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A CALIBRATION METHOD FOR HJM MODELS BASED ON THE LEVENBERG-MARQUARDT OPTIMIZATION ALGORITHM

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Abstract

This work presents a particular approach to the calibration procedure for Heath-Jarrow-Morton models. Are analyzed precisely payer swap options for which have been computed prices through Monte Carlo simulations of the HJM dynamics. Results are compared with prices obtained using a Black like formula focusing on qualitative aspects due to the characteristic of the framework. The main problem has been to specify the volatility structure for practical purpose. So that has been taken under consideration the Hull-White model to provide it. This leads to the creative and interesting part of the work, which is the calibration procedure of the parameters of the model. For this aim, using Matlab is originally implemented the Levenberg-Marquardt optimization algorithm which minimize iteratively the sum of the square of the difference between Black and HW prices, paying particular attention to the behavior of the Hull-White model. The variables considered are precise and the parameters encountered respect the usual range of values present in the literature, confirming the goodness of the approach.

The first three chapters consist in a theoretical part necessary for a better comprehension of the following development. Are presented the basis of the interest rate theory, then the Heath-Jarrow-Morton framework and its characteristic, and thirdly a description of interest rate derivatives. In chapter four is calculated the volatility structure for HJM through a calibration process of HW parameters involving the implementation of the Levenberg-Marquardt algorithm. The fifth chapter deals with the pricing of a payer swap option within the HJM dynamics involving Monte Carlo simulation. Finally are presented possible developments and conclusions, while in the appendices is exposed the Matlab code.

Abstract

Questo lavoro nasce dalla volontà di investigare riguardo a modelli su tassi di interesse e come questi possano essere combinati per ottenere prezzi di derivati adeguati, che rappresenta uno dei principali scopi di un ingegnere finanziario. In particolare viene presa in considerazione la dinamica di Heath-Jarrow-Morton per prezzare payer swaptions, con la quale si ottengono, attraverso simulazioni Monte Carlo, risultati solamente qualitativi a causa delle caratteristiche del framework; è infatti nota una sua certa tendenza a sovraestimare. HJM deve la sua importanza al fatto che teoricamente qualsiasi modello di tassi di interesse può essere derivato a partire da esso. D'altro canto la difficoltà nell'ottenere una funzione di volatilità adatta conduce spesso a calcoli onerosi.
Per l'obiettivo appena presentato è quindi necessario fornire una matrice di volatilità, la cui definizione rappresenta il problema principale nel caso pratico. Questo conduce all'anima di questo studio, che può essere identificata con il processo di calibrazione dei parametri del modello di Hull-White extended Vasicek, a partire dai quali si possono ricavare le volatilità necessarie in maniera abbastanza semplice per via del fatto che il modello possiede struttura affine. Per ottenere i giusti parametri si utilizza l'algoritmo di ottimizzazione di Levenberg-Marquardt appositamente adattato alla situazione in questione. Questo risulta essere uno strumento potente in grado di raggiungere rapidamente soluzioni ottimali, precise e che rispettano i canoni incontrati in letteratura. La sua implementazione consiste in minimizzare iterativamente la somma dei quadrati delle differenze dei prezzi ottentuti attraverso una formula di tipo Black e tramite il modello di Hull-White.

I primi tre capitoli sono dedicati a una presentazione della parte teorica del lavoro, necessaria per meglio comprendere il seguito. Nel primo vengono introdotte le basi della teoria sui tassi di interesse, vengono fornite definizioni e formule dei principali strumenti, viene dato qualche accenno sulla teoria di non arbitraggio e sui modelli unifattoriali per i tassi di interesse.
Nel secondo capitolo si analizza accuratamente HJM, proposto da Heath, Jarrow e Morton per far fronte alle inadempienze dei modelli preesistenti. Essi derivarono il framework libero da arbitraggio per l'evoluzione stocastica dell'intera

curva dei rendimenti, dove la dinamica dei tassi forward è interamente specificata attraverso la struttura di volatilità istantanea, la quale presenta delle difficoltà nella sua definizione. Sono esposti quindi vantagi e svantaggi, e la relazione con alcuni modelli unifattoriali. A partire da HJM infatti teoricamente qualsiasi modello su tassi di interesse può essere ricavato.
Nel terzo capitolo vengono presentati i derivati, strumenti finanziari che hanno aquisito grande importanza e popolarità negli ultimi anni. In particolare sono presi in questione derivati su tassi di interesse, a partire dagli swaps, passando per caps e floors, per giungere agli swaptions.
Nel quarto capitolo si tratta il procedimento per ottenere la matrice di volatilità cercata. Per questo ci sono diverse possibilità, tra cui si è scelto il modello già esistente di Hull-White che fornisce una formula esplicita dovuta al fatto che possiede struttura affine. Altri modelli, come i multifattoriali, possono adattarsi meglio ai dati e essere più performanti ma conducono spesso a calcoli eccessivamente complicati. Per quanto trattato in questo lavoro si è ritenuto sufficiente il modello considerato, che è semplice ma efficace. Viene presentata la sua dinamica e vengono esposti i dati di mercato utilizzati per l'analisi. Si procede quindi alla calibrazione dei parametri attraverso l'implementazione in Matlab dell'algoritmo di ottimizazzione di Levenberg-Marquardt adattandolo al caso corrente e ponendo particolare attenzione alle caratteristiche di Hull-White, i cui prezzi vengono messi a confronto con quelli ottenuti tramite una formula di tipo Black.
Nel quinto capitolo si prezzano payer swaption per diverse maturities e tenors a partire dalla dinamica di Heath-Jarrow-Morton, per la quale viene realizzata un'analisi specifica che mostra come i valori tendano a risultare sovraestimati. Per effettuare le simulazioni necessarie si utilizza il metodo di Monte Carlo in quanto non è possibile ottenere una forma esplicita e, per una convergenza più rapida, si ricorre alla tecnica delle variabili antitetiche.
L'argomento trattato permette alcune modifiche che vengono commentate nelle conclusioni, dove sono presentate diverse osservazioni e le varie possibilità di sviluppo del lavoro. In quanto ai codici implementati sono riassunti negli appendici.

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Contents

Introduction

This work arises from the keen wish of investigate on some interest rate models, trying to figure out how they can be combined in order to obtain derivative prices. The latter in fact is one of the major challenges that a financial engineer has to cope with. In particular Hull-White model is taken under consideration to provide a volatility structure to the Heath-Jarrow-Morton framework, used to study precisely the case of payer swaptions. This is a topic which is still under process and not fully explained in the literature. The principal reason lies in the still actual problem of defining a suitable volatility function for HJM in practical cases, which generally means that burdensome procedures are necessary to price interest rate derivatives. However the framework assumes particular importance due to the fact that, in theory, allows to derive any interest rate model.
The subject is widely treated, especially by Carl Chiarella. In his numerous papers are considered classes of HJM term structure models either characterized by time deterministic volatilities for the instantaneous forward rate or with stochastic volatility; the latter models admit transformations to Markovian systems and from them is possible to get well solution techniques for the bond and bond option pricing, and in some situations are included numerical simulations. Furthermore the model has been estimated via the maximum likelihood method, obtaining estimators for observable future prices, and applied to interbank rates in different markets. In this work swap options have been taken under analysis and has been implemented an apposite code through the language of technical computing Matlab to obtain the prices, recurring to Monte Carlo simulations.
Numerous books present as well the affine term structure model of Hull-White and its dynamics. As well can be found analytical formulas for the principal financial instruments, while complex derivatives are valued using a tree or partial differential equation approach. Furthermore are given some indications on the usual range of the the parameters and how to chose them through a cali-

bration procedure. To obtain their best values, in this study, has been used a Levenberg-Marquardt approach. This is an optimization algorithm which has been adapted to the current situation in an original way. The implementation is easy to understand and it takes under consideration the characteristic of the HW model. It is powerful, it allows to well replicate swaption prices and can readily be modified to be used for different models and other derivatives.
The aim of the work is to supply the reader, which is supposed to own already a basic knowledge of stochastic differential equations, a qualitative and quantitative analysis of the models under question. This is done with the fundamental support of a list of figures which often offers a better comprehension.

So that, in the first chapter are presented the basis of the interest rate theory. Firstly are given definitions and formulas of the principal instruments, starting from discount factors and zero coupon bond to get to spot and forward interest rates. It is then briefly presented the absence of arbitrage theory introducing concepts as the equivalent martingale measure and the numeraire. To conclude is given an hint of short interest rate models. This is a short introduction to the real topic, important to understand the language and symbols used in the work. In the second chapter is taken under accurate observation the HJM framework, built up by Heath, Jarrow and Morton in order to solve drawbacks arising in short rate models. They proposed the idea to capture the full dynamics of the entire yield curve, developing a complete model which does not involve the market price of risk. Furthermore, at least in theory, it allows to derive any kind of interest rate model. As a consequence is shown the relation with some short interest rate models. It is as well exposed the way undertaken by Heath, Jarrow and Morton to get the final dynamics. However the HJM framework is not free of disadvantages, in fact can be complicated define a suitable volatility for practical purpose and it does not provide exact solution for pricing derivatives. So that a numerical method is necessary and it will be explained how to use it. The third chapter presents a large argumentation of derivatives, which are financial instruments that recently became increasingly popular. A large variety of these objects exists then in the market in order to supply the demand. Particularly the attention is focused on interest rate derivatives, for which are given definitions and explicit formulas. A swap is a contract between two counterparties that exchange series of future cash flows dependent on the type of financial instrument involved. Cap and Floor, which are strictly related, are contracts designed to guarantee to the holder that floating rates do not exceed a specific level, in the first case , and the opposite for the second contract. These derivatives and swap options represent the main product of interest rate. A swaption is contract that gives the holder the right but not the obligation to enter into an underlying interest rate swap at a certain future time. It can be distinguished into two types, a peyer swap option giving the right but not the obligation to pay fixed rate and receive floating rate in the underlying swap, while is the contrary for a receiver swaption.

Chapter four is the soul of the work. Its aim is to provide the volatility structure necessary for the HJM dynamics. This can be done in different ways, here has been taken the already existent short rate model of Hull-White extended Vasicek, which gives explicit formula for absolute volatilities due to the fact that it is an affine term structure model. Are exposed the available data of discount factors and at the money volatilities and strikes. Then the dynamics of the model is presented and through a calibration procedure involving a Black like formula the parameters of the model are estimated. For this purpose a Matlab code has been implemented, and graphs are useful for a specific analysis. An original modification of the Levenberg-Marquardt algorithm has been computed in order to obtain calibration results. It has been adapted to the current situation and has been improved to accommodate drawbacks deriving from HW.

The fifth chapter deals with the particular case of pricing a payer swap option through the Heath-Jarrow-Morton framework, whose dynamics allows to derive the important financial instruments necessary to compute the prices. Monte Carlo methods are then introduced for the simulation of the dynamics, in fact there is no way to get an explicit pricing formula. In respect to it has been used a simple technique to accelerate the convergence of the price to the final solution. After a specific analysis on the dynamics of HJM, from which is possible to get important information, are exposed and commented the results deriving from the simulations.

Finally are given conclusions and observations about the results. In the last chapter are exposed as well possible alternatives and future developments. The argument touched in this study is wide enough to leave many opportunities to modify and extend models and cases analyzed in the work. For these reasons an apposite section is devoted to them.

In order to give a clear explanation, in the appendixes is exposed most of the code used during the research which has been implemented through the language of technical computing Matlab. They are allocated in two different chapters following the respective argument of which they refer.

Chapter 1

Interest rate theory

This chapter present concepts of the interest rate world. Nowadays interest rate is a notion that enter in the daily life of many people. If someone goes to a bank in order to deposit some money, then expects the amount grows in time. Consequently it is already in his mind the fact that the value of a certain amount of money is not the same today and tomorrow. Holding a capital before someone else allowed to make a profit out of it, hence this opportunity has a value.

The most famous interest rate is the LIBOR (London InterBank Offered Rate) that is the rate at which an international bank lends money to another international bank. It is often considered as a reference for contracts and it can be used as the risk free rate when derivatives are valued. An analogous interest rate present in other markets is the EURIBOR fixing in Brussels.

1.1 Money-market account and discount factor

The money-market account or bank account is a process that describe the evolution of a riskless investment, where the profit is compounded continuously at the risk free rate present in the market at every moment.
Let $B(t)$ be the value of the bank account at time $t \geq 0$ and assume that its evolution progresses according to the following differential equation:

$$dB(t) = r_t B(t)dt, \qquad B(0) = 1$$

where r_t is the *instantaneous rate* at which the money account accumulates capital. Integrating we have

$$B(t) = exp\left\{\int_0^t r_s ds\right\}$$

with r_s positive function of time. As a consequence, investing a unit of currency today we obtain at time t an amount given from the above definition.

Frequently, in financial mathematic, is necessary to use the inverse operation, which is the discount of a certain amount of money from a future date T to present.
The *discount factor* $D(t,T)$ between t and T, is the invested amount at time t that give back a unit of currency at time T:

$$D(t,T) = \frac{B(t)}{B(T)} = exp\left\{-\int_t^T r_s ds\right\}.$$

In many pricing application r_t can be assumed deterministic. However, the evolution of r_t is modeled through a stochastic process when dealing with interest rate products.

1.2 Structures of spot and forward interest rates

In the financial market is possible to distinguish two different kind of operations, they can be identified as spot and forward operations.
With the term spot operation we define all the processes in which there is an immediate payment caused from the sale of a financial product, the exchange of money between the two counterparts arises at same moment of the stipulation of the contract.
It turns out useful to define a T-maturity *zero-coupon bond* or pure discount bond, as a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. We denote the price of this contract at time $t < T$ by $p(t,T)$, as a consequence $p(T,T) = 1$ for all T. Hence a zero-coupon bond establishes the present value of one unit of currency to be paid at time T. As the future discount rate is unknown it is only possible to calculate the bond expected price

$$p(t,T) = \mathbb{E}\left[exp\left\{-\int_t^T r_s ds\right\}\right]. \tag{1.1}$$

In order to present a forward contract it is necessary to identify three different instants of time, as in the spot case the moment t at which the rate is considered and T that sign the end of the exchange of money; however, in this occasion, we introduce as well S as the time that signs the beginning of the money trading. We denote by $p(t, S, T)$ the forward price defined at t for a zero-coupon bond with money exchange at time S and ending at T. Summarizing, the contract it is stipulated at t, one part pay the amount $p(t, S, T)$ at S and the counterpart give back a unit of currency at T. Note that in the particular case when $S = t$ the forward contract become a spot contract, $p(t, t, T) = p(t, T)$.

Turns out natural now define the *time to maturity* $T - t$ as the amount of time from the present t to the maturity $T > t$. Arise necessary to identify this amount of time in terms of the number of days between the two dates. It is calculated according to the relevant market convention, this choice is not unique. Here are mentioned three examples:

- Actual/365. Assuming the right length of each month and a year of 365 days long.
- Actual/360. With this convention is considered a year 360 days long.
- 30/360. Months are in this case assumed to be 30 days long.

1.3 From zero-coupon bonds to spot interest rates

With the hypothesis that in the market there are exchanged zero-coupon bonds for every maturity T, the price $p(t, T)$ of a zero-coupon bond exchanged at t, that give one unit of currency at time T, leads towards the definition of three different measure of rate:

1. *Simply-compounded spot rate* in the interval $[t, T]$

$$L(t, T) = \frac{1 - p(t, T)}{(T - t)p(t, T)}.$$

2. *Yield rate* or continuously-compounded spot rate in the interval $[t, T]$

$$y(t, T) = -\frac{logp(t, T)}{T - t}.$$

3. *Annually compounded spot rate* in the interval $[t, T]$

$$Y(t,T) = \frac{1}{p(t,T)^{1/(T-t)}} - 1.$$

It is easy to see that the first is the constant rate at which an investment of $p(t,T)$ at time t produce an amount of un unit of currency at maturity, the second is the constant rate at which an investment has to be made to accrue continuously until yield a unit of currency at maturity, and the last one is the constant rate at which an investment of $p(t,T)$ units of currency at t, when reinvesting the gained amounts every year, gives back one unit of the currency at maturity. Interest rates are more useful then price of zero-coupon bonds when comparing between operations with different maturity.

1.4 Forward rates

Forward rates are interest rates that can be locked in today for an investment in a future time period. Can be demonstrate that in a arbitrage-free market, where subsists the impossibility to make money from nothing (see [2] for further explanations) the following holds

$$p(t,S,T) = \frac{p(t,T)}{p(t,S)}.$$

Since the above relation holds it is possible to define two different forward rates:

1. *Simply-compounded forward rate* defined in t for the interval $[t, T]$

$$F(t,S,T) = \frac{1 - p(t,S,T)}{(T-S)p(t,S,T)} = \frac{p(t,S) - p(t,T)}{(T-S)p(t,T)}.$$

2. *Continuously-compounded forward rate* defined in t for the interval $[t, T]$

$$y(t,S,T) = -\frac{logp(t,S,T)}{T-S} = -\frac{logp(t,T) - logp(t,S)}{T-S}.$$

These formulas lead to spot interest rates for $S = t$.
When the maturity of the simply-compounded forward rate collapses to towards its expiry date S, we obtain the *instantaneous forward rate* at T defined at time t

$$f(t,T) = -\frac{\partial logp(t,T)}{\partial T}$$

which denotes the interest strength at time T implicit in the price $p(t,T)$. From the above follows

$$p(t,T) = exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Considering at this point the 1.1, by letting T tend to t the expectation becomes known at t and comparing with the latter equation results $r(t) = f(t,t)$.

It is now possible define a *forward rate agreement*:

$$FRA(t,S,T,N,R) = Np(t,T)(T-S)(R - F(t,S,T)). \tag{1.2}$$

The latter is a contract made on a nominal value N where the parties, a lander and a borrower, agree to exchange a fixed rate R with a simply-compounded spot rate $L(S,T)$. It means that the holder at maturity receives a fixed amount and pays $N(T-S)L(S,T)$. This contract can be generalized to the *interest rate swap* (IRS), where the parties exchange a series of cash flows starting at a future time $S = T_\alpha$. In this case, the two counterparts pay the respectively amounts $N(T_i - T_{i-1})R$ and $N(T_i - T_{i-1})L(T_{i-1}, T_i))$ at every instant T_i for $i = \alpha+1, \ldots, \beta$ where $T = T_\beta$. Consequently we define the *forward swap rate* as the fixed rate R for which the IRS is a fair contract

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t,T_i)}.$$

1.5 Absence of arbitrage and change of numeraire

In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (in order to have a detailed explanation of mathematical objects presented, the interested reader can have a look at [6], [18] or [16]) are given $n+1$ non-dividend paying trading assets, whose prices are indicated by $B = S^0, \ldots, S^n$ where $B(t)$ is a money-market account. It is then possible to define a *trading strategy* as a process $(h_t)_{t\in[0,T]}$ with predictable components $h^0, \ldots, h^n$, and its associated *value process*

$$V_t(h) = h_t S_t = \sum_{i=0}^{n} h_t^i S_t^i.$$

Hence, we say that h is *self-financing* if its value changes only due to changes in the asset prices, which means that after the initial time cash flows do not take places.

A *contingent claim* H is a random variable belonging to $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and, if exists a self-financing portfolio h that $V_T(h) = H$, the contingent claim is said to be *attainable*. The price of h at t associated with it is denoted by $\pi_t = V_t(h)$.

Let introduce a new probability measure $\mathbb{Q}$ on the space $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$, which means that they share the same set of null probability. Then, if the Radon-Nikodin derivative $d\mathbb{Q}/d\mathbb{P}$ is square integrable with respect to $\mathbb{P}$ and the discounted asset price process $D(0, \cdot)S$ is a $\mathbb{Q}$-martingale, $\mathbb{Q}$ is said to be an *equivalent martingale measure*. If it does exist the market is free of arbitrage and taking now an attainable H, the associated price π_t is unique for $t \in [0, T]$ and it is given by $\pi_t = \mathbb{E}[D(t, T)H \mid \mathcal{F}_t]$. When every contingent claim is attainable the market is said to be complete and the martingale measure is unique.

A *numeraire* Z is any positive non-dividend-paying asset chosen to normalize all other asset prices with respect to it. A self-financing portfolio does not change after modify the numeraire. The measure $\mathbb{Q}^T$ associated with the zero coupon bond whose maturity is T is known as *T-forward measure*. Then a price at time t of a derivative with maturity T whose payoff is H_T is given by

$$\pi_t = p(t, T)\mathbb{E}^T[H_T \mid \mathcal{F}_t] \qquad \forall t \in [0, T]$$

which can be calculated under the risk-neutral measure $\mathbb{Q}$ using

$$\pi_t = \mathbb{E}\left[exp\left\{-\int_t^T r_s ds\right\} H_T \mid \mathcal{F}_t\right] \qquad \forall t \in [0, T].$$

1.6 Short rate models

As seen above in order to calculate the price processes of a zero-coupon bond and of a derivative with maturity T is necessary to specify the instantaneous spot interest rate. The hypothesis that the latter is constant can be accepted only for contracts with short life. So that arise necessary to figure out which is its behavior over the interval $[0, T]$. The classical approach is to assume that the dynamics of r_t evolves under the measure $\mathbb{P}$ as follow:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW^P(t).$$

In an arbitrage free market where exist bonds for all maturities, through the Ito's formula, the price dynamics for the T-bond it is given by

$$dp(t,T) = \alpha^T(t, r(t))p(t,T)dt + \sigma^T(t, r(t))p(t,T)dW^P(t)$$

then there exists a stochastic process λ such that

$$\frac{\alpha^T(t, r(t)) - r(t)}{\sigma^T(t, r(t))} = \lambda(t) \tag{1.3}$$

holds for all t and for every choice of maturity T. Considering the risk-neutral measure $\mathbb{Q}$ the instantaneous spot rate evolves according to

$$dr(t) = [\mu(t, r(t)) - \lambda(t)\sigma(t, r(t))]dt + \sigma(t, r(t))dW(t).$$

Taking a closer look to the process λ we see that the numerator of the formula 1.3 is called *risk premium* and represents the excess rate of return for the risky bond over the risk-free rate. Then λ is defined by the risk premium per unit of volatility, for this reason the process is known as *risk free rate*.

The given market is not complete as in the Black-Scholes model. As a consequence there is not only one equivalent martingale measure and the bond prices are not uniquely determined by the dynamics under $\mathbb{P}$ of the short rate. When the market has determined the dynamics of a price bond process then λ is determined and all the bond prices can be calculated. The market price of risk is implicit in the considered dynamics and has to be specified by using market data.

In the literature have been proposed numerous way on how to specify the dynamic of the short rate under the equivalent martingale measure. See [9] to obtain a deep knowledge with respect to this argument.
Here are presented some of the most popular short rate models:

Vasicek : $dr(t) = k[\theta - r(t)]dt + \rho dW(t)$

Cox-Ingersoll-Ross (CIR) : $dr(t) = k[\theta - r(t)]dt + \rho\sqrt{r(t)}dW(t)$

Ho-Lee : $dr(t) = \theta(t)dt + \rho dW(t)$

Hull-White extended Vasicek : $dr(t) = k(t)[\theta(t) - r(t)]dt + \rho(t)dW(t)$

Hull-White extended CIR : $dr(t) = k(t)[\theta(t) - r(t)]dt + \rho(t)\sqrt{r(t)}dW(t)$

These model are referred as *single factor models* because they present a single source of randomness. The main advantage of these methods lies in the fact that it is allowed to specify the short rate as a solution of a stochastic differential equation and through Markov theory use the associated partial differential equation. A drawback is that they can not reproduce satisfactorily the market curve which seems to depend on all the rate and not only on the short interest rate considered to construct the model. A consistent alternative to short rate models is the Heath-Jarrow-Morton framework, an application incorporating the entire yield curve that will be present in the next chapter.

Chapter 2

Heath - Jarrow - Morton Framework

As stated in the last chapter modeling interest rate using short rate models gives often the possibility to obtain analytical formulas for bond prices and derivatives. However, they present some drawbacks, indeed does not seem reasonable use only one explanatory variable to describe the market, furthermore a complicated model is necessary to reconstruct properly the volatility observed in forward rates and as it becomes more realistic increase the difficulty to obtain the inverse of the yield curve.

The Heath-Jarrow-Morton framework (HJM) arises as the most straightforward solution to disadvantages occurred in short rate models. It is a general framework to model the evolution of interest rates built up by Heath, Jarrow and Morton in the late 1980s (see [3]). They developed the idea to capture the full dynamics of the entire yield curve, proposed firstly by Ho and Lee, extending it in continuous time. Practically, this framework is based on focusing on the instantaneous forward rate as fundamental quantity to be modeled and this is one of the most general way to express absence of arbitrage opportunities. Precisely, the drift in the forward rate dynamics can be expressed as function of the diffusion coefficient in the same dynamics. Differently from the one factor short rate model, no drift estimation is needed, it is completely determined by the chosen volatility coefficient.

The framework for the pricing of interest rate derivatives is consistent under HJM, it models the evolution of the entire forward rate curve while the short rate models only capture a point on the curve. Such a framework is important because in theory allows to derive any interest rate model; analogies with particular short rate models will be show later in this chapter. Furthermore the model is complete, which means that does not involve the market price of risk, this is an important result sought by earlier studies.

However, HJM presents some disadvantages, in fact it is not trivial to define a suitable volatility for practical purposes, and only a restricted class of volatilities is known to imply a markovian short rate process. As a consequence, determine prices of derivatives can be very complicated and particularly slow. Precisely, the model does not provide an exact solution, which means that, in order to be implemented, a numerical method, such as Monte Carlo, is necessary.

2.1 The dynamics of HJM

The foundation on which Heath-Jarrow-Morton framework is built, for a fixed maturity T, is the dynamics of the instantaneous forward rate $f(t,T)$ which evolves according to

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f'(0,T), \end{aligned}$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are adapted processes, W is an N-dimensional Wiener process under the measure $\mathbb{P}$ (see [1] and [15] to have a full argumentation of stochastic processes), while as the initial condition is used the market forward rated curve $\{f'(0,T) : T \geq 0\}$ which provides a perfect fit between theoretical and observed bond prices at present time.

This dynamics does not provide necessarily a situation without arbitrage possibilities. In order to obtain a unique equivalent martingale measure, the approach followed by Heath, Jarrow and Morton was to model the continuously compounded instantaneous forward rate $f(t,T)$, through the basic arbitrage relationship encountered in the bond pricing theory:

$$f(t,T) = -\frac{\partial logp(t,T)}{\partial T}$$

from which follows

$$p(t,T) = exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{2.1}$$

They found out that the drift is completely determined by the vector diffusion process and cannot be chosen arbitrarily. Particularly it has to be equal to an object depending on σ and the drift rates in the dynamics of N zero coupon bond prices. Furthermore, as we will see, the short rate process does not require to be modeled but is not a Markov process in general.

2.1.1 The instantaneous forward rate dynamics under the risk-neutral measure

Go by Black's theory (see [12] for further explanations) Heath Jarrow and Morton assumed the following dynamics for typical bonds under the risk-neutral measure:

$$dp(t,T) = r(t)p(t,T)dt + \nu(t,T)p(t,T)dW(t) \tag{2.2}$$

where the risk free instantaneous spot rate is the same for all bonds and assets. For notational convenience the path dependance has been omitted, so the vector volatility is written $\nu(t,T)$ rather then $\nu(t,T,p(t,T))$.

Considering a pair of zero coupon bonds with different maturities, which prices are denoted by $p(t,T)$ and $p(t,U)$. From the arbitrage relationship 2.1 we obtain

$$\frac{p(t,T)}{p(t,U)} = exp\left\{\int_T^U f(t,s)ds\right\}$$

and rearranging this expression

$$log\frac{p(t,T)}{p(t,U)} = \int_T^U f(t,s)ds.$$

Taking infinitesimal interval δ, it is possible to write the above in terms of simply forward rate (non instantaneous) as follow

$$d[logp(t,T)] - d[logp(t,T+\delta)] = F(t,T,T+\delta)\delta,$$

then through Ito's lemma we have

$$d[logp(t,T)] = \frac{dp(t,T)}{p(t,T)} - \frac{1}{2}\frac{(\nu(t,T)p(t,T))^2}{p(t,T)^2}dt$$

and similarly

$$d[logp(t,T+\delta)] = \frac{dp(t,T+\delta)}{p(t,T+\delta)} - \frac{1}{2}\nu(t,T+\delta)^2 dt.$$

Replacing 2.2 and subtracting the last expression from the one before the short rate disappear

$$d[logp(t,T)] - d[logp(t,T+\delta)] = \frac{1}{2}[\nu^2(t,T+\delta) - \nu^2(t,T)]dt +$$
$$+[\nu(t,T) - \nu(t,T+\delta)]dW(t).$$

This is due to the fact that we are in a risk-neutral environment, therefore r_t is the same in the two equations.
As a consequence

$$\begin{aligned} F(t,T,T+\delta) &= \frac{d[logp(t,T)] - d[logp(t,T+\delta)]}{\delta} \\ &= \frac{\nu^2(t,T+\delta) - \nu^2(t,T)}{2\delta}dt - \frac{\nu(t,T+\delta) - \nu(t,T)}{\delta}dW(t). \end{aligned}$$

Recalling the formula of partial derivative

$$\frac{\partial g}{\partial x} = \lim_{\delta \to 0} \frac{g(x+\delta) - g(x)}{\delta}$$

and letting $\delta \to 0$, the terms of the above become

$$\lim_{\delta \to 0} F(t,T,T+\delta) = df(t,T),$$
$$\begin{aligned} \lim_{\delta \to 0} \frac{(\nu(t,T+\delta))^2 - (\nu(t,T))^2}{2\delta} &= \lim_{\delta \to 0} \left\{ \frac{\nu(t,T+\delta) + \nu(t,T)}{2} \frac{\nu(t,T+\delta) - \nu(t,T)}{\delta} \right\} \\ &= \nu(t,T)\frac{\partial \nu(t,T)}{\partial T} \end{aligned}$$

and

$$\lim_{\delta \to 0} \frac{\nu(t,T+\delta) - \nu(t,T)}{\delta} = \frac{\partial \nu(t,T)}{\partial T}$$

therefore we obtain

$$df(t,T) = \nu(t,T)\frac{\partial \nu(t,T)}{\partial T}dt - \frac{\partial \nu(t,T)}{\partial T}dW(t).$$

At this point, simply defining $\sigma(t,T) := -\partial\nu(t,T)/\partial T$ we find the famous HJM drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds,$$

so that the integrated dynamics of the instantaneous forward rate under the risk-neutral measure is

$$f(t,T) = f(0,T) + \int_0^t \sigma(u,T)\int_u^T \sigma(u,s)dsdu + \int_0^t \sigma(s,T)dW(s)$$

which is completely specified when the vector volatility process is provided. Finally, the instantaneous short rate at time t does not need to be modeled with a diffusion process but can be derived from the instantaneous forward rate as follow

$$r(t) = f(t,t) = f(0,T) + \int_0^t \sigma(u,t)\int_u^t \sigma(u,s)dsdu + \int_0^t \sigma(s,t)dW(s).$$

As earlier mentioned the above does not fulfill the Markov property, in fact the time t appears inside the integral function and as extreme of integration. However, it is possible to identify particular volatilities that make it a Markov process. A special case was detected by Carverhill, who considered in his studies vector volatility of the type

$$\sigma(t,T) = \phi(t)\psi(T),$$

where ϕ and ψ are strictly positive and deterministic functions of time. As a consequence, the short rate process assumes the form

$$r(t) = f(0,T) + \psi(t)\int_0^t \phi^2(u)\int_u^t \psi(s)dsdu + \psi(t)\int_0^t \phi(s)dW(s). \qquad (2.3)$$

Other studies were made by Ritchken and Sankarasubramanian for the one dimensional case (see [17] for details). They found out a necessary and sufficient condition on the volatility structure of forward rate for the price of any interest rate derivative to be completely determined by a Markov process of the form $\chi(t) = (r(t), \xi(t))$. Taking an adapted process η and a deterministic function k the following condition holds

$$\sigma(t,T) := \eta(t)exp\left\{-\int_t^T k(v)dv\right\}. \qquad (2.4)$$

In this situation, ξ is defined by

$$\xi(t) = \int_0^t \sigma^2(s,t)ds$$

whose stochastic differential equation is given by

$$d\xi(t) = [\eta^2(t) - 2k(t)\xi(t)]dt,$$

while the dynamics of the short rate evolves according to

$$\begin{aligned} dr(t) = {} & \frac{\partial}{\partial t} f(0,t)dt + \left[\sigma(t,t)\int_t^t \sigma(t,s)ds\right]dt - \left[\int_0^t k(t)\sigma(u,t)\int_u^t \sigma(u,s)dsdu\right]dt+ \\ & + \left[\int_0^t \sigma^2(u,t)du\right]dt - \left[\int_0^t k(t)\sigma(s,t)dW(s)\right]dt + \sigma(t,t)dW(t) \\ = {} & \left[\frac{\partial}{\partial t} f(0,t) + k(t)[f(0,t) - r(t)] + \xi(t)\right]dt + \eta(t)dW(t) \end{aligned}$$

where we used the equation for the dynamic of the short rate and

$$\sigma(t,t) = \eta(t), \qquad \frac{\partial}{\partial t}\sigma(u,t) = -k(t)\sigma(u,t),$$

$$\frac{\partial}{\partial t}\left(\sigma(u,t)\int_u^t \sigma(u,s)ds\right) = -k(t)\sigma(u,t)\int_u^t \sigma(u,s)ds + \sigma^2(u,t).$$

We have then calculated the evolution of the Markov process:

$$d\chi(t) = \begin{pmatrix} dr(t) \\ d\xi(t) \end{pmatrix}.$$

If the volatility function is as given in 2.4, then zero coupon bond prices are given by the formula

$$p(t,T) = \frac{p(0,T)}{p(0,t)} exp\left\{-\frac{1}{2}\gamma^2(t,T)\xi(t) + \gamma(t,T)[f(0,t) - r(t)]\right\}$$

where

$$\gamma(t,T) = \int_t^T exp\left\{-\int_t^u k(v)dv\right\}du.$$

Similar results were obtained by Inui and Kijima for the N-dimensional case in [10].

As a conclusion, an arbitrarily chosen of the forward rate volatility leads to a short rate process that is not Markovian. This implies major computational problems making a discretization of the short rate dynamics for the pricing of a derivative. Beyond the over exposed, further proposes for a suitable volatility structure where made, an example is given by the Mercurio and Moraleda Model presented in [13], where an analytical formula for the price of a European call option on pure discount bond is achieved.

2.2 How to use HJM

This section is devoted to briefly describe step by step how to use the Heath-Jarrow-Morton framework in order to compute prices of derivatives through numerical method such as Monte Carlo (see [20] for a closer examination).

1. Firstly is necessary to specify the volatility structure $\sigma(t,T)$ by using one of the models present in the literature or considering a new own choice. Then observe the market instantaneous-forward curve.

2. Simulate the evolution of the entire forward rate curve in the risk neutral world until the date needed

$$df(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds + \sigma(t,T)dW(t).$$

3. Compute bond pricing for all dates through the formula

$$p(t,T) = exp\left\{-\int_t^T f(t,s)ds\right\}.$$

4. Obtain cash flows using forward rates.

5. Consider the short rate in order to calculate the present value of the cash flows.

6. Go back at point 2 in order to make enough realizations to have the discounted expected value with the desired precision.

2.3 Relation with short rate models

As said in the first part of the chapter, it is possible to obtain classes of interest models under the HJM framework. In theory, every short rate model can be equivalently formulated in term of the forward rate.

2.3.1 Hull-White extended Vasicek

Let us go back to the markovian case, precisely consider the Carverhill's formulation 2.3 for the short rate (have a look at [3]). In the case it is considered only one factor, it is possible to define the deterministic and differentiable function G by

$$G(t) := f(0,t) + \psi(t)\int_0^t \phi^2(u)\int_u^t \psi(s)dsdu$$

in order to obtain

$$\begin{aligned} dr(t) &= G'(t)dt + \psi'(t)\int_0^t \phi(s)dW(s) + \psi(t)\phi(t)dW(t) \\ &= \left[G'(t) + \psi'(t)\frac{r(t)-G(t)}{\psi(t)}\right]dt + \psi(t)\phi(t)dW(t). \end{aligned}$$

Now define

$$\vartheta(t) = G'(t) - \frac{\psi'(t)}{\psi(t)}G(t), \qquad k(t) = -\frac{\psi'(t)}{\psi(t)}, \qquad \rho(t) = \psi(t)\phi(t)$$

the model reduces to Hull-White encountered at the end of the last chapter

$$dr(t) = [\vartheta(t) - k(t)r(t)]dt + \rho(t)dW(t)$$

where $\vartheta(t) = k(t)\theta(t)$. With trivial definition of the variables and considering θ as a constant we find the model stated by Vasicek.

2.3.2 Hull-White extended CIR

The dynamics of the short rate in the Hull-White extended CIR evolves according to

$$dr(t) = k(t)[\theta(t) + r(t)]dt + \rho(t)\sqrt{r(t)}dW(t).$$

In order to achieve such equation under the HJM framework (see [4]) it is necessary to define the volatility function as follow

$$\sigma(s,t) = \eta(s)\sqrt{r(s)}\exp\left\{-\int_s^t k(v)dv\right\}.$$

Recalling Ritchken and Sankarasubramanian framework we take care about the evolution of the Markov process $\chi(t)$. Simply differentiating, the dynamics of ξ takes the form

$$d\xi(t) = [\eta^2(t)r(t) - 2k(t)\xi(t)]dt$$

while for the short rate

$$\begin{aligned}dr(t) &= \frac{\partial}{\partial t}f(0,t)dt + \left[\sigma(t,t)\int_t^t \sigma(t,s)ds\right]dt - \left[\int_0^t k(t)\sigma(u,t)\int_u^t \sigma(u,s)dsdu\right]dt+\\ &+\left[\int_0^t \sigma^2(u,t)du\right]dt - \left[\int_0^t k(t)\sigma(s,t)dW(s)\right]dt + \sigma(t,t)dW(t) =\\ &= \left[\frac{\partial}{\partial t}f(0,t) + k(t)[f(0,t)-r(t)] + \xi(t)\right]dt + \eta(t)\sqrt{r(t)}dW(t)\end{aligned}$$

where, with respect to the case seen above only change

$$\sigma(t,t) = \eta(t)\sqrt{r(t)}.$$

Finally we get the expected result defining

$$\vartheta = \frac{\partial}{\partial t}f(0,t) + k(t)f(0,t) + \xi(t),$$

$$\eta(t) = \rho(t), \qquad \vartheta(t) = k(t)\theta(t).$$

As before, it is possible to obtain the model stated by Cox, Ingersoll and Ross adopting some variable as constants instead of functions of time.

2.3.3 Ho-Lee

In the Ho-Lee model the evolution of the spot rate is determined by the equation

$$dr(t) = \theta(t)dt + \rho dW(t).$$

This time, the trick is to consider the process σ as a deterministic constant in the drift condition. As a consequence it results for the forward rate

$$\begin{aligned}f(t,T) &= f(0,T) + \int_0^t \sigma^2(T-u)du + \int_0^t \sigma dW(s)\\ &= f(0,T) + \sigma^2 t\left(T-\frac{t}{2}\right) + \sigma W(t),\end{aligned}$$

in particular

$$r(t) = f(t,t) = f(0,T) + \sigma^2 t\frac{t^2}{2} + \sigma W(t),$$

so that the dynamics becomes

$$dr(t) = \left[\frac{\partial}{\partial t}f(0,t) + \sigma^2 t\right]dt + \sigma dW(t),$$

and now defining

$$\theta = \frac{\partial}{\partial t}f(0,t) + \sigma^2 t, \qquad \rho = \sigma$$

it is founded what we were looking for.

Chapter 3

Derivatives

In the last years derivatives have become increasingly popular, the growth of their markets represents an important development in finance. They are traded on exchanged and over the counter (OTC) markets, where the procedures become easier by the time passing due to the electronic and technological expansion. In the first case contracts are standardized and defined by the exchange, while in OTC are traded directly between two parties. The latter, is the largest market for derivatives, products such swaps, FRA and exotic options are almost always traded in this way.

A *derivative* is a financial instrument whose value is dependent on an underlying variable already existent in the market. Normally the underlying variables are asset prices, however it can be almost anything that is measurable. This contracts are therefore agreements between two parties with values linked to the expected future price of some assets. They are commonly used to reduce or eliminate the risk, provide leverage or speculate and make profit from the underlying movements.
It is important to highlight that in order to avoid inconsistency between the derivative and the underlying price, derivatives can not be priced arbitrarily in an absolute sense, but their prices are determined in terms of the market price of the underlying assets.

Exists a large variety of derivatives, we proceed here with a briefly introduction of the most notables. A *forward* contract is an agreement between two

parties where who assumes a long position commits to buy an asset at a certain future date for a determined price established at present time, while who assumes a short position sells the underlying. This simple derivative is not standardized and is therefore traded in over the counter markets. This is the main difference with a *future* which is indeed usually traded on an exchange. The characteristic of the contracts are the same, but in the latter there is a daily exchange of money.
An *option* is a contract that give the owner the right, but not the obligation, to buy (in the case of a *call* option) or to sell (*put* option) the underlying asset by a certain prefixed date, denoted *maturity*, for a certain price called *strike price* or *exercise price*. Exact specifications may differ depending on the option style. A *European* option allows the holder to exercise the option only at maturity while an *American* option can be exercised at any time during the life of the option.
Nonstandard derivatives that may include complex financial structures are sometimes identified as *exotic* options, the others considered standard are termed as *plain vanilla*. In this work will be emphasized interest rate derivatives, already introduced in the first chapter and explained in more details in the rest of this chapter.

3.1 Interest rate derivatives

The purpose of this part of the work is to give a general presentation of products inherent in interest rates. *Interest rate derivatives* are financial instruments whose payoff depend somehow on interest rates. Trade this kind of financial objects became popular at the end of the past century, market exchanges considerably increased and the pricing of these products became an interested challenge for investors. Indeed the valuation of interest rate derivatives presents some difficulties not encountered for other financial instruments. Interest rates behaviors are generally more complicated and they are used for discounting as well as for defining the payoff from the derivative. Furthermore in many cases could be necessary to develop a model, such HJM, to describe the entire yield curve.
In finance the most famous tool for pricing is the model proposed by Black and Scholes, which guarantees closed solutions for some products. Here are presented

the formulas at time t for European options with maturity T,

$$Call = S\phi(d1) - Ke^{-r(T-t)}\phi(d2) \tag{3.1}$$

$$Put = Ke^{-r(T-t)}\phi(-d2) - S\phi(-d1) \tag{3.2}$$

where S is the price of the stock with volatility σ, K the strike, r the risk-free rate, $\phi(\cdot)$ the standard normal cumulative function and

$$d1 = \frac{log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d2 = d1 - \sigma\sqrt{T-t}.$$

How these expression have been derived is out of the aim of this work, interested readers may refer to [2]. The Black and Scholes model has been modified and extended during the years adapting to requirements of traders, up to cover some interest rate derivatives.

3.2 Interest rate swaps

A swap is a contract between two counterparts that exchange series of future cash flows dependent on the type of financial instrument involved. The agreement defines the dates at which the streams have to be paid and how they are calculated. These cash flows, called *legs* of the swap, are calculated over a notional amount which is usually not exchanged between the parties, but it is only used for calculating the size of the interests. Usually at the initial time at least one of the two streams series is determined by an uncertain variable such as a rate.

Here the attention is over the most common type of swap, the interest rate swap of which has been given a hint in the first chapter. Starting again from a FRA contract, at maturity T a floating payment based on the spot rate $L(S,T)$ is exchanged against a fixed payment based on a fixed rate R, where interests are calculated from time S. In T the value is

$$N(T-S)(R - L(S,T)),$$

rewriting the spot rate in function of the zero coupon bond price becomes

$$N\left[(T-S)R - \frac{1}{p(S,T)} + 1\right]$$

and through some substitutions the value at time t is given by

$$FRA(t, S, T, N, R) = N\left[p(t, T)(T - S)R - p(t, S) + p(t, T)\right]$$

which is equivalent to 1.2.

As already stated, an interest rate swap (IRS) can be seen as a generalization of the forward rate agreement. Counterparts exchange cash flows equal to interests at a predetermined fixed rate (the swap rate) against a floating rate (typically the LIBOR) on a notional principal for a future set of dates $T_{\alpha+1}, \ldots, T_\beta$ where $T_\alpha = S$ and $t_\beta = T$. Usually between two stream exchanges pass six months and due to its definition as an interest rate with a fixed expiry, stated at present time and that determines the payment at the end of the period, the LIBOR is fixed six months before the payment. The floating payment can be identified by $N(T_i - T_{i-1})L(T_{i-1}, T_i)$ while the correspondent fixed leg by $N(T_i - T_{i-1})R$. It is better to point out that this is a simple argumentation of swaps, payment can occur at different dates and different year fractions.
An investor is said to hold a payer swap (PFS) if he pays at fixed rate and receives the floating leg, whereas in the opposite case the investor is said to hold a receiver swap (RFS). Consider the first circumstance, the value of a PFS at time $t < S$ can be expressed as

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)N(T_i - T_{i-1})(L(T_{i-1}, T_i) - R)$$

while the discounted payoff of a RFS at present time is given by

$$\sum_{i=\alpha+1}^{\beta} D(t, T_i)N(T_i - T_{i-1})(R - L(T_{i-1}, T_i)).$$

It is possible to obtain the value of the latter considering the contract as a portfolio of forward rate agreements

$$RFS(t, S, T, N, R) = \sum_{i=\alpha+1}^{\beta} FRA(t, T_{i-1}, T_i, N, R).$$

In some occasion turns out that companies have a comparative advantage going in fixed rate market to obtain a loan while others going to floating rate markets. As a consequence a company borrow money in the market where is more convenient, but in this way maybe the kind of rate used for calculated the interest is not the desired one. Thus a swap results to be useful to transform fixed payments into floating payments and vice versa.

As a conclusion of the paragraph, it is better to stress the fact that swap are highly liquid instruments and for this reason they can be used to determine the bond prices through the forward rate swap quoted on the market for different maturities, therefore it is possible to obtain the yield curve.

3.3 Caps and floors

A common interest rate option, exchanged mainly in over the counter markets is the *interest rate cap*, which is a contract designed to guarantee to the holder that floating rate does not exceed a specific level, known as *cap rate*. In other words, the buyer receives payments at the end of each period in which the interest rate rises above the strike. Analogously a *floor* contract provides that the interest rate on a floating rate loan will never be below a certain level, denoted as *floor rate*. A company enters a cap contract if has a loan at a floating rate of interest L and it is afraid that this will increase in the future, so would like to guarantee itself that L does not exceed a maximum cap rate R. It means that at each payment date the company pais no more then R

$$L - max(L - R) = min(L, R).$$

An interest rate cap can be analyzed as a series of basic contracts, called *caplets* which is defined as a contingent claim whose discounted payoff on a notional value N is given by

$$D(t, T_i)N(T_i - T_{i-1})(L(T_{i-1}, T_i) - R)^+$$

where has been introduced the notation $(\cdot)^+ = max(\cdot, 0)$. It has been defined for this product the Black's formula as follow

$$Cpl_i^B(t, T_{i-1}, T_i, N, R, \sigma_i) = (T_i - T_{i-1})p(t, T_i)(F(t, T_{i-1}, T_i)\phi(d1) - R\phi(d2))$$

for $i = 1, \ldots, T$ where the forward rates are lognormal and

$$d1 = \frac{log\left(\frac{F}{R}\right) + \frac{\sigma_i^2}{2}(T_{i-1} - t)}{\sigma_i\sqrt{T_{i-1} - t}}$$

$$d2 = d1 - \sigma_i\sqrt{T_{i-1} - t}.$$

The constant volatilities σ_i are known as Black's volatilities for caplets.
For a cap is then defined the Black's price formula as

$$Cap^B(t, T_{i-1}, T_i, N, R, \sigma) = N \sum_{i=\alpha+1}^{\beta} (T_i - T_{i-1}) p(t, T_i)(F(t, T_{i-1}, T_i)\phi(d1) - R\phi(d2))$$

for $i = 1, \ldots, T$ with analogous $d1$ and $d2$ where σ_i becomes $\sigma_{\alpha,\beta}$.

Similarly is possible to decompose additively a floor contract, whose discounted payoff is a sum of terms of the form

$$D(t, T_i)N(T_i - T_{i-1})(R - L(T_{i-1}, T_i))^+$$

which denotes a contract called *floorlet*.
In this case the Black's formula for the price is given by

$$Flr^B(t, T_{i-1}, T_i, N, R, \sigma) = N \sum_{i=\alpha+1}^{\beta} (T_i - T_{i-1}) p(t, T_i)(-F(t, T_{i-1}, T_i)\phi(-d1) + R\phi(-d2)).$$

In the market, cap prices are quoted in terms of implied Black volatilities $\sigma'_\alpha, \ldots, \sigma'_\beta$ defined as

Implied flat volatilities solution of $Cap^M(t, T_i) = \sum_{k=1}^{i} Cpl_k^B(t, \sigma'_i)$, $i = \alpha, \ldots, \beta$.

Implied forward volatilities solution of $Cpl_i^M(t) = Cpl_i^B(t, \sigma'_i)$, $i = \alpha, \ldots, \beta$.

A sequence of implied volatilities as above is termed *volatility structure*.

Another way to see a cap is as a PFS where exchange payments occur only if are positive. It is possible to write the discounted payoff as

$$\sum_{i=1}^{\alpha} D(t, T_i)N(T_i - T_{i-1})(L(T_{i-1}, T_i) - R)^+.$$

A floor, similarly, can be viewed as a RFS therefore its discounted payoff is given by

$$\sum_{i=1}^{\alpha} D(t, T_i)N(T_i - T_{i-1})(L(T_{i-1}, T_i) - R)^+.$$

To conclude this part, is given the definition of cap *at the money* (ATM). Considering the forward swap rate given in the first chapter, a cap is said to be ATM when

$$R = R_{ATM} := S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(0,T_i)}.$$

If $R < R_{ATM}$ the cap is said to be *in the money* (ITM) and if $R_{ATM} < R$ the cap is said to be *out of the money* (OTM).
Finally the difference between a cap and the corresponding floor is equivalent to a swap

$$(L - R)^+ - (R - L)^+ = L - R.$$

This is known as cap-floor parity. From the equation above it is possible to see that a cap is ATM if and only if its price is the same as the price of the corresponding floor.

3.4 Swaptions

Swaptions represent with caps the main derivative products on interest rates. They will be utilized as a sample of pricing with the model considered, for this reason they assume a relevant importance in this work. A *swaption*, short form of swap option, is an option on interest rate swap. This contract gives the holder the right but not the obligation to enter into an underlying IRS at a certain future time. They can be used to guarantee that the rate which the holder will pay on a loan at some future time will not rise above a specific level. Considered in this way a swaption can be used as an alternative to forward swap; the difference lies in the fact that the swap option give the holder the possibility to choose if enter or not in a swap, the drawback is that this advantage has a cost.

There are two types of swaption contracts:

- A *payer swaption* gives the owner the right to enter into a payer interest rate swap at a given future time T, the maturity, that normally coincides with the date which the IRS starts.

- A *receiver swaption* allows the holder to enter into a receiver swap, paying floating cash flows and receiving fixed quantities for a specified period of time $T_\beta - T_\alpha$, termed *tenor*.

As a consequence of what presented above, a swaption must specify not only the expiry time of the option but even the tenor of the underlying IRS.

Consider a payer swaption written on a notional amount N with maturity T_α coinciding with the first date of the tenor and strike rate R, then is possible to define its discounted payoff as follow

$$ND(t,T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha,T_i)(T_i-T_{i-1})(F(T_\alpha,T_{i-1},T_i)-R)\right)^+ ;$$

an alternative expression in terms of the forward swap rate can be considered:

$$ND(t,T_\alpha)(S_{\alpha,\beta}(T_\alpha)-R)^+ \sum_{i=\alpha+1}^{\beta} p(T_\alpha,T_i)(T_i-T_{i-1}).$$

A similar formula to the above is used for the receiver-swaption payoff discounted to present time t from the maturity

$$ND(t,T_\alpha)\left(\sum_{i=\alpha+1}^{\beta} p(T_\alpha,T_i)(T_i-T_{i-1})(R-F(T_\alpha,T_{i-1},T_i))\right)^+ .$$

It is important to stress on the fact that the payer-swaption payoff cannot be split into more elementary products as caplet for cap. This difference implies that the value of a payer swaption will be always smaller than the value of the corresponding cap contract.

The Black model is the market practice to compute swaption prices. It started being used before has being justified by the theory. The formula for a payer swaption at time $t < T_\alpha$ is given by

$$PS^B = N(S_{\alpha,\beta}(t)\phi(d1) - R\phi(d2)) \sum_{i=\alpha+1}^{\beta} (T_i - T_{i-1})p(t,T),$$

$$d1 = d1 = \frac{log\left(\frac{S_{\alpha,\beta}(t)}{R}\right) + \frac{\sigma_{\alpha,\beta}^2}{2}(T_\alpha - t)}{\sigma_i\sqrt{T_\alpha - t}}$$

$$d2 = d1 - \sigma_{\alpha,\beta}\sqrt{T_\alpha - t}.$$

Note that the volatility parameter $\sigma_{\alpha,\beta}$ is different from the one encountered for caps an floors. Given the market swaption price, the Black volatility implied by

the formula is termed *implied Black volatility*.
Similarly the Black formula for a receiver swaption is given by

$$RS^B = N(-S_{\alpha,\beta}(t)\phi(-d1) + R\phi(-d2)) \sum_{i=\alpha+1}^{\beta} (T_i - T_{i-1})p(t,T).$$

As for cap and floors, given strike R, maturity T_α, underlying swap payment dates $T_{\alpha+1}, \dots, T_\beta$, a payer or receiver swaption is said to be at the money if and only if

$$R = R_{ATM} := S_{\alpha,\beta}(0) = \frac{P(0,T_\alpha) - P(0,T_\beta)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(0,T_i)}.$$

If $R < R_{ATM}$ a payer swaption is said to be in the money while a receiver swaption is said to be out of the money. The opposite occurs when $R > R_{ATM}$. Even for swaption subsists the parity relation:

$$PayerSwaption - ReceiverSwaption = PayerSwap.$$

Chapter 4

Volatility structure

As seen in the second chapter, in order to compute prices of derivatives through numerical method using the HJM framework, is necessary to specify the volatility structure $\sigma(t, T)$. Define a suitable σ is not trivial, it is possible to take an own choice due to some particular consideration or alternatively use one of the models present in literature.

This work is oriented on using a model already existent in order to achieve the above purpose. In theory, as said in the second chapter, every short rate model can be properly related with the HJM framework. In general, whenever the correlations of different rates considerably influence the product to be priced is better use multifactor models, they provide higher precision but loose efficiency in the numerical implementation as the number of factors involved grows. Neglecting the capability of these models to fit satisfactorily market data and to realistic represent correlation patterns, here have been considered one factor short rate models, briefly introduced at the end of the first chapter, which are easier to understand and do not lead to heavy computational statement. They are still acceptable mainly when the rates that jointly influence the payoff at every instant are close.

In paragraph 1.5 has been shown that under the risk neutral measure the price at time t of a contingent claim with maturity T and payoff H_T is given by

$$\pi_t = \mathbb{E}\left[exp\left\{-\int_t^T r_s ds\right\} H_T \mid \mathcal{F}_t\right] \qquad \forall t \in [0, T].$$

A special example is offered by the zero coupon bond, whose payoff at time T reduces to the unit amount of money. In this case the price at t takes the form of the following expression

$$p(t,T) = \mathbb{E}\left[exp\left\{-\int_t^T r_s ds\right\} \mid \mathcal{F}_t\right] \qquad \forall t \in [0,T].$$

Whenever the distribution of the latter is characterized in terms of the dynamics of the short rate is then possible to obtain bond prices and hence all rates can be computed. As it will be shown some models provide analytical formulas, not only for zero coupon bonds but even for some more sophisticated derivative.

4.1 Affine term structure

This argument acquires importance due to the fact that from an analytical and computational point of view the existence of an affine term structure extremely simplifies the content.

A model is said to possess an *affine term structure* (ATS) if the zero coupon bond can be written in the form

$$p(t,T) = A(t,T)exp\left\{-B(t,T)r(t)\right\} \tag{4.1}$$

where A and B are deterministic functions.

Assume the following dynamics for the short rate under the risk neutral measure $\mathbb{Q}$

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t), \tag{4.2}$$

then it is natural to wonder which choices of the parameters lead to an affine term structure. The answer can be found for instance in [5], the coefficients μ and σ^2 needs to be both affine functions of r, which means linear plus a constant as follow:

$$\mu(t,r) = \alpha(t)r + \beta(t)$$
$$\sigma^2(t,r) = \gamma(t)r + \delta(t).$$

Then the model admits an ATM where A and B satisfy the system

$$\frac{\partial B(t,T)}{\partial t} + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1, \quad B(T,T) = 0,$$
$$\frac{\partial A(t,T)}{\partial t} = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \qquad A(T,T) = 0.$$

The first expression is a Riccati differential equation that can be solved without involving A. Then using B is possible to get the result of the second expression. In general this is not a the only way to obtain an ATS but if μ and σ^2 are time independent then they need to be affine. An example of affine term structure model is the Hull-White extended Vasicek model which will be presented in the next paragraph.

An important result deriving from affine models is the simply form gained for the absolute volatility of the instantaneous forward rate, which is the input of the HJM dynamics. The latter can be written as

$$f(t,T) = -\frac{\partial logA(t,T)}{\partial T} + \frac{\partial B(t,T)}{\partial T}r(t),$$

so that, using the general expression of the short dynamics 4.2,

$$df(t,T) = \left(\mu(t,r(t))\frac{\partial B(t,T)}{\partial T} - \frac{\partial logA(t,T)}{\partial T}\right)dt + \frac{\partial B(t,T)}{\partial T}\sigma(t,r(t))dW(t).$$

Therefore the absolute volatility of the instantaneous forward rate is given by

$$\sigma_f(t,T) = \frac{\partial B(t,T)}{\partial T}\sigma(t,r(t)) \tag{4.3}$$

which is a deterministic function of time and maturity, it has an extremely simple form and leads to easy computation.

4.2 Market data

The data available for this work have been gained from Bloomberg software, they are quoted on 29th of October 2010 and are displayed on table 4.1. The first column represents the data in which the variables are measured, in the second column is shown the instantaneous forward rate, in the next column the continuously-compounded spot interest rate, then the discount factors and last the daily time interval between two data.

Mty/Term	Market Rate	Spot Rate	Discount	dt
29/10/2010	0	0	1	0
01/11/2010	0,22563	0,22563	0,998121517	3
02/11/2010	0,14	0,14	0,995945685	1
09/11/2010	0,24841	0,24841	0,999952	7
16/11/2010	0,25	0,25	0,999903	7
02/12/2010	0,25375	0,25375	0,999789	16
04/01/2011	0,26766	0,26766	0,999532	33
02/02/2011	0,28594	0,28594	0,99927	29
16/03/2011	0,33	0,30742	0,998857	42
15/06/2011	0,374	0,33742	0,997914	91
21/09/2011	0,413	0,36281	0,996793	98
21/12/2011	0,476	0,38894	0,995595	91
21/03/2012	0,554	0,41988	0,994203	91
20/06/2012	0,636	0,45482	0,992607	91
19/09/2012	0,748	0,49564	0,990734	91
19/12/2012	0,875	0,54137	0,988547	91
20/03/2013	1,041	0,59445	0,985953	91
19/06/2013	1,222	0,6561	0,982917	91
18/09/2013	1,428	0,72527	0,979382	91
18/12/2013	1,648	0,8006	0,975319	91
03/11/2014	1,10561	1,11427	0,956497	320
02/11/2015	1,4726	1,49324	0,928315	364
02/11/2016	1,81625	1,85422	0,895171	366
02/11/2017	2,114	2,17265	0,85962	365
02/11/2018	2,357	2,43681	0,82385	365
04/11/2019	2,56	2,661	0,788157	367
02/11/2020	2,732	2,85414	0,753221	364
02/11/2021	2,869	3,00945	0,71995	365
02/11/2022	2,999	3,16022	0,686422	365
03/11/2025	3,266	3,47488	0,596395	1097
04/11/2030	3,497	3,75051	0,475513	1827
02/11/2035	3,61	3,8823	0,382405	1824
02/11/2040	3,673	3,95224	0,309094	1827
02/11/2050	3,699	3,93328	0,210548	3652
02/11/2060	3,625	3,69643	0,160198	3653

Table 4.1: Market data

The exposed data are not convenient for a computational treatment, therefore have been carried out linear interpolations and the resulting instantaneous forward rate curve and discount curve are represented in figure 4.1 as function of time expressed in years.

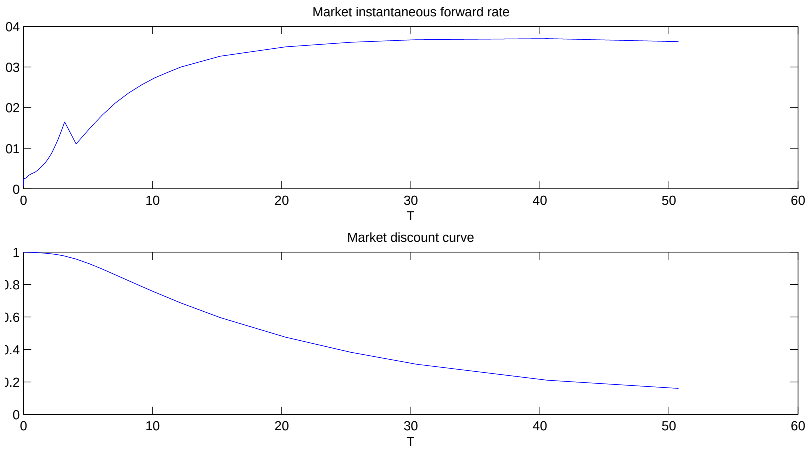

Figure 4.1: Instantaneous forward rate curve and discount evolution.

4.3 Hull-White extended Vasicek model

Hull and White developed in the nineties a model of future interest rates based on the one proposed by Vasicek, improving the poor fitting of the initial term structure of interest rates. They used a linear stochastic differential equation in order to describe the short rate dynamics. As a consequence the related process is normally distributed. This is an enormous advantage from a computational point of view but, on the other hand, implies that the short rate can assume negative values and this is unreasonable in economic field.

The objective here is to use a calibration procedure in order to obtain the parameters for which is given the best fit to the prices obtained by today swaption market volatilities for different tenor and maturities, then derive the volatility structure for future times at various maturities (the input for the HJM dynamics). For this aim has been developed a code using the language of technical computing MATLAB.

4.3.1 Dynamics

The need for an exact fit to market data led Hull and White to the introduction of time dependent parameter in the Vasicek model. There is a degree of ambiguity amongst practitioners about exactly which parameters are constant and which time-varying.
Here, following [2], is considered the dynamics under the measure $\mathbb{Q}$ which evolves according to

$$dr(t) = [\theta(t) - ar(t)]dt + \sigma dW(t), \tag{4.4}$$

known as Hull-White extension of the Vasicek model, where a and σ are the constant parameters to calibrate and θ is a deterministic function of time given by

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}),$$

chosen so as to exactly fit the market term structure of interest rates. Here it has been considered the market instantaneous forward rate at present time for the maturity T as

$$f^M(0,t) = -\frac{\partial log p^M(0,t)}{\partial T}.$$

As mentioned above Hull-White is an affine term structure model. It is then necessary to specify A and B to have 4.1. From the literature it is possible to see that through some burdensome calculation have been obtained the following expressions

$$B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right),$$
$$A(t,T) = \frac{p^M(0,T)}{p^M(0,t)} exp\left\{B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1 - e^{-2at})B(t,T)^2\right\}.$$

Furthermore, integrating the 4.4, results

$$r(t) = x(t) + \alpha(t),$$

where

$$\alpha(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1 - e^{-at})^2,$$

and x is a process which dynamics evolves according to

$$dx(t) = -ax(t)dt + \sigma dW(t), \qquad x(0) = 0.$$

The latter under the forward measure $\mathbb{Q}^T$, necessary to compute the prices if European options written on zero coupon bonds, becomes

$$dx(t) = [-B(t,T)\sigma^2 - ax(t)]dt + \sigma dW^T(t),$$

where the $\mathbb{Q}^T$-Brownian motion is defined by $dW^T(t) = dW(t) + \sigma B(t,T)dt$.

Given the market discount factors, has been written a Matlab function denoted *hw.m* (see appendix A) in order to compute the calculations necessary for the current aim, which is, at this point of the work, the volatility structure. It takes as inputs the constants a and σ, the market instantaneous forward rate and the discount factors at present time for different maturities, the last maturity expressed in year fraction and the year fraction intended as the time passing between two subsequent rates or discount (1/12 for monthly data). It is therefore assumed that intervals between market data introduced here are constants. The outputs are represented by the above A and B, and by the short rate and the bond prices at any future time.
In figure 4.2 are shown the short rate evolution and the bond prices as functions of maturity for a monthly-progressing time.
For this realizations have been considered $a = 0.05$, $sig = 0.02$ due to market observations; furthermore, with these values, the curves seem to be more realistic and the discount curve at present time exactly fit the market curve. The other parameters are chosen as follow, $T = 40$ years, while $f0$ and $p0$ refer to market data quoted on October 29, 2010[1]. The short rate grows in time and as expected from theory it can assume negative values close to current time. This drawback does not affect substantially consequent results but does not allowed to consider the evolution fully reasonable from an economic point of view. As a consequence, close to present even bond prices can assume values higher then the unit. However, bond curve trends reflect what observed in the market and for the purposes of this work these errors become negligible.

[1]Special thanks are addressed to the friend Andrea Bafundi for kindly provide such data.

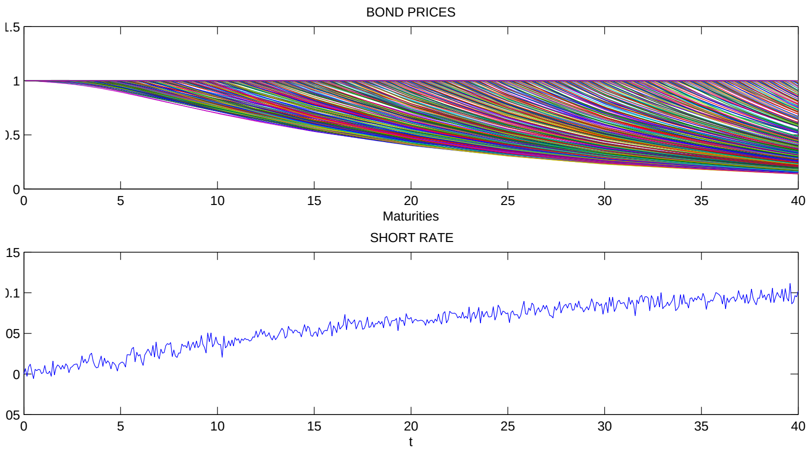

Figure 4.2: Bond Prices and short rate evolution for $a = 0.05$, $\sigma = 0.02$.

4.3.2 Pricing

After having analyzed the dynamics of the model, it is shown how to price a swap option, either for a payer or a receiver swaption. The prices for different tenor and maturities given by analytical formulas are then compared with the prices obtained from market data using Black like formula in order to calibrate a and σ for Hull-White model.

An European swaption can be viewed as an option on a coupon bearing bond. The latter can be explicitly priced using Jamshidian's decomposition (see [11] for details). For this purpose, take a European option with strike X and maturity T, written on a bond paying s coupons after the option maturity. Let t_i and c_i be the payment time and the value of the i-th cash flow after option maturity for $i = 1, \dots, s$. Denote by r^* the value of the short rate at time T for which

$$\sum_{i=1}^{s} c(i)A(T,t_i)exp(-B(T,t_i)r^*) - 1 = 0,$$

and set

$$X_i := A(T,t_i)exp(-B(T,t_i)r^*).$$

The value of r^* is obtained using the bisection method, the drawback is that for Hull-White it can assume negative values. Although this is unreasonable

economically speaking, it does not affect substantially the consequent results. Therefore it is still useful for computation. Furthermore observing deeply the case analyzed in this work it never happens.
The price of the option at a chosen time t before maturity is given by

$$CBO(t, T, s, c, X) = \sum_{i=1}^{s} c(i) ZBO(t, T, t_i, X_i),$$

where ZBO represents the price at t of a European option with maturity T written on a zero coupon bond maturing at time s.
Since, under the $\mathbb{Q}^T$ measure, the short rate is still normal, we have

$$ZBC(t, T, s, X) = p(t, s)\phi(h) - Xp(t, T)\phi(h - \sigma_p),$$

for a call with maturity T and strike X, while for a correspondent put

$$ZBP(t, T, s, X) = Xp(t, T)\phi(-h + \sigma_p) - p(t, s)\phi(-h),$$

where

$$\sigma_p = \sigma \sqrt{\frac{1 - exp\{-2a(T - t)\}}{2a}} B(T, s),$$
$$h = \frac{1}{\sigma_p} log \frac{p(t, s)}{p(t, T)X} + \frac{\sigma_p}{2}.$$

To compute this formulas has been created the Matlab function europeopthw.m, shown in appendix A, where the meanings of the symbols are of easy comprehension.
It is now possible to get the analytical formula for a European swaption with strike rate X, maturity T and nominal value N, which gives to the owner the right to enter in a swap where are exchanged the fixed rate X against the LIBOR. The cash flows are given by $c_i = X(t_i - t_{i-1})$ for $i = 1, \ldots, s - 1$ and $c_s = 1 + X(t_i - t_{i-1})$, where $t_0 = T$.
The payer swaption price at time t is then given by

$$PS(t, T, s, N, X) = N \sum_{i=1}^{s} c_i ZBP(t, T, t_i, X_i),$$

and the price of the corresponding receiver swaption is therefore

$$RS(t, T, s, N, X) = N \sum_{i=1}^{s} c_i ZBC(t, T, t_i, X_i).$$

The resultant Matlab code is exposed in appendix A and it is denoted by swaptionhw.m, where d represents the vector of payment times of the interest rate swap.

In figure 4.3 are shown the prices of a payer swaption obtained for ATM

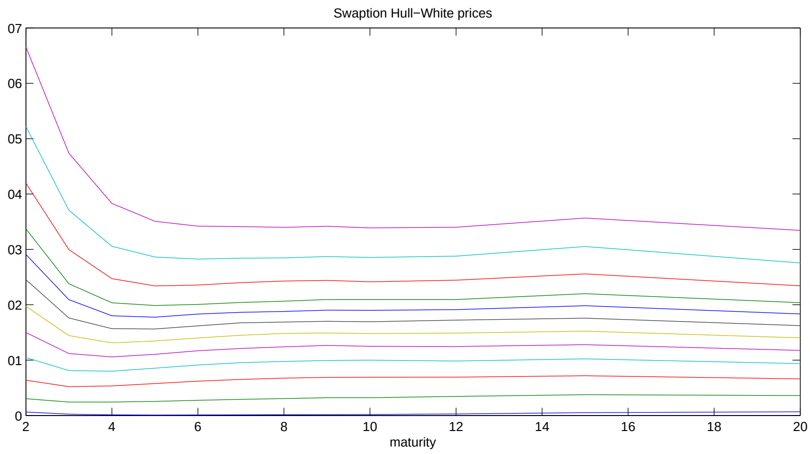

Figure 4.3: Swaption prices for maturities and tenors given by the vectors mat and d, $a = 0.05$, $\sigma = 0.02$, using Hull-White model.

strikes, $N = 1$, while the maturity and tenor vectors are given by $mat = [2 : 10\ 12\ 15\ 20]$ and $d = [2 : 10\ 12\ 15\ 20]$. It is possible to observe a decreasing trend for short maturity, which becomes essentially flat as maturity grows. On the other hand the prices value increases as the tenor becomes greater, especially for small maturity, so that these functions assume more significantly convex form as the tenor grows. From this figure the price seems to be more sensible to the change of tenor then to a maturity modification. The range of values assumed in this case varies between zero and 0.1. It will be show that similar results are obtained using a Black like formula.

4.4 Parameters calibration

This paragraph is devoted to the calibration procedure of the coefficients a and σ of the Hull-White model. For this aim have been calculated the prices for different maturities and tenor through a Black like formula, and this has been done using Black market volatilities and strikes of at the money swaption for

Expiry		1 YR	2 YR	3 YR	4 YR	5 YR	6 YR	7 YR	8 YR	9 YR	10 YR	12 YR	15 YR	20 YR	25 YR	30 YR
1 MO	Vol	104,05	92,55	77,78	63,75	55,23	47,85	45,65	42,6	40,55	38,98	37,45	34,35	32,25	31,8	31,68
	Strike	0,3981	0,5483	0,8034	1,1599	1,5283	1,8719	2,1652	2,4059	2,6053	2,7728	3,0352	3,2971	3,5268	3,6378	3,6978
3 MO	Vol	91,55	77,83	70,58	57,85	50,35	45,6	42,85	39,45	37,8	36,43	34,85	30,95	29,95	28,85	28,7
	Strike	0,437	0,6084	0,8953	1,2638	1,6334	1,9713	2,2554	2,4906	2,6835	2,8433	3,0972	3,3486	3,5686	3,6739	3,7297
6 MO	Vol	88,9	69,05	62,15	51,95	45,8	41,53	38,75	37	35,55	33,95	32,5	29,35	27,7	27,15	26,73
	Strike	0,4956	0,7018	1,0353	1,4201	1,7896	2,1163	2,3884	2,6113	2,7974	2,9454	3,1866	3,4234	3,629	3,7265	3,7762
9 MO	Vol	87,66	67,41	57,15	48,75	43,85	39,63	37,13	35,85	34,5	32,75	31,13	28,65	26,93	26,55	25,93
	Strike	0,5711	0,8197	1,1944	1,587	1,9552	2,2701	2,5284	2,7388	2,914	3,0508	3,2792	3,5001	3,6913	3,7804	3,8239
1 YR	Vol	82,35	65,31	52,82	45,45	41,73	37,8	35,48	33,8	33,3	31,48	29,95	27,9	26,25	25,9	25,05
	Strike	0,6646	0,9609	1,3603	1,7613	2,126	2,4272	2,6704	2,8687	3,0315	3,157	3,3718	3,5766	3,7532	3,8339	3,8712
2 YR	Vol	60,76	46,98	40,25	36,08	33,7	31,83	30,15	29,95	29,15	27,98	26,8	25,5	24,2	23,95	23,05
	Strike	1,2588	1,7148	2,1391	2,5091	2,8031	3,0332	3,2165	3,3648	3,4751	3,5887	3,7372	3,877	3,9961	4,0438	4,0572
3 YR	Vol	41,05	34,42	31,38	29,27	28,87	27,57	26,46	26,56	25,56	25,04	24,41	23,4	22,45	22,2	21,88
	Strike	2,1836	2,5979	2,9502	3,2187	3,4223	3,5809	3,7079	3,7973	3,896	3,9576	4,0672	4,147	4,2123	4,2305	4,2219
4 YR	Vol	31,02	28,24	26,59	25,96	25,36	24,86	24,08	24,3	23,6	23,13	22,75	21,95	20,98	21,05	20,45
	Strike	3,024	3,3507	3,5859	3,7583	3,8899	3,9948	4,0631	4,148	4,1946	4,2406	4,3006	4,3475	4,3704	4,366	4,3401
5 YR	Vol	26,87	25,07	24,49	23,93	23,48	23,16	22,55	22,66	22,35	21,7	21,45	20,6	19,8	19,9	19,55
	Strike	3,6882	3,8818	4,0213	4,1272	4,2124	4,2614	4,3358	4,3694	4,4055	4,4386	4,4581	4,482	4,4728	4,4525	4,4132
6 YR	Vol	24,22	23,91	23,46	22,95	22,7	22,45	22,05	21,8	21,55	21,25	20,7	19,95	19,55	19,4	19,55
	Strike	4,0832	4,1983	4,2857	4,3577	4,3912	4,4608	4,4846	4,5138	4,5416	4,5443	4,5534	4,556	4,5312	4,4967	4,4486
7 YR	Vol	23,1	22,4	21,65	21,6	21,43	21,18	20,65	21	20,85	20,08	19,8	19,25	18,58	18,8	18,58
	Strike	4,3183	4,3934	4,4574	4,477	4,5468	4,5625	4,587	4,6113	4,608	4,6074	4,6094	4,5962	4,5601	4,5151	4,4603
8 YR	Vol	22,55	21,95	21,65	21,35	21,1	20,95	20,7	20,55	20,45	20,3	19,9	19,35	18,95	18,9	18,85
	Strike	4,4713	4,5318	4,5348	4,6106	4,6182	4,6392	4,6613	4,6522	4,6475	4,6458	4,6416	4,6144	4,5707	4,5171	4,4574
9 YR	Vol	22,01	21,41	21,25	20,91	20,6	20,5	20,1	20	19,9	19,8	19,55	19,2	18,95	18,75	18,5
	Strike	4,5957	4,5691	4,6617	4,6596	4,6779	4,6985	4,6833	4,6746	4,6702	4,666	4,6502	4,6176	4,5683	4,5079	4,4437
10 YR	Vol	20,47	20,55	20,08	20,08	19,85	19,48	19,5	19,85	19,7	19,15	18,65	18,3	17,6	17,95	17,45
	Strike	4,5414	4,6969	4,6829	4,7008	4,722	4,7003	4,688	4,6816	4,6757	4,6704	4,6446	4,6073	4,5549	4,4882	4,4207
12 YR	Vol	20,19	20,23	19,85	19,88	19,74	19,28	19,29	19,45	19,33	18,96	18,51	18,09	17,55	17,86	17,5
	Strike	4,6527	4,7051	4,7408	4,7023	4,6838	4,6755	4,6682	4,6621	4,6452	4,631	4,6022	4,567	4,5015	4,428	4,342
15 YR	Vol	19,99	19,95	19,8	19,75	19,65	19,1	19,1	19	18,9	18,8	18,4	17,9	17,6	17,85	17,75
	Strike	4,575	4,5876	4,6003	4,6042	4,6053	4,5865	4,5719	4,5566	4,5417	4,5269	4,5065	4,4687	4,3888	4,309	4,1955
20 YR	Vol	19,2	19,5	19,3	19,15	19	18,9	18,1	18,35	18,25	18,35	18	17,5	17,3	17,25	17,15
	Strike	4,4783	4,4734	4,4589	4,4441	4,4285	4,4243	4,4144	4,4021	4,3882	4,374	4,3356	4,2787	4,1855	4,0472	3,935
25 YR	Vol	18,15	17,95	17,7	17,5	17,4	17,3	17,25	17,2	17,2	17,25	17,7	18,4	16,5	16,55	16,6
	Strike	4,4002	4,3732	4,3497	4,3269	4,3063	4,277	4,25	4,2243	4,2005	4,1764	4,1297	4,0644	3,8924	3,7622	3,528
30 YR	Vol	16,95	16,9	16,9	16,9	16,9	16,95	16,85	16,8	16,95	17,1	18,3	20,1	15,95	15,84	15,69
	Strike	4,1116	4,0872	4,0628	4,0412	4,0167	3,9927	3,969	3,9471	3,9241	3,9016	3,8047	3,6909	3,5481	3,2667	3,0563

Table 4.2: Swaption volatility data

different maturities and tenors. Then these prices have been compared with the prices calculated through the Hull-White model; finally has been implemented a Matlab code in order to search for the values of the coefficients which minimize the residuals.

4.4.1 Black like formula

The data necessary for the prices calculation are obtained again from Bloomberg and are reported in table 4.2. They refer to ATM swaption volatilities and strikes for different maturities and tenors, given in data 29th of October 2010.

These data are inputs for the Matlab function swaptionblack.m implemented in order to compute the payer swaption pricing. They are displayed in figure 4.4

for $mat = [2:10\ \ 12\ \ 15\ \ 20]$ and $d = [2:10\ \ 12\ \ 15\ \ 20]$. It is possible to state that volatilities have a decreasing trend for either tenors or maturities growth, particulary the change is pronounced for small maturity and minor values of tenor, while curves are almost flat after $T = 10$. The range of their values sets up between 0.1 and 0.5. The strikes have almost the opposite behavior, in fact for short maturities the values grows rapidly concordant with the tenor, while slightly decreases when the parameters increase. The range of the strikes is given by $[0.01, 0.05]$.

The prices at time t are calculated through Black like formula introduced in the

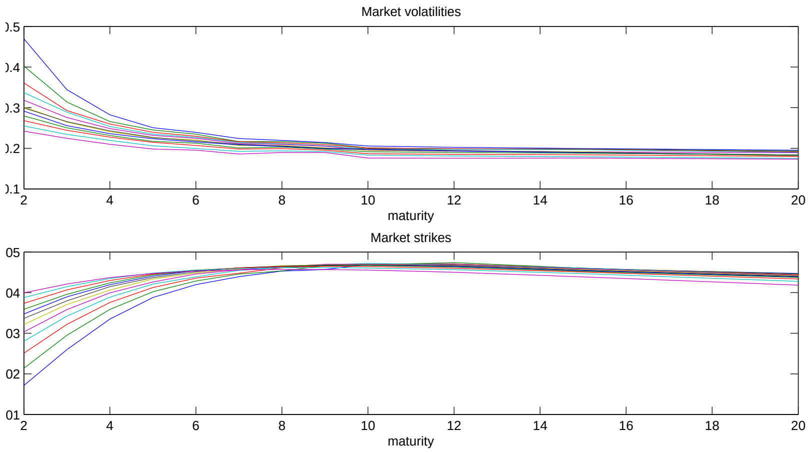

Figure 4.4: Market swaption volatilities and strikes for maturities and tenors given by the vectors mat and d.

chapter above, which is written with the Matlab editor as shown in appendix A through swaptionblack.m, where z and u represent respectively the lengths of the maturity vector mat and the tenor vector d; $p0$ is the market discount vector while X is the strike rate and R is the strike at the money. Again N is the nominal value while vol is the volatility matrix given from the market interpolated data.

In figure 4.5 are shown the prices of a payer swaption obtained for X and R both ATM strike matrix, $N = 1$, while the maturity and tenor vectors are given by $mat = [2:10\ \ 12\ \ 15\ \ 20]$ and $d = [2:10\ \ 12\ \ 15\ \ 20]$. The trends conversely to the Hull-White case, are upward sloping for short maturities but then assume a similar decreasing progress. This behavior suggest to be careful with small

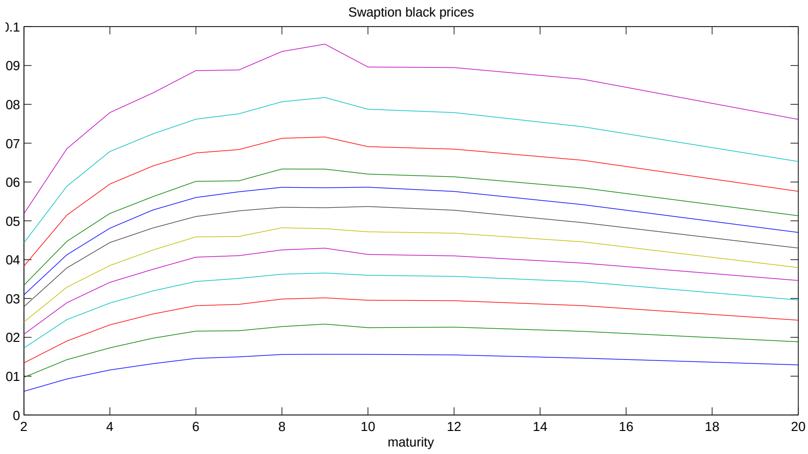

Figure 4.5: Swaption prices for maturities and tenors given by the vectors mat and d using the Black like formula.

maturities. The approach followed in this work is to give less importance to prices deriving from short maturities assigning them smaller weights during the calibration procedure as shown below. Again the prices increase when tenor increases and they seem to be more affected by changes of the latter then by changes of the maturity. The range of values is like the one encountered using the Hull-White model set between 0 and 0.1. As a consequence it is possible to say that the values of the parameters a and σ are close to the optimal solution.

4.4.2 The Levenberg-Marquardt algorithm

In order to find the best values of the parameters in the Hull-White model, has been implemented the Matlab function LMFsolveswaptionhw.m (see appendix A) based on the Levenberg-Marquardt (LM) algorithm solving multi-variables optimization problems (for a better understanding of this technique see [14]). It can be thought as a combination of gradient descent (steepest descent) and the Gauss-Newton method, acting as the first one when the current solution is far from the correct one, and as the second when the current solution is close to the correct solution.

LM is an iterative technique that locates the minimum of a function that is expressed as the sum of squares of nonlinear functions. Its principal application is optimize parameters x of the model curve $f(y, x)$, where y is a given variable, so that the sum of the squares of the deviations

$$S(x) = \sum_{i=1}^{n} [pbl_i - f(y_i, x)]^2$$

becomes minimal. In this work pbl_i represent Black prices while the function f gives HW prices phw_i, which are dependent on known variables y_i and on the guessed parameters x.
The basis of LM is a linear approximation to f in the neighborhood of x that can be expressed as

$$f(y_i, x + d) \simeq f(y_i, x + d) + J_i d,$$

where

$$J_i = \frac{\partial f(y_i, x)}{\partial x}$$

is the gradient of the function respect to x. In the follow the term y, which represents all the given variable necessary to compute HW prices, will be omitted for simplicity without causing any effect on the results. In each iteration step, the parameter vector x is replaced by a new estimate xd starting from the guessed vector xc, trying to minimize the square distance $S = r'r$ with $r = pbl - phw$. The latter can be written in vector notation as

$$\|pbl - f(xd)\|^2 \simeq \|pbl - f(x) - Jd\|^2 = \|r - Jd\|^2,$$

where J represent the Jacobian matrix. Taking the derivatives with respect to d and setting the result to zero follows

$$(J'J)d = J'r,$$

from which is possible to obtain the increments d. Note that in the Matlab function $J'J$ has been named A while $J'r$ has been denoted v.
Due to Levenberg there is a damped version of the equation above given by

$$(J'J - \lambda I)d = J'r,$$

where I is the identity matrix and λ is referred to as the *damping term*, which is adjusted at each iteration. LM controls its own damping, so that is adaptive. In fact it raises the damping if a step fails to reduce the error, otherwise the

damping decreases. So that a smaller value of λ is used when the error reduction is rapid leading the algorithm closer to the Gauss-Newton method, whereas the damping term can be increase if the error reduction is slow giving a step closer to the gradient descent algorithm.
A disadvantage shows up when the dumping term increases too much and, as a consequence, the inversion $JJ' - \lambda I$ is not involved significantly in the computation. So that Marquardt proposed to replace the identity matrix with the diagonal of $J'J$ obtaining

$$(J'J - \lambda diag(J'J))d = J'r,$$

which allowed to avoid slow convergence in the direction of small gradient. Each component of the gradient is scaled according to the curvature implying a larger movement along the direction where the gradient is smaller.
The choice of the damping parameter at each step is based on a preexistent Matlab function. As mention above, the procedure reduces λ if the predicted error is to high, and raises the term if the predicted error is too low. The algorithm terminates when the residuals are all smaller than a prefixed tolerance or when the maximum number of iterations allowed is reached.

4.4.3 Calibration procedure

In order to contrast the inclination of the Hull-White model to provide bad estimations for small maturities and tenors, have been introduced weights on the residuals in order to give more importance to long maturities. The choice has been made following the usual form deriving from the experience and testing its goodness relative to the current case.

The function LMFsolveswaptionhw.m takes as input the vector of initial guesses for a and σ, the relative prices obtained through the Hull-White method, the prices calculated using Black like formula, and the usual parameters necessary to recall the function cicloswaption.m which gives back the swaption prices using Hull-White for new guessed values. The algorithm consists on compute model prices of at the money swap options at current time and compare the result with the correspondent prices obtained through Black market formula, then a and σ for which the market trend is better reproduced are founded minimizing the difference between the prices. As already mentioned a variable named

peso has been introduced to give more weight to prices correspondent to higher maturities and tenors. The reason of this adjustment is a direct consequence of what observed in figure 4.3 and 4.5, where prices calculated with Hull-White model had the contrary slope to the ones encountered using Black like formula when maturities and tenors are small. Furthermore this inappropriate behavior is pointed out as well in the literature.
LMFsolveswaption.m allows to specify the maximum number of iterations permitted, the tolerance for final sum of residuals and the tolerance on difference of solutions when the guess parameter vector changes. The output arguments are the approximations of the final solution, the sum of residual squares, the number of iterations carried out, which is displayed as negative if the algorithm does not converge before the maximum of allowed iterations is reached, and the number of iterations necessary to get to the final solution. Are then specified Hull-White prices and ATM strikes calculated using the final a and σ parameters.

Using a number of maximum iterations equals fifty, residual tolerance 10^{-5} and solution tolerance 10^{-3}, with initial guesses $a = 0.05$ and $\sigma = 0.02$, is gained a resultant vector xf which values are 0.0640 and 0.0385 approximated at the fourth digit number. The choice of the initial values is due to an investigation of the values observed in the market related with the function implemented in this work, even though in some papers have been used rather higher values, for instance look at [19]. The value of the sum of residual squares is 0.0368, the reason why is quite high lies in the fact that Hull-White does not perform exactly for short maturities and small tenors. Therefore is clear why is reached the maximum number of iterations allowed. Furthermore it is interesting to notice that the final solution is reached in just nine iterations, this proves the good performances supplied by the algorithm.
Finally in figure 4.6 are compared the prices obtained through the Black like formula with the prices obtained through the Hull-White model using the final values of a and σ. It is evident now that the prices are better approximated as maturities and tenors increase.

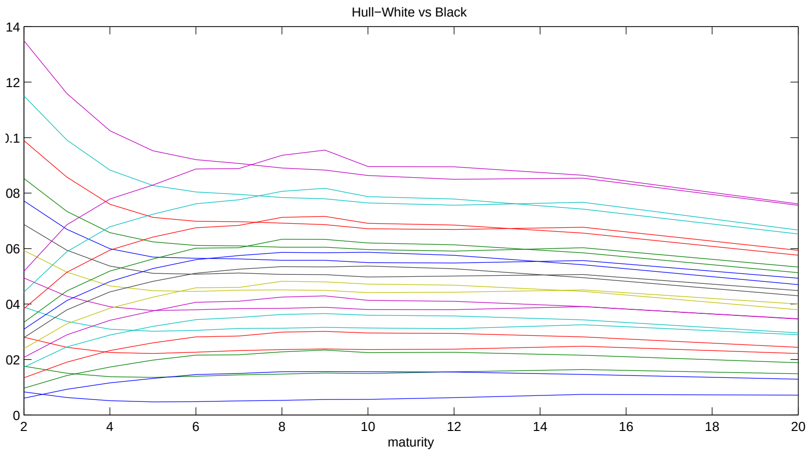

Figure 4.6: Comparison between Hull-White and Black like swaption prices.

4.5 Absolute volatility

At last is possible to derive the implied swaption volatility structure that will be use in the HJM dynamics in order to have the price of a swaption with particular maturity and tenor. This is done using the formula 4.3 presented above for affine term structure models, which gives in the Hull-White case the following deterministic function

$$\sigma_f(t,T) = e^{-a(T-t)}\sigma.$$

Again a Matlab function has been implemented, it is named sigmaabs.m and it is displayed in appendix A. It gives back the volatility matrix, and the inputs in this circumstance are the market values of the instantaneous forward rate already interpolated and the parameters of Hull-White model derived in the last section.

In figure 4.7 are represented the absolute volatilities at four different times, $t = 0$, $t = 10$, $t = 20$ and $t = 30$ years, for all maturities up to forty years. The curves are downward sloping as expected. In fact either from the literature or in the market is possible to observe such a trend.

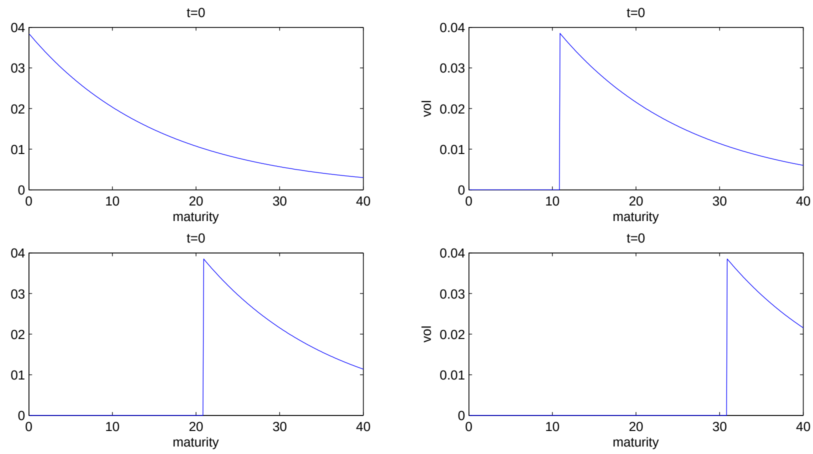

Figure 4.7: Absolute volatility deriving from Hull-White model at four different times.

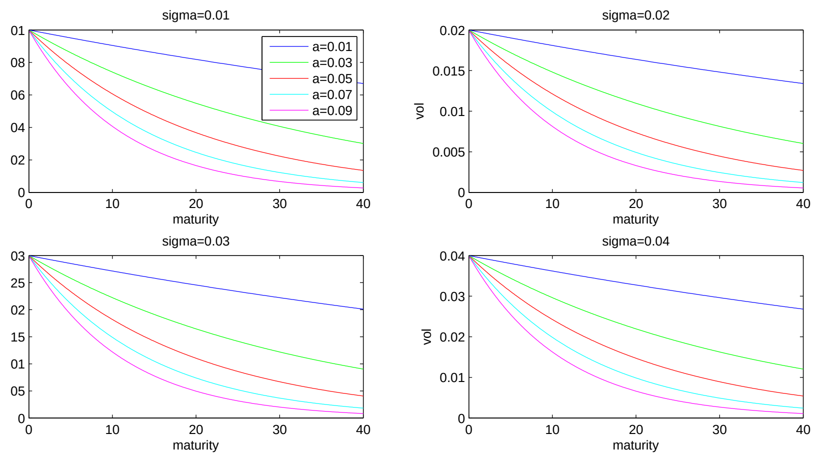

Figure 4.8: Absolute volatility at current time for $a \in [0.01, 0.09]$, $\sigma \in [0.01, 0.04]$.

4.5.1 Term structure analysis

The purpose of this paragraph is to give to the reader a clear explanation of how different choices of the parameters of the Hull-White model affect the volatility structure. This is done through the analysis of the curves displayed in the figure 4.8 where, according to market observations, a varies in $[0.01, 0.09]$, while the range of σ is $[0.01, 0.04]$. Each graph is plotted by keeping the σ parameter fixed and varying a. These graphs suggest that absolute volatilities for swaption decrease as a increases and increase as σ increases. In addition a has a bigger influence on longer maturities.

Chapter 5

Pricing

This chapter presents a manner of pricing a European payer swaption based on the results obtained in the last chapters. Particularly, is considered the Monte Carlo method. The dynamics of the Heath-Jarrow-Morton framework is simulated until the necessary date taking as input the volatility structure specified above; once the entire forward curve has been gained, all rates and bond prices can be compute, then all cash flows are obtained and discounted. Finally, when enough realizations have been made, the expected value is calculated with the desired precision.

In order to achieve the aim of pricing the derivative considered have been implemented functions through the language of technical computing Matlab. It is then given an analysis of the results with the support of explanatory graphs. Most of the objects used here have been presented in earlier chapters and there is no claim to keep the chapter self-contained, however it has been tried to explain satisfactorily and clearly the argument .

5.1 Monte Carlo method

Monte Carlo methods are a class of computational algorithms that relies on repeated random sampling to compute their results. They are useful for modeling phenomena with significant uncertainty inputs and tend to be used when it is impossible to get an exact result through a deterministic algorithm. These models have a wide area of applications, principally it is common recur to them in simulating physical and mathematical systems. Monte Carlo methods have also proven efficient in solving financial issues, such as evaluating derivatives.

Therefore, in mathematical finance (see [8]), Monte Carlo methods are used to analyze instruments by simulating the random sources affecting their value, and then determining their expected value over the range of resultant outcomes. Usually the products considered have to be exercised at final time; typically their final payoffs involve the history of the underlying variable up to the end, and not only the final value of the underlying.

Use these methods brings some advantages. First of all simulations are ease to compute, then many software are available to compute the realizations, and it is possible to incorporate path-dependency. Furthermore is often simple to modify the models in order to have one suitable for the considered case. However Monte Carlo methods are not free of drawbacks. In fact, since trajectories are propagated forward in time, there is no clue whether at a certain point in time is optimal continue or exercise. Therefore it can be difficult manage products involving early exercise, like American option. Furthermore, if an analytical technique exists, these methods are usually too slow to be competitive.

5.2 Rates calculation

In this work has been paid particular attention to the payer swap option, introduced in the third chapter, where turned out that is necessary to know the evolution of the forward swap rate in order to compute its discounted payoff given by

$$ND(t, T_\alpha)(S_{\alpha,\beta}(T_\alpha) - R)^+ \sum_{i=\alpha+1}^{\beta} p(T_\alpha, T_i)(T_i - T_{i-1}). \tag{5.1}$$

For an exact comprehension of the formulas introduced here the reader can refer to the corresponding argumentation presented in former chapters.
For the prefixed aim, has been considered the HJM framework (see the relative chapter), which assumed the following dynamics for the instantaneous forward rate, for a fixed maturity T,

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ f(0,T) &= f'(0,T). \end{aligned}$$

Having computed the above using the volatility structure previously obtained, is possible to achieve zero-coupon bond prices $p(t, T)$. As a consequence, is possible to calculate the other interest rates, such as the short rate, the Libor, the simply-compounded forward rate, and the discount factor D, through the formulas introduced in the first chapter. As well, the forward swap rate can be calculated, so that to compute the discounted payoff 5.1.

For this procedure has been implemented as usual a Matlab function named rates.m (see appendix B), which take as inputs the market instantaneous forward rate curve at time $t = 0$, the volatility structure, the year fraction and a random matrix which values are normally distributed.
Using the Monte Carlo method, this function is recalled at every simulation and gives as output for every time and maturity, the bond prices, the instantaneous forward rate, the short rate, the Libor, the simply-compounded forward rate and the discount factor.

Consider now a single representation in which the volatility is the term structure calculated with the sigmaabs.m function in the last chapter and the instantaneous forward rate at current time is given from the market as usual. The HJM dynamics $f(t, T)$ is displayed in figure 5.1 where is possible to ob-

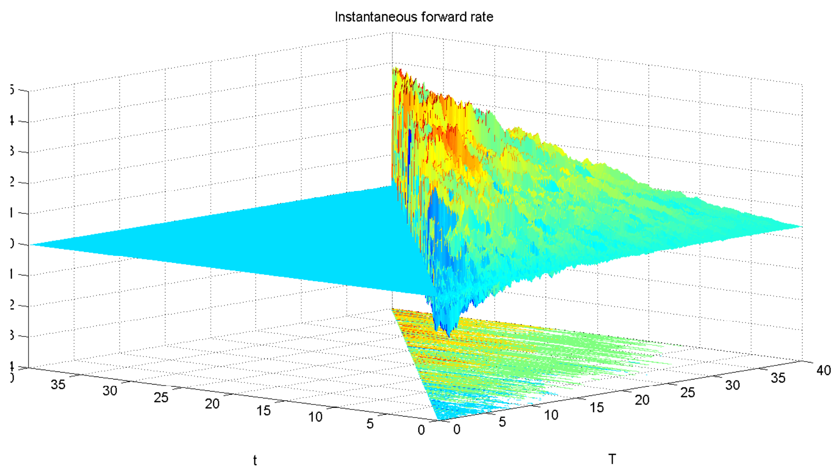

Figure 5.1: Instantaneous forward rate dynamics.

serve that it grows either as the time or the maturity increase. Obviously it is null for $t > T$. Furthermore it can assume negative values due to the presence of the Brownian motion, and the resultant range is given by $[-0.4, 0.5]$. It is interesting to notice that curves get smooth as the maturity becomes greater, which means they get less affected by the randomness.

In figure 5.2 are represented the bond price trends for different time and maturities.

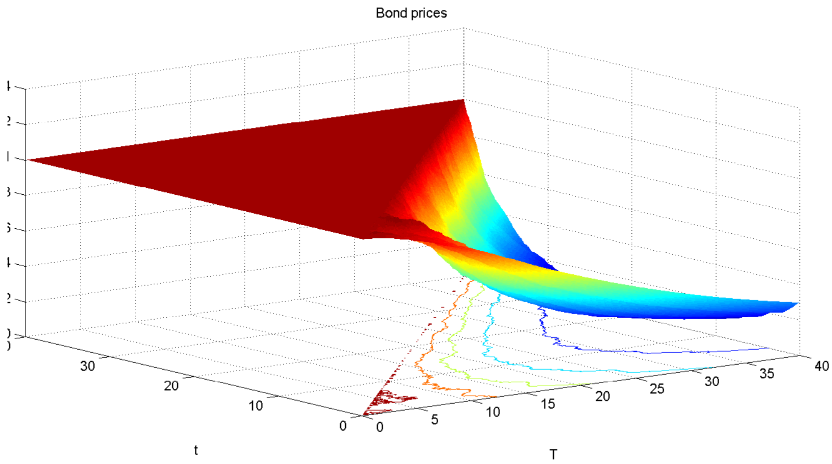

Figure 5.2: Zero coupon bond prices.

They are correctly downward sloping and they keep values between zero and the unit. They assume a remarkable importance because they are necessary for the calculation of the forward swap rate.

The figure 5.3 shows the resultant short rate, which is noticeably influenced by the Brownian motion of the instantaneous forward rate dynamics. Has expected it has an upward sloping trend while the range is [-0.3,0.5]. Therefore it can assume negative values due to the high variability, which are not conceivable from an economic point of view. Again this drawback affects slightly the aim of the work because doing many Monte Carlo simulation the bad effect is reduced considerably.

Then the discount factors, which have a trend similar to the bonds, are displayed in figure 5.4. Their values are fundamental to compute the prices, in fact they discount to present time the cash flows of the derivative considered. They lie between zero and the unit as expected.

The Libor, even if is not directly implicated in the final purpose of this work, it will be involved in the further analysis of the next section. It is possible to see in

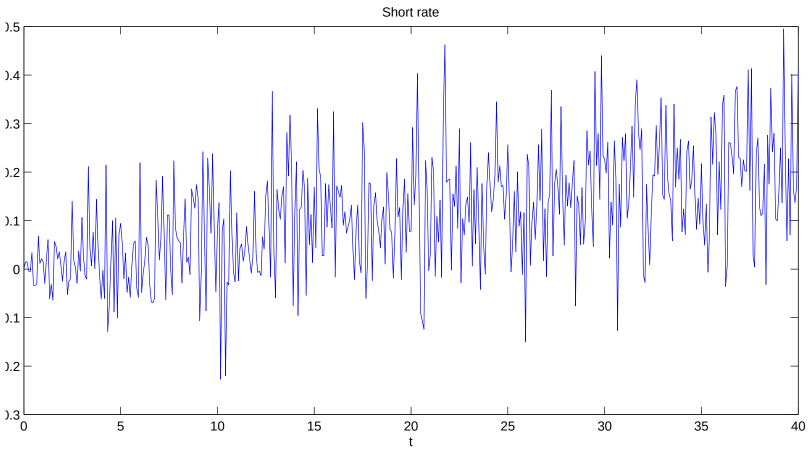

Figure 5.3: Short rate.

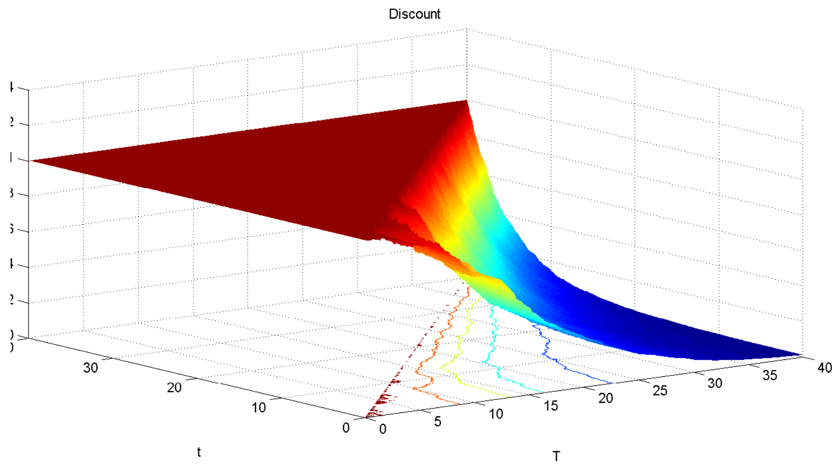

Figure 5.4: Stochastic discount factors.

figure 5.5 that it is direct proportional to the maturity while assumes a concave form as function of time, behavior that is more evident for higher maturities. The range varies between -0.4 and 0.6.

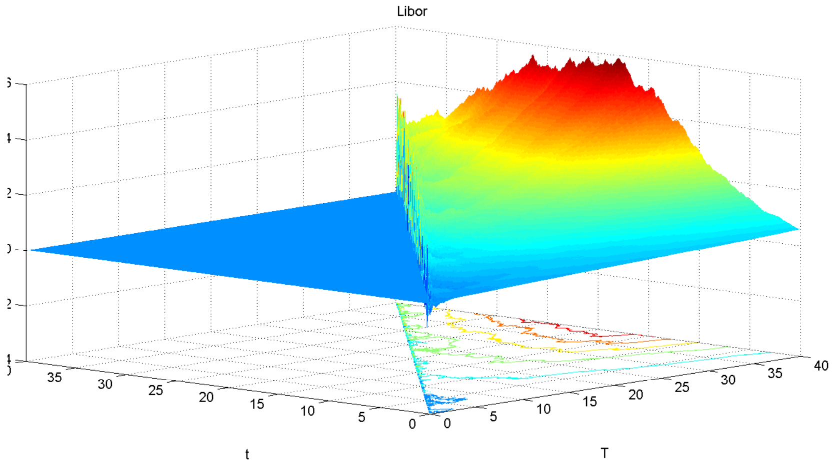

Figure 5.5: Libor.

5.2.1 Analysis

In order to test the goodness of the function rates.m have been considered data available on [21]. These data, displayed in table 5.1, refer to the instantaneous forward rate and Libor evolutions, for monthly and semi-annual intervals of times.

For this analysis the volatility structure has been taken constant and equals 0.2, this choice derives from the values observed in the market. The uncertain matrix *phi* is normally distributed and the time is taken in respect to the data. Firstly have been considered monthly data, available for the next five years; giving to the Matlab function the instantaneous forward rate has been calculated the Libor for $t = 0$, then the latter has been compared with the market Libor data. The result is shown in figure 5.6 where is possible to state that the function gives a nice approximation although overestimates the Libor trend. The sum of squares of residuals, equals 0.0037, is not negligible, therefore is necessary to pay attention to the consequent analysis of the prices. This realization leads to think that all objects computed in this way suffer in term of accuracy. The reason lie in the characteristics of the Heath-Jarrow-Morton framework, which dynamics presumes to be continuous while here are considered at least monthly data.

f monthly	L monhtly	f semi-annual	L semi-annual
0,63	0,57	1,29	0,87
0,76	0,64	1,65	1,25
0,86	0,70	1,28	1,30
0,97	0,75	1,50	1,32
1,11	0,81	1,85	1,39
1,29	0,87	2,24	1,50
1,47	0,95	2,59	1,63
1,63	1,02	2,91	1,77
1,73	1,10	3,20	1,91
1,76	1,16	3,45	2,06
1,73	1,21	3,68	2,19
1,65	1,25	3,88	2,33
1,55	1,28	4,05	2,45
1,45	1,30	4,19	2,57
1,37	1,30	4,31	2,68
1,32	1,31	4,41	2,79
1,29	1,31	4,49	2,89
1,28	1,30	4,55	2,98
1,29	1,30	4,60	3,06
1,32	1,30	4,63	3,14
1,36	1,31	4,65	3,21
1,40	1,31	4,67	3,28
1,45	1,31	4,67	3,34
1,50	1,32	4,67	3,39
1,55	1,33	4,67	3,44
1,61	1,34	4,66	3,49
1,67	1,35	4,65	3,53
1,73	1,36	4,64	3,57
1,79	1,38	4,62	3,61
1,85	1,39	4,60	3,64
1,92	1,41	4,58	3,67
1,98	1,42	4,56	3,70
2,05	1,44	4,54	3,73
2,11	1,46	4,52	3,75
2,17	1,48	4,49	3,77
2,24	1,50	4,47	3,79
2,30	1,52	4,44	3,81
2,36	1,54	4,42	3,83
2,42	1,56	4,39	3,84
2,48	1,59	4,36	3,85
2,54	1,61	4,33	3,87
2,59	1,63	4,30	3,88
2,65	1,65	4,27	3,89
2,70	1,68	4,24	3,90
2,76	1,70	4,20	3,90
2,81	1,72	4,17	3,91
2,86	1,75	4,14	3,91
2,91	1,77	4,11	3,92
2,96	1,80	4,07	3,92
3,01	1,82	4,04	3,92
3,06	1,84		
3,11	1,87		
3,16	1,89		
3,20	1,91		
3,24	1,94		
3,29	1,96		
3,33	1,99		
3,37	2,01		
3,41	2,03		
3,45	2,06		

Table 5.1: Bank of England yield curves

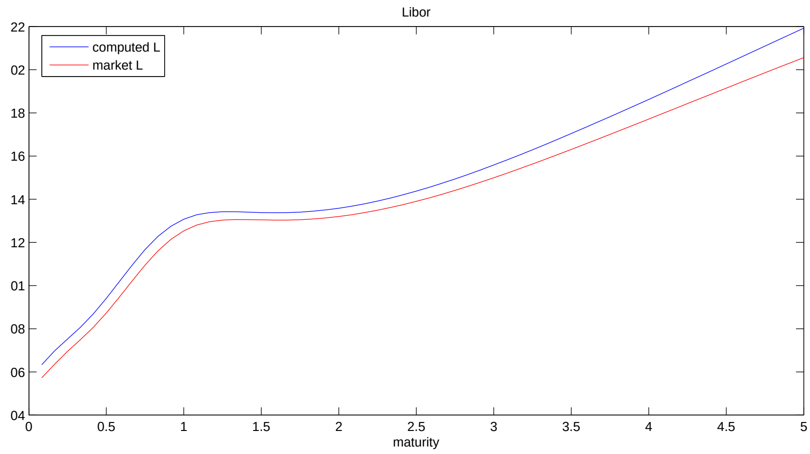

Figure 5.6: Monthly Libor estimated and market data.

Now are taken as inputs semi-annual instantaneous forward rate data for the next twenty-five years. Again the resultant Libor is compared with market Libor data in the table 5.1. Observing the figure 5.7 it is clear that the situation is getting worse as the maturity goes forward. Until T equals five years the difference between the two curves remains the same as the previous monthly case, but the sum of squares of residual grows substantially going farer, reaching the total amount of 0.0052.

As a conclusion of this analysis it is better to remark that using the dynamics of the instantaneous forward rate deriving from the HJM framework the resultant objects necessary for pricing can be involved in some estimation errors. As a consequence becomes very important to keep an eye on the price values that can be significantly affected using this procedure. As a matter of fact, it is known from literature that HJM overestimates financial products.

5.3 Simulations

This paragraph is devoted to the analysis and calculation of payer swaption prices through the Heath-Jarrow-Morton framework. For this purpose have been considered several maturities and tenors according to the available data. As already mentioned the procedure used in this work tends to return overestimated

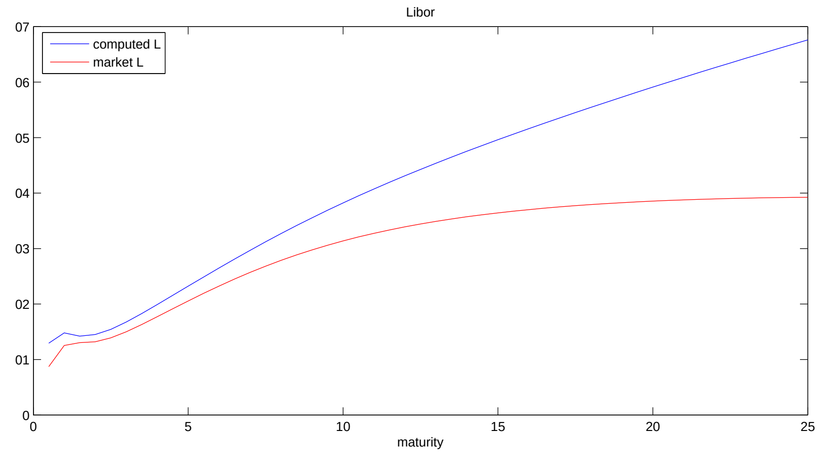

Figure 5.7: Semi-annual Libor estimated and market data.

values, implying the analyst to be very careful with the result obtained. Here the focus is pointed on the analysis of the models taken under consideration, it is carried out an investigation computing prices of payer swap options, for which the prices have been already estimated through the Black like formula and the Hull-White model during the calibration process. This is done in order to give a program fully understandable for studies of more complex derivatives.

The work follow the procedure exposed in paragraph 2.2, where it is explained how to adapt the variety of Monte Carlo methods to the case under question. Having seen how get rate values, is now the time to have a look to the function implemented to compute the prices for a single realization. Through Matlab has been created simulation.m which is displayed in appendix B.

5.3.1 Results

As inputs of simulation.m have been taken the market instantaneous forward rate, the year fraction and the present time, the volatility matrix from the previous chapter and a random matrix which values are normally distributed, while the nominal value equals the unit. Maturity and tenor both assume values of five, ten, fifteen and twenty years.
These values are the same passed to the Matlab function pricing.m (see ap-

pendix B) for which is necessary to specify also the number n of Monte Carlo simulations. This function returns the final price of the payer swap option as a mean of all the simulations executed for the specific tenor and maturity chosen. In literature exist various methods capable to accelerate the convergence. The one used here is called *antithetic variates (AV) method* and consists on double the simulations considering each random matrix twice, changing the signs of all elements.

In figure 5.8 are represented the evolutions of the payer swaption prices calcu-

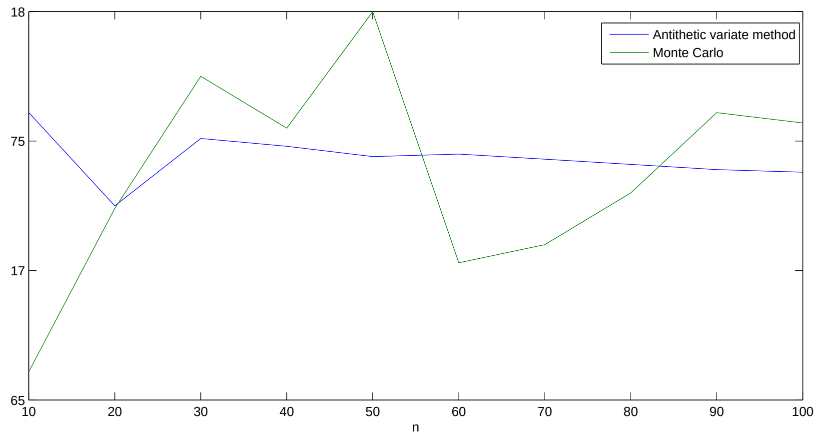

Figure 5.8: Comparison of prices obtained through Monte Carlo with and without using the antithetic variates method.

lated through Monte Carlo for $n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$, $tenor = 20$ and $T = 20$ years, recurring or not to the AV method. It turns out that in the first case the algorithm converges more rapidly and it get close to the final solution even for relative small values of n, while, in the second case, results present high variability and don't seem to assure a rapid convergence.

Then final price obtained for $n = 1000$ using AV is displayed in table 5.2, where are shown as well the results for different number of simulations. It is evident the better performance offered by the antithetic variates method respect to the normal Monte Carlo. Results for different maturities and tenor with the higher n are presented in table 5.2.

In table 5.3 are shown for each maturity and tenor the prices obtained with

n	AV	MC
10	0.1761	0.1661
20	0.1725	0.1724
30	0.1751	0.1775
40	0.1748	0.1755
50	0.1744	0.1800
60	0.1745	0.1703
70	0.1743	0.1710
80	0.1741	0.1730
90	0.1739	0.1761
100	0.1738	0.1757
1000	0.1736	0.1753

Table 5.2: Comparison between prices using MC with and without AV.

mat-tenor	price	pbl	phw	a	σ	Ssq	iter
5-5	0.0910	0.0319	0.0313	0.0240	0.0328	0.0009	5
5-10	0.1397	0.0562	0.0474	0.0718	0.0350	0.0059	14
5-15	0.1722	0.0724	0.0577	0.0878	0.0373	0.0106	16
5-20	0.2060	0.0829	0.0689	0.0729	0.0344	0.0140	15
10-5	0.1829	0.0360	0.0418	0.0299	0.0358	0.0018	12
10-10	0.2779	0.0620	0.0656	0.0402	0.0337	0.0124	5
10-15	0.3340	0.0787	0.0803	0.0492	0.0348	0.0237	18
10-20	0.3695	0.0896	0.0906	0.0502	0.0347	0.0330	15
15-5	0.1516	0.0343	0.0406	0.0555	0.0405	0.0023	15
15-10	0.2298	0.0585	0.0644	0.0551	0.0369	0.0141	11
15-15	0.2673	0.0742	0.0793	0.0585	0.0372	0.0259	7
15-20	0.2882	0.0864	0.0879	0.0585	0.0369	0.0352	18
20-5	0.0915	0.0296	0.0347	0.0636	0.0422	0.0025	16
20-10	0.1415	0.0513	0.0549	0.0635	0.0390	0.0153	17
20-15	0.1643	0.0653	0.0674	0.0630	0.0384	0.0270	16
20-20	0.1736	0.0761	0.0756	0.0640	0.0385	0.0368	9

Table 5.3: Results for payer swaptions.

the three different models. The results reflect what expected from previous considerations, in fact it is possible to see that prices computed through Black like formula and Hull-White are really close (as already seen in the last chapter with the figure 4.6). This means that the parameters a and σ are really well calibrated, note that they vary with maturity and tenor due to the fact that for the calibration procedure has been considered only data until the necessary time for calculation (for example for the first line of the table are taken only data with maturity and tenor up to five years). In the table is reported also the number of iterations to get to the best solution, it is good to point out that are all low numbers, which imply that the implemented method used for the calibration has excellent performances.

Finally it is possible to compare prices obtained using the Heath-Jerrow-Morton

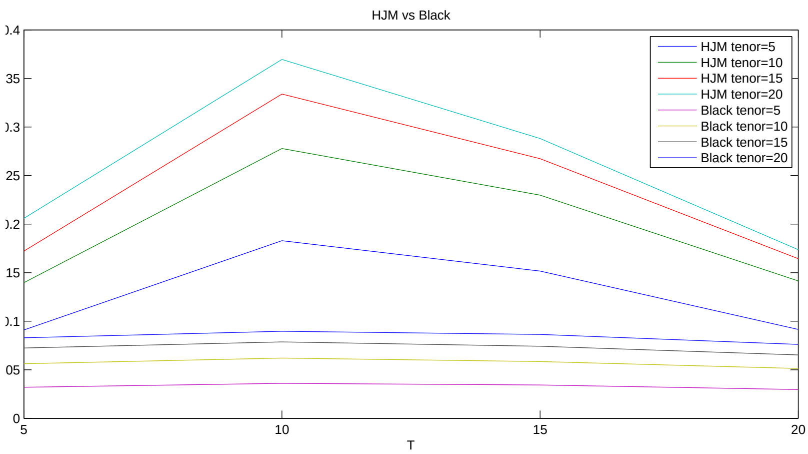

Figure 5.9: Comparison between prices obtained through HJM and Black like formula.

framework with prices calculated through Black like formula. In figure 5.9 are represented the differences between the results. It is easy to observe that as expected HJM gives overestimated values of prices, this poor estimation is greater for intermediate values of maturity while the inaccuracy is quite reduced for long maturities where HJM prices can still reach twice as much the Black's values. A possible explanation can be found in the characteristics of the models. In fact Hull-white doesn't work well for short maturities, so that in such a case the variables passed to HJM suffer of imprecise valuation. While for long matu-

mat/tenor	5	10	15	20
5	0.0894	0.1789	0.2408	0.2786
10	0.1470	0.2546	0.3167	0.3502
15	0.1375	0.2187	0.2594	0.2794
20	0.0945	0.1422	0.1638	0.1736

Table 5.4: HJM payer swaption prices.

rities, when Hull-White provides optimal solution, is the HJM dynamics which is involved in estimation problems. As a consequence for mid-term maturities, prices are subjected to the influence of both of these factors, resulting as in the figure.

A similar trend is obtained using constant parameters $a = 0.0640$ and $\sigma = 0.0385$ (values deriving from the analysis done with much data as possible) for all maturities and tenors. The prices resultant from the HJM dynamics are displayed in table 5.4 and it can be stated that they are similar to the prices obtained before varying the parameters. In figure 5.10 can be observed that in this case curves become a bit smoother for maturity values equal ten and fifteen years.

From this analysis it is possible to say that through this procedure can be

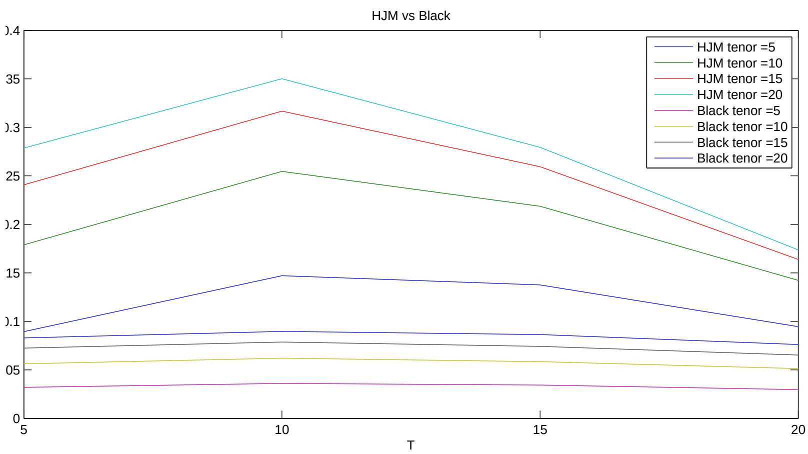

Figure 5.10: Comparison between prices obtained through HJM and Black like formula for constant parameters $a = 0.0640$ and $\sigma = 0.0385$.

obtained quite good approximations for long maturities; being careful with the overestimation of the results it could be a nice development try to analyse more complex derivatives. Although inaccuracy for short maturities can not be neglected, trends follow predictions based on literature.

Chapter 6

Summary and conclusions

Pricing derivatives represents nowadays one of the most interesting areas for financial engineers. As the exchange of these instrument has been increasing rapidly, arises the need to formulate mathematical models able to reproduce price trends. In fact prediction and analysis are becoming fundamental in the financial sector, the first mostly to detect the best strategy to adopt, while the second is necessary for a better comprehension. Particularly the interest rate field is still "work in progress". For these reasons have been created many models trying to satisfy the general needs, most of them are well explained in literature and they give some hints for further evolutions. However does not exist one model able to cover satisfactorily the entire wide variety of derivative pricing procedures, so that is often necessary to combine two or more models in order to obtain exhaustive results.

The objective of this work has been to supply a specific evaluation of the pricing process of payer swap options, providing useful information on the considered models. Particularly, the choice land on the Heath-Jarrow-Morton framework able to capture the full dynamics of the entire yield curve. For its realization have been done Monte Carlo simulations through the language of technical computing Matlab, which helped as well with explanatory graphs the qualitative analysis carried out. The values obtained were slightly overestimated due to the dynamics considered. Has been found out that these errors tend to be more consistent for long maturities even increasing the number of simulations. As a consequence the mentioned framework is probably not the most appropriate for the derivative considered, at least used combined with the Hull-White model.

In fact it does not give results with the precision expected for all the cases taken under analysis.
HJM led to the investigation over a model able to furnish the volatility structure necessary, so that has been realized a deep analysis of the Hull-White model, which enabled to obtain quantitative answers and consequently allowed to give important considerations. Particular attention has been focused on the calibration procedure for which has been made up an original function based on the Levenberg-Marquardt algorithm. Respect to it many tests and researches have been executed in order to find the best values for the questioned parameters, imposing prices obtained through Hull-White and Black like formula close as possible. Precisely, knowing the Hull-White drawbacks from the literature, has been decided to give less importance to short maturity and tenor data. The final solutions were in completely agreement with values observed in the market and confirmed the goodness of the initial choices. As a consequence all the functions involved in the calibration procedure allow to have nice performance and lead to satisfactorily results. Then it is possible to state that the Black like formula for payer swaption can be a good approach for the Hull-White parameters calibration, when at the money data are available.

6.1 Possible alternatives and future developments

During the evolution of the work have arisen some natural opportunities of alternatives to the procedures and models used.
Firstly the market data has been interpolated linearly, which is a quite good way for the prefixed aims, but could be possible to consider a different interpolation especially for more advanced analysis.
Secondly in order to achieve the volatility structure has been used the Hull-White model, fundamentally for practical reason. For this purpose it is possible to take own choices due to particular considerations or alternatively use another model present in literature, has been show that Hull-White model does not perform exactly for short maturities. This inconvenient has been solved imposing different weights to the variables under question but it could be possible directly consider other single or multi-factor models which give a better fit, however these models often lead to lose efficiency in the numerical implementation.
Thirdly the data available for the calibration were the ATM volatilities and

strikes, having out of the money and in the money data could be possible to get more realistic results. However the process could become burdensome from a computational point of view and difficult to achieve with standard computers. Then has been created a particular Matlab function based on Levenberg-Marquardt algorithm for the calibration procedure, which resulted to be good for the case under consideration. However many new alternatives could be covered, either own choices or models existent in the literature. Furthermore to compare the prices has been used a Black like formula, while it could be possible to find directly market prices or use other models to compute the necessary results. An interesting idea that came out during the work evolution is to try to adapt the SABR model (see [7] for details) to this survey, for example using it instead of the Black like formula for the calibration. However for this advanced study it is necessary to be careful in considering dynamics under different measures.

Furthermore, as mentioned in the pricing chapter, various techniques to accelerate the convergence when facing Monte Carlo exist in literature. Here has been used one of quite simple implementation with satisfactory results, for this reason have not been taken under consideration other methods as for example the control variates technique.

Finally this work focused the attention on payer swap option, but starting from what has been shown here other derivatives could be analyzed. Especially the reference is to complex derivatives for which there is no analytical solution. For example having obtained the volatility structure as in this work could be possible to analyze Bermudan swaption, in which the owner is allowed to enter the swap only on certain specific dates that fall between the start date of the contract and the option maturity.

Appendix A

Volatility structure code

In this appendix are exposed all the functions implemented through the language of technical computing Matlab relative to chapter four.

A.1 Hull-White

Here is displayed the function hw.m computing the principal elements of the Hull-White model, the coefficients A and B, the short rate and the prices of zero coupon bond.

```
function [A,B,r,p] = hw(a,sig,f0,p0,T,dt)
% Hull-White model
% a, sig -> parameters
% f0, p0 market instantaneous forward rate and discount factors
% T -> maturity
% dt -> year fraction

% Initializations
B = zeros (T+1,T);                % parameter
A = zeros (T+1,T);                % parameter
r = zeros (T+1,1);                % short rate
p = ones (T+1,T);                 % bond prices

f0 = [0 f0];                      % computational adjustment
p0 = [1 p0];
```

```
for j = 1:T                          % parameters calculation
    for i = 1:j+1
        B(i,j) = 1/a * ( 1 - exp( -a*(j-(i-1))*dt ) );
        A(i,j) = p0(j+1)/p0(i) * exp( B(i,j) * f0(i) - (sig^2)/(4*a) * ...
            ( 1 - exp(-2*a*(i-1)*dt) ) * B(i,j)^2  );
    end
end

% Spot rate
h = randn(T,1);                      % random source
x = zeros(T+1,1);                    % parameter
alpha = zeros(T+1,1);                % parameter
for i = 2:T+1
    x(i) = ( -B(i-1,T) * sig^2 - a * x(i-1) ) * dt + ...
        sig * sqrt(dt) * ( h(i-1)  );
    alpha(i) = f0(i) + (sig^2)/(2*a^2) * ( 1 - exp(-a*i*dt) )^2;
    r(i) = x(i) + alpha(i);
end

for j = 1:T                          % bond prices
    for i = 1:j
        p(i,j) = A(i,j) * exp( - B(i,j) * r(i) );
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

A.2 Swaption prices

The next function is necessary to compute the price of a swap option through Hull-White model. It gives the price of a European option.

```
function price = europeopthw(a,sig,t,T,s,dt,X,w,B,p)
% European option price through Hull-White model
% w=1 call
% w=-1 put
```

```
sigp = sig * sqrt( ( 1 - exp( -2*a*(T-(t-1))*dt ) )/( 2*a ) ) * B(T+1,s);
h = 1/sigp * log( p(t,s)/( p(t,T)*X ) ) + sigp/2;

price = w * p(t,s) * ( 0.5 + 0.5 * erf( (w*h)/sqrt(2) ) ) - ...
    w * X * p(t,T) * ( 0.5 + 0.5 * erf( (w*(h-sigp))/sqrt(2) ) );
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The function swaptionhw.m use the above to return the price of a swaption.

```
function price = swaptionhw(a,sig,t,T,d,s,dt,N,c,w,A,B,p)
Xn = zeros(s,1);
suma = 0;

for i = 1:s
    Xn(i) = A(T+d(1),T+d(i)) * exp( -B(T+d(1),T+d(i)) * rstar );
    suma = suma + c(i) * europeopthw(a,sig,t,T,T+d(i),dt,Xn(i),-w,B,p);
end

price =N * suma;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The next function use Black like formula to compute the price of a swaption.

```
function pbl = swaptionblack(z,u,t,T,d,dt,p0,X,R,vol,w,N)
% Swaption price through Black formula at time t
% w=1 payer
% w=-1 receiver
pbl = zeros(z,u);
for i = 1:z
    for j = 1:u
        suma = 0;
        dtderiv = [0 d'];
        for m = 1:j
            suma = suma + p0(T(i)+ d(m)) * (dtderiv(m+1)-dtderiv(m)) *dt;
        end
        pbl(i,j) = N * black(X(i,j),R(i,j),vol(i,j),w,t,T(i)*dt) * suma;
    end
```

```
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

A.3 Calibration

The function LMFsolveswaption.m is an original development of the Levenberg-Maquardt algorithm which provides a numerical solution to the problem of minimizing a function (the difference between HW prices and Black prices) over a space of parameters (a and σ) of the function. Note that is supported by another function finjac.m which gives numerical approximation to Jacobi matrix.

```
function [xf, Ssq, cnt, phw, R, iter] = ...
    LMFsolveswaption(xc,phw,pbl,z,u,f0,p0,T,tenor,dt,N,w)
% LMFSOLVE for swaption at initial time
% xc -> vector of initial guesses
% Output Arguments:
% xf         final solution approximation
% Ssq        sum of squares of residuals
% cnt        >0          count of iterations
%            -MaxIter,   did not converge in MaxIter iterations
% phw        final Hull-White prices
% R          Hull-White ATM strikes
% iter       number of good iterations

%               Default Options
   MaxIter  = 50;        %   maximum number of iterations allowed
   FunTol   = 1e-5;      %   tolerance for final function value
   XTol     = 1e-3;      %   tolerance on difference of x-solutions

t=1;
x  = xc;
lx = length(x);

peso = (T*dt)*(tenor*dt)';  % Weights
peso = sort(1./sqrt(1+peso));
```

```
phw = phw .* peso;
pbl = pbl .* peso;

r  = (phw-pbl);              % Residuals at starting point
% Procedure to have residuals as a vector
lrT = length(r(:,1));
lrd = length(r(1,:));
r1 = zeros(lrT*lrd,1);
count = 0;
for i = 1:lrT
    for j = 1:lrd
        r1(count+j) = r(i,j);
    end
    count = count + lrd;
end
r = r1;

%~~~~~~~~~~~~~~~~~
S  = r'*r;
epsx = XTol;
epsf = FunTol;
if length(epsx)<lx, epsx=epsx*ones(lx,1); end
J = finjac(r,x,epsx,z,u,f0,p0,T,t,tenor,dt,N,w,pbl);
%~~~~~~~~~~~~~~~~~~~~~~~

A = J.'*J;                    % System matrix
v = J.'*r;

D = diag(diag(A));            % automatic scaling
for i = 1:lx
    if D(i,i)==0, D(i,i)=1; end
end

Rlo = 0.25;
Rhi = 0.75;
```

```
l=1;      lc=.75;
cnt = 0;

maxit = MaxIter;        %   maximum permitted number of iterations
iter=0;
while cnt<maxit && ...          %   MAIN ITERATION CYCLE
    any(abs(r) >= epsf)

    d  = (A+l*D)\v;             %   negative solution increment
    for i = 1:length(d)         %   evolution control
        if (x(i)-d(i))<0
            d(i)=0;
        end
    end
    xd = x-d;
    phw = cicloswaption(z,u,xd(1),xd(2),f0,p0,T,t,tenor,dt,N,w);
    phw = phw .* peso;

    rd = phw - pbl;              %   new residuals
    lrT = length(rd(:,1));
    lrd = length(rd(1,:));
    r1 = zeros(lrT*lrd,1);
    count = 0;
    for i = 1:lrT
        for j = 1:lrd
            r1(count+j) = rd(i,j);
        end
        count = count + lrd;
    end
    rd = r1;

%   ~~~~~~~~~~~~~~~~~~~

    Sd = rd.'*rd;
    dS = d.'*(2*v-A*d);         %   predicted reduction
```

```
    R  = (S-Sd)/dS;
    if R>Rhi                        %   halve lambda if R too high
        l = l/2;
        if l<lc, l=0; end
    elseif R<Rlo                    %   find new nu if R too low
        nu = (Sd-S)/(d.'*v)+2;
        if nu<2
            nu = 2;
        elseif nu>10
            nu = 10;
        end
        if l==0
            lc = 1/max(abs(diag(inv(A))));
            l  = lc;
            nu = nu/2;
        end
        l = nu*l;
    end

    cnt = cnt+1;

    if Sd<S
        S = Sd;
        x = xd;
        r = rd;
        J = finjac(r,x,epsx,z,u,f0,p0,T,t,tenor,dt,N,w,pbl);
%       ~~~~~~~~~~~~~~~~~~~~~~~~~
        iter = iter+1;
        A = J'*J;
        v = J'*r;
    end

end %   while
```

```
xf = x;                         %   final solution
if cnt==maxit
    cnt = -cnt;
end                             %   maxit reached

[phw,R] = cicloswaption(z,u,x(1),x(2),f0,p0,T,t,tenor,dt,N,w);
pbl = pbl./peso;
r  = (phw-pbl);                 % Final residuals
lrT = length(r(:,1));
lrd = length(r(1,:));
r1 = zeros(lrT*lrd,1);
count = 0;
for i = 1:lrT
    for j = 1:lrd
        r1(count+j) = r(i,j);
    end
    count = count + lrd;
end
r = r1;
%~~~~~~~~~~~~~~~~~
Ssq  = r'*r;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%   FINJAC       numerical approximation to Jacobi matrix
%   %%%%%%
function J = finjac(r,x,epsx,z,u,f0,p0,T,t,tenor,dt,N,w,pbl)
%~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~
lx=length(x);
J=zeros(length(r),lx);
for k=1:lx
    dx=.25*epsx(k);
    xd=x;
    xd(k)=xd(k)+dx;
    phw = cicloswaption(z,u,xd(1),xd(2),f0,p0,T,t,tenor,dt,N,w);
```

```
    peso = (T*dt) * (tenor*dt)';
    peso = sort(1./sqrt(1+peso));
    phw = phw .* peso;

    rd = ( phw - pbl );
    lrT = length(rd(:,1));
    lrd = length(rd(1,:));
    r1 = zeros(lrT*lrd,1);
    count = 0;
    for i = 1:lrT
        for j = 1:lrd
            r1(count+j) = rd(i,j);
        end
        count = count + lrd;
    end
    rd = r1;
%   ~~~~~~~~~~~~~~~~
    J(:,k)=((rd-r)/dx);
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

A.4 Absolute volatility

The next function return the volatility structure necessary for the HJM dynamics. It is derived from Hull-White model considering that the latter has affine term structure.

```
function sigma = sigmaabs(f0,a,sig)
% Absolute instantaneous volatilities through Hull-White model

T = length(f0);
dt=1/12;
sigma = zeros(T+1,T);

for j = 1:T
    for i = 1:j+1
```

```
        sigma(i,j) = exp(-a*(j-(i-1))*dt) * sig;
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Appendix B

Pricing code

This appendix is devoted to the Matlab functions relative to chapter five.

B.1 Rates

In this section is presented the function rates.m which compute rates, bond prices and discount factors starting from the HJM dynamics.

```
function [p,f,r,L,F,D] = rates (f0,vol,dt,phi)

% f0 market values of forward rates at t for all maturity
% vol volatility
% dt year fraction
% phi random variable

T = length(f0);          % number of time units from present to maturity
f = zeros(T+1,T);        % initialization instantaneous forward rates
r = zeros(T+1,1);        % initialization short rate
p = ones(T+1,T);         % initialization bond prices
alpha = zeros(T,T);      % initialization drift
h = zeros(T,1);          % initialization random variable
L = zeros(T+1,T);        % initialization LIBOR
F = zeros(T+1,T,T);      % initialization forward rates
D = ones(T+1,T);         % initialization discount
```

```
sigma = vol;            % volatility assignment
f(1,1:T) = f0;          % market value of forward rates at present time

for j = 1:T              % instantaneous forward rates
    h(:) = phi(j,:);
    for i = 2:j+1
        alpha(i-1,j) = sigma(i-1,j) * sum( sigma(i-1,i-1:j) * dt );
        f(i,j) = f(i-1,j) + alpha(i-1,j) * dt + sigma(i-1,j) * sqrt(dt) * h(i-1);
    end
end

for j = 1:T              % bond prices
    for i = 1:j
        p(i,j) = exp( -sum( f(i,i:j) * dt ) );
    end
end

for i = 1:T            % short rates
    r(i+1) = f(i+1,i);
end

for j = 1:T            % discount
    for i = 1:j
        D(i,j) = exp(- sum( r(i:j) * dt) );
    end
end

for j = 1:T              % libor
    for i = 1:j
        L(i,j) = (1 - p(i,j)) / (dt * (j-(i-1)) * p(i,j));
    end
end

for j = 2:T              % forward rates
    for k = 1:j-1
```

```
        for i = 1:k
            F(i,k,j) = ( ( p(i,k) / p(i,j) ) - 1) / ( dt * (j-k) );
            F(i,j,j) = f(i,j);
        end
    end
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

B.2 Monte Carlo

The next function computes the final price of a payer swaption for particular tenor and maturity. It recurs to the simulation.m displayed below.

```
function price = pricing (t,T,f0,N,s,n,vol,dt)

% t initial time
% T maturity + tenor
% s option maturity
% f0 % f0 market values of instantaneous forward rates
% N nominal value
% n number of simulations
% vol volatility
% dt year fraction

t = t+1;   % Index adjustment
s = s/dt+1;
T = T/dt;

simul = zeros(n,1);  % Initialization prices vector
for k = 1:n          % Price simulations
    phi = randn(length(f0));   %random variable
    simul1 = simulation(t,T,f0,N,s,vol,dt,phi);
    simul2 = simulation(t,T,f0,N,s,vol,dt,-phi);
    simul(k) = mean([simul1,simul2]);
end
price = mean(simul);  % Price calculation
```

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

The following function represents a single simulation of the Monte Carlo methods used to price the specific payer swaption considered through the HJM dynamics.

```
function sim = simulation (t,T,f0,N,s,vol,dt,phi)

% phi random variable

% rates calculation
[p,f,r,L,F,D] = rates (f0,vol,dt,phi);
% forward swap rate at t=0
K = ( p(t,s) - p (t,T) ) / ( dt * sum(p(t,s+1:T)) );
%forward swap rate at maturity
R = ( p(s,s) - p (s,T) ) / ( dt * sum(p(s,s+1:T)) );

% payer swaption price
val = 0;
for i = s+1:T
    val = val + p(s,i) * dt ;
end
ps = N * D(t,s) * max( R - K , 0 ) * val;
sim = ps;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

List of Figures

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