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Numerical solution of elliptic problems by virtual control methods

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Abstract

The *virtual control* approach is based on the optimal control theory that has been introduced in domain decomposition method with overlapping subdomains to treat both heterogeneous couplings, involving Navier-Stokes and full potential operators ([1]), and homogeneous problems, either elliptic and parabolic (see [2, 3, 4, 5]). In the pioneering papers by Glowinski et al. ([1, 2]), this method was referred to as a *Least Square* formulation of the multi domain problem.
The basic idea of this approach consists in introducing two "virtual" controls which play the role of unknown Dirichlet data on the interfaces of the decomposition and then minimizing the L^2-norm of the difference between solutions (defined inside the two subdomains) on the overlap. A recent general description of this approach can be found in [6].

In this work we focus on the homogeneous domain decomposition method for a scalar elliptic problem in two dimensions. The approach is extended to the case of Neumann boundary controls on the interfaces and the difference between the solutions on the overlap is minimized in the L^2-norm, the H_0^1-norm and the H^1-norm. An *augmented* H_0^1-norm is also considered and the behavior of the method is studied in both the cases of a penalized cost functional and of a non-penalized cost functional. Well posedness is proved for all these choices of the cost functional. The optimality system is derived analytically and then it is numerically approximated by the *Galerkin-Finite Element* method. The numerical simulations allow the validation of the theoretical model and a comparison between the different control strategies: we vary the choice of boundary controls and the choice of cost functional.

Even in the simple case of the Poisson equation, the numerical solution of such PDE-constrained optimization problems is usually quite expensive. The large dimensional linear systems which result from discretization and which need to be solved are of saddle-point type. The fact that we are solving a coupled problem on two domains introduces some additional difficulties and a high number of iterations is needed to achieve convergence. We study the issue of preconditioning the linear system that arises from a coupled virtual control problem. For that we take inspiration from the results derived in the contest of the domain decomposition and those concerning the preconditioning of optimization problems.

Finally, the method is applied to the solution of the Stokes problem. The numerical simulations permit the identification of a suitable cost functional to be minimized on the overlapping region. Again, the virtual controls can play both the role of a Dirichlet condition on the velocity or that of the normal component of the Cauchy stress tensor. A preconditioning approach is also tested.

In what follows, for each chapter, the content of the work is outlined.
In *Chapter 1* the theoretical setting of the problem is presented. First, we present a general introduction to the optimal control problem. The derivation of the optimality system is reported following both the *Lions* and the *Lagrangian* approach, focusing on the theory concerning the elliptic equations. Then we introduce the specific virtual control problem in the case of the scalar elliptic equation. The state equations depend on both Dirichlet and Neumann boundary controls and different choices of the cost functional are introduced. The well posedness of the problem is proved and the optimality system (made of state, adjoint and optimality equations) is derived for each type of control and each choice of the cost functional.

In *Chapter 2* we introduce the discretization of the problem using finite elements. The discrete version of the optimality systems is derived. In this chapter we describe three possible approaches for the solution of the optimality system: two of them iterate between the equations and one regards the coupled virtual control problem as a global linear system. In the latter case, the issue of preconditioning is studied, for Neumann boundary controls and the L^2-norm cost functional.

In *Chapter 3* we report the numerical results obtained with our methods. The theoretical results concerning the model are validated on a variety of test cases. We verify the well posedness of the optimization problem and carry out a comparison between the different control approaches introduced in *Chapter 1*. In a different section we report the numerical results concerning the preconditioning of the global matrix: we compare the efficiency of the different approaches introduced in *Chapter 2*.

In *Chapter 4* the virtual control with overlap method is studied in the case of the Stokes problem. The optimality system is derived for different choices of the cost functional and of the space of the boundary controls. A preconditioning matrix is introduced and we report numerical results concerning the validation of the theoretical model and of the preconditioning approach.

The proofs of the well posedness of the virtual control method with Neumann boundary controls applied to the solution of the elliptic equation are not found in literature and represent an original contribution of this work. Moreover, the study of several cost functionals for the minimization of the difference between the solutions on the overlap for both the elliptic and the Stokes problems is new. Finally, the analysis of preconditioning the virtual control method is inspired to the results for the control problems and domain decomposition methods and it also represents an original part of the work.

The numerical simulations in this work have been carried out in the *Matlab* environment. A great part of this work concerned the implementation of finite elements code.

This work has been carried out in the *Chair of Modeling and Scientific Computing* (CMCS) of the École Polytechnique Fédérale de Lausanne (CH), in the contest of the international exchange program *Erasmus* of the European Union.

Lausanne, September 2010.

Sommario

Il metodo dei *controlli virtuali* è basato sulla teoria del controllo ottimo introdotta nell'ambito delle tecniche di decomposizione di dominio con ricoprimento per trattare problemi di accoppiamento eterogeneo, per esempio Navier-Stokes e operatori potenziali [1] e problemi omogenei di tipo ellittico e parabolico (si vedano [2, 3, 4, 5]). Nei primi lavori in cui il metodo viene trattato [1, 2], si parla di formulazione ai *minimi quadrati* del problema multi-dominio.
Questo approccio si basa sull'idea di introdurre due controlli "virtuali" che rappresentano i dati di Dirichlet incogniti sulle interfacce della decomposizione e successivamente di minimizzare la norma L^2 della differenza tra le due soluzioni (definite sui due sottodominii) nella regione di ricoprimento. Per una descrizione generale dell'approccio si veda [6].

In questo lavoro, ci siamo concentrati su un problema di decomposizione di dominio omogeneo per un'equazione scalare ellittica in due dimensioni. Il metodo è esteso al caso di controlli virtuali di tipo Neumann e la differenza tra le soluzioni sulla regione di ricoprimento viene minimizzata nella norma L^2, nella norma H_0^1 e nella norma H^1. Inoltre viene introdotta una norma H_0^1 *aumentata* e il metodo viene studiato in presenza e in assenza di penalizzazione del funzionale. La buona posizione del problema è dimostrata per tutte queste scelte del funzionale costo. Il sistema di ottimalità viene derivato analiticamente e successivamente viene discretizzato con il metodo degli *elementi finiti*. Attraverso le simulazioni numeriche è possibile convalidare il modello teorico ed effettuare un paragone tra i vari approcci al variare della scelta del tipo di controlli, Neumann o Dirichlet, e della scelta del funzionale costo.

Anche nel semplice caso dell'equazione di Poisson, la risoluzione numerica del problema di controllo virtuale è piuttosto costosa dal punto di vista computazionale. Il sistema lineare che deriva dalla discretizzazione è di dimensioni elevate e presenta una struttura di tipo punto-sella. Il fatto che si sta risolvendo un problema accoppiato su due sottodominii introduce delle difficoltà aggiuntive e un grande numero di iterazioni è necessario per arrivare a convergenza. In questo lavoro viene studiato il problema di trovare un precondizionatore efficace per un sistema lineare che deriva da un problema di controllo virtuale. Per fare questo ci ispiriamo ai risultati derivanti dalla decomposizione di dominio e dal precondizionamento di problemi di controllo ottimo.

Il metodo viene poi applicato al problema di Stokes. Attraverso le simulazioni numeriche vogliamo identificare il funzionale costo adatto per la minimizzazione della differenza tra le soluzioni sulla zona di ricoprimento. Anche in questo caso i controlli possono rappresentare le condizioni al bordo di Dirichlet per la velocità o la derivata normale del tensore degli sforzi di Cauchy. Inoltre viene testato un approccio di precondizionamento.

La tesi è organizzata come segue.
Nel *Capitolo 1* viene descritta l'ambientazione teorica del problema. Dopo un'introduzione generale ai problemi di controllo ottimo, vengono riportati i risultati sulla derivazione del sistema di ottimalità secondo l'approccio *alla Lions* e l'approccio *Lagrangiano*, facendo particolare attenzione ai risultati sulle equazioni di tipo ellittico. In seguito, viene introdotto nello specifico il problema di controllo virtuale. Le equazioni di stato possono dipendere sia da controlli di tipo Dirichlet che da controlli di tipo Neumann e diversi funzionali costo sono introdotti. La buona posizione del problema è dimostrata e il sistema di ottimalità (costituito dalle equazioni di stato, aggiunte e di ottimalità) viene derivato per ogni scelta dello spazio dei controlli e del funzionale costo.

Nel *Capitolo 2* introduciamo la discretizzazione del problema mediante gli elementi finiti e presentiamo la versione discreta del sistema di ottimalità. In questo capitolo vengono descritti tre possibili approcci per la risoluzione del sistema di ottimalità: due di questi iterano tra le equazioni del sistema e nel rimanente si costruisce il sistema lineare globale associato al problema di ottimalità. In quest'ultimo caso, studiamo il problema di trovare un precondizionatore efficace per il sistema, nel caso di controlli di tipo Neumann e del funzionale che osserva la norma L^2 della differenza tra le soluzioni.

Nel *Capitolo 3* riportiamo i risultati delle simulazioni numeriche. I risultati teorici sul modello di controllo virtuale vengono testati su diversi casi tesi. Verifichiamo la buona posizione del problema di controllo ed effettuiamo un confronto tra i diversi approcci descritti nel *Capitolo 1*. In un secondo momento, riportiamo i risultati sul precondizionamento della matrice globale: verifichiamo l'efficienza dei vari approcci introdotti nel *Capitolo 2*.

Nel *Capitolo 4* il metodo di controllo virtuale con ricoprimento viene applicato al problema di Stokes. Il sistema di ottimalità è derivato per diverse scelte del funzionale costo e dello spazio dei controlli. Anche per questo problema viene introdotto un precondizionatore e si presentano i risultati numerici per la validazione del modello e dell'approccio di precondizionamento.

Le dimostrazioni di buona posizione del problema di controllo virtuale applicato all'equazione ellittica nel caso di controlli di tipo Neumann rappresentano un contributo originale di questo lavoro. Anche lo studio di diversi funzionali costo per la minimizzazione della differenza tra le soluzioni sulla regione di ricoprimento rappresenta una novità. Inoltre, il precondizionamento di problemi di controllo virtuale non si trova in letteratura, e rappresenta anch'esso una parte originale della tesi.

Le simulazioni numeriche sono state effettuate con *Matlab*. La programmazione di codici a elementi finiti ha costituito una cospicua parte del lavoro.

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Contents

Chapter 1

Coupled Virtual Control for the Scalar Elliptic Problem

The virtual control method can be regarded as a special case of overlapping *domain decomposition* method. The idea consists in introducing a control function (called virtual control in [7]) on the subdomain interfaces which have the role of guaranteeing that the two solutions match on the region of overlap. In this section we treat the case of the scalar elliptic equation and concentrate on the case of two overlapping domains: in this situation we have to deal with two control interfaces, one for each subdomain, simultaneously.

In the first section of this chapter we briefly review some basic theoretical results concerning the control theory of partial differential equations. The results are presented in a general framework and the specific problem that we have to solve is not taken into account. In the second section we will adapt these concepts to the case of the coupled virtual control and some new results will have to be introduced. This subdivision justifies the change of notations between the two sections: in the first we will use the classical notation of the optimal control theory while in the second section we will use a notation closer to the one that is typically used in the contest of virtual control.

1.1 Optimal Control for Partial Differential Equations

In this section we will recall the basic concepts of the optimal control theory of systems governed by partial differential equations (PDE's). The classical setting of optimal control problems is based on the work by *J.L. Lions* in [8] and [9]. We will concentrate our description on problems governed by linear elliptic equations. An alternative analysis of optimal control problems makes use of the *Lagrangian* formalism and is based on the solution of the optimization problem as a problem of constrained minimization.
A detailed description of the theoretical setting of the optimal control presented in this section can be found in [8] and in [10], and the functional analysis results can be found in [11] and [12].

1.1.1 The Problem Setting

In order to introduce the results concerning the optimal control theory we need to list the mathematical entities that are necessary for the definition of the model:

- The *control function* u, belonging to a functional space $\mathcal{U}_{ad}$ of the *admissible controls*. The space of the controls is chosen according to the role of the control function (distributed control or boundary control) and to the possible constraints that u needs to satisfy.
- The *state of the system* $y(u)$, depending on the control u and satisfying the *state equation*

$$\Lambda y(u) = f.$$

  This problems is defined by the physical system subject to the control variable and has to be coupled with suitable boundary conditions.

- The *observation equation*, denoted as $z(u)$, depending on the control variable u through the state variable $y(u)$ in the following way

$$z(u) = Cy(u).$$

  This function belongs to the space of the *observed functions* $\mathcal{Z}$.

- The *cost functional* $J(u)$, defined on the space $\mathcal{Z}$, with $J(z(u)) \in \mathbb{R}$ and $J(u) \geq 0$.
- The *control problem*: determine an *optimal control* $u \in \mathcal{U}_{ad}$ such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v),$$

  which is equivalent to

$$J(u) \leq J(v) \qquad \forall v \in \mathcal{U}_{ad}.$$

At this point it is necessary to study the issue of the existence and of the unicity of the optimal control and it is necessary to determine the optimality conditions. Moreover we need to analyze the structure and the properties of the equations that govern the control problem.

An Existence and Uniqueness Result

In this section we report some results on the unicity and on the existence of the minima of cost functionals, concentrating our analysis on the results related to the optimal control theory.
We consider a *Hilbert* functional space $\mathcal{U}$ on $\mathbb{R}$, with the norm $\|\cdot\|$ defined by the scalar product on $\mathcal{U}$: $\|v\| = \sqrt{(v, v)}$. We define:

$$u, v \to \pi(u, v) \qquad \forall u, v \in \mathcal{U},$$

and we assume that π is a continuous, symmetric and bi-linear form on $\mathcal{U}$ (for an explanation of these definitions see [11]). Moreover we define:

$$v \to L(v), \qquad \forall v \in \mathcal{U},$$

with L continuous linear form on $\mathcal{U}$. We want to minimize over the space of the admissible controls $\mathcal{U}_{ad} \subset \mathcal{U}$ the quadratic functional J, which can be written as:

$$J(v) = \pi(v, v) - 2L(v) \quad \forall v \in \mathcal{U}_{ad}. \tag{1.1}$$

The existence and the unicity of the minimum of (1.1) is given by the following result.

Theorem 1. *Let $\pi(u, v)$ be coercive on $\mathcal{U}$:*

$$\pi(v, v) \geq c\|v\|^2 \quad \forall v \in \mathcal{U}, \qquad c > 0.$$

Then there exists a unique element $u \in \mathcal{U}$ such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v). \tag{1.2}$$

Theorem 2. *In the same hypothesis of Theorem 1, the solution $u \in \mathcal{U}_{ad}$ is characterized by the following variational inequality:*

$$\pi(u, v - u) \geq L(v - u), \qquad \forall \in \mathcal{U}_{ad}.$$

Theorem 3. *If the function $v \to L(v)$ is strictly convex and satisfies the condition $J(v) \to \infty$ with $\|v\| \to \infty, \forall v \in \mathcal{U}_{ad}$, the only element $u \in \mathcal{U}$ that satisfies (1.2) is characterized by the following inequality:*

$$J'(u) \cdot (v - u) \geq 0, \qquad \forall v \in \mathcal{U}_{ad}. \tag{1.3}$$

This means that the optimal control problem can be reformulated conveniently as follows: find $u \in \mathcal{U}_{ad}$ such that (1.3) holds.

Choice of Functional Spaces and Definition of Adjoint Operator

We now want to define the functional spaces in which to set the variables of the control problem. Moreover we give the definition of the adjoint of an operator that will be used in the following derivation of the optimal control system. We give here an outline of the results, a more detailed description can be found in [11] and [12].

We consider two Hilbert spaces V and H that verify the following statements:

- $V \subset H$;
- the injection of V in H is continuous;
- V is dense in H.

If these hypothesis hold true, denoting V^* as the dual space of V, we have that

$$V \subset H \subset V^*,$$

and we refer to the triplet V, H and V^* as an *Hilbert triplet*. In what follows we will chose the spaces V and H as $H^1(\Omega)$ and $L^2(\Omega)$ respectively, where we recall that

$$L^2(\Omega) = \{f : \Omega \to \mathbb{R} : \int_\Omega f(x)^2 d\Omega < +\infty\},$$
$$H^1(\Omega) = \{f \in L^2(\Omega) : \mathcal{D}_i f \in L^2(\Omega), \quad \forall i = 1, ..., n\}.$$

$\Omega \subset \mathbb{R}^n$ is the domain on which the problem is studied and $\mathcal{D}_i f$ denotes the distributional derivative of f with respect to x_i.
We present here the definition of the *adjoint operator*.

Definition 1. Given a linear and continuous operator A and $\phi, \psi \in V$, the *adjoint operator* of A is defined as the operator A^* for which the following identity holds true:

$$_V\langle A\phi, \psi\rangle_{V^*} =_{V^*} \langle \phi, A^*\psi\rangle_V,$$

where the notation $_V\langle A\cdot, \cdot\rangle_{V^*}$ refers to the duality product on V.

In the same hypothesis of Definition 1 we have the following:

Definition 2. A is said to be *self-adjoint* if

$$A \equiv A^*.$$

In what follows we will give a detailed description of the optimal control theory in the case of elliptic problems.

1.1.2 Control of Elliptic Variational Problems

In this section we report the basic results of existence and uniqueness for optimal control problems governed by elliptic partial differential equations. We chose the Hilbert spaces V and H as in the previous section. We consider

- a bilinear form $a(u, v)$, continuous and coercive on V;
- a linear form $L(v) = (f, v)$, continuous on V, where $f \in V^*$.

The *Lax-Milgram* Lemma guarantees the existence and the uniqueness of a solution $y \in V$ such that

$$a(y, v) = (f, v) \qquad \forall v \in V. \tag{1.4}$$

The equation (1.4) can be interpreted in the following way: since the form $v \to a(u, v)$ is linear and continuous, thanks to the *Riesz Representation* Theorem, we can rewrite

$$a(u, v) = (Au, v), \qquad Au \in V^*,$$

which defines $A \in \mathscr{L}(V, V^*)$. Hence (1.4) is equivalent to

$$Ay = f.$$

This equation represents the physical system and it is called state equation. We may now formulate the first control problem. On the Hilbert space of the controls $\mathcal{U}$ we define the operator B with $B \in \mathscr{L}(\mathcal{U}, V^*)$. For a control $u \in \mathcal{U}$, the state of the system y is given by the solution of

$$Ay = f + Bu, \qquad y(u) \in V, \tag{1.5}$$

this equation defines $y(u)$ uniquely by virtue of the *Lax-Milgram* Lemma. For example, B can be the identity or the trace opertator.
We are also given an *observation equation*

$$z(u) = Cy(u),$$

where $C \in \mathscr{L}(V, H)$. Often C is a restriction operator, in the sense that one is interested in controlling the state of the system in a part of the domain or of the boundary. Finally we are given $N \in \mathscr{L}(\mathcal{U},\mathcal{U})$, where N is symmetric and positive definite and such that

$$(Nu, u)_V \geq \nu \|u\|^2_{\mathcal{U}}, \qquad \nu > 0, \tag{1.6}$$

we will refer to this term as a penalization term. With every control $u \in \mathcal{U}$ we associate the cost:

$$J(u) = \|Cy(u) - z_d\|^2_H + (Nu, u)_{\mathcal{U}}, \tag{1.7}$$

where z_d is a given element in H. Let $\mathcal{U}_{ad}$ be a closed, convex subset of $\mathcal{U}$. We can now state the control problem: find $\inf_{v \in \mathcal{U}_{ad}} J(v)$.

According to (1.5) the map $u \to y(u)$ is an affine map of $\mathcal{U} \to V$. In order to be able to write the cost functional (1.7) in the form defined in (1.1) we write:

$$J(u) = J(u) = \|C(y(u) - y(0)) + Cy(0) - z_d\|^2_H + (Nu, u)_{\mathcal{U}}.$$

We obtain a cost functional of the form (1.1) by setting

$$\begin{aligned} \pi(u, v) &= (C(y(u) - y(0)), C(y(v) - y(0)))_H + (Nu, v)_{\mathcal{U}}, \\ L(v) &= (z_d - Cy(0), C(y(v) - y(0)))_H, \end{aligned}$$

so that we have

$$J(v) = \pi(v, v) - 2L(v) + \|z_d - Cy(0)\|^2_H.$$

Since $\|z_d - Cy(0)\|^2_H$ is clearly ≥ 0, from (1.6) we have that

$$\pi(v, v) \geq \nu \|v\|^2_{\mathcal{U}} \qquad \forall v \in \mathcal{U}.$$

So Theorem 1 can be applied and there exists a unique element $u \in \mathcal{U}_{ad}$ such that

$$J(u) = inf_{v \in \mathcal{U}_{ad}} J(v).$$

Definition 3. The element $u \in \mathcal{U}_{ad}$ for which $J(v)$ attains its minimum is called the optimal control.

If $N = 0$ one can in general only conclude that $\pi(v, v) \geq 0$. In this case the demonstration of the existence of the optimal control is straightforward, the unicity is not guaranteed. We will be particularly interested in the non penalized case and in some particular cases we will be able to prove the coercivity of the cost functional even if $N = 0$.

The Set of Inequalities Defining the Optimal Control

Once that, under suitable hypotheses, the unicity and the existence of the control u is proved, we need to study the structure of the equations that define the optimal control problem in order to extract from these the information needed for the solution of the problem. We recall that, in virtue of Theorem 3, if u is an optimal control then

$$J'(u) \cdot (v-u) \geq 0 \qquad \forall v \in \mathcal{U}_{ad} \tag{1.8}$$

and conversely. Since A is an isomorphism of V onto V^* (according to the *Lax-Milgram* Lemma) we may write

$$y(u) = A^{-1}(f + Bu),$$

and we have that

$$y'(u) \cdot (v-u) = A^{-1}B(v-u) = y(v) - y(u).$$

Therefore the condition (1.8) is equivalent to

$$(Cy(u) - z_d, Cy(v) - Cy(u)))_H + (Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}. \tag{1.9}$$

We now need to introduce the dual H^* of H and we denote by $\Lambda = \Lambda_H$ the canonical isomorphism of H on to H^*, moreover we denote by $C^* \in \mathscr{L}(H^*, V^*)$ the adjoint of C. We can rewrite:

$$(C^*\Lambda(Cy(u) - z_d), y(v) - y(u))_H + (Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}, \tag{1.10}$$

and we transform (1.10) through the *adjoint state*. We denote the adjoint of the operator A as $A^* \in \mathscr{L}(V, V^*)$. For a control v the adjoint state $p(v) \in V$ is defined by:

$$A^*p(v) = C^*\Lambda(Cy(v) - z_d).$$

So we can deduce that the inequality (1.10) can be transformed as

$$\begin{aligned}
&(C^*\Lambda(Cy(u) - z_d), y(v) - y(u))_H + (Nu, v-u)_{\mathcal{U}} = \\
&(A^*p(u), y(v) - y(u))_H + (Nu, v-u)_{\mathcal{U}} = \\
&(p(u), Ay(v) - Ay(u))_H + (Nu, v-u)_{\mathcal{U}} = \\
&(p(u), B(v-u))_H + (Nu, v-u)_{\mathcal{U}} = \\
&(B^*p(u), v-u)_H + (Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad},
\end{aligned}$$

where $B^* \in \mathscr{L}(V, \mathcal{U}^*)$ is the adjoint of B, $\mathcal{U}^*$ being the dual of $\mathcal{U}$. If we now denote as $\Lambda_{\mathcal{U}}$ the canonical isomorphism of $\mathcal{U}$ onto $\mathcal{U}^*$, we can finally reformulate (1.10) as

$$(\Lambda_{\mathcal{U}}^{-1}B^*p(u) + Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}, \tag{1.11}$$

We remark that we have implicitly shown that $\frac{1}{2}J'(u) = \Lambda_{\mathcal{U}}^{-1}B^*p(u) + Nu$. All the results can be summarized in the following

Theorem 4. *A necessary and sufficient condition for u to be an optimal control is that the following equations and inequalities are satisfied:*

$$Ay(u) = f + Bu, \tag{1.12}$$

$$A^*p(u) = C^*\Lambda(C(y(u) - z_d)), \tag{1.13}$$

$$(\Lambda_{\mathcal{U}}^{-1}B^*p(u) + Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}. \tag{1.14}$$

If N satisfies (1.6) the optimal control is unique, if $N = 0$ there exists at least one solution. We refer to the system formed by (1.12), (1.13), (1.14) as the optimality system.

1.1.3 Lagrangian Approach to Control Problems

In this section we describe the *Lagrangian* approach for the resolution of an optimal control problem. This method is broadly used in many applications because the derivation of the optimality system is typically more straightforward in this case, since one does not have to pass through the definition of the adjoints in order to be able to formulate the problem.
In general, the use of the Lagrangian permits the determination of the extreme point x of a certain function f subject to the constraint expressed by the function g. In an optimal control problem one wants to find a function $u \in \mathcal{U}$ such that

$$J(y, u) \text{ is minimized,}$$

where J is the cost functional and y is the solution of the following state equation

$$Ay(u) = f + Bu, \quad y \in V,$$

that has to be coupled with suitable boundary conditions. The operators A and B have the same meaning that was given them in the previous section. We remark that this problem can be seen as a constrained optimization problem where the function to be minimized is represented by the cost functional, the constraint is represented by the state equation and the minimum is the optimal control u. The resolution of the optimal control problem can be reduced to the search of the extreme points of the *Lagrangian functional* defined in the following way

$$\mathcal{L}(y, p, u) = J(y, u) + \langle p, f + Bu - Ay(u) \rangle, \tag{1.15}$$

where the variable p represents the *Lagrange multiplier* and the symbol $\langle \cdot, \cdot \rangle$ denotes the duality product. In this contest the extreme points of $\mathcal{L}(y, p, u)$ are the functions $\hat{y}$, $\hat{u}$ and $\hat{p}$ that correspond to the optimum. The problem is:

$$\text{find } (\hat{y}, \hat{p}, \hat{u}) : \nabla \mathcal{L}(\hat{y}, \hat{p}, \hat{u}) = 0,$$

that can also be expressed as: find the solution $(\hat{y}, \hat{p}, \hat{u})$ of

$$\begin{cases} \mathcal{L}_y(\hat{y}, \hat{p}, \hat{u}) = 0, \\ \mathcal{L}_p(\hat{y}, \hat{p}, \hat{u}) = 0, \\ \mathcal{L}_u(\hat{y}, \hat{p}, \hat{u}) = 0. \end{cases}$$

We now consider the elliptic state equation in its weak formulation; with $y \in V$, $u \in \mathcal{U}$ and $f \in H$, we have:

$$a(y, \varphi) = (f, \varphi) + b(u, \varphi), \quad \forall \varphi \in V,$$

where $a(\cdot, \cdot)$ represents the linear elliptic operator and the bilinear form $b(\cdot, \cdot)$ introduces in the weak formulation the control term. The cost functional that has to be minimized can be expressed as

$$J(y, u) = \frac{1}{2} \|Cy - z_d\|^2 + \frac{1}{2} n(u, u).$$

We remark that no hypotheses were made on the choice of the boundary conditions, on the type of control problem (distributed or boundary control) and on the observation of the system; this is in order to be able to use the most general expression for the equations.
We now consider the *Lagrange multiplier* $p \in V$ and (1.15) becomes:

$$\mathcal{L}(y, p, u) = J(y, u) + b(u, p) + (f, p) - a(y, p)$$

and we want to solve the problem

$$\text{find } (\hat{y}, \hat{p}, \hat{u}) \in V \times V \times \mathcal{U} : \nabla\mathcal{L}(\hat{y}, \hat{p}, \hat{u})[(\varphi, \phi, \psi)] = 0, \quad \forall(\varphi, \phi, \psi) \in V \times V \times \mathcal{U}.$$

We obtain the following set of equations

$$\begin{cases} \mathcal{L}_y[\varphi] &= (Cy - z_d, C\varphi) - a(p, \varphi) = 0 & \forall\varphi \in V, \\ \mathcal{L}_p[\phi] &= (f, \phi) + b(u, \phi) - a(y, \phi) = 0 & \forall\phi \in V, \\ \mathcal{L}_u[\psi] &= b(p, \psi) + n(u, \psi) = 0 & \forall\psi \in \mathcal{U}, \end{cases}$$

and rearranging the various term we obtain

$$\begin{cases} a(p, \varphi) &= (Cy - z_d, C\varphi) & \forall\varphi \in V, \\ a(y, \phi) &= (f, \phi) + b(u, \phi) & \forall\phi \in V, \\ b(p, \psi) &= n(u, \psi) & \forall\psi \in \mathcal{U}. \end{cases}$$

This system is complementary to the optimality system introduced in Theorem 4, in fact the three equations represent respectively the adjoint equation, the state equation and the optimality condition. The adjoint variable, seen as a Lagrange multiplier, is related to the *sensitivity* of the cost functional to the variations of the observation equation, which depend of the variations of the control variable u. The last equation represents the derivative of the cost functional $J'(u)$ in virtue of the *Riesz Representation* Theorem (see [11]).

In the theory developed by *Lions* the derivation of the adjoint equations is obtained through the introduction of suitable adjoint operators, while in this approach the adjoints are obtained by derivation of the Lagrangian functional with respect to the state variable. The main difference resides in the fact that *Lions'* approach formulates the equations in their strong form while the *Lagrangian formalism* generates the equations of the optimality system in their weak formulation. These approaches do not lead to the definition of the same adjoint problems but this does not imply that a method is better than the other. It is important however, when solving an optimization problem, to be coherent to the theoretical setting taken into account.

1.1.4 Some Results on Non-Homogeneous Dirichlet Problems

We consider a second-order elliptic operator with coefficients sufficiently regular in $\bar{\Omega}$. Let f and g be functions or distributions on Ω and Γ respectively. We search for y in a suitable class, that will be specified later, such that

$$\begin{aligned} Ay &= f \qquad \text{in } \Omega, \\ y &= g \qquad \text{on } \Gamma. \end{aligned} \tag{1.16}$$

We consider a test function Ψ in $\bar{\Omega}$ and we multiply the first equation by Ψ and through the use of *Green*'s Formula we obtain

$$\int_\Omega (Ay)\Psi dx = \int_\Omega f\Psi dx = -\int_\Gamma \frac{\partial y}{\partial \nu_A}\Psi d\Gamma + \int_\Gamma y\frac{\partial \Psi}{\partial \nu_{A^*}} d\Gamma + \int_\Omega y(A^*\Psi)dx,$$

and, using the Dirichlet boundary condition we deduce

$$\int_\Omega y(A^*\Psi)dx = \int_\Omega f\Psi dx - \int_\Gamma g\frac{\partial \Psi}{\partial \nu_{A^*}} d\Gamma + \int_\Gamma \frac{\partial y}{\partial \nu_A}\Psi d\Gamma.$$

At this point it is natural to impose the condition $\Psi|_\Gamma = 0$, so that we obtain

$$\int_\Omega y(A^*\Psi)dx = \int_\Omega f\Psi dx - \int_\Gamma g\frac{\partial \Psi}{\partial \nu_{A^*}} d\Gamma. \tag{1.17}$$

The idea is to use (1.17) as an equation that defines a function y which by definition is a *weak solution* of (1.16). Let φ be a given element in $L^2(\Omega)$. The solution of

$$\begin{aligned} A^*\Psi &= \varphi, \\ \Psi|_\Gamma &= 0 \end{aligned}$$

belongs to the space $H^2(\Omega)$, thanks to the *elliptic regularity* results (see [11]). This means that A^* is an isomorphism of $H^2(\Omega)\cap H_0^1(\Omega)$ onto $L^2(\Omega)$. By transposing the isomorphism (thus the name of this approach: *transposition method*) we also see that if $\Psi \to \mathscr{L}(\Psi)$ is a continuous linear form on $H^2(\Omega)\cap H_0^1(\Omega)$, then there exists a unique $y \in L^2(\Omega)$ such that

$$\int_\Omega y(A^*\Psi)dx = \mathscr{L}(\Psi), \quad \forall \Psi \in H^2(\Omega)\cap H_0^1(\Omega).$$

We can now take $f \in L^2(\Omega)$ and $g \in H^{-\frac{1}{2}}(\Gamma)$, so that the linear form

$$\mathscr{L}(\Psi) = \int_\Omega f\Psi dx - \int_\Gamma g\frac{\partial \Psi}{\partial \nu_{A^*}} d\Gamma$$

is continuous on $H^2(\Omega)\cap H_0^1(\Omega)$. We have thus proved the following

Theorem 5. *For $f \in L^2(\Omega)$ and $g \in H^{-\frac{1}{2}}(\Gamma)$, there exists a unique $y \in L^2(\Omega)$ such that (1.17) is true.*

1.2 Coupled Virtual Control: the Poisson Problem

Virtual control is a powerful technique that has been introduced in the domain decomposition method with overlapping subdomains. In this section we want to prove the well posedness of the coupled virtual control method on two overlapping subdomains for the solution of the Poisson equation. We will test the method for different choices of the control space and for different choices of the functional that is minimized on the overlapping region.
We consider a two dimensional domain Ω and we adopt the following notation (see Figure 1.1): Ω_1 and Ω_2 are two overlapping subdomains of Ω such that

$$\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset,$$

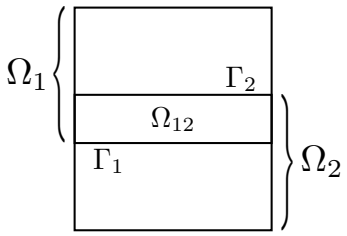


Figure 1.1: *Reference domain.*

$$\partial\Omega_{12} = \Gamma_1 \cup \Gamma_2 \cup (\partial\Omega_1 \cap \partial\Omega_2),$$
$$\partial\Omega_i = \Gamma_1 \cup \Gamma_i^D \cup \Gamma_i^N, \text{ with } \Gamma_i = \partial\Omega_i \backslash \partial\Omega,$$

where $\{\Gamma_i^D, \Gamma_i^N\}$ represents a suitable partition of $\partial\Omega_i \cap \partial\Omega$, $i = 1, 2$. We will refer to Γ_1 and Γ_2 as the control interfaces of the problem. Moreover we have that

$$\Gamma = \partial\Omega, \quad \Gamma^D = \Gamma_1^D \cup \Gamma_2^D, \quad \Gamma^N = \Gamma_1^N \cup \Gamma_2^N.$$

1.2.1 The Formulation of the Problem

In this section we present the setting of the coupled virtual control method. In particular, we introduce the state problems and the relative cost functional. In the following sections we will place the formulation of this problem in the framework of optimal control and derive the optimality system related to it.

The homogeneous domain decomposition is formulated through two unknown functions λ_1 and λ_2 in the following way: we want to solve on the two overlapping domains Ω_1 and Ω_2 the following Poisson problems, that will be referred to as the *state problems*:
for $i = 1, 2$, find $\varphi^i \in H^1(\Omega_i)$ so that

$$\begin{cases} -\nabla \cdot (K\nabla\varphi^i) = f^i & in \quad \Omega_i \\ K\nabla\varphi^i \cdot \mathbf{n}_i = \Psi_N^i & on \quad \Gamma_i^N \\ \varphi^i = \Psi_D^i & on \quad \Gamma_i^D \\ \varphi^i = \lambda_i & on \quad \Gamma_i. \end{cases} \tag{1.18}$$

Let $f^i \in L^2(\Omega_i)$, $\Psi_N^i \in H^{-\frac{1}{2}}(\Omega_i)$ and $\Psi_D^i \in H^{\frac{1}{2}}(\Omega_i)$. We seek for the unknown control variables λ_1 and λ_2 in the following space of *admissible controls*:

$$\Lambda_i^D = \Big\{ \mu \in H^{\frac{1}{2}}(\Gamma_i) : \exists \Psi \in H^1(\Omega_i), \Psi = \mu \text{ on } \Gamma_i, \nabla\Psi \cdot \mathbf{n}_i = 0 \text{ on } \Gamma_i^N \ , \\ \Psi = 0 \text{ on } \Gamma_i^D \Big\}.$$

The virtual controls can also play the role of Neumann boundary condition; in this case the set of *admissible controls* is

$$\Lambda_i^N = \Big\{ \mu \in H^{-\frac{1}{2}}(\Gamma_i) : \exists \Psi \in H^1(\Omega_i), \nabla\Psi \cdot \mathbf{n}_i = \mu \text{ on } \Gamma_i, \nabla\Psi \cdot \mathbf{n}_i = 0 \text{ on } \Gamma_i^N \ , \\ \Psi = 0 \text{ on } \Gamma_i^D \Big\},$$

and, for $i = 1, 2$, the *state problems* assume the following form:

$$\begin{cases} -\nabla \cdot (K\nabla\varphi^i) &= f^i & in \quad \Omega_i \\ K\nabla\varphi^i \cdot \mathbf{n}_i &= \Psi_N^i & on \quad \Gamma_i^N \\ \varphi^i &= \Psi_D^i & on \quad \Gamma_i^D \\ K\nabla\varphi^i \cdot \mathbf{n}_i &= \lambda_i & on \quad \Gamma_i. \end{cases} \tag{1.19}$$

In what follows we will refer to the spaces of the virtual controls as Λ_1 and Λ_2 when there is no need to distinguish whether the virtual controls are of "Dirichlet" or of "Neumann" type.

The virtual controls are determined as solution of a minimization problem. In fact we want to minimize a suitable cost functional, depending on the difference between the two solutions φ^1 and φ^2 on the overlapping region $\Omega_{12} = \Omega_1 \cap \Omega_2$. The aim is to be able to recover the global solution of the Poisson problem on Ω when the cost functional is minimized. We want to test the good position of the minimization problem when observing the difference $(\varphi^1 - \varphi^2)|_{\Omega_{12}}$ in the following ways:

Minimization in the $L^2(\Omega_{12})$ norm

$$J_{L^2}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^1 - \varphi^2)^2. \tag{1.20}$$

Minimization in the $H_0^1(\Omega_{12})$ semi-norm

$$J_{H_0^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^1 - \nabla\varphi^2)^2. \tag{1.21}$$

Minimization in the $H^1(\Omega_{12})$ norm

$$J_{H^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^1 - \varphi^2)^2 + \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^1 - \nabla\varphi^2)^2. \tag{1.22}$$

Minimization in an augmented $H_0^1(\Omega_{12})$ norm

$$\widehat{J_{H_0^1}}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^1 - \nabla\varphi^2)^2 + \frac{1}{2}\int_{\partial\Omega} \chi_{12}(\varphi^1 - \varphi^2)^2. \tag{1.23}$$

We remark that we referred to (1.21) as a semi-norm. The choice of this notation will be clarified in the following section as we will show that $J_{H_0^1}(\lambda_1, \lambda_2)$ is a norm or a semi-norm according to the boundary conditions that are imposed on $\partial\Omega_{12}$.
We will analyze the different choices separately, showing theoretically that not all the approaches guarantee the uniqueness of the solution. The implementation of the different choices in our code will then allow us to validate our conclusions and to make a comparison between the different cost functionals.

A penalization term can be added to all the cost functionals ((1.20)-(1.23)):

$$\frac{1}{2}\beta_1\int_{\Gamma_1}\lambda_1^2 + \frac{1}{2}\beta_2\int_{\Gamma_2}\lambda_2^2. \tag{1.24}$$

This term serves a regularization purpose and we will see that all the cost functionals, when penalized, are coercive. In a sequent analysis we will show that the conditioning of the problem improves for high values of the coefficients β_1 and β_2. It is important to remark that the penalization of the cost functional implies a modification of the original problem. In the case of a penalized cost functional the fact that the minimum is reached does not guarantee that the difference between the solutions is actually zero on the overlap. We recall that we are studying a homogeneous domain decomposition problem and we want to be able to recover the exact solution of the global problem. For this reason, the most interesting cases are the non penalized ones and the ones corresponding to very low values of the penalization coefficients β_1 and β_2. In the non penalized case $\inf_{\lambda_1,\lambda_2} J(\lambda_1,\lambda_2) = 0$, which is achieved by taking the controls λ_1 and λ_2 as the restrictions (traces) of the global solution on Γ_1 and Γ_2 respectively.

In order to be able to prove the well posedness of the virtual control problem, applying the results presented in section 1.1, it is useful to split the problem in a part depending on the controls and a part depending on the data. Thanks to the linearity of the original problem we have that:

$$\varphi^i = \varphi^{i,\lambda_i} + \varphi^{i,f}, \tag{1.25}$$

where $\varphi^{i,f}(i=1,2)$ is the solution of a problem depending only on the given data while $\varphi^{i,\lambda_i}(i=1,2)$ depends only on the virtual controls. More precisely, for $i=1,2$, $\varphi^{i,\lambda_i} \in H^1(\Omega_i)$ satisfies

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\varphi^{i,\lambda_i}) &=& 0 & in\quad \Omega_i \\ K\nabla\varphi^{i,\lambda_i}\cdot\mathbf{n}_i &=& 0 & on\quad \Gamma_i^N \\ \varphi^{i,\lambda_i} &=& 0 & on\quad \Gamma_i^D \\ K\nabla\varphi^{i,\lambda_i}\cdot\mathbf{n}_i \;/\; \varphi^{i,\lambda_i} &=& \lambda_i & on\quad \Gamma_i, \end{array}\right. \tag{1.26}$$

and for $i=1,2$, $\varphi^{i,f} \in H^1(\Omega_i)$ satisfies:

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\varphi^{i,f}) &=& f^i & in\quad \Omega_i \\ K\nabla\varphi^{i,f}\cdot\mathbf{n}_i &=& \Psi_N^i & on\quad \Gamma_i^N \\ \varphi^{i,f} &=& \Psi_D^i & on\quad \Gamma_i^D \\ K\nabla\varphi^{i,f}\cdot\mathbf{n}_i \;/\; \varphi^{i,f} &=& 0 & on\quad \Gamma_i, \end{array}\right. \tag{1.27}$$

where we considered both the choices for the controls (Neumann or Dirichlet) on the interfaces Γ_1 and Γ_2.
Correspondingly the cost functionals ((1.20)-(1.23)) can be split into the sum of a quadratic and of an affine functional in the following way:

- $J_{L^2}(\lambda_1,\lambda_2) = J^0_{L^2}(\lambda_1,\lambda_2) + \mathscr{A}_{L^2}(\lambda_1,\lambda_2)$,

  where

$$J^0_{L^2}(\lambda_1,\lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2})^2,$$
$$\mathscr{A}_{L^2}(\lambda_1,\lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^{1,f}-\varphi^{2,f})^2 + \int_\Omega \chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2})\cdot(\varphi^{1,f}-\varphi^{2,f}),$$

- $J_{H_0^1}(\lambda_1, \lambda_2) = J^0_{H_0^1}(\lambda_1, \lambda_2) + \mathscr{A}_{H_0^1}(\lambda_1, \lambda_2)$,

  where

$$J^0_{H_0^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})^2,$$
$$\mathscr{A}_{H_0^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f})^2 + \int_\Omega \chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})\cdot(\nabla\varphi^{1,f} - \nabla\varphi^{2,f}),$$

- $J_{H^1}(\lambda_1, \lambda_2) = J^0_{H^1}(\lambda_1, \lambda_2) + \mathscr{A}_{H^1}(\lambda_1, \lambda_2)$,

  where

$$J^0_{H_0^1}(\lambda_1, \lambda_2) = J^0_{L^2}(\lambda_1, \lambda_2) + J^0_{H_0^1}(\lambda_1, \lambda_2),$$
$$\mathscr{A}_{H_0^1}(\lambda_1, \lambda_2) = \mathscr{A}_{L^2}(\lambda_1, \lambda_2) + \mathscr{A}_{H_0^1}(\lambda_1, \lambda_2),$$

- $\widehat{J_{H_0^1}}(\lambda_1, \lambda_2) = \widehat{J^0_{H_0^1}}(\lambda_1, \lambda_2) + \widehat{\mathscr{A}_{H_0^1}}(\lambda_1, \lambda_2)$,

  where

$$\widehat{J^0_{H_0^1}}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})^2 + \frac{1}{2}\int_{\partial\Omega} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})^2,$$
$$\widehat{\mathscr{A}_{H_0^1}}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f})^2 + \int_\Omega \chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})\cdot(\nabla\varphi^{1,f} - \nabla\varphi^{2,f}) + \int_{\partial\Omega} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\cdot(\varphi^{1,f} - \varphi^{2,f}).$$

We remark that the splitting of $J_{H^1}(\lambda_1, \lambda_2)$ as been obtained by linear combination of the splitting of $J_{L^2}(\lambda_1, \lambda_2)$ and $J_{H_0^1}(\lambda_1, \lambda_2)$. If added, the penalization term is obviously part of the functional depending on the sole controls.

1.2.2 Existence and Uniqueness of the Solution

The aim of this section is to place the coupled virtual control problem in the general framework of optimal control presented in the previous section. We want to prove the well posedness of the state problems, and we want to verify the hypothesis of the coercivity of the cost functional in order to be able to use the results of Theorem 1.

The Well Posedness of the Boundary Value Problem

Let's consider problems (1.26) and (1.27) in the case of Neumann boundary controls. The choice of Neumann boundary controls is motivated by the fact that natural boundary conditions are included in a more straightforward way in the weak formulation of the problem. Moreover, this choice allows us to follow the approach presented in [8]. Proofs concerning the problem with Dirichlet boundary controls can be found in [3], [4] and [5]. We denote by V^i the Hilbert space $H^1_{\Gamma_D}(\Omega_i)$, $i = 1, 2$.

In virtue of the splitting of the solutions (1.25) we can study the well posedness of the problems depending on the data and of the problems depending on the controls separately. In order to write the weak formulation of problems (1.27) we introduce, for each problem, a function R^i so that $R^i \in H^1(\Omega_i)$ and $R^i|_{\Gamma_i^D} = \Psi_D^i$ and so that there exists a constant $C > 0$ so that

$$\|R^i\|_{H^i(\Omega_i)} \leq C\|\Psi_D^i\|_{H^{\frac{1}{2}}(\Gamma_i^D)} \qquad i = 1, 2. \tag{1.28}$$

The functions $R^i, i = 1, 2$ represent the *lifting* of the Dirichlet data of the state problems. We consider

$$\hat{\varphi}^{i,f} = \varphi^{i,f} - R^i, i = 1, 2. \tag{1.29}$$

The weak formulation of the problem for $\hat{\varphi}^{i,f}$ reads as follows: for $i = 1, 2$, find $\hat{\varphi}^{i,f} \in V^i$ so that

$$\int_{\Omega_i} K\nabla\hat{\varphi}^{i,f}\nabla\psi = -\int_{\Omega_i} K\nabla R^i\nabla\psi + \int_{\Omega_1} f^i\psi + \int_{\Gamma_i^N} \Psi_N^i\psi \qquad \forall\psi \in V^i. \tag{1.30}$$

Introducing the bi-linear form

$$a^i(\varphi, \psi) = \int_{\Omega_i} K\nabla\varphi\nabla\psi \qquad \forall\varphi, \psi \in V^i,$$

and the linear form

$$F^i(\psi) = -\int_{\Omega_i} K\nabla R^i\nabla\psi + \int_{\Omega_i} f^i\psi + \int_{\Gamma_i^N} \Psi_N^i\psi \qquad \forall\psi \in V^i,$$

we can write problems (1.30) as:
for $i = 1, 2$, find $\hat{\varphi}^{i,f} \in V^i$ so that

$$a^i(\hat{\varphi}^{i,f}, \psi) = F^i(\psi) \qquad \forall\psi \in V^i. \tag{1.31}$$

For $i = 1, 2$, the bi-linear form $a^i(\cdot, \cdot)$ is continuous and coercive on V^i and thanks to the hypothesis on the data the linear form $F^i(\cdot)$ is continuous on V^i. So the *Lax-Milgram* Lemma guarantees that problems (1.31) admit a unique solution $\hat{\varphi}^{i,f} \in V^i$, for $i = 1, 2$. Moreover we have that

$$\|\hat{\varphi}^{i,f}\|_{V^i} \leq M\big(\|f^i\|_{L^2(\Omega_i)} + \|\Psi_N^i\|_{H^{-\frac{1}{2}}(\Gamma_i^N)}\big) \qquad M > 0, \qquad i = 1, 2,$$

and recovering $\varphi^{i,f}$, thanks to (1.28) we obtain that

$$\|\varphi^{i,f}\|_{V^i} \leq M\big(\|f^i\|_{L^2(\Omega_i)} + \|\Psi_N^i\|_{H^{-\frac{1}{2}}(\Gamma_i^N)} + \|\Psi_D^i\|_{H^{\frac{1}{2}}(\Gamma_i^D)}\big) \qquad M > 0, \qquad i = 1, 2. \tag{1.32}$$

In an analogous way given $\lambda_i \in \Lambda_i^N$, problems (1.26) admit a unique solution $\varphi^{i,\lambda_i} \in V^i$ so that

$$\|\varphi^{i,\lambda_i}\|_{V^i} \leq M\|\lambda_i\|_{\Lambda_i} \qquad M > 0, \qquad i = 1, 2. \tag{1.33}$$

In accordance to the notations introduced in section 1.1 we introduce, for $i = 1, 2$, the following operators

$$A_i \in \mathscr{L}(V^i, (V^i)^*),$$
$$B_i \in \mathscr{L}(H^{-\frac{1}{2}}(\Gamma_i), (V^i)^*).$$

The definition of these linear operators is analogous to the one given in section 1.1. These operators are continuous in virtue of the well posedness of the state problems. In particular, the operators A_i define an isomorphism of V^i onto $(V^i)^*$, $i = 1, 2$. We are now able to write the problems (1.27) and the problems (1.26) in the following operatorial form: for $i = 1, 2$

$$A_i\varphi^{i,f} = f,$$
$$A_i\varphi^{i,\lambda_i} = B_i\lambda_i.$$

The Well Posedness of the Control Problem

In this section we want to verify the well posedness of the cost functionals (1.20)-(1.23). We first rewrite the expressions of the cost functionals given in section 1.2.1 in the generic form (1.1), then we will prove the coercivity of all the cost functionals in the penalized case and the coercivity of all the cost functionals, with some restrictions, in the non-penalized case.
We give here a generic expression of a cost functional that minimizes the difference between the solutions on the overlapping region. Later we will specify the form the operators that define the choice of $J(\lambda_1, \lambda_2)$ between (1.20)-(1.23). The generic expression of the observation operators is the following:

$$z^i(\lambda_i) = C_i\varphi^i, \quad \text{where} \qquad C_i \in \mathcal{L}(V^i, H) \quad i = 1, 2,$$

H being an Hilbert space. As we will see, the choice of the Hilbert space in which the observation is set depends on the choice of the cost functional. Then we are given the operator $N_i \in \mathscr{L}(\Lambda_i^N, \Lambda_i^N)$, which is symmetric and verifies the following coercivity assumption

$$(N_i\lambda_i, \lambda_i)_{\Lambda_i^N} \geq \nu_i\|\lambda_i\|^2_{\Lambda_i^N} \qquad \nu_i > 0, \quad i = 1, 2. \tag{1.34}$$

This operator is related to the penalization term and its definition is analogous for all the different choices of the cost functional to be minimized. In fact, the operators N_i are identified as

$$(N_i\lambda_i, \lambda_i)_{\Lambda_i^N} = \frac{1}{2}\beta_i \int_{\Gamma_i} \lambda_i^2 \qquad i = 1, 2,$$

and we recognize that $\nu_1 = \frac{1}{2}\beta_1$ and $\nu_2 = \frac{1}{2}\beta_2$. The choice $N_i = 0$, $i = 1, 2$, corresponds to the non-penalized case. In what follows we will treat the penalized and the non-penalized case separately.

In the virtual control problem we seek for the couple of controls (λ_1, λ_2) that minimizes the difference between the solutions of (1.18) on Ω_{12}. We remark that this setting is different than the one presented in section 1.1 because we want to find, simultaneously the optimal boundary values to be assigned on the control interfaces of two distinct state problems. For this reason we need to introduce the space of *admissible controls* ($\mathcal{U}_{ad} = \Lambda^N$)

$$\Lambda^N = \Lambda_1^N \times \Lambda_2^N.$$

On this space we define the norm $\|(\cdot, \cdot)\|_{\Lambda^N}$ in the following way:

$$\|(\mu_1, \mu_2)\|_{\Lambda^N} = \sqrt{\|\mu_1\|^2_{\Lambda_1^N} + \|\mu_2\|^2_{\Lambda_2^N}}. \tag{1.35}$$

Thanks to the inequalities

$$\sqrt{a^2+b^2} \leq a+b \leq \sqrt{2}\sqrt{a^2+b^2} \qquad \forall a \geq 0, b \geq 0,$$

a norm equivalent to (1.37), sometimes easier to use is

$$\|(\mu_1,\mu_2)\|_{\Lambda^N} = \|\mu_1\|_{\Lambda_1^N} + \|\mu_2\|_{\Lambda_2^N}. \tag{1.36}$$

The fact that (1.36) is actually a norm is easily verified, in fact:

- $\|(\mu_1,\mu_2)\|_{\Lambda^N} \geq 0$ and $\|(\mu_1,\mu_2)\|_{\Lambda^N} = 0 \Leftrightarrow (\mu_1,\mu_2) = (0,0)$;
- $\|\alpha(\mu_1,\mu_2)\|_{\Lambda^N} = \|\alpha\mu_1\|_{\Lambda_1^N} + \|\alpha\mu_2\|_{\Lambda_2^N} = |\alpha|(\|\mu_1\|_{\Lambda_1^N} + \|\mu_2\|_{\Lambda_2^N}) = |\alpha|\|(\mu_1,\mu_2)\|_{\Lambda^N}$;
- $\|(\mu_1,\mu_2)+(\nu_1,\nu_2)\|_{\Lambda^N} = \|\mu_1+\nu_1\|_{\Lambda_1^N} + \|\mu_2+\nu_2\|_{\Lambda_2^N} \leq \|\mu_1\|_{\Lambda_1^N} + \|\nu_1\|_{\Lambda_1^N} + \|\mu_2\|_{\Lambda_2^N} + \|\mu_2\|_{\Lambda_2^N}$
  $= \|(\mu_1,\mu_2)\|_{\Lambda^N} + \|(\nu_1,\nu_2)\|_{\Lambda^N}$.

To every couple of controls we associate the cost functional

$$J(\lambda_1,\lambda_2) = \frac{1}{2}\|C_1\varphi^1 - C_2\varphi^2\|_H^2 + (N_1\lambda_1,\lambda_1)_{\Lambda_1^N} + (N_2\lambda_2,\lambda_2)_{\Lambda_2^N}.$$

According to (1.25) the cost functional $J(\lambda_1,\lambda_2)$ can be decomposed in a quadratic and into an affine part as follows:

$$J((\lambda_1,\lambda_2)) = \frac{1}{2}\pi((\lambda_1,\lambda_2),(\lambda_1,\lambda_2)) - L((\lambda_1,\lambda_2)) + \|C_1\varphi^{1,f} - C_2\varphi^{2,f}\|_H^2,$$

where the quadratic form π is defined as

$$\begin{aligned}\pi((\lambda_1,\lambda_2),(\mu_1,\mu_2)) = {} & (C_1\varphi^{1,\lambda_1} - C_2\varphi^{2,\lambda_2}, C_1\varphi^{1,\mu_1} - C_2\varphi^{2,\mu_2})_H \\ & + (N_1\lambda_1,\mu_1)_{\Lambda_1^N} + (N_2\lambda_2,\mu_2)_{\Lambda_2^N},\end{aligned}$$

and the linear form L is defined as

$$L((\mu_1,\mu_2)) = -(C_1\varphi^{1,\mu_1} - C_2\varphi^{2,\mu_2}, C_1\varphi^{1,f} - C_2\varphi^{2,f})_H.$$

We remark that the quadratic linear form π is equivalent to the quadratic part of the generic cost functional that has been referred to as J^0 and the linear form L corresponds to the affine part of the generic cost functional that we have denoted as $\mathscr{A}$.

The control problem is: find $(\lambda_1,\lambda_2) \in \Lambda^N$ such that

$$J(\lambda_1,\lambda_2) = \inf_{(\mu_1,\mu_2)\in\Lambda^N} J(\mu_1,\mu_2).$$

We want to verify the hypotheses of Theorem 1, in order to be able to guarantee the existence and the uniqueness of the optimal controls (λ_1,λ_2) that minimize the cost functional. In particular we have to verify that

- $L((\mu_1,\mu_2))$ is continuous on Λ^N,
- $\pi((\lambda_1,\lambda_2),(\mu_1,\mu_2))$ is continuous and coercive on Λ^N.

Before proceeding we have to identify the observation operators C_1 and C_2 for each specific choice of the cost functional and analogously we need to identify the Hilbert space in which the observation is placed. For every choice of the cost functional the observation operator is the composition of the restriction operator $\chi_{12} \in L^\infty(\Omega)$ with the injection map of V into H.

We can identify the Hilbert space H for the different cost functionals that we have previously introduced.

- $J_{L^2}(\lambda_1, \lambda_2) \qquad \longleftrightarrow \qquad H = L^2(\Omega_{12})$
- $J_{H_0^1}(\lambda_1, \lambda_2) \qquad \longleftrightarrow \qquad H = H_0^1(\Omega_{12})$
- $J_{H^1}(\lambda_1, \lambda_2) \qquad \longleftrightarrow \qquad H = H^1(\Omega_{12})$
- $\widehat{J_{H_0^1}}(\lambda_1, \lambda_2) \qquad \longleftrightarrow \qquad H = H^1(\Omega_{12})$

The derivation of these injection is straightforward and derives explicitly from the definition of the cost functional.

The proof of the continuity of the linear functional $L((\mu_1, \mu_2))$ can be given independently of the distinction between a penalized and a non-penalized cost functional ($N_i \neq 0$ or $N_i = 0$), because the penalization terms appear only in the definition of the quadratic part of the cost functional. We want to verify the hypothesis of Theorem 1. We first show that $L((\mu_1, \mu_2))$ is continuous on Λ_N:

$$\begin{aligned}
|L((\mu_1, \mu_2))| &\leq \|(C_1\varphi^{1,\mu_1} - C_2\varphi^{2,\mu_2}\|_H \|C_1\varphi^{1,f} - C_2\varphi^{2,f}\|_H \\
&\leq (\|C_1\|_{\mathscr{L}(V^1,H)}\|\varphi^{1,\mu_1}\|_{V^1} + \|C_2\|_{\mathscr{L}(V^2,H)}\|\varphi^{2,\mu_2}\|_{V^2}) \\
&\quad (\|C_1\|_{\mathscr{L}(V^1,H)}\|\varphi^{1,f}\|_{V^1} + \|C_2\|_{\mathscr{L}(V^2,H)}\|\varphi^{2,f}\|_{V^2}) \\
&\leq \tilde{C}(\|\mu_1\|_{\Lambda_1^N} + \|\mu_2\|_{\Lambda_2^N}) \\
&\leq \tilde{C}\|(\mu_1, \mu_2)\|_{\Lambda^N} \qquad\qquad \forall (\mu_1, \mu_2) \in \Lambda^N,
\end{aligned}$$

where we have used (1.32) and (1.33).
In analogous way we can prove the continuity of the quadratic form $\pi((\lambda_1, \lambda_2), (\mu_1, \mu_2))$ on Λ_N:

$$\begin{aligned}
|\pi((\lambda_1, \lambda_2), (\mu_1, \mu_2))| &\leq \|(C_1\varphi^{1,\lambda_1} - C_2\varphi^{2,\lambda_2}\|_H \|C_1\varphi^{1,\mu_1} - C_2\varphi^{2,\mu_2}\|_H + \|N_1\lambda_1\|_{\Lambda_1^N}\|\mu_1\|_{\Lambda_1^N} + \|N_2\lambda_2\|_{\Lambda_2^N}\|\mu_2\|_{\Lambda_2^N} \\
&\leq (\|C_1\|_{\mathscr{L}(V^1,H)}\|\varphi^{1,\lambda_1}\|_{V^1} + \|C_2\|_{\mathscr{L}(V^2,H)}\|\varphi^{2,\lambda_2}\|_{V^2}) \\
&\quad (\|C_1\|_{\mathscr{L}(V^1,H)}\|\varphi^{1,\mu_1}\|_{V^1} + \|C_2\|_{\mathscr{L}(V^2,H)}\|\varphi^{2,\mu_2}\|_{V^2}) + \\
&\quad \|N_1\|_{\mathscr{L}(\Lambda_1^N,\Lambda_2^N)}\|\lambda_1\|_{\Lambda_1^N}\|\mu_1\|_{\Lambda_1^N} + \|N_2\|_{\mathscr{L}(\Lambda_1^N,\Lambda_2^N)}\|\lambda_2\|_{\Lambda_2^N}\|\mu_2\|_{\Lambda_2^N} \\
&\leq C\|(\lambda_1, \lambda_2)\|_{\Lambda^N}\|(\mu_1, \mu_2)\|_{\Lambda^N} \qquad\qquad \forall (\mu_1, \mu_2) \in \Lambda^N.
\end{aligned}$$

We proceed to prove the coercivity of the quadratic part of the cost functionals (1.20)-(1.23). We now need to distinguish between the penalized and the non-penalized case: in the first case the proof is straightforward and is valid for all the choices of the cost functional, in the second case we see that one obtains a weaker result and a distinction has to be done between the different cost functionals.

The Penalized Case

Since $\|C_1\varphi^{1,\mu_1} - C_2\varphi^{2,\mu_2}\|_H^2$ is clearly ≥ 0, from (1.34) follows the coercivity of the bi-linear form on Λ_N:

$$\begin{aligned}\pi((\mu_1,\mu_2),(\mu_1,\mu_2)) &\geq (N_1\lambda_1,\lambda_1)_{\Lambda_1^N} + (N_2\lambda_2,\lambda_2)_{\Lambda_2^N} \\ &\geq \nu_1\|\lambda_1\|^2_{\Lambda_1^N} + \nu_2\|\lambda_2\|^2_{\Lambda_2^N} \\ &= \min(\nu_1,\nu_2)\|(\lambda_1,\lambda_2)\|^2_{\Lambda^N} \qquad \forall(\mu_1,\mu_2)\in\Lambda^N.\end{aligned}$$

It is obvious that this argument is no longer valid in the case $N_i = 0$, $i = 1, 2$. We observe that in this proof the term related to the definition of the cost functional is ignored so that the choice between (1.20)-(1.23) is irrelevant (this term should verify the continuity assumptions but does not define the coercivity of the cost functional).

The Non-Penalized Case

In this case the verification of the coercivity of the bi-linear form π is not straightforward. If the control functions λ_2 and λ_2 are smooth enough one can define the semi-norm

$$|||(\mu_1,\mu_2)||| = \pi((\mu_1,\mu_2),(\mu_1,\mu_2)) = \|C_1\varphi^{1,\mu_1} - C_2\varphi^{2,\mu_2}\|_H^2 \tag{1.37}$$

on the space of the controls. Expression (1.37) is a semi-norm thanks to its symmetry and because the solutions of the state problems depending on the sole controls φ^{i,λ_i} are identically zero if $\lambda_i = 0$, $i = 1, 2$ because they are the solution of a homogeneous Poisson problem. If we can prove that (1.37) is actually a norm for the virtual controls, then the optimization problem will admit a unique solution in the space of λ_1 and λ_2 obtained by completion of this norm. This abstract space is of course "very large" and cannot be identified explicitly.

The semi-norm (1.37) is a norm is the following holds

$$\pi((\mu_1,\mu_2),(\mu_1,\mu_2)) = 0 \qquad \Rightarrow \qquad \mu_1 = \mu_2 = 0. \tag{1.38}$$

In order to be able to prove this assumption for the different cost functionals we need the following result

Proposition 1. If the solutions φ^{1,μ_1} and φ^{2,μ_2} of (1.26) are such that $\varphi^{1,\mu_1} = \varphi^{2,\mu_2}$ *a.e. in* Ω_{12}, then $\mu_1 = 0$ on Γ_1 and $\mu_2 = 0$ on Γ_2.

Proof. We define

$$w = \begin{cases} \varphi^{1,\mu_1} & \text{in} \quad \Omega_1\backslash\Omega_{12} \\ \varphi^{1,\mu_1} = \varphi^{2,\mu_2} = \varphi & \text{in} \quad \Omega_{12} \\ \varphi^{2,\mu_2} & \text{in} \quad \Omega_2\backslash\Omega_{12}. \end{cases}$$

Since φ^{1,μ_1} and φ^{2,μ_2} are solutions of the global state problems on Ω_1 and Ω_2 respectively, and because of the continuity between these solutions on the overlap Ω_{12} (derived from the fact that the cost functional is minimized) w is the solution of the global Darcy problem on Ω. This problem is

homogeneous and this implies $w \equiv 0$ on Ω so that necessarily $w|_{\Gamma_1}$, $w|_{\Gamma_2}$ and $\frac{\partial w}{\partial n_1}|_{\Gamma_1}$, $\frac{\partial w}{\partial n_2}|_{\Gamma_2}$, which proves that $\lambda_1 = \lambda_2 = 0$ in both the cases of Dirichlet and Neumann boundary controls. □

This results guarantees that if the solutions are matching on the overlapping region the virtual controls are necessarily the null controls. So now, for each different cost functional we have to prove that its minimization assures that the solutions of the state problems are equal on the overlap:

$$\pi((\mu_1, \mu_2), (\mu_1, \mu_2)) = 0 \qquad \Rightarrow \qquad \varphi^{1,\mu_1} = \varphi^{2,\mu_2} \quad a.e. \quad in \quad \Omega_{12}. \tag{1.39}$$

We will see that this result is not always true and that we need to distinguish between the different choices for the cost functional.

Proposition 2. The cost functionals $J^0_{L^2}(\mu_1, \mu_2)$, $J^0_{H^1}(\mu_1, \mu_2)$ and $\widehat{J^0_{H^1_0}}(\mu_1, \mu_2)$ define a norm on the space of the controls (λ_1, λ_2). The cost functional $J^0_{H^1_0}(\mu_1, \mu_2)$ defines a norm on the space of (λ_1, λ_2) if $\partial\Omega_{12} \cap \Gamma^D \neq 0$, otherwise it only defines a seminorm.

Proof. For each cost functional, we want to prove that (1.39) is verified. Then, from Proposition 1 follows (1.38).
In the case of $J^0_{L^2}(\mu_1, \mu_2)$ we have that

$$J^0_{L^2}(\mu_1, \mu_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^{1,\mu_1} - \varphi^{2,\mu_2})^2 = \frac{1}{2}\|\varphi^{1,\mu_1} - \varphi^{2,\mu_2}\|^2_{L^2(\Omega_{12})} = 0$$

$$\Rightarrow \quad \varphi^{1,\mu_1} = \varphi^{2,\mu_2} \quad a.e. \quad in \quad \Omega_{12}.$$

Analogously, if minimizing $J^0_{H^1}(\mu_1, \mu_2)$ we have that

$$J^0_{H^1}(\mu_1, \mu_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^{1,\mu_1} - \varphi^{2,\mu_2})^2 + \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,\mu_1} - \nabla\varphi^{2,\mu_2})^2 = \frac{1}{2}\|\varphi^{1,\mu_1} - \varphi^{2,\mu_2}\|^2_{H^1(\Omega_{12})} = 0$$

$$\Rightarrow \quad \varphi^{1,\mu_1} = \varphi^{2,\mu_2} \quad a.e. \quad in \quad \Omega_{12}.$$

In the case of $\widehat{J^0_{H^1_0}}(\mu_1, \mu_2)$ the following holds

$$\begin{aligned}\widehat{J^0_{H^1_0}}(\mu_1, \mu_2) =& \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,\mu_1} - \nabla\varphi^{2,\mu_2})^2 + \frac{1}{2}\int_{\partial\Omega} \chi_{12}(\varphi^{1,\mu_1} - \varphi^{2,\mu_2})^2 = \\ & \frac{1}{2}\|\nabla\varphi^{1,\mu_1} - \nabla\varphi^{2,\mu_2}\|^2_{L^2(\Omega_{12})} + \frac{1}{2}\|\varphi^{1,\mu_1} - \varphi^{2,\mu_2}\|^2_{L^2(\partial\Omega\cap\partial\Omega_{12})} = 0\end{aligned}$$

$$\Rightarrow \|\varphi^{1,\mu_1} - \varphi^{2,\mu_2}\|^2_{L^2(\partial\Omega\cap\partial\Omega_{12})} = 0 \quad \Rightarrow \quad \varphi^{1,\mu_1} - \varphi^{2,\mu_2} = 0 \quad a.e.\ on\ \partial\Omega \cap \partial\Omega_{12}.$$

This implies that the *Poincaré inequality* holds for the difference between the solutions $(\varphi^{1,\mu_1} - \varphi^{2,\mu_2})|_{\Omega_{12}}$ and this guarantees that

$$(\varphi^{1,\mu_1} - \varphi^{2,\mu_2})|_{\Omega_{12}} \in H^1_{\Gamma_D}(\Omega_{12}),$$

so that the following holds

$$\|\nabla\varphi^{1,\mu_1} - \nabla\varphi^{2,\mu_2}\|^2_{L^2(\Omega_{12})} \simeq \|\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}\|^2_{H^1_{\Gamma_D}(\Omega_{12})} \quad \Rightarrow \quad \varphi^{1,\mu_1} = \varphi^{2,\mu_2} \quad a.e. \quad in \quad \Omega_{12}.$$

When considering the cost functional $J^0_{H^1_0}(\lambda_1, \lambda_2)$ we observe that

$$J^0_{H^1_0}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})^2 = \frac{1}{2}\|\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}\|^2_{L^2(\Omega_{12})} = 0,$$

now if

$$(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})|_{\Omega_{12}} \in H^1_{\Gamma_D}(\Omega_{12}) \quad i.e. \quad (\partial\Omega_{12} \cap \Gamma^D) \neq \emptyset$$

we obtain that

$$\|\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}\|^2_{L^2(\Omega_{12})} \simeq \|\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}\|^2_{H^1_{\Gamma_D}(\Omega_{12})} \quad \Rightarrow \quad \varphi^{1,\lambda_1} = \varphi^{2,\lambda_2} \quad a.e. \quad in \quad \Omega_{12}.$$

□

We observe that the cost functional that minimizes the difference between the solutions in $H^1_0(\Omega_{12})$ does not guarantee the uniqueness of the optimal controls if a full Neumann condition is applied on the boundary of the overlap. The point is that the difference between the two state solutions, restricted on the overlap, has to belong to the space $H^1_0(\Omega_{12})$, in order to have $J^0_{H^1_0}$ defining a norm on the space of the controls (λ_1, λ_2). We remark that adding a term that controls the difference between the solutions on a part of the boundary ($\widehat{J_{H^1_0}}$) we recover the coercivity of the cost functional. This results will be confirmed by the numerical simulations.

We have shown that adding a penalization term of the form (1.24) is not necessary for the well posedness of the problem. We observed that if $\partial\Omega_{12} \cap \Gamma^D = \emptyset$, $J_{H^1_0}$ can be modified as in (1.23) in order to obtain a coercive cost functional. With this choice one still controls the difference the solutions and does not need to add a penalization term on the single controls λ_1 and λ_2 as in (1.24). In what follows the penalized formulation will be used only in the algebraic *one-shot* approach for the solution of the problem, that will be described in the contest of preconditioning.

1.2.3 The Optimality System

We want to compute $\nabla J = \nabla J^0 + \nabla\mathscr{A}$ for $J = J_{L^2}$, $J = J_{H^1_0}$, $J = J_{H^1}$ and $J = \widehat{J_{H^1_0}}$ in order to find the optimal controls (λ_1, λ_2) corresponding to the minimum of the cost functional. For this reason, we compute the *Gateaux* derivatives of the different cost functionals with respect to the two controls λ_1 and λ_2. Then we show that for each cost functional these derivatives can be rewritten in terms of the solution of the dual problems of (1.26) and (1.27), with forcing term and boundary conditions depending on the solution of the state problems (1.26) and (1.27).

In the case of the minimization of J_{L^2} we also present the derivation of the optimality system according to the *Lagrangian* formalism.

Minimization in the $L^2(\Omega_{12})$ Norm

In the case of this cost functional we will develop the formulation of the optimality system in presence of the penalization term (1.24). The non penalized formulation can be obtained by simply setting $\beta_1 = \beta_2 = 0$. The penalized formulation for the remaining cost functionals can be derived in an

analogous way.

The derivatives of functional (1.20), split in the part depending on the controls and the part depending on the data, assume the following form

$$\begin{aligned}
\langle\frac{\partial J^0_{L^2}}{\partial\lambda_1},\mu_1\rangle &= \lim_{\delta\to0}\Big(\frac{1}{2}\int_\Omega\chi_{12}(\varphi^{1,\lambda_1}+\delta\varphi^{1,\mu_1}-\varphi^{2,\lambda_2})^2-\frac{1}{2}\int_\Omega\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2})^2\Big)\\
&\quad+\lim_{\delta\to0}\Big(\frac{1}{2}\int_{\Gamma_1}(\lambda_1+\delta\mu_1)^2-\frac{1}{2}\int_{\Gamma_1}\lambda_1^2\Big)\\
&=\int_\Omega\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2})\varphi^{1,\mu_1}+\int_{\Gamma_1}\lambda_1\mu_1 \qquad \forall\mu_1\in\Lambda_1, \qquad (1.40a)\\
\langle\frac{\partial J^0_{L^2}}{\partial\lambda_2},\mu_2\rangle &=-\int_\Omega\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2})\varphi^{2,\mu_2}+\int_{\Gamma_2}\lambda_2\mu_2 \qquad \forall\mu_2\in\Lambda_2, \qquad (1.40b)\\
\langle\frac{\partial \mathscr{A}_{L^2}}{\partial\lambda_1},\mu_1\rangle &=\int_\Omega\chi_{12}(\varphi^{1,f}-\varphi^{2,f})\varphi^{1,\mu_1} \qquad \forall\mu_1\in\Lambda_1, \qquad (1.40c)\\
\langle\frac{\partial \mathscr{A}_{L^2}}{\partial\lambda_2},\mu_2\rangle &=-\int_\Omega\chi_{12}(\varphi^{1,f}-\varphi^{2,f})\varphi^{2,\mu_2} \qquad \forall\mu_2\in\Lambda_2. \qquad (1.40d)
\end{aligned}$$

We now consider the following dual problem

$$\begin{cases}
-\nabla\cdot(K\nabla\Psi^1) &= \chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}) & in\quad\Omega_1\\
K\nabla\Psi^1\cdot\mathbf{n}_1 &= 0 & on\quad\Gamma_1^N\\
\Psi^1 &= 0 & on\quad\Gamma_1^D\\
K\nabla\Psi^1\cdot\mathbf{n}_1\ /\ \Psi^1 &= 0 & on\quad\Gamma_1,
\end{cases} \qquad (1.41)$$

and show how we can express the partial derivative (1.40a) as a function of the solution of this problem Ψ^1. In fact we have

$$\begin{aligned}
&\int_{\Omega_1}\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2})\varphi^{1,\mu_1}=\int_{\Omega_1}-\nabla\cdot(K\nabla\Psi^1)\varphi^{1,\mu_1}=\\
&\int_{\Omega_1}-\nabla\cdot(K\nabla\varphi^{1,\mu_1})\Psi^1-\int_{\partial\Omega_1}(K\nabla\Psi^1\cdot\mathbf{n}_1)\varphi^{1,\mu_1}+\int_{\partial\Omega_1}(K\nabla\varphi^{1,\mu_1}\cdot\mathbf{n}_1)\Psi^1,
\end{aligned}$$

and remarking that φ^{1,μ_1} is solution of (1.26) and that Ψ^1 is solution of (1.41) we obtain that

$$\langle\frac{\partial J^0_{L^2}}{\partial\lambda_1},\mu_1\rangle=-\int_{\Gamma_1}(K\nabla\Psi^1\cdot\mathbf{n}_1)\mu_1+\int_{\Gamma_1}\lambda_1\mu_1 \qquad \forall\mu_1\in\Lambda_1^D, \qquad (1.42)$$

in the case of Dirichlet boundary controls, and we obtain that

$$\langle\frac{\partial J^0_{L^2}}{\partial\lambda_1},\mu_1\rangle=\int_{\Gamma_1}\Psi^1\mu_1+\int_{\Gamma_1}\lambda_1\mu_1 \qquad \forall\mu_1\in\Lambda_1^N, \qquad (1.43)$$

in the case of Neumann boundary controls.
In an analogous way, considering the following dual problems

$$\begin{cases}
-\nabla\cdot(K\nabla\Psi^2) &= -\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}) & in\quad\Omega_2\\
K\nabla\Psi^2\cdot\mathbf{n}_2 &= 0 & on\quad\Gamma_2^N\\
\Psi^2 &= 0 & on\quad\Gamma_2^D\\
K\nabla\Psi^2\cdot\mathbf{n}_2\ /\ \Psi^2 &= 0 & on\quad\Gamma_2,
\end{cases} \qquad (1.44)$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^1) &= \chi_{12}(\varphi^{1,f}-\varphi^{2,f}) & in \quad \Omega_1 \\ K\nabla\tilde{\Psi}^1\cdot\mathbf{n}_1 &= 0 & on \quad \Gamma_1^N \\ \Psi^1 &= 0 & on \quad \Gamma_1^D \\ K\nabla\tilde{\Psi}^1\cdot\mathbf{n}_1 \ / \ \tilde{\Psi}^1 &= 0 & on \quad \Gamma_1, \end{cases} \tag{1.45}$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^2) &= -\chi_{12}(\varphi^{1,f}-\varphi^{2,f}) & in \quad \Omega_2 \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 &= 0 & on \quad \Gamma_2^N \\ \tilde{\Psi}^2 &= 0 & on \quad \Gamma_2^D \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 \ / \ \tilde{\Psi}^2 &= 0 & on \quad \Gamma_2, \end{cases} \tag{1.46}$$

we can reformulate the remaining partial derivatives as

$$\langle\frac{\partial J^0_{L^2}}{\partial\lambda_2},\mu_2\rangle = -\int_{\Gamma_2}(K\nabla\Psi^2\cdot\mathbf{n}_2)\mu_2+\int_{\Gamma_2}\lambda_2\mu_2 \qquad \forall\mu_2\in\Lambda_2^D, \tag{1.47a}$$

$$\langle\frac{\partial \mathscr{A}_{L^2}}{\partial\lambda_1},\mu_1\rangle = -\int_{\Gamma_1}(K\nabla\tilde{\Psi}^1\cdot\mathbf{n}_1)\mu_1 \qquad \forall\mu_1\in\Lambda_1^D, \tag{1.47b}$$

$$\langle\frac{\partial \mathscr{A}_{L^2}}{\partial\lambda_2},\mu_2\rangle = -\int_{\Gamma_2}(K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_1)\mu_1 \qquad \forall\mu_2\in\Lambda_2^D, \tag{1.47c}$$

in the case of Dirichlet controls, or as

$$\langle\frac{\partial J^0_{L^2}}{\partial\lambda_2},\mu_2\rangle = \int_{\Gamma_2}\Psi^2\mu_2+\int_{\Gamma_2}\lambda_2\mu_2 \qquad \forall\mu_2\in\Lambda_2^N, \tag{1.48a}$$

$$\langle\frac{\partial \mathscr{A}_{L^2}}{\partial\lambda_1},\mu_1\rangle = \int_{\Gamma_1}\tilde{\Psi}^1\mu_1 \qquad \forall\mu_1\in\Lambda_1^N, \tag{1.48b}$$

$$\langle\frac{\partial \mathscr{A}_{L^2}}{\partial\lambda_2},\mu_2\rangle = \int_{\Gamma_2}\tilde{\Psi}^2\mu_2 \qquad \forall\mu_2\in\Lambda_2^N, \tag{1.48c}$$

in the case of Neumann boundary controls.

Lagrangian Derivation of the Optimality System

In order to define the *Lagrangian functional* (as in (1.15)) of this optimization problem we need to distinguish between Dirichlet and Neumann boundary controls. In fact, we observe that the state equations represents the constraint of the Lagrangian and its expression varies according to the choice of boundary controls that has been implemented. Moreover, in this case we do not want to split the solution in a part depending on the data and a part depending on the controls as in (1.25) but we work with the global solutions φ^1 and φ^2.
We recall the expression of the cost functional

$$J(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\varphi^1-\varphi^2)^2+\frac{1}{2}\beta_1\int_{\Gamma_1}\lambda_1^2+\frac{1}{2}\beta_2\int_{\Gamma_2}\lambda_2^2.$$

The bilinear forms related to the state problems (1.18) and (1.19) read as follows:

$$a(\varphi^i,\rho)=(f^i,\rho)+b(\lambda,\rho) \qquad \forall\rho\in V^i, i=1,2,$$

In the case of Dirichlet boundary controls, through the use of the *transposition* approach presented in section 1.1.4 we have that

$$a(\varphi^i, \rho) = -\int_{\Omega_i} \varphi^i \nabla \cdot (K \nabla \rho),$$
$$b(\lambda, \rho) = \int_{\Gamma_i} \lambda_i \nabla \rho \mathbf{n}_i,$$

where we have to assume that $\rho \in H^2(\Omega_i) \cap H^1_{\Gamma^D}(\Omega_i)$ and $\varphi^i \in L^2(\Omega^i)$. In the case of Neumann boundary controls the bilinear forms related to the differential operator of the state problem and to the weak formulation of the control term are

$$a(\varphi^i, \rho) = \int_{\Omega_i} K \nabla \varphi^i \nabla \rho,$$
$$b(\lambda, \rho) = -\int_{\Gamma_i} \lambda_i \rho,$$

We are now able to derive the specific expression of the Lagrangian cost functional in both the cases of Dirichlet and Neumann boundary controls, for $i = 1, 2$:

$$\mathcal{L}^D(\varphi^i, \Psi^i, \lambda_i) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^1 - \varphi^2)^2 + \frac{1}{2}\beta_1 \int_{\Gamma_1} \lambda_1^2 + \frac{1}{2}\beta_2 \int_{\Gamma_2} \lambda_2^2 + \int_{\Omega_i} \varphi^i \nabla \cdot (K \nabla \Psi^i) - \int_{\Gamma_i} \lambda_i \nabla \Psi^i \mathbf{n}_i,$$
$$\mathcal{L}^N(\varphi^i, \Psi^i, \lambda_i) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^1 - \varphi^2)^2 + \frac{1}{2}\beta_1 \int_{\Gamma_1} \lambda_1^2 + \frac{1}{2}\beta_2 \int_{\Gamma_2} \lambda_2^2 - \int_{\Omega_i} K \nabla \varphi^i \nabla \Psi^i + \int_{\Gamma_i} \lambda_i \Psi^i.$$

We see how the dual variable plays the role of the Lagrangian multiplier. We remark that because of the regularity conditions on the test function of the weak formulation of the state equation in the case of Dirichlet boundary controls, one has to make sure that the adjoint variable is sufficiently regular. This is obviously true in the case of the adjoint of the Poisson operator. We now proceed to the derivation of the Lagrangian functional, following the approach presented in section 1.1.3. It is important to remark that in this approach the state, adjoint and control variable are seen as independent, so that the dependencies between these three variables have to be neglected when differentiating the functional. In this sense, the differentiation approach used in this contest differs from the derivatives computed for example in (1.40a).
In the case of $(\lambda_1, \lambda_2) \in \Lambda^D$ and $i = 1$ we obtain the following expressions:

$$\mathcal{L}^D_{\lambda_1}[\mu] = \beta_1 \int_{\Gamma_1} \lambda_1 \mu - \int_{\Gamma_1} \mu \nabla \Psi^1 \mathbf{n}_1 = 0 \qquad \forall \mu \in \Lambda_1^N,$$
$$\mathcal{L}^D_{\varphi_1}[\rho] = \int_{\Omega_1} \chi_{12}(\varphi^1 - \varphi^2)\rho + \int_{\Omega_1} \rho \nabla \cdot (K \nabla \Psi^1) = 0 \qquad \forall \rho \in H^2(\Omega_1) \cap V^1,$$
$$\mathcal{L}^D_{\Psi^1}[\xi] = \int_{\Omega_1} \varphi^1 \nabla \cdot (K \nabla \xi) - \int_{\Gamma_1} \lambda_1 \nabla \xi \mathbf{n}_1 = 0 \qquad \forall \xi \in V^1,$$

and for $i = 2$ the complementary expressions:

$$\mathcal{L}^D_{\lambda_2}[\mu] = \beta_2 \int_{\Gamma_2} \lambda_2 \mu - \int_{\Gamma_2} \mu \nabla \Psi^2 \mathbf{n}_2 = 0 \qquad \forall \mu \in \Lambda_1^N,$$
$$\mathcal{L}^D_{\varphi_2}[\rho] = -\int_{\Omega_2} \chi_{12}(\varphi^1 - \varphi^2)\rho + \int_{\Omega_2} \rho \nabla \cdot (K \nabla \Psi^2) = 0 \qquad \forall \rho \in H^2(\Omega_2) \cap V^1,$$
$$\mathcal{L}^D_{\Psi^2}[\xi] = \int_{\Omega_{12}} \varphi^2 \nabla \cdot (K \nabla \xi) - \int_{\Gamma_2} \lambda_2 \nabla \xi \mathbf{n}_2 = 0 \qquad \forall \xi \in V^1,$$

where we have repetitively used the property that the derivative of a linear continuous application is given by the application itself.
We recognize that we have obtained the optimality equation, the adjoint equation and the state equation respectively for the two problems on Ω_1 and Ω_2. These equations are derived in their weak formulation and are equivalent to the strong formulations of the equations derived in the previous section. Analogously we can derive the optimality systems on the two domains with the Lagrangian approach in the case of Neumann boundary controls: for $i = 1$

$$\mathcal{L}^D_{\lambda_1}[\mu] = \beta_1 \int_{\Gamma_1} \lambda_1 \mu + \int_{\Gamma_1} \mu \Psi^1 = 0 \qquad \forall \mu \in \Lambda_1^N,$$
$$\mathcal{L}^D_{\varphi_1}[\rho] = \int_{\Omega_1} \chi_{12}(\varphi^1 - \varphi^2)\rho - \int_{\Omega_1} K \nabla \rho \nabla \Psi^1 = 0 \qquad \forall \rho \in V^1,$$
$$\mathcal{L}^D_{\Psi^1}[\xi] = -\int_{\Omega_1} \nabla \varphi^1 \nabla \xi + \int_{\Gamma_1} \lambda_1 \xi = 0 \qquad \forall \xi \in V^1,$$

and for $i = 2$:

$$\mathcal{L}^D_{\lambda_2}[\mu] = \beta_2 \int_{\Gamma_2} \lambda_2 \mu + \int_{\Gamma_2} \mu \Psi^2 = 0 \qquad \forall \mu \in \Lambda_1^N,$$
$$\mathcal{L}^D_{\varphi_2}[\rho] = -\int_{\Omega_2} \chi_{12}(\varphi^1 - \varphi^2)\rho - \int_{\Omega_2} \nabla \rho \nabla \Psi^2 = 0 \qquad \forall \rho \in V^1,$$
$$\mathcal{L}^D_{\Psi^2}[\xi] = \int_{\Omega_{12}} \nabla \varphi^2 \nabla \xi + \int_{\Gamma_2} \lambda_2 \xi = 0 \qquad \forall \xi \in V^1.$$

We observe how the two approaches have led to the same optimality system at the weak differential level. We remark that the *transposition* approach permits a natural inclusion of the Dirichlet control variable in the weak formulation of the state problem. In order to be able to do this, a double integration by parts of the state equation has been performed and we remark how the boundary term $\int_{\Gamma_i} \lambda_i \rho$, apart from defining the Dirichlet boundary condition on the state problem, defines the expression of the partial derivative of the non penalized part of the cost functional.

We have derived the optimality system for the virtual control Poisson problem in both the cases of Dirichlet and Neumann virtual controls and following a *Lions* and a *Lagrangian* approach. The first led to a strong formulation of the optimality system and the second to a weak formulation of it. The two approaches are complemetar and lead to the same approximation of the coupled virtual control problem.

Minimization in the $H_0^1(\Omega_{12})$ Norm

As said before, in the case of this cost functional we develop the formulation of the optimality system omitting the penalization term (1.24). We first determine the expressions of the partial derivatives:

$$\begin{aligned}
\langle \frac{\partial J^0_{H_0^1}}{\partial \lambda_1}, \mu_1 \rangle &= \lim_{\delta \to 0} \Big(\frac{1}{2} \int_\Omega \chi_{12} (\nabla \varphi^{1,\lambda_1} + \delta \nabla \varphi^{1,\mu_1} - \nabla \varphi^{2,\lambda_2})^2 \; - \frac{1}{2} \int_\Omega \chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})^2 \Big) \\
&= \int_\Omega \chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2}) \nabla \varphi^{1,\mu_1} \\
&= - \int_\Omega \nabla \cdot (\chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})) \varphi^{1,\mu_1} \\
&+ \int_{\partial\Omega} (\chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})) \cdot \mathbf{n}_1 \varphi^{1,\mu_1} \qquad \forall \mu_1 \in \Lambda_1, \\
\langle \frac{\partial J^0_{H_0^1}}{\partial \lambda_2}, \mu_2 \rangle &= \int_\Omega \nabla \cdot (\chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})) \varphi^{2,\mu_2} \\
&- \int_{\partial\Omega} (\chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})) \cdot \mathbf{n}_2 \varphi^{2,\mu_2}, \qquad \forall \mu_2 \in \Lambda_2, \\
\langle \frac{\partial \mathscr{A}_{H_0^1}}{\partial \lambda_1}, \mu_1 \rangle &= - \int_\Omega \nabla \cdot (\chi_{12} (\nabla \varphi^{1,f} - \nabla \varphi^{2,f})) \varphi^{1,\mu_1} \\
&+ \int_{\partial\Omega} (\chi_{12} (\nabla \varphi^{1,f} - \nabla \varphi^{2,f})) \cdot \mathbf{n}_1 \varphi^{1,\mu_1}, \qquad \forall \mu_1 \in \Lambda_1, \\
\langle \frac{\partial \mathscr{A}_{H_0^1}}{\partial \lambda_2}, \mu_2 \rangle &= \int_\Omega \nabla \cdot (\chi_{12} (\nabla \varphi^{1,f} - \nabla \varphi^{2,f})) \varphi^{2,\mu_2} \\
&- \int_{\partial\Omega} (\chi_{12} (\nabla \varphi^{1,f} - \nabla \varphi^{2,f})) \cdot \mathbf{n}_2 \varphi^{2,\mu_2} \qquad \forall \mu_2 \in \Lambda_2.
\end{aligned}$$

We observe that in this case the partial derivatives include a term on the boundary due to to the integration by parts that we had to perform because of the presence of the derivation term in the expression of the cost functional. This term introduces a complication in the formulation of the dual problem, for this reason, we will treat separately the formulation where we seek for the controls in $\Lambda_1^D \times \Lambda_2^D$ and the formulation where we seek for the controls in $\Lambda_1^N \times \Lambda_2^N$.
We remark the presence of a term of the form $\nabla \cdot (\chi_{12}(...))$. The divergence of a discontinuous function cannot be performed, so that one should refer to this term as a formal notation. We have used the integral notation to describe a duality product that has to be interpreted in a *distributional* sense. The well posedness of this expression is guaranteed by the regularity of the functions φ^{i,μ_i}, $i = 1, 2$, which are solutions of a Poisson problem. In fact the integration by parts will not be actually performed, and the dual problem (1.49) in its weak formulation will have as forcing term the partial derivative in its form before the integration.

When dealing with Dirichlet boundary controls we consider the following dual problems

$$\left\{ \begin{array}{rcll}
-\nabla \cdot (K \nabla \Psi^1) & = & -\nabla \cdot (\chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})) & in \quad \Omega_1 \\
K \nabla \Psi^1 \cdot \mathbf{n}_1 & = & (\chi_{12} (\nabla \varphi^{1,\lambda_1} - \nabla \varphi^{2,\lambda_2})) \cdot \mathbf{n}_1 & on \quad \Gamma_1^N \\
\Psi^1 & = & 0 & on \quad \Gamma_1^D \\
\Psi^1 & = & 0 & on \quad \Gamma_1,
\end{array} \right. \tag{1.49}$$

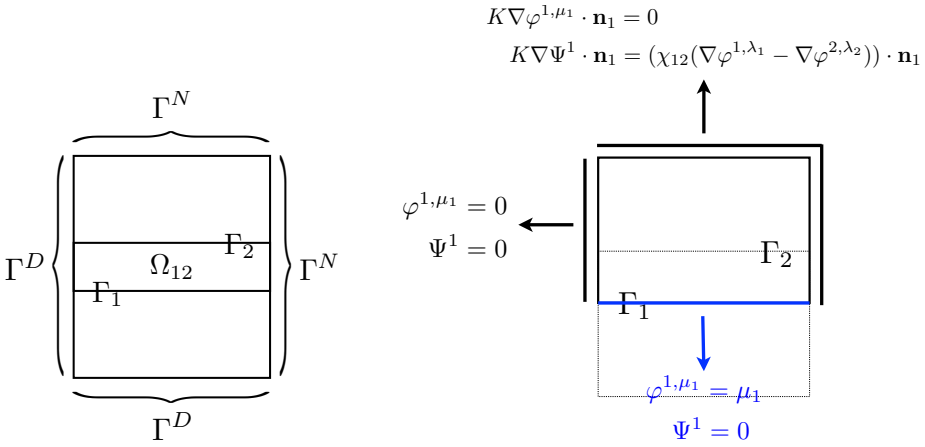


Figure 1.2: *Boundary conditions for the test function φ^{1,μ_1} and the solution of the dual problem Ψ^1.*

$$\begin{cases} -\nabla\cdot(K\nabla\Psi^2) &= \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})) & in \quad \Omega_2 \\ K\nabla\Psi^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_2 & on \quad \Gamma_2^N \\ \Psi^2 &= 0 & on \quad \Gamma_2^D \\ \Psi^2 &= 0 & on \quad \Gamma_2, \end{cases} \tag{1.50}$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^1) &= -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})) & in \quad \Omega_1 \\ K\nabla\tilde{\Psi}^1\cdot\mathbf{n}_1 &= (\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_1 & on \quad \Gamma_1^N \\ \tilde{\Psi}^1 &= 0 & on \quad \Gamma_1^D \\ \tilde{\Psi}^1 &= 0 & on \quad \Gamma_1, \end{cases} \tag{1.51}$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^2) &= \nabla\cdot(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})) & in \quad \Omega_2 \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_2 & on \quad \Gamma_2^N \\ \tilde{\Psi}^2 &= 0 & on \quad \Gamma_2^D \\ \tilde{\Psi}^2 &= 0 & on \quad \Gamma_2. \end{cases} \tag{1.52}$$

We now show how we can rewrite the first partial derivative $\langle\frac{\partial J_{H_0^1}^0}{\partial\lambda_1},\mu_1\rangle$ as a function of the solution of problem (1.49) (refer to Figure 1.2):

$$\begin{aligned} &-\int_\Omega\nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\varphi^{1,\mu_1}+\int_{\partial\Omega}(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1\varphi^{1,\mu_1}= \\ &-\int_{\Omega_1}\nabla\cdot(K\nabla\Psi^1)\varphi^{1,\mu_1}+\int_{\partial\Omega}(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1\varphi^{1,\mu_1}= \\ &\int_{\Omega_1}-\nabla\cdot(K\nabla\varphi^{1,\mu_1})\Psi^1-\int_{\partial\Omega_1}(K\nabla\Psi^1\cdot\mathbf{n}_1)\varphi^{1,\mu_1}+\int_{\partial\Omega_1}(K\nabla\varphi^{1,\mu_1}\cdot\mathbf{n}_1)\Psi^1 \\ &+\int_{\partial\Omega}(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1\varphi^{1,\mu_1}= \\ &-\int_{\Gamma_1}(K\nabla\Psi^1\cdot\mathbf{n}_1)\mu_1+\int_{\Gamma_1}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})\cdot\mathbf{n}_1\mu_1. \end{aligned}$$

We have obtained that

$$\langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle = -\int_{\Gamma_1} (K\nabla\Psi^1 \cdot \mathbf{n}_1)\mu_1 + \int_{\Gamma_1} (\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}) \cdot \mathbf{n}_1\mu_1 \qquad \forall \mu_1 \in \Lambda_1^D, \quad (1.53a)$$

$$\langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle = -\int_{\Gamma_2} (K\nabla\Psi^2 \cdot \mathbf{n}_2)\mu_2 + \int_{\Gamma_2} (\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}) \cdot \mathbf{n}_2\mu_2 \qquad \forall \mu_2 \in \Lambda_2^D, \quad (1.53b)$$

$$\langle \frac{\partial \mathscr{A}_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle = -\int_{\Gamma_1} (K\nabla\tilde{\Psi}^1 \cdot \mathbf{n}_1)\mu_1 + \int_{\Gamma_1} (\nabla\varphi^{1,f} - \nabla\varphi^{2,f}) \cdot \mathbf{n}_1\mu_1 \qquad \forall \mu_1 \in \Lambda_1^D, \quad (1.53c)$$

$$\langle \frac{\partial \mathscr{A}_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle = -\int_{\Gamma_2} (K\nabla\tilde{\Psi}^2 \cdot \mathbf{n}_2)\mu_2 + \int_{\Gamma_2} (\nabla\varphi^{1,f} - \nabla\varphi^{2,f}) \cdot \mathbf{n}_2\mu_2 \qquad \forall \mu_2 \in \Lambda_2^D. \quad (1.53d)$$

When dealing with Neumann boundary controls we consider the following dual problems

$$\begin{cases} -\nabla\cdot(K\nabla\Psi^1) &= -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})) & in \quad \Omega_1 \\ K\nabla\Psi^1\cdot\mathbf{n}_1 &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1 & on \quad \Gamma_1^N \\ \Psi^1 &= 0 & on \quad \Gamma_1^D \\ K\nabla\Psi^1\cdot\mathbf{n}_1 &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1 & on \quad \Gamma_1, \end{cases} \quad (1.54)$$

$$\begin{cases} -\nabla\cdot(K\nabla\Psi^2) &= \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})) & in \quad \Omega_2 \\ K\nabla\Psi^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_2 & on \quad \Gamma_2^N \\ \Psi^2 &= 0 & on \quad \Gamma_2^D \\ K\nabla\Psi^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_2 & on \quad \Gamma_2, \end{cases} \quad (1.55)$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^1) &= -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f})) & in \quad \Omega_1 \\ K\nabla\tilde{\Psi}^1\cdot\mathbf{n}_1 &= (\chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f}))\cdot\mathbf{n}_1 & on \quad \Gamma_1^N \\ \tilde{\Psi}^1 &= 0 & on \quad \Gamma_1^D \\ K\nabla\tilde{\Psi}^1\cdot\mathbf{n}_1 &= (\chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f}))\cdot\mathbf{n}_1 & on \quad \Gamma_1, \end{cases} \quad (1.56)$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^2) &= \nabla\cdot(\chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f})) & in \quad \Omega_2 \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f}))\cdot\mathbf{n}_2 & on \quad \Gamma_2^N \\ \tilde{\Psi}^2 &= 0 & on \quad \Gamma_2^D \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,f} - \nabla\varphi^{2,f}))\cdot\mathbf{n}_2 & on \quad \Gamma_2. \end{cases} \quad (1.57)$$

In an analogous way we show how we can rewrite the first partial derivative $\langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle$ as a function of the solution of problem (1.54):

$$\begin{aligned} &-\int_\Omega \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\varphi^{1,\mu_1} + \int_{\partial\Omega} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1\varphi^{1,\mu_1} = \\ &-\int_{\Omega_1} \nabla\cdot(K\nabla\Psi^1)\varphi^{1,\mu_1} + \int_{\partial\Omega} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1\varphi^{1,\mu_1} = \\ &\quad \int_{\Omega_1} -\nabla\cdot(K\nabla\varphi^{1,\mu_1})\Psi^1 - \int_{\partial\Omega_1} (K\nabla\Psi^1\cdot\mathbf{n}_1)\varphi^{1,\mu_1} + \int_{\partial\Omega_1} (K\nabla\varphi^{1,\mu_1}\cdot\mathbf{n}_1)\Psi^1 \\ &+\int_{\partial\Omega} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1\varphi^{1,\mu_1} = \int_{\Gamma_1} \Psi^1\mu_1. \end{aligned}$$

We have obtained that

$$\langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle = \int_{\Gamma_1} \Psi^1 \mu_1 \qquad \forall \mu_1 \in \Lambda_1^N, \tag{1.58a}$$

$$\langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle = \int_{\Gamma_2} \Psi^2 \mu_2 \qquad \forall \mu_2 \in \Lambda_2^N, \tag{1.58b}$$

$$\langle \frac{\partial \mathscr{A}_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle = \int_{\Gamma_1} \tilde{\Psi}^1 \mu_1 \qquad \forall \mu_1 \in \Lambda_1^N, \tag{1.58c}$$

$$\langle \frac{\partial \mathscr{A}_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle = \int_{\Gamma_1} \tilde{\Psi}^2 \mu_2 \qquad \forall \mu_2 \in \Lambda_2^N. \tag{1.58d}$$

Minimization in the $H^1(\Omega_{12})$ Norm

In this case, because of the linearity of the problem, the results derive from a linear combination of the previous ones. The forcing terms imposed to the dual problems are obtained by linear combination of the ones imposed in the case of the minimization of J_{L^2} and of $J_{H^1_0}$. The partial derivatives are given by the sum of the partial derivatives of the functionals that minimize the L^2 and the H^1_0 norm of the difference between the two solutions on the overlap. In this case we do not need to distinguish between the case of Neumann boundary controls and the case of Dirichlet boundary controls.

$$\langle \frac{\partial J^0_{H^1}}{\partial \lambda_1}, \mu_1 \rangle = \langle \frac{\partial J^0_{L^2}}{\partial \lambda_1}, \mu_1 \rangle + \langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle \qquad \forall \mu_1 \in \Lambda_1,$$

$$\langle \frac{\partial J^0_{H^1}}{\partial \lambda_2}, \mu_2 \rangle = \langle \frac{\partial J^0_{L^2}}{\partial \lambda_2}, \mu_2 \rangle + \langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle \qquad \forall \mu_2 \in \Lambda_2,$$

$$\langle \frac{\partial \mathscr{A}_{H^1}}{\partial \lambda_1}, \mu_1 \rangle = \langle \frac{\partial \mathscr{A}^0_{L^2}}{\partial \lambda_1}, \mu_1 \rangle + \langle \frac{\partial \mathscr{A}^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle \qquad \forall \mu_1 \in \Lambda_1,$$

$$\langle \frac{\partial \mathscr{A}_{H^1}}{\partial \lambda_2}, \mu_2 \rangle = \langle \frac{\partial \mathscr{A}^0_{L^2}}{\partial \lambda_2}, \mu_2 \rangle + \langle \frac{\partial \mathscr{A}^0_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle \qquad \forall \mu_2 \in \Lambda_2.$$

Minimization in an Augmented $H^1_0(\Omega_{12})$ Norm

The partial derivatives of the cost functional are as follows

$$\langle \frac{\partial \widehat{J^0_{H^1_0}}}{\partial \lambda_1}, \mu_1 \rangle = \langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle + \int_{\partial\Omega} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\varphi^{1,\mu_1} \qquad \forall \mu_1 \in \Lambda_1,$$

$$\langle \frac{\partial \widehat{J^0_{H^1_0}}}{\partial \lambda_2}, \mu_2 \rangle = \langle \frac{\partial J^0_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle - \int_{\partial\Omega} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\varphi^{2,\mu_2} \qquad \forall \mu_2 \in \Lambda_2,$$

$$\langle \frac{\partial \widehat{\mathscr{A}_{H^1_0}}}{\partial \lambda_1}, \mu_1 \rangle = \langle \frac{\partial \mathscr{A}^0_{H^1_0}}{\partial \lambda_1}, \mu_1 \rangle + \int_{\partial\Omega} \chi_{12}(\varphi^{1,f} - \varphi^{2,f})\varphi^{1,\mu_1} \qquad \forall \mu_1 \in \Lambda_1,$$

$$\langle \frac{\partial \widehat{\mathscr{A}_{H^1_0}}}{\partial \lambda_2}, \mu_2 \rangle = \langle \frac{\partial \mathscr{A}^0_{H^1_0}}{\partial \lambda_2}, \mu_2 \rangle - \int_{\partial\Omega} \chi_{12}(\varphi^{1,f} - \varphi^{2,f})\varphi^{2,\mu_2} \qquad \forall \mu_2 \in \Lambda_2.$$

In the case of Dirichlet controls we now consider the following dual problems

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\Psi^1) &=& -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})) & in \quad \Omega_1 \\ K\nabla\Psi^1\cdot\mathbf{n}_1 &=& (\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1+\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}) & on \quad \Gamma_1^N \\ \Psi^1 &=& 0 & on \quad \Gamma_1^D \\ \Psi^1 &=& 0 & on \quad \Gamma_1, \end{array}\right. \tag{1.59}$$

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\Psi^2) &=& \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})) & in \quad \Omega_2 \\ K\nabla\Psi^2\cdot\mathbf{n}_2 &=& -(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_2+\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}) & on \quad \Gamma_2^N \\ \Psi^2 &=& 0 & on \quad \Gamma_2^D \\ \Psi^2 &=& 0 & on \quad \Gamma_2, \end{array}\right. \tag{1.60}$$

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\tilde\Psi^1) &=& -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})) & in \quad \Omega_1 \\ K\nabla\tilde\Psi^1\cdot\mathbf{n}_1 &=& (\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_1+\chi_{12}(\varphi^{1,f}-\varphi^{2,f}) & on \quad \Gamma_1^N \\ \tilde\Psi^1 &=& 0 & on \quad \Gamma_1^D \\ \tilde\Psi^1 &=& 0 & on \quad \Gamma_1, \end{array}\right. \tag{1.61}$$

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\tilde\Psi^2) &=& \nabla\cdot(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})) & in \quad \Omega_2 \\ K\nabla\tilde\Psi^2\cdot\mathbf{n}_2 &=& -(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_2+\chi_{12}(\varphi^{1,f}-\varphi^{2,f}) & on \quad \Gamma_2^N \\ \tilde\Psi^2 &=& 0 & on \quad \Gamma_2^D \\ \tilde\Psi^2 &=& 0 & on \quad \Gamma_2. \end{array}\right. \tag{1.62}$$

We obtain that

$$\langle\frac{\partial\widehat{J^0_{H^1_0}}}{\partial\lambda_1},\mu_1\rangle = -\int_{\Gamma_1}(K\nabla\Psi^1\cdot\mathbf{n}_1)\mu_1+\int_{\Gamma_1}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})\cdot\mathbf{n}_1\mu_1 \qquad \forall\mu_1\in\Lambda_1^D, \tag{1.63a}$$

$$\langle\frac{\partial\widehat{J^0_{H^1_0}}}{\partial\lambda_2},\mu_2\rangle = -\int_{\Gamma_2}(K\nabla\Psi^2\cdot\mathbf{n}_2)\mu_2+\int_{\Gamma_2}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})\cdot\mathbf{n}_2\mu_2 \qquad \forall\mu_2\in\Lambda_2^D, \tag{1.63b}$$

$$\langle\frac{\partial\widehat{\mathscr{A}_{H^1_0}}}{\partial\lambda_1},\mu_1\rangle = -\int_{\Gamma_1}(K\nabla\tilde\Psi^1\cdot\mathbf{n}_1)\mu_1+\int_{\Gamma_1}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})\cdot\mathbf{n}_1\mu_1 \qquad \forall\mu_1\in\Lambda_1^D, \tag{1.63c}$$

$$\langle\frac{\partial\widehat{\mathscr{A}_{H^1_0}}}{\partial\lambda_2},\mu_2\rangle = -\int_{\Gamma_2}(K\nabla\tilde\Psi^2\cdot\mathbf{n}_2)\mu_2+\int_{\Gamma_2}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})\cdot\mathbf{n}_2\mu_2 \qquad \forall\mu_2\in\Lambda_2^D. \tag{1.63d}$$

In the case of Neumann controls we solve the following dual problems

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\Psi^1) &=& -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})) & in \quad \Omega_1 \\ K\nabla\Psi^1\cdot\mathbf{n}_1 &=& (\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1+\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}) & on \quad \Gamma_1^N \\ \Psi^1 &=& 0 & on \quad \Gamma_1^D \\ K\nabla\Psi^1\cdot\mathbf{n}_1 &=& (\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_1 & on \quad \Gamma_1, \end{array}\right. \tag{1.64}$$

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\Psi^2) &=& \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})) & in \quad \Omega_2 \\ K\nabla\Psi^2\cdot\mathbf{n}_2 &=& -(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_2+\chi_{12}(\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}) & on \quad \Gamma_2^N \\ \Psi^2 &=& 0 & on \quad \Gamma_2^D \\ K\nabla\Psi^2\cdot\mathbf{n}_2 &=& -(\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}))\cdot\mathbf{n}_2 & on \quad \Gamma_2, \end{array}\right. \tag{1.65}$$

$$\left\{\begin{array}{rcll} -\nabla\cdot(K\nabla\tilde\Psi^1) &=& -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})) & in \quad \Omega_1 \\ K\nabla\tilde\Psi^1\cdot\mathbf{n}_1 &=& (\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_1+\chi_{12}(\varphi^{1,f}-\varphi^{2,f}) & on \quad \Gamma_1^N \\ \tilde\Psi^1 &=& 0 & on \quad \Gamma_1^D \\ K\nabla\tilde\Psi^1\cdot\mathbf{n}_1 &=& (\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_1 & on \quad \Gamma_1, \end{array}\right. \tag{1.66}$$

$$\begin{cases} -\nabla\cdot(K\nabla\tilde{\Psi}^2) &= \nabla\cdot(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f})) & in \quad \Omega_2 \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_2+\chi_{12}(\varphi^{1,f}-\varphi^{2,f}) & on \quad \Gamma_2^N \\ \tilde{\Psi}^2 &= 0 & on \quad \Gamma_2^D \\ K\nabla\tilde{\Psi}^2\cdot\mathbf{n}_2 &= -(\chi_{12}(\nabla\varphi^{1,f}-\nabla\varphi^{2,f}))\cdot\mathbf{n}_2 & on \quad \Gamma_2, \end{cases} \tag{1.67}$$

and we obtain

$$\langle\widehat{\frac{\partial J^0_{H^1_0}}{\partial\lambda_1}},\mu_1\rangle = \int_{\Gamma_1}\Psi^1\mu_1 \qquad \forall\mu_1\in\Lambda_1^N, \tag{1.68a}$$

$$\langle\widehat{\frac{\partial J^0_{H^1_0}}{\partial\lambda_2}},\mu_2\rangle = \int_{\Gamma_2}\Psi^2\mu_2 \qquad \forall\mu_2\in\Lambda_2^N, \tag{1.68b}$$

$$\langle\widehat{\frac{\partial \mathscr{A}_{H^1_0}}{\partial\lambda_1}},\mu_1\rangle = \int_{\Gamma_1}\tilde{\Psi}^1\mu_1 \qquad \forall\mu_1\in\Lambda_1^N, \tag{1.68c}$$

$$\langle\widehat{\frac{\partial \mathscr{A}_{H^1_0}}{\partial\lambda_2}},\mu_2\rangle = \int_{\Gamma_2}\tilde{\Psi}^2\mu_2 \qquad \forall\mu_2\in\Lambda_2^N. \tag{1.68d}$$

1.2.4 A Scalar Elliptic Problem

We have previously observed that, in the non penalized case, the unicity of the solution is not guaranteed if one minimizes the cost functional:

$$J_{H^1_0}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\nabla\varphi^1-\nabla\varphi^2)^2,$$

when pure Neumann boundary conditions are applied on $\partial\Omega_{12}\cap\Gamma$. In fact, in this case, the fact that the gradients of the state solutions are equal on the overlap does not guarantee that the same holds for the non derived solutions. The problem was solved considering the cost functional (1.23), that imposes the equivalence of the solutions of the state problems on a part of the boundary of the overlapping region.

In this section we want to show that if the bilinear forms defined by the state problems are coercive in the $H^1(\Omega_{12})$ norm, the fact that the cost functional $J_{H^1_0(\lambda_1,\lambda_2)}$ is minimized guarantees the unicity of the solution. Consequently the problem associated to the optimality system is well posed in this case. We consider the following problem:
for $i=1,2$, find $\varphi^i\in H^1(\Omega_i)$ so that

$$\begin{cases} \alpha\varphi^i-\nabla\cdot(K\nabla\varphi^i) &= f^i & in \quad \Omega_i \\ K\nabla\varphi^i\cdot\mathbf{n}_i &= \Psi^i_N & on \quad \Gamma_i^N \\ \varphi^i &= \Psi^i_D & on \quad \Gamma_i^D \\ K\nabla\varphi^i\cdot\mathbf{n}_i/\varphi^i &= \lambda_i & on \quad \Gamma_i, \end{cases} \tag{1.69}$$

with $\alpha>0$. The cost functionals considered in this case are the same as those presented in the previous section ((1.20)-(1.23)), so that the form of the partial derivatives is unaltered and so will be the forcing terms and the boundary conditions imposed to the dual problems. As the systems (1.69) are self-adjoint the dual problem will be of the same form of (1.69).

The proofs of the well posedness of the control problem are also equivalent if we consider the following bilinear form and linear functional in the weak formulation of the state problem:

$$a^i(\varphi,\psi)=\int_{\Omega_i}\varphi\psi+\int_{\Omega_i}K\nabla\varphi\nabla\psi\qquad\forall\varphi,\psi\in V^i,$$

$$F^i(\psi)=-\int_{\Omega_i}R^i\psi-\int_{\Omega_i}K\nabla R^i\nabla\psi+\int_{\Omega_i}f^i\psi+\int_{\Gamma_i^N}\Psi_N^i\psi\qquad\forall\psi\in V^i,$$

where the spaces V^i and the liftings R^i are the same that have been defined in section 1.2.2. We can see that in this case the bilinear forms a^i, $i=1,2$ define the H^1 norm $\|\cdot\|_{H^1}=\|\cdot\|_{L^2}+\|\nabla\cdot\|_{L^2}$. Now we want to show that if we consider the problems (1.69) one is able to prove that (1.21) is actually a norm on Ω_{12}. In the case where $\alpha=0$, if $(\partial\Omega_{12}\cap\Gamma^D)\neq\emptyset$ one was not able to prove that

$$J_{H_0^1}(\lambda_1,\lambda_2)=0\quad\Rightarrow\quad\varphi^{1,\lambda_1}=\varphi^{2,\lambda_2}\quad a.e.\quad in\quad\Omega_{12}.$$

Again, the solutions of the problems (1.69) φ^1 and φ^2 can be split in a part depending on the given data of the problem and in a part depending on the sole controls λ^1 and λ^2 as in (1.25), and our analysis is restricted to the case $(\lambda_1,\lambda_2)\in\Lambda^N$, for the same motivations that were previously given. Considering the part of the cost functional that depends on the sole controls $J^0_{H_0^1}$, analogously as before we have that

$J^0_{H_0^1}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2})^2=\frac{1}{2}\|\nabla\varphi^{1,\lambda_1}-\nabla\varphi^{2,\lambda_2}\|^2_{L^2(\Omega_{12})}=0\ \Rightarrow$

$\nabla\varphi^{1,\lambda_1}=\nabla\varphi^{2,\lambda_2}\quad a.e.\quad in\quad\Omega_{12}.$

This result is now sufficient to prove that the solutions that depend on the control variables are almost everywhere equal on the overlapping region Ω_{12}. Defining $w=\varphi^{1,\lambda_1}-\varphi^{2,\lambda_2}$, we have that $\nabla w=0$ *a.e. in* Ω_{12}. We can subtract the first equation in the system (1.69), when $i=2$ to the same equation when $i=1$. We obtain a diffusion-reaction equation for w on the overlap

$$\alpha w-\nabla\cdot(K\nabla w)=0\qquad in\qquad\Omega_{12},\tag{1.70}$$

which reduces to $\alpha w=0$ in Ω_{12} and this implies $\varphi^{1,\lambda_1}=\varphi^{2,\lambda_2}$ almost everywhere in Ω_{12}.

Chapter 2

Discretization and Preconditioning

2.1 The Discretization of the Problem

We now introduce the discretization of the optimization problem. It is obvious that the problem cannot be solved exactly, so we have to look for an approximate solution.
There is one philosophical issue that arises when effecting the discretization of an optimal control problem: the choice between the differentiate-then-discretize and the discretize-then-differentiate approach.

2.1.1 Differentiate-then-Discretize or Discretize-then-Differentiate

In an *differentiate-then-discretize* approach one obtains the adjoint equations and the optimality conditions at the partial differential equations level and then discretizes the result. In a *discretize-then-differentiate* approach one first discretizes the continuous state equations and the cost functional. Then one obtains a discrete adjoint problem and discrete optimality conditions.
Thus we have two paths for arriving at the discrete adjoint and at the discrete partial derivatives of the cost functional. In general the two paths lead to different approximations because the differentiation and the discretization steps do not commute. Evidently for both the differentiate-then-discretize and the discretize-then-differentiate approaches, as the grid size goes to zero, the discretized gradients of the cost functional all converge to the same thing (if the solutions are smooth enough). However for finite values of the grid size (that are used in practice) there can be a substantial difference.

One advantage of the discretize-then-differentiate approach is that one obtains the exact gradient of the discretized functional. This is not the case when one differentiates and then discretizes because the approximate gradient obtained in this case is neither the gradient of the continuous cost functional nor the gradient of the discretized cost functional and this can lead to inconsistencies when solving the problem. Moreover in the discretize-then-differentiate approach an automatic

differentiation software can be used to compute the derivatives of the cost functional. This simplifies the implementation of the code but the drawback is that such software usually requires more storage and CPU time with respect to a handwritten code.
In general the choice between the two strategies has been, and remains, more a matter of taste, then a result made on the basis of clear advantages possessed by one approach over the other. Both strategies have been used with great success. A more detailed analysis of the two strategies can be found in [13].

Our choice was to implement a differentiate-then-discretize approach because our analysis is oriented towards a theoretical approach. We are interested in deriving the differential forms of the adjoints and of the derivatives of the cost functional in order to be able to point out the differences between the various choices for the cost functional.

2.1.2 The Finite Elements Approximation of the Problem

We consider a regular triangulation $\mathcal{T}_h$ of the domains $\bar{\Omega}_1$ and $\bar{\Omega}_2$, depending on a positive parameter $h > 0$, made up of non overlapping triangles K. We set $h_K = diam(K), \forall K \in \mathcal{T}_h$ where $diam(K) = max_{x,y\in K}|x-y|$ is the *diameter* of the element K, we define $h = max_{K\in\mathcal{T}_h} h_K$. Moreover we assume the grid to be *regular* and *quasi uniform* (see [14]).
We assume that the triangulations $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ induced on the subdomains Ω_1 and Ω_2 are compatible on the overlapping region Ω_{12}, that is they share the same triangles therein.

We denote by $\mathbb{P}_r$ the space of polynomials of global degree less or equal to r, for $r = 1, 2, ...$:

$$\mathbb{P}_r = \Big\{ p(x_1, x_2) = \sum_{i,j\geq 0, i+j\leq r} a_{ij} x_1^i x_2^j \qquad \text{with} \qquad a_{ij} \in \mathbb{R} \Big\}.$$

We introduce on each domain Ω_1 and Ω_2 the spaces of finite elements:

$$X_h^{r,i} = \big\{ v_h \in C^0(\Omega_i) : v_h|_K \in \mathbb{P}_r \in \mathcal{T}_h^i \big\} \qquad r = 1, 2, ... \qquad i = 1, 2.$$

Moreover we define

$$\overset{\circ}{X}{}_h^{r,i} = \big\{ v_h \in X_h^{r,i} : v_h|_{\Gamma_i^D} = 0 \big\}.$$

The spaces $X_h^{r,i}$ and $\overset{\circ}{X}{}_h^{r,i}$ are a suitable approximation of $H^1(\Omega_i)$ and $H^1_{\Gamma_i^D}(\Omega_i)$, $i = 1, 2$.
We set $V_h^i = \overset{\circ}{X}{}_h^{r,i}$.
In a generic domain Ω, each function $v_h \in V_h$ is characterized univocally by the values it takes in the nodes N_j, ($j = 1, ...N_h$, where N_h is the total number of nodes of the grid $\mathcal{T}_h$, excluding the boundary nodes where $v_h = 0$). A basis on the space V_h can be the set of characteristic Lagrangian functions $\psi_j \in V_h$, $j = 1, ..., N_h$ such that

$$\psi_k(N_j) = \delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k, \end{cases} \qquad j, k = 1, ..., N_h.$$

A general function $v_h \in V_h$ can be expanded as a linear combination of the basis functions of V_h in the following way:

$$v_h(x) = \sum_{k=1}^{N_h} v_k \psi_k(x) \qquad \forall x \in \Omega, \quad v_k = v_h(N_k).$$

The Discretization of the Primal Problems

We want to introduce the Galerkin formulation of the problems that depend on the data (1.27). We first have to explain how we treat the Dirichlet boundary data in the discretized problem. The functions Ψ_D^1 and Ψ_D^2 are assigned on the Dirichlet boundary of Ω_1 and Ω_2, respectively. For $i = 1, 2$, we denote by N_h^i the internal nodes of the grid $\mathcal{T}_h^i$ and by $N_h^{t,i}$ the total number of nodes, thus including the boundary nodes that we suppose to be numbered last. $\{N_j^{b,i}, j = N_h^i + 1, ...N_h^{t,i}\}$ is the set of boundary nodes.
The extension of the Dirichlet boundary data is constructed as follows:

$$R_h^i \in X_h^{r,i} : \; R_h^i(x) = \sum_{k=N_h^i+1}^{N_h^{t,i}} \Psi_D^i(N_k)\psi_k(x) \qquad \forall x \in \Omega_i, \qquad i = 1, 2.$$

We are now ready to write the finite elements formulation of problems (1.27): for $i = 1, 2$, find $\overset{\circ}{\varphi}_h^{\,if} \in V_h^i$ so that

$$\int_{\Omega_i} \nabla \overset{\circ}{\varphi}_h^{\,if} \nabla \psi_h = \int_{\Omega_i} f^i \psi_h + \int_{\partial\Omega_i} \Psi_N^i \psi_h - \int_{\partial\Omega_i} \nabla R_h^i \nabla \psi_h \qquad \forall \psi_h \in V_h^i.$$

The approximate solution will then be provided by $\varphi_h^{\,if} = \overset{\circ}{\varphi}_h^{\,if} + R_h$, $i = 1, 2$.
Expressing the discrete solution $\overset{\circ}{\varphi}_h^{\,if}$, $i = 1, 2$, in terms of the basis $\{\psi_j\}$, in the following way

$$\overset{\circ}{\varphi}_h^{\,if} = \sum_{k=1}^{N_h^i} \overset{\circ}{\varphi}_k^{\,if} \psi_k(x) \qquad \text{with} \qquad \overset{\circ}{\varphi}_k^{\,if} = \overset{\circ}{\varphi}_h^{\,if}(N_k) \qquad k = 1, ..., N_h^i, \quad i = 1, 2,$$

we obtain, for $j = 1, ..., N_h$, the following linear system of N_h^i equations in the N_h^i unknowns $\overset{\circ}{\varphi}_k^{\,if}$:

$$\sum_{k=1}^{N_h^i} \overset{\circ}{\varphi}_k^{\,if} \int_{\Omega_i} \nabla\psi_k \nabla\psi_j = \sum_{k=1}^{N_h^i} f_k^i \int_{\Omega_i} \psi_k\psi_j + \sum_{k=1}^{N_h^i} \Psi_{N,k}^i \int_{\partial\Omega_i} \hat{\psi}_k\hat{\psi}_j - \sum_{k=N_h+1}^{N_h^{t,i}} \Psi_{D,k}^i \int_{\Omega_i} \nabla\hat{\psi}_k \nabla\psi_j,$$

where we defined as $\{\hat{\psi}_j\}$ the analogous basis functions on the boundary.

We introduce the following matrices and vectors: for $i = 1, 2$

$$C_i = [c^i_{jk}] \in \mathbb{R}^{N^i_h \times N^i_h} \qquad \text{with} \quad c^i_{jk} = \int_{\Omega_i} \nabla\psi_k \nabla\psi_j$$

$$\varphi^f_i = [\mathring{\varphi}^{if}_j] \in \mathbb{R}^{N^i_h \times 1} \qquad \text{with} \quad \mathring{\varphi}^{if}_j = \mathring{\varphi}^{if}(N_j)$$

$$M_i = [m^i_{jk}] \in \mathbb{R}^{N^i_h \times N^i_h} \qquad \text{with} \quad m^i_{jk} = \int_{\Omega_i} \psi_k \psi_j$$

$$f_i = [f^i_j] \in \mathbb{R}^{N^i_h \times 1} \qquad \text{with} \quad f^i_j = f^i(N_j)$$

$$N_i = [n^i_{jk}] \in \mathbb{R}^{N^i_h \times N^{N,i}_h} \qquad \text{with} \quad n^i_{jk} = \int_{\Gamma^N_i} \hat{\psi}_k \psi_j$$

$$\Psi_{N,i} = [\Psi^i_{N,j}] \in \mathbb{R}^{N^{N,i}_h \times 1} \qquad \text{with} \quad \Psi^i_{N,j} = \Psi^i_N(N_j)$$

$$D_i = [d^i_{jk}] \in \mathbb{R}^{N^i_h \times N^{b,i}_h} \qquad \text{with} \quad d^i_{jk} = \int_{\Omega_i} \nabla\hat{\psi}_{k+N^i_h} \nabla\psi_j$$

$$\Psi_{D,i} = [\Psi^i_{D,j}] \in \mathbb{R}^{N^{b,i}_h \times 1} \qquad \text{with} \quad \Psi^i_{D,j} = \Psi^i_D(N_j)$$

where, for $i = 1, 2$, we defined as $N^{N,i}_h$ the total number of nodes on Γ^N_i.
The discretization of problems (1.27) can be written as the following linear problem: for $i = 1, 2$, find φ^f_i so that

$$C_i \varphi^f_i = M_i f_i + N_i \Psi_{N,i} - D_i \Psi_{D,i}. \tag{2.1}$$

We remark that the discretization of the problem depending on the data is independent of the choice of the space for the virtual controls (Dirichlet or Neumann controls), since the boundary data imposed on Γ_1 and Γ_2 are homogeneous in both cases.

We proceed in an analogous way for the discretization of problems (1.26) depending on the sole controls (λ_1, λ_2). We define

$$\lambda_i = [\lambda_{i,j}] \in \mathbb{R}^{N^{\Gamma_i}_h \times 1} \qquad \text{with} \quad \lambda_{i,j} = \lambda_i(N_j),$$

and

$$\varphi^\lambda_i = [\mathring{\varphi}^{i,\lambda_i}_j] \in \mathbb{R}^{N^i_h \times 1} \qquad \text{with} \quad \mathring{\varphi}^{i,\lambda_i}_j = \mathring{\varphi}^{i,\lambda_i}(N_j),$$

where $N^{\Gamma_i}_h$, $(i = 1, 2)$ is the number of nodes on Γ_1 and Γ_2.
If the controls correspond to a Neumann boundary condition, for each interface, we introduce the matrix

$$N_{\Gamma_i} = [n^{\Gamma_i}_{jk}] \in \mathbb{R}^{N^i_h \times N^{\Gamma_i}_h} \qquad \text{with} \quad n^{\Gamma_i}_{jk} = \int_{\Gamma_i} \hat{\psi}_k \psi_j,$$

so that the discretization of the problems reads as: for $i = 1, 2$, find φ^λ_i so that

$$C_i \varphi^\lambda_i = N_{\Gamma_i} \lambda_i, \tag{2.2}$$

whereas if the controls correspond to a Dirichlet boundary condition we need to introduce

$$D_{\Gamma_i} = [d^{\Gamma_i}_{jk}] \in \mathbb{R}^{N^i_h \times N^{\Gamma_i}_h} \qquad \text{with} \quad d^{\Gamma_i}_{jk} = \int_{\Omega_i} \nabla\hat{\psi}_k \nabla\psi_j,$$

so that the discretization of the problems reads as: for $i = 1, 2$, find φ^λ_i so that

$$C_i \varphi^\lambda_i = -D_{\Gamma_i} \lambda_i. \tag{2.3}$$

We remark that in the case of Neumann boundary controls, the nodes where the controls are computed are part of the unknowns of the state problem. In the case of Dirichlet boundary controls, the nodes on the control interfaces are not part of the internal nodes of the primal problem, and an extension of the controls has to be computed, through the matrices D_{Γ_i}, $i = 1, 2$.

The Discretization of the Dual Problems

The discretization of the dual problems is formulated in an analogous way.
We observe that all the dual problems of the optimality systems presented in the previous section are of the following form: for $i = 1, 2$, find $\Psi^i \in H^1_{\Gamma_i^D}(\Omega_i)$ so that:

$$\begin{cases} -\nabla \cdot (K\nabla\Psi^i) &= f^i & in \quad \Omega_i \\ K\nabla\Psi^i \cdot n_i &= g_{\Gamma_i^N} & on \quad \Gamma_i^N \\ \Psi^i &= 0 & on \quad \Gamma_i^D, \end{cases} \tag{2.4}$$

with the following boundary condition on the control interface:

$$\Psi^i = 0 \qquad on \quad \Gamma_i,$$

if we are considering Dirichlet boundary controls, or with

$$K\nabla\Psi_i \cdot n_i = g_{\Gamma_i} \qquad on \quad \Gamma_i,$$

if considering Neumann boundary controls.
We denoted the forcing term through a generic function f^i which varies according to which cost functional we are minimizing. The same is valid for the boundary data $g_{\Gamma_i^N}$ and g_{Γ_i}.

For $i = 1, 2$, the weak formulation of problems (2.4) reads as: find $\Psi^i \in H^1_{\Gamma_i^D}(\Omega_i)$ such that

$$\int_{\Omega_i} \nabla\Psi^i \nabla\psi = \int_{\Omega_i} f^i\psi + \int_{\Gamma_i^N} g_{\Gamma_i^N}\psi \qquad \forall\psi \in H^1_{\Gamma_i^D}(\Omega_i)$$

if $(\lambda_1, \lambda_2) \in \Lambda_i^D$ and as: find $\Psi^i \in H^1_{\Gamma_i^D}(\Omega_i)$ such that

$$\int_{\Omega_i} \nabla\Psi^i \nabla\psi = \int_{\Omega_i} f^i\psi + \int_{\Gamma_i^N} g_{\Gamma_i^N}\psi + \int_{\Gamma_i} g_{\Gamma_i}\psi \qquad \forall\psi \in H^1_{\Gamma_i^D}(\Omega_i)$$

if $(\lambda_1, \lambda_2) \in \Lambda_i^N$. Since the Dirichlet data of the problem is always homogeneous there is no need to split the solution as the sum of two terms as in (1.29).

We can define the following linear functionals on $H^1_{\Gamma_i^D}(\Omega_i)$: for $i = 1, 2$,

$$F_i^D(\psi) = \int_{\Omega_i} f^i\psi + \int_{\Gamma_i^N} g_{\Gamma_i^N}\psi \qquad \forall\psi \in H^1_{\Gamma_i^D}(\Omega_i),$$

$$F_i^N(\psi) = \int_{\Omega_i} f^i\psi + \int_{\Gamma_i^N} g_{\Gamma_i^N}\psi + \int_{\Gamma_i} g_{\Gamma_i}\psi \qquad \forall\psi \in H^1_{\Gamma_i^D}(\Omega_i),$$

corresponding, respectively, to Dirichlet and Neumann controls.

The finite elements approximation of the generic dual problem reads as follows: for $i = 1, 2$, find $\Psi_h^i \in V_h^i$ such that

$$\int_{\Omega_i} \nabla \Psi_h^i \nabla \psi_h = F_i(\psi_h) \qquad \forall \psi_h \in V_h^i, \tag{2.5}$$

where F_i is either equal to F_i^D or to F_i^N according to the choice of the space in which we seek the virtual controls.

As we did when discretizing the primal problem, for $i = 1, 2$ we can reformulate the solution ψ^i as decomposed on the Lagrangian basis $\{\Psi_j\}$, $j = 1, ..., N_h$ in the following way

$$\Psi_h^i = \sum_{k=1}^{N_h^i} \Psi_k^i \psi_k(x) \qquad \text{with} \qquad \Psi_k^i = \Psi_h^i(N_k^i) \qquad k = 1, ..., N_h,$$

and substituting these expressions in (2.5) we obtain the following linear systems:

$$C_i \Psi_i = f_i, \tag{2.6}$$

where

$$C_i = [c_{jk}^i] \in \mathbb{R}^{N_h^i \times N_h^i} \qquad \text{with} \quad c_{jk}^i = \int_{\Omega_i} \nabla \psi_k \nabla \psi_j$$
$$\Psi_i = [\Psi_j^i] \in \mathbb{R}^{N_h^i \times 1} \qquad \text{with} \quad \Psi_j^i = \Psi(N_j^i)$$

and the vectors $f_i \in \mathbb{R}^{N_h^i \times 1}$ assume a form that we will specify according to the different cost functionals.

We now want to specify the form assumed by the right hand side and the Neumann boundary data of the different dual problems according to which cost functional is being minimized. We will write the explicit forms of the linear functionals $F^D(\psi)$ and $F^N(\psi)$ and introduce the matrices and vectors needed for their finite element approximations. As usual we will proceed following the order of the four different cost functionals considered in the previous section. For the sake of simplicity we consider only the duals of the state problems depending on the sole controls (1.26). As a further simplification we focus our analysis on the problems corresponding to domain Ω_1. It is clear, from section 1.2.3, that the formulations corresponding to the problems depending on the data will be analogous and that the formulations corresponding to the second domain Ω_2 differ only in sign.

Before proceeding in this analysis we introduce the matrices that we will use in the following part of this section. We observe that all the data that appear on the right hand side of the dual problems are multiplied by the characteristic function of the overlapping domain: χ_{12}. For this reason it is clear that we need to identify the nodes of the triangulation corresponding to the overlap Ω_{12}. We denote this set of nodes as $\{N_j^{12}, j = 1...N_h^{12}\}$, where we indicated as N_h^{12} the total number of nodes

on the overlap. We need to consider the following matrices: for $i = 1, 2$,

$$C_i^{12} = [c_{jk}^{12,i}] \in \mathbb{R}^{N_h^i \times N_h^i} \quad \text{with} \quad c_{jk}^{12,i} = \int_{\Omega_i} \chi_{12}(\nabla\psi_k \nabla\psi_j)$$

$$M_i^{12} = [m_{jk}^{12,i}] \in \mathbb{R}^{N_h^i \times N_h^i} \quad \text{with} \quad m_{jk}^{12,i} = \int_{\Omega_i} \chi_{12}(\psi_k \psi_j)$$

$$N_i^{12} = [n_{jk}^{12,i}] \in \mathbb{R}^{N_h^i \times N_h^{N12,i}} \quad \text{with} \quad n_{jk}^{12,i} = \int_{\Gamma_i^N \cap \partial\Omega_i} \chi_{12}(\psi_k \hat{\psi}_j),$$

where $N_h^{N12,i}$ indicates the number of nodes on the boundary $(\partial\Omega_i \cap \partial\Omega_{12}) \cap \Gamma_i^N$. Since the basis functions that appear in the previous integrals are multiplied by the characteristic function of the overlap, the matrices C_i^{12} and M_i^{12} will have non null entries only in correspondence of the rows and columns associated to the vector of nodes N^{12}. The same holds for matrix N_i^{12} whose non null entries will be in correspondence of the nodes belonging simultaneously to the Neumann boundary of the domain taken into consideration and to the boundary of the overlap.

Minimization in the $L^2(\Omega_{12})$ norm In this case the data of the dual problem (1.41) assume the following form:

$$\begin{aligned} f &= \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}), \\ g_{\Gamma^N} &= 0, \\ g_{\Gamma} &= 0, \end{aligned}$$

So that

$$F^D(\psi) = F^N(\psi) = \int_{\Omega_1} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1).$$

It is straightforward to deduce that the algebraic formulation of this term, which goes on the right hand side of the linear system (2.6) is

$$f_1 = M_1^{12}\varphi^{1,\lambda_1} - M_1^{12}\varphi^{2,\lambda_2}.$$

Minimization in the $H_0^1(\Omega_{12})$ norm Referring to problem (1.49) if we consider Dirichlet boundary controls and to problem (1.54) if we consider Neumann boundary controls, we observe that in this case the data imposed on the dual problem are:

$$\begin{aligned} f &= -\nabla \cdot (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})), \\ g_{\Gamma^N} &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})) \cdot n_1, \\ g_{\Gamma} &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})) \cdot n_1. \end{aligned}$$

Consequently the linear functionals appearing in the weak formulation of the dual problem assume the following form:

$$\begin{aligned} F^D(\psi) = &- \int_{\Omega_1} \nabla \cdot (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\psi \\ &+ \int_{\Gamma_1^N} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})) \cdot n_1\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1), \end{aligned}$$

$$F^N(\psi) = -\int_{\Omega_1} \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\psi + \int_{\Gamma_1^N U \Gamma_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1).$$

By integration by parts we obtain that

$$\begin{aligned} F^D(\psi) &= \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi - \int_{\Gamma_1^N} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \\ &\quad + \int_{\Gamma_1^N} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \\ &= \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1), \end{aligned}$$

for what concerns the problem deriving from the use of Dirichlet boundary controls, and analogously we obtain that

$$\begin{aligned} F^N(\psi) &= \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi - \int_{\Gamma_1^N U \Gamma_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \\ &\quad + \int_{\Gamma_1^N U \Gamma_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \\ &= \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi = F^D(\psi) \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1), \end{aligned}$$

for what concerns the problem deriving from the use of Neumann boundary controls.
We see how the contribution deriving from the boundary terms is canceled when considering the weak formulation of the problem. We can write the algebraic formulation of this term using matrix C_{12}:

$$f_1 = C_1^{12}\varphi^{1,\lambda_1} - C_1^{12}\varphi^{2,\lambda_2}.$$

Minimization in the $H^1(\Omega_{12})$ norm Now the data of the dual problem is as follows:

$$\begin{aligned} f &= \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}) - \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})), \\ g_{\Gamma^N} &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1, \\ g_\Gamma &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1, \end{aligned}$$

By linear combination of the previous results we obtain:

$$F^D(\psi) = \int_{\Omega_1} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi - \int_{\Omega_1} \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\psi + \int_{\Gamma_1^N} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1).$$

$$F^N(\psi) = \int_{\Omega_1} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi - \int_{\Omega_1} \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\psi + \int_{\Gamma_1^N U \Gamma_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1).$$

Again, by integration by parts we obtain:

$$F^D(\psi) = \int_{\Omega_1} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi - \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1).$$

$$F^N(\psi) = \int_{\Omega_1} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi - \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi = F^D(\psi) \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1),$$

and this term is discretized as

$$f_1 = M_1^{12}\varphi^{1,\lambda_1} - M_1^{12}\varphi^{2,\lambda_2} + C_1^{12}\varphi^{1,\lambda_1} - C_1^{12}\varphi^{2,\lambda_2}.$$

Minimization in an augmented $H^1_0(\Omega_{12})$ norm We now refer to problems (1.59) and (1.64). This is the data assigned to dual problems:

$$\begin{aligned} f &= -\nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2})), \\ g_{\Gamma^N} &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1 + \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}), \\ g_{\Gamma} &= (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1, \end{aligned}$$

so that we obtain the following linear functionals

$$\begin{aligned} F^D(\psi) = &- \int_{\Omega_1} \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\psi \\ &+ \int_{\Gamma_1^N} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \\ &+ \int_{\Gamma_1^N} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1), \end{aligned}$$

$$\begin{aligned} F^N(\psi) = &- \int_{\Omega_1} \nabla\cdot(\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\psi \\ &+ \int_{\Gamma_1^N \cup \Gamma_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\cdot n_1\psi \\ &+ \int_{\Gamma_1^N} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1). \end{aligned}$$

By integration by parts

$$\begin{aligned} F^D(\psi) = &- \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi \\ &+ \int_{\Gamma_1^N} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1), \end{aligned}$$

$$\begin{aligned} F^N(\psi) = &- \int_{\Omega_1} (\chi_{12}(\nabla\varphi^{1,\lambda_1} - \nabla\varphi^{2,\lambda_2}))\nabla\psi \\ &+ \int_{\Gamma_1^N} \chi_{12}(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2})\psi = F^D(\psi) \qquad \forall\psi \in H^1_{\Gamma_D}(\Omega_1). \end{aligned}$$

Now we still have a contribution from the boundary on the right hand side of the weak formulation of the problem. The algebraic formulation of the right hand side is now:

$$f_1 = C_1^{12}\varphi^{1,\lambda_1} - C_1^{12}\varphi^{2,\lambda_2} + N_1^{12}\varphi^{1,\lambda_1}|_{\Gamma_1^N\cap\partial\Omega_{12}} - N_1^{12}\varphi^{2,\lambda_2}|_{\Gamma_1^N\cap\partial\Omega_{12}}.$$

The Discretization of the Optimality Conditions

Once again we will follow the order of the previously introduced cost functionals. In order to discretize the partial derivatives of the objective functional we do not need to introduce any new matrix. In order to simplify the notation we refer again to domain Ω_1 and to the quadratic part of the cost functional, that depends only on the virtual controls.

Minimization in the $L^2(\Omega_{12})$ norm In the case of Dirichlet controls the partial derivatives assume the following algebraic form:

$$-D_{\Gamma_1}^T \Psi_1,$$

while in the case of Neumann boundary controls we have that

$$N_{\Gamma_1}^T \Psi_1.$$

Minimization in the $H_0^1(\Omega_{12})$, $H^1(\Omega_{12})$ and $H_0^1(\Omega_{12})$ norm In this case we obtain, for Dirichlet and Neumann controls respectively, the following expressions

$$-D_{\Gamma_1}^T \Psi_1 + D_{\Gamma_1}^T (\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}),$$
$$N_{\Gamma_1}^T \Psi_1.$$

When adding a penalization term to the cost functional (see (1.24)), a term is added to the optimality conditions. To introduce the algebraic discretization of this term we need to define , for $i = 1, 2$, the following interface mass matrices

$$M_{\Gamma_i} = [m_{jk}^{\Gamma_i}] \in \mathbb{R}^{N_h^{\Gamma_i} \times N_h^{\Gamma_i}} \qquad \text{with} \quad m_{jk}^{\Gamma_i} = \int_{\Gamma_i} (\hat{\psi}_k \hat{\psi}_j).$$

The penalization term is added to the optimality condition related to the dual problem that depends only on the virtual controls, and not on the given data of the problem. This term has, for $i = 1, 2$, the form $\beta M_{\Gamma_1} \lambda_i$.

In the following pages we report, in Table 2.1 and Table 2.2, the algebraic formulations of the optimality systems that we introduced. The first table refers to Dirichlet virtual controls and the second table to Neumann virtual controls. In both tables, we vary the choice of the cost functional that is being minimized. In order to give an exhaustive description we consider, for each cost functional, the penalized case. The non penalized case can be easily obtained setting $\beta_1 = \beta_2 = 0$.

The Discretization of the Scalar Elliptic Problem

When solving the state problems of the form (1.69) one should consider the discretization of the mass term $\alpha\varphi^i$, for $i = 1, 2$. The formulations of the primal problems and of the adjoints will be modified by taking this term into account. The optimality conditions are unaltered. The discretization of state

problems depending on the data (2.1), and of the state problems depending on the sole controls, for both Neumann (2.2) and Dirichlet (2.3) boundary controls, read as follows: for $i = 1, 2$

$$\begin{aligned} M_i\varphi_i^f + C_i\varphi_i^f &= M_i f_i + N_i\Psi_{N,i} - \tilde{D}_i\Psi_{D,i}, \\ M_i\varphi_i^\lambda + C_i\varphi_i^\lambda &= N_{\Gamma_i}\lambda_i, \\ M_i\varphi_i^\lambda + C_i\varphi_i^\lambda &= -\tilde{D}_{\Gamma_i}\lambda_i, \end{aligned}$$

where M_i is the mass matrix assembled on Ω_i, for $i = 1, 2$, while the matrices responsible of the lifting of the Dirichlet boundary conditions $\tilde{D}_i$ and $\tilde{D}_{\Gamma_i}$ are now defined as

$$\begin{aligned} \tilde{D}_i = [d_{jk}^i] \in \mathbb{R}^{N_h^i \times N_h^{b,i}} \qquad & \text{with} \quad d_{jk}^i = \int_{\Omega_i} \hat{\psi}_{k+N_h^i}\psi_j + \int_{\Omega_i} \nabla\hat{\psi}_{k+N_h^i}\nabla\psi_j \\ \tilde{D}_{\Gamma_i} = [d_{jk}^{\Gamma_i}] \in \mathbb{R}^{N_h^i \times N_h^{\Gamma_i}} \qquad & \text{with} \quad d_{jk}^{\Gamma_i} = \int_{\Omega_i} \hat{\psi}_k\psi_j + \int_{\Omega_i} \nabla\hat{\psi}_k\nabla\psi_j, \end{aligned}$$

where for all the notations one should refer to the section on the discretization of the problem in the case of the Poisson equation. For what concerns the discretization of the dual problems, the generic form of the linear system associated to the adjoint (2.6) now reads as

$$M_i\Psi_i + C_i\Psi_i = f_i,$$

2.2 Solution Algorithms

We have derived for each model a system of partial differential equations whose solution leads to the optimal controls (λ_1, λ_2), the optimal states (φ_1, φ_2) and the optimal adjoint states (Ψ_1, Ψ_2). One cannot solve the state systems independently since the boundary conditions (λ_1, λ_2) are not known. Similarly, one cannot solve the adjoint systems independently because the states (φ_1, φ_2) that appear in the forcing terms are not known and one cannot solve the optimality conditions because the adjoint variables (Ψ_1, Ψ_2) are not known. Otherwise said, the optimality system is fully coupled.

In what follows we will describe three different approaches for the solution of the discretized optimality systems presented in Table 2.1 and in Table 2.2. All the three approaches have been implemented in our code, and all converge to the same finite elements solution. Differences, advantages and disadvantages of one approach over the other will be pointed out when describing the results obtained with the numerical simulations.

2.2.1 Solving the Extremality Equation: Iterative Algorithm

We iterate between the equations in the optimality system in order to find the couple of controls (λ_1, λ_2) that minimize the cost functional. We aim to solve $\nabla J = \nabla J^0 + \nabla\mathscr{A} = 0$, where

$$\begin{aligned} \langle\nabla J^0, (\mu_1, \mu_2)\rangle &= (\langle\frac{\partial J^0}{\partial\lambda_1}, \mu_1\rangle, \langle\frac{\partial J^0}{\partial\lambda_2}, \mu_2\rangle), \\ \langle\nabla\mathscr{A}, (\mu_1, \mu_2)\rangle &= (\langle\frac{\partial\mathscr{A}}{\partial\lambda_1}, \mu_1\rangle, \langle\frac{\partial\mathscr{A}}{\partial\lambda_2}, \mu_2\rangle). \end{aligned}$$

STATE	J	DUAL	PARTIAL DERIVATIVE
$C_1\varphi_1^\lambda = -D_{\Gamma_1}\lambda_1$	J_{L^2}	$C_1\Psi_1 = M_1^{12}\varphi_1^{\lambda_1} - M_1^{12}\varphi_2^{\lambda_2}$	$-D_{\Gamma_1}^T\Psi_1 + \beta_1 M_{\Gamma_1}\lambda_1$
	$J_{H_0^1}$	$C_1\Psi_1 = C_1^{12}\varphi^{1,\lambda_1} - C_1^{12}\varphi^{2,\lambda_2}$	$-D_{\Gamma_1}^T\Psi_1 + D_{\Gamma_1}^T(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}) + \beta_1 M_{\Gamma_1}\lambda_1$
	J_{H^1}	$C_1\Psi_1 = M_1^{12}\varphi_1^{\lambda_1} - M_1^{12}\varphi_2^{\lambda_2} + C_{12}\varphi^{1,\lambda_1} - C_{12}\varphi^{2,\lambda_2}$	$-D_{\Gamma_1}^T\Psi_1 + D_{\Gamma_1}^T(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}) + \beta_1 M_{\Gamma_1}\lambda_1$
	$\widehat{J_{H_0^1}}$	$C_1\Psi_1 = C_1^{12}\varphi_1^{\lambda_1} - C_1^{12}\varphi_2^{\lambda_2} + N_1^{12}\varphi^{1,\lambda_1}\vert_{\Gamma_1^N\cap\partial\Omega_{12}} - N_1^{12}\varphi^{2,\lambda_2}\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-D_{\Gamma_1}^T\Psi_1 + D_{\Gamma_1}^T(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}) + \beta_1 M_{\Gamma_1}\lambda_1$
$C_2\varphi_2^\lambda = -D_{\Gamma_2}\lambda_2$	J_{L^2}	$C_2\Psi_2 = -M_2^{12}\varphi_1^{\lambda_1} + M_2^{12}\varphi_2^{\lambda_2}$	$-D_{\Gamma_2}^T\Psi_2 + \beta_2 M_{\Gamma_2}\lambda_2$
	$J_{H_0^1}$	$C_2\Psi_2 = -C_2^{12}\varphi^{1,\lambda_1} - C_2^{12}\varphi^{2,\lambda_2}$	$-D_{\Gamma_2}^T\Psi_2 - D_{\Gamma_2}^T(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}) + \beta_2 M_{\Gamma_2}\lambda_2$
	J_{H^1}	$C_2\Psi_2 = -M_2^{12}\varphi_1^{\lambda_1} + M_2^{12}\varphi_2^{\lambda_2} + -C_2^{12}\varphi^{1,\lambda_1} + C_2^{12}\varphi^{2,\lambda_2}$	$-D_{\Gamma_2}^T\Psi_2 - D_{\Gamma_2}^T(\varphi^{1,\lambda_1} - \varphi^{2,\lambda_2}) + \beta_2 M_{\Gamma_2}\lambda_2$
	$\widehat{J_{H_0^1}}$	$C_2\Psi_2 = -C_2^{12}\varphi_1^{\lambda_1} + C_2^{12}\varphi_2^{\lambda_2} - N_2^{12}\varphi^{1,\lambda_1}\vert_{\Gamma_1^N\cap\partial\Omega_{12}} + N_2^{12}\varphi^{2,\lambda_2}\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-D_{\Gamma_2}^T\Psi_2 - D_{\Gamma_2}^T(\varphi^{2,\lambda_2} - \varphi^{2,\lambda_2}) + \beta_2 M_{\Gamma_2}\lambda_2$
$C_1\varphi_1^f = M_1 f_1 + N_1\Psi_{N,1} - D_1\Psi_{D,1}$	J_{L^2}	$C_1\tilde{\Psi}_1 = M_1^{12}\varphi_1^f - M_1^{12}\varphi_2^f$	$-D_{\Gamma_1}^T\tilde{\Psi}_1$
	$J_{H_0^1}$	$C_1\tilde{\Psi}_1 = C_1^{12}\varphi_1^f - C_1^{12}\varphi_2^f$	$-D_{\Gamma_1}^T\tilde{\Psi}_1 + D_{\Gamma_1}^T(\varphi_1^f - \varphi_2^f)$
	J_{H^1}	$C_1\tilde{\Psi}_1 = M_1^{12}\varphi_1^f - M_1^{12}\varphi_2^f + C_{12}\varphi_1^f - C_{12}\varphi_2^f$	$-D_{\Gamma_1}^T\tilde{\Psi}_1 + D_{\Gamma_1}^T(\varphi_1^f - \varphi_2^f)$
	$\widehat{J_{H_0^1}}$	$C_1\tilde{\Psi}_1 = C_1^{12}\varphi_1^f - C_1^{12}\varphi_2^f + N_1^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}} - N_1^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-D_{\Gamma_1}^T\tilde{\Psi}_1 + D_{\Gamma_1}^T(\varphi_1^f - \varphi_2^f)$
$C_2\varphi_2^f = M_2 f_2 + N_2\Psi_{N,2} - D_2\Psi_{D,2}$	J_{L^2}	$C_2\tilde{\Psi}_2 = -M_2^{12}\varphi_1^f + M_2^{12}\varphi_2^f$	$-D_{\Gamma_2}^T\tilde{\Psi}_2$
	$J_{H_0^1}$	$C_2\tilde{\Psi}_2 = -C_2^{12}\varphi_1^f + C_2^{12}\varphi_2^f$	$-D_{\Gamma_2}^T\tilde{\Psi}_2 - D_{\Gamma_2}^T(\varphi_1^f - \varphi_2^f)$
	J_{H^1}	$C_2\tilde{\Psi}_2 = -M_2^{12}\varphi_1^f + M_2^{12}\varphi_2^f - C_{12}\varphi_1^f + C_{12}\varphi_2^f$	$-D_{\Gamma_2}^T\tilde{\Psi}_2 - D_{\Gamma_1}^T(\varphi_1^f - \varphi_2^f)$
	$\widehat{J_{H_0^1}}$	$C_2\tilde{\Psi}_2 = -C_2^{12}\varphi_1^f + C_2^{12}\varphi_2^f - N_2^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}} + N_2^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-D_{\Gamma_2}^T\tilde{\Psi}_2 - D_{\Gamma_1}^T(\varphi_1^f - \varphi_2^f)$

Table 2.1: *The algebraic formulation of the optimality systems, with Dirichlet boundary controls.*

STATE	J	DUAL	PARTIAL DERIVATIVE
$C_1\varphi_1^\lambda = N_{\Gamma_1}\lambda_1$	J_{L^2}	$C_1\Psi_1 = M_1^{12}\varphi_1^{\lambda_1} - M_1^{12}\varphi_2^{\lambda_2}$	$-N_{\Gamma_1}^T\Psi_1 + \beta_1 M_{\Gamma_1}\lambda_1$
	$J_{H_0^1}$	$C_1\Psi_1 = C_1^{12}\varphi^{1,\lambda_1} - C_1^{12}\varphi^{2,\lambda_2}$	$-N_{\Gamma_1}^T\Psi_1 + \beta_1 M_{\Gamma_1}\lambda_1$
	J_{H^1}	$C_1\Psi_1 = M_1^{12}\varphi_1^{\lambda_1} - M_1^{12}\varphi_2^{\lambda_2} + C_{12}\varphi^{1,\lambda_1} - C_{12}\varphi^{2,\lambda_2}$	$-N_{\Gamma_1}^T\Psi_1 + \beta_1 M_{\Gamma_1}\lambda_1$
	$\widehat{J_{H_0^1}}$	$C_1\Psi_1 = C_1^{12}\varphi_1^{\lambda_1} - C_1^{12}\varphi_2^{\lambda_2} + N_1^{12}\varphi^{1,\lambda_1}\vert_{\Gamma_1^N\cap\partial\Omega_{12}} - N_1^{12}\varphi^{2,\lambda_2}\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-N_{\Gamma_1}^T\Psi_1 + \beta_1 M_{\Gamma_1}\lambda_1$
$C_2\varphi_2^\lambda = N_{\Gamma_2}\lambda_2$	J_{L^2}	$C_2\Psi_2 = -M_2^{12}\varphi_1^{\lambda_1} + M_2^{12}\varphi_2^{\lambda_2}$	$-N_{\Gamma_2}^T\Psi_2 + \beta_2 M_{\Gamma_2}\lambda_2$
	$J_{H_0^1}$	$C_2\Psi_2 = -C_2^{12}\varphi^{1,\lambda_1} - C_2^{12}\varphi^{2,\lambda_2}$	$-N_{\Gamma_2}^T\Psi_2 + \beta_2 M_{\Gamma_2}\lambda_2$
	J_{H^1}	$C_2\Psi_2 = -M_2^{12}\varphi_1^{\lambda_1} + M_2^{12}\varphi_2^{\lambda_2} + -C_2^{12}\varphi^{1,\lambda_1} + C_2^{12}\varphi^{2,\lambda_2}$	$-N_{\Gamma_2}^T\Psi_2 + \beta_2 M_{\Gamma_2}\lambda_2$
	$\widehat{J_{H_0^1}}$	$C_2\Psi_2 = -C_2^{12}\varphi_1^{\lambda_1} + C_2^{12}\varphi_2^{\lambda_2} - N_2^{12}\varphi^{1,\lambda_1}\vert_{\Gamma_1^N\cap\partial\Omega_{12}} + N_2^{12}\varphi^{2,\lambda_2}\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-N_{\Gamma_2}^T\Psi_2 + \beta_2 M_{\Gamma_2}\lambda_2$
$C_1\varphi_1^f = M_1 f_1 + N_1\Psi_{N,1} - D_1\Psi_{D,1}$	J_{L^2}	$C_1\tilde{\Psi}_1 = M_1^{12}\varphi_1^f - M_1^{12}\varphi_2^f$	$-N_{\Gamma_1}^T\tilde{\Psi}_1$
	$J_{H_0^1}$	$C_1\tilde{\Psi}_1 = C_1^{12}\varphi_1^f - C_1^{12}\varphi_2^f$	$-N_{\Gamma_1}^T\tilde{\Psi}_1$
	J_{H^1}	$C_1\tilde{\Psi}_1 = M_1^{12}\varphi_1^f - M_1^{12}\varphi_2^f + C_{12}\varphi_1^f - C_{12}\varphi_2^f$	$-N_{\Gamma_1}^T\tilde{\Psi}_1$
	$\widehat{J_{H_0^1}}$	$C_1\tilde{\Psi}_1 = C_1^{12}\varphi_1^f - C_1^{12}\varphi_2^f + N_1^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}} - N_1^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-N_{\Gamma_1}^T\tilde{\Psi}_1$
$C_2\varphi_2^f = M_2 f_2 + N_2\Psi_{N,2} - D_2\Psi_{D,2}$	J_{L^2}	$C_2\tilde{\Psi}_2 = -M_2^{12}\varphi_1^f + M_2^{12}\varphi_2^f$	$-N_{\Gamma_2}^T\tilde{\Psi}_2$
	$J_{H_0^1}$	$C_2\tilde{\Psi}_2 = -C_2^{12}\varphi_1^f + C_2^{12}\varphi_2^f$	$-N_{\Gamma_2}^T\tilde{\Psi}_2$
	J_{H^1}	$C_2\tilde{\Psi}_2 = -M_2^{12}\varphi_1^f + M_2^{12}\varphi_2^f - C_{12}\varphi_1^f + C_{12}\varphi_2^f$	$-N_{\Gamma_2}^T\tilde{\Psi}_2$
	$\widehat{J_{H_0^1}}$	$C_2\tilde{\Psi}_2 = -C_2^{12}\varphi_1^f + C_2^{12}\varphi_2^f - N_2^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}} + N_2^{12}\varphi_1^f\vert_{\Gamma_1^N\cap\partial\Omega_{12}}$	$-N_{\Gamma_2}^T\tilde{\Psi}_2$

Table 2.2: *The algebraic formulation of the optimality systems, with Neumann boundary controls.*

Given an initial guess for the control variables we solve the state equations, then we solve the adjoints, then through the optimality condition we obtain a new approximation of the controls. The process is repeated until satisfactory convergence is achieved. It can be shown that this simple iterative method is equivalent to a steepest descent algorithm (with a fixed step size) for the cost functional (see [15]).

In the initialization part we compute the term that depends on the data of the problem $\nabla\mathscr{A}$, as follows:

1. compute $\varphi^{1,f}$ and $\varphi^{2,f}$;
2. compute $\tilde{\Psi}^1$ and $\tilde{\Psi}^2$;
3. compute $\nabla\mathscr{A}$.

In the main loop we solve by an iterative method the linear system $\nabla J^0(\lambda_1, \lambda_2) = -\nabla\mathscr{A}$. We choose the *Bi-CGStab* algorithm, because the matrix $J^0(\lambda_1, \lambda_2)$ is non symmetric in the case of Dirichlet virtual controls (for a detailed description of the solution of linear systems with *Bi-CGStab* see [16]). We start with an initial guess for the virtual controls $(\lambda_1^0, \lambda_2^0)$ and at the generic iteration k we:

1. compute φ^{1,λ_1} and φ^{2,λ_2};
2. compute Ψ^1 and Ψ^2;
3. compute $\nabla J^0(\lambda_1^k, \lambda_2^k)$.

The method stops when the relative increment between two consecutive iterates $(\lambda_1^k, \lambda_2^k)$ and $(\lambda_1^{k-1}, \lambda_2^{k-1})$ is lower than a certain tolerance.

The gradients of the cost functional $\nabla\mathscr{A}$ and ∇J^0 are computed in the weak form, but we are looking for the controls variables in the strong form. The gradient of the cost functionals lives on the interfaces Γ_1 and Γ_2, so we want to multiply it by the inverse of the interface mass matrix. Moreover, the gradient is a bidimensional vector, so we need to constuct the following mass matrix, associated to both the interfaces of the problem:

$$M_\Gamma = \begin{bmatrix} M_{\Gamma_1} & 0 \\ 0 & M_{\Gamma_2} \end{bmatrix}$$

We multiply $\nabla\mathscr{A}$ by M_Γ^{-1} and the same is implemented for ∇J^0 at each iteration of the main loop.

This is the algorithm that was first implemented and it was used to test the theoretical model and the theoretical results.

2.2.2 A Descent Method

Another approach consists of directly minimizing the functional, using its gradient as a direction of descent. The step length can be automatically selected by the algorithm, for example using the *Armijo Rule*. The gradient algorithm for this method reads as follows:

initialization:

1. choose τ, α : $0 < \alpha < 1$ and $(\lambda_1^0, \lambda_2^0)$. Set $k = 0$, $s = 1$;
2. solve the non homogeneous problems (1.27) to compute $\varphi^{1,f}$ and $\varphi^{2,f}$;
3. solve the dual problems of (1.27) to compute $\tilde{\Psi}^1$ and $\tilde{\Psi}^2$;
4. compute $\nabla\mathscr{A}$;
5. solve the homogeneous problems (1.26) with $(\lambda_1^0, \lambda_2^0)$ to compute $\varphi^{\lambda_1^0}$ and $\varphi^{\lambda_2^0}$;
6. compute $\varphi^{1,0} = \varphi^{\lambda_1^0} + \varphi^{1,f}$ and $\varphi^{2,0} = \varphi^{\lambda_2^0} + \varphi^{2,f}$;
7. evaluate $J^0(\varphi^{1,0}, \varphi^{2,0})$;

main loop:

1. set $k = k + 1$;
2. solve the dual problems of (1.26) with $(\varphi^{\lambda_1^{k-1}}, \varphi^{\lambda_2^{k-1}})$ to compute $(\Psi^{1,k}, \Psi^{2,k})$;
3. evaluate $\nabla J^k = \nabla({J^0}^k + \mathscr{A})$;
4. update $\langle(\lambda_1^k, \lambda_2^k), (\mu_1, \mu_2)\rangle = \langle(\lambda_1^{k-1}, \lambda_2^{k-1}), (\mu_1, \mu_2)\rangle - s\langle(\frac{\partial(J^0+\mathscr{A})}{\partial\lambda_1}, \frac{\partial(J^0+\mathscr{A})}{\partial\lambda_2}), (\mu_1, \mu_2)\rangle$;
5. solve the homogeneous problems (1.26) with $(\lambda_1^k, \lambda_2^k)$ to compute $\varphi^{\lambda_1^k}$ and $\varphi^{\lambda_2^k}$;
6. compute $\varphi^{1,k} = \varphi^{\lambda_1^k} + \varphi^{1,f}$ and $\varphi^{2,k} = \varphi^{\lambda_2^k} + \varphi^{2,f}$;
7. evaluate $J^k(\varphi^{1,k}, \varphi^{2,k})$;
8. if $J^k \geq J^{k-1}$ set $s = \alpha s$ and go to step (4); otherwise, continue;
9. if $\frac{|J^k - J^{k-1}|}{|J^k|} > \tau$ set $s = \alpha^{-1}s$ and go to step (1); otherwise, stop;

The bulk of the computational costs are found in the solution of the state equations and of the adjoint problems (steps (2) and (5) of the main loop). Steps (8) and (9) define the automatic step length determination sub-algorithm.

This algorithm was not used for in our work. The criterion used for the choice of the step length is not optimal and its behavior varies according to the different problems that we considered. A further analysis could lead to the definition of descent methods that work better than the one that we presented. Inspecting the various possibilities for the minimization algorithm was not the main goal of this work, so we did not further investigate this aspect.

2.2.3 The One-Shot Approach

In the so-called one-shot approach one constructs a global linear system whose solution gives simultaneously the optimal controls, the adjoint variables and the state variables.

The motivation of this approach derives from the need to speed up the convergence of the solution of the problem. As we will see in chapter 3, concerning the numerical results, the number of loops

needed to achieve convergence iterating between the equations is excessively high, and this number increases with the refinement level of the mesh.

We constructed the global linear system in order to be able to study a preconditioner for the global matrix. The issue of preconditioning a problem that derives from optimization is analyzed in literature following the one shot approach (a preconditioner for the martix deriving from an optimality system is studied).

The assembling of the global matrix will be presented in section 2.3, where the preconditioning issue is analyzed.

2.2.4 A Comparison Between the Solution Algorithms

We now discuss some advantages and disadvantages of the minimization algorithms that have been previously introduced. The iterative algorithm for the solution of the *extremality equation* and the *descent* method follow the same approach in which the state problem, the adjoint problem and the optimality condition (solve $\nabla J = 0$ in the first case and minimizing functional J in the direction of ∇J in the second case) are handled independently. In the *one shot* approach the solution is obtained by a single solve of the optimality system.

In general, the *one shot* approach is more straightforward but it can be difficult to implement in the case of complicated problems (e.g. for the Stokes equations we would obtain a very big matrix). In some cases just solving for the state problem may involve a high number of variables or a non linearity. In these circumstances an optimization algorithm is to be preferred. Moreover, the global system is almost always solved by an iterative method and the number of iterations needed to achieve convergence is often significantly higher than the number of iterations needed by the first iterative method and the *descent* approach. A preconditioner for the global matrix is indispensable.

We implemented the first method where the extremality equation is solved by an iterative algorithm and the *one shot* approach successfully. The first approach is preferable when one is not interested in preconditioning the problem: the number of iterations needed to achieve convergence is smaller than the one needed by a non preconditioned iterative method used for the solution of the global system. As said before, we used this method to test the theoretical model in our first simulations. The second approach allowed us to investigate the issue of preconditioning.

2.3 Preconditioning the Optimality System

The matrix deriving from an optimization problem is usually ill conditioned, so any iterative algorithm for the solution of the linear system has to be preconditioned to assure a satisfactory convergence rate. The issue of preconditioning the matrix of an optimality system is an open subject of research and many approaches have been proposed in the past years. Most of our difficulties will derive from the fact that we are treating a coupled problem and that the observation is restricted to Ω_{12}, subdomain of both Ω_1 and Ω_2.

In this section we will analyze different approaches of preconditioning. We will take in consideration both the theoretical results concerning the preconditioning of optimization problems and the theoretical results concerining the preconditioning of domain decomposition problems. We start by a description of the theoretical setting.

2.3.1 The Optimality System as a Global Linear System

We started our analysis motivated by the work on preconditioners for the solution of optimization problems presented in [17]. For this reason, in this section, we will follow some of the choices that have been made in this work:

- the solutions of the state problems (1.18) are no longer split in a part depending on the controls and in a part depending on the data, so from now on we will refer to the global solutions φ^1 and φ^2;
- the cost functional minimized over the overlapping region is

$$J_{L^2}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\varphi^1 - \varphi^2)^2 + \frac{1}{2}\beta_1 \int_{\Gamma_1} \lambda_1^2 + \frac{1}{2}\beta_2 \int_{\Gamma_2} \lambda_2^2.$$

  We remark that the penalization term is present in this choice for the cost functional (we will also consider the non penalized case, when setting $\beta_1 = \beta_2 = 0$);
- the control variables are imposed as a Neumann boundary condition: $(\lambda_1, \lambda_2) \in \Lambda^N$.

We first present a brief summary of the problem setting.The problems satisfied by the global solutions of the optimality system (φ^1 and φ^2) can be obtained by linear combination of the problems that have been considered in the previous sections. The same holds for the solutions of the dual problems and of the optimality conditions; we give here their differential and algebraic formulation.

The state problems on Ω_1 and Ω_2, respectively, are the following:
find $\varphi^1 \in H^1(\Omega_1)$ so that

$$\begin{cases} -\nabla\cdot(K\nabla\varphi^1) &= f^1 & in & \Omega_1 \\ K\nabla\varphi^1\cdot n_1 &= \psi_N^1 & on & \Gamma_1^N \\ \varphi^1 &= \psi_D^1 & on & \Gamma_1^D \\ K\nabla\varphi^1\cdot n_1 &= \lambda_1 & on & \Gamma_1, \end{cases} \tag{2.7}$$

and find $\varphi^2 \in H^1(\Omega_2)$ so that

$$\begin{cases} -\nabla\cdot(K\nabla\varphi^2) &= f^2 & in & \Omega_2 \\ K\nabla\varphi^2\cdot n_2 &= \psi_N^2 & on & \Gamma_2^N \\ \varphi^2 &= \psi_D^2 & on & \Gamma_2^D \\ K\nabla\varphi^2\cdot n_2 &= \lambda_2 & on & \Gamma_2, \end{cases} \tag{2.8}$$

where λ_1 and λ_2 are the solutions of the minimization problem

$$\inf J_{L^2}(\lambda_1, \lambda_2),$$

with

$$J_{L^2}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega \chi_{12}(\varphi^1-\varphi^2)^2+\frac{\beta_1}{2}\int_{\Gamma_1}\lambda_1^2+\frac{\beta_2}{2}\int_{\Gamma_2}\lambda_2^2.$$

As previously said, this is the only cost functional that will be considered in this section so from now on we will refer to $J_{L^2}(\lambda_1,\lambda_2)$ as $J(\lambda_1,\lambda_2)$.
The derivatives of the cost functional assume the form:

$$\langle\frac{\partial J}{\partial\lambda_1},\mu_1\rangle=\int_\Omega \chi_{12}(\varphi^1-\varphi^2)\varphi^{\mu_1}+\beta_1\int_{\Gamma_1}\lambda_1\mu_1,$$
$$\langle\frac{\partial J}{\partial\lambda_2},\mu_2\rangle=-\int_\Omega \chi_{12}(\varphi^1-\varphi^2)\varphi^{\mu_2}+\beta_2\int_{\Gamma_2}\lambda_2\mu_2.$$

We introduce the dual problems:
find $\varphi^1\in H^1(\Omega_1)$ so that

$$\begin{cases} -\nabla\cdot(K\nabla\Psi^1) &= -\chi_{12}(\varphi^1-\varphi^2) & in\quad \Omega_1\\ K\nabla\Psi^1\cdot n_1 &= 0 & on\quad \Gamma_1^N\\ \Psi^1 &= 0 & on\quad \Gamma_1^D\\ K\nabla\Psi^1\cdot n_1 &= 0 & on\quad \Gamma_1, \end{cases} \tag{2.9}$$

find $\varphi^2\in H^1(\Omega_2)$ so that

$$\begin{cases} -\nabla\cdot(K\nabla\Psi^2) &= \chi_{12}(\varphi^1-\varphi^2) & in\quad \Omega_2\\ K\nabla\Psi^2\cdot n_2 &= 0 & on\quad \Gamma_2^N\\ \Psi^2 &= 0 & on\quad \Gamma_2^D\\ K\nabla\Psi^2\cdot n_2 &= 0 & on\quad \Gamma_2. \end{cases} \tag{2.10}$$

We remark that the formulation of the couple of dual problems is similar to (1.41) and (1.44). In the definition of the forcing terms we find the global solutions of the dual problems, instead of the solutions of the problems depending on the sole controls. Moreover we remark that the sign of the forcing term is inverted with respect to the previous formulation. This is again to follow the approach of [17]; this choice implies a different sign in the formulation of the optimality conditions, but apart from this the global formulation of the optimality system is equivalent to the previous one.
The optimality conditions read as follows:

$$-\int_{\Gamma_1}\Psi^1\mu_1+\beta_1\int_{\Gamma_1}\lambda_1\mu_1=0\qquad\forall\mu_1\in\Lambda_1^N, \tag{2.11}$$

$$-\int_{\Gamma_2}\Psi^1\mu_2+\beta_2\int_{\Gamma_2}\lambda_2\mu_2=0\qquad\forall\mu_2\in\Lambda_2^N. \tag{2.12}$$

After introducing the weak formulation of the optimality system and its finite elements approximation, we can rewrite the state equations, the dual problems and the optimality conditions as discrete linear systems as follows:
the state equations

$$C_1\varphi_1-N_{\Gamma_1}\lambda_1=M_1f_1+N_1\Psi_{N,1}-D_1\Psi_{D,1}=d_1 \tag{2.13}$$

$$C_2\varphi_2-N_{\Gamma_2}\lambda_2=M_2f_2+N_2\Psi_{N,2}-D_2\Psi_{D,2}=d_2 \tag{2.14}$$

the dual problems

$$C_1\Psi_1 + M_1^{12}\varphi_1 - M_2^{12}\varphi_2 = 0 \tag{2.15}$$

$$C_2\Psi_2 - M_1^{12}\varphi_1 + M_2^{12}\varphi_2 = 0 \tag{2.16}$$

the optimality conditions

$$-N_{\Gamma_1}^T\Psi^1 + \beta_1 M_{\Gamma_1}\lambda_1 = 0 \tag{2.17}$$

$$-N_{\Gamma_2}^T\Psi^2 + \beta_2 M_{\Gamma_2}\lambda_2 = 0 \tag{2.18}$$

After reordering the equations the global problem can be formulated as follows:

$$\begin{bmatrix} \beta_1 M_{\Gamma_1} & 0 & 0 & 0 & -N_{\Gamma_1}^T & 0 \\ 0 & \beta_2 M_{\Gamma_2} & 0 & 0 & 0 & -N_{\Gamma_1}^T \\ 0 & 0 & M_1^{12} & -M_2^{12} & C_2 & 0 \\ 0 & 0 & -M_1^{12} & M_2^{12} & 0 & C_2 \\ -N_{\Gamma_1} & 0 & C_1 & 0 & 0 & 0 \\ 0 & -N_{\Gamma_2} & 0 & C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \varphi_1 \\ \varphi_2 \\ \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix}, \tag{2.19}$$

where the equations that define the optimality system have been written in the following order: optimality conditions, dual problems, state problems.

Because of the symmetry of the problem we take $\beta_1 = \beta_2 = \beta$. In order to write the system in a more compact form we define the following matrices:

$$M_\Gamma = \begin{bmatrix} M_{\Gamma_1} & 0 \\ 0 & M_{\Gamma_2} \end{bmatrix} \in \mathbb{R}^{(N_h^{\Gamma_1}+N_h^{\Gamma_2})\times(N_h^{\Gamma_1}+N_h^{\Gamma_2})},$$

$$M_{12} = \begin{bmatrix} M_1^{12} & -M_2^{12} \\ -M_2^{12} & M_1^{12} \end{bmatrix} \in \mathbb{R}^{(N_h^1+N_h^2)\times(N_h^1+N_h^2)},$$

$$E = \begin{bmatrix} N_{\Gamma_1} & 0 \\ 0 & N_{\Gamma_2} \end{bmatrix} \in \mathbb{R}^{(N_h^1+N_h^2)\times(N_h^{\Gamma_1}+N_h^{\Gamma_2})},$$

$$K = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \in \mathbb{R}^{(N_h^1+N_h^2)\times(N_h^1+N_h^2)},$$

and the following vectors

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \mathbb{R}^{(N_h^{\Gamma_1}+N_h^{\Gamma_2})\times 1},\ \varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \in \mathbb{R}^{(N_h^1+N_h^2)\times 1},\ \Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \in \mathbb{R}^{(N_h^1+N_h^2)\times 1}.$$

Now we rewrite (4.29) as

$$\begin{bmatrix} \beta M_\Gamma & 0 & -E^T \\ 0 & M_{12} & K^T \\ -E & K & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \varphi \\ \Psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}. \tag{2.20}$$

We observe that the coupled optimality system has now the form of a generic optimality system. We hid the contributions of the variables corresponding to the two different domains Ω_1 and Ω_2 and we refer now to one control, one state and one adjoint variable. The block responsible of the coupling between the two problems is the mass matrix M_{12} assembled on the overlap Ω_{12}.

The matrix is symmetric indefinite and has a saddle-point structure of the form

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}, \tag{2.21}$$

where $A = \begin{bmatrix} \beta M_\Gamma & 0 \\ 0 & M_{12} \end{bmatrix}$, $B = \begin{bmatrix} -E & K \end{bmatrix}$, $C = 0$.
We have proved the well posedness of the differential optimality system and the matrix $\mathcal{A}$ derives from the Galerkin approximation of the differential problem. This is sufficient to guarantee the non singularity of the matrix $\mathcal{A}$, so that the system (2.20) admits a unique solution.

Introducing $x = [\lambda,\ \varphi]^T$ and $x = \Psi$, we can write (2.20) in the following generic form:

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ d \end{bmatrix}. \tag{2.22}$$

A Brief Analysis of the Numerical Solution of Saddle Point Problems

Linear systems of saddle point type arise in many applications of computational science and engineering (for example, fluid and solid mechanics and optimization). Many methods and results on the numerical solution of saddle point problems have been proposed, although many of these solvers have been developed with respect to specific applications. In fact, when choosing a particular approach (or developing a new one) understanding the specifics of the problem at hand is essential. A review of many of these approaches can be found in [18].

Besides the distinction between direct and iterative methods, solution algorithms for generalized saddle point problems can be subdivided into two categories: *segregated* and *coupled* methods. Segregated methods compute the unknown vectors (x, y in (2.22)) separately. This approach involves the solution of two linear systems of smaller size (called *reduced systems*), one for each variable. Segregated methods can be either direct or iterative, or involve a combination of the two; for example, one of the reduced systems could be solved with a direct method and the other one iteratively. The main representative of the segregated approach is the Schur complement reduction.
Coupled methods deal with the system (2.22) as a whole, computing x and y simultaneously and without making explicit use of the reduced systems. These methods include both direct solvers based on triangular factorizations of the global matrix $\mathcal{A}$, and iterative algorithms like Krylov subspace methods applied to the entire system, usually in presence of a preconditioner.
The segregated approach is typical of the domain decomposition: a Schur complement matrix is constructed and one solves a reduced problem on the sole interface between the separate domains. On the contrary, in the solution of optimization problems the coupled method is tipically preferred.

Most of the theoretical results concerning the solution of saddle point linear systems are derived with the assumption that the $(1,1)$ block of the saddle point matrix (the matrix A in (2.21)) is nonsingular. This property is not verified in (2.20) because the matrix M_{12} is singular. This matrix

Method	Required $\mathcal{A}$	Required $\mathcal{P}$
CG	symmetric positive definite	symmetric positive definite
MINRES	symmetric	symmetric definite
GMRES	general	general
BiCGStab	general	general

Table 2.3: *Summary of Krylov subspace methods.*

is assembled on the sole overlapping region of the two domains Ω_1 and Ω_2, so that all the entries belonging to the rows and columns corresponding to $\Omega_1 \backslash \Omega_{12}$ and $\Omega_2 \backslash \Omega_{12}$ will be equal to zero. In (2.20) the matrix A is only positive semidefinite.

We will also consider a different reordering of the equations appearing in (2.20) in order to deal with a positive definite $(1,1)$ block. In this case the matrix $\mathcal{A}$ will be non symmetric. The fact that the matrix A is non singular permits different factorizations of the matrix $\mathcal{A}$ and these will be used in the contest of preconditioning.

To solve (2.20), or a reordering of it, we will mainly use *Krylov subspace* methods. These methods require to form an orthogonal basis of the sequence of successive matrix powers times the initial residual (the so called *Krylov subsequence*). The approximations to the solution are then formed by minimizing the residual over the subspace formed. The prototypical method in this class is the conjugate gradient (CG). Other methods that we will use are the *generalized minimal residual method* (GMRES), the *minimal residual method* (MINRES) and the *biconjugate gradient stabilized method* (BiCGStab). A detailed analysis of these methods can be found in [19]. The choice of the solution algorithm is motivated by the properties satisfied by the matrix $\mathcal{A}$, as we can see from Table 2.3.

An iterative method has to be coupled with a preconditioner to get satisfactory convergence. We want to find a matrix $\mathcal{P}$ for which $\mathcal{P}^{-1}\mathcal{A}$ has better spectral properties than $\mathcal{A}$ (and such that $\mathcal{P}^{-1}v$ is cheap to evaluate for any given vector v). In fact, for symmetric problems, the (worst-case) rate of convergence of Krylov subspace methods like CG or MINRES depends on the distribution of the eigenvalues of $\mathcal{A}$. For non-symmetric problems the situation is more complicated, and the eigenvalues may not describe the convergence of non-symmetric matrix iterations like GMRES or BICGStab. Nevertheless, a clustered spectrum (away from 0) often results in rapid convergence.

We then solve a symmetric preconditioned system equivalent to

$$\mathcal{P}^{-1}\mathcal{A}x = \mathcal{P}^{-1}b.$$

Generally speaking, there are two approaches to constructing preconditioners. The first is based on purely algebraic techniques, like incomplete factorizations and sparse approximate inverses. These preconditioners require little knowledge of the problem at hand and can be applied in a more or less black-box fashion. When applied to saddle point linear systems these preconditioners are often found to perform poorly. The second approach develops preconditioners that are tailored to the particular application at hand. This approach requires knowledge of the specific characteristics of the problem and the more information one can use, the better the quality of the resulting preconditioner. The drawback of this approach is that the range of problems that can be treated with a particular

preconditioner will necessarily be narrow.
For saddle point problems, the construction of high-quality preconditioners requires the exploitation of the block structure of the problem, together with detailed knowledge of the origin and of the structure of the various blocks. Because this varies from application to application the choice of the preconditioner is strongly problem dependent.

In what follows we list the preconditioners that have been implemented and tested in this work. We will motivate our choices and discuss the advantages and disadvantages of one approach over the other. The numerical results obtained with our numerical simulations will be presented in section 3.1.2.

2.3.2 A Block Diagonal Preconditioner

The matrix $\mathcal{A}$ is symmetric indefinite and can be solved with a preconditioned MINRES algorithm. Our first approach was to construct a block diagonal preconditoner (as in [17]) of the following form:

$$\mathcal{P}_d = \begin{bmatrix} A & 0 \\ 0 & BA^{-1}B^T \end{bmatrix}.$$

In [20] it is shown that the preconditioned matrix $\mathcal{P}^{-1}\mathcal{A}$ has exactly two or exactly three distinct eigenvalues. The drawback of this approach is that it can be shown that the forming the preconditioned system is essentially as expensive as computing the the inverse of $\mathcal{A}$ using an appropriate factorization of the matrix. For this reason, the exact preconditioner needs to be replaced by an approximation. Moreover the presence of the Schur complement $BA^{-1}B^T$ in the $(2,2)$ block of the preconditioner implies that A has to be invertible, which is not the case in (2.20). In fact, the block diagonal preconditioner for $\mathcal{A}$ reads as:

$$\begin{bmatrix} \beta M_\Gamma & 0 & 0 \\ 0 & M_{12} & 0 \\ 0 & 0 & \beta^{-1}EM_\Gamma^{-1}E^T + KM_{12}^{-1}K^T \end{bmatrix},$$

and, as it has been previously remarked, the matrix M_{12} in not positive definite since most of its columns and rows are zeros. This problem could be overcome by considering the approximate inverse of M_{12} or by inverting its restriction on the indices of the nodes belonging to the overlapping region and then re-extending the result to the whole domain. However, as we can see from Table 2.3, MINRES requires a positive definite preconditioner and we need to modify $\mathcal{P}$ in order to satisfy this constraint. In order to deal with a positive definite preconditioner we introduced the following modifications of the mass matrix assembled on the overlap M_{12}.
We tested the following modifications of M_{12}:

$$\tilde{M}_{12} = M_{12} + \alpha I, \tag{2.23}$$

$$\hat{M}_{12} = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}, \tag{2.24}$$

where M_i is the mass matrix assembled on the whole domain Ω_i, $i = 1, 2$. $\hat{M}_{12}$ is the matrix that would appear in the optimality system if the overlap had been extended to the whole domain

$\Omega = \Omega_1 \cup \Omega_2$. We refer to the preconditioning matrices in which these modifications have been implemented as $\tilde{\mathcal{P}}_d$ and $\hat{\mathcal{P}}_d$.
Because of the presence of βM_Γ in the upper-left part of the precondtioner we observe that this approach cannot be used in the non penalized case $\beta = 0$. The computational cost of the assembling of this preconditioner is of two solutions of a linear system and two matrix-vector modification.

2.3.3 Gauss-Seidel and Jacobi Preconditioners

We can reorder the equations that constitute the optimality system in order to deal with a positive definite $(1,1)$ block. In particular we can interchange the second and the third rows of $\mathcal{A}$ in (2.20), obtaining the following linear system:

$$\begin{bmatrix} \beta M_\Gamma & 0 & -E^T \\ -E & K & 0 \\ 0 & M_{12} & K^T \end{bmatrix} \begin{bmatrix} \lambda \\ \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} 0 \\ d \\ 0 \end{bmatrix}. \tag{2.25}$$

The matrix associated to this linear system also has a saddle point structure

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{B} & -\tilde{C} \end{bmatrix},$$

where

$$\tilde{A} = \beta M_g, \quad \tilde{B} = \begin{bmatrix} -E \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} -K & 0 \\ -M_{12} & -K^T \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} 0 & -E^T \end{bmatrix}.$$

We observe that now both the diagonal blocks appearing in the global matrix are positive definite, but the symmetry of the matrix is lost. Moreover, the matrix C is no longer zero. In this case we will use a GMRES solver instead of MINRES.
A common preconditioning approach comes from the block LU factorization

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{B} & -\tilde{C} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}\tilde{A}^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{D} \\ 0 & -\tilde{C} - \tilde{B}\tilde{A}^{-1}\tilde{D} \end{bmatrix}.$$

Another common approach is to use a block *Gauss-Seidel* preconditioner which consists in neglecting part of the coupling:

$$\mathcal{P}_{GS} = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{B} & -\tilde{C} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \tilde{B}\tilde{A}^{-1} & I \end{bmatrix} \begin{bmatrix} \tilde{A} & 0 \\ 0 & -\tilde{C} \end{bmatrix}.$$

This is a block triangular preconditioner, in fact $\mathcal{P}_{GS}$ is equal to the lower triangular part of $\mathcal{A}$. This preconditioner has been successfully used in many applications (for an example see [21]). One could also consider a simpler preconditioner extracting the diagonal part of $\mathcal{A}$; this is called the *Jacobi* preconditioner:

$$\mathcal{P}_J = \begin{bmatrix} \tilde{A} & 0 \\ 0 & -\tilde{C} \end{bmatrix}.$$

The explicit form of the Gauss-Seidel and of the Jacobi preconditioners for the iterative solution of (2.25) is the following:

$$\mathcal{P}_{GS} = \begin{bmatrix} \beta M_g & 0 & 0 \\ -E & K & 0 \\ 0 & M_{12} & K^T \end{bmatrix}, \qquad \mathcal{P}_J = \begin{bmatrix} \beta M_g & 0 & 0 \\ 0 & K & 0 \\ 0 & M_{12} & K^T \end{bmatrix}.$$

In both approaches, in order to deal with an invertible preconditioner, the penalization term has to be present ($\beta \neq 0$). Because of the presence of the matrix $\tilde{C}$, the cost of the solution of a linear system on the preconditioner is comparable to the resolution of the original problem, since one has to solve for the primals and the duals. This shows that this approach cannot be used in practice unless one finds a suitable approximation of $\mathcal{P}_{GS}$ and $\mathcal{P}_J$.

2.3.4 Preconditioning of Saddle Point Linear Systems with a Matrix Featuring a Singular (1,1) Block

We once again concentrate on the linear system (2.20). The matrix of this problem is symmetric semidefinite, invertible, and with an ill-conditioned $(1,1)$ block.

In [22] and [23] a preconditioner for saddle point matrices with singular $(1,1)$ block is presented. The motivation of these works is the solution of linear systems deriving from interior point optimization methods. The attractive property of the proposed approach is that eigenvalue clustering improves with increasing ill conditioning of the singular block. As we previously said, for symmetric problems, the Krylov subspace methods converge at a rate dependent on the number of distinct eigenvalues of the preconditioned matrix, so it is desirable to have a small number of distinct eigenvalues or at least a small number of clusters because in this case the convergence will be rapid.

These preconditioners have been developed with respect to a very specific problem; we found out that (2.20) verifies all the hypotheses necessary for the definition of this preconditioner. Here we present a brief summary of the problem setting.
We want to find a preconditoner for the saddle point matrix:

$$\mathcal{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix},$$

where $A \in \mathbb{R}^{n \times n}$ is a symmetric and positive semidefinite matrix with nullity (dimension of the kernel of A) p, and the matrix $B \in \mathbb{R}^{m \times n}$ has full row rank. It can be proven (see [18]) that the assumption that $\mathcal{A}$ is nonsingular implies that $ker(A) \cap ker(B) = 0$, which is used in the derivation of the results concerning the spectral clustering of the preconditioned matrix.
In the case of the matrix in (2.20) the non singularity of $\mathcal{A}$ is guaranteed by the well posedness of the differential formulation and we recall that $B = \begin{bmatrix} -E & K \end{bmatrix}$: the presence of the stiffness matrix K guarantees that B has full row rank.

The approach is based on the augmentation of the $(1,1)$ block using a weight matrix W ($W \in \mathbb{R}^{m \times m}$ is symmetric positive definite). In [22] the following preconditioner is proposed.

$$\mathcal{P}_t = \begin{bmatrix} A + B^T W^{-1} B & tB^T \\ 0 & W \end{bmatrix}. \tag{2.26}$$

The preconditioned matrix $\mathcal{P}_t^{-1}\mathcal{A}$ will have

$$\lambda = 1$$

with algebraic multiplicity $n - m$,

$$\lambda_{\pm} = \frac{-t \pm \sqrt{t^2 + 4}}{2}$$

with multiplicity $2p$, and the $2(m-p)$ remaining eigenvalues verify

$$\lambda_{\pm} = \frac{-t \pm \sqrt{1 + \frac{4\mu^2}{1+\mu^2}}}{2},$$

where μ are some $m-p$ eigenvalues of the following generalized eigenvalue problem:

$$B^T W^{-1} B x = \mu^2 A x. \tag{2.27}$$

For example in the case $k=-1$ we have $\lambda = 1$ with multiplicity $n-m$ and $\lambda_{\pm} = \frac{1\pm\sqrt{5}}{2}$ with multiplicity $2p$. Since λ_+ is a strictly increasing function of μ on $(0,\infty)$ (and λ_- is strictly decreasing) the intervals containing the remaining eigenvalues can be found using $\lim_{\mu\to 0,\infty} \lambda_{\pm}(\mu)$ and we easily obtain

$$\lambda_+ \in (1, \frac{1+\sqrt{5}}{2}) \qquad \lambda_- \in (\frac{1-\sqrt{5}}{2}, 0). \tag{2.28}$$

When A is highly singular many of the generalized eigenvalues μ of (2.27) are large, so that the corresponding eigenvalues $\lambda_{\pm}$ are bounded away from zero.

The simplest choice for the weight matrix is $W = I$. As proposed in [22] we can take $W = \gamma I$ with γ chosen so that the augmentation term $\frac{1}{\gamma} B^T B$ is of norm comparable to the one of A. For this reason we tested the case $1/\gamma = max(A)$.
Since computing the $(1,1)$ block of the preconditioner at each iteration of the minimization algorithm is expensive, we considered a *Cholesky* factorization ($HH^T = A + B^T W^{-1} B$). The preconditioning matrix can be factorized as follows:

$$\mathcal{P}_t = \begin{bmatrix} H & tB^T W^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} H^T & 0 \\ 0 & W \end{bmatrix}.$$

To reduce the computational cost one should use an inexact factorization (LU or Cholesky), even if the number of iterations of GMRES will increase.

We also present the preconditioners introduced in [23], with the corresponding LU factorization of the $(1,1)$ block. The first one reads as

$$\tilde{\mathcal{P}}_t = \begin{bmatrix} A + B^T W^{-1} A B & (1-t)B^T \\ 0 & tW \end{bmatrix}, \quad t \neq 0$$

$$\tilde{\mathcal{P}}_t = \begin{bmatrix} L & \frac{1-t}{t} B^T W^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & tW \end{bmatrix}.$$

We remark that for $t=1$ we obtain a block diagonal preconditioner. It is possible to demonstrate that the eigenvalues of the preconditioned matrix $\tilde{\mathcal{P}}_t^{-1}\mathcal{A}$ satisfy

- $\lambda = 1$ with multiplicity n
- $\lambda = -\frac{1}{t}$ with multiplicity p
- $\lambda = -\frac{\mu}{t(\mu+1)}$ with multiplicity $m-p$, where μ is the solution of a generalized eigenvalue problem.

Moreover the following bounds can be obtained for λ

$$(0, -\frac{1}{t}) \quad (t < 0), \qquad (-\frac{1}{t}, 0) \quad (t > 0).$$

The second preconditioner reads as

$$\hat{\mathcal{P}}_t = \begin{bmatrix} G + tB^T W^{-1} B & tB^T \\ 0 & \frac{1-t}{t} W \end{bmatrix}, \quad 1 \neq t > 0$$

$$\hat{\mathcal{P}}_t = \begin{bmatrix} L & \frac{t^2}{1-t} B^T W^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & \frac{1-t}{t} W \end{bmatrix}.$$

In this case, for $\hat{\mathcal{P}}_t^{-1}\mathcal{A}$ we have that

- $\lambda = 1$ with multiplicity n
- $\lambda = -\frac{1}{t-1}$ with multiplicity p
- $\lambda = -\frac{\mu t}{(t-1)(\mu t+1)}$ with multiplicity $m - p$, where μ is the solution of a generalized eigenvalue problem.

Moreover the following bounds can be obtained for λ

$$(0, \frac{1}{t-1}) \quad (t > 10), \qquad (\frac{1}{t-1}, 0) \quad (t < 1).$$

In this section we have presented a preconditioner that has the property that the more ill-conditioned the (1,1) block of the saddle point matrix is, the faster a minimum residual solver such as MINRES converges. In fact the more the size of the nullity of A increases, the more the eigenvalues will be clustered and bounded away from 0.
This property is extremely interesting in the framework of our application, where we want to use low values of the penalization coefficient β, in order to modify the original problem as least as possible. In fact for low values of β the $(1, 1)$ block of the matrix $\mathcal{A}$ is more ill-conditioned. In particular, in the case $\beta = 0$ (no modification of the original problem) the size of the nullity of A increases of a value equal to the size of the problem on the interfaces. We remark how in the previous application the value of β had to be different from 0. On the other hand, these preconditioners are not optimal in the case of high values of the penalization coefficient. We can conclude that this approach is recommended in the case of no or very small penalization, which is the most common and most interesting situation in the practice of optimization.

2.3.5 Preconditioning The Schur Complement Interface Equation

We introduce an approach belonging to the class of segregated methods for the solution of saddle point linear systems. The inspiration for this approach derives from the domain decomposition theory (see [24]). We want to uncouple the equations that characterize the linear system (2.20) in order to reduce the the global problem to an equation of the form $\Sigma\lambda = f$ for the vector of the controls

λ. In order to define the interface equation we write the optimality, adjoint and state equations of the control problem for the coupled variables λ, φ and Ψ:

$$\beta M_\Gamma \lambda - E^T \Psi = 0, \tag{2.29}$$

$$M_{12}\varphi + K^T \Psi = 0, \tag{2.30}$$

$$-E\lambda + K\varphi = d. \tag{2.31}$$

We substitute in the equation (2.29) representing the optimality condition the dual variable Ψ derived from the adjoint equation (2.30):

$$\beta M_\Gamma \lambda + E^T (K^T)^{-1} M_{12}\varphi = 0,$$

then we substitute the expression of the state variable φ derived from the state equation (2.31):

$$\beta M_\Gamma \lambda + E^T (K^T)^{-1} M_{12}(K^{-1}(d + E\lambda)) = 0.$$

Rearranging the various terms we obtain the following equation on the interface:

$$\left(\beta M_\Gamma + E^T (K^T)^{-1} M_{12} K^{-1} E\right)\lambda = -E^T (K^T)^{-1} M_{12} K^{-1} d, \tag{2.32}$$

that can be rewritten as $\Sigma\lambda = f$, where

$$\Sigma = \beta M_\Gamma + S, \qquad f = -E^T (K^T)^{-1} M_{12} K^{-1} d,$$

and

$$S = E^T (K^T)^{-1} M_{12} K^{-1} E.$$

Once that one has obtained λ from (2.32), the state variable φ can be obtained by the resolution of the state equation (2.31):

$$\varphi = K^{-1}(d + E\lambda).$$

The dual variable Ψ can be derived in an analogous way from (2.30); however we remark that this variable is defined in the optimality system as an intermediate step between the optimality conditions and the state equations. Unless one is interested in evaluating the explicit form of the solution of the dual problem, this step can be skipped. The fact that the solution of the adjoints does not need to be determined explicitly represents an advantage of the segregated approach.

The matrix Σ associated to the problem on the interface is the sum of two terms: the first is the mass matrix on the interface weighted with the coefficient β, related to the penalization and the second (S) is a matrix related to the solution of the optimality system. This matrix takes the Neumann data on the control boundary (E), solves the state equation (K^{-1}), restricts the solution of the state equation on the overlapping region (M_{12}), solves the dual problem ($(K^T)^{-1}$) and takes this data to the boundary (E^T). We will refer to Σ as the *Schur Complement* matrix.

We implemented the solution of the interface equation. The matrix Σ is symmetric positive definite, so the system is solved with a *Conjugate Gradient* algorithm. Since the assembling of Σ and of the right hand side f is very costly (two inversions of a stiffness matrix and two matrix-vector multiplications for Σ and two inversions of a stiffness matrix and two matrix-vector multiplications for the right hand side), we introduced an LU factorization of the matrix K.

In order to devise a possible preconditioner for Σ, we can rewrite the Schur complement matrix showing the contributions of the matrices corresponding to the two domains Ω_1 and Ω_2:

$$\Sigma = \begin{bmatrix} \beta M_{\Gamma_1} + N_{\Gamma_1}^T (C_1^T)^{-1} M_1^{12} C_1^{-1} N_{\Gamma_1} & -N_{\Gamma_1}^T (C_1^T)^{-1} M_2^{12} C_2^{-1} N_{\Gamma_2} \\ -N_{\Gamma_2}^T (C_2^T)^{-1} M_1^{12} C_1^{-1} N_{\Gamma_1} & \beta M_{\Gamma_2} + N_{\Gamma_2}^T (C_2^T)^{-1} M_2^{12} C_2^{-1} N_{\Gamma_2} \end{bmatrix}.$$

On the diagonal of this matrix we recognize two matrices with the same structure of Σ and corresponding to the two different overlapping domains. We can call them Σ_1 and Σ_2:

$$\Sigma = \begin{bmatrix} \Sigma_1 & -N_{\Gamma_1}^T (C_1^T)^{-1} M_{12} C_2^{-1} N_{\Gamma_2} \\ -N_{\Gamma_2}^T (C_2^T)^{-1} M_{12} C_1^{-1} N_{\Gamma_1} & \Sigma_2 \end{bmatrix}.$$

Similarly we define $S_1 = N_{\Gamma_1}^T (C_1^T)^{-1} M_1^{12} C_1^{-1} N_{\Gamma_1}$ and $S_2 = N_{\Gamma_2}^T (C_2^T)^{-1} M_2^{12} C_2^{-1} N_{\Gamma_2}$. One idea is to take the block diagonal preconditioner

$$P_\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} = \begin{bmatrix} \beta M_{\Gamma_1} + S_1 & 0 \\ 0 & \beta M_{\Gamma_2} + S_2 \end{bmatrix} \tag{2.33}$$

neglecting the terms related to the coupling.
When β is high the penalization terms βM_{Γ_1} and βM_{Γ_2} dominate the matrices S_1 and S_2, therefore there is no need to assemble S_1 and S_2 in the case $\beta = 1$. For this reason we consider the following preconditioner

$$P = \beta M_\Gamma + (1 - \beta) P_S, \tag{2.34}$$

where

$$P_S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} \qquad \text{and} \qquad P_\Sigma = \beta M_\Gamma + S.$$

We remark that the preconditioner (2.34) corresponds to a linear combination of M_Γ and P_Σ that weights the magnitude of the penalization. The cost of assembling this precoditioner is given by the cost of assembling S_1 and S_2 so twice the cost of two matrix-vector multiplication and of two system resolutions. In practice, since we have already assembled the interface matrix Σ, we can obtain this preconditioner by simply extracting the diagonal blocks of this matrix.

We also thought of a simplified version of this preconditioner. If we neglect in S the part related to the solution of the dual problem, we still have a map from the interface to the interface which passes through the solution of a stiffness problem. For this reason we considered

$$\tilde{P} = \beta M_g + (1 - \beta) \tilde{P}_S, \tag{2.35}$$

where

$$\tilde{P}_S = \begin{bmatrix} \tilde{S}_1 & 0 \\ 0 & \tilde{S}_2 \end{bmatrix} = \begin{bmatrix} N_{\Gamma_1}^T C_1^{-1} N_{\Gamma_1} & 0 \\ 0 & N_{\Gamma_2}^T C_2^{-1} N_{\Gamma_2} \end{bmatrix}.$$

The cost of assembling for $\tilde{P}$ is of two matrix vector multiplications and two system resolutions. So it requires half of the computations required for P.

A further approximation consists of assembling the stiffness matrices that are inverted in the

preconditioner on a restriction of the two overlapping domains Ω_1 and Ω_2, close to the interfaces. For this reason in the latter preconditioner we consider $C_1 = C_2 = C_{12}$, where C_{12} is the stiffness matrix assembled only on the overlapping region Ω_{12}, and the interface mass matrices are restricted accordingly. We refer to this preconditioner as $\tilde{P}_{12}$ and we have:

$$\tilde{P}_{12} = \beta M_\Gamma + (1-\beta)\tilde{P}_{S_{12}}, \tag{2.36}$$

where

$$\tilde{P}_{S_{12}} = \begin{bmatrix} (\tilde{S}_1)_{12} & 0 \\ 0 & (\tilde{S}_2)_{12} \end{bmatrix} = \begin{bmatrix} N_{\Gamma_1}^T C_{12}^{-1} N_{\Gamma_1} & 0 \\ 0 & N_{\Gamma_2}^T C_{12}^{-1} N_{\Gamma_2} \end{bmatrix}.$$

The last approximation we introduce is to precondition on both interfaces with the same matrix, for example $\tilde{S}_1$. By doing this we again reduce of a factor two the number of computations needed to assemble the preconditioning matrix. In this case

$$\tilde{P}_{12}^1 = \beta M_g + (1-\beta)\tilde{P}_{S_{12}^1}, \tag{2.37}$$

where

$$\tilde{P}_{S_{12}^1} = \begin{bmatrix} (\tilde{S}_1)_{12} & 0 \\ 0 & (\tilde{S}_1)_{12} \end{bmatrix} = \begin{bmatrix} N_{\Gamma_1}^T C_{12}^{-1} N_{\Gamma_1} & 0 \\ 0 & N_{\Gamma_1}^T C_{12}^{-1} N_{\Gamma_1} \end{bmatrix}.$$

In this section we have presented a series of preconditioners for the Suhr complement matrix Σ. The matrices are presented in decreasing order of assemblage complexity. We have adopted a *Jacobi* preconditioning approach: all the preconditioning matrices, from (2.33) to (2.37), are block diagonal. The two blocks refer to the two overlapping subdomains. This shows that the preconditioner acts in a decoupled way and neglects the terms related to the coupling between the two problems.

Equivalence Between the Extremality Equation and the Shur Complement System

Through the Schur complement reduction, we have derived a linear system for the vector of the controls λ of the following form:

$$\Sigma\lambda = f. \tag{2.38}$$

We want to show the equivalence between this approach and the solution of the extremality equation presented in section 2.2.1. We can prove that solving the linear system related to the optimality condition

$$\nabla J^0(\lambda) = -\nabla \mathscr{A} \tag{2.39}$$

iterating between the equations as shown in section 2.2.1 is equivalent to solving (2.38) iteratively.

In fact, the algorithm of minimization of the cost functional corresponds to the following iterative scheme: given λ^0, solve $\forall k > 0$

$$\lambda^{(k+1)} - \lambda^{(k)} = s M_\Gamma^{-1}\big(f - \Sigma\lambda^{(k)}\big), \tag{2.40}$$

where s is a scalar related to the length of the descent step. In order to show this we denote

$$\lambda^{(k)} = \begin{bmatrix} \lambda_1^{(k)} \\ \lambda_2^{(k)} \end{bmatrix}, \quad \varphi^{(k)} = \begin{bmatrix} \varphi_1^{(k)} \\ \varphi_2^{(k)} \end{bmatrix}, \quad \Psi^{(k)} = \begin{bmatrix} \Psi_1^{(k)} \\ \Psi_2^{(k)} \end{bmatrix},$$

and we refer to the gradient minimization algorithm (presented in section 2.2.2) because the steps are easier to interpret, but the results obtained with the BiCGStab algorithm (section 2.2.1) are analogous, since the latter is also equivalent to a descent method.
At the step $(k+1)$ of the algorithm we compute $\lambda^{(k+1)}$ from $\lambda^{(k)}$ in the following way:

1. $\lambda^{(k)} \to \varphi^{(k)}$:
$$\varphi^{(k)} = K^{-1}(d + E\lambda^{(k)})$$

2. $\varphi^{(k)} \to \psi^{(k)}$:
$$\psi^{(k)} = -(K^T)^{-1}M_{12}\varphi^{(k)} = -(K^T)^{-1}M_{12}K^{-1}(d + E\lambda^{(k)})$$

3. $\psi^{(k)} \to \lambda^{(k+1)}$:
$$\begin{aligned}\lambda^{(k+1)} &= \lambda^{(k)} - sM_\Gamma^{-1}(\beta M_\Gamma \lambda^{(k)} - E^T\lambda^{(k)}) \\ &= \lambda^{(k)} - sM_\Gamma^{-1}(\beta M_\Gamma \lambda^{(k)} + E^T(K^T)^{-1}M_{12}K^{-1}(d + E\lambda^{(k)})\end{aligned}$$

Reordering the terms appearing in the last equation we obtain (2.40). Denoting by $r^{(k)} = f - \Sigma\lambda^{(k)}$ the residual at the k-th step of the algorithm, we can rewrite iteration (2.40) as
$$\lambda^{(k+1)} = \lambda^{(k)} + sM_\Gamma^{-1}r^{(k)}.$$
This corresponds to an iterative algorithm for the solution of the Schur complement equation (2.38). The interface mass matrix M_Γ is needed to pass from the weak formulation in which the optimality condition is stated to the strong formulation in which we are looking for the controls. For high values of β, this matrix behaves as a good preconditioner for Σ, due to the fact that the interface mass matrix that appears in the definition of the Schur complement is the dominant term in this case. However, as previously remarked, the non preconditioned problem is well conditioned for high values of the penalization coefficient so that preconditioning is not needed.

2.3.6 The Optimality System and the Schur Complement Equation in the Case of Dirichlet Boundary Controls

All our analysis on preconditioning has been carried out in the case of Neumann boundary controls, so we have derived the global expression of the optimality system and the expression of the Schur complement equation in this particular case. In order to be exhaustive, in this section we want to show the algebraic formulation of the global system and of the interface equation in the case of Dirichlet boundary controls. Again we treat the case of a penalized $L^2(\Omega_{12})$ cost functional, and we are not splitting the cost functional into a part depending on the data and into a part depending on the sole controls. We present here the discretized state, adjoint and optimality equations corresponding to this particular problem. This description will be synthetic because the derivation of these results is analogous to what we have shown in the case of Neumann boundary controls in the previous section.
All the matrices have already been introduced in the section concerning the discretization of the problem. The state equations are as follows
$$C_1\varphi_1 + D_{\Gamma_1}\lambda_1 = d_1, \tag{2.41}$$

$$C_2\varphi_2 + D_{\Gamma_2}\lambda_2 = d_2, \tag{2.42}$$

the dual problems are analogous to the case of Neumann boundary controls

$$C_1\Psi_1 + M_1^{12}\varphi_1 - M_2^{12}\varphi_2 = 0, \tag{2.43}$$

$$C_2\Psi_2 - M_1^{12}\varphi_1 + M_2^{12}\varphi_2 = 0, \tag{2.44}$$

and the optimality conditions are

$$D_{\Gamma_1}^T\Psi^1 + \beta_1 M_{\Gamma_1}\lambda_1 = 0, \tag{2.45}$$

$$D_{\Gamma_2}^T\Psi^2 + \beta_2 M_{\Gamma_2}\lambda_2 = 0. \tag{2.46}$$

After reordering the equations the global problem can be formulated as follows:

$$\begin{bmatrix} \beta_1 M_{\Gamma_1} & 0 & 0 & 0 & D_{\Gamma_1}^T & 0 \\ 0 & \beta_2 M_{\Gamma_2} & 0 & 0 & 0 & D_{\Gamma_1}^T \\ 0 & 0 & M_1^{12} & -M_2^{12} & C_2 & 0 \\ 0 & 0 & -M_1^{12} & M_2^{12} & 0 & C_2 \\ D_{\Gamma_1}^T & 0 & C_1 & 0 & 0 & 0 \\ 0 & D_{\Gamma_2}^T & 0 & C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \varphi_1 \\ \varphi_2 \\ \Psi_1 \\ \Psi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_1 \\ d_2 \end{bmatrix}, \tag{2.47}$$

where the equations that define the optimality system have been written in the following order: optimality conditions, dual problems, state problems.
We can group the various blocks defined in (2.47) in order to obtain the following linear system:

$$\begin{bmatrix} \beta M_\Gamma & 0 & D^T \\ 0 & M_{12} & K^T \\ D & K & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \varphi \\ \Psi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}. \tag{2.48}$$

This linear system is analogous to (2.20). The only difference is represented by the presence of the matrix D that is needed to lift the Dirichlet boundary data and to state the optimality condition. The definition of this matrix is the following:

$$D = \begin{bmatrix} D_{\Gamma_1} & 0 \\ 0 & D_{\Gamma_2} \end{bmatrix} \in \mathbb{R}^{(N_h^1+N_h^2)\times(N_h^{\Gamma_1}+N_h^{\Gamma_2})},$$

where the matrices D_{Γ_1} and D_{Γ_2} have been defined in section 2.1. Again, the system can be reduced to the following equation on the interfaces for the vector of controls λ:

$$\left(\beta M_\Gamma + D^T(K^T)^{-1}M_{12}K^{-1}D\right)\lambda = -D^T(K^T)^{-1}M_{12}K^{-1}d.$$

We can define $\Sigma_D = \beta M_\Gamma + D^T(K^T)^{-1}M_{12}K^{-1}D$ and decomposing this matrix into blocks depending on the contributions of the two subproblems on Ω_1 and Ω_2 we can see that Σ_D is made up of the following terms:

$$\Sigma = \begin{bmatrix} \beta M_{\Gamma_1} + D_{\Gamma_1}^T(C_1^T)^{-1}M_1^{12}C_1^{-1}D_{\Gamma_1} & -D_{\Gamma_1}^T(C_1^T)^{-1}M_2^{12}C_2^{-1}D_{\Gamma_2} \\ -D_{\Gamma_2}^T(C_2^T)^{-1}M_1^{12}C_1^{-1}D_{\Gamma_1} & \beta M_{\Gamma_2} + D_{\Gamma_2}^T(C_2^T)^{-1}M_2^{12}C_2^{-1}D_{\Gamma_2} \end{bmatrix}.$$

The diagonal terms of Σ_D are related to the two uncoupled subproblems while outside the diagonal are represented the coupling terms.
In this case the matrices Σ_1 and Σ_2 behave as the *Steklov-Poincaré* operator of the domain decomposition theory (see [24]). In fact the matrices map a Dirichlet control in the space of the Neumann controls.

Chapter 3

Numerical Results

3.1 Numerical Results

In this section we will present the numerical results obtained with our simulations, using *Matlab*. The first part of the results concerns the validation of the theoretical model described in section 1.2. The second part concerns the numerical solution of the problem in presence of the different preconditioning approaches presented in section 2.3.

3.1.1 The Model

In this section we want to test the theoretical results that have been derived in section 1.2. We solve the discrete problems presented in Table 2.1 and in Table 2.2 via the the iterative minimization algorithm described in section 2.2.1. We are interested in testing the well posedness of the model and the convergence orders of the finite elements approximation of the solution. Moreover, through

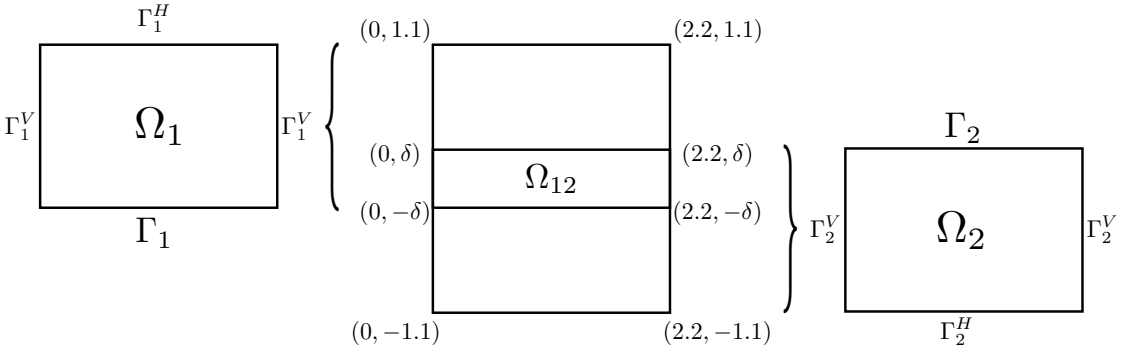


Figure 3.1: *Domain and boundary conditions*

these tests, we can study the convergence behavior of the algorithm with respect to the different choices of the cost functional and with respect to the different choices of Dirichlet or Neumann boundary controls.

The solution of the optimization problem is the couple of vectors (λ_1, λ_2) that define the controls on the interfaces. Then, solving the linear systems related to the state problems one finds the approximations of the states φ_h^1 and φ_h^2, on Ω_1 and Ω_1 respectively. We build the solution on the whole domain in the following way

$$\varphi_h^G = \begin{cases} \varphi_h^1 & on \quad \Omega_1 \\ \varphi_h^2 & on \quad \Omega_2 \backslash \Omega_{12}. \end{cases} \tag{3.1}$$

When the algorithm has reached convergence the value of the discretized cost functional is very close to 0 (10^{-15}-10^{-20}), so that $\varphi_h^1|_{12}$ and $\varphi_h^2|_{12}$ can be considered equivalent. Another approach could be to build φ_h^G as

$$\varphi_h^G = \begin{cases} \varphi_h^1 & on \quad \Omega_1 \backslash \Omega_{12} \\ \frac{\varphi_h^1 + \varphi_h^2}{2} & on \quad \Omega_{12} \\ \varphi_h^2 & on \quad \Omega_2 \backslash \Omega_{12}. \end{cases} \tag{3.2}$$

It is important to make sure that φ_h^G is equivalent to the finite elements approximation of the following global problem

$$\begin{cases} -\nabla \cdot (K \nabla \varphi) = f & in \quad \Omega \\ K \nabla \varphi \cdot \mathbf{n} = \Psi_N & on \quad \Gamma^N \\ \varphi = \Psi_D & on \quad \Gamma^D. \end{cases} \tag{3.3}$$

that we will denote as φ_h.

We initialize the algorithm with the controls $(\lambda_1^0, \lambda_2^0) = (0, 0)$ and we set a tolerance of 10^{-10} on the relative residual between the subsequent iterations as a stopping criterion. As a first analysis we test the convergence of the algorithm when minimizing the following cost functionals on the overlapping region: $J = J_{L^2}$, $J = J_{H_0^1}$, $J = J_{H^1}$. We remark that all the cost functionals are minimized without the presence of the penalization term (1.24).

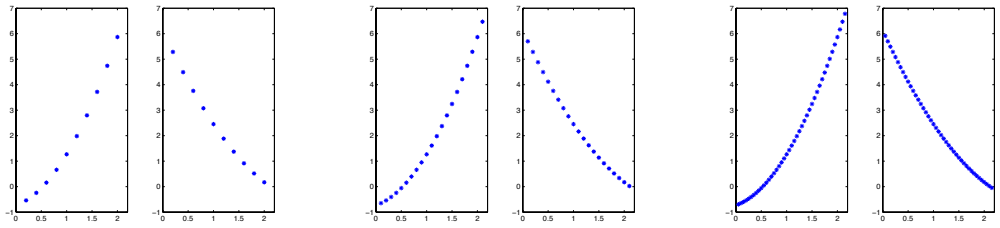
We want to test that the numerical solution of the coupled virtual control method converges as predicted by the finite elements estimates:

$$\|\varphi - \varphi_h^G\|_{H^1(\Omega)} \leq C h^r |\varphi|_{H^{r+1}(\Omega)}, \tag{3.4}$$

$$\|\varphi - \varphi_h^G\|_{L^2(\Omega)} \leq C h^{r+1} |\varphi|_{H^{r+1}(\Omega)}, \tag{3.5}$$

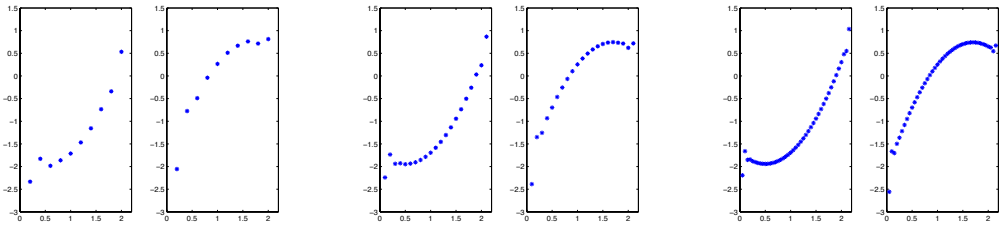
where h denotes the level of the mesh refinement and r is the degree of the polynomials used to approximate the discrete finite elements solution.
Moreover we check that the errors between the numerical and the exact solution in the $L^2(\Omega)$ and in the $H^1(\Omega)$ norm are the same that one obtains solving the global elliptic problem on Ω:

$$\|\varphi - \varphi_h^G\|_{H^1(\Omega)} = \|\varphi - \varphi_h\|_{H^1(\Omega)},$$
$$\|\varphi - \varphi_h^G\|_{L^2(\Omega)} = \|\varphi - \varphi_h\|_{L^2(\Omega)}$$



(a) $h = 0.2$ (b) $h = 0.1$ (c) $h = 0.05$

Figure 3.2: *The Dirichlet virtual controls at the final iteration, for the three levels of mesh refinement.*



(a) $h = 0.2$ (b) $h = 0.1$ (c) $h = 0.05$

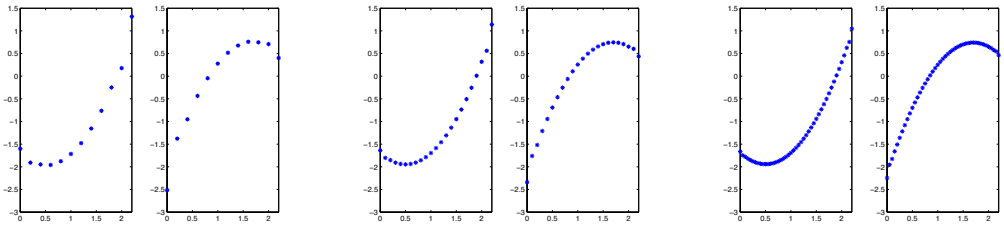
Figure 3.3: *The Neumann virtual controls at the final iteration, for the three levels of mesh refinement.*

The algorithm was tested on a regular mesh with an overlap defined by $\delta = 0.3$. The domain is represented in Figure 3.1 and we applied Dirichlet boundary conditions on Γ_i^V, $i = 1, 2$ and Neumann boundary conditions on Γ_i^H, $i = 1, 2$. The boundary conditions and the forcing term are imposed in order to obtain the numerical approximation of the following cubic exact solution:

$$\varphi^{ex} = x(1-x)(y-1) + \frac{1}{3}(y-1)^3 + 2x. \tag{3.6}$$

We interpolated the numerical solution with both linear and quadratic finite elements, and in correspondence of three levels of mesh refinement defined by $h = 0.2$, $h = 0.1$ and $h = 0.05$. We remark that in this test the boundary conditions are imposed so that $(\partial\Omega_{12} \cap \Gamma^N) \neq \emptyset$. As shown in section 1.2.2, this choice guarantees the well posedness of the optimization problem with all the choices for the cost functional (1.20)-(1.22).

In figure (4.3) and in figure (3.3) we report the virtual controls (λ_1, λ_2) obtained by our minimization algorithm in the case of linear finite elements. The first group of images shows the Dirichlet boundary controls while the second group the Neumann boundary controls.The solutions are represented for the three levels of mesh refinement. For each choice of the parameter h the figure on the left represents the solution λ_1 on the interface Γ_1 $(y = -\delta)$ and the figure on the right represents the solution λ_2 on the interface Γ_2 $(y = \delta)$. We observe that with Dirichlet boundary controls the exact



(a) $h = 0.2$ (b) $h = 0.1$ (c) $h = 0.05$

Figure 3.4: *The Neumann virtual controls at the final iteration, for the three levels of mesh refinement, when Neumann boundary conditions are applied on Γ_1^V and Γ_2^V.*

solutions on the interfaces

$$\varphi|_{\Gamma_1} = x(1-x)(-\delta-1) + \frac{1}{3}(-\delta-1)^3 + 2x,$$
$$\varphi|_{\Gamma_2} = x(1-x)(\delta-1) + \frac{1}{3}(\delta-1)^3 + 2x,$$

with $\delta = 0.3$, are recovered without spurious oscillations. In the case of Neumann boundary, there is a small deviation from the exact conditions on the control boundaries

$$(\nabla\varphi \cdot \mathbf{n}_1)|_{\Gamma_1} = -x^2 + x + (-\delta-1)^2,$$
$$(\nabla\varphi \cdot \mathbf{n}_2)|_{\Gamma_2} = x^2 - x - (\delta-1)^2,$$

with $\delta = 0.3$, close to the edges of the interface. We should recall that Dirichlet boundary conditions are applied on Γ_1^V and on Γ_2^V: the oscillations are due to the fact that a mixed boundary condition is applied to the subdomains and the controls should manage this discontinuity. In fact, as we can see from Figure 3.4, when a Neumann boundary condition is applied to the sides of the overlap the controls converge smoothly to the exact solution. We remark that in both cases the oscillations do not affect the errors between the numerical and the exact solution.

In figure (3.5) we report the solution of the state problems: φ_h^1 and φ_h^2. The solutions are calculated on Ω_1 and Ω_2 respectively and then extended to zero on the remaining part of the global domain Ω. In Figure 3.6 we represent the global solution φ_h^G, constructed as in (3.1). Moreover, we report in Figure 3.7 the convergence history of the three cost functionals (in logarithmic scale) that minimize the difference between the solutions in the $L^2(\Omega_{12})$ norm, the $H_0^1(\Omega_{12})$ and the $H^1(\Omega_{12})$ norm, at each iteration of the *BiCGStab* algorithm. The graphs are represented in the case of Dirichlet and Neumann boundary controls. In the case of J_{L^2}, with both choices of boundary controls (Dirichlet and Neumann) the convergence is non monotonic and we observe that in the first iterations (before the $10th$ iteration) the cost functional reaches a value lower than 10^{-6}, and by the time that the algorithm has achieved convergence the cost functional reaches a value of the order of 10^{-18} in the case of Dirichlet boundary controls and of the order of 10^{-15} in the case of Neumann boundary controls. In the case of $J_{H_0^1}$ we observe a smoother convergence and a reduced number of iterations needed to achieve convergence (especially in the Dirichlet case). Moreover, in the Dirichlet case, the cost functional reaches a value of the order of 10^{-23} and, in the case of Neumann boundary controls, a value of 10^{-20}, when convergence is achieved. The case corresponding to J_{H^1} is similar

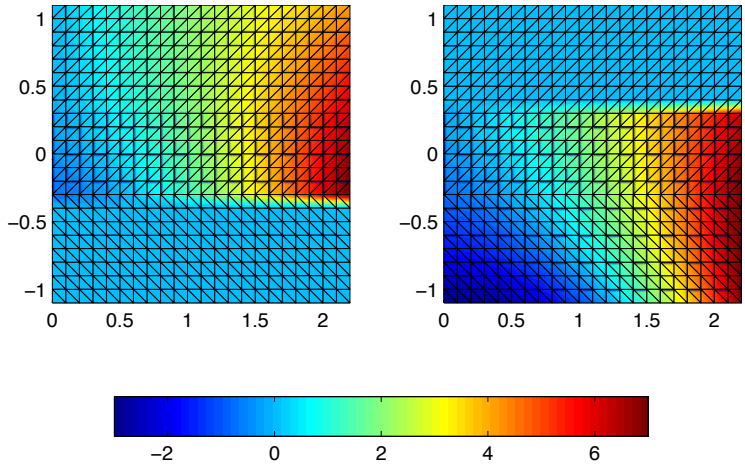


Figure 3.5: *The state solutions φ^1 and φ^2 on Ω_1 and Ω_2, for $h = 0.1$.*

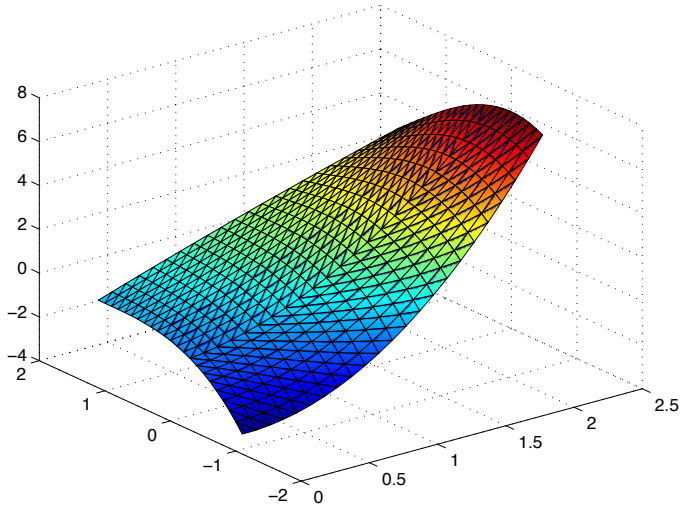


Figure 3.6: *The global state solution φ on Ω, for $h = 0.1$.*

	h	$\#D$	$\#N$	$\|\varphi_h^G - \varphi\|_{H^1}$	$\|\varphi_h^G - \varphi^{ex}\|_{L^2}$	$\|\varphi_h - \varphi^{ex}\|_{H^1}$	$\|\varphi_h - \varphi^{ex}\|_{L^2}$
J_{L^2}	0.2	17	27	0.6677	0.0236	0.6677	0.0236
	0.1	34	72	0.3345	0.0059	0.3345	0.0059
	0.05	42	165	0.1673	0.0015	0.1673	0.0015
$J_{H_0^1}$	0.2	5	14	0.6677	0.0236	0.6677	0.0236
	0.1	11	24	0.3345	0.0059	0.3345	0.0059
	0.05	26	36	0.1673	0.0015	0.1673	0.0015
J_{H^1}	0.2	10	14	0.6677	0.0236	0.6677	0.0236
	0.1	20	26	0.3345	0.0059	0.3345	0.0059
	0.05	35	37	0.1673	0.0015	0.1673	0.0015

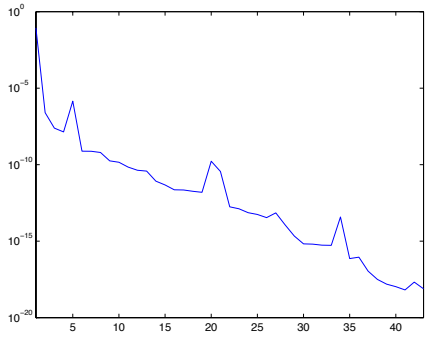
Table 3.1: *Convergence with $\mathbb{P}_1$ elements* ($tol = 10^{-10}, \delta = 0.3$).

	h	$\#D$	$\#N$	$\|\varphi_h^G - \varphi\|_{H^1}$	$\|\varphi_h^G - \varphi^{ex}\|_{L^2}$	$\|\varphi_h - \varphi^{ex}\|_{H^1}$	$\|\varphi_h - \varphi^{ex}\|_{L^2}$
J_{L^2}	0.2	35	82	0.0150	$3.4558e-04$	0.0150	$3.4558e-04$
	0.1	58	179	0.0038	$4.3508e-05$	0.0038	$4.3508e-05$
	0.05	70	397	$9.4476e-04$	$5.4581e-06$	$9.4476e-04$	$5.4581e-06$
$J_{H_0^1}$	0.2	13	24	0.0150	$3.4558e-04$	0.0150	$3.4558e-04$
	0.1	27	35	0.0038	$4.3508e-05$	0.0038	$4.3508e-05$
	0.05	44	46	$9.4476e-04$	$5.4581e-06$	$9.4476e-04$	$5.4581e-06$
J_{H^1}	0.2	21	26	0.0150	$3.4558e-04$	0.0150	$3.4558e-04$
	0.1	35	35	0.0038	$4.3508e-05$	0.0038	$4.3508e-05$
	0.05	59	47	$9.4476e-04$	$5.4581e-06$	$9.4476e-04$	$5.4581e-06$

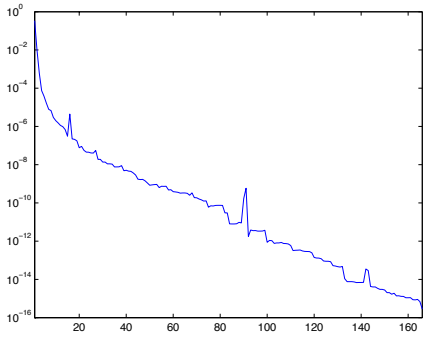
Table 3.2: *Convergence with $\mathbb{P}_2$ elements* ($tol = 10^{-10}, \delta = 0.3$).

to the latter case: in the case of Neumann controls we have almost the same number of iterations and a very similar convergence history, in the case of Dirichlet boundary controls the convergence is less smooth and a higher number of iterations is required.

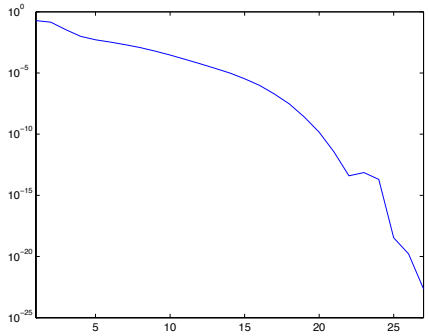
In Table 3.1 and in Table 3.2 we report the convergence history of the algorithm with $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements respectively. The results concern the cost functionals $J = J_{L^2}$, $J = J_{H_0^1}$, $J = J_{H^1}$ and both the choices of the space for the controls, Λ^D and Λ^N. The column marked as $\#D$ indicates the number of iterations needed to achieve convergence in the case of Dirichlet controls; analogously the column marked as $\#N$ indicates the number of iterations needed to achieve convergence in the case of Neumann controls. We report the errors obtained by the virtual control iterative algorithm and the ones obtained solving the global problem (3.3). In each case we verified that the errors obtained between the solution of the iterative virtual control algorithm and the exact solution and the errors between the finite elements solution and the exact solution are the same. The convergence orders are the ones predicted by the finite elements theory. The number of iterations needed to achieve convergence is strongly dependent on the level of the mesh refinement (we remark that these results are obtained without preconditioning) and it is highest in the case of J_{L^2} with Neumann controls (both with linear and quadratic finite elements). In general the number of iterations needed to achieve convergence is higher when controlling the L^2 norm of the difference between the solutions. In the remaining cases the number of iterations is lower and not dependent on the choice of the space for the control and of the choice between the H_0^1 and the H^1 norms.



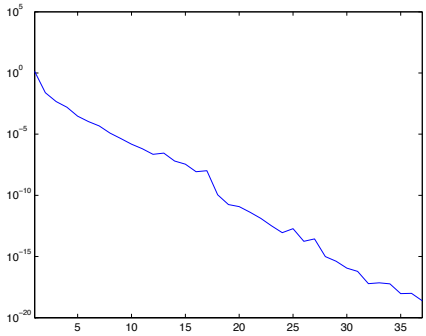
(a) J_{L^2}, *Dirichlet controls*.



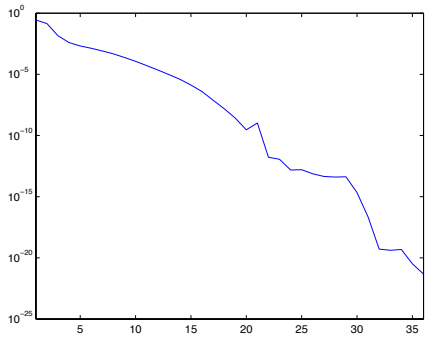
(b) J_{L^2}, *Neumann controls*.



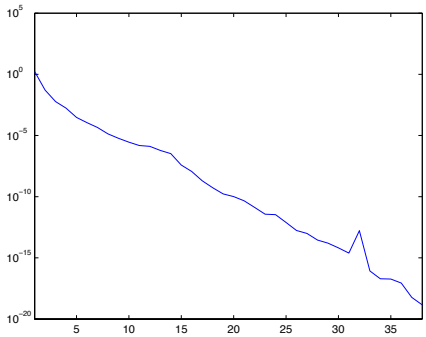
(c) $J_{H_0^1}$, *Dirichlet controls*.



(d) $J_{H_0^1}$, *Neumann controls*.



(e) J_{H^1}, *Dirichlet controls*.



(f) J_{H^1}, *Neumann controls*.

Figure 3.7: *Convergence history of the different cost functionals using the BiCGStab algorithm ($h = 0.05$, $\delta = 0.3$, $\mathbb{P}_1$ finite elements).*

On the Coercivity of the Cost Functionals

We now want to verify the theoretical results derived in section 1.2.2 about the coercivity of the cost functionals (1.20)-(1.23), when there are no Dirichlet boundary conditions imposed on $\partial\Omega_{12} \cap \partial\Omega$. For this reason, we now apply Dirichlet boundary conditions on Γ_i^H, $i = 1, 2$ and Neumann boundary conditions on Γ_i^V, $i = 1, 2$ (refer to Figure 3.1 for the notation of the boundaries). In particular we want to validate the following results:

- the choice of the cost functionals J_{L^2} and J_{H^1} always guarantees the well posedness of the optimization problem;
- the choice of the cost functional $J_{H_0^1}$ does not guarantee the uniqueness of the solution, since it only defines a semi-norm for the difference between the states on Ω_{12};
- penalizing $J_{H_0^1}$ as in $\widehat{J_{H_0^1}}$ one obtains a coercive problem;
- when solving the coupled problem defined by (1.69) the optimality system is well posed with all of the choices for the cost functional that were previously described, thanks to the coercivity of the bilinear forms related to the state problems.

One way to make sure that these results are true, is to impose the boundary conditions and the forcing terms of the state problems so that one approximates an exact solution whose degree is of the same order of the the finite elements used to interpolate it. If the uniqueness of the solution is guaranteed, the errors between the numerical and the exact solution will be very close to zero. For this reason we want to recover the following exact linear solution

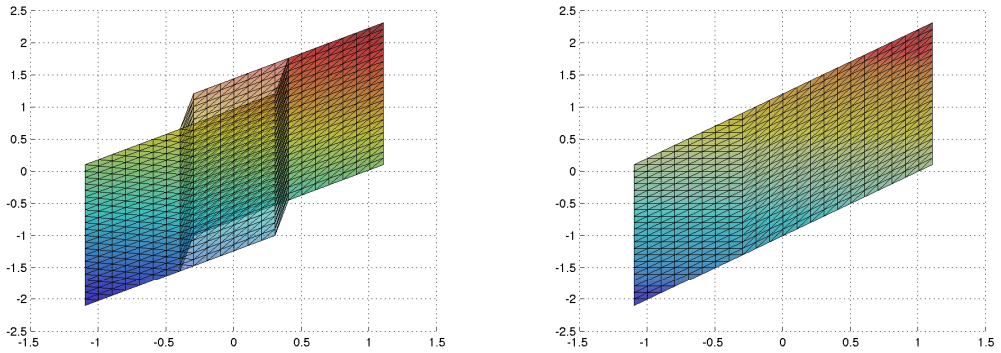
$$\varphi_1^{ex} = (y - 1) + x, \tag{3.7}$$

when using $\mathbb{P}_1$ finite elements, and the following exact quadratic solution

$$\varphi_2^{ex} = (1 - x)(y - 1) + x, \tag{3.8}$$

when using $\mathbb{P}_2$ finite elements.

In Table 3.3 we report the results concerning the convergence to the linear and of the quadratic exact solutions φ_1^{ex} and φ_2^{ex} with linear and quadratic finite elements, respectively. In the first column we report the choice of the cost functional that is being minimized; then we wrote the number of iterations needed to achieve convergence, the order of the errors obtained between the numerical and the exact solution and the values reached by the cost functional in both the cases of Dirichlet and Neumann boundary controls. We remark that the tolerance on the residual imposed as a stopping criterion was lowered to a value of 10^{-12}; this is because the numerical solution has to be approximated very strictly in order to be able to recover such small errors (the error is not dominated by the error on the finite elements, when interpolating a solution of the same order of the finite elements).
We observe that when a full Neumann boundary condition is applied to $\partial\Omega_{12} \cap \partial\Omega$ the algorithm based on the minimization of the gradient (functional (1.21)) is not able to recover the exact solution (highlighted results). This is because no data is fixed on the boundary of the overlap. The solution exists (i.e. the functional is minimized, which means that $\nabla\varphi_1 = \nabla\varphi_2$ q.o. in Ω_{12}) but it is not unique. The point is that if no Dirichlet data is applied on $\partial\Omega \cap \partial\Omega_{12}$ the difference of the two solutions on the overlap $(\varphi_1 - \varphi_2)|_{\Omega_{12}}$ does not belong to $H^1_{\Gamma_D}(\Omega_{12})$ but only to $H^1(\Omega_{12})$ so



(a) $J_{H_0^1}$. (b) $\widehat{J_{H_0^1}}$.

Figure 3.8: *The linear solution obtained minimizing a non coercive and a coercive cost functional, in the case* $\partial\Omega_{12} \cap \Gamma^D = \emptyset$.

functional (1.21) only represents a seminorm for the space and its minimization does not guarantee that $\varphi_1 = \varphi_2$ almost everywhere in Ω_{12}.
On the contrary, when minimizing the difference between the states in the full $H^1(\Omega_{12})$ norm (J_{H^1}) or in the augmented semi-norm ($\widehat{J_{H_0^1}}$), we observe that the quadratic and the linear solutions are interpolated exactly.
In Figure 3.8 we report the solutions of the state problems in the case of the minimization of the semi-norm ($J_{H_0^1}$) and of the augmented semi-norm ($\widehat{J_{H_0^1}}$), when a Dirichlet condition is applied only on parts of the boundaries of the domains $\partial\Omega_1$ and $\partial\Omega_2$ that do not belong to the boundary of the overlap $\partial\Omega_{12}$. One can see that the gradients of the solutions are the same on the overlapping region, and this is confirmed by the results stated in Table 3.3 that show that the value of the cost functional is very close to zero. When the cost functional is penalized with the addition of a term depending on the difference between the solutions on $\partial\Omega_{12} \cap \partial\Omega$ the exact solution is recovered.

The Scalar Elliptic Problem

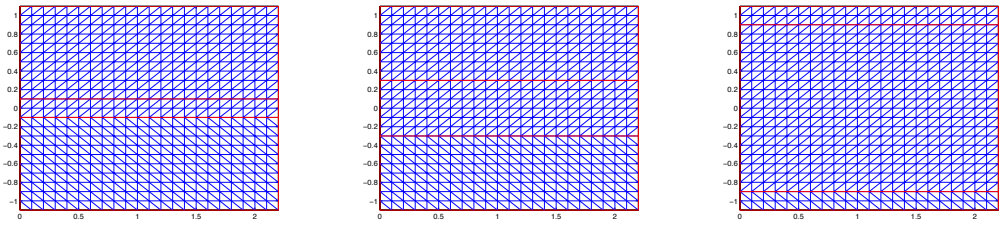
We now test the results concerning the scalar elliptic problem (1.69). Again, we applied Dirichlet boundary conditions on $\Gamma_i^H, i = 1, 2$ and Neumann boundary conditions on $\Gamma_i^V, i = 1, 2$ so that $\partial\Omega_{12} \cap \Gamma^D = \emptyset$. We want to verify that now the exact solution is recovered even with the functional minimizing the difference between the gradients of the states: $J_{H_0^1}$. In Table 3.4 we report the numerical results associated to this problem. In this case we consider exclusively the minimization of the cost functional (1.21). The first part of the table concerns the approximation of the linear and quadratic exact solutions (φ_1^{ex} and φ_2^{ex}) with linear and quadratic finite elements respectively, in the second part of the table we report the error when the exact solution is the cubic function (φ^{ex}) for both choices of finite elements. As predicted by the theory the algorithm converges to the exact solution in this case.

	J	$\#D$	$error \sim$	$J \sim$	$\#N$	$error \sim$	$J \sim$
$\mathbb{P}_1$, φ_1^{ex}	J_{L^2}	68	10^{-11}	10^{-25}	250	10^{-10}	10^{-21}
	$J_{H_0^1}$	17	1	10^{-16}	210	10	10^{-14}
	J_{H^1}	45	10^{-12}	10^{-25}	35	10^{-12}	10^{-23}
	$\widehat{J_{H_0^1}}$	40	10^{-12}	10^{-26}	24	10^{-12}	10^{-23}
$\mathbb{P}_2$, φ_2^{ex}	J_{L^2}	84	10^{-12}	10^{-24}	559	10^{-10}	10^{-20}
	$J_{H_0^1}$	51	1	10^{-17}	63	1	10^{-16}
	J_{H^1}	99	10^{-12}	10^{-24}	53	10^{-12}	10^{-23}
	$\widehat{J_{H_0^1}}$	142	10^{-12}	10^{-24}	60	10^{-12}	10^{-23}

Table 3.3: $\partial\Omega_{12} \cap \Gamma^D = \emptyset$, $h = 0.1$, $tol = 10^{-12}$, $\delta = 0.3$.

	h	$\#D$	$\#N$	$\|\varphi_h^G - \varphi^{ex}\|_{H^1}$	$\|\varphi_h^G - \varphi^{ex}\|_{L^2}$	$\|\varphi_h - \varphi^{ex}\|_{H^1}$	$\|\varphi_h - \varphi^{ex}\|_{L^2}$
$\mathbb{P}_1$, φ_1^{ex}	0.2	30	22	0.0502	0.0051	0.0502	0.0051
	0.1	49	31	0.0246	0.0013	0.0246	0.0013
	0.05	73	43	0.0123	$3.2259e-04$	0.0123	$3.2259e-04$
$\mathbb{P}_2$, φ_2^{ex}	0.2	78	35	$1.2969e-04$	$5.2796e-06$	$1.2969e-04$	$5.2796e-06$
	0.1	112	51	$1.6456e-05$	$3.3203e-07$	$1.6456e-05$	$3.3203e-07$
	0.05	125	63	$2.0715e-06$	$2.0812e-08$	$2.0715e-06$	$2.0812e-08$
$\mathbb{P}_1$, φ^{ex}	0.2	29	22	0.6454	0.0197	0.6454	0.0197
	0.1	53	33	0.3226	0.0049	0.3226	0.0049
	0.05	82	42	0.1613	0.0012	0.1613	0.0012
$\mathbb{P}_2$, φ^{ex}	0.2	59	34	0.0152	$3.5060e-04$	0.0152	$3.5060e-04$
	0.1	114	46	0.0038	$4.3826e-05$	0.0038	$4.3826e-05$
	0.05	225	59	$9.4784e-04$	$5.4782e-06$	$9.4784e-04$	$5.4782e-06$

Table 3.4: Convergence minimizing $J_{H_0^1}$:$\partial\Omega_{12} \cap \Gamma^D = \emptyset$ *and coercive bilinear form* ($\alpha = 1$), $tol = 10^{-12}$, $\delta = 0.3$.



(a) $\delta = 0.1$. (b) $\delta = 0.3$. (c) $\delta = 0.9$.

Figure 3.9: *The mesh on the overlapping domains for* $\delta = 0.1$, $\delta = 0.3$ *and* $\delta = 0.9$.

On the Dependence on the Size of the Overlapping Region

As a further analysis on the behavior of the method, we varied the size of the overlapping region Ω_{12} between the two domains. It is important to compare the results on domains with different sizes of the overlap (different δ, see Figure 3.1) but with matching discretization. To this aim we implemented a converter from a mesh file in the `.mesh` format (generated using GMSH) to the *Matlab* format. This approach permits a greater flexibility in the generation of the mesh. In Figure 3.9 we report three meshes with different size of the overlapping region. We remark that for $\delta = 0.1$ the two domains only have two layers of triangles in common, while in the case corresponding to $\delta = 0.9$ the overlap is almost as big as the whole domain.

In Table 3.5 we report the results related to the algorithm tested on the meshes described in Figure 3.9. We report the number of iterations needed to achieve convergence for the different choices of the cost functional, of the mesh refinement and of the boundary controls: Dirichlet or Neumann. In every case that we have considered the algorithm has converged and we remark that, for every choice of the cost functional, and for every size of the overlapping region, the number of iterations needed to achieve convergence increases when refining the grid. For increasing sizes of the overlapping region, for a fixed value of h, we remark that

- when minimizing J_{L^2} with Dirichlet controls the number of iterations decreases when using $\mathbb{P}_1$ finite elements and is more or less stable when using $\mathbb{P}_2$ finite elements;
- when minimizing J_{L^2} with Neumann controls the number of iterations needed to achieve convergence seems to be increasing, with both the choices of $\mathbb{P}_1$ and $\mathbb{P}_2$ finite elements;
- the minimization of the cost functionals $J_{H_0^1}$ and J_{H^1} is independent of the size of the overlapping region.

In order to further investigate these results we have generated a mesh where the size of the overlap reduces to the grid size (i.e. $\delta = h$). The corresponding results are listed in Table 3.6. We observe that when considering J_{L^2} the number of iterations needed to achieve convergence increases significantly when refining the mesh and consequently the size of the overlap, especially if $\mathbb{P}_2$ elements are used. In the case of $J_{H_0^1}$ and J_{H^1} the behavior is analogous but the number of iterations, when reducing h, increases in a very small amount.

These results need further investigation: in this work we limit ourselves to their description.

			D			N		
			$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.2$	$h = 0.1$	$h = 0.05$
$\mathbb{P}_1$	$\delta = 0.2$	J_{L^2}	12	54	138	27	55	130
		$J_{H_0^1}$	5	12	27	12	20	33
		J_{H^1}	10	23	39	14	21	31
	$\delta = 0.6$	J_{L^2}	17	34	42	27	72	165
		$J_{H_0^1}$	5	11	26	14	24	36
		J_{H^1}	10	20	35	14	26	37
	$\delta = 1.8$	J_{L^2}	11	31	42	24	66	201
		$J_{H_0^1}$	5	11	25	14	24	30
		J_{H^1}	10	22	34	14	25	36
$\mathbb{P}_2$	$\delta = 0.2$	J_{L^2}	28	56	74	68	146	330
		$J_{H_0^1}$	14	28	50	24	31	43
		J_{H^1}	23	40	66	23	32	45
	$\delta = 0.6$	J_{L^2}	35	58	70	82	179	397
		$J_{H_0^1}$	13	27	44	24	35	46
		J_{H^1}	21	35	59	26	35	47
	$\delta = 1.8$	J_{L^2}	35	54	78	92	217	437
		$J_{H_0^1}$	13	27	43	24	37	45
		J_{H^1}	20	34	60	27	35	47

Table 3.5: *Number of iterations* ($tol = 10^{-10}$).

		D			N		
		$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.2$	$h = 0.1$	$h = 0.05$
$\mathbb{P}_1, \quad \delta = h$	J_{L^2}	12	25	42	25	57	112
	$J_{H_0^1}$	5	14	31	12	20	23
	J_{H^1}	10	26	57	14	21	24
$\mathbb{P}_2, \quad \delta = h$	J_{L^2}	28	313	615	75	159	283
	$J_{H_0^1}$	14	30	62	22	29	33
	J_{H^1}	23	47	90	25	32	34

Table 3.6: *Number of iterations* ($tol = 10^{-10}$).

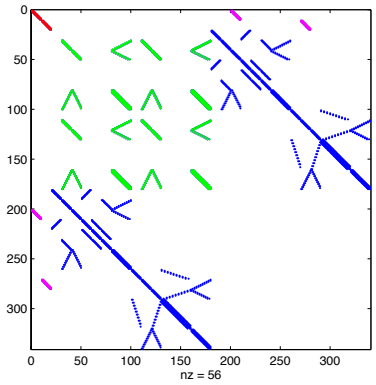


Figure 3.10: *Pattern of the global matrix,* $h = 0.2$, $\delta = 0.3$.

3.1.2 Preconditioning

In this section we report the results obtained with the various preconditioning approaches presented in section 2.3. We want to test the efficiency of the different choices that have been implemented, as well as the advantages and disadvantages of the various techniques. We are dealing with the problem of finding an appropriate preconditioner for the saddle point linear system (2.20). A representation of the structure of the global matrix is helpful to understand the characteristics of $\mathcal{A}$: in Figure 3.10 we report the pattern of the matrix. This matrix is symmetric and is of saddle point structure. The central block (the M_{12} matrix in (2.20), colored in green) represents the coupling term of the problem. We observe that this matrix is positive semi-definite: only the rows and columns corresponding to the nodes on the overlapping region are different from zero. The (red) block on the upper-left part of the matrix represents the penalization term that is added for the regularization of the cost functional. When the penalization coefficient β is equal to zero the ill-posedness of the matrix increases.

The analysis has been carried on for values of β ranging from 0 to 1. The case corresponding to $\beta = 0$ represents the non penalized case in which the consistency of the original problem is preserved. The opposite situation, $\beta = 1$, is implemented in order to study the behavior of the limit case of a highly penalized cost functional. This choice is never recommended in practice because this term implies a modification of the original problem and one looses the possibility to recover the exact solution.

The domain of the problem is again the one represented in Figure 3.1, with overlap defined by $\delta = 0.3$ and with the Dirichlet boundary conditions applied on the vertical sides of the domain. All the tests have been carried out with $\mathbb{P}_1$ finite elements and in all the approaches we have fixed a tolerance of 10^{-10}. We are considering Neumann boundary controls and the minimization of the cost functional J_{L^2}.
We study the convergence of the numerical solution of the coupled virtual control problem to the exact cubic solution (3.6) introduced in the previous section. In Table 3.7 we report some preliminary results on the resolution of the non preconditioned problem. We report the number of iterations

β			#	$J \sim$	errors $\|\varphi_h^G - \varphi^{ex}\|_{H^1}$	errors $\|\varphi_h^G - \varphi^{ex}\|_{L^2}$
10^{-8}	MINRES	$h = 0.2$	2240	10^{-8}	0.6677	0.0236
		$h = 0.1$	17376	10^{-8}	0.3345	0.0059
		$h = 0.05$	85816	10^{-8}	0.1673	0.0015
	BiCGStab	$h = 0.2$	27	10^{-8}	0.6677	0.0236
		$h = 0.1$	73	10^{-8}	0.3345	0.0059
		$h = 0.05$	158	10^{-8}	0.1673	0.0015
10^{-4}	MINRES	$h = 0.2$	2114	10^{-4}	0.6671	0.0237
		$h = 0.1$	9108	10^{-4}	0.3412	0.0069
		$h = 0.05$	28397	10^{-4}	0.2141	0.0040
	BiCGStab	$h = 0.2$	23	10^{-4}	0.6671	0.0237
		$h = 0.1$	28	10^{-4}	0.3412	0.0069
		$h = 0.05$	29	10^{-4}	0.2141	0.0040
1	MINRES	$h = 0.2$	492	0.2707	7.2823	0.7743
		$h = 0.1$	1744	0.2689	13.0570	0.7475
		$h = 0.05$	6027	0.2684	24.6345	0.7329
	BiCGStab	$h = 0.2$	3	0.2707	7.2823	0.7743
		$h = 0.1$	3	0.2689	13.0570	0.7475
		$h = 0.05$	3	0.2684	24.6345	0.7329

Table 3.7: *Convergence history without preconditioning: MINRES on the global system and BiCGStab to solve the extremality equation.*

β		$\tilde{\mathcal{P}}_d$	$\hat{\mathcal{P}}_d$
10^{-8}	$h = 0.2$	126	141
	$h = 0.1$	209	191
	$h = 0.05$	257	226
10^{-4}	$h = 0.2$	208	246
	$h = 0.1$	417	476
	$h = 0.05$	572	976
1	$h = 0.2$	28	34
	$h = 0.1$	30	33
	$h = 0.05$	38	33

Table 3.8: *Convergence history: Preconditioned MINRES with $\tilde{P}_{12}$ and $\hat{P}_{12}$.*

needed for the resolution of the global linear system (2.20) to the number of iterations needed to solve the extremality equation (2.39). The first linear system, being symmetric indefinite, is solved with MINRES, the second linear system being non symmetric is solved by the use of BiCGStab. With these results we want to show the consistency of the two approaches: in both cases the cost functional is minimized and its lower bound is dominated by the presence of the penalization term. The errors between the numerical and the exact solution are the same in both cases and we observe that with $\beta = 10^{-8}$ the errors $\|\varphi_h^G - \varphi^{ex}\|_{H^1}$ and $\|\varphi_h^G - \varphi^{ex}\|_{L^2}$ are equal to the ones recovered by the algorithm in the case of absence of penalization with a precision of 10^{-4} (see Table 3.1). This means that for $\beta \sim 10^{-8}$ the variation from the original problem caused by the presence of the penalization term is negligible if one is satisfied by this level of accuracy. Moreover we observe that, in both the approaches, the number of iterations that are needed to achieve convergence lowers significantly as the penalization coefficient is augmented. This behavior is particularly evident in the case of the extremality equation and can be explained through the results obtained in section 2.3 on the equivalence between the resolution of the extremality equation and the Schur complement equation. In fact we have shown that for high values of the penalization term the iterations needed to solve the extremality equation are equivalent to the iterations of a preconditioned Richardson algorithm.

A Block Diagonal Preconditioner

In this section we describe the results obtained with the block diagonal preconditioners presented in section 2.3.2. We are solving the linear system (2.20) with MINRES and we precondition the algorithm with the matrices $\tilde{\mathcal{P}}_d$ and $\hat{\mathcal{P}}_d$. When considering $\tilde{\mathcal{P}}_d$ we set the coefficient $\alpha = 0.1$ in order to deal with a positive definite preconditioner. The results obtained with this approach are presented in Table 3.8. The results have to be compared with the ones obtained with the non preconditioned MINRES algorithm presented in Table 3.7. The number of iterations needed to achieve convergence is significantly reduced when preconditioning. For $\beta = 10^{-8}$ the preconditioners behave in a similar way, the method requires a lower number of iterations to converge with respect to the non preconditioned case but it is still dependent on the level of the mesh refinement. For $\beta = 10^{-4}$ the number of iterations is higher with respect to what happens in less penalized case of

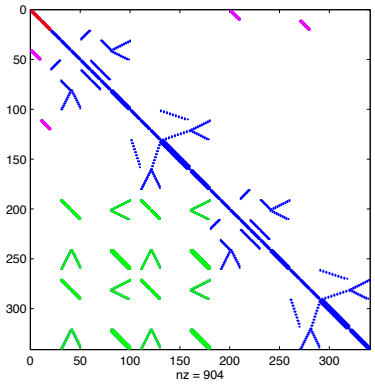


Figure 3.11: *Pattern of the global reordered matrix,* $h = 0.2$, $\delta = 0.6$.

$\beta = 10^{-8}$, this aspect still needs further investigation. For a highly penalized problem, $\beta = 1$, we observe that the choice of $\hat{\mathcal{P}}_d$ is more efficient and the method achieves convergence independently of the level of the mesh refinement. We have recovered the behavior of the non preconditioned BiCGStab for the resolution of the optimality equation. Preconditioning the problem with $\tilde{\mathcal{P}}_d$ the behavior is similar but there is still some dependency on the level of discretization.

We can conclude that this choice for preconditioning the linear system (2.20) is not optimal. First of all, we are interested in finding an effective preconditioner for small values of the penalization coefficient β because we want to solve a problem that preserves the consistency with the original one; and we have shown that this approach is effective only for a highly penalized problem. Moreover the costs for the resolution of a linear system for the preconditioners is so high that this option is not successfully implementable in practice.

Gauss-Seidel and Jacobi Preconditioners

In this section we solve the linear system related to the optimization problem presented in section 2.3.3. The linear system (2.25) corresponds to a reordering of the rows of (2.20) so it is equivalent to the latter although it presents some peculiarities. In fact the matrix in (2.25) is not symmetric but presents a positive definite $(1, 1)$ block. The pattern of this matrix is represented in Figure 3.11, where the colored block represent the corresponding blocks in Figure 3.10. Because of the lack of symmetry the problem is solved with GMRES.
The first observation that arises from an examination of Table 3.9 regards the non preconditioned case. In fact, we observe that the number of iterations needed to achieve convergence in the case of the resolution of the non symmetric linear system with GMRES is lower than the number of iterations needed to achieve convergence when solving the symmetric linear system with MINRES (results presented in Table 3.7). This means that without preconditioning, it is better to deal with a non symmetric problem whose diagonal blocks are all positive definite. The results concerning the preconditioned problem are also reported in Table 3.9. We observe that in presence of preconditioning the number of iterations needed to achieve convergence is significantly reduced, but with neither

β		$\mathcal{P} = I$	$\mathcal{P} = \mathcal{P}_{GS}$	$\mathcal{P} = \mathcal{P}_J$
10^{-8}	$h = 0.2$	294	21	41
	$h = 0.1$	784	42	78
	$h = 0.05$	1768	70	123
10^{-4}	$h = 0.2$	284	21	39
	$h = 0.1$	634	26	50
	$h = 0.05$	1205	28	50
1	$h = 0.2$	99	5	9
	$h = 0.1$	182	5	9
	$h = 0.05$	332	5	9

Table 3.9: *Converge history for $\tilde{\mathcal{A}}$: GMRES and preconditioned GMRES with $\mathcal{P}_{GS}$ and $\mathcal{P}_J$.*

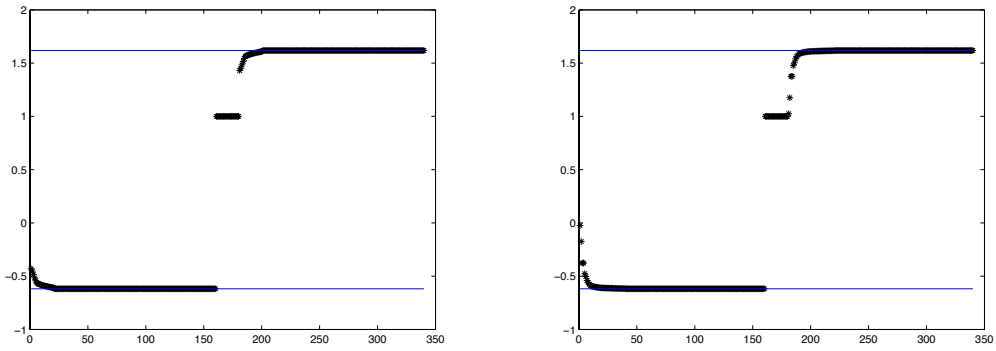
	n	m	p	$n - m$	$2p$	$2(m - p)$
$\beta \neq 0$	180	160	120	20	240	80
$\beta = 0$	180	160	140	20	280	40

Table 3.10: *Dimension of the problem, $h = 0.2$.*

preconditioner we obtain a number of iterations independent of the mesh size. The *Gauss-Seidel* preconditioner performs better than the *Jacobi* preconditioner, even if the two matrices only differ for the presence of the E block, related to the lifting of the Neumann boundary conditions. We observe that when $\beta = 1$ the number of iterations (with both preconditioners) is low and independent of the level of the mesh refinement. It is hard to make a comparison between this case and the the case of the block diagonal preconditioner introduced in the previous section, because of the difference that already exists between the non preconditioned approaches. Both the block diagonal preconditoner presented in section 2.3.2 and the *Gauss-Seidel* and *Jacobi* preconditioner presented in this section are very costly to assemble and do not perform well in the case of low penalization. The conclusion is that these approaches are far from optimal in the particular case of a linear system deriving from a coupled optimization problem.

A Preconditioner for Highly Singular $(1, 1)$ Block

We now concentrate on the case of the preconditioning approach presented in section 2.3.4. As previously shown, these preconditioners are tailored for the case of symmetric saddle point matrices with a highly singular $(1, 1)$ block. For this reason we test them on the optimality linear system in the original form (2.20).
We first concentrate on the results concerning the first preconditioner of this kind, $\mathcal{P}_t$, whose definition is given in (2.26). As a preliminary analysis we want to verify the theoretical bounds of the eigenvalue distribution of the preconditioned matrix. In Figure 3.12 we report these eigenvalues when $t = -1$ and $h = 0.2$ (mesh refinement), for two values of the penalization coefficient β. The numbers referring to the dimension of the problem in this particular case are shown in Table 3.10, for the notation refer to section 2.3.4. As predicted, we observe $n - m$ eigenvalues equal to 1,



(a) $\beta = 10^{-8}$.

(b) $\beta = 1$.

Figure 3.12: *Eigenvalue distribution.*

p eigenvalues equal to $\frac{1-\sqrt{5}}{2}$, p eigenvalues equal to $\frac{1+\sqrt{5}}{2}$ and the remaining are bounded in the intervals given in (2.28). Moreover we confirm that the spectral clustering is stronger and the eigenvalues are bounded away from 0 when A is more ill-conditioned (for small values of β).

In Table 3.11 we report the convergence history with the following preconditioners: $\mathcal{P}_t$ with $t = -1$ and with $\gamma = 1$, $\mathcal{P}_t$ with $t = -1$ and with the choice $1/\gamma = max(A)$ (we refer to this preconditioner as $\mathcal{P}_{-1}^{\gamma}$), and $\mathcal{P}_t$ with $t = 1$ and $t = 2$ and $\gamma = 1$. We verify the property that all the preconditioners are more effective for an ill-conditioned A; in particular, when $\beta = 0$, so that the dimension of the kernel of the matrix is augmented, we attain convergence in two iterations. Moreover we remark that the choice $\gamma = 1$ gives better results with respect to the choice in which the augmentation term is weighted in order to be in norm comparable to A, in the case of the preconditioner $\mathcal{P}_{-1}$. We remark that for high values of the penalization term the convergence is not independent of the mesh refinement but the number of iterations needed to achieve convergence is still significantly reduced with respect to the non preconditioned case. We recall that, since we are solving a virtual control problem, we are interested in a non penalized cost functional. Small values of the penalization term are acceptable as long as the modification induced on the original problem is negligible. For this reason this preconditioning approach is optimal for a matrix deriving from a coupled virtual control problem.

In Table 3.12 we make a comparison between the preconditioner $\mathcal{P}_t$, presented in [22] and the preconditioners presented in [23]. The values of the parameter t are chosen as in the cited works. $\tilde{\mathcal{P}}_{-1}$ behaves better than $\mathcal{P}_{-1}$ because it guarantees a stronger spectral clustering, as predicted by the eigenvalues bounds. Moreover we remark the following properties, in agreement with what is observed in [23]: the number of iterations with $\tilde{\mathcal{P}}_{-1}$ is smaller than the number of iterations with $\tilde{\mathcal{P}}_1$ and $\tilde{\mathcal{P}}_2$ and the number of iterations with $\hat{\mathcal{P}}_t$ is smallest when $t = 2$.
Because of the fact that we are interested in the case of very small or inexistent penalization we can conclude that the various preconditioning approaches, for the different values of the parameter t are equivalent. In fact the number of iterations needed to achieve convergence differs, in the various cases, only in correspondence of high values of the penalization term ($\beta = 0.4$ and $\beta = 1$).

β		$P = I$	$P = \mathcal{P}_{-1}$	$P = \mathcal{P}_{-1}^{\gamma}$	$P = \mathcal{P}_1$	$P = \mathcal{P}_2$
0	$h = 0.2$	338	2	2	2	2
	$h = 0.1$	–	2	2	2	2
	$h = 0.05$	–	3	2	2	3
10^{-8}	$h = 0.2$	338	4	7	5	4
	$h = 0.1$	–	5	9	6	4
	$h = 0.05$	–	6	7	7	7
10^{-4}	$h = 0.2$	337	8	15	8	8
	$h = 0.1$	–	9	35	9	9
	$h = 0.05$	–	11	103	12	13
1	$h = 0.2$	329	14	24	16	15
	$h = 0.1$	–	20	43	21	21
	$h = 0.05$	–	29	85	29	31

Table 3.11: *Convergence history: GMRES and preconditioned GMRES. – means that the method did not converge in the first* 1000 *iterations.*

β		$\tilde{\mathcal{P}}_{-1}$	$\tilde{\mathcal{P}}_1$	$\tilde{\mathcal{P}}_2$	$\hat{\mathcal{P}}_{\frac{1}{2}}$	$\hat{\mathcal{P}}_2$	$\hat{\mathcal{P}}_4$	$\hat{\mathcal{P}}_8$
0	$h = 0.2$	2	2	2	2	2	2	2
	$h = 0.1$	2	2	2	2	2	2	2
	$h = 0.05$	2	3	3	2	2	3	3
10^{-8}	$h = 0.2$	3	3	3	5	4	4	4
	$h = 0.1$	4	4	5	5	4	4	5
	$h = 0.05$	4	6	6	8	5	5	6
10^{-4}	$h = 0.2$	5	5	5	8	7	7	7
	$h = 0.1$	6	6	7	9	9	9	9
	$h = 0.05$	7	9	9	12	10	12	11
1	$h = 0.2$	8	9	9	15	14	15	15
	$h = 0.1$	11	13	12	19	17	18	20
	$h = 0.05$	15	17	18	28	24	26	27

Table 3.12: *Convergence history: preconditioned GMRES.*

β		$P = I$	$K(\Sigma)$	P_Σ	$K(P_\Sigma^{-1}\Sigma)$	$\tilde{P}$	$\tilde{P}_{12}$	$\tilde{P}_{12}^1$
	$h = 0.2$	31	10^4	13	~ 25	20	20	20
0	$h = 0.1$	84	10^5	13	~ 25	34	32	32
	$h = 0.05$	204	10^6	13	~ 25	55	52	57
	$h = 0.2$	31	10^4	13	~ 25	20	20	20
10^{-8}	$h = 0.1$	85	10^5	13	~ 25	35	32	32
	$h = 0.05$	203	10^6	13	~ 25	55	55	56
	$h = 0.2$	31	10^3	13	~ 25	16	16	16
10^{-4}	$h = 0.1$	46	10^4	13	~ 25	19	18	18
	$h = 0.05$	50	10^4	13	~ 25	21	21	19
	$h = 0.2$	12	4.25	5	1.69	5	5	5
1	$h = 0.1$	12	4.36	5	1.69	5	5	5
	$h = 0.05$	10	4.41	5	1.66	5	5	5

Table 3.13: *Shur complement equation. Convergence history: CG and preconditioned CG.*

	$\beta = 0$		$\beta = 10^{-8}$		$\beta = 10^{-4}$		$\beta = 1$	
	B	C	B	C	B	C	B	C
$h = 0.2$	27	31	27	31	23	31	3	12
$h = 0.1$	72	84	73	85	28	46	3	12
$h = 0.05$	165	204	158	203	29	50	3	10

Table 3.14: *Comparison between the resolution of the extremality equation with the BiCGStab algorithm (B) and the resolution of the Schur complement equation using the Conjugate Gradient algorithm (C), for different values of β.*

Preconditioning the Interface Equation

We test here the various preconditioning approaches for the Schur complement equation presented in section 2.3.5. Because of the symmetry and of the positive definiteness of the Schur complement matrix we solve the system by the *conjugate gradient* method. In the first column of Table 3.13 we report the number of iterations needed to achieve convergence for the non preconditioned linear system, for different values of the mesh refinement and for different values of the penalization coefficient β. We remark that for strong penalizations the rate of convergence is independent of the level of refinement of the mesh. We observe (column two) that the condition number of this matrix is independent of h for $\beta = 1$. As a matter of fact, being the interface matrix $\beta M_\Gamma + E^T(K^T)^{-1}M_{12}K^{-1}E$, when β is high the first term is dominant and, since M_Γ is a mass matrix on the control interfaces, the condition number is independent of the mesh refinement.

The block diagonal precondtioner P_Σ (2.33), is very efficient, for all the levels of penalization of the cost functional. In fact, the number of iterations needed to achieve convergence is small and independent of h, because the condition number of the preconditioned matrix is independent of h. The drawback of this preconditioner is the high computational cost needed for its assembling. For this reason we tested the following simplifications of P_Σ: $\tilde{P}$, $\tilde{P}_{12}$ and $\tilde{P}_{12}^1$ defined in (2.35), (2.36)

and (2.37) respectively. The convergence history of the various approaches is reported in Table 3.13. The approximations of P behave similarly and the results we obtain are satisfactory for values of β starting from 10^{-4}. Since the results for $\tilde{P}$, $\tilde{P}_{12}$ and $\tilde{P}^1_{12}$ are analogous, a natural choice is to choose as a preconditioner $\tilde{P}^1_{12}$, which is the less costly to assemble.

In Table 3.14 we report a comparison between the resolution of the extremality equation and of the Schur complement equation. The equivalence between the two has been proved in section 2.3.5 and here we report the number of iterations needed to achieve convergence of the two non-preconditioned algorithms. The first was used in the numerical results presented in section 3.1.1 and the second has been implemented in the contest of preconditioning. We remark that the behavior of the two is analogous. The speed of convergence increases more significantly with β in the case of the iterative algorithm that solves the extremality equation: in fact we have shown that this approach corresponds to a Richardson method preconditioned with the matrix M_Γ, which is an effective preconditioner for high values of the penalization parameter.

Chapter 4

Coupled Virtual Control for the Stokes Problem

In this chapter we want to apply the virtual control method to the Stokes system. We will formulate the problem and introduce the different cost functionals that will be minimized on the overlapping region. Then we will present the algebraic formulation of the problem and the numerical results obtained with our numerical simulations.

4.1 Coupled Virtual Control: the Stokes Problem

4.1.1 Formulation of the Problem

We want to solve on the two overlapping domains Ω_1 and Ω_2 the following Stokes problems:
for $i = 1, 2$, find $(\mathbf{u}^i, p^i) \in [H^1(\Omega_i)]^2 \times L^2(\Omega_i)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}^i + \nabla p^i &=& \mathbf{f}^i & in \quad \Omega_i \\ \nabla\cdot\mathbf{u}^i &=& 0 & in \quad \Omega_i \\ \nu\nabla\mathbf{u}^i\cdot\mathbf{n}_i - p^i\mathbf{n}_i &=& \mathbf{u}_N^i & on \quad \Gamma_i^N \\ \mathbf{u}^i &=& \mathbf{u}_D^i & on \quad \Gamma_i^D \\ \mathbf{u}^i &=& \lambda_i & on \quad \Gamma_i, \end{array}\right. \tag{4.1}$$

where, for $i = 1, 2$, $\mathbf{f}^i$, $\mathbf{u}_N^i$, $\mathbf{u}_N^i$ are suitably chosen regular data, Γ_i represent the control interfaces and Γ_i^D and Γ_i^N are suitable partitions of $\partial\Omega_i$: $\Gamma_i^D \cup \Gamma_i^N = \partial\Omega_i \backslash \Gamma_i$ and $\Gamma_i^D \cap \Gamma_i^N = \emptyset$.
We look for the control variables λ_1 and λ_2 in the following spaces

$$\Lambda_i^D = \left\{\mu \in [H^{\frac{1}{2}}(\Gamma_i)]^2 : \exists \mathbf{v} \in [H^1(\Omega_i)]^2, \mathbf{v} = \mu \ on \ \Gamma_i, \mathbf{v} = \mathbf{0} \ on \ \Gamma_i^D\right\}, \qquad i = 1, 2.$$

The virtual controls can also play the role of natural boundary conditions, in this case λ_1 and λ_2 correspond to the normal components of the Cauchy stress tensors on the interfaces, Γ_1 and Γ_2. In

that case, the state problems to be solved are of the following form:
for $i = 1, 2$, find $(\mathbf{u}^i, p^i) \in [H^1(\Omega_i)]^2 \times L^2(\Omega_i)$ so that

$$\begin{cases} -\nu\Delta\mathbf{u}^i + \nabla p^i &= \mathbf{f}^i & in \quad \Omega_i \\ \nabla \cdot \mathbf{u}^i &= 0 & in \quad \Omega_i \\ \nu\nabla\mathbf{u}^i \cdot \mathbf{n}_i - p^i\mathbf{n}_i &= \mathbf{u}_N^i & on \quad \Gamma_i^N \\ \mathbf{u}^i &= \mathbf{u}_D^i & on \quad \Gamma_i^D \\ \nu\nabla\mathbf{u}^i \cdot \mathbf{n}_i - p^i\mathbf{n}_i &= \lambda_i & on \quad \Gamma_i, \end{cases} \tag{4.2}$$

and we seek for the controls in the spaces

$$\Lambda_i^N = \Big\{\mu \in [H^{-\frac{1}{2}}(\Gamma_i)]^2 : \exists \mathbf{v} \in [H^1(\Omega_i)]^2, \exists p \in L^2(\Omega_i), (\nu\nabla\mathbf{u}^i - p^i)\mathbf{n}_i = \mu \text{ on } \Gamma_i, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_i^D, \\ (\nu\nabla\mathbf{u}^i - p^i)\mathbf{n}_i = \mathbf{0} \text{ on } \Gamma_i^N\Big\}, \qquad i = 1, 2.$$

We define $\Lambda^D = \Lambda_1^D \times \Lambda_2^D$ and $\Lambda^N = \Lambda_1^N \times \Lambda_2^N$ and we will refer to the spaces Λ_1 and Λ_2 if there is no need to specify which type of boundary conditions are applied on the control interfaces.
The optimal controls are determined as solutions of a minimization problem; in particular, we want to minimize a suitable cost functional depending on the difference between the two solutions $(\mathbf{u}^1, p^1)$ and $(\mathbf{u}^2, p^2)$ on the overlapping region Ω_{12}.
The issue of the choice of the cost functional is critical. We have to make sure that we are controlling, on the overlapping region, the difference between both the components of the velocities $((u_1^1 - u_1^2)|_{\Omega_{12}}$ and $(u_2^1 - u_2^2)|_{\Omega_{12}})$ and the difference between the pressures $((p^1 - p^2)|_{\Omega_{12}})$ of the state problems.
The choice of the optimal cost functional will be discussed in what follows. We now want to derive the general formulation of the optimality system, for both choices of the spaces of the virtual controls: Λ^D and Λ^N. The velocities can be controlled by minimizing the following cost functionals:

$$J_{\mathrm{u}L^2}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\mathbf{u}^1 - \mathbf{u}^2)^2, \tag{4.3}$$

$$J_{\mathrm{u}H_0^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)^2, \tag{4.4}$$

and the pressure can be controlled through the cost functional.

$$J_{pL^2}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(p^1 - p^2)^2. \tag{4.5}$$

We will now derive the expressions of the optimality conditions and of the adjoint problems associated to (4.3), (4.4) and (4.5). A cost functional that controls both the pressures and the velocities can be obtained by linear combination of (4.3), (4.4) and (4.5), and analogously one can obtain the formulation of the optimality equations and of the adjoints. For example, in the numerical simulations, the following cost functionals will be tested:

$$J_{totH^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\mathbf{u}^1 - \mathbf{u}^2)^2 + \frac{1}{2}\int_\Omega \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)^2 + \frac{1}{2}\int_\Omega \chi_{12}(p^1 - p^2)^2, \tag{4.6}$$

$$J_{totH_0^1}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)^2 + \frac{1}{2}\int_\Omega \chi_{12}(p^1 - p^2)^2, \tag{4.7}$$

$$\widehat{J_{totH_0^1}}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega \chi_{12}(\nabla\mathbf{u}^1-\nabla\mathbf{u}^2)^2+\frac{1}{2}\int_{\partial\Omega}\chi_{12}(\mathbf{u}^1-\mathbf{u}^2)^2+\frac{1}{2}\int_\Omega\chi_{12}(p^1-p^2)^2. \quad (4.8)$$

In the case of (4.6) one minimizes the difference between the velocities in the $H^1(\Omega_{12})$ norm and the difference between the pressures in the $L^2(\Omega_{12})$ norm. With the choice (4.7), the $H_0^1(\Omega_{12})$ semi-norm of the difference between the velocities is considered and with (4.8) one considers an augmented $H_0^1(\Omega_{12})$ norm. We can remark the analogy of (4.8) with the augmented cost functional related to the solution of the Poisson problem (1.23): a term controlling the difference between the solutions (the velocities, in this case) on part of the boundary of the overlap is added.

Again we can split both problems in one problem depending on the controls (λ_1,λ_2):
for $i=1,2$, find $(\mathbf{u}^{i,\lambda_i},p^{i,\lambda_i})\in[H^1(\Omega_i)]^2\times L^2(\Omega_i)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}^{i,\lambda_i}+\nabla p^{i,\lambda_i} &=& \mathbf{0} & in\quad \Omega_i\\ \nabla\cdot\mathbf{u}^{i,\lambda_i} &=& 0 & in\quad \Omega_i\\ \nu\nabla\mathbf{u}^{i,\lambda_i}\cdot\mathbf{n}_i-p^{i,\lambda_i}\mathbf{n}_i &=& \mathbf{0} & on\quad \Gamma_i^N\\ \mathbf{u}^{i,\lambda_i} &=& \mathbf{0} & on\quad \Gamma_i^D\\ \nu\nabla\mathbf{u}^{i,\lambda_i}\cdot\mathbf{n}_i-p^{i,\lambda_i}\mathbf{n}_i/\mathbf{u}^{i,\lambda_i} &=& \lambda_i & on\quad \Gamma_i, \end{array}\right. \quad (4.9)$$

and one problem depending on the data:
for $i=1,2$, find $(\mathbf{u}^{i,\mathrm{f}},p^{i,f})\in[H^1(\Omega_i)]^2\times L^2(\Omega_i)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}^{i,f}+\nabla p^{i,f} &=& \mathbf{f}^i & in\quad \Omega_i\\ \nabla\cdot\mathbf{u}^{i,f} &=& 0 & in\quad \Omega_i\\ \nu\nabla\mathbf{u}^{i,f}\cdot\mathbf{n}_i-p^{i,f}\mathbf{n}_i &=& \mathbf{u}_N^i & on\quad \Gamma_i^N\\ \mathbf{u}^{i,f} &=& \mathbf{u}_D^i & on\quad \Gamma_i^D\\ \nu\nabla\mathbf{u}^{i,f}\cdot\mathbf{n}_i-p^{i,f}\mathbf{n}_i/\mathbf{u}^{i,f} &=& \mathbf{0} & on\quad \Gamma_i, \end{array}\right. \quad (4.10)$$

so that, for $i=1,2$

$$\mathbf{u}^i=\mathbf{u}^{i,\lambda_i}+\mathbf{u}^{i,f}\qquad p^i=p^{i,\lambda_i}+p^{i,f}. \quad (4.11)$$

The cost functionals (4.3), (4.4) and (4.5) can be split in a quadratic part and in an affine part in the following way:

- $J_{uL^2}(\lambda_1,\lambda_2)=J^0_{uL^2}(\lambda_1,\lambda_2)+\mathscr{A}_{uL^2}(\lambda_1,\lambda_2)$,

  where

$$J^0_{uL^2}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\mathbf{u}^{1,\lambda_1}-\mathbf{u}^{2,\lambda_2})^2,$$
$$\mathscr{A}_{uL^2}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\mathbf{u}^{1,f}-\mathbf{u}^{2,f})^2+\int_\Omega\chi_{12}(\mathbf{u}^{1,\lambda_1}-\mathbf{u}^{2,\lambda_2})\cdot(\mathbf{u}^{1,f}-\mathbf{u}^{2,f}),$$

- $J_{uH_0^1}(\lambda_1,\lambda_2)=J^0_{uH_0^1}(\lambda_1,\lambda_2)+\mathscr{A}_{uH_0^1}(\lambda_1,\lambda_2)$,

  where

$$J^0_{uH_0^1}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\nabla\mathbf{u}^{1,\lambda_1}-\nabla\mathbf{u}^{2,\lambda_2})^2,$$
$$\mathscr{A}_{uH_0^1}(\lambda_1,\lambda_2)=\frac{1}{2}\int_\Omega\chi_{12}(\nabla\mathbf{u}^{1,f}-\nabla\mathbf{u}^{2,f})^2+\int_\Omega\chi_{12}(\nabla\mathbf{u}^{1,\lambda_1}-\nabla\mathbf{u}^{2,\lambda_2})\cdot(\nabla\mathbf{u}^{1,f}-\nabla\mathbf{u}^{2,f}),$$

- $J_{pL^2}(\lambda_1, \lambda_2) = J^0_{pL^2}(\lambda_1, \lambda_2) + \mathscr{A}_{pL^2}(\lambda_1, \lambda_2)$,

  where

$$J^0_{pL^2}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(p^{1,\lambda_1} - p^{2,\lambda_2})^2,$$
$$\mathscr{A}_{pL^2}(\lambda_1, \lambda_2) = \frac{1}{2}\int_\Omega \chi_{12}(p^{1,f} - p^{2,f})^2 + \int_\Omega \chi_{12}(p^{1,\lambda_1} - p^{2,\lambda_2})\cdot(p^{1,f} - p^{2,f}),$$

4.1.2 The Optimality System

We are now interested in deriving the expression of the optimality systems (state and adjoint problems, optimality condition) for the different choices of the cost functionals (4.3), (4.4) and (4.5). The partial derivatives of the cost functional that minimizes the L^2 norm of the difference between the velocities on the overlap read as follows:

$$\begin{aligned}
\langle\frac{\partial J^0_{\mathbf{u}L^2}}{\partial\lambda_1}, \mu_1\rangle &= \int_\Omega \chi_{12}(\mathbf{u}^{1,\lambda_1} - \mathbf{u}^{2,\lambda_2})\mathbf{u}^{1,\mu_1} && \forall\mu_1 \in \Lambda_1,\\
\langle\frac{\partial J^0_{\mathbf{u}L^2}}{\partial\lambda_2}, \mu_2\rangle &= -\int_\Omega \chi_{12}(\mathbf{u}^{1,\lambda_1} - \mathbf{u}^{2,\lambda_2})\mathbf{u}^{2,\mu_2} && \forall\mu_2 \in \Lambda_2,\\
\langle\frac{\partial \mathscr{A}_{\mathbf{u}L^2}}{\partial\lambda_1}, \mu_1\rangle &= \int_\Omega \chi_{12}(\mathbf{u}^{1,f} - \mathbf{u}^{2,f})\mathbf{u}^{1,\mu_1} && \forall\mu_1 \in \Lambda_1,\\
\langle\frac{\partial \mathscr{A}_{\mathbf{u}L^2}}{\partial\lambda_2}, \mu_2\rangle &= -\int_\Omega \chi_{12}(\mathbf{u}^{1,f} - \mathbf{u}^{2,f})\mathbf{u}^{2,\mu_2} && \forall\mu_2 \in \Lambda_2.
\end{aligned}$$

In the case of the minimization of $J_{H^1_0}$ we have that:

$$\begin{aligned}
\langle\frac{\partial J^0_{\mathbf{u}H^1_0}}{\partial\lambda_1}, \mu_1\rangle &= -\int_\Omega \nabla\cdot(\chi_{12}(\nabla\mathbf{u}^{1,\lambda_1} - \nabla\mathbf{u}^{2,\lambda_2}))\mathbf{u}^{1,\mu_1} + \int_{\partial\Omega}(\chi_{12}(\nabla\mathbf{u}^{1,\lambda_1} - \nabla\mathbf{u}^{2,\lambda_2}))\cdot\mathbf{n}_1\mathbf{u}^{1,\mu_1} && \forall\mu_1 \in \Lambda_1,\\
\langle\frac{\partial J^0_{\mathbf{u}H^1_0}}{\partial\lambda_2}, \mu_2\rangle &= \int_\Omega \nabla\cdot(\chi_{12}(\nabla\mathbf{u}^{1,\lambda_1} - \nabla\mathbf{u}^{2,\lambda_2}))\mathbf{u}^{2,\mu_2} - \int_{\partial\Omega}(\chi_{12}(\nabla\mathbf{u}^{1,\lambda_1} - \nabla\mathbf{u}^{2,\lambda_2}))\cdot\mathbf{n}_2\mathbf{u}^{2,\mu_2} && \forall\mu_2 \in \Lambda_2,\\
\langle\frac{\partial \mathscr{A}_{\mathbf{u}H^1_0}}{\partial\lambda_1}, \mu_1\rangle &= -\int_\Omega \nabla\cdot(\chi_{12}(\nabla\mathbf{u}^{1,f} - \nabla\mathbf{u}^{2,f}))\mathbf{u}^{1,\mu_1} + \int_{\partial\Omega}(\chi_{12}(\nabla\mathbf{u}^{1,f} - \nabla\mathbf{u}^{2,f}))\cdot\mathbf{n}_1\mathbf{u}^{1,\mu_1} && \forall\mu_1 \in \Lambda_1,\\
\langle\frac{\partial \mathscr{A}_{\mathbf{u}H^1_0}}{\partial\lambda_2}, \mu_2\rangle &= \int_\Omega \nabla\cdot(\chi_{12}(\nabla\mathbf{u}^{1,f} - \nabla\mathbf{u}^{2,f}))\mathbf{u}^{2,\mu_2} - \int_{\partial\Omega}(\chi_{12}(\nabla\mathbf{u}^{1,f} - \nabla\mathbf{u}^{2,f}))\cdot\mathbf{n}_2\mathbf{u}^{2,\mu_2} && \forall\mu_2 \in \Lambda_2,
\end{aligned}$$

The expressions of the form $\nabla\cdot(\chi_{12}(..))$ must be looked at as formal notation. In fact, it describes a duality product in a *distributional* sense. This notation derives from an integration by parts and it is needed to be able to give the strong formulation of the adjoint problems. In practice, this expression of the partial derivative will not be used, and we will take into account only the partial derivative before its integration by parts.

When observing the pressures the partial derivatives read as:

$$\begin{aligned}
\langle\frac{\partial J^0_{pL^2}}{\partial\lambda_1},\mu_1\rangle &= \int_\Omega \chi_{12}(p^{1,\lambda_1}-p^{2,\lambda_2})p^{1,\mu_1} && \forall\mu_1\in\Lambda_1,\\
\langle\frac{\partial J^0_{pL^2}}{\partial\lambda_2},\mu_2\rangle &= -\int_\Omega \chi_{12}(p^{1,\lambda_1}-p^{2,\lambda_2})p^{2,\mu_2} && \forall\mu_2\in\Lambda_2,\\
\langle\frac{\partial \mathscr{A}_{pL^2}}{\partial\lambda_1},\mu_1\rangle &= \int_\Omega \chi_{12}(p^{1,f}-p^{2,f})p^{1,\mu_1} && \forall\mu_1\in\Lambda_1,\\
\langle\frac{\partial \mathscr{A}_{pL^2}}{\partial\lambda_2},\mu_2\rangle &= -\int_\Omega \chi_{12}(p^{1,f}-p^{2,f})p^{2,\mu_2} && \forall\mu_2\in\Lambda_2.
\end{aligned}$$

The generic expression of the duals of the state problems (4.9) and (4.10) is the following: for $i=1,2$, $j=\lambda,f$, find $(\mathbf{v}^{i,j},q^{i,j})\in[H^1(\Omega_i)]^2\times L^2(\Omega_i)$ so that

$$\left\{\begin{array}{rcll}
-\nu\Delta\mathbf{v}^{i,j}+\nabla q^{i,j} &=& \mathbf{F}^1 & in\quad \Omega_i\\
\nabla\cdot\mathbf{v}^{i,j} &=& F^2 & in\quad \Omega_i\\
\nu\nabla\mathbf{v}^{i,j}\cdot\mathbf{n}_i-q^{i,j}\mathbf{n}_i &=& \mathbf{g} & on\quad \Gamma_i^N\\
\mathbf{v}^{i,j} &=& \mathbf{0} & on\quad \Gamma_i^D
\end{array}\right. \tag{4.12}$$

The index i refers to the domain on which the problem is solved: Ω_1 or Ω_2. The index j denotes whether the right hand side of the problem is function of the states depending on the sole controls (solutions of (4.9)) or of the states depending on the given data of the problem (solutions of (4.10)).

In Table 4.1, we report, for the three cost functionals (4.3)-(4.5), the specific data assigned to the dual problems and the final expressions of the optimality conditions. The table has to be read in the following way: we solve for $(\mathbf{v},q)$ an adjoint system with forcing term of the continuity equation $\mathbf{F}^1$, forcing term of the equation derived from the conservation of the mass F^2 and boundary condition on $\partial\Omega_{12}\cap\Gamma^N$ equal to $\mathbf{g}$; we rewrite the weak formulations of the partial derivatives of the cost functional ($\langle\frac{\partial J}{\partial\lambda},\mu\rangle^D$ in the case of Dirichlet boundary controls and as $\langle\frac{\partial J}{\partial\lambda},\mu\rangle^N$ in the case of Neumann boundary controls) as functions of $(\mathbf{v},q)$. From an analysis of Table 4.1 we can state the following observations:

- the adjoint in the case of the minimization of J_{pL^2} represent a non divergence-free Stokes problem, in particular the mass is not conserved on the overlap region Ω_{12};
- in the case of the minimization of $J_{uH^1_0}$ a boundary term is applied to the adjoint problem on the part of the boundary $\partial\Omega_{12}\cap\Gamma^N$, this part includes the interfaces Γ_1 and Γ_2 in the case of Neumann boundary controls (in the case of Dirichlet controls the condition imposed on the interfaces is homogeneous);
- in the case of the minimization of $J_{uH^1_0}$, if the controls belong to the space Λ^D, an extra term appears in the expression of the partial derivative, depending on the solutions of the state problems;
- in general, the choice of the space of the controls Λ^N simplifies the expression of the partial derivatives, in fact, in this case, the partial derivatives are symmetric and depend on the sole solution of the adjoint $\mathbf{v}$ and not on the normal component of the Cauchy stress tensor.

J	(v,q)	F^1	F^2	g $(on\ \partial\Omega_{12}\cap\Gamma^N)$	$\langle\frac{\partial J}{\partial\lambda},\mu\rangle^D$	$\langle\frac{\partial J}{\partial\lambda},\mu\rangle^N$
J_{uL^2}	$(v^{1,\lambda},q^{1,\lambda})$	$\chi_{12}(u^{1,\lambda}-u^{2,\lambda})$	0	0	$-\int_{\Gamma_1}(\nu\nabla v^{1,\lambda}\cdot n_1-q^{1,\lambda}n_1)\cdot\mu_1$	$\int_{\Gamma_1}v^{1,\lambda}\cdot\mu_1$
	$(v^{2,\lambda},q^{2,\lambda})$	$-\chi_{12}(u^{1,\lambda}-u^{2,\lambda})$	0	0	$-\int_{\Gamma_2}(\nu\nabla v^{2,\lambda}\cdot n_2-q^{2,\lambda}n_2)\cdot\mu_2$	$\int_{\Gamma_2}v^{2,\lambda}\cdot\mu_2$
	$(v^{1,f},q^{1,f})$	$\chi_{12}(u^{1,f}-u^{2,f})$	0	0	$-\int_{\Gamma_1}(\nu\nabla v^{1,f}\cdot n_1-q^{1,f}n_1)\cdot\mu_1$	$\int_{\Gamma_1}v^{1,f}\cdot\mu_1$
	$(v^{2,f},q^{2,f})$	$-\chi_{12}(u^{1,f}-u^{2,f})$	0	0	$-\int_{\Gamma_2}(\nu\nabla v^{2,f}\cdot n_2-q^{2,f}n_2)\cdot\mu_2$	$\int_{\Gamma_2}v^{2,f}\cdot\mu_2$
$J_{uH_0^1}$	$(v^{1,\lambda},q^{1,\lambda})$	$-\nabla\cdot(\chi_{12}(u^{1,\lambda}-u^{2,\lambda}))$	0	$(\chi_{12}(u^{1,\lambda}-u^{2,\lambda}))\cdot n_1$	$-\int_{\Gamma_1}(\nu\nabla v^{1,\lambda}\cdot n_1-q^{1,\lambda}n_1)\cdot\mu_1$ $+\int_{\Gamma_1}(\chi_{12}(u^{1,\lambda}-u^{2,\lambda}))\cdot n_1\mu_1$	$\int_{\Gamma_1}v^{1,\lambda}\cdot\mu_1$
	$(v^{2,\lambda},q^{2,\lambda})$	$\nabla\cdot(\chi_{12}(u^{1,\lambda}-u^{2,\lambda}))$	0	$-(\chi_{12}(u^{1,\lambda}-u^{2,\lambda}))\cdot n_2$	$-\int_{\Gamma_2}(\nu\nabla v^{2,\lambda}\cdot n_2-q^{2,\lambda}n_2)\cdot\mu_2$ $-\int_{\Gamma_2}(\chi_{12}(u^{1,\lambda}-u^{2,\lambda}))\cdot n_2\mu_2$	$\int_{\Gamma_2}v^{2,\lambda}\cdot\mu_2$
	$(v^{1,f},q^{1,f})$	$-\nabla\cdot(\chi_{12}(u^{1,f}-u^{2,f}))$	0	$(\chi_{12}(u^{1,f}-u^{2,f}))\cdot n_1$	$-\int_{\Gamma_1}(\nu\nabla v^{1,f}\cdot n_1-q^{1,f}n_1)\cdot\mu_1$ $+\int_{\Gamma_1}(\chi_{12}(u^{1,f}-u^{2,f}))\cdot n_1\mu_1$	$\int_{\Gamma_1}v^{1,f}\cdot\mu_1$
	$(v^{2,f},q^{2,f})$	$\nabla\cdot(\chi_{12}(u^{1,f}-u^{2,f}))$	0	$-(\chi_{12}(u^{1,f}-u^{2,f}))\cdot n_2$	$-\int_{\Gamma_2}(\nu\nabla v^{2,f}\cdot n_2-q^{2,f}n_2)\cdot\mu_2$ $-\int_{\Gamma_2}(\chi_{12}(u^{1,f}-u^{2,f}))\cdot n_2\mu_2$	$\int_{\Gamma_2}v^{2,f}\cdot\mu_2$
J_{pL^2}	$(v^{1,\lambda},q^{1,\lambda})$	0	$-\chi_{12}(p^{1,\lambda}-p^{2,\lambda})$	0	$-\int_{\Gamma_1}(\nu\nabla v^{1,\lambda}\cdot n_1-q^{1,\lambda}n_1)\cdot\mu_1$	$\int_{\Gamma_1}v^{1,\lambda}\cdot\mu_1$
	$(v^{2,\lambda},q^{2,\lambda})$	0	$\chi_{12}(p^{1,\lambda}-p^{2,\lambda})$	0	$-\int_{\Gamma_2}(\nu\nabla v^{2,\lambda}\cdot n_2-q^{2,\lambda}n_2)\cdot\mu_2$	$\int_{\Gamma_2}v^{2,\lambda}\cdot\mu_2$
	$(v^{1,f},q^{1,f})$	0	-$\chi_{12}(p^{1,f}-p^{2,f})$	0	$-\int_{\Gamma_1}(\nu\nabla v^{1,f}\cdot n_1-q^{1,f}n_1)\cdot\mu_1$	$\int_{\Gamma_1}v^{1,f}\cdot\mu_1$
	$(v^{2,f},q^{2,f})$	0	$\chi_{12}(p^{1,f}-p^{2,f})$	0	$-\int_{\Gamma_2}(\nu\nabla v^{2,f}\cdot n_2-q^{2,f}n_2)\cdot\mu_2$	$\int_{\Gamma_2}v^{2,f}\cdot\mu_2$

Table 4.1: *Data of the adjoint problems and optimality conditions, for three choices of the cost functional.*

4.1.3 The Generalized Stokes Problem

As we did in section 1.2 we consider the following generalized Stokes problem: for $i = 1, 2$, find $(\mathbf{u}^i, p^i) \in H^1(\Omega_i) x L^2(\Omega_i)$ so that

$$\begin{cases} \alpha\mathbf{u} - \nu\Delta\mathbf{u}^i + \nabla p^i &= \mathbf{f}^i & in \quad \Omega_i \\ \nabla \cdot \mathbf{u}^i &= 0 & in \quad \Omega_i \\ \nu\nabla\mathbf{u}^i \cdot \mathbf{n}_i - p^i\mathbf{n}_i &= \mathbf{u}_N^i & on \quad \Gamma_i^N \\ \mathbf{u}^i &= \mathbf{u}_D^i & on \quad \Gamma_i^D \\ \nu\nabla\mathbf{u}^i \cdot \mathbf{n}_i - p^i\mathbf{n}_i/\mathbf{u}^i &= \lambda_i & on \quad \Gamma_i. \end{cases} \tag{4.13}$$

We are interested in testing if the results that we have obtained in the case of the scalar elliptic problem are extendible to the Stokes system. In particular we want to verify if taking $\alpha > 0$ guarantees the well posedness of the problem when considering the minimization of the H_0^1 semi-norm of the difference between the velocities on the overlap.

The derivation of the optimality system in this case is analogous to the case when $\alpha = 0$, apart from considering as dual problem a system of equations of the same form as (4.13).

4.1.4 Minimization Algorithms

As we have done in the case of the scalar elliptic equation we consider a couple of solution algorithms for the optimality system deriving from the virtual control approach applied to the Stokes problem.

Solving the Extremality Equation: Iterative Algorithm

We iterate between the equations in the optimality system in order to find the couple of controls (λ_1, λ_2) that minimize the cost functional. We aim to solve $\nabla J = \nabla J^0 + \nabla \mathscr{A} = 0$, where

$$\langle \nabla J^0, (\mu_1, \mu_2) \rangle = (\langle \frac{\partial J^0}{\partial \lambda_1}, \mu_1 \rangle, \langle \frac{\partial J^0}{\partial \lambda_2}, \mu_2 \rangle),$$
$$\langle \nabla \mathscr{A}, (\mu_1, \mu_2) \rangle = (\langle \frac{\partial \mathscr{A}}{\partial \lambda_1}, \mu_1 \rangle, \langle \frac{\partial \mathscr{A}}{\partial \lambda_2}, \mu_2 \rangle).$$

Given an initial guess for the control variables we solve the state equations, then we solve the adjoints, then through the optimality condition we obtain a new approximation of the controls. The process is repeated until satisfactory convergence is achieved.

In the initialization part we compute the term that depends on the data of the problem $\nabla \mathscr{A}$, as follows:

1. compute $(\mathbf{u}^{1,f}, p^{1,f})$ and $(\mathbf{u}^{2,f}, p^{2,f})$;
2. compute $(\mathbf{v}^{1,f}, q^{1,f})$ and $(\mathbf{v}^{2,f}, q^{2,f})$;
3. compute $\nabla \mathscr{A}$.

In the main loop we solve by an iterative method the linear system $\nabla J^0(\lambda_1, \lambda_2) = -\nabla \mathscr{J}$. We choose the *BiCGStab* algorithm, because the matrix $J^0(\lambda_1, \lambda_2)$ is non symmetric in the case of Dirichlet virtual controls. We start with an initial guess for the virtual controls $(\lambda_1^0, \lambda_2^0)$ and at the generic iteration k we:

1. compute $(\mathbf{u}^{1,\lambda_1}, p^{1,\lambda_1})$ and $(\mathbf{u}^{2,\lambda_2}, p^{2,\lambda_2})$;
2. compute $(\mathbf{v}^{1,\lambda_1}, q^{1,\lambda_1})$ and $(\mathbf{v}^{2,\lambda_2}, q^{2,\lambda_2})$;
3. compute $\nabla J^0(\lambda_1^k, \lambda_2^k)$.

The method stops when the relative increment between two consecutive iterates $(\lambda_1^k, \lambda_2^k)$ and $(\lambda_1^{k-1}, \lambda_2^{k-1})$ is lower than a certain tolerance.

The One Shot Approach

In the *one-shot* approach one constructs a global system for the resolution of the coupled virtual control problem. The formal expression of this system will be derived in the following section, when dealing with the issue of preconditioning, which is designed with the aim of reducing the high number of iterations needed for the solution of the linear system.
The global matrix is composed with four matrices corresponding to the Stokes problem (a state problem and an adjoint problem for each domain), plus the terms related to the coupling, the boundary conditions and the optimality equations.

4.2 Preconditioning the Optimality System

In this section we will first present the expression of the global optimality linear system, then we will derive the expression of the corresponding *Schur Complement* equation on the interface and propose a preconditioner for it, analogously to what has been done in section 2.3.

4.2.1 The Optimality System as a Global Linear System

We assemble the global matrix with respect to a problem with the following specifications:

- the virtual controls represent a Neumann boundary condition ($(\lambda_1, \lambda_2) \in \Lambda^N$));
- the solutions $(\mathbf{u}^1, p^1)$ and $(\mathbf{u}^2, p^2)$ are not split in a part depending on the controls and in a part depending on the data (consequently, this decomposition will not be present in the cost functional and in the adjoint problems);
- we minimize the following cost functional:

$$\begin{aligned} J_{tot}(\lambda_1, \lambda_2) =& \frac{1}{2}\int_\Omega \chi_{12}(\mathbf{u}^1 - \mathbf{u}^2)^2 + \frac{1}{2}\int_\Omega \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)^2 + \frac{1}{2}\int_\Omega \chi_{12}(p^1 - p^2)^2 + \\ & \frac{1}{2}\beta_1 \int_{\Gamma_1} \lambda_1{}'\lambda_1 + \frac{1}{2}\beta_2 \int_{\Gamma_2} \lambda_2{}'\lambda_2, \end{aligned} \tag{4.14}$$

which is a linear combination of (4.3), (4.4) and (4.5). With this choice we are sure to control the $H^1(\Omega_{12})$ norm of the difference between the velocities and the $L^2(\Omega_{12})$ norm of the difference between the pressures. We also remark the presence of a penalization term on the cost functional: this term was added in order to be able to study the behavior of the optimization problem with respect to this additional control. We take $\beta_1 = \beta_2 = \beta$.

On Ω_1 the state problem read as follows: find $(\mathbf{u}^1, p^1) \in [H^1(\Omega_1)]^2 \times L^2(\Omega_1)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}^1 + \nabla p^1 &=& \mathbf{f}^1 & in \quad \Omega_1 \\ \nabla\cdot\mathbf{u}^1 &=& 0 & in \quad \Omega_1 \\ \nu\nabla\mathbf{u}^1\cdot\mathbf{n}_1 - p^i\mathbf{n}_1 &=& \mathbf{u}_N^1 & on \quad \Gamma_1^N \\ \mathbf{u}^i &=& \mathbf{u}_D^1 & on \quad \Gamma_1^D \\ \nu\nabla\mathbf{u}^1\cdot\mathbf{n}_1 - p^1\mathbf{n}_1 &=& \lambda_1 & on \quad \Gamma_1, \end{array}\right. \tag{4.15}$$

and on Ω_2 as: find $(\mathbf{u}^2, p^2) \in [H^1(\Omega_2)]^2 \times L^2(\Omega_2)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{u}^2 + \nabla p^2 &=& \mathbf{f}^2 & in \quad \Omega_2 \\ \nabla\cdot\mathbf{u}^2 &=& 0 & in \quad \Omega_2 \\ \nu\nabla\mathbf{u}^2\cdot\mathbf{n}_2 - p^2\mathbf{n}_2 &=& \mathbf{u}_N^2 & on \quad \Gamma_2^N \\ \mathbf{u}^2 &=& \mathbf{u}_D^2 & on \quad \Gamma_2^D \\ \nu\nabla\mathbf{u}^2\cdot\mathbf{n}_2 - p^2\mathbf{n}_2 &=& \lambda_i & on \quad \Gamma_2. \end{array}\right. \tag{4.16}$$

The forcing terms and the boundary data that have to be imposed on the right hand side of the adjoints when minimizing (4.14) can be derived by linear combination of the results obtained for (4.3), (4.4) and (4.5). In particular one obtains the following problem on Ω_1: find $(\mathbf{v}^1, q^1) \in [H^1(\Omega_1)]^2 \times L^2(\Omega_1)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{v}^1 + \nabla q^1 &=& -\chi_{12}(\mathbf{u}^1 - \mathbf{u}^2) + \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2) & in \quad \Omega_1 \\ \nabla\cdot\mathbf{v}^1 &=& \chi_{12}(p^1 - p^2) & in \quad \Omega_1 \\ \nu\nabla\mathbf{v}^1\cdot\mathbf{n}_1 - q^1\mathbf{n}_1 &=& -\chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)\cdot\mathbf{n}_1 & on \quad \Gamma_1^N \\ \mathbf{v}^1 &=& \mathbf{0} & on \quad \Gamma_1^D \\ \nu\nabla\mathbf{v}^1\cdot\mathbf{n}_1 - q^1\mathbf{n}_1 &=& -\chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)\cdot\mathbf{n}_1 & on \quad \Gamma_1, \end{array}\right. \tag{4.17}$$

and the following system on Ω_2: find $(\mathbf{v}^2, q^2) \in [H^1(\Omega_2)]^2 \times L^2(\Omega_2)$ so that

$$\left\{\begin{array}{rcll} -\nu\Delta\mathbf{v}^2 + \nabla q^2 &=& \chi_{12}(\mathbf{u}^1 - \mathbf{u}^2) - \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2) & in \quad \Omega_2 \\ \nabla\cdot\mathbf{v}^2 &=& -\chi_{12}(p^1 - p^2) & in \quad \Omega_2 \\ nu\nabla\mathbf{v}^2\cdot\mathbf{n}_2 - q^2\mathbf{n}_2 &=& \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)\cdot\mathbf{n}_2 & on \quad \Gamma_2^N \\ \mathbf{v}^2 &=& \mathbf{0} & on \quad \Gamma_2^D \\ \nu\nabla\mathbf{v}^2\cdot\mathbf{n}_2 - q^2\mathbf{n}_2 &=& \chi_{12}(\nabla\mathbf{u}^1 - \nabla\mathbf{u}^2)\cdot\mathbf{n}_2 & on \quad \Gamma_2. \end{array}\right. \tag{4.18}$$

We can express the optimality conditions (partial derivatives of (4.14) set to zero) as functions of the solutions of the dual problems (4.17) and (4.18):

$$\langle\frac{\partial J_{tot}}{\partial\lambda_1}, \mu_1\rangle = -\int_{\Gamma_1}\mathbf{v}^1\cdot\mu_1 + \beta\int_{\Gamma_1}\lambda_1\cdot\mu_1 = 0 \qquad \forall\mu_1 \in \Lambda_1, \tag{4.19}$$

$$\langle\frac{\partial J_{tot}}{\partial\lambda_2}, \mu_2\rangle = -\int_{\Gamma_2}\mathbf{v}^2\cdot\mu_2 + \beta\int_{\Gamma_2}\lambda_2\cdot\mu_2 = 0 \qquad \forall\mu_2 \in \Lambda_2. \tag{4.20}$$

After the discretization of the problem with the use of finite elements (a detailed description can be found in [14]) the state and dual problems and the optimality conditions can be rewritten as discrete linear systems in the following way:

$$\begin{bmatrix} A_1 & B_1^T \\ B_1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ p_1 \end{bmatrix} - \begin{bmatrix} M_{\Gamma_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix} \tag{4.21}$$

$$\begin{bmatrix} A_2 & B_2^T \\ B_2 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ p_2 \end{bmatrix} - \begin{bmatrix} M_{\Gamma_2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_2 \\ 0 \end{bmatrix} = \begin{bmatrix} f_2 \\ 0 \end{bmatrix} \tag{4.22}$$

$$\begin{bmatrix} A_1 & B_1^T \\ B_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ q_1 \end{bmatrix} + \begin{bmatrix} M_{12}+C_{12} & 0 \\ 0 & -M_{12}^p \end{bmatrix} \begin{bmatrix} u_1 \\ p_1 \end{bmatrix} - \begin{bmatrix} M_{12}+C_{12} & 0 \\ 0 & -M_{12}^p \end{bmatrix} \begin{bmatrix} u_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.23}$$

$$\begin{bmatrix} A_2 & B_2^T \\ B_2 & 0 \end{bmatrix} \begin{bmatrix} v_2 \\ q_2 \end{bmatrix} - \begin{bmatrix} M_{12}+C_{12} & 0 \\ 0 & -M_{12}^p \end{bmatrix} \begin{bmatrix} u_1 \\ p_1 \end{bmatrix} + \begin{bmatrix} M_{12}+C_{12} & 0 \\ 0 & -M_{12}^p \end{bmatrix} \begin{bmatrix} u_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{4.24}$$

$$\beta_1 M_{\Gamma_1} \lambda_1 - M_{\Gamma_1} v_1 = 0 \tag{4.25}$$

$$\beta_2 M_{\Gamma_2} \lambda_2 - M_{\Gamma_2} v_2 = 0 \tag{4.26}$$

By reordering the equations (optimality conditions, adjoints, states) we can obtain the following global matrix:

$$\left[\begin{array}{cc|cccc|cccc} \beta_1 M_{\Gamma_1} & & & & & & -M_{\Gamma_1} & & & \\ & \beta_2 M_{\Gamma_2} & & & & & & -M_{\Gamma_2} & & \\ \hline & & M_{12}+C_{12} & 0 & -M_{12}-C_{12} & 0 & A_1 & B_1^T & & \\ & & 0 & -M_{12}^p & 0 & M_{12}^p & B_1 & 0 & & \\ & & -M_{12}-C_{12} & 0 & M_{12}+C_{12} & 0 & & & A_2 & B_2^T \\ & & 0 & M_{12}^p & 0 & -M_{12}^p & & & B_2 & 0 \\ \hline -M_{\Gamma_1} & & A_1 & B_1^T & & & & & & \\ & -M_{\Gamma_2} & B_1 & 0 & & & & & & \\ & & & & A_2 & B_2^T & & & & \\ & & & & B_2 & 0 & & & & \end{array}\right], \tag{4.27}$$

corresponding to the vector of unknowns

$$\left[\begin{array}{cc|cccc|cccc} \lambda_1 & \lambda_2 & u_1 & p_1 & u_2 & p_2 & v_1 & q_1 & v_2 & q_2 \end{array}\right]^T \tag{4.28}$$

and the right hand side

$$\left[\begin{array}{cc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & d_1 & 0 & d_2 & 0 \end{array}\right]^T. \tag{4.29}$$

We can see that the global system has the following form

$$\begin{bmatrix} \beta M_\Gamma & 0 & -E^T \\ 0 & M_{12} & K^T \\ -E & K & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ u^* \\ v^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix}, \tag{4.30}$$

where $\lambda = \left[\begin{array}{cc} \lambda_1 & \lambda_2 \end{array}\right]^T$, $u^* = \left[\begin{array}{cccc} u_1 & p_1 & u_2 & p_2 \end{array}\right]^T$ and $v^* = \left[\begin{array}{cccc} v_1 & q_1 & v_2 & q_2 \end{array}\right]^T$. The structure of this linear system in analogous to (2.20); it is of saddle point type, so that the same considerations that have been made in section 2.3.1 hold true. The main difference between (2.20) and (4.30) resides in the bigger dimension of the matrix related to the coupled Stokes problem.

4.2.2 Preconditioning the Schur Complement Interface Equation

As we have done in section 2.3.5 we can eliminate the unknowns corresponding to the state's solutions and to the adjoint's solutions in (4.30) in order to obtain the following equation on the interface

$$\beta M_\Gamma \lambda + E^T (K^T)^{-1} M_{12} K^{-1} E\lambda = -E^T (K^T)^{-1} M_{12} K^{-1} d. \tag{4.31}$$

We define $\Sigma = \beta M_\Gamma + S$ where $S = E^T (K^T)^{-1} M_{12} K^{-1} E$. M_Γ is the mass matrix on the control interfaces and is related to the penalization term and S is the matrix related to the solution of the problem. We refer to Σ as the *Schur complement* matrix. The matrix S maps the vector of the controls λ from the Neumann boundary condition on the interfaces to the optimality equation back on the interfaces, through the resolution of the state and adjoint problems.

Analogously to what has been done in section 2.3.5, we want to derive the expression of a preconditioner for the matrix Σ. It is helpful to highlight the contributions coming form the two different subdomains, in the following way:

$$\Sigma = \begin{bmatrix} \beta M_{\Gamma_1} + N_{\Gamma_1}^T (D_1^T)^{-1} M_1^{12} D_1^{-1} N_{\Gamma_1} & -N_{\Gamma_1}^T (D_1^T)^{-1} M_2^{12} D_2^{-1} N_{\Gamma_2} \\ -N_{\Gamma_2}^T (D_2^T)^{-1} M_1^{12} D_1^{-1} N_{\Gamma_1} & \beta M_{\Gamma_2} + N_{\Gamma_2}^T (D_2^T)^{-1} M_2^{12} D_2^{-1} N_{\Gamma_2} \end{bmatrix}, \tag{4.32}$$

where we have defined, for $i = 1, 2$ the matrix related to the Stokes problem as

$$D_i = \begin{bmatrix} A_i & B_i^T \\ B_i & 0 \end{bmatrix}.$$

On the diagonal of the matrix (4.32) we recognize two matrices with the same structure of Σ and corresponding to the two different overlapping domains, we can call them Σ_1 and Σ_2:

$$\Sigma = \begin{bmatrix} \Sigma_1 & -N_{\Gamma_1}^T (D_1^T)^{-1} M_{12} D_2^{-1} N_{\Gamma_2} \\ -N_{\Gamma_2}^T (D_2^T)^{-1} M_{12} D_1^{-1} N_{\Gamma_1} & \Sigma_2 \end{bmatrix}.$$

We tested the following block diagonal preconditioner

$$P_{12} = \beta M_\Gamma + (1-\beta) P_{S_{12}}, \tag{4.33}$$

where the matrices M_Γ and $P_{S_{12}}$ are weighted with respect to the amount of penalization, and where

$$P_{S_{12}} = \begin{bmatrix} M_{\Gamma_1}^T D^{-1} M_{\Gamma_1} & 0 \\ 0 & M_{\Gamma_1}^T D^{-1} M_{\Gamma_1} \end{bmatrix}.$$

This preconditioner represents an iterface-to-interface map that passes through the resolution of the Stokes operator. The solution of the adjoint has been neglected and consequently we have omitted the matrices related to the coupling M_i^{12} $(i = 1, 2)$.

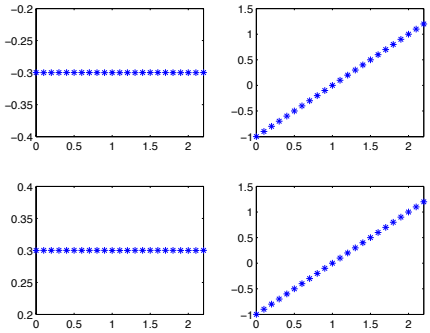


Figure 4.1: *Dirichlet controls at the final iteration. In clockwise order: first component of the velocity on Γ_1, second component of the velocity on Γ_1, first component of the velocity on Γ_2, second component of the velocity on Γ_2.*

4.3 Numerical Results

The model

On the Coercivity of the Cost Functional

In analogy with the approach used for the solution of the scalar elliptic equation, we want to test the coercivity of the different cost functionals presented in section 4.1.1. The domain is again the one represented in Figure 3.1, and we use the minimization algorithm that iterates between the problems to solve the extremality equation. We impose the boundary conditions and the forcing terms in order to recover the following exact linear solution:

$$\begin{cases} u_1^{ex} &= y \\ u_2^{ex} &= x-1 \\ p^{ex} &= x+y-1. \end{cases} \tag{4.34}$$

We use $\mathbb{P}^2$-$\mathbb{P}^1$ finite element spaces, so that we expect to interpolate (4.34) exactly. We use Dirichlet boundary conditions on $\partial\Omega_{12} \cap \partial\Omega$. We first test the minimization of (4.3)-(4.5): none of the resolutions of the optimality systems related to these cost functionals led to the correct approximation of (4.34). In particular, we observe the following behavior:

- the minimization of J_{uL^2} or $J_{uH_0^1}$ both lead to the correct approximation of one component of the velocities, the other component converges to a solution different from the exact one and the pressures do not coincide on the overlap;
- the minimization of J_{pL^2} is not sufficient to guarantee the convergence;
- the minimization of $J_{uL^2} + J_{pL^2}$ tends to recover the exact solution but we observe some spurious oscillations.

A stronger control is necessary when dealing with the Stokes problem. The functionals depending on the sole velocity do not guarantee the uniqueness of the solution. In fact, since the pressure is

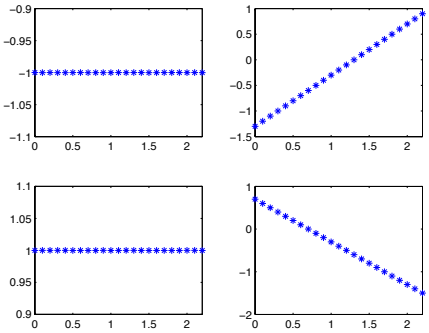


Figure 4.2: *Neumann controls at the final iteration. In clockwise order: first component of the stress tensor on Γ_1, second component of the stress tensor on Γ_1, first component of the stress tensor on Γ_2, second component of the stress tensor on Γ_2.*

imposed only through the stress tensor on the boundary, different couples of velocity and pressure satisfy the the local problems on Ω_1 and Ω_2. The minimization of the difference between the pressures on the overlap is also not sufficient. For this reason we have to use a functional that depends on both the velocities and the pressures. We implemented the minimization of the cost functional (4.6), which minimizes the difference in $H^1(\Omega_{12})$ of the velocities and the difference between the pressures in $L^2(\Omega_{12})$, and we obtained satisfactory results. The same holds for the cost functionals (4.7) and (4.8).

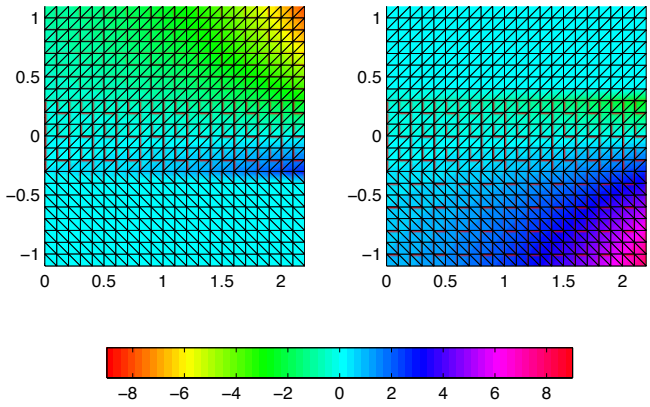
In Table 4.2 we report the results corresponding to these choices for the cost functional, varying the conditions that are applied on the boundary of the overlap: the algorithm has converged to the exact linear solution, with the exception of the case of the minimization of $J_{totH_0^1}$ when $\partial\Omega_{12} \cap \Gamma_D = \emptyset$. In fact, in this case, the cost functional only represents a semi-norm for the difference between the velocities on the overlap, and is not sufficient to recover the exact solution. In Figure 4.1 and in Figure 4.2 we report the controls obtained with the minimization algorithm in the case of Dirichlet and Neumann boundary controls respectively.

In Table 4.3 we report the convergence history of the different functionals when a Neumann boundary condition is applied on the boundary of the overlap and we solve the generalized Stokes problem (4.13) with $\alpha = 1$. In this case the exact solution is recovered for all of the choices (4.6)-(4.8).

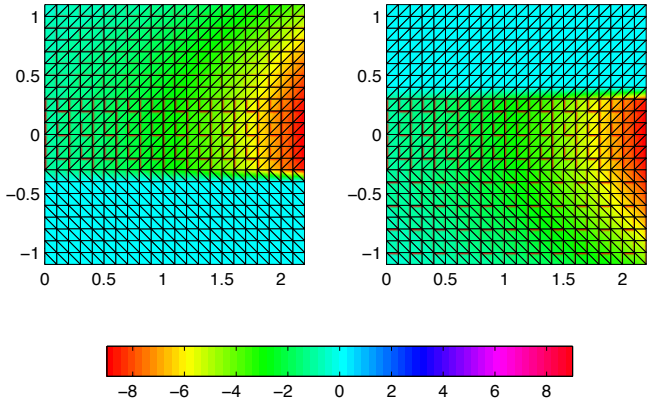
We remark that in Table 4.2 and in Table 4.3 Dirichlet boundary controls have been considered. In the case where $\partial\Omega_{12} \cap \Gamma^D \neq 0$ one should pay particular attention to the initial values imposed to the problems that depend on the given data. In fact imposing an homogeneous Dirichlet boundary condition on the interfaces Γ_1 and Γ_2 as we do in (4.10) might not be the best choice.
We have to assure that the Dirichlet data imposed on the two subdomains Ω_1 and Ω_2 is continuous otherwise some spurious oscillations are generated for the pressure p^f. These errors will be propagated along the solution of the algorithm since the solution of the problem depending on the data goes on the right hand side of the extremality equation $\nabla J^0 = -\nabla \mathscr{A}$.

For this reason we choose a non homogeneous Dirichlet value on the interfaces, imposing

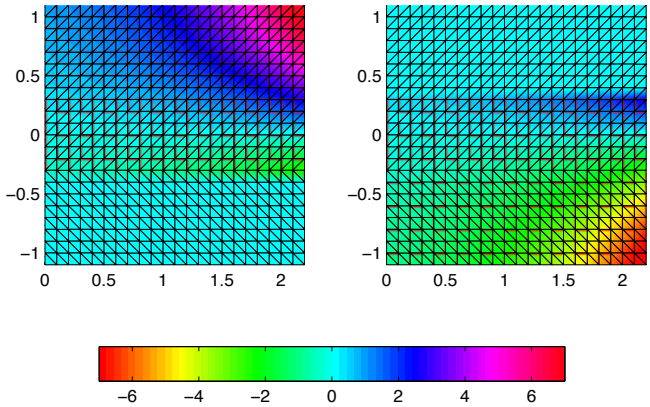
$$\mathbf{u}^{i,f} = \mathbf{u}^{\Gamma_i} \quad on \quad \Gamma_i, \qquad i = 1, 2,$$



(a) *First component of the velocity.*



(b) *Second component of the velocity.*



(c) *Pressure.*

Figure 4.3: *The state solution of the Stokes problem on Ω_1 and Ω_2, for $h = 0.1$.*

	J	$\#iter$	$error \sim$	$J \sim$
$\partial\Omega_{12} \cap \Gamma_D \neq \emptyset$	J_{totH^1}	189	10^{-10}	10^{-17}
	$J_{totH^1_0}$	157	10^{-10}	10^{-17}
	$\widetilde{J_{totH^1_0}}$	157	10^{-10}	10^{-16}
$\partial\Omega_{12} \cap \Gamma_D = \emptyset$	J_{totH^1}	257	10^{-10}	10^{-16}
	$J_{totH^1_0}$	–	1	10^{-10}
	$\widetilde{J_{totH^1_0}}$	456	10^{-10}	10^{-16}

Table 4.2: *Convergence of the different functionals with different boundary conditions,* $h = 0.1$, $tol = 10^{-10}$.

	J	$\#iter$	$error \sim$	$J \sim$
$\partial\Omega_{12} \cap \Gamma_D = \emptyset$	J_{totH^1}	323	10^{-10}	10^{-17}
	$J_{totH^1_0}$	428	10^{-8}	10^{-16}
	$\widetilde{J_{totH^1_0}}$	292	10^{-10}	10^{-17}

Table 4.3: *Convergence of the different functionals, Generalized Stokes* ($\alpha = 1$), $h = 0.1$, $tol = 10^{-10}$.

so that $\mathbf{u}^{i,f}|_{\Gamma_i^D \cup \Gamma_i} \in H^{\frac{1}{2}}(\Gamma_i^D \cup \Gamma_i)$. The value assumed by the solution on the virtual control interfaces is obviously unknown: what we do is to impose $\mathbf{u}^{\Gamma_i}$ as a parabolic function that matches the value of the Dirichlet boundary data on the edges of the interfaces.
Moreover if the problem on Ω_i is a "full Dirichlet problem" ($\partial\Omega_i = \Gamma_i^D \cup \Gamma_i$), $\mathbf{u}^{\Gamma_i}$ has to ensure the compatibility condition:

$$\int_{\partial\Omega_i} \mathbf{u}^{i,f} \cdot \mathbf{n}_i = 0 \qquad i = 1, 2,$$

which derives from the fact that the solution must be divergence free.

Testing the Convergence Orders

As a further analysis, we want to test the convergence orders to the exact solution, when we impose the boundary conditions and the forcing terms in order to approximate:

$$\begin{cases} u_1^{ex} &= -e^x sin(y) \\ u_2^{ex} &= -e^x cos(y) \\ p^{ex} &= e^x sin(y). \end{cases} \tag{4.35}$$

In Table 4.4 we report the errors obtained solving the Stokes problem globally in Ω. We want to make sure that the finite elements approximation of (4.35) with the virtual control minimization algorithm leads to the same approximation of the solution of the global problem. In Table 4.5 and in Table 4.6 we report the number of iterations needed to achieve convergence and the errors with respect to the exact solution obtained through the minimization of (4.6) with Dirichlet and Neumann boundary conditions, respectively.
It is interesting to observe that the functional never attains the value of 0, which is in contrast with what happens when converging to the linear solution. There is a dependence of the minimum on the grid size. Consequently the errors obtained with the iterative algorithm are similar but not exactly

h	$\|u_1 - u_1^{ex}\|_{H^1}$	$\|u_1 - u_1^{ex}\|_{L^2}$	$\|u_2 - u_2^{ex}\|_{H^1}$	$\|u_2 - u_2^{ex}\|_{L^2}$	$\|p - p^{ex}\|_{L^2}$
0.2	0.0328	$8.8143e-04$	0.0325	$7.3640e-04$	0.0240
0.1	0.0082	$1.0853e-04$	0.0081	$8.5491e-05$	0.0059
0.05	0.0020	$1.3492e-05$	0.0020	$1.0198e-05$	0.0015

Table 4.4: *Convergence of the global finite element solution.*

h	#	$J \sim$	$\|u_1 - u_1^{ex}\|_{H^1}$	$\|u_1 - u_1^{ex}\|_{L^2}$	$\|u_2 - u_2^{ex}\|_{H^1}$	$\|u_2 - u_2^{ex}\|_{L^2}$	$\|p - p^{ex}\|_{L^2}$
0.2	223	10^{-5}	0.0332	$9.2855e-04$	0.0331	$9.7799e-04$	0.0248
0.1	405	10^{-7}	0.0082	$1.1132e-04$	0.0082	$1.1492e-04$	0.0059
0.05	792	10^{-8}	0.0021	$1.3850e-05$	0.0020	$1.4268e-05$	0.0015

Table 4.5: *Convergence history, Neumann controls,* $tol = 10^{-12}$.

the same as those obtained when solving the global problem. In Figure 4.3 we report the solutions of the state problems on Ω_1 and Ω_2.

h	#	$J \sim$	$\|u_1 - u_1^{ex}\|_{H^1}$	$\|u_1 - u_1^{ex}\|_{L^2}$	$\|u_2 - u_2^{ex}\|_{H^1}$	$\|u_2 - u_2^{ex}\|_{L^2}$	$\|p - p^{ex}\|_{L^2}$
0.2	258	10^{-5}	0.0332	$9.3098e-04$	0.0332	$9.8312e-04$	0.0249
0.1	678	10^{-7}	0.0082	$1.1139e-04$	0.0082	$1.1505e-04$	0.0059
0.05	1548	10^{-8}	0.0021	$1.3855e-05$	0.0020	$1.4276e-05$	0.0015

Table 4.6: *Convergence history, Dirichlet controls,* $tol = 10^{-12}$.

Preconditioning

Finally, we present the results concerning the preconditioning approach presented in section 4.2. We tested this preconditioner for different values of β and the results are reported in Table (4.7). We remark that the preconditioned linear system takes less iterations to converge, with respect to the non-preconditioned one. Analogously to what we have observed when preconditioning the linear system deriving from the virtual control applied to the Poisson problem, we remark that the global matrix is well conditioned for high values of the penalization coefficient β. In order to apply this approach effectively one needs to find a suitable approximation for the preconditioner, because the computational cost needed for its assemblage and for its resolution is excessively high.

	$\beta = 0$		$\beta = 10^{-8}$		$\beta = 10^{-4}$		$\beta = 1$	
	I	P	I	P	I	P	I	P
$h = 0.2$	73	39	73	33	78	34	27	13
$h = 0.1$	120	41	121	41	137	41	27	13
$h = 0.05$	187	47	187	45	208	46	26	13

Table 4.7: *Solution of the global matrix (4.30) for different values of β, with (P) and without preconditioning (I).*

Conclusions

In this work we have investigated the virtual control approach in the contest of domain decomposition problems. The method has been extended to the case of Neumann boundary controls and different cost functionals have been considered for the observation of the difference between the solutions. The well posedness of the optimization problem has been proved and validated with numerical simulations. The method has been applied efficiently for the homogeneous domain decomposition of the scalar elliptic equation and of the Stokes problem.

The issue of preconditioning the optimality system that arises from a coupled virtual control problem has been studied and we have shown that a recent preconditioning approach, developed in the contest of interior point optimization problems, applies efficiently to the case of the scalar elliptic equation. Moreover, we have studied a Schur complement approach for the construction of the preconditioning matrix as it is typically done in the contest of domain decomposition. This preconditioner has been tested in both the cases of the elliptic equation and of the Stokes system. The preconditioners are shown to be efficient, but an approximate version of the matrices has to be introduced due to the high computational costs needed for their assemblage and resolution.

A good understanding of the coupling of the homogeneous problems is the first step towards a heterogeneous domain decomposition approach. The method can be applied to the solution of the Stokes/Darcy problem that describes the process of the filtration of a fluid in porous media; in fact this problem treats the coupling of the Stokes equation for the fluid with the Darcy scalar elliptic equation for the pressure in the porous media. The analysis of the well posedness of the different cost functionals has been carried out for the homogeneous problems and can be extended to the multi-physics problem. We have shown which cost functionals guarantee convergence in the case of the elliptic equation and of the Stokes problem. In particular we have shown that in the case of the Stokes problem the cost functional should depend on the difference between the velocities and the pressures simultaneously. The choice of the space of the controls, either Neumann or Dirichlet, permits a flexible approach and the results on preconditioning obtained in this work represent a starting point for the analysis of the issue of preconditioning the coupled heterogeneous problem.

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