POLITECNICO DI MILANO

Facoltà di Ingegneria dei Sistemi

POLO REGIONALE DI COMO

Master of Science in
Management, Economics and Industrial Engineering

On the use of Phase-Type distributions for modeling general distributed events in manufacturing systems

Supervisor: Prof. Marcello Colledani

Master Graduation Thesis by: Dariush Ebrahimi Azarbayejan

Student Id. Number: 734941

Academic Year 2010/2011

Contents

List of Figures

List of Tables

Abstract

This thesis investigates the application of Phase-Type (PH) distributions for modeling non-exponential repair times when evaluating the performance of manufacturing lines where machines can fail in multiple modes, with analytical techniques. In fact, existing analytical methods for the performance evaluation of manufacturing systems almost always rely on the assumption of exponentially distributed failure and repair times. However, while the assumption of exponential distributions for time to failure is generally verified in real plants, the assumption of exponentially distributed times to repair is not matched, and a modeling approximation is typically introduced.

In this thesis, the proper way to characterize Phase-Type distributions to match generally distributed repair times is investigated. The impact of the moments of the repair time distribution on the system throughput is investigated, showing that only the first and the second moment of the distribution have a significant impact on the system production rate. Higher impacts are negligible. This finding simplifies the Phase-Type distribution characterization drastically, since only few alternative PH distributions need to be considered for a proper characterization of all the possible repair events. The advantages of using the characterized set of PH distributions for modeling generally distributed repair times are proved through the application of a PH-compliant performance evaluation method to a real case study in the automotive industry. The results show that the analysis carried out with the use of PH distributed repair time models captures much better the dynamics of the material flow in the system, thus providing a prediction of the performance of the system that is much closer to the system historical data.

1. Introduction to Phase-Type distributions

1.1 Introduction

Approximate analytical methods have been developed in the last three decades to evaluate the performance of transfer lines [12]. The goal to develop these methods, from practical point of view, is to provide support for decision makers dealing with the problem of configuring, controlling and improving transfer line performance. Transfer lines are specific types of production systems where work piece move sequentially from one stage to another. In case the operational machines start and finish processing parts simultaneously, they are called the synchronous transfer lines. Transfer lines have a wide range of applications in high volume industries such as automobile manufacturing, mechanical component manufacturing, white goods industry and consumer electronics production. Early works have focused on unreliable serial transfer lines with finite capacity buffers where machines are characterized by a unique failure mode [13]. Since the complexity is increasing with the number of stages in the line and the number of failure modes of each machine, decomposition methods, which recommend the exact solution to performance measurements, is proposed for large systems [14].

More recently, analytical models were extended to deal with machines with multiple failure modes [15]. In fact, multiple failure modes come to pass when every single machine in the line can go to downstate, which is failure state, in several modes. Each failure mode is characterized by specific times to failure and times to repair. Practically points of view, machines of production lines are composed of several parts and components which might fail frequently. Consequently, these different failure modes of machines might make sever problems in the production flow. For instance, in [15] it is proved that, in case of having the multiple failure modes with significant different time to repairs of each single failure mode while we have small buffers, simplifying the multiple failure mode by a single average failure mode may cause the relevant error in the throughput estimation. Therefore, we need to reflect on multiple failure modes.

Although there are studies related to analytical methods to approximate the performances of transfer lines, it is still a challenge to consider the actual system with these approaches. In fact, the studies

till now are available in case we have the assumption of exponentially distributed time to repair and time to failure. While these existing approaches to the performance measurement of manufacturing lines have an acceptable accuracy, the times to repair and failure are rarely distributed exponentially. Given the nature of failure, which is mechanically or electronically, time to repair is related to managerial issues, because the small amount of the repair period is dedicated to physical repair process and the other portion is related to managerial decisions.

In this project, we focus on considering the decomposition method for evaluating the performance of multi-stage transfer lines in which machines can fail in multiple modes, and, for each mode, Phase-Type distributed repair times are considered . Many theoretical studies have been conducted related to Phase-Type distributions which is been shown in the literature. In fact, these interesting distributions can cover most of general distributed times to repair and failures.

1.2 Characterization of Phase-Type Distribution

1.2.1 Continuous phase-type distributions

The Continuous phase-type distribution (CPH) is the distribution of time to absorption in a discrete time Markovian chain (CTMC) with n transient states and one absorbing state. This process can be written in the form of a transition rate matrix, as follow:

$$Q = \begin{bmatrix} S & S^0 \\ 0 & 0 \end{bmatrix}$$

Where S is a $(k \times k)$ matrix that describes the transient behavior of the CTMC and S^0 is a column vector of length k that groups the transition rates to the absorbing state. Let $\alpha = [\alpha, \alpha_{n+1}]$ is of length $(k + 1)$ with $\sum_{i=1}^{n} \alpha_i = 1 - \alpha_{n+1}$ .Thus, α is the initial state probability vector. The tuple (α, Q) is called the representation of the Continuous phase-type distribution, and m is the order [1].

Continuous nonnegative random variable describing time until absorption, X, has the following cumulative function:

3

$$(1)\, F(x) = 1 - \boldsymbol{\alpha}\, exp(Sx)\mathbf{1}$$

Therefore, the density function defined for positive X is,

$$(2)\, f(x) = \boldsymbol{\alpha} e^{xS} \boldsymbol{S}^0$$

Mean of the X is,

$$(3)\, E(x) = -\boldsymbol{\alpha} S^{-1}\mathbf{1}$$

n^{th} moment around zero is,

$$(4)\, E(X^n) = (-1)^n\, n!\, \boldsymbol{\alpha}\, S^{-n}\, \mathbf{1}$$

And varianc of X is given by,

$$(5)\, Var(x) = 2\, \boldsymbol{\alpha} S^{-2}\mathbf{1} - (\boldsymbol{\alpha}\, S^{-1}\mathbf{1})^2$$

Since the introduction of phase-type distribution by Neuts [2] in 1975, phase-type (PH) distributions have been used in a wide range of stochastic modeling applications in areas as diverse as telecommunications, finance, teletraffic modeling, biostatistics, queuing theory, drug kinetics, reliability theory, and survival analysis. PH distributions have enjoyed such popularity because they constitute a very versatile class of distributions defined on the nonnegative real numbers that lead to models which are algorithmically tractable. Their formulation also allows the Markov structure of stochastic models to be retained when they replace the familiar exponential distribution.

Phase-type distribution usage in modeling systems in the healthcare industry has been conducted by Mark Fackrell [3]. He has fitted the length of stay and inter arrival times to General Phase-type distribution. Consequently, according to his comparisons to other distribution's fittings, he has concluded that the general PH-type distribution fit is the best followed by the Coxian distribution (which is a type of phase-type distribution) fit.

1.2.2 Mostly used Continuous Phase-Type Distributions

The following probability distributions are considered all types of a continuous phase-type distribution. In fact, changing the transitions between states and their rates cause differences in the distributions. Moreover, each of them has been used widely in literatures.

1.2.2.1 Exponential distribution

The exponential distribution is everywhere in stochastic modeling, mainly because of its simplicity and ability to model random lengths of time reasonably well. For example, it has been used to model the length of stay in a hospital bed, or the time between presentations to an emergency department. In this short part we introduce the exponential distribution and list some of its properties.

A continuous nonnegative random variable X is distributed according to an exponential distribution with parameter $\lambda > 0$, if its cumulative function, defined for $x \geq 0$, is given by

$$(6)\, F(X) = 1 - e^{-\lambda x}$$

The density function of X, defined for $x \geq 0$, is given by

$$(7)\, f(X) = \lambda e^{-\lambda x}$$

The expected value of X, or its mean, is,

$$(8)\, E(X): {}^{1}/_{\lambda}$$

And   its   variance   is

$$(9)\, Var\,(X): {}^{1}/_{\lambda^2}.$$

An exponential distribution can be also a phase-type distribution which has only one phase. Consequently, processing time till the absorbing state would be just moving from initial state to the absorbing state, which is one transition.

In the following we have SCV and transition rate matrix of exponential distribution:

$$(10) \quad SCV = 1$$

$$(11) \quad S = [-\lambda]$$

According to [4], applying the exponential distribution simply in stochastic modeling is not only due to the fact that there is only one parameter, λ, but also there is another unique property which is so called memory less property. That is, for *s, t* $\geq$ 0, *P (T > s + t|T > t) = P (T > s)*. The memory less property enables simple expressions for many performance measures of stochastic models.

Although the exponential distribution has been used widely in stochastic modeling, its main drawback is its lack of versatility, being characterized by only one parameter. Consequently, we need to define other kinds of phase-type distribution which has more favorable properties compare to the exponential distribution.

1.2.2.2 Erlang distribution

Regarding to [5], Erlang was the first person to extend the exponential distribution with his "method of stages". He defined a nonnegative random variable as the time taken to move through a fixed number of states, spending an exponential amount of time with a fixed rate in each one. Nowadays we refer to distributions defined in this manner as Erlang distributions.

In other words, Erlang distribution is the distribution of the time when process starts from initial state to absorbing state in such a way that system passes all the states skip-free with an identical rate. Besides, we can claim that each of those phases have an exponential distribution with the same rate.

The nonnegative continuous random variable *X* is distributed according to Erlang distribution with the parameter *m* which is so called the shape parameter and the parameter λ which is so called the rate parameter, if its cumulative function, defined for $x \geq 0$, is given by

$$(12) \quad F(X) = \frac{\gamma(k, \lambda x)}{(k-1)!}$$

Consequently, the density function of *X*, defined for $x \geq 0$, is given by

$$(13) \quad f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$$

The mean is,

$$(14) \quad E(X): k/\lambda$$

And the variance is,

$$(15) \quad Var\ (X): k/\lambda^2$$

Moreover, initial state probability vector is,

$$(16) \quad \alpha: (1,0,0,0,\ldots 0)$$

And the Squared coefficient of variation is,

$$(17) \quad SCV: 1/k \text{ , which is always less than one.}$$

Figure1 depicts the state transition diagram for an order K continuous time Erlang distributions.
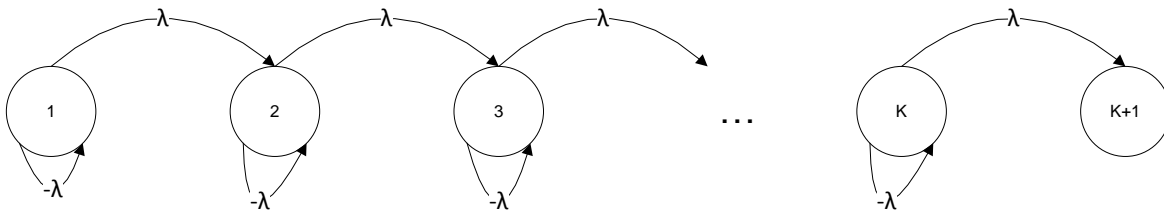
Figure 1, states transition diagram for Erlang distribution

States 1 to k are transient states and state K+1 are the absorbing state. Note that a state is *transient* if once it has been reached, the probability of returning to it is less than one, and a state is *absorbing* if once it has been reached the process stops.

Following matrices are transition rates between states in Erlang distributions,

$$S = \begin{bmatrix} -\lambda & \lambda & \ldots & 0 & 0 \\ 0 & -\lambda & \lambda & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & -\lambda \end{bmatrix}$$

$$S^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda \end{bmatrix}$$

Figure 2 shows the density function for an order 5 Erlang distribution (that is, all rates are equal to one).
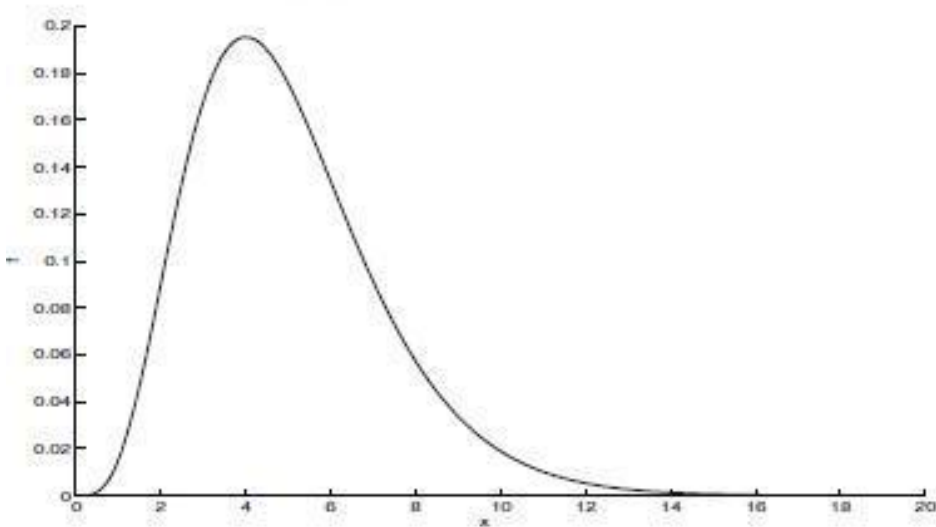
Figure2, density function curve for an order 5 Erlang distribution

Because of the factorial function in the denominator, the Erlang distribution is only defined when the parameter k is a positive integer. In fact, this distribution is sometimes called the Erlang-k distribution (e.g., an Erlang-2 distribution is an Erlang distribution with $k=2$). The Gamma distribution generalizes the Erlang by allowing k to be a real number, using the gamma function instead of the factorial function, though the Gamma distribution is not the derivation of phase-type distribution.

1.2.2.3 Mixture of Erlang distribution

The mixture of two Erlang distribution with parameter $E(3,\beta_1)$, $E(3,\beta_2)$ and (α_1,α_2) (such that $\alpha_1 + \alpha_2 = 1$ and for each i, $\alpha_i \geq 0$) can be represented as a phase type distribution with

$$\boldsymbol{\alpha} = (\alpha_1, 0, 0, \alpha_2, 0, 0),$$

And

$$S = \begin{bmatrix} -\beta_1 & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & -\beta_1 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & -\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta_2 & \beta_2 & 0 \\ 0 & 0 & 0 & 0 & -\beta_2 & \beta_2 \\ 0 & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}.$$

1.2.2.4 Hyper-exponential distribution

Probability distribution of two or more non-identical phases that each has a probability of occurring in a mutually exclusive, parallel manner is Hyper-exponential. The continuous nonnegative random variable X is distributed according to Hyper-exponential distribution if its density function is defined as follow:

$$(18) \quad f(x) = \Sigma_{i=1}^n f_{Y_i}(y) p_i$$

Where Y_i is an exponentially distributed random variable with rate parameter λ_i, and p_i is the probability that X will take on the form of the exponential distribution with rate λ_i [4]. In fact p_i is the probability of being in state i. It is named the Hyper-exponential distribution since its coefficient of variation is greater than that of the exponential distribution, whose coefficient of variation is 1. $(SCV > 1)$

According to [4], the cumulative function is given by,

$$(19) \quad F(x) = 1 - \Sigma_{i=1}^k p_i e^{-\mu_i x}$$

And the mean time till absorption is,

$$(20) \quad E(x) = \Sigma_{i=1}^n \frac{p_i}{\lambda_i}$$

The second moment around zero is,

$$(21) \quad E(x^2) = \sum_{i=1}^{n} \frac{2p_i}{\lambda_i^2}$$

Generally, the moment formulation is,

$$(22) \quad E(e^{tx}) = \sum_{i=1}^{n} \frac{\lambda_i p_i}{\lambda_i - t}$$

The squared coefficient of variation is given by,

$$(23) \quad SCV = \frac{E(x^2) - E^2(x)}{E^2(x)}, \text{(SCV>1)}$$

The transition rates matrices, S and S^0 is given in the following;

$$S = \begin{bmatrix} -\lambda_1 & 0 & ... & 0 \\ 0 & -\lambda_2 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & -\lambda_k \end{bmatrix}$$

$$S^0 = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix}$$

$$\boldsymbol{\alpha} = (\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, ..., \boldsymbol{\alpha_k})$$

In the following figure we have depicted the transition rates diagram between states of Hyper exponential distribution.
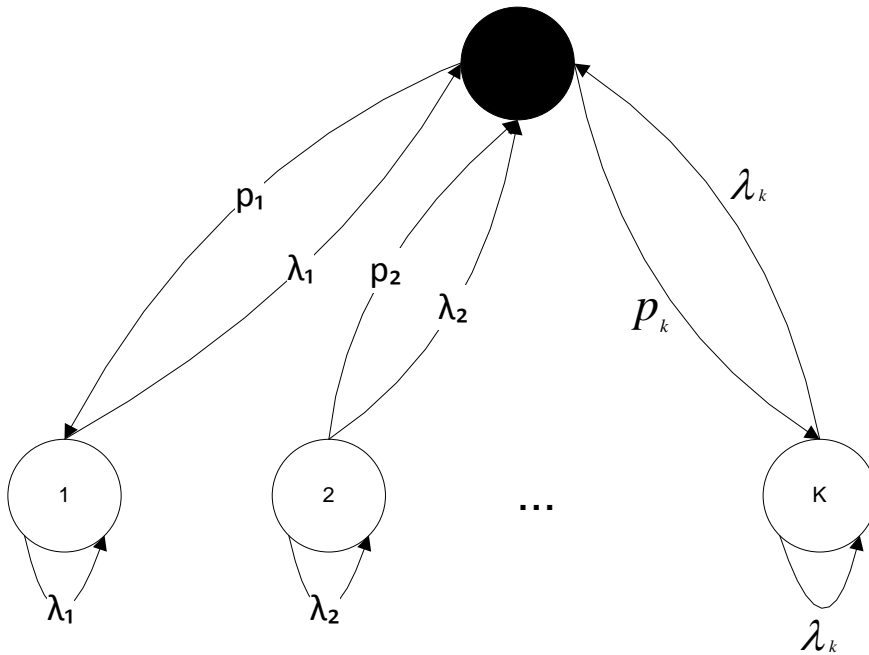
Figure3, transition rate diagram for hyper exponential distribution

10

The bold black state is absorbing state. Note that if X is distributed according to Two-phase Hyper exponential distribution the cumulative function is,

$$(24) \quad F(x) = 1 - p_1 e^{-\mu_1 x} - p_2 e^{-\mu_2 x}, x > 0$$

And when we have $p_1=p_2=0.5$ we get a two phase Balanced Hyper exponential distribution with the following cumulative function for X,

$$(25) \quad F(x) = 1 - 1/2 e^{-\mu_1 x} - 1/2 e^{-\mu_2 x}, x > 0$$

When we have $1/\mu_2=0$, we get the generalized exponential distribution,

$$(26) \quad F(x) = 1 - pe^{-\mu x} , x>0$$

Where

$$(27) \quad E(x) = \frac{p}{\mu}$$

and

$$(28) \quad SCV = \frac{(2-p)}{p}$$

Obviously the case $p=1$ correspond to the exponential distribution.

1.2.2.5 Coxian distribution

The Coxian distribution is a generalization of the Hypo exponential. Instead of only being able to enter the absorbing state from state k it can be reached from any phase. The phase-type representation is given by,

$$S = \begin{bmatrix} -\lambda_1 & p_1\lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & p_2\lambda_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & -\lambda_{k-2} & p_{k-2}\lambda_{k-2} & 0 \\ 0 & 0 & \cdots & 0 & -\lambda_{k-1} & p_{k-1}\lambda_{k-1} \\ 0 & 0 & \cdots & 0 & 0 & -\lambda_k \end{bmatrix}$$

$$S^0 = \begin{bmatrix} 1 - p_1 \\ 1 - p_2 \\ \vdots \\ 1 - p_{k-1} \\ \lambda_k \end{bmatrix}$$

And

$$\boldsymbol{\alpha} = (1, 0, \ldots, 0),$$

Where $0 < p_1,\ldots,p_{k-1} \leq 1$. In the case where all $p_i = 1$ we have the Hypo exponential distribution. The Coxian distribution is extremely important as an acyclic phase-type distribution.

The nonnegative continuous variable X is distributed according to Coxian distribution if it has the cumulative function of:

$$(29) \quad F(X) = \alpha \, exp\{Sx\}1$$

And the densities function of:

$$(30) \quad f(X) = \alpha \, exp\{S.x\}q \,,$$

Where q is as follow:

$$q = -S.I = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$$

Figure 4 depicts the transition rates diagram between states of Coxian distributions:
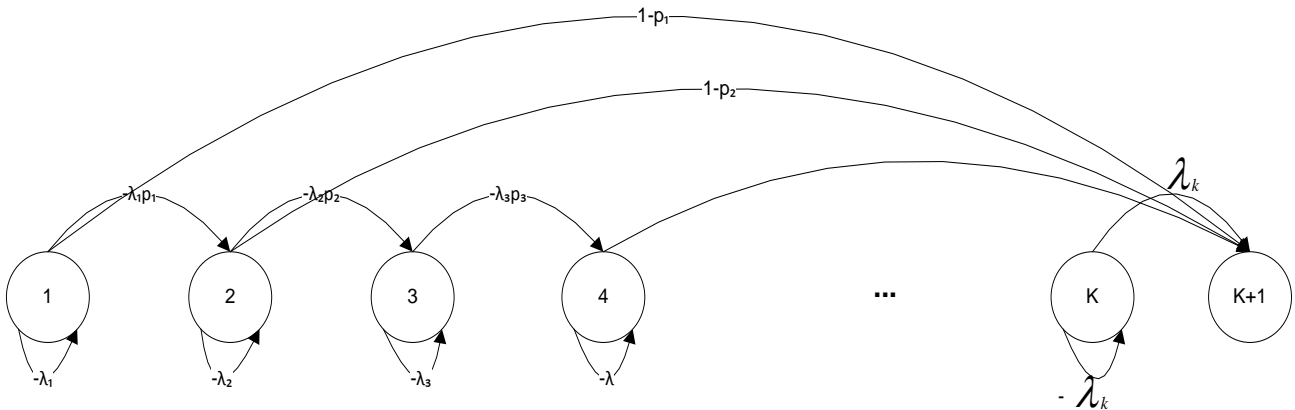
Figure 4, transition rate diagram for Coxian distributions

Lots of researches have been conducted around applying of Coxian distribution. For instance, in [6] the author has done a research on Using Coxian Phase-Type Distributions to Identify Patient Characteristics for Duration of Stay in Hospital. They have considered Coxian distribution as a phase-type distribution to analyze the Markov model that describes duration until an event occurs in terms of a process consisting of a sequence of latent phases. Their paper considers the use of Coxian phase-type distributions for modeling patient duration of stay for the elderly in hospitals. The identification of common characteristics for patient length of stay groups would offer hospital managers and clinicians possible insights into the overall management and bed allocation of the hospital wards. Accordingly, the process begins in the first phase and may either progress through the phases sequentially or enter into the absorbing state (the terminating event). Such phases may then be used to describe stages of a process which terminates at some point. For example, duration of stay in hospital can be thought of as a series of transitions through phases such as: acute illness, intervention, recovery or discharge. This may capture how a domain expert conceptualizes the process. In the paper, Coxian phase-type distribution relates directly to survival analysis where the survivor function is the duration of time until a certain event takes place. The event could be leaving hospital due to transfer, discharge or death. Consequently, applying Coxian distribution helped the author to analyze the duration of stay of patients in hospitals and at last helping managers to have more precise idea about their bed capacity allocations in their hospitals.

Figure 5 shows the density function for an order 4 Coxian distribution [3]. Its shape exemplifies the extra flexibility Coxian distributions exhibit over generalized Erlang and Hyper exponential distributions.
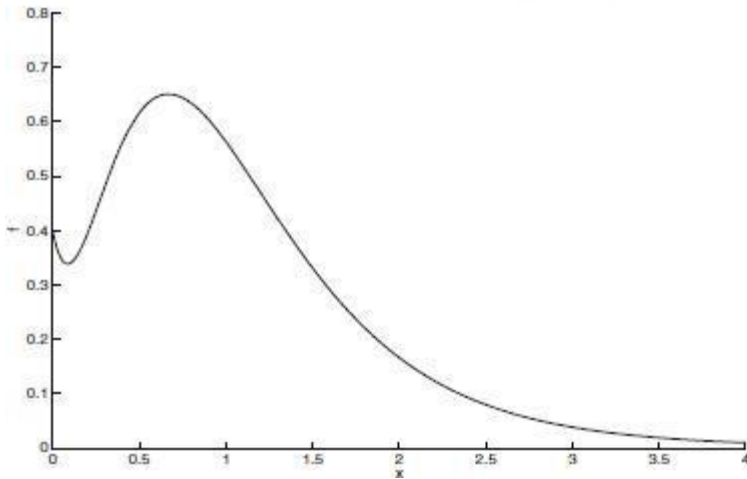
Figure5, Density function of an order 4 Coxian distribution with $\alpha = (0.1, 0.8, 0, 0.1)$, and

$\lambda 1 = 1, \lambda 2 = 2, \lambda 3 = 3, \lambda 4 = 4$

Unlike several other works, such as [8], which obtain the steady-state probabilities and evaluates the performance of the system under the assumption that the processing times follow an exponential as well as Erlang distribution, **[9]** extends the work of [8] by incorporating setup cost under the assumption that the processing time follows a Two phase Coxian (Cox-2) distribution. It is well known that the 2-phase Coxian distribution is very useful to model processing times in practice.

As shown in **[8]** which Considers a two-phase production system that produces a single item to stock, All items are processed at the first phase. Then, the items are processed at the second phase with probability **α**; or are passed without processing at the second phase with probability **(1-α):** Note that two items cannot be processed at both phases simultaneously. The processing times of the two phases are exponentially distributed with rate of $\mu_i (i = 1,2)$, respectively. He assumes that the unit production times follow a Two- phase Coxian distribution as shown in figure 6. A 2-phase Coxian distribution is known as a good approximation of several arbitrary distributions. Due to imbedded memory less property, the Two-phase Coxian distribution is computationally tractable, and has been extensively used.
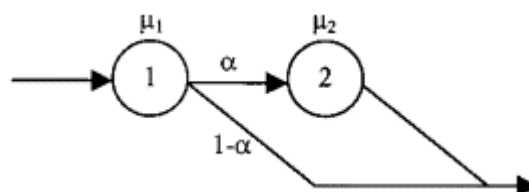
Figure6

Transition rates for Two phase Coxian distribution

14

In most of the manufacturing systems, the process involves performing several tasks of different nature such as flexible assembly systems where the amount of time taken to produce an item has a general distribution. Probability distributions commonly used in modeling such systems are exponential, Erlang, generalized exponential, Hyper exponential, phase type, Cox etc., of which Coxian distribution is most general that generalizes all the well-known distributions. Cox demonstrated that any distribution with rational LST and square coefficient of variation (SCV) lying in $(^1/_k, \infty)$ can be approximated by a sequence of k fictitious independent exponential stages as shown in figure7;
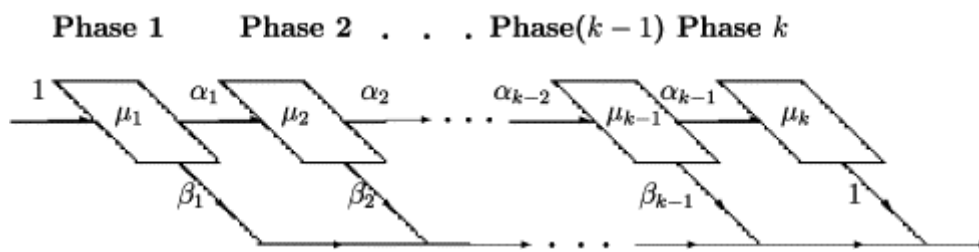
Figure7, k fictitious independent exponential stages

The generalized Coxian distribution relaxes the constraint of starting the transitions from initial state. In other words, the initial state does not have to be the first state.

1.2.2.6 Two phase Coxian distribution: Cox-2

Cox-2 has been widely used in stochastic analysis of manufacturing lines, since there are several processes, such as repair process, which can be assumed as a two phase process. For example, we can assume the calling phase for repair as the first phase and the repair process as the second phase, considering each of them as an exponential distribution. Consequently, we have the two phase Coxian distribution for the total time to repair. Moreover, two phase Coxian distribution has got simpler formulations and parameters to match compare to Coxian distribution.

If the nonnegative continuous random variable, X, has the following cumulative function, it is distributed according to Cox-2 distribution,

$$(31) \quad F(s) = p_1(\frac{\mu_1}{\mu_1+s}) + p_2(\frac{\mu_1}{\mu_1+s})(\frac{\mu_2}{\mu_2+s})$$

The mean of X variable is,

$$(32) \quad E(x) = \frac{1}{\mu_1} + p_2 \frac{1}{\mu_2}$$

And we have,

$$(33) \quad p_1 = 1 - p_2$$

Squared coefficient of variance is given by,

$$(34) \quad SCV = \frac{\mu_2^2 + p_2(1+p_1)\mu_1^2}{(\mu_2 + p_2\mu_2)^2} \quad (\geq 0.5)$$

Figure 8 depicts the transitions diagram between states of Cox-2 distribution.
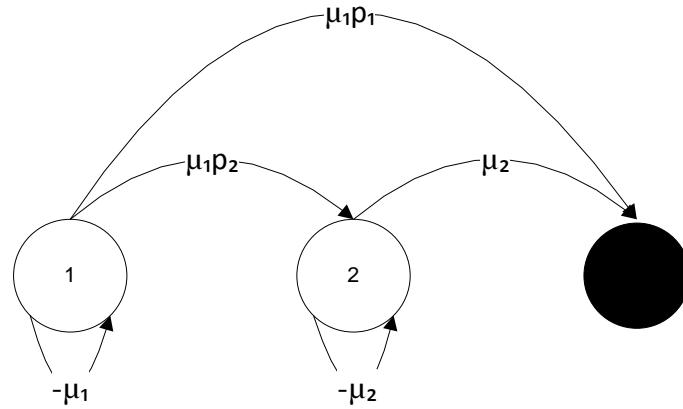
Figure 8, transition rate diagram for Cox-2 distribution

State 1 and 2 are transient states while the black state is the absorbing state.

Matrix of Transition rates is as follow:

$$(35) \quad S = \begin{bmatrix} -\mu_1 & \mu_1 p_2 \\ 0 & -\mu_2 \end{bmatrix}$$

$$(36) \quad S^0 = \begin{bmatrix} \mu_1 p_1 \\ \mu_2 \end{bmatrix}$$

And the vector of initial state probability is as follow,

$$(37) \quad \alpha = (1,0)$$

Having S and S^0 and α we can define all the properties of the Cox-2 distribution.

1.2.2.7 Balanced Mean Two Phase Coxian distribution (Cox-2:b)

Suppose in the two-phase Coxian (Cox-2) distribution we set $1/\mu_1 = (p_2) / (\mu_2) = \frac{1}{\mu}$. Then the cumulative function becomes,

$$(38) \quad F(s) = p_1\left(\frac{\mu}{\mu+s}\right) + p_2\left(\frac{\mu}{\mu+s}\right)\left(\frac{\mu}{\mu+s}\right)$$

The mean of the variable becomes,

$$(39) \quad E(X) = \frac{2}{\mu}$$

Finally, the squared coefficient of variation becomes,

$$(40) \quad SCV = 1/(2p_2) \ (\geq 0.5)$$

Given the mean and the squared coefficient of variation of random variable, the preceding relationships can be used to find a balanced mean two-phase Coxian distribution. It should be noted that the squared coefficient of variation of a *PH* random variable cannot be smaller than $1/K$. [7]

1.2.2.8 Hypo exponential

In probability theory the Hypo exponential distribution or the generalized Erlang distribution is a continuous distribution, that has been used in the same fields as the Erlang distribution, such as queuing theory, tale traffic engineering and more generally in stochastic processes. It is called the Hypo exponential distribution as it has a coefficient of variation less than one (*SCV<1*), compared to the hyper-exponential distribution which has coefficient of variation greater than one and the exponential distribution which has coefficient of variation of one.

While the Erlang distribution is a series of k exponential distributions all with rate λ, The Hypo exponential is a series of k exponential distributions each with their own rate λ_i, the rate of the ith exponential distribution. If we have k independently distributed exponential random variables X_i, then the random variable X,

$$(41) \quad X = \sum_{i=1}^{k} x_i$$

is Hypo exponentially distributed.

Moreover phase-type distribution becomes Hypo exponential distribution if we start in first phase and move skip-free from state i to state $i+1$ with the rate of λ_i until state k and with the rate of λ_k to absorbing state. Transition rates are shown in matrix S.

$$S = \begin{bmatrix} -\lambda_1 & \lambda_1 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\lambda_k \end{bmatrix}$$

$\boldsymbol{\alpha}$, initial state vector probability is as follow:

$$(42) \quad \alpha = (1,0,\dots,0)$$

The nonnegative continuous random variable, X, is distributed according to Hypo exponential distribution if it has cumulative function as follow,

$$(43) \quad F(x) = 1 - \alpha e^{-xS} 1$$

Accordingly, the density function is,

$$(44) \quad f(x) = -\alpha e^{-xS} S1$$

and n^{th} moment around zero and second moment around zero are as follow,,

$$(45) \quad E(X^n) = (-1^n)n!\,\alpha S^{-n} 1$$

$$(46) \quad E(X^2) = 2\sum_{i=1}^{k} 1/\lambda_i \sum_{n=1}^{i} 1/\lambda_n$$

Mean of the X is,

$$(47) \quad E(X) = \sum_{i=1}^{k} 1/\lambda_i$$

Therefore, variance is,

$$(48) \quad Var(X) = E(X^2) - E^2(X)$$

Finally, the SCV becomes,

$$(49) \quad SCV = Var^2(X)/E^2(X), (SCV<1)$$

In figure 2 we have depicted the transition rates between states of Hypo-Exponential distribution.
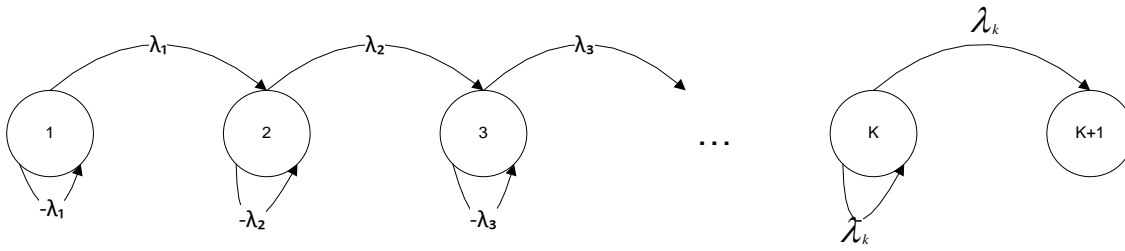
Figure9, transition rates diagram of Hypo-Exponential distribution

As we see here in the figure9, transitions between states occur with different rates. In fact, if we put all the λs the same, it becomes the Erlang distributions.

1.2.2.9 Degenerate Distribution

Point mass at zero or the empty phase-type distribution - 0 phases. In other words the system is already in absorbing state.

1.2.2.10 Deterministic distribution (or constant)

The limiting case of an Erlang distribution, as the number of phases becomes infinite, while the time in each state becomes zero.

2. Phase-Type Distribution for modeling generally distributed repair times in manufacturing systems

In the phase-type characterization which has been shown in the first chapter, we observed several phase-type distributions which are reasonably different in terms of applying for real data. In fact, in most of the researches regarding the analytical performance measurement of the manufacturing transfer lines we observe associating the exponential distribution for mean time to repair (which is the average time of repair process) or mean time to failure (which is the up time of the machine) as a variable distribution. However, we know the fact that repair process of the manufacturing machines consists of several phases, which each of them is potentially a random variable in terms of time. On the other hand, as we have conclude in the first chapter, each of several type of phase-type distributions has got its own characteristics, both in terms of type of transitions and in terms of mathematical formulas. As a matter of fact, differences in their mathematical formulas result in dissimilarity between their mean, variance and consequently, Squared coefficient of variance. Therefore, we have to know which of those mentioned Phase-Type distributions are more proper to fit in different cases.

One of the characteristics of the mentioned phase-type distributions which have been considered in our thesis is the squared coefficient of variance. In fact, SCV of phase-type distributions would be very helpful to shed lights on their differences since each of them covers a specific range in SCV values.

In this part of our thesis, we have mentioned those studies which have used the Phase-Type distribution in their analytical approaches to evaluate the systems with generally distributed events. A two-valve fluid-flow system with finite storage in the between is considered in the analysis of Two-valve Fluid-Flow with General repair times [16]. Valves are components which are subjected to failures. Failures are distributed exponentially but the repair times distributions are assumed to be phase-type. In this study they have investigate the impact of considering the phase-type distribution for time to repair in case there are two valves and a finite storage in between. They have developed a continuous time Markov Chain approach to study the steady-state behavior of the valves and the

material in the storage. Although in this research there are investigations around using the phase-type distribution for time to repair, but there are no expansions for multiple failure or multi stage systems.

A device that is a subject to three modes of failures is assumed to investigate the repairable models with repair times governed by Phase-Type distributions [17]. Failures are considered to be in three modes: repairable, not-repairable and failures due to wear-out. Actually, the last one is assumed to be not repairable since the components is assumed to be consumable. In their study, the life times between failures are stochastically decreasing according to geometric process. Indeed, after a previously fixed number of repairs for a single component, the device is replaced by a new one. Therefore, the functioning of the device can be modeled by a Markov process. Phase-Type distributions supply a versatile model for the treatment of maintenance and the reliability of the repairable systems. Two models have been investigated throughout the paper [17], the transitions among operating and repair states and vice versa take into account the Phase-Type occupied when the device failed.

Although there are a lot of researches around the Phase-Type distributions in the literature, but there is no specific set of data regarding this applicable distribution. Prior to our thesis, we decided to gather all the information related to the Phase-Type distribution because we are aimed to focus on applying this distribution in the modeling of general distribution events in manufacturing systems and in order to apply them we need a more depth in the formulas regarding this distribution. To get to know the Phase-type distribution we brought other studies around its application after explaining each type of Phase-Type distribution.

3. Impact of higher moments of the repair time on the system performance

3.1. Impact of SCV on the Throughput and WIP in the two machine manufacturing system

In this part we have carried out a test to show the impact of SCV on the mean performance of the line. In this test, we compare the throughput and WIP of the line while we have associate phase-type distribution and exponential distribution to repair time. We assumed a two machine, one buffer transfer line which modeled with the Tan-Gershwin [11] method. We compare the throughput of the line with those obtained by approximating the behavior of the machines with the single down phase, which is an exponential behavior. The upstream machine, M1, has single failure mode and the repair has 5 phases to skip. The machine has efficiency in isolation e_1=0.861, mean time to repair MTTR$_1$=5.36 and SCV$_1$=0.04. according to the Tan-Gershwin method, we have defined the transition rates between Markovian states, through a matrix. In fact, as we have SCV$_1$ less than one, we have associate a Hypo exponential distribution, as a type of Phase-Type distribution, to repair time. Downstream machine, M2, has two failure modes with three phases for each failure. The efficiency in isolation, e_2=0.856, the time to repair, MTTR$_2$= 3.16 and the squared coefficient of variation SCV$_2$= 0.02. for M2 as well as M1, we have associate a Hypo exponential distribution to repair process time since the SCV$_2$ is less than one. Although there are other Phase-type distributions to associate to repair time, but there are constraints to choose. For instance, we have Erlang distribution with SCV less than one, but it must be specific values, such as 0.5 for 2 phases, or 0.25 for 4 phases. As depicted in the figure 10, the error in approximation increases as the buffer size increases in our test. Moreover, the approximation always underestimates the throughput. Consequently, according to the test result, we observe the enormous impact of second moment on the mean throughput of the line.
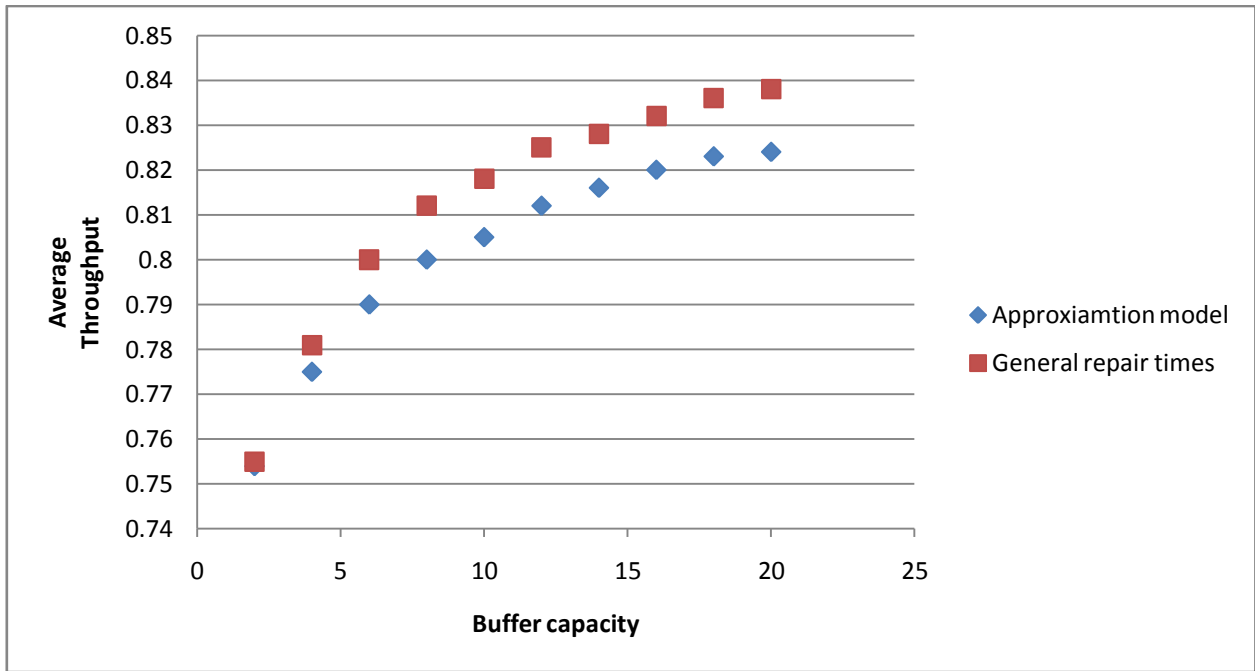
Figure 10, Throughput as a function of the buffer size.

Impact of approximation with an exponential distribution

In the figure 11 we have depicted the effect of an exponential approximation on the WIP of the mentioned line. As we observe there are some differences which may cause many problems in the real cases such as underestimating the mean level of work in progress and lacking enough space for parts during the manufacturing process.
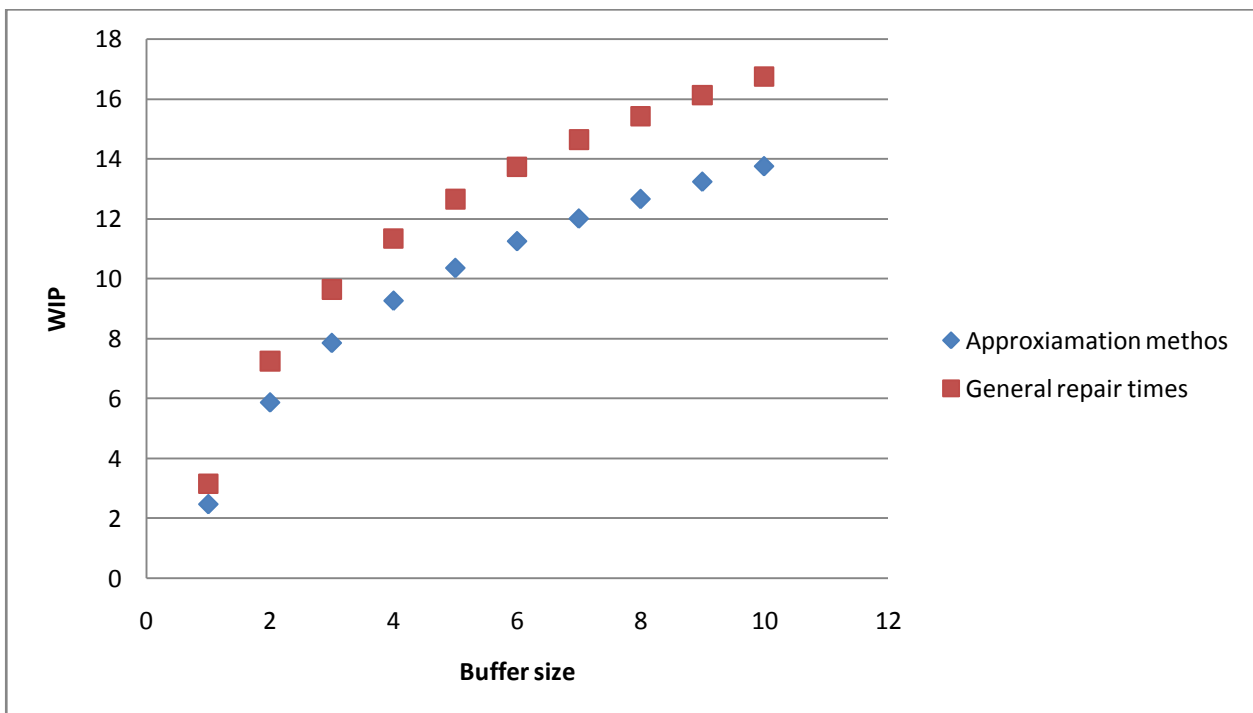
Figure 11, WIP as a function of buffer size,

3.2. Impact of Skewness on the TH and WIP

In the Tan-Gershwin study [11], a Markovian fluid flow system with two stages separated by a finite buffer is considered. There are researches around the fluid flow models which have been analyzed extensively to evaluate the performance of production, telecommunication and computer systems, however this method in developed to cover general Markovian continuous systems with finite buffer. The flexibility of this Methodology allows us to analyze a wide range of systems by specifying the transition rates and the flow rates associated with each state of each stage. The study we carry out around general model include systems with Phase-Type failure and repair time distributions, systems with machines that have multiple up and down states. Although this study have analyzed all the Markovian fluid flow systems with two stages separated by a finite buffer, but we still do not know which of Phase-Type distributions should be selected to fit to real data, regarding their different skewness. For instance, we have a two machine single buffer manufacturing transfer line, clearly this methodology allows us to associate a Phase-Type distribution for time to failure and time to repair of each machine, but we are not able to choose the proper Phase-Type distribution to fit the data.

In our experiments, we have considered the modeling of various two-stage continuous flow systems with a finite capacity buffer. The dynamics of each stage are described by a continuous time discrete state Markov chain where a different flow rates is associated with each state. In fact, Tan-Gershwin model covers a wide range of systems such as portion of manufacturing line which is composed of two unreliable machines with Phase-type up and down distributions. Actually, there are a vast number of researches around the analysis of two machines and a finite buffer line, but as this method considered a general Phase-type distributions, it covers all those methods as a special cases.

Design of experiments have been carried out through some steps,

- We have defined those types of phase type distribution which have been mostly used in the analyzing the performances of manufacturing lines. Those distributions are Erlang, Hypo Exponential, Coxian distribution, Cox-2 balanced mean, Hyper Exponential and finally, the exponential distribution.

- Clustering those distributions which have the same range of SCV (squared coefficient of variance), in terms of values. According to the formulas of each distribution which have been conducted, we came out to the range of each distribution in terms of values which have been shown in figure 12.

- Defining a hypothetic production system, its assumptions and distributions of time to failures and time to repairs. In this step we have considered a two machine one buffer production line. Furthermore, failures are time dependant which means failure happens just while machines are in upstate and they are working on a part (these kinds of failures are mostly mechanical failures). Nonetheless, other assumptions of hypothetic line depend on the designed experiment.

- Each cluster is defined according to the range of squared coefficient of variance. Considering a fixed point in the SCV values vector, which is depicted in the figure1, we have found two distributions which could have the mentioned SCV. So, we have found mean of those two distributions in such a way that they were totally similar (through designed excel file). Finally, we have conducted this step for each range of the SCV. Consequently, we have two totally different associated distributions to time to repair while the same SCV and the same mean in each range of SCV.

  Needless to say, analyzing the mentioned distributions is enough to be done only for a single variable, which is been considered for mean time to repair of up-stream machine.

- Having all values of parameters for each of those mentioned experiments, we defined the input matrices of transition rates between states and the production rates matrices. Clearly we have defined the matrices of transitions according to each distribution. Matrices and their values are calculated in each experiment separately.

- Carrying out experiments in each cluster by means of Matlab written M-file of Tan-Gershwin. As a result, Outputs are parameters which contribute to our goal of experiments. In fact, the output parameters are mean parameters of performance measurement of the assumed manufacturing line.

- Comparing two sets of output parameters, which come up according to two diverse distributions with the same SCV and mean, in each cluster such as Throughput, level of Work in progress (WIP), probability of blocking of down-stream machine and the probability of starvation of up-stream machine.

- Analyzing the result of each cluster, in order to understand if considering third moment, SCV, affects the first moment in the analysis of performances of manufacturing lines.

Proceeding to go through details and result of each cluster, we have defined the method to classify clusters.

According to definition of phase-type distribution, we have achieved the range of Squared Coefficient of Variance (SCV) for each of them. Consequently, there are at least two distributions which fall into the same range. Here we brought the range of SCV and corresponding distributions:
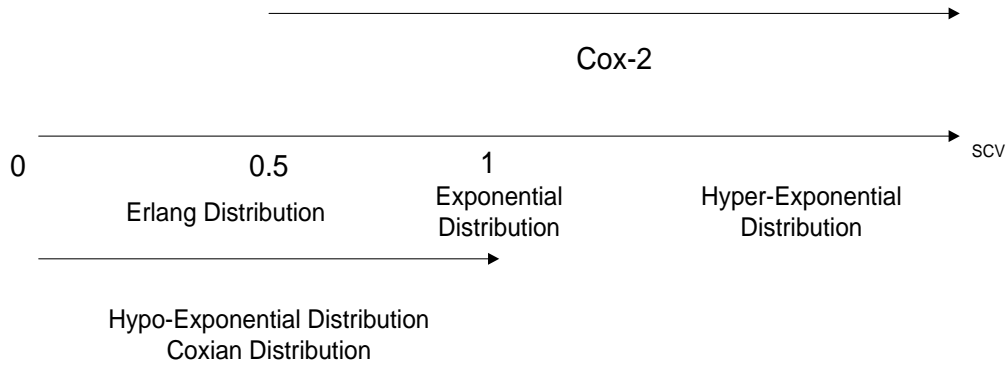
Figure 12, SCV range perspective

For instance we have clustered Cox-2 distribution and an Exponential distribution, fixing SCV to 1.

3.2.1. First Cluster

First cluster is composed of two distributions, two phase Coxian distribution (Cox-2) and Exponential distribution, fixing SCV equal to one. Here we introduce a manufacturing line composed of two machines and one buffer. There are some assumptions regarding the mentioned line, as below:

- Up-stream machine has once, exponential time to repair, then Two phase Coxian distribution.(in order to compare outputs, based on two different input data sets)
- Up-stream machine has exponential time to failure.
- Down-stream machine has Erlang distributed time to repair.
- Down-stream machine has exponentially distributed time to failure.
- Buffer size in the Tan-Gershwin model is being considered as finite. Consequently, we have decided to make experiments in the buffer level of 1 to 14.

26

According to the fact that repair processes of manufacturing machines are composed of a number of phases, considering simply exponential distribution for this variable is a little challenging. Consequently, we have designed three sets of experiments based on different configuration of so called manufacturing line.

3.2.1.1 Experiment 1

Inputs parameters and configurations:

In the first sets of experiments we assumed two different statistical distributions for time to repair of up-stream machine, first one is an exponential distribution and the second is two phase Coxian distribution (Cox-2). In whole sets of experiments in this thesis we assumed the time to failure has an exponential distribution (rates are different according to the requirement of each experiment) both for up-stream and down-stream machine. Furthermore, down-stream machine has an Erlang distribution for time to repair for entire experiments.

Up-stream machine

We have come up with two figures of transitions between states for up-stream machine. Figure 13 and figure 3 show us these two transitions rates.
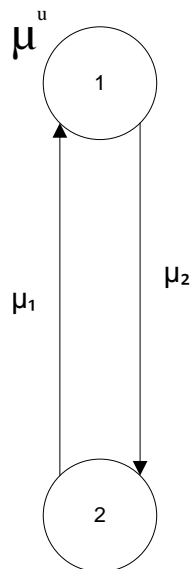
Figure 13

The transition rates between states of up-stream machine when both time to failure and time to repair are exponentially distributed

Parameter's definitions are as below:

μ^u: operating rate of up-stream machine

μ_1: transition rate from state 2 (down state) to state 1 (up state), which is repair rate.

μ_2: transition rate from state 1 (up state) to state 2 (down state), which is failure rate.

Table 1 shows the values assigned to those parameters,

Parameter	Value assigned in our experiment
μ^u	2
μ_1	0.4
μ_2	0.5

Table 1, values of parameters of upstream machine's distribution for experiment 1, considering exponential distribution for repair time

As shown in the figure 13, we see that the up-stream machine's time to failure and time to repair have both an exponential distribution with rates of μ_2 and μ_1 respectively.

The matrix **λ-u**, which is the transition rates between two states of up-stream machine, and vector **Mu-u**, which is the processing rate of up-stream machine have been shown as bellow:

$$\lambda\text{-u} = \begin{bmatrix} -\mu_2 & \mu_2 \\ \mu_1 & -\mu_1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 \\ 0.4 & -0.4 \end{bmatrix}$$

,

$$\text{Mu-u} = [\mu^u \quad 0 \quad 0 \quad 0] = [2 \quad 0 \quad 0 \quad 0]$$

Figure 14 shows up-stream machine while we have assigned Coxian distribution for time to repair.
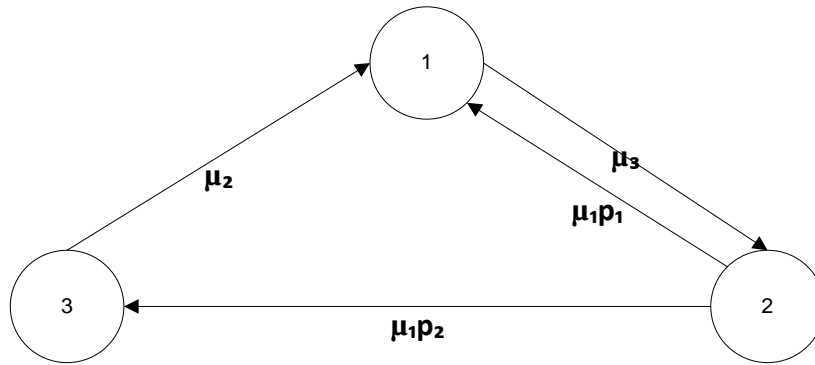
Figure 14

The transition rates between states of up-stream machine when repair time has two phase Coxian distribution and failure time is distributed exponentially

Parameter's definitions are as below:

μ_1: rate of staying in the state 2, which is in fact staying in the down state. In other words, this is the rate system neither goes to next phase of failure nor goes to absorbing state which is the up state.

μ_2: transition rate from state 3 which is down state to state 1 which operating state. Definitely, we have considered the up state as an absorbing state in the two phase Coxian distribution.

$p_1 = 1 - p_2$

μ_3: transition rate from state 1, which is up state, to state 2, which is down state. As a matter of fact, failure rate of up-stream machine, which is a rate of the exponential distribution assigned for time to failure of up-stream machine.

Needless to say, we need to define an absorbing state in each of phase type distributions. So, here we have considered state number 1 as an absorbing state. In addition, Rates which have been shown in the figure 14 are collected from Stochastic Models Of Manufacturing Systems by John A. Buzacott [7].

Parameter	Value assigned in our experiment
p_2	0.8
μ_2	0.4
p_1	0.2

μ_1	2
μ_3	0.5

Table 2, values of parameters of upstream machine's distribution for experiment 1, considering Coxian distribution for repair time

Note: values of input parameters, which is depicted in table 2, are assigned in such way that provide us Squared Coefficient of Variance (SCV) equal to one. Simultaneously, having same Mean Time To Repair (MTTR) for up-stream machine as we have the exponential distribution for time to repair.

Considering the Cox-2 distribution for time to repair of up-stream machine, The matrix **λ-u**, which is the transition rates between three states of up-stream machine, and vector **Mu-u**, which is the processing rate of up-stream machine have been shown as bellow:

$$\lambda\text{-u} = \begin{bmatrix} -\mu_3 & \mu_3 & 0 \\ \mu_1 p_1 & -\mu_1 & \mu_1 p_2 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} = \begin{bmatrix} -0.5 & 0.5 & 0 \\ 0.4 & -2 & 1.6 \\ 0.4 & 0 & -0.4 \end{bmatrix}$$

,

$$\text{Mu-u} = \begin{bmatrix} \mu^u & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

The efficiency in isolation of up-stream machine (in both figures) is being calculated as below:

e = MTTF/ (MTTF+MTTR) =2/ (2+2.5) =0.44

Down-stream machine

In the whole sets of experiments of all clusters we have assumed Down-stream machine with an Erlang distributed time to repair and exponential distributed time to failure, while changing only the parameter's values according to the requirements of each experiment. Because changing just the distribution of time to repair of up-stream machine in each set of experiments would be sufficient to approach the purpose of these experiments.

We have come up with figure14 to show the state of down-stream machine and its transition rates.
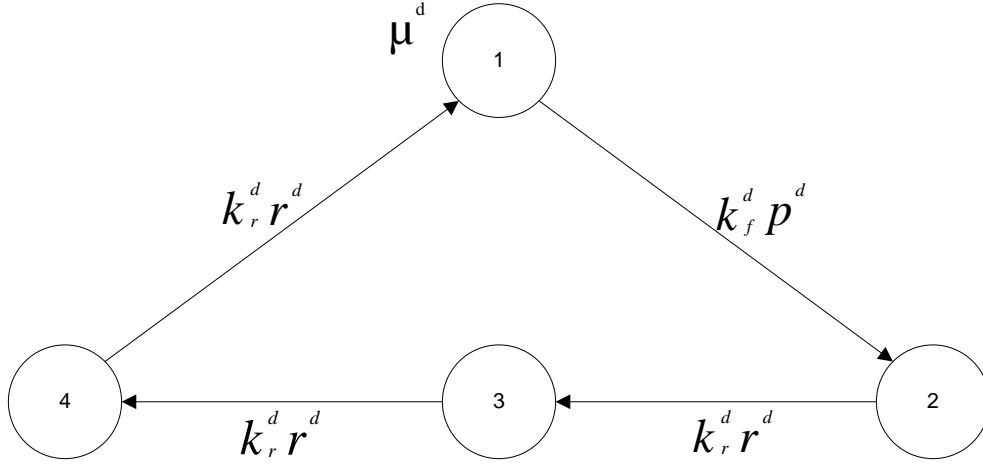
Figure 15

Down-stream machine's transition rates when time to repair has Erlang distribution and time to failure has an exponential distribution

Parameters definitions are as below:

μ^d: operating rate of down-stream machine.

K_f^d: Number of failure phases.

p^d: Failure rate of down-stream machine.

K_r^d: Number of repair phases of down-stream machine.

r^d: Repair rate of down-stream machine.

Note: $1/r^d$ is the expected value of repair time and $1/p^d$ is the expected value of failure time. Defiantly, in the real world data, there are no rates of failures and repairs, while we have the mean time to repair and the mean time to failure which can be gathered from the database of manufacturing lines, which are the expected time of the activity.

Apparently, state number 1 is the operating state and states number 2, 3 and 4 are down states. The time machine spends in upstate (state number 1), which is Time To Failure, is exponentially distributed with the rate of $K_f^d p^d$. Besides, the time down-stream machine spends in down state, which is Time To Repair, has an Erlang distribution with the rate of $K_r^d r^d$. There is an assumption which notifies machine is working only in upstate with operating rate of μ^d. Table 3 demonstrate the values assigned to those parameters,

Parameter	Value assigned in our experiment
μ^d	2
K_f^d	1
p^d	0.6
K_r^d	3
r^d	0.2

Table 3, values of parameters for downstream machine, experiment 1

The matrix λ-d, which is the transition rates between four states of down-stream machine, and vector Mu-d, which is the processing rate of down-stream machine have been shown as bellow:

$$\lambda - \mathbf{d} = \begin{bmatrix} -k_f^d p^d & k_f^d p^d & 0 & 0 \\ 0 & -k_r^d r^d & k_r^d r^d & 0 \\ 0 & 0 & -k_r^d r^d & k_r^d r^d \\ k_r^d r^d & 0 & 0 & -k_r^d r^d \end{bmatrix} = \begin{bmatrix} -0.6 & 0.6 & 0 & 0 \\ 0 & -0.6 & 0.6 & 0 \\ 0 & 0 & -0.6 & 0.6 \\ 0.6 & 0 & 0 & -0.6 \end{bmatrix}$$

$$\mathbf{Mu\text{-}d} = \begin{bmatrix} \mu^d & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

The efficiency in isolation of down-stream machine is being calculated as below:

MTTF/ (MTTF+MTTR) =1.67/ (1.67+5) =0.25

As we see here efficiency in isolation of up-stream machine is larger than efficiency in isolation of down-stream machine, so the down-stream machine is bottle neck machine of the line.

Outputs:

Finally, we have two sets of input parameters. In one hand, up-stream machine with exponential distribution for both tome to failure and time to repair (figure 13), in the other hand, up-stream machine with exponential time to failure and two phase Coxian distribution for time to repair, while in both situations down-stream machine has an Erlang distribution for repair time and an

Exponential distribution for failure time. Consequently, we have two sets of outputs which are to be compared.

Table 4 summarizes all output data related to the first experiment including the probability of starvation of down-stream machine and the probability of blocking of up-stream machine.

	1	2	3	4	5	6
N	TH(1)	TH(2)	Difference	WIP(1)	WIP(2)	Difference
14	0.4955	0.4955	0	12.0709	12.0709	0
13	0.4944	0.4944	0	11.1155	11.1155	0
12	0.493	0.493	0	10.1678	10.1678	0
11	0.4913	0.4913	0	9.2289	9.2289	0
10	0.4891	0.4891	0	8.3001	8.3001	0
9	0.4863	0.4863	0	7.3829	7.3829	0
8	0.4829	0.4829	0	6.4787	6.4787	0
7	0.4785	0.4785	0	5.5893	5.5893	0
6	0.4729	0.4729	0	4.7166	4.7166	0
5	0.4658	0.4658	0	3.8627	3.8627	0
4	0.4566	0.4566	0	3.0303	3.0303	0
3	0.4446	0.4446	0	2.2226	2.2226	0
2	0.4289	0.4289	0	1.4438	1.4438	0
1	0.4083	0.4083	0	0.7002	0.7002	0

Table 4, output parameters for experiment 1

As we see in the table 4, we have 4 sets of output parameters. First column is the throughput of hypothetic line while the time to repair of up-stream machine is exponentially distributed. Second column is the same as first column, but with the assumption of Cox-2 distribution for time to repair of up-stream machine. Third column compares two first columns in terms of values. Column 4 and 5 show the mean level of work in progress while we have exponential distribution and Cox-2 distribution respectively. Column 6 depicts the distance between the values of fourth and fifth column. Noticeably, we have considered the same position for all experiments to ease the analyzing approaches.

As depicted in the table 4, which are result of Tan-Gershwin model, there are no differences in any of those output parameters despite the fact that we were changing the distribution of time to repair for up-stream machine. In fact, even changing the buffer level, as we demonstrated in the table 1, did not influence the former declaration. Consequently, we can declare that if we have the data set of repair time of a manufacturing machine which has the SCV equal to one, we are precisely able to

substitute the exponential distribution with the Coxian distribution in the fitting phase of performance measurements

Result:

As a conclusion for the first experiment, we can state that using an exponential distribution instead of two phase Coxian (Cox-2) distribution, while having same SCV and the same Mean Time of the process that we want to consider, does not influence the mean parameters of analytical performance measurement of the manufacturing lines. Thus, for instance, if we gathered a dataset of repair process time, which has the SCV equal to one, no matter how many phases it has, we are able to simply fit an exponential distribution them. In fact, fitting an exponential distribution enable us to ease the fitting process because with an exponential distribution we need only a simple rate, which is easily the contrast of the mean time to repair.

Indeed the conclusion is based only on the mentioned experiment, so in the next chapter DOE will help us to make this result much more generally.

3.2.1.2 Experiment 2

Inputs parameters and configurations:

In the second experiment we have the same goal as first experiment and all assumptions remain the same, while we have changed parameters in such a way that give us same efficiency in isolation for both machines(no bottle neck machine). Consequently, we have the same configurations and the same distributions with different parameters of distributions.

Up-stream machine

So, parameters definitions stay the same and the values for up-stream machine when we have the same configuration as figure 13 (an exponential distribution for both time to failure and time to repair) are as follow:

Parameter	Value assigned in our experiment
μ^u	2
μ_1	0.4
μ_2	0.125

Table 5, values of parameters of upstream machine's distribution for experiment 2, considering exponential distribution for repair time

The matrix of λ-u and the vector of **Mu-u** are also the same while the values change consequently as follow:

$$\lambda\text{-u} = \begin{bmatrix} -\mu_2 & \mu_2 \\ \mu_1 & -\mu_1 \end{bmatrix} = \begin{bmatrix} -0.125 & 0.125 \\ 0.4 & -0.4 \end{bmatrix}$$

$$\text{Mu-u} = \begin{bmatrix} \mu^u & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

Accordingly, parameters definitions of figure 14 (an exponential time to failure and Cox-2 distribution time to repair) remain the same but the values change as shown in table 6:

Parameter	Value assigned in our experiment
$\mu_1 p_2$	1.6
μ_2	0.4
$\mu_1 p_1$	0.4
p_1	0.2
μ_1	2
μ_3	0.125

Table 6, values of parameters of upstream machine's distribution for experiment 2, considering Cox-2 distribution for repair time

The matrix of λ-u and the vector of **Mu-u** are also the same while the values changed consequently as follow:

$$\lambda\text{-u} = \begin{bmatrix} -\mu_3 & \mu_3 & 0 \\ \mu_1 p_1 & -\mu_1 & \mu_1 p_2 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} = \begin{bmatrix} -0.125 & 0.125 & 0 \\ 0.4 & -2 & 1.6 \\ 0.4 & 0 & -0.4 \end{bmatrix}$$

$$\text{Mu-u} = \begin{bmatrix} \mu^u & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

The efficiency in isolation of up-stream machine is finally as above:

$$\text{MTTF/ (MTTF+MTTR)} = 8/ (8+2.5) = 0.76$$

Down-stream machine

As well as up-stream machine, down-stream machine remains with the same distributions of time to failure and time to repair, while changing simply the values of parameters in order to have the same Mean Time To Repair (MTTR) and the same Mean Time To failure (MTTF) which cause to having the same efficiency in isolation. Consequently, we have come up with the values shown in table 7:

Parameter	Value assigned in our experiment
μ^d	2
K_f^d	1
p^d	0.125
K_r^d	3
r^d	0.4

Table 7, values of parameters for downstream machine, experiment 2

The matrix of **λ-d** and the vector of **Mu-d** have also the same formation while the values changed consequently as follow:

$$\lambda - \mathbf{d} = \begin{bmatrix} -k_f^d p^d & k_f^d p^d & 0 & 0 \\ 0 & -k_r^d r^d & k_r^d r^d & 0 \\ 0 & 0 & -k_r^d r^d & k_r^d r^d \\ k_r^d r^d & 0 & 0 & -k_r^d r^d \end{bmatrix}$$

$$= \begin{bmatrix} -0.125 & 0.125 & 0 & 0 \\ 0 & -1.2 & 1.2 & 0 \\ 0 & 0 & -1.2 & 1.2 \\ 1.2 & 0 & 0 & -1.2 \end{bmatrix}$$

$$\mathbf{Mu\text{-}d} = [\mu^d \quad 0 \quad 0 \quad 0] = [2 \quad 0 \quad 0 \quad 0]$$

The efficiency in isolation of down-stream machine is finally as below:

MTTF/ (MTTF+MTTR) =8/ (8+2.5) =0.76

As we see, we have come up with the same efficiency in isolation for both up-stream and down-stream machine, having no bottle neck in the line.

Outputs:

Output parameters are as there were in the first experiment, while changing values. Table 8 shows the summery of second experiment and the results of output values.

	1	2	3	4	5	6
N	TH(1)	TH(2)	Difference	WIP(1)	WIP(2)	Difference
14	1.4176	1.4176	0	7.3308	7.3308	0
13	1.4125	1.4125	0	6.8194	6.8194	0
12	1.4068	1.4068	0	6.3067	6.3067	0
11	1.4006	1.4006	0	5.7927	5.7927	0
10	1.3936	1.3936	0	5.2771	5.2771	0
9	1.3858	1.3858	0	4.7597	4.7597	0
8	1.377	1.377	0	4.2399	4.2399	0
7	1.367	1.367	0	3.7175	3.7175	0
6	1.3556	1.3556	0	3.1917	3.1917	0
5	1.3423	1.3423	0	2.662	2.662	0
4	1.3268	1.3268	0	2.1278	2.1278	0
3	1.3085	1.3085	0	1.5893	1.5893	0
2	1.2867	1.2867	0	1.0493	1.0493	0
1	1.2609	1.2609	0	0.5149	0.5149	0

Table8, output parameters for experiment 2

As we can see in Table 8, we have 4 output parameters of Tan-Gershwin model.

Result:

Regarding the result of second experiment, we can conclude that even having same efficiency in isolation of machines does not affect the substitution of the two phase Coxian distribution with an exponential distribution for repair process time, while having same SCV and the same Mean Time To Repair.

3.2.1.3 Experiment 3

Inputs parameters and configurations:

In the third sets of experiments all assumptions remain the same as two last experiments though we have changed the processing rates. In other words, we have the same distribution for time to failures and time to repairs with the same values for parameters while changing the operating rates of machines. Consequently, we are going to investigate the situation which machines have the same efficiency in isolation but different processing rates.

Up-stream machine

Accordingly, parameters definition and the values would be the same as second experiment for up-stream machine when we have the same configuration as both in figure 13and figure 14. Matrices of transition rates between states, which are called λ-u, are as well the same as those of second experiment. The only thing that changes in both figure 13 and figure 14 is the vector of processing rate, **Mu-u**.

$$\text{Mu-u} = [\mu^u \quad 0 \quad 0 \quad 0] = [3 \quad 0 \quad 0 \quad 0]$$

Down-stream machine

Accordingly, parameters definitions and their value of figure 14 remain the same as second experiment. Matrix of transitions rates and vector of processing rates are kept the same consequently.

Outputs:

Output parameters of Tan-Gershwin model through having two different phase-type distribution have been gathered and compared in the Table 6.

N	1 TH(1)	2 TH(2)	3 Difference	4 WIP(1)	5 WIP(2)	6 Difference
14	1.5038	1.5038	0	12.2786	12.2786	0
13	1.5005	1.5005	0	11.3488	11.3488	0
12	1.4965	1.4965	0	10.4263	10.4263	0
11	1.4919	1.4919	0	9.5115	9.5115	0
10	1.4864	1.4864	0	8.6051	8.6051	0
9	1.4799	1.4799	0	7.7075	7.7075	0
8	1.4723	1.4723	0	6.8194	6.8194	0
7	1.4632	1.4632	0	5.9412	5.9412	0
6	1.4524	1.4524	0	5.0731	5.0731	0
5	1.4394	1.4394	0	4.2151	4.2151	0
4	1.4238	1.4238	0	3.3665	3.3665	0
3	1.4048	1.4048	0	2.5256	2.5256	0
2	1.3815	1.3815	0	1.6892	1.6892	0
1	1.3524	1.3524	0	0.851	0.851	0

Table 9, output parameters of experiment 3

Result:

As we can see in the value of output parameters in table 9, the differences between values of parameters, while considering two different phase-type distributions for repair time of up-stream machine, are all zero. Therefore, we can conclude that simplifying the Coxian distribution with an exponential distribution for a repair time does not change the output parameters even if the up-stream machine operates faster than the down-stream machine, having same efficiency in isolation.

Result of the first cluster:

Through these three experiments of first cluster, we have revealed that having same efficiency in isolation of up-stream machine and down-stream machine with the same operating rates of machines or even different operating rates, cannot affect the theory of simplifying Two phase Coxian distribution with an exponential distribution for processes with more than one phase, such as repair processes.

According to our result, we can conclude that if we face to the data sets of duration of multiple phase process with specifications aligned with Two phase Coxian distribution requirements, we are able to substitute the distribution with an exponential distribution.

Consequently, in this cluster we have carried out several experiments which shed lights on the effect of third moment on the mean parameters of performance measurement of manufacturing lines while there are some doubt regarding having other range of parameters that we will examine in the next chapter through DOE analysis.

3.2.2. Second Cluster

Second cluster is designed in order to compare the output parameters of Tan-Gershwin method while we consider Balanced mean two phase Coxian distribution (Cox-2:b) and Hyper exponential distribution for time to repair. In fact, In the second cluster we are aimed to investigate the differences between mean parameters of performance measurement of manufacturing lines while we consider two different phase-type distribution having the SCV more than one (SCV>1).

Here we introduce a manufacturing line composed of two machines and one buffer to carry out our experiments on that, as we had in the first cluster. There are some assumptions regarding the mentioned line, as follow:

- Up-stream machine has first, Balanced Mean Two Phase Coxian distribution (Cox-2:b) time to repair then, Hyper-Exponential distribution time to repair.
- Up-stream machine has an exponential time to failure.
- Down-stream machine has Erlang distributed time to repair.
- Down-stream machine has exponentially distributed time to failure.
- Processing rate of both up-stream and Down-stream machine are constant and equal to 2.
- Up-stream machine is bottle neck of the system.
- Buffer size in the Tan-Gershwin model is being considered as finite. Consequently, we have decided to make experiments in the range of 1 to 14.

Input parameters:

Considering the data sets with the same squared coefficient of variance (SCV), do we have to be concerned for taking into account the Balanced mean two phase Coxian distribution(Cox-2:b) or Hyper-exponential distribution? Having same rang of SCV for these two distributions may impose doubt which one to consider. Consequently, we have considered these two distributions for time to

repair of up-stream machine in order to find the solution, however the solution might not be reliable in terms of expansion of parameters values.

Up-stream machine

Assuming the two mentioned distributions for Time to repair of up-stream machine, generates two different figures of states of up-stream machine. Figure 16 and 17 show these two modes of states and their transition rates.
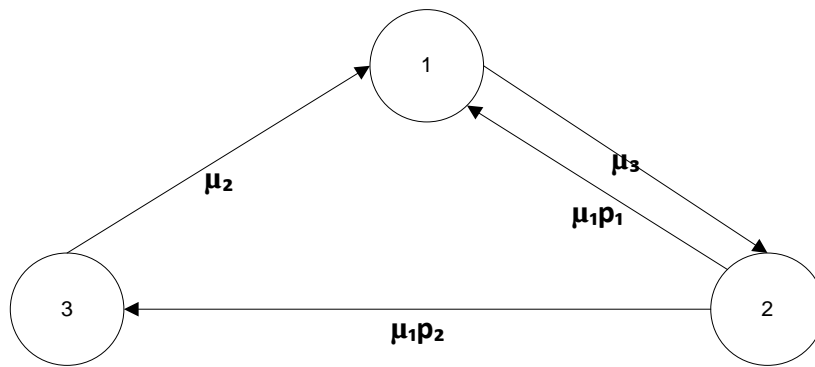
Figure 16

Transition rates between states of up-stream machine while it has Cox-2:b distribution

Parameter's definitions are as below:

μ_1: transition rate of up-stream machine staying in the down state 2. In fact, the transition rate of up-stream machine neither getting repair nor moving to the next failure phase.

μ_2: transition rate from down state (state 3) to state up state.

$p_1 = 1-p_2$

μ_3: transition rate from state 1, which is up state, to down state. As a matter of fact, failure rate of up-stream machine. Besides, $1/\mu_3$ is the expected operating time before failures.

According to definition of Cox-2:b distribution, $\mu_1 p_2$ is the transition rate of going from state 2 of failures to state 3 . Moreover, $\mu_1 p_1$ is the transition rate if the machine gets repaired in the state 2 and there is no need to go to state 3 before moving to operating state.

In this experiment we have considered Balanced Mean two phase Coxian distribution (Cox2:b), since by fixing the SCV and Mean Time To Repair we are able to achieve other parameters with less effort compare to Coxian distribution. Therefore we have fixed the SCV and MTTR of up-stream machine and we have achieved values of parameters as we shown in table 10,through following equations:

$$(50)\ p_2=1/(2*SCV)$$

$$(51)\ \mu= 2/(E(x))$$

$$(52)\ \mu_2=p_2 * \mu$$

Parameter	Value assigned in our experiment
$\mu_1 p_2$	0.799
μ_2	0.79
$\mu_1 p_1$	1.718
p_1	0.2
μ_1	2.51
μ_3	1.13

Table 10, values of parameters of upstream machine's distribution for second cluster experiments, considering Cox-2:b distribution for repair time

The matrix **λ-u**, which is the transition rates between three states of up-stream machine, and vector **Mu-u**, which is the processing rate of up-stream machine have been shown as bellow:

$$\lambda\text{-u} = \begin{bmatrix} -\mu_3 & \mu_3 & 0 \\ \mu_1 p_1 & -\mu_1 & \mu_1 p_2 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} = \begin{bmatrix} -1.13 & 1.13 & 0 \\ 1.718 & -2.51 & 0.79 \\ 0.79 & 0 & -0.79 \end{bmatrix}$$

$$\text{Mu-u} = \begin{bmatrix} \mu^u & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

The efficiency in isolation of up-stream machine is being calculated as below:

$$\text{MTTF/ (MTTF+MTTR)} = 0.88/ (0.88+0.79) = 0.52$$

Figure 17 shows us the up-stream machine with the Hyper-Exponential Time to repair.
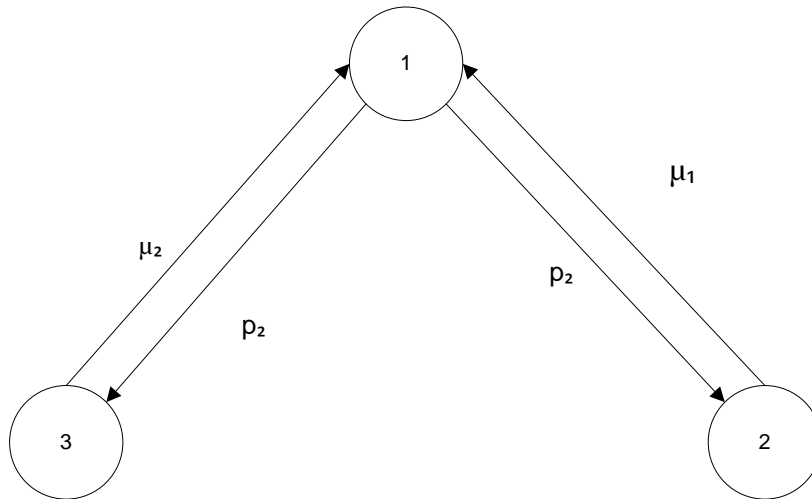
Figure 17

Transition rates of up-stream machine while it has Hyper-Exponential time to repair and Exponential time to failure

According to the Hyper-Exponential distribution definitions, up-stream machine can go from up-state (state number 1) to each of down states (state 2 and 3) with an exponential rate. In fact, there are two components which getting failed and repaired separately with an exponential rate, but the failure of each of them cause the system going to down state.

Parameters are defined as follow:

p_1: transition rate from up-state (state 1) to down state (state 2). In fact this parameter is the parameter of the exponential distribution of time to failure. In other words, $1/p_1$ is the mean time to failure number 2.

p_2: transition rate from up-state to down-state (state 3). Actually $1/p_2$ is the mean time to failure number 3, since the distribution of time to failure number 3 is an exponential distribution with the rate of p_2.

μ_1: transition rate from down state (state 2) to up-state (state 1). Therefore, mean time to repair of failure number 2 is $1/\mu_1$, since the distribution of repair time is an exponential distribution with rate of μ_1.

$\mathbf{\mu_2}$: transition rate from down state (state 3) to up-state (state 1). Therefore, mean time to repair of failure number 3 is $1/\mu_2$.

Value assigned to above mentioned parameters are shown in table 11,

Parameter	Value assigned in our experiment
p_2	0.689
μ_2	3.5
p_1	0.44
μ_1	4.5

Table 11, values of parameters of upstream machine's distribution for second cluster experiments, considering hyper exponential distribution for repair time

The matrix $\boldsymbol{\lambda}$-$\mathbf{u}$, which is the transition rates between three states of up-stream machine, and vector $\mathbf{Mu\text{-}u}$, which is the processing rate of up-stream machine have been shown as bellow:

$$\boldsymbol{\lambda}\text{-}\mathbf{u} = \begin{bmatrix} -(\mathbf{p}_1 + \mathbf{p}_2) & \mathbf{p}_1 & \mathbf{p}_2 \\ \boldsymbol{\mu}_1 & -\boldsymbol{\mu}_1 & \mathbf{0} \\ \boldsymbol{\mu}_2 & \mathbf{0} & -\boldsymbol{\mu}_2 \end{bmatrix} = \begin{bmatrix} -1.129 & 0.44 & 0.689 \\ 1.2 & -1.2 & 0 \\ 1.3 & 0 & -1.3 \end{bmatrix}$$

$$\mathbf{Mu\text{-}u} = \begin{bmatrix} \boldsymbol{\mu}^{\mathbf{u}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

The efficiency in isolation of up-stream machine is being calculated as below:

MTTF/ (MTTF+MTTR) =0.88/ (0.88+0.79) =0.52

Note: MTTR and MTTF have been calculated in the Excel file, according to formula of Hyper exponential distribution and exponential distribution which have been mentioned before.

Down-stream machine

The distribution of the time to failure and time to repair of down-stream machine of this cluster stays the same as it was in the first cluster. Consequently, as we have in figure14, the time to repair has an Erlang Distribution and time to failure has an Exponential distribution with the same values as the first cluster.

Outputs:

Output parameters of Tan-Gershwin model through having two different phase-type distribution have been gathered and compared in the Table 9.

	1	2	3	4	5	6
N	TH(1)	TH(2)	Difference	WIP(1)	WIP(2)	Difference
1	0.945	0.9435	0.0015	0.192	0.1959	0.0039
2	0.9759	0.9736	0.0023	0.384	0.3955	0.0115
3	0.9985	0.9959	0.0026	0.5614	0.5816	0.0202
4	1.0146	1.012	0.0026	0.7168	0.7457	0.0289
5	1.026	1.0235	0.0025	0.8484	0.8858	0.0374
6	1.034	1.0318	0.0022	0.9575	1.003	0.0455
7	1.0397	1.0378	0.0019	1.0467	1.1	0.0533
8	1.0438	1.0422	0.0016	1.1189	1.1795	0.0606
9	1.0468	1.0453	0.0015	1.1769	1.2442	0.0673
10	1.0489	1.0477	0.0012	1.2231	1.2965	0.0734
11	1.0504	1.0494	0.001	1.2598	1.3387	0.0789
12	1.0515	1.0507	0.0008	1.2887	1.3725	0.0838
13	1.0523	1.0516	0.0007	1.3114	1.3995	0.0881
14	1.0529	1.0523	0.0006	1.3291	1.421	0.0919

Table 12, output parameters for second cluster experiments

Result of the second cluster:

As illustrated in table 12, in the third column we observe the difference between mean throughput values considering the two different phase-type distribution while the SCV is greater than one. Actually, this column shows us the positive values which might have some reasonable meanings.

On the other hand, we have the sixth column as differences between the WIPs. In fact, in the sixth column also there are some meaningful positive values.

Consequently, we came to the point that there are no enough reasons to make us convinced that there are no reasonable differences between mean parameters of performance measurements of manufacturing lines through having two different phase-type distributions. Thus, we need to expand our experiments through DOE method, which will be explained in the next chapter.

3.2.3. Third Cluster

Within this cluster we are going to investigate the affect of third moment on the mean performance measurement of manufacturing lines while we have the SCV lower than one (SCV<1). Thus, as we have mentioned in the first chapter there are several different phase-type distributions which have the SCV lower than one. So, we decided to run the experiments once considering the Hypo exponential distribution and then the Erlang distribution for time to repair of up-stream machine. In fact, In the third cluster we have decided to fix the SCV equal to 0.25 (SCV in this cluster should be lower or equal to 0.5). Therefore, we faced to the parameters which made two identical distributions with the same parameters. So, there is no further reason in order to go through this cluster because these two distributions are exactly the same when the SCVs are equal. In fact, there is a unique way to have the hypo exponential distribution with the same SCV as an Erlang distribution, which gives the identical transition rates. As we mentioned in chapter one, having the Hypo exponential distribution with the identical transition rates is equal to Erlang distribution.

3.2.4. Forth Cluster

Within the forth cluster we aimed to investigate the affect of third moment on the mean performance measurement of manufacturing lines while we have the SCV lower than one (SCV<1), as we had in the third cluster, but here we want to consider another distributions. In fact, forth

cluster is designed to compare the output parameters of Tan-Gershwin model while we have once the Hypo exponential and then the Cox-2:b distribution for time to repair of up-stream machine. Accordingly, we have considered the same production line as we had in last clusters. The line's assumptions are as follows:

- Up-stream machine has first, Balanced Mean Two Phase Coxian (Cox 2:b) distribution time to repair, then Hypo Exponential distribution time to repair.
- Up-stream machine has an exponential time to failure.
- Down-stream machine has Erlang distributed time to repair.
- Down-stream machine has exponentially distributed time to failure.
- Processing rate of both up-stream and Down-stream machine are constant and equal to 2.
- Down-stream machine is bottle neck of the system.
- Buffer size in the Tan-Gershwin model is being considered as finite. Consequently, we have decided to make experiments in the range of 1 to 8.

Input parameters

As we had in last experiments, we need to assign values for parameters of distributions in such a way that give us the fixed SCV.

Up-stream machine

In the forth cluster we have considered the up-stream machine, first with Cox2:b distribution for time to repair and then, with Hypo Exponential time to repair. In both of them the time to failure is considered an exponential distribution.

Therefore, we have to define two figures for these two different configurations of states. Figure 18 and 18 depict the states and their transition rates.
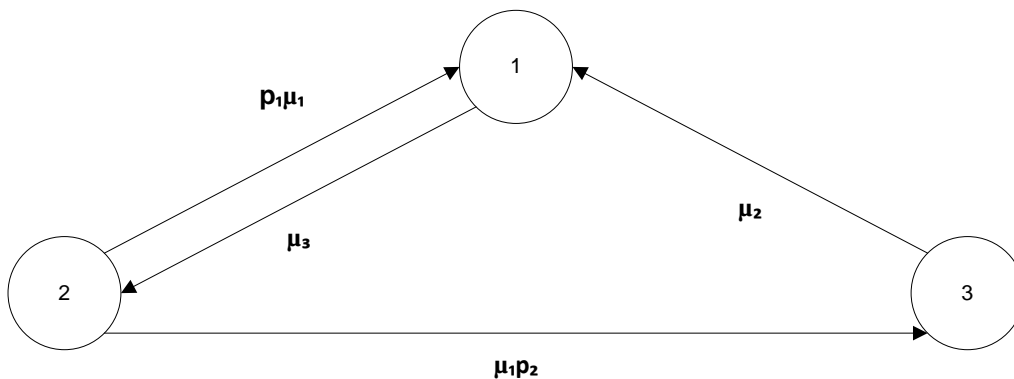
Figure 18

Transition rates of up-stream machine when the repair time is distributed with Cox-2:b distribution and the time to failure is distributed Exponentially

We have assumed that the up state (state number 1) is the absorbing state, and the other two states are down states.

Parameters regarding the Cox-2:b distribution are defined as bellow:

p_1: in fact 1- p_1, is the rate of transition from state 2, which is down state, to absorbing state (state 1)

p_2: in fact 1- p_2, is the rate of transition from state 3, which is down state, to absorbing state (state 1)

μ_1: transition rate of staying in the first state.

μ_2: transition rate of staying in the second state.

μ_3: transition rate of staying in the third state.

Regarding to definition of Cox-2:b distribution, transition between states are composed of parameters which have been defined above. According to the figure 18, transition rate from state 1(operating state) to state 2(down state), which is actually the failure rate, is μ_3. Based on definition of the Cox-2: b distribution, from one transition state (which is down states in this figure), system can move to the absorbing state (in the figure with rate of $p_1\mu_1$ to state 1) or moves to the next transition state (in the figure with the rate of $\mu_1 p_2$ to state 3). Similarly, when the system is in the last transition state (state 3), there is no further way except going to the absorbing state (state 1) with the rate of μ_2.

Table 13 illustrate the values assigned to the parameters,

Parameter	Value assigned in our experiment
p_1	0
p_2	1
μ_1	2.518
μ_2	2.518
μ_3	0.25

Table 13, values of parameters of upstream machine's distribution for forth cluster experiments, considering Cox-2:b distribution for repair time

Then, transition matrix, **λ-u**, between states and the vector of processing rates, **Mu-u** are as below:

$$\lambda\text{-}u = \begin{bmatrix} -\mu_3 & \mu_3 & 0 \\ \mu_1 p_1 & -\mu_1 & \mu_1 p_2 \\ \mu_2 & 0 & -\mu_2 \end{bmatrix} = \begin{bmatrix} -0.25 & 0.25 & 0 \\ 0 & -2.51 & 2.51 \\ 2.51 & 0 & -2.51 \end{bmatrix}$$

$$\text{Mu-u} = \begin{bmatrix} \mu^u & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

Figure 19 depicts the upstream machine's state while we have associated the hypo-exponential distribution to time to repairs,
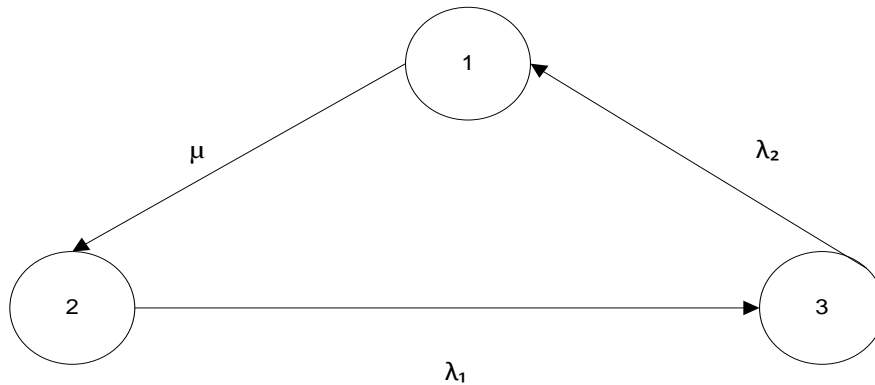
Figure 19, upstream machine while the repair time has hypo-exponential distribution

Parameters defined as below,

μ:the failure rate, from upstate, state 1, to down state, which is state 2.

λ_1: transition rate from downstate 2 to downstate 3. Actually, this rate is the repair rate between two successive phases of repair process.

λ_2: transition rate from downstate 3, the last phase of repair, to upstate.

Table 14 shows the values calculated for theses parameters according to the following equations:

$$(53)\ Y = -12 * E(X)^2 + 8 * E(X^2)$$

$$(54)\ \lambda_1 = (2 * E(X) + \sqrt{Y})/4$$

$$(55)\ \lambda_2 = E(X) - \lambda_1$$

Parameters	Value assigned in our experiment
λ_1	2.51
λ_2	2.51
μ	0.25

Table 14, values of parameters of upstream machine's distribution for forth cluster experiments, considering hypo exponential distribution for repair time

Transition rate matrix and the operating matrix is shown below:

$$\lambda\text{-u} = \begin{bmatrix} -\mu & \mu & 0 \\ 0 & -\lambda_1 & \lambda_1 \\ \lambda_2 & 0 & -\lambda_2 \end{bmatrix} = \begin{bmatrix} -0.25 & 0.25 & 0 \\ 0 & -2.51 & 2.51 \\ 2.51 & 0 & -2.51 \end{bmatrix}$$

$$\text{Mu-u} = \begin{bmatrix} \mu^u & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}$$

The efficiency in isolation of up-stream machine is being calculated as below:

$$\text{MTTF/ (MTTF+MTTR)} = 4/ (4+0.794) = 0.83$$

Down-stream machine

As we had in the first and second cluster, the down-stream machine stays with the same distribution of time to failure and time to repair.

Output

Output parameters of Tan-Gershwin model through having two different phase-type distribution have been gathered and compared in the Table 15.

N	TH(C2:b)	TH(Hypo)	Difference	WIP(C2:b)	WIP(hypo)	Difference
	1	2	3	4	5	6
1	1.3657	1.2897	0.076	0.4675	0.4221	0.0454
2	1.3984	1.3236	0.0748	1.0094	0.8895	0.1199
3	1.423	1.3503	0.0727	1.6035	1.3859	0.2176
4	1.4415	1.3713	0.0702	2.2329	1.8973	0.3356
5	1.4558	1.3882	0.0676	2.889	2.4172	0.4718
6	1.4669	1.402	0.0649	3.5677	2.9425	0.6252
7	1.4759	1.4134	0.0625	4.2669	3.4721	0.7948
8	1.4831	1.423	0.0601	4.9852	4.0056	0.9796

Table 15, output parameters for forth cluster experiments

Result of the forth cluster

As illustrated in the table 12, looking at third and sixth column, we see the reasonable differences between mean parameters of performance measurements. Actually, in some cases we see quite significant differences. For instance, we have 0.9796 as a difference of WIPs when we have the buffer size of 8.

Actually, there are some analytical reasons which contribute to understand these differences. For instance, having Cox-2:b distribution for time to repair will allow the chain to have the transitions from the first down state to the up state without passing the second, which means better performance of up-stream machine compare to having the Hypo exponential distribution for time to repair. Since Hypo exponential distribution make the chain's transitions pass through all down states skip-free, consequently, machine spends in the average more time in the down state. In fact, having the Cox-2:b distribution for time to repair enable the up-stream machine operate with better performance which is depicted in the table 5. Needless to say, more throughputs and higher level of WIP in front, proves the better performance.

However there are quite meaningful differences between considering these two distributions, but we cannot rely simply on the experiments which have been carried out because one may come to other values for parameters which comes to the different conclusions. So, we decided to expand our experiments levels. In fact, applying DEO in the following chapter will demonstrate the general solution.

3.2.5. DOE

In the last part we have done some experiments in order to realize the effect of third moment on the mean performance measurement of two machines and one finite buffer manufacturing lines through Tan-Gershwin model [11]. In fact, in the last part we have designed experiments through clusters to clarify the effects in each area of SCV range; however, parameters might not be enough in terms of expansion of values. Thus, we decide to use the DOE technique in order to expand the solution more generally. Consequently, we can defiantly decide to choose the proper Phase-Type distribution to associate to general events manufacturing lines.

According to the fact that there are parameters, such as squared coefficient of variance (SCV), Mean Time To Repair (MTTR) and buffer size (N), which affect the throughput of the line and the mean level of buffer which is called Work In Progress (WIP), we want to get to know how changing these parameters and at the same time having different phase-type distributions can affect the TH and the WIP of hypothetic line. To be more clear, after doing these experiments we will be able to observe whether associating different phase-type distribution to time to repair of a manufacturing line might affect the mean performance measurement or not. Clearly, throughout the experiments, different phase-type distributions are assigned to variables according to the SCV level of each experiment, since each of phase-type distributions covers specific rang of SCV, as explained in chapter two, figure1.

Consequently, we need to design an experiment to understand the effect considering different phase-type distribution on throughput and the WIP. Since we have three factors that are of interest, there are two options of experimental design,

- a fractional factorial design of resolution III with 4 runs, or
- a full factorial design with 8 runs

A two-level design with three factors has eight (2^3) possible factor combinations. By choosing a design with all possible combinations, called a full factorial design, we would get results that show effects free from confounding, that is, all effects are distinguishable from other effects. However, we may also be able to obtain meaningful results by doing fewer runs or combinations, which is called fractional factorial design.

We decided that the full factorial design with 3 factors and 8 runs is more appropriate than the fractional factorial design. Since, runs that manipulate the factors of interest - MTTR, SCV and N- are not expensive or time-consuming. In fact, the experiments results are not random data in our case because the output of the each experiment is a result of running the MATLAB written program by Gershwin and Tan [11] with the defined matrices according to designed experiments. Therefore experiments design turn to full factorial with single replicate experiments.

Model assumptions and parameters

We have assumed a two unreliable machine, which have the phase-type repair and failure time, and one finite buffer manufacturing line to apply our experiments. Since we have to verify the impact of three factors, SCV, MTTR and N, we assumed the down-stream machine to be constant in terms of distributions of time to repair, distribution of time to failure and its production rate. From the other hand, we changed the distribution of time to repair for up-stream machine according to the experiments levels. Consequently, we changed the Mean Time To Repair (MTTR) and SCV of time to repair for up-stream machine while changing the buffer size, according to the experiments designed levels. Furthermore, according to our response variable, which will be described after, we associate two different phase-type distribution to the time to failure according to the SCV level.

First of all we need to define the numeric levels of each factor. Clearly, the smallest possible value of each factor is the lower level of the factor while the largest possible is the higher level. Defining the range of each parameter must be in such a way that covers all the possible situations which may occur, such as having once first machine being bottleneck or vice versa. Thus, we have defined the range of three parameters as below:

Mean time to repair, MTTR= [1.06, 4.1576], to design the level of MTTR we have considered the lower level and the higher level in such a way that give us once Up-stream machine as a bottleneck of the line and then Down-stream as a bottleneck of the line. As we want to cover a wide range of potential situations which may occur in reality, we have also considered the bottleneck.

SCV= [0.629, 1.7608],

N= [2, 14],

Now, according to the Tan-Gershwin model [11], we have to identify the matrices of transitions between states, which are called Lambda matrices, as we had in the second chapter. As we have

mentioned in the model assumptions, we need to define two matrices, representing two different Phase-type distributions for time to repair, for each level of DOE. In the following we have brought matrices for each level;

When we have MTTR=1.016 and SCV=1.7608, considering two phase balanced mean distribution (Cox-2:b):

$$\text{Lambda-u} = \begin{bmatrix} -1 & 1 & 0 \\ 1.409 & -1.96 & 0.5589 \\ 0.5589 & 0 & -0.5589 \end{bmatrix}$$

When we have MTTR=1.016 and SCV=1.7608, considering Hyper exponential distribution:, since we have SCV more than one:

$$\text{Lambda-u} = \begin{bmatrix} -1 & 0.2375 & 0.7624 \\ 0.4675 & -0.4675 & 0 \\ 1.5 & 0 & -1.5 \end{bmatrix}$$

When we have MTTR=1.016 and SCV=0.629, considering two phase balanced mean distribution (Cox-2:b):

$$\text{Lambda-u} = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0.4037 & -1.96 & 1.5642 \\ 1.5642 & 0 & -1.5642 \end{bmatrix}$$

When we have MTTR=1.016 and SCV=0.629, considering Hypo exponential distribution, since we have SCV lower than one:

$$\text{Lambda-u} = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & -4 & 4 \\ 1.305 & 0 & -1.305 \end{bmatrix}$$

When we have MTTR=4.1576 and SCV=1.7608, considering two phase balanced mean distribution (Cox-2:b):

$$\text{Lambda-u} = \begin{bmatrix} -1 & 1 & 0 \\ 0.344 & -0.481 & 0.1366 \\ 0.1366 & 0 & -0.1366 \end{bmatrix}$$

When we have MTTR=4.1576 and SCV=1.7608, considering Hyper exponential distribution:, since we have SCV more than one:

$$\text{Lambda-u} = \begin{bmatrix} -1 & 0.2375 & 0.7624 \\ 0.1142 & -0.1142 & 0 \\ 0.366 & 0 & -0.366 \end{bmatrix}$$

When we have MTTR=4.1576 and SCV=0.629, considering Cox2:b distribution:

$$\text{Lambda-u} = \begin{bmatrix} -0.333 & 0.333 & 0 \\ 0.09867 & -0.48105 & 0.3823 \\ 0.3823 & 0 & -0.3823 \end{bmatrix}$$

When we have MTTR=4.1576 and SCV=0.629, considering Hypo exponential distribution, having SCV lower than one:

$$\text{Lambda-u} = \begin{bmatrix} -0.333 & 0.333 & 0 \\ 0 & -0.319 & 0.319 \\ 0.977 & 0 & -0.977 \end{bmatrix}$$

Subsequently, we have done two experiments for each level of DEO for two different phase-type distributions associated to time to repair of upstream machine.

Finally, we have two sets of output data for each level, which are Throughput and WIP with different phase-type distributions. According to our purpose, which is comparing the mean performances of the line with different phase-type distributions for Time to repair, we have assumed the following formula for response data in DOE,

%WIP= [WIP (considering C2:b distribution)- WIP(considering Hyper or Hypo exponential, according to level of experiment)]/ WIP(considering C2:b distribution)

And,

%TH== [TH (considering C2:b distribution)- TH(considering Hyper or Hypo exponential, according to level of experiment)]/ TH(considering C2:b distribution)

Considering the factors levels, we have applied Lenth's method through MINITAB, given that we have single replicate experiments. Table 16 depicts the experiments results while changing the experiment's levels.

Standard order	SCV	MTTR	N	Efficiency in isolation	%Throughput	%WIP
1	0.629	1.016	2	$e_1 > e_2$	0.0002	-0.0009

2	0.629	1.016	14	$e_1 > e_2$	0.000	-0.0003
3	1.7608	1.016	2	$e_1 > e_2$	0.0007	-0.0027
4	1.7608	1.016	14	$e_1 > e_2$	0.0004	0.0046
5	0.629	4.1576	2	$e_1 < e_2$	0.0007	-0.0028
6	0.629	4.1576	14	$e_1 < e_2$	0.0001	0.0145
7	1.7608	4.1576	2	$e_1 < e_2$	0.0005	-0.003
8	1.7608	4.1576	14	$e_1 < e_2$	-0.003	-0.0265

Table 16, DOE values

As illustrated in table 16, we observe the fact that experiments one to four are designed in such a way that we have downs ream machine as a bottleneck and for others vice versa.

Having the created factorial design and collected response data, we have fitted the model to the results and generated some graphs to evaluate the effects. Therefore, we used two graphical methods to help see which factors have significant effect the response data which is explained above.

We have fitted the full model, which includes the three main effects, two-way interactions, and one three-way interaction. Taking into consideration the P column of the estimated effects and coefficients table in the MINITAB session window output, we can determine which effects are significant.

3.2.5.1. %WIP AS A RESPONS

Here we illustrated the normal probability plot and the Pareto chart of the effects of three factors to demonstrate how factors influence the %WIP as the response.

Normal Plot of the Effects
(response is WIP%, Alpha = 0.05)
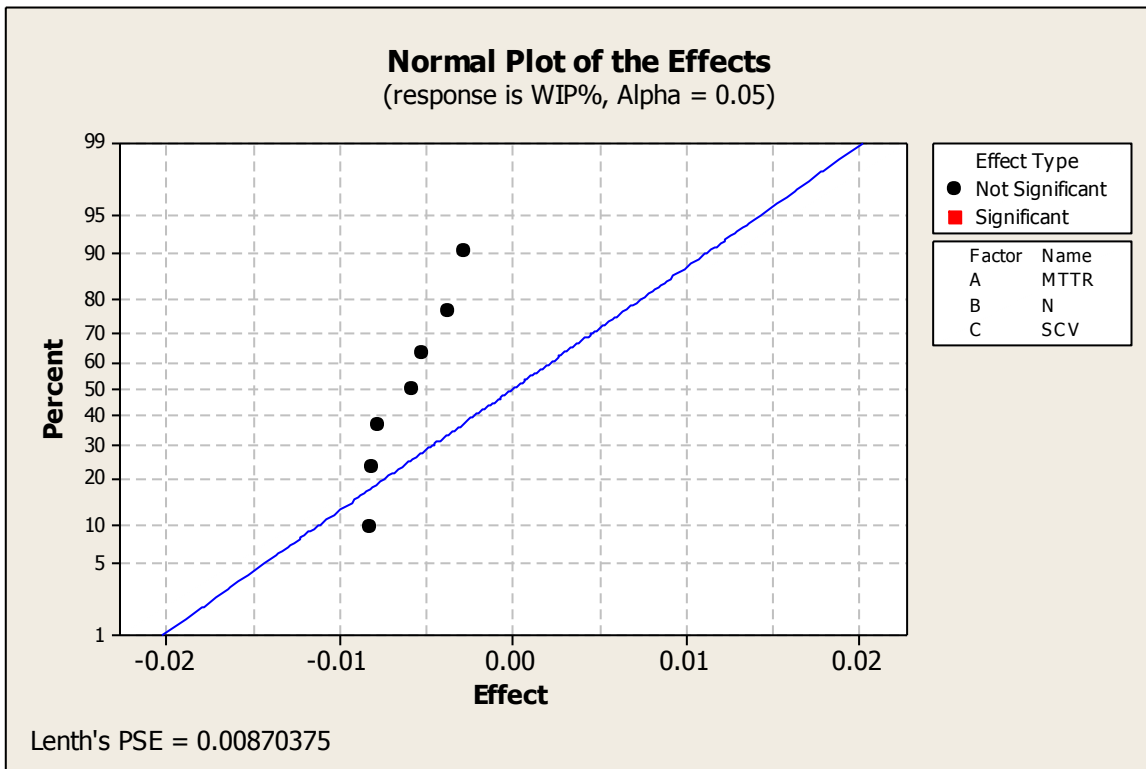
Lenth's PSE = 0.00870375

Figure 20

Normal plot of the effects while the response is WIP, Alpha=0.05

The normal probability plot labels effects that are lower than the α level we have chosen in the Analyze Factorial Design-Graphs. As we see in the figure20, none of those factors has the significant effect on the response data.

Pareto Chart of the Effects
(response is WIP%, Alpha = 0.05)

0.02714

Factor	Name
A	MTTR
B	N
C	SCV

Lenth's PSE = 0.00721125

Figure 21, the Pareto chart of the Effects, while response is %WIP and Alpha=0.05.

The Pareto chart uses the same α as the normal plot to determine the significance of effects. So again, we witness no significant effect on %WIP.

3.2.5.2 %TH AS A RESPONSE

At this point we brought the normal probability plot and the Pareto chart of the effects of three factors to show how factors influence the %TH as response of the experiment.

Normal Plot of the Effects
(response is TH%, Alpha = 0.05)

Lenth's PSE = 0.000486375
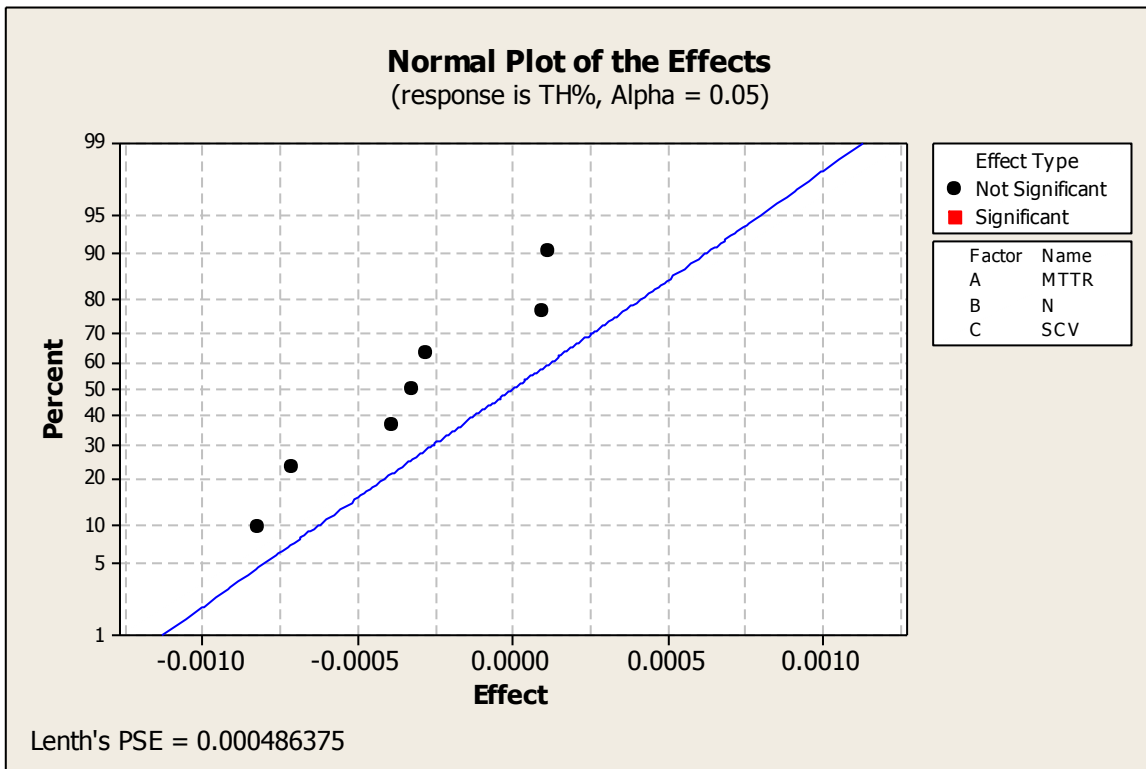
Figure 22

N normal plot of the Effects while the response is TH, Alpha=0.05

As we have mentioned in figure1, The normal probability plot shows the factors which have the P-value lower than the α level we have chosen in the Analyze Factorial Design-Graphs. As illustrated in the figure 22, there is no factor which has the significant affect on the response data.

In fact The Pareto chart uses the same α as the normal plot to determine the significance of effects. The same result came up while looking at the Pareto plot, figure 23,
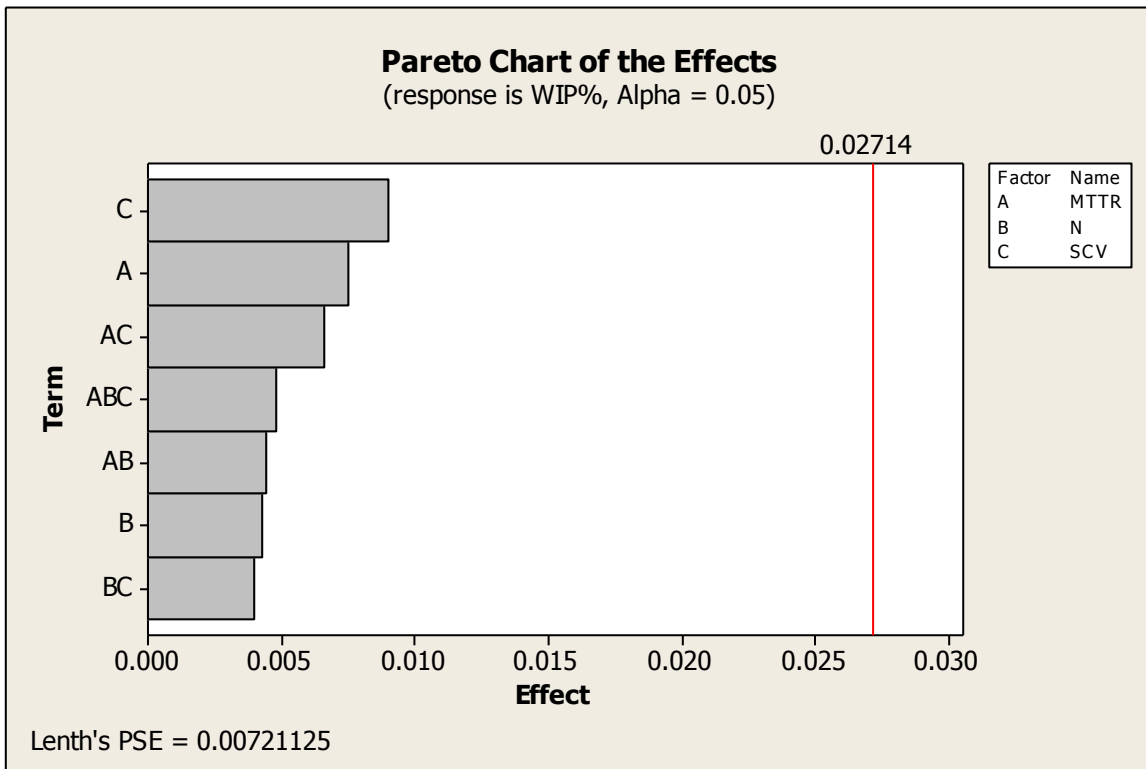
Figure 23

Pareto chart of the Effects, while response data is TH and α=0.05

In fact The Pareto chart uses the same α as the normal plot to determine the significance of effects.
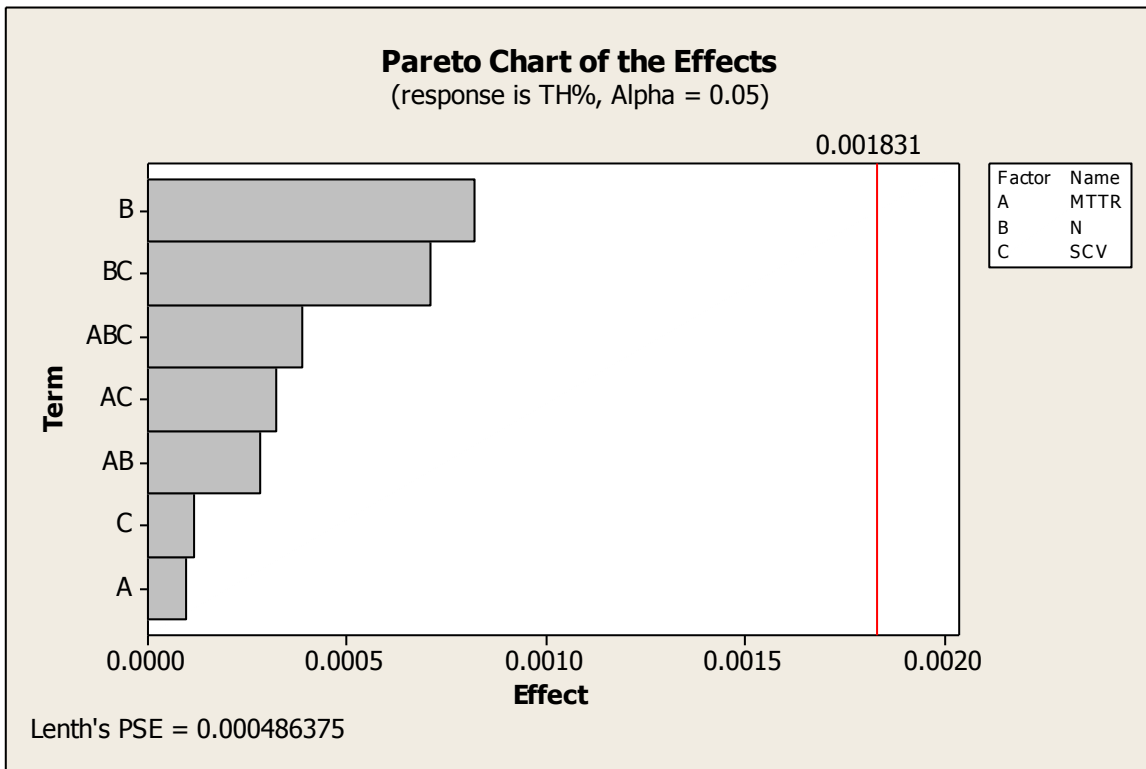
3.3 Conclusion

In the first part of this chapter we have analyzed the impact of second moment on the mean throughput of the line at the beginning. The results, which have been shown, prove the enormous effect of second moment on the throughput. Consequently, we can claim that in the analytical modeling of transfer lines we have to associate the phase-type distributions to the process's time, such as repair time.

In the second part of this chapter we have analyzed the effect of third moment, skewness, on the mean performance measurements of the transfer lines. According to the figures 18 to 21, for both %TH and %WIP, we see no significant effect within factors on the response data. In fact, having no significant influential factor's effects means considering one of phase-type distributions instead of

another according to the ease of utilization, while considering the rang of SCV, does not influence the mean performance measurement of manufacturing lines. For instance, considering the Coxian distribution instead of Hyper exponential distribution in case we have the SCV more than one, would have no significant effect on the throughput or WIP level while measuring the mean performances of manufacturing lines.

Generally, we have understood that considering the phase-type distributions for generally distributed events make a huge difference in the results though associating one of phase-type distributions instead of another may help us to fit the distribution to the real data much easier because having fewer number of parameters might contribute to ease the utilization of the distribution and performance measurement, which is really worthwhile to know. For example, if we know that there are no significant effects when we consider the different distributions of phase-type, we can use the Two-phase balanced mean Coxian distribution for all the data covering the SCV range from 0.5 to greater than 0.5. Consequently, for modeling the general events manufacturing line, we can apply the proper Phase-Type distribution only considering the level of SCV of the data set.

4.Industrial Case Study

In this section we have analyzed a real automotive manufacturing transfer line system. The goal of this chapter is measuring the mean performances of the mentioned transfer line through considering the Phase-type distributions for time to repair and comparing the method described in the thesis to simulation method and the analytical method without considering the Phase-type distributions for repair process. This part also sheds lights on the application method of Phase-type distributions. In the following, we will describe the system description, layout of the system, line cycle time, failure and repair time's distributions.

4.1 System description

The analyzed assembly system is an automotive production line with a linear layout composed of 22 stations in series. Two Buffers are present. Stations are clustered in 15 Protection Areas (or safety areas). The Protection Area identifies a group of machines which are simultaneous stopped in case of failure to guarantee the repairmen to operate safely. In turn, the Protection Areas are

clustered in 3 Stages St, with t=1,..,3. The Stage identifies a group of Protection Areas which are not decoupled by buffers and represent unbuffered parts of the line. The block system diagrams representing the system layout are reported in Figure 23.

S1: from OP005 to OP090
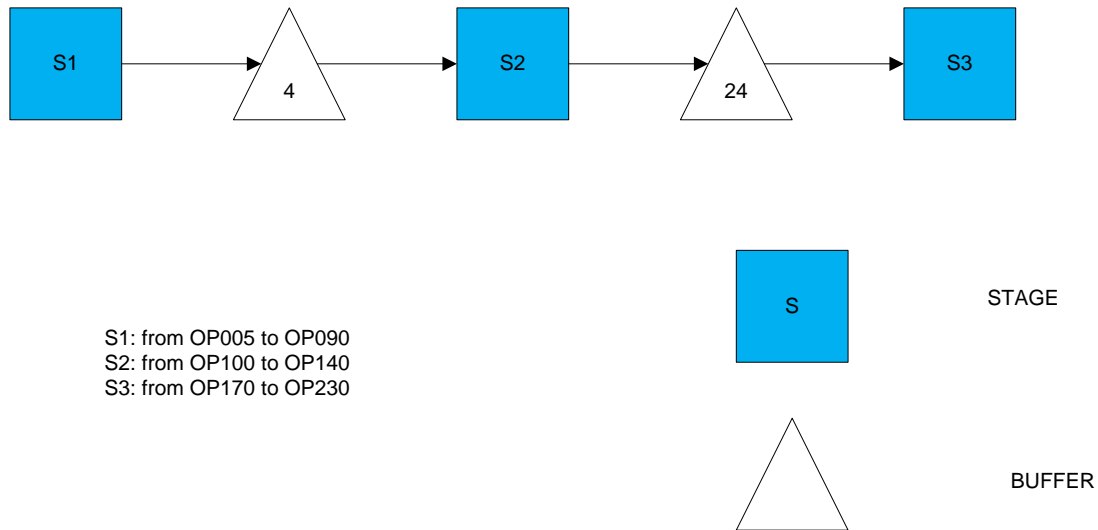S2: from OP100 to OP140
S3: from OP170 to OP230

Figure 24, The block system diagrams representing the system layout

The protection area identifies a group of machines which are simultaneously stopped in case of failure. In other words, if one station in the protection area fails, the gate bounding the safety area must be opened to allow the repairman to fix the problem through corrective maintenance and reset the machine to operational conditions. During this period, all the stations in the protection area are stopped to guarantee the repairman to operate safely.

Each station OP processes material with cycle time CT. Moreover, each station is subject to F different causes for failure, each one related to a specific equipment or component. In each Stage, if a station OP is failed, its Protection Area is stopped; the Stations following the Protection Area get starved and the Stations preceding the Protection Area get blocked after the corresponding cycle time CT.

In summary, we can observe the system in these ways:

- **Component level** (e.g. robots, machines, transporters, etc.). Each component has specific MTBF and MTTR values.

- **Station level**. Each Station is composed of one or more components.

- **Protection Area level**. Each PA is composed of one or more Stations.
- **Stage level**. Each Stage S is composed of one or more Protection Areas.

Figure 25 shows a detailed layout of the whole line. As we see in the figure, the line is composed by three separate stages which are protection areas. In fact, two buffers separate the protection areas, as we mentioned in the figure 24 which is block system diagram of the same line.

Figure 25, layout of the automative transfer line

At the beginning of each cycle, the generic PA_t, if operational, takes one part from its upstream Protection Area PA_{t-1} or from outside the system, in case of PA1. If the upstream PA_t has not

delivered the part under processing yet, the PA_t has to wait until PA_{t-1} finishes its operations before loading the part. In this case, PA_t is in the idle state. As soon as PA_t loads the part, it gets busy and the operation begins. If PA_t fails, the protection area enters into a down state and requires the repairman intervention. In this case, the part under processing is not delivered to the downstream protection area. After a repair time, the PA is turned busy again and is ready for performing the operation. The operation requires a deterministic time CTPA and during this period the machine is in a working state. After this time, the machine is ready to unload the worked part and to start a new operational cycle. However, if the downstream PA_{t+1} is not idle, the PA_t enters into a blocking state (i.e. the machine is blocked) and has to wait before unloading the processed part.

Further assumptions are listed below:

- PA_1 is never starved and PA_k (the last protection area) is never blocked
- The blocking rule of PA_s is the BAS (Blocking After Service), i.e. first the PA processes a product and then it enters into the blocking state
- Failures are operational dependent, for instance, a PA can fail only during an operational state when it is in the working state. In fact, there are no failures during the blocking states.
- Failures do not overlap; a PA cannot be down due to more than one failure. In other words a PA cannot fail in more than one model at the same time.
- Transportation times are not modeled. They are already included into the cycle times.
- Products are not destroyed or rejected at any PA in the system.

Protection area characterization

Cycle Time

The cycle time of each protection area CTPA is obtained as the maximum cycle time of the stations composing the PA.

Failure Mode

We have assumed the multiple failures for each stage, which is composed of some PAs. Actually, each failure that causes (related to a specific station) the stopping of a protection area is modeled with a mode of failure, and thus a PA will have a number of failure modes equal to its number of stations. In this case the station is modeled with a single mode of failure, which models in an aggregate way its component downtimes. Consequently, for each stage we have failures equal to the

number of stations, considering each station with the single failure. Here is the part that we have to apply the Phase-type association approach to the real data of repair process. According to the fact that each machine has a failure mode, we have a data set which stands for each machine's repair process. As an example, we have shown the repair time histogram for machine OP040 in the figure 26.
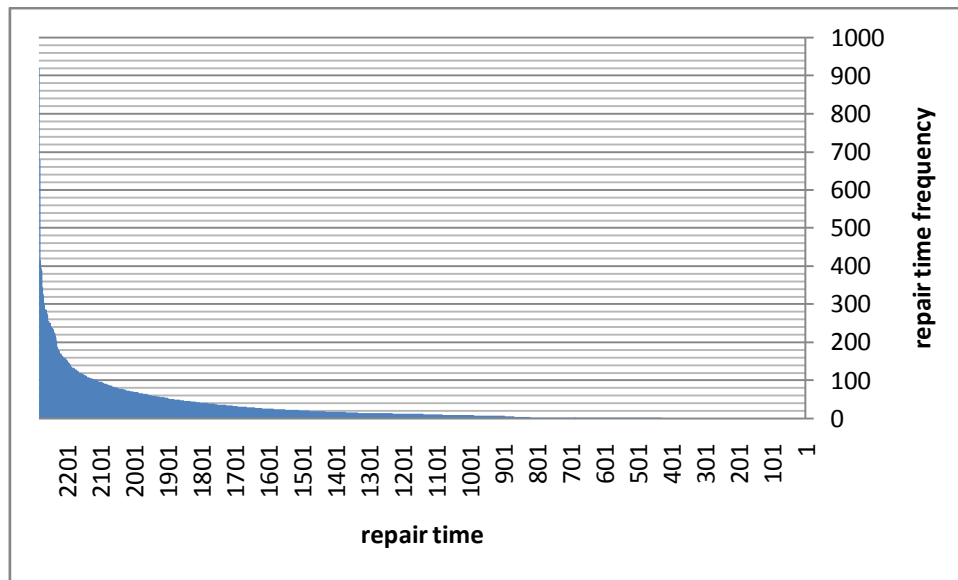

Figure 26, repair time for machine OP040

In the figure 26 we just have the repair time for a single machine which the first stage is composed of. So, for each single failures of the each stage we have a similar histogram of repair and failure time which are illuminated because of frequency of similar figures.

4.2 Simulation Model

There is a simulation model which is done by POLITECNICO DI MILANO researchers over this real case of automotive manufacturing transfer line. There are some assumptions regarding the simulation model which are taken into considerations. In fact, the time to repairs, time to failures and cycle times are all the same, as we explained above. However, the distributions associated to the repair processes are differently considered. In fact, the repair processes are considered to be exponentially and log normally distributed while we have associated Phase-Type and exponential distributions to them. We have taken the Throughput of the line as a simulation result parameter in order to compare to the analytical approach.

4.3 Analytical method

In this part, we have applied the Performance evaluation of transfer lines with general repair times and multiple failure modes proposed by Marcello Colledani-Tullio Tolio [18]. Two analytical approaches have been considered. First, associating Phase-Type distribution to repair process time and then associating exponential distribution to repair process time. According to the confidential data which is given by the company, we have achieved the mean throughput of the line, considering the line assumptions, through following steps:

- Assuming each stage as a multiple failure mode machine composed by decoupled protection areas. Thus, each stage has TTF and TTR equal to the number of stations which are composed of.

- Cleaning the TTF and TTR data of each station. Since there are real data, sometimes we have faced some significant different data, for example for repair time, which have been illuminated through histogram approach.

- Calculating mean, variance and SCV of TTR and TTF of each failure mode of each stage.

- Associating the proper Phase-Type distribution to the repair times according to the results we have from chapter two. Furthermore, we have calculated the parameters of each Phase-Type distribution regarding the time to repair.

- Associating the exponential distribution to repair process time and calculating the repair rates.

- Associating the exponential distribution for time to failure, computing the parameter for each single failure time.

- Developing the transition rate matrices between states of Markov chain, according to Tan-Gershwin model. In this step, we have considered the upstate as an absorbing state of Phase-Type distribution.

- Developing the production rate matrices, for each stage.

- Running the developed program which calculates the mean throughput and the WIP according to decomposition method.

We have applied theses steps for each single failure mode for each stage. For example stage one has 20 failure modes. Diagram 1 shows the steps more clearly,
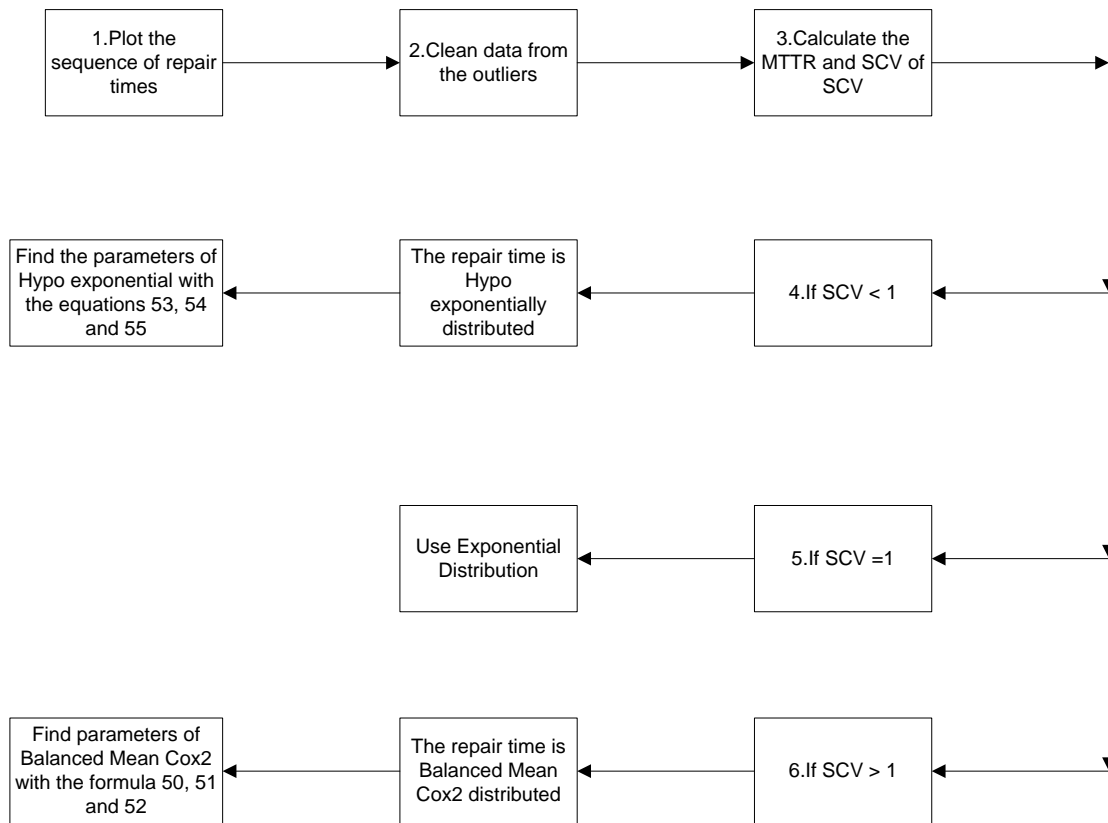
Diagram1,

Analyzing steps for repair time of each failure mode

As we have mentioned in the above steps to run the program, we have associate different Phase-Type distributions to the repair processes. The associating method is rooted to the last chapter. According to the level of SCV, we have two or more Phase-Type distribution to associate to the real data, depending on the SCV value. For instance, if there is a SCV more than one for repair process we decided to associate Cox2:b distribution while if the SCV is less than one, we associate Hypo Exponential distribution. Nonetheless there are more distributions to associate in case of SCV less than one; we associate the Phase-Type distribution which has less parameter to find, since we found no significant effect on the mean performance measurement parameters in chapter two. Finding all parameters contributes to develop the transition rate matrices. For each single stage, which is assumed to be a machine with multiple failure modes, we have a transition rate matrix.

In the other hand, we have associated the exponential distribution to repair process time. Consequently, there are two analytical approaches which will be clarifying the effect of considering Phase-Type distributions on the mean performance measurement of manufacturing transfer lines.

Finally, we have two different sets of transition rates between states of Markovian system which will be the inputs of our model of performance measurement of manufacturing transfer lines. In the following part we have brought the output data and the comparisons between the outputs of different models.

4.4 Comparison between Simulation Model and the Analytical Method

In this part we compare the throughput of the described manufacturing line as an output variable of the simulation and two different analytical method with the actual throughput of the line. The data which have been given by the company should have been reviewed in order to find the throughput of the line with the same dimensions we got through analytical and simulation methods. The actual throughput of the line which has been given is only parts which are passed from quality department's criteria although the data sets which we have applied, such as repair and failure time, belong to whole period's time. In fact the period we have considered is three months of September, October, and November. Consequently, the throughputs coming from analytical and simulation methods are the whole throughput including defected parts. So, we need to modify the given throughput by multiplying the throughput by the efficiency of the line for each month. In the table 17 we see the modification:

	SEP	OCT	NOV
TH	55.5555556	55.555556	55.55556
O.E.E	0.6814	0.6301	0.6099
Modified TH	37.8555556	35.005556	33.88333

Table 17, Actual throughput of the transfer line

In order to compare the results from three approaches to the actual one, we have made an average of three numbers of throughputs for three months. Then in the following table the actual throughput is the average one.

Table 18 depicts the comparisons between approaches,

	Analytical method (exp)	Analytical method (Phase-Type)	Simulation method
TH(JPH)	21.45	32.736	30.81
Actual Throughput	35.58	35.58	35.58
% Deviation	39.713	7.993254637	13.40640809

Table 18, Comparison between methods

As we see in the table 18, associating the exponential distribution to the both repair and the failure time cause a enormous deviation of approximately 39% while the analytical methods considering the Phase-Type distributions for repair time and exponential distribution for failure time comes up with the much lower deviation of around 8%. On the other hand, simulation method gives an approximately 13% deviation from the actual throughput which is more than that of analytical method with Phase-Type association approach.

As a conclusion for the industrial case study, we can claim that the analytical method will have the more precise results if we associate the Phase-Type distributions to the repair process. Furthermore, the analytical method comes to the solution much faster than the simulation, so we can apply the method in the online needed industrial cases. In fact, using the decomposition method and associating the proper Phase-Type distribution to the Markovian systems instead of exponential distribution, results in more precise performance measurements.

Conclusion

In this thesis we have analyzed the modeling of manufacturing transfer lines while we the repair time of machines is generally distributed. Repair process is mostly considered as an exponential distribution in the literature although in real systems we rarely observe this situation. In our thesis, we have carried several experiments to realize the effect of third moment on the mean performance measurement of manufacturing lines. According to the fact that we have designed our experiment to machines with multiple failure mode, the decomposition method is chosen in our work to model the transfer lines through Markovian systems. In our experiment, the down times, which are repair

process time, are considered as Phase-type distributions. Results of experiment shows us no significant impact of third moment on the mean performance measurement of manufacturing lines. Consequently, we can claim there are no significant effect of considering different Phase-Type distributions on the mean performance parameters of the transfer lines, such as mean throughput or WIP, although associating Phase-Type distribution to general event systems has significant impact on the more precise solution to estimation models. As we have shown in the real case, considering the Phase-Type distributions for general event Markovian systems contributes more precise estimation compare to other models.

Future work

According to the fact that we have analyzed the analytical method through real case and also hypothetic manufacturing lines, still there are few issues to consider. Developing the analytical method of performance measurement for transfer lines with more machines would be an idea for future work. In the thesis, we have compare the result of real case through associating the phase-type distribution to the repair times of three stage machine line, but there are few lines which have the same characterization, so applying the method for more than three stages would be very useful in order to get to know about the precision on the method for more stages lines. Although through the thesis we have analyzed the effect of the Skewness of the phase-type distributions on the mean performance measurements of the transfer lines, but there is still an issues left, which is the number of machines in the transfer line. Finally, developing the available codes of analytical method of performance measurement of transfer lines to more machines would be very useful in order to contribute more researches for greater lines.

Bibliography

1) Andres Horvath, Miklos Telek , A general Phase-type Fitting Tool
2) Neuts MF, Probability distributions of phase-type (1975)
3) Mark Fackrell ,Modeling healthcare systems with phase-type distributions(2008)
4) Ross SM, Introduction to probability models (1993)

5) Erlang AK, Solution of some problems in the theory of probabilities of significance in automatic telephone exchanges (1917–1918)

6) Adele H. Marshall and Sally I. Mcclean , Using Coxian Phase-Type Distributions to Identify Patient Characteristics for Duration of Stay in Hospital (2004)

7) John a. Buzacott J. Gerge Shanthikumar, Stochastic Models of Manufacturing systems, page 540

8) Ha, the steady state probabilities in performance measurement of manufacturing lines(1997)

9) Jeong Eun Lee and Yushin Hong,controlled stochastic production system with 2-phase Coxian processing times and lost sales (1998)

10) Altiok, Performance Analysis of Manufacturing Systems (1985)

11) Tan-Gershwin, Modeling and Analysis of Markovian Continuous Flow Systems with a Finite Buffer (2009)

12) Altiok, Performance Analysis of Manufacturing Systems, Springer, New York (1997)

13) Gershwin, S.B., Manufacturing Systems Engineering, (1994)

14) Gershwin, S.B., An efficient Decomposition Method for the Approximate Evaluation of Tandem Queues with Finite Storage Space and Blocking. Operations Research, 35(2), 291-304 (1987)

15) Gershwin, S.B., Matta, A., Tolio, T., Analysis of Two Machine Lines with Multiple Failure Modes (2002)

16) Tayfur Altiok, Unsal Ozdogru, Analysis of Two-Valve Fluid-Flow Systems with General Repair Times (2001)

17) Marcel F. Neuts, Rafael Perez ocon, Inmaculada Torres-Castro ,Repairable models with operating and repair times governed by Phase-Type distributions (2000)

18) Marcello Colledani , Tullio Tolio, Performance evaluation of transfer lines with general repair times and multiple failure modes (2009)