ANNEX A: ANALYSIS METHODOLOGIES

A.1 Introduction

Before discussing supplemental damping devices, this annex provides a brief review of the seismic analysis methods used in the optimization algorithms considered in this thesis.

First, time domain analyses are described in relation to different possible structural systems. Particular attention is given to Newmark's time-step method which was implemented in order to carry out linear analyses required in two optimization methods. Then frequency domain analysis is described, explaining how the transfer function can be used to obtain the structural response. Finally a brief presentation on the stochastic response of linear structures is presented in order to better understand the meaning of power spectral density and the way to get the statistical response used in one of the analyzed methods.

A.2 Time domain

The dynamic response of a system in time domain can be determined in different ways. In a linear system with classical damping it is possible to apply classical modal analysis. Natural frequencies and mode shapes are computed and the equations of motion, when transformed to modal coordinates, become uncoupled. Thus the response of each vibration mode can be computed independently from the others, and the modal responses can be superposed, in different ways, to determine the total response. Time-stepping methods can solve the single degree of freedom equations of each dynamic coordinate in case that the excitation is in the form of real accelerograms.

This classical modal analysis can't be applied to such systems containing very different levels of damping and also in presence of viscous damping devices. In this latter case in fact the matrix relative to the added damping for example for two-dimensional systems is a tridiagonal one which is not always diagonal if transformed in modal coordinates. For such non-classically damped systems modal analysis is not possible because classical vibration modes do not exist and the equations of motion can't be uncoupled. There are two possible ways to find the dynamic response. It is possible to apply a complex analysis, that is transforming the equations of motion to eigenvectors of the complex eigenvalue problem which includes also damping matrix. The second possibility consists on solving the coupled system of differential equations. In this case numerical methods are required even in the case of simple dynamic excitations because solutions in closed-form are not possible. For usual practice this latter way is mostly used, also because numerical methods are required to solve the inelastic problem which is of primary interest in earthquake engineering.

In the present research, for the optimization methods which require time history analysis, this latter approach has been chosen using Newmark ‘s method. In the next paragraph the mathematical description of the algorithm is provided.

Newmark’s method

Developed by N. M. Newmark in 1959 this method is surely the most used in time history analysis because of its stability (under certain conditions) and accuracy. It is based on a finite different approximation which takes the form:

(A..2.1) $$\dot{u}_{i+1} = \dot{u}_i + [(1-\gamma)\Delta t]\ddot{u}_i + (\Delta t\,\gamma)\ddot{u}_{i+1}$$

(A..2.2) $$u_{i+1} = u_i + \Delta t\,\dot{u}_i + [(0.5-\beta)(\Delta t)^2]\ddot{u}_i + [\beta(\Delta t)^2]\ddot{u}_{i+i}$$

The factor γ provide a weighting between the influence of the initial (i) and the final (i+1) acceleration on the change of velocity. The factor β instead provides a weighting of the initial and final accelerations to the displacement.
The different studies that have been carried on this formulation showed that the factor γ influences the amount of artificial damping induced by the numerical method, so that it is recommended to take this value equal to 0.5.
Moreover, choosing a factor β=1/4 the formulation becomes:

$$\frac{\dot{u}_{i+1} - \dot{u}_i}{\Delta t} = \frac{1}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

It is possible to see that the acceleration corresponds to the average value of the acceleration at the two time steps considered. For this reason Newmark’s method with β=1/4 is also referred to as constant average acceleration method.
On the other hand, considering β=1/6 and substituting it in the Newmark’s formulation results in the same equations as if a linear acceleration over the time was chosen. Thus the Newmark β=1/6 method is also known as the linear acceleration method.

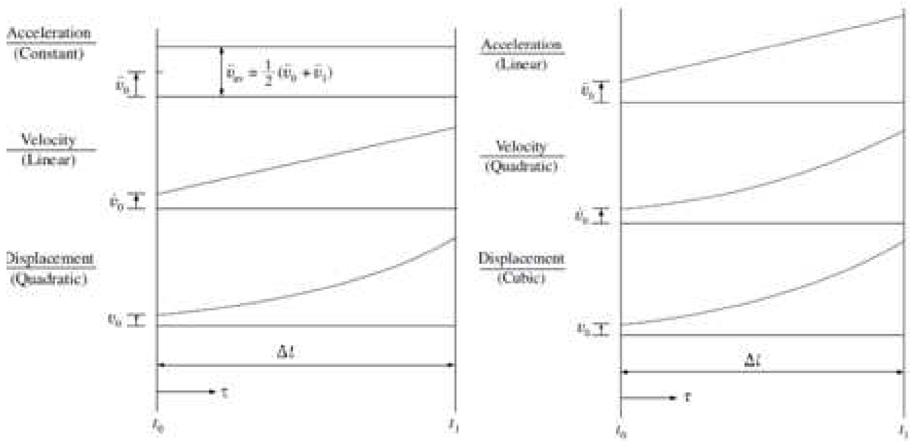

Figure A- 1 Variations with time in case of average and linear acceleration assumption

An important characteristic to know about numerical methods is the stability, that is the capacity to lead to bounded solutions without diverging. If it happens only for a limited range of time-increment these methods are *conditionally stable*, if, instead, there are no restrictions on the choice of the length of the steps they are called *unconditionally stable*.
Linear acceleration method belongs to this last category and is stable if $\Delta t / T_n < 0.55$ where T_n is the natural period. This stability criteria is not restrictive for singular degree of freedom system because values of the time-increment are sensibly less than the natural period. It becomes influent in case of multi degree of freedom system so that the adoption of unconditionally stable methods is necessary.
It is to note that Newmark's formulation requires iterations in order to be implemented, because the unknown $\ddot{u}_{i+1}$ appears also in the second member of the equations. It is possible to avoid this iterative process using incremental quantities:

$$\Delta\ddot{u}_i = \ddot{u}_{i+1} - \ddot{u}_i \qquad \Delta\dot{u}_i = \dot{u}_{i+1} - \dot{u}_i \qquad \Delta u_i = u_{i+1} - u_i$$

$$\Delta p_i = p_{i+1} - p_i$$

Moreover the original formulation can be rewritten as:

$$\Delta\dot{u}_i = (\Delta t)\ddot{u}_i + (\gamma\,\Delta t)\Delta\ddot{u}_i$$

$$\Delta u_i = \Delta t\,\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i + \beta(\Delta t)^2\,\Delta\ddot{u}_i$$

Solving the second of these equations brings:

$$\Delta\ddot{u}_i = \frac{1}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta\,\Delta t}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i$$

Substituting this equation derived from the second one in the first one gives:

$$\Delta\dot{u}_i = \frac{\gamma}{\beta\,\Delta t}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i$$

It is now possible to substitute the two obtained equations in the incremental equation of motion:

$$m\,\Delta\ddot{u}_i + c\,\Delta\dot{u}_i + k\,\Delta u_i = \Delta p_i$$

The substitution brings:

$$\hat{k}\Delta u_i = \Delta\hat{p}_i$$

Where:

$$\hat{k} = k + \frac{\gamma}{\beta\,\Delta t}c + \frac{1}{\beta(\Delta t)^2}m$$

And

$$\Delta\hat{p}_i = \Delta p_i + \left(\frac{1}{\beta\,\Delta t}m + \frac{\gamma}{\beta}c\right)\dot{u}_i + \left[\frac{1}{2\,\beta}m + \Delta t\left(\frac{\gamma}{2\beta} - 1\right)c\right]\ddot{u}_i$$

With $\hat{k}$ and $\Delta\hat{p}_i$ known from the system properties *m*, *c*, and *k* algorithm parameters γ and β, and the $\dot{u}_i$ and $\ddot{u}_i$ at the beginning of the time step, the incremental displacement is computed from:

$$\Delta u_i = \frac{\Delta\hat{p}_i}{\hat{k}}$$

Once Δu_i is known, $\Delta \dot{u}_i$ and $\Delta \ddot{u}_i$ can be computed from the previous equations. In order to obtain the value of the acceleration to start the analysis the equation of motion can be used:

$$\ddot{u}_i = \frac{p_{i+1} + c\,\dot{u}_{i+1} - k\,u_{i+1}}{m}$$

Hereafter the implementation of Newmark's method in MATLAB is shown. This numerical algorithm has been used for solving time-histories in the optimization problem which require it. Note that average acceleration method has been adopted since multi degree of freedom systems must be analyzed.

MATLAB code

```
function [u,u1,u2]= Newmark(Dt,nomefile,fact,M,C,K,IC);
% Input
parameters_________________________________________________________
% Dt          time step
% nomefile    name of the file containing the accelerogram
% 1/fact      scale factor for the accelerogram
% M           mass matrix
% C           damping matrix
% K           stiffness matrix
% IC          matrix containing the initial conditions of the system
%             i.e. velocities and displacements
% Output
parameters________________________________________________________
% u           displacement time-history
% u1          velocity time-history
% u2          acceleration time-history
% Number of degree of freedom
gdl=max(size(M));
% Modal Analysis
[S,w2]=eig(K,M);
w=sqrt(w2);
Periodi=2*pi*inv(w);
% Ground motion
leggo=fopen(nomefile,'r');
accelerazione=fscanf(leggo,'%g',[1 inf]);
tmax=Dt*max(size(accelerazione));
acc=accelerazione/fact;
fclose(leggo);
p=-M*diag(ones(gdl))*acc;
% Newmark parameters
beta=1/4;       %mean acceleration: beta=1/4
gamma=0.5;      %linear acceleration: beta=1/6;
% Parameters for the Newmark's method
KK=K+gamma/(beta*Dt)*C+1/(beta*Dt^2)*M;
a=1/(beta*Dt)*M+gamma/beta*C;
b=1/(2*beta)*M+Dt*(gamma/(2*beta)-1)*C;
% Initializing matices
u=zeros(gdl,max(size(p)));
u1=zeros(gdl,max(size(p)));
u2=zeros(gdl,max(size(p)));
```

```
du1=zeros(gdl,max(size(p)));
du2=zeros(gdl,max(size(p)));
% Initial conditions
u(:,1)=(IC(:,1));
u1(:,1)=(IC(:,2));
u2(:,1)=inv(M)*(p(:,1)-C*u1(:,1)-K*u(:,1));
% Newmark method
passi=max(size(p));
for i=1:(passi-1)
    DP=p(:,i+1)-p(:,i);
    DPP=DP+a*u1(:,i)+b*u2(:,i);
    du(:,i)=inv(KK)*DPP;
    u(:,i+1)=u(:,i)+du(:,i);
    du1(:,i)=gamma/(beta*Dt)*du(:,i)-gamma/beta*u1(:,i);
    u1(:,i+1)=u1(:,i)+du1(:,i);
    du2(:,i)=1/(beta*Dt^2)*du(:,i)-1/(beta*Dt)*u1(:,i)-
1/(2*beta)*u2(:,i);
    u2(:,i+1)=u2(:,i)+du2(:,i);
end
end
```

A.3 Frequency domain

The main limit of time history analysis is the dependency of the results from the ground motion considered. In fact not only the acceleration amplitude or the duration of an earthquake has to be taken into account, but also its frequency content, that is the Fourier transform of the accelerogram. An earthquake can in fact excites a structure in different ranges of frequency which could match the natural frequencies of the system. Analyses in frequency domain allow a better understanding of the response of the structure without any dependency on input ground motions.

A..3.1 Oscillator with sinusoidal excitation

The simplest case to analyze to better understand the frequency domain analysis is a single degree of freedom system subjected to a sinusoidal excitation having an amplitude $\ddot{u}_g$ and a circular frequency $\overline{\omega}$:

$$\ddot{u}_g(t) = \ddot{u}_g \sin(\overline{\omega} t)$$

The equation of motion is the well known:

$$m\,\ddot{u}(t) + c\,\dot{u}(t) + k\,u(t) = -m\,\ddot{u}_g(t) = -m\,\ddot{u}_g \sin(\overline{\omega} t)$$

which can be written also as:

$$\ddot{u}(t) + 2\,\xi\,\omega_n\,\dot{u}(t) + {\omega_n}^2\,u(t) = -\,\ddot{u}_g \sin(\overline{\omega} t)$$

As in all differential equations of second order a particular and a general solution can be found. The general solution is related with the initial conditions of the oscillator. With damped systems it is possible to demonstrate that this part of the response reduces

quickly also if low damping ratios are present. In this case it is possible to neglect this transient response derived from the general solution and focalize the attention on the steady state part of it.
Figure A- 2 shows the different parts of the solution.

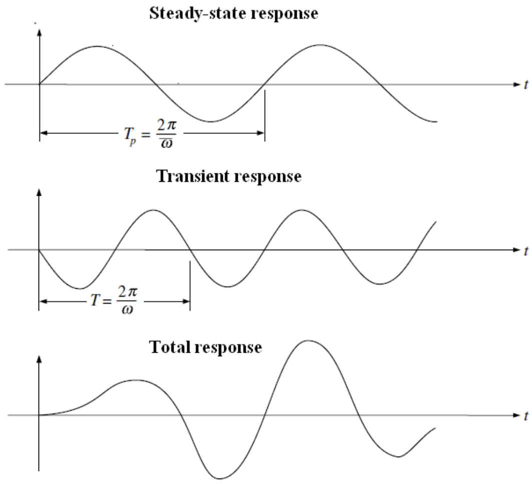

Figure A- 2 Combination of steady-state and transient responses

Due to the presence of damping, in the formulation of the particular solution, a cosine wave must be added (or a phase angle) because the response is generally not in phase with the excitation. the particular solution takes the form:

$$u_{part} = A\sin(\overline{\omega}t) + B\cos(\overline{\omega}t)$$

Substituting the derivatives of the particular solution in the equation of motion leads to:

$$\sin(\overline{\omega}t)\left[-\overline{\omega}^2 A - 2\,\xi\,\omega_n\overline{\omega}\,B + \omega_n^2 A + u_g\right] +$$
$$+\cos(\overline{\omega}t)\left[-\overline{\omega}^2 B - 2\,\xi\,\omega_n\overline{\omega}\,A + \omega_n^2 B\right] = 0$$

In order to satisfy this equation for all values of t, it is necessary that each of the two square bracket quantities equal zero; thus, one obtains:

$$A(1-\beta^2) - B\,2\,\xi\,\beta = \frac{m\,u_g}{k}$$
$$B(1-\beta^2) + A\,2\,\xi\,\beta = 0$$

where:

$$\beta = \frac{\overline{\omega}}{\omega_n}$$

Solving these two equations simultaneously yields:

$$A = \frac{m\,u_g}{k}\left[\frac{1-\beta^2}{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}\right]$$
$$B = \frac{m\,u_g}{k}\left[\frac{-2\,\xi\,\beta}{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}\right]$$

Introducing these expressions in the steady state solution brings:

$$u = \frac{m\,u_g}{k}\left[\frac{1}{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}\right]\left[(1-\beta^2)^2\sin(\overline{\omega}t) - 2\,\xi\,\beta\cos(\overline{\omega}t)\right]$$

This formulation of the response can be more easily understood if rewritten in the form of a unique sinusoidal wave with a phase angle:

$$u = \hat{u}\ \sin(\overline{\omega}t - \Phi)$$

It can be demonstrated that the amplitude takes the form:

$$\hat{u} = \frac{m\,u_g}{k}\left[\frac{1}{\sqrt{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}}\right]$$

while the phase angle is:

$$\Phi = arctan\left(\frac{2\,\xi\,\beta}{1-\beta^2}\right)$$

It is interesting at this point plotting the so called dynamic magnification factor *R*, that is the ratio between the dynamic response to the equivalent static load $\frac{m\,u_g}{k}$ for different values of damping:

$$R = \frac{\hat{u}}{\frac{m\,u_g}{k}} = \frac{1}{\sqrt{(1-\beta^2)^2 + (2\,\xi\,\beta)^2}}$$

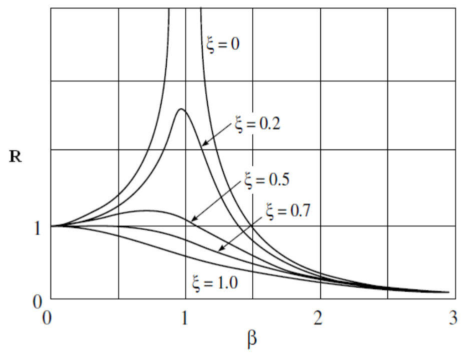

Figure A- 3 Magnification response factor spectrum for different values of damping

This frequency response curve shows that sinusoidal ground accelerations with different circular frequency have a different impact on the oscillation of the singular degree of freedom system. The same discussion is valid for different type of ground motions acting on a structure. Of course real earthquakes have a more complex frequency content than that of a sinusoidal wave, but their response can be found with a simple superposition of the responses at all the different frequencies of the earthquake. This procedure is explained hereafter.

A..3.2 Frequency domain superposition

This procedure consist on expressing the applied loading in terms of harmonic components, evaluating the response of the structure to each component, and then superposing the harmonic responses to obtain total structural response.

In order to do that it is necessary to introduce Fourier transforms which allow the transformation from time dependent variables to frequency dependent. In doing so, the Fourier series representation is in exponential and integral form:

$$p(t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} P(i\overline{\omega}_n) e^{i\overline{\omega}_n t}\, d\overline{\omega}_n$$

$$P(i\overline{\omega}_n) = \int_{-\infty}^{\infty} p(t)\, e^{-i\overline{\omega}_n t} dt$$

The first expression is the inverse Fourier transform instead the second is the direct one. With this latter equation the frequency content of the ground motion can be computed.
If ground motion is expressed, using the direct transform, in terms of individual harmonics the *n-th* harmonic steady state response of a singular degree of freedom system will be:

$$u_n = H_n\, P(i\overline{\omega}_n)\, e^{i\overline{\omega}_n t}$$

Using the exponential form for Fourier transforms also the frequency response coefficient H_n must be exponential. This is easily obtained expressing an unitary wave excitation and the steady state response in exponential form and then substituting them in the equations of motions:

$$\ddot{u}_g = \ddot{U}_g\, e^{i\overline{\omega} t}$$

$u = H\, e^{i\overline{\omega} t}$ $\quad \dot{u} = i\overline{\omega}\, H\, e^{i\overline{\omega} t}$ $\quad \ddot{u} = -\overline{\omega}^2\, H\, e^{i\overline{\omega} t}$

Substituting them in the equation of motion:

$$m\, \ddot{u}(t) + c\, \dot{u}(t) + k\, u(t) = -m\, \ddot{u}_g(t)$$

brings:

$$H(i\overline{\omega}) = \frac{m}{m\, \overline{\omega}^2 - i\, \overline{\omega}\, c - k}$$

Having the response for an individual harmonic it is now possible to use the principle of superposition to evaluate the total response of the singular degree of freedom system:

$$u(t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} H(i\overline{\omega}) P(i\overline{\omega}_n) e^{i\overline{\omega}_n t}\, d\overline{\omega}_n$$

A..3.3 Frequency domain analysis for multi degree of freedom systems

All that explained since now for a single degree of freedom system can be expanded to multi degree of freedom systems. The concepts are the same, the formulation will be matricial. The equations of motion for such systems are:

$$\boldsymbol{m}\, \ddot{\boldsymbol{u}}(t) + \boldsymbol{c}\, \dot{\boldsymbol{u}}(t) + \boldsymbol{k}\, \boldsymbol{u}(t) = -\boldsymbol{m}\, \boldsymbol{1}\, \ddot{u}_g(t)$$

where e is a vector made of zeros and ones that point out where the dynamic excitation is applied.
As done before it is now possible to define the steady state response and the dynamic load in the exponential form:

$$\ddot{u}_g = \ddot{U}_g\, e^{i\varpi t}$$

$$\boldsymbol{u} = \boldsymbol{H}\, e^{i\varpi t} \qquad \dot{\boldsymbol{u}} = i\varpi\, \boldsymbol{H}\, e^{i\varpi t} \qquad \ddot{\boldsymbol{u}} = -\varpi^2\, \boldsymbol{H}\, e^{i\varpi t}$$

Substituting in the equations of motion the matrix of the frequency response transfer function **H** is derived:

$$\boldsymbol{H}(i\varpi) = inv(\boldsymbol{m}\,\varpi^2 - i\,\varpi\,\boldsymbol{c} - \boldsymbol{k})\boldsymbol{m}$$

Each component of this matrix H_{ij} give the effect in the *i-th* degree of freedom due to a unit load applied in coordinate *j*.

The response of the structure can be obtained expressing the load in frequency content using the direct Fourier transform and then, using the indirect transform and the frequency response transfer function:

$$\boldsymbol{P}(i\varpi_n) = \int_{-\infty}^{\infty} -\boldsymbol{1}\,\ddot{u}_g(t) e^{-i\varpi_n t} dt$$

$$\boldsymbol{u}(t) = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \boldsymbol{H}(i\varpi)\,\boldsymbol{P}(i\varpi_n)\, e^{i\varpi_n t}\, d\varpi_n$$

The concepts here developed are usually not used to obtain the response of structures but are useful for the analysis of the stochastic response as will be explained in the next paragraphs.

A.4 Stochastic response of linear structures

This paragraph develops input-output relationships for linear systems and characterizes the output stochastic response in terms of stochastic input and their transfer function.

Firstly a brief review of the main probability function is reported in order to better understand the concept of power spectral density, developed in the second part.

A..4.1 <u>*Averages*</u>

Averages most commonly used in nondeterministic analysis are defined hereafter:

mean value

$$\mu_u = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)dt$$

mean square value

$$\psi_u^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T u^2(t)dt$$

variance

$$\sigma_u^2 = \lim_{T\to\infty} \frac{1}{T}\int_0^T [u(t) - \mu_u]^2 dt$$

standard deviation

$$\sigma_u = \sqrt{\sigma_u^2}$$

Properties of random data are mainly described by three different types of function: the probability density function, the autocorrelation function and the spectral power density.

A..4.2 *Probability density function*

Probability density function is defined as the quantity *p(u)* which represents the probability that the variable *u(t)* equals a value on the range between *u* and *u+Δu*. It is described with the following relation:

$$p(x) = \lim_{\Delta\bar{u}\to 0} \frac{prob(\bar{u} < u(t) < \bar{u} + \Delta\bar{u})}{\Delta\bar{u}}$$

In terms of probability density function the mean value μ_u and the mean square value ψ_u^2 are related with the following relationships:

$$\mu_u = \int_{-\infty}^{\infty} u\, p(u)\, du$$

$$\psi_u^2 = \int_{-\infty}^{\infty} u^2\, p(u)\, du$$

The most commonly used probability density function of a single random variable is the so called normal, or gaussian, distribution which is defined by the symmetric relation:

$$p(u) = \frac{1}{\sqrt{2\pi}\sigma_u} e^{-\frac{(u-\mu_u)^2}{2{\sigma_u}^2}}$$

A..4.3 *Autocorrelation function*

Autocorrelation function defines the dependency of one value of the random process at a defined time to the value of the same random process at the others time instants. For a given time history u(t) the autocorrelation between its values at time t and t+τ is:

$$R_u(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\, u(t+\tau)\, dt$$

Clearly this function has a maximum in τ=0, that is at the instant to which all values are referred to. Moreover it is a symmetrical function.

The relationships with the mean value μ_u and the mean square value ψ_u^2 follow:

$$\mu_u = \sqrt{R_u(\infty)}$$

$$\psi_u^2 = R_u(0)$$

In order to better understand the meaning of this function it is useful to consider it with respect with two different signals: a sinusoidal one mixed with a low random disturb and a casual narrow banded signal as shown in Figure A- 4 and Figure A- 5.

Figure A- 4 Sinusoidal signal with disturb

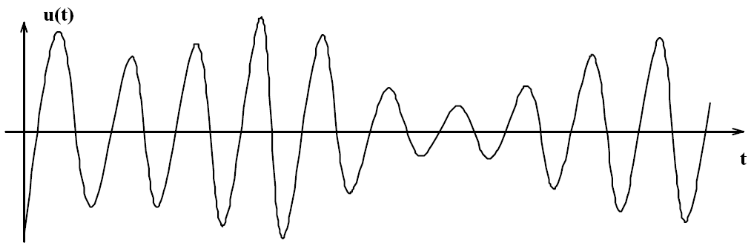

Figure A- 5 Narrow-banded signal

The relative autocorrelation functions follow.

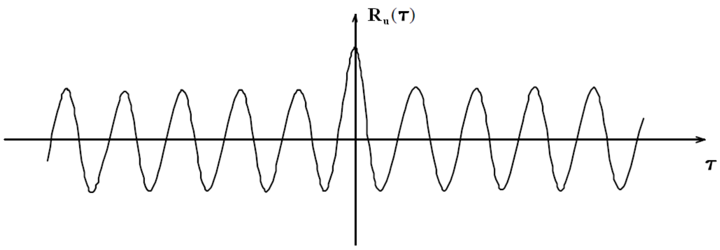

Figure A- 6 Autocorrelation function for sinusoidal signal with disturb

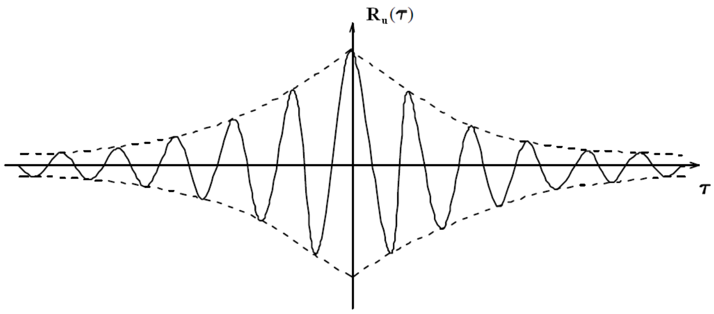

Figure A- 7 Autocorrelation function for narrow-banded signal

It can be noticed that a sinusoidal wave, like any other deterministic function, has an autocorrelation function which doesn't vanish with the time like happens for random data. The autocorrelation is useful to understand if exists a deterministic component in a signal.

A..4.4 Power spectral density function

The power spectral density function of random data describes the frequency content of the mean square value of this data. In the next paragraph the mathematical derivation and the properties of this function are reported.

It is known that any accelerogram can be separated in its frequency components using standard Fourier analysis. The duration of the ground motion the Fourier integral representation can be used:

$$u(t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} U(i\overline{\omega}) e^{i\overline{\omega}t}\, d\overline{\omega}$$

$$U(i\overline{\omega}) = \int_{-\infty}^{\infty} u(t)\, e^{-i\overline{\omega}t} dt$$

Usually the quantity of most interest in analyzing stationary random processes is the mean square value of u(t) over the interval s, which can be obtained by substituting the first of Fourier equations into the relation:

$$\psi_u^2 = \frac{1}{s} \int_{-s/2}^{s/2} u(t)^2\, dt$$

to obtain:

$$\psi_u^2 = \int_{-\infty}^{\infty} U(i\overline{\omega})^2\, d\overline{\omega}$$

Using the second of the Fourier equations it becomes:

$$\psi_u^2 = \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} u(t)\, e^{-i\overline{\omega}t} dt \right|^2 d\overline{\omega} = \int_{-\infty}^{\infty} S_u(\overline{\omega})\, d\overline{\omega}$$

where the function:

$$S_u(\overline{\omega}) = \left| \int_{-\infty}^{\infty} u(t)\, e^{-i\overline{\omega}t} dt \right|^2$$

is defined as power spectral density function for the random data u(t). This function yields the mean squared value of u(t) when integrated over the frequencies.

The power spectral density and the autocorrelation functions are related with the Fourier transforms as shows:

$$R_u(\tau) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} S_u(\bar{\omega})\, e^{i\bar{\omega}t} d\bar{\omega}$$

$$S_u(\bar{\omega}) = \int_{-\infty}^{\infty} R_u(\tau)\, e^{-i\bar{\omega}t} d\tau$$

Relationships can be found also with the mean value and the mean squared value:

$$\mu_u = \sqrt{\int_{0^-}^{0^+} S_u(\bar{\omega})\ d\bar{\omega}}$$

$$\psi_u^2 = \int_{0}^{\infty} S_u(\bar{\omega})\ d\bar{\omega}$$

In relation with the previous plotted signals, the correspondent power spectral density functions are presented hereafter:

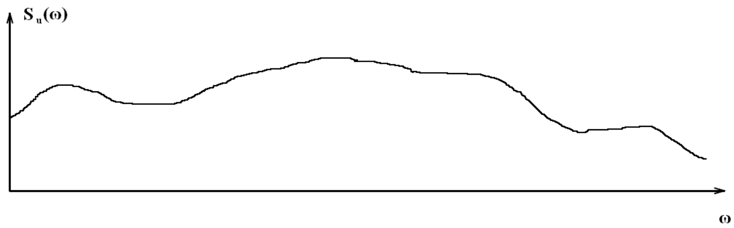

Figure A- 8 Power spectral density function of a sinusoidal signal with disturb

Figure A- 9 Power spectral density function for a narrow-banded signal

A..4.5 <u>Stochastic response for singular degree of freedom systems</u>

It is now possible, knowing all the functions already explained, to describe how to obtain stochastic output measures using the power spectral density of the chosen ground motion. In order to give a mathematical explanation on the relationship between the input and output statistic quantities autocorrelation function and convolution integral must be used.
The convolution integral is a way to obtain response through time domain superposing all the consecutive responses of a pure impulse input. It is also called Duhamel integral and

it is suitable only for linear analysis due to the superposition of effects. It can be written in the form:

$$u(t) = \int_{-\infty}^{\infty} h(\tau)\, p(t-\tau)\, d\tau$$

where *h(t)* is known as the unit impulse response function because it expresses the response of the SDOF system to a pure impulse of unit magnitude applied at time $t = \tau$.

The response obtained with this time domain approach can be related with the one obtained from the frequency domain approach explained previously using Fourier transforms. The frequency transfer function $H(i\bar{\omega})$ in fact is the Fourier direct transform of the unit impulse response function *h(t)* as shown:

$$H(\bar{\omega}) = \int_{-\infty}^{\infty} h(\tau)\ e^{-i\bar{\omega}t} d\tau$$

Consider now two different instants *t* and $t+\tau$, it is possible to determine their responses with the convolution integral in the following way:

$$u(t) = \int_{-\infty}^{\infty} h(\epsilon)\, p(t-\epsilon)\, d\epsilon$$

$$u(t+\tau) = \int_{-\infty}^{\infty} h(\mu)\, p((t+\tau)-\mu)\, d\mu$$

Their autocorrelation functions are given by:

$$u(t)\, u(t+\tau) = \int_{-\infty}^{\infty} h(\epsilon)\, h(\mu)\, p(t-\epsilon)\, p((t+\tau)-\mu)\, d\epsilon\, d\mu$$

that is:

$$R_u(\tau) = \int_{-\infty}^{\infty} h(\epsilon)\, h(\mu)\, R_p(\tau+\epsilon-\mu)\, d\epsilon\, d\mu$$

which relates the input R_p and output R_u autocorrelation functions.

Fourier transform can be applied to each of the functions that appear in the previous equation:

$$S(\bar{\omega}) = \int_{-\infty}^{\infty} R(\tau)\, e^{-i\bar{\omega}t} d\tau$$

$$H(\bar{\omega}) = \int_{-\infty}^{\infty} h(\tau)\ e^{-i\bar{\omega}t} d\tau$$

to obtain:

$$S_u(\bar{\omega}) = |H(i\bar{\omega})|^2 \, S_p(\bar{\omega})$$

This is the important relationship which relates input and output power spectral density.

A..4.6 Input power spectral density functions

In usual practice there are three main types of power spectral density functions: the Gaussian white noise, the Kanai-Tajimi and Clough-Penzien models.

Gaussian white noise is the most simple in that it assumes that power spectral density function is constant over frequency. Although it is not realistic this model is suitable for narrow banded systems, that is for systems having reasonably low damping ($\xi<0.1$). In this case in fact the area of the response power spectral density is concentrated near the natural frequencies of the system as shown in Figure A- 10. The most part of existing structural systems can be classified as narrow banded.

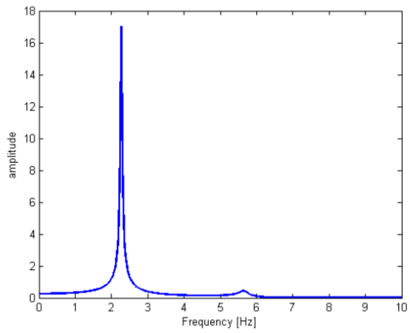

Figure A- 10 Frequency transfer for narrow-banded systems

Kanai-Tajimi model is obtained by passing a Gaussian white noise process through a single degree of freedom filter. In this way high frequencies contribution is partially removed yielding to a more realistic shape. The output power spectral density takes the form:

$$S_{KT}(\bar{\omega}) = |H(i\bar{\omega})|^2 \, S_p(\bar{\omega}) = \frac{1 + 4\xi^2 \left(\frac{\bar{\omega}}{\omega_n}\right)^2}{\left(1 - \left(\frac{\bar{\omega}}{\omega_n}\right)^2\right)^2 + 4\xi^2 \left(\frac{\bar{\omega}}{\omega_n}\right)^2} S_0$$

where S_0^2 is a measure of the intensity of the ground motion while ω_n and ξ influence the position of the peak of the power spectral density function with respect to frequencies.

As can be seen from Figure A- 11, Kanai-Tajimi function has nonzero value in the origin of the axis, which is not realistic.

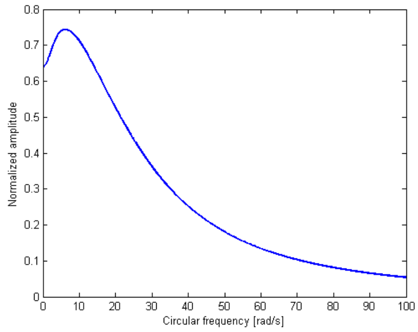

Figure A- 11 Kanai-Tajimi function

In order to avoid this problem Clough-Penzien model use two different filters, always derived from singular degree of freedom system transfer function. One of this filter removes the low frequency content of the Gaussian white noise so that the final function has zero values at the origin. Based on the above, Clough-Penzien spectrum function can be written as:

$$S_{CP}(\overline{\omega}) = \frac{1 + 4\xi^2 \left(\frac{\overline{\omega}}{\omega_n}\right)^2}{\left[\left(1 - \left(\frac{\overline{\omega}}{\omega_n}\right)^2\right)^2 + 4\xi^2 \left(\frac{\overline{\omega}}{\omega_n}\right)^2\right]\left[\left(1 - \left(\frac{\overline{\omega}}{\omega_g}\right)^2\right)^2 + 4\xi_g^2 \left(\frac{\overline{\omega}}{\omega_g}\right)^2\right]} S_0$$

where ω_g and ξ_g represents the natural frequency and the critical damping ratio of the second filter respectively. The plot of this function is shown in Figure A- 12.

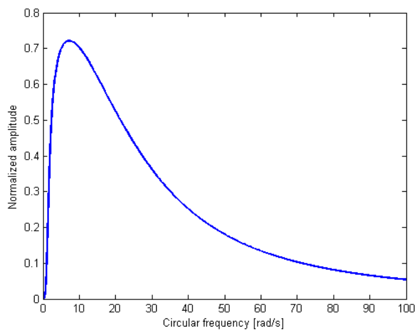

Figure A- 12 Clough-Penzien function

The parameters of these models can be computed by fitting the Fourier transform of design accelerogram. In order to obtain a significant stochastic representation of the phenomena several ground motions must be considered. A practical way to do it is using the fast Fourier transform. This function is computed for each accelerogram and then its mean value is considered. The fitting is based on this mean value.

A..4.7 <u>Stochastic response for multi degree of freedom systems</u>

The same procedure seen for singular degree of freedom systems can be carried on for multi degrees of freedom systems bringing at the same kind of results, but in matricial formulation. Hence the output power spectral density can be written as:

$$\boldsymbol{S_u}(\varpi) = |\boldsymbol{H}(i\varpi)|^2 \, \boldsymbol{1} \, S_p(\varpi)$$

where $\boldsymbol{S_u}(\varpi)$ is a vector which represents the power spectral response at each degree of freedom, $\boldsymbol{H}(i\varpi)$ is the frequency transfer function matrix and $S_p(\varpi)$ is the ground motion power spectral density. In this case a ones vector denotes that excitation works at all degrees of freedom.

$$\begin{bmatrix} S_{u_1}(\varpi) \\ S_{u_2}(\varpi) \\ \vdots \\ S_{u_n}(\varpi) \end{bmatrix} = \left\| \begin{bmatrix} H_{11}(i\varpi) & H_{21}(i\varpi) & \cdots & H_{n1}(i\varpi) \\ H_{12}(i\varpi) & \ddots & \ddots & H_{n2}(i\varpi) \\ \vdots & \ddots & \ddots & \vdots \\ H_{1n}(i\varpi) & H_{2n}(i\varpi) & \cdots & H_{nn}(i\varpi) \end{bmatrix} \right\|^2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} S_p(\varpi)$$

Once the output power spectral densities for each degree of freedom are known it is possible to find the mean square response of the structure by integrating each of them over the frequencies:

$$\boldsymbol{\psi_u^2} = \int_0^\infty \boldsymbol{S_u}(\varpi) \, d\varpi$$

Moreover if one is interested in finding power spectral density of other quantities, like interstorey drifts or velocities, related with the structural response by the relationship:

$$\boldsymbol{z} = \boldsymbol{T} \, \boldsymbol{u}$$

the spectral density for vector **z** is given by:

$$\boldsymbol{S_z}(\varpi) = \boldsymbol{T}|\boldsymbol{H}(i\varpi)|^2 \, \boldsymbol{1} \, S_p(\varpi) \, \boldsymbol{T}^T$$

A..4.8 *State space notation*

In order to use optimization tools for linear systems the formulation of the equations of motion has to be modified in the so called *state space* notation, i.e. in first order differential equation. This reformulation is done by introducing an additional unknown vector $\boldsymbol{z}(t)$:

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{u}(t) \\ \dot{\boldsymbol{u}}(t) \end{bmatrix}$$

It is now possible to write in matrix formulation the system composed by the equations of motion and the velocity identity:

$$\dot{\boldsymbol{u}}(t) = \dot{\boldsymbol{u}}(t)$$
$$\ddot{\boldsymbol{u}}(t) = -\boldsymbol{m}^{-1}\boldsymbol{c}\,\dot{\boldsymbol{u}}(t) - \boldsymbol{m}^{-1}\boldsymbol{k}\,\boldsymbol{u}(t) - \boldsymbol{1}\ddot{u}_g(t)$$

bringing to the compact form:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}\,\boldsymbol{z}(t) + \boldsymbol{H}\,\ddot{u}_g(t)$$

where:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ -\boldsymbol{m}^{-1}\boldsymbol{k} & -\boldsymbol{m}^{-1}\boldsymbol{c} \end{bmatrix} \qquad \boldsymbol{H} = \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{1} \end{bmatrix}$$

All the quantities related with the response, such as acceleration and forces, can be derived by $\boldsymbol{z}(t)$ using proper matrices. In the case of inter-story drifts for example the following relationship is used:

$$\boldsymbol{\delta}(t) = \boldsymbol{D}\,\boldsymbol{z}(t)$$

where:

$$\boldsymbol{D} = [\boldsymbol{T}^{-1} \quad \boldsymbol{0}] \text{ and } \quad \boldsymbol{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A..4.9 Lyapunov equations

In case of a first order differential equation system written as:

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{A}\,\boldsymbol{z}(t) + \boldsymbol{H}\,\ddot{u}_g(t)$$

the response $\boldsymbol{z}(t)$ is given by:

$$\boldsymbol{z}(t) = \boldsymbol{B}e^{\boldsymbol{A}t}$$

The stochastic response for a random load is described by the covariance matrix $\boldsymbol{Q}(t)$ and the mean value response $\bar{\boldsymbol{z}}(t)$. Since the mean value of input ground motion can be considered equal zero also the mean value of the structural response can be neglected. The covariance matrix is defined as:

$$\boldsymbol{Q}(t) = \int_0^\infty \boldsymbol{z}^{\boldsymbol{T}}(t)\;\boldsymbol{z}(t)dt = \int_0^\infty e^{\boldsymbol{A}^{\boldsymbol{T}}t}\,\boldsymbol{B}^{\boldsymbol{T}}\,\boldsymbol{B}\,e^{\boldsymbol{A}t}dt$$

It can be demonstrated (see Robust and Optimal Control by Kemin Zhou et al.) that for white noise input the covariance response $\boldsymbol{Q}(t) = E(\boldsymbol{z}\cdot\boldsymbol{z}^{\boldsymbol{T}})$ becomes constant and its values can be derived from Lyapunov's equation.

$$\boldsymbol{A}^T\boldsymbol{Q} + \boldsymbol{Q}\boldsymbol{A} + \boldsymbol{B}\boldsymbol{W}\boldsymbol{B}^{\boldsymbol{T}} = \boldsymbol{0}$$

In order to evaluate the response in terms of inter-story drifts $\boldsymbol{Q}_\delta = E(\boldsymbol{\delta}\cdot\boldsymbol{\delta}^{\boldsymbol{T}})$ the following transformation can be applied:

$$\boldsymbol{Q}_\delta = \boldsymbol{D}\boldsymbol{Q}_z\boldsymbol{D}^{\boldsymbol{T}}$$

The values on the diagonal of matrix $\boldsymbol{Q}_\delta$ represent the mean squared of the inter-story drifts. These values are taken as control values since they control both the achievement of the objective function and the updating of the damping matrix, through the performance index described hereafter.