + C
\end{aligned}
$$

where in the last passage we have used the Poincarè inequality which holds for $\Delta\widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})$.

This last estimate allows us to determine the parameter δ previously introduced: we chose $\delta < \frac{1}{4}$ so small that $2\delta|\mathbf{d}|^2_{\mathbf{H}^3} \leq \frac{1}{4}|\nabla(\Delta\widehat{\mathbf{d}}-\mathbf{f}(\mathbf{d}))|^2_2 + O.T.$ and we observe that δ depends only on the domain Ω and on the boundary data $\mathbf{h}$. Using these bounds in (2.24) we finally obtain:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla\mathbf{u}|^2_2 + |\Delta\widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})|^2_2\right) + \frac{\nu}{4}|\Delta\mathbf{u}|^2_2 + \frac{1}{4}|\nabla(\Delta\widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d}))|^2_2 \\
\leq \widetilde{K}^2|\Delta\widehat{\mathbf{d}} - \mathbf{f}(\mathbf{d})|^4_2 + \widetilde{K}1|\nabla\mathbf{u}|^4_2 + |\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3} + \frac{2}{\nu}|\mathbf{g}|^2_2 + \widetilde{K}. \quad (2.25)
\end{aligned}
$$

Setting $A(t) = |\nabla\mathbf{u}(t)|^2_2 + |\Delta\widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))|^2_2$, this inequality can be rewritten as:

$$
\frac{\mathrm{d}}{\mathrm{d}t}A(t) \leq \widetilde{K}^2 A^2(t) + |\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3} + \frac{2}{\nu}|\mathbf{g}|^2_2 + \widetilde{K} \quad (2.26)
$$

Thanks to estimate (2.19) we have $A \in L^1(0,T)$ for all $T > 0$. We can therefore apply Gronwall's inequality and get:

$$
\begin{aligned}
&|\nabla\mathbf{u}(t)|^2_2 + |\Delta\widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))|^2_2 \\
&\quad\leq \left(|\nabla\mathbf{u}_0|^2_2 + |\mathbf{f}(\mathbf{d}_0)|^2_2 + \widetilde{K}t + \int_0^t |\widetilde{\mathbf{d}}(s)|^2_{\mathbf{H}^3}\,ds + \frac{2}{\nu}\int_0^t |\mathbf{g}(s)|^2_2\,ds\right) \\
&\qquad\qquad e^{\widetilde{K}^2\int_0^t |\nabla\mathbf{u}(s)|^2_2 + |\Delta\widehat{\mathbf{d}}(s) - \mathbf{f}(\mathbf{d}(s))|^2_2\,ds}
\end{aligned}
$$

from which we easily prove that $\mathbf{u} \in L^2(0,T;\mathbf{H}^2) \cap L^\infty(0,T;\mathbf{H}^1)$ and $\mathbf{d} \in L^2(0,T;\mathbf{H}^3) \cap L^\infty(0,T;\mathbf{H}^2)$ for all $T > 0$ as claimed.

With these results we have completed the study of the well-posedness of model (2.1). In the next chapters we will continue our analysis of equation (2.1) by studying the long term behavior of solutions and focusing on the existence and properties of attractors for the corresponding non-autonomous dynamical system.

Chapter 3

Global Attractors (after Chepyzhov and Vishik)

In this chapter we will study the existence of a global attractor for system (2.1). We will follow the approach of Chepyzhov and Vishik (see [6, Part 2]) as developed for less regular forcing terms by Lu et al. in [25] and [26].

We start by recalling some basic notions for autonomous dynamical systems which will be useful when studying the non-autonomous case. We will try to cast these results in a rather general setting, avoiding, when possible, the use of metric notions. Next we will quickly resume Chepyzhov and Vishik's results in order to emphasize the differences with respect to the new approach by Lu. We will conclude this chapter by applying these results to system (2.1).

3.1 Autonomous global attractors in Hausdorff spaces

The theory of autonomous global attractors seems to be cast in its outmost generality when stated for semigroups acting on Hausdorff spaces (see [6, chapter XI]). With this approach we can easily state the main results on the existence of a global attracting set under very weak hypothesis that apply in a wide variety of situations arising from applications. Actually, we will be able to obtain the existence of the global attractor for groups acting on Banach spaces endowed with various topologies (both strong and weak). We also remind that the much quicker and common approach to attracting sets through metric space cannot be applied to weak topologies of Banach spaces which are not metrizable (see, for example, [33]). For a more classical approach the reader is referred to, e.g., [1, Chapter 2] or [17, Chapter 3].

We begin by recalling some general facts about Hausdorff spaces (see [5] for a complete presentation of this subject).

Definition 3.1.1. Let X be a topological space. If for any two distinct points $x, y \in X$ there exists a neighbourhood V of x and a neighbourhood W of y such that $V \cap W = \emptyset$, X will be said to satisfy the *second separability axiom*, and we will call X a *Hausdorff space*.

Notation. In the following of this section X will always indicate a Hausdorff space, unless otherwise explicitly stated.

We now define semigroups. Informally speaking, a semigroup is a one-dimensional family of maps parameterized by a "time gap" variable δt which describes the evolution of a certain system after a time δt.

Definition 3.1.2. A family of mappings $S(t) : X \to X$ defined on a topological space X and depending on a real parameter $t \geq 0$ is called a *semigroup acting on* X if

- the semigroup identity holds

$$S(t_1)S(t_2) = S(t_1 + t_2) \qquad \forall t_1, t_2 \geq 0$$

- the fixed time condition $S(0) = \mathrm{Id}$ is verified.

Notation. In the sequel we will denote simply by $\{S(t)\}$ a semigroup acting on X.

Absorbing and attracting sets are the next two important notions in the analysis of long time behaviour of dynamical systems we are now going to introduce. These concepts translate in a mathematical rigourous setting the notion of dissipation and loss of energy usually associated with the evolution of open physical systems. We observe that the notion of absorbing set is not strictly necessary in the development of the abstract theory here presented. However, it will result very useful in applications.

Definition 3.1.3. A set $B_0 \subset X$ *absorbs* a set $B \subset X$ with respect to the semigroup $\{S(t)\}$ if there exists a time $t_0 = t_0(B) > 0$ such that $S(t)B \subset B_0$ for all $t \geq t_0$.

Definition 3.1.4. A set $K \subset X$ *attracts* a set $B \subset X$ with respect to the semigroup $\{S(t)\}$ if for every neighbourhood V of K, there exists a time $t_0 = t_0(B, V)$ such that $S(t)B \subset V$ for all $t \geq t_0$.

Remark. Obviously an absorbing set for B also attracts B. Moreover, if K attracts (absorbs) B, then any set $V \supset K$ attracts (absorbs) B.

We now introduce a description of the ultimate evolution of a certain set under the action of the semigroup of interest. The basic idea is to focus on those states of the system which are asymptotically relevant starting from a given set of initial data. To fix ideas, consider a point mass whose motion is subject to a quadratic potential and to a dissipative drag force. Intuition tells us that the fate of the system is to asymptotically reach a rest state with the point mass lying at the point of minimum potential. The notion of ω-limit set, will formalize this natural idea.

Definition 3.1.5. The set

$$\omega(B) \doteq \bigcap_{t\geq 0} \overline{\bigcup_{h\geq t} S(h)B}$$

is the ω-*limit set* of $B \subset X$.

We can now introduce the central notion of this chapter: the global attractor. Always stressing on the intuitive point of view, a global attractor of a system contains the descriptions of all the intrinsic motions of a physical system which can be observed after all transient dynamics have decayed starting from any set of initial conditions.

Definition 3.1.6. A set $\mathcal{A} \subset X$ is called the *global attractor* or, more simply, *attractor* of the family $\mathfrak{B}$ of subsets of X if:

- $\mathcal{A}$ is compact;
- $\mathcal{A}$ attracts every set in $\mathfrak{B}$;
- $\mathcal{A}$ is contained in any other compact attracting set for the family $\mathfrak{B}$ (minimality property).

Remark. The minimality property ensures uniqueness of the attractor. In less general situations, the minimality property can be substituted by the strict invariance of the attractor: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ (see, for example [1, Chapter 2] or [6, Chapter II]). Moreover, in those contexts one can also show that the global attractor is also the maximal compact invariant subset of X.

There exists a strong connection between global attractors and ω-limit sets which can be summarized with the following to theorems (see [6, Chapter XI] for a detailed proof of these results). We stress on the fact that no regularity assumption is needed on the semigroup in order to guarantee the existence of the attractor.

Theorem 3.1.1. *Let $K \subset X$, with X a Hausdorff space, be a compact attracting set for the family $\mathfrak{B}$ of subsets of X under the action of the semigroup $S(t)$. Then $\mathfrak{B}$ has the attractor $\mathcal{A}$:*

$$\mathcal{A} = \overline{\bigcup_{B\in\mathfrak{B}} \omega(B)}.$$

Theorem 3.1.2. *Let the assumption of theorem 3.1.1 hold. If in addition the semigroup $\{S(t)\}$ is continuous (that is, for every $t \geq 0$ $S(t) : X \to X$ is a continuous map), then the attractor $\mathcal{A}$ is also strictly invariant.*

We now show how these rather abstract results can be applied in the more familiar setting of Banach spaces endowed with strong or weak topologies.

Usually, one wishes to study the fate of bounded sets of initial conditions in some metric vector space (for example L^2). This simply amounts to considering all the bounded sets of initial conditions as the family $\mathfrak{B}$ of subsets of X previously introduced. In this setting it is easy to give a more explicit description of the structure of the global attractor. We still need a couple of definitions.

Definition 3.1.7. Let X be a Banach space. A curve $\gamma : \mathbb{R} \to X$ is said to be a *complete trajectory* of the semigroup $\{S(t)\}$ if $S(t)\gamma(s) = \gamma(t+s)$, for all $t \geq 0$ and for all $s \in \mathbb{R}$. A *bounded complete trajectory* is a complete trajectory such that $\sup_{t\in\mathbb{R}} |\gamma(t)|_X < \infty$.

Definition 3.1.8. The *kernel* $\mathcal{K}$ of $\{S(t)\}$ is the union of all bounded complete trajectories of $\{S(t)\}$. The *kernel section* at time t is the set:

$$\mathcal{K}(t) \doteq \{\gamma(t) \mid \gamma \in \mathcal{K}\}.$$

We can now state theorem 3.1.2 in the particular form we will need when dealing with Banach spaces.

Theorem 3.1.3. *Let X be a Banach space endowed with an Hausdorff topology (which is not necessarily the one induced by the natural norm) and let $\{S(t)\}$ be a continuous semigroup acting on X. Assume that there exists a compact set $K \subset X$ attracting all bounded (in the natural norm) subsets of X and that K is bounded in X. Then there exists a unique global attractor $\mathcal{A} \subset X$ which is also bounded and which verifies:*

$$\mathcal{A} = \mathcal{K}(t) = \mathcal{K}(0) \qquad \forall t \in \mathbb{R}.$$

Notation. In the following we will write $\mathcal{B}(X)$ to denote the family of all bounded subsets of the Banach space X.

3.2 The non-autonomous case - Chepyzhov and Vishik's theory

We now wish to extend the theory developed in the previous section to non-autonomous dynamical systems. In this section we will restrict the exposition of the abstract theory to the more usual setting of Banach spaces. As reference problem we will consider the non-autonomous evolution equation:

$$\partial_t u = A(u,t) \qquad \forall t \in \mathbb{R} \tag{3.1}$$

completed by suitable initial (and boundary) conditions. For example, if $A(u,t) = \Delta u + f(t)$, one gets the usual heat equation with a time-dependent source term.

We will usually suppose that the time dependency can be completely described through a finite set of functions that we shall denote by $\sigma(t)$. We will call $\sigma(t)$ the *time symbol* or simply the *symbol* of equation (3.1). In order to emphasize this time dependence, we will also write:

$$\partial_t u = A_{\sigma(t)}(u) \qquad \forall t \in \mathbb{R}, \sigma \in \Sigma$$

instead of (3.1), where Σ is the set of all symbols of interest (we will shortly discuss this aspect in detail). Considering again the previous example, the time symbol of the just introduced heat equation will simply be $f(t)$.

First of all we need some definitions in order to deal with families of (solution-) operators which now have a much richer structure than before. The central notion will be that of processes.

Definition 3.2.1. A two parameter family of mappings $\{U(t,\tau)\}$, $U(t,\tau) : X \to X$ (where X is a Banach space) is said to be a *process* in X if:

- the process identity holds:

$$U(t,s)U(s,\tau) = U(t,\tau) \qquad \forall t,s \geq 0, \forall \tau \in \mathbb{R}$$

- the fixed time condition $U(\tau,\tau) = \mathrm{Id}$ is verified for all $\tau \in \mathbb{R}$.

We now need to introduce some notions of dissipativeness also in this context. However, here the problem is much more delicate since many definitions of absorbing and attracting sets are possible depending on the uniformity properties one wishes to guarantee (for example, with respect to the symbols or to time). Under suitable hypothesis, these uniformity properties are, however, equivalent (see for example [6, Chapter 4]).

Definition 3.2.2. A set $B_0 \subset X$ is said to be *uniformly (with respect to $\sigma \in \Sigma$) absorbing* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if for any $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(X)$ there exists an absorbtion time $t_0 = t_0(\tau, B) \geq \tau$ such that $\cup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq t_0$.

Definition 3.2.3. A set $K \subset X$ is *uniformly (w.r.t. $\sigma \in \Sigma$) attracting* for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if it satisfies, for any fixed $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(X)$, the following relation:

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \mathrm{dist}_X(U_\sigma(t,\tau)B, K) = 0$$

Notation. We will write $\mathrm{dist}_X(A,B)$ to indicate the usual Hausdorff semidistance between subsets of a metric space (X, d_X):

$$\mathrm{dist}_X(A,B) = \sup_{a\in A} \inf_{b\in B} d_X(a,b).$$

Definition 3.2.4. A family of processes possessing a compact uniformly (w.r.t. $\sigma \in \Sigma$) attracting set will be said to be *uniformly asymptotically compact*.

As in the preceding section, we can now define the (uniform) ω-limit set of a set $B \subset X$ and the (uniform) global attractor for a family of processes.

Definition 3.2.5. The set:

$$\omega_{\tau,\Sigma}(B) = \bigcap_{t \geq \tau} \overline{\bigcup_{\sigma \in \Sigma} \bigcup_{s \geq t} U_\sigma(s,\tau)B}^X$$

is the *uniform (w.r.t.* $\sigma \in \Sigma$*)* ω*-limit set* of $B \subset X$.

Definition 3.2.6. A closed set $\mathcal{A}_\Sigma \subset X$ is the *uniform (w.r.t.* $\sigma \in \Sigma$*) attractor* of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ if:

- $\mathcal{A}_\Sigma$ is uniformly (w.r.t. $\sigma \in \Sigma$) attracting (*attracting property*);
- $\mathcal{A}_\Sigma$ is contained in every other closed uniformly attracting set (*minimality property*).

Remark. We observe that in general $\cup_{\sigma \in \Sigma} \mathcal{A}_\sigma \subseteq \mathcal{A}_\Sigma$ and that strict inclusion may be verified. See, for example [6, Section IV.4].

As in the previous section, the following existence result follows under very weak assumptions.

Theorem 3.2.1. *A uniformly (w.r.t.* $\sigma \in \Sigma$*) asymptotically compact family of processes* $\{U_\sigma(t,\tau)\}$ *possesses the uniform (w.r.t* $\sigma \in \Sigma$*) attractor* $\mathcal{A}_\Sigma$*. Moreover,* $\mathcal{A}_\Sigma$ *is compact in* X *and, if* B_0 *is a closed bounded uniformly (w.r.t.* $\sigma \in \Sigma$*) absorbing set for* $\{U_\sigma(t,\tau)\}$*, then:*

$$\mathcal{A}_\Sigma = \overline{\bigcup_{\tau \in \mathbb{R}} \omega_{\tau,\Sigma}(B_0)}.$$

In order to better understand how the autonomous theory can be extended to the non-autonomous case, we shortly digress and consider the well-known finite dimensional case. In this setting, one possible way to deal with the time dependent terms is to consider an extended phase space. Consider a generic first order non-autonomous ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = F(y,t). \tag{3.2}$$

It is easily seen that this equation (3.2) is equivalent to the autonomous extended system to which apply all the results we already know:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} y = F(y,t) \\ \frac{\mathrm{d}}{\mathrm{d}s} t = 1 \end{cases}$$

In an analogous way, when dealing with infinite dimensional non-autonomous systems we can think of defining a semigroup on an extended phase space. If the symbol space Σ is invariant under time-shifts:

$$T(t) : \Sigma \to \Sigma \qquad \sigma(s) \mapsto \sigma(t+s),$$

then we can define the so called *skew product semigroup*:

$$S(t) : X \times \Sigma \to X \times \Sigma, \qquad S(t)(u,\sigma) = (U_\sigma(t,0), T(t)\sigma), \qquad \forall t \geq 0. \quad (3.3)$$

As for semigroups, we can introduce a notion of kernel also for processes.

Definition 3.2.7. A curve $u(s)$, $s \in \mathbb{R}$ is a *complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t) \qquad \forall t,\tau \in \mathbb{R}, t \geq \tau.$$

Definition 3.2.8. The *kernel* $\mathcal{K}$ of the process $\{U_\sigma(t,\tau)\}$ is the set of all its bounded complete trajectories.

We still need a continuity hypothesis in order to apply theorem 3.1.3 to the extended semigroup.

Definition 3.2.9. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ acting in X is said to be $(X \times \Sigma, X)$-*continuous* if for all fixed $t,\tau \in \mathbb{R}$, $t \geq \tau$ the mapping $(u,\sigma) \mapsto U_\sigma(t,\tau)u$ is continuous.

Theorem 3.1.3 then easily extends to the following non-autonomous version [6, theorem IV.5.1].

Theorem 3.2.2. *Let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ be a uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact and $(X \times \Sigma, X)$-continuous family of processes acting in X, let K be a compact uniformly (w.r.t. $\sigma \in \Sigma$) attracting set for $\{U_\sigma(t,\tau)\}$, let Σ be a compact metric space and let $\{T(t)\}$ be a continuous invariant ($T(t)\Sigma = \Sigma$) semigroup on Σ satisfying the translation identity:*

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau) \qquad \forall \sigma \in \Sigma, t,s,\tau \in \mathbb{R}, t \geq \tau, s \geq 0. \quad (3.4)$$

Then the skew product semigroup $\{S(t)\}$ defined by (3.3) *possesses the compact attractor $\mathfrak{A} = \omega(K \times \Sigma)$ which is strictly invariant with respect to $\{S(t)\}$: $S(t)\mathfrak{A} = \mathfrak{A}$. Moreover:*

- $\Pi_X \mathfrak{A} = \mathcal{A}_\Sigma$ *is the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ (here $\Pi_X : X \times \Sigma \to X$ is the natural projection of the product space $X \times \Sigma$ on the first factor);*

- $\Pi_\Sigma \mathfrak{A} = \Sigma$ *(here $\Pi_\Sigma : X \times \Sigma \to \Sigma$ is the natural projection on the second factor);*

- *the global attractor satisfies:*

$$\mathfrak{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\};$$

- *the uniform attractor satisfies:*

$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0)$$

where $\mathcal{K}_\sigma(0)$ is the section at time $t = 0$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t, \tau)\}$.

3.3 Weaker symbol compactness

Although theorem 3.2.2 can be useful in many applications, it has some drawbacks. Most importantly compactness of the symbol space is required and this reduces greatly the kind of forcing terms (external forces in physical problems) one can consider. The range of applicability is actually reduced to the so called *translation compact* function, that is functions whose translation hull is compact in the symbol space.

Definition 3.3.1. The *hull* of a function $f : \mathbb{R} \to E$ is the set:

$$\mathcal{H}(f) = \overline{\{T(s)f \mid s \in \mathbb{R}\}}^{\Xi}$$

where E and Ξ are metric spaces such that $E \subset \Xi$.

Common choices for the enveloping space Ξ are $\mathbf{C}(\mathbb{R}, E)$ or $L^p(\mathbb{R}, E)$. In these settings translation compact functions have been widely studied. On behalf of completeness we recall some characterization theorems.

Proposition 3.3.1. *A function f is translation compact in $\mathbf{C}(\mathbb{R}, E)$ if and only if:*

- *the set $\{f(t) \mid t \in \mathbb{R}\}$ is precompact in E;*
- *f is uniformly continuous on $\mathbb{R}$.*

Proposition 3.3.2. *A function f is translation compact in $L^p_{\mathrm{loc}}(\mathbb{R}, E)$ if and only if:*

- *for any $h \in \mathbb{R}$ the set $\{\int_t^{t+h} f(s)\, ds \mid t \in \mathbb{R}\}$ is precompact in E;*
- *the function:*

$$y \mapsto \sup_{t \in \mathbb{R}} \int_t^{t+1} |f(s) - f(s+y)|_E^p \, ds$$

is uniformly continuous on $\mathbb{R}$.

The interested reader can find a more detailed discussion of translation compact functions in [6, Chapter V] and many interesting compactness results in [34].

Usually, in applications, one wants to construct an attractor in the strong topology of a certain Banach space (for example L^2 or H^1). The necessity of considering translation compact functions is a somewhat strong limitation and leaves out of the analysis some kinds of forcing terms which can be interesting in applications. For example, if $E = \mathbb{R}$, any forcing function which oscillates at an arbitrary high frequency cannot be considered.

To obtain asymptotic results for larger classes of symbols, one could ideally work in the weak topology of the Banach space of interest. However, this kind of results is usually too weak to be of practical interest. A possible way around this problem is given by the recent results of Lu and coworkers (see [25] and [26]). This extension of the theory of Chepyzhov and Vishik consists in optimally weakening the asymptotic compactness hypothesis and then considering the interaction of strong and weak topologies in order to prove that the attractor in the weak topology is indeed the sought attractor in the norm convergence.

We start by slightly weakening the uniform asymptotic compactness of the family of processes. This can be done by introducing the Kuratowski measure of noncompactness (see [10] for more details and properties).

Definition 3.3.2. Let B be a bounded subset of some metric space E. Its *Kuratowski measure of noncompactness* is defined by:

$$\alpha(B) \doteq \inf\{\delta > 0 \mid B \text{ admits a finite cover by sets of diameter} \leq \delta\}.$$

Definition 3.3.3. A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is *uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact* if for any $\tau \in \mathbb{R}$ and any set $B \in \mathcal{B}(X)$ the set:

$$B_t = \bigcup_{\sigma\in\Sigma} \bigcup_{s\geq t} U_\sigma(s,\tau)B$$

is bounded for every t and $\lim_{t\to\infty} \alpha(B_t) = 0$.

Remark. We observe that although a uniformly (w.r.t. $\sigma \in \Sigma$) asymptotically compact process is uniformly ω-limit compact, the converse is not in general true. In order to fix ideas, consider the family of sets in $\mathbb{R}^2$:

$$B_t = (0,1) \times ((2t)^{-1}, t^{-1}) \bigcup \{1\} \times [-1,1].$$

Then $\omega(B) = \{1\} \times [-1,1]$ and we have $\alpha(B_t) = 0$, but $\mathrm{dist}_{\mathbb{R}^2}(B_t, \omega(B)) = 1$ as $t \to \infty$.

With this weaker notion of asymptotic compactness we can state a more strict existence result (see [25, Theorem 2.2]).

Theorem 3.3.3. *Let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ be a family of processes and let Σ be a symbol space invariant under the action of the time shift operator $T(t)$ such that the translation identity* (3.4) *holds. Then the process $\{U_\sigma(t,\tau)\}$ possesses the (compact) uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$ if and only if:*

- *$\{U_\sigma(t,\tau)\}$ has a uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0;*
- *$\{U_\sigma(t,\tau)\}$ is uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact.*

As before, by adding a suitable continuity hypothesis we can obtain also some information on the structure of the attractor (see [25, Section 2.3]).

Theorem 3.3.4. *Let $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ be a uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact and $(X \times \Sigma, X)$-weakly continuous family of processes acting in X, let B_0 be a weakly compact (i.e. bounded) uniformly (w.r.t. $\sigma \in \Sigma$) weakly attracting set for $\{U_\sigma(t,\tau)\}$, let Σ be a weakly compact subset of some Banach space and let $\{T(t)\}$ be a weakly continuous invariant ($T(t)\Sigma = \Sigma$) semigroup on Σ satisfying the translation identity:*

$$U_\sigma(t+s,\tau+s) = U_{T(s)\sigma}(t,\tau) \qquad \forall \sigma \in \Sigma, t, s, \tau \in \mathbb{R}, t \geq \tau, s \geq 0.$$

Then the skew product semigroup $\{S(t)\}$ defined by (3.3) *possesses the compact attractor $\mathfrak{A} = \omega(B_0 \times \Sigma)$ (in the weak topology) which is strictly invariant with respect to $\{S(t)\}$: $S(t)\mathfrak{A} = \mathfrak{A}$. Moreover:*

- *$\Pi_X \mathfrak{A} = \mathcal{A}_\Sigma$ is the uniform (w.r.t. $\sigma \in \Sigma$) attractor (in the strong topology!) of the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$;*
- *$\Pi_\Sigma \mathfrak{A} = \Sigma$;*
- *the global attractor satisfies:*

$$\mathfrak{A} = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) \times \{\sigma\};$$

- *the uniform attractor satisfies:*

$$\mathcal{A}_\Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{K}_\sigma(0) = \omega_{0,\Sigma}(B_0)$$

  where $\mathcal{K}_\sigma(0)$ is the section at time $t = 0$ of the kernel $\mathcal{K}_\sigma$ of the process $\{U_\sigma(t,\tau)\}$.

We now state a useful result that gives us a criterion to prove the uniform ω-limit compactness for a given process (see [25, Theorem 2.3]).

Proposition 3.3.5. *Let X be a uniformly convex Banach space (see [4], any L^p space with $p \neq 1, \infty$ will be suitable). Then the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is uniformly (w.r.t. $\sigma \in \Sigma$) ω-limit compact if and only if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(X)$ and $\epsilon > 0$ there exists $t_0 = t_0(\tau, B, \epsilon) \geq \tau$ and a finite-dimensional subspace X_1 of X such that:*

- $\Pi\left(\cup_{\sigma\in\Sigma}\cup_{t\geq t_0} U_\sigma(t,\tau)B\right)$ *is bounded;*
- $|(\mathrm{Id}-\Pi)\left(\cup_{\sigma\in\Sigma}\cup_{t\geq t_0} U_\sigma(t,\tau)u\right)|_X \leq \epsilon$, $\forall u\in B$

where $\Pi : X \to X_1$ *is a bounded projector.*

Before concluding this section, we introduce the class of functions (which are not translation compact) for which we will prove the existence of uniform attractors for system (2.1) in the rest of this chapter. See [25] and [26] for a more detailed treatment.

Definition 3.3.4. Let E be a Banach space. A function $f \in L^2_{loc}(\mathbb{R}; E)$ is *normal* if for every $\epsilon > 0$ there exists $\eta > 0$ such that:

$$\sup_{t\in\mathbb{R}} \int_t^{t+\eta} |\varphi(s)|_E^2\, ds \leq \epsilon.$$

With $L^2_n(\mathbb{R}; E)$ we shall indicate the space of all normal functions taking values in E.

Remark. We note that normal functions are obviously not necessary translation compact whilst all translation compact functions are also normal. This new class is a proper subset of the set of all translation bounded functions (i.e. weakly translation compact functions).

3.4 Back to our system: bounded absorbing sets

In order to apply the abstract theory of the previous sections to the simplified model of liquid crystals we are studying, we need some preliminary estimates. Our first goal will be to find some absorbing sets for the trajectories of our system in various function spaces. Most of the results of this section are simple consequences of the estimates obtained in the previous chapter which can be given a deeper meaning.

First of all, we have to define the symbol spaces and the phase spaces we will consider in the following sections. We set:

$$\Sigma = \mathcal{H}(\mathbf{g}) \times \mathcal{H}(\mathbf{h}) \tag{3.5}$$

where $\mathcal{H}(\mathbf{f})$ is the (weak) hull of $\mathbf{f}$ defined as follows.

Definition 3.4.1. The set:

$$\mathcal{H}_\mathcal{T}(f) \doteq \overline{\{T(h)f \mid h \in \mathbb{R}\}}^{\mathcal{T}}$$

is the *hull* of f in the topology $\mathcal{T}$.

In particular, in this chapter we will consider two different symbol spaces, one with optimal regularity conditions and one with stronger regularity assumptions. Actually we will use:

- a "weaker" symbol space Σ_0, defined as in (3.5) considering a forcing term $\mathbf{g} \in L^2_n(\mathbb{R}, \mathbf{V}^*)$ and boundary conditions such that $\mathbf{h} \in L^2_n(0,\infty; \mathbf{H}^{3/2}(\partial\Omega))$ and $\partial_t \mathbf{h} \in L^2_n(0,\infty; \mathbf{H}^{-1/2}(\partial\Omega))$.
- a more regular symbol space Σ_1 for which we suppose $\mathbf{g} \in L^2_n(\mathbb{R}, \mathbf{H})$, $\mathbf{h} \in L^2_n(0,\infty; \mathbf{H}^{5/2}(\partial\Omega))$ and $\partial_t \mathbf{h} \in L^2_n(0,\infty; \mathbf{H}^{1/2}(\partial\Omega))$.

We now have to define an appropriate phase space. By reviewing the "weak" existence result of the previous chapter (theorem 2.1.1), we see that the natural phase space for our system is $\mathbf{H} \times \mathbf{H}^1(\Omega)$ with the additional constraint $|\mathbf{d}| \leq 1$ a.e. $\mathbf{x} \in \Omega$ which follows from the weak maximum principle stated above.

Finally, thanks again to theorem 2.1.1 and to the dissipation result which follows, we can also define the process associated with the solution operator of equation (2.1) acting on the phase space $\mathbf{H} \times \mathbf{H}^1(\Omega)$ indexed by a symbol $\sigma \in \Sigma_0$ (or $\sigma \in \Sigma_1$).

Theorem 3.4.1. *Under the regularity assumptions of theorem 2.1.1, system* (2.1) *admits a uniformly (w.r.t.* $\sigma \in \Sigma_0$*) absorbing set* $B_0 \subset \mathbf{H} \times \mathbf{H}^1$*:*

$$B_0 = \{(\mathbf{u}, \mathbf{d}) \in \mathbf{H} \times \mathbf{H}^1 \mid |\mathbf{u}|_2^2 + |\mathbf{d}|^2_{\mathbf{H}^1} \leq \rho_0\}$$

where

$$\rho_0 = |\Omega| + 2\left(\tfrac{8}{27 C_0 \epsilon^4} + \tfrac{e^{C_0}}{e^{C_0}-1}\left(\tfrac{1}{\nu}|\mathbf{g}|^2_{L^2_b(\mathbf{V}^*)} + C_\Omega |\partial_t \mathbf{h}|^2_{L^2_b(\mathbf{H}^{-1/2})}\right) + C|\mathbf{h}|_{\mathbf{H}^{1/2}}\right).$$

Moreover we have:

$$\int_t^{t+1} |\mathbf{u}(s)|^2_{\mathbf{V}}\, ds + \int_t^{t+1} |\mathbf{d}(s)|^2_{\mathbf{H}^2}\, ds \leq \rho_1$$

with

$$\rho_1 = \max\{1, \nu\}\left(2\rho_0 + C_\Omega |\mathbf{h}|^2_{L^2_b(\mathbb{R}, \mathbf{H}^{3/2})}\right).$$

Proof. Consider estimate (2.20) again. We only need to show that the two integrals are bounded if $(\mathbf{g}, \mathbf{h}) \in \Sigma_0$. Actually this can be easily shown under more general assumptions, namely it is enough that $\mathbf{g}$ and $\mathbf{h}$ are $\mathbf{L}^2$ translation bounded. In the general case we have the following definition (see [6, Section V.4]).

Definition 3.4.2. A function $\varphi \in \mathbf{L}^2_{loc}(\mathbb{R}; \mathcal{E})$ (where $\mathcal{E}$ is a Banach space) is said *translation bounded* if

$$|\varphi|^2_{\mathbf{L}^2_b(\mathbb{R}, \mathcal{E})} \doteq \sup_{t \in \mathbb{R}} \int_t^{t+1} |\varphi(s)|^2_{\mathcal{E}}\, ds < \infty.$$

The function space made up by all translation bounded functions with values in $\mathcal{E}$ will be indicated with $\mathbf{L}^2_b(\mathbb{R}; \mathcal{E})$.

The sought uniform estimate then simply is:

$$\begin{aligned} e^{-C_0 n}\int_0^n e^{C_0 s}|\mathbf{g}(s)|^2_{\mathbf{V}^*}\,ds &= e^{-C_0 n}\sum_{i=0}^{n-1}\int_i^{i+1} e^{C_0 s}|\mathbf{g}(s)|^2_{\mathbf{V}^*}\,ds \\ &\le e^{-C_0 n}\sum_{i=0}^{n-1} e^{C_0(i+1)}\int_i^{i+1}|\mathbf{g}(s)|^2_{\mathbf{V}^*}\,ds \\ &\le e^{-C_0 n}e^{C_0}|\mathbf{g}|^2_{L^2_b(\mathbf{V}^*)}\sum_{i=0}^{n-1} e^{C_0 i} \le \frac{e^{C_0}}{e^{C_0}-1}|\mathbf{g}|^2_{L^2_b(\mathbf{V}^*)} \end{aligned} \tag{3.6}$$

Recalling a standard elliptic regularity estimate (cf. (2.16)) $|\nabla\mathring{\mathbf{d}}|_2 \le |\mathbf{h}|_{\mathbf{H}^{1/2}}$ and remembering that, under the present hypothesis, $\mathbf{h}$ is continuous with values in $\mathbf{H}^{1/2}(\partial\Omega)$, we obtain the absorbing set B_0 as claimed. We will denote by $t_0(B)$ the absorbtion time of the bounded set B in B_0.

In order to prove the second part of theorem 3.4.1, we only need to integrate equation (2.19) from t to $t+1$ with t sufficiently large (it is enough to consider $t \ge t_0$). This immediately gives the estimate we claimed. □

Analogously, starting from the result of section 2.3, we can prove the existence of absorbing sets bounded in the stronger topology of $\mathbf{V}\times\mathbf{H}^2$.

Theorem 3.4.2. *Under the regularity assumptions of theorem 2.3.1, system* (2.1) *admits a uniformly (w.r.t. $\sigma\in\Sigma_1$) absorbing set $B_2\subset\mathbf{V}\times\mathbf{H}^2$:*

$$B_2 = \{(\mathbf{u},\mathbf{d})\in\mathbf{V}\times\mathbf{H}^2 \mid |\mathbf{u}|^2_{\mathbf{H}^1} + |\mathbf{d}|^2_{\mathbf{H}^2} \le \rho_2\}$$

and we have:

$$\int_t^{t+1}|\mathbf{u}(s)|^2_{\mathbf{H}^2}\,ds + \int_t^{t+1}|\mathbf{d}(s)|^2_{\mathbf{H}^3}\,ds \le \rho_3$$

where ρ_2 and ρ_3 depend only on ν, ϵ, Ω, $|\mathbf{h}|_{L^2_b(\mathbb{R},\mathbf{H}^{5/2})}$, $|\partial_t\mathbf{h}|_{L^2_b(\mathbb{R},\mathbf{H}^{1/2})}$ and $|\mathbf{g}|_{L^2_b(\mathbb{R},\mathbf{L}^2)}$.

Proof. Consider estimate (2.26). If we use the uniform Gronwall's inequality (see, for example, [38, Chap. 3, Sec. 1.1.3]), we can deduce for all $t\in\mathbb{R}$:

$$\begin{aligned} &|\nabla\mathbf{u}(t+\epsilon)|_2^2 + |\Delta\widehat{\mathbf{d}}(t+\epsilon) - \mathbf{f}(\mathbf{d}(t+\epsilon))|_2^2 \\ &\quad\le \left(\frac{1}{\epsilon}\int_t^{t+\epsilon} A(s)\,ds + K\epsilon + \int_t^{t+\epsilon}|\widetilde{\mathbf{d}}|^2_{\mathbf{H}^3}\,ds + \frac{2}{\nu}\int_t^{t+\epsilon}|\mathbf{g}(s)|_2^2\,ds\right) \\ &\qquad\qquad\cdot e^{\int_t^{t+\epsilon} A(s)\,ds} = C(\epsilon). \end{aligned}$$

where $A(t) = |\nabla\mathbf{u}(t)|_2^2 + |\Delta\widehat{\mathbf{d}}(t) - \mathbf{f}(\mathbf{d}(t))|_2^2$ as we set in the previous chapter satisfies $A\in L^1(0,T)$ for all $T>0$. By choosing $\epsilon = 1$ and by recalling the regularity estimate (2.6) for the lifting solution $\widetilde{\mathbf{d}}$, we then easily deduce the existence of the strong absorbing set B_2.

As in the proof of theorem 3.4.1, in order to obtain the second part of the theorem we only need to integrate estimate (2.25) from t to $t+1$ with t sufficiently large (again it is sufficient to suppose t greater than the absorbtion time in B_2). □

We end this section introducing another absorbing set which will prove to be useful when dealing with exponential attractors in the next chapter.

Corollary 3.4.3. *Under the regularity assumptions of theorem 2.3.1, system* (2.1) *admits the following uniform (w.r.t. $\sigma \in \Sigma_1$) long-term bound:*

$$|\partial_t \mathbf{d}|_2^2 \leq \rho_4$$

where

$$\rho_4 = C\rho_0\rho_2 + 3\rho_2^2 + \frac{4}{9\epsilon^4}|\Omega|.$$

Proof. This estimate can be obtained directly from the equation for the order parameter field in (2.1). Indeed we have:

$$\begin{aligned}|\partial_t \mathbf{d}|_2^2 &\leq 3|(\mathbf{u}\cdot\nabla)\mathbf{d}|_2^2 + 3|\Delta\mathbf{d}|_2^2 + 3|\mathbf{f}(\mathbf{d})|_2^2 \\ &\leq 3|\mathbf{u}|_4^2|\nabla\mathbf{d}|_4^2 + 3|\Delta\mathbf{d}|_2^2 + 3|\mathbf{f}(\mathbf{d})|_2^2 \\ &\leq C|\mathbf{u}|_2|\nabla\mathbf{u}|_2|\nabla\mathbf{d}|_2|\mathbf{d}|_{\mathbf{H}^2} + 3|\Delta\mathbf{d}|_2^2 + 3|\mathbf{f}(\mathbf{d})|_2^2 \\ &\leq C\rho_0\rho_2 + 3\rho_2 + \frac{4}{9\epsilon^4}|\Omega|\end{aligned}$$

which is the desired result. □

3.5 A smooth attractor

The goal of this section is to apply theorem 3.3.4 to system (2.1) under strong regularity assumptions. We will consider the less regular setting in the next section.

First of all we have to define the symbol space Σ for our problem. In the following we will consider $\Sigma = \Sigma_1$ as defined in the previous section. The aim of this section is to prove the following result.

Theorem 3.5.1. *Given $\mathbf{g} \in L^2_n(\mathbb{R}, \mathbf{H})$, $\mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{5/2}(\partial\Omega))$, and $\partial_t\mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{1/2}(\partial\Omega))$, the process $\{U_{(\mathbf{g},\mathbf{h})}(t,\tau)\}$ generated by the solution operator of problem* (2.1) *possesses a compact uniform (w.r.t. $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$) attractor $\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})}$ in $\mathbf{V} \times \mathbf{H^2}$ which uniformly (w.r.t. $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$) attracts the bounded sets in $\mathbf{H} \times \mathbf{H}^1$ in the norm of $\mathbf{H} \times \mathbf{H}^1$. Moreover we have:*

$$\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})} = \bigcup_{(\mathbf{g},\mathbf{h})\in\mathcal{H}_w(\mathbf{g})\times\mathcal{H}_w(\mathbf{h})} \mathcal{K}_{(\mathbf{g},\mathbf{h})}(0)$$

where $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ is the kernel of the process $\{U_{(\mathbf{g},\mathbf{h})}(t,\tau)\}$ and where $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ is nonempty for all $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$.

Remark. From this result we deduce that all the solution of (2.1) belonging to the kernel of the solution process are strong and globally bounded. We therefore deduce that system (2.1) holds a.e. on the kernel.

We observe that these regularity assumptions on the boundary term $\mathbf{h}$ immediately imply that $\mathbf{h} \in L^\infty(0,\infty;\mathbf{H}^{3/2}(\partial\Omega))$.

Actually, with the above assumptions, most of the hypothesis of theorem 3.3.4 are automatically verified. In order to get the result just stated, we will have only to prove ω-limit compactness and weak continuity of the process defined by the solution operator.

We start by proving ω-limit compactness for our system. To obtain this result we will consider again the lifted system (2.17) (always forgetting all m's). First of all, however, we have to prove that the lifted term $\partial_t \nabla \mathring{\mathbf{d}}$ is bounded and in particular normal itself under these assumptions. From the problem (2.16) tested against a function $\mathbf{v} \in \mathbf{H}^1(\Omega)$, a simple integration by parts yields:

$$\int_\Omega \nabla \mathring{\mathbf{d}} : \nabla \mathbf{v} = \int_{\partial\Omega} \partial_\nu \mathring{\mathbf{d}} \cdot \mathbf{v}.$$

By choosing $\mathbf{v} = \mathring{\mathbf{d}}$ we eventually get:

$$\begin{aligned}
|\nabla \mathring{\mathbf{d}}|_2^2 &\le |\partial_\nu \mathring{\mathbf{d}}|_{\mathbf{H}^{-1/2}} |\mathbf{h}|_{\mathbf{H}^{1/2}} \\
&\le C|\nabla \mathring{\mathbf{d}}|_2 |\mathbf{h}|_{\mathbf{H}^{1/2}} + C|\mathring{\mathbf{d}}|_2 |\mathbf{h}|_{\mathbf{H}^{1/2}} \\
&\le \frac{1}{2}|\nabla \mathring{\mathbf{d}}|_2^2 \ + C\left(1 + |\mathbf{h}|^2_{\mathbf{H}^{1/2}}\right).
\end{aligned}$$

We then easily obtain:

$$\sup_{t\in\mathbb{R}} \int_t^{t+\epsilon} |\nabla \mathring{\mathbf{d}}|_2^2 \le C \sup_{t\in\mathbb{R}} \int_t^{t+\epsilon} |\mathbf{h}|^2_{\mathbf{H}^{1/2}} + C\epsilon.$$

If we apply this last estimate to $\partial_t \mathring{\mathbf{d}}$ instead of $\mathbf{d}$ with $\partial_t \mathbf{h}$ substituted to $\mathbf{h}$, we have then proved that $\partial_t \nabla \mathbf{d} \in L^2_n(\mathbb{R}, \mathbf{L}^2)$ since $\partial_t \mathbf{h} \in L^2_n(\mathbb{R}, \mathbf{H}^{1/2})$.

We now recall proposition 3.3.5, which gives a straightforward way to check ω-limit compactness for the process. Thanks to the Hilbert setting which provides a natural norm-reducing projection onto any linear subspace and thanks to the estimates of the previous section, the first assumption of proposition 3.3.5 has already been verified. We now have to control for the “dissipativeness” of the higher modes. As subspaces we will consider those generated by the first n eigenvalues of Stokes's problem (the space $\mathbf{V}^n$ already introduced in chapter 2) for the velocity field and $\mathbf{D}^m$ spanned by the first m eigenfunctions of the Laplacian with Dirichlet homogeneous boundary conditions in Ω. Let $\{\lambda_n\}$ and $\{\mu_m\}$ be the ascending sequences of eigenvalues respectively for Stokes's problem and Laplace's problem on Ω and let P_n and Q_m be the projections respectively in the subspace generated by the first n eigenfunctions of Stokes's problem and the first m eigenfunctions of Laplace's one. Let $\mathbf{u}_1 \doteq P_n\mathbf{u}$ (respectively $\mathbf{d}_1 \doteq Q_m\mathbf{d}$) and

$\mathbf{u}_2 \doteq \mathbf{u} - \mathbf{u}_1$ (respectively $\mathbf{d}_2 \doteq \mathbf{d} - \mathbf{d}_1$) be the projections of $\mathbf{u}$ (respectively $\mathbf{d}$) on $\mathbf{V}^n$ (respectively $\mathbf{D}^m$) and its orthogonal complement.

Consider again the equation for the velocity field in (2.17) and take its scalar product in $\mathbf{L}^2$ with $-\Delta\mathbf{u}_2$. Using the orthogonality of the chosen base (notice, for example that $(\Delta\mathbf{u}, \Delta\mathbf{u}_2) = |\Delta\mathbf{u}_2|_2^2$), we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}_2|_2^2 + \nu|\Delta\mathbf{u}_2|_2^2 = ((\mathbf{u}\cdot\nabla)\mathbf{u}, \Delta\mathbf{u}_2) + \Big((\nabla\mathbf{d})^t\Delta\check{\mathbf{d}}, \mathbf{u}_2\Big) - (\mathbf{g}(t), \Delta\mathbf{u}_2)\,. \tag{3.7}$$

As usual, we have to estimate all terms on the right hand side of this last expression. In order to obtain the desired estimates we recall a useful interpolation result of [3].

Lemma 3.5.2. *Let $f \in \mathbf{H}^2(\Omega)$, let $\Omega \subset \mathbb{R}^2$ have a compact smooth boundary, then*

$$|f|_{L^\infty} \le C|f|_{H^1}\left(1 + \ln\frac{|f|_{H^2}^2}{|f|_{H^1}^2}\right)^{1/2}$$

where the constant C depends only on the domain Ω.

We start by analyzing the well-known trilinear term of Navier-Stokes equations. We have:

$$\begin{aligned}|((\mathbf{u}\cdot\nabla)\mathbf{u}, \Delta\mathbf{u}_2)| &\le |((\mathbf{u}_1\cdot\nabla)\mathbf{u}, \Delta\mathbf{u}_2)| + |((\mathbf{u}_2\cdot\nabla)(\mathbf{u}_1+\mathbf{u}_2), \Delta\mathbf{u}_2)|\\ &\le |\mathbf{u}_1|_\infty|\nabla\mathbf{u}|_2|\Delta\mathbf{u}_2|_2 + |\mathbf{u}_2|_\infty|\nabla\mathbf{u}_1|_2|\Delta\mathbf{u}_2|_2\\ &\le C|\nabla\mathbf{u}_1|_2\left(1 + \ln\frac{|\Delta\mathbf{u}_1|_2^2}{|\nabla\mathbf{u}_1|_2^2}\right)^{1/2}|\Delta\mathbf{u}_2|_2|\nabla\mathbf{u}|_2\\ &\qquad + C|\mathbf{u}_2|_2^{1/2}|\Delta\mathbf{u}_2|_2^{3/2}\left(|\nabla\mathbf{u}_1|_2 + |\nabla\mathbf{u}_2|_2\right).\end{aligned}$$

Recalling the absorbing sets identified in the previous sections and noticing that $|\Delta\mathbf{u}_2|_2^2 \le \lambda_{n+1}|\nabla\mathbf{u}_2|_2^2$, we finally obtain:

$$\begin{aligned}|((\mathbf{u}\cdot\nabla)\mathbf{u}, \Delta\mathbf{u}_2)| &\le C\rho_2\left(1 + \ln\lambda_{n+1}\right)^{1/2}|\Delta\mathbf{u}_2|_2 + C\rho_0^{1/4}\rho_2^{1/2}|\Delta\mathbf{u}_2|_2^{3/2}\\ &\le \frac{\nu}{12}|\Delta\mathbf{u}_2|_2^2 + \frac{CL\rho_2^2}{\nu} + \frac{C\rho_0\rho_2^2}{\nu^3},\end{aligned}$$

where we have set $L \doteq 1 + \ln\lambda_{n+1}$ for easiness.

The other nonlinear term can be estimated analogously as follows:

$$\begin{aligned}\left|\Big((\nabla\mathbf{d})^t\Delta\check{\mathbf{d}}, \mathbf{u}_2\Big)\right| &\le |\nabla\mathbf{d}_1|_\infty|\Delta\check{\mathbf{d}}|_2|\Delta\mathbf{u}_2|_2 + |\nabla\mathbf{d}_2|_4\left(|\Delta\check{\mathbf{d}}_1|_4 + |\Delta\check{\mathbf{d}}_2|_4\right)|\Delta\mathbf{u}_2|_2\\ &\le C|\nabla\mathbf{d}_1|_{\mathbf{H}^1}\left(1 + \ln\frac{|\nabla\mathbf{d}_1|_{\mathbf{H}^2}^2}{|\nabla\mathbf{d}_1|_{\mathbf{H}^1}^2}\right)^{1/2}|\Delta\mathbf{u}_2|_2|\Delta\check{\mathbf{d}}_2|_2\\ &\quad + C|\nabla\mathbf{d}_2|_2^{1/2}|\nabla\mathbf{d}_2|_{\mathbf{H}^1}^{1/2}\left(|\Delta\check{\mathbf{d}}_1|_2^{1/2}|\Delta\check{\mathbf{d}}_1|_{\mathbf{H}^1}^{1/2} + |\Delta\check{\mathbf{d}}_2|_2^{1/2}|\Delta\check{\mathbf{d}}_2|_{\mathbf{H}^1}^{1/2}\right)|\Delta\mathbf{u}_2|_2\\ &\le C\rho_2 M^{1/2}|\Delta\mathbf{u}_2|_2\\ &\quad + C\rho_0^{1/4}\rho_2^{1/2}\left(|\nabla\Delta\check{\mathbf{d}}_1|_2^{1/2} + |\nabla\Delta\check{\mathbf{d}}_2|_2^{1/2}\right)|\Delta\mathbf{u}_2|_2 + C\rho_0^{1/4}\rho_2^{3/4}|\Delta\mathbf{u}_2|_2\end{aligned}$$

where $M \doteq 1 + \ln \mu_{m+1}$. By recalling that $|\nabla\Delta\check{\mathbf{d}}_1|_2 \le \mu_{m+1}^{1/2}|\Delta\check{\mathbf{d}}_1|_2$, we finally get:

$$\left|\Big((\nabla\mathbf{d})^t\Delta\check{\mathbf{d}}, \mathbf{u}_2\Big)\right| \le \frac{\nu}{12}|\Delta\mathbf{u}_2|_2^2 + \frac{1}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + C(\rho_0,\rho_2,M,\nu) + C\frac{\rho_0^{1/2}\rho_2^{3/2}}{\nu}\mu_{m+1}^{1/2}$$

The last term on the right hand side of equation (3.7) is easily dealt with:

$$|(\mathbf{g}, \Delta\mathbf{u}_2)| \le |\mathbf{g}|_2|\Delta\mathbf{u}_2|_2 \le \frac{\nu}{12}|\Delta\mathbf{u}_2|_2^2 + \frac{3}{\nu}|\mathbf{g}|_2^2.$$

Putting everything together, we get the first half of the desired estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{u}_2|_2^2 + \frac{3}{2}\nu|\Delta\mathbf{u}_2|_2 \le C(\rho_0,\rho_2,L,M,\nu) + \frac{6}{\nu}|\mathbf{g}|_2^2 + \frac{1}{6}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + C\frac{\rho_0^{1/2}\rho_2^{3/2}}{\nu}\mu_{m+1}^{1/2}. \quad (3.8)$$

We now turn our attention to the equation for the order parameter. Multiplying the second equation in (2.17) by $\Delta\Delta\check{\mathbf{d}}_2$, integrating by parts and using the orthogonality of the eigenbasis of the Laplacian, we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\check{\mathbf{d}}_2|_2^2 + |\nabla\Delta\check{\mathbf{d}}_2|_2^2 = \Big((\nabla\mathbf{u}^t\cdot\nabla)\mathbf{d}, \nabla\Delta\check{\mathbf{d}}_2\Big) + \Big(\nabla\nabla\mathbf{d}\cdot\mathbf{u}, \nabla\Delta\check{\mathbf{d}}_2\Big) + \Big(\nabla\mathbf{f}(\mathbf{d}), \nabla\Delta\check{\mathbf{d}}_2\Big) + \Big(\partial_t\nabla\mathring{\mathbf{d}}, \nabla\Delta\check{\mathbf{d}}_2\Big) - {}_{\mathbf{H}^{1/2}}\Big\langle\partial_t\mathring{\mathbf{d}}, \partial_\nu\Delta\check{\mathbf{d}}_2\Big\rangle_{\mathbf{H}^{-1/2}}.$$

As with the equation for the velocity field, we now have to bound all terms on the right hand side of this last equality.

$$\begin{aligned}\left|\Big((\nabla\mathbf{u}^t\cdot\nabla)\mathbf{d}, \nabla\Delta\check{\mathbf{d}}_2\Big)\right| &\le |\nabla\mathbf{u}|_2|\nabla\mathbf{d}_1|_\infty|\nabla\Delta\check{\mathbf{d}}_2|_2 + (|\nabla\mathbf{u}_1|_4 + |\nabla\mathbf{u}_2|_4)\,|\nabla\mathbf{d}_2|_4|\nabla\Delta\check{\mathbf{d}}_2|_2\\ &\le C\rho_2M^{1/2}|\nabla\Delta\check{\mathbf{d}}_2|_2 + C\rho_0^{1/4}\rho_2^{1/2}\left(|\Delta\mathbf{u}_1|_2^{1/2} + |\Delta\mathbf{u}_2|_2^{1/2}\right)|\nabla\Delta\check{\mathbf{d}}_2|_2\\ &\le \frac{1}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + \frac{\nu}{10}|\Delta\mathbf{u}_2|_2^2 + C\rho_2^2M + \frac{C}{\nu}\rho_0\rho_2^2 + C\rho_0^{1/2}\rho_2^{3/2}\lambda_{n+1}^{1/2}.\end{aligned}$$

The second term is dealt with in a similar way. We only remember that, due to the elliptic regularity results that apply to problem (2.16), we have $|\mathbf{d}|_{\mathbf{H}^3}^2 \le C(|\nabla\Delta\check{\mathbf{d}}|_2^2 + |\mathbf{h}|_{\mathbf{H}^{5/2}}^2)$.

$$\begin{aligned}\left|\Big(\nabla\nabla\mathbf{d}\cdot\mathbf{u}, \nabla\Delta\check{\mathbf{d}}_2\Big)\right| &\le |\mathbf{d}|_{\mathbf{H}^2}|\mathbf{u}_1|_\infty|\nabla\Delta\check{\mathbf{d}}_2|_2 + |\mathbf{d}|_{\mathbf{H}^2}|\mathbf{u}_2|_\infty|\nabla\Delta\check{\mathbf{d}}_2|_2\\ &\le C\rho_2^{1/2}L^{1/2}|\nabla\mathbf{u}_1|_2|\nabla\Delta\check{\mathbf{d}}_2|_2 + C\rho_0^{1/2}\rho_2^{1/4}|\Delta\mathbf{u}_2|_2^{1/2}|\nabla\Delta\check{\mathbf{d}}_2|_2\\ &\le \frac{1}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + C\rho_2^2L + \frac{\nu}{12}|\Delta\mathbf{u}_2|_2^2 + \frac{C}{\nu}\rho_0^2\rho_2.\end{aligned}$$

Finally the last two bulk terms can be estimated quite easily:

$$\left|\left(\nabla \mathbf{f}(\mathbf{d}), \nabla\Delta\check{\mathbf{d}}_2\right)\right| \leq C|\nabla \mathbf{d}|_2|\nabla\Delta\check{\mathbf{d}}_2|_2 \leq \frac{1}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + C\rho_0$$

and

$$\left|\left(\partial_t\nabla\mathring{\mathbf{d}}, \nabla\Delta\check{\mathbf{d}}\right)\right| \leq \frac{1}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + C|\partial_t\nabla\mathring{\mathbf{d}}|_2^2.$$

In order to control the boundary term, we can write:

$$\begin{aligned}\left|{}_{\mathbf{H}^{1/2}}\left\langle \partial_t\mathring{\mathbf{d}}, \partial_\nu\Delta\check{\mathbf{d}}_2\right\rangle_{\mathbf{H}^{-1/2}}\right| &\leq C|\partial_t\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}|\Delta\check{\mathbf{d}}_2|_{\mathbf{H}^1} \\ &\leq \frac{1}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 + C\rho_2 + C|\partial_t\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}.\end{aligned}$$

Adding everything together, we eventually get:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\check{\mathbf{d}}_2|_2^2 + \frac{7}{12}|\nabla\Delta\check{\mathbf{d}}_2|_2^2 \\ &\qquad\leq C(\rho_0,\rho_2,L,M,\nu) + C\rho_0^{1/2}\rho_2^{3/2}\lambda_{n+1}^{1/2} + C\rho_0^{1/2}\rho_2^{3/2}\mu_{m+1}^{1/2} \\ &\qquad\qquad + \frac{\nu}{4}|\Delta\mathbf{u}_2|_2^2 + C|\partial_t\nabla\mathring{\mathbf{d}}|_2^2 + C|\partial_t\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}.\end{aligned}$$

By recalling estimate (3.8) and adding the last inequality we have obtained, we find the desired bound on the higher modes of our solution:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla\mathbf{u}_2|_2 + |\Delta\check{\mathbf{d}}_2|_2^2\right) + \nu|\Delta\mathbf{u}_2|_2 + |\nabla\Delta\check{\mathbf{d}}_2|_2^2 \\ &\qquad\leq C(\rho_0,\rho_2,\nu) + C(\rho_0,\rho_2,\nu)(M+L) + C(\rho_0,\rho_2,\nu)(\lambda_{n+1}^{1/2} + \mu_{m+1}^{1/2}) \\ &\qquad\qquad + \frac{6}{\nu}|\mathbf{g}|_2^2 + C|\partial_t\nabla\mathring{\mathbf{d}}_2|_2^2 + C|\partial_t\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}.\end{aligned}$$

From Poincarè's inequality in $\mathbf{V}$ and $\mathbf{H}_0^1$ we have $|\Delta\mathbf{u}_2|_2^2 \geq \lambda_{n+1}|\nabla\mathbf{u}_2|_2^2$ and $|\nabla\Delta\check{\mathbf{d}}_2|_2^2 \geq \mu_{m+1}|\Delta\check{\mathbf{d}}_2|_2^2$. By setting $\kappa = \min\{\nu\lambda_{n+1}, \mu_{m+1}\}$ we can rewrite the last estimate as:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(|\nabla\mathbf{u}_2|_2 + |\Delta\check{\mathbf{d}}_2|_2^2\right) + \kappa\left(|\nabla\mathbf{u}_2|_2 + |\Delta\check{\mathbf{d}}_2|_2^2\right) \\ &\qquad\leq C(\rho_0,\rho_2,\nu) + C(\rho_0,\rho_2,\nu)(M+L) + C(\rho_0,\rho_2,\nu)(\lambda_{n+1}^{1/2} + \mu_{m+1}^{1/2}) \\ &\qquad\qquad + \frac{6}{\nu}|\mathbf{g}|_2^2 + C|\partial_t\nabla\mathring{\mathbf{d}}_2|_2^2 + C|\partial_t\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}.\end{aligned}$$

By using Gronwall's inequality we finally get:

$$\begin{aligned}&|\nabla\mathbf{u}_2(t)|_2^2 + |\Delta\check{\mathbf{d}}_2(t)|_2^2 \leq \left(|\nabla\mathbf{u}_2(t_0)|_2^2 + |\Delta\check{\mathbf{d}}_2(t_0)|_2^2\right)e^{-\kappa(t-t_0)} \\ &\quad + \frac{C(\rho_0,\rho_2,\nu)}{\kappa} + \frac{C(\rho_0,\rho_2,\nu)}{\kappa}(M+L) + \frac{C(\rho_0,\rho_2,\nu)}{\kappa}(\lambda_{n+1}^{1/2} + \mu_{m+1}^{1/2}) \\ &\qquad + \frac{6}{\nu}\int_{t_0}^t e^{-\kappa(t-s)}|\mathbf{g}(s)|_2^2\,ds + C\int_{t_0}^t e^{-\kappa(t-s)}|\partial_t\nabla\mathring{\mathbf{d}}(s)|_2^2\,ds \\ &\qquad\qquad + C\int_{t_0}^t e^{-\kappa(t-s)}|\partial_t\mathbf{h}(s)|^2_{\mathbf{H}^{1/2}}\,ds.\end{aligned}$$

All terms on the right hand side of last inequality can be made arbitrarily small by choosing n and m sufficiently large and so that $\nu\lambda_n \approx \mu_m$. In particular, recalling estimate (3.6) for translation bounded functions, the integral terms can be bounded uniformly as follows:

$$\int_{t_0}^{t} e^{-\kappa(t-s)}|\mathbf{g}(s)|_2^2\,ds \le \frac{1}{1-e^{-\kappa}} \sup_{t\in\mathbb{R}} \int_0^1 e^{-\kappa(1-s)}|\mathbf{g}(s+t)|_2^2\,ds.$$

Moreover, by the normality assumption, for every $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that

$$\sup_{t\in\mathbb{R}} \int_{1-\eta(\epsilon)}^{1} |\mathbf{g}(s+t)|_2^2\,ds \le \frac{\epsilon}{2}.$$

Standard estimates then lead to the desired result:

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \int_0^1 e^{-\kappa(1-s)}|\mathbf{g}(s+t)|_2^2\,ds \\
&\quad\le \sup_{t\in\mathbb{R}} \int_{1-\eta(\epsilon)}^{1} e^{-\kappa(1-s)}|\mathbf{g}(s+t)|_2^2\,ds + \sup_{t\in\mathbb{R}} \int_0^{1-\eta(\epsilon)} e^{-\kappa(1-s)}|\mathbf{g}(s+t)|_2^2\,ds \\
&\quad\le \sup_{t\in\mathbb{R}} \int_{1-\eta(\epsilon)}^{1} |\mathbf{g}(s+t)|_2^2\,ds + e^{-\kappa\eta(\epsilon)} \sup_{t\in\mathbb{R}} \int_0^1 |\mathbf{g}(t+s)|_2^2\,ds \\
&\quad\le \frac{\epsilon}{2} + e^{-\kappa\eta(\epsilon)} \sup_{t\in\mathbb{R}} \int_t^{t+1} |\mathbf{g}(s)|_2^2\,ds.
\end{aligned}$$

We have therefore proved ω-limit compactness for the process generated by equation (2.1) under strong regularity assumptions. To complete the proof of the existence of the global attractor we still have to control weak continuity of the process with respect to initial data and to the symbol. In the last part of this section we will follow [32, Lemma 2.1] with the obvious changes.

Consider again the lifted problem (2.17). Obviously the lifting problem is weakly continuous with respect to the initial boundary data so we have to care only of the lifted equation. Let $\{\mathbf{u}_{0n}\} \subset \mathbf{V}$, $\mathbf{u}_{0n} \rightharpoonup \mathbf{u}_0$, $\{\mathbf{d}_{0n}\} \subset \mathbf{H}^2$, $\mathbf{d}_{0n} \rightharpoonup \mathbf{d}_0$, $\{\mathbf{g}_n\} \subset L^2_{\text{loc}}(\mathbb{R},\mathbf{H})$, $\mathbf{g}_n \rightharpoonup \mathbf{g}$ and $\{\mathbf{h}_n\} \subset L^2_{\text{loc}}(\mathbb{R},\mathbf{H}^{5/2}) \cap L^\infty(\mathbb{R},\mathbf{L}^\infty)$, $\{\partial_t\mathbf{h}_n\} \subset L^2_{\text{loc}}(\mathbb{R},\mathbf{H}^{1/2})$, $\mathbf{h}_n \rightharpoonup \mathbf{h}$, $\partial_t\mathbf{h}_n \rightharpoonup \partial_t\mathbf{h}$ be all weakly convergent sequences of initial data, forcing terms and boundary data. We want to prove $U_{(\mathbf{g}_n,\mathbf{h}_n)}(t,\tau)(\mathbf{u}_{0n},\mathbf{d}_{0n}) \rightharpoonup U_{(\mathbf{g},\mathbf{h})}(t,\tau)(\mathbf{u}_0,\mathbf{d}_0)$ in $\mathbf{V}\times\mathbf{H}^2$.

Let $(\mathbf{u}_n(t),\mathbf{d}_n(t)) = U_{(\mathbf{g}_n,\mathbf{h}_n)}(t,\tau)(\mathbf{u}_{0n},\mathbf{d}_{0n})$. From the absorbing estimates of section 3.4 we already know that $\{(\mathbf{u}_n(t),\check{\mathbf{d}}_n(t))\}$ is bounded in $L^\infty(\tau,\infty;\mathbf{V})\times L^\infty(\tau,\infty;\mathbf{H}^2)$ and in $L^2_{\text{loc}}(\tau,\infty;\mathbf{H}^2)\times L^2_{\text{loc}}(\tau,\infty;\mathbf{H}^3)$. Moreover $\{\mathbf{d}_n\}$ is bounded in $L^\infty(\tau,\infty;\mathbf{L}^\infty)$. Directly from the equation we also get that $\{(\partial_t\mathbf{u}_n,\partial_t\check{\mathbf{d}}_n)\}$ is bounded in $L^2_{\text{loc}}(\tau,\infty;\mathbf{H})\times L^2_{\text{loc}}(\tau,\infty;\mathbf{H}^1)$.

The next step will be of proving the precompactness of $\{(\mathbf{u}_n(t),\check{\mathbf{d}}_n(t))\}$ in $L^2_{\text{loc}}(\tau,\infty;\mathbf{V})\times L^2_{\text{loc}}(\tau,\infty;\mathbf{H}^2)$. Actually we have:

$$\begin{aligned}
(\mathbf{u}_n(t+a)-\mathbf{u}_n(t),\mathbf{v}) &= \int_t^{t+a} (\partial_t\mathbf{u}_n(s),\mathbf{v})\,ds \\
&\le a^{1/2}|\mathbf{v}|_2|\partial_t\mathbf{u}_n|_{L^2_{loc}(\mathbf{L}^2)} \le Ca^{1/2}|\mathbf{v}|_2
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{L}^2$ and for a.e. $t \in [\tau, T]$. By choosing $\mathbf{v} = -\Delta(\mathbf{u}_n(t+a) - \mathbf{u}_n(t)) \in \mathbf{H}$ a.e. $t \geq \tau$ and integrating by parts, we obtain:

$$\int_\tau^{T-a} |\nabla(\mathbf{u}_n(t+a) - \mathbf{u}_n(t))|_2^2 \, dt \leq C_T a^{1/2} \int_\tau^{T-a} |\Delta(\mathbf{u}_n(t+a) - \mathbf{u}_n(t))|_2 \, dt$$
$$\leq C_T a^{1/2} T^{1/2} \int_\tau^{T-a} |\Delta(\mathbf{u}_n(t+a) - \mathbf{u}_n(t))|_2^2 \, dt \leq C_T a^{1/2}.$$

Since $\{\mathbf{u}_n\}$ is bounded in $L^2(\tau, T; \mathbf{H}^2)$ it follows from [34, theorem 3][1]that $\{\mathbf{u}_n\}$ is precompact in $L^2(\tau, T; \mathbf{V})$ for all $T > \tau$.

We can proceed analogously for the order parameter field by considering:

$$\Big(\nabla(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n(t)), \mathbf{w}\Big) = \int_t^{t+a} \Big\langle \partial_t \nabla \check{\mathbf{d}}_n(s), \mathbf{w} \Big\rangle \, ds$$
$$\leq a^{1/2} |\mathbf{w}|_2 |\partial_t \nabla \check{\mathbf{d}}_n|_{L^2_{\text{loc}}(\mathbf{L}^2)} \leq C a^{1/2} |\mathbf{w}|_2 \quad (3.9)$$

for any $\mathbf{w} \in \mathbf{L}^2$ and a.e. $t \in [\tau, T]$. If we take $\mathbf{w} = -\nabla\Delta(\mathbf{d}_n(t+a) - \mathbf{d}_n) \in \mathbf{L}^2$ a.e. $t \geq \tau$, we get:

$$\int_\tau^{T-a} |\Delta(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n(t))|_2^2 \, dt$$
$$\leq \int_\tau^{T-a} {}_{\mathbf{H}^{-1/2}}\Big\langle \partial_\nu(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n), \Delta(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n) \Big\rangle_{\mathbf{H}^{1/2}} \, dt$$
$$+ C a^{1/2} \int_\tau^{T-a} |\nabla\Delta(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n)|_{\mathbf{H}^1} \, dt$$

However, noting that $\Delta\check{\mathbf{d}} - \mathbf{f}(\mathbf{h}) - \partial_t \mathbf{h}|_{\partial\Omega} = \mathbf{0}$, the first term on the right hand side of this inequality can be estimated as follows:

$$\int_\tau^{T-a} {}_{\mathbf{H}^{-1/2}}\Big\langle \partial_\nu(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n(t)), \Delta(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n(t)) \Big\rangle_{\mathbf{H}^{1/2}} \, dt$$
$$\leq C \int_\tau^{T-a} |\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n(t)|_{\mathbf{H}^1}$$
$$\cdot |\mathbf{f}(\mathbf{h}_n(t+a)) - \mathbf{f}(\mathbf{h}_n(t)) + \partial_t \mathbf{h}_n(t+a) - \partial_t \mathbf{h}_n(t)|_{\mathbf{H}^{1/2}} \, dt.$$

From estimate (3.9) we immediately have:

$$|\nabla(\check{\mathbf{d}}_n(t+a) - \check{\mathbf{d}}_n(t))|_2^2 \leq C_T a^{1/2}$$

---

[1]We recall here, for the ease of the reader, the result we used.

Theorem. *Let X, B be Banach spaces and let X be compactly embedded in B. Let $F \subset L^p(0, T; B)$, $1 \leq p \leq \infty$ such that:*

- *F is bounded in $L^1_{\text{loc}}(0, T; X)$;*
- *$|f(t+a) - f(t)|_{L^2(0,T-a;B)} \to 0$ as $a \to 0$ uniformly for $f \in F$.*

Then F is precompact in $L^p(0, T; B)$ (and in $\mathbf{C}(0, T; B)$ if $p = \infty$).

and therefore, on account of the regularity assumptions on $\mathbf{h}$, we deduce:

$$\int_\tau^{T-a} {}_{\mathbf{H}^{-1/2}}\Big\langle \partial_\nu(\check{\mathbf{d}}_n(t+a)-\check{\mathbf{d}}_n(t)), \Delta(\check{\mathbf{d}}_n(t+a)-\check{\mathbf{d}}_n(t))\Big\rangle_{\mathbf{H}^{1/2}} \le C_T a^{1/4}.$$

Since we already know that $\{\mathbf{d}_n\}$ is bounded in $L^2(\tau, T-a;\mathbf{H}^3)$, using the same lemma as before, we conclude that $\{\mathbf{d}_n\}$ is precompact in $L^2(\tau, T-a;\mathbf{H}^2)$.

From the boundedness and compactness of the sequences just proved, by means of a diagonal extraction process, we can find two subsequences of $\{\mathbf{u}_n\}$ and $\{\mathbf{d}_n\}$ such that:

- $\{\mathbf{u}_n\}$ converges weakly* in $L^\infty(\tau,\infty;\mathbf{V})$, weakly in $L^2_{\text{loc}}(\tau,\infty;\mathbf{H^2})$ and strongly in $L^2_{\text{loc}}(\tau,\infty;\mathbf{V})$ to $\mathbf{u}$;
- $\{\mathbf{d}_n\}$ converges weakly* in $L^\infty(\tau,\infty;\mathbf{H}^2)$ and in $L^\infty(\tau,\infty;\mathbf{L}^\infty)$, weakly in $L^2_{\text{loc}}(\tau,\infty;\mathbf{H^3})$ and strongly in $L^2_{\text{loc}}(\tau,\infty;\mathbf{H}^2)$ to $\mathbf{d}$,

where $\mathbf{u}$ and $\mathbf{d}$ solve equation (2.1) (the passage to the limit in the equation being completely analogous to that treated in chapter 2, we shall skip the detail here).

From the strong convergence we have $\mathbf{u}_n(t)\to\mathbf{u}(t)$ strongly in $\mathbf{V}$ and $\mathbf{d}_n(t)\to\mathbf{d}(t)$ strongly in $\mathbf{H}^2$ for a.e. $t\ge\tau$. We therefore have:

$$(\nabla\mathbf{u}_n(t),\mathbf{v})\to(\nabla\mathbf{u}(t),\mathbf{v})$$
$$(\Delta\mathbf{d}_n(t),\mathbf{w})\to(\Delta\mathbf{d}(t),\mathbf{w})$$

for almost every $t\ge\tau$ and any regular pair of functions $(\mathbf{v},\mathbf{w})$. We note that, from the estimates of the previous paragraphs, the functions $(\nabla\mathbf{u}_n(t),\mathbf{v})$ and $(\Delta\mathbf{d}_n(t),\mathbf{w})$ are equibounded and equicontinuous as functions of t. Therefore the previous convergences hold for all $t\ge\tau$, i.e. we have obtained weak continuity for the solution process we are studying.

Thanks to this result we can apply proposition 3.3.5 and theorem 3.3.4. This finishes the proof of theorem 3.5.1.

3.6 A less regular attractor

We now want to extend the results of the previous section to a more general setting. In particular, we want to investigate what happens when we consider less regular forcing terms of critical regularity. In this section we will suppose $\Sigma=\Sigma_1=\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})$ where $\mathbf{g}\in L^2_n(\mathbb{R},\mathbf{V}^*)$ and where $\mathbf{h}\in L^2_n(\mathbb{R},\mathbf{H}^{3/2}(\partial\Omega))$, $\partial_t\mathbf{h}\in L^2_n(\mathbb{R},\mathbf{H}^{-1/2}(\partial\Omega))$. The main result we shall obtain is the following.

Theorem 3.6.1. *Given* $\mathbf{g}\in L^2_n(\mathbb{R},\mathbf{V}^*)$*,* $\mathbf{h}\in L^2_n(\mathbb{R},\mathbf{H}^{3/2}(\partial\Omega))$*, and* $\partial_t\mathbf{h}\in L^2_n(\mathbb{R},\mathbf{H}^{-1/2}(\partial\Omega))$*, the process* $\{U_{(\mathbf{g},\mathbf{h})}(t,\tau)\}$ *corresponding to problem* (2.1) *possesses a compact uniform (w.r.t.* $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*) attractor* $\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})}$ *in* $\mathbf{H} \times \mathbf{H^1}$ *which uniformly (w.r.t.* $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*) attracts the bounded sets in* $\mathbf{H} \times \mathbf{H}^1$ *in the norm of* $\mathbf{H} \times \mathbf{H}^1$*. Moreover we have:*

$$\mathcal{A}_{\mathcal{H}(\mathbf{g})\times\mathcal{H}(\mathbf{h})} = \bigcup_{(\mathbf{g},\mathbf{h})\in\mathcal{H}_w(\mathbf{g})\times\mathcal{H}_w(\mathbf{h})} \mathcal{K}_{(\mathbf{g},\mathbf{h})}(0)$$

where $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ *is the kernel of the process* $\{U_{(\mathbf{g},\mathbf{h})}(t,\tau)\}$ *and where* $\mathcal{K}_{(\mathbf{g},\mathbf{h})}$ *is nonempty for all* $(\mathbf{g},\mathbf{h}) \in \mathcal{H}_w(\mathbf{g}) \times \mathcal{H}_w(\mathbf{h})$*.*

Starting from the discussion of the previous section, we observe that most of the assumptions of the abstract theorem 3.3.4 have already been verified. As for the stronger regularity setting we only have to check ω-limit compactness and weak continuity. Although the proof of the weak continuity can be carried over to the current setting with no significant changes (and therefore we skip here the details), checking ω- limit compactness involves a slightly more refined estimate. We note that, due to the structure of the nonlinear terms and in particular to the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$, the direct approach adopted in the last section does not lead to any useful estimate in this case. All this section will therefore be devoted to checking this last assumption.

We start by recalling some results from section 3.4. In particular, we proved that there exists an absorbing set for $|\mathbf{u}(t)|_2^2 + |\mathbf{d}|^2_{\mathbf{H}^1}$ and that:

$$\nu \int_t^{t+\delta t} |\nabla\mathbf{u}(s)|_2^2\, ds + \int_t^{t+\delta t} |\Delta\mathbf{d}(s)|_2^2\, ds$$

is uniformly bounded (w.r.t. $t \geq \tau$) for sufficiently large t.

As usual we now look for a bound on the time derivative of the solution fields in the natural weak norms. By standard estimates we have:

$$\begin{aligned} |\partial_t\mathbf{u}|_{\mathbf{V}^*} &\leq |(\mathbf{u}\cdot\nabla)\mathbf{u}|_{\mathbf{V}^*} + \nu|\Delta\mathbf{u}|_{\mathbf{V}^*} + |\nabla\mathbf{d}^t\Delta\mathbf{d}|_{\mathbf{V}^*} + |\mathbf{g}|_{\mathbf{V}^*} \\ |\partial_t\mathbf{d}|_2 &\leq |(\mathbf{u}\cdot\nabla)\mathbf{d}|_2 + |\Delta\mathbf{d}|_2 + |\mathbf{f}(\mathbf{d})|_2. \end{aligned}$$

where the nontrivial terms on the right hand side of the last inequalities can be bounded as follows:

$$\begin{aligned} |\Delta\mathbf{u}|_{\mathbf{V}^*} &= |\nabla\mathbf{u}|_2 \\ |\nabla\mathbf{d}^t\Delta\mathbf{d}|_{\mathbf{V}^*} &\leq C|\mathbf{d}|_2^{1/2}|\mathbf{d}|_{\mathbf{H}^1}|\mathbf{d}|^{1/2}_{\mathbf{H}^2} \leq C\rho_0^{1/2}|\mathbf{d}|^{1/2}_{\mathbf{H}^2} \\ |(\mathbf{u}\cdot\nabla)\mathbf{d}|_2 &\leq |\mathbf{u}|_4|\nabla\mathbf{d}|_4 \leq C|\mathbf{u}|_2^{1/2}|\nabla\mathbf{u}|_2^{1/2}|\nabla\mathbf{d}|_2^{1/2}|\mathbf{d}|^{1/2}_{\mathbf{H}^2} \\ &\leq C\rho_0|\nabla\mathbf{u}|_2 + C\rho_0|\mathbf{d}|_{\mathbf{H}^2}. \end{aligned}$$

Remembering that $|\mathbf{d}|^2_{\mathbf{H}^2} \leq C|\Delta\check{\mathbf{d}}|_2^2 + |\mathbf{h}|^2_{\mathbf{H}^{3/2}}$, we easily obtain that $\partial_t\mathbf{u} \in L^2_{\text{loc}}(\tau,\infty;\mathbf{V}^*)$ and $\partial_t\mathbf{d} \in L^2_{\text{loc}}(\tau,\infty;\mathbf{L}^2)$. We observe that the bound on the norm over the interval $[t, t+\delta t]$ of both derived fields does not depend on the time t. Thanks to [34, Corollary 4][2] we have that

$$B_{[t,t+\delta t]} = \{(\mathbf{u}(s), \mathbf{d}(s)) = U_{(\mathbf{g},\mathbf{h})}(t,\tau)(\mathbf{u}_\tau, \mathbf{d}_\tau), (\mathbf{u}_\tau, \mathbf{d}_\tau) \in B_0\}\Big|_{s\in[t,t+\delta t]}$$

is precompact in $L^2(t, t+\delta t; \mathbf{H}\times\mathbf{H}^1)$.

From the precompactness of $B_{[t,t+\delta t]}$ we deduce that there exists a finite number of pairs $(\mathbf{u}_1, \mathbf{d}_1), \ldots, (\mathbf{u}_N, \mathbf{d}_N)$ such that for any $(\mathbf{u}, \mathbf{d}) \in B_{[t,t+\delta t]}$ there exists an i that verifies:

$$\int_t^{t+\delta t} \left(|\mathbf{u}-\mathbf{u}_i|_2^2 + |\nabla\mathbf{d} - \nabla\mathbf{d}_i|_2^2\right) \le \epsilon$$

Therefore there exists a time $\tilde{t} \in [t, t+\delta t]$ such that:

$$|\mathbf{u}(\tilde{t}) - \mathbf{u}_i(\tilde{t})|_2^2 + |\nabla\mathbf{d}(\tilde{t}) - \nabla\mathbf{d}_i(\tilde{t})|_2^2 \le \frac{\epsilon}{\delta t}.$$

We now use the continuous dependence estimate (2.21) proven in chapter 2 and get:

$$\begin{aligned}
|\mathbf{u}(t+\delta t) - \mathbf{u}_i(t+\delta t)|_2^2 &+ |\nabla\mathbf{d}(t+\delta t) - \nabla\mathbf{d}_i(t+\delta t)|_2^2 \\
&\le C\left(|\mathbf{u}(\tilde{t}) - \mathbf{u}_i(\tilde{t})|_2^2 + |\nabla\mathbf{d}(\tilde{t}) - \nabla\mathbf{d}_i(\tilde{t})|_2^2\right) \\
&+ C\left(\sup_{t\ge\tau}\int_{\tilde{t}}^{t+\delta t}\left(\tfrac{3}{\nu}|\mathbf{g}|_{\mathbf{V}^*}^2 + C|\partial_t\mathbf{h}|_{\mathbf{H}^{-1/2}}^2 + C|\mathbf{h}|_{\mathbf{H}^{3/2}}^2\right) ds\right)
\end{aligned}$$

where all constants depend only on ρ_0, ρ_1 and are bounded with respect to $\delta t \le 1$. Using the normality assumption on the forcing and boundary terms and the precompactness of the trajectories just proven, we can therefore bound the left hand side of the last inequality by a fixed constant times ϵ. We have so proven that $B_T = U_{T,\tau}B_0$ is compact for sufficiently big $T-\tau$, uniformly w.r.t. $\tau \in \mathbb{R}$. Since B_0 is absorbing, this also proves the ω-limit compactness for the process and ends our proof of theorem 3.6.1.

In this chapter we have studied the existence of the global attractor for system (2.1) under very general assumptions. However, as it is well known from the abstract theory (see, e.g., [27]), global attractors suffer a serious drawback: the rate of attraction of orbits can indeed be arbitrarily slow.

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[2] As before for reader's convenience, we recall the result we used.

Theorem. *Let X, B, Y be Banach spaces, $X \subset B \subset V$ and let X be compactly embedded in B. Let $F \subset L^p(0,T;X)$, $1 \le p < \infty$ such that:*

- *F is bounded in $L^p(0,T;X)$;*
- *$\partial_t F$ is bounded in $L^1(0,T;Y)$.*

Then F is precompact in $L^p(0,T;B)$.

In order to avoid this problem, it is necessary to construct larger attracting sets that are still finite dimensional, but which guarantee an exponential rate of attraction. This can actually be done by establishing the existence of exponential attractors which will be the goal of the next chapter.

Chapter 4

Exponential Attractors

As it was noted at the end of the previous chapter, global attractors are not necessarily the only description of a dissipative dynamical system. A feature one usually wants to guarantee is an exponential attracting rate of trajectories to the attractor preserving the finite-dimension. This corresponds, from a physical point of view, to an observable quick decay of high order modes and to a fast reduction of the variety of the expected dynamics. Moreover, some sort of continuous dependence of the attractor on the external data is hoped for. This means that small changes in the form of the forcing terms should not cause any relevant modification in the characteristics of the attracting sets. An updated review of these issues can be found in [27].

One possible solution to these problem was proposed in the 90's by Eden, Foias, Nicolaenko and Temam (see [11]), who first introduced the notion of exponential attractor. Although also this kind of objects suffers some important drawbacks (notably the lack of uniqueness) and its application was initially limited to Hilbert settings involving somehow tortuous computations, it soon became clear (see the works of Efendiev, Miranville and Zelik as [12]) that the basic ideas could indeed be applied more generally in Banach space settings by using rather natural estimates on the solutions.

In this chapter we will quickly review the theory of exponential attractors for nonautonomous dissipative PDEs with particular attention to the case of quasiperiodic forcing data (see [6, Section V.1] for more details).

4.1 The Hilbert space setting

We start by introducing exponential attractors for semigroups.

Definition 4.1.1. Let E be a Banach space. A compact set $\mathcal{M} \subset E$ is an *exponential attractor* for the semigroup $\{S(t)\}$ if:

- it has finite fractal dimension;
- it is positively invariant, i.e. $S(t)\mathcal{M} \subset \mathcal{M}$, $\forall t \geq 0$;
- it attracts exponentially the bounded subsets of E in the following sense:

$$\forall B \subset E \text{ bounded }, \text{dist}_E(S(t)B, \mathcal{M}) \leq Q(|B|_E)e^{-\alpha t}, t \geq 0,$$

  where α is a positive number and Q is a monotonic function both independent of B.

We start by observing that, if an exponential attractor exists, being a compact absorbing set, it surely contains the global attractor defined in the previous chapter. It also immediately follows from the above definition that exponential attractors are not uniquely defined. Indeed, one can always enlarge a given exponential attractor by simply adding the forward orbit under S starting from any point $u \in E$: invariance and exponential attraction of bounded sets is immediately verified, whereas it is straightforward to check that the fractal dimension of the attractor undergoes no relevant changes.

In the early stages of the development of the theory of exponential attractors (see [11]), the following squeezing property played a central role in proving the existence of such sets.

Definition 4.1.2. A mapping $S : X \to X$, where X is a compact subset of an Hilbert space E, enjoys the *squeezing property* on X if for some $\delta \in (0, \frac{1}{4})$ there exists an orthogonal projection $P = P(\delta)$ with finite rank such that for every $u, v \in X$ either

$$|(I - P)(Su - Sv)|_E \leq |P(Su - Sv)|_E$$

or

$$|Su - Sv|_E \leq \delta|u - v|_E$$

holds.

From a geometrical point of view, the squeezing property corresponds to requiring that higher modes of solutions are exponentially decaying, or, if this is not the case, that lower modes are dominating. Indeed it implies that only a finite number of degree of freedom are necessary in order to fully describe the asymptotic behaviour of the system we are studying (see [31, Part IV] for more details).

One can usually guarantee the existence of exponential attractors under very mild assumptions. In particular, the following result was proved in [11].

Theorem 4.1.1. *Let $\{S(t)\}$, $S(t) : X \to X$ be a semigroup and let t^* be a positive real number. If the mapping $S(t^*)$ enjoys the squeezing property,* *then the discrete time semigroup generated by $S(t^*)$ has an exponential attractor $\mathcal{M}^*$. Moreover, if the mapping $S(t)u$ is Lipschitz (or Hölder) on $[0,t^*] \times X$, then the set:*

$$\mathcal{M} \doteq \bigcup_{t \in [0,t^*]} S(t)\mathcal{M}^*$$

is an exponential attractor for the continuous-time semigroup $\{S(t)\}$ on X.

Here we will only give a short sketch of the ideas involved in the proof of this result, referring to the original work of Eden et al. for further details. We start by considering a closed ball B^0 of radius R which contains an absorbing set for our system. Since, by assumption, the squeezing property holds, we can find a finite number of closed balls $\{B_i^1\}_{i=1,\dots,N}$ of radius $R/2$ such that $S(t^*)B^0 \subset \cup_i B_i^1$. Intuitively this is possible since, under the action of our semigroup, B^0 is shrunk in all but a finite number of dimensions. Moreover, by proceeding in exactly the same way for all the B_i^1, we can find a finite number of closed balls of radius $R/4$ which cover $S(t^*) \circ S(t^*)B^0$. This same reasoning can easily be continued by induction giving a countable infinity of balls of decreasing radius. By considering the closure of the forward orbits starting from all the centres of the balls just constructed, we can build a compact set which is proved to be an exponential attractor for the discrete time semigroup. The last part of the statement of the theorem can be obtained by standard continuity estimates. Further details on this point can be found, e.g., in [13].

4.2 The Banach space setting

The theory briefly reviewed in the last section and, in particular, the squeezing property introduced above, is strongly bind to the Hilbert setting. At a first glance it seems impossible to extend it to the more general setting of Banach spaces, although appealing such an extension could be.

The key idea for this passage was given by Efendiev, Miranville and Zelik and brought at the same time a simplification in the estimates needed to verify the assumptions of the existence theorems. We now introduce the smoothing property as in [12].

Definition 4.2.1. Let E, E_1 be Banach spaces with E_1 compactly embedded in E, let X be a bounded subset of E_1 and let $S : E \to E$. Then S enjoys the *smoothing property* on X if

$$|Su - Sv|_{E_1} \leq C|u - v|_E \qquad \forall u, v \in X.$$

As it can easily be seen, this time the "squeezing" feature of the semigroup is guaranteed by the compactness assumption.

Before stating the main result we introduce, as in [13], the following class of mappings.

Definition 4.2.2. Let E and E_1 be Banach spaces with E_1 compactly embedded in E and let X be a bounded subset of E_1. Given positive constants δ and K, a (nonlinear) operator $S : E \to E$ belongs to the class of *smoothing operators* $\mathbb{S}_{\delta,K}(X)$ if:

- $S\mathcal{O}_\delta(X) \subset X$ where $\mathcal{O}_\delta(X)$ is a neighbourhood of X of radius δ in the topology of E_1;
- S enjoys the smoothing property on $\mathcal{O}_\delta(X)$, that is:

$$|Su - Sv|_{E_1} \le C|u - v|_E \qquad \forall u, v \in \mathcal{O}_\delta(X).$$

We observe that any map S in $\mathbb{S}_{\delta,K}$ naturally gives rise to a discrete-time semigroup simply by iteration: $S(n)u = S^{\circ n}u$.

In order to study the continuous dependence of exponential attractors on the chosen semigroup, we also introduce a metric on $\mathbb{S}_{\delta,K}(X)$ by setting:

$$|S_1 - S_2|_{\mathbb{S}} \doteq \sup_{u \in \mathcal{O}_\delta(X)} |S_1 u - S_2 u|_{E_1}$$

We can now state the following basic result for discrete time semigroups (see [12] for a proof). The straightforward extension to the continuous time case can be performed exactly as in the previous section.

Theorem 4.2.1. *For every $S \in \mathbb{S}_{\delta,K}(X)$, there exists an exponential attractor $\mathcal{M}_S$ in the topology of E_1, that is:*

1. $\dim_F(\mathcal{M}_S) \le C$;
2. $S\mathcal{M}_S \subset \mathcal{M}_S$;
3. $\text{dist}_{E_1}(S(n)X, \mathcal{M}_S) \le Ce^{-\alpha n}, n \in \mathbb{N}$.

Moreover, the map $S \mapsto \mathcal{M}_S$ can be chosen such that it is Hölder continuous in the following sense:

$$\text{dist}^{symm}_{E_1}(\mathcal{M}_{S_1}, \mathcal{M}_{S_2}) \le C|S_1 - S_2|^\kappa_{\mathbb{S}}.$$

Finally, α, κ and all other constants which appear in the preceding estimates depend only on X, δ and K, but they are otherwise independent of the particular semigroup $S \in \mathbb{S}_{\delta,K}(X)$.

We immediately observe that the particular (constructive) choice of the map $S \mapsto \mathcal{M}_S$ is the key passage required to prove the continuity of the attracting set under small perturbations of the semigroup. This result has indeed been one of the major concerns of research during the last decades: for global attractors, only upper semicontinuity can be proved since this attracting set can suddenly implode (see [31, Chapter 10] for some results concerning this problem). The original idea in the work of Efendiev, Miranville e Zelik is to fix a canonical way to cover the unit ball in E_1 by a finite number of balls of radius $1/2K$ in E. This standard covering, chosen once for all semigroups $S \in \mathbb{S}_{\delta,K}(X)$, characterizes the map $S \mapsto \mathcal{M}_S$ and leads to all the estimates stated above.

The just developed theory can quite easily be extended to processes. In particular, two different approaches to the exponential attractors of non-autonomous evolution equations are possible: one can fix a particular forcing term (or symbol) and study the evolution of solutions in the particular chosen case, or it is possible to consider a full class of forcing terms which share some particular property. In the first case, it is often possible (see [13] again for the details) to identify a time varying exponential attractor $\mathcal{M}(t)$ which exponentially attracts all the trajectories of the system. Although this approach is very appealing, we shall not pursue it in this thesis. We only observe that one usually shows the existence of an $N+1$-dimensional time-varying exponential attractor where N is the dimension of any of its fixed-time sections. In contrast to the alternative approach we will follow, however, once this attracting set is projected on the original phase space (as we did in chapter 3 for the global attractors of skew product semigroups) its finite-dimensionality is lost.

In the following section we will consider a peculiar family of forcing terms, namely quasi-periodic functions, in order to apply the results of this section to the extended semigroup associated to the process of interest as in the previous chapter. We will therefore show that, if the space of the symbols for the non-autonomous equation is finite-dimensional, all the theory developed so far carries easily over to the non-autonomous case.

4.3 Quasi-periodic functions and extended phase spaces

In this section we will introduce quasi-periodic functions and then study how they can represent a useful class of symbols for non-autonomous evolution equations.

Definition 4.3.1. Let Ξ be a Banach space, let $(\alpha^1, \dots, \alpha^k)$ be a k-tuple of rationally independent real numbers and let $\phi : \mathbb{R}^k \to \Xi$ be a continuous function which is 2π-periodic in each argument, that is:

$$\phi(\omega^1, \dots, \omega^i + 2\pi, \dots, \omega^k) = \phi(\omega^1, \dots, \omega^i, \dots, \omega^k).$$

Then $\sigma(s) \doteq \phi(\alpha^1 s, \alpha^2 s, \dots, \alpha^k s) \doteq \phi(\alpha s)$ is said to be a *quasi-periodic* function with values in Ξ.

Given a quasi-periodic function σ in the sense of definition 4.3.1, its hull $\mathcal{H}(\sigma)$ (see definition 3.3.1) can easily be characterized as follows (see [6, Section V.1] for a proof of this result).

Lemma 4.3.1. *The hull $\mathcal{H}(\sigma)$ of the function σ in $\mathbf{C}(\mathbb{R};\Xi)$ satisfies:*

$$\mathcal{H}(\sigma) = \{\phi(\alpha\omega + \theta) \mid \theta \in \mathbb{T}^k\}$$

where $\mathbb{T}^k = \bigotimes_{i=1}^k \mathbb{S}^1$ is the k-dimensional real torus and where for any fixed element in $\mathcal{H}(\sigma)$, θ is said to be its initial phase*.*

This last lemma implies that if we wish to consider a quasi-periodic function as symbol for an evolution equation, we can simplify the analysis by reducing the symbol space just to $\mathbb{T}^k$. We immediately observe that $\mathbb{T}^k$ is finite dimensional, bounded and closed and therefore compact by the Heine-Borel theorem.

The torus model for quasi-periodic forcing terms can be further enhanced, by noting that it is possible to define a natural metric on $\mathbb{T}^k$. Indeed we will set:

$$|\omega_1 - \omega_2|^2_{\mathbb{T}^k} \doteq \sum_{i=1}^{k} \Big(|\omega_1^i - \omega_2^i| \mod 2\pi\Big)^2$$

We immediately notice that the natural identification introduced above of elements in the hull of a quasi-periodic function σ and points on the torus $\mathbb{T}^k$ endowed with this metric is indeed continuous, as can be easily seen using Heine-Cantor theorem. Actually the natural identification with $\mathbb{T}^k$ is a diffeomorfism of $\mathbb{T}^k$ onto a subset of Ξ. In the following analysis, when dealing with quasi-periodic forcing terms, we will therefore always consider $\mathbb{T}^k$ as phase space noting here, once and for all, that we have:

$$|\sigma_1 - \sigma_2|_\Xi \leq C|\omega_1 - \omega_2|_{\mathbb{T}^k}, \tag{4.1}$$

where $\sigma_i = \phi(\alpha\omega_i)$, $i = 1, 2$.

If we consider a non-autonomous evolution equation whose symbol is assumed to be quasi-periodic and whose solution operator generates a process on a Banach space E (as in the previous chapter), we can easily reduce the solution process acting on E to a semigroup acting on the larger phase space given by $E \times \mathbb{T}^k$. To this particular semigroup we can then apply the results of the previous section, noting that the smoothing property for the variables in the extended part of the phase space is automatically verified thanks to the compactness of the symbol space (and in particular of $\mathbb{T}^k$). In the following sections we will actually reduce the process associated with equation (2.1) to a semigroup using the observations of this section and then apply theorem 4.2.1 to obtain the existence of an exponential attractor for our system.

4.4 Back to our system: a discrete-time exponential attractor

This section and the next are devoted to prove the existence of an exponential attractor for system (2.1). Thanks to theorem 4.2.1, setting $E = \mathbf{H} \times \mathbf{H}^1$

and $E_1 = \mathbf{V} \times \mathbf{H}^2$, we will only need to prove that the extended semigroup $S : \mathbf{H} \times \mathbf{H}^1 \times \mathbb{T}^k \to \mathbf{H} \times \mathbf{H}^1 \times \mathbb{T}^k$ belongs to the class of operators $\mathbb{S}_{\delta,K}(X)$ for suitable δ, K and X.

We start by stating our main result whose proof will occupy the remaining part of this section.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^2$ be a regular bounded domain, let $\mathbf{g}$ and $\mathbf{h}$ be quasi-periodic functions with values in $\mathbf{L}^2$ and $\mathbf{H}^{5/2}(\partial\Omega)$, respectively, such that also $\partial_t \mathbf{h}$ is quasi-periodic with values in $\mathbf{H}^{1/2}(\partial\Omega)$. Let $\{S(t)\}$ be the extended semigroup associated to the solution operator of problem* (2.1) *acting on the extended phase space $\mathbf{H} \times \mathbf{H}^1 \times \mathbb{T}^k$ (here k is equal to the sum of the different irrationally independent periods of $\mathbf{h}$ and $\mathbf{g}$)*[1]*. Then there exists a finite time t^* such that the discrete-time semigroup generated by $S(t^*)$ possesses a uniform (w.r.t. the initial phase $\theta \in \mathbb{T}^k$) exponential attractor.*

Remark. Theorem 4.2.1 gives us also continuous dependence of the exponential attractor on the semigroup considered. When considering quasi-periodic symbols, it is easy to see that the extended semigroup continuously depends on the frequencies of the different periods characterizing the forcing terms. we therefore have continuity of the exponential attractor with respect to the frequencies of the symbols.

In order to prove that $S \in \mathbb{S}_{\delta,K}(X)$ for suitable δ, K and X it is enough to show that there exists an absorbing bounded set $X \subset \mathbf{V} \times \mathbf{H}^2 \times \mathbb{T}^k$ and that the smoothing property holds (cf. definition 4.2.2).

The first requirement has already been verified thanks to the results of section 3.4. Namely we proved that (see theorem 3.4.2) the "strong" regularity assumptions imply the existence of a bounded absorbing set in $\mathbf{V} \times \mathbf{H}^2$. Since $\mathbb{T}^k$ is invariant under the action of the extended semigroup, it immediately follows that the extended semigroup $S(t)$ possesses the required absorbing set. Therefore, in what follows we can choose strong absorbing sets given by theorem 3.4.2 and its corollary as the set X which appears in definition 4.2.2.

We now need to show that the smoothing property holds. Since the proof of this result is quite lengthy, we give here a short overview of the standard main argument that will follow. The main estimate we shall prove will be made up of three major contribution: the first arising from the difference equation for the velocity field obtained from (2.8) (without ms!), the second coming from the difference lifted order parameter equation deduced again from (2.8) while the third and last derives from the difference time-dependent lifted problem got from (2.4). In any of the three cases, our aim

[1]We observe that here we have implicitly substituted the natural extension $\mathbb{T}^l \oplus \mathbb{T}^m$ of the phase space with the algebraically and geometrically equivalent space $\mathbb{T}^{l+m}$. This equivalence can immediately be proven by considering the standard coordinate description of a k-dimensional torus through vectors of $\mathbb{R}^k$ with the proper identification.

will be to obtain inequalities of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\delta|_{E_1}^2 + |\delta|_{E_2}^2 \leq C|\delta|_E^2 + C|\delta|_{E_1}^2$$

where E_2 will be a Banach space (compactly) embedded in E_1 and where δ is any difference of solutions.

From an inequality of this kind we can easily use the uniform Gronwall inequality to get a time dependent bound on $|\delta(t)|_{E_1}$ of the form:

$$|\delta(t)|_{E_1}^2 \leq C(t)\int_0^t |\delta(s)|_{E_1}^2\, ds.$$

The smoothing property can then be obtained by recalling once more the results of section 3.4.

We now start the proof by considering the equation for the velocity field in (2.8). Let $(\mathbf{u}_1, \mathbf{d_1})$ and $(\mathbf{u}_2, \mathbf{d_2})$ be two solutions to (2.8). Taking the difference of the equations for the velocity field, we obtain:

$$\begin{aligned}\partial_t \mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u}_1 + (\mathbf{u}_2\cdot\nabla)\mathbf{w} - \nu\Delta\mathbf{w} + \nabla(p_1 - p_2)&\\ = -(\nabla\mathbf{e})^t\Delta\mathbf{d}_1 - (\nabla\mathbf{d}_2)^t\Delta\mathbf{e} + \mathbf{g}_1 - \mathbf{g}_2&\end{aligned}$$

for a.e. $t \in \mathbb{R}$, where we have set $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$ and $\mathbf{e} = \mathbf{d}_1 - \mathbf{d}_2$. We observe that, since $\mathbf{u}_i|_{\partial\Omega} = \mathbf{0}$, $i = 1, 2$, we also have $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$. Multiplying this last equation by $-\Delta\mathbf{w}$ and integrating by parts, we deduce:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{w}|_2^2 - ((\mathbf{w}\cdot\nabla)\mathbf{u}_1, \Delta\mathbf{w}) - ((\mathbf{u}_2\cdot\nabla)\mathbf{w}, \Delta\mathbf{w}) + \nu|\Delta\mathbf{w}|_2^2&\\ = \Big((\nabla\mathbf{e})^t\Delta\mathbf{d}_1, \Delta\mathbf{w}\Big) + \Big((\nabla\mathbf{d}_2)^t\Delta\mathbf{e}, \Delta\mathbf{w}\Big) + (\mathbf{g}_1 - \mathbf{g}_2, \Delta\mathbf{w})\,.&\end{aligned}$$

Arguing as in the previous chapters, we now try to bound all the non-linear terms appearing in this last relation. We start by considering the usual trilinear term coming from the Navier-Stokes equation. We have:

$$\begin{aligned}|((\mathbf{w}\cdot\nabla)\mathbf{u}_1, \Delta\mathbf{w})| &\leq |\mathbf{w}|_\infty|\nabla\mathbf{u}_1|_2|\Delta\mathbf{w}|_2\\ &\leq C|\mathbf{w}|_2^{1/2}|\nabla\mathbf{u}_1|_2|\Delta\mathbf{w}|_2^{3/2}\\ &\leq \frac{\nu}{14}|\Delta\mathbf{w}|_2^2 + \frac{C}{\nu}|\mathbf{w}|_2^2|\nabla\mathbf{u}_1|_2^4\\ &\leq \frac{\nu}{14}|\Delta\mathbf{w}|_2^2 + \frac{C}{\nu}\rho_2^2|\mathbf{w}|_2^2\end{aligned}$$

where the last estimate follows from the results of section 3.4 (and in particular it is a consequence of theorem 3.4.2). For the second term arising from the convective contribution in Navier-Stokes equations, we can write:

$$\begin{aligned}|((\mathbf{u}_2\cdot\nabla)\mathbf{w},\Delta\mathbf{w})| &\le |\mathbf{u}_2|_4|\nabla\mathbf{w}|_4|\Delta\mathbf{w}|_2\\ &\le C|\mathbf{u}_2|_2^{1/2}|\nabla\mathbf{u}_2|_2^{1/2}|\nabla\mathbf{w}|_2^{1/2}|\Delta\mathbf{w}|_2^{3/2}\\ &\le \frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\frac{C}{\nu}|\mathbf{u}_2|_2^2|\nabla\mathbf{u}_2|_2^2|\nabla\mathbf{w}|_2^2\\ &\le \frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\frac{C}{\nu}\rho_0\rho_2|\nabla\mathbf{w}|_2^2.\end{aligned}$$

We now consider the two contributions coming from the nonlinear coupling with the order parameter equation. We have:

$$\begin{aligned}\left|\Big((\nabla\mathbf{e})^t\Delta\mathbf{d}_1,\Delta\mathbf{w}\Big)\right| &\le |\nabla\mathbf{e}|_\infty|\Delta\mathbf{d}_1|_2|\Delta\mathbf{w}|_2\\ &\le C|\nabla\mathbf{e}|_2^{1/2}|\mathbf{e}|_{\mathbf{H}^3}^{1/2}|\Delta\mathbf{d}_1|_2|\Delta\mathbf{w}|_2\\ &\le \frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\beta|\mathbf{e}|_{\mathbf{H}^3}^2+\frac{C}{\nu^2\beta}|\nabla\mathbf{e}|_2^2|\Delta\mathbf{d}_1|_2^4\\ &\le \frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\beta|\mathbf{e}|_{\mathbf{H}^3}^2+C\frac{\rho_2^2}{\nu^2\beta}|\nabla\mathbf{e}|_2^2\end{aligned}$$

where β is a positive real number that will be determined later. Similarly we get:

$$\begin{aligned}\left|\Big((\nabla\mathbf{d}_2)^t\Delta\mathbf{e},\Delta\mathbf{w}\Big)\right| &\le |\nabla\mathbf{d}_2|_4|\Delta\mathbf{e}|_4|\Delta\mathbf{w}|_2\\ &\le C|\nabla\mathbf{d}_2|_2^{1/2}|\mathbf{d}_2|_{\mathbf{H}^2}^{1/2}|\Delta\mathbf{e}|_2^{1/2}|\mathbf{e}|_{\mathbf{H}^3}^{1/2}|\Delta\mathbf{w}|_2\\ &\le \frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\beta|\mathbf{e}|_{\mathbf{H}^3}^2+\frac{C}{\nu^2\beta}|\nabla\mathbf{d}_2|_2^2|\mathbf{d}_2|_{\mathbf{H}^2}^2|\Delta\mathbf{e}|_2^2\\ &\le \frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\beta|\mathbf{e}|_{\mathbf{H}^3}^2+C\frac{\rho_0\rho_2}{\nu^2\beta}|\Delta\mathbf{e}|_2^2.\end{aligned}$$

In order to obtain the first contribution to the main estimate of this section, we still need an estimate for the non-autonomous forcing term, namely:

$$|(\mathbf{g}_1-\mathbf{g}_2,\Delta\mathbf{w})|\le|\mathbf{g}_1-\mathbf{g}_2|_2|\Delta\mathbf{w}|_2\le\frac{\nu}{14}|\Delta\mathbf{w}|_2^2+\frac{7}{2\nu}|\mathbf{g}_1-\mathbf{g}_2|_2^2.$$

By putting all these estimates together, we deduce the following inequality:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{w}|_2^2+\nu\left(1-\frac{5}{14}\right)|\Delta\mathbf{w}|_2^2\\ &\qquad\le 2\beta|\mathbf{e}|_{\mathbf{H}^3}^2+C\frac{\rho_2^2}{\nu}|\mathbf{w}|_2^2+C\frac{\rho_0\rho_2}{\nu}|\nabla\mathbf{w}|_2^2\\ &\qquad\quad+C\frac{\rho_2^2}{\nu^2\beta}|\nabla\mathbf{e}|_2^2+C\frac{\rho_0\rho_2}{\nu^2\beta}|\Delta\mathbf{e}|_2^2+\frac{7}{2\nu}|\mathbf{g}_1-\mathbf{g}_2|_2^2. \qquad (4.2)\end{aligned}$$

We now consider the lifted equation for the order parameter field (see system (2.8)). By taking the difference of the equations satisfied by the same two solutions $(\mathbf{u}_1, \mathbf{d}_1)$ and $(\mathbf{u}_2, \mathbf{d}_2)$ as above, we easily obtain the following identity:

$$\partial_t \widehat{\mathbf{e}} - (\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)) + (\mathbf{u}_1 \cdot \nabla)\mathbf{e} + (\mathbf{w} \cdot \nabla)\mathbf{d}_2 = 0. \tag{4.3}$$

If we multiply this last equation by $\Delta(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))$ and if we integrate by parts noting that $\widehat{\mathbf{e}}|_{\partial\Omega} = \mathbf{0}$ and that $\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_{\partial\Omega} = \mathbf{0}$, we have:

$$\begin{aligned}
&\langle \partial_t \Delta\widehat{\mathbf{e}}, \Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\rangle + |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1)) + \mathbf{f}(\mathbf{d}_2)|_2^2 \\
&\qquad = \left\langle (\nabla \mathbf{u}_1)^t \nabla \mathbf{e} + \nabla\nabla\mathbf{e} \cdot \mathbf{u}_1, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\right\rangle \\
&\qquad\qquad + \left\langle (\nabla \mathbf{w})^t \nabla \mathbf{d}_2 + \nabla\nabla\mathbf{d}_2 \cdot \mathbf{w}, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\right\rangle.
\end{aligned}$$

We now complete the first term on the left hand side of this relation and obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1)) + \mathbf{f}(\mathbf{d}_2)|_2^2 \\
&\qquad = \left\langle (\nabla \mathbf{u}_1)^t \nabla \mathbf{e} + \nabla\nabla\mathbf{e} \cdot \mathbf{u}_1, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\right\rangle \\
&\qquad\qquad + \left\langle (\nabla \mathbf{w})^t \nabla \mathbf{d}_2 + \nabla\nabla\mathbf{d}_2 \cdot \mathbf{w}, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\right\rangle \\
&\qquad\qquad\qquad + \left\langle \partial_t(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)), \Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right\rangle.
\end{aligned} \tag{4.4}$$

As usual, we have to estimate all the nonlinear terms appearing on the right hand side of this last equation. We start by dealing with the four terms arising from the transport term. We have:

$$\begin{aligned}
&\left|\left\langle (\nabla \mathbf{u}_1)^t \nabla \mathbf{e}, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\right\rangle\right| \\
&\qquad \le |\nabla \mathbf{u}_1|_2 |\nabla \mathbf{e}|_\infty |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le C|\nabla \mathbf{u}_1|_2 |\nabla \mathbf{e}|_2^{1/2} |\mathbf{e}|_{\mathbf{H}^3}^{1/2} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le \frac{1}{16}|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \beta|\mathbf{e}|_{\mathbf{H}^3}^2 + \frac{C}{\beta}|\nabla \mathbf{u}_1|_2^4 |\nabla \mathbf{e}|_2^2 \\
&\qquad \le \frac{1}{16}|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \beta|\mathbf{e}|_{\mathbf{H}^3}^2 + C\frac{\rho_2^2}{\beta}|\nabla \mathbf{e}|_2^2
\end{aligned}$$

and, analogously, we get:

$$\begin{aligned}
&|\langle \nabla\nabla\mathbf{e} \cdot \mathbf{u}_1, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\rangle| \\
&\qquad \le |\mathbf{e}|_{\mathbf{W}^{2,4}} |\mathbf{u}_1|_4 |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le C|\mathbf{e}|_{\mathbf{H}^2}^{1/2} |\mathbf{e}|_{\mathbf{H}^3}^{1/2} |\mathbf{u}_1|_2^{1/2} |\nabla \mathbf{u}_1|_2^{1/2} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le \frac{1}{16}|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \beta|\mathbf{e}|_{\mathbf{H}^3}^2 + \frac{C}{\beta}|\mathbf{u}|_2^2 |\nabla \mathbf{u}|_2^2 |\mathbf{e}|_{\mathbf{H}^2}^2 \\
&\qquad \le \frac{1}{16}|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \beta|\mathbf{e}|_{\mathbf{H}^3}^2 + C\frac{\rho_0\rho_2}{\beta}|\mathbf{e}|_{\mathbf{H}^2}^2.
\end{aligned}$$

We can also proceed in a similar way for the following two terms, obtaining:

$$
\begin{aligned}
&\left|\left\langle (\nabla \mathbf{w})^t \nabla \mathbf{d}_2, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\right\rangle\right| \\
&\qquad \le |\nabla \mathbf{w}|_4 |\nabla \mathbf{d}_2|_4 |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le C|\nabla \mathbf{w}|_2^{1/2} |\Delta \mathbf{w}|_2^{1/2} |\nabla \mathbf{d}_2|_2^{1/2} |\mathbf{d}_2|_{\mathbf{H}^2}^{1/2} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le \frac{1}{16} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2^2 \\
&\qquad\qquad + \frac{\nu}{14} |\Delta \mathbf{w}|_2^2 + \frac{C}{\nu} |\nabla \mathbf{d}_2|_2^2 |\mathbf{d}_2|_{\mathbf{H}^2}^2 |\nabla \mathbf{w}|_2^2 \\
&\qquad \le \frac{1}{16} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \frac{\nu}{14} |\Delta \mathbf{w}|_2^2 + C\frac{\rho_0\rho_2}{\nu} |\nabla \mathbf{w}|_2^2
\end{aligned}
$$

and deducing:

$$
\begin{aligned}
&|\langle \nabla\nabla \mathbf{d}_2 \cdot \mathbf{w}, \nabla\left(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right)\rangle| \\
&\qquad \le |\mathbf{d}_2|_{\mathbf{H}^2} |\mathbf{w}|_\infty |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le C|\mathbf{d}_2|_{\mathbf{H}^2} |\mathbf{w}|_2^{1/2} |\Delta \mathbf{w}|_2^{1/2} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2 \\
&\qquad \le \frac{1}{16} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \frac{\nu}{14} |\Delta \mathbf{w}|_2^2 + \frac{C}{\nu} |\mathbf{d}_2|_{\mathbf{H}^2}^2 |\mathbf{w}|_2 \\
&\qquad \le \frac{1}{16} |\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d_1}) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \frac{\nu}{14} |\Delta \mathbf{w}|_2^2 + C\frac{\rho_2^2}{\nu} |\mathbf{w}|_2.
\end{aligned}
$$

We now have to consider the last term in (4.4). We start by observing that the following identity holds:

$$
\begin{aligned}
\partial_t(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)) &= \nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_1) \cdot \partial_t \mathbf{d}_1 - \nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2) \cdot \partial_t \mathbf{d}_2 \\
&= (\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_1) - \nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2)) \cdot \partial_t \mathbf{d}_1 + \nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2) \cdot \partial_t \mathbf{e}
\end{aligned}
$$

where with $\nabla_{\mathbf{d}}$ we denote the gradient with respect to $\mathbf{d}$. Before going on, we recall that the tensor norm we use throughout this work (also known as Frobenius norm) is compatible with the standard euclidean norm of vectors. Therefore we have:

$$
\begin{aligned}
|\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2) \cdot \partial_t \mathbf{e}|_2^2 &= \int_\Omega |\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2) \cdot \partial_t \mathbf{e}|^2 \\
&\le \int_\Omega |\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2)|^2 \, |\partial_t \mathbf{e}|^2 \le |\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2)|_\infty^2 |\partial_t \mathbf{e}|_2^2.
\end{aligned}
$$

By using these results we obtain:

$$
\begin{aligned}
&\left|\left\langle \partial_t(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)), \Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right\rangle\right| \\
&\qquad \le |\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_1) - \nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2)|_\infty |\partial_t \mathbf{d}_1|_2 |\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2 \\
&\qquad\qquad + |\nabla_{\mathbf{d}} \mathbf{f}(\mathbf{d}_2)|_\infty |\partial_t \mathbf{e}|_2 |\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2.
\end{aligned}
$$

We now observe that, thanks to lemma 2.1.3, we have:

$$|\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_1) - \nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_2)|_\infty \le \frac{2\sqrt{10}}{\epsilon^2}|\mathbf{e}|_\infty$$

and

$$|\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_2)|_\infty \le \frac{\sqrt{7}}{\epsilon^2}.$$

In order to finish this part of our argument, we still have to find an appropriate estimate for $|\partial_t\mathbf{e}|_2$. By considering equation (4.3) again, we obtain:

$$\begin{aligned}
|\partial_t\mathbf{e}|_2 &\le |(\mathbf{u}_1\cdot\nabla)\mathbf{e}|_2 + |(\mathbf{w}\cdot\nabla)\mathbf{d}_2|_2 + |\Delta\mathbf{e}|_2 + |\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)|_2 \\
&\le |\mathbf{u}_1|_4|\nabla\mathbf{e}|_4 + |\mathbf{w}|_4|\nabla\mathbf{d}_2|_4 + |\Delta\mathbf{e}|_2 + |\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)|_2 \\
&\le C|\mathbf{u}_1|_2^{1/2}|\nabla\mathbf{u}_1|_2^{1/2}|\nabla\mathbf{e}|_2^{1/2}|\mathbf{e}|_{\mathbf{H}^2}^{1/2} + C|\mathbf{w}|_2^{1/2}|\nabla\mathbf{w}|_2^{1/2}|\nabla\mathbf{d}_2|_2^{1/2}|\mathbf{d}_2|_{\mathbf{H}^2}^{1/2} \\
&\qquad + |\Delta\mathbf{e}|_2 + |\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)|_2 \\
&\le C\rho_0^{1/4}\rho_2^{1/4}|\mathbf{e}|_{\mathbf{H}^2} + C\rho_0^{1/4}\rho_2^{1/4}|\nabla\mathbf{w}|_2 + |\mathbf{e}|_{\mathbf{H}^2} + \frac{2}{\epsilon^2}|\mathbf{e}|_2.
\end{aligned}$$

We can now obtain the desired estimate for the last term in (4.4). By using all the results of the previous paragraphs and remembering corollary 3.4.3, we get:

$$\begin{aligned}
&\left|\left\langle \partial_t(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)), \Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right\rangle\right| \\
&\quad \le \frac{C}{\epsilon^2}\left(\rho_4^{1/2}|\mathbf{e}|_\infty + \left(\rho_0^{1/4}\rho_2^{1/4} + 1\right)|\mathbf{e}|_{\mathbf{H}^2} + \rho_0^{1/4}\rho_2^{1/4}|\nabla\mathbf{w}|_2 + \frac{2}{\epsilon^2}|\mathbf{e}|_2\right) \\
&\qquad\qquad |\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2.
\end{aligned}$$

On account of $|\mathbf{e}|_\infty \le C|\mathbf{e}|_2^{1/2}|\mathbf{e}|_{\mathbf{H}^2}^{1/2} \le C|\mathbf{e}|_{\mathbf{H}^2}$, we therefore obtain:

$$\begin{aligned}
\left|\left\langle \partial_t(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)), \Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)\right\rangle\right| &\le \frac{1}{2}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 \\
&+ \frac{C}{\epsilon^4}\left(\rho_4 + \rho_0^{1/2}\rho_2^{1/2} + 1 + \frac{1}{\epsilon 4}\right)|\mathbf{e}|_{\mathbf{H}^2}^2 + C\frac{\rho_0^{1/2}\rho_2^{1/2}}{\epsilon^4}|\nabla\mathbf{w}|_2^2.
\end{aligned}$$

We can now substitute these estimates in (4.4) obtaining:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + \left(1 - \frac{1}{4}\right)|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2^2 \\
&\le 2\beta|\mathbf{e}|_{\mathbf{H}^3} + \frac{\nu}{7}|\Delta\mathbf{w}|_2^2 + \frac{1}{2}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 \\
&\quad + C\left(\frac{\rho_2^2}{\beta} + \frac{\rho_0}{\beta}\rho_2 + \frac{1}{\epsilon^4}\rho_4 + \frac{1}{\epsilon^4}\rho_0^{1/2}\rho_2^{1/2} + \frac{1}{\epsilon^8}\right)|\mathbf{e}|_{\mathbf{H}^2}^2 \\
&\qquad + C\left(\frac{\rho_0\rho_2}{\nu} + \frac{\rho_2^2}{\nu} + \frac{1}{\epsilon^4}\rho_0^{1/2}\rho_2^{1/2}\right)|\nabla\mathbf{w}|_2^2. \qquad (4.5)
\end{aligned}$$

However, we still have to deal with the term $|\mathbf{e}|_{\mathbf{H}^3}$ which appears on the right hand side of this last inequality. Since $\widehat{\mathbf{e}}|_{\partial\Omega} = \mathbf{0}$, we have:

$$\begin{aligned}|\mathbf{e}|^2_{\mathbf{H}^3} &\le 2|\widehat{\mathbf{e}}|^2_{\mathbf{H}^3} + 2|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^3} \le C|\nabla\Delta\mathbf{e}|^2_2 + C|\Delta\mathbf{e}|^2_2 + 2|\widetilde{\mathbf{e}}|_{\mathbf{H}^3} \\ &\le C|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|^2_2 + C|\nabla(\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2))|^2_2 + C|\mathbf{e}|^2_{\mathbf{H}^2} + 2|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^3}\end{aligned}$$

and we observe that $|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|^2_2$ can easily be estimated as above by writing:

$$\begin{aligned}\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)) &= \nabla_{\mathbf{d}}\mathbf{d}(\mathbf{d}_1)\cdot\nabla\mathbf{d}_1 - \nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_2)\cdot\nabla\mathbf{d}_2 \\ &= (\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_1) - \nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_1))\cdot\nabla\mathbf{d}_1 + \nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_2)\cdot\nabla\mathbf{e}.\end{aligned}$$

Therefore we have:

$$\begin{aligned}&|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|^2_2 \\ &\qquad\le 2|\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_1) - \nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_1)|^2_\infty|\nabla\mathbf{d}_1|^2_2 + 2|\nabla_{\mathbf{d}}\mathbf{f}(\mathbf{d}_2)|^2_\infty|\nabla\mathbf{e}|^2_2 \\ &\qquad\le C\frac{\rho_0}{\epsilon^4}|\mathbf{e}|^2_\infty + C\frac{1}{\epsilon^4}|\nabla\mathbf{e}|^2_2 \le \frac{C}{\epsilon^4}(\rho_0+1)|\mathbf{e}|^2_{\mathbf{H}^2}\end{aligned}$$

from which we deduce:

$$|\mathbf{e}|^2_{\mathbf{H}^3} \le C|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|^2_2 + C\left(1 + \frac{1}{\epsilon^4} + \frac{\rho_0}{\epsilon^4}\right)|\mathbf{e}|^2_{\mathbf{H}^2} + 2|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^3}. \quad (4.6)$$

We have now to deal with the $\mathbf{H}^3$ norm of $\widetilde{\mathbf{e}}$. This leads to the third and last preparatory step in the proof of theorem 4.4.1. We start by considering the difference of the equation satisfied by two solutions $\widetilde{\mathbf{d}}_1$ and $\widetilde{\mathbf{d}}_2$ of the lifting problem (2.4). Multiplying by $-\Delta\widetilde{\mathbf{e}}$ the resulting equation and writing $\widetilde{\mathbf{e}} \doteq \widetilde{\mathbf{d}}_1 - \widetilde{\mathbf{d}}_2$ for the difference of the two solutions, we get:

$$-\langle\partial_t\widetilde{\mathbf{e}}, \Delta\widetilde{\mathbf{e}}\rangle + |\Delta\widetilde{\mathbf{e}}|^2_2 = 0.$$

Integrating by parts, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\widetilde{\mathbf{e}}|^2_2 + |\Delta\widetilde{\mathbf{e}}|^2_2 &= {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\langle\partial_t(\mathbf{h}_1 - \mathbf{h}_2), \partial_\nu\widetilde{\mathbf{e}}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)} \\ &\le C|\partial_t(\mathbf{h}_1 - \mathbf{h}_2)|_{\mathbf{H}^{-1/2}(\partial\Omega)}|\widetilde{\mathbf{e}}|_{\mathbf{H}^2} \\ &\le \frac{1}{2}|\Delta\widetilde{\mathbf{e}}|^2_2 + |\mathbf{h}_1 - \mathbf{h}_2|^2_{\mathbf{H}^{3/2}(\partial\Omega)} + C|\partial_t(\mathbf{h}_1 - \mathbf{h}_2)|^2_{\mathbf{H}^{-1/2}(\partial\Omega)}.\end{aligned}$$

We can now easily deduce:

$$\begin{aligned}&|\nabla\widetilde{\mathbf{e}}(t)|^2_2 + \int_0^t |\Delta\widetilde{\mathbf{e}}(s)|^2_2\,ds \\ &\qquad\le |\nabla\mathbf{e}_0|^2_2 + \int_0^t |\delta\mathbf{h}(s)|^2_{\mathbf{H}^{3/2}(\partial\Omega)}\,ds + C\int_0^t |\partial_t\delta\mathbf{h}(s)|^2_{\mathbf{H}^{-1/2}(\partial\Omega)}\,ds\end{aligned}$$

where we have used the self explanatory notation $\delta\mathbf{h} \doteq \mathbf{h}_1 - \mathbf{h}_2$ to simplify the expression.

In order to obtain the desired estimate for $|\widetilde{\mathbf{e}}|_{\mathbf{H}^3}$, we need also a higher regularity result. We start again from the equation satisfied by the difference of two solutions of the lifted problem, but this time we take its laplacian (in the sense of distributions) and then multiply it by $\Delta\widetilde{\mathbf{e}}$. After a simple integration by parts we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widetilde{\mathbf{e}}|_2^2 + |\nabla\Delta\widetilde{\mathbf{e}}|_2^2 = {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\langle\partial_\nu\Delta\widetilde{\mathbf{e}}, \Delta\widetilde{\mathbf{e}}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}\,.$$

Using directly equation (2.4) we then deduce:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widetilde{\mathbf{e}}|_2^2 + |\nabla\Delta\widetilde{\mathbf{e}}|_2^2 &= {}_{\mathbf{H}^{-1/2}(\partial\Omega)}\langle\partial_\nu\Delta\widetilde{\mathbf{e}}, \partial_t\widetilde{\mathbf{e}}\rangle_{\mathbf{H}^{1/2}(\partial\Omega)}\\ &\leq C|\widetilde{\mathbf{e}}|_{\mathbf{H}^3}|\partial_t\delta\mathbf{h}|_{\mathbf{H}^{1/2}(\partial\Omega)}\\ &\leq \frac{1}{4}|\nabla\Delta\widetilde{\mathbf{e}}|_2^2 + |\delta\mathbf{h}|^2_{\mathbf{H}^{5/2}(\partial\Omega)} + C|\partial_t\delta\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}\end{aligned}$$

whence we finally obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widetilde{\mathbf{e}}|_2^2 + \left(1 - \frac{1}{4}\right)|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^3} \leq C|\delta\mathbf{h}|^2_{\mathbf{H}^{5/2}(\partial\Omega)} + C|\partial_t\delta\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}. \tag{4.7}$$

We can now use the three main estimates we have obtained in the preceding pages. Summing together estimates (4.2), (4.5) and (4.7) we have:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{w}|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widetilde{\mathbf{e}}|_2^2\\ &\qquad + \frac{\nu}{2}|\Delta\mathbf{w}|_2^2 + \frac{3}{4}|\nabla(\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2))|_2^2 + \frac{3}{4}|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^3}\\ &\leq 4\beta|\mathbf{e}|^2_{\mathbf{H}^3} + C\left(\frac{\rho_2^2}{\nu\beta} + \frac{\rho_0\rho_2}{\nu\beta} + \frac{\rho_0^{1/2}\rho_2^{1/2}}{\epsilon^4}\right)|\nabla\mathbf{w}|_2^2 + \frac{1}{2}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2\\ &\qquad + C\left(\frac{\rho_2^2}{\nu^2} + \frac{\rho_0\rho_2}{\nu^2} + \frac{\rho_2^2}{\beta} + \frac{\rho_0^{\rho_2}}{\beta} + \frac{\rho_4}{\epsilon^4} + \frac{\rho_0^{1/2}\rho_2^{1/2}}{\epsilon^4} + \frac{1}{\epsilon^8}\right)|\mathbf{e}|^2_{\mathbf{H}^2}\\ &\qquad\qquad + \frac{C}{\nu}|\delta\mathbf{g}|_2^2 + C|\delta\mathbf{h}|^2_{\mathbf{H}^{5/2}(\partial\Omega)} + C|\partial_t\delta\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}.\end{aligned}$$

Recalling now estimate (4.6) and choosing β sufficiently small (and, in particular, smaller than a constant which depends only on the domain Ω) we obtain:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{w}|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widetilde{\mathbf{e}}|_2^2\\ &\qquad \leq C|\nabla\mathbf{w}|_2^2 + |\Delta\mathbf{e} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + C|\mathbf{e}|^2_{\mathbf{H}^2}\\ &\qquad\qquad + C|\delta\mathbf{g}|_2^2 + C|\delta\mathbf{h}|^2_{\mathbf{H}^{5/2}(\partial\Omega)} + C|\partial_t\delta\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)}\end{aligned}$$

where all the constants appearing on the right hand side of this expression depend only on ρ_0, ρ_2, ρ_4, Ω, ν and ϵ. We now recall inequality (4.1) introduced above. In particular we observe that:

$$\begin{aligned} |\delta\mathbf{g}|_2^2 &\le C|\delta\theta|^2, \\ |\delta\mathbf{h}|^2_{\mathbf{H}^{5/2}(\partial\Omega)} &\le C|\delta\phi|^2, \\ |\partial_t\delta\mathbf{h}|^2_{\mathbf{H}^{1/2}(\partial\Omega)} &\le C|\delta\phi|^2 \end{aligned}$$

where $(\theta_i, \phi_i) \in \mathbb{T}^k$, $i = 1, 2$ and where we have set $\delta\theta = \theta_1 - \theta_2$ and $\delta\phi = \phi_1 - \phi_2$. We also note that, under the action of the extended semigroup, the quantities $|\delta\theta|$ and $|\delta\phi|$ are conserved and therefore we can write:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\mathbf{w}|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\Delta\widetilde{\mathbf{e}}|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\delta\theta|^2 + \frac{\mathrm{d}}{\mathrm{d}t}|\delta\phi|^2 \\ &\qquad \le C|\nabla\mathbf{w}|_2^2 + |\Delta\mathbf{e} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + C|\mathbf{e}|^2_{\mathbf{H}^2} + C|\delta\theta|^2 + C|\delta\phi|^2. \end{aligned} \tag{4.8}$$

On the other hand, we have:

$$\begin{aligned} |\mathbf{e}|^2_{\mathbf{H}^2} &\le 2|\widehat{\mathbf{e}}|^2_{\mathbf{H}^2} + 2|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^2} \\ &\le C|\Delta\widehat{\mathbf{e}}|_2^2 + C|\Delta\widetilde{\mathbf{e}}|_2 + |\delta\mathbf{h}|^2_{\mathbf{H}^{3/2}(\partial\Omega)} \\ &\le C|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + C|\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)|_2^2 + C|\Delta\widetilde{\mathbf{e}}|_2^2 + C|\delta\phi|^2 \\ &\le C|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 + C\frac{C}{\epsilon^4}|\mathbf{e}|_2^2 + C|\Delta\widetilde{\mathbf{e}}|_2^2 + C|\delta\phi|^2. \end{aligned} \tag{4.9}$$

Before applying the uniform Gronwall inequality, we still have to verify the integrability on the interval $[0, T]$ of all the arguments of the time derivatives on the left hand side of estimate (4.8). This can be partly deduced from theorem 2.2.1 and partly by considering the following estimate:

$$\begin{aligned} |\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 &\le 2|\Delta\widehat{\mathbf{e}}|_2^2 + 2|\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)|_2^2 \\ &\le 2|\widehat{\mathbf{e}}|^2_{\mathbf{H}^2} + 2|\mathbf{f}(\mathbf{d}_1) - \mathbf{f}(\mathbf{d}_2)|_2^2 \\ &\le 4|\mathbf{e}|^2_{\mathbf{H}^2} + 4|\widetilde{\mathbf{e}}|^2_{\mathbf{H}^2} + \frac{C}{\epsilon^4}|\mathbf{e}|_2^2. \end{aligned}$$

Applying now Gronwall's inequality we get:

$$\begin{aligned} &|\nabla\mathbf{w}(t)|_2^2 + |\Delta\widehat{\mathbf{e}}(t) - \mathbf{f}(\mathbf{d}_1(t)) + \mathbf{f}(\mathbf{d}_2(t))|_2^2 + |\Delta\widetilde{\mathbf{e}}(t)|_2^2 + |\delta\theta|^2 + |\delta\phi|^2 \\ &\quad \le \Bigg(\frac{1}{t}\int_0^t |\nabla\mathbf{w}(s)|_2^2 + |\Delta\widehat{\mathbf{e}}(s) - \mathbf{f}(\mathbf{d}_1(s)) + \mathbf{f}(\mathbf{d}_2(s))|_2^2 + |\Delta\widetilde{\mathbf{e}}(s)|_2^2\, ds \\ &\qquad\qquad + |\delta\theta|^2 + |\delta\phi|^2 + C\int_0^t |\mathbf{e}(s)|_2^2\, ds\Bigg)e^{Ct}. \end{aligned}$$

Recalling estimate (2.21) and using again estimate (4.9), we eventually obtain:

$$|\nabla \mathbf{w}|_2^2 + |\mathbf{e}|_{\mathbf{H}^2}^2 + |\delta\theta|^2 + |\delta\phi|^2 \leq C(t)\left(|\mathbf{w}_0|_2^2 + |\mathbf{e}_0|_{\mathbf{H}^1}^2 + |\delta\theta|^2 + |\delta\phi|^2\right)$$

where we observe that we have also used the following estimate:

$$|\Delta\widehat{\mathbf{e}} - \mathbf{f}(\mathbf{d}_1) + \mathbf{f}(\mathbf{d}_2)|_2^2 \leq 4|\Delta\mathbf{e}|_2^2 + 4|\Delta\widetilde{\mathbf{e}}|_2^2 + C|\mathbf{e}|_2^2.$$

These estimates conclude our proof of the smoothing property for system (2.1). We have therefore also completed the proof of theorem 4.4.1. In the next section we will show how it is possible to extend the exponential attractor constructed in this section for a discrete-time semigroup to the continuous-time case.

4.5 The continuous-time attractor

In the last section we established the existence of an exponential attractor for the discrete-time semigroup generated by $S(t^*)$. We now want to prove that also a continuous-time exponential attractor exists. To do this, we simply have to apply theorem 4.1.1 to system (2.1) (see [13] or [11, Chapter 3] for more details). Actually, the following result holds.

Theorem 4.5.1. *Let the same assumptions of theorem 4.4.1 be verified. Then there exists an exponential attractor $\mathcal{M}$ for the extended semigroup $\{S(t)\}$ on $\mathbf{H}\times\mathbf{H}^1\times\mathbb{T}^k$. Moreover, if Π_1 and Π_2 are the projections of the extended phase space on $\mathbf{H}\times\mathbf{H}^1$ and $\mathbb{T}^k$ respectively, then $\Pi_1\mathcal{M}$ is the uniform (w.r.t. $\theta\in\mathbb{T}^k$) exponential attractor for the family of processes and $\Pi_2\mathcal{M}=\mathbb{T}^k$.*

Thanks to the study of global attractors in chapter 3, as an immediate consequence of this result, we have the following corollary.

Corollary 4.5.2. *Let the same assumptions of theorem 4.4.1 be verified. Then the global attractor of system* (2.1) *has finite fractal dimension.*

In order to prove theorem 4.1.1 we only need to prove that the extended semigroup $S(t)$ is Lipschitz continuous on the phase space and Hölder continuous in time. Indeed, the first statement follows easily from theorem 2.2.1, whereas for the second statement the following estimates hold:

$$\begin{aligned}
&|\mathbf{u}(t)-\mathbf{u}(\tau)|_2 + |\mathbf{d}(t)-\mathbf{d}(\tau)|_{\mathbf{H}^1}\\
&\qquad\leq \left|\int_\tau^t \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{u}(s)\,ds\right|_2 + \left|\int_\tau^t \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{d}(s)\,ds\right|_{\mathbf{H}^1}\\
&\qquad\leq \int_\tau^t |\partial_s\mathbf{u}(s)|_2\,ds + \int_\tau^t |\partial_s\mathbf{d}(s)|_{\mathbf{H}^1}\,ds\\
&\qquad\leq (t-\tau)^{1/2}\left(|\partial_t\mathbf{u}|_{L^2(\tau,t;\mathbf{H})}^2 + |\partial_t\mathbf{d}|_{L^2(\tau,t;\mathbf{H}^1)}^2\right)
\end{aligned}$$

where $(\mathbf{u}(s), \mathbf{d}(s))$ is any solution of (2.1).

We start by considering $\partial_t \mathbf{u}$. From the equation for the velocity field in (2.1) we obtain:

$$\begin{aligned}
|\partial_t \mathbf{u}|_2 &\leq |(\mathbf{u} \cdot \nabla)\mathbf{u}|_2 + \nu|\Delta \mathbf{u}|_2 + |(\nabla \mathbf{d})^t \Delta \mathbf{d}|_2 \\
&\leq |\mathbf{u}|_4 |\nabla \mathbf{u}|_4 + \nu |\Delta \mathbf{u}|_2 + |\nabla \mathbf{d}|_4 |\Delta \mathbf{d}|_4 \\
&\leq C|\mathbf{u}|_2^{1/2} |\nabla \mathbf{u}|_2 |\Delta \mathbf{u}|_2^{1/2} + \nu|\Delta \mathbf{u}|_2 + C|\nabla \mathbf{d}|_2^{1/2} |\mathbf{d}|_{\mathbf{H}^2} |\mathbf{d}|_{\mathbf{H}^3}^{1/2} \\
&\leq C\rho_0^{1/4} \rho_2^{1/2} |\Delta \mathbf{u}|_2^{1/2} + \nu |\Delta \mathbf{u}|_2 + C\rho_0^{1/4} \rho_2^{1/2} |\mathbf{d}|_{\mathbf{H}^3}^{1/2}.
\end{aligned}$$

By squaring and integrating between τ and t, remembering the results of section 3.4, we easily find that on the absorbing set (that is on a neighbourhood of the exponential attractor) $|\partial_t \mathbf{u}|_{L^2(\tau,t;\mathbf{H})}$ is bounded.

An analogous estimate can be obtained by taking the gradient of the equation for the order parameter. In particular we have:

$$\begin{aligned}
|\partial_t \nabla \mathbf{d}|_2 &\leq |\nabla \mathbf{u}|_4 |\nabla \mathbf{d}|_4 + |\mathbf{u}|_4 |\mathbf{d}|_{\mathbf{W}^{2,4}} + |\mathbf{d}|_{\mathbf{H}^3} + |\nabla \mathbf{f}(\mathbf{d})|_2 \\
&\leq C|\nabla \mathbf{u}|_2^{1/2} |\Delta \mathbf{u}|_2^{1/2} |\nabla \mathbf{d}|_2^{1/2} |\mathbf{d}|_{\mathbf{H}^2}^{1/2} \\
&\qquad + C|\mathbf{u}|_2^{1/2} |\nabla \mathbf{u}|_2^{1/2} |\mathbf{d}|_{\mathbf{H}^2}^{1/2} |\mathbf{d}|_{\mathbf{H}^3}^{1/2} + |\mathbf{d}|_{\mathbf{H}^3} + C|\nabla \mathbf{d}|_2 \\
&\leq C\rho_0^{1/2} \rho_2^{1/2} |\Delta \mathbf{u}|_2^{1/2} + C\rho_0^{1/4} \rho_2^{1/2} |\mathbf{d}|_{\mathbf{H}^3}^{1/2} + |\mathbf{d}|_{\mathbf{H}^3} + C|\nabla \mathbf{d}|_2.
\end{aligned}$$

Again simple calculations give a uniform bound on $|\partial_t \mathbf{d}|_{L^2(\tau,t;\mathbf{H}^1)}$ on a neighbourhood of the exponential attractor. With these estimates we have obtained the results needed to prove theorem 4.5.1.

We end here the mathematical treatment of system (2.1). In the last three chapter we have proved that the system is well posed and that it possesses many different kinds of attractors depending on the nature of the forcing terms. In the last chapter of this work we will try to design some numerical methods for the simulation of a nematic liquid crystal flow under a (quasi-)periodic forcing term in a simple though interesting test case.

Chapter 5

Some numerical experiments

In this chapter we intend to apply some of the results obtained in the previous chapters to the numerical simulation of system (2.1). Our main goal will be to show how those results can be used to design and improve some simple numerical methods for the full evolution system (2.1).

The consistency and convergence analysis of standard finite elements schemes for system (2.1) has already been studied in detail in the available literature (see, for instance, [23] and [24]). The main results are consistent with the intuition arising from the regularity results of chapter 2: a couple of spaces satisfying the Ladyzhenskaya-Babuška-Brezzi condition (see [30]) is necessary to approximate correctly the velocity and pressure fields, while a finite elements space with one order of approximation more is needed for the order parameter field to obtain optimal convergence. In particular, unless otherwise stated, we will choose the couple $\mathbb{P}_1$-bubble $\times$ $\mathbb{P}_1$ for the velocity and pressure fields and the space $\mathbb{P}_2$ for the order parameter field (here we use the standard notation meaning with $\mathbb{P}_i$ the space of continuous functions which are piecewise polynomial of degree i on a fixed simplicial grid).

For all the simulations of this chapter we used FreeFem++ free software running on a Windows Vista PC (Intel® Dual Core™ 4400, 2.2 GHz, 4Gb RAM). We refer to its extensive guide [18] for the details.

5.1 The test case

We start by introducing the test case we will mainly study in this chapter. We will consider the flow of a nematic liquid crystal in a rectangular domain $\Omega = [0, 3] \times [0, 0.5]$ with homogeneous Dirichlet boundary conditions for the velocity field and the following constant Dirichlet boundary conditions for

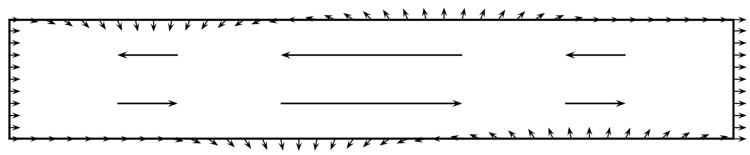


Figure 5.1: A sketch of the test case. We note that the driving force is represented as it is exerted during its first half period: during the second half these arrows should be reversed.

the order parameter field:

$$\begin{aligned}
\mathbf{d}(0,y) &= (1,0)^T && \text{for } y \in [0,0.5] \\
\mathbf{d}(x,0.5) &= \begin{pmatrix} \cos\left(\frac{2\pi x}{2.4}\right)\sqrt{1+\sin^2\left(\frac{2\pi x}{2.4}\right)} \\ -\sin\left(\frac{2\pi x}{2.4}\right)\left|\sin\left(\frac{2\pi x}{2.4}\right)\right| \end{pmatrix} && \text{for } x \in [0,2.4] \\
\mathbf{d}(x,0.5) &= (1,0)^T && \text{for } x \in [2.4,3] \\
\mathbf{d}(3,y) &= (1,0)^T && \text{for } y \in [0,0.5] \\
\mathbf{d}(x,0) &= \begin{pmatrix} \cos\left(\frac{2\pi(3-x)}{2.4}\right)\sqrt{1+\sin^2\left(\frac{2\pi(3-x)}{2.4}\right)} \\ \sin\left(\frac{2\pi(3-x)}{2.4}\right)\left|\sin\left(\frac{2\pi(3-x)}{2.4}\right)\right| \end{pmatrix} && \text{for } x \in [0.6,3] \\
\mathbf{d}(x,0) &= (1,0)^T && \text{for } x \in [0,0.6]
\end{aligned}$$

We notice that this boundary condition belongs to $\mathbf{H}^{5/2}(\partial\Omega)$. Moreover, we will consider $\nu = 1$ for most of our tests and $\nu = 0.1$ in the next to last section of this chapter when we will study a Taylor-like instability which arises in the flow.

When dealing with the evolution equation we will consider the following periodic forcing term $\mathbf{g} \in L^2_n(0,\infty;\mathbf{L}^2)$:

$$\mathbf{g}(x,y;t) = \begin{pmatrix} 2000\nu x(3-x)\sin\left(\frac{2\pi y}{.5}\right)\sin(t) \\ 0 \end{pmatrix}.$$

As initial data for our problem we will take the equilibrium solution of the equation for the order parameter field and $\mathbf{u}_0 = \mathbf{0}$ for the velocity field. A schematic representation of the configuration we will study is given in figure 5.1.

Thanks to the results of chapter 2, we conclude that under the above assumptions our system has a global strong solution.

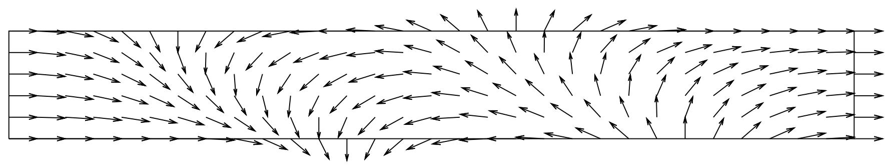


Figure 5.2: The equilibrium solution of the test case computed on a mesh with 12470 elements and $\epsilon = 0.05$.

5.2 The equilibrium solution

In order to approximate the solution to system (2.1) with the above forcing and boundary terms, we first need to compute the equilibrium initial configuration for the initial order parameter field. We remember that the equation we have to solve in this case reduces to:

$$\Delta \mathbf{d} = \mathbf{f}(\mathbf{d}) \qquad \text{in } \Omega \tag{5.1}$$

subject to the Dirichlet boundary condition specified above. This is indeed the Euler-Lagrange equation arising from Frank's free energy with a Ginzburg-Landau correction to approximate the constraint $|\mathbf{d}| = 1$ (see section 1.5).

We start by noting that, in the general case, the usual representations of the order parameter as a field of (nearly-)unitary vectors may lead to some numerical difficulties. Actually, from a physical point of view, vectors as $(0, 1)$ and $(0, -1)$ are undistinguishable while from a mathematical and numerical viewpoint a jump in the field from $(0, 1)$ to $(0, -1)$ cannot be accepted. One possibility to cope with this problem is to consider a suitable polar parametrization of the order vectors writing the corresponding evolution equations for these new variables (in terms of an angle θ and an intensity ρ). However, the resulting system would be fully nonlinear leading to much more difficult numerical problems.

An alternative, although less trivial, approach, could be to cut the domain in subdomains on which the usual description of the order parameter field could be locally acceptable and then glue together different portions of the domain by simply reversing the order parameter at their interfaces. This is indeed the strategy we are pursuing in this short introduction to the numerical simulation of system (2.1). In our case, however, the order parameter field is quite regular and does not exhibit topological defects or similar features. We can therefore consider a unique parametrization of the order parameter through vectors as if we were forgetting of the above mentioned identification.

The easiest approach to the numerical approximation of equation (5.1) is given by a fixed point argument similar to the one arising in the proof of the

ϵ	min $\lvert\mathbf{d}\rvert$	max $\lvert\mathbf{d}\rvert$	energy	iterations $\Delta\mathbf{d}^{(0)}=\mathbf{0}$	iterations $\mathbf{d}^{(0)}=\mathbf{d}_{\epsilon=0.1}$
0.1	0.881446	1	17.9143	10	NA
0.05	0.969184	1	19.457	12	6
0.01	0.998557	1	20.1593	54	8
0.005	0.999623	1	20.186	62	14
0.001	0.99998	1	20.1949	NA	11

Table 5.1: Summary of the calculation of the equilibrium solution for different values of the parameter ϵ. All the results are the same for the algorithm starting both from the Laplace solution and from the equilibrium solution computed for $\epsilon = 0.1$. For $\epsilon = 0.001$ the algorithm did not converge to the equilibrium solution, but numerically got stuck in a non-optimal configuration.

well posedness (see section 2.1). However, the success of the linearization $\mathbf{d}^{(n+1)} = \mathbf{f}(\mathbf{d}^{(n)})$ heavily depends on the magnitude ϵ of Ginzburg-Landau's potential.

In order to design a method sufficiently robust w.r.t. the choice of ϵ, we propose a Newton fixed point algorithm (see [30, Section 10.3] for details for the general case). We remember that for a general operator $\mathcal{F}$ between two Banach spaces, the Newton method is given by:

$$D\mathcal{F}(\mathbf{d}^{(n+1)} - \mathbf{d}^{(n)}) = -\mathcal{F}\mathbf{d}^{(n)}$$

where $D\mathcal{F}$ is the Gâteaux differential of $\mathcal{F}$. In this case, the iteration step of Newton method involves the solution of the following linearized problem:

$$\Delta\mathbf{d}^{(n+1)} - \frac{1}{\epsilon^2}\left(|\mathbf{d}^{(n)}|^2 - 1\right)\mathbf{d}^{(n+1)} - \frac{2}{\epsilon^2}\left(\mathbf{d}^{(n)}\cdot\mathbf{d}^{(n+1)}\right)\mathbf{d}^{(n)} = -\frac{2}{\epsilon^2}|\mathbf{d}^{(n)}|^2\mathbf{d}^{(n)}.$$

As stopping criterion we have used the $\mathbf{H}^1$ norm of the difference $\mathbf{d}^{(n+1)} - \mathbf{d}^{(n)}$. As it is well known from the general theory, this criterion is optimal for Newton method. Moreover, in this case, it corresponds to evaluating the change in the free energy after a step of the fixed-point algorithm. We also observe that the maximum principle stated in chapter 2 for the order parameter vector no longer holds for this linearized problem: therefore in our simulations it will be possible for the absolute value of the order parameter filed to be slightly larger than 1 in magnitude.

In figure 5.2 we represent the approximated equilibrium solution for the case we are studying obtained by setting $\epsilon = 0.01$ and calculated on a grid of 12470 elements. In figure 5.3 we report the decay in the energy of the approximation calculated in the successive iterations of the proposed method. We note that, as initial guess, we used the solution of the Laplace problem with the prescribed boundary conditions. Finally in table 5.1 we

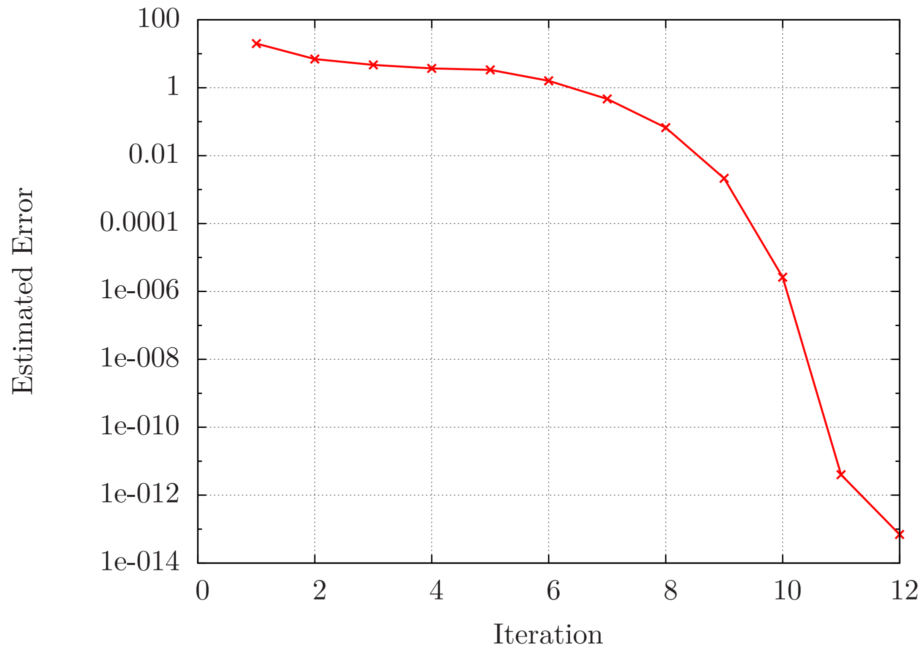


Figure 5.3: The estimated error on the equilibrium solution after each iteration of the algorithm of section 5.2, for $\epsilon = 0.05$, on a mesh of 12470 elements starting from the solution of Laplace problem. We observe that the algorithm gets initially stuck since at the first iteration some defects in the order parameter field appear due to the coarse initial guess. Once these singularities annihilate (at iteration 6) the convergence to the equilibrium solution is very quick.

summarize the behaviour of this algorithm with different choices of ϵ. As it can be seen, the solution seems satisfactory under a wide range of physically meaningful values. We also observe that for very small ϵ it is more convenient to use the approximate solution easily obtained with a greater ϵ (for example $\epsilon = 0.1$) as initial guess for the iterative algorithm. This choice makes the convergence much faster.

5.3 The Newton method for the complete system

Since we have obtained an accurate approximation of the initial condition for system (2.1), we now turn our attention to the simulation of the full evolution problem. We will denote with $(\mathbf{u}_j, \mathbf{d}_j)$ the solution at the jth timestep and we will write $(\mathbf{u}_j^{(n)}, \mathbf{d}_j^{(n)})$ for the nth iterative solution of the linearized problem approximating $(\mathbf{u}_j, \mathbf{d}_j)$.

We start by considering an explicit Euler approximation for the time derivative. By applying the Newton method to the resulting system, we

Elements	Newton	fixed point
638	18.571	5.935
1556	43.538	15.209
2880	76.898	27.578
5794	153.824	55.894
7912	NA	100.625
12470	NA	118.472

Table 5.2: Average execution time in seconds per time-step for the two methods proposed for the simulation of the full system of evolution equations (2.1) on meshes of varying coarseness. The data were obtained by solving the test problem proposed in section 5.1 on grids of different coarseness with timestep equal to 0.1 from $t = 0$ to $t = 12.6 \approx 4\pi$. The missing values in the table are due to the excessively long computation times required to complete these simulations. As can easily be seen, the proposed splitting method is far quicker than the full Newton iterative approximation.

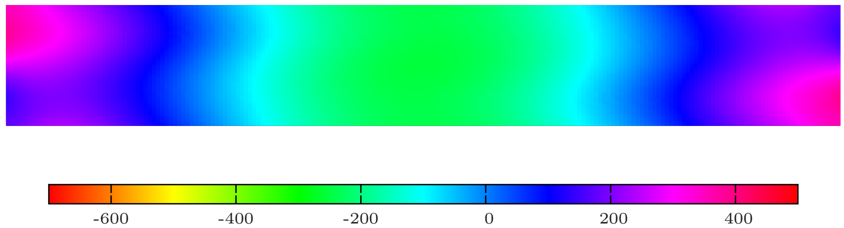
obtain the following iterative numerical scheme:

$$\begin{aligned}
&\mathbf{u}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n)}\cdot\nabla\right)\mathbf{u}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n+1)}\cdot\nabla\right)\mathbf{u}_j^{(n)} - dt\Delta\mathbf{u}_j^{(n+1)} \\
&\qquad - dt\left(\mathbf{d}_j^{(n)}\right)^T\Delta\mathbf{d}_j^{(n+1)} - dt\left(\mathbf{d}_j^{(n+1)}\right)^T\Delta\mathbf{d}_j^{(n)} \\
&\quad = dt\left(\mathbf{u}_j^{(n)}\cdot\nabla\right)\mathbf{u}_j^{(n)} - dt\left(\mathbf{d}_j^{(n)}\right)^T\Delta\mathbf{d}_j^{(n)} + dt\,\mathbf{g}_j + \mathbf{u}_{j-1}
\end{aligned}$$

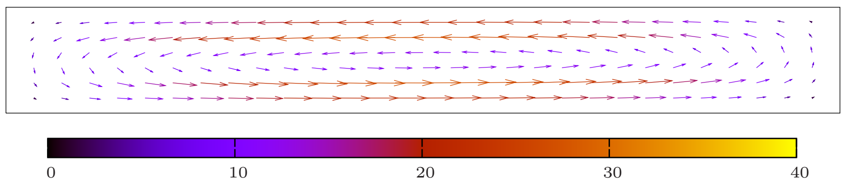
$$\begin{aligned}
&\mathbf{d}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n)}\cdot\nabla\right)\mathbf{d}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n+1)}\cdot\nabla\right)\mathbf{d}_j^{(n)} - dt\Delta\mathbf{d}_j^{(n+1)} \\
&\qquad + \frac{dt}{\epsilon^2}\left(|\mathbf{d}_j^{(n)}|^2 - 1\right)\mathbf{d}_j^{(n+1)} + \frac{2dt}{\epsilon^2}\left(\mathbf{d}_j^{(n)}\cdot\mathbf{d}_j^{(n+1)}\right)\mathbf{d}_j^{(n)} \\
&\quad = dt\left(\mathbf{u}_j^{(n)}\cdot\nabla\right)\mathbf{d}_j^{(n)} + \frac{2dt}{\epsilon^2}|\mathbf{d}_j^{(n)}|^2\mathbf{d}_j^{(n)} + \mathbf{d}_{j-1}.
\end{aligned}$$

We observe that, for the case studied where we have chosen a fixed timestep equal to 0.1 for grids of varying coarseness, each successive time iteration usually requires the solution of four or five Newton linearized problems. Since this problem involves the simultaneous solution of five different scalar equations (two for the velocity field, two for the order parameter field and one for the pressure), this method results in quite lengthy simulations (see table 5.2).

In figures 5.4 and 5.5 we show a comparison between the solution for the full system and the one obtained considering only the Navier-Stokes equations with the same external force. As it can be easily seen, the qualitative evolution of the velocity and pressure fields are quite similar: the main difference is to be found in the variations of the pressure field, greater in the Navier-Stokes system than in our full model. This is due to the additional



(a) The pressure field.



(b) The velocity field.

Figure 5.4: The solution of the Navier-Stokes equations at $t = 1.6$ ($\nu = 1$) just after the first peak in the intensity of the forcing field. We observe that the field is completely laminar.

viscous effect given by the nonlinear coupling with the order parameter field, whose contribute is indeed responsible for part of the expected pressure gradient.

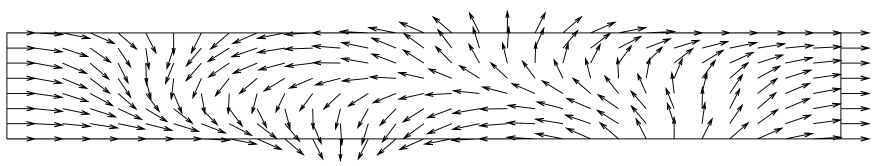
5.4 An alternative splitting method

As it was pointed out in the previous section, the performance of Newton's method for the iterative solution of the full system (2.1) is rather time consuming. Inspired by the proof of section 2.1, we also discuss a different iterative scheme based on the linearization of the splitting (2.9). Numerically this corresponds to solving iteratively the finite element formulation of the following partial differential equations.

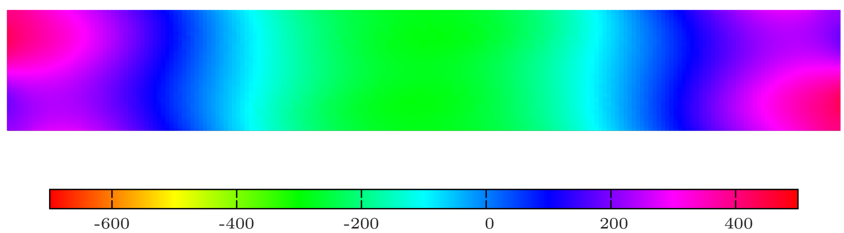
- The first step is to solve the following problem for the velocity field:

$$\begin{aligned}\mathbf{u}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n)} \cdot \nabla\right)\mathbf{u}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n+1)} \cdot \nabla\right)\mathbf{u}_j^{(n)} - dt\Delta\mathbf{u}_j^{(n+1)} \\ = dt\left(\mathbf{u}_j^{(n)} \cdot \nabla\right)\mathbf{u}_j^{(n)} + dt\left(\mathbf{d}_j^{(n)}\right)^T \Delta\mathbf{d}_j^{(n)} + dt\,\mathbf{g}_j + \mathbf{u}_{j-1}.\end{aligned}$$

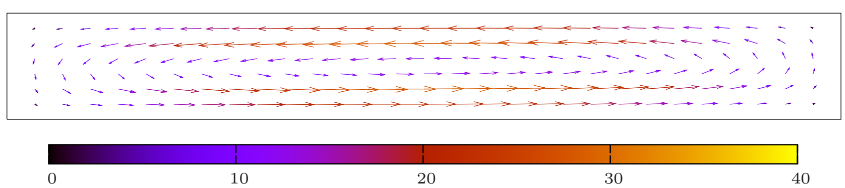
- The second step consists in solving the following linearized equation



(a) The order parameter field.

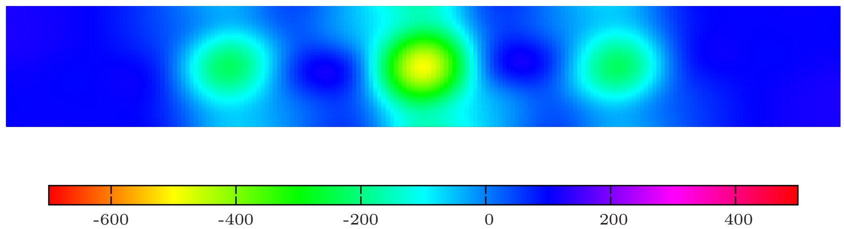


(b) The pressure field.

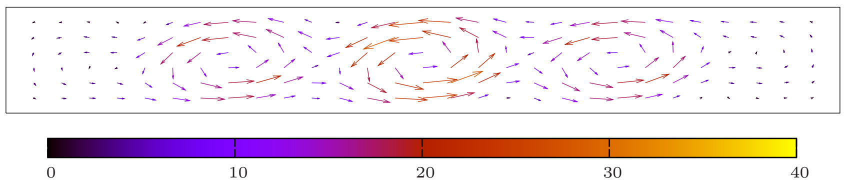


(c) The velocity field.

Figure 5.5: The solution of the full system at $t = 1.6$ ($\nu = 1$) just after the first peak in the intensity of the forcing field. We note that the velocity and pressure fields are quite similar to the Navier-Stokes case, whereas some differences can be seen between the order parameter field represented here and the one depicted above (see figure 5.2). Also notice the difference in the pressure gradients.



(a) The pressure field.



(b) The velocity field.

Figure 5.6: The solution of the Navier-Stokes equations at $t = 1.6$ ($\nu = 0.1$) just after the first peak in the intensity of the forcing field. Three vortices are clearly visible.

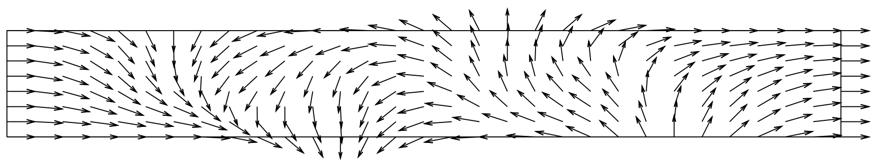
for the order parameter field:

$$\begin{aligned}
&\mathbf{d}_j^{(n+1)} + dt\left(\mathbf{u}_j^{(n+1)} \cdot \nabla\right)\mathbf{d}_j^{(n+1)} - dt\Delta\mathbf{d}_j^{(n+1)} \\
&\qquad + \frac{dt}{\epsilon^2}\left(|\mathbf{d}_j^{(n)}|^2 - 1\right)\mathbf{d}_j^{(n+1)} + \frac{2dt}{\epsilon^2}\left(\mathbf{d}_j^{(n)} \cdot \mathbf{d}_j^{(n+1)}\right)\mathbf{d}_j^{(n)} \\
&= \frac{2dt}{\epsilon^2}|\mathbf{d}_j^{(n)}|^2\mathbf{d}_j^{(n)} + \mathbf{d}_{j-1}.
\end{aligned}$$

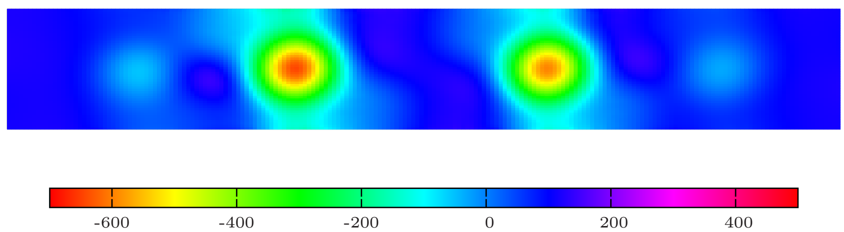
The numerical results are exactly the same as in the previous section. However, as can be seen in table 5.2, the overall computation time is much shorter despite the increased number of iterations per time step (now usually seven or eight).

5.5 Increasing the Reynolds number

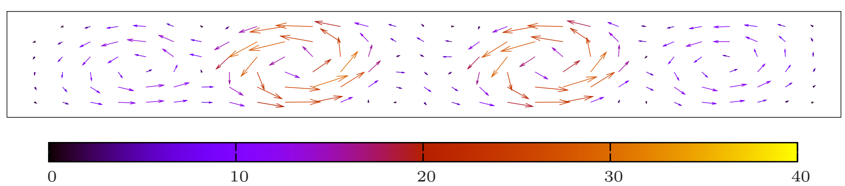
We now want to study the evolution of our system under slightly greater Reynolds numbers. In this section we choose $\nu = 0.1$, a viscosity for which a Taylor-like instability is clearly visible in the solution of Navier-Stokes equations (see figure 5.6). By using the same numerical scheme presented in the previous section, we can solve the full system also in this case without any additional problem. We only observe that the convergence of the scheme is slightly slower, requiring, in the average, a tenfold of iterations per time step.



(a) The order parameter field.



(b) The pressure field.



(c) The velocity field.

Figure 5.7: The solution of the full system at $t = 1.6$ ($\nu = 0.1$) just after the first peak in the intensity of the forcing field. We note that in this case some qualitative relevant differences appear when comparing this solution to the simpler Navier-Stokes equations (see figure 5.6 above). In particular, only two vortices can be observed in the velocity field. Moreover there seems to be an additional time instability connected with their motion which does not appear in the simpler case: during the first cycle these two vortices move from right to left leaving space at the end to a smaller vortex on the right which has not enough time to fully develop, whereas, at the beginning of the second period through which the simulation was run, a reversed motion (from left to right) appears.
We also note, comparing these pictures with figure 5.5, that the vortices have a homogenizing effect on the order parameter field.

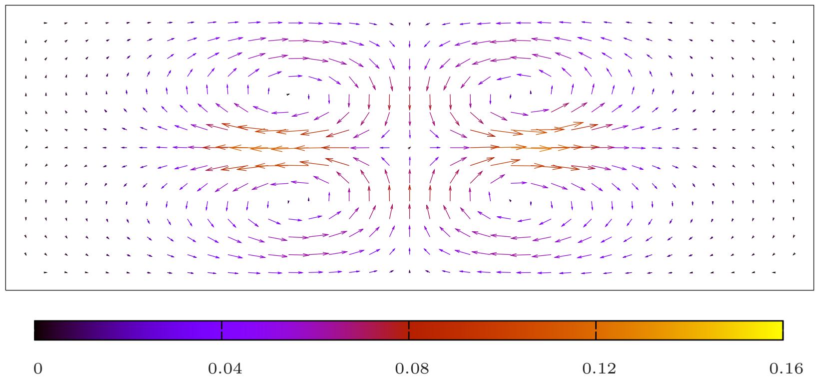


Figure 5.8: The velocity field at time $t = 0.15$ around the two moving singularities. See text for description.

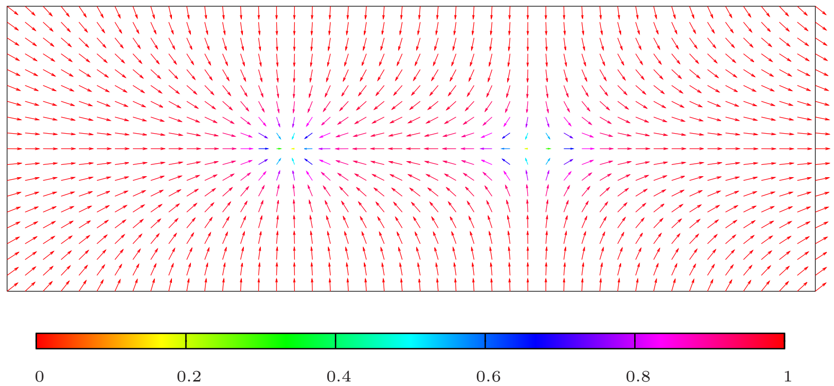
As it can be seen by a sample step of the computed solution, in this case the velocity and pressure field are quite different from the basic Navier-Stokes solution: indeed we notice some major asymmetries in the field which were not to be seen either in the Navier-Stokes case or in the lower Reynold's number case. We conclude that in this case the use of the full system is necessary to obtain correct information on the evolution.

5.6 Coalescence of singularities

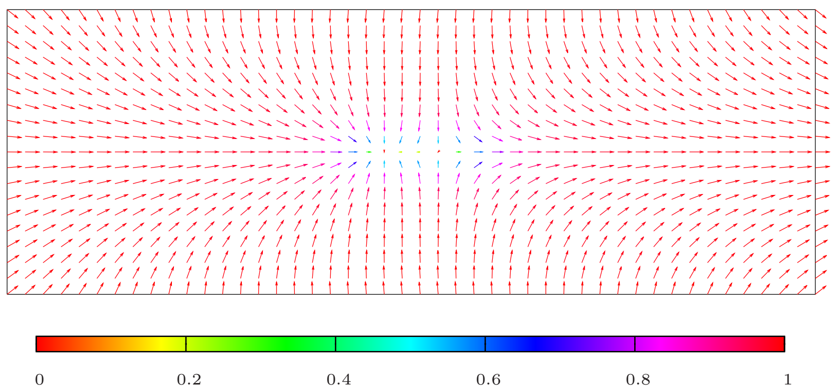
We end this chapter by showing some numerical results obtained by considering the evolution and annihilation of two singularities of opposed sign (respectively $+1$ and -1) in the order parameter field originally at rest. In this case the domain of computation is $[0, 2] \times [0, 0.8]$ and the viscosity is assumed to be 0.33.

After a short time, the two singularities begin to attract each other and eventually collapse (see figure 5.9). The induced velocity field shows a pair of vortices around the moving singularities (see figure 5.8). This numerically confirms the analytical results in [2].

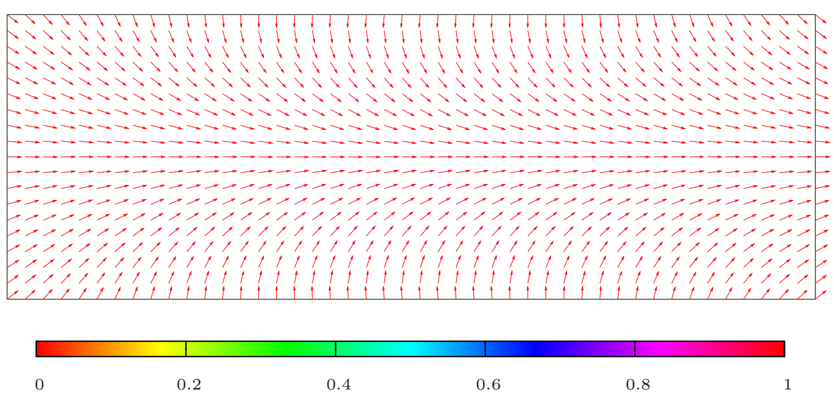
We conclude this chapter by observing that the analytical study of system (2.1) was crucial in designing a simple and efficient numerical method for its approximate solution. The solution of the full system consisting of five different unknowns is still today a computational expensive task, which can be made more feasible by a suitable abstract study.



(a) The order parameter at $t = 0$.



(b) The order parameter at $t \approx 0.5$.



(c) The order parameter at $t = 0.65$.

Figure 5.9: The annihilation of two singularities of opposed signs. See text for description.

Conclusions

At the end of the redaction of any important document there is always an unavoidable moment in which one looks back at what he has written. Immediately many question arise: Was the initially prefigured goal reached? Were there any sudden development departing far from the original plan? Was any unexpected difficulty encountered? Remembering how a work was initially conceived always brings some surprise with: initially important targets may have fallen apart while unexpected and interesting arguments, once never imagined, may form, at the end, a consistent part of the final work.

These considerations apply also in the present case: although many could argue on its importance, this thesis more or less underwent the same tortuous path from its initial concept to the present final result. Some results on a finer description of the long terms dynamics of our systems did not find ultimately a place in the present work (for example an estimate of the fractal dimension of the global and exponential attractors) and still await to be written, although the silent work they absorbed. On the other hand, entire chapters were introduced in the overall plan only along the way as was the case for the numerical examples presented in chapter 5. In particular, it has been quite a surprise when the hints given by the analytical study of system (2.1) brought such interesting numerical results.

However difficult the story of this document could have been, I cannot finish it without remembering the most important rails which guided me all along this trip. Although complex the analysis of a mathematical model could be, the secrets we are able to worm out of it are of incredible importance: not only they give us a deeper understanding of the nature surrounding us, but they also bring an immediate and priceless wealth of information for possible applications.

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