

POLITECNICO DI MILANO

4<sup>th</sup> School of Engineering

Course of Mechanical Engineering



**A comprehensive method for the choice  
of electrical motor and transmission in  
mechatronic applications**

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# Abstract

The thesis deals with the proper choice of brushless electrical motor and transmission. The choice is based on  $J_M$ - $M_{m,rms}$  and  $J_M$ - $M_{m,max}$  curves, that are obtained from the “reference task” for giving transmission.

The method permits the designer to link the electrical motor with transmission in order to guarantee that the motor’s dynamic working range and continuous duty working range is suitable for the driven load. Particularly, the thesis takes into account the direct and inverse efficiencies of the transmission and its inertia.

In addition, the analytic expressions and figures, with the help of which the choices of electrical motors will be achieved, will be analyzed.

Finally, an algorithm is proposed to elaborate the choice of the proper electrical motor. The algorithm is based upon analytical considerations and has been developed to minimize its calculation time.

**Keywords** Coupling of the motor and transmission; direct and inverse efficiencies; transmission inertia; brushless electrical motor



# Sommario

La tesi si occupa della scelta corretta di un motore elettrico brushless e relativa trasmissione. La scelta è basata su curve  $J_M-M_{m, rms}$  e  $J_M-M_{m, max}$ , che vengono ottenute dalla "missione di riferimento", per data trasmissione.

Il metodo permette al progettista di collegare il motore elettrico con la trasmissione al fine di garantire il campo di lavoro dinamico del motore. E il campo di lavoro servizio continuo adeguati alla macchina operatrice. In particolare, la ricerca tiene in conto i rendimenti diretto e inverso della trasmissione e la sua inerzia.

Inoltre, saranno analizzati le espressioni analitiche e le figure, con l'aiuto delle quali la scelta dei motori elettrici sarà effettuata.

Infine, un algoritmo si propone di elaborare la scelta corretta del motore elettrico. L'algoritmo è basato su considerazioni analitiche ed è stato sviluppato per ridurre al minimo il tempo di calcolo.

**Parole chiave** Accoppiamento motore-trasmissione; Rendimento diretto e inverso; Inerzia della trasmissione; motore elettrico brushless

# Chapter 1: Introduction

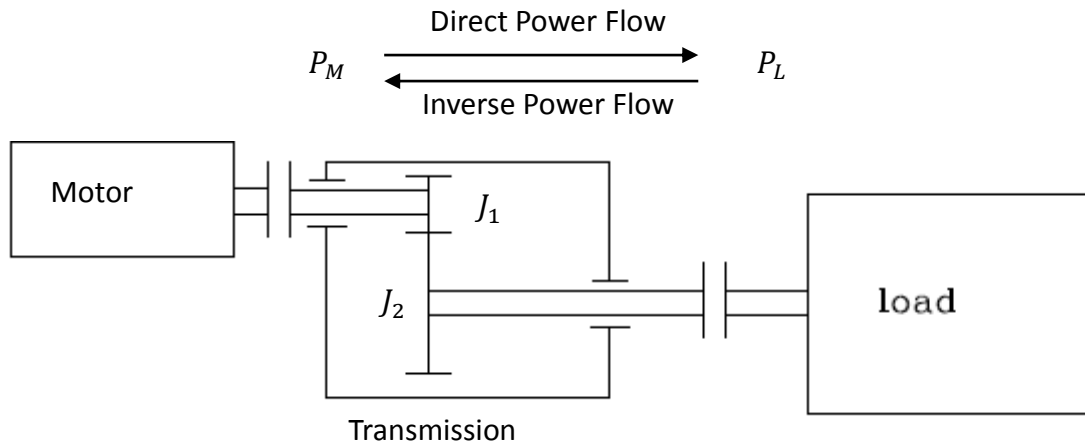
Symbol	Description
$M_m$	Motor torque
$M_{m,max}$	Maximum torque exerted by the motor
$M_{m,rms}$	Root mean square torque of the motor
$J_M$	Moment of inertia of the motor
$M_{M,rated}$	Motor rated torque
$M_{M,S1}$	Limit torque of the continuous duty range
$M_{M,dyn}$	Limit torque of the dynamic range
$\omega_m$	Motor angular speed
$\alpha_m$	Motor angular acceleration
$M_L$	Load torque
$J_L$	Moment of inertia of the load
$M_L^*$	Generalized load torque
$M_{L,rms}^*$	Generalized root mean square torque of the load
$M_{L,max}$	Load maximum torque
$\omega_L$	Load angular speed
$\alpha_L$	Load angular acceleration
$\alpha_{L,rms}$	Root mean square acceleration of the load
$T$	Cycle time
$\tau$	Transmission ratio
$\eta_d$	Direct transmission efficiency
$\eta_i$	Inverse transmission efficiency
$J_1 \quad J_2$	Moment of inertia of the transmission
$J_T$	Generalized moment of inertia of the transmission
$\omega_{m,max}$	Maximum speed achievable by the motor
$\omega_{L,max}$	Maximum speed achieved by the load
$P_M$	Power on motor side
$P_L$	Power on load side

**Table 1.1** Nomenclature

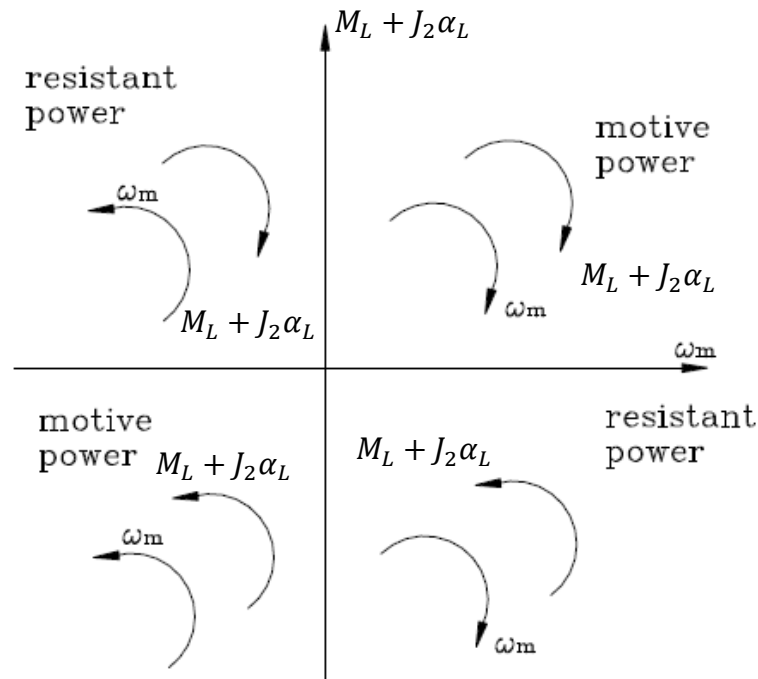
In servo-actuated machines, the choice of the electrical motor and transmission is usually related to a dynamic load. The difficulty of matching them depends on several constraints due to the continuous duty working range and the dynamic working range of the motor, the transmission ratio  $\tau$ , mechanical efficiency  $\eta$  and moment of inertia  $J_T$ .

In previous researches, the transmission is approximated to an ideal system in which power losses are neglected or the direct and inverse efficiencies are taken into account individually. In this paper a methodology for choosing the motor and

transmission which considers both direct and inverse efficiencies and the transmission inertia at the same time, is introduced.



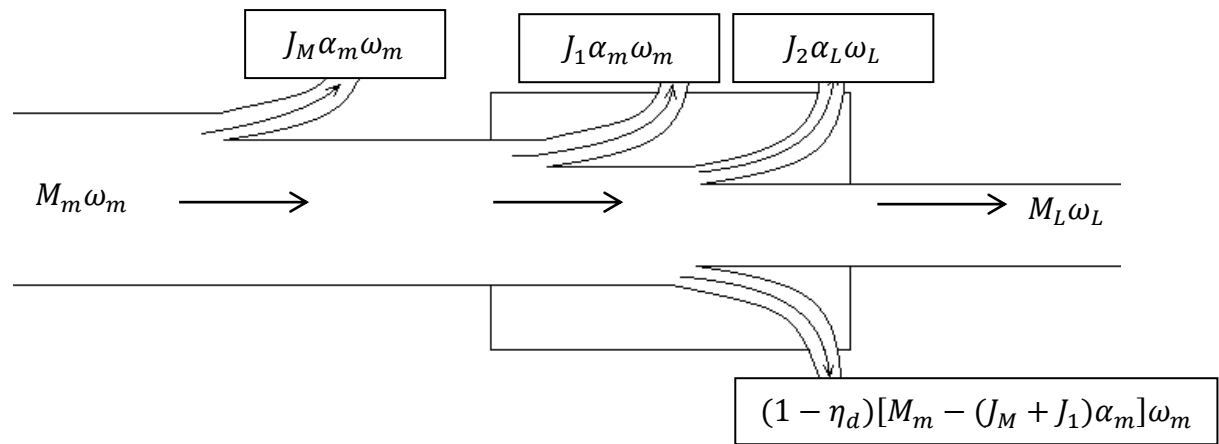
**Fig. 1.1** Scheme of the actuation part of the machine



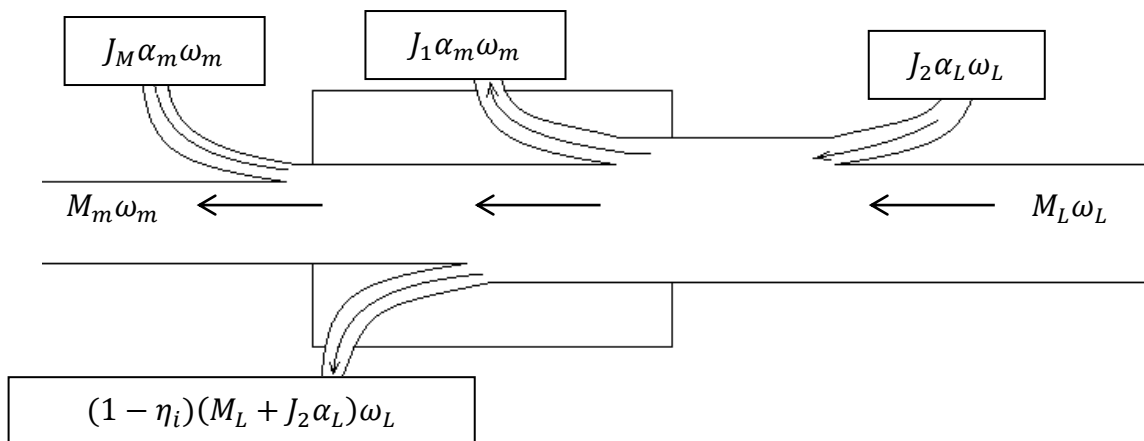
**Fig. 1.2** Sign conventions regarding the load

In this paper it is assumed that the transmission has two inertias:  $J_1$  on motor side and  $J_2$  on load side. In **Fig. 1.1**, if  $M_L + J_2 \alpha_L$  and  $\omega_m$  have the same sign, the load introduces motive power to the motor; if they have opposite signs, the load introduces resistant power to the motor.

And the corresponding power flows are shown in **Fig. 1.3** and **Fig. 1.4**.



**Fig. 1.3** Direct power flow



**Fig. 1.4** Inverse power flow

In **Fig. 1.3** and **Fig. 1.4** the power flow loss due to the direct and inverse efficiency is shown. In order to choose a brushless motor, apart from transmission ratio, we should take into account both direct and inverse efficiencies. In the second and fourth quadrant, the efficiency is direct, while in the first and third quadrant, the efficiency is inverse. That is, when  $M_L$  and  $\omega_L$  have the same sign, the efficiency is direct; but when they have opposite signs, the efficiency is inverse.

In this paper, we will apply a new mathematic method to find out the right motor for a specific load.

In chapter 2, some basic knowledge, which will be very useful, are introduced, such as brushless motor, harmonic drive transmission, conic section.

In chapter 3, a method which takes into account both direct and inverse efficiencies and the inertia of transmission is proposed.

In chapter 4, such a method is applied to continuous duty operating range.

In chapter 5, this method is applied to dynamic operating range.

In chapter 6, a simple example which applies this method to choose the appropriate transmission and motor is presented.

In chapter 7, a conclusion will be given.

# Chapter 2: Drive system characterization

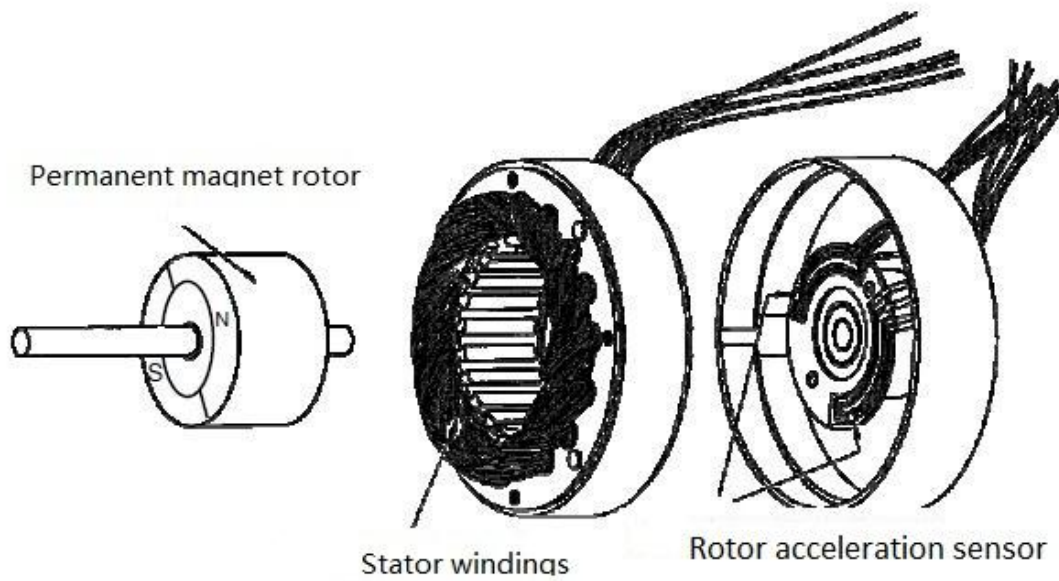
## 2.1 AC Brushless Motor

### 2.1.1 Construction of a AC brushless motor

The brushless motor here is assembled with permanent magnets, and it is an electric motor driven by an alternating current. The brushless motors are based on magnetic rotating field.

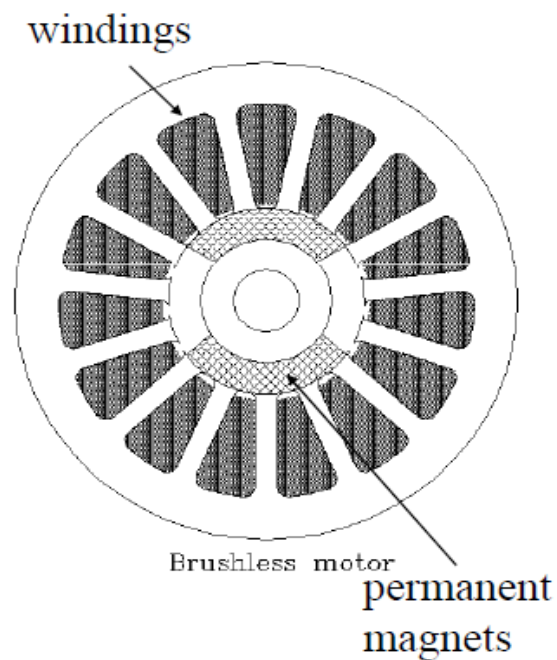
A brushless motor having permanent magnets that can be used as a prime mover for automobiles, in place of internal combustion engines, since the motor can yield high torque during low speed rotation, as in the case of conventional types of brushless motors and can be used at high torque and with excellent motor efficiency at rotations three times as high as that of conventional types.

The brushless motor having permanent magnets according to the invention comprises a stator having a plurality of stator magnetic poles and a winding for generating a rotating field in the stator magnetic poles, a rotor having a rotating shaft and field permanent magnets rotating with respect to the stator magnetic poles, a control circuit for detecting the position of magnetic poles of the field permanent magnet with respect to the stator and feeding current to the winding in accordance with the position; where in the field permanent magnets comprise a first field permanent magnet having magnetic poles of different polarities alternately arranged in the direction of rotation, and a second field permanent magnet that is adapted to be rotatable with respect to the first field permanent magnet and has magnetic poles of different polarities alternately arranged in the direction of rotation; the first and second field permanent magnets facing the stator magnetic poles, and a mechanism for changing the phase of the synthesized magnetic poles of the first and second field permanent magnets in accordance with the rotation of the rotor is provided. (see **Fig. 2.1** and **Fig. 2.2**)



**Fig. 2.1** An overall construction of a AC brushless motor

**Fig. 2.1** shows the section structure of a brushless motor. We can see that the windings are in the stator. The permanent magnets are in the rotor. A position transducer allows us to know the position  $\theta_m$  of the rotor, i.e. of the magnetic field.



**Fig. 2.2** Section of AC brushless motor

### 2.1.2 Alternating Current (AC) Motor

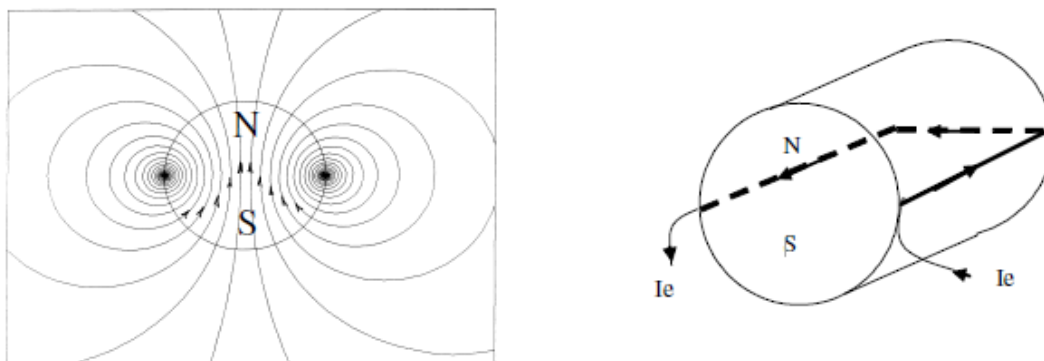
There is no brush or sliding contact in the AC brushless motors. Heat is generated in the stator winding and the thermal resistance, to transmit heat to the environment, is very small. So they have advantages compared to the DC motors:

- Higher current limit due to the commutator contact.
- Higher maximum speed than DC machine
- Light rotor and consequently low inertia. The brushless motor has great response quickness.
- High power and torque density.
- Less maintenance problem and sparking problems due to no commutator.

The AC motors are based on magnetic rotating field.

### 2.1.3 The basic electro-magnet

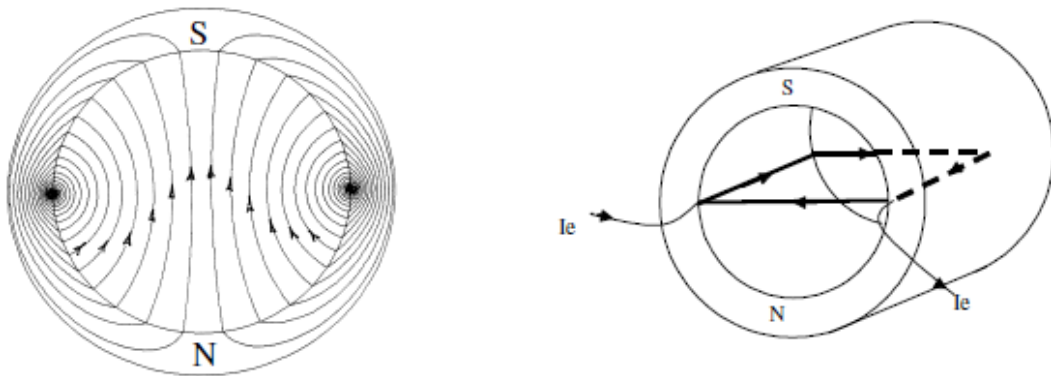
The magnetic permeability of an iron cylinder (diameter  $D$ , length  $l$ ) is much larger than that of the air; an  $N$  turn winding is placed on diametrical plane; a constant current  $I_e$  flows into the winding. This device acts like a permanent magnet with cylindrical shape. The north pole is the surface from which the magnetic field lines come out, and the south pole is the surface in which the magnetic field lines enter (shown in **Fig. 2.3**).



**Fig. 2.3** A ferromagnetic cylinder surrounded

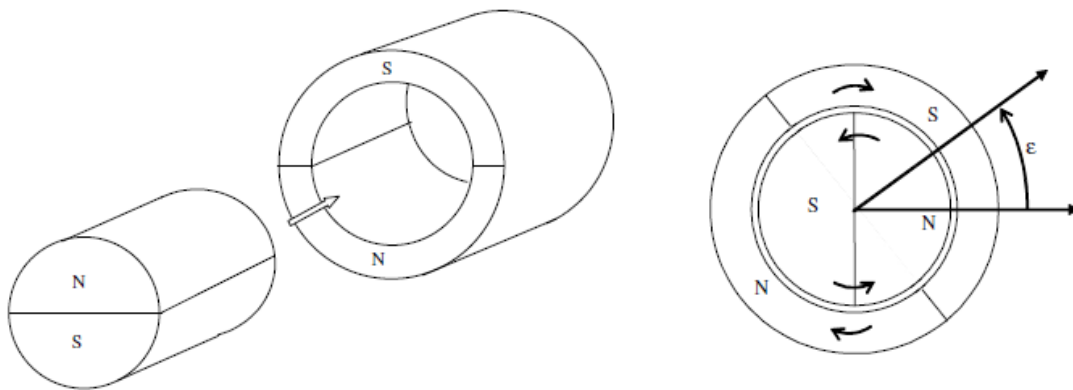
An  $N$  turn winding is placed on the diametrical plane of a hollow cylinder (stator); a constant current flows into the winding; this device acts as a permanent magnet (see **Fig. 2.4**).





**Fig. 2.4** A hollow cylinder (stator)

### 2.1.4 Electro-magnetic joint: the alignment law

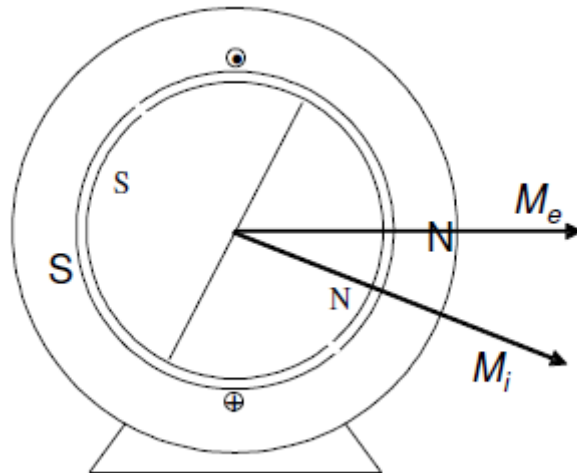


**Fig. 2.5** Electro-magnetic joint

In **Fig. 2.5**, if the two basic permanent-magnets or electromagnets are put one inside the other, an aligning torque arises.

The torque depends on the value of the angle as a sine function:  $T = -K \sin \varepsilon$ . And the torque acts so as to align the opposite magnetic polarities.

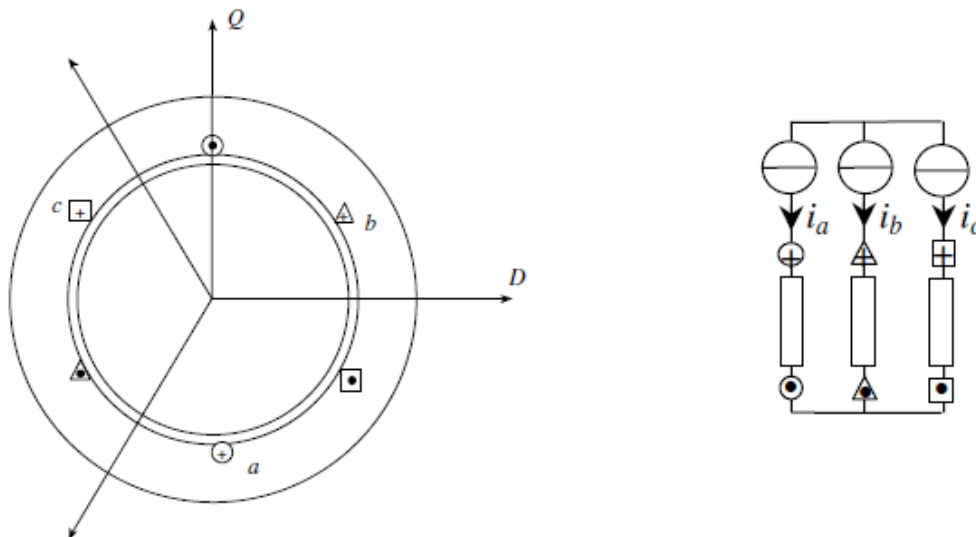
From joint to motor, if the magnetic polarities of the external magnet could rotate around the cylinder axis, this would make the alignment torque drag the inner permanent magnet thus providing a rotation. The rotation of the polarities of the external magnets can be achieved by using an external three-phase winding and placing a permanent magnet on the rotor. And the magnetic field produced by magnets or electro-magnets of cylindrical shape can be represented by vector oriented in the direction of the field line. As shown in **Fig. 2.6**, the vectors are called space phasors, and  $M_e$ ,  $M_i$  are the magneto motive force space phasors of external and internal magnets.



**Fig. 2.6** Electro-magnetic joint in motor

### 2.1.5 The three-phase windings

Three windings  $a$ ,  $b$  and  $c$  are displaced by  $120^\circ$  one from each other; three ideal current generators feed the windings with balanced AC three-phase currents  $i_a$ ,  $i_b$  and  $i_c$ ; the resulting magnetic field is the sum of the field produced by the single windings. Then a rotating magnetic field is obtained (shown in **Fig. 2.7**).



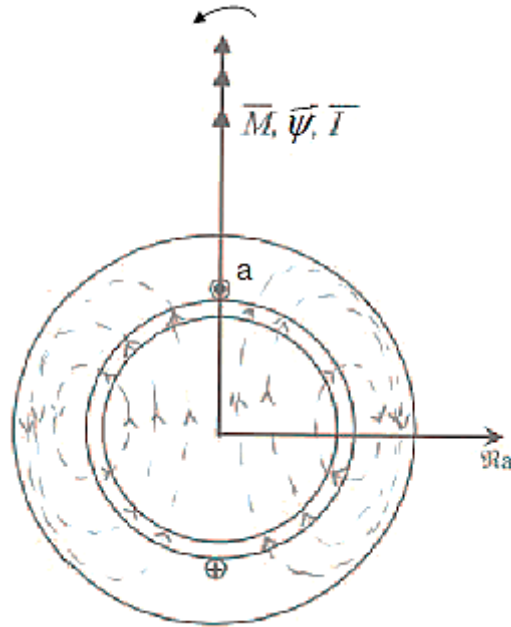
**Fig. 2.7** Views of the three-phase windings

### 2.1.6 The voltage vector model

The rotating magnetic motive force vector (space phasor) is:

$$\underline{f} = \frac{3}{2} N I_M \cdot e^{j\omega t} \quad (2.1)$$

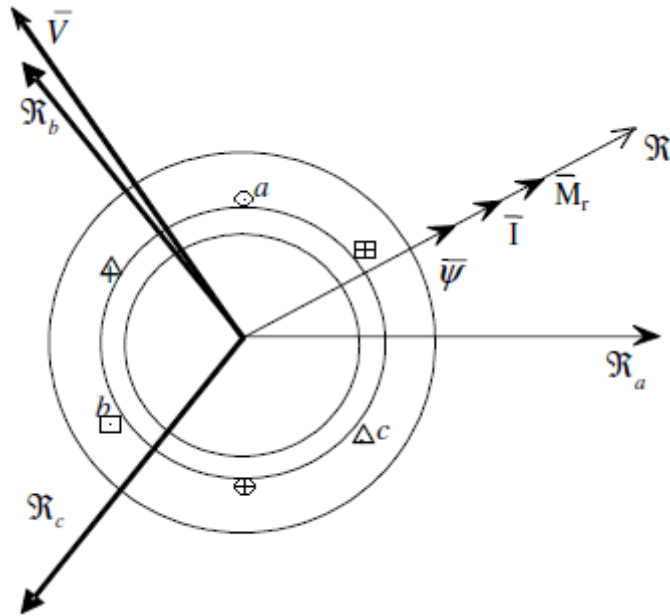
Also the balanced three-phase current system can be represented by a rotating current space phasor  $\underline{I}$ , and the three-phase flux linkage can be represented by a rotating space phasor  $\underline{\Psi}$  too (see **Fig. 2.8**).



**Fig. 2.8** Representation of space phasors

Then we know that the voltage can be represented by a space phasor, after project the rotating space phasor on each phase axes we can obtain the instantaneous value of phase variable (shown in **Fig. 2.9**), and the expressions are:

$$\begin{cases} v_a = R_a i_a + \frac{d\psi_a}{dt} \\ v_b = R_b i_b + \frac{d\psi_b}{dt} \\ v_c = R_c i_c + \frac{d\psi_c}{dt} \end{cases} \quad \longrightarrow \quad \underline{v}_s = R_s \underline{i}_s + \frac{d\underline{\psi}_s}{dt} \quad (2.2)$$

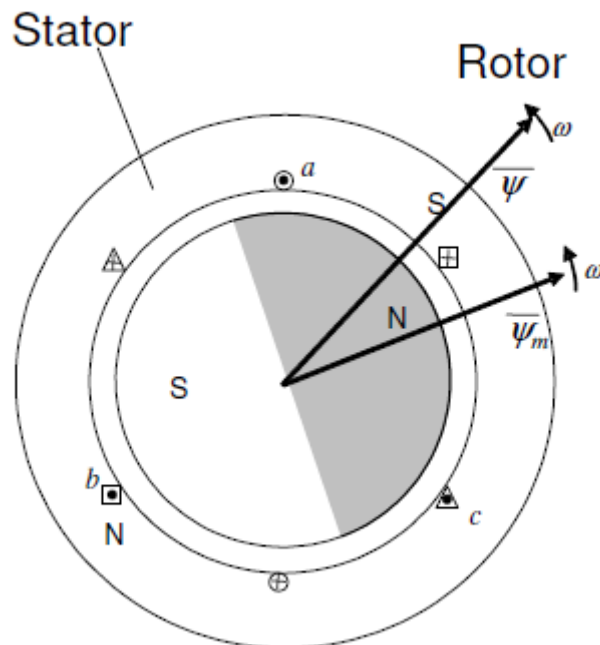


**Fig. 2.9** The voltage space phasor

The flux and current are related by a constant: the synchronous (self) inductance  $L_s$  , and we have:

$$\underline{\Psi} = L_s \underline{i}_s \quad (2.3)$$

### 2.1.7 AC Permanent Magnet Synchronous Motor (AC brushless motor)



**Fig. 2.10** Two fields of AC brushless motor

As shown in **Fig. 2.10**, the stator rotating field (produced by the three-phase stator windings) produces an aligning torque that interacts with the rotor permanent magnet field, and the two fields run synchronously, i.e. the rotor's angular speed is equal to the stator's rotating field.

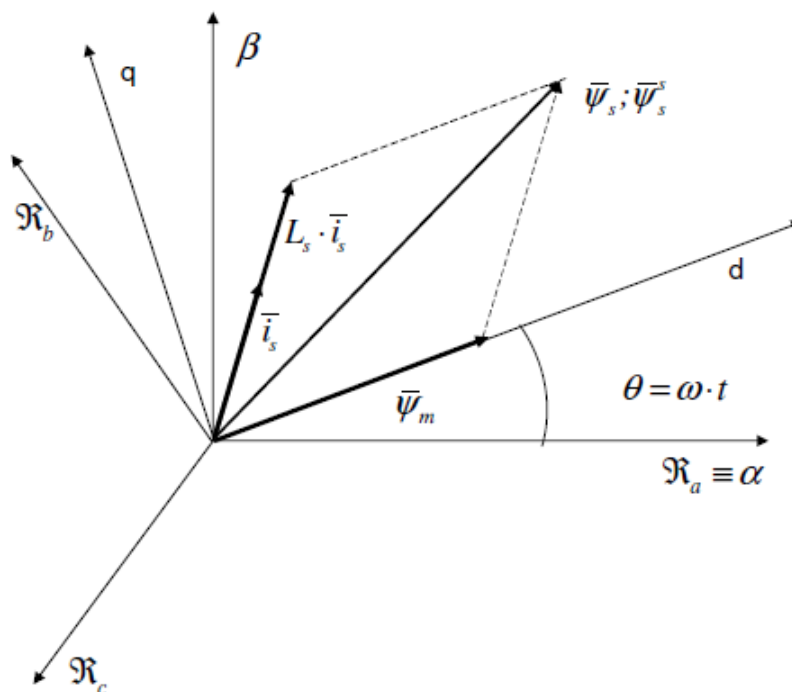
As we mentioned in **2.1.6**, the voltages depend on the total flux  $\underline{\Psi}_s$  produced by the stator (rotating field) and by the rotor's permanent magnet. Then we project the space phasors on a fixed axes reference frame  $\alpha$ - $\beta$  or on a synchronous rotating frame  $d$ - $q$ , that is:

$$\begin{cases} \underline{v}_s^s = \underline{v}_s e^{i\omega t} \\ \underline{i}_s^s = \underline{i}_s e^{i\omega t} \\ \underline{\Psi}_s^s = \underline{\Psi}_s e^{i\omega t} \end{cases} \quad (2.4)$$

A rotating reference frame  $d$ - $q$  is chosen having axes that are synchronous with the rotor's permanent magnet flux space phasor  $\underline{\Psi}_m$ ,

$$\underline{\Psi}_s^s = \underline{\Psi}_s \cdot e^{j\omega} \quad (2.5)$$

Where  $\underline{\Psi}_s = \underline{\Psi}_m + L_s \underline{i}_s$ , and it's the total flux referred to a fixed reference frame  $\alpha$ - $\beta$ ;  $\underline{\Psi}_s^s$  is the total flux referred to a rotating reference frame  $d$ - $q$  (shown in **Fig. 2.11**).



**Fig. 2.11** The total flux vector projected on two frames

The total flux vector (referred to a fixed frame  $\alpha$ - $\beta$ ) is:

$$\underline{\Psi}_s^s = \underline{\Psi}_m^s + L_s \underline{i}_s^s \quad (2.6)$$

The total flux vector (referred to a rotating frame  $d$ - $q$ ) is:

$$\underline{\Psi}_s e^{i\omega t} = \underline{\Psi}_m e^{i\omega t} + L_s \underline{i}_s e^{i\omega t} \quad (2.7)$$

Taking the total time derivative:

$$\frac{d\underline{\Psi}_s e^{i\omega t}}{dt} = i\omega \underline{\Psi}_m e^{i\omega t} + L_s \frac{d\underline{i}_s}{dt} e^{i\omega t} + i\omega L_s \underline{i}_s e^{i\omega t} \quad (2.8)$$

Thus

$$\underline{v}_s e^{i\omega t} = R_s \underline{i}_s e^{i\omega t} + \frac{d\underline{\Psi}_s}{dt} e^{i\omega t} = \left( R_s \underline{i}_s + i\omega \underline{\Psi}_m + L_s \frac{d\underline{i}_s}{dt} + i\omega L_s \underline{i}_s \right) e^{i\omega t} \quad (2.9)$$

Simplifying the  $e^{i\omega t}$  term we obtain:

$$\underline{v}_s = R_s \underline{i}_s + i\omega \underline{\Psi}_m + L_s \frac{d\underline{i}_s}{dt} + i\omega L_s \underline{i}_s \quad (2.10)$$

Then it is possible to project the vectors along the reference axes  $d$  (real) and  $q$  (imaginary):

$$\underline{\Psi}_m = \Psi_m \quad (2.11)$$

$$\underline{i}_s = i_{sd} + i \cdot i_{sq} \quad (2.12)$$

The following two-axis model is therefore obtained:

$$\begin{cases} v_{sd} = R_s i_{sd} + L_s \frac{di_{sd}}{dt} - \omega L_s i_{sq} \\ v_{sq} = R_s i_{sq} + L_s \frac{di_{sq}}{dt} + \omega (L_s i_{sd} + \Psi_m) \end{cases} \quad (2.13)$$

Because the driving torque  $M_m$  can be obtained from an energy balance, the mechanical equation has to be considered to complete the model:

$$J\dot{\omega} = M_m - M_L \quad (2.14)$$

The power entering the motor is equal to:

$$W = \underline{v}_s * \underline{i}_s = v_{sd} i_{sd} + v_{sq} i_{sq} \quad (2.15)$$

Substituting the expressions of  $v_{sd}$  and  $v_{sq}$  we obtain

$$\begin{aligned} W &= R_s i_{sd}^2 + L_s \frac{di_{sd}}{dt} i_{sd} - \omega L_s i_{sq} i_{sd} + R_s i_{sq}^2 + L_s \frac{di_{sq}}{dt} i_{sq} + \omega (L_s i_{sd} + \Psi_m) i_{sq} \\ &= R_s i_{sd}^2 + L_s \frac{di_{sd}}{dt} i_{sd} + R_s i_{sq}^2 + L_s \frac{di_{sq}}{dt} i_{sq} + \omega \Psi_m i_{sq} \\ &= R_s (i_{sd}^2 + i_{sq}^2) + L_s \left( \frac{di_{sd}}{dt} i_{sd} + \frac{di_{sq}}{dt} i_{sq} \right) + \omega \Psi_m i_{sq} \end{aligned} \quad (2.16)$$

In Eq. (2.16) we can see that the power entering the motor is made up of three parts: The power losses in the windings  $R_s(i_{sd}^2 + i_{sq}^2)$ ; the power of magnetic energy variation  $L_s \left( \frac{di_{sd}}{dt} i_{sd} + \frac{di_{sq}}{dt} i_{sq} \right)$  and the mechanical power exiting the motor  $\omega \Psi_m i_{sq}$ . The mechanical power exiting the motor is therefore equal to:

$$W_m = M_m \cdot \omega = \omega \Psi_m i_{sq} \quad (2.17)$$

Thus, the electro-magnetic torque is:

$$M_m = \Psi_m i_{sq} \quad (2.18)$$

If windings with  $N$  pole pairs are used, the rotor angular speed  $\omega_m$  is different from the current frequency  $\omega_{el}$  by a factor  $N$ . Also the torque varies (increases) by a factor  $N$ :

$$\omega_m = \frac{\omega_{el}}{N} = \frac{\omega}{N} \quad (2.19)$$

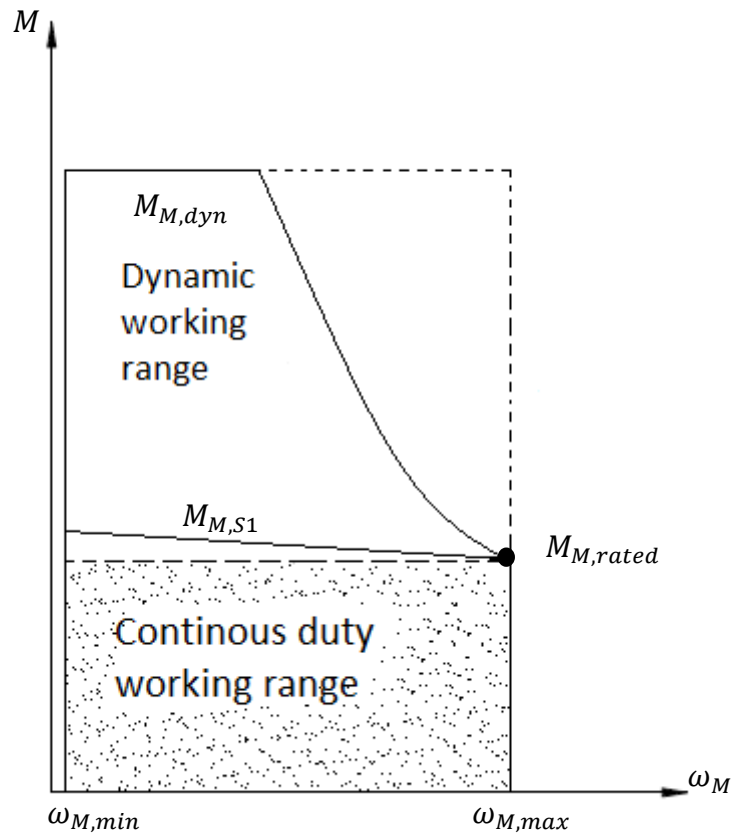
$$M_m = N \Psi_m i_{sq} \quad (2.20)$$

So the complete model is:

$$\begin{cases} v_{sd} = R_s i_{sd} + L_s \frac{di_{sd}}{dt} - \omega L_s i_{sq} \\ v_{sq} = R_s i_{sq} + L_s \frac{di_{sq}}{dt} + \omega (L_s i_{sd} + \Psi_m) \\ J \omega_m = J \frac{\dot{\omega}}{N} = N \Psi_m i_{sq} - M_L \end{cases} \quad (2.21)$$

### 2.1.8 The working range of AC brushless motor

The working range of a brushless motor can reach higher speeds and torques and is nearly rectangular.



**Fig. 2.12** Working range of brushless motor

As shown in the **Fig. 2.12** the working range can be approximately subdivided into a continuous working zone (delimited by the motor rated torque  $M_{M,S1}$ ) and a dynamic zone (delimited by the maximum motor torque  $M_{M,dyn}$ ). Usually the motor rated torque decreases slowly with the motor speed  $\omega_m$ . In this paper, it is considered constant and equal to  $M_{M,S1}$ .

The nominal motor torque  $M_{M,rated}$  is usually specified by the manufacturer in the catalogues. The  $M_{M,S1}$  is defined as the torque that can be supplied by the motor for an infinite time without overheating. The trend of maximum torque of the dynamic working range  $M_{M,dyn}$  is very complex. Because of the sparkles in the commutator it will go down when the speed is very high near the  $\omega_{M,max}$ . The trend of  $M_{M,dyn}$  also depends on many other factors and it is difficult to express it with an equation. So in this paper we assume it is constant and equal to  $M_{M,dyn}$ .

Here we discuss the constraints of the brushless motor.

As the motor torque is proportional to the current, the wasted power is mainly attributable to the Joule effect and it is supposed that the machine cycle has a period much shorter than the motor thermal constant. The thermal behavior of the motor can be analyzed with the root mean square torque  $M_{m,rms}$ .



The root mean square torque generates the same energetic dissipation which is actually present in the cycle.

$$M_{m,rms}(\omega) = \sqrt{\frac{1}{T} \int_0^T M_m^2(t) dt} \quad (2.22)$$

In the operation the maximum torque  $M_{m,max}$  is:

$$M_{m,max} = \max |M_m(t)| \quad \text{Where } 0 \leq t \leq T \quad (2.23)$$

The following set is the condition that must be satisfied by the motor:

$$\begin{cases} M_{m,max} \leq M_{M,dyn} \\ M_{m,rms} \leq M_{M,S1} \\ \omega_{m,max} \leq \omega_{M,max} \end{cases} \quad (2.24)$$

Because  $M_{m,rms}$  and  $M_{m,max}$  are univocal functions of  $J_M$ . Therefore we can plot the corresponding  $M_{m,rms}$  and  $M_{m,max}$  diagrams.

## 2.2 Transmission

The mechanical power is the product of a torque for a speed. Generally speaking, it is easier to produce mechanical power with small torques at high speeds; the transmission performs the task of changing the distribution of power, adjusting the optimal conditions for its production to the ones for its optimum use. This work done by the transmission usually involves reducing speed while increasing the available torque.

The transmission depends on the transmission ratio  $\tau$  and on the transmission efficiency  $\eta$  and the inertia  $J_T$ . if the transmission is an ideal one, the efficiency  $\eta$  would be 1 and the inertia  $J_T$  equals to 0.

A realistic model of the transmission has to consider the inevitable loss of power. The power dissipated affects the resulting performance of the machine and the choosing of correct motor.

It is defined that the transmission ratio  $\tau$  :

$$\tau = \frac{\omega_{out}}{\omega_{in}} \quad (2.25)$$

The relationship between the input acceleration and the output is shown below:

$$\alpha_{in} = \dot{\omega}_{in} = \frac{\dot{\omega}_{out}}{\tau} = \frac{\alpha_{out}}{\tau} \quad (2.26)$$

From the definition of the efficiency  $\eta$ , we can easily obtain:

$$\eta = \frac{P_{out}}{P_{in}} = \frac{M_{out} \omega_{out}}{M_{in} \omega_{in}} = \frac{M_{out}}{M_{in}} \tau \leq 1$$

Where the  $M_{out}$  and  $M_{in}$  represent output and input torque of the transmission and the  $\omega_{out}$  and the  $\omega_{in}$  are the corresponding angular speeds.

In a more realistic model, i.e. at the situation:  $\eta \neq 1$  as shown in Eq. (2.27) and Eq. (2.28), the power flow is sometimes from the motor to the load which is said to work with direct power flow, otherwise from the load to the motor which is said to work with inverse power flow. The transmission power losses are described by two different mechanical efficiency values  $\eta_i$  and  $\eta_d$ .

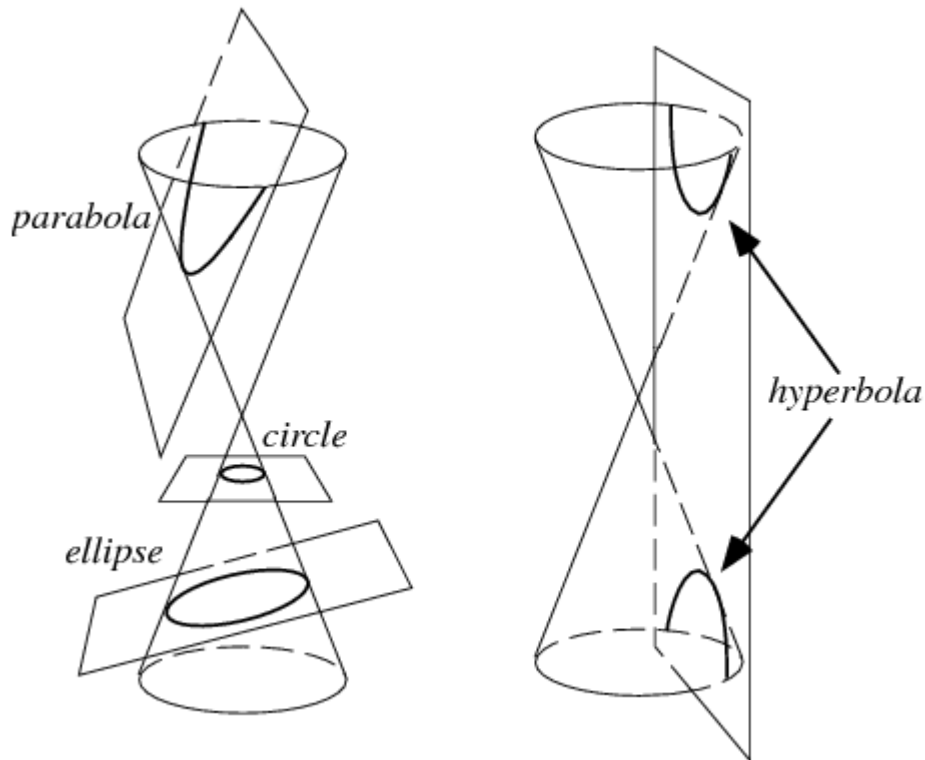
$$\eta_d = \frac{P_{out,L}}{P_{in,m}} \quad (\text{Direct power flow})(2.27)$$

$$\eta_i = \frac{P_{out,m}}{P_{in,L}} \quad (\text{Inverse power flow})(2.28)$$

In Eq. (2.27), the symbol  $P_{in,m}$  represents the power generated by motor flowing into the transmission and  $P_{out,L}$  represents the power flowing out from the transmission into the load.

Similarly, in Eq. (2.28), the symbol  $P_{in,L}$  represents the power generated by the load flowing into the transmission and  $P_{out,m}$  represents the power flowing out from the transmission into the motor.

## 2.3 Conic sections



**Fig. 2.13** Conic sections

The conic sections (shown in **Fig. 2.13**) are the non-degenerate curves generated by the intersections of a plane with one or two nappes of a cone. For a plane perpendicular to the axis of the cone, a circle is produced. For a plane that is not perpendicular to the axis and that intersects only a single nappe, the curve produced is either an ellipse or a parabola.

The curve produced by a plane intersecting both nappes is a hyperbola. The ellipse and hyperbola are known as central conics.

In the Cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section – though it may be degenerate, and all conic sections arise in this way. The equation will be of the form:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0 \quad (2.29)$$

In Eq. (2.29)  $a_{11}$ ,  $a_{12}$ ,  $a_{22}$  are not all zero.

The above equation can be written in matrix notation as

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

So we can introduce the discriminant  $\delta$ :

$$\delta = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -a_{22}^2 + a_{11}a_{12}$$

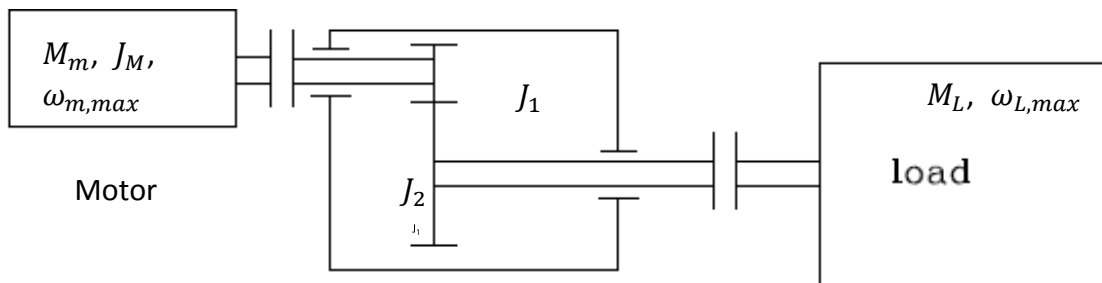
If the conic is non-degenerate, then we have below conditions:

1. If  $\delta > 0$ , the equation represents an ellipse;
  - ◆ If  $a_{11} = a_{22}$  and  $a_{12} = 0$ , the equation represents a circle, which is a special case of an ellipse;
2. If  $\delta = 0$ , the equation represents a parabola;
3. If  $\delta < 0$ , the equation represents a hyperbola;
  - ◆ If we also have  $a_{11} + a_{22} = 0$ , the equation represents a rectangular hyperbola.

## Chapter 3: Method to solve direct and inverse efficiencies

In the following section, we will take into consideration the use of the method which can solve both direct and inverse efficiencies and the transmission inertia.

### 3.1 Mathematical model of a machine



**Fig. 3.1** A generic machine considered inertia in transmission

A generic machine is shown in the **Fig. 3.1**. It is assumed that the moment of inertia of transmission has two parts:  $J_1$  and  $J_2$ . On the left side a motor is connected to the gear in the transmission which has the moment of inertia  $J_1$ , while on the right side the load is connected to the gear in the transmission whose inertia is  $J_2$

In the transmission from motor point of view, we can get the equation of power balance:

$$\frac{1}{2}J_2'\omega_L^2 = \frac{1}{2}J_2\omega_M^2 \quad (3.1)$$

Then, bearing in mind Eq. (2.25), we can obtain  $\omega_M = \frac{\omega_L}{\tau}$ , so combining with Eq. (3.1), we obtain

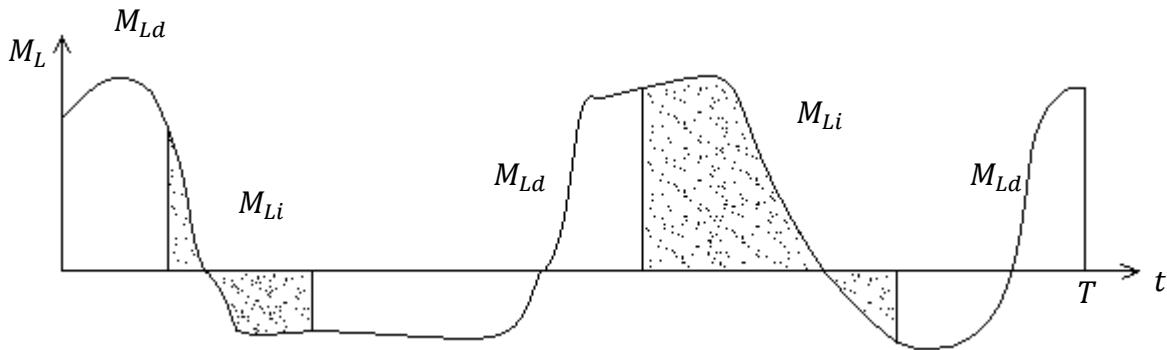
$$J_2' = J_2\tau^2$$

So the total inertia of transmission is:

$$J_T = J_1 + J_2' = J_1 + J_2\tau^2 \quad (3.2)$$

The inertia of the transmission is known. The acceleration law  $\alpha_L(t)$  is designed according to the task motion we need.

We know there are direct and inverse power flows. In general, the power flows and their relations with the efficiencies are very complex. In this paper we assume that the power flow is sometimes direct and sometimes inverse.



**Fig. 3.2**  $M_L$ -time curve

As shown in **Fig. 3.2**, the  $M_L$ -time curve is divided into 2 curves (shown in **Fig. 3.3** and **Fig. 3.4**):  $M_{Ld}$  which means the torque produces a direct power flow; while the other one is  $M_{Li}$  represent the torque for inverse power flow.

We can easily achieve the inequality from the **Fig. 1.2**:

$$\begin{cases} (M_L + J_2 \alpha_L) \omega_L > 0 & \text{direct efficiency} \\ (M_L + J_2 \alpha_L) \omega_L = 0 \\ (M_L + J_2 \alpha_L) \omega_L < 0 & \text{inverse efficiency} \end{cases} \quad (3.3)$$

If  $(M_L + J_2 \alpha_L) \omega_L > 0$ , the load torque has the same rotating direction with the angular speed. Find the time domain of this zone; let the  $M_{Ld}$  to represent the torque of direct efficiency. The system has the direct efficiency.

Similarly, if  $(M_L + J_2 \alpha_L) \omega_L < 0$ , the load torque has the opposite rotating direction with respect to the angular speed. So the system has the inverse efficiency. Let the  $M_{Li}$  to represent the torque of inverse efficiency.

However, the situation when  $(M_L + J_2 \alpha_L) \omega_L = 0$  is a little complex. It is assumed that the motor works periodically. The time  $T$  is the cycle time. Here two possibilities come:

If the angular speed is not zero, i.e.  $\omega_L \neq 0$ , it is obvious that  $(M_L + J_2 \alpha_L) = 0$ .

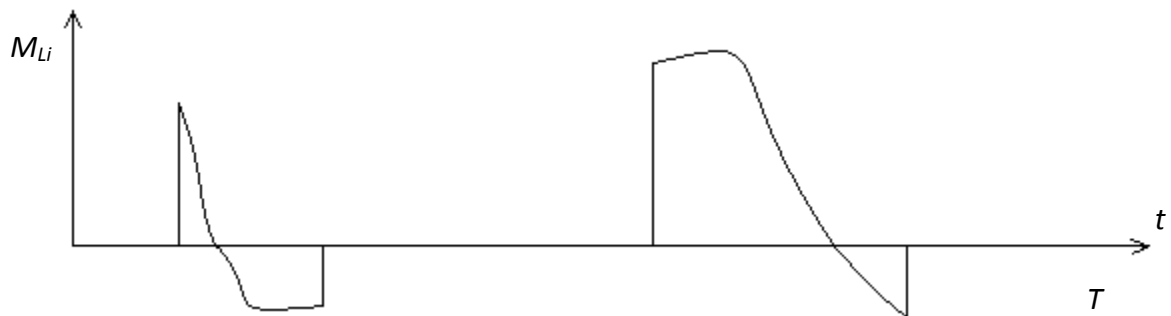
If the angular speed is zero, which indicates that  $\omega_L = 0$ , the power flow state will be the same as the previous conditions. If the previous state has direct efficiency, then during this time period the system has direct efficiency. Otherwise it will have the inverse efficiency.

If at the initial time of the motion ( $t = 0$ ) the situation is  $(M_L + J_2 \alpha_L) \omega_L = 0$  and  $\omega_L = 0$ . The system will have the same power flow efficiency as the situation at the end of the period. For example, for  $t = T$ , the system has direct efficiency, the system will have the direct efficiency at  $t = 0$  when  $(M_L + J_2 \alpha_L) \omega_L = 0$ .

After the analysis, the  $M_L$ -time curve will be split into two curves:  $M_{Ld}$ -time curve and  $M_{Li}$ -time curve, so we can obtain the direct and inverse components separately, and it is very convenient for further research.



**Fig. 3.3**  $M_{Ld}$ -time curve



**Fig. 3.4**  $M_{Li}$ -time curve

In this paper we assume the transmission is already known:  $J_1$ ,  $J_2$ ,  $\tau$ ,  $\eta_d$  and  $\eta_i$ . Due to the transmission efficiency, the inertia of transmission will have some changes. The following shows the equilibriums from the motor point of view.

When the system has direct transmission efficiency, from the **Fig. 1.3** and Eq. (2.27) we can have the equation of the power balance:

$$\eta_d P_{in,m} = P_{out,L} \quad (3.4)$$

From the **Fig. 1.3**, we can easily find out the  $P_{in,m}$  in Eq. (3.4) represents the input power flow coming from the motor on the left side of the transmission which includes three parts: the positive power produced by the motor, the negative power loss due to the inertia  $J_M$  of the motor and the negative power loss due to the inertia  $J_1$  of the transmission.

$$P_{in,m} = M_m \omega_m - J_M \alpha_m \omega_m - J_1 \alpha_m \omega_m \quad (3.5)$$

And  $P_{out,L}$  in Eq. (3.4) represents the output power flow on the right side of the transmission which includes two parts: the positive power loss due to the inertia  $J_2$  of the transmission and the final output power to the load.

$$P_{out,L} = M_L \omega_L + J_2 \alpha_L \omega_L \quad (3.6)$$

Use these two equations to substitute in Eq. (3.4), we can obtain:

$$\eta_d [M_m \omega_m - (J_M + J_1) \alpha_m \omega_m] = (M_L + J_2 \alpha_L) \omega_L \quad (3.7)$$

Remembering in mind Eq. (2.26), we achieve the relation between the acceleration of motor and the load:

$$\alpha_m = \frac{\alpha_L}{\tau} \quad (3.8)$$

We replace Eq. (3.8) in Eq. (3.7):

$$\eta_d \left[ M_m - (J_M + J_1) \frac{\alpha_L}{\tau} \right] \omega_m = (M_L + J_2 \alpha_L) \omega_L \quad (3.9)$$

Dividing all the terms in Eq. (3.9) by  $\tau$ , we obtain

$$\frac{M_m}{\tau} = \frac{J_M}{\tau^2} \alpha_L + \frac{J_1}{\tau^2} \alpha_L + \frac{J_2}{\eta_d} \alpha_L + \frac{M_L}{\eta_d} \quad (3.10)$$

Then, combining the inertia parts of transmission, we obtain

$$\frac{M_m}{\tau} = \frac{J_M}{\tau^2} \alpha_L + \left( \frac{J_1}{\tau^2} + \frac{J_2}{\eta_d} \right) \alpha_L + \frac{M_L}{\eta_d} \quad (3.11)$$

We indicate by  $J_{T,d}$ :

$$J_{T,d} = \frac{J_1}{\tau^2} + \frac{J_2}{\eta_d} \quad (3.12)$$

Similarly, when the system has direct transmission efficiency, from the **Fig. 1.4** and Eq. (2.28) we can have the equation of the power balance:

$$\eta_i P_{in,L} = P_{out,m} \quad (3.13)$$

From the **Fig. 1.4**, we can easily find out the  $P_{in,L}$  in Eq. (3.13) represents the input power flow coming from the load on the right side of the transmission which includes two parts: the power generated by the load and inertia  $J_2$  of the transmission

$$P_{in,L} = M_L \omega_L + J_2 \alpha_L \omega_L \quad (3.14)$$

In Eq. (3.13)  $P_{out,m}$  represents the output power flow going to the motor on the left side of the transmission, which includes three parts: the power due to the inertia  $J_1$  of the transmission and to the inertia  $J_M$  of the motor and the final output power to



the motor.

$$P_{out,m} = M_m \omega_m - J_1 \alpha_m \omega_m - J_M \alpha_m \omega_m \quad (3.15)$$

Substitute them in Eq. (3.13), we can achieve:

$$\eta_i (M_L + J_2 \alpha_L) \omega_L = \left[ M_m - (J_M + J_1) \frac{\alpha_L}{\tau} \right] \omega_m \quad (3.16)$$

By dividing by  $\tau$  and leaving  $\frac{M_m}{\tau}$  to the first member we obtain

$$\begin{aligned} \frac{M_m}{\tau} &= \frac{J_M}{\tau^2} \alpha_L + \frac{J_1}{\tau^2} \alpha_L + \eta_i J_2 \alpha_L + M_L \eta_i \\ \frac{M_m}{\tau} &= \frac{J_M}{\tau^2} \alpha_L + \left( \frac{J_1}{\tau^2} + \eta_i J_2 \right) \alpha_L + M_L \eta_i \end{aligned} \quad (3.17)$$

We indicate by  $J_{T,i}$ :

$$J_{T,i} = \frac{J_1}{\tau^2} + \eta_i J_2 \quad (3.18)$$

It is easy to get:

$$M_m = \frac{J_m}{\tau} \alpha_L + \tau \left( J_{T,d} \alpha_L + \frac{M_L}{\eta_d} \right) \quad (3.19)$$

$$M_m = \frac{J_m}{\tau} \alpha_L + \tau (J_{T,i} \alpha_L + \eta_i M_L) \quad (3.20)$$

In order to simplify them, we introduce  $M_L^*$

$$M_L^* = \begin{cases} J_{T,d} \alpha_L + \frac{M_L}{\eta_d} & \text{if } (M_L + J_2 \alpha_L) > 0 \\ J_{T,i} \alpha_L + \eta_i M_L & \text{if } (M_L + J_2 \alpha_L) < 0 \end{cases} \quad (3.21)$$

So we obtain

$$M_m = \frac{J_m}{\tau} \alpha_L + \tau M_L^* \quad (3.22)$$

# Chapter 4: Apply the method to continuous duty operating range

In this chapter, we will introduce a method to deal with the continuous duty working range of the motor.

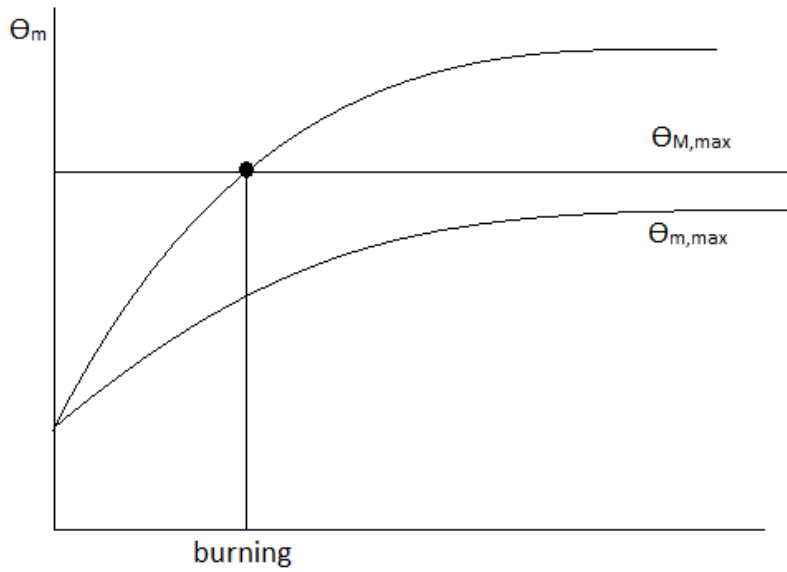
## 4.1 Introduction to the $M_{M,s1}$ and $\tau_{th}$

$M_{M,s1}$  depends on the thermal characteristics of the motor. The motor warms up because of:

- Joule effect (copper losses);
- Parasitic currents and Hysteresis (iron losses);
- Mechanical losses (bearings, etc.).

The motor can suffer a maximum internal temperature  $\Theta_{M,max}$ , above which the sheathing “burns”. Let us consider  $M_m$  and  $\omega_m$  constant, which indicates that the motor is in mechanical steady state. And then the wasted power  $W_w$  in the motor is constant.

However, it is possible that the motor is not at thermal steady state. After a thermal transient, the motor tends to reach a steady state temperature  $\Theta_{M,max}$ . If  $\Theta_{m,max} \leq \Theta_{M,max}$ , the motor reaches  $\Theta_{m,max}$ . On the contrary, if  $\Theta_{m,max} > \Theta_{M,max}$ , the motor burns before to reach  $\Theta_{m,max}$  (shown in **Fig. 4.1**).



**Fig. 4.1** Temperature profile versus time

The nominal torque  $M_{M,S1}$  corresponds to the limit condition:  $\Theta_{m,max} = \Theta_{M,max}$ . Therefore, the nominal torque  $M_{M,S1}$  is the maximum torque that the motor can exert without burning in mechanical and thermal steady state. So the constraint inequality is  $|M_{mo}| \leq M_{M,S1}$ .

Then we introduce  $\tau_{th}$ , which is the thermal time constant of the motor.

If the mechanical behavior of the motor is periodic (with period  $T$ ), we can make a good approximation if the following three conditions are satisfied:

1)  $T \ll \tau_{th}$ : in this condition the motor warming up depends on the average wasted power  $\bar{W}_w$  in the period, and not on the instantaneous power (which varies significantly in the period);

2) The motor torque is proportional to the current:

$$M_m = K_T i \quad (4.1)$$

3) The motor wasted power  $W_w$  is only due to the Joule Effect:

$$W_w = Ri^2 \quad (4.2)$$

Then bearing in mind Eq. (2.22), and the average wasted power is

$$\begin{aligned} \bar{W}_w &= \frac{E_w}{T} = \frac{1}{T} \int_0^T W_w(t) dt = \frac{1}{T} \int_0^T Ri^2(t) dt = R \frac{1}{T} \int_0^T \frac{M_m^2(t)}{K_T^2} dt \\ &= \frac{R}{K_T^2} \frac{1}{T} \int_0^T M_m^2(t) dt = \frac{R}{K_T^2} M_{m,rms}^2 \end{aligned} \quad (4.3)$$

## 4.2 The $J_M$ versus $M_{m,rms}$ curve

By applying Eq.(3.22), now the root mean square of the torque due to both direct and inverse efficiencies can be expressed as:

$$\begin{aligned}
 M_{m,rms}^2 &= \frac{1}{T} \int_0^T M_m^2 dt = \frac{1}{T} \int_0^T \left( \frac{J_m^2}{\tau^2} + \tau^2 M_L^{*2} + 2J_m \alpha_L M_L^* \right) dt \\
 &= \left\{ \frac{1}{T} \int_0^T \frac{J_m^2}{\tau^2} \alpha_L^2 dt + \frac{1}{T} \int_0^T \tau^2 M_L^{*2} dt + \frac{1}{T} \int_0^T 2J_m \alpha_L M_L^* dt \right\} \\
 &= \frac{J_m^2}{\tau^2} \alpha_{L,rms}^2 + \tau^2 M_{L,rms}^{*2} + 2J_m G_L \tag{4.4}
 \end{aligned}$$

Where

$$G_L = \frac{1}{T} \int_0^T \alpha_L M_L^* dt$$

And  $\alpha_{L,rms}$  and  $M_{L,rms}^*$  are the root mean square of the angular acceleration and the generalized load.

$$\begin{aligned}
 \alpha_{L,rms} &= \sqrt{\frac{1}{T} \int_0^T \alpha_L^2 dt} \\
 M_{L,rms}^* &= \sqrt{\frac{1}{T} \int_0^T M_L^{*2} dt}
 \end{aligned}$$

In the previous studies, the transmission is an unknown factor while the motors are known. The old method is from motor point of view to choose the transmission.

However in this case all the parameters of transmission are known, which means the transmission ratio  $\tau$  and the transmission inertia  $J_1, J_2$  are known. Now we choose motor from transmission point of view.

For this situation, it is reasonable to draw the  $J_M$ - $M_{m,rms}$  curve and to check if the motor satisfies the curve conditions. Each motor has the parameters of  $M_{m,rms}$  and  $J_M$ . It is easy to plot the motor on the curve using the point  $(J_M, M_{m,rated})$ . Each point represents a motor. If the point is above the  $J_M$ - $M_{m,rms}$  curve, then the motor is feasible; if not, then the motor is unaccepted.

Form Eq. (4.4) we have:

$$M_{m,rms} = \sqrt{J_M^2 \left( \frac{\alpha_{L,rms}^2}{\tau^2} \right) + 2J_M G_L + \tau^2 M_{L,rms}^{*2}} \tag{4.5}$$

After having simplified:

$$M_{m,rms} = \sqrt{AJ_M^2 + 2BJ_M + C} \quad (4.6)$$

Where

$$A = \frac{\alpha_{L,rms}^2}{\tau^2} = \frac{1}{\tau^2} \frac{1}{T} \int_0^T \alpha_L^2 dt \quad (4.7)$$

$$B = G_L = \frac{1}{T} \int_0^T \alpha_L M_L^* dt \quad (4.8)$$

$$C = \tau^2 M_{L,rms}^{*2} = \tau^2 \frac{1}{T} \int_0^T M_L^{*2} dt \quad (4.9)$$

Now let us discuss Eq. (4.6). From the formulas (4.7) and (4.9), we can know that:

$$A \geq 0 \text{ and } C \geq 0$$

And if  $A = 0$  then  $B = 0$ , because

$$A = \frac{\alpha_{L,rms}^2}{\tau^2} = 0$$

So

$$\alpha_{L,rms} = \sqrt{\frac{1}{T} \int_0^T \alpha_L^2 dt} = 0$$

Then  $\alpha_L(t) = 0 \forall t$ .

Then bearing in mind Eq. (4.8), we can prove that  $B = 0$ .

In order to have a better analysis of Eq. (4.6), a variable D is induced:

$$D = B^2 - AC$$

It is possible to show that:

$$D = B^2 - AC \leq 0$$

To sum up, the system must satisfy these conditions:

$$\begin{cases} A \geq 0 \\ C \geq 0 \\ D \leq 0 \end{cases}$$

If  $A = 0$ ,  $B = 0$ .

In fact, Eq. (4.6) is a conic section formula. We can rewrite it into the form:

$$a_{11}J_M^2 + 2a_{12}J_M M_{m,rms} + a_{22}M_{m,rms}^2 + 2a_{13}J_M + 2a_{23}M_{m,rms} + a_{33} = 0$$

The matrix of the conic section is  $M = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

So Eq. (4.6) becomes:

$$AJ_M^2 - M_{m,rms}^2 + 2BJ_M + C = 0 \quad (4.10)$$

So

$$M = \begin{bmatrix} A & 0 & B \\ 0 & -1 & 0 \\ B & 0 & C \end{bmatrix}$$

And the discriminant of the conic section is

$$\delta = \det \begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix} = -A$$

From the formula it is known that:

If  $A > 0$  then the curve will be a hyperbola.

If  $A = 0$  then it will become a parabola.

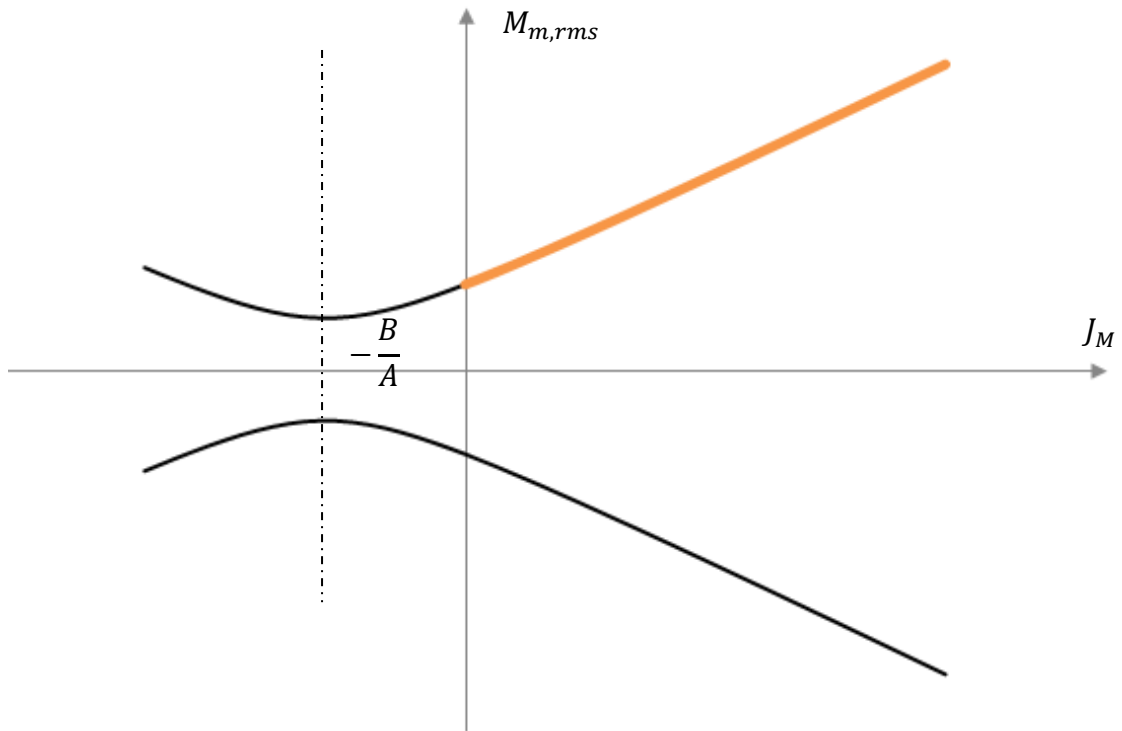
If  $C = 0$  then the curve will pass through the origin point.

If  $D = 0$  which means  $= \pm\sqrt{AC}$ , the curve will become two straight lines.

$$\begin{aligned} M_{m,rms} &= \sqrt{AJ_m^2 \pm 2\sqrt{AC}J_m + C} \\ &= \sqrt{(\sqrt{A}J_m \pm \sqrt{C})^2} \\ &= \sqrt{A}J_m \pm \sqrt{C} \end{aligned} \quad (4.11)$$

The detail cases are shown below:

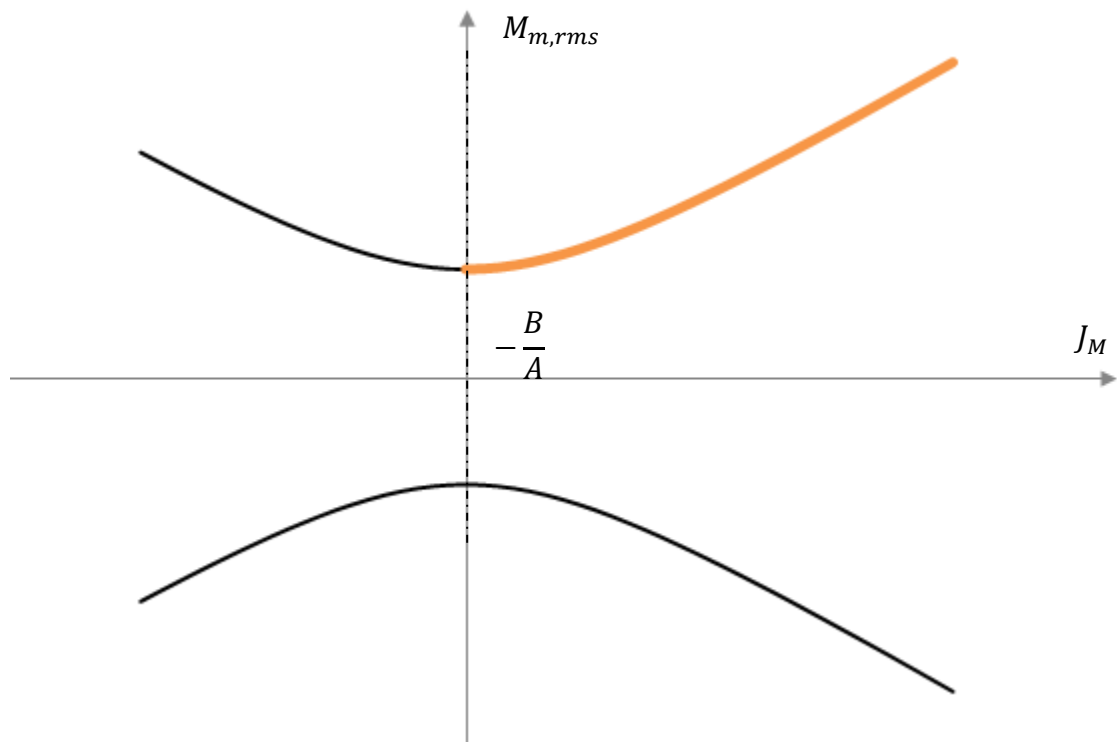
1. If  $A > 0$ ,  $C > 0$ ,  $B > 0$  and  $D < 0$  then the  $J_M - M_{m,rms}$  curve is as below:



**Fig. 4.2**  $J_M - M_{m,rms}$  curve

As the figure shown above, Eq. (4.10) is a hyperbola curve. The orange part of the curve shows the real  $J_M - M_{m,rms}$  curve because both  $J_M$  and  $M_{m,rms}$  must be greater than zero. And the directrix of the curve is always at the line  $J_M = -\frac{B}{A}$ , which is on the left of the ordinate, so in this case, the minimum  $M_{m,rms}$  is useless. And the minimum  $M_{m,rms}$  of the orange curve is bigger, and it is  $\sqrt{C}$ , at the abscissa  $J_M = 0$ . And the orange part is ascending.

2. If  $A > 0$ ,  $C > 0$ ,  $B = 0$  and  $D < 0$  then the  $J_M - M_{m,rms}$  curve is like below:



**Fig. 4.3**  $J_M - M_{m,rms}$  curve

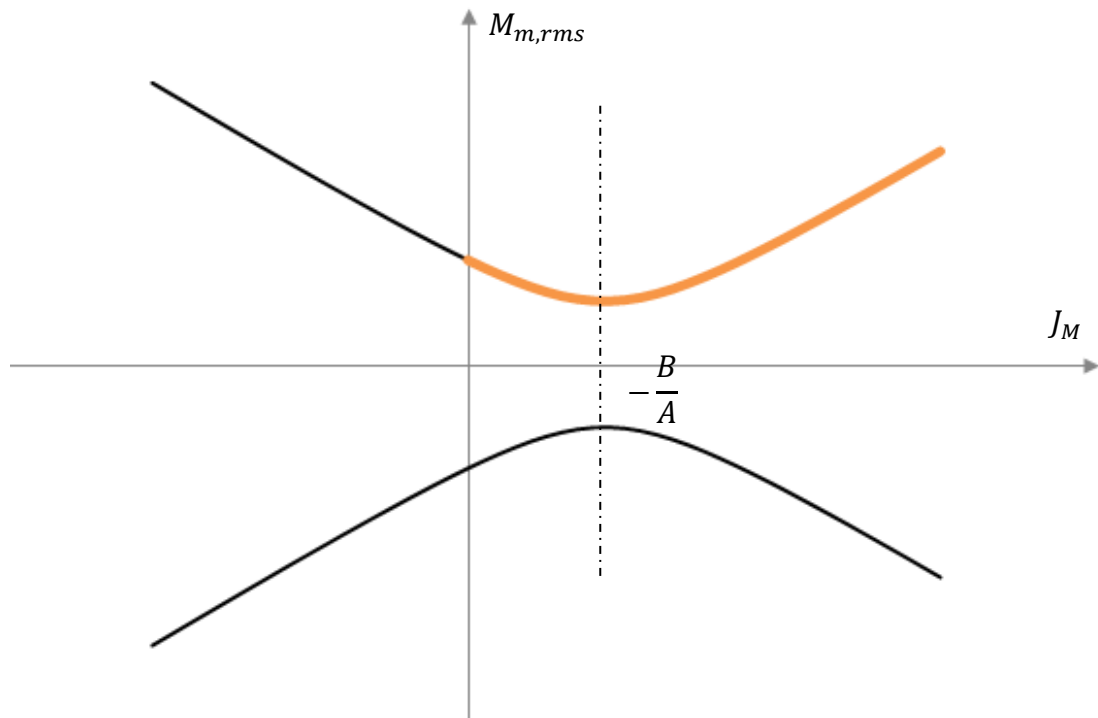
As the figure shows above, Eq. (4.10) is also a hyperbola curve. The directrix of the curve is at the abscissa  $J_M = 0$  because

$$-\frac{B}{A} = 0$$

The minimum  $M_{m,rms}$  is at where  $J_M = -\frac{B}{A} = 0$ , so in this case, the orange part that we need contains the real minimum  $M_{m,rms}$ . And the orange part is ascending.



3. If  $A > 0$ ,  $C > 0$ ,  $B < 0$  and  $D < 0$  then the  $J_M - M_{m,rms}$  curve is like below:



**Fig. 4.4**  $J_M - M_{m,rms}$  curve

As **Fig. 4.4** shows above, Eq. (4.10) is a hyperbola curve. The directrix of the curve lies on right side of  $J_M = 0$  because  $J_M = -\frac{B}{A} > 0$ , so the directrix is on the right of the ordinate axis, and minimum  $M_{m,rms}$  is at the abscissa  $J_M = -\frac{B}{A}$ , and is useful, because it's also the minimum point of the orange part that we need. And we can also see that the orange curve descends before  $J_M = -\frac{B}{A}$ , and then ascends after  $J_M = -\frac{B}{A}$ .

4. If  $A > 0$ ,  $D = 0$ ,  $B > 0$  .

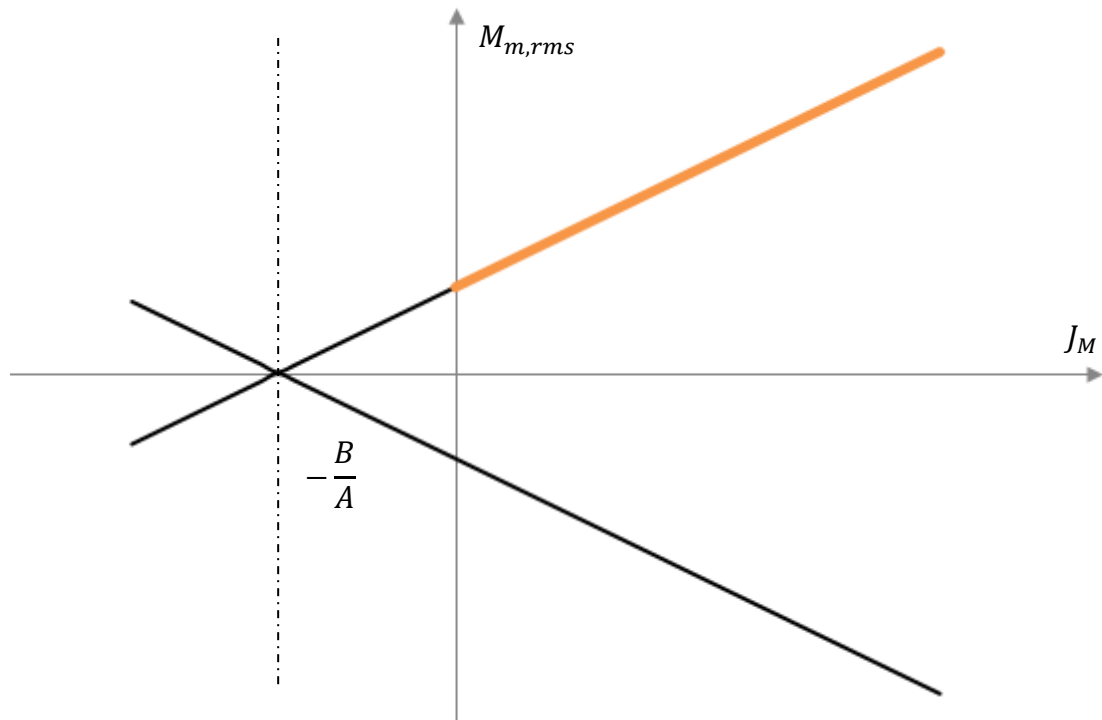
Because we have  $D = B^2 - AC = 0$ , Eq. (4.10) becomes two straight lines.

$$B^2 = AC$$

And then:

$$C = \frac{B^2}{A} > 0$$

Then, the  $J_M - M_{m,rms}$  curve is as below:



**Fig. 4.5**  $J_M$  -  $M_{m,rms}$  curve

The directrix of the curve lies on left side of the ordinate axis, because  $J_M = -\frac{B}{A} < 0$ . And the orange  $J_M$  -  $M_{m,rms}$  curve becomes a straight line, and its minimum point is not at the abscissa  $J_M = -\frac{B}{A}$ , but at the abscissa  $J_M = 0$ . So the orange part is ascending.

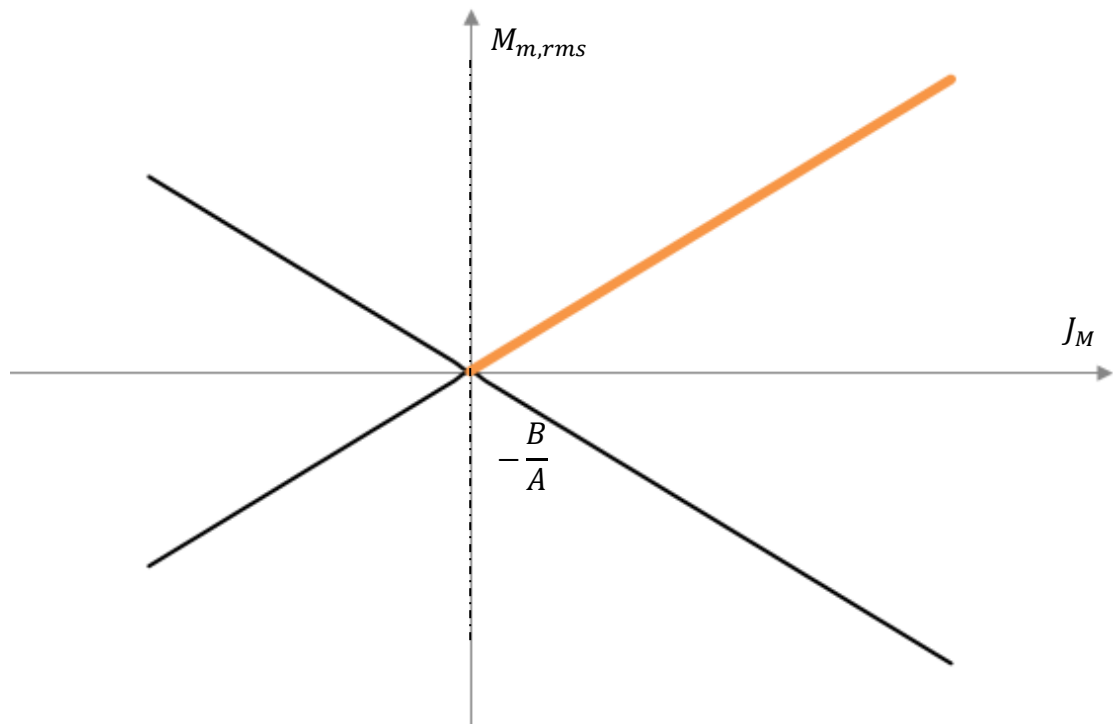
5. If  $A > 0$ ,  $D = 0$ ,  $B = 0$

Because we have  $D = B^2 - AC = 0$ , Eq. (4.10) becomes two straight lines.

From  $B^2 = AC = 0$  then

$$C = 0$$

Then, the  $J_M$  -  $M_{m,rms}$  curve is shown in **Fig. 4.6**:



**Fig. 4.6**  $J_M$  -  $M_{m,rms}$  curve

As the figure shows, Eq. (4.10) becomes two straight lines which pass through the origin point  $(0, 0)$ , which is also the minimum point. The directrix of the curve is at  $J_M = 0$  because  $B = 0$ . The orange real  $J_M$  -  $M_{m,rms}$  curve becomes a line which start from the origin point. And the orange part is ascending.

6. If  $A > 0$ ,  $D = 0$ ,  $B < 0$

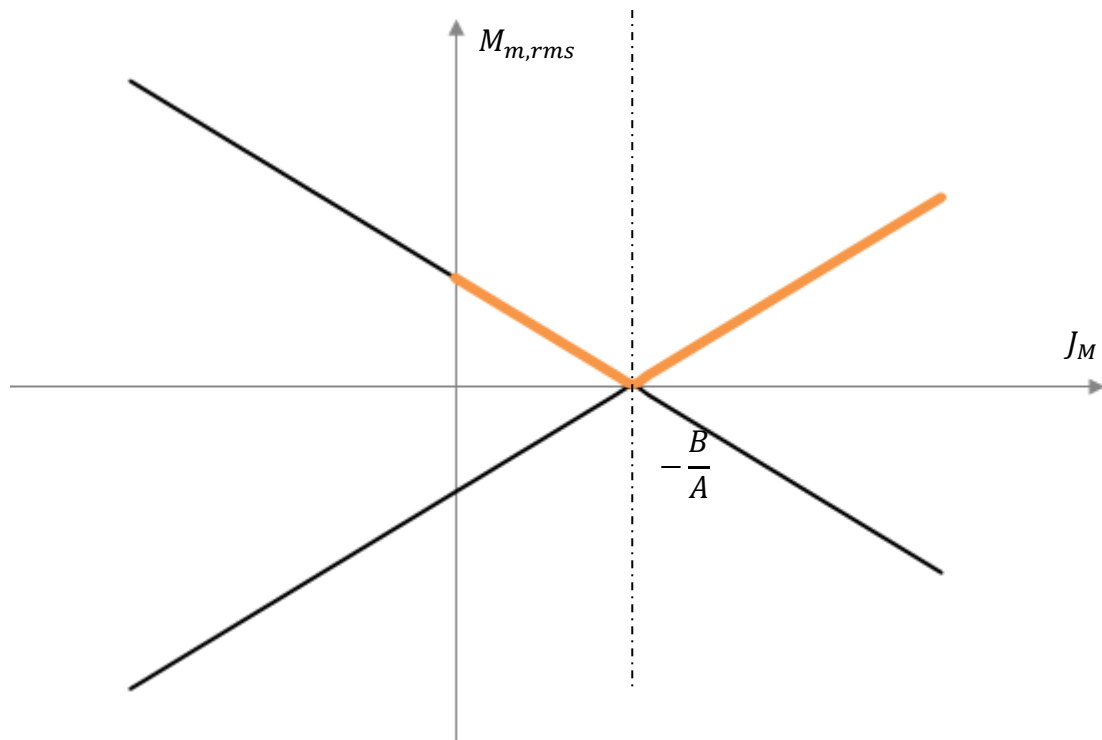
As before because  $D = B^2 - AC = 0$ , Eq. (4.10) becomes two straight lines

$$B^2 = AC > 0$$

And then:

$$C = \frac{B^2}{A} > 0$$

The  $J_M$  -  $M_{m,rms}$  curve is as below in **Fig. 4.7**:



**Fig. 4.7**  $J_M$ - $M_{m,rms}$  curve

As the figure shows, Eq. (4.10) also consists of two straight lines due to  $D = B^2 - AC = 0$ . The directrix of the curve lies on the right side of ordinate, because  $J_M = -\frac{B}{A} > 0$ . The orange real  $J_M$ - $M_{m,rms}$  curve becomes a polyline, which descends before  $J_M = -\frac{B}{A}$ , and then ascends after  $J_M = -\frac{B}{A}$ .

7. If  $A = 0$ ,  $D = 0$ ,  $B = 0$  and  $C > 0$  then the  $J_M - M_{m,rms}$  curve is as below in **Fig. 4.8**:



**Fig. 4.8**  $J_M - M_{m,rms}$  curve

As the figure shows, Eq. (4.10) becomes two parallel straight lines. In fact the formula becomes:

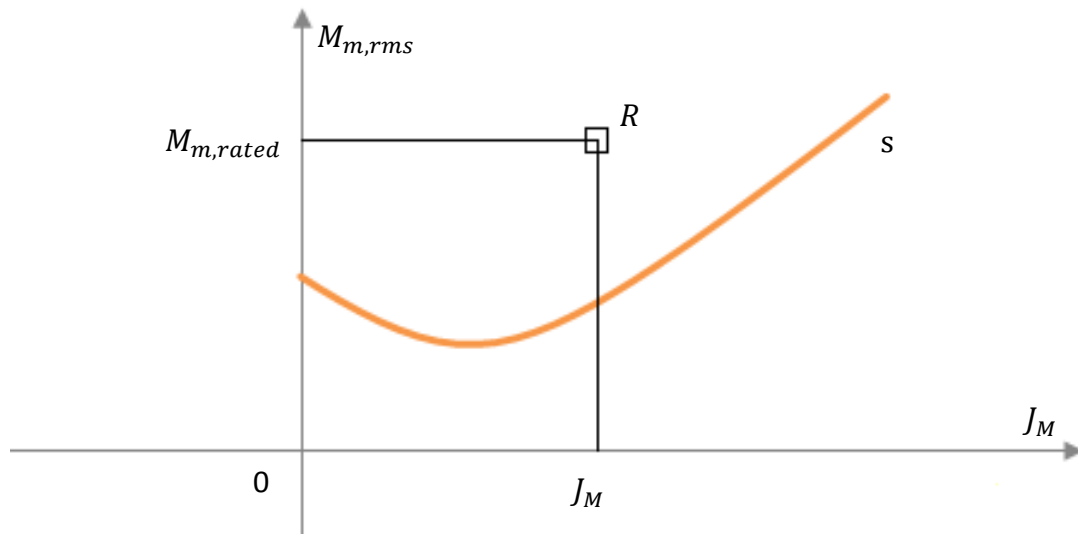
$$M_{m,rms}^2 = C$$

When  $C > 0$

$$M_{m,rms} = \pm\sqrt{C}$$

So the curves are two parallel lines. But we only care about the line which is in the first quadrant, which is the orange part shown in **Fig. 4.7**.

We use these  $J_M - M_{m,rms}$  curves to choose the motor.



**Fig. 4.9**  $J_M - M_{m,rms}$  curve

As shown in **Fig. 4.9**, a point  $R \equiv (J_M, M_{m,rated})$  represents a motor, where  $M_{m,rated}$  is the rated torque of a motor. If the point is above the curve  $s$  shown in the figure, the motor satisfies the condition:

$$M_{m,rms} \leq M_{m,rated}$$

# Chapter 5: Apply the method to dynamic operating range

## 5.1 Dependence of $M_m$ on $J_M$ in a given instant [1]

$M_{M,dyn}$  depends on the electronic converter driving the motor. The converter transistors can suffer a maximum peak current without burning for a small time range. Because of the proportionality between  $M_m$  and  $i$ , the torque  $M_{M,dyn}$  corresponds to this current.

From previous analysis, we know that in order to let the motor become feasible it must satisfy three conditions, i.e. it does not only satisfy  $M_{m,rms} \leq M_{M,S1}$  and  $\omega_{m,max} \leq \omega_{M,max}$ , but also

$$M_{m,max} \leq M_{M,dyn}$$

So it is necessary to check if the maximum torque of the motor during the cycle is smaller than the maximum torque the motor can provide.

In order to obtain the  $J_M$ - $M_{m,max}$  curve we need to adopt a different approach, which, apart from achieving the same results, allows us to focus attention on the most exacting condition for the system. We know according to the machine design and the definition of the reference task, during the system working there are lots of pairs of angular accelerations and the generalized torques of the load like  $(\alpha_L, M_L^*)$ . Therefore we wish to identify them in the reference task, which are directly responsible for the limitations in the motor maximum torque, i.e.  $M_{m,max}$ .

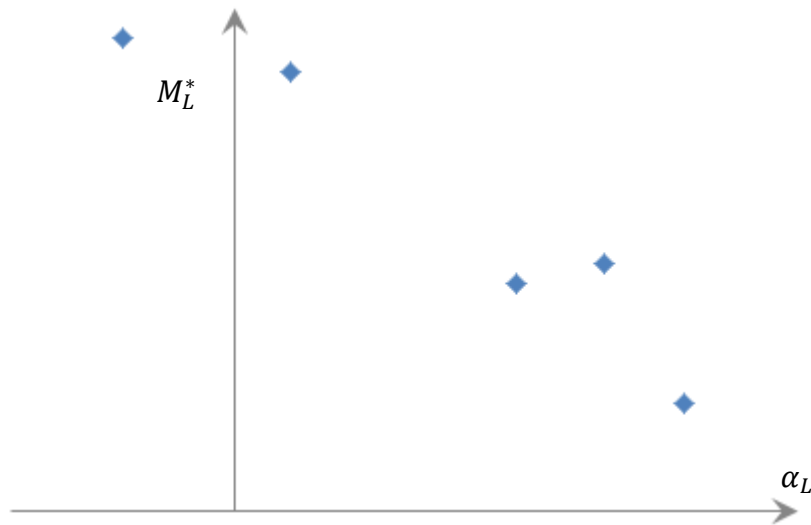
For a given instant  $t$ , we consider about the pair  $(\alpha_L, M_L^*)$ . In the same instant  $t$ , we take into account Eq. (3.22) written in the form where  $M_m$  is a function of  $J_M$ , and expressed in Eq.(5.4)

$$M_m = \frac{J_M}{\tau} \alpha_L + M_L^* \tau = \frac{\alpha_L}{\tau} J_M + M_L^* \tau \quad (5.1)$$

In fact the pair  $(\alpha_L, M_L^*)$  is time consumed, which means  $\alpha_L(t)$  and  $M_L^*(t)$  are functions of time. Considering the plane  $(\alpha_L, M_L^*)$ , it is advisable to use a diagram that presents  $\alpha_L$  in abscissa and  $M_L^*$  in ordinate. Once  $\alpha_L(t)$  and  $M_L^*(t)$  are known, because the transmission ratio  $\tau$  is given,  $M_m$  is a linear function of  $J_M$ . In fact, the inertia of the motor is always positive. So  $J_M$  satisfies the condition:

$$J_M > 0$$

During a time period, it is possible to use the discrete points to show the angular accelerations and the generalized torques of the load. As **Fig. 5.1** shown below, the discrete points represent the pairs of  $(\alpha_L, M_L^*)$ .



**Fig. 5.1** The discrete points of  $\alpha_L - M_L^*$

Since the transmission ratio  $\tau$  and the pair  $(\alpha_L, M_L^*)$  are known, Eq. (5.1) will become several straight lines that use  $J_M$  as the abscissa and  $M_m$  as ordinate. Each pair of angular acceleration and generalized torque forms a linear function. The slope of the line is  $\frac{\alpha_L}{\tau}$  and the point of intersection between the graph of the function and the  $M_m$ -axis is  $M_L^* \tau$ .

We now consider the  $J_M$  versus  $M_m$  curves. There are four cases.

1.  $\alpha_L \geq 0$  and  $M_L^* \geq 0$

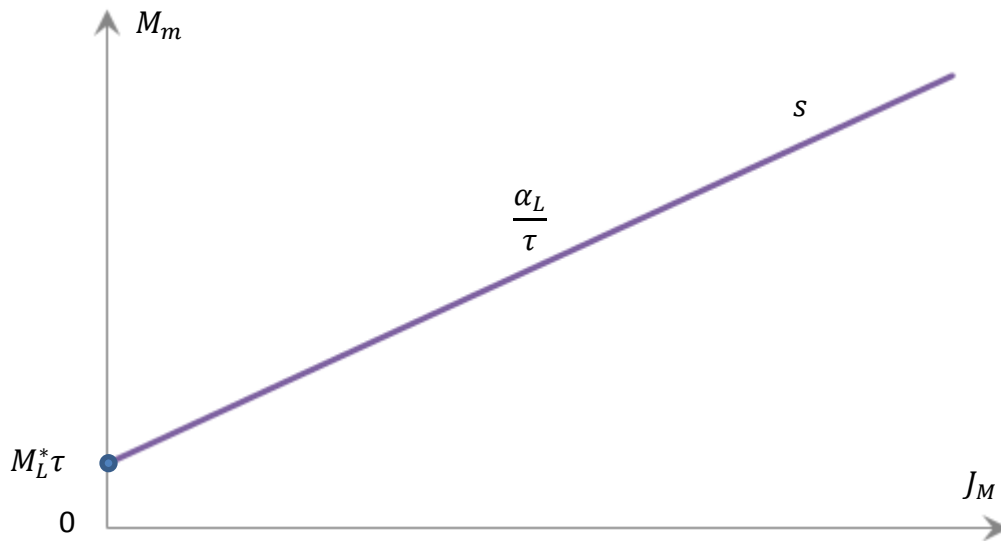
In this case, in the plane  $\alpha_L - M_L^*$  the corresponding point  $S$  lies in the first quadrant.

From Eq. (5.1) we can know that both the slope  $\frac{\alpha_L}{\tau}$  and the  $M_m$ -intercept are

positive. In the  $J_M - M_m$  plane we obtain a corresponding curve  $s$  lying in the first quadrant because  $J_M$  is positive (shown in **Fig. 5.2**).

In fact, because of  $J_M > 0$ , the curve  $s$  has a minimum point  $(0, M_L^* \tau)$ .





**Fig. 5.2** Curve  $s$  in plane  $J_M$ - $M_m$  when  $\alpha_L > 0$  and  $M_L^* > 0$

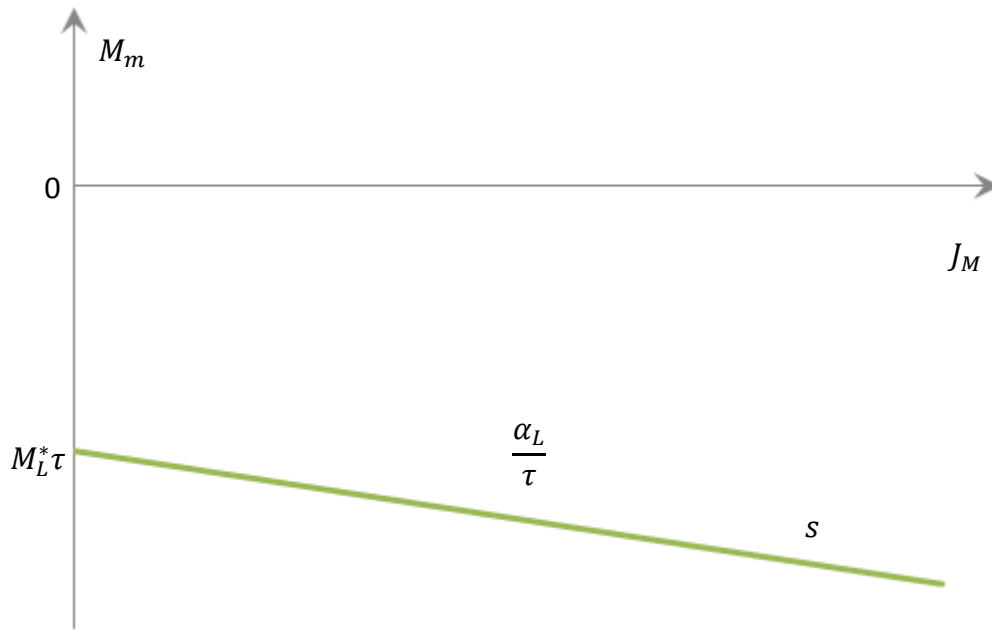
If  $\alpha_L = 0$  then the curve  $s$  will become a horizontal line which parallels to the abscissa  $J_M$ , passing the point  $(0, M_L^* \tau)$ .

If  $M_L^* = 0$  then the curve  $s$  remains a linear function with the slope  $\frac{\alpha_L}{\tau}$ , passing through the origin point  $(0, 0)$ .

If  $\alpha_L = 0$  and  $M_L^* = 0$  then the curve  $s$  will overlap the abscissa  $J_M$ .

## 2. $\alpha_L \leq 0$ and $M_L^* \leq 0$

In this case, in the plane  $\alpha_L$ - $M_L^*$  the corresponding point  $S$  lies in the third quadrant. From Eq. (5.1) we can know that both the slope  $\frac{\alpha_L}{\tau}$  and the  $M_m$ -intercept are negative. In the plane  $J_M$ - $M_m$  we obtain a corresponding curve  $s$  lying in the fourth quadrant due to  $J_M$  is always positive (shown in **Fig. 5.3**).

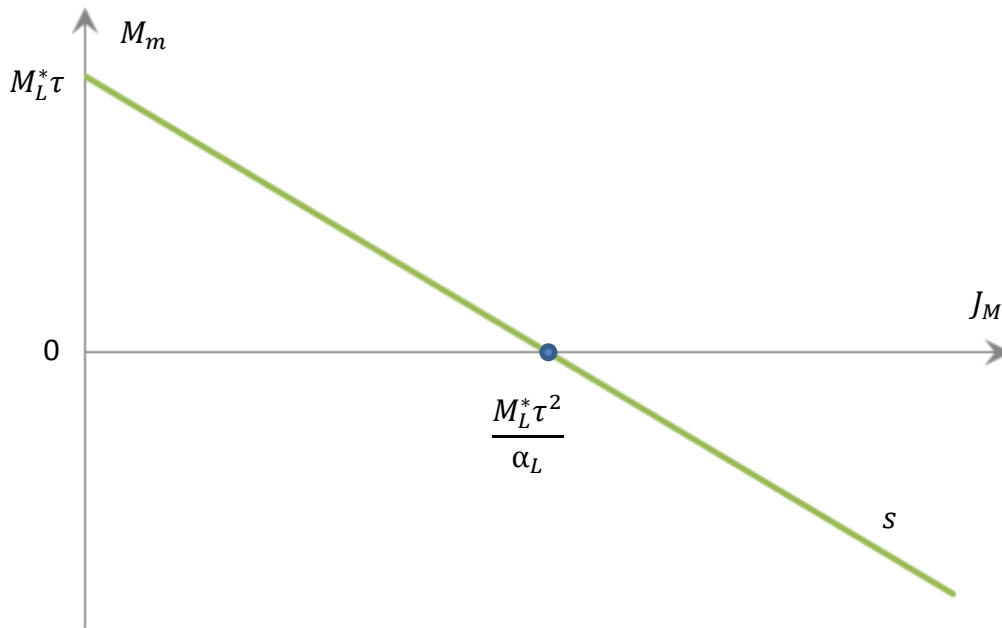


**Fig. 5.3** Curve  $s$  in plane  $J_M$ - $M_m$  when  $\alpha_L < 0$  and  $M_L^* < 0$

Obviously this curve is symmetric of the curve corresponding to the coordinates  $(|\alpha_L|, |M_L^*|)$  which respect to the abscissa axis.

3.  $\alpha_L < 0$  and  $M_L^* > 0$

In this case, in the plane  $\alpha_L$ - $M_L^*$  the corresponding point  $S$  lies in the second quadrant. From Eq. (5.1) we can know that the slope  $\frac{\alpha_L}{\tau}$  is negative, and the  $M_m$ -intercept is positive. In the plane  $J_M$ - $M_m$  we obtain a corresponding curve  $s$  lying in the first quadrant and in the fourth quadrant (shown in **Fig. 5.4**).



**Fig. 5.4** Curve  $s$  in plane  $J_M$ - $M_m$  when  $\alpha_L < 0$  and  $M_L^* > 0$

The curve  $s$  is monotonically descending and it is easy to solve the intersection point between the curve and the abscissa  $J_M$ . According to Eq. (5.1):

$$M_m = \frac{\alpha_L}{\tau} J_M + M_L^* \tau = 0$$

So we can get:

$$J_M = -\frac{M_L^* \tau^2}{\alpha_L} \quad (5.2)$$

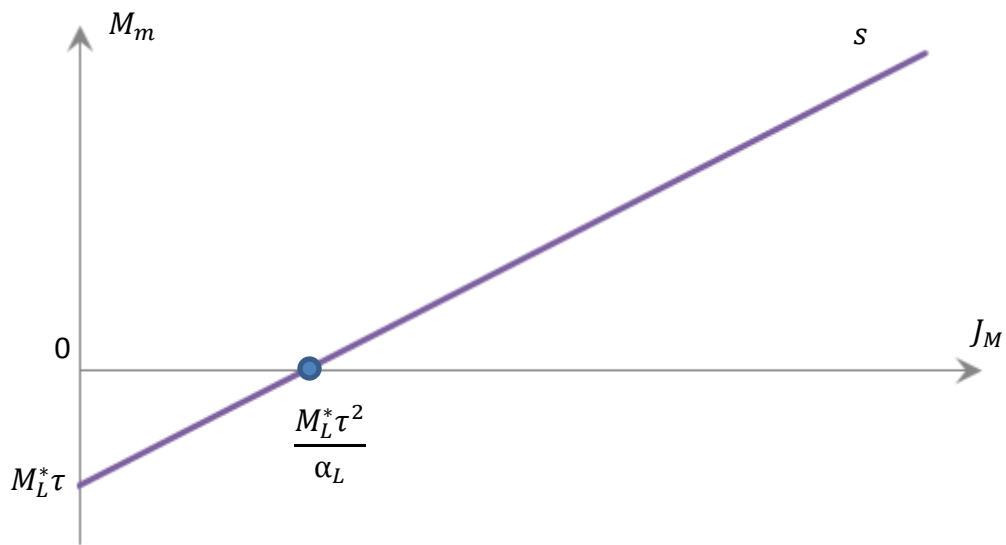
The formula indicates the curve  $s$  passing through the point  $(-\frac{M_L^* \tau^2}{\alpha_L}, 0)$ . In this case, the inertial torque of the motor is used for balancing the load, while the motor does not exert any torque  $M_m$ .

Furthermore, both  $\alpha_L$  and  $M_L^*$  vary over time, so that the point  $S$  changes in the second quadrant of the plane  $\alpha_L$ - $M_L^*$ . The corresponding curve  $s$  and its intersection with the abscissa axis in the plane  $J_M$ - $M_m$  also vary.

#### 4. $\alpha_L > 0$ and $M_L^* < 0$

In this case, in the plane  $\alpha_L$ - $M_L^*$  the corresponding point  $S$  lies in the fourth quadrant. From Eq. (5.1) we can know that the slope  $\frac{\alpha_L}{\tau}$  is positive, and the  $M_m$ -intercept is negative. In the plane  $J_M$ - $M_m$  we obtain a corresponding curve  $s$  lying in the first quadrant and in the fourth quadrant (shown in **Fig. 5.5**).

Obviously, this curve is symmetric of the curve corresponding to the coordinates  $(-|\alpha_L|, |M_L^*|)$  with respect to the abscissa axis.



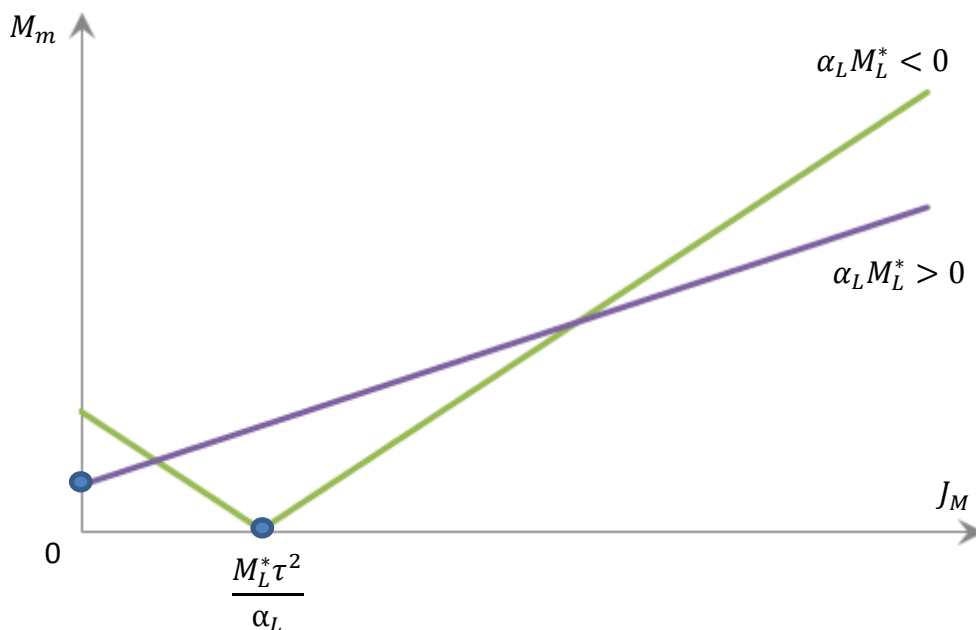
**Fig. 5.5** Curve  $s$  in plane  $J_M$ - $M_m$  when  $\alpha_L > 0$  and  $M_L^* < 0$

Obviously, both  $\alpha_L$  and  $M_L^*$  vary over time, so that the point  $S$  changes in the second quadrant of the plane  $\alpha_L$ - $M_L^*$ . The corresponding curve  $s$  and its intersection with the abscissa axis in the plane  $J_M$ - $M_m$  also vary.

#### 5. $|M_m|$ versus $J_M$

In **Fig. 5.6** there are some curves representing the absolute value  $|M_m|$  versus  $J_M$  with different values of  $\alpha_L$  and  $M_L^*$ . From Eq. (5.1) we obtain

$$|M_m| = \left| \frac{\alpha_L}{\tau} J_M + M_L^* \tau \right| \quad (5.3)$$



**Fig. 5.6** Curve  $s$  in plane  $J_M$ - $M_m$  when  $\alpha_L > 0$  and  $M_L^* < 0$

If  $\alpha_L > 0$  and  $M_L^* > 0$ , the curve in **Fig. 5.6** remains unaffected with respect to that in **Fig. 5.2**.

If  $\alpha_L < 0$  and  $M_L^* < 0$ , the curve in **Fig. 5.6** is symmetric of that in **Fig. 5.3** with respect to the abscissa axis.

If  $\alpha_L < 0$  and  $M_L^* > 0$ , as the **Fig. 5.6** shows, the curve does not lie in the first and in the fourth quadrant, but rather only in the first one.  $|M_m|$  has a minimum value, which is an edge point, on the abscissa  $J_M = -\frac{M_L^* \tau^2}{\alpha_L}$  with an ordinate equal to 0. On the left of the minimum point the curve descends monotonically. This branch is equal to the corresponding branch in **Fig. 5.4**, while on the right of the minimum point the curve ascends monotonically when  $J_M$  increases. This branch is symmetric of the corresponding branch in **Fig. 5.4** with respect to the abscissa axis.

If  $\alpha_L > 0$  and  $M_L^* < 0$ , as the **Fig. 5.6** shows, the curve no longer lies in the first and in the fourth quadrant, but rather only in the first one.  $|M_m|$  has a minimum value, which is an edge point, on the abscissa  $J_M = -\frac{M_L^* \tau^2}{\alpha_L}$  with an ordinate equal to 0. On the left of the minimum point the curve descends monotonically when  $J_M$  increases. This branch is symmetric of the corresponding branch in **Fig. 5.5** with respect to the abscissa axis, while on the right of the minimum point the curve ascends monotonically when  $J_M$  increases. This branch is equal to the corresponding branch in **Fig. 5.5**.

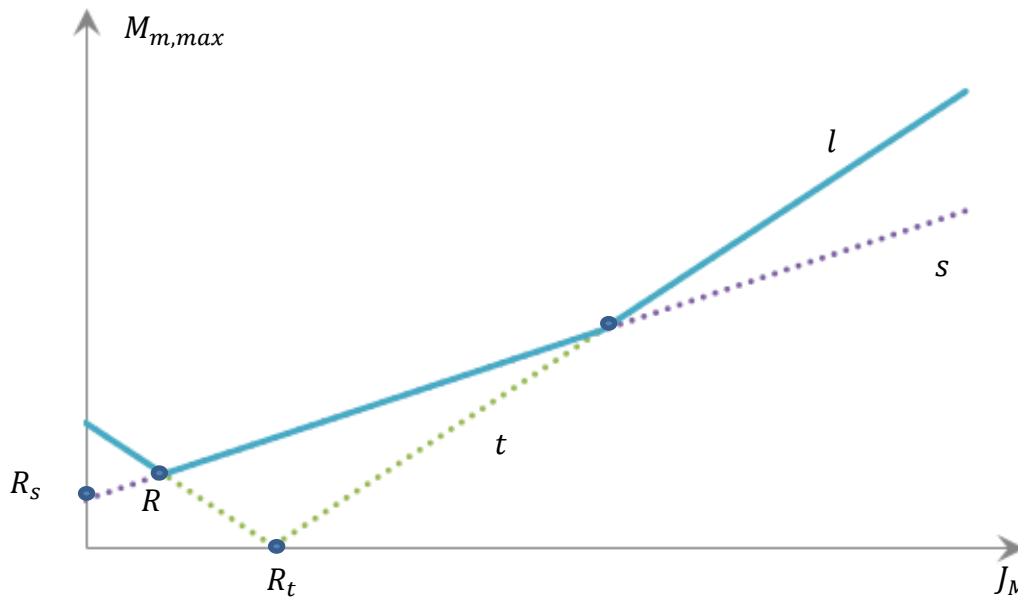
In any case, the curve  $|M_m|$  versus  $J_M$  is continuous.

## 5.2 The $M_{m,max}$ versus $J_M$ curve<sup>[1]</sup>

We wish to find the maximum torque among the linear lines of  $J_M$ - $M_m$ . The curve  $M_{m,max}$  versus  $J_M$  depends on  $\alpha_L$  and  $M_L^*$ , both of which assume different values over time. For each value of  $J_M$ , our aim is to obtain the maximum value assumed over time by  $M_m$ . In presence of many pairs  $(\alpha_L, M_L^*)$ , we now consider, for each value of  $J_M$ , the curve having the maximum value of  $M_m$ . Therefore we obtain a new curve, which is the curve  $J_M$ - $M_{m,max}$ . The calculation of  $M_{m,max}$  can be expressed:

$$M_{m,max} = \max_{J_M} [|M_m|] = \max_t \left[ \left| \frac{\alpha_L(t)}{\tau} J_M + M_L^*(t) \tau \right| \right] \quad (5.4)$$

We can now start by considering only two curves  $s$  and  $t$ , corresponding to two generic points  $S \equiv (\alpha_{L,S}, M_{L,S}^*)$  and  $T \equiv (\alpha_{L,T}, M_{L,T}^*)$ . These curves are shown in **Fig. 5.7**:



**Fig. 5.7** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

We intend to show the  $J_M$ - $M_{m,max}$  curve is continuous, and has only one the minimum value of  $M_{m,max}$ .

It is possible to think that in some abscissas ranges the curve  $s$  lies above the curve  $t$  and the  $J_M$ - $M_{m,max}$  curve  $l$  coincides with the corresponding continuous branches of  $s$ . On the contrary, in other abscissas ranges the curve  $t$  lies above the curve  $s$  and the curve  $l$  coincides with the corresponding continuous branches of  $t$ . The transition from a branch of  $s$  to a branch  $t$  happens with continuity in the intersection points between  $s$  and  $t$ , just as **Fig. 5.7** shows. Because of the reasons mentioned above, the  $J_M$ - $M_{m,max}$  curve is continuous.

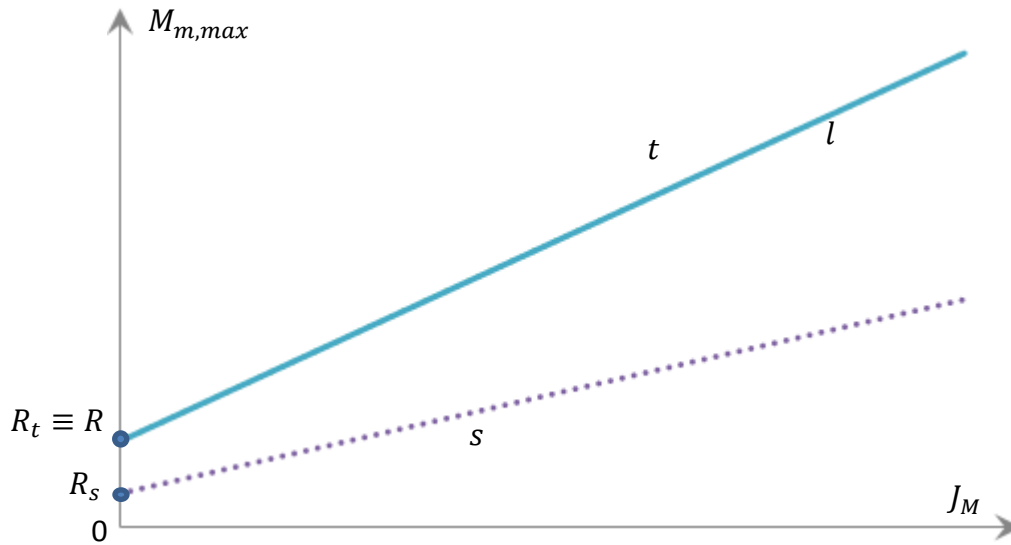
Now let us discuss the minimum value of the curve  $l$ .

There are three main cases of the  $J_M$ - $M_{m,max}$  curve

### 5.2.1 The two pairs of value $(\alpha_{L,S}, M_{L,S}^*)$ and $(\alpha_{L,T}, M_{L,T}^*)$ in the first or third quadrant:

According to discussion before, if  $\alpha_L$  and  $M_L^*$  have the same sign, the curve  $s$  and  $t$  would ascend monotonically. Here we have two different situations:

### 5.2.1.1 $|\alpha_{L,S}| < |\alpha_{L,T}|$ and $|M_{L,S}^*| < |M_{L,T}^*|$



**Fig. 5.8** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

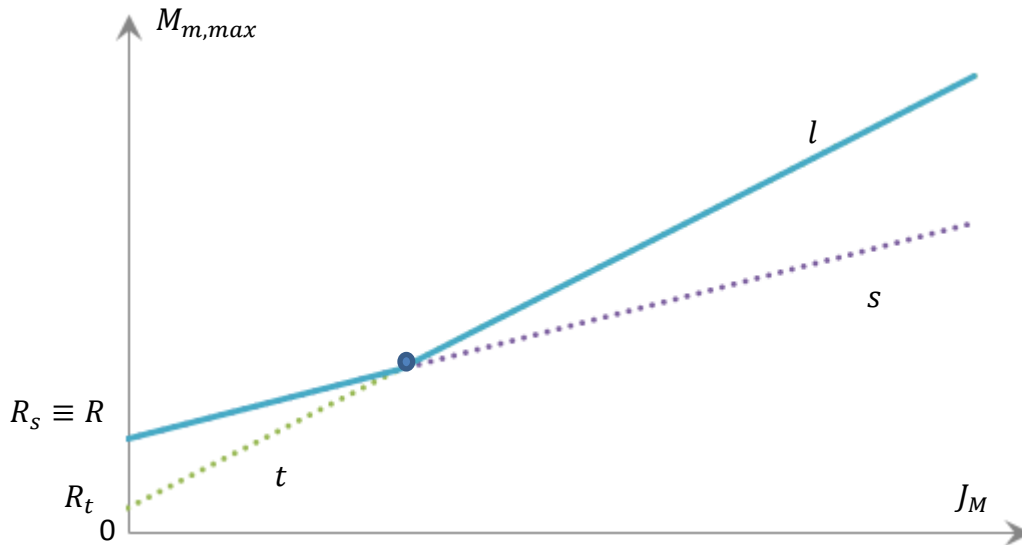
As **Fig. 5.8** shows above, the curve  $s$  and  $t$  have no intersection points. And the minimum point of one curve (for example, curve  $t$  in **Fig. 5.8**) lies above the other curve (curve  $s$  in **Fig. 5.8**), then the minimum point  $R_t$  of the curve  $t$  is a minimum point of the  $J_M$ - $M_{m,max}$  curve  $l$ . In fact, since the curve  $t$  is above curve  $s$ , the curve  $l$  is overlapping the whole curve  $t$ . Because of this reason, the minimum point of curve  $l$  coincides with the curve  $t$ , which means  $R_t \equiv R$ .

We can also consider the plane  $\alpha_L$  versus  $M_L^*$  to analyze the relative position of the two points  $S_s \equiv (|\alpha_{L,S}|, |M_{L,S}^*|)$  and  $S_t \equiv (|\alpha_{L,T}|, |M_{L,T}^*|)$ . It is obvious that  $S_s$  lies on the left side of  $S_t$  which means that the curve  $s$  does not give contribution to the curve  $l$ .

Therefore in this situation, the point  $R_t$  is the only minimum point of the curve  $l$ .

### 5.2.1.2 $|\alpha_{L,S}| < |\alpha_{L,T}|$ and $|M_{L,S}^*| > |M_{L,T}^*|$

In this case, as **Fig. 5.9** shows, the curve  $s$  and  $t$  have one intersection point. Because both slopes are greater than zero, the two curves are monotonically ascending.



**Fig. 5.9** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

On the left side of the intersection point, the curve  $s$  lies above the curve  $t$  and this branch of  $s$  coincides with the curve  $l$ ; while on the right side of the intersection point, the curve  $t$  lies above the curve  $s$  because the slope of  $t$  is greater and that branch of  $t$  makes up an ascending branch of curve  $l$ .

The curve  $l$  is monotonically ascending. Therefore the minimum point  $R$  of the curve  $l$  coincides with the minimum point  $R_s$  of curve  $s$ , which is at abscissa 0.

In both situations, the curve  $l$  is monotonically ascending. And the curve  $l$  only has one minimum point which is at the axis of ordinate. The minimum value of the maximum torque of the motor is:

$$\min(M_{m,max}) = \max(|M_{L,S}^* \tau|, |M_{L,T}^* \tau|) \text{ if } \alpha_{L,S} M_{L,S}^* > 0 \text{ and } \alpha_{L,T} M_{L,T}^* > 0$$

### 5.2.2 The point $(\alpha_{L,S}, M_{L,S}^*)$ in the first or third quadrant; the point $(\alpha_{L,T}, M_{L,T}^*)$ in the second or fourth quadrant.

According to  $(\alpha_{L,S}, M_{L,S}^*)$  in the first or third quadrant, which means  $\alpha_{L,S}$  and  $M_{L,S}^*$  have the same sign, we can know that the curve  $s$  is monotonically ascending.

Since  $\alpha_{L,T}$  and  $M_{L,T}^*$  have the different sign, no matter  $\alpha_{L,T}$  is greater than zero or not, the curve  $t$  descends from the abscissa 0 to the  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$ . In fact, if the

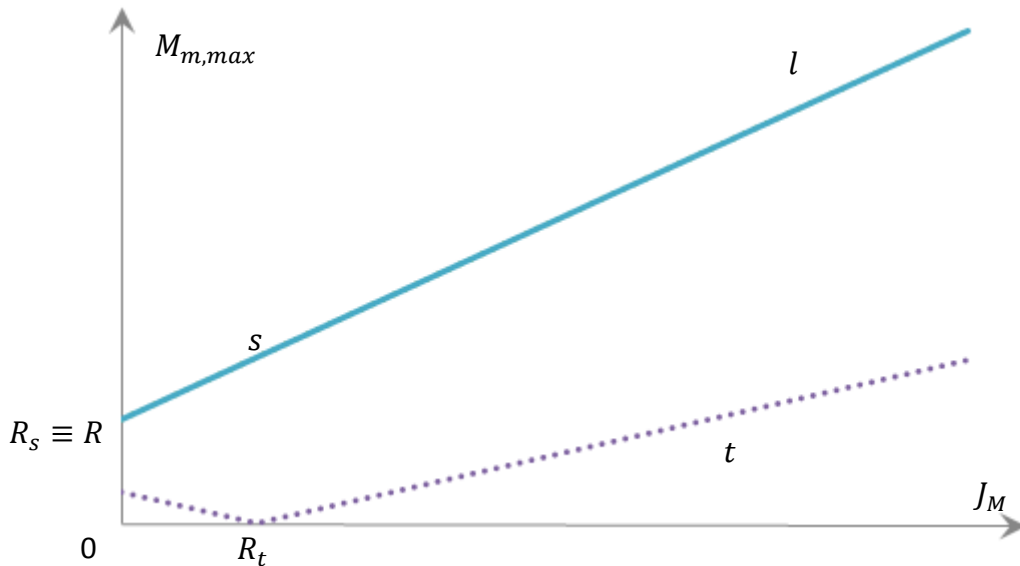


point of  $(\alpha_{L,T}, M_{L,T}^*)$  is in the second quadrant, from abscissa 0 to abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  the curve remains unaffected with respect to **Fig. 5.4**, this means it monotonically descends. On the contrary, if the point of  $(\alpha_{L,T}, M_{L,T}^*)$  is in the fourth quadrant, from abscissa 0 to abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  the branch of the curve is symmetric of the corresponding original branch in **Fig. 5.5** with respect to the abscissa axis, which means that it also monotonically descends.

The curve  $t$  should have an edge point on the abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  with an ordinate equal to 0. The curve  $t$ , no matter the point  $(\alpha_{L,T}, M_{L,T}^*)$  is in the second or the fourth quadrant, becomes monotonically ascending from abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  to  $+\infty$ . In fact if  $\alpha_{L,T} < 0$  and  $M_{L,T}^* > 0$ , the branch of the curve  $t$  from abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  to abscissa  $+\infty$  is symmetric of the corresponding branch in **Fig. 5.4** with respect to the abscissa axis which is monotonically ascending; while if  $\alpha_{L,T} > 0$  and  $M_{L,T}^* < 0$ , the branch of the curve  $t$  from abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  to abscissa  $+\infty$  remains unaffected, which means that it monotonically ascends.

And now we have four different situations:

### 5.2.2.1 $|\alpha_{L,S}| \geq |\alpha_{L,T}|$ and $|M_{L,S}^*| \geq |M_{L,T}^*|$



**Fig. 5.10** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

As the **Fig. 5.10** shows above, if  $|M_{L,S}^*| > |M_{L,T}^*|$ , the intersection between the curve  $s$  and the ordinate is higher than that of the curve  $t$ , the curve  $s$  and  $t$  have no intersection points before the abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$ . After the minimum point  $R_t$  of curve  $t$ , because the slope of curve  $t$  is not greater than that of curve  $s$ , these two curves have no intersection points.

Especially when  $|M_{L,S}^*| = |M_{L,T}^*|$ , which means that the two curves have the same  $M_{m,max}$ -intercept, the curve  $s$  and  $t$  have one intersection points at the ordinate.

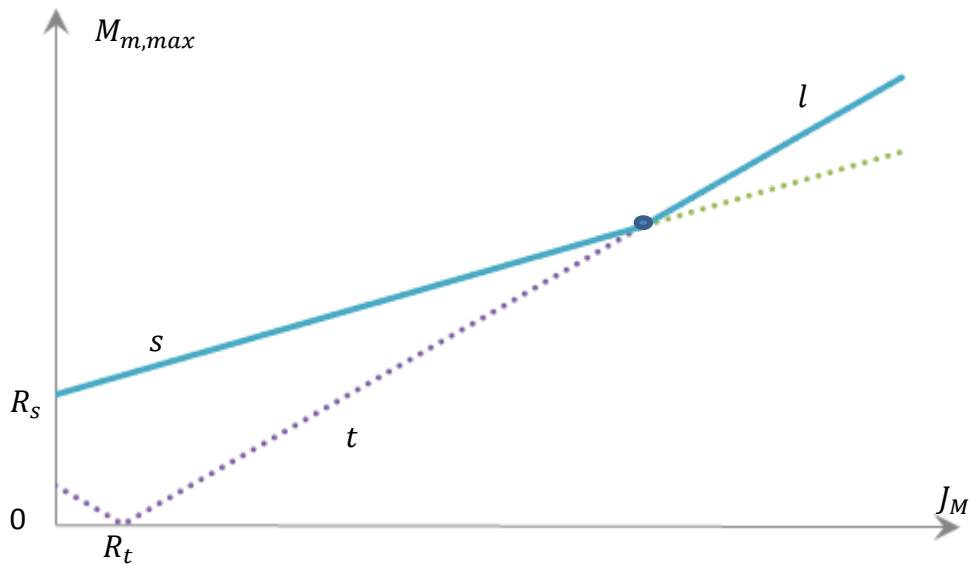
We can easily find that every point of curve  $s$  lies above the curve  $t$  (except the point at the ordinate when  $|M_{L,S}^*| = |M_{L,T}^*|$ ). And the minimum point  $R_s$  of curve  $s$  is a minimum point of the  $J_M$ - $M_{m,max}$  curve  $l$ . In fact, since the curve  $s$  is above  $t$ , the curve  $l$  is overlapping the whole curve  $s$ . Because of this reason, the minimum point of curve  $l$  coincides with the curve  $s$ , which means  $R_s \equiv R$ .

We can also consider the plane of  $\alpha_L$  versus  $M_L^*$  to analyze the relative position of the two points  $S_s \equiv (|\alpha_{L,S}|, |M_{L,S}^*|)$  and  $S_t \equiv (|\alpha_{L,T}|, |M_{L,T}^*|)$ . It is obvious that  $S_s$  lies on the upper right side of  $S_t$  which means that the curve  $t$  does not have

contribution to the curve  $l$ .

Therefore in this situation, the point  $R_s$  is the only minimum point of the curve  $l$ .

### 5.2.2.2 $|\alpha_{L,S}| < |\alpha_{L,T}|$ and $|M_{L,S}^*| \geq |M_{L,T}^*|$



**Fig. 5.11** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

In this case, as the **Fig. 5.11** shows curve  $s$  is monotonically ascending and the curve  $t$  descends before the abscissa  $J_M = \frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$ . If  $|M_{L,S}^*| > |M_{L,T}^*|$ , which means that the intersection between the curve  $s$  and the ordinate is higher than that of the curve  $t$ , the curve  $s$  and  $t$  have no intersection points before the abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$ . After that abscissa of curve  $t$ , because  $|\alpha_{L,S}| < |\alpha_{L,T}|$ , which means that the slope of curve  $t$  is greater than that of curve  $s$ , these two curves have one intersection point.

Especially when  $|M_{L,S}^*| = |M_{L,T}^*|$ , which means that the two curves have the same  $M_{m,max}$ -intercept, the curve  $s$  and  $t$  have one intersection point at the ordinate axis. In this case, the two curves have two intersection points.

From the left side of the intersection point, the curve  $s$  lies above the curve  $t$  and this branch of  $s$  coincides with the curve  $l$  (except the point at the ordinate when  $|M_{L,S}^*| = |M_{L,T}^*|$ ); while to the right side of the intersection point, the curve  $t$  lies

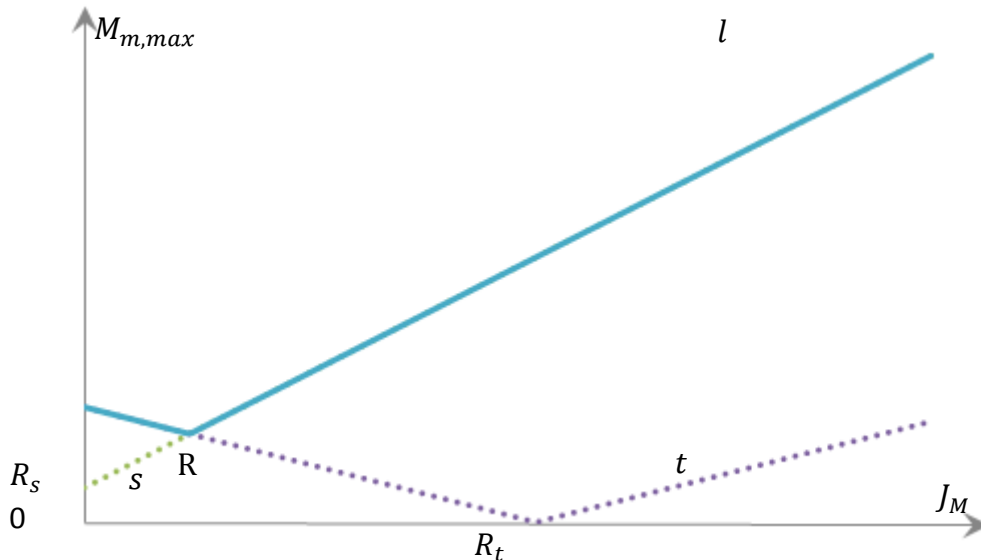
above the curve  $s$  because the slope of  $t$  is greater and that branch of  $t$  makes up an ascending branch of curve  $l$ .

These two branches contribute the curve  $l$ . The curve  $l$  is monotonically ascending. Therefore the minimum point  $R$  of the curve  $l$  coincides with the minimum point  $R_s$  of curve  $s$ , which is at position of the abscissa 0.

Therefore the only minimum point of curve  $l$  is  $R_s$ .

### 5.2.2.3 $|\alpha_{L,S}| \geq |\alpha_{L,T}|$ and $|M_{L,S}^*| < |M_{L,T}^*|$

In this case, as the **Fig. 5.12** shows curve  $s$  is monotonically ascending and the curve  $t$  descends before the abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$ . The condition of  $|M_{L,S}^*| < |M_{L,T}^*|$  means the intersection between the curve  $s$  and the ordinate is lower than that of the curve  $t$ . Because of that reason the curve  $s$  and  $t$  have one intersection point before the abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$ . Due to  $|\alpha_{L,S}| \geq |\alpha_{L,T}|$ , which means the slope of curve  $t$  is not greater than that of curve  $s$ , these two curves have no intersection points after that edge point of curve  $t$ .



**Fig. 5.12** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

It is shown in **Fig. 5.12**, to the left side of the intersection point, the curve  $t$  lies above the curve  $s$  and this branch of  $t$  coincides with the curve  $l$ ; while to the right

side of the intersection point, the curve  $t$  lies below the curve  $s$  and the branch of  $s$  makes up an ascending branch of curve  $l$ .

These two branches contribute the curve  $l$ . The curve  $l$  descends before the intersection point and after that it ascends. Therefore the minimum point  $R$  of the curve  $l$  is the intersection point and it is the exclusive minimum point.

#### 5.2.2.4 $|\alpha_{L,S}| < |\alpha_{L,T}|$ and $|M_{L,S}^*| < |M_{L,T}^*|$

In this case, as the **Fig. 5.7** shows, the curve  $s$  is monotonically ascending; the curve  $t$  descends before the abscissa  $J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$  and it ascends after the minimum

point  $R_t$ . The condition of  $|M_{L,S}^*| < |M_{L,T}^*|$  means the intersection between the curve  $s$  and the ordinate is lower than that of the curve  $t$ . Therefore the curve  $s$  and  $t$  have one intersection point before the abscissa  $J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$ . Due to

$|\alpha_{L,S}| < |\alpha_{L,T}|$ , which means the slope of curve  $t$  is greater than that of curve  $s$ , to

the right side of the abscissa  $J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$ , which means from the abscissa

$J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$  to the positive infinite the two curves must have one intersection

point. In total, there are two intersections of these two curves.

It can be seen in **Fig. 5.7** that to the left side of the intersection point, the curve  $t$  lies above the curve  $s$  and this branch of  $t$  coincides with the curve  $l$ . Because from

the abscissa 0 to abscissa  $J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$  the curve  $t$  descends and the curve  $s$  ascends, there must be only one intersection point. And from the right side of

abscissa  $J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$  to the other intersection point, the curve  $s$  lies above the  $t$ ,

so this branch of  $s$  makes up curve  $l$ . After that, every point of curve  $t$  is above the point of  $s$ , so this branch of  $t$  coincides with curve  $l$ .

These three branches contribute the curve  $l$ . The curve  $l$  descends before the first intersection point and after that it ascends. Therefore the minimum point  $R$  of the curve  $l$  is the first intersection point and it is the only one minimum point.

To sum up, in the case 5.2.2, there is only one minimum point of  $M_{m,max}$ .

### 5.2.3 The $(\alpha_{L,S}, M_{L,S}^*)$ and $(\alpha_{L,T}, M_{L,T}^*)$ in the second or fourth quadrant:

In this case, both  $\alpha_{L,S}M_{L,S}^*$  and  $\alpha_{L,T}M_{L,T}^*$  are smaller than zero, i.e. they have different sign to each other. No matter  $\alpha_{L,T}$  or  $\alpha_{L,S}$  is greater than zero or not, the curve  $t$  and  $s$  descends from the abscissa 0 to the minimum points of themselves.

The curve  $t$  has an edge point on the abscissa  $J_M = -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$  with an ordinate equal to 0; while the curve  $s$  has an edge point on the abscissa  $J_M = -\frac{M_{L,S}^*\tau^2}{\alpha_{L,S}}$  with an ordinate equal to 0. And both of them monotonically ascend from the edge points to the abscissa positive infinite.

Because the curve  $t$  and curve  $s$  can replace each other, there is no difference between  $|M_{L,S}^*|$  is greater than  $|M_{L,T}^*|$  or not. So here we assume that  $|M_{L,S}^*| > |M_{L,T}^*|$ , which means the  $M_{m,max}$ -intercept point of curve  $s$  is higher than the point of curve  $t$ . And we are interested in the relative position of the edge points of the two curves. So we have following three situations.

$$5.2.3.1 \quad -\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}} > -\frac{M_{L,S}^*\tau^2}{\alpha_{L,S}}$$

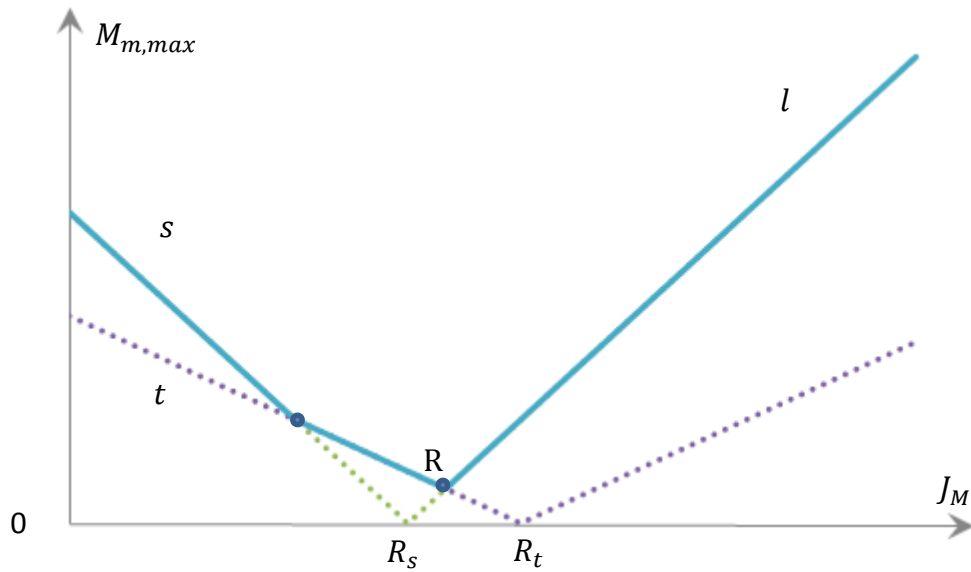
In this case, we first discuss the relation between  $|\alpha_{L,S}|$  and  $|\alpha_{L,T}|$ . Because the general load torques  $M_L^*$  and the load angular accelerations  $\alpha_L$  of both curves have different sign,  $-\frac{M_{L,T}^*\tau^2}{\alpha_{L,T}}$  should be smaller than zero. Therefore the relation of absolute value of them would be

$$\left| \frac{M_{L,T}^*\tau^2}{\alpha_{L,T}} \right| > \left| \frac{M_{L,S}^*\tau^2}{\alpha_{L,S}} \right|$$

It is assumed that  $|M_{L,S}^*| > |M_{L,T}^*|$  and we can obtain that:

$$|\alpha_{L,T}| < |\alpha_{L,S}|$$

The example curves of  $s$  and  $t$  is shown in the **Fig. 5.13**:



**Fig. 5.13** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

Because  $|\alpha_{L,T}| < |\alpha_{L,S}|$ , the slope of curve  $s$  is greater than curve  $t$ . The

$M_{m,max}$ -intercept point of curve  $s$  is higher than the point of curve  $t$ . And also the minimum point  $R_s$  of curve  $s$  is to the left side of  $R_t$ , so there are two intersections in the descending branch of curve  $t$ . After the edge point  $R_t$  of curve  $t$ , because the slope of  $t$  is always smaller than  $s$ , there are no intersections.

To the left side of the first intersection point, the curve  $s$  lies above the curve  $t$  and this branch of  $s$  coincides with the curve  $l$ ; while from the first to the second intersection point,  $t$  is above  $s$  whose branch makes up of the curve  $l$ . These two branches monotonically descend. To the right side of the second intersection, branch of  $s$  contribute the curve  $l$ , which is ascending. So the minimum point  $R$  of the curve  $l$  is the second intersection point and it is the exclusive minimum point.

$$5.2.3.2 \quad -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}} = -\frac{M_{L,S}^* \tau^2}{\alpha_{L,S}}$$

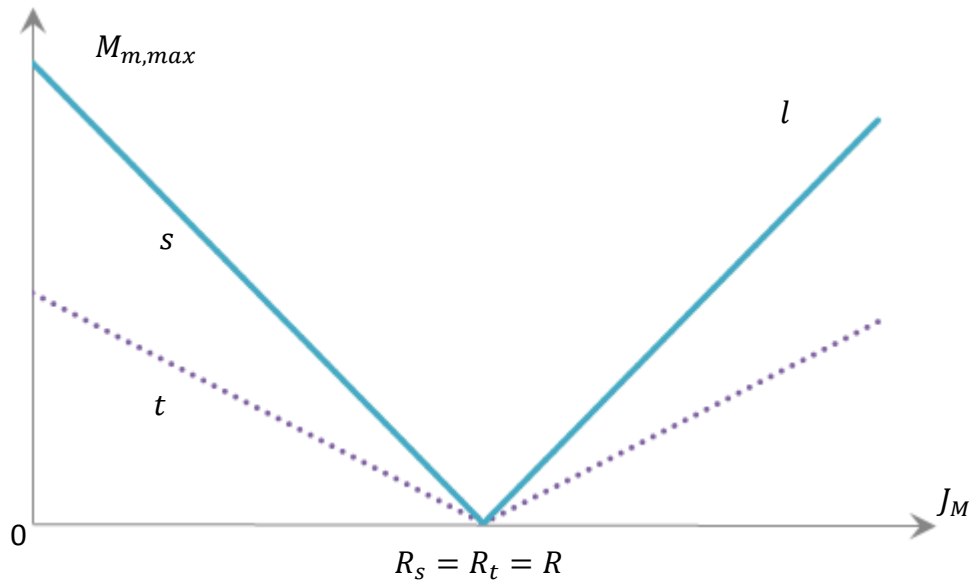
In this special case, the minimum points of two curves coincide in the same point. So we have

$$\left| \frac{M_{L,T}^* \tau^2}{\alpha_{L,T}} \right| = \left| \frac{M_{L,S}^* \tau^2}{\alpha_{L,S}} \right|$$

And because of  $|M_{L,S}^*| > |M_{L,T}^*|$  we can easily achieve the inequality:

$$|\alpha_{L,T}| < |\alpha_{L,S}|$$

The inequality shows the slope of the curve  $s$  is greater than that of curve  $t$ .



**Fig. 5.14** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

As the **Fig. 5.14** shows above, to the left side and to the right side of the minimum point there is no intersection point. And we can see curve  $s$  is above  $t$ , except the minimum point  $R_s$  is overlapping with  $R_t$ . The curve  $l$  is totally coinciding with the curve  $s$ . Therefore, the curve has only one minimum point  $R$  which is at the abscissa

$$J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}} = -\frac{M_{L,S}^* \tau^2}{\alpha_{L,S}} \text{ with ordinate equal to zero.}$$

$$5.2.3.3 \quad -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}} < -\frac{M_{L,S}^* \tau^2}{\alpha_{L,S}}$$

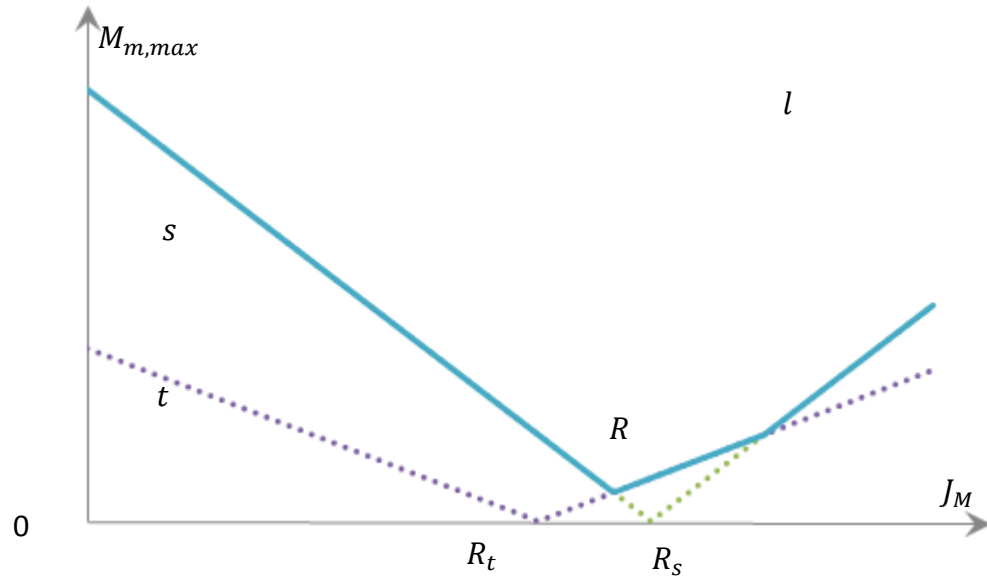
In this case, the relation of absolute value of them would be

$$\left| \frac{M_{L,T}^* \tau^2}{\alpha_{L,T}} \right| < \left| \frac{M_{L,S}^* \tau^2}{\alpha_{L,S}} \right|$$

It is assumed that  $|M_{L,S}^*| > |M_{L,T}^*|$  and we cannot tell if  $|\alpha_{L,T}|$  is greater than  $|\alpha_{L,S}|$  or not. So here come two different situations.

If  $|\alpha_{L,T}| < |\alpha_{L,S}|$  the example curves of  $s$  and  $t$  is shown in the following figure:





**Fig. 5.15** Example of  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

From **Fig. 5.15** we can see that the slope of curve  $s$  is greater than curve  $t$ . And also the minimum point  $R_s$  of curve  $s$  is to the right side of  $R_t$ , so there are two intersections in the ascending branch of curve  $t$ .

To the left side of the first intersection point, the curve  $s$  lies above the curve  $t$  and this branch of  $s$  coincides with the curve  $l$  and it monotonically descends. While from the first to the second intersection point, the ascending branch of  $t$  is above  $s$  which makes up of the curve  $l$ . To the right side of the second intersection, branch of  $s$  contribute the curve  $l$ , which is also ascending. So the minimum point  $R$  of the curve  $l$  is the first intersection point and it is the exclusive minimum point.

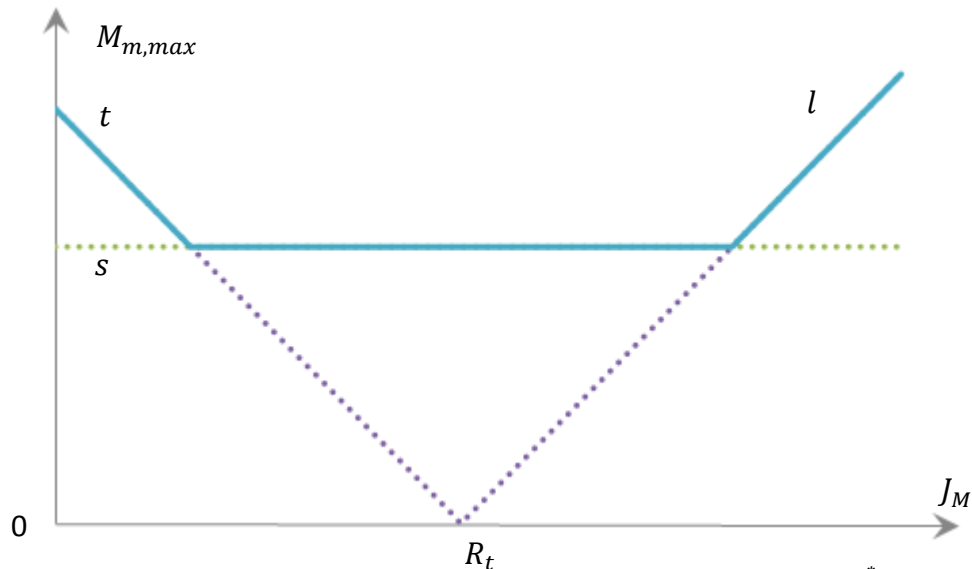
If  $|\alpha_{L,T}| \geq |\alpha_{L,S}|$ , due to the slope of curve  $s$  is not greater than the slope of  $t$ , there would be no intersection after the minimum point  $R_s$ . To the left side of the point  $R_s$ , the situation is the same as before. Therefore the minimum point  $R$  of the curve  $l$  is the intersection point and it is the exclusive minimum point.

#### 5.2.4 When $\alpha_{L,S}$ is equal to zero:

This case is a very special situation because the  $J_M$ - $M_{m,max}$  curve may have a flat branch which will lead to infinite minimum points.

For example, we assume that  $|M_{L,S}^*| < |M_{L,T}^*|$  and  $\alpha_{L,T}M_{L,T}^* < 0$ .

In this case, the point of the pair of value  $(\alpha_{L,T}, M_{L,T}^*)$  is in the second or the fourth quadrant. So the curve  $t$  will descend from abscissa 0 to the abscissa  $J_M = -\frac{M_{L,T}^* \tau^2}{\alpha_{L,T}}$  and ascend after the edge point. And the  $M_{m,max}$ -intercept of curve  $s$  is below curve  $t$ . So there are two intersection points.



**Fig. 5.16**  $J_M$ - $M_{m,max}$  curve corresponding to two pairs of values  $(\alpha_{L,S}, M_{L,S}^*)$  and  $(\alpha_{L,T}, M_{L,T}^*)$

**Fig. 5.16** shows the curve  $l$  coincides with curve  $s$  between the two intersection points. Because curve  $s$  is flat, there are infinite minimum points in curve  $l$ . However the minimum value of  $M_{m,max}$  are the same. So even in this special case the  $M_{m,max}$  has only one minimum value.

To sum up, the minimum point  $R$  of the  $J_M$ - $M_{m,max}$  curve can be either a regular point or an edge point. Especially when the angular acceleration of the load is equal to zero, the minimum point  $R$  would become a flat line which is parallel to the abscissa  $J_M$ . And there is only one minimum value of  $J_M$ - $M_{m,max}$  curve.

It is simple to extend this demonstration to the case of more than two curves.

It is clear that when considering all the pairs  $(\alpha_L, M_L^*)$ , the corresponding  $J_M$ - $M_{m,max}$  curve  $l$  presents the following characteristics:

- When the abscissa  $J_M$  is zero, the  $J_M$ - $M_{m,max}$  curve  $l$  coincides with the maximum value of  $M_{m,max}$ -intercept of the curves.
- When the abscissa  $J_M$  approach to  $+\infty$ , the  $J_M$ - $M_{m,max}$  curve  $l$  coincides with the line whose slope has the maximum value among the curves.
- The  $J_M$ - $M_{m,max}$  curve  $l$  has only one minimum value.

## 5.3 The points $(\alpha_L, M_L^*)$ defines the $J_M$ - $M_{m,max}$ curve<sup>[1]</sup>

### 5.3.1 New points $(\alpha_L, M_L^*)$ for the choice of motor

For the choice of motor and transmission, the analysis leads us to consider in the plane  $\alpha_L - M_L^*$  there are new points  $S' \equiv (\alpha_L', M_L^{*'})$  that are different from the original points  $S \equiv (\alpha_L, M_L^*)$ .

If the original point  $S_1$  lies in the first quadrant, i.e.  $\alpha_L \geq 0$  and  $M_L^* \geq 0$ , as we discuss before, the  $M_{m,max}$  versus  $J_M$  curve remain the same respect to that in **Fig. 5.2**. We must consider a new point  $S_{1,1}' = S_1 = (\alpha_L, M_L^*)$ , that lies also in the first quadrant.

If the original point  $S_3$  lies in the third quadrant, i.e.  $\alpha_L \leq 0$  and  $M_L^* \leq 0$ , as **Fig. 5.6** shows, the  $M_{m,max}$  versus  $J_M$  curve is symmetric of that in **Fig. 5.3** with respect to the abscissa axis. We must consider a new point  $S_{3,1}' = (-\alpha_L, -M_L^*)$ , that lies in the first quadrant.

If the original point  $S_2$  lies in the second quadrant, i.e.  $\alpha_L \leq 0$  and  $M_L^* \geq 0$ , as **Fig. 5.6** shows, On the left of the minimum point the curve descends monotonically. This branch is equal to the corresponding branch in **Fig. 5.4**, while on the right of the minimum point the curve ascends monotonically when  $J_M$  increases. This branch is symmetric of the corresponding branch in **Fig. 5.4** with respect to the abscissa axis. Therefore together with the point  $S_{2,2}' \equiv S_2$  which lies in the second quadrant representing the descending branch of the  $M_{m,max}$  versus  $J_M$  curve, we must also consider another point  $S_{2,4}' = (-\alpha_L, -M_L^*)$  that lies in the fourth quadrant. It represents the ascending branch of the same curve. These two points  $S_{2,2}'$  and  $S_{2,4}'$  are symmetric with respect to the origin.

If the original point  $S_4$  lies in the fourth quadrant, i.e.  $\alpha_L \leq 0$  and  $M_L^* \leq 0$ . As **Fig. 5.6** shows, on the left of the minimum point the curve descends monotonically when  $J_M$  increases. This branch is symmetric of the corresponding branch in **Fig. 5.5** with respect to the abscissa axis, while on the right of the minimum point the curve ascends monotonically when  $J_M$  increases. This branch is equal to the corresponding branch in **Fig. 5.5**. Therefore together with the point  $S_{4,4}' \equiv S_4$  which lies in the fourth quadrant representing the ascending branch of the  $M_{m,max}$  versus  $J_M$  curve, we must also consider another point  $S_{4,2}' = (-\alpha_L, -M_L^*)$  that lies in the second quadrant. It represents the ascending branch of the same curve. These two points  $S_{4,2}'$  and  $S_{4,4}'$  are symmetric with respect to the origin.

To sum up,

- If  $\alpha_L$  and  $M_L^*$  have the same sign, we will consider one set of points  $S_{1,1}'$  and  $S_{3,1}'$  lying in the first quadrant;

- If  $\alpha_L$  and  $M_L^*$  have the opposite signs, we will consider two sets of symmetric points:  
 $S_{2,2}'$  and  $S_{4,2}'$  in the second quadrant only contribute to the descending branch of the  $J_M$ - $M_{m,max}$  curve.  
 $S_{2,4}'$  and  $S_{4,4}'$  in the fourth quadrant only contribute to the ascending branch of the  $J_M$ - $M_{m,max}$  curve.

### 5.3.2 General consideration

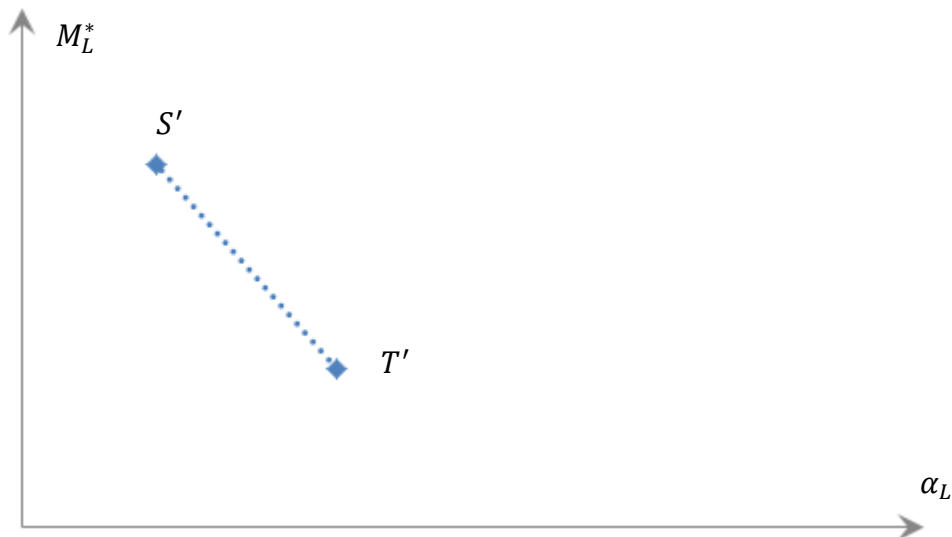
We now consider the plane  $\alpha_L$ - $M_L^*$  and the points  $S'$  which lie in the first, second and fourth quadrant. We now identify the points able to contribute to this curve.

In any case, not all these points play a part in the definition of the  $J_M$ - $M_{m,max}$  curve. Now let us identify the points able to contribute to this curve.

It is easy to observe that in the plane  $J_M$ - $M_{m,max}$ , corresponding to two generic points  $S'$  and  $T'$  of the plane  $\alpha_L$ - $M_L^*$ , the two curves  $s'$  and  $t'$  intersect at the abscissa:

$$J_M = -\frac{M_{L,S'}^* - M_{L,T'}^*}{\alpha_{L,S'} - \alpha_{L,T'}} \tau^2 \quad (5.5)$$

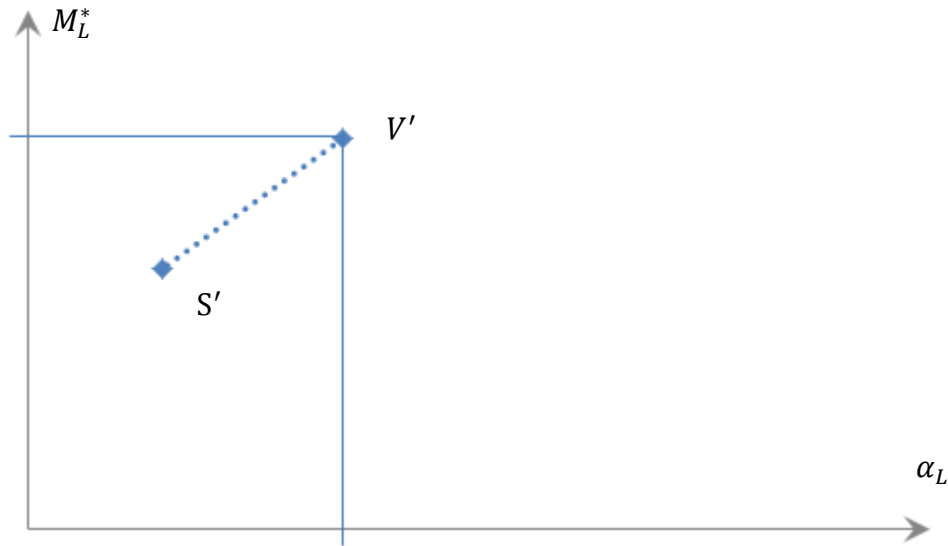
Because  $J_M \geq 0$ , the intersection exists only if the slope of the segment  $S'T'$  is negative or equal to zero. (see **Fig. 5.17**)



**Fig. 5.17** Two points  $S'$  and  $T'$  whose curve intersect

If the point  $V' \equiv (\alpha_{L,V'}, M_{L,V'}^*)$  as the vertex of the quadrant below and on its left,

just as **Fig. 5.18** shows below: the point  $V'$  excludes any other point  $S'$  belonging to this quadrant from contributing the  $J_M-M_{m,max}$  curve  $l$ .



**Fig. 5.18** The point  $T'$  prevents the point  $S'$  from contributing to the  $J_M-M_{m,max}$  curve

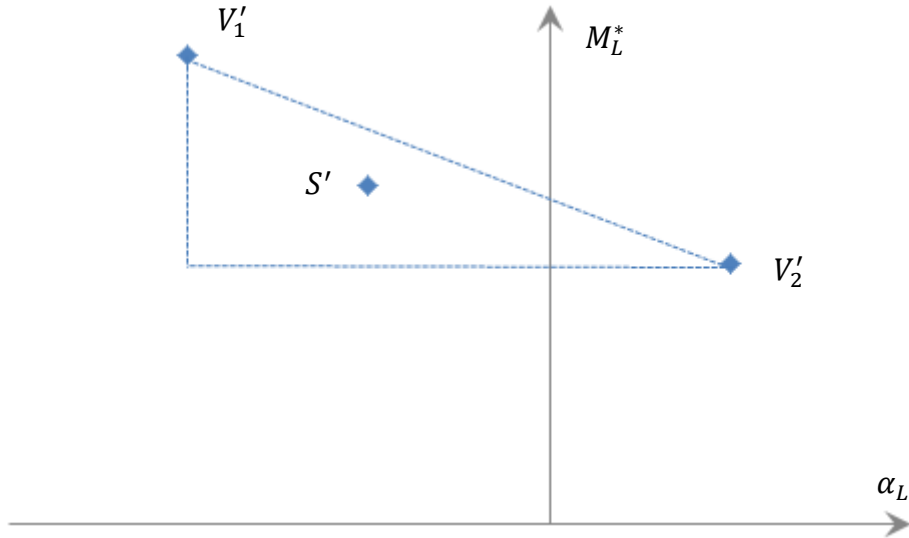
In fact, as we have discussed in the chapter 5.2, if the point  $S'$  is at the left and down to the point  $V'$ , the segment  $S'V'$  shows a positive slope. Because of Eq. (5.4) the corresponding two curves  $s'$  and  $v'$  in the plane  $J_M-M_{m,max}$  do not intersect (see **Fig. 5.8** and **Fig. 5.10**). The curve  $v'$  lies above the curve  $s'$  so the curve  $s'$  is excluded from giving its contribution to the  $J_M-M_{m,max}$  curve  $l$ .

In the plane  $\alpha_L-M_L^*$ , the points which are able to contribute to the  $J_M-M_{m,max}$  curve are positioned along a line which is a monotonous descending line, because the slope between two near points must be negative.

Therefore, in order to draw the  $J_M-M_{m,max}$  curve, we can find the point having the maximum ordinate as the start point and find the point having maximum abscissa as the end point.

Now we consider three points:  $V'_1$ ,  $V'_2$  and  $S'$ .  $V'_1$  and  $S'$  are located in the second quadrant, while  $V'_2$  is located in the first quadrant.

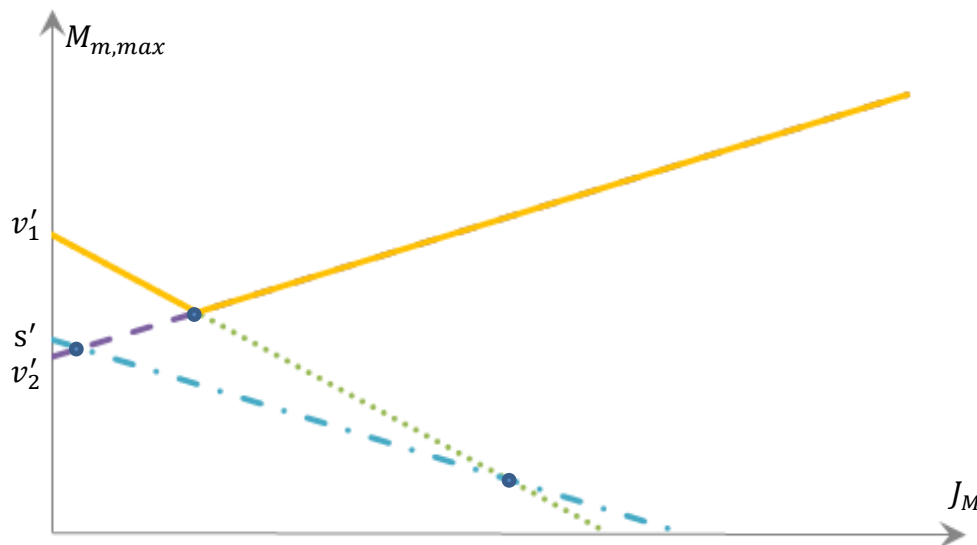
The segment  $V'_1V'_2$  have negative slope. And the point  $S'$  is located in the right-angled triangle having  $V'_1V'_2$  as hypotenuse. The curves corresponding to the three points are  $v'_1$ ,  $v'_2$  and  $s'$  which are both located in the first quadrant.



**Fig. 5.19** The point  $V_1'$  and  $V_2'$  prevents the point  $S'$  from contributing to the  $J_M$ - $M_{m,max}$  curve

From Eq. (5.5) we can know that due to the slope of segment  $V_1'V_2'$  is smaller than the slope of segment  $V_1'S'$  so the abscissa of intersection of curves  $v_1'$  and  $s'$  lies to the right side of the abscissa of intersection of curves  $v_1'$  and  $v_2'$ . On the other hand, the slope of segment  $V_1'V_2'$  is greater than the slope of segment  $S'V_2'$  so the abscissa of intersection of curves  $v_2'$  and  $s'$  lies to the left side of the abscissa of intersection of curves  $v_1'$  and  $v_2'$ .

In **Fig. 5.20** it can be seen that the points  $V_1'$ ,  $V_2'$  exclude any other point  $S'$  lying in the above-mentioned triangle from contributing the  $J_M$ - $M_{m,max}$  curve  $l$ .

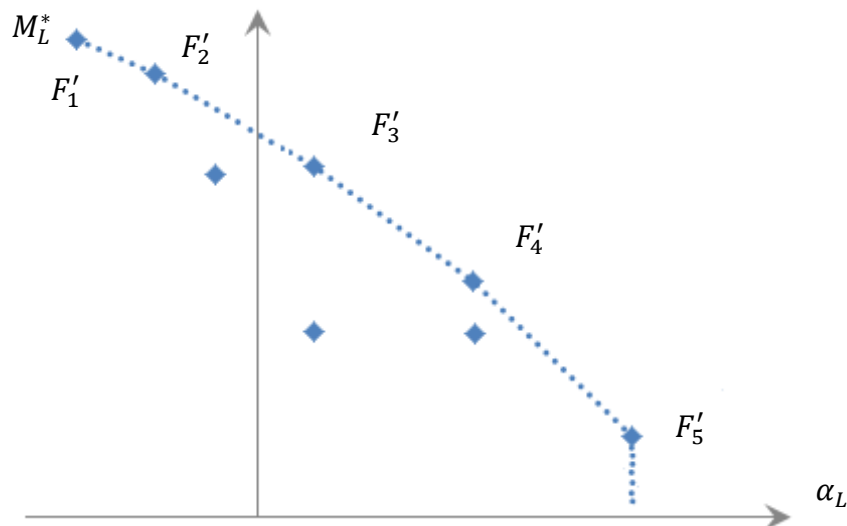


**Fig. 5.20** The  $J_M$ - $M_{m,max}$  curves corresponding to the points in **Fig. 5.19**

The rules demonstrated above can identify easily the points taking part in the

definition of  $J_M-M_{m,max}$  curve  $l$ , as **Fig. 5.21** shows, by tracing an open polygon according to the following procedure:

1. For the all new points in the first or second quadrant we find the point with maximum ordinate  $M_{L,F'}^*$  and call it  $F'_1$ ;
2. We use the point  $F'_1$  as a center, make a half line to the right of  $F'_1$ . Let it rotate clockwise to touch another point  $F'_2$ . Pay attention that the half line should not passing through the vertical;
3. With center in  $F'_2$ , we consider a half line that rotates clockwise, starting from the direction of the segment  $F'_1F'_2$ , until it touches another point without passing through the vertical.
4. Repeat the procedure until the half line passes through the vertical.



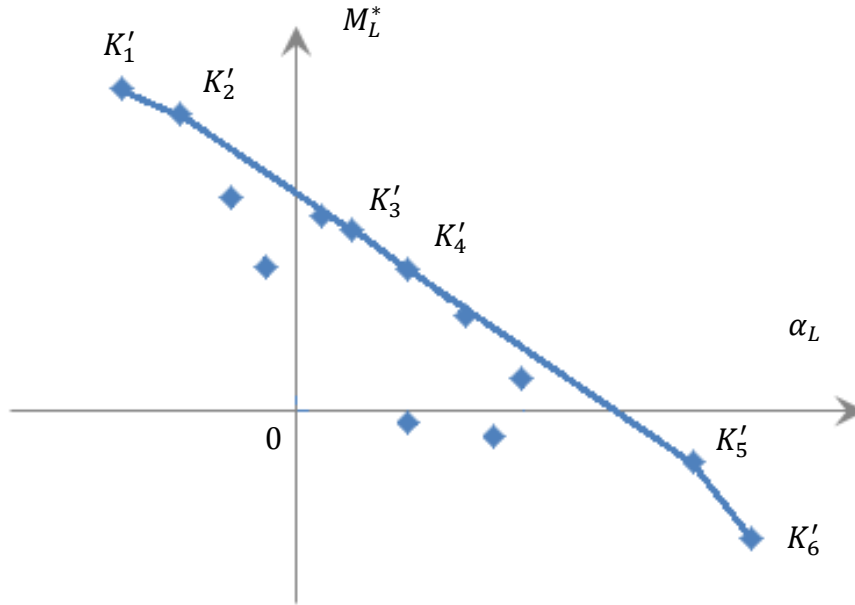
**Fig. 5.21** The points  $F'_i$  that contribute to the  $J_M-M_{m,max}$  curve

In this way we obtain an open polygon whose concavity is downwards. And it is simple to find the points  $F'_i$  in the first quadrant of the plane  $\alpha_L-M_L^*$  which make contribute to draw the  $J_M-M_{m,max}$  curve.

### 5.3.3 Global $J_M-M_{m,max}$ curve

Similarly to the analysis before, among the new points lying in the first, second and fourth quadrant of the plane  $\alpha_L-M_L^*$ , we can identify the few points playing a role in the definition of the global  $J_M-M_{m,max}$  curve, and we name them  $K'_i$ . We can use

almost the same procedure to trace the points.



**Fig. 5.22** The points  $K'_i$  that contribute to the global  $J_M$ - $M_{m,max}$  curve

As **Fig. 5.22** shows above, the points  $K'_i$  are positioned along a monotonous descending line. This open polygon has vertices belonging to the tree polygons already found in the first, second and fourth quadrant. We can find the points  $K'_i$  directly, without finding the points  $F'_i$ ,  $G'_i$  and  $H'_i$  before.

Now we wish to find the minimum point R of the  $J_M$ - $M_{m,max}$  curve.

According to Eq. (5.4) the equation of the branch of  $J_M$ - $M_{m,max}$  curve corresponding to the point  $K'_i \equiv (\alpha_{L,K'_i}, M_{L,K'_i}^*)$  is:

$$M_{m,max} = \frac{\alpha_{L,K'_i}}{\tau} J_M + M_{L,K'_i}^* \tau \quad (5.6)$$

The intersection with the next branch has abscissa:

$$J_M = -\frac{M_{L,K'_i}^* - M_{L,K'_{i+1}}^*}{\alpha_{L,K'_i} - \alpha_{L,K'_{i+1}}} \tau^2 \quad (5.7)$$

We use this to calculate the ordinate:

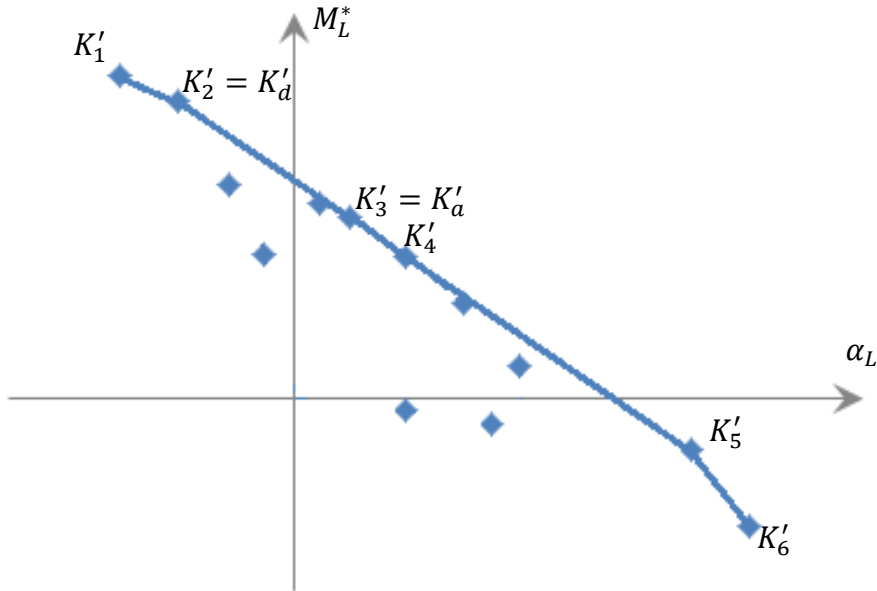
$$\begin{aligned} M_{m,max} &= \frac{\alpha_{L,K'_i} - M_{L,K'_i}^* + M_{L,K'_{i+1}}^*}{\tau \alpha_{L,K'_i} - \alpha_{L,K'_{i+1}}} \tau^2 + M_{L,K'_i}^* \tau \\ &= \left[ \frac{\alpha_{L,K'_i} (M_{L,K'_{i+1}}^* - M_{L,K'_i}^*)}{\alpha_{L,K'_i} - \alpha_{L,K'_{i+1}}} + M_{L,K'_i}^* \right] \tau \end{aligned}$$



$$= \frac{\alpha_{L,K'_i} M_{L,K'_{i+1}}^* - \alpha_{L,K'_{i+1}} M_{L,K'_i}^*}{\alpha_{L,K'_i} - \alpha_{L,K'_{i+1}}} \tau \quad (5.8)$$

The  $J_M - M_{m,max}$  curve is made up of consecutive branches, corresponding to the consecutive points  $K'_i$ . Each branch belongs to a different linear line or a polyline.

From previous analysis we can know that each point  $K'_i$  in the second quadrant brings the descending branch to the  $J_M - M_{m,max}$  curve; while each point in the first and fourth quadrant contributes the ascending branch to the  $J_M - M_{m,max}$  curve. So we name the last point of the polygon in the second quadrant  $K'_d$ , and the first point of the polygon in the first or in the fourth quadrant  $K'_a$ . The  $J_M - M_{m,max}$  curve monotonically descends from start to the branch of  $k'_d$ ; then it ascends from the branch of  $k'_a$  to the abscissa positive infinite. So the minimum point is the intersection of the two branches  $k'_a$  and  $k'_d$ .



**Fig. 5.23** The points  $K'_d$  and  $K'_a$  that determine the minimum point  $R$

And then we can obtain the minimum point  $R$  which has the abscissa:

$$J_M|_R = -\frac{M_{L,K'_d}^* - M_{L,K'_a}^*}{\alpha_{L,K'_d} - \alpha_{L,K'_a}} \tau^2 \quad (5.9)$$

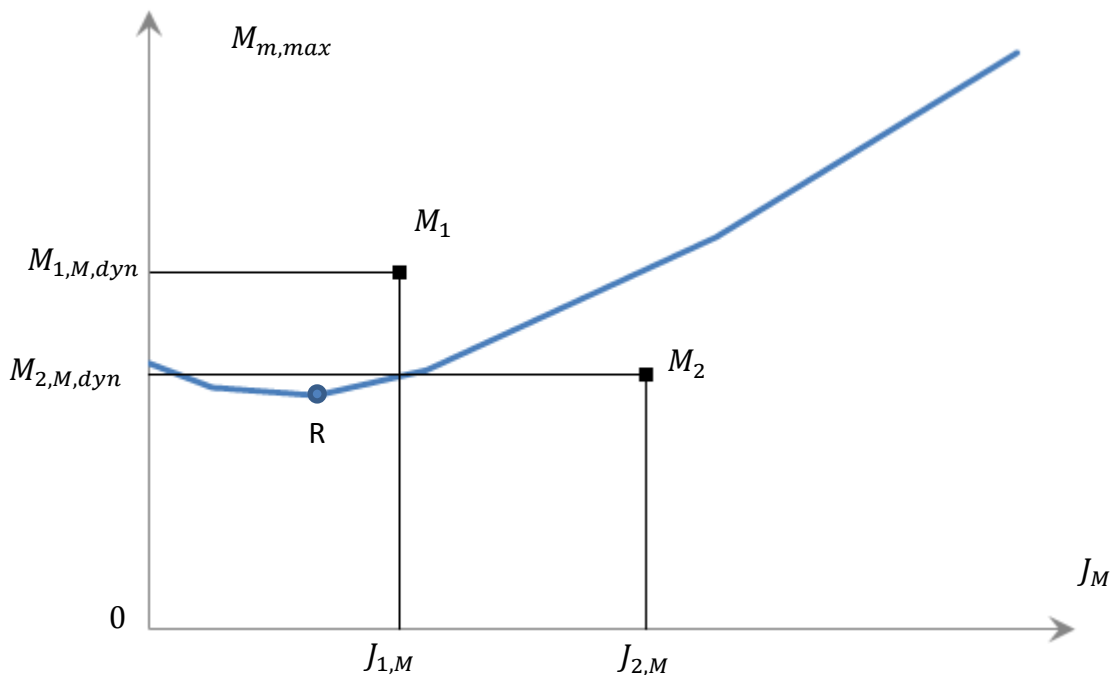
And the ordinate:

$$M_{m,max}|_R = \frac{\alpha_{L,K'_d} M_{L,K'_a}^* - \alpha_{L,K'_a} M_{L,K'_d}^*}{\alpha_{L,K'_d} - \alpha_{L,K'_a}} \tau \quad (5.10)$$

## 5.4 Choosing admissible motors according to the

### $J_M$ - $M_{m,max}$ curve

We have plotted the  $J_M$ - $M_{m,max}$  curve  $l$ , so it is easy to find motors suitable to move the load. As the shows below, in the same diagram we can plot the drive system representative points. A motor is represented by a point  $M_i$  whose abscissa is  $J_M$  and the ordinate is the  $M_{M,dyn}$ . The points representing the different drive systems are reported definitively.



**Fig. 5.24** The  $J_M$ - $M_{m,max}$  curve and the drive system representative points

As the analyze before, in order to let the motor drive the load the maximum torque of the motor in dynamic working range should be larger than that of the load under the same inertia condition. We should check for the motor  $i$

$$M_{m,max}(J_{i,M}) \leq M_{M,dyn}(J_{i,M})$$

It means the representative point of the drive system should under the  $J_M$ - $M_{m,max}$  curve. As **Fig. 5.24** shows, the representative point  $M_1$  of motor 1 is above the curve  $l$ , it is capable of driving the load under the given transmission parameters. On the contrary, the point  $M_2$  of the motor 2 is under the curve  $l$ , so it cannot move the load. In this case, we only choose the motor 1.

To sum up, once chosen the transmission, the knowledge of its direct efficiency  $\eta_d$ , the inverse efficiency  $\eta_i$  and the transmission ratio  $\tau$  together with the reference task allow us to achieve the  $J_M$ - $M_{m,max}$  curve and  $J_M$ - $M_{m,rms}$  curve. If the

representative point of the motor is above the two curves and also satisfies the inequality  $\omega_{M,max} \geq \omega_{m,max}$ , the motor is admissible.

## Chapter 6: A simple case study

In this chapter, we use the above-mentioned method to choose a motor.

### 6.1 Data of the catalogs of the motor and transmission

In the following, we report the specific data related to six electric brushless motors equipped with an inverter.

For example in **Table 6.1** the commercial characterized code of the motor is “1FK7040-5AK71 CT”, while the motor belongs to the series with the following nomenclature: “SINAMICS S120 SLM 400V”<sup>[2]</sup>. The specific data in **Table 6.1** are just a part of the complete data sheet. The example shows three major parts of the technical data: design data, optimal operating point and limit data. In this diagram, the angular velocity, torque and some characteristic of the operation of the engine are delivered. The values of current and torque from braking have been provided for two different temperature rise compared to the environment equal to 297° K, i.e. a temperature difference of 60° K in the first case while at 100 ° K in the second case. And the optimal operating point at maximum power and high efficiency are also defined in the table. It generally does not coincide with the point of maximum torque, while it may match the nominal point. For the limit data, however, define the parameters causing an irreversible damage of the machine. For example, the maximum current is the root mean square value of the current causing a demagnetization of the permanent magnet rotor.

Technical data	Abbreviation	Unit	-5AK71
Nominal velocity	$n_N$	Rpm	6000
Number of poles	$2p$		8
Rated torque(100K)	$M_N$	Nm	1.1
Nominal current	$I_N$	A	1.7
Braking torque (60 K)	$M_o(60 K)$	Nm	1.3
Braking torque (100 K)	$M_o(100 K)$	Nm	1.6
Braking current (60 K)	$I_o(60 K)$	A	1.8
Braking current (100 K)	$I_o(100 K)$	A	2.3
Inertia (with brake)	$J_{motFr}$	$10^{-4}$ kgm <sup>2</sup>	2.41
Inertia(without brake)	$J_{mot}$	$10^{-4}$ kgm <sup>2</sup>	1.7
Optimal velocity	$n_{opt}$	rpm	6000
Optimal power	$P_{opt}$	kW	0.69
Max. allowed velocity (mecc.)	$n_{max\ mecc}$	rpm	9000
Max. allowed velocity (converter)	$n_{max\ Inv}$	rpm	8000
Max. allowed moment	$M_{max}$	Nm	5.1
Maximum current	$I_{max}$	A	7.7

**Table 6.1** Specific data of the motor 1FK7040-5AK71 CT

In this paper, the designed motion law is periodic. And the smooth deceleration is required. Therefore the motor need not brake to stop. The useful data are the maximum allowed velocity, inertia (without brake), rated torque (100K) and the maximum allowed moment. In **Table 6.2** the key specific data of six motors are presented:

Commercial Code	Max. allowed Velocity [rad/s]	Inertia(without brake) [kgm <sup>2</sup> ]	Rated torque [Nm]	Max. allowed moment [Nm]
1FK7011-5AK21 CT	837.333	6.40E-06	0.08	0.5
1FK7015-5AK21 CT	837.333	8.30E-06	0.16	1
1FK7022-5AK21 CT	1046.67	2.80E-05	0.6	3.4
1FK7032-5AF21 CT	1046.67	6.10E-05	0.8	4.5
1FK7034-5AF21 CT	1046.67	9.00E-05	1	6.5
1FK7040-5AK71 CT	942	1.70E-04	1.1	5.1

**Table 6.2** Specific data of the motors

From the analysis in Chapter 3 we know that in order to use the method to choose the motor, all the parameters of transmission should be given first, i.e. the transmission ratio  $\tau$ , the direct and the inverse efficiencies of the transmission  $\eta_d$  and  $\eta_i$  and the inertia of the transmission  $J_1$  and  $J_2$  should be known.

For the sake of simplification, here we present part of the data sheet of one type of

harmonic planetary gearbox whose commercial name is “HPG-14A-45-BL3-F0-E14.20-SP”<sup>[3]</sup>.

Size	Ratio	Limit for repeated peak torque $T_R$ [Nm]	Limit for average torque $T_A$ [Nm]	Rated torque $T_N$ [Nm]	Moment of inertia with output flange $J_T$ [kgm <sup>2</sup> ]	Maximum input speed [rpm]
14	45	23	15	10	2.06E-4	6000

**Table 6.3** Specific data of the transmission “HPG-14A-45-BL3-F0-E14.20-SP”

**Table 6.3** above is just presented the useful part of the data. We can get the transmission ratio is  $\tau = 1/45$ .

From the data of the transmission we can see there is only one value of inertia in the catalog. The moment of inertia is for standard coupling referred to the input shaft. Therefore, due to Eq. (3.2), we can easily design the inertia connecting to the motor side  $J_1$  and to the load side  $J_2$ .

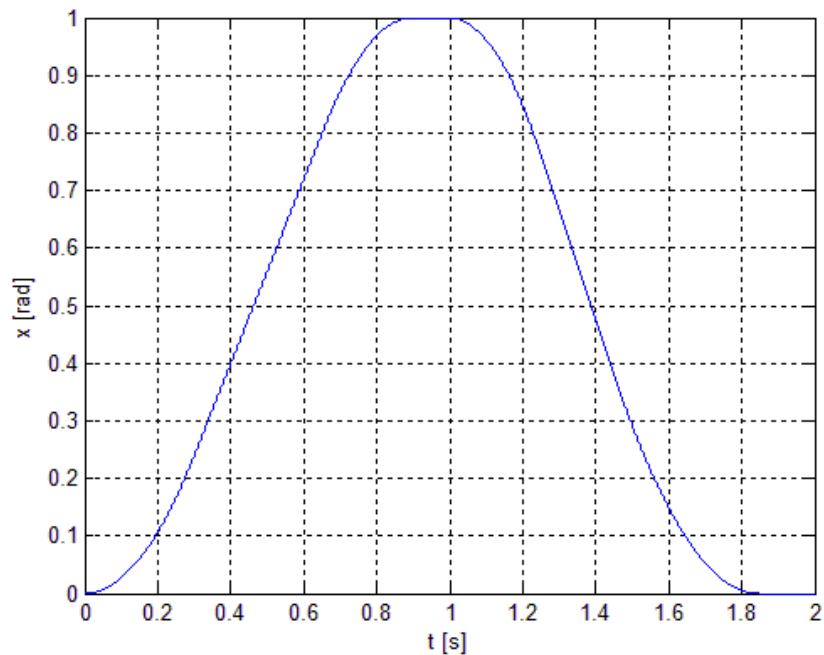
Here we assume that  $J_1 = 1.57 \times 10^{-4}$  [kgm<sup>2</sup>] and  $J_2 = 0.1$  [kgm<sup>2</sup>].

For the sake of simplicity, we assume that the curves of direct and inverse efficiencies are constant values which remain the same value for all speeds, ambient temperatures and the output torques. So we can assume the efficiencies:

$$\begin{aligned}\eta_d &= 0.85 \\ \eta_i &= 0.80\end{aligned}$$

## 6.2 Design motion laws

The motion device is used to realize a periodically rotary motion by means of a motor and a transmission. In this case, we use a harmonic drive transmission to reduce the speed of the rotation motion. The task movement mission we designed is a linear motion whose position law is characterized by the plotted in **Fig. 6.1**.



**Fig. 6.1** The position law of the device

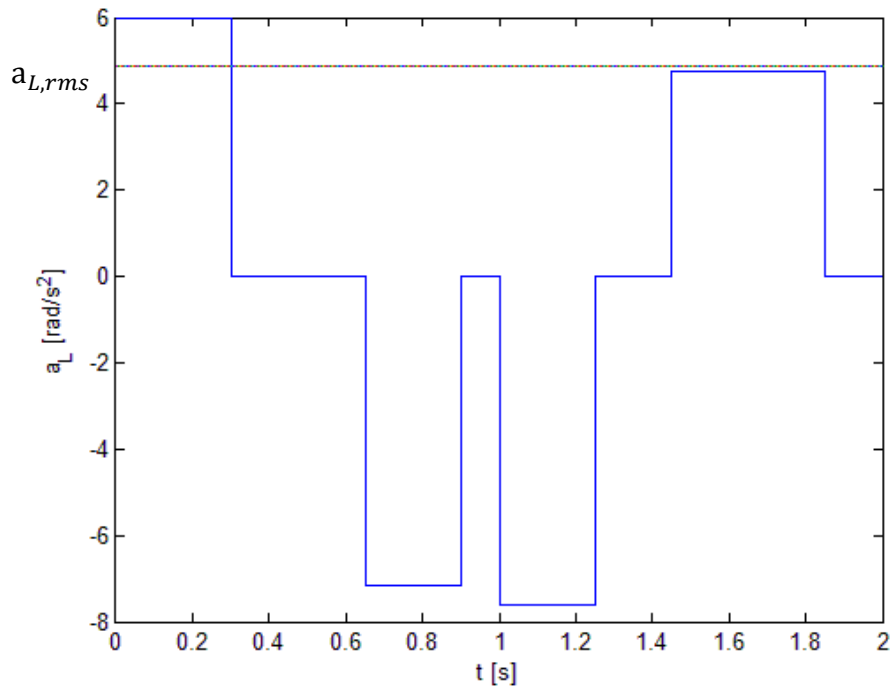
We assume that there are three types of movement in the sample motion: rise movement, the first dwell and the return movement and the final dwell. Both the rise phase and the return phase of the load have a general constant acceleration law.

The time period  $t_{m1}$  is the motion time for rise phase and  $t_{m2}$  is for return phase. Each constant acceleration law has a time range  $t_a^+$  with constant and positive acceleration, an intermediate time range with null acceleration and a third time range  $t_a^-$  at constant and negative acceleration.  $t_{m1}$  is the motion time for rise phase and  $t_{m2}$  is for return phase. The detail position law is listed below:

Time period	Symbols	Value
Rise phase	$t_{m1}$	0.9 [s]
First dwell	$t_{r1}$	0.1 [s]
Return phase	$t_{m2}$	0.85 [s]
Second dwell	$t_{r2}$	0.15 [s]
Total time	$T$	2 [s]
Stroke of the slide	$x$	1 [rad]
Starting position	$x_0$	0 [rad]

**Table 6.4** Task motion law

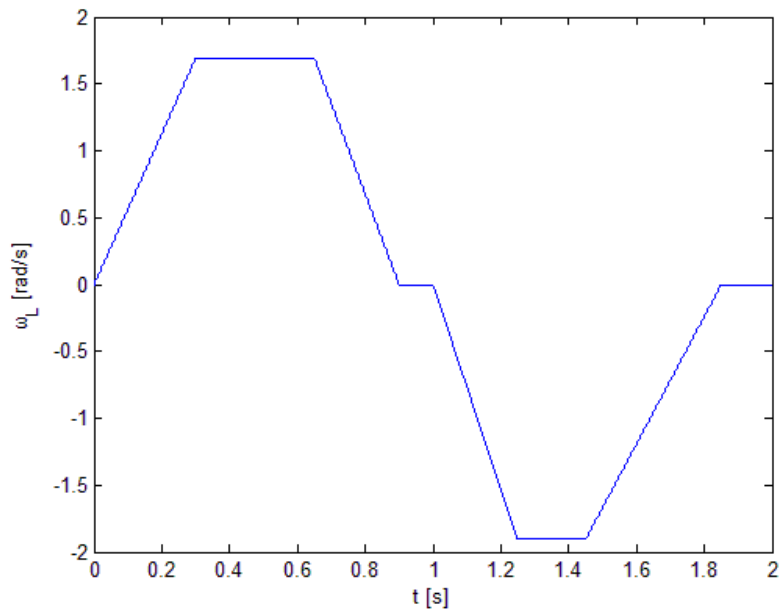
We can easily obtain the acceleration law of the load in **Fig. 6.2**.



**Fig. 6.2** The acceleration law of the load

And the dotted line represents the root mean square value of the acceleration of the load.

We can also calculate the speed law of the load in **Fig. 6.3**.



**Fig. 6.3** The speed law of the load

From the speed law we can check the feasibility of the motors. The motor must satisfy their maximum allowed speed is larger than the maximum speed of the



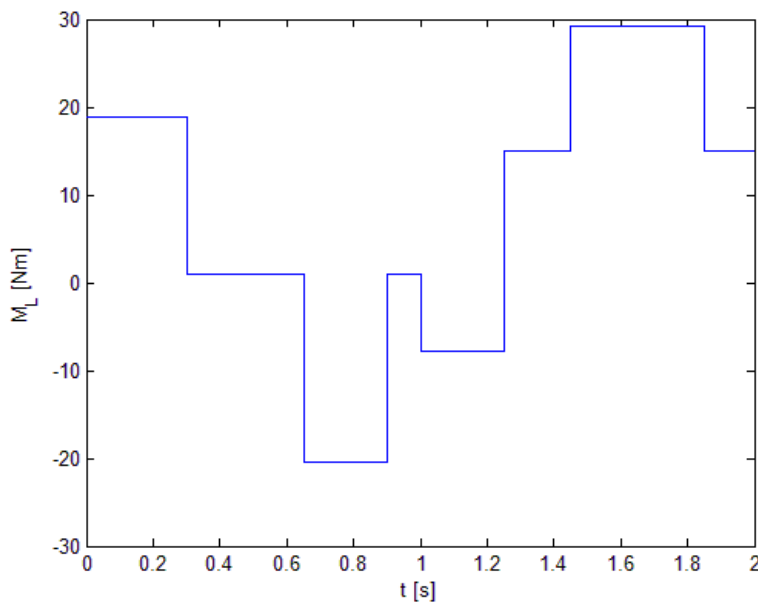
designed speed law, i.e.  $\omega_{m,max} \leq \omega_{M,max}$ . And this is the first feasibility condition.

For the sake of simplicity, we assume the total moment load is:

$$M_L = J_L * \alpha_L(t) + M_{L0} \quad (6.1)$$

In Eq. (6.1),  $M_{L0}$  is a constant moment applied on the load side. According to Eq. (3.2), substituting the expressions of  $\alpha_L(t)$  with previous designed value, we can obtain the moment of the load (see **Fig. 6.4**):

$$M_L(t) = (J_1 + J_2\tau^2) * \alpha_L(t) + M_{L0} \quad (6.2)$$



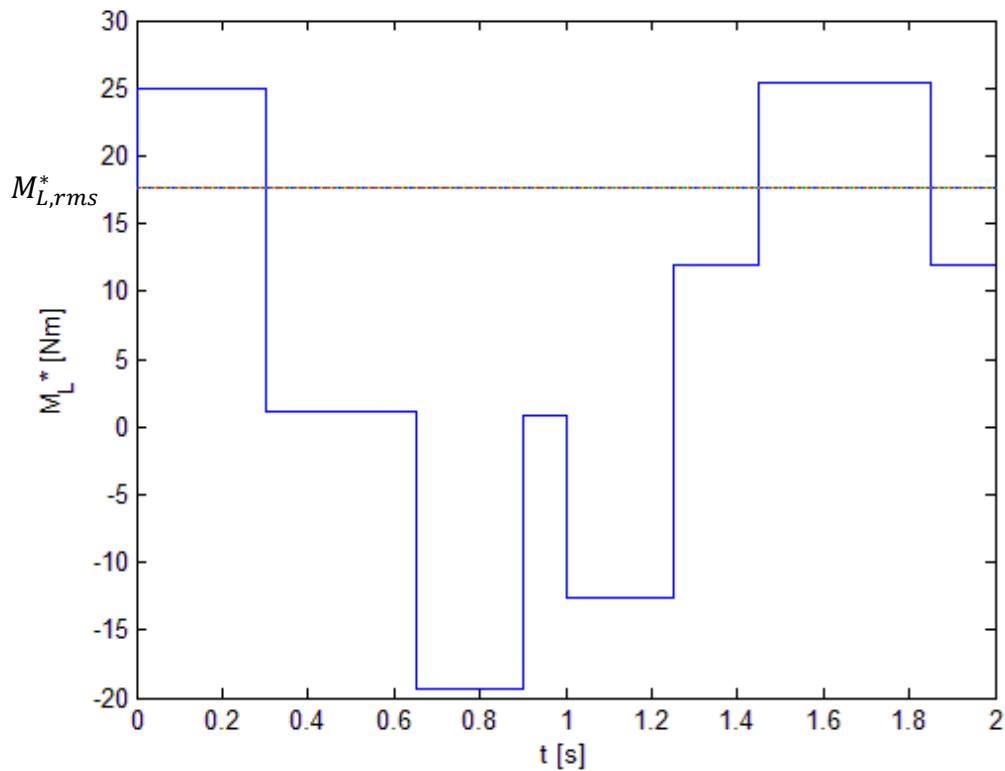
**Fig. 6.4** The moment of the load

Remembering Eq. (3.12) and Eq. (3.18), we can calculate the equivalent inertias of the transmission “HPG-14A-45-BL3-FO-E14.20-SP” due to the direct efficiency  $J_{T,d} =$

$$\frac{J_1}{\tau^2} + \frac{J_2}{\eta_d}$$

and the inverse efficiency  $J_{T,i} = \frac{J_1}{\tau^2} + \eta_i J_2$  where the efficiencies are presented in chapter 6.1.

Substituting the expressions of  $J_{T,d}$  and  $J_{T,i}$  in Eq. (3.21), we calculate the generalized moment of the load  $M_L^*$  (**Fig. 6.5**). And the dotted line in the figure shows the root mean square value of the generalized moment of the load.

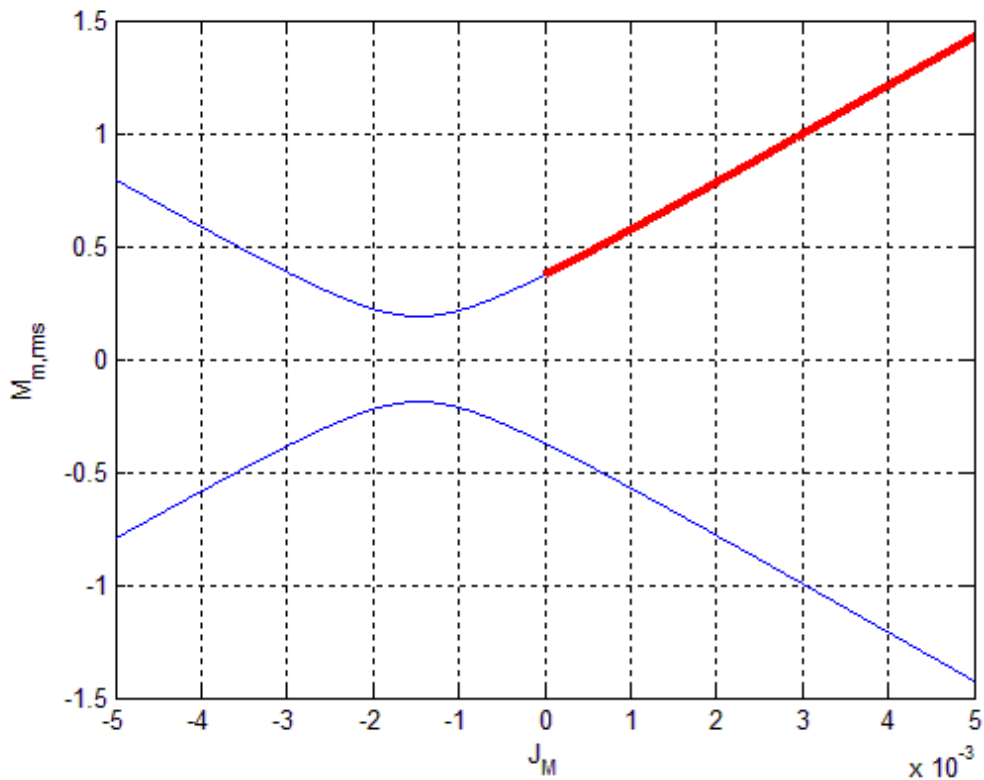


**Fig. 6.5** The generalized moment of the load

### 6.3 Check the motors in the continuous duty operating range

Bearing in mind Eq. (4.5) we can get the curve for the inertia of the motor versus the root mean square value of the moment of the motor, i.e. the  $J_M$ - $M_{m,rms}$  curve.

As the discussion shown in the chapter 4, the blue curves in **Fig. 6.6** regard the defined intervals of  $M_{m,rms}$  and  $J_M$  as from  $-\infty$  to  $+\infty$ . The red curve represents the real  $J_M$ - $M_{m,rms}$  curve with inertia of the motor and  $M_{m,rms}$  positive.

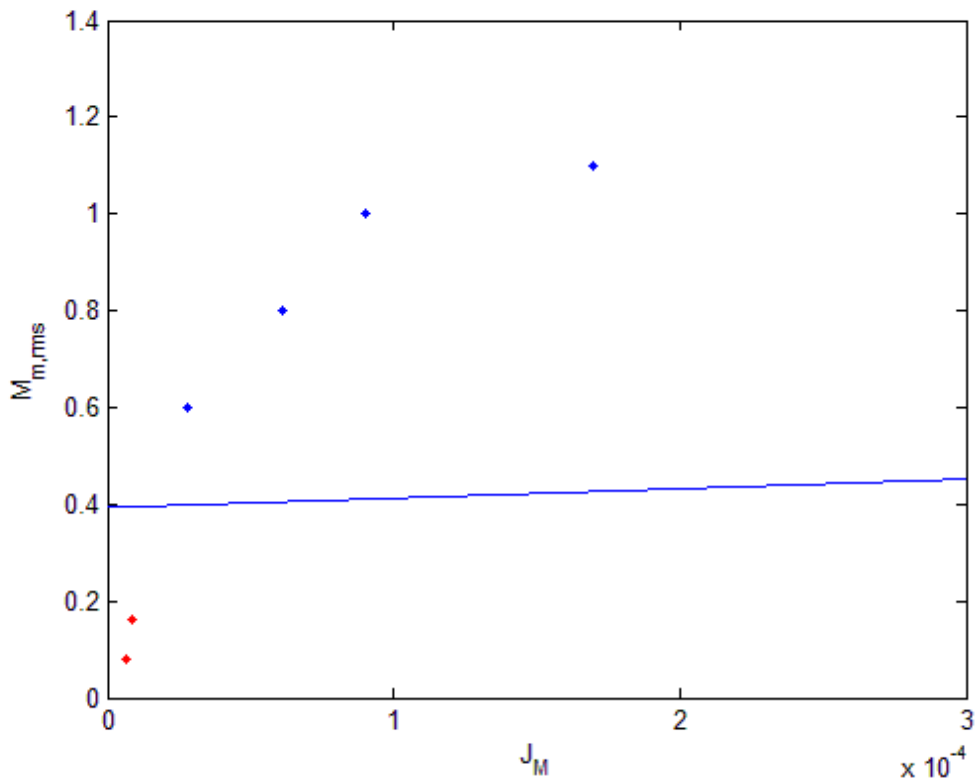


**Fig. 6.6** The  $J_M - M_{m,rms}$  curve with defined intervals from  $-\infty$  to  $+\infty$

Now we take into account the catalogs of the motors. The points representing the characteristic of a motor are plotted on the figure of the  $J_M - M_{m,rms}$  curve. Using the moment of inertia of the motor as abscissa and the rated torque of a motor as ordinate, we can easily check if the motor satisfies the inequality:

$$M_{m,rms} \leq M_{M,Rated}$$

From **Fig. 6.7** we can see that the first two red points which represent the motors *1FK7011-5AK21* and *1FK7015-5AK21* are below the  $J_M - M_{m,rms}$  curve. Therefore these two motors do not satisfy the required continuous duty working range condition. While the other four blue points are beyond the  $J_M - M_{m,rms}$  curve, those motors satisfy the condition.

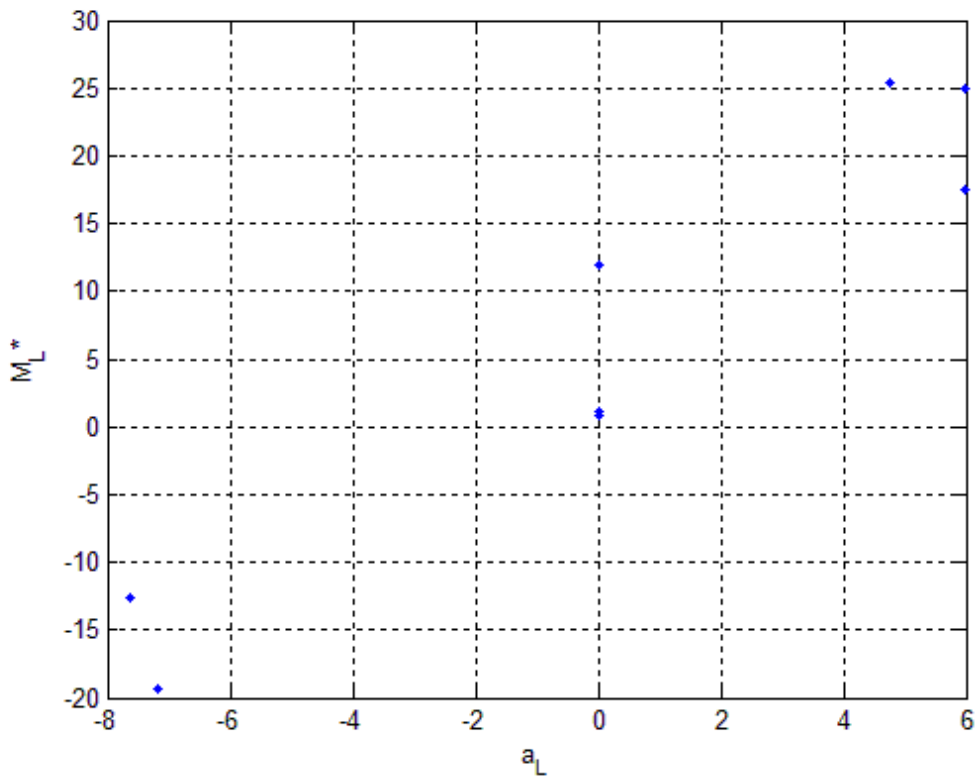


**Fig. 6.7** Check the feasibility of the motors in the  $J_M$ - $M_{m,rms}$  curve

## 6.4 Check the motors in the dynamic operating range

Remembering the algorithm in chapter 5, we should find the pair of value  $(\alpha_L, M_L^*)$  to define the  $J_M$ - $M_{m,rms}$  curve. We have obtained the acceleration law of the load  $\alpha_L(t)$  and the generalized torque of the load  $M_L^*(t)$  in the previous calculations. Let the time vary from zero to the cycle time  $T$ , check every pair  $(\alpha_L(t), M_L^*(t))$  at each time instant. If the pair of value does not appear before, we record this pair of value into the set of points  $S$ . We should also count the number of the points.

In this case, according the motion law we designed in chapter 6.2, there are 11 points representing the pairs of value  $(\alpha_{L,i}, M_{L,i}^*)$ . As shown in **Fig. 6.8** these points are located not only in the first and third quadrant but also on the ordinate axis.



**Fig. 6.8** The discrete points of  $\alpha_L - M_L^*$

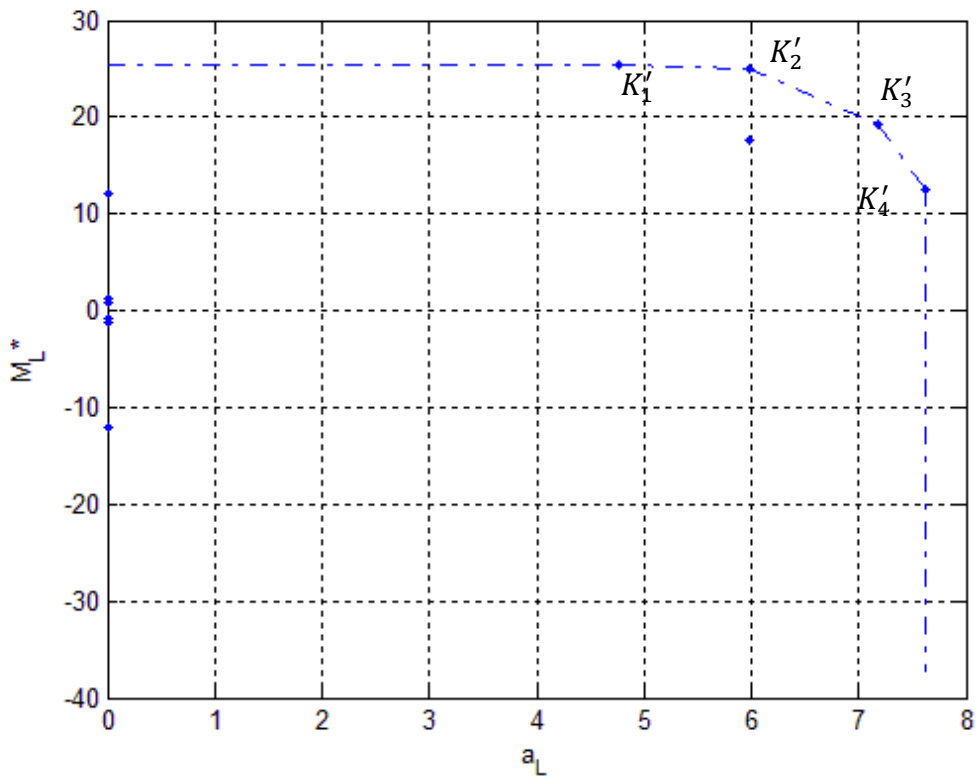
In order to calculate the  $M_{m,max}$  versus  $J_M$  curve, we should find the new points  $S'$  that contribute to that curve directly. As the discussion shows in the chapter 5.3, the relationships between the points  $S$  are:

If the point  $S_i$  is in the first quadrant, we consider a point  $S_{i,1}'$  in the first quadrant at the exactly same position with  $S_i$ .

If the point  $S_i$  is in the third quadrant, we consider a point  $S_{i,1}'$  located in the first quadrant which is symmetric with  $S_i$ .

If the point  $S_i$  is in the second or fourth quadrant, we consider two points  $S_{i,2}'$  and  $S_{i,4}'$ .  $S_{i,2}'$  is located in the second quadrant while  $S_{i,4}'$  is in the fourth quadrant. They are at the exactly same position with  $S_i$  or symmetric with  $S_i$ .

After using the method in chapter 5, we can find few points  $K_i'$  among the points  $S_i'$  and connect the points  $K_i'$  to achieve a monotonically descending polygon. Each segment of the polygon represent a branch of the  $J_M - M_{m,max}$  curve. In this case, as **Fig. 6.9** shows, there are 4  $K_i'$  points.



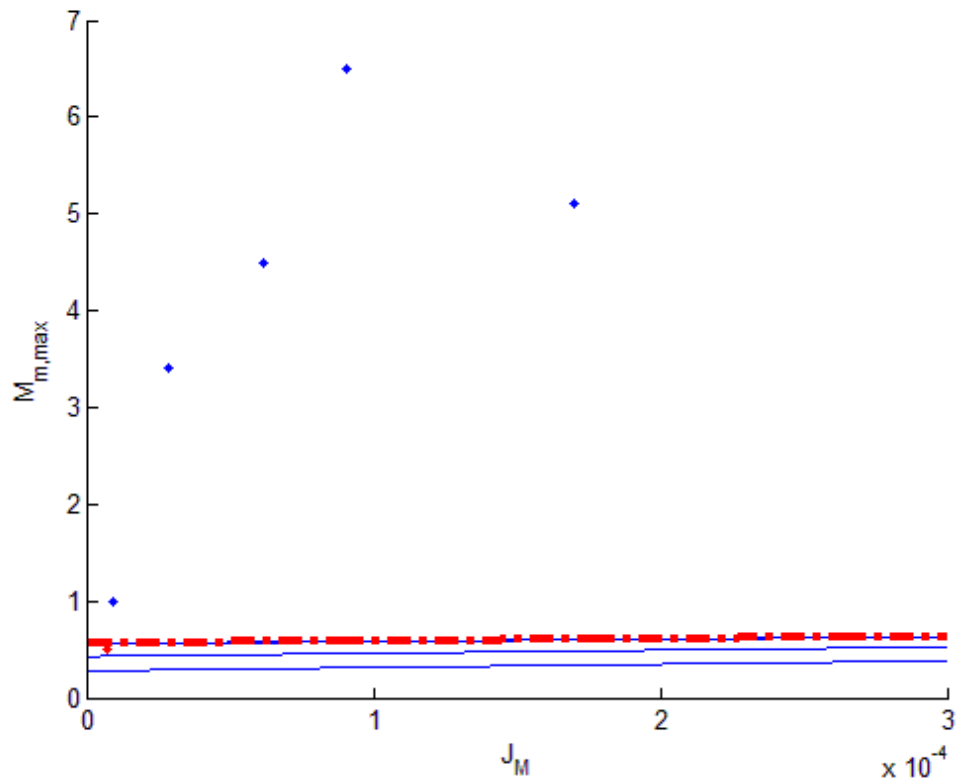
**Fig. 6.9** The points  $K'_i$  contributing to the  $J_M$ - $M_{m,max}$  curve

After that we can plot the drive system representative points on the same diagram of the  $J_M$ - $M_{m,max}$  curve. Taking into account the catalogs of the motors, we use the moment of inertia of the motor as abscissa and the maximum allowed torque as ordinate. In this way we can easily check if the motor is satisfied the inequality:

$$M_{m,max} \leq M_{M,max}$$

From **Fig. 6.10** we can see that there are four blue lines. And we can easily obtain the red line with the maximum value among those four lines.

There is only one red point which represents the motor *1FK7011-5AK21* is below the  $J_M$ - $M_{m,max}$  curve. Therefore this motor does not satisfy the required dynamic duty working range condition. The other five blue points which are beyond the  $J_M$ - $M_{m,max}$  curve represent their motors satisfy the condition.



**Fig. 6.10** The  $J_M$ - $M_{m,rms}$  curve and the drive system representative points

To sum up, after checking the three conditions of the feasibility of the motor, we can draw the conclusions that the first and second motors *1FK7011-5AK21 CT* and *1FK7015-5AK21 CT* is not suitable for the desired motion law with the transmission *HPG-14A-45-BL3-F0-E14.20-SP*. The other motors listed in chapter 6.1 have the feasibility.

## Chapter 7: Conclusion

In this paper, we study how to choose a brushless permanent synchronous motor and related transmission in mechatronic applications, when the motor is working under specific load.

Moreover, comparing with previous studies, this paper is more comprehensive and closer to real case, because we take in account both direct and inverse efficiencies of the transmission and the inertia of the transmission. Therefore there is no necessity of further verifications by simulating the system with the motor and the transmission chosen.

Once the “reference task” for given transmission has been defined, we can obtain the acceleration, the speed and the torque of the load. From the reference task, we can achieve the  $J_M-M_{m,rms}$  and  $J_M-M_{m,max}$  curves which help us to choose all the adequate electrical motors for given transmission. The first curve chooses the motors with reference to the continuous duty operating range, while the latter one selects the motors with reference to the dynamic operating range. The points representing the characteristic of a motor are plotted on both figures. We discard the motors whose representative points are below the curves. The motors must also satisfy that their maximum allowed speed is greater than the maximum speed of the designed speed law.

If a motor satisfies the three conditions mentioned above, it will be chosen.

The method is implemented with an algorithm solving, which does a feasibility check. After importing the characteristic parameters of a particular motor connected to the transmission and the “reference task”, the user can achieve the proper motors that can be used.

This method is not sufficient only if the dynamic range of the motor is not rectangular.



# Acknowledgment

First and foremost we offer our sincerest gratitude to our Professor CUSIMANO GIANCARLO who has supported us throughout the thesis with his enlivening, patience and in depth knowledge. Without him this thesis would not have been completed.

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Last but not least we thank all our friends who have shared us their prosperous suggestions, inspirations and comments in our work.

Thank all of you!

## Appendix: Algorithm solution scripts

```

clear all
close all
clc

%% input data

% the data of design motion law
T= 2;           %s
t=0:0.0001:T;  %s
h=1;           %rad
tm1=0.9;       %s
tr1=0.1;       %s
tm2=0.85;      %s
tr2=T-tm1-tm2-tr1; %s
tap1=0.3;      %tap1/tm
tan1=0.25;     %tan1/tm
tap2=0.4;      %tap2/tm
tan2=0.25;     %tan2/tm

% the specific data of the load
JL=3; % kg*M^2
ML01=1; %N*M the constant resistant moment for the direct power flow
ML02=15; %N*M the constant resistant moment for the inverse power flow

% the parameters of the transmission
etad=0.85;
etai=0.8;

tau=1/45;

J1=1.57e-4; % kg*M^2
J2=0.1; % kg*M^2
JM=0:1e-7:3e-4; % kg*M^2

%% calculate the designed motion law

x=zeros(1,length(t));
wL=zeros(1,length(t));
aL=zeros(1,length(t));

% use the function Required_Motion_Law() to solve the designed motion law

```

```
for ii=1:length(t)

    OUT=Required_Motion_Law (t(ii), h,T, tml,
tap1,tan1,tr1,tm2,tap2,tan2,tr2);
    x(ii)=OUT(1);
    wL(ii)=OUT(2);
    aL(ii)=OUT(3);
end

figure(1)
plot(t,x)
ylabel('x [rad]')
xlabel('t [s]')
grid on

figure(2)
plot(t,wL)
ylabel('|θL [rad/s]')
xlabel('t [s]')

figure(3)
plot(t,aL)
ylabel('aL [rad/s2]')
xlabel('t [s]')
hold on
aLrms=rms(aL,T);
plot(t,aLrms,'k-.')

ML=zeros(1,length(aL));
for ii=1:length(ML)
    if t(ii)<=tml+tr1;
        ML(ii)=JL*aL(ii)+ML01;
    else
        ML(ii)=JL*aL(ii)+ML02;
    end
end

figure(4)
plot(t,ML)
xlabel('t [s]')
ylabel('ML [Nm]')
```

```

%% assume etai, etad, tau, J1, J2 to find motor
% there is not exist the KP curve, we draw only Mm,rms-JM Mm,max-JM curve

disp('For transmission::')
disp([' J1=',num2str(J1),' [kg*M^2]'])
disp([' J2=',num2str(J2),' [kg*M^2]'])

disp('at the conditions::')
disp([' etai=',num2str(etai)])
disp([' etad=',num2str(etad)])
disp([' tau=',num2str(tau),])

dt=T/length(t);
wLmax=max(abs(wL));
MLstar=zeros(1,length(t));

JTd=J1/tau^2+J2/etad;
JTi=J1/tau^2+J2*etai;
previous=0;
initial=0;
for ii=1:length(t)
    if (ML(ii)+J2*aL(ii))*wL(ii)>0
        %direct
        MLstar(ii)=JTd*aL(ii)+ML(ii)/etad;
        previous=1;
    elseif (ML(ii)+J2*aL(ii))*wL(ii)<0
        MLstar(ii)=JTi*aL(ii)+ML(ii)*etai;
        previous=-1;
    elseif (ML(ii)+J2*aL(ii))*wL(ii)==0 && wL(ii)~=0
        MLstar(ii)=0;
    elseif (ML(ii)+J2*aL(ii))*wL(ii)==0 && wL(ii)==0
        if previous==1
            %means direct
            MLstar(ii)=JTd*aL(ii)+ML(ii)/etad;
        elseif previous==-1
            %means inverse
            MLstar(ii)=JTi*aL(ii)+ML(ii)*etai;

        else
            initial=1;% means we should deal with the initial condition
        end
    end
end
end

```

```

end
% the initial condition coincide with the direction of final power flow

if initial==1
    zz=1;
    while (ML(zz)+J2*aL(zz))*wL(zz)==0
        if previous==1
            %means direct
            MLstar(zz)=JTd*aL(zz)+ML(zz)/etad;

        elseif previous==-1
            %means inverse
            MLstar(zz)=JTi*aL(zz)+ML(zz)*etai;
        end
        zz=zz+1;
    end
end

figure(5)
plot(t,MLstar)
xlabel('t [s]')
ylabel('M_L* [Nm]')
hold on
MLstarrms=rms(MLstar,T);
plot(t,MLstarrms,'k-.')
A=aLrms^2/tau^2;

sumB=0;
for ii=1:length(t)
    sumB=sumB+MLstar(ii)*aL(ii)*dt;
end
B=sumB/T;
C=tau^2*MLstarrms^2;
D=B^2-A*C;
Mm_rms=zeros(1,length(JM));

xx=-5e-3:1e-5:5e-3;
yy1=length(xx);
yy2=length(xx);
for ii=1:length(xx)
    yy1(ii)=sqrt(A*xx(ii)^2+2*B*xx(ii)+C);

```

```

yy2(ii)=-sqrt(A*xx(ii)^2+2*B*xx(ii)+C);
end

figure(6)
plot(xx,yy1)
xlabel('J_M')
ylabel('M_m,_r_m_s')
hold on
plot(xx,yy2)
p=find(xx==0);

plot(xx(p:length(xx)),yy1(p:length(yy1)),'r-','LineWidth',3)
grid on
%

for ii=1:length(JM)
    Mm_rms(ii)=sqrt(A*JM(ii)^2+2*B*JM(ii)+C);
end

figure(7)
plot(JM,Mm_rms)
xlabel('J_M')
ylabel('M_m,_r_m_s')

%% draw the Mm,max JM curve
z=zeros(length(t),2);
count=1;
z(1,1)=aL(1);
z(1,2)=MLstar(1);
for ii=1:length(t)
    verified=1;
    for jj=1:count
        if aL(ii)==z(jj,1) && MLstar(ii)==z(jj,2)
            verified=0;
        end
    end
    if verified==1
        count=count+1;

        z(count,1)=aL(ii);
        z(count,2)=MLstar(ii);

    end
end

```

```

end

figure(8)
plot(z(1:count,1),z(1:count,2),'.')
xlabel('a_L')
ylabel('M_L*')
grid on
% move the points of third quadrant to the first quadrant
count2=0; %count for the points in second and fourth quadrant
for ii=1:count

    if z(ii,1)<0 && z(ii,2)<0
        z(ii,1)=-z(ii,1);
        z(ii,2)=-z(ii,2);

    elseif z(ii,1)<=0 && z(ii,2)>0 % in second quadrant
        count2=count2+1;
        z(count+count2,1)=-z(ii,1);
        z(count+count2,2)=-z(ii,2);
    elseif z(ii,1)>0 && z(ii,2)<=0 % in fourth quadrant
        count2=count2+1;
        z(count+count2,1)=-z(ii,1);
        z(count+count2,2)=-z(ii,2);
    end
end
count=count+count2;

figure(9)
plot(z(1:count,1),z(1:count,2),'.')
xlabel('a_L')
ylabel('M_L*')
grid on
hold on

temK=zeros(count,2);
temK(1,2)=max(z(:,2));
p1=find(z(:,2)==temK(1,2));

temK(1,1)=z(p1,1); %find the temK(1)

jj=1;
while(temK(jj,1)~=max(z(:,1)))
    maxs1=0;
    for ii=1:count

```

```

if z(ii,1)>temK(jj,1)
    % calc the slope
    if maxsl==0
        maxsl=(z(ii,2)-temK(jj,2))/(z(ii,1)-temK(jj,1));
        temK(jj+1,1)=z(ii,1);
        temK(jj+1,2)=z(ii,2);
    else
        sl=(z(ii,2)-temK(jj,2))/(z(ii,1)-temK(jj,1));
        if sl>maxsl
            maxsl=sl;
            temK(jj+1,1)=z(ii,1);
            temK(jj+1,2)=z(ii,2);
        end
    end
end

end
end
jj=jj+1;

end
K=temK(1:jj,1:2);
for ii=1:length(K(:,1))+1
    if ii==1
        plot([0 K(1,1)], [K(1,2) K(1,2)], 'b-.')
    elseif ii==length(K(:,1))+1 % draw the vertical line
        plot([K(ii-1,1) K(ii-1,1)], [K(ii-1,2) K(ii-1,2)-50], 'b-.')
    else
        plot([K(ii-1,1) K(ii,1)], [K(ii-1,2) K(ii,2)], 'b-.')
    end
end

end
hold off

% calc the Mm,max JM line

figure(10)
xlabel('J_M')
ylabel('M_m, m_a_x')

hold on
Mmtemp=zeros(length(K), length(JM));
Mmmax=zeros(1, length(JM));

```



```

for ii=1:length(K(:,1))
    for jj=1:length(JM)
        Mmtemp(ii,jj)=K(ii,1)/tau*JM(jj)+K(ii,2)*tau;

    end

    plot(JM,Mmtemp(ii,:))

end
for ii=1:length(JM)
    Mmmax(ii)=max(Mmtemp(:,ii));
end
plot(JM,Mmmax,'r-.','LineWidth',3)

%% Take into account the specific data of motors

% use the Motor matrix to storage the data in order to easily access
% to all the data.

%      JM      Mm,rms, rated M_Mmax      wM,max
Motor=[6.4e-6      0.08      0.5      837;          %motor1 1FK7011 CT
      8.3e-6      0.16      1      837;          %motor2 1FK7015 CT
      2.8e-5      0.6      3.4      1047;        %motor3 1FK7022 CT
      6.1e-5      0.8      4.5      1047 ;      %motor4 1FK7032 CT
      9e-5      1      6.5      1047;          %motor5 1FK7034 CT
      1.7e-4      1.1      5.1      942];      %motor6 1FK7040 CT

% use a vector fM to store the results of feasibility of the motors

fM=zeros(1,length(Motor));

% the motor must satisfy 3 conditions
% 1. wm,max<=wMmax
wmmax=max(wL)/tau;
disp(['wm,max=', num2str(wmmax)])
for ii=1:length(Motor(:,1))
    if wmmax<=Motor(ii,4)
        fM(ii)=1;
    else
        fM(ii)=0;
    end
end

% 2. the Mm,rms, rated of the motor must be above the JM-Mm,rms curve

```

```

% polt on the JM-Mm_rms curve directly

figure(7)
hold on
for ii=1:length(Motor)
    for jj=1:length(Motor)
        if JM(ii)-Motor(ii,1)<1e-8
            p=jj;
        end

        end

        if Mm_rms(p)<=Motor(ii,2)
            plot(Motor(ii,1),Motor(ii,2),'b.')
        else
            fM(ii)=0; % delete the unfeasible points
            plot(Motor(ii,1),Motor(ii,2),'r.')
        end

    end

end
hold off

% 3.the M_Mmax must be above the JM-Mm,max curve

% polt on the JM-Mm_max curve directly
figure(10)
hold on
for ii=1:length(Motor)
    for jj=1:length(Motor)
        if JM(ii)-Motor(ii,1)<1e-8
            p=jj; %to find the position of JM
        end

        end

        if Mmmax(p)<=Motor(ii,3)
            plot(Motor(ii,1),Motor(ii,3),'b.')
        else
            fM(ii)=0;
            plot(Motor(ii,1),Motor(ii,3),'r.')
        end

    end

end

for ii=1:length(Motor)
    if fM(ii)==1

```

```
        disp(['Motor ', num2str(ii), ' is feasible'])
    else
        disp(['Motor ', num2str(ii), ' is unfeasible'])
    end
end

end
```

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