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Time-domain modelling of the wind forces acting on a bridge deck with indicial functions

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Index

Introduction

In the last decades a constant effort has been made in the design of bridges to increase their span length and slenderness. Their lighter mass and reduced stiffness have made them more vulnerable to dynamic effects, especially due to wind forces. The well known collapse of the original Tacoma Narrows Bridge, that took place in 1940, is a good illustration of the possible devastating effects of wind on structures. Its amplitude of oscillation reached several meters due to a positive feedback phenomenon between the motion induced by the wind forces and the wind forces themselves, called flutter. This eventually led first to failure of some hangers and then to the complete collapse of this suspended bridge. In order to prevent structures from incurring in such phenomena, it is important to consider the wind as a dynamic load and to understand how it may get coupled with the mechanical behavior of the structure. To design a bridge deck profile for suspended or cable-stayed bridges, the undesirable aerodynamic effects that may be created by such a profile always have to be studied.

The path inhere chosen to model the coupling of wind forces on a bridge deck cross-section with its motion is the indicial function theory. This has been developed starting in the 30's for airplane wings profiles and has proved useful in flight-mechanics. Its main underlying assumption is that the wind forces are linear with respect to the motion of the airfoil. Based on the flutter derivatives that link the wind forces and the airfoil motion in the frequency-domain, and on the abovementioned linearity assumption, the indicial functions approach can be derived. This is a way to model wind forces in the time-domain accounting for the fluidodynamic phenomena that render wind forces amplitude and frequency dependent. Once the wind forces expressed theoretically in time domain, they can be evaluated numerically and introduced in a larger computing framework that models also the mechanical and dynamic behavior of the bridge/structure.

Purpose of this work is to develop and implement a computer code that is able to evaluate the wind forces acting on a bridge deck at a given time using the indicial functions approach. To do so, in the *1st Part: Theory*, the theoretical background of indicial functions is explained and the analytical formulation of the corresponding wind forces is obtained. In the *2nd Part: Implementation*, the implemented procedure is explained in some detail. Finally, in the *3rd Part: Practical applications*, the implementation is validated and tested on different application cases.

In Chapter I), starting from the Navier-Stokes equations, the hypotheses under which the wind forces may be considered linear with respect to the bridge deck motion are detailed. The basic principles of fluid mechanics that are used in this work are introduced and explained.

In Chapter II), aerodynamics applied to bridges is introduced within the steady-state aerodynamics framework. The coupling phenomena that arise between the air flow and a moving deck profile, as well as the conditions needed to encounter them, are detailed.

In Chapter III), once assumed linearity between the profile motion and the wind forces, flutter derivatives are defined as impedances linking linearly these two sets of physical variables. Flutter derivatives may be constants or functions of the oscillation frequency of the bridge deck. In this chapter, it is also explained how the various aerodynamic data of a profile can be obtained in practice with wind tunnel experiments.

In Chapter IV), the concept and the definition of the indicial functions are introduced. In this chapter it is explained how wind forces can be computed in time-domain accounting for the effects expressed by the flutter derivatives, which are instead given in the frequency-domain. Three approaches to indicial functions are made that allow a better understanding of their physical meaning and properties.

In Chapter V) the complete expression of the wind forces acting on a bridge deck is formulated. It is explained how the different components of the wind forces are modeled: quasi-steady forces, self-excited forces and buffeting forces. The practical use of the aerodynamic data with the interpolation procedures is also exposed.

In Chapter VI), the procedure, coded in FORTRAN90, is explained along with the various input and output files and the data flow scheme. The different components of the wind forces are presented one after the other.

In Chapter VII), the implementation is tested and validated two cases: first, the Akashi Kaikyō Bridge deck that is well-known in the indicial functions literature and then on the Tsurumi Fairway Bridge deck.

In the 1st Appendix, the Fourier transform is defined and some of its properties, like the response to an impulse are studied.

In the 2nd Appendix, the implemented FORTRAN code that has been developed for this work is enclosed.

In the 3rd Appendix, the implemented MATLAB procedure that has been developed for the validation of the main procedure is enclosed

In the 4th Appendix, the notations commonly used in the work are listed with their physical meaning. The different variables, but also the mathematical operators are described.

1st Part: Theory

I) From the Navier-Stokes equations to the linear equation governing an air flow

The Navier-Stokes equations govern the fluid mechanics from a macroscopic point of view. It is based on the principles of mass conservation and motion quantity conservation. The Navier-Stokes equations are the fundamental equations of fluid mechanics. They have been proved to be true for many liquids and gases. It governs the behavior of air even under intense conditions, such as the flow of air around the wing of a supersonic jet plane.

a) The Navier-Stokes equations

- Mass conservation:

$$\frac{\partial \rho}{\partial t} = div(\rho \underline{V}) \qquad (1.1)$$

Where ρ is the density of air and $\underline{V}$ is the velocity of air. div is the divergence operator (cf. definition in the 4th Appendix).

- Momentum conservation:

$$\frac{\partial \underline{V}}{\partial t} + \underline{\underline{\nabla V}}.\underline{V} = g + \frac{1}{\rho}\underline{\nabla}[-p + (\lambda + \mu).div(\underline{V})] + \frac{1}{\rho}\underline{div}\left(\mu \underline{\underline{\nabla V}}\right) \qquad (1.2)$$

Where g represents the volumetric forces and λ and μ are the Lame's constants for viscosity of air. μ is also called the dynamic viscosity. p is the pressure of air taken as $p = P - P_0$ with P absolute pressure and P_0 pressure at rest. $\underline{\nabla}$ is the gradient operator and . , called "dot" is the scalar product operator (cf. definitions in the 4th Appendix).

The Navier-Stokes equations are conservation equations. For the mass conservation, this means that considering a finite volume of space, the mass entering equals that exiting it. The momentum conservation equation comes from the equilibrium of forces on the same finite volume of space.

They can be initially casted as integrals on a given element of space; by assuming that these integral equations hold true for any element of space it is derived that they are valid for any particle of fluid and so, the equations (1.1) and (1.2) are obtained.

b) Air flow acting on a bridge deck

As seen in the previous paragraph, the Navier-Stokes equations have a large field of application. For the studied case of air flow around a bridge deck, some simplifications can be made:

- perfect fluid: i.e. incompressibility $d\rho/dt = 0$ and non-viscous fluid (inviscid flow)
- no external volumetric forces: i.e. $g = 0$

The first assumption is justified since the velocity of the studied flow is, at maximum, in the range 30-70m/s. The Mach number, *M*, helps understand whether the flow can be considered incompressible.

$$M = \frac{U}{c}$$

where *U* is the flow velocity and *c* is the speed of sound in the medium flowing. In atmospheric conditions (15°C) the speed of sound is normally assumed *c* = 340 m/s.

For the velocity interval that is considered, *M* ranges from 0.088 to 0.206. As reported in [1], it is considered by experience that the limit under which a flow can be considered incompressible is 0.4. Since this inequality is verified in the studied case, it can rightfully be assumed that:

$$\frac{\partial \rho}{\partial t} = 0 \qquad (1.3)$$

And therefore, from the mass conservation equation (1.1), it comes that:

$$div(\underline{V}) = 0 \qquad (1.4)$$

As any external volumetric forces are neglected, $g = 0$. So, the momentum conservation equation (1.2) can be rewritten as:

$$\frac{\partial \underline{V}}{\partial t} + \underline{\underline{\nabla V}}.\underline{V} = -\frac{1}{\rho}\underline{\nabla p} + \frac{1}{\rho}\underline{div}\left(\mu \underline{\underline{\nabla V}}\right) \qquad (1.5)$$

In order to understand better the importance of the different terms in equation (1.5), a dimensional analysis of it will be made. Making a dimensional analysis of equations is a technique frequently used in fluid mechanics in order to evaluate whether some terms can be considered negligible or not. In a dimensional analysis, the physical variables and the mathematical operators (derivate, divergence, etc…) are approximated by typical constants in order to maintain the dimensional consistency of the equations. For example, the term $\partial \underline{V} / \partial t$ can be dimensionally approximated by V/T, where V and T are typical constants respectively for the speed of the air flow and for time in the studied problem. In the previous equation, the ratio of convection forces over viscous forces is evaluated, by a dimensional analysis, as made in [2].

The viscous forces can be approximated by:

$$\frac{1}{\rho}\underline{div}\left(\mu\underline{\underline{\nabla V}}\right) \cong V\mu/\rho L^2$$

where the term L^2 at the denominator is due to the derivation, involved by the divergence operator, acting on the derivation implied by the gradient operator.

And the convection term can be approximated by:

$$\underline{\underline{\nabla V}}.\underline{V} \cong V^2/L$$

So, the ratio is approximated as:

$$\frac{\underline{\underline{\nabla V}}.\underline{V}}{\frac{1}{\rho}\underline{div}(\mu\underline{\underline{\nabla V}})} \cong \frac{V^2/L}{V\mu/\rho L^2} = \frac{VL\rho}{\mu} = \mathcal{R}_e$$

The constant $\mathcal{R}_e$ is the well-known Reynolds number which is dimensionless:

$$\frac{[L][T]^{-1}.[L].[M][L]^{-3}}{[M][T]^{-1}[L]^{-1}} = [1]$$

In the case of an air flow in atmospheric conditions the viscosity of air can be taken as μ=17.1 10^{-6} Pas (or kg m^{-1} s^{-1}) (cf. [1]), the density of air is worth ρ=1.29 kg m^3. For a bridge deck, a typical velocity of air is in the range V=10-70 m s^{-1} and a typical length is the width of a bridge deck which is worth 10-30 m.

The limit $\mathcal{R}_e$ = 1 marks the boundary of laminar flows, for which viscous effects are as important as inertial effects. In the case of a bridge deck, the Reynolds number ranges from $\frac{10*10*1.29}{17.1*10^{-6}}$=7.5 10^6 to $\frac{70*30*1.29}{17.1*10^{-6}}$=1.6 10^8. These values are largely above the limit $\mathcal{R}_e$ = 1 and the flow can be defined as "turbulent", for which the inertial effects are much predominant above the viscous effects and therefore the term $1/\rho\ \underline{div}\left(\mu\underline{\underline{\nabla V}}\right)$ is negligible.

The momentum conservation equation (1.5) under the hypothesis of non-viscous fluid becomes:

$$\frac{\partial \underline{V}}{\partial t} + \underline{\underline{\nabla V}}.\underline{V} = -\frac{1}{\rho}\underline{\nabla} p \qquad (1.6)$$

The convective term, $\underline{\underline{\nabla V}}.\underline{V}$, can be simplified thanks to a mathematical analysis [1]. From the vector fields proprieties, it is known that (cf. [3]):

$$\underline{\nabla}\left(\underline{A}.\underline{B}\right) = \underline{\underline{\nabla A}}.\underline{B} + \underline{A}.\underline{\underline{\nabla B}} + \underline{A} \wedge \underline{rot}\left(\underline{B}\right) + \underline{rot}\left(\underline{A}\right) \wedge \ \underline{B} \qquad (1.7)$$

where ∧ is the cross-product.

If $\underline{A} = \underline{B} = \underline{V}$, the equation becomes:

$$\underline{\nabla}(V^2) = 2\underline{\underline{\nabla V}}.\underline{V} + 2\underline{V} \wedge \underline{rot}\left(\underline{V}\right)$$

Therefore:

$$\underline{\underline{\nabla V}}.\underline{V} = \underline{\nabla}(V^2/2) - \underline{V} \wedge \underline{rot}\left(\underline{V}\right) \qquad (1.8)$$

So the Navier-Stokes equation for incompressible inviscid flow (1.6), becomes:

$$\frac{\partial \underline{V}}{\partial t} + \underline{\nabla}\left(\frac{V^2}{2}\right) + \underline{rot}\left(\underline{V}\right) \wedge \underline{V} = -\frac{1}{\rho}\underline{\nabla} p \qquad (1.9)$$

The last hypothesis that can be made is considering only two directions of space in the study. Indeed, the study of the wind forces is made on sections of bridge decks and not on entire bridge decks. Therefore, the direction oriented along the bridge span is supposed to have no influence in the wind forces calculations. This allows for developing only a "sectional" theory.

Up to this point, only hypotheses on the velocity of the air flow and on the dimension of the bridge deck have been assumed. Therefore, equation (1.9) governs the air flow around a bridge deck. This equation, however, is too complex to be solved in this form, as the goal of the study is looking for a coupling of the aerodynamic and the mechanic behaviors of a bridge deck and not only the analysis of the air flow around a bridge deck. It would be too complicated to solve the last equation each time the wind forces have to be evaluated. A way to simplify the air flow analysis is to assume that the bridge deck can be considered as a "thin-airfoil". Under the thin-airfoil hypothesis, further simplifications on the Navier-Stokes equations can be made. In the next paragraph, the implications connected to considering an airfoil as thin and whether it can be correct to consider a bridge deck as thin-airfoil will be explained.

c) The thin-airfoil theory

The major hypothesis made when considering an airfoil as thin is the irrotationality of the air flow around the airfoil. The flow far ahead of the airfoil is parallel: the wind velocity is always oriented along one single direction, even if this direction may vary in time. When the wind comes across an obstacle, it is deviated and it is observed that the flow develops components perpendicular to the wind orientation in order to pass the obstacle (cf. Figure 2 and the streamlines around an obstacle). If the flow after the obstacle is parallel again, the flow is said to be “irrotational” and it may be considered that the vortex part is worth zero [1]. So $\underline{rot}(\underline{V}) = \underline{0}$ and equation (1.7) becomes:

$$\frac{\partial \underline{V}}{\partial t} + \underline{\nabla}\left(\frac{V^2}{2}\right) = -\frac{1}{\rho}\underline{\nabla} p \qquad (1.8)$$

The “irrotational” behavior of the air flow means that the vortexes created by the profile (as a vertical speed component always appears around the profile) vanish without the need of viscosity. This means that there is no separation of the flow from the profile and therefore, that vortexes are not created in the wake of the profile. The irrotationality of an air flow around an obstacle depends only on the obstacle itself (geometry, aspect ratio, surface material).

The analysis of the different components of the wind velocity allows making some additional simplifications. The air flow around the bridge deck due to the wind is composed of four elements, cf. [4]:

- the average horizontal wind velocity $U\,\underline{e}_x$

- the variations due to the turbulence of the wind far away ahead of the bridge deck ($u\,\underline{e}_x$, $w\,\underline{e}_z$)

- the speed variations of the air flow due to the bridge deck geometry ($w_1'\,\underline{e}_x$, $w_3'\,\underline{e}_z$)

- the speed variations of the bridge deck position due to its motion ($\dot{x}\,\underline{e}_x$, $\dot{z}\,\underline{e}_z$)

The referential ($\underline{e}_x$, $\underline{e}_z$) in which all these speeds have been written is fixed and horizontally/vertically oriented.

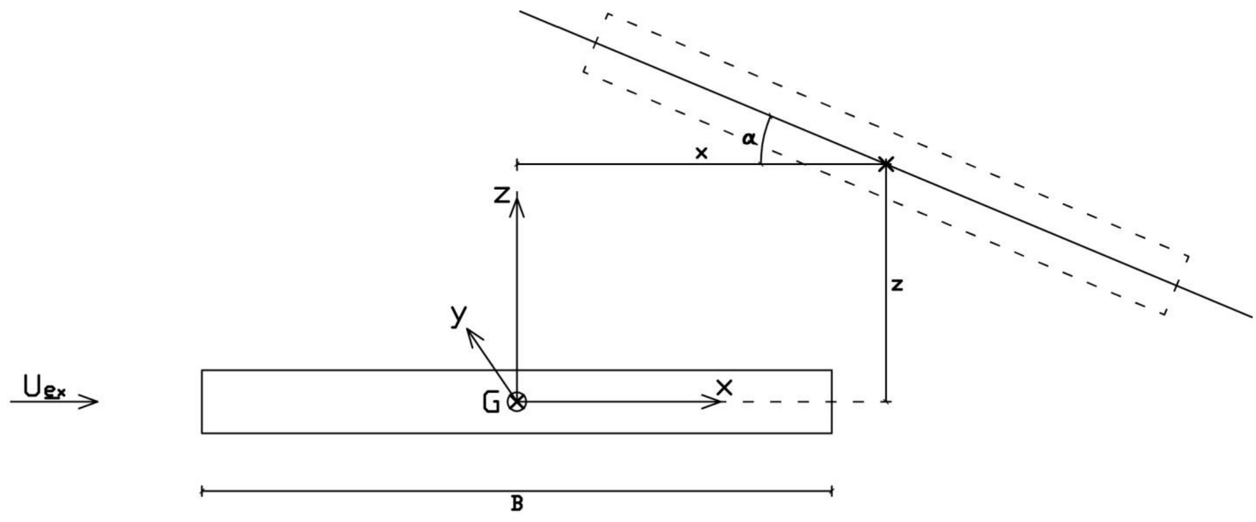

Figure 1: Scheme of the cross-section of a bridge deck and definition of the motion variables

In order to be able to make the additional simplifications, the wind has to be considered in a special way, as in [5] and [6]: the wind is horizontal and constant with respect to the bridge deck. The wind turbulence and the motion of the bridge deck are included in the angle of attack of the bridge deck, by the formula, as in [4]:

$$\gamma_r = arctg\left(\frac{w_1 - \dot{x}}{U + w_3 - \dot{z}}\right) \qquad (1.9)$$

The term γ_r is added to the "regular" angle of attack of the bridge deck, α.

Therefore, all the calculations will be done in the absolute referential ($\underline{e}_x$, $\underline{e}_y$), which corresponds to the bridge deck referential at rest and not in the referential of the bridge deck when oscillating.

Under the new formulation of the air flow speeds, the velocity of the air around the bridge deck is equal to $\underline{V}=(U+w_1')\ \underline{e}_x + w_3'\ \underline{e}_z$. In order to linearize the equation, the following inequalities are assumed: $|w_1'| \ll U$ and $|w_3'| \ll U$. This is the hypothesis of small perturbation.

As the flow is irrotational, the velocity of air can be derived from a potential, Φ. The introduction of the potential is made possible because if $\underline{V} = \underline{\nabla}\Phi$, then it comes that $\underline{rot}(\underline{V}) = \underline{rot}(\underline{\nabla}\Phi) = \underline{0}$ (the rotational of a gradient is worth zero, cf. [3] for the demonstration).

The flow is incompressible ($div(\underline{V}) = 0$, as seen in I.a)) and so, $div(\underline{\nabla}\Phi) = \Delta\Phi = 0$.

The equation $\underline{V} = \underline{\nabla}\Phi$ in two-dimension under the previous hypotheses can be written:

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = U + {w_1}' \\ \dfrac{\partial \Phi}{\partial z} = w_3' \end{cases} \qquad (1.10)$$

And the equation $\Delta\Phi = 0$ can be expressed as:

$$\frac{\partial^2 \Phi}{\partial^2 x} + \frac{\partial^2 \Phi}{\partial^2 z} = 0 \qquad (1.11)$$

Rewriting the equation of mass quantity conservation (1.8) with the potential Φ, it comes:

$$\frac{\partial \underline{\nabla}\Phi}{\partial t} + \underline{\nabla}\left(\frac{\underline{\nabla}\Phi^2}{2}\right) = -\frac{1}{\rho}\underline{\nabla}p$$

From the Schwarz theorem, the temporal and spatial derivates can be exchanged:

$$\underline{\nabla}\left(\frac{\partial \Phi}{\partial t} + \frac{\underline{\nabla}\Phi^2}{2} - \frac{p}{\rho}\right) = \underline{0} \qquad (1.12)$$

So the term inside the gradient is a space constant called *F(t)*.

$$\frac{\partial \Phi}{\partial t} + \frac{\underline{\nabla}\Phi^2}{2} - \frac{p}{\rho} = F(t) \qquad (1.13)$$

The space constant *F(t)* is evaluated considering being far away from the bridge deck. With the hypotheses made before, it is known that far away from the bridge deck, the pressure is constant and worth zero, because the pressure, p, is the difference between absolute pressure (far away from the bridge deck it is the pressure at rest) and pressure at rest. The velocity is a time and space constant, *U*, so Φ is a space constant too.

The unique term which is a non-zero far away from the bridge deck is:

$$\frac{\underline{\nabla}\Phi^2}{2} = \frac{\underline{V_\infty}^2}{2} = \frac{U^2}{2}$$

So, *F(t)* can be expressed as:

$$\frac{U^2}{2} = F(t) \qquad (1.14)$$

Therefore, equation (1.13) becomes:

$$\frac{\partial \Phi}{\partial t} + \frac{\underline{\nabla}\Phi^2}{2} - \frac{p}{\rho} = \frac{U^2}{2} \qquad (1.15)$$

Under the hypothesis of small perturbation, the term $\underline{\nabla}\Phi^2$ becomes:

$$\underline{\nabla}\Phi^2 = (U + w_1')^2 + w_3'^2 \cong U^2 + 2Uw_1' \qquad (1.16)$$

Changing the expression of the speed potential Φ into Φ' by:

$$\Phi' = \Phi - Ux$$

it comes:

$$\frac{\partial \Phi'}{\partial x} = w_1' \quad \text{and} \quad \frac{\partial \Phi'}{\partial t} = \frac{\partial \Phi}{\partial t}$$

Therefore equation (1.15) becomes:

$$\frac{\partial \Phi'}{\partial t} + 2U\frac{\partial \Phi'}{\partial x} - \frac{p}{\rho} = 0 \qquad (1.17)$$

A linear equation (1.17) has been obtained, as the pressure (which is the force by unit of surface applied by the wind on the profile) is linear with respect to the speed potential. This equation is the basis for the linear aeroelastic theory developed by [5] and [6] about oscillations of thin-airfoils in an incompressible flow.

The final linear equation is true under the following hypothesis:

- subsonic flow (*M*<0.4)

- perfect fluid: its Lame’s constants are zero so it is non-viscous and incompressible

- irrotational flow (known as the Kutta condition)

- some wind speed approximations

- small perturbation of the flow around the bridge deck

Without studying all the mathematical steps required to pass from this equation of the pressure as a function of the air velocity potential to the integrated forces due to an air flow on an airfoil, the theoretical results obtained from the last equation will be developed and explained in chapter III).

In the case of a bridge deck, the two biggest approximations are the small perturbation and the irrotationality of the flow. These hypotheses depend both largely on the shape of the airfoil (here the bridge deck) and so and on the fact that the airfoil can be considered as a thin-airfoil or not. A thin-airfoil is an airfoil with the following geometric properties:

- a little aspect ratio that ensures little perturbations of the flow around the airfoil. The aspect ratio is defined by the height divided by the length.

- no singularities (such as angles) that may create a separation of the flow from the airfoil and violate the Kutta condition

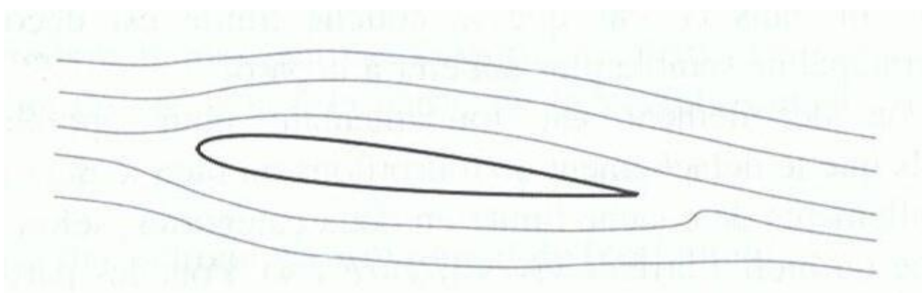

Figure 2: *Air flow respecting the Kutta condition, without separation. Figure from [7]*

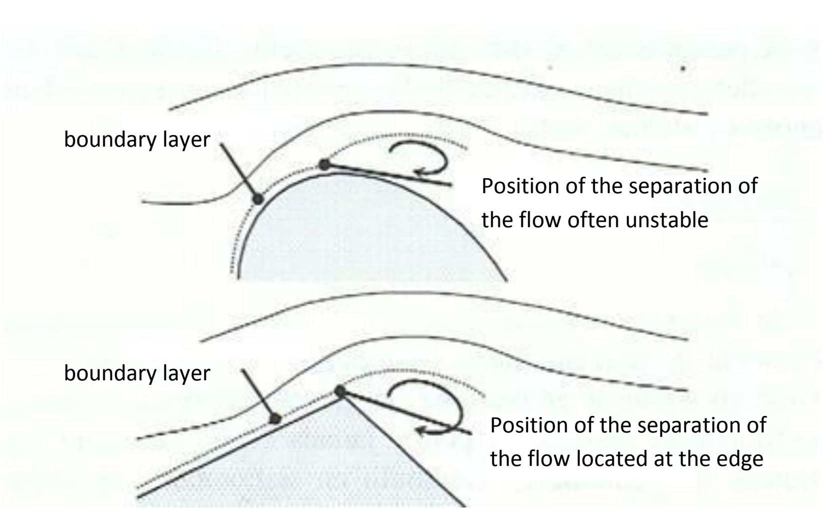

Figure 3: Air flow not respecting the Kutta conditions. A separation of the boundary layer from the profile can be noticed. Figure from [7]

Clearly, a “classic” bridge deck (cf. the Akashi Kaikyō Bridge cross-section, Figure 4) does not verify the geometrical conditions required to be considered as a thin-airfoil because it always has singularities and an aspect ratio too high. Some modern suspension bridges have decks that are so thin that they become close to the thin-airfoil conditions. The thinner a bridge deck cross-section is, the more it can be considered that the air flow is governed by a linear equation such as (1.17). For example, the Normandy Bridge (Figure 5) has a very aerodynamic shape. In some cases, having a bridge deck that can nearly be considered a thin-airfoil is done by purpose, in order to better control the effect of the wind on the structure, by evaluating it thanks to the thin-airfoil theory. In other cases, the philosophy may be, on the opposite, to create as many singularities as possible, in order to avoid being to close to the Kutta conditions and so avoid any phenomenon of excessive wind forces. But a bridge deck never perfectly respects the Kutta conditions, as it always needs road equipments for the safety of the people crossing the bridge that create a separation of the flow from the bridge deck on its extrados.

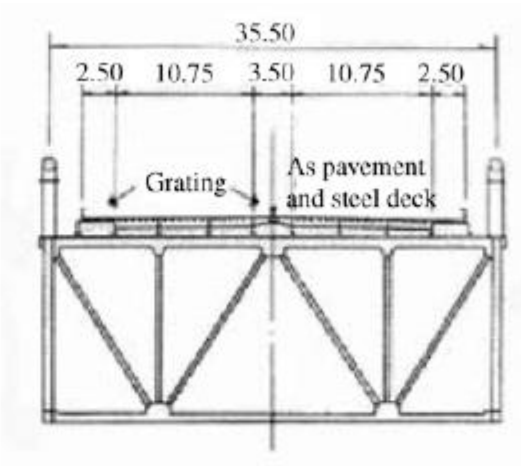

Figure 4: The Akashi Kaikyō Bridge cross-section: a clearly non-aerodynamic profile. Figure from [8]

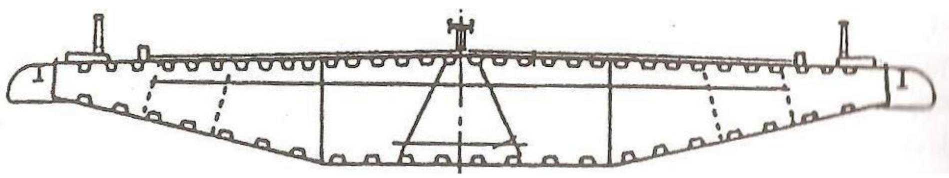

Figure 5: The Normandy Bridge cross-section: an aerodynamic profile. Figure from [7]

The only forces that are considered in the study are the pressure forces applied by the air on the profile. These elementary forces are oriented perpendicularly to the surface of the profile. The shearing forces (also called skin friction) which are the tangential forces are neglected as the viscosity of the fluid has been neglected previously

d) Dimensional analysis of the Navier-Stokes equation for an incompressible flow

Another way to simplify the Navier-Stokes equations is to transform it into a dimensionless equation. This means dividing every physical variable by a typical constant, i.e., a constant that best approximate the variable.

The Navier-Stokes equation for an incompressible flow

$$\frac{\partial \underline{V}}{\partial t} + \underline{\underline{\nabla V}}.\underline{V} = -\frac{1}{\rho}\underline{\nabla p} + g + \frac{1}{\rho}\underline{div}\left(\mu\underline{\underline{\nabla V}}\right) \qquad (1.18)$$

can be transformed into a dimensionless one, using the following changes of variable, where the highlighted variable, $\overline{}$, are the dimensionless ones.

$$x = L\overline{x} \quad y = L\overline{y} \quad z = L\overline{z} \quad g = g_\infty\overline{g}$$

$$V = V_\infty\overline{V} \quad p = \rho V_\infty^2\overline{p} \quad t = \frac{L}{V_\infty}\overline{t}$$

Where *L* is a typical length. It could be the bridge deck width, *B*. V_∞ is a typical velocity. It could be *U*, the mean wind velocity. g_∞ is a typical volumetric force, in this case, it is worth 0. And ρ is the density of air.

By replacing the variable by the dimensionless ones, it comes the following equation:

$$\frac{\partial \underline{\overline{V}}}{\partial \overline{t}} + \underline{\underline{\overline{\nabla V}}}.\underline{\overline{V}} = -\underline{\overline{\nabla p}} + \frac{\overline{g}}{\mathcal{F}^2} + \frac{1}{\mathcal{R}_e}\underline{\overline{div}}\left(\mu\underline{\underline{\overline{\nabla V}}}\right) \qquad (1.19)$$

Where $\mathcal{R}_e = V_\infty L\rho/\mu$ is the Reynolds number and $\mathcal{F} = V_\infty/\sqrt{g_\infty l}$ is the Froude number.

In the studied cases, bridge decks, the Reynolds number is much bigger than one and so it can be considered infinite. The volumetric forces are worth zero and so the Froude number is infinite. Therefore it comes the same equation than the one found in I.b), under a dimensionless form:

$$\frac{\partial \underline{\overline{V}}}{\partial \overline{t}} + \underline{\underline{\overline{\nabla V}}}.\underline{\overline{V}} = -\underline{\overline{\nabla p}} \qquad (1.20)$$

- Conclusion:

In this chapter, the hypotheses under which one can pass from the fundamental equations of fluid mechanics, the Navier-Stokes equations, to the equation governing an air flow around a bridge deck have been highlighted. In paragraph c), a linear equation has been obtained in the thin-airfoil theory. This equation is the basis of the aeroelastic theory that will be developed further, in chapter III). It has also been exposed that a bridge deck cross-section can never be considered as thin. Therefore, the fundamental idea of aeroelasticity applied to bridges is that even if the decks are not thin, the linearity of the wind forces with respect to the motion of the bridge can still be assumed.

II) An introduction to aerodynamics

After having considered only the mechanics of air in chapter I), in this chapter, the aerodynamic theories will be introduced. Aerodynamics is the branch of physics that studies the interaction arising from relative motion between air and a moving structure. This interaction can be considered as a weak coupling which means that only the influence of the air on the structure (or the opposite) is considered or as a strong coupling in which we have to consider the mechanics of both phases (gaseous and solid, i.e. air and structure) at the same time because they interact. Aerodynamics theories have been developed especially for studying the flight of airplanes but can also be applied to civil structures to study the influence of the wind on deformable structures such as a slender bridge deck or a cable (power line or bridge cable or stay).

a) Steady-state aerodynamics

In the steady-state aerodynamics, the interaction of an air flow horizontal and constant on a profile is studied, assuming that the profile moves slowly enough to consider it fixed and to assume that the flow is steady. The profile moves so slowly that the profile can be considered as fixed at any time from the point of view of the fluid. This means that when the profile motion parameters change, the flow reaches instantaneously the time-domain equilibrium. This is a weak coupling: the forces on the profile depend only on the air parameters and on the profile present configuration. It is not considered that the wind forces may induce a change of the profile parameters and is it not considered that the time-history of the profile motion may have an influence on the wind forces at the present time. In this case, the wind forces can be defined in the simplest way:

$$\begin{cases} F_x = \frac{1}{2}\rho c U^2 C_x \\ F_z = \frac{1}{2}\rho c U^2 C_z \\ M_y = \frac{1}{2}\rho c^2 U^2 C_M \end{cases} \qquad (2.1)$$

Where F_x is the drag (horizontal) force considered positive if the force is in the same direction as the air flow. F_z is the lift (vertical) force considered positive if the force is directed upwards. And M_y is the pitching-moment considered positive if rotating clockwise. For a scheme with the different definitions of lift and drag, see Figure 17, in IV.a). ρ is the air density, U is the speed of the air flow and c is a typical length of the structure, for example the chord of the profile. C_x, C_z and C_M are the dimensionless static aerodynamic coefficients respectively of drag, lift and pitching-moment. As the static aerodynamic coefficients are dimensionless, the wind forces are expressed in *N/m* and the pitching-moment is in *Nm/m*. These are forces on a section of bridge deck and so, forces per unit of length. The forces of lift and drag and the pitching-moment are applied on the elastic center of the profile, *G*. In the studied case, a bridge deck, the profile is symmetrical, so the elastic center is in the middle of the profile.

From [1], one knows that the static aerodynamic coefficients are functions of the Reynolds number, $\mathcal{R}_e = Uc\rho/\mu$, the Strouhal number, $k=\omega c/U$, also called reduced frequency, the Mach number, $M=U/c$, the shape of the profile and the angle of attack. We have made the hypothesis that the flow is perfect, so the influence of $\mathcal{R}_e$ is negligible. The Strouhal number is equal to zero as the profile is considered fixed. As noticed experimentally, the dependency on the speed of the flow is neglected. So in steady-state aerodynamics, the static aerodynamic coefficients are functions only of the angle of attack and of the shape of the profile. So, for a given profile, the pitching-moment and the two forces acting on the profile can be evaluated with the three static aerodynamic coefficients which are functions only of the angle of attack.

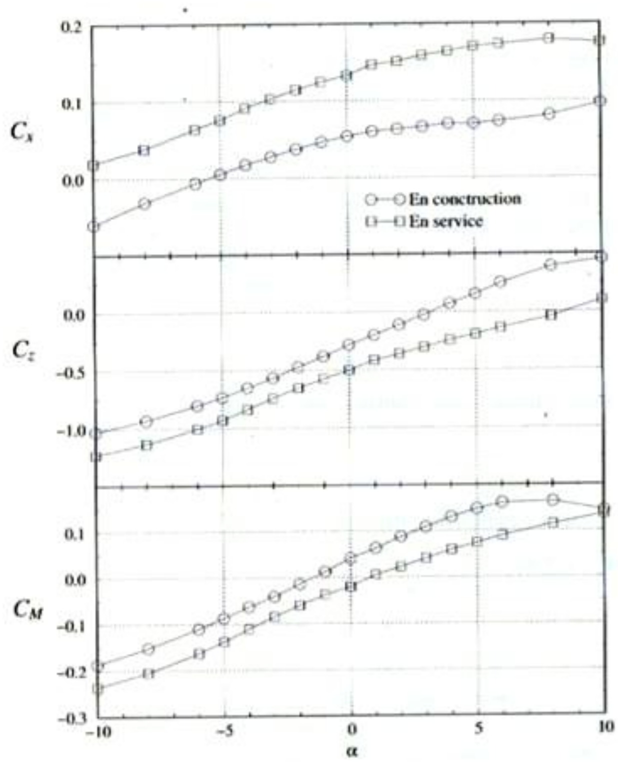

Figure 6: Static aerodynamic coefficients for the "Viaduc de Millau" deck as a function of the angle of attack in degrees. The circles represent the value during the construction, the squares, once finished. Figure from [7]

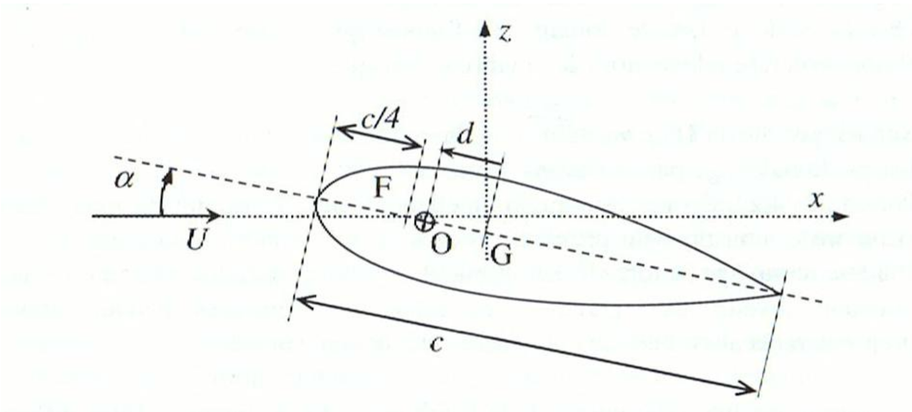

Figure 7: a thin-airfoil scheme with classic nomenclature. G is the elastic center. O is the point of application of the wind forces. F is the aerodynamic center.

By defining some specific points of the profile, such as the pressure center and the aerodynamic center and evaluating their position on the horizontal axis of the profile (cf. Figure 7), one may better understand how are acting the forces on the profile.

The center of pressure, *0*, is defined as the point of application of the aerodynamic forces, considering only the lift forces. Therefore, the pitching-moment evaluated with respect to *P* is worth zero. Respecting the conventions of sign, the distance from the leading edge to the pressure center, x_P is equal to:

$$M_y = -x_0 F_z$$

$$x_0 = -\frac{M_y}{F_z} = -c\frac{C_M}{C_z} \qquad (2.2)$$

The aerodynamic center, *F*, is the point on the chord about which the aerodynamic moment is independent of the angle of attack. With x_F the distance from the leading edge to the aerodynamic center, the pitching-moment with respect to *F* is:

$$M_F = M_0 - (x_0 - x_F)F_z \qquad (2.3)$$

Introducing the aerodynamic coefficients, it comes the following expression:

$$C_{M\,F} = -\frac{(x_0 - x_F)}{c}C_Z$$

Using the expression of x_0:

$$C_{M\,F} = C_M + \frac{x_F}{c}C_Z$$

Deriving this expression and using the fact that the moment M_F does not depend on the angle of attack, in *F*:

$$0 = \frac{\partial C_{M\,F}}{\partial \alpha} = \frac{\partial C_M}{\partial \alpha} + \frac{x_F}{c}\frac{\partial C_Z}{\partial \alpha}$$

As verified experimentally (cf. [7]) for little values of the angle of attack (not more than 5° of amplitude), it can be assumed that the derivates of the static aerodynamic coefficients in $\alpha=0$ is a constant. Therefore, the position of the aerodynamic center does not vary with the angle of attack. And so:

$$\frac{x_F}{c} = -\left.\frac{\partial C_M}{\partial \alpha}\right|_0 \left.\frac{\partial \alpha}{\partial C_Z}\right|_0 = -\frac{\partial C_M}{\partial C_Z} \qquad (2.4)$$

If the thin-airfoil conditions are respected, it is known from theory, cf. [9], that:

$\frac{\partial C_M}{\partial \alpha} = 2\pi$ and that $x_F = \frac{c}{4}$

Therefore, the following relation between the aerodynamic coefficients of lift and pitching moment is obtained:

$$\frac{\partial C_M}{\partial \alpha} = \left(\frac{1}{4} + a\right)\frac{\partial C_Z}{\partial \alpha} \qquad (2.5)$$

Where *a* is the dimensionless distance between the middle of the profile and the elastic center of the profile. In the case of a symmetrical profile, as in the case of a bridge deck, *a=0*.

b) <u>Notion of reduced frequency</u>

If now one considers that the profile is moving, in order to estimate if the coupling between the air flow and the profile is weak or strong, a dimensionless quantity is introduced: the reduced frequency which depends on the frequency of oscillation, the dimension of the studied profile and on the air speed.

$$f_r = \frac{{}^{c}/_{U}}{T_S}$$

Where f_r is the reduced frequency, *c* is a typical length in the problem (like the chord of the profile), *U* is the speed of the flow and T_S is the frequency of oscillation of the profile.

A high reduced frequency, $f_r \gg 1$, means that the structure configuration changes faster than the wind one: if a perturbation in the air flow is created at the leading edge of the profile, the profile motion will have substantially changed when the perturbation will arrive at the end of the profile. This kind of interaction may happen only for high frequencies of oscillation of the solid body. This field of study is called vibro-acoustics.

A reduced frequency close to one means a high coupling between air flow and structure motion. One may observe high motion amplitudes in air and in the structure. The behavior of both may become non-linear.

A low reduced frequency $f_r \ll 1$ means that the structure configuration changes more slowly than the wind one. The behavior of the air influences the structure motion. The air flow can be considered as steady, and the quasi-steady theory of aerodynamics may be used, with the static aerodynamic coefficients as introduced in the previous paragraph.

In the case of a bridge deck, a typical length is between 10 and 30 m, a typical velocity, *U* between 10 and 70 m/s and a frequency of oscillation around 1s. This corresponds to a reduced frequency in the range 0.14-3, which means that a strong coupling of the air and of the structure motion has to be considered.

Different kind of interactions between air and structure may happen in the case of a bridge deck subjected to wind:

- The Turbulence Induced Vibrations, also called TIV. The air turbulences (vertical or horizontal gusts) create the phenomenon of buffeting that may induce vibrations of the bridge deck.
- The Movement Induced Vibrations, also called MIV. When moving, the structure modifies the air flow around itself. By its motion, the bridge deck may create wind forces even superior and induce a phenomenon of resonance, with high amplitudes of oscillation of the bridge deck.
- The Vortex Induced Vibrations, also called VIV. In this case, the forces are due only to the air flow and not the bridge deck motion. A vortex trail appears behind the structure, at a given frequency, like in Figure 8. This vortex trail is not too dangerous if the frequency of the vortex trail does not correspond to a natural frequency of oscillation of the bridge deck. If it is so, the structure and the vortex trail may create a phenomenon of resonance, with high amplitudes of oscillation of the structure.

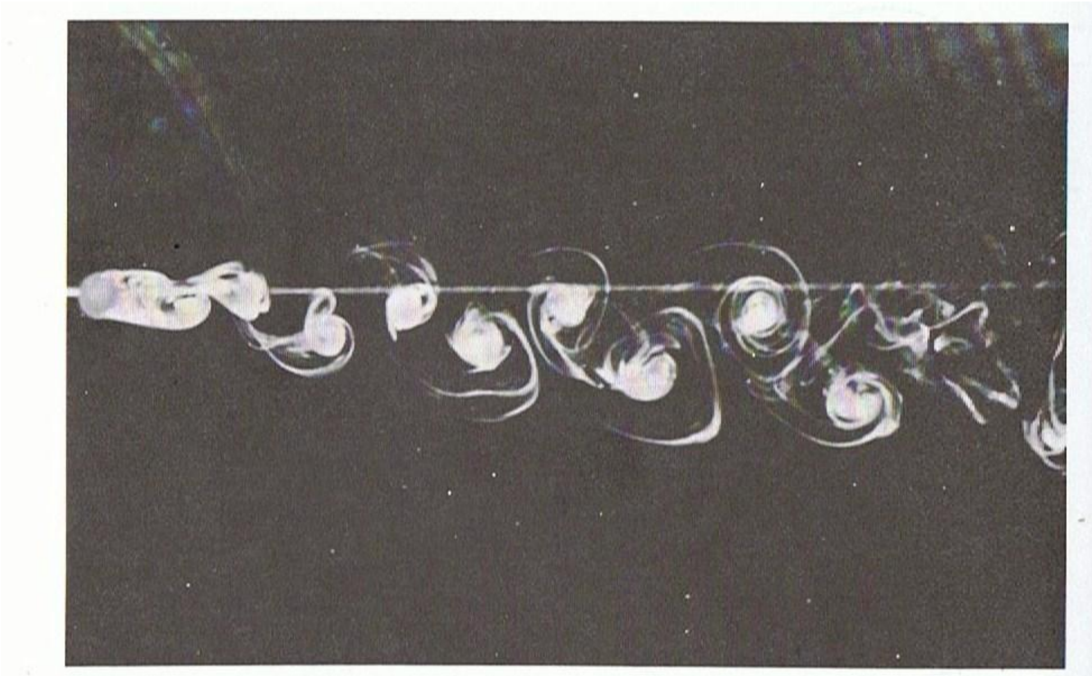

Figure 8: Vortex trail in water tunnel behind a circular cylinder. Figure from [10]

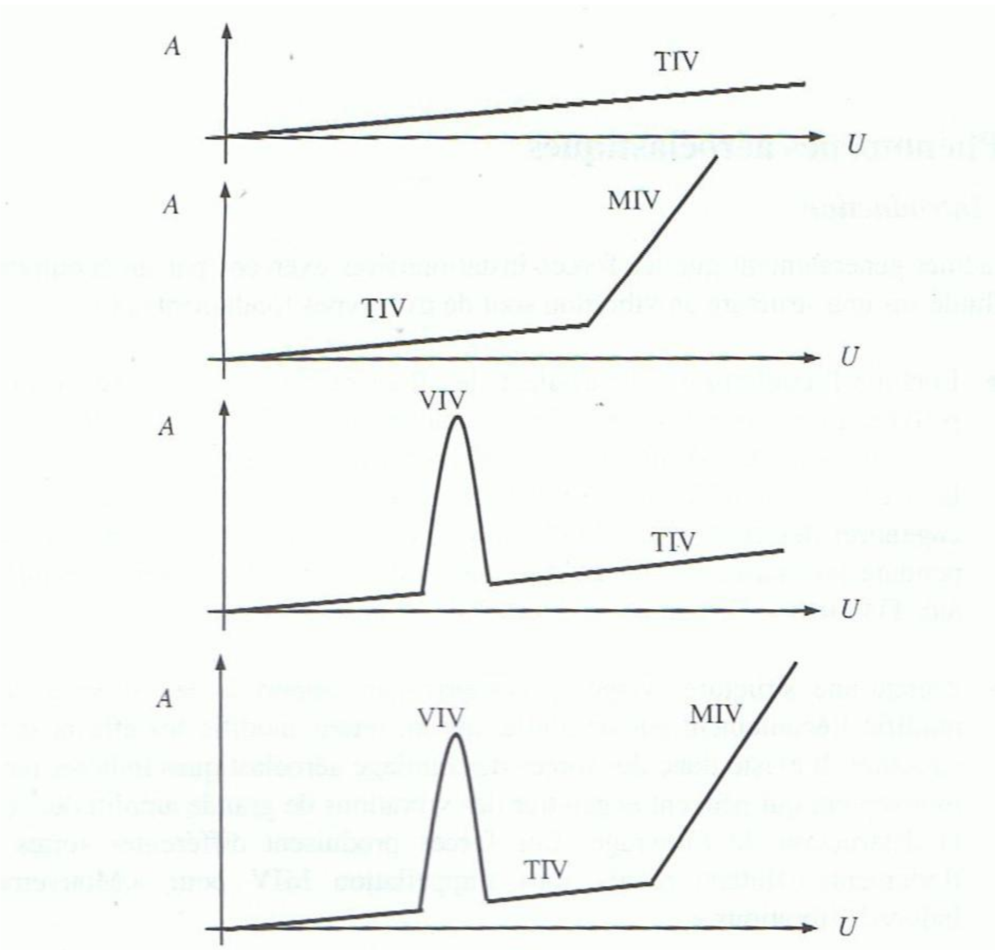

Figure 9: Different kind of behaviors for the law amplitude of oscillation as a function of the wind speed, for different profiles. Figure from [7]

In Figure 9, a reassuming of the different behaviors of the bridge decks that may be observed is exposed. All of the three kinds of interactions will not exist for all the bridges. For the wind speeds they are subjected to, some bridge decks, will never encounter the VIV, for example.

In practice, bridge decks are designed in order to try to stay in the case represented by the first graph in Figure 9. To do so, it must be verified that the MIV would appear only for wind speeds that are not reached on the site and that the mechanical frequencies and the vortex frequencies do not correspond, to avoid the VIV.

III) Wind forces in the frequency-domain: flutter derivatives

In the previous chapter, the steady-state aerodynamic theory has been introduced and the limits of the steady-state approximation for an air flow on a moving structure have been detailed thanks to the reduced frequency. For a bridge deck, such a hypothesis is not valid; the coupling of air and structure has to be studied by taking into account the motion of the structure and not only its present angle of attack. The flutter derivatives are functions that link the motion of the structure with the induced wind forces, assuming that the forces are linear with respect to the structure motion parameters.

Two approaches are possible at this point. The first one is the quasi-steady theory under which the flutter derivatives are constants. In the quasi-steady theory, the flutter derivatives are calculated using the static aerodynamic coefficients, considering only the present motion parameters (position and velocity) of the bridge deck. In the second one, the coupling that is considered is more complex, so, the wind forces at the present time depend on the present motion of the bridge deck but also on its time-history motion. In this case, a way to link the complete motion of the bridge deck with the induced forces is to evaluate the forces due to forced harmonic oscillations of the bridge deck. In the second case, the flutter derivatives are functions of the frequency of the structure oscillation. If the flow respects the Kutta condition, the expression of the flutter derivatives can be found analytically with the Fung theory. If it does not, the expression has to be found experimentally as done by Scanlan.

a) Steady forces and forced linearization of the equation

In a steady flow, as seen in II.a), the wind forces and moment applied on a profile by a constant and horizontal air flow are:

$$\begin{cases} F_x = \dfrac{1}{2}\rho c U^2 C_x(\alpha) \\ F_z = \dfrac{1}{2}\rho c U^2 C_z(\alpha) \\ M_O = \dfrac{1}{2}\rho c^2 U^2 C_M(\alpha) \end{cases} \qquad (3.1)$$

Where ρ is the density of air, U is the speed of the steady flow, c is a typical length of the structure (A special care has to be taken in order to always consider the same length, like, for example, the width of the bridge deck) and $C_x(\alpha)$, $C_z(\alpha)$ and $C_M(\alpha)$ are the dimensionless static aerodynamic coefficients of drag, lift and pitching-moment respectively.

The wind forces are assumed to be functions of only two parameters of the bridge deck: the vertical motion, z, and the rotation, α, also called angle of attack. The horizontal motion of the bridge deck, *x*, is neglected as it has a much smaller influence than the vertical motion and the angle of attack. Having a horizontal motion of the bridge deck is like increasing or reducing the incident horizontal wind velocity on the bridge deck:

$$U_{bridge} = U - \dot{x} \qquad (3.2)$$

Where U_{bridge} is the incident wind velocity as felt by the bridge deck, *U* is the absolute incident wind velocity and $\dot{x}$ is the horizontal bridge velocity. In practice, as $|\dot{x}| \ll U$, the influence of $\dot{x}$ is neglected and it is considered that $U_{bridge} = U$. Moreover, the stiffness of a bridge deck is much bigger in the horizontal direction and so its displacements are smaller than in the vertical direction.

The drag is often not considered either in the flutter derivatives theories applied to bridges. It is evaluated with the static aerodynamic coefficients. This difference of treatment is due to the fact that the drag does not create phenomenon of resonance whereas the lift or the pitching moment may resonate with the torsional and heaving modes of the bridge deck. So, one does not look for the same sensitivity for the interaction between drag and mechanical behavior of the bridge. As no resonance phenomenon is encountered for the drag, a static evaluation of the wind forces is enough to size the bridge deck for horizontal wind forces.

In the flutter derivatives theory, the wind forces linearity with respect to the structure motion is assumed, so one may write that:

$$\begin{cases} F_z = \dfrac{1}{2}\rho c U^2 (H_1 \dot{z} + H_2 \dot{\alpha} + H_3 z + H_4 \alpha) \\ M_O = \dfrac{1}{2}\rho c^2 U^2 (A_1 \dot{z} + A_2 \dot{\alpha} + A_3 z + A_4 \alpha) \end{cases} \qquad (3.3)$$

Where H_i and A_i are the flutter derivatives. They link the lift, F_z and the pitching-moment, M_O, with the motion parameters of the bridge deck, α, the angle of attack, $\dot{\alpha}$, the velocity of the angle of attack, z, the vertical displacement of the bridge deck and $\dot{z}$, its vertical velocity. With $\dot{ }$ is indicated a time derivation.

One may see this approximation as a Taylor series approximation, as the functions are linearized around the equilibrium position. The equilibrium position corresponds to the position at rest. So, the linearization is made with respect to the four motions parameters at $z = 0$, $\alpha = 0$, $\dot{z} = 0$ and $\dot{\alpha} = 0$ (cf. [7]):

$$\begin{cases} F_z = F_z(\alpha, z, \dot{\alpha}, \dot{z}) = \left.\dfrac{\partial F_z}{\partial \dot{z}}\right|_{\dot{z}=0} \dot{z} + \left.\dfrac{\partial F_z}{\partial \dot{\alpha}}\right|_{\dot{\alpha}=0} \dot{\alpha} + \left.\dfrac{\partial F_z}{\partial z}\right|_{z=0} z + \left.\dfrac{\partial F_z}{\partial \alpha}\right|_{\alpha=0} \alpha \\ M_O = M_0(\alpha, z, \dot{\alpha}, \dot{z}) = \left.\dfrac{\partial M_0}{\partial \dot{z}}\right|_{\dot{z}=0} \dot{z} + \left.\dfrac{\partial M_0}{\partial \dot{\alpha}}\right|_{\dot{\alpha}=0} \dot{\alpha} + \left.\dfrac{\partial M_0}{\partial z}\right|_{z=0} z + \left.\dfrac{\partial M_0}{\partial \alpha}\right|_{\alpha=0} \alpha \end{cases} \qquad (3.4)$$

Therefore, there is a correspondence between the flutter derivatives in equation (3.3) and the derivates of the wind forces with respect to the motion parameters in equation (3.4). For example, considering the lift:

$$H_1 = \frac{\partial F_z}{\partial \dot{z}}\bigg|_{\dot{z}=0} \quad H_2 = \frac{\partial F_z}{\partial \dot{\alpha}}\bigg|_{\dot{\alpha}=0} \quad H_3 = \frac{\partial F_z}{\partial z}\bigg|_{z=0} \quad H_4 = \frac{\partial F_z}{\partial \alpha}\bigg|_{\alpha=0} \qquad (3.5)$$

b) <u>An example of the quasi-steady theory: forces on a galloping cylinder with flutter derivatives for a low reduced frequency</u>

In order to illustrate the quasi-steady theory, a practical case will be analyzed: the wind forces on a cable of diameter *D*. The quasi-steady theory is much more appropriate for a cable than for a bridge deck, as for the same frequency of oscillation of the cable and the same mean velocity of the wind, the reduced frequency is lower, because the typical dimension is smaller. For a bridge deck, the typical dimension is in the range 10-30m, for a cable, it is around 20 cm. So, the reduced frequency, $f_r = D/UT_S$, is 100 times smaller and can be considered as much smaller than 1.

The hypotheses considered are similar to the ones adopted in [4]. The quasi-steady theory for a galloping cylinder is also developed in [7] and [10]. The hypotheses are:

- The incident wind has a constant speed
- The cylinder moves in the *(Oxz)* plane with a speed $\underline{u} = (\dot{x}, \dot{y})$
- Quasi-steady approach

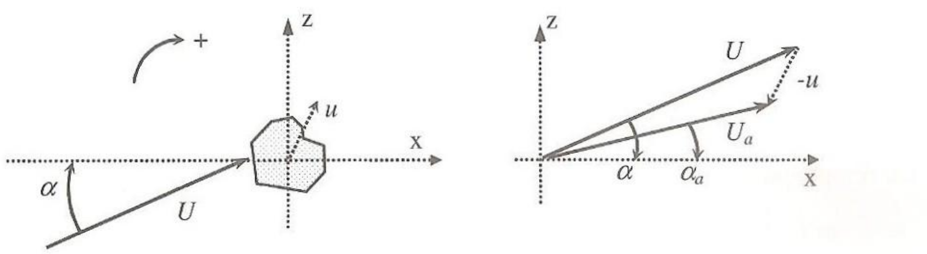

Figure 10: Galloping cylinder in translation and corresponding speeds triangle. Figure from [10]

Therefore, the wind speed with respect to the center of the cylinder is $\underline{U_a} = Ue_x - \underline{u}$:

$$\underline{U_a} = \begin{pmatrix} Ucos(\alpha) - \dot{x} \\ Usin(\alpha) - \dot{z} \end{pmatrix} \qquad (3.6)$$

So:

$$\alpha_a = arctg\left(\frac{Usin(\alpha) - \dot{z}}{Ucos(\alpha) - \dot{x}}\right) \qquad (3.7)$$

The wind forces considered are the static ones:

$$\begin{cases} F_x = \frac{1}{2}\rho D {U_a}^2 C_x(\alpha_a) \\ F_z = \frac{1}{2}\rho D {U_a}^2 C_z(\alpha_a) \end{cases} \qquad (3.8)$$

Where α_a is the angle of attack as "seen" by the cylinder, as defined in Figure 10. Whereas α is the "classic" angle of attack, defined as the angle between the mean wind speed and the horizontal axis of the cylinder. So, the motion of the cylinder, $\underline{u}$, is added in the calculation of the angle α_a.

The forces F_x and F_y depend on U_a and α_a that depend both on $(\dot{x}, \dot{y})$ which are the degrees of freedom considered in the problem. One wants to linearize the forces with respect to the degrees of freedom. The horizontal force is assumed to depend only on the horizontal speed component and the vertical one to depend only on the vertical speed. Therefore, it comes the following expression for F_x and F_y:

$$\begin{cases} F_x = \frac{1}{2}\rho D U^2 P \dot{x} = \left.\frac{\partial F_x}{\partial \dot{x}}\right|_{\dot{x}=0} \dot{x} \\ F_z = \frac{1}{2}\rho D U^2 H \dot{z} = \left.\frac{\partial F_z}{\partial \dot{z}}\right|_{\dot{z}=0} \dot{z} \end{cases} \qquad (3.9)$$

where *P* and *H* are the flutter derivatives in the quasi-steady theory.

Deriving the forces F_x and F_y respectively with respect to $\dot{x}$ and $\dot{z}$, it comes:

$$U^2 P = \left(\frac{\partial {U_a}^2}{\partial \dot{x}} C_x(\alpha_a) + {U_a}^2 \frac{\partial C_x(\alpha_a)}{\partial \alpha_a}\frac{\partial \alpha_a}{\partial \dot{x}}\right)_{\dot{x}=0} \qquad (3.10)$$

$$U^2 H = \left(\frac{\partial {U_a}^2}{\partial \dot{z}} C_z(\alpha_a) + {U_a}^2 \frac{\partial C_z(\alpha_a)}{\partial \alpha_a}\frac{\partial \alpha_a}{\partial \dot{z}}\right)_{\dot{z}=0} \qquad (3.11)$$

After calculations of the derivatives:

$$P = \frac{1}{U}\left(-2\cos(\alpha)\, C_x(\alpha) + \sin(\alpha) {C_x}'(\alpha)\right) \qquad (3.12)$$

$$H = \frac{1}{U}\left(-2\sin(\alpha)\, C_z(\alpha) - \cos(\alpha) {C_z}'(\alpha)\right) \qquad (3.13)$$

An expression of the flutter derivatives depending only on the static aerodynamic coefficients of the profile and on the instantaneous value of the angle of attack has been obtained.

This procedure is similar to the one used in [4]. Indeed, one could also derive the forces with respect to the turbulences of the wind (*u, w*), considering them in the evaluation of ${U_a}^2$ and α_a, by defining the wind speed as:

$$\underline{U} = U\underline{e}_x + u\underline{e}_x + w\underline{e}_z$$

Where *U* is the mean wind velocity and *u* and *w* are the horizontal and vertical air gusts.

Doing so, one would have found an expression of F_x and F_y under the following form:

$$\begin{cases} F_x = \dfrac{1}{2}\rho D U^2 P\dot{x} + \dfrac{1}{2}\rho D U^2 P_2 u = \left.\dfrac{\partial F_x}{\partial \dot{x}}\right|_{\dot{x}=0} \dot{x} + \left.\dfrac{\partial F_x}{\partial w_1}\right|_{w_1=0} u \\ F_z = \dfrac{1}{2}\rho D U^2 H\dot{z} + \dfrac{1}{2}\rho D U^2 H_2 w = \left.\dfrac{\partial F_z}{\partial \dot{z}}\right|_{\dot{z}=0} \dot{z} + \left.\dfrac{\partial F_z}{\partial w_2}\right|_{w_2=0} w \end{cases} \qquad (3.14)$$

still considering separately the horizontal and vertical forces and motion parameters.

The crossed components which express the dependency of the forces in a direction with respect to the motion in the other direction may also be added to this expression, like the dependency of F_x with respect to $\dot{z}$ and *w*, for example.

c) <u>Coupling of aerodynamic and mechanical behavior for a reduced frequency close to 1</u>

As seen in II.b), for a reduced frequency around 1, like for a bridge deck, the coupling between aerodynamic and mechanical behaviors requires a more complex formulation than the quasi-steady one to be modeled. Therefore, the aerodynamic forces have to be considered as depending on the time-history motion of the airfoil. A way to understand this dependency is to consider the wind forces as a function of the frequency of the bridge deck oscillation.

The expression of the linearized forces remains the same for forced harmonic oscillations if the problem respects the Kutta conditions, as demonstrated by Theodorsen in 1935 and detailed by Fung in [6]:

$$\begin{cases} F_z = \dfrac{1}{2}\rho c U^2 \left(H_1^{\#}(k)\dfrac{\dot{z}}{U} + H_2^{\#}(k)\dfrac{c\dot{\alpha}}{U} + H_3^{\#}(k)\alpha + H_4^{\#}(k)\dfrac{z}{c}\right) \\ M_O = \dfrac{1}{2}\rho c^2 U^2 \left(A_1^{\#}(k)\dfrac{\dot{z}}{U} + A_2^{\#}(k)\dfrac{c\dot{\alpha}}{U} + A_3^{\#}(k)\alpha + A_4^{\#}(k)\dfrac{z}{c}\right) \end{cases} \qquad (3.15)$$

$$\alpha = \alpha_0 e^{i\omega t}$$

$$z = z_0 e^{i\omega t}$$

This expression is valid only for harmonic oscillations of the bridge deck parameters, α, the angle of attack and z, the vertical displacement of the bridge deck. Now, the flutter derivatives which are named with an $^{\#}$, depend on the dimensionless pulsation of oscillation, $k = \omega b/U$, where b is the half-chord of the profile and ω is the pulsation of oscillation of the profile. The half-chord of a profile is half the distance between the leading edge and the trailing edge.

The flutter derivatives $H_3^{\#}$, $H_4^{\#}$ and $A_3^{\#}$, $A_4^{\#}$ are the "in phase" components. The flutter derivatives $H_1^{\#}$, $H_2^{\#}$ and $A_1^{\#}$, $A_2^{\#}$ are the "quadrature" components. This means that the response to a bridge deck motion has a part which oscillates proportionally to the motion of the bridge deck and a part that oscillates with a difference of phase of $\pi/2$. If the motion is a cosine, the in phase component will be proportional to a cosine too and the quadrature component will be proportional to a sine. Therefore, the response (the wind forces) to an oscillation of the motion parameters of the bridge deck for a given frequency can also be defined by an amplitude and a phase differences. Cf. VII.a) for the steps required to pass from the flutter derivatives to the expression of the amplitude and the phase difference.

The exact expression of these flutter derivatives can be found with complicated calculations, under the hypothesis of small perturbation of the flow and irrotationality of the flow (Kutta condition). This hypotheses allow to do a potential approach for solving the aerodynamic equation (1.17) that has been found in I.c). These calculations are done in [6] by Fung assuming harmonic oscillations of the airfoil and using the Theodorsen function $C(k)$:

$$C(k) = F(k) + iG(k) = \frac{K_1(ik)}{K_0(ik) + K_1(ik)} \qquad (3.16)$$

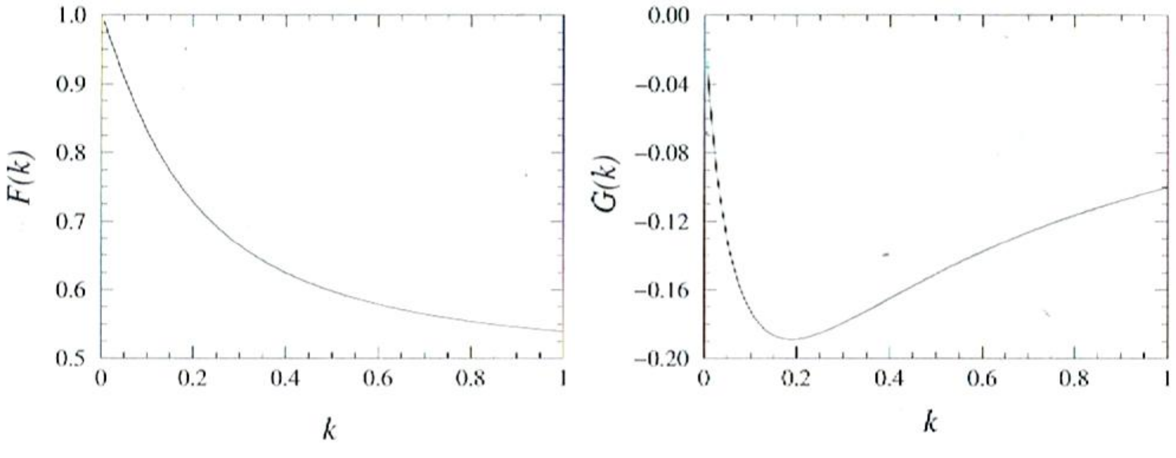

Figure 11: Real part F(k) and imaginary part G(k) of the Theodorsen function. Figure from [7]

$K_0(ik)$ and $K_1(ik)$ are the modified Bessel functions of order 0 and 1. They appear in the Fung's formulas because of the calculation of integrals in the resolution of the linear equation.

Fung finds out the following expressions of the flutter derivatives:

$$H_1^{\#} = -C_z' F$$

$$H_2^{\#} = C_z' \left[\left(\frac{1}{4} - a\right) F + \frac{G}{K}\right] + \frac{C_z'}{4}$$

$$H_3^{\#} = C_z' \left[F - KG\left(\frac{1}{4} - a\right)\right] + C_z' K^2 \frac{a}{4}$$

$$H_4^{\#} = C_z' KG + C_z' \frac{K^2}{4}$$

$$A_1^{\#} = -C_M' F$$

$$A_2^{\#} = C_M' \left[\left(\frac{1}{4} - a\right) F + \frac{G}{K}\right] + \frac{C_z'}{4}(a - \frac{1}{4})$$

$$A_3^{\#} = C_M' \left[F - KG\left(\frac{1}{4} - a\right)\right] + C_z' K^2 \frac{a^2}{4} + C_z' \frac{K^2}{128}$$

$$A_4^{\#} = -C_M' KG - C_z' a \frac{K^2}{4}$$

$$(3.17)$$

Where C_z' and C_M' are respectively the derivative of the static aerodynamic coefficients for the lift and the pitching moment, evaluated in *α=0*, the position at rest. *a* is the distance between the middle of the profile and the elastic center of the profile. It is worth 0 for a symmetrical profile, such as a bridge deck.

If the thin airfoil conditions are not respected, one may still assume a linear behavior of the aerodynamic forces, as done by Scanlan (1971), [10]. So the same form for the expression of the lift and the pitching moment than in equation (3.15) is kept:

$$\begin{cases} F_z = \dfrac{1}{2}\rho B U^2 (K H_1^* \dfrac{\dot{z}}{U} + K H_2^* \dfrac{B\dot{\alpha}}{U} + K^2 H_3^* \alpha + K^2 H_4^* \dfrac{z}{B}) \\ M_O = \dfrac{1}{2}\rho B^2 U^2 (K A_1^* \dfrac{\dot{z}}{U} + K A_2^* \dfrac{B\dot{\alpha}}{U} + K^2 A_3^* \alpha + K^2 A_4^* \dfrac{z}{B}) \end{cases} \quad (3.18)$$

but the flutter derivatives written with an * have to be determined by an experimental way and cannot be found with the Fung's approach. They are found applying a harmonic motion to the airfoil in each degree of freedom, under a constant horizontal air flow of speed *U*.

d) Determination of bridge deck aerodynamic coefficients

The determination of bridge deck aerodynamics coefficients can be done in two ways: an experimental way, using a wind tunnel or a computational way, using a CFD (Computational Fluid Dynamic) software. The accuracy of the evaluation of the flutter derivatives and of the static aerodynamic coefficients is determinant for the quality of the analysis of the wind effects on structures.

1) Wind tunnels experiments

The most common way for the determination of the aerodynamic coefficients is the wind tunnel analysis. This analysis requires the construction of a reduced geometric scale model which is subjected to an air flow in a wind tunnel. In order to find the true wind forces on the bridge deck even if one works on a model, some characteristics of the air flow and bridge deck should be kept similar to reality. The physical similitude needed for a proper correspondence between the reality and the experimentation results is called the "similarity" criteria. These criteria are based on a dimensional analysis of the wind forces.

The principle of the dimensional analysis for the similarity criteria applied to wind tunnel experiments is more detailed in [1] and [10].

From the Navier-Stokes equations, it is known that the wind forces depend on 6 parameters, ρ, the air density, *D*, a typical dimension of the bridge deck, *U*, the flow velocity, *n*, some frequency, μ, the fluid viscosity and *g*, the gravitational acceleration. *F*, the wind force is assumed to be dimensionally related to all of the 6 parameters through the following dimensional relation:

$$F \triangleq \rho^{\alpha} U^{\beta} D^{\gamma} n^{\delta} \mu^{\varepsilon} g^{\xi} \quad (3.19)$$

Where the symbol $\triangleq$ means "dimensionally consistent". α, β, γ, δ, ε, ξ are exponents to be determined. All these 6 parameters are dimensionally functions of only 3 basic quantities: mass, *M*, length, *L* and time, *T*. So the equation (3.19) becomes:

$$\frac{ML}{T^2} \triangleq \left(\frac{M}{L^3}\right)^{\alpha} \left(\frac{L}{T}\right)^{\beta} (L)^{\gamma} \left(\frac{1}{T}\right)^{\delta} \left(\frac{M}{LT}\right)^{\varepsilon} \left(\frac{L}{T^2}\right)^{\xi} \qquad (3.20)$$

From this dimensional relation, by equating corresponding exponents for mass, length and time:

$$1 = \alpha + \varepsilon$$

$$1 = -3\alpha + \beta + \gamma - \varepsilon + \xi$$

$$-2 = -\beta - \delta - \varepsilon - 2\xi$$

Considering some coefficients as functions of others

$$\alpha = 1 - \varepsilon$$

$$\beta = 2 - \varepsilon - \delta - 2\xi$$

$$\gamma = 2 + \delta - \varepsilon + \xi$$

The force *F* can be dimensionally expressed as:

$$F \triangleq \rho^{1-\varepsilon} U^{2-\varepsilon-\delta-2\xi} D^{2+\delta-\varepsilon+\xi} n^{\delta} \mu^{\varepsilon} g^{\xi} \qquad (3.21)$$

or

$$F \triangleq \rho U^2 D^2 \left(\frac{Dn}{U}\right)^{\delta} \left(\frac{\mu}{\rho U D}\right)^{\varepsilon} \left(\frac{Dg}{U^2}\right)^{\xi} \qquad (3.22)$$

From this analysis, it follows that the dimensionless force $F/\rho U^2 D^2$ is a function of the following dimensionless numbers:

- Dn/U, also known as the Strouhal number if *n* corresponds to the frequency of vortex shedding from a bluff objective. If *n* corresponds to n_m, a mechanical frequency, Dn_m/U is called the reduced frequency.
- $\mu/\rho UD$ is the reciprocal of the Reynolds number.
- Dg/U is the Froude number

This analysis helps understand which coefficients are important to be considered when passing from a reduced geometry scale model to a full-scale model.

If one wants the aerodynamic behavior of the reduced scale geometry to correspond exactly to the reality of the bridge deck, the three dimensionless numbers should have the same value in both cases. Passing from the model to the real bridge deck, three scale factors appear: a length scale (geometric scale of the model), a velocity scale (ratio real wind speed over speed of the air flow in the tunnel) and a density scale (ratio air over the liquid or gas used in the tunnel). For example, if one uses a wind tunnel working with air in atmospheric conditions, the density scale will be worth one and, necessarily, one of the three dimensionless numbers will not be equal in both cases. Therefore, it is often not possible to have the three dimensionless numbers equal in both cases (reality and wind tunnel) as only one or two scale factors can be adjusted. Anyway, the experience of the laboratory engineers is fundamental to control the suitability of the experiment.

2) Computational fluid dynamic

This paragraph is inspired by a document from the INRIA [11] and an interview with Mr. Frédéric Bourquin [12].

Another way to determine the aerodynamic characteristics of a profile is to develop a model of the air flow around the profile with a CFD (Computational fluid dynamic) software. Once known the air flow around the model, the pressure around the profile can be integrated and the air forces on the profile evaluated. One can do static or dynamic model, by moving or not the profile.

The CFD softwares solve numerically the Navier-stokes equations in two dimensions, as the studied profiles are assumed to be infinitely extended. Even if solving numerically the Navier-Stokes equations is quite heavy in term of calculation and time requested, some interesting results can be obtained with this tool. The INRIA (Institut National de Recherche en Informatique et en Automatique) has tried to develop it with simple profile such as cylinders (Figure 12) or rectangular prisms (Figure 13).

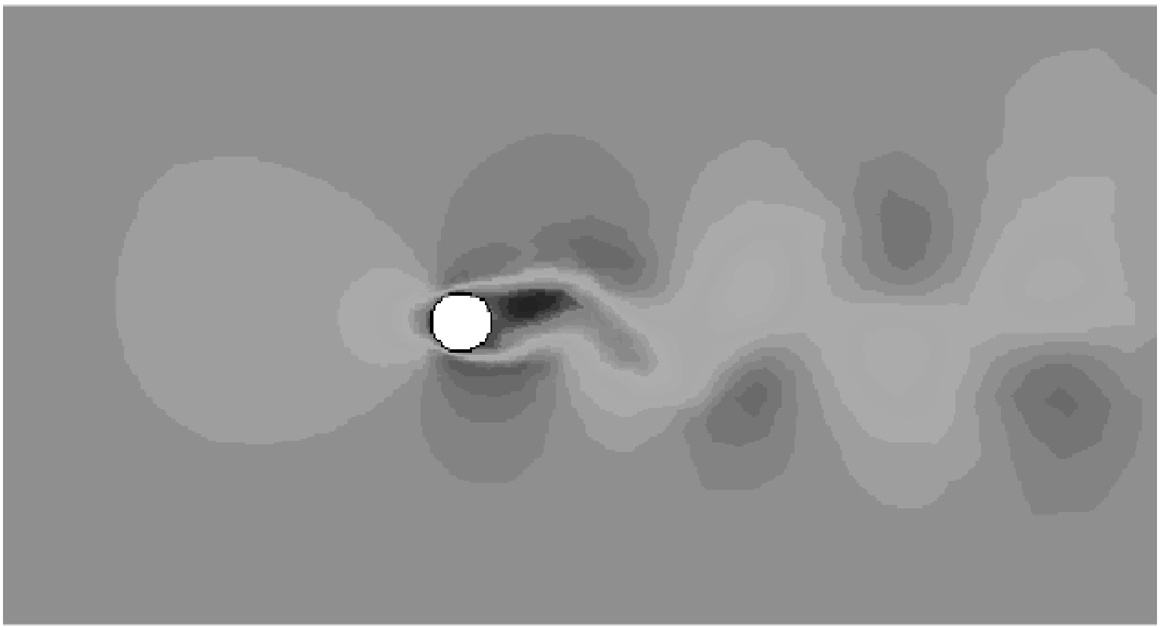

Figure 12: Periodic vortex shedding created by a cylinder in an incompressible air flow after a transient phase.

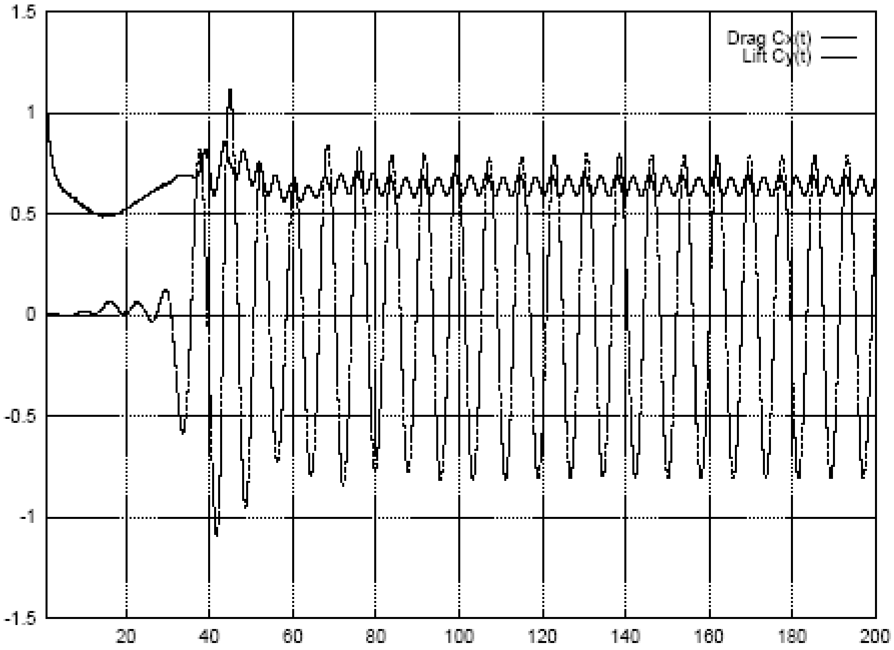

Figure 13: Drag and lift of a rectangular profile oscillating harmonically in an incompressible air flow. After a transient time, it can be noticed that the drag and lift oscillate harmonically, with the same period as the rectangular profile.

- Conclusion :

In this chapter, two different approaches to the flutter derivatives have been exposed: the quasi-steady one, in which the flutter derivatives depend on the static aerodynamic coefficients and the frequency-dependency one, in which the flutter derivatives depend on the frequency of oscillation of the bridge deck. The quasi-steady theory has been applied to a cable to understand how the wind forces could be linearized to define the flutter derivatives. The equations for the flutter derivatives as a function of the frequency of oscillation have been found by Thedorsen in the 30's; in the thin-airfoil conditions, for a harmonic oscillation of the airfoil, the flutter derivatives link the wind forces and the airfoil motion linearly, by creating amplitude and phase differences between the forced oscillation and the wind forces. These are the flutter derivatives named with an $^{\#}$ which can be found theoretically. For a bridge deck which cannot be considered as thin, Scanlan has made the hypothesis that the wind forces are still linked to the bridge deck motion by the same analytical expression as the Theodorsen's one. But the flutter derivatives are named with an * and should be found experimentally. Finally, in paragraph III.d), experimental and numerical ways to evaluate the aerodynamic coefficients have been outlined.

IV) Wind forces in the time-domain: indicial functions

In the previous chapter, an approach to the wind forces in the frequency-domain with the flutter derivatives has been studied. Since purpose of the work is to make a time-domain modelling of the wind force, in the following chapter, the notion of indicial functions will be introduced. These functions link the wind forces with the time-history motion of profiles, like the flutter derivatives link the wind forces with the frequency of oscillation of profiles.

In paragraph a), an infinitesimal approach to the indicial functions will be done. It helps understand and define what an indicial function is. In paragraphs b), c) and d), the lift as a function of the vertical motion of a bridge deck will be approached with indicial functions by three different ways. The first approach in paragraph b) is inspired by the article of Scanlan [13]; the definition of indicial functions on the basis of the flutter derivatives will be given. After having defined the Fourier transform (cf. 1st Appendix), in paragraph c), the wind forces will be defined as a convolution product between the indicial functions and the time-history motion of the bridge deck. In paragraph d), a final approach to the indicial functions will be made with the Duhamel's integral. In this approach, it will be clarified under which hypotheses the theory of the indicial functions can be considered true. In paragraph e), a reassuming of the three ways used to define the indicial functions and a comparison of the approaches are made. Finally, in paragraph f), the complete wind forces will be expressed with the Fourier transform: the lift and the pitching-moment as a function of both the angle of attack and the vertical motion of the bridge deck.

a) Infinitesimal approach

With the flutter derivatives, the fact that wind forces do not depend only on the instantaneous parameters of the airfoil motion anymore has already been considered. In the case of a single degree of freedom (the angle of attack of the airfoil, for example), the lift acting on the airfoil is the sum of contributes given by the previous angles. In the frequency-dependent flutter derivatives theory, the forces at a time t can be obtained only if the angle of attack is oscillating harmonically. In this chapter, the angle of attack time-history is considered random.

Going backwards to the steady approach, the lift is equal to:

$$L(\alpha) = \frac{1}{2}\rho U^2 B C_L(\alpha) \qquad (4.a.1)$$

This corresponds with an airfoil motion where $|\dot{\alpha}| \ll 1$, which means that the airfoil is moving slowly. This allows making a quasi-steady development. It is also assumed that the variations of the angle of attack of the bridge deck are in the range ±5° and so that the derivate of the static aerodynamic coefficients in α=0 are constant, as made in II.a). Therefore, a Maclaurin expansion of the static aerodynamic coefficient can be made for the lift that allows transforming the lift as a function of the angle of attack into a Taylor series limited to the first term:

$$L(\alpha) = \frac{1}{2}\rho U^2 B(C_L(0) + C_L'\alpha) \qquad (4.a.2)$$

Where C_L' is a constant:

$$C_L' = \left.\frac{dC_L}{d\alpha}\right|_{\alpha=0}$$

Doing this Maclaurin expansion is like assuming that the lift as a function of the angle of attack is a linear function.

Now, the airfoil is considered moving too fast and so, the quasi-steady theory is not valid anymore. For example, if one applies an abrupt step-function change from a zero lift condition to an incremental angle of attack worth α_0 at the airfoil at the time $t=0$, the lift will undergo a transient change (cf. Figure 14 for the pitching-moment) because the flow needs a certain time before it can be considered as steady again. The vortexes created at the edges because of the abrupt change of angle of attack need to flow along the profile before reaching stabilization of the flow again (cf. Figure 15). In this non-steady case, a certain linearity of the lift with respect to the angle of attack may still be considered, by modelling the transient phase by the following equation:

$$L(s) = \frac{1}{2}\rho U^2 B C_L' \alpha_0 \varphi(s) \qquad (4.a.3)$$

Where $s=2Ut/B$ is the dimensionless time as defined by Theodorsen and $\varphi(s)$ is an indicial lift-growth function. Now, the linearity is between the amplitude of the angle of attack, α_0, and the transient response to it. In the thin-airfoil hypotheses, in 1925, Wagner has proved that equation (4.a.3) is valid and has also found the theoretical expression of the indicial function $\varphi(s)$ (cf. Figure 16 for the shape of the indicial function). The indicial function as defined by Wagner is consistent with the quasi-steady approach, as for s tending to ∞, $\varphi(s)$ tends to *1*. So, for an infinite time value which corresponds to a stabilization of the flow, the lift is worth:

$$L(s) = \frac{1}{2}\rho U^2 B C_L' \alpha_0$$

as in equation (4.a.2).

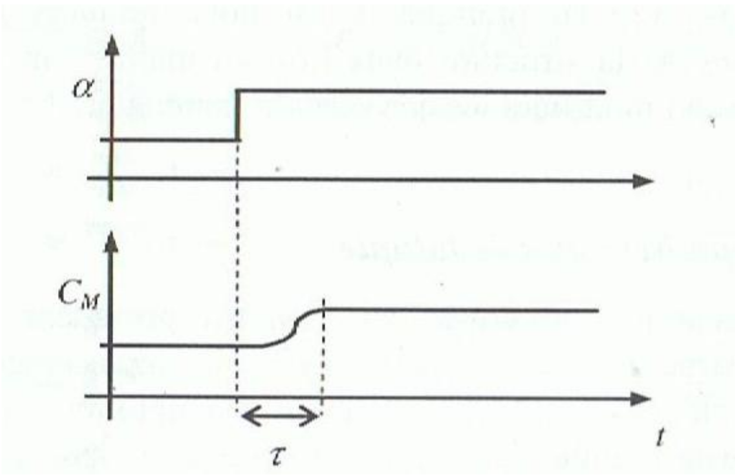

Figure 14: Highlighting of the phase-shift between the motion history of the angle of attack and the pitching-moment represented by the static aerodynamic coefficient of pitching-moment. Figure from [7]

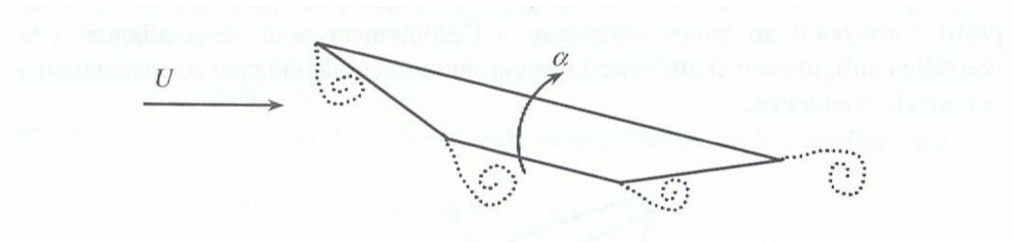

Figure 15: Highlighting of the existence of a stabilization time of the flow after a change of the angle of attack. The vortexes created by the rotation must flow along the bridge deck before the flow can be considered as steady again. Figure from [7]

If the time-history of the angle of attack is more complex than just a step function, this history can be decomposed into a sum of step functions:

$$\alpha(s) = \int_{-\infty}^{s} \underline{1_\sigma}\, d\alpha(\sigma) = \int_{-\infty}^{s} \underline{1_\sigma} \frac{d\alpha}{d\sigma} d\sigma \qquad (4.a.4)$$

Where $\underline{1_\sigma}(s)$ is a step function, such as:

$$\begin{cases} \underline{1_\sigma}(s) = 1 \quad if\ \sigma < s \\ \underline{1_\sigma}(s) = 0 \quad if\ \sigma > s \end{cases}$$

If the system respects the linear superposition principle, the lift at the present time can be expressed as a function of the time-history of the angle of attack, using equation (4.a.3) and the decomposition of the time-history of the angle of attack (4.a.4):

$$L(s) = \frac{1}{2}\rho U^2 B C_L' \int_{-\infty}^{s} \varphi(s-\sigma)\alpha'(\sigma)d\sigma \qquad (4.a.5)$$

Cf. pargraph IV.d), about the Duhamel integral applied to the indicial functions, for more explanations about the decomposition of the time-history of the angle of attack, the superposition principle and the underlying hypotheses.

Through integration by part, it comes:

$$L(s) = \frac{1}{2}\rho U^2 B C_L' \left[\varphi(0)\alpha(s) + \int_{-\infty}^{s} \varphi'(s-\sigma)\alpha(\sigma)d\sigma\right] \qquad (4.a.6)$$

And, with a change of variable, $\sigma_1 = s - \sigma$:

$$L(s) = \frac{1}{2}\rho U^2 B C_L' \left[\varphi(0)\alpha(s) + \int_{0}^{\infty} \varphi'(\sigma_1)\alpha(s-\sigma_1)d\sigma_1\right] \qquad (4.a.7)$$

This expression written only for the lift forces in function of the angle of attack may be also assumed for the moment, under a similar form.

Assuming that the forces are linear with respect to the vertical velocity of the airfoil, as well as they are linear with respect to the angle of attack, both lift and pitching moment can be expressed as function of the angle of attack and the vertical velocity in the same way.

In doing this reasoning, it is assumed that the system respects the superposition principle which is the condition for integrating the infinitesimal forces and assuming that the indicial functions are constants in time. These hypotheses have been proved to be true by Wagner in 1925, for the thin-airfoil condition. Even if the thin-airfoil condition is not respected, it is assumed that the expression of the lift can be written in the same way, with indicial functions. But new expressions of the indicial functions should be found, as the ones found by Wagner theoretically are no longer exact in the studied case:

$$L(s) = qBC_L' \left[\Phi_{L\alpha}(0)\alpha(s) + \Phi_{Lz}(0)\dot{z}(s) + \int_{-\infty}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma + \int_{-\infty}^{s} \Phi_{Lz}{}'(s-\sigma)\dot{z}(\sigma)d\sigma\right]$$

$$M(s) = qB^2C_M' \left[\Phi_{M\alpha}(0)\alpha(s) + \Phi_{Mz}(0)\dot{z}(s) + \int_{-\infty}^{s} \Phi_{M\alpha}{}'(s-\sigma)\alpha(\sigma)d\tau + \int_{-\infty}^{s} \Phi_{Mz}{}'(s-\tau)\dot{z}(\tau)d\tau\right]$$

$$(4.a.8)$$

Where Φ_i are the indicial response functions in the case of a bridge deck section which is not a thin-airfoil. They should be found experimentally. *$q=1/2.\rho U^2$* is the dynamic pressure. Equation (4.a.8) is also assumed by Costa and Borri in [14].

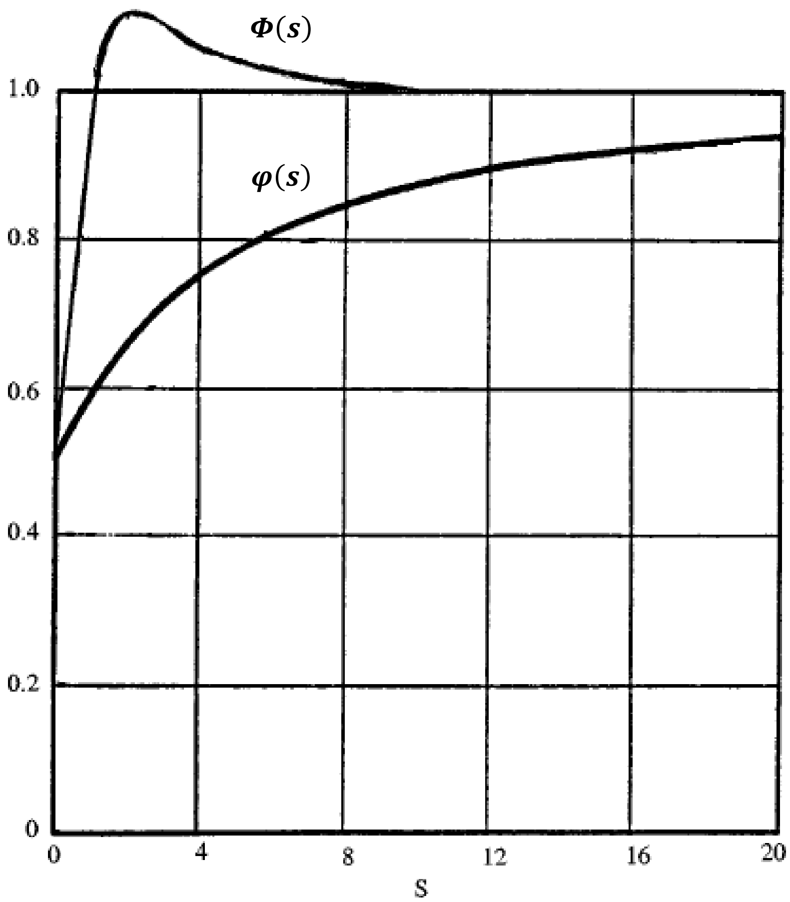

Figure 16: Indicial response functions: Wagner ($\boldsymbol{\varphi}$) and real function in the case of a bridge deck ($\boldsymbol{\Phi}$). Figure from 13

In the next paragraph, it will be proved that the analytical expressions of the indicial functions can be obtained from the flutter derivatives. But these analytical expressions are not easy to use in practice as they contain integrals that should be calculated each time it is required to evaluate the value of the functions for a given time. According to [14] and [5], in order to fasten the use of an indicial function, an empiric expression that approximates the indicial function can be used. This empiric function, as given by Scanlan, is a sum of exponential groups depending on five parameters:

$$\Phi_i(s) = c_{1,i} - c_{2,i}\exp(-d_{1,i}s) - c_{3,i}\exp(-d_{2,i}s) \qquad (4.a.9)$$

If the derivative of the indicial functions is need, one simply derives expression (4.a.9) and obtains:

$$\Phi_i'(s) = c_{2,i}d_{1,i}\exp\left(-d_{1,i}s\right) + c_{3,i}d_{2,i}\exp\left(-d_{2,i}s\right) \qquad (4.a.10)$$

The Wagner's indicial function, *φ(s)*, that gives the lift for an abrupt change of the value of the angle of attack has been approximated by R.T. Jones in 1940 under the form (4.a.10) with the following values for the 4 coefficients used in the derivative formula: $c_2 = 0.165$, $d_1 = 0.0455$, $c_3 = 0.335$ and $d_2 = 0.300$, cf. [13].

Depending on the complexity of the shape of the indicial function, one exponential group or, on the contrary, three or four groups, may be required to approximate an indicial function under the form of a sum of exponential groups in a proper way.

Going backwards to equation (4.a.8), it may be surprising that the wind forces need to be defined as an integral from $-\infty$ to the present time *s*. This would mean that the present forces are linked to the motion of the bridge deck even a long time before the present time. This may create a problem of convergence of the result, because an infinite time-history is needed to evaluate the true forces acting on the bridge deck at the present time. In fact, the shape of the indicial forces allows making a simplification on the integral. Considering the lift as a function of the angle of attack, one can notice from Figure 16 that the indicial function, Φ, tends to 1 for *s* tending to ∞ and so, that the derivative of the indicial function, Φ', tends to 0 for *s* tending to ∞. Therefore, one may assume that after a certain time that will be called s_L, the integration time, the indicial function may be approximated by the constant 1 and its derivative by the constant 0. Therefore, equation (4.a.8) may be re-written as:

$$L(s) = qBC_L'\left[\Phi_{L\alpha}(0)\alpha(s) + \int_{s-s_L}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma\right] \qquad (4.a.11)$$

If the integrated form of the lift is considered, as in equation (4.a.5),

$$L(s) = \frac{1}{2}\rho U^2 BC_L' \int_{-\infty}^{s} \varphi(s-\sigma)\alpha'(\sigma)d\sigma \qquad (4.a.12)$$

the same simplification on the integral may be made:

$$L(s) = \frac{1}{2}\rho U^2 BC_L'\left[\int_{-\infty}^{s} (\varphi(s-\sigma)-1)\alpha'(\sigma)d\sigma + \int_{-\infty}^{s} \alpha'(\sigma)d\sigma\right] \qquad (4.a.13)$$

$$L(s) = \frac{1}{2}\rho U^2 BC_L'\left[\int_{s-s_L}^{s} (\varphi(s-\sigma)-1)\alpha'(\sigma)d\sigma + \alpha(s)\right] \qquad (4.a.14)$$

as for an infinite time, the airfoil is assumed at rest.

According to [15], a correct integration time is $s_L = 10$. For such an integration time, the convolution integral is always convergent. The fading memory effect allows forgetting what happened in the time-history motion of the bridge deck before this integration time.

b) Finding the expression of indicial functions – Scanlan's approach

In this paragraph, an expression of the Fourier transform of indicial functions as a function of the flutter derivatives will be determined.

In the case of the study of the wind force as a function of the angle of attack and of the vertical motion of the airfoil, for harmonic oscillations, the following relation (from the Scanlan's flutter derivatives formula) can be used, as seen in paragraph III.c):

$$L(\omega,t) = \frac{1}{2}\rho B U^2 \left(K H_1^*(K)\frac{\dot{z}(\omega,t)}{U} + K H_2^*(K)\frac{B\dot{\alpha}(\omega,t)}{U} + K^2 H_3^*(K)\alpha(\omega,t) + K^2 H_4^*(K)\frac{z(\omega,t)}{B}\right)$$

$$M(\omega,t) = \frac{1}{2}\rho B^2 U^2 \left(K A_1^*(K)\frac{\dot{z}(\omega,t)}{U} + K A_2^*(K)\frac{B\dot{\alpha}(\omega,t)}{U} + K^2 A_3^*(K)\alpha(\omega,t) + K^2 A_4^*(K)\frac{z(\omega,t)}{B}\right)$$

$$(4.b.1)$$

Where *K=Bω/U* is the reduced velocity and *ω* is the pulsation of oscillation.

As the motion of the airfoil is harmonic, the expression may be re-formulated by noticing that:

$$\dot{z}(\omega,t) = i\omega z(\omega,t) \;\text{ as }\; z(t) = z_0 exp\,(i\omega t)$$

$$\dot{\alpha}(\omega,t) = i\omega \alpha(\omega,t) \;\text{ as }\; \alpha(t) = \alpha_0 exp\,(i\omega t)$$

$$(4.b.2)$$

Therefore, the lift can be expressed as a function only of the vertical motion of the bridge deck and the angle of attack:

$$L(\omega,t) = \frac{1}{2}\rho B U^2 \left(K H_1^*\frac{i\omega z}{U} + K H_2^*\frac{Bi\omega\alpha}{U} + K^2 H_3^*\alpha + K^2 H_4^*\frac{z}{B}\right) \qquad (4.b.3)$$

As *ω=UK/B*, the expression of the lift may be rearranged:

$$L(\omega,t) = \frac{1}{2}\rho B U^2 \left(K^2 H_1^*\frac{iz}{B} + K^2 H_2^* i\alpha + K^2 H_3^*\alpha + K^2 H_4^*\frac{z}{B}\right)$$

$$L(\omega,t) = \frac{1}{2}\rho B U^2 K^2 \left([iH_1^* + H_4^*]\frac{z}{B} + [H_3^* + iH_2^*]\alpha\right) \qquad (4.b.4)$$

By analogy, the pitching moment becomes:

$$M(\omega,t)=\frac{1}{2}\rho B^{2U^2}K^2\left([iA_1^*+A_4^*]\frac{z}{B}+[A_3^*+iA_2^*]\alpha\right)\qquad(4.b.5)$$

In paragraph IV.a), the expression of the lift and the moment as functions of the angle of attack and the vertical motion of the airfoil has been assumed by analogy with the thin-airfoil theory. It gives the wind forces for generic motions of the airfoil at the present dimensionless time, *s*:

$$L(s)=qBC_L'\left[\Phi_{L\alpha}(0)\alpha(s)+\Phi_{Lz}(0)\dot{z}(s)+\int_{-\infty}^{s}\Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma+\int_{-\infty}^{s}\Phi_{Lz}{}'(s-\sigma)\dot{z}(\sigma)d\sigma\right]$$

$$M(s)=qB^2C_M'\left[\Phi_{M\alpha}(0)\alpha(s)+\Phi_{Mz}(0)\dot{z}(s)+\int_{-\infty}^{s}\Phi_{M\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma+\int_{-\infty}^{s}\Phi_{Mz}{}'(s-\sigma)\dot{z}(\sigma)d\sigma\right]$$

$$(4.b.6)$$

With $\dot{}$, one indicates the derivative with respect to the time, t. With ', one indicates the derivative with respect to the dimensionless time, *s*.

At this point, two expressions of the wind forces have been obtained: one expression that is the Scanlan equation, valid only for harmonic oscillations of the bridge deck, equation (4.b.4-5), and one expression that has been assumed with the indicial functions, equation (4.b.6). If a harmonic motion of the bridge motion parameters is also assumed in the second equation, one will be able to equal between both expressions of the wind forces as functions of the motion parameters of the bridge deck.

To simplify the understanding on how the expression of the indicial functions can be found thanks to these two expressions of the wind forces, only the lift as the function of the vertical motion of the airfoil will be considered. So equation (4.b.4) becomes:

$$L(\omega,t)=\frac{1}{2}\rho BU^2K^2[iH_1^*+H_4^*]\frac{z}{B}\qquad(4.b.7)$$

Which can also be expressed as a function of $\dot{z}/_U$:

$$L(\omega,t)=\frac{1}{2}\rho BU^2K[H_1^*-iH_4^*]\frac{\dot{z}}{U}\qquad(4.b.8)$$

Under a dimensionless form, and so, as a function of $\dot{z}/_U$ and with a change of variable as done for equation (4.a.7), equation (4.b.6) becomes:

$$L(s)=qBC_L'\left[\Phi_{Lz}(0)\frac{\dot{z}(s)}{U}+\int_0^{\infty}\Phi_{Lz}{}'(\sigma)\frac{\dot{z}(s-\sigma)}{U}d\sigma\right]\qquad(4.b.9)$$

One can notice in Figure 16 that Φ is not continuous in 0. It is worth 0 in 0^{-} and 0.5 in 0^{+}. So, as Φ_{Lz} is discontinuous in 0, one must decide whether 0 means 0^{+} or 0^{-}.

If one decides that it means 0^{+}, the exact equation is, in fact:

$$L(s) = qBC_L'\left[\Phi_{Lz}(0^{+})\frac{\dot{z}(s)}{U} + \int_{0^{+}}^{\infty} \Phi_{Lz}{}'(\sigma)\frac{\dot{z}(s-\sigma)}{U}d\sigma\right] \qquad (4.b.10)$$

The arbitrary choice made about the meaning of σ worth *0* will be discussed later in this paragraph.

As said previously, in order to compare these two expressions, a harmonic motion of the vertical motion of the bridge deck is assumed, so:

$$\dot{z}(t) = U\alpha_0 e^{i\omega t} = U\alpha_0 e^{iks} \qquad (4.b.11)$$

Where ω is the pulsation of the harmonic motion, *k=Bω/2U=K/2* is the reduced velocity, *s=2Ut/B*, the dimensionless time and α_0 is the dimensionless amplitude of oscillation of the vertical speed motion of the bridge deck:

Therefore, equation (4.b.8) becomes:

$$L(k,s) = \frac{1}{2}\rho KBU^2\alpha_0(H_1^* - iH_4^*)e^{iks} \qquad (4.b.12)$$

Calculating the lift with the harmonic motion for $\dot{z}$, as defined in equation (4.b.11), equation (4.b.10) may be transformed:

$$L(s) = qBC_L'\left[\Phi_{Lz}(0^{+})\alpha_0 e^{iks} + \int_{0^{+}}^{\infty} \Phi_{Lz}{}'(\sigma)\alpha_0 e^{ik(s-\sigma)}d\sigma\right]$$

$$L(s) = \alpha_0 e^{iks} qBC_L'\left[\Phi_{Lz}(0^{+}) + \int_{0^{+}}^{\infty} \Phi_{Lz}{}'(\sigma)e^{-ik\sigma}d\sigma\right]$$

$$L(s) = \alpha_0 e^{iks} qBC_L'\left[\Phi_{Lz}(0^{+}) + \sqrt{2\pi}.\overline{\Phi_{Lz}{}'}\right] \qquad (4.b.13)$$

Where $\overline{\Phi_{Lz}{}'}$ is defined as: $\overline{\Phi_{Lz}{}'} = \frac{1}{\sqrt{2\pi}}\int_{0^{+}}^{\infty} \Phi_{Lz}{}'(\sigma)e^{-ik\sigma}d\sigma$. It is not the exact Fourier transform of the derivative of the indicial function, $\Phi_{Lz}{}'$, as the exact Fourier transform, $FO(\Phi_{Lz}')$, of $\Phi_{Lz}{}'$ would be:

$$FO(\Phi_{Lz}') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Phi_{Lz}{}'(\sigma)e^{-ik\sigma}d\sigma = \frac{1}{\sqrt{2\pi}}\int_{0^{-}}^{\infty} \Phi_{Lz}{}'(\sigma)e^{-ik\sigma}d\sigma \qquad (4.b.14)$$

Cf. 1st Appendix on the Fourier transform for the definition of the Fourier transform used in this work. From *-∞* to *0⁻*, Φ_{Lz} is worth *0*, so it s derivative is worth *0* too. The link between the exact Fourier transform of $\Phi_{Lz}{}'$ and $\overline{\Phi_{Lz}{}'}$ is just an additive constant:

$$FO(\Phi'_{Lz}) = \frac{1}{\sqrt{2\pi}}\int_{0^+}^{\infty}\Phi_{Lz}{}'(\sigma)e^{-ik\sigma}d\sigma + \frac{1}{\sqrt{2\pi}}\int_{0^-}^{0^+}\Phi_{Lz}(0^+)\delta_0(s)e^{-ik\sigma}d\sigma = \overline{\Phi_{Lz}{}'} + \frac{1}{\sqrt{2\pi}}\Phi_{Lz}(0^+)$$

So:

$$\overline{\Phi_{Lz}{}'} = FO(\Phi'_{Lz}) - \frac{1}{\sqrt{2\pi}}\Phi_{Lz}(0^+) \qquad (4.b.15)$$

And, finally, the notation $FO(\Phi'_{Lz})$ can be introduced in equation (4.b.13):

$$L(s) = \alpha_0 e^{iks} qBC'_L\left[\sqrt{2\pi}.FO(\Phi'_{Lz})\right] \qquad (4.b.16)$$

Previously, one had decided in the definition of the lift with the indicial functions, that if σ, the integration variable was worth *0*, it meant that it was worth *0⁺*. If it had been decided that *0* meant *0⁻*, one would have obtained:

$$L(s) = qBC'_L\left[\Phi_{Lz}(0^-)\frac{\dot{z}(s)}{U} + \int_{0^-}^{\infty}\Phi_{Lz}{}'(\sigma)\frac{\dot{z}(s-\sigma)}{U}d\sigma\right] \qquad (4.b.17)$$

As $\Phi_{Lz}(0^-) = 0$, it comes that:

$$L(s) = qBC'_L\left[\int_{0^-}^{\infty}\Phi_{Lz}{}'(\sigma)\frac{\dot{z}(s-\sigma)}{U}d\sigma\right] \qquad (4.b.18)$$

Assuming the same harmonic motion and defining $\overline{\Phi_{Lz}{}'}$ as $\overline{\Phi_{Lz}{}'} = \frac{1}{\sqrt{2\pi}}\int_{0^-}^{\infty}\Phi_{Lz}{}'(\sigma)e^{-ik\sigma}d\sigma = FO(\Phi'_{Lz})$ which is the real Fourier transform of $\Phi_{Lz}{}'$, it comes:

$$L(s) = \alpha_0 e^{iks} qBC'_L\left[\sqrt{2\pi}.FO(\Phi'_{Lz})\right] \qquad (4.b.19)$$

As it was predictable, the same equation as equation (4.b.16) for $L(s)$ is obtained whether one decides that 0 means 0^+ or 0^-.

For the rest of the study, the first definition of *0* will be kept, so *0* means *0⁺*. So, as in equation (4.b.13):

$$L(s) = \alpha_0 e^{iks} qBC'_L\left[\sqrt{2\pi}.\overline{\Phi'_{Lz}} - \Phi_{Lz}(0^+)\right]$$

Equaling both expressions of the lift, equation (4.b.12) and (4.b.13), it comes:

$$\alpha_0 e^{iks} q B C_L' \left[\sqrt{2\pi}.\overline{\Phi_{Lz}'} - \Phi_{Lz}(0^+)\right] = \frac{1}{2}\rho K B U^2 \alpha_0 \big(H_1^*(k) - iH_4^*(k)\big) e^{iks} \qquad (4.b.20)$$

As $q = \frac{1}{2}\rho U^2$, and after some simplifications, the same equation as in the article by Scanlan [13] is obtained:

$$C_L' \left[\sqrt{2\pi}.\overline{\Phi_{Lz}'} - \Phi_{Lz}(0^+)\right] = K[H_1^*(k) - iH_4^*(k)] \qquad (4.b.21)$$

Therefore, $\overline{\Phi_{Lz}'}$ takes as a final writing:

$$\sqrt{2\pi}.\overline{\Phi_{Lz}'}(k) - \Phi_{Lz}(0^+) = \left[C_L'^{-1} K[H_1^* - iH_4^*]\right] \qquad (4.b.22)$$

Using the notation *FO* for the definition of the Fourier transform:

$$\sqrt{2\pi}.FO(\Phi_{Lz}') = \left[C_L'^{-1} K[H_1^* - iH_4^*]\right] \qquad (4.b.23)$$

And by applying the inverse Fourier transform, the expression of the indicial function is obtained (cf. 1st Appendix about the definitions of the Fourier transform):

$$\Phi_{Lz}'(s) = \frac{1}{2\pi C_L'}\left[\int_{-\infty}^{+\infty} K[H_1^*(K) - iH_4^*(K)] e^{iks} dk\right] \qquad (4.b.24)$$

In this paragraph, an equation for the Fourier transform of an indicial function as a function of the flutter derivatives has been obtained, cf. equations (4.b.22-23). This equation is in the frequency-domain. By an inverse Fourier transform, the expression of the indicial function in the time-domain has also be found under the form of a complex integral, equation (4.b.24). To obtain these equations, equality has been made between the Scanlan's formulation of the wind forces in the case of harmonic oscillations of a bridge deck and an expression of the wind forces that has been assumed. These two formulations have been proved to be consistent in the case of harmonic oscillations, and, from equaling them, the expression of an indicial function has been found for the lift as a function of the vertical motion of the bridge deck.

c) <u>Finding the expression of indicial functions – Fourier transform</u>

In this paragraph, the Scanlan's formula for the wind forces in the frequency domain will be used, as defined in paragraph III.c), for harmonic oscillations. In order to obtain the wind forces in the case of non-harmonic oscillations of the bridge deck, the time-history of the bridge deck motion will be transformed into a frequency spectrum with a Fourier transform of it. Once in the frequency domain, from the Scanlan's formula, the resulting wind forces for each frequency of the frequency spectrum is known. Therefore, the frequency spectrum of the resulting wind forces will be obtained. In order to get back to the time-domain, an inverse Fourier transform of the frequency spectrum of the wind forces has to be made. Then, by analogy with the definition of the indicial functions as made in IV.a), a formula of the wind forces is obtained as a function of the time-history motion of the bridge deck and of the indicial functions. As in IV.b), to make the development easier, only the lift as a function of the vertical motion of the bridge deck will be considered.

In the case of the study of the lift as a function of the vertical motion of the airfoil, for a harmonic oscillation, the following relation is known (from the Scanlan's flutter derivatives):

$$L(\omega,t) = \frac{1}{2}\rho BKU^2[H_1^* - iH_4^*]\frac{\dot{z}(\omega,t)}{U} \qquad (4.c.1)$$

With $\dot{}$, one indicates the derivative with respect to the time, *t*. With ', one indicates the derivative with respect to the dimensionless time, *s*.

One wants to find the expression of the lift as a function of the acceleration of the vertical motion of the bridge deck, $\dot{z}'$. So:

$$\dot{z} = i\omega z$$

$$\dot{z}' = ik\dot{z} = -k\omega z$$

These derivates are due to the fact that *z* is harmonic, so:

$$z = \exp(i\omega t) = \exp\,(iks)$$

k and *s* are the dimensionless pulsation and time.

Expressing the lift as a function of the acceleration, $\dot{z}'$, in equation (4.c.1), it comes:

$$L(\omega,t) = \frac{1}{2}\rho BU^2\left(-\frac{K}{Uk}[iH_1^* + H_4^*]\dot{z}'\right)$$

$$L(\omega,t) = \frac{1}{2}\rho BU^2\left(-\frac{2}{U}[iH_1^* + H_4^*]\dot{z}'\right) \qquad (4.c.2)$$

The Fourier transform of $\dot{z}'$, $FZ2(k)$ is worth:

$$FZ2(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} \dot{z}'(\sigma) e^{-ik\sigma} d\sigma$$

$FZ2(k)$ is defined according to the 1st Appendix of this work. The time integration is done only till the present time, s, because the time-history of the vertical motion is known only till s. $FZ2(k)$ is the frequency spectrum of the vertical motion of the bridge deck. It represents the weight of each frequency present in the time-history of the vertical motion.

Using the results found in the 1st Appendix and the equation in the frequency-domain (4.c.2), the Fourier transform of the lift, *FL*, as a function of the frequency spectrum of the vertical motion is equal to:

$$FL(k) = \frac{1}{2}\rho B U^2 \left(-\frac{2}{U}[iH_1^*(k) + H_4^*(k)]\, FZ2(k)\right) \qquad (4.c.3)$$

Equation (4.c.5) is similar to the theoretical results obtained in the article by Borri and Hoeffer in 2000, [16].

Now that the frequency spectrum of the lift, *FL* has been obtained to get back to the time-domain, an inverse Fourier transform of *FL* should be made, as defined in the 1st Appendix. So, the lift force equals to:

$$L(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} FL(k) e^{iks} dk \qquad (4.c.4)$$

$$L(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} -\left[\frac{1}{2}\rho B U^2 \frac{2}{U}[iH_1^* + H_4^*]\, FZ2\right] e^{iks} dk \qquad (4.c.5)$$

The studied functions and variables are smooth, as they are physical, so we can do some modifications on the integral that has just been found,

$$L(s) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{s} \left[\frac{1}{2}\rho B U^2 \frac{2}{U}[iH_1^* + H_4^*]\right] \dot{z}'(\sigma) e^{ik(s-\sigma)} d\sigma dk$$

$$L(s) = -\frac{1}{2\pi}\frac{1}{2}\rho B U^2 \int_{-\infty}^{s} \left[\int_{-\infty}^{+\infty} [2[iH_1^* + H_4^*]] e^{ik(s-\sigma)} dk\right] \frac{\dot{z}'(\sigma)}{U} d\sigma \qquad (4.c.6)$$

Thanks to the modification on the integrals, now, the main integral is on the time-domain, instead of an integral on the frequency-domain, as in (4.c.5). The integral on the frequency-domain still exists, but should be evaluated first. The analytical form that has just been found in (4.c.6) for the lift, is a convolution product between the inverse Fourier transform of $-\frac{1}{2\pi}\left[2[iH_1^* + H_4^*]\right]$ and the vertical acceleration of the airfoil.

If one puts down that the integral on the frequency-domain is a function of time, it comes:

$$\Phi_{Lz}(\sigma) = -\frac{1}{2\pi C_L'}\left[\int_{-\infty}^{+\infty}\left[2[iH_1^*(k) + H_4^*(k)]\right]e^{ik\sigma}dk\right] \qquad (4.c.7)$$

This formulation with such multiplicative constants has been chosen in order to be consistent with the definition of the indicial functions as in IV.a). Therefore, Φ_{Lz} is an indicial function.

With the introduction of the indicial function as defined in (4.c.7), equation (4.c.6) may be transformed into:

$$L(s) = \frac{1}{2}\rho BU^2 C_L' \int_{-\infty}^{s} \Phi_{Lz}(s-\sigma)\frac{\dot{z}'(\sigma)}{U}d\sigma \qquad (4.c.8)$$

With integration by part of equation (4.c.8):

$$L(s) = [\Phi_{Lz}(s-\sigma)\dot{z}(\sigma)]_{-\infty}^{s} + \int_{-\infty}^{s} \Phi'_{Lz}(s-\sigma)\dot{z}(\sigma)d\sigma$$

$$L(s) = \Phi_{Lz}(0)\dot{z}(s) + \int_{-\infty}^{s} \Phi'_{Lz}(s-\sigma)\dot{z}(\sigma)d\sigma \qquad (4.c.9)$$

$\Phi_{Lz}(-\infty) = 0$, as at an infinite time, the airfoil is at rest. This result is similar to the formulation of the lift as defined in IV.a).

In this paragraph, thanks to a Fourier transform of the vertical motion of the bridge deck, and using the Scanlan's formula in the frequency-domain, the Fourier transform of the lift has been obtained. Then, with an inverse Fourier transform, the expression of the lift in the time-domain has been found. Then, in order to find the same formulation for the lift as the one assumed in IV.a), the indicial function, $\Phi_{Lz}(s)$, has been identified. In this approach, it has not been assumed a particular formulation for the lift. The final formulation is due to the Fourier transform and inverse transform steps that can be written under the form of a convolution product.

d) Finding the expression of indicial functions – Duhamel's integral

In the following paragraph, it will be explained how one can pass from the response of a system to an impulse to the response of this system to any forced motion of the bridge deck. In doing so, two steps are required. First, the hypotheses under which we may pass from the response to an impulse to the response to any forced functions have to be detailed. Secondly, the response to an impulse using the flutter derivatives has to be determined. This theoretical procedure that is called the Duhamel's integral, is quite similar to the one developed in the lecture notes of the course "Structural dynamics" dispensed by Professor Perotti at Politecnico di Milano, [17].

In this paragraph, the lift will be considered only as function of the vertical motion of the bridge deck, as made in paragraphs b) and c). The response of the system to an impulse of acceleration of the vertical motion of the bridge deck is studied. The impulse is called *I*:

$$I = \lim_{\Delta s \to 0} \int_{-\Delta s/2}^{\Delta s/2} \frac{\dot{z}'(\sigma)}{U} d\sigma \qquad (4.d.1)$$

So, $I = \dot{z}(\Delta s/2)/U - \dot{z}(-\Delta s/2)/U$. Rewriting it, with Δs approaching 0 and $\dot{z}(0^-) = 0$:

$$\dot{z}(\Delta s/2)/U = \dot{z}(0^+)/U = I \qquad (4.\mathrm{d}.2)$$

And so,

$$\frac{\dot{z}(s)}{U} = I \text{ for } s > 0 \qquad (4.d.3)$$

The impulse of acceleration *I* means passing from a bridge deck at rest, where $\dot{z}(s) = 0$, to a bridge deck moving at a speed $\dot{z}(s) = UI$.

The dimensionless velocity of the vertical motion matches the Heaviside function, *1(s)*, multiplied by the constant *I*:

$$\frac{\dot{z}(s)}{U} = I \cdot \underline{1}(s) \qquad (4.d.4)$$

And the acceleration of the vertical motion matches the Dirac function, $\delta_0(s)$, multiplied by the constant I:

$$\frac{\dot{z}'(s)}{U} = I \cdot \delta_0(s) \qquad (4.d.5)$$

For the first part of the analysis, it is assumed that the response to an impulse of acceleration of the vertical motion at a time 0 is known. This indicial response is called *h(s)*. So:

$$\begin{cases} L(s) = Ih(s) & \text{for } s > 0 \\ L(s) = 0 & \text{for } s < 0 \end{cases} \qquad (4.d.6)$$

The first major approximation that is made at this point is assuming that the response is proportional to the amplitude of the impulse. This may be considered correct only for little oscillations of the bridge deck. This is called the hypothesis of linearity.

If the impulse happens at a time σ, the response of the system will be the same as in equation (4.d.4), with a translation of the time axis origin:

$$\begin{cases} L(s) = Ih(s-\sigma) & \text{for } s > \sigma \\ L(s) = 0 & \text{for } s < \sigma \end{cases} \qquad (4.d.7)$$

The second major hypothesis made by doing this translation in time is assuming that the system is invariable in time. In practice, the wind forces on a bridge deck are studied on intervals of nearly 1 minute to 10 minutes maximum. So, one may assume that the bridge deck does not change in such an interval of time.

The vertical velocity can always be approximated into an infinite sum of Heaviside functions changing the continuous signal into a discrete one. To make this easier, we choose constant sampling time, Δs.

The velocity of the vertical motion $\dot{z}(s)$ becomes:

$$\frac{\dot{z}(s)}{U} = \sum_{k=0}^{+\infty} \frac{\dot{z}'(k\Delta s)}{U} \cdot \underline{1}(s - k\Delta s)\Delta s \qquad (4.d.8)$$

The impulses *I* defined before now equal $(\dot{z}'(k\Delta s)/U)\Delta s$.

So, the acceleration is a sum of Dirac functions at the times $k\Delta s$:

$$\frac{\dot{z}'(s)}{U} = \Delta s \sum_{k=0}^{+\infty} \frac{\dot{z}'(k\Delta s)}{U} \delta_0(s - k\Delta s) \qquad (4.d.9)$$

It has been assumed that we know the response to an impulse. It has also been assumed that the system is invariable in time and linear, so the response to the discrete vertical motion is equal to the sum of the responses to each of the impulses at the times $k\Delta s$:

$$L(n\Delta s) = \Delta s \sum_{k=0}^{+\infty} \frac{\dot{z}'(k\Delta s)}{U} \cdot h(s - k\Delta s) \qquad (4.d.10)$$

In doing this operation, the superposition principle is applied: the response to various stimuli is equal to the sum of the responses which would have been caused by each stimulus individually. This principle is linked to the linearity and to the invariance in time of the system.

The sum is limited to the time $(n-1)\Delta s$, as the response at a time $s = n\Delta s$ depends only on the previous configurations of the angle of attack (causality principle).

By transforming the sum into an integral and transforming the time intervals Δs into infinitesimal ones, $d\sigma$, the Duhamel's integral is obtained:

$$L(s) = \int_{O}^{s} \frac{\dot{z}'(\sigma)}{U} h(s-\sigma) d\sigma \qquad (4.d.11)$$

In equation (4.d.11), it has been found that the lift can be written in the form of a convolution product between the time-history motion of the bridge deck and the response to an impulse, *h(s)*. With this approach, the indicial function corresponds with the response to a unitary impulse of the velocity of the vertical motion, *h(t)*.

Now, the response to an impulse has to be evaluated. To evaluate it, the results obtained in paragraph b) of the 1st Appendix about the Fourier transform will be used. The impulse corresponds to the Heaviside function, therefore, the response to an impulse is worth:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{Z(i\omega)} e^{i\omega(t-\tau)} d\omega \qquad (4.d.12)$$

y(t) is the response to an impulse. In this case, it corresponds to *h(t)*. The complex impedance, $Z(i\omega)$ is defined as:

$$Z(i\omega) = \frac{L(k,t)}{\dot{z}'(k,t)/U} \qquad (4.d.13)$$

Using, equation (4.c.2), it comes that:

$$Z^{-1}(i\omega) = -\frac{1}{2}\rho B U^2 (2[iH_1^*(K) + H_4^*(K)]) \qquad (4.d.14)$$

Therefore, using dimensionless variables, equation (4.d.12) may be re-expressed for the studied case:

$$h(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} -\frac{1}{2}\rho B U^2 (2[iH_1^* + H_4^*]) e^{ik(s-\sigma)} dk \qquad (4.d.15)$$

The function *h(s)* that has just been found is the response to an impulse. As equation (4.d.11) is similar to the equation of the wind forces expressed with the indicial functions, in order to be consistent with equation (4.a.8), $\Phi_{Lz}(s)$ is defined as:

$$\Phi_{Lz}(\sigma) = -\frac{1}{2\pi C_L'}\int_{-\infty}^{+\infty}(2[iH_1^* + H_4^*])e^{ik\sigma}dk = \frac{h(s)}{\frac{1}{2}\rho BU^2C_L'} \qquad (4.d.16)$$

$\Phi_{Lz}(s)$ and $h(s)$ are proportional. The indicial function corresponds indeed to the response to an impulse, as defined in paragraph IV.a).

Inserting the indicial function Φ_{Lz} as defined in equation (4.d.16), the expression of the lift that has just found can be re-written:

$$L(s) = \frac{1}{2}\rho BU^2C_L'\int_0^s \Phi_{Lz}(s-\sigma)\frac{\dot{z}'(\sigma)}{U}d\sigma \qquad (4.d.17)$$

e) <u>Comparison of the three ways used for the calculation of indicial functions</u>

The lift as a function of the vertical motion of a bridge deck has been expressed in the time-domain in three different ways: the Scanlan's approach, the Fourier transform and the Duhamel's integral. In the three developments, the same expression for the lift has finally been obtained. The indicial functions have always been defined in the same way,

$$\Phi_{Lz}(\sigma) = -\frac{1}{2\pi C_L'}\int_{-\infty}^{+\infty}(2[iH_1^* + H_4^*])e^{ik\sigma}dk \qquad (4.e.1)$$

for the indicial function and

$$\Phi_{Lz}'(s) = \frac{1}{2\pi C_L'}\left[\int_{-\infty}^{+\infty}K[H_1^*(K) - iH_4^*(K)]e^{iks}dk\right] \qquad (4.e.2)$$

for the derivate of the indicial function.

This definition of the indicial function that is consistent with the definition made in IV.a), allows expressing the lift under the form of a convolution product:

$$L(s) = \frac{1}{2}\rho BU^2C_L'\left[\Phi_{Lz}(0)\frac{\dot{z}(s)}{U} + \int_0^\infty \Phi_{Lz}{}'(\sigma)\frac{\dot{z}(s-\sigma)}{U}d\sigma\right] \qquad (4.e.3)$$

These three different approaches are a way to ensure that the formulation of the problem is true: first it has been verified that the wind forces could be expressed under the form as assumed in IV.a) and secondly that the formulation of the indicial functions with the flutter derivatives and the different multiplicative constants was true. But these different approaches reflect also different ways to perceive the indicial functions.

In the Scanlan's approach, the form of the wind forces has been assumed under the form of a convolution product with indicial functions. Enforcing equality with the wind forces derived from the flutter derivatives for an harmonic motion, the indicial function where identified as functions of the flutter derivatives.

Since the wind forces were defined in the frequency-domain by the flutter derivatives in chapter III), in the second approach, it has been decided to use the Fourier transform. Seeking the wind forces at time t, the time-domain history-motion of the bridge deck has been transformed into the frequency one. Then, the frequency spectrum thus obtained has been multiplied by the flutter derivatives for each frequency of the spectrum. A back transformation in the time-domain with an inverse Fourier transform provides the wind forces at time t. In this approach, it has been possible to identify the indicial function and define it a posteriori, once the final expression of the lift forces is found in time domain.

In the first part of the third approach, the Duhamel's integral, it has been assumed that the response of the system to an impulse was known, as done in paragraph IV.a). The different hypotheses that allow passing from this indicial response valid for any time-history motions of the bridge deck to the Duhamel's convolution integral have been detailed. In the second part, the response to an impulse with the Fourier transform has been evaluated. This third approach, through its hypotheses, allowed perceiving the limits of the indicial functions theory which are the linearity and the time invariance of the studied system.

f) Complete expression of lift and pitching moment with indicial functions

In this part, the lift and pitching moment will be expressed with indicial functions thanks to the second approach studied, the Fourier transform. As done in the second approach, a Fourier transform of the time-history motion of the bridge deck will be done. Then, the frequency spectrum will be multiplied by the flutter derivatives. The Fourier transformation of the wind forces is transformed back in the time-domain by an inverse Fourier transform.

Using the well-known Scanlan's formula of the flutter derivatives for harmonic oscillation of the airfoil:

$$\begin{cases} L = \frac{1}{2}\rho B U^2 (K H_1^* \frac{\dot{z}}{U} + K H_2^* \frac{B\dot{\alpha}}{U} + K^2 H_3^* \alpha + K^2 H_4^* \frac{z}{B}) \\ M = \frac{1}{2}\rho B^2 U^2 (K A_1^* \frac{\dot{z}}{U} + K A_2^* \frac{B\dot{\alpha}}{U} + K^2 A_3^* \alpha + K^2 A_4^* \frac{z}{B}) \end{cases} \qquad (4.f.1)$$

The motion of the bridge deck is harmonic, therefore, the different motion parameters of the bridge deck can be expressed only with the angle of attack and vertical velocity of the bridge deck:

$$z = -i\frac{\dot{z}}{\omega}, \text{as } z = z_0 \exp{(i\omega t)}$$

$$\dot{\alpha} = i\omega\alpha, \text{as } \alpha = \alpha_0 \exp{(i\omega t)}$$

With $\dot{}$, one indicates the derivative with respect to the time, *t*.

As *ω=UK/B*, the lift becomes:

$$L = \frac{1}{2}\rho B U^2 \left[K(H_1^* - iH_4^*)\frac{\dot{z}}{U} + K^2(H_3^* + iH_2^*)\alpha\right] \qquad (4.f.2)$$

Expressing z and α as functions of the reduced velocity *k=Bω/2U* and *s=2Ut/B*:

$$z = z_0 \exp{(iks)}$$

$$\alpha = \alpha_0 \exp{(iks)}$$

$$\dot{z}' = ik\dot{z}$$

$$\alpha' = ik\alpha$$

With ', one indicates the derivative with respect to the dimensionless time, *s*.

Expressing the lift as a function of $\dot{z}'$ and α', with 2*k=K*, it comes:

$$L = \frac{1}{2}\rho B U^2 \left[2(-iH_1^* - H_4^*)\frac{\dot{z}'}{U} + 4k(-iH_3^* + H_2^*)\alpha'\right] \qquad (4.f.3)$$

By analogy, for the pitching moment:

$$M = \frac{1}{2}\rho B^2 U^2 \left[2(-iA_1^* - A_4^*)\frac{\dot{z}'}{U} + 4k(-iA_3^* + A_2^*)\alpha'\right] \qquad (4.f.4)$$

Therefore, the Fourier transform of the lift, *FL*, and of the moment, *FM*, can be expressed with the Fourier transform of $\dot{z}'/U$, *FZ2*, and of α', *FA1*. These expressions correspond with the article by Borri and Hoeffer, [16]:

$$FL(k) = \frac{1}{2}\rho B U^2 [2(-iH_1^* - H_4^*)FZ2 + 4k(-iH_3^* + H_2^*)FA1] \qquad (4.f.5)$$

$$FM(k) = \frac{1}{2}\rho B^2 U^2 [2(-iA_1^* - A_4^*)FZ2 + 4k(-iA_3^* + A_2^*)FA1] \qquad (4.f.6)$$

Considering the lift:

$$FL(k) = \frac{1}{2}\rho B U^2 [2(-iH_1^* - H_4^*)FZ2 + 4k(-iH_3^* + H_2^*)FA1]$$

where,

$$FZ2(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} \frac{\dot{z}'(\sigma)}{U} e^{-ik\sigma} d\sigma$$

$$FA1(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} \alpha'(\sigma) e^{-ik\sigma} d\sigma$$

Because of the causality principle, the bridge deck motion is known only till the present time, *s*. So the time integration of the Fourier transform of the time-history motion of the bridge deck is limited from $-\infty$ to *s*.

Applying the inverse Fourier transform:

$$L(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} FL(k) e^{iks} dk \qquad (4.f.7)$$

After calculation, the time-domain lift can be expressed as a function of the motion of the bridge deck and the flutter derivatives:

$$L(s) = \frac{1}{\sqrt{2\pi}}\frac{1}{2}\rho B U^2 \left\{ \int_{-\infty}^{+\infty} 2(-iH_1^* - H_4^*)FZ2 e^{iks} dk + \int_{-\infty}^{+\infty} 4k(-iH_3^* + H_2^*)FA1\, e^{iks} dk \right\}$$

$$L(s) = \frac{1}{2\pi}\frac{1}{2}\rho B U^2 \left\{ \int_{-\infty}^{+\infty} 2(-iH_1^* - H_4^*) \int_{-\infty}^{s} \frac{\dot{z}'(\sigma)}{U} e^{-ik\sigma} d\sigma\, e^{iks} dk \right.$$
$$\left. + \int_{-\infty}^{+\infty} 4k(-iH_3^* + H_2^*) \int_{-\infty}^{s} \alpha'(\sigma) e^{-ik\sigma} d\sigma\; e^{iks} dk \right\}$$

$$(4.f.8-9)$$

As the studied functions are smooth, some modifications can be done on the integral that has just been found.

$$L(s) = \frac{1}{2\pi}\frac{1}{2}\rho BU^2 \left\{ \int_{-\infty}^{s}\int_{-\infty}^{+\infty} 2(-iH_1^* - H_4^*)e^{ik(s-\sigma)}dk \, \frac{\dot{z}'(\sigma)}{U}d\sigma + \int_{-\infty}^{s}\int_{-\infty}^{+\infty} 4k(-iH_3^* + H_2^*)e^{ik(s-\sigma)}dk \, \alpha'(\sigma)d\sigma \right\}$$

$$(4.f.10)$$

As lift and pitching moment almost have the same expression, by analogy, the expression of the pitching moment can also be obtained:

$$M(s) = \frac{1}{2\pi}\frac{1}{2}\rho B^2U^2 \left\{ \int_{-\infty}^{s}\int_{-\infty}^{+\infty} 2(-iA_1^* - A_4^*)e^{ik(s-\sigma)}dk \, \frac{\dot{z}'(\sigma)}{U}d\sigma + \int_{-\infty}^{s}\int_{-\infty}^{+\infty} 4k(-iA_3^* + A_2^*)e^{ik(s-\sigma)}dk \, \alpha'(\sigma)d\sigma \right\}$$

$$(4.f.11)$$

In order to simplify the calculation, the following expressions are put down:

$$\Phi_{Lz}(\sigma) = \frac{1}{2\pi C_L'}\int_{-\infty}^{+\infty} 2(-iH_1^* - H_4^*)\, e^{ik\sigma}dk$$

$$\Phi_{L\alpha}(\sigma) = \frac{1}{2\pi C_L'}\int_{-\infty}^{+\infty} 4k(-iH_3^* + H_2^*)\, e^{ik\sigma}dk$$

$$\Phi_{Mz}(\sigma) = \frac{1}{2\pi C_M'}\int_{-\infty}^{+\infty} 2(-iA_1^* - A_4^*)\, e^{ik\sigma}dk$$

$$\Phi_{M\alpha}(\sigma) = \frac{1}{2\pi C_M'}\int_{-\infty}^{+\infty} 4k(-iA_3^* + A_2^*)\, e^{ik\sigma}dk$$

With these functions that have been introduced, the lift and the pitching moment can be written in a more simple way:

$$L(t) = \frac{1}{2}\rho BU^2C_L'\left[\int_{-\infty}^{s} \Phi_{Lz}(s-\sigma)\frac{\dot{z}'(\sigma)}{U}d\sigma + \int_{-\infty}^{s} \Phi_{L\alpha}(s-\sigma)\alpha'(\sigma)d\sigma\right] \qquad (4.f.12)$$

$$M(t) = \frac{1}{2}\rho B^2U^2C_M'\left[\int_{-\infty}^{s} \Phi_{Mz}(s-\sigma)\frac{\dot{z}'(\sigma)}{U}d\sigma + \int_{-\infty}^{s} \Phi_{M\alpha}(s-\sigma)\alpha'(\sigma)d\sigma\right] \qquad (4.f.13)$$

With integration by part:

$$L(s) = \frac{1}{2}\rho BU^2C_L'\left[\left[\Phi_{Lz}(s-\sigma)\frac{\dot{z}(\sigma)}{U}\right]_{-\infty}^{s} + \int_{-\infty}^{s} \Phi_{Lz}{}'(s-\sigma)\frac{\dot{z}(\sigma)}{U}d\sigma + [\Phi_{L\boldsymbol{\alpha}}(s-\sigma)\alpha(\sigma)]_{-\infty}^{s} + \int_{-\infty}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma\right]$$

In $-\infty$, the system is at rest, the bridge deck is steady, so:

$$L(s) = \frac{1}{2}\rho B U^2 C_L' \left[\Phi_{Lz}(0)\frac{\dot{z}(s)}{U} + \Phi_{L\boldsymbol{\alpha}}(0)\alpha(s) + \int_{-\infty}^{s} \Phi_{Lz}{}'(s-\sigma)\frac{\dot{z}(\sigma)}{U}d\sigma + \int_{-\infty}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma \right]$$

$$(4.f.14)$$

By analogy, for the pitching moment, it comes:

$$M(s) = \frac{1}{2}\rho B^2 U^2 C_M' \left[\Phi_{Mz}(0)\frac{\dot{z}(s)}{U} + \Phi_{M\boldsymbol{\alpha}}(0)\alpha(s) + \int_{-\infty}^{s} \Phi_{Mz}'(s-\sigma)\frac{\dot{z}(\sigma)}{U}d\sigma + \int_{-\infty}^{s} \Phi_{M\alpha}'(s-\sigma)\alpha(\sigma)d\sigma \right]$$

$$(4.f.15)$$

This final expression of the wind forces corresponds to the definition of the wind forces made in IV.a) with the indicial functions. The function Φ_i that have been identified are the indicial functions.

- Conclusion:

In this chapter, it has been shown how the wind forces can be expressed in the time-domain thanks to the indicial functions. All through the chapter, different ways to approach the indicial functions and different properties of these functions have been discovered. The link between the indicial functions and the flutter derivatives has been made, in the frequency-domain and in the time-domain, in three different ways. The limits of this linear formulation that has been assumed being true by analogy with the thin-airfoil theory, have been tackled. Finally, in the last paragraph, the complete wind forces have been expressed by defining the four different indicial functions.

V) Practical expression of the wind forces acting on a bridge deck

In the last chapter, we have expressed the wind forces due to the motion of the bridge deck. These forces have to be integrated in the complete system of the wind forces acting on the bridge deck. Some approximations have to be made to define the wind and the wind forces in order to make the analysis easier and faster. The choices we have made are similar to the choices made in [14]. They offer a good approximation for a first approach to the indicial functions. The main hypotheses made are to consider the horizontal part of the wind field constant, to neglect the horizontal movement of the bridge deck and to consider the drag as a steady-state load.

a) Wind model

We assume that the bridge deck is immersed in a turbulent, incompressible and non-viscous wind field. In the wind velocity, $\vec{U}_{wind}$, we distinguish the mean wind velocity which depends only of the position of the measurement of the wind field and the turbulent wind velocity which depends on the position and the time of the measurement and whose mean is equal to zero.

$$\vec{U}_{wind}(M,t) = \vec{U}(M) + \vec{u}(M,t) \qquad (5.1)$$

$$\overline{\vec{U}_{wind}}(M) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \vec{U}_{wind}(M,t)dt = \vec{U}(M) \qquad (5.2)$$

Where $\overline{\vec{U}_{wind}}(M)$ is the mean of the wind velocity. $\vec{U}(M)$ is the mean wind speed and $\vec{u}(M,t)$ is the turbulent perturbation. *M* is a general point of coordinates *(x,y,z)*.

So,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \vec{u}(M,t)\,dt = 0 \qquad (5.3)$$

The mean velocity is oriented along the x-axis which corresponds to the horizontal direction:

$$\vec{U}(M) = U(M)\vec{\imath}_x \qquad (5.4)$$

The turbulent velocity has components on the three direction of space. So,

$$\vec{U}_{wind}(M,t) = U(M)\vec{\imath}_x + u(M,t)\vec{\imath}_x + v(M,t)\vec{\imath}_y + w(M,t)\vec{\imath}_z \qquad (5.5)$$

where *u, v* and *w* are the turbulent components oriented respectively along the x-axis, the y-axis and the z-axis.

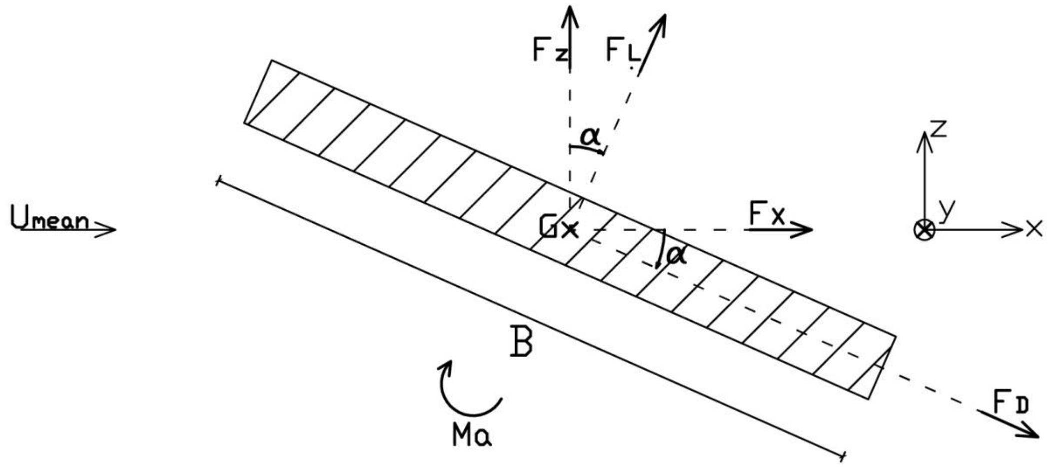

Figure 17: Section scheme. With drag, F_D, and lift, F_L, defined in the referential of the profile

The y-axis is defined as the axis along the bridge span. The z-axis is defined as the vertical axis, perpendicular to the bridge deck at rest. The x-axis is the horizontal axis along the bridge deck at rest. *G*, the origin, is the elastic center of the bridge deck.

For the evaluation of the wind forces on the bridge deck, we consider the bridge deck as a profile which means that we consider only two dimensions of space. We study a cross-section which is assumed infinitely extended. Therefore, the y-axis direction is not taken into account. The forces that we have to evaluate are three: the horizontal force on the profile along the x-axis, also called drag, the vertical force along the z-axis, also called the lift force and a moment around the y-axis, also called pitching moment.

b) <u>Wind forces</u>

The wind load acting on the bridge deck cross-section is expressed as a sum of three components:

- the quasi-static part
- the self-excited part which depends on the motion of the bridge deck
- the buffeting part which depends on the turbulence of the incoming wind velocity

$$F_z(t) = F_{Z0} + F_{Zse} + F_{Zb}$$

$$M_y(t) = M_{Y0} + M_{Yse} + F_{Yb}$$

$$F_x(t) = F_{X0}$$

$$(5.6)$$

Where $F_z(t)$ is the vertical wind force, along the z-axis, $F_x(t)$ is the horizontal wind force, along the x-axis and $M_y(t)$ is the aerodynamic moment. The subscripts *(0)*, *(se)* and *(b)* indicate respectively the quasi-static, self-excited and buffeting components.

Depending on the definition of $F_z(t)$ and $F_x(t)$, $F_z(t)$ and $F_x(t)$ may not correspond respectively to the lift, *L* and the drag, *D*. What is called lift may whether be the force "vertical" with respect to the moving bridge deck, so its orientation is rotated of the angle of attack α with respect to the z-axis (as defined on Figure 17), or whether be the absolute vertical force, along the z-axis. For the drag, we can find both definitions too, whether it is the "horizontal" force with respect to the bridge deck, rotated of an angle α, or whether it is the absolute horizontal force, along the x-axis. In order to know what the definitions of the drag and lift are, we should consider carefully the definition of the data of the aerodynamic coefficients (static aerodynamic coefficients and flutter derivatives). Commonly, in the study of aerodynamics applied to bridges, the forces are defined in the absolute referential, whereas in classical aerodynamics, the forces are defined in the airfoil referential, rotated of an angle α.

The aerodynamic moment, called M_a, corresponds to the moment about the y-axis, $M_y(t)$. The forces are defined with respect to the elastic center of the profile, G. Therefore, no additive moment due to a possible change of the point of application of forces that may create a lever arm is to be considered.

If drag and lift are orientated along with the bridge deck, we should rotate them in order to express them in the absolute referential:

$$\begin{cases} F_z = \cos(\alpha)\, L - \sin(\alpha) D \\ F_x = \sin(\alpha)\, L + \cos(\alpha)\, D \\ M_y = M_a \end{cases} \qquad (5.7)$$

As the angle of attack remains small (some degrees), we consider that the cosine of α is equal to 1 and that the sine of α is equal to 0. So, no matter the definition of the drag and lift forces, we may always consider that:

$$\begin{cases} F_z = L \\ F_x = D \\ M_y = M_a \end{cases} \qquad (5.8)$$

The quasi static part depends only on the angle of attack α and the mean horizontal velocity, U:

$$\begin{cases} D_0 = \frac{1}{2}\rho U^2 B C_D(\alpha) \\ L_0 = \frac{1}{2}\rho U^2 B C_L(\alpha) \\ M_0 = \frac{1}{2}\rho U^2 B^2 C_M(\alpha) \end{cases} \qquad (5.9)$$

Where U is the average wind speed, ρ is the air density and C_D, C_l and C_M are the aerodynamic coefficients.

The self-excited forces are linked to the motion of the bridge deck and can be expressed as seen in IV) with indicial functions:

$$F_{Lse}(s) = \frac{1}{2}\rho U^2 B C'_L \left[\Phi_{Lz}(0)\frac{z'(s)}{B/2} + \Phi_{L\boldsymbol{\alpha}}(0)\alpha(s) + \int_{-\infty}^{s} \Phi_{Lz}'(s-\sigma)\frac{z'(\sigma)}{B/2}d\sigma + \int_{-\infty}^{s} \Phi_{L\alpha}'(s-\sigma)\alpha(\sigma)d\sigma \right]$$

$$M_{ase}(s) = \frac{1}{2}\rho U^2 B^2 C'_M \left[\Phi_{Mz}(0)\frac{z'(s)}{B/2} + \Phi_{M\boldsymbol{\alpha}}(0)\alpha(s) + \int_{-\infty}^{s} \Phi_{Mz}'(s-\sigma)\frac{z'(\sigma)}{B/2}d\sigma + \int_{-\infty}^{s} \Phi_{M\alpha}'(s-\sigma)\alpha(\sigma)d\sigma \right]$$

$$(5.10)$$

We use the same definition for the derivates than in IV): z' is the derivate of z with respect to s, the dimensionless time. The relation to pass from the real vertical velocity of the bridge deck $\dot{z}$ to the dimensionless one, z', is:

$$z' = \frac{\partial z}{\partial s} = \frac{\partial z}{\partial t}\frac{\partial t}{\partial s}$$

Where $\dot{z} = \partial z/\partial t$ and $s = 2Ut/B$, so:

$$\frac{\partial t}{\partial s} = \frac{B}{2U}$$

And:

$$z' = \frac{B}{2U}\dot{z} \qquad (5.11)$$

The buffeting forces depend only on the turbulent part of the wind velocity. Considering that $u \ll U$, we can neglect the horizontal component, *u*. As we do not consider the forces along the y-axis, the only turbulence that we consider is *w*, the vertical one. The exact formulation of the buffeting load is a Küssner-type formulation:

$$F_{Lb}(s) = qBC'_L\left[\psi_{Lz}(0)w(s) + \int_{-\infty}^{s}\psi_{Lz}'(s-\sigma)w(\sigma)d\sigma\right]$$

$$M_{ab}(s) = qB^2C'_M\left[\psi_{Mz}(0)w(s) + \int_{-\infty}^{s}\psi_{Mz}'(s-\sigma)w(\sigma)d\sigma\right]$$

$$(5.12)$$

Where ψ_{Lz} and ψ_{Mz} are indicial functions similar to the indicial functions for the self-excited forces, but for which, $\psi_{Lz}(0) = 0$ and $\psi_{Mz}(0) = 0$.

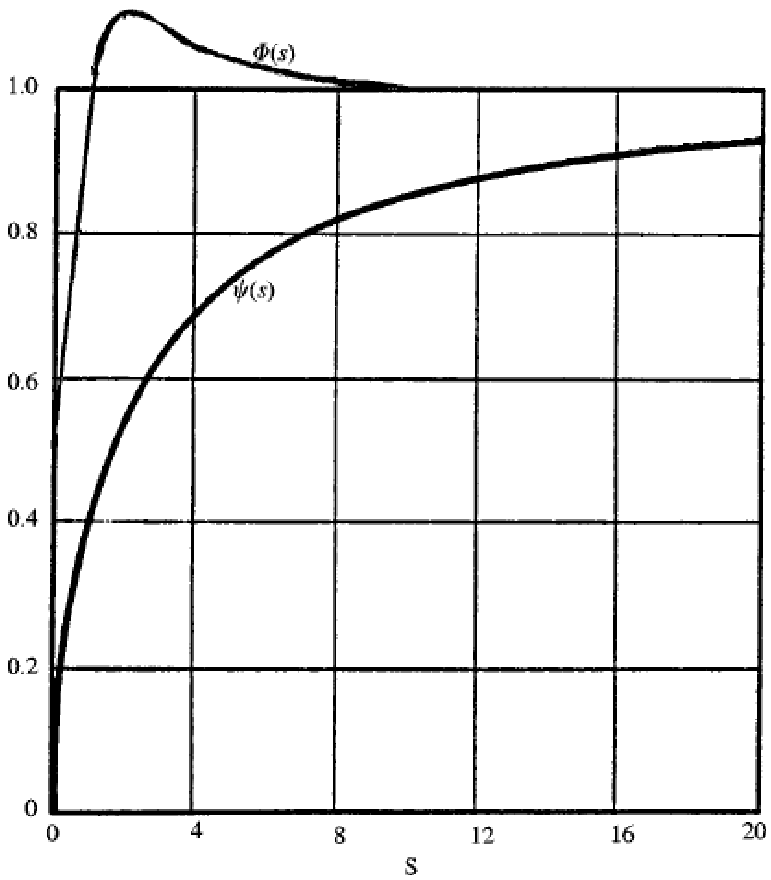

Figure 18: comparison between the Küssner's-type indicial function, ψ, and the self-excited type function, Φ, as a function of dimensionless time, s. Figure from [13]

As done by Borri and Costa [14], we may approximate these Küssner-type indicial functions with the Wagner-type ones: we consider that the turbulence is added to the vertical motion of the bridge deck. So, to the vertical motion of the bridge, *z'*, we should add the vertical turbulence, as if the bridge had a vertical motion equal to *z'+w* in a horizontal steady flow of velocity *U*. A particular attention has to be taken when adding these two physical variables. One should control how they are defined before adding them in order to not mistake with the signs.

Therefore, the buffeting is expressed with the same formulation as for the self-excited forces by the expression:

$$F_{Lb}(s) = qBC_L' \left[\Phi_{Lz}(0)w(s) + \int_{-\infty}^{s} \Phi_{Lz}'(s-\sigma)w(\sigma)d\sigma \right]$$

$$M_{ab}(s) = qB^2C_M' \left[\Phi_{Mz}(0)w(s) + \int_{-\infty}^{s} \Phi_{Mz}'(s-\sigma)w(\sigma)d\sigma \right]$$

(5.13)

c) <u>Presentation of the first cased studied for the validation of the program: the Akashi Kaikyō Bridge</u>

For the first set of validations of the program, we study the well-known Akashi Kaikyō Bridge. We will explain the validation method used and the different steps needed with this case. This bridge is one of the most studied bridges in the literature of aeroelasticity applied to bridges. It has the longest central span of any suspension bridge at 1,991 meters. Located in Japan, it was completed in 1998.

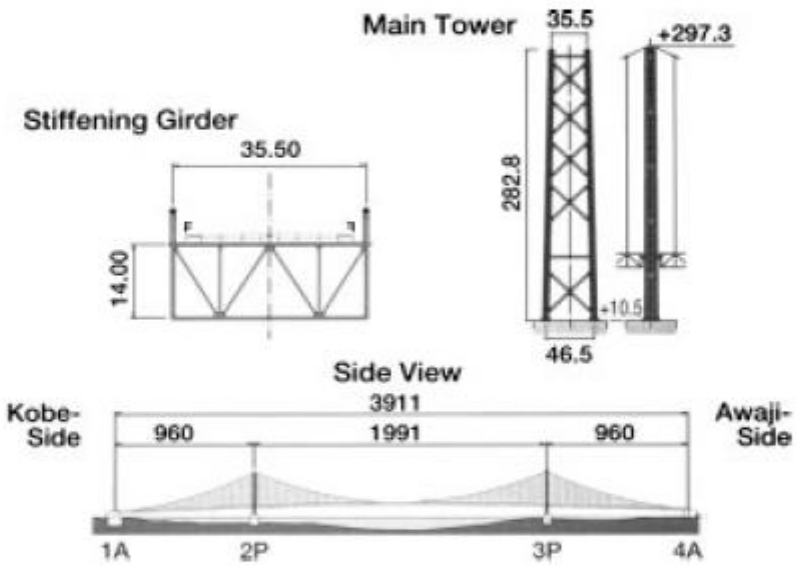

Figure 19: General View of the Akashi Kaikyō Bridge. Figure from [8]

1) Aerodynamic data

The data of the static aerodynamic coefficients have been taken from [18]: "*Wind tunnel test of Naboky-Bisan-Seto Bridges with stiffening truss*" by Shin Narui. The static aerodynamic coefficients of drag, lift and pitching moment are defined in the steady and absolute referential. So, in order to obtain the forces in the mechanical referential, *(Oxz)*, which is fixed, we do not need to rotate the forces of an angle worth the angle of attack:

$$L = F_z$$

$$D = F_x$$

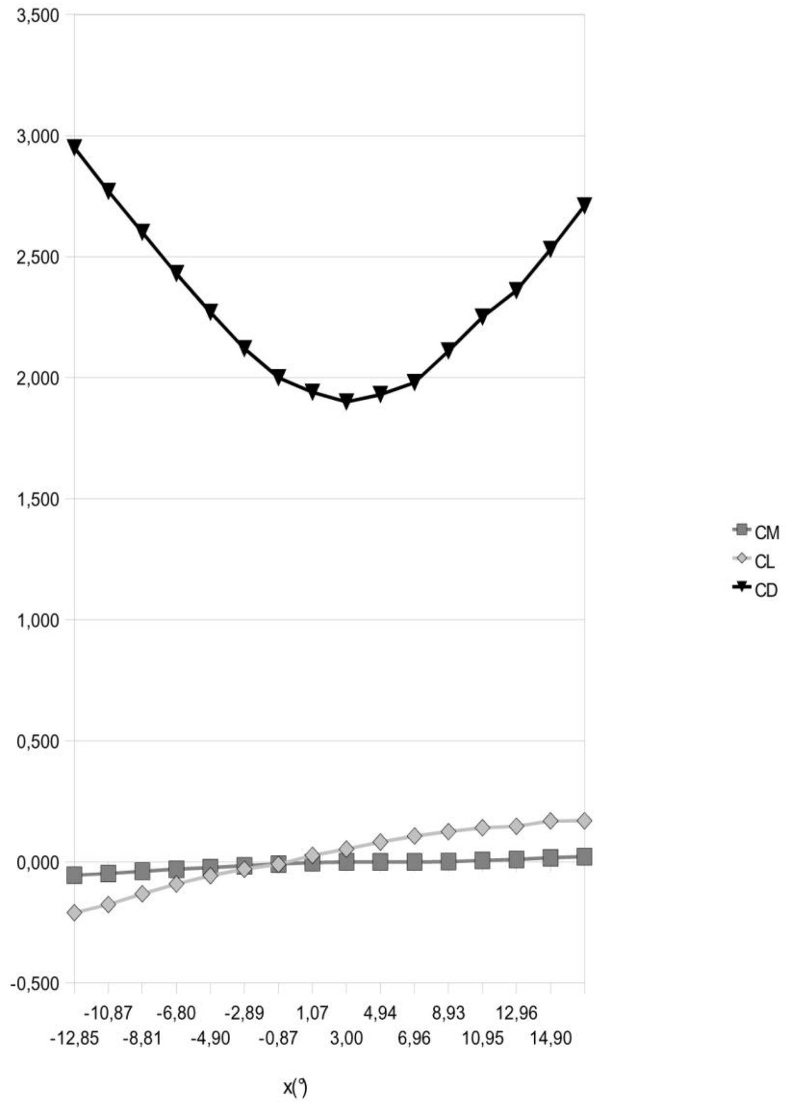

Figure 20: static aerodynamic coefficients of drag (C_D), lift (C_L) and pitching moment (C_M) of the bridge deck cross section of the Akashi Kaikyō Bridge central span as a function of the angle of attack α

For the flutter derivatives, we have taken the data from the article [19], "*Multi-mode flutter and buffeting analysis of the Akashi-Kaikyō bridge*", by H. Katsuchi et al. In this case too, the wind forces are defined in the absolute referential, *(Oxz)*.

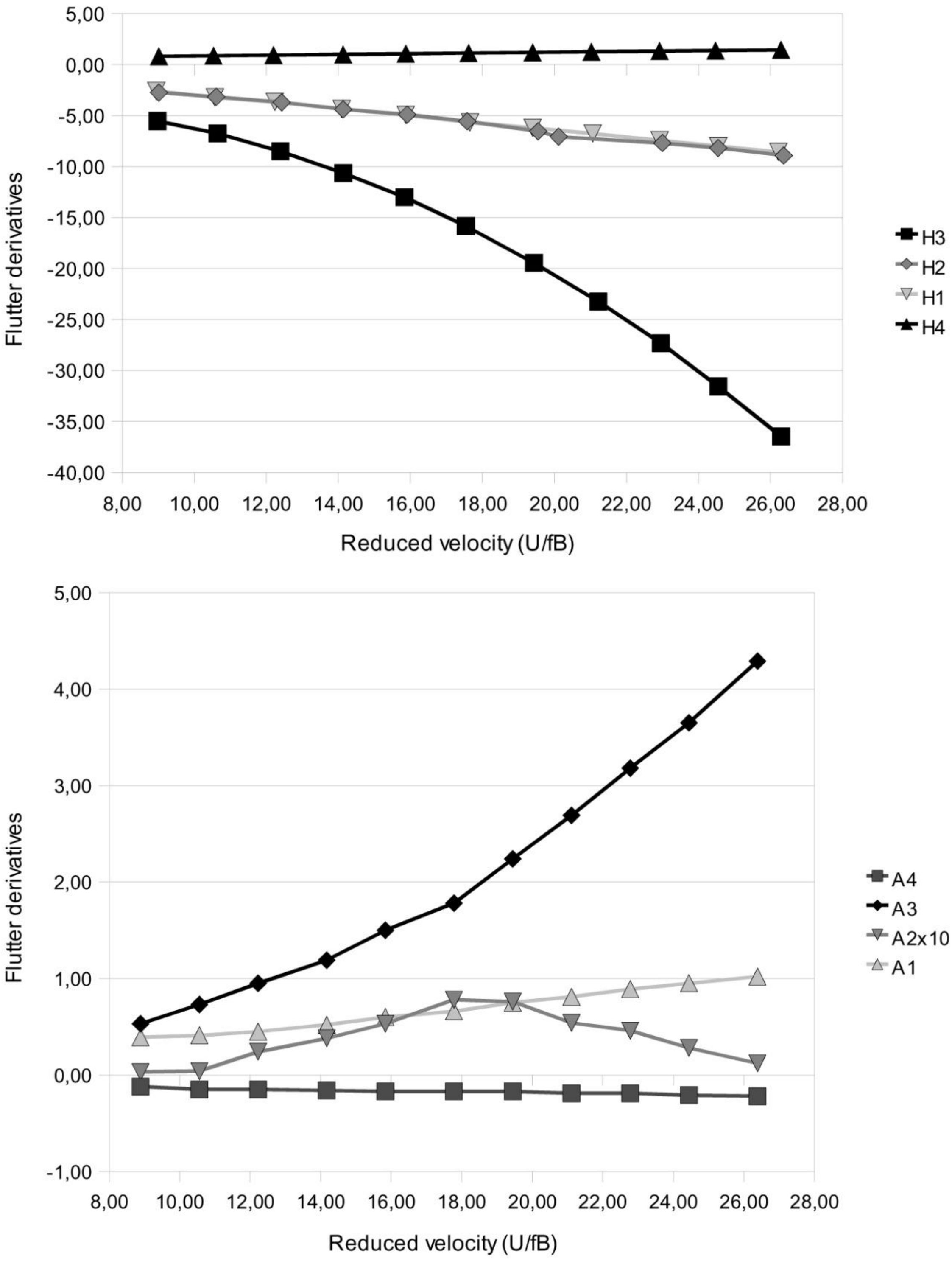

Figure 21: Flutter derivatives of lift (H_i) and pitching moment (A_i) of the bridge deck cross section of the Akashi Kaikyō Bridge central span.

2) Indicial functions data

We have seen in chapter IV.g) how we can express the indicial functions on the basis of the flutter derivatives. We have also seen in IV.a), how we may approximate the indicial functions by a sum of exponential groups:

$$\Phi_M(s) = a_0 - \sum_{i=k}^{n} a_k \exp(-b_k s) \qquad (5.14)$$

In the article by Caracoglia and Jones [20], the values of the parameters of the exponential groups are given. The values in this article are calculated with the flutter derivatives from article [19].

It will be explained in the next paragraph, V.d), how we can obtain the coefficients of the exponential groups in practice with a minimization algorithm.

In order to keep a consistent definition of Φ_M for s approaching infinity, the value of a_0 has been fixed at 1.

	Indicial Functions	a_1	b_1	a_2	b_2
Akashi Bridge	Φ_{Lz}	-0.365	0.021	-11.652	7.235
	$\Phi_{L\alpha}$	-0.392	0.008	-3.653	1.155
	Φ_{Mz}	0.039	0.000	-	-
	$\Phi_{M\alpha}$	0.073	0.025	1.758	7.098

Values of the exponential coefficients for the approximate expressions of the indicial functions as in [20]

The other values we need in order to be able to use the indicial functions are:

- The bridge deck width: $B = 35.50$ m
- The derivates of the static coefficients with respect to the angle of attack, for $\alpha=0$: $C_L' = -1.192$ and $C_M' = 0.307$.

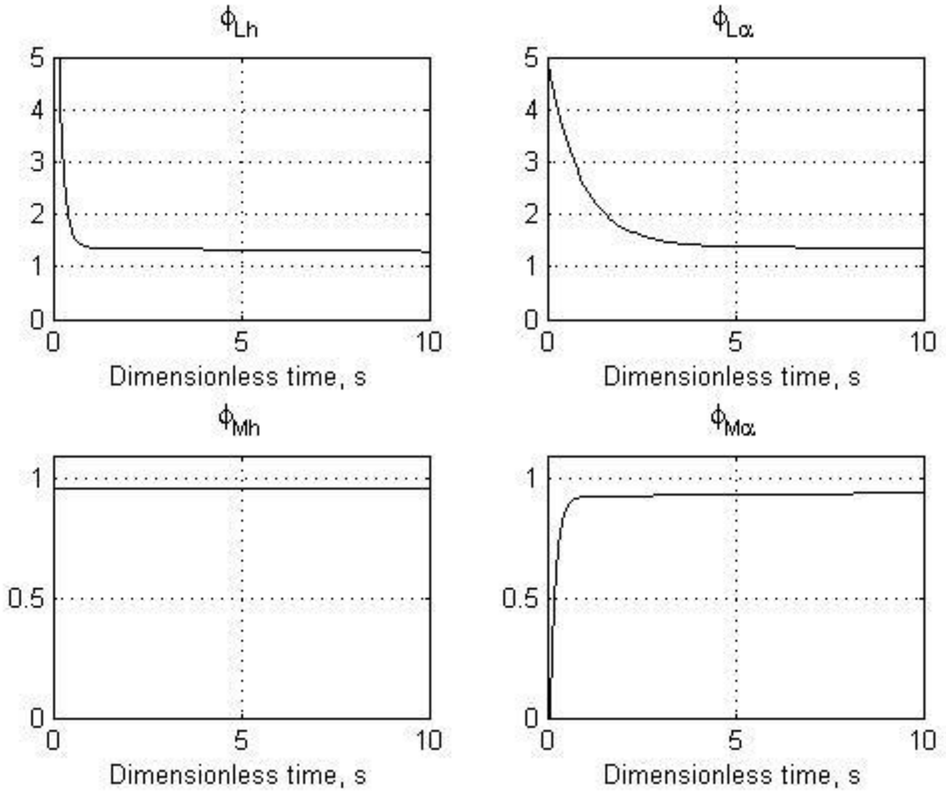

Figure 22: Shape of the four indicial functions for the Akashi-Kaikyō Bridge deck

d) Determination of the parameters of the approximated indicial functions

The indicial functions are calculated with the experimental flutter derivatives data. The formal expression that we have obtained for the indicial functions in chapter IV.g) is an integration on the frequency domain to obtain a function of time through an inverse Fourier transform. For example, for the lift as a function of the vertical motion of the bridge deck:

$$\Phi_{Lz}(\sigma) = \frac{1}{2\pi C_L'} \int_{-\infty}^{+\infty} 2(-iH_1^*(k) - H_4^*(k))\, e^{ik\sigma} dk \qquad (5.15)$$

In practice, this expression cannot be used, as we do not know the values of the flutter derivatives for *k* equal from *0* to *∞*. Therefore, the idea is not to evaluate the indicial functions with this formula and then to interpolate it as a function of time, but to interpolate the curves of the flutter derivatives on the basis of the coefficients of the indicial function.

In IV.c), the Scanlan's approach to indicial functions, we have obtained the following equation for the lift as a function of the vertical motion of the bridge deck:

$$C_L' \left[\sqrt{2\pi}.\overline{\Phi_{Lz}'} + \Phi_{Lz}(0)\right] = K[H_1^*(k) - iH_4^*(k)] \qquad (5.16)$$

Where $\overline{\Phi_{Lz}{}'} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \Phi_{Lz}{}'(\sigma) e^{-ik\sigma} d\sigma$.

If we use the approximate expression of the indicial function,

$$\Phi_{Lz}(s) = a_0 - \sum_{j=1}^{n} a_j \exp(-b_j s)$$

$$\Phi_{Lz}'(s) = \sum_{j=1}^{n} a_j b_j \exp(-b_j s)$$

and apply the Fourier transform to this approximate expression:

$$\overline{\Phi_{Lz}{}'} = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} a_i b_i \int_0^\infty e^{-ik\sigma - b_j\sigma} d\sigma \qquad (5.17)$$

$$\overline{\Phi_{Lz}{}'} = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{n} \frac{a_j {b_j}^2}{k^2 + {b_j}^2} - \frac{ik a_j b_j}{k^2 + {b_j}^2} \qquad (5.18)$$

The term $\Phi_{Lz}(0)$ is worth:

$$\Phi_{Lz}(0) = a_0 - \sum_{j=1}^{n} a_j \qquad (5.19)$$

Separating the real part and the imaginary part:

$$\sqrt{2\pi}.\overline{\Phi_{Lz}'} + \Phi_{Lz}(0) = F_{Lh}(k) + iG_{Lh}(k) \qquad (5.20)$$

With,

$$F_{Lh}(k) = a_0 + \sum_{j=1}^{n} -a_j + \frac{a_j {b_j}^2}{k^2 + {b_j}^2}$$

$$G_{Lh}(k) = -k\left[\sum_{j=1}^{n} \frac{a_j b_j}{k^2 + {b_j}^2}\right]$$

and using equation (5.16), we have two expressions of the same quantity, $\sqrt{2\pi}.\overline{\Phi'_{Lz}} + \Phi_{Lz}(0)$:

$$F_{Lh}(k) + iG_{Lh}(k)$$

$$\frac{K}{C'_L}[H_1^*(k) - iH_4^*(k)]$$

The flutter derivatives are determined experimentally for a given number of frequencies, k_l. We have to determine the coefficients of the indicial functions on the basis of these data. The identification of the coefficients is performed in [20] through a non-linear least squares method, by minimizing the squared-error function, also called residual. Still for the same case, the lift as a function of the vertical motion, the residual is defined this way [21]:

$$R(a_k, b_k) = \sum_{l=1}^{M}\left[\frac{F_{Lh}(k_l)}{2k_l} - \frac{H_1^*(k_l)}{C'_L}\right]^2 + \left[\frac{G_{Lh}(k_l)}{2k_l} + \frac{H_4^*(k_l)}{C'_L}\right]^2 \qquad (5.21)$$

With M, the number of frequencies for which the flutter derivatives have been measured experimentally.

In the article [20], the number of exponential groups of the indicial function, n, may vary from 1 to 4. It is determined by a statistical analysis. Sometimes, it is simply fixed at 2.

Now, we have an interpolation of the flutter derivatives. Still for the same example, the interpolations $\widetilde{H}_1^*$ and $\widetilde{H}_4^*$ as a function of the exponential coefficients are worth:

$$\widetilde{H}_1^*(k) = \frac{C'_L}{2k}\left[a_0 + \sum_{j=1}^{n} -a_j + \frac{a_j {b_j}^2}{k^2 + {b_j}^2}\right]$$

$$\widetilde{H}_4^*(k) = -\frac{C'_L}{2}\left[\sum_{j=1}^{n} \frac{a_j b_j}{k^2 + {b_j}^2}\right]$$

$$(5.22)$$

In Figure 23, we have traced the graphs of the experimental flutter derivatives data and of the interpolated flutter derivatives. The interpolation of the flutter derivatives is not the best that we could make because we interpolate these curves with the coefficients of the sum of the exponential groups used for interpolating the indicial functions. Anyway, it gives good results except for the flutter derivatives A_2 and A_3.

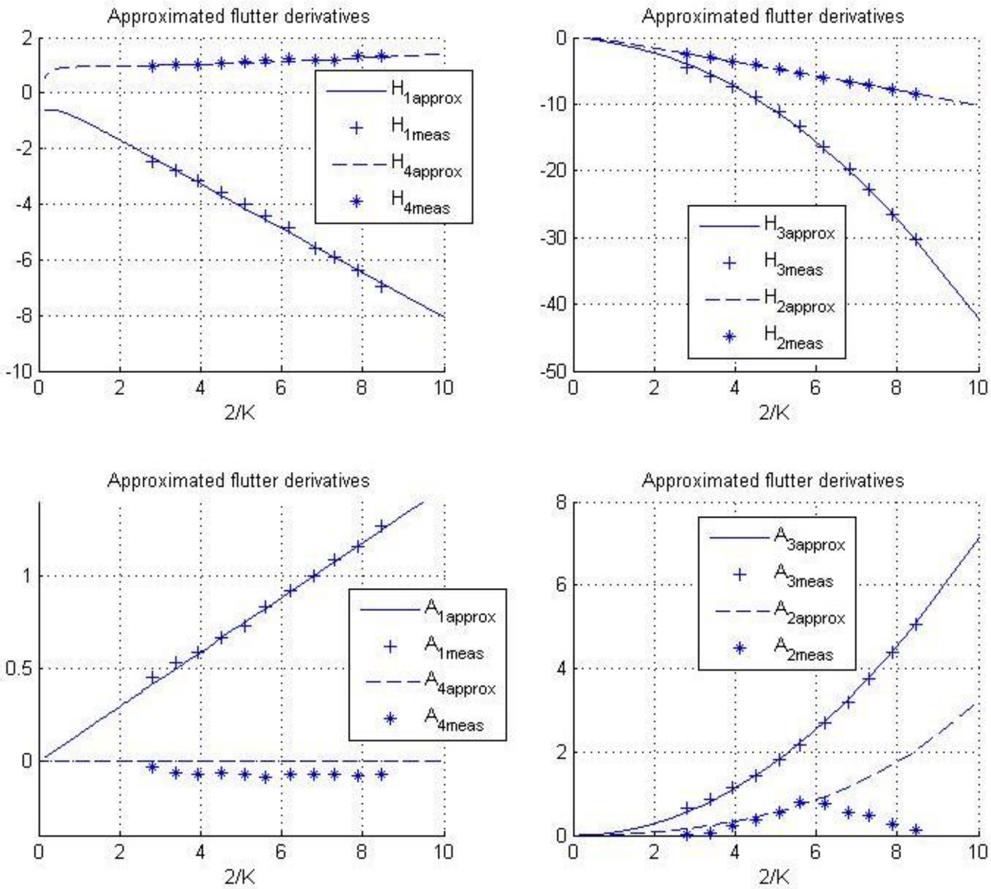

Figure 23: Experimental flutter derivatives (with "+") and interpolated flutter derivatives with continuous line.

2^{nd} Part: Implementation

VI) Implementation of the procedure

Purpose of the program is to create an executable file that evaluates the wind forces acting on a bridge deck at a given time, knowing the aerodynamic characteristics of the bridge deck and the time-history of its motion. The executable file is supposed to be only a module of larger computing framework like a finite elements software: it is called by this software at each time step to evaluate the wind forces. So, from one step to another, it is supposed to neither store any information nor interact with the main software.

a) Functioning of the main program

The main executable file reads in input the following information stored as text files:

- The bridge deck static aerodynamic coefficients: C_D, C_L, C_M
- The bridge deck indicial functions parameters, under the form of the exponential approximation (a_i, b_i), the bridge deck width, B, and the derivates of the static aerodynamic coefficients for $\alpha=0$, C_L', C_M'.
- The bridge deck present position ($\alpha(t)$ and $\dot{z}(t)$)
- The bridge deck time-history motion

It provides in output the wind forces of drag, lift and pitching moment in the fixed referential (*Oxy*), as defined in Figure 17. Cf. the 2nd Appendix for the FORTRAN code.

The procedure has been implemented under the hypotheses made in chapter V). According to these hypotheses, the wind forces can be divided in three parts: the quasi static component, the self-excited component and the buffeting component.

The input information is organized in five input files that are:

- In an input file, we store the bridge static aerodynamic coefficients and the mean velocity. As they are functions of the angle of attack, they are defined under the form of the coefficients of a polynomial of degree 7.

- In another input file, we store the bridge deck indicial functions parameters which are 5 for each of the 4 functions, the bridge deck width and the derivate of the static aerodynamic coefficients for $\alpha=0$.
- A last input file provides us the air gust vertical velocity and the mean velocity of the wind.

These first three files are not changed when executing the compiled file. These are data related to a specific problem, i.e. a given deck section, and a given wind configuration.

- The file of the bridge deck present position can be provided from different sources. If we work with ANSYS, it will be given by ANSYS each time we have to evaluate the forces. If we are in a control phase of the development, it can be given by the user under the form of a forced motion of the bridge deck.
- The bridge deck time-history has to be loaded each time we execute the compiled file. This file is managed by the program itself. Inside, we store the information on the bridge deck motion at each time path in order to keep the history of its motion. This file is re-written with the last position of the bridge deck before closing the procedure.

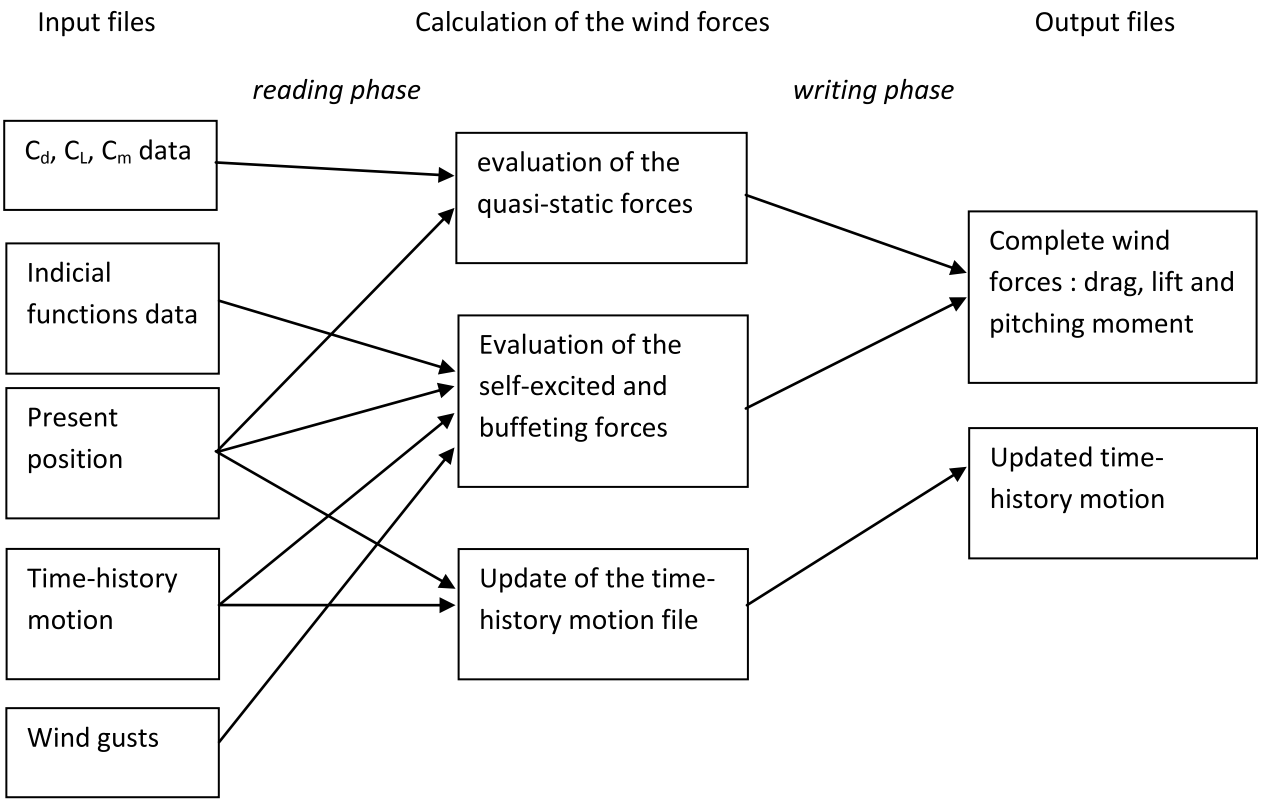

Figure 24: Data flow diagram of the implemented procedure

This data flow has been chosen in order to have the program evaluate the wind forces with given information on the bridge deck at the present time of the problem. Since, in practice, for one studied bridge deck motion, we always evaluate the wind forces at different times, we could have chosen to store the time-history motion of the bridge deck inside the program under the form of a matrix variable that we update at every time step when we evaluate the wind forces. By updating an internal variable of the program, instead of reading and writing an external file at each time step, we could save some calculation time, but we would lose the modular functioning of the program. Indeed, the program has been created in order to work as a module of a main routine or program, such as ANSYS. In this case, ANSYS executes the compiled file at each time step and closes it once the forces evaluated. As ANSYS does not store the time-history motion and that the program is closed at each time step, the only way to keep memory of the history of the motion is to store it in a file that is read at each time step when the compiled file is executed and re-written updated with the present position of the bridge deck before closing the file.

The dimension of the time-history input file is limited. As we have seen in IV.a), we do not need the history of the bridge deck motion from an infinite time till the present time. For this reason, we can limit the size of this file: each time we update it with the last position of the bridge deck, we erase the oldest position that was still kept in memory. One line in this input file corresponds to one time in the time-history. Since the convolution product is done on the time-history motion of the bridge deck, the number of lines of this input file, N, is linked with the integration time, s_L, as s_L is worth $N*ds$.

The functioning of this module with ANSYS does not allow a complete coupling of both entities. Indeed, at each time step, first, we evaluate the wind forces and, then, ANSYS solves the dynamic equation and evaluates the present position of the bridge deck. Therefore for a time step t, the present position that is the output of ANSYS does not correspond with the input information used in our procedure. We evaluate the wind forces at a time $t-\Delta t$, whereas ANSYS solves the dynamic equation at a time t. In order to solve this conflict, even by keeping both entities separated, we should go from solving the equations in one entity to solving them in the other, in a while loop, till the wanted convergence is reached. So, we would have integrated both entity into a while loop that would stop once convergence of the present position of the bridge deck reached.

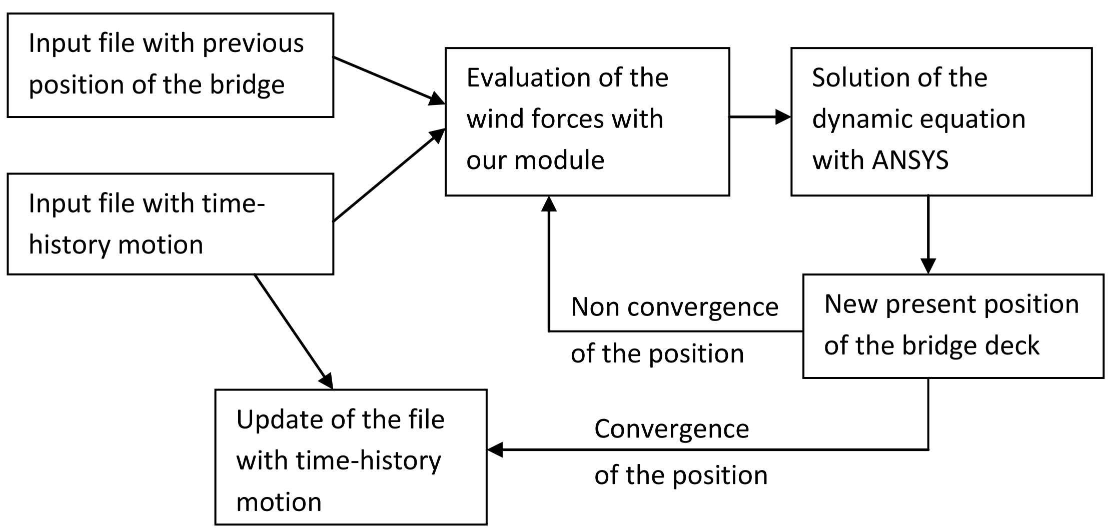

Figure 25: Complete coupling between our module and ANSYS

This coupling, as explained in Figure 25, is not possible with ANSYS. Indeed, ANSYS does not understand that it should not pass to the next time step when solving the dynamic equation. Therefore, when solving the dynamic equation, automatically, ANSYS assumes also that we pass from a present time t to time $t+\Delta t$.

b) Quasi static forces

The quasi static forces are defined with the aerodynamic static coefficients. We have chosen to approximate the experimental data with a polynomial of degree 7, because the number of measurements on the interval that concerns us is 8 for the Akashi Kaikyō Bridge. If the number of measurements is bigger, this interpolation procedure still works. Our interpolation of the static aerodynamic coefficients has been made only for angles of attack in the range -6° ; +6°, because we know that the theory of aeroelasticity does not work for angles of attack bigger than ±5°.

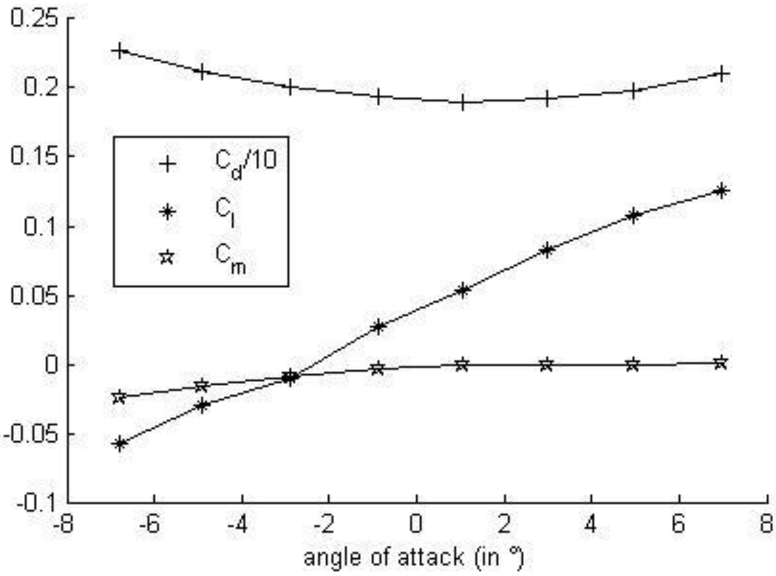

Figure 26: aerodynamic static coefficients. With the crosses are the measured values, with the continuous line, the interpolations

The evaluation of the quasi-static forces is quite simple, as it depends only on the mean velocity, the bridge deck width, the air density and the angle of attack that is worth the rotation of the bridge deck with respect to the mean velocity. A subroutine has been created in the program that evaluates the static aerodynamic coefficients for a given value of the angle of attack.

c) Self-excited forces and buffeting

As we have seen in V.a), we evaluate the self-excited forces and the buffeting forces together, by assuming that the vertical gust velocities are added to the vertical motion of the bridge deck. To evaluate the self-excited and buffeting forces, we have to consider the time history of the bridge deck through a convolution product of the time-history and of the indicial functions. This convolution requires an operation of integration. If we consider the case of the lift as a function of the angle of attack, the expression is:

$$F_{Lse}(s) = \frac{1}{2}\rho U^2 B C'_L \left[\Phi_{L\boldsymbol{\alpha}}(0)\alpha(s) + \int_{-\infty}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma \right] \qquad (6.1)$$

We know from IV.a), that this integral can be transformed in practice into:

$$L(s) = qBC_L' \left[\Phi_{L\alpha}(0)\alpha(s) + \int_{s-s_L}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma \right] \qquad (6.2)$$

By transforming the integral into a discrete sum and translating the time axis in order that the present time is worth 0:

$$L(s) = qBC_L' \left[\Phi_{L\alpha}(0)\alpha(0) + \sum_{k=-k_L}^{0} \Phi_{L\alpha}{}'(-k\Delta s)\ \alpha(k\Delta s)\ \Delta s \right] \qquad (6.3)$$

Where k_L is defined as the floor function of $s_L/\Delta s$.

A subroutine evaluates $\Phi_{L\alpha}(0)$, the value of the indicial function in 0 and another subroutine evaluates the derivatives of the indicial functions, $\Phi_{L\alpha}{}'(-k\Delta s)$, for every integration time step, with the exponential groups formula.

d) Initialization of the time-history input file

As the time-history input file is called in the main program of the module, it has to be initialized. This operation is realized in another procedure that has to be executed before executing the main program. This procedure creates the file that contains the time history-motion of the bridge deck and sets to zero the time-history motion. The files that are created while initializing are called "angolodatacco.txt" for the angle of attack, "dangolodatacco.txt", for the derivative of the angle of attack, "dspostvert.txt", for the vertical velocity and "d2spostvert.txt", for the vertical acceleration. Cf. the 2nd Appendix, paragraph b), to see the procedure coded.

3^{rd} Part: Practical applications

VII) Validation of the program

In order to ensure that the program we have realized gives true results, we have to validate it. The validation procedure that has been adopted is the following one:

- we assume forced harmonic oscillations of the degrees of freedom of the bridge deck section for a given frequency in input of the program
- we have the program evaluate the wind forces on a time interval worth at least twice the integration time ($s_l=20$)
- we compare the obtained results with the flutter derivatives for the oscillation frequency chosen.

a) Validation procedure

The validation procedure we use to control the program is to assume a harmonic motion of the bridge deck in input of the program and have the program evaluate the self excited forces induced. Using the program this way is like recreating the wind tunnel experience, by forcing the oscillations of the bridge deck and measuring the wind forces induced.

In order to create these forced oscillations, the program implemented in VI) has been slightly modified. We have inserted the procedure initially implemented into a loop on time that generates the present position of the bridge deck for the different times of the study. In the 2^{nd} Appendix, paragraph a), the program used for the validation of the procedure has been enclosed. The loop on the variable *J* is the loop that we have added.

1) Forced oscillations of the angle of attack

We assume a unitary harmonic motion of the bridge deck attack angle under the form:

$$\alpha(s) = \cos(ks) \qquad (7.1)$$

This unitary oscillation is harmonic and function of the dimensionless time, *s*.

In output of our program, we obtain the following forces, lift, F_z and pitching moment, M_y:

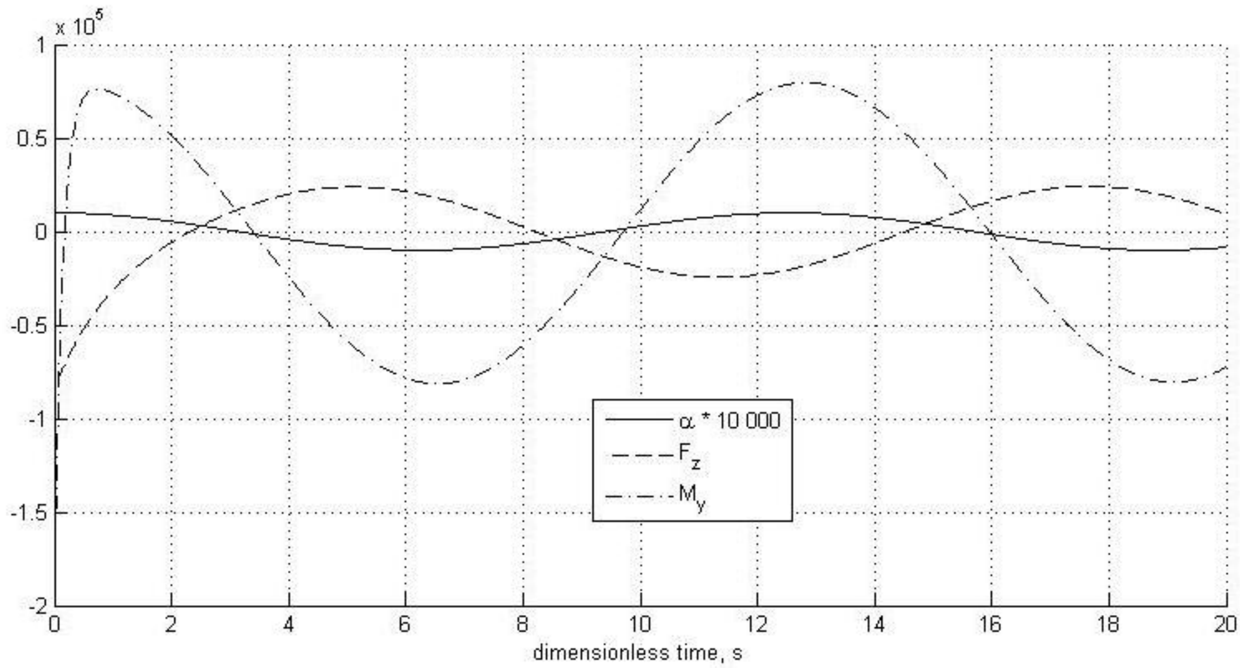

Figure 27: Input and output of the program for an oscillating angle of attack

To obtain Figure 27, we have fixed the following parameters:

- Aerodynamic coefficients for the indicial functions of the Akashi- Kaikyō Bridge
- reduced frequency: $k = 0.5$
- dimensionless time path, $ds = 0.01$
- integration time, $s_L = 20$
- mean wind speed, $U = 20m/s$
- the lift force, F_Z is expressed in N/m
- the pitching-moment, M_y, is expressed in Nm/m

These conditions are the one corresponding with the FORTRAN code in Appendix 2.b).

After a brief transient time (circa *s=2*), we note that the induced wind forces are harmonic too, with the same frequency of oscillation. The phase and the amplitude are different between these two signals.

In order to compare these results with the expected one, we use the well-known equation by Scanlan and Simiu [10], considering only the angle of attack:

$$\begin{cases} L = \frac{1}{2}\rho B U^2 \left(K H_2^* \frac{B\dot{\alpha}}{U} + K^2 H_3^* \alpha \right) \\ M = \frac{1}{2}\rho B^2 U^2 \left(K A_2^* \frac{B\dot{\alpha}}{U} + K^2 A_3^* \alpha \right) \end{cases} \tag{7.2}$$

The angle of attack is worth: $\alpha(s) = \cos(ks)$ or, as a function of dimensional parameters: $\alpha(t) = \cos(\omega t)$, as $s = 2Ut/B$ and $k = \omega B/2U = K/2$.

Therefore,

$$\dot{\alpha} = \frac{\partial \alpha}{\partial t} = -\omega \sin(ks) = -\frac{KU}{B} sin(ks) \tag{7.3}$$

And so,

$$\begin{cases} L = \frac{1}{2}\rho B U^2 K^2 \left(-\widetilde{H}_2^* \sin(ks) + \widetilde{H}_3^* \cos(ks) \right) \\ M = \frac{1}{2}\rho B^2 U^2 K^2 \left(-\tilde{A}_2^* \sin(ks) + \tilde{A}_3^* \cos(ks) \right) \end{cases} \tag{7.4}$$

Where $\widetilde{H}_i^*$ and $\tilde{A}_i^*$ are the interpolated flutter derivatives, as defined in chapter V.d). We use these values instead of the measured ones, because the indicial functions coefficients are defined on the basis of the interpolated flutter derivatives.

We can reformulate the lift and pitching moment as a function of an amplitude and phase differences between both signals:

$$\begin{cases} L = \frac{1}{2}\rho B U^2 (\mathrm{A_L} \cos(ks + \varphi_L)) \\ M = \frac{1}{2}\rho B^2 U^2 (\mathrm{A_M} \cos(ks + \varphi_M)) \end{cases} \tag{7.5}$$

Where:

- A_L is the lift amplitude defined as: $A_L = K^2 \sqrt{\widetilde{H}_2^{*2} + \widetilde{H}_3^{*2}}$
- A_M is the pitching-moment amplitude defined as: $A_L = K^2 \sqrt{\widetilde{H}_2^{*2} + \widetilde{H}_3^{*2}}$

- φ_L is the lift phase difference defined as: $\varphi_L = arg(\widetilde{H}_2^*, \widetilde{H}_3^*)$
- φ_M is the pitching-moment phase difference defined as: $\varphi_L = arg(\tilde{A}_2^*, \tilde{A}_3^*)$.

We have to pay a particular attention in the use of the function *arg*. This function is defined on the base of the function *atan*, also called *arctan* or *tan*$^{-1}$ that is the arctangent function. The function *arg* is defined as follows:

$$\arg(y, x) = \begin{cases} \operatorname{atan}(y/x) & if\ x > 0 \\ \pi + \operatorname{atan}(y/x) & if\ x < 0 \text{ and } y \geq 0 \\ -\pi + \operatorname{atan}(y/x) & if\ x < 0 \text{ and } y < 0 \\ \pi \cdot sign(y) & if\ x = 0 \end{cases} \qquad (7.6)$$

In the case of the phase difference of the lift, $x = \widetilde{H}_2^*$ and $y = \widetilde{H}_3^*$. The amplitude and phase differences are functions of the frequency of oscillation. To validate the program, we have the program run for a certain number of frequencies of oscillation. Then, we determine the amplitude and phase differences of the resulting forces in output with respect to the input harmonic signal. And finally, we compare the amplitude and phase differences of these two pieces of information: they should be equal.

2) Forced oscillations of the vertical velocity

By doing the same reasoning for the lift and the moment as a function of the vertical motion of the airfoil, we have the following equations for the vertical motion of the bridge deck and for the lift and the pitching moment:

$$z'(s) = \cos(ks) \qquad (7.7)$$

$$z(s) = \frac{\sin(ks)}{\mathrm{k}}$$

$$\dot{z}(s) = \frac{\partial z}{\partial t} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial t} = \frac{2\mathrm{U}}{\mathrm{B}}\cos(ks) \qquad (7.8)$$

$$\begin{cases} L = \frac{1}{2}\rho B U^2 \left(K H_1^* \frac{\dot{z}}{U} + K^2 H_4^* \frac{z}{B} \right) \\ M = \frac{1}{2}\rho B^2 U^2 \left(K A_1^* \frac{\dot{z}}{U} + K^2 A_4^* \frac{z}{B} \right) \end{cases} \qquad (7.9)$$

Modifying the expression of the lift and of the moment:

$$\begin{cases} L = \frac{1}{2}\rho B U^2 \frac{2K}{B}\left(H_1^* \cos{(ks)} + H_4^* \sin(ks)\right) \\ M = \frac{1}{2}\rho B^2 U^2 \frac{2K}{B}\left(A_1^* \cos{(ks)} + A_4^* \sin(ks)\right) \end{cases} \quad (7.10)$$

which can also be written under the form of an amplitude coefficient and of a phase difference:

$$\begin{cases} L = \frac{1}{2}\rho B U^2 \left(A_L \cos{(ks + \varphi_L)}\right) \\ M = \frac{1}{2}\rho B^2 U^2 \left(A_M \cos{(ks + \varphi_M)}\right) \end{cases} \quad (7.11)$$

Where:

- $A_L = 2K/B\sqrt{H_1^{*2} + H_4^{*2}}$
- $A_M = 2K/B\sqrt{A_1^{*2} + A_4^{*2}}$
- $\varphi_L = arg(-H_4^*, H_1^*)$
- $\varphi_M = arg(-A_4^*, A_1^*)$

To validate the self-excited wind forces as functions of the vertical velocity of the bridge deck, we should assume the angle of attack being equal to zero, and create forced oscillations of the vertical velocity of the bridge deck. Finally, as with the angle of attack, we can just compare both results by comparing the amplitude and phase differences.

For both of the examples treated below, the parameters of the program have been fixed:

- the dimensionless time path is worth: *ds*=0.01
- the dimension of the time-history input file is 2000 lines: the integration time is worth s_L= 20

b) <u>Akashi Kaikyō Bridge: forced oscillations of the bridge deck</u>

1) Forced oscillations of the angle of attack

Using the formula obtained in VII.a.1), we evaluate what should be the amplitude and phase differences for the lift as a function of the angle of attack, in theory. Cf. 3rd Appendix, paragraph a), for the MATLAB code used for the determination of the graphs of amplitude and phase difference.

Then, using the MATLAB code enclosed in the 3rd Appendix, paragraph b), we can recover the amplitude and phase differences from the wind forces in output of the FORTRAN procedure.

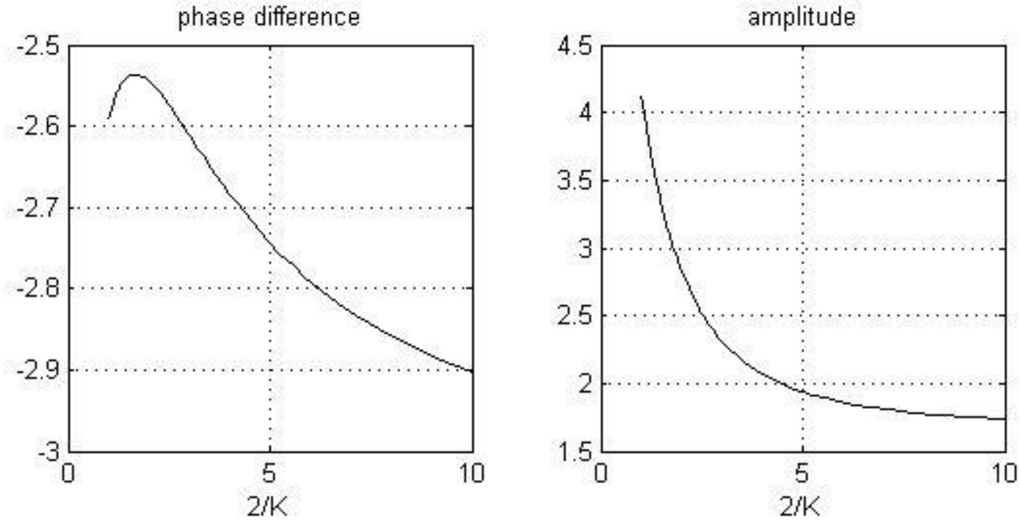

Figure 28: phase difference and amplitude for the lift as a function of the reduced velocity of oscillation of the angle of attack

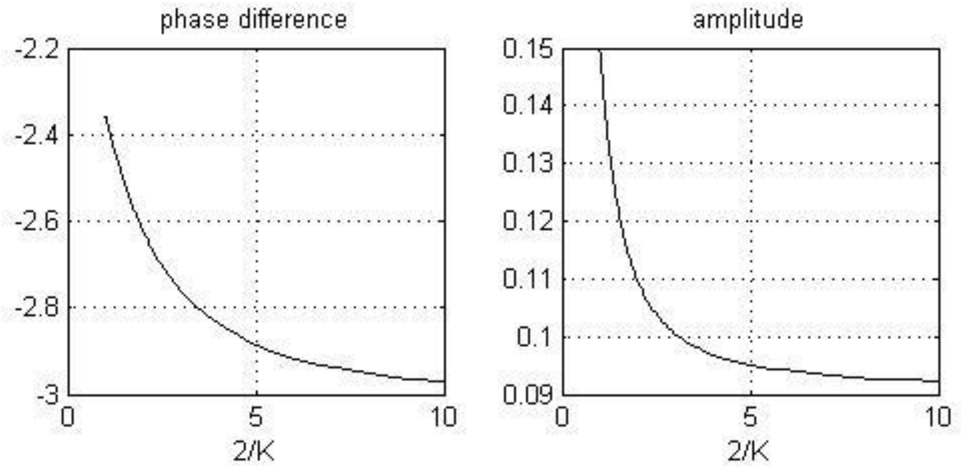

Figure 29: phase difference and amplitude for the moment as a function of the reduced velocity of oscillation of the angle of attack

For *k=0.2*:

	From the flutter derivatives	In output of the program	Validated
Amplitude of the lift	1.94	1.97	Yes
Phase difference of the lift	-2.74	-2.75	Yes
Amplitude of the moment	0.298	0.265	Yes
Phase difference of the moment	0.095	-0.07	Yes

For *k=0.5*:

	From the flutter derivatives	In output of the program	Validated
Amplitude of the lift	2.84	2.85	Yes
Phase difference of the lift	-2.54	-2.56	Yes
Amplitude of the moment	0.29	0.27	Yes
Phase difference of the moment	-0.27	-0.15	Yes

For *k=1*:

	From the flutter derivatives	In output of the program	Validated
Amplitude of the lift	4.13	4.12	Yes
Phase difference of the lift	-2.59	-2.60	Yes
Amplitude of the moment	0.28	0.27	Yes
Phase difference of the moment	-0.27	-0.28	Yes

2) Forced oscillations of the vertical velocity of the bridge deck

Using the formula obtained in VII.a.1), we evaluate what should be the amplitude and phase differences for the lift as a function of the vertical velocity of the bridge deck, in theory.

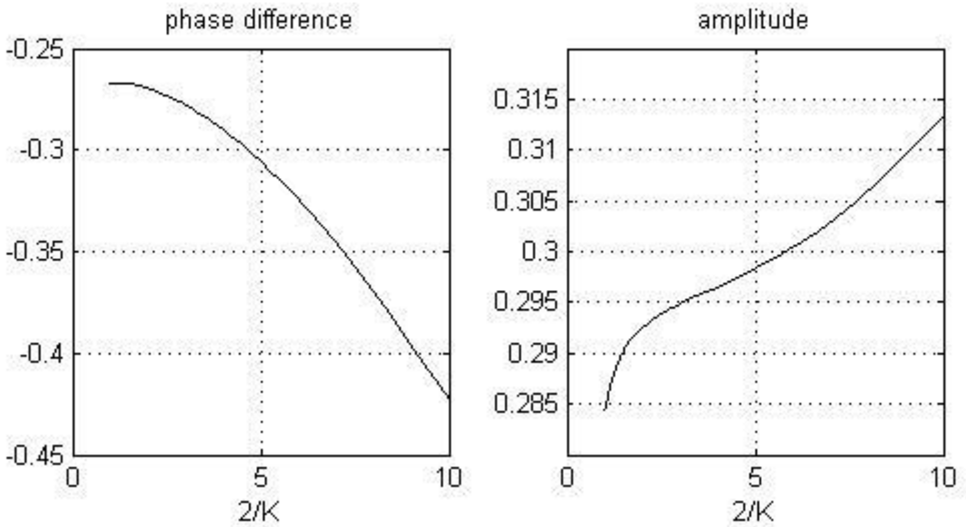

Figure 30: phase difference and amplitude for the lift as a function of the reduced velocity of oscillation of the vertical motion

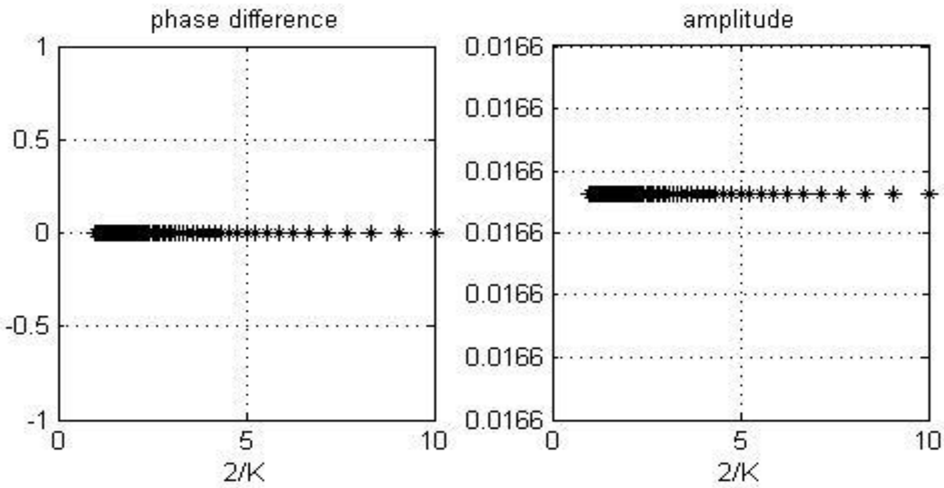

Figure 31: phase difference and amplitude for the moment as a function of the reduced velocity of oscillation of the vertical motion

For *k=0.2*:

	From the flutter derivatives	In output of the program	Validated
Amplitude of the lift	0.095	0.123	Yes (30% error)
Phase difference of the lift	-2.73	-2.88	Yes
Amplitude of the moment	0.017	0.017	Yes
Phase difference of the moment	0	0	Yes

For *k=0.5*:

	From the flutter derivatives	In output of the program	Validated
Amplitude of the lift	0.11	0.138	Yes (30% error)
Phase difference of the lift	-2.62	-2.73	Yes
Amplitude of the moment	0.017	0.017	Yes
Phase difference of the moment	0	-0.01	Yes

For *k=1*:

	From the flutter derivatives	In output of the program	Validated
Amplitude of the lift	0.15	0.17	Yes
Phase difference of the lift	-2.35	-2.51	Yes
Amplitude of the moment	0.017	0.017	Yes
Phase difference of the moment	0	-0.01	Yes

c) Tsurumi Bridge: Forced oscillations of the bridge deck

We proceed in the same way to validate the program with the Tsurumi bridge cross-section. First, we evaluate the amplitude of oscillation and the phase for the lift and the pitching-moment as a function of the angle of attack and of the vertical velocity of the bridge deck with the interpolated flutter derivatives. Then we have the program run with forced oscillation for a given set of frequency values and we evaluate the amplitude and the phase difference of the response. Finally, we compare the expectable results and the obtained ones.

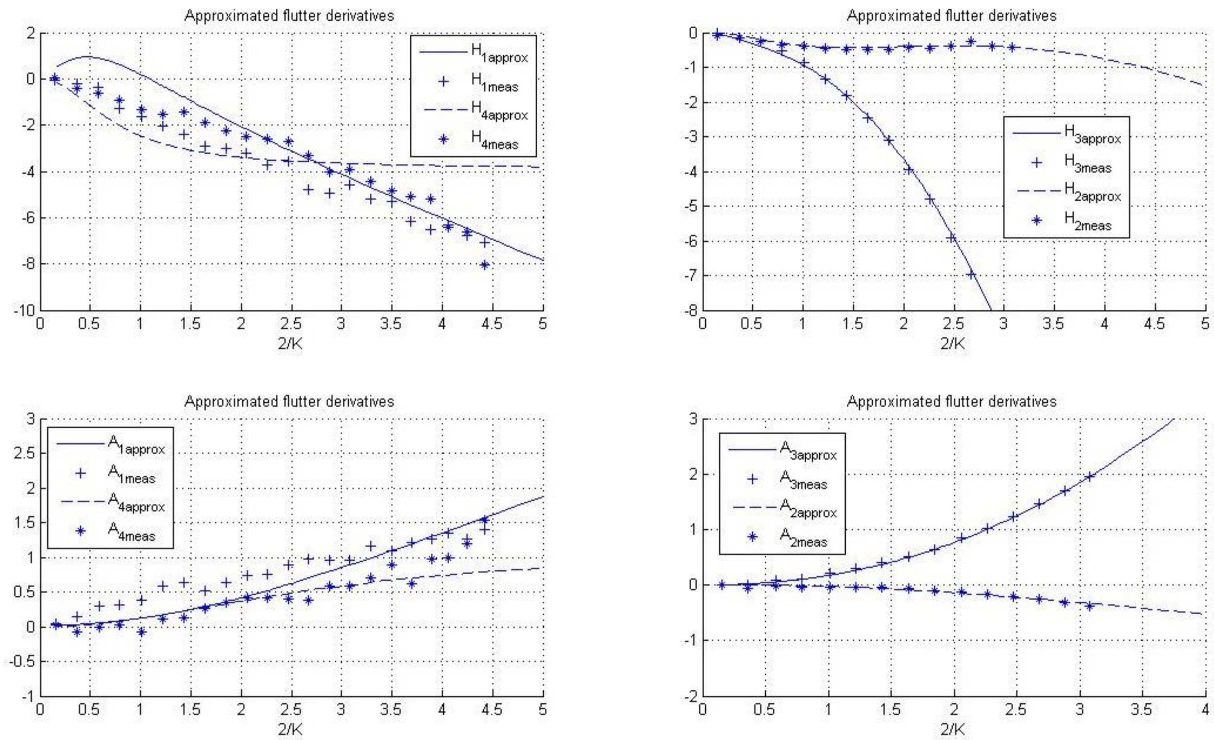

Figure 32: Flutter derivatives of the Tsurumi bridge deck

Exponential coefficients for the approximation of the indicial functions of the Tsurumi Fairway Bridge:

	Indicial Functions	a_1	b_1	a_2	b_2	a_3	b_3
Tsurumi Fairway	Φ_{Lz}	3.035	1.316	-	-	-	-
	$\Phi_{L\alpha}$	-1.868	1.978	0.784	0.559	-0.334	0.101
	Φ_{Mz}	0.829	0.348	-	-	-	-
	$\Phi_{M\alpha}$	0.305	0.390	-	-	-	-

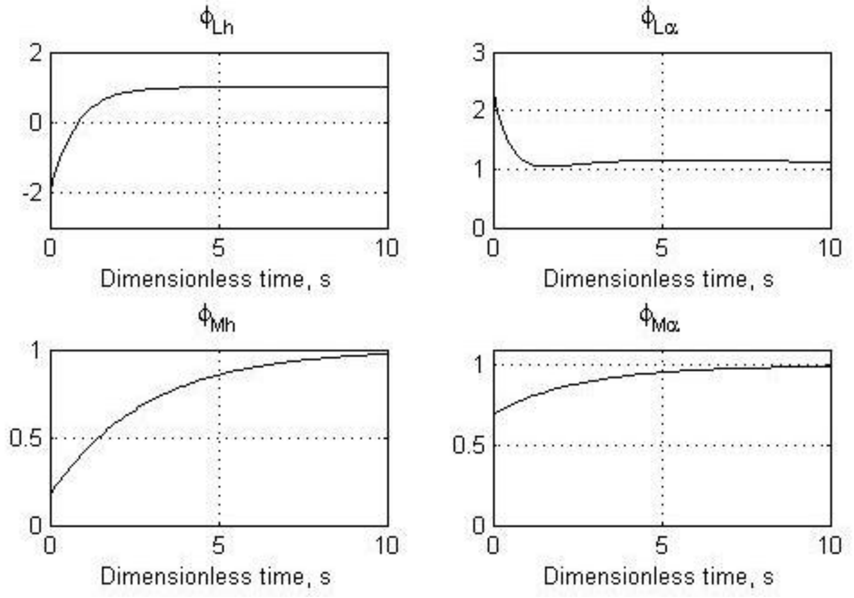

Figure 33: Shape of the four indicial functions of the Tsurumi Fairway Bridge

The other values we need for the evaluation of the self-excited forces are:

- B = 38m
- C_L'=-3.370
- C_M'=0.943

1) Forced oscillations of the bridge deck attack angle

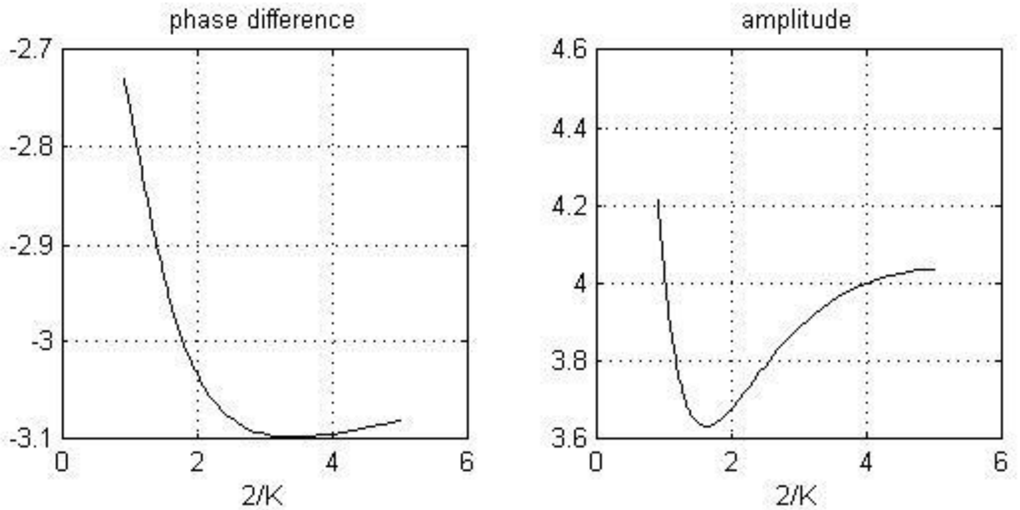

Figure 34: Amplitude of oscillation and phase difference for the lift for a harmonic angle of attack

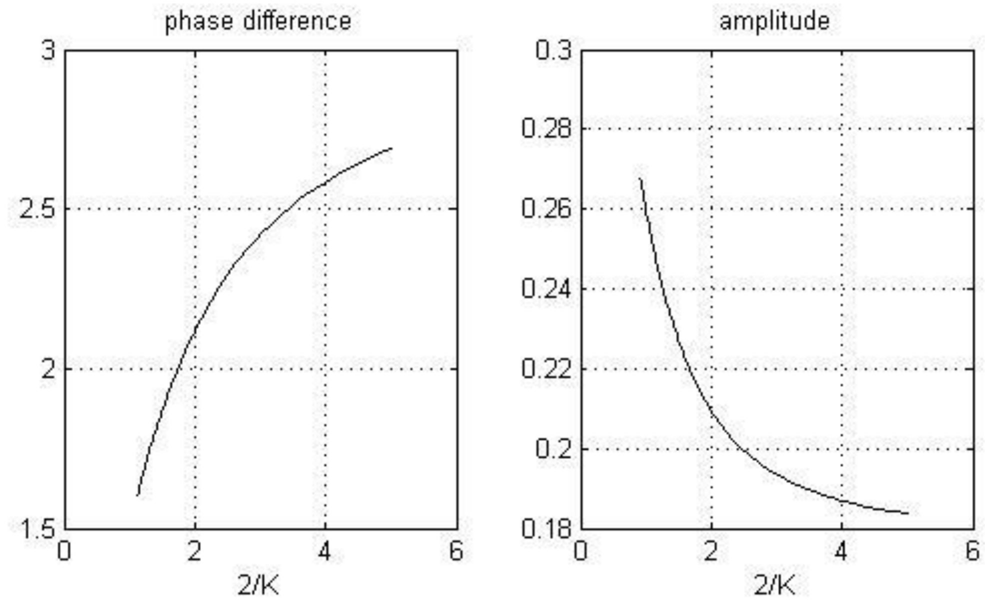

Figure 35: Amplitude of oscillation and phase difference for the pitching moment for a harmonic angle of attack

For *k=0.2*:

	From the flutter derivatives	In output of the program	Validation
Amplitude of the lift	4.04	4.12	Yes
Phase difference of the lift	-3.08	-3.07	Yes
Amplitude of the moment	0.89	0.89	Yes
Phase difference of the moment	-0.13	-0.13	Yes

For *k=0.5*:

	From the flutter derivatives	In output of the program	Validation
Amplitude of the lift	3.68	3.74	Yes
Phase difference of the lift	-3.03	-3.04	Yes
Amplitude of the moment	0.78	0.78	Yes
Phase difference of the moment	-0.18	-0.19	Yes

For *k=1*:

	From the flutter derivatives	In output of the program	Validation
Amplitude of the lift	4.07	4.05	Yes
Phase difference of the lift	-2.78	-2.76	Yes
Amplitude of the moment	0.70	0.70	Yes
Phase difference of the moment	-0.15	-0.14	Yes

2) Forced oscillations of the bridge deck vertical velocity

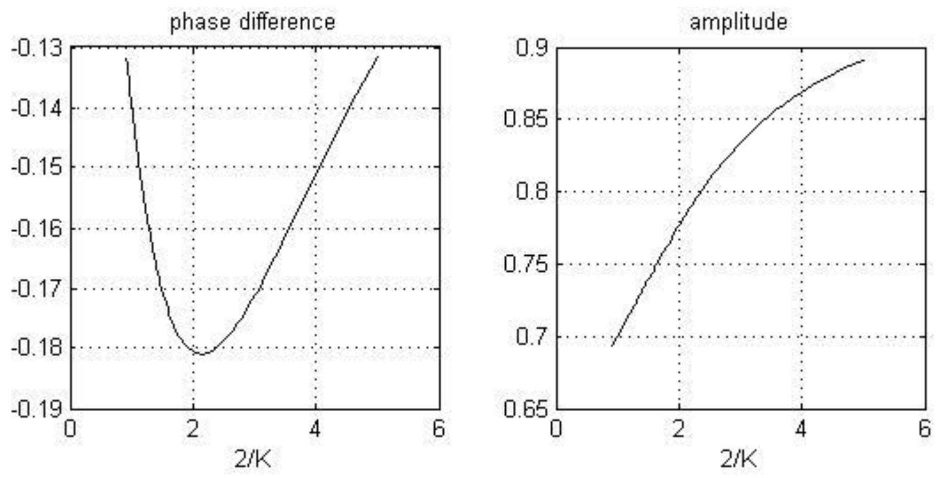

Figure 36: Amplitude and phase difference of the lift for a harmonic vertical velocity of the bridge deck

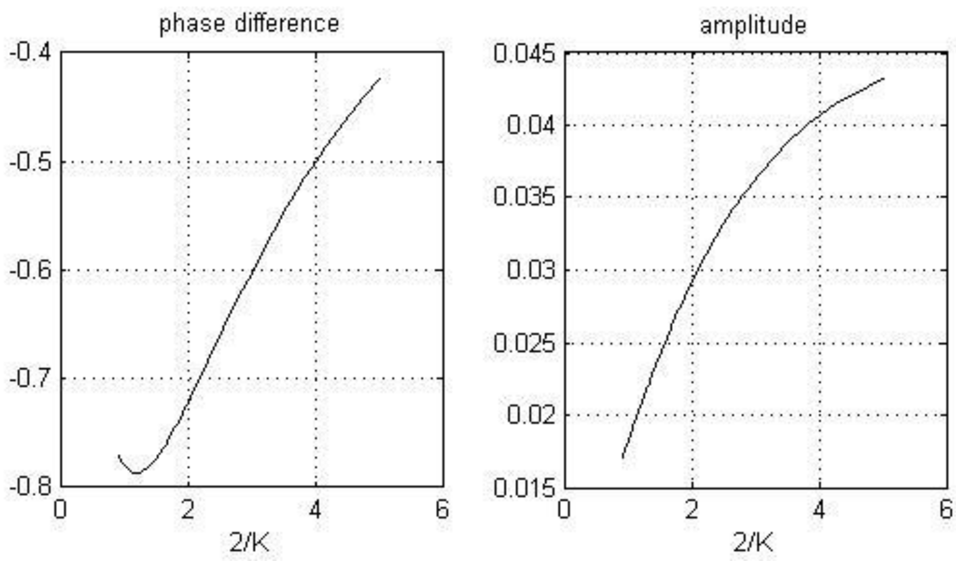

Figure 37: Amplitude and phase difference for the pitching-moment for a harmonic vertical velocity of the bridge deck

For *k=0.2*:

	From the flutter derivatives	In output of the program	Validation
Amplitude of the lift	0.18	0.18	Yes
Phase difference of the lift	2.69	2.69	Yes
Amplitude of the moment	0.043	0.043	Yes
Phase difference of the moment	-0.42	-0.42	Yes

For *k=0.5*:

	From the flutter derivatives	In output of the program	Validation
Amplitude of the lift	0.21	0.21	Yes
Phase difference of the lift	2.12	2.11	Yes
Amplitude of the moment	0.029	0.029	Yes
Phase difference of the moment	-0.72	-0.73	Yes

For *k=1*:

	From the flutter derivatives	In output of the program	Validation
Amplitude of the lift	0.26	0.26	Yes
Phase difference of the lift	1.50	1.43	Yes
Amplitude of the moment	0.018	0.018	Yes
Phase difference of the moment	-0.78	-0.78	Yes

- Conclusion

In the last two paragraphs, we have validated the procedure implemented in VI) by evaluating the self-excited forces on two cases, the Akashi-Kaikyō Bridge and the Tsurumi Fairway Bridge. The difference between the results that we expected from the flutter derivatives interpolation and the results obtained in output of the program is always under 10% of error. The biggest error that is made is for the pitching-moment as a function of the angle of attack for the Akashi-Kaikyō Bridge. In this case, the error is of nearly 30%. It is due to the shape of the corresponding indicial function (cf. Figure 22): the exponential representing the indicial function varies so quickly at the beginning (for *s* close to *0*) that the time discretization used for integrating the convolution product is probably too small to give a precise result. In the last paragraph of this chapter, we will test the program, by changing some variables and see if the evaluated wind forces are more precise and/or converge better .

d) Tests on the program

Some parameters of the program have been fixed for the validation process. In this paragraph, we will see what influence they have on the quality of the results. The parameters that have been fixed are the integration time path of the convolution product, *ds*, and the integration time, s_L. In order to better understand their influence, we will always study the same case in this paragraph: the moment a function of the angle of attack for the Akashi Kaikyō Bridge at a frequency of oscillation of the angle of attack worth k = 0.5.

1) Influence of the dimensionless time path

This parameter has been fixed at 0.01 for the previous validation process. To test its importance, we have fixed it at three different values, 1, 0.1 and 0.01. The integration time is worth s_L=20.

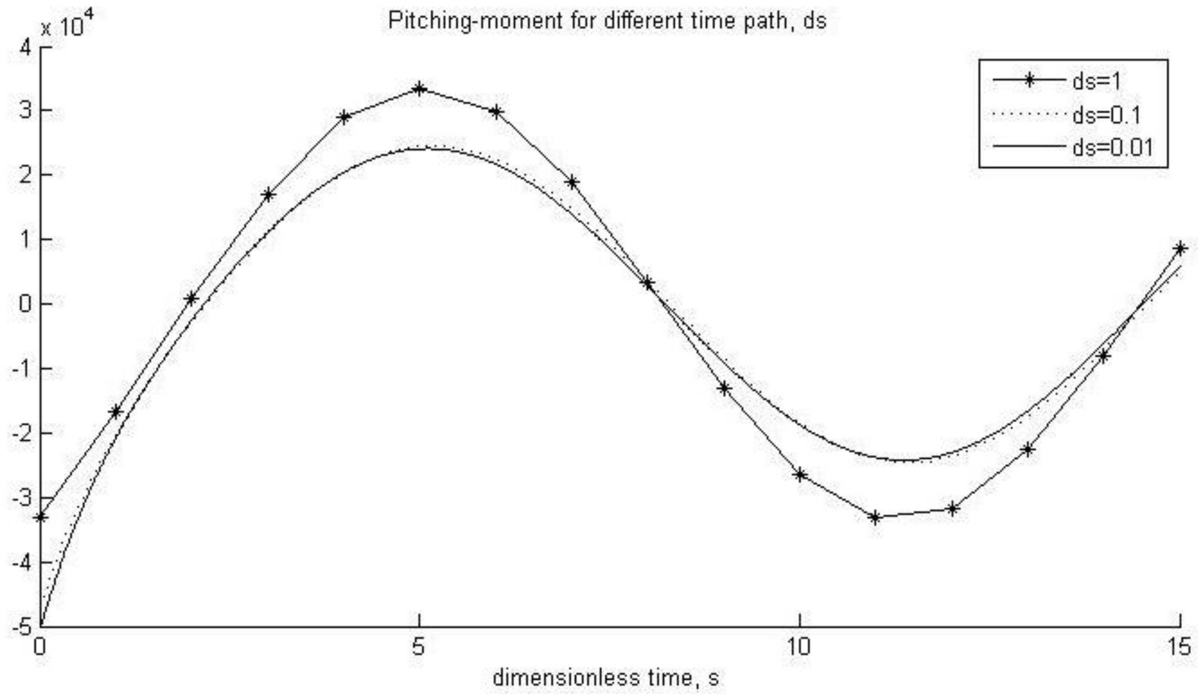

Figure 38: influence of the integration time path, ds

When the dimensionless time path is reduced, the results get more accurate. It seems that in this case, a time path worth *ds*=0.1 is sufficient to have a good evaluation of the lift. By reducing the time path, we increase the precision of the results but we also increase the run time of the program. Indeed, in our program, because of the reading and re-writing of the time-history motion at each time step, dividing the time step by 10 is like increasing the run time of the program by 100.

Normally, the time step is not set up by the user; it is set up by the computing framework that executes the module. The time step is a piece of data that is fixed when solving the dynamic equation. Anyway, when we use this module, we have to take care that the dimensionless time path is not smaller than 0.1. If it is smaller than this value, it means that the time step should be reduced for the need of the module.

2) Influence of the integration time, s_L

The integration time is the time defined in IV.a):

$$L(s) = qBC_L' \left[\Phi_{L\alpha}(0)\alpha(s) + \int_{s-s_L}^{s} \Phi_{L\alpha}{}'(s-\sigma)\alpha(\sigma)d\sigma \right] \qquad (4.a.11)$$

The integration time is the time interval on which we have to integrate the convolution product. It corresponds to the time-history interval that we have to consider to evaluate the present wind forces. To evaluate its influence, as done in VII.d.1), we will consider different values of the integration time: 1, 5 and 20. The integration time path is fixed at 0.1, as we have seen previously that it is small enough.

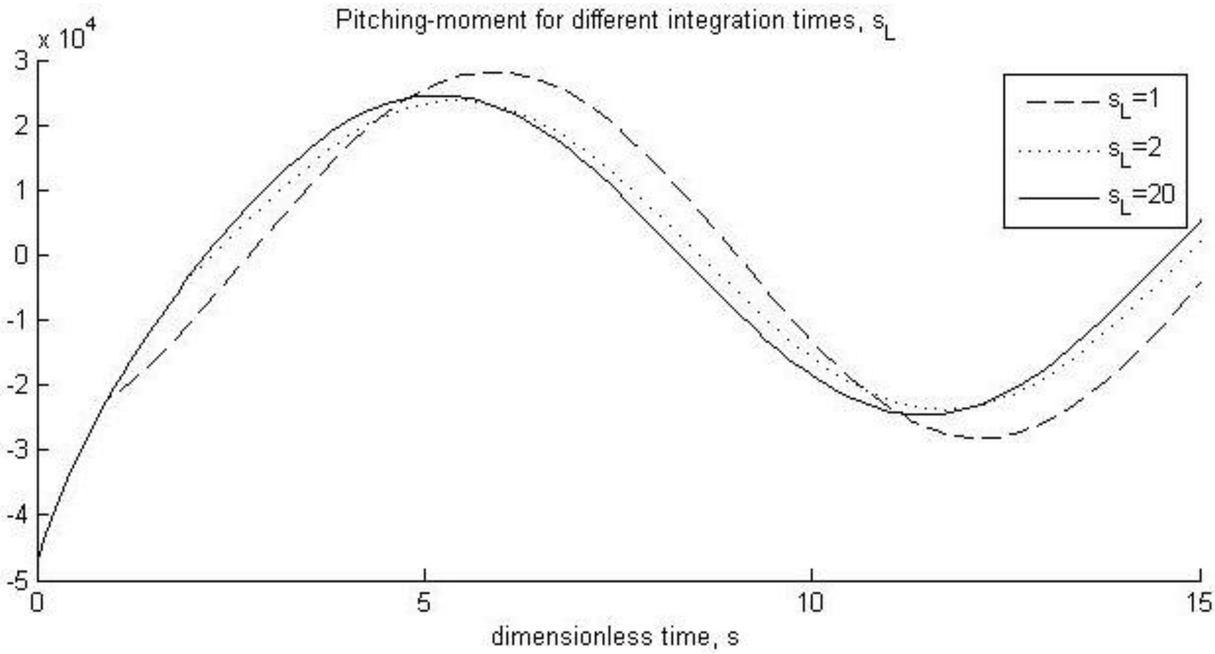

Figure 39: influence of the integration time, s_L

In the article [15], it is said that an integration time that should always be big enough is s_L worth 10. It is verified in our case, as for s_L=2, we already nearly have convergence.

Conclusions

The present work has been developed along three lines: 1) the theoretical background required to understand and formulate the wind forces expressed through indicial functions, 2) the implementation of a computer procedure for the evaluation of said wind forces and, finally, 3) the validation of that procedure.

The theoretical part developed in this thesis resumes different approaches to the indicial functions derivations, some of them pursued in literature and some other newly formulated, showing their substantial equivalence from the point of view of the results. This part is of paramount importance in order to correctly formulate the problem and understand the limits of the theories adopted.

In the implementation part, a particular attention has been made with regards to the future use of code developed. This has been implemented as a FORTRAN90 module that has to be called from a bigger computing structure, such as a finite element software or, in this initial cases a smaller, independent, driver program. The program that has been implemented works for one cross-section of a bridge deck. It has been tested and validated on the Akashi-Kaikyō Bridge and the Tsurumi Fairway Bridge cases, by studying the self-excited component of the wind forces.

Further work is needed on the subject of indicial functions applied to bridges, not only from the theoretical point of the validity of the linearity hypothesis but also for the mundane implementation part. The procedure here developed, that has been used only on one cross-section, can easily be extended to a whole bridge span. In this case, in several sections along the bridge deck the wind forces can be independently evaluated. The procedure can easily be modified so that it can be called within a main ANSYS data file, by means of APDL interface routines, and its result can be reintroduced in the ANSYS structural case in the form of external nodal forces. This allows for the evaluation of the dynamic behavior of a whole bridge without requiring the cumbersome development of a specific ANSYS aerodynamic element. This would help understanding better the possible phenomenon of flutter from the point of view of the whole bridge, and not only of a bridge deck cross-section.

Finally, the buffeting component of the wind forces may be also further developed; from the theoretical point view, by trying to obtain the Küssner's type indicial functions for a bridge deck cross-section, and also from the practical point view of its implementation.

1^{st} Appendix: Fourier analysis, definition and properties

a) Introduction to the Fourier transform

We assume that we are in the aeroelastic conditions. Therefore, the wind forces are linked linearly to the motion of the bridge deck for harmonic oscillations, through the flutter derivatives which are function of the frequency of oscillation of the airfoil. Using the typical notation for a Fourier transform analysis, we will call *Z(iω)*, the complex impedance, function of the pulsation of oscillation, *ω*, that links proportionally the motion and the wind forces.

If the forced function *F(t)* is harmonic:

$$F(t) = F_0 e^{i\omega t}$$

Using the linearity of the problem, we obtain a response, *y(t)*, which is harmonic too and at the same pulsation:

$$y(t) = \frac{F(t)}{Z(i\omega)}$$

If the forced function is a sum of harmonic motions, using the superposition principle that corresponds with the linearity of the studied problem:

$$F(t) = \sum_{n=-\infty}^{+\infty} C_n e^{in\omega t} \qquad y(t) = \sum_{n=-\infty}^{+\infty} \frac{C_n}{Z(in\omega)} e^{in\omega t}$$

For a non periodic function, we use the Fourier transform, to transform our forced function into an integral depending on the frequencies of oscillation:

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(t) e^{-i\omega t} dt$$

$$F'(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G(\omega) e^{i\omega t} d\omega$$

$G(\omega)$ is the Fourier transform of *F*. And *F'* is the inverse Fourier transform of $G(\omega)$.

It can be demonstrated that if the function *F(t)* is smooth enough, the inverse Fourier transform of the Fourier transform is equal to the function itself, and therefore that *F'(t)=F(t)*. Cf. [22] for details on the demonstration. For a physical case, as ours, we will assume that this property is always verified.

Using the impedance *Z*, we know that the response to the forced function *F* is equal to the inverse Fourier transform of *G(ω)* multiplied by the complex impedance, so:

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{G(\omega)}{Z(i\omega)} e^{i\omega t} d\omega$$

b) Response to an impulse

If we consider as the forced function, a step-function, $F(t) = \underline{1}(t)$

$$\begin{cases} F(t) = 1 \; for \; t \geq 0 \\ F(t) = 0 \; for \; t < 0 \end{cases}$$

This function is also called Heaviside function. It corresponds to a unitary abrupt change of the value of the forced function.

The derivative of such a function is the Dirac distribution, δ_0:

$$\dot{F}(t) = \delta_0(t)$$

Where $\dot{}$ means a derivate with respect to time.

If the abrupt change happens at time *τ* different from *t=0*:

$$F(t) = \underline{1}(t - \tau)$$

$$\dot{F}(t) = \delta_\tau(t) = \delta_0(t - \tau)$$

The Dirac distribution can be seen as a function equal to zero along all the axis of real numbers, except on the interval $[-\Delta t/2 \; ; \Delta t/2]$, such as:

$$1 = \lim_{\Delta t \to 0} \int_{-\Delta t/2}^{\Delta t/2} \delta_0(t) dt$$

This means that even if δ_0 tends to an infinite value for *t=0*, its integral remains convergent and equal to one.

We look for the response to such an impulse happening at a time τ, considering our system still linear and invariable in time. If we know the complex impedance of our system, *Z(iω)*, we first calculate the Fourier transform of the Dirac δ_τ:

$$G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(t-\tau) e^{-i\omega t} dt$$

$$G(\omega) = \frac{1}{\sqrt{2\pi}} e^{-i\omega\tau}$$

Assuming that the Dirac function is still smooth enough to do the reasoning we have done before, we obtain, that the Fourier inverse transform response to the Dirac function is equal to:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{G(\omega)}{Z(i\omega)} e^{i\omega t} d\omega$$

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{Z(i\omega)} e^{i\omega(t-\tau)} d\omega$$

Ths function is equal to the Fourier transform of the inverse of the complex impedance, with a translation on the time axis. We evaluate it not at *t* but at the time *t-τ*.

2nd Appendix: Implemented code in FORTRAN

The present code is the code used for the validation of the procedure. This case corresponds with the case *k=0.2*, s_L=20 and *ds*=0.1, for evaluating the lift and pitching-moment corresponding to a harmonic oscillation of the bridge deck, as done in VII.a). To learn how to code in FORTRAN90, I have consulted references [23] and [24].

The main program is presented first. Further on, is also presented the initialization code.

a) Main program

```
program forzefunzindiciali

implicit none

   double precision,dimension(4) :: X                ! vettore posizione (3 spost (x,y,z) + una rot)
   double precision :: L=1.0                                 ! lunghezza d'influenza del nodo
   double precision :: Umed=20.0                             ! velocità media del vento orrizontale lungo x
   double precision :: ro=1.2                                ! densità aria
   double precision :: B                                             ! B maiuscolo, dim dell'impalcato di ponte
   double precision,dimension(4) :: v_nodo         ! vettore velocità del nodo (3 spost + una rot)
   double precision,dimension(3) :: vflut_nodo ! vettore fluttazione velocità vento al nodo (wind gusts)
   double precision,dimension(4,5) :: coeff_funz ! array with the coefficients ai and bi of the indicial funtions.
                                                             ! 1st number: function, 2nd number: coefficients a0,a1,a2,b1,b2

          double precision :: dCl,dCm                ! derivatives of the lift and pitching moment with respect to the angle of
attack in 0
          double precision :: ds=0.01                ! passo temporale
          double precision :: t,s                            ! time and dimensionless time
          double precision :: coeffmultipL           ! coeff multiplicativo davanti all'integrale nella formulazione di Jones per il
lift
          double precision :: coeffmultipM           ! coeff multiplicativo davanti all'integrale nella formulazione di Jones per il
momento
          integer :: N=2000,M=5000                   ! N is number of time-history position we keep in memory
                                                     ! M is the number of iteration done in the validation phase
          double precision, dimension(1000) :: alpha         ! storia angolo d'attaco
          double precision, dimension(1000) :: dalpha        ! storia deriv angolo d'attaco
          double precision, dimension(1000) :: dh                    ! storia deriv spost verticale
          double precision, dimension(1000) :: ddh                   ! storia deriv seconda spost verticale


   double precision,dimension(3,7000) :: Fe                  ! Fe(1): Fx. Fe(2): Fz. Fe(3): My


!    variabili locali
   double precision :: Fd=0d0
   double precision :: Fl=0d0
   double precision :: Fm=0d0
   double precision :: Cd
   double precision :: Cl
   double precision :: Cm
          double precision :: modulo
          double precision :: pLh,pLalpha,pMh,pMalpha
          integer :: I,J
```

```
!     variabili locali per forze aerodinamiche linearizzate

            double precision :: gamma_r,attack_angle
            double precision :: k=0.2


do J=1,M,1              ! we create a loop inside the program only for the validation phase
            open(34,file='alpha.txt')                   ! File dove viene scritto l'angolo d'attaco instantaneo
            open(35,file='dalpha.txt')                  ! File dove viene scritto la derivate dell'angolo d'attaco instantaneo
            open(36,file='dh.txt')                      ! File dove viene scritte la velocità verticale instantanea
            open(37,file='ddh.txt')                     ! File dove viene scritto l'accelerazione verticale instantanea

            write(34,*)  cos(k*J*ds)                    ! Imponiamo un moto armonico all'angolo d'attaco
            write(35,*) –k*sin(k*J*ds)
            write(36,*) 0.000                           ! moto vertical bloccato
            write(37,*) 0.000
            close(34)
            close(35)
            close(36)
            close(37)


            v_nodo(:)=0d0
            vflut_nodo(:)=0d0
            X(:)=0d0
            coeff_funz(:,:)=1.0

!-----Azzeramento variabili
   Fe(:,J)=0d0
   Fd=0d0
   Fl=0d0
   Fm=0d0
   Cd=1.0
   Cl=0d0
   Cm=0d0
   gamma_r=0d0
  modulo=0d0

            open(20,file='Coefficientsfunzioniindiciali.txt')                 !lettura dati sulle funzioni indiciali
   read(20,*)
   read(20,*)
   read(20,*) (coeff_funz(1,I),I=2,5)
   read(20,*)
   read(20,*) (coeff_funz(2,I),I=2,5)
   read(20,*)
   read(20,*) (coeff_funz(3,I),I=2,5)
   read(20,*)
   read(20,*) (coeff_funz(4,I),I=2,5)
   read(20,*)
   read(20,*)
   read(20,*) B
   read(20,*)
   read(20,*) dCl
   read(20,*)
   read(20,*) dCm
            close(20)

            coeffmultipL=0.5*ro*B*Umed*Umed*dCl
            coeffmultipM=0.5*ro*B*B*Umed*Umed*dCm

if(ds*N<10) print*, "error: integration time, sL, too short"   ! the integration time is worth N*ds, the number of position
                                                                ! kept in memory multiplied by the dimensionless time step
```

```
open (30, file='angolodatacco.txt')
open (31, file='dangolodatacco.txt')
open (32, file='dspostvert.txt')
open (33, file='d2spostvert.txt')

! lettura della time-history motion dell'impalcato di ponte, nei file di input
do I=2,N,1
  read(30,*) alpha(I)
  read(31,*) dalpha(I)
  read(32,*) dh(I)
  read(33,*) ddh(I)
end do

close(30)
close(31)
close(32)
close(33)

! lettura della posizione attuale dell'impalcato di ponte. Quest'informazione può provenire da ANSYS. In questo caso, viene
! definita da noi ad ogni instante.
open(34,file='alpha.txt')
open(35,file='dalpha.txt')
open(36,file='dh.txt')
open(37,file='ddh.txt')

read(34,*) alpha(1)
read(35,*) dalpha(1)
read(36,*) dh(1)
read(37,*) ddh(1)

close(34)
close(35)
close(36)
close(37)

attack_angle=alpha(1)
! Calcoliamo le forze del vento su un nodo corrispondente ad una lunghezza L

! Calcolo modulo del vettore Vr
          modulo=(Umed+(vflut_nodo(1)-v_nodo(1)))**2+(vflut_nodo(3)-v_nodo(3))**2

!--------Calcola Cd, Cl e Cm in funzione dell'angolo di attacco

! Calcolo angolo tra Vr ed asse di riferimento solidale con la sezione, in radianti

          gamma_r=atan((vflut_nodo(3)-v_nodo(3))/(Umed+(vflut_nodo(1)-v_nodo(1))))
!         attack_angle=gamma_r+X(4)

! Calcolo il Cd ed il Cl dato l'angolo di attacco in radianti
!         call evaluate_CdClCm(attack_angle,Cd,Cl,Cm)

! Calcolo forze di drag e di lift
!   Fd=0.5*ro*modulo*B*Cd                   ! Non calcoliamo le forze quasi-statiche, vogliamo solo le self-excited forces
!   Fl=0.5*ro*modulo*B*Cl
!   Fm=0.5*ro*modulo*B*B*Cm

t=0d0
s=0d0

call evaluatePhi(coeff_funz,t,pLh,pLalpha,pMh,pMalpha)          ! Valutazione della funzione indiciale in 0
Fl=Fl+coeffmultipL*(pLalpha*alpha(1)+pLh*dh(1)*2/B)             ! calcolo della costante presente nell'espressione delle !
                                                                self-excited forces
Fm=Fm+coeffmultipM*(pMalpha*alpha(1)+pMh*dh(1)*2/B)
```

```
s=-N*ds;
do I=1,N,1

! Valutazione della funzione indiciale al tempo -sigma
  call evaluatedPhi(coeff_funz,-s,pLh,pLalpha,pMh,pMalpha)
  Fl=Fl+coeffmultipL*(pLalpha*alpha(N+1-I)+pLh*dh(N+1-I)*2/B)*ds     ! Calcolo del prodotto di convoluzione discretizzato
  Fm=Fm+coeffmultipM*(pMalpha*alpha(N+1-I)+pMh*dh(N+1-I)*2/B)*ds
  s=s+ds
end do


! rotazione per ricavare le forze lungo (0x), (Oz) e il momento attorno a (Oy),
! con angolo pari all'angolo d'attacco. NON NECESSARIO NEL NOSTRO CASO

!Fe(1,J)=L*(Fd*cos(attack_angle)-Fl*sin(attack_angle))
!Fe(2,J)=L*(Fd*sin(attack_angle)+Fl*cos(attack_angle))
!Fe(3,J)=L*Fm

Fe(1,J)=L*Fd
Fe(2,J)=L*Fl
Fe(3,J)=L*Fm

open (30, file='angolodatacco.txt')
open (31, file='dangolodatacco.txt')
open (32, file='dspostvert.txt')
open (33, file='d2spostvert.txt')

! Update del file di time-history motion
do I=1,N,1
  write(30,*) alpha(I)
  write(31,*) dalpha(I)
  write(32,*) dh(I)
  write(33,*) ddh(I)
end do

close(30)
close(31)
close(32)
close(33)
end do                    ! Fine del ciclo di modelazione delle oscillazione armoniche forzate

! Scrittura delle forze in output
open(100,file='resultFe.txt')
do J=1,M,1
  write(100,*) Fe(:,J)
end do
close(100)
end program forzefunzindiciali



! Valutazione dei coefficient aerodinamici statici per un certo valore dell'angolo d'attaco
subroutine evaluate_CdClCm(attack_angle,Cd,Cl,Cm)

        implicit none

        double precision, intent(IN):: attack_angle
        double precision, intent(OUT):: Cd,Cl,Cm

! variables locales

        double precision, dimension(8):: coeffCd, coeffCl, coeffCm
```

```
            integer:: i=1

   ! lettura del file di input con i dati dei coefficienti aerodinamici statici
    open(10,file='CdClCmcoeffs.txt')
    read(10,*)
    read(10,*) coeffCd(:)
    read(10,*)
    read(10,*)
    read(10,*) coeffCl(:)
    read(10,*)
            read(10,*)
            read(10,*) coeffCm(:)
            close(10)

! !!!! Metodo di calcolo dei coefficienti aerodinamici valido solo per angoli compresi fra -6° e +6°

Cd=0d0
Cl=0d0
Cm=0d0

if (abs(attack_angle)<0.00001)then
            Cd=coeffCd(1)
            Cl=coeffCl(1)
            Cm=coeffCm(1)
else

            do i=1,8,1
            Cd=Cd+coeffCd(i)*attack_angle**(i-1)
            Cl=Cl+coeffCl(i)*attack_angle**(i-1)
            Cm=Cm+coeffCm(i)*attack_angle**(i-1)
            end do
endif
endsubroutine



subroutine evaluatedPhi(coeff_funz,s,pLh,pLalpha,pMh,pMalpha)

implicit none
            double precision,dimension(4,5), intent(IN):: coeff_funz
            double precision, intent(IN) :: s
            double precision, intent(OUT) :: pLh,pLalpha,pMh,pMalpha
pLh=coeff_funz(1,3)*coeff_funz(1,2)*exp(-coeff_funz(1,3)*s)+coeff_funz(1,4)*coeff_funz(1,5)*exp(-coeff_funz(1,5)*s)
pLalpha=coeff_funz(2,3)*coeff_funz(2,2)*exp(-coeff_funz(2,3)*s)+coeff_funz(2,4)*coeff_funz(2,5)*exp(-coeff_funz(2,5)*s)
pMh=coeff_funz(3,3)*coeff_funz(3,2)*exp(-coeff_funz(3,3)*s)+coeff_funz(3,4)*coeff_funz(3,5)*exp(-coeff_funz(3,5)*s)
pMalpha=coeff_funz(4,3)*coeff_funz(4,2)*exp(-coeff_funz(4,3)*s)+coeff_funz(4,4)*coeff_funz(4,5)*exp(-coeff_funz(4,5)*s)

endsubroutine


subroutine evaluatePhi(coeff_funz,s,pLh,pLalpha,pMh,pMalpha)

implicit none
            double precision,dimension(4,5), intent(IN):: coeff_funz
            double precision, intent(IN) :: s
            double precision, intent(OUT) :: pLh,pLalpha,pMh,pMalpha

pLh=coeff_funz(1,1)-coeff_funz(1,2)*exp(coeff_funz(1,3)*s)-coeff_funz(1,4)*exp(coeff_funz(1,5)*s)
pLalpha=coeff_funz(2,1)-coeff_funz(2,2)*exp(coeff_funz(2,3)*s)-coeff_funz(2,4)*exp(coeff_funz(2,5)*s)
pMh=coeff_funz(3,1)-coeff_funz(3,2)*exp(coeff_funz(3,3)*s)-coeff_funz(3,4)*exp(coeff_funz(3,5)*s)
pMalpha=coeff_funz(4,1)-coeff_funz(4,2)*exp(coeff_funz(4,3)*s)-coeff_funz(4,4)*exp(coeff_funz(4,5)*s)

endsubroutine
```

b) Initialization program

```
program ini

implicit none

integer :: I
integer :: N=2000

open(10,file='angolodatacco.txt')
open(11,file='dangolodatacco.txt')
open(12,file='dspostvert.txt')
open(13,file='d2spostvert.txt')

do I=1,N,1
  write(10,*) 0d0
  write(11,*) 0d0
  write(12,*) 0d0
  write(13,*) 0d0

end do

end program ini
```

3^{rd} Appendix: MATLAB code for the validation of the procedure

a) Code for the evaluation of amplitude and phase differences using the flutter derivatives

```
k=[0.10:0.01:1.2];

% aerodynamic coefficients of the Akashi Kaikyō Bridge deck
dCl=-1.192;
dCm=0.307;
B=35.5;
aLh=[1;-0.365;-11.652];
bLh=[0.021;7.235];
aLa=[1;-0.392;-3.653];
bLa=[0.008;1.155];
aMh=[1;0.039;0.0];
bMh=[0.0;0.0];
aMa=[1;0.073;1.758];
bMa=[0.025;7.098];

% Evaluation of the flutter derivatives interpolated with the coefficients of the
% indicial functions.

H1approx=dCl./(2*k).*((aLh(1)-aLh(2)-
aLh(3))+(aLh(2)*bLh(1)^2./(k.^2+bLh(1)^2))+(aLh(3)*bLh(2)^2./(k.^2+bLh(2)^2)));

H4approx=dCl/2*(aLh(2)*bLh(1)./(k.^2+bLh(1)^2)+aLh(3)*bLh(2)./(k.^2+bLh(2)^2));

H3approx=dCl./((2*k).^2).*((aLa(1)-aLa(2)-
aLa(3))+(aLa(2)*bLa(1)^2./(k.^2+bLa(1)^2))+(aLa(3)*bLa(2)^2./(k.^2+bLa(2)^2)));

H2approx=-dCl./(4*k).*(aLa(2)*bLa(1)./(k.^2+bLa(1)^2)+aLa(3)*bLa(2)./(k.^2+bLa(2)^2));

A1approx=dCm./(2*k).*((aMh(1)-aMh(2)-
aMh(3))+(aMh(2)*bMh(1)^2./(k.^2+bMh(1)^2))+(aMh(3)*bMh(2)^2./(k.^2+bMh(2)^2)));


A4approx=dCm./(2*k).*(aMh(2)*bMh(1)./(k.^2+bMh(1)^2)+aMh(3)*bMh(2)./(k.^2+bMh(2)^2)
);

A3approx=dCm./((2*k).^2).*((aMa(1)-aMa(2)-
aMa(3))+(aMa(2)*bMa(1)^2./(k.^2+bMa(1)^2))+(aMa(3)*bMa(2)^2./(k.^2+bMa(2)^2)));

A2approx=dCm./((2*k).^2).*((aMa(2)*bMa(1)./(k.^2+bMa(1)^2)+aMa(3)*bMa(2)./(k.^2+bMa
(2)^2)));


% evaluation of the amplitude differences

AMPLa=4*k.^2.*sqrt(H2approx.^2+H3approx.^2);
AMPLh=4*k.*sqrt(H1approx.^2+H4approx.^2)/B;
AMPMa=4*k.^2.*sqrt(A3approx.^2+A2approx.^2);
AMPMh=4*k.*sqrt(A1approx.^2+A4approx.^2)/B;

% evaluation of the phase differences

if (H3approx(30)>0)
    PHASELa=atan(H2approx./H3approx);
else
if (H3approx(30)<0)
    if (H2approx(30)>=0)
        PHASELa=pi+atan(H2approx./H3approx);
```

```
    else
        PHASELa=-pi+atan(H2approx./H3approx);
    end
else
    if (H2approx(30)>0)
        PHASELa=pi*ones(1,length(k));
    else
        PHASELa=-pi*ones(1,length(k));
    end
end
end


if (H4approx(30)>0)
    PHASELh=atan(H1approx./H4approx);
else
if (H4approx(30)<0)
    if (H1approx(30)>=0)
        PHASELh=pi+atan(H1approx./H4approx);
    else
        PHASELh=-pi+atan(H1approx./H4approx);
    end
else
    if (H1approx(30)>0)
        PHASELh=pi*ones(1,length(k));
    else
        PHASELh=-pi*ones(1,length(k));
    end
end
end



if (A3approx(30)>0)
    PHASEMa=atan(-A2approx./A3approx);
else
if (A3approx(30)<0)
    if (A2approx(30)>=0)
        PHASEMa=pi+atan(-A2approx./A3approx);
    else
        PHASEMa=-pi+atan(-A2approx./A3approx);
    end
else
    if (A2approx(30)>0)
        PHASEMa=pi*ones(1,length(k));
    else
        PHASEMa=-pi*ones(1,length(k));
    end
end
end

if (A4approx(30)>0)
    PHASEMh=atan(A1approx./A4approx);
else
if (A4approx(30)<0)
    if (A1approx(30)>=0)
        PHASEMh=pi+atan(A1approx./A4approx);
    else
        PHASEMh=-pi+atan(A1approx./A4approx);
    end
else
    if (A1approx(30)>0)
        PHASEMh=pi*ones(1,length(k));
    else
        PHASEMh=-pi*ones(1,length(k));
    end
end
end
```

```
figure('color','w')
subplot(1,2,1)
plot(1./k(1:90),PHASELa(1:90),'k')
grid on
title('phase difference')
xlabel('2/K')
subplot(1,2,2)
plot(1./k(1:90), AMPLa(1:90),'k')
grid on
xlabel('2/K')
title('amplitude')

figure('color','w')
subplot(1,2,1)
plot(1./k(1:90),PHASEMa(1:90),'k')
grid on
title('phase difference')
xlabel('2/K')
subplot(1,2,2)
plot(1./k(1:90), AMPMa(1:90),'k')
grid on
xlabel('2/K')
title('amplitude')

figure('color','w')
subplot(1,2,1)
plot(1./k(1:90),PHASELh(1:90),'k')
grid on
title('phase difference')
xlabel('2/K')
subplot(1,2,2)
plot(1./k(1:90), AMPLh(1:90),'k')
grid on
xlabel('2/K')
title('amplitude')

figure('color','w')
subplot(1,2,1)
plot(1./k(1:90),PHASEMh(1:90),'*k')
grid on
title('phase difference')
xlabel('2/K')
subplot(1,2,2)
plot(1./k(1:90), AMPMh(1:90),'*k')
grid on
xlabel('2/K')
title('amplitude')


I=11;
k(I)      % I is the position of the studied frequency in the vector k. In this
          % case, we are studying the frequency k=0.2

% values of the amplitude and phase differences for k=0.2
AMPLalpha=AMPLa(I)
AMPLspost=AMPLh(I)
AMPMalpha=AMPMa(I)
AMPMspost=AMPLh(I)

PHASELalpha=PHASELa(I)
PHASELspost=PHASELh(I)
PHASEMalpha=PHASEMa(I)
PHASEMspost=PHASELh(I)
```

b) Code developed to obtain the amplitude and phase differences of the wind forces in output of the program

```
k=0.2;
B=35.5;
U=20;
ds=0.01;

% resultFe is the output file of our program in which are stored the wind
% forces

WForces=resultFe(1500:end,:); % we do not consider the beginning of the wind forces
                              % to avoid studying the transient time
lift=WForces(:,2);
moment=WForces(:,3);
[aL,bL]=max(lift);
[aM,bM]=max(moment);

AMPL=aL/(0.5*1.2*B*U^2)    % amplitude of the lift
AMPM=aM/(0.5*1.2*B^2*U^2)    % amplitude of the moment

DSL=-(bL+1500)*ds;
phiL=k*DSL;
while (phiL<-pi)
    phiL=phiL+2*pi;
end
phiL  % phase difference for the lift

DSM=-(bM+1500)*ds;
phiM=k*DSM;
while (phiM<-pi)
    phiM=phiM+2*pi;
end
phiM   % phase difference for the moment
```

4^{th} Appendix: Notations

$\mathfrak{F}$	Froude number
$\underline{1}(t)$	Heaviside function, also called step function
$\mathfrak{R}_e$	Reynolds number
$A_i^{\#}$	torsional flutter derivative, as expressed by Fung (theoretical)
A_i^{*}	torsional flutter derivative, as expressed by Scanlan (experimental)
c	speed of sound in atmospheric conditions
c	chord of the airfoil
C_D, C_L, C_M	aerodynamic static coefficients for drag and lift and pitching moment
C_L', C_M'	derivate of the static aerodynamic coefficients with respect to α in $\alpha=0$
C_X, C_Z, C_M	aerodynamic static coefficients for horizontal, vertical forces and pitching moment
D, L	drag and lift forces
div	divergence operator: $div(\underline{U})=(\partial U_x)/\partial x+(\partial Uy)/\partial y+(\partial Uz)/\partial z$
Φ	indicial response function for a bridge deck (determined experimentally)
F_x, F_z, M_y	horizontal and vertical wind forces and pitching-moment
$H_i^{\#}$	vertical flutter derivative, as expressed by Fung (theoretical)
H_i^{*}	vertical flutter derivative, as expressed by Scanlan (experimental)
φ	indicial response function for a thin-airfoil (Wagner)
K	dimensionless pulsation also called reduced velocity, $B\omega/U$
k	reduced velocity as defined by Theodorsen, $k=K/2$
M	Mach number
μ	dynamic viscosity
M_0 or M_a	pitching moment
$\underline{\nabla}$	gradient operator: _ $F=(\partial F/\partial x; \partial F/\partial y; \partial F/\partial z)$
p	relative pressure
P	absolute pressure
P_0	air pressure at rest
ρ	air density (1,2 kg/m^3 in atmospheric condiditons)
$\underline{rot}$	rotational operator: $\underline{rot}\ \underline{A} = (\partial A_z/\partial y-\partial A_y/\partial z; \partial A_x/\partial z-\partial A_z/\partial x; \partial A_y/\partial x-\partial A_x/\partial y)$
s	dimensionless present time, $2Ut/B$
s_L	integration time for the convolution integral
σ	dimensionless time, integration variable
t	present time

τ	time integration variable
U	average horizontal wind speed
u	horizontal air gust velocity
w	vertical air gust velocity
x,y,z	degrees of freedom for the displacement of the bridge deck
ψ	Küssner's type indicial function: response to a vertical gust.
α	angle of attack
ω	pulsation
$(\ )'$	derivate with respect to the dimensionless time, *s*
$(\dot{\ })$	derivate with respect to the time, *t*
. ("dot")	scalar product of two vectors
Λ	cross-product of two vectors

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