POLITECNICO DI MILANO

SCUOLA DI INGEGNERIA DEI SISTEMI

CORSO DI STUDI IN INGEGNERIA MATEMATICA

Tesi di laurea magistrale

Interface Control Domain Decomposition Problems

Relatore:
Prof. Quarteroni Alfio

Correlatrice:
Prof.ssa Gervasio Paola

Tesi di laurea di:
Signorini Marianna
matr. 766131

ANNO ACCADEMICO 2011-2012

alla mia famiglia
ad Andrea

Acknowledgements

Desidero ringraziare innanzitutto il Professor Alfio Quarteroni, per avermi permesso di lavorare su questi argomenti sotto la sua supervisione. Nonostante i numerosi impegni, ha sempre trovato il tempo per indirizzare e seguire il mio lavoro.

Un ringraziamento particolare va alla mia correlatrice, la Professoressa Paola Gervasio, per la sua costante disponibilità e gentilezza, nonché per le sue competenze. Un grazie di cuore per avermi dato l'opportunità di crescere, con i suoi utili consigli, ed aver sempre risposto in modo tempestivo alle mie domande.

Ringrazio anche al Professor Marco Discacciati, per la sua grande disponibilità e per i risultati dell'ultimo capitolo, nonché per l'utile discussione su questi ultimi.

Vorrei ringraziare anche i Professori Sandro Salsa e Davide Ambrosi per il loro aiuto e la loro gentilezza.

Un ringraziamento speciale va alla mia famiglia, ai miei genitori che mi hanno sempre sostenuto, sia con il loro affetto ed orgoglio, sia economicamente, e alle mie sorelle, Federica e Silvia, che mi sono sempre state vicine.

Grazie Andrea, per essere la mia vera forza. Grazie per tutti i momenti che abbiamo condiviso in questi anni, e per il tuo continuo sostegno. Grazie perché mi hai sempre motivata e spronata a dare il meglio.

Infine un grazie ai miei amici e compagni di corso, che mi hanno sempre incoraggiata e sostenuta.

Abstract

The scope of this thesis is to analytically and numerically analyze the so-called interface control domain decomposition methods. These methods are the subject of current research within the academic world.
They represent a viable alternative to more classical methods for the resolution of heterogeneous problems, without the necessity of setting neither the interface between the different sub-domains in which should be solved different problems (not always easily identifiable), nor the coupling conditions, but rather minimizing a suitable cost functional.
The basic idea of these methods, actually, consists in splitting the original domain into two or more overlapping sub-domains, in introducing the interface variables $\boldsymbol{\lambda}_i$ on the different interfaces and in minimizing a cost functional depending on the introduced variables. In particular, the solution of the problems in the different sub-domains depends on $\boldsymbol{\lambda}_i$ and we try to minimize the difference of meaningful quantities measured on adjacent domains in an appropriate norm.
In this thesis we discuss the methods mentioned above in the case of two sub-domains. In particular, at first some methods for the resolution of elliptic problems are presented. These methods are analyzed in the case of both continuous and discontinuous coefficients, in order to test their suitability and efficiency. Then, we propose two methods for the solution of the heterogeneous Stokes-Darcy problem, we report the obtained numerical results and assess their properties of well-posedness and stability.

Keywords: domain decomposition, optimal control, interface control, elliptic problems, finite element method, Stokes-Darcy coupling

Sommario

L'obiettivo di questa tesi è quello di analizzare dal punto di vista analitico e numerico i cosiddetti metodi di decomposizione dei domini con controllo all'interfaccia. Tali metodi sono oggetto di ricerche attuali all'interno del mondo accademico.
Essi rappresentano una valida alternativa ai metodi più classici per la risoluzione di problemi eterogenei, senza la necessità di fissare né l'interfaccia tra i diversi sottodomini su cui devono essere risolti i diversi problemi (non sempre facilmente identificabile), né le condizioni di accoppiamento, ma minimizzando un opportuno funzionale costo.
L'idea fondamentale di questi metodi, infatti, è quella di scomporre il dominio originario in due o più sottodomini con sovrapposizione, introdurre delle variabili di interfaccia $\boldsymbol{\lambda}_i$ sulle diverse interfacce e minimizzare un funzionale costo dipendente dalle variabili introdotte. In particolare, la risoluzione dei problemi nei sottodomini dipende dalle $\boldsymbol{\lambda}_i$ e si cerca di minimizzare la differenza di quantità significative misurate su domini adiacenti in un' opportuna norma.
In questa tesi vengono trattati i metodi suddetti nel caso di due sottodomini. In particolare, dapprima sono presentati dei metodi per la risoluzione di problemi di tipo ellittico. Tali metodi vengono analizzati sia nel caso di coefficienti continui che discontinui, al fine di testare la bontà e la robustezza dei metodi proposti. In secondo luogo vengono esposti due metodi per la risoluzione del problema eterogeneo Stokes-Darcy, riportati i risultati numerici ottenuti e valutate le proprietà di buona positura e stabilità dei metodi.

Parole Chiave: decomposizione di domini, controllo ottimo, controllo d'interfaccia, problemi ellittici, metodo degli elementi finiti, accoppiamento Stokes-Darcy

Contents

List of Figures

Introduction

The aim of this thesis is to analytically and numerically analyze the so-called interface control domain decomposition methods (ICDD). These methods are the subject of current research within the academic world.
They represent a viable alternative to more classical methods for the resolution of heterogeneous problems, without the necessity of setting neither the interface between the different sub-domains in which should be solved different problems (not always easily identifiable), nor the coupling conditions, but rather minimizing a suitable cost functional.
The basic idea of these methods, actually, consists in splitting the original domain into two or more overlapping sub-domains, in introducing the so-called interface controls $\boldsymbol{\lambda}_i$ which play the role of unknown boundary data on the different interfaces and in minimizing a cost functional depending on the introduced variables. In particular, the solution of the problems in the different sub-domains depends on $\boldsymbol{\lambda}_i$ and we try to minimize the difference of meaningful quantities measured on adjacent domains in an appropriate norm.
In particular, we discuss these methods in the case of two sub-domains. We first consider elliptic differential problems, then we consider the case of Stokes-Darcy coupling.
ICDD methods for elliptic problems were developed in [8]. There the case of Dirichlet interface control with homogeneous boundary conditions was analyzed. We extend the results of [8] to a more general setting, by considering mixed (Dirichlet and Neumann) boundary conditions. Moreover we discuss with more details the case of Robin interface control. This part represents an original contribution of this work. Moreover, we discretize the proposed methods using piecewise linear finite elements $\mathbb{P}_1$, while in [8], piecewise bilinear $\mathbb{Q}_1$ finite elements and spectral elements $\mathbb{Q}_N$, $N > 1$ were used.
In the second part of this thesis we apply ICDD methods to an heterogeneous problem. We propose two different functionals to be minimized, already considered in [10], to solve the Stokes-Darcy coupling. No theoretical results of well-posedness of these methods are available for this kind of problems. We numerically analyze the proposed methods in order to understand whether they are well-posed and robust or not.
Our work is organized as follows: in the first three Chapters we introduce the theoretical background necessary to develop ICDD methods, in Chapters 4 and 5 we present theoretical and numerical results for ICDD methods applied to elliptic differential problems and in Chapter 6 we formulate the Stokes-Darcy coupling with two ICDD methods, and show numerical results.

More in detail:

Chapter 1 We introduce and give the basic results concerning optimal control problems. This theory is necessary for the development of ICDD methods.

Chapter 2 We formulate elliptic problems, we recall the statements of the Galerkin Finite Element Method (FEM) and we present the main convergence results of this method.

Chapter 3 We describe one of the classical Domain Decomposition Methods (DDM), the Schwarz method, and we present theoretical and numerical results. We are interested in these kinds of methods because the Interface Control Domain Decomposition methods are based on the domain decomposition of the computational domain into overlapping sub-domains.

Chapter 4 We present the theory of ICDD methods to solve elliptic differential problem, with two different kinds of interface controls (Dirichlet and Robin interface controls), and three different cost functionals to be minimized, and we discuss the well-posedness of these methods. Finally we obtain the optimality systems to be solved for every method.

Chapter 5 We discretize the optimality systems obtained in the previous Chapter with finite element method and we present the numerical resolution and results obtained with ICDD methods in order to choose the better among these mehods. We compare the numbers of iterations required by these methods also with the ones of Schwarz method, so that we can understand how they compare with classical Schwarz method. The numerical simulations are made for different test cases and with both continuous and discontinuous coefficients, in order to test the robustness of these methods. For our comparisons we fix the overlap thickness and vary the mesh size, then we fix the mesh size and vary the overlap thickness.

Chapter 6 After introducing the Stokes and the Darcy problem and their numerical approximations used in this work, we describe the classical theory of Stokes-Darcy coupling with sharp interface and we give its dimentionless formulation. Finally we present two different ICDD methods to solve this heterogeneous problem and we try to understand with numerical sinulations whether they are well-posed and give a solution consistent with the real solution of the problem.

All our numerical simulations have been carried out using MATLAB®[15]. This numerical investigation represents a very substantial part of our thesis.

Milano, December 2012.

Chapter 1

Optimal Control Problems for Partial Differential Equations

In this Chapter we recall the general theory of optimal control problems (see [17]), which is needed to develop the interface control domain decomposition method.

1.1 The Problem Setting

In order to give the basic results concerning optimal control, we need to introduce the following fundamental data:

- The *control function* u, belonging to a certain functional space U_{ad} (the *space of admissible controls*), usually a closed and bounded subspace of a Hilbert space U;

- The *state function* $y = y(u) \in V$, with V Hilbert space, which is the state of the system to be controlled by u. The function $y(u)$ is the solution of the *state system*:

$$Ay(u) = f(u)$$

  where $A : V \to V'$ is supposed to be a known operator, and $f \in V'$ a function of u;

- The *observation function* $z = z(u) \in \mathcal{H}$, with $\mathcal{H}$ Hilbert space, which depends on the control function u through the state function and an *observation operator* $C : V \to \mathcal{H}$ which is linear and continuous:

$$z(u) = Cy(u) \ .$$

  This function is to be compared with the desired observation z_d, which is the desired objective to be reached.

- The *cost functional* $J : U_{ad} \to \mathbb{R}$

$$u \in U_{ad} \mapsto J(u) \in \mathbb{R}$$

- The *control problem*: find the *optimal control* $u \in U_{ad}$ such that

$$J(u) = \inf_{v \in U_{ad}} J(v)$$

or, equivalently, such that

$$J(u) \leq J(v) \quad \forall\, v \in U_{ad}$$

1.2 Functional Minimization: Existence and Uniqueness Results

Let U be a Hilbert space, endowed with the inner product $(\cdot,\cdot)$, and with the norm $\|\cdot\| := \sqrt{(\cdot,\cdot)}$. We consider the continuous bilinear form on U

$$u, v \mapsto \pi(u,v) \quad \forall u, v \in U \tag{1.1}$$

that is supposed to be symmetric, and the continuous linear functional on U

$$v \mapsto L(v) \quad \forall v \in U \tag{1.2}$$

Moreover, let U_{ad} be a closed and convex subspace of U. Now we consider the cost functional of the form:

$$J(v) = \pi(v,v) - 2L(v) \quad \forall v \in U_{ad} \tag{1.3}$$

The existence and uniqueness of the minimum of (1.2) is given by the following Theorems. The proofs can be found in [14].

Theorem 1. *Let $\pi(\cdot,\cdot)$ be coercive on U:*

$$\exists \alpha : \pi(v,v) \geqslant \alpha \|v\|^2 \quad \forall v \in U\ .$$

Then there exists a unique element $u \in U_{ad}$ such that

$$J(u) = \inf_{v \in U_{ad}} J(v)\ . \tag{1.4}$$

Theorem 2. *Let us assume the hypothesis of the previous Theorem is valid. Then:*

1. *The minimizing element $u \in U_{ad}$ is characterized by*

$$\pi(u, v-u) \geqslant L(v-u) \quad v \in U_{ad} \tag{1.5}$$

2. *If we assume also that the functional $v \mapsto L(v)$ is strictly convex and Gâteaux-differentiable in U_{ad}, and that the cost functional satisfies the condition*

$$J(v) \longrightarrow +\infty \quad \text{as } \|v\| \longrightarrow +\infty \tag{1.6}$$

then, the unique $u \in U_{ad}$ satisfying (1.4) is characterized by the following inequality:

$$J'(u)[v-u] \geqslant 0 \quad \forall v \in U_{ad} \tag{1.7}$$

where $J'(u)$ stands for the Gâteaux derivative of J in u, defined as the linear operator $J'(u) : U_{ad} \to \mathbb{R}$ such that

$$J'(u)[\phi] = \lim_{\delta \to 0} \frac{J(u+\delta\phi) - J(u)}{\delta} \qquad \forall \phi \in U_{ad}$$

Corollary 1. *Under the assumptions of Theorem 2, in the case that $U_{ad} = U$, the* (1.5) *becomes:*

$$\pi(u, \phi) = L(\phi) \quad \forall \phi \in U$$

and it is named Euler equation associated to the minimization problem (1.4)*.*

Corollary 2. *Under the assumptions of Theorem 2, the characterization* (1.7) *is equivalent to:*

$$J'(v)[v - u] \geqslant 0 \quad \forall v \in U_{ad}$$

1.3 Control of Systems Governed by Elliptic Partial Differential Equations

In this Section first we introduce the mathematical formalism needed to treat these kinds of problems, and then try to apply to these problems the general theory introduced above, in order to find results of existence and uniqueness of the solutions also in these cases.

Problem Statements

Let V and H be two Hilbert spaces on $\mathbb{R}$, and let us assume that $V \hookrightarrow H$, with continuous and dense injection. This means that $V \subset H$, that $\exists c > 0$ such that

$$\|v\|_H \leqslant c\|v\|_V \quad \forall v \in V$$

and that V is dense in H.
In this case, denoting the dual spaces of V and H respectively with V' and H', it results that

$$V \hookrightarrow H \backsimeq H' \hookrightarrow V' \tag{1.8}$$

and we refer to (V, H, V') as *Hilbert Triplet*.
Finally, let

$$a : V \times V \to \mathbb{R}$$

be a continuous and coercive bilinear form on V, and for a given $f \in V'$ let F be the continuous linear form on V

$$F : V \to \mathbb{R}\,, \quad F(v) = {}_{V'}\langle f, v\rangle_V\,.$$

The following holds true.

Lemma 1. (Lax-Milgram's) *Under the previous hypothesis on $a(\cdot, \cdot)$, for a given $f \in V'$, there exists a unique $u \in V$ such that:*

$$a(u, v) = F(v) \quad \forall v \in V. \tag{1.9}$$

We may rewrite (1.9) noting that the form $v \mapsto a(u, v)$ is linear and continuous on V, and so we can define the linear operator $A \in \mathcal{L}(V, V')$ as

$$a(u, v) = {}_{V'}\langle Au, v\rangle_V \tag{1.10}$$

Hence (1.9) is equivalent to

$$Au = f \qquad in\ V'. \tag{1.11}$$

We are now ready to formulate the *control problem*, according to the general setting shown in Section 1.1.
Let the Hilbert space U be the *space of the controls*, and let us define the operator $B \in \mathcal{L}(U, V')$[1]. The *state variable* $y(u) \in V$ is governed by the following equation, called the *state equation*:

$$Ay(u) = f + Bu \qquad in\ V'. \tag{1.12}$$

We note that this equation uniquely defines $y(u)$, thanks to Lemma 1. The operator B can be, for example, the identity or a restriction operator, such as the trace operator.
The *observation equation* is instead of the form:

$$z(u) = Cy(u) \tag{1.13}$$

where $C \in \mathcal{L}(V, H)$.
Finally, we introduce the operator $N \in \mathcal{L}(U, U)$, with N symmetric and positive definite and such that $\exists \nu > 0$:

$$(Nu, u)_U \geq \nu \|u\|_U^2 \tag{1.14}$$

The term $(Nu, u)_U$ is called *penalization term.*
To every control $u \in U$, we can associate the *cost functional*

$$J(u) = \|Cy(u) - z_d\|_H^2 + (Nu, u)_U \tag{1.15}$$

where $z_d \in H$ is the desired observation.

Some Remarks on the Control Problem

Let $U_{ad} \subset U$ be the space of *admissible controls.* It can be easily shown that the following Theorem holds:

Theorem 3. *Under all the hypotheses and definitions given above, there exists a unique element $u \in U_{ad}$ such that*

$$J(u) = \inf_{v \in U_{ad}} J(v)\ . \tag{1.16}$$

The element u is said to be the optimal control.

Proof. Let us write $J(u)$ in the following form:

$$J(u) = \|C(y(u) - y(0)) + Cy(0) - z_d\|_H^2 + (Nu, u)_U.$$

We now set

$$\pi(u, v) := (C(y(u) - y(0)), C(y(v) - y(0)))_H + (Nu, v)_U$$
$$L(v) := (z_d - Cy(0), C(y(v) - y(0)))_H\ .$$

We can note that the form $\pi(u, v)$ is a continuous and coercive bilinear form on U, that $L(v)$ is a continuous linear functional on U and that the functional $J(v)$ can be written in the following way:

$$J(v) = \pi(v, v) - 2L(v) + \|z_d - Cy(0)\|_H^2$$

Thus, thanks to Theorem 1, the thesis follows. □

[1]Given two Banach spaces U and V, the space of the linear and continuous operators from U into V is called $\mathcal{L}(U, V')$

Corollary 3. *Under all the hypotheses of the previous Theorem except on N, if $N = 0$ but there exist a constant $\alpha_0 > 0$ such that*

$$\exists \alpha_0 > 0 \ : \ \|Cy(u) - z_d\|_H^2 \geqslant \alpha_0 \|u\|_U^2 \quad \forall u \in U_{ad} \ ,$$

then the thesis of the previous Theorem is still valid.

The set of inequalities defining the optimal control

In the previous paragraph we have shown that, under certain hypotheses, the existence and uniqueness of the optimal control can be proved. Now we need to study the structure of the problem in order to be able to solve it.
We recall that, thanks to Theorem 2, the *optimal control* $u \in U_{ad}$ is characterized by:

$$J'(u)[v - u] \geqslant 0 \quad \forall v \in U_{ad}, \tag{1.17}$$

where

$$\begin{aligned} &J'(u)[\phi] = 2\pi(u, \phi) - 2L(\phi) \\ &= 2\left((C(y(u) - y(0)), C(y(\phi) - y(0)))_H + (Nu, \phi)_U\right) - 2(z_d - Cy(0), C(y(\phi) - y(0)))_H \ . \end{aligned}$$

Moreover, since $u \mapsto y(u)$ is an affine linear map, we can write

$$Ay(v - u) = f + B(v - u) = f + A(y(v) - y(u))$$

and, given that $f = Ay(0)$, we have

$$A(y(v - u)) = A(y(0) + y(v) - y(u))$$

and so

$$y(v - u) - y(0) = y(v) - y(u)$$

since A is an isomorphism from V into V'. Therefore (1.17) is equivalent to

$$(Cy(u) - z_d, C(y(v) - y(u)))_H + (Nu, v - u)_U \geqslant 0 \quad \forall v \in U_{ad}. \tag{1.18}$$

Let us denote by $\Lambda = \Lambda_H$ the canonical isomorphism from H into H', and by $C^* \in \mathcal{L}(H', V')$ the adjoint operator of C.
We can now rewrite (1.18), obtaining:

$${}_{V'}\langle C^*\Lambda(Cy(u) - z_d), y(v) - y(u)\rangle_V + (Nu, v - u)_U \geqslant 0 \quad \forall v \in U_{ad}. \tag{1.19}$$

We shall now transform (1.19), using the *adjoint state*.
Actually, let $A^* \in \mathcal{L}(V, V')$ be the adjoint operator of A. It is associated to the bilinear form $a(.,.)$ in the following way:

$${}_{V'}\langle A^*u, v\rangle_V = {}_{V'}\langle u, Av\rangle_V = a(u, v) \quad \forall u, v \in V \ .$$

For any control $v \in U$, the *adjoint state* $p(v) \in V$ is defined by the following equation:

$$A^*p(v) = C^*\Lambda(Cy(v) - z_d) \ .$$

Therefore, the terms on the left of the inequality (1.19) becomes:

$$
\begin{aligned}
&{}_{V'}\langle C^*\Lambda(Cy(u)-z_d), y(v)-y(u)\rangle_V + (Nu, v-u)_U = \\
&{}_{V'}\langle A^*p(u), y(v)-y(u)\rangle_V + (Nu, v-u)_U = \\
&{}_{V}\langle p(u), A(y(v)-y(u))\rangle_{V'} + (Nu, v-u)_U = \\
&{}_{V}\langle p(u), B(v-u)\rangle_{V'} + (Nu, v-u)_U = \\
&{}_{U'}\langle B^*p(u), v-u\rangle_U + (Nu, v-u)_U \ ,
\end{aligned}
$$

where $B^* \in \mathcal{L}(V, U')$ is the adjoint operator of B. We can also introduce the canonical isomorphism $\Lambda_U : U \to U'$, and deduce:

$$
{}_{U'}\langle B^*p(u), v-u\rangle_U = (\Lambda_U^{-1}B^*p(u), v-u)_U \ .
$$

So, we can finally rewrite (1.17) as:

$$
\frac{1}{2}J'(u)[v-u] = (\Lambda_U^{-1}B^*p(u) + Nu, v-u)_U \geqslant 0 \quad \forall v \in U_{ad} \ .
$$

We have also shown that the *Riesz' element*[2] (belonging to U) that represents the Gâteaux derivative of J in u is:

$$
J'(u) = 2(\Lambda_U^{-1}B^*p + Nu)
$$

All the results can be summarized in the following

Theorem 4. *Assume that all the hypotheses and definitions given in Section 1.3 hold, except hypothesis* (1.14)*. A necessary and sufficient condition for u to be an optimal control is that the following equations and inequalities (called optimality system) are satisfied:*

$$
\begin{cases}
Ay(u) = f + Bu & \text{in } V' \\
A^*p(u) = C^*\Lambda(Cy(u) - z_d) & \text{in } V' \\
(\Lambda_U^{-1}B^*p(u) + N(u), v-u)_U \geqslant 0 & \forall v \in U_{ad} \ .
\end{cases}
\tag{1.20}
$$

If $U_{ad} = U$, the last inequality is an equality.
Moreover, if N satisfies (1.14)*, the optimal control is unique.*
On the other hand, if $N = 0$ and U_{ad} is bounded, there exists at least one solution.
Finally, if $N = 0$ and

$$
\exists \alpha_0 > 0 \ : \ \|Cy(u) - z_d\|_H^2 \geqslant \alpha_0 \|u\|_U^2 \quad \forall u \in U_{ad} \ ,
$$

then the optimal control is unique.

[2] Riesz representation Theorem can be found, for example, in [16]

Chapter 2

Elliptic Problems and Galerkin Finite Element Method

In this Chapter we recall the general formulation of elliptic problems and their numerical approximation by finite elements.
Elliptic equations describe diffusion, reaction and transport phenomena in the case when there is no dependence on the time variable. Moreover they can be used to model electrostatic and electromagnetic potentials, and the deformation of elastic structures. Besides, also the time-dependent problems, after being discretized, can be seen as stationary problems to be solved at each temporal step.
The general theory of these kinds of problems can be found in [16] while their numerical approximation by Galerkin finite element method is described in [17]. In this Chapter we will base our analysis on these two references.

2.1 The Problem Setting

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^d$ (d=1,2,3) with a Lipschitz boundary $\partial\Omega$.
Let L be the linear elliptic operator:

$$Lu = div(-K\nabla u + \mathbf{b}u) + b_0 u \tag{2.1}$$

where $K = K(\mathbf{x})$ is a symmetric positive definite tensor $K = (K_{ij})_{i=1,\dots,d,j=1,\dots,d}$ such that $K_{ij} \in L^\infty(\Omega)$, $K_{ij} = K_{ji}$. Moreover, we will assume that K satisfies the elliptic constraint:

$$\sum_{i,j=1}^{d} K_{ij}\xi_i\xi_j \geq \overline{K}|\xi|^2 \quad \forall \xi \in \mathbb{R}^d\,, \tag{2.2}$$

for a certain $\overline{K} > 0$.
Moreover $\mathbf{b} \in [W^{1,\infty}(\Omega)]^d$ and $b_0 \in L^\infty(\Omega)$ with $b_0(\mathbf{x}) \geqslant 0$ in Ω, are such that:

- if $\Gamma_N = \emptyset$, $\exists \sigma_0 \geqslant 0$ such that

$$b_0(\mathbf{x}) + \frac{1}{2} div(\mathbf{b}(\mathbf{x})) \geqslant \sigma_0 \quad \forall \mathbf{x} \in \Omega; \tag{2.3}$$

- if $\Gamma_D = \emptyset$, $\exists \sigma_0 > 0$ such that (2.3) is satisfied and $\exists \varepsilon_0 \geqslant 0$ such that:

$$\textit{either} \quad \|\mathbf{b}\|_{L^\infty(\Gamma_N)} \leqslant \frac{2[min\{\overline{K}, \sigma_0\} - \varepsilon_0]}{C^*} \ , \quad \textit{or} \quad \mathbf{b} \cdot \mathbf{n} \leqslant 0 \ \textit{a.e. on} \ \Gamma_N \ , \tag{2.4}$$

  where C^* is the constant of trace inequality $\|v\|^2_{L^2(\partial\Omega)} \leqslant C^* \|v\|^2_{H^1(\Omega)}$, $\forall v \in H^1(\Omega)$;

- if both $\Gamma_D \neq \emptyset$ and $\Gamma_N \neq \emptyset$, $\exists \sigma_0 \geqslant 0$ such that (2.3) is satisfied and $\exists \varepsilon_0 \geqslant 0$ such that:
  - if $\sigma_0 = 0$, $\mathbf{b}$ satisfies:

$$\textit{either} \quad \|\mathbf{b}\|_{L^\infty(\Gamma_N)} \leqslant \frac{2\overline{K} C_\Omega - \varepsilon_0}{C^*} \ , \quad \textit{or} \quad \mathbf{b} \cdot \mathbf{n} \leqslant 0 \ \textit{a.e. on} \ \Gamma_N \ , \tag{2.5}$$

    where C_Ω is the constant of the Poincaré inequality $\|v\|_{L^2(\Omega)} \leqslant C_\Omega \|\nabla v\|_{L^2(\Omega)}$, $\forall v \in H^1_{0,\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \ \textit{on} \ \Gamma_D\}$,
  - if $\sigma_0 > 0$, $\mathbf{b}$ satisfies (2.4).

Let us consider the following *elliptic problem* $\mathcal{P}$ on Ω:

$$\begin{cases} Lu = f & \textit{in} \ \Omega \\ u = \phi_D & \textit{on} \ \Gamma_D \\ \partial_{n_L} u = \phi_N & \textit{on} \ \Gamma_N \end{cases} , \tag{2.6}$$

where $f \in L^2(\Omega)$, $\phi_D \in H^{1/2}(\Gamma_D)$, $\phi_N \in H^{-1/2}(\Gamma_N)$ and $\partial_{n_L} u$ is the conormal derivative of u with respect to the operator L:

$$\partial_{n_L} u = \sum_{i,j=1}^{d} K_{ij} \frac{\partial u}{\partial x_j} n_i - \mathbf{b} \cdot \mathbf{n} u \ ,$$

where n_i are the components of $\mathbf{n}$, the unit normal vector external to $\partial\Omega$.
The weak formulation of (2.6) is: find $u \in H^1(\Omega)$, $u = \phi_D$ on Γ_D such that

$$a(u, v) = F(v) \quad \forall v \in V \ , \tag{2.7}$$

where $V = \{v \in H^1(\Omega) \ : \ v = 0 \ \textit{on} \ \Gamma_D\}$, while the bilinear form $a(\cdot, \cdot)$, and the linear functional $F(\cdot)$ are defined as follows:

$$a(u, v) = \int_\Omega K \nabla u \cdot \nabla v d\Omega - \int_\Omega u \mathbf{b} \cdot \nabla v d\Omega + \int_\Omega b_0 u v d\Omega \quad \forall u, v \in H^1(\Omega) \tag{2.8a}$$

$$F(v) = \int_\Omega f v d\Omega + \int_{\Gamma_N} \phi_N v d\Gamma \quad \forall v \in H^1(\Omega) \tag{2.8b}$$

It can be shown that the following Theorem holds:

Theorem 5. *Let us consider the weak formulation* (2.7) *of the Elliptic problem* $\mathcal{P}$*: under the definitions and hypotheses made, the problem has a unique weak solution* $u \in V = H^1(\Omega)$.
Moreover the following inequality holds:

$$\|u\|_V \leqslant \frac{1}{\alpha} \|F\|_{V'} \ ,$$

where $\alpha > 0$ *is the coercivity constant of* $a(\cdot, \cdot)$.

Let us show the basic ideas of the proof. For details, see [16]. The proof of this Theorem is based on Lax-Milgram's Lemma. It is sufficient to show that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive on V and the thesis follows.

2.2 Galerkin Finite Element Method for Elliptic Equations

In this Section we recall the statements of the Galerkin Finite Element Method (FEM) and we present the main convergence results of this method in the case of problems governed by elliptic equations defined in a domain $\Omega \subset \mathbb{R}^2$. This method is useful to numerically solve the problem as an algebraic system.
Let us consider the generic elliptic problem (2.6), and let all the hypotheses of Section 2.1 be considered valid.
Let us introduce a function $R_{\phi_D} \in H^1(\Omega)$ which is the extension of the Dirichlet data into the whole domain, namely it is such that $R_{\phi_D} = \phi_D$ on Γ_D. Setting $w = u - R_{\phi_D}$, we can define a new problem, equivalent to problem (2.6), but with homogeneous boundary conditions:

$$\begin{cases} Lw = f - LR_{\phi_D} & in\ \Omega \\ \partial_{n_L} w = \phi_N - \partial_{n_L} R_{\phi_D} & on\ \Gamma_N \\ w = 0 & on\ \Gamma_D \end{cases} . \tag{2.9}$$

The weak formulation of problem (2.6) is equivalent to:
find $u = w + R_{\phi_D}$ such that $w \in V = \{v \in H^1(\Omega) : w = 0\ on\ \Gamma_D\}$ and

$$a(w,v) = F(v) - a(R_{\phi_D}, v) \quad \forall v \in V\ , \tag{2.10}$$

where $a(\cdot,\cdot)$ is defined in (2.8), while

$$F(v) = \int_\Omega f v d\Omega + \int_{\Gamma_N} \phi_N v d\Gamma\ .$$

Moreover, let us set $\widehat{F}(v) = F(v) - a(R_{\phi_D}, v)$.
We first consider the case when the Dirichlet data is zero. In this case no extension is required and we can find u directly.
Let us define a triangulation $\mathcal{T}_h$ of the domain Ω. $\mathcal{T}_h$ represents the covering of the domain Ω with non overlapping triangles. In particular, if we set $h_K = diam\{K\}$ the diameter of the generic triangle K belonging to $\mathcal{T}_h$, h is defined as $h = \max_{K\in\mathcal{T}_h} h_K$. Moreover, let us assume that the grid $\mathcal{T}_h$ satisfies the following regularity condition. Let ρ_K be the diameter of the inscribed circle in the triangle K. A family of triangulation $\{\mathcal{T}_h,\ h > 0\}$ is regular if

$$\exists \delta > 0 :\ \frac{h_k}{\rho_h} \leqslant 0 \quad \forall K \in \mathcal{T}_h\ .$$

Let us denote by $\mathbb{P}_r$ the space of the polynomials of global degree r and let us define

$$X_h^r = \{v_h \in C^0(\overline{\Omega}) : v_{h|_K} \in \mathbb{P}_r\ \forall K \in \mathcal{T}_h\}\ .$$

X_h^r is the space of global continuous functions that are polynomials of degree r on each single triangle. Moreover, we set

$$\widehat{X}_h^r = \{v_h \in X_h^r : v_h = 0\ on\ \Gamma_D\}\ .$$

These definitions allow us to approximate the problem 2.10 with a finite element problem of the following form:
find $u_h \in V_h = \widehat{X}_h^r$ such that:

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \ . \tag{2.11}$$

We want to highlight that the dimension of the generic space $\mathbb{P}_r$ defined on $\mathbb{R}^2$ is

$$dim\mathbb{P}_r = \frac{(r+1)(r+2)}{2} \ .$$

In particular, the $\mathbb{P}_1$ space has dimension three. Therefore, in each triangle of the triangulation we can fix three nodes (the vertices) such that each function belonging to $V_h = X_h^1$ is uniquely defined once its nodal values are known in the nodes $\mathbf{x}_i, i = 1, ..., N_h^t$ of the triangulation $\mathcal{T}_h$.
Moreover, we can define a basis of the space V_h as the set of $\varphi_i \in V_h,\ i = 1, ..., N_h^t$ such that

$$\varphi_i(\mathbf{x}_j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \quad i, j = 1, ..., N_h^t \ .$$

This definition allows us to represent a generic function $v_h \in V_h$ as a linear combination of the basis functions of V_h in the following way:

$$v_h(\mathbf{x}) = \sum_{i=1}^{N_h} v_i \varphi_i(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega, \quad v_i = v_h(\mathbf{x}_i) \ ,$$

where N_h is the number of nodes of $\mathcal{T}_h$, except the boundary ones where a Dirichlet condition is imposed. It is supposed that the Dirichlet nodes are numbered at the end. Let us set $N_h^b = N_h^t - N_h$. In particular, also the solution u_h of the discretized problem (2.11) can be expressed in this way. Moreover (2.11) is satisfied for any $v_h \in V_h$ if and only if it is satisfied for all basis functions $\varphi_i, i = 1, ..., N_h$. It can be easily shown that problem (2.11) is equivalent to:
find $u_j = u_h(\mathbf{x}_j),\ j = 1, ..., N_h$ such that

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i) \quad \forall i = 1, ..., N_h \ . \tag{2.12}$$

Introducing the matrix A and the vectors $\mathbf{f}$ and $\mathbf{u}$ such that $A_{ij} = a(\varphi_j, \varphi_i)$, $\mathbf{f}_i = F(\varphi_i)$ and $\mathbf{u}_i = u_i$ for $i, j = 1, ..., N_h$, the problem (2.12) is equivalent to the following algebraic system:

$$A\mathbf{u} = \mathbf{f} \ , \tag{2.13}$$

where the unknown vector to be found $\mathbf{u}$ holds the nodal values of the approximate solution $u_h \in V_h$.
We now describe how to treat non homogeneous Dirichlet data in the case of finite element approximation and in particular how to choose its extension. If we have $u = \phi_D$ on Γ_D in the continuous problem, we can discretize this boundary condition by setting

$$\phi_{Dh}(\mathbf{x}) = \sum_{i=N_h+1}^{N_h^t} \phi_D(\mathbf{x}_i) \varphi_{i|_{\Gamma_D}}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_D \ ,$$

and then

$$u_h(\mathbf{x}) = \phi_{Dh}(\mathbf{x}) \quad \forall \mathbf{x} \in \Gamma_D \ .$$

We can define the extension of the discretized Dirichlet data in the following way:

$$R_{\phi_{Dh}}(\mathbf{x}) = \sum_{i=N_h+1}^{N_h^t} \phi_{Dh}(\mathbf{x}_i)\varphi_i(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega$$

Therefore, in this case, problem (2.10) is equivalent to the algebraic system

$$A\mathbf{w} = \mathbf{f} - B\boldsymbol{\phi}_D \ ,$$

where $\{\boldsymbol{\phi}_D\}_i = \phi_D(\mathbf{x}_{i+N_h})$, for $i = 1, ..., N_h^b$ and $B_{ij} = a(\varphi_{j+N_h}, \varphi_i)$ for $i = 1, ..., N_h$ and $j = 1, ..., N_h^b$, while A and $\mathbf{f}$ have been already defined above. We highlight that both A and B are sparse matrices. Let us now give the most important convergence results, without proving them. The proofs can be found in [17].

Theorem 6. *Let us assume that all the assumptions made above are valid. Then the Galerkin method is stable, uniformly with respect to h. Therefore, the following inequality holds true:*

$$\|u_h\|_V \leqslant \frac{1}{\alpha}\|\widehat{F}\|_{V'} \ ,$$

where $u_h = w_h + R_{\phi_{Dh}}$, $V = H^1(\Omega)$ and α is the coercivity constant of the bilinear form $a(\cdot, \cdot)$.

Theorem 7. *Let $u \in V$ be the weak solution of the problem (2.7), and let $u_h = w_h + R_{\phi_{Dh}}$ be its approximate solution with finite element method of degree r. If there exists $p > 0$ such that $u \in H^{p+1}(\Omega)$, the following convergence estimates hold true:*

$$\begin{aligned} &\|u - u_h\|_{L^2(\Omega)} \leqslant Ch^{s+1}|u|_{H^{s+1}(\Omega)} \\ &\|u - u_h\|_{H^1(\Omega)} \leqslant Ch^{s}|u|_{H^{s+1}(\Omega)} \end{aligned} \quad s = min\{r, p\} \ , \tag{2.14}$$

where C is a constant independent of both h and u.

It is possible to discretize the problem also on a grid of quadrilaterals and to define on it a new space of degree 1 that is the space of continuous functions on the whole domain which are polynomials of degree one with respect to each of the variables on each quadrilateral. This space is the finite elements space $\mathbb{Q}_1$. A detailed description of this space and the discretization of the problem with these elements can be found in [5]. We will focus on the spaces $\mathbb{P}_r$ and we will use the space $\mathbb{Q}_1$ in order to compare the numerical results when necessary.

Chapter 3

Domain Decomposition: Classical Schwarz Method

In this chapter we describe one of the classical Domain Decomposition Methods (DDM), the Schwarz method. DDM enable fast resolution of a problem governed by differential partial equations if using parallel computing. Actually DDM enable us to rewrite the original problem into subproblems, one in each sub-domain of the original one. Moreover, DDM allow us to easily formulate and solve heterogeneous problems, namely problems where we have to solve different partial differential equations in two or more sub-domains. Interface Control Domain Decomposition methods are based on the domain decomposition of the computational domain, as the name suggests, into overlapping sub-domains; this motivates our interest.
The theory of DDM is presented in [18] and [20] and the theory of Schwarz method can be found also in [17]. In this Chapter we will refer to these books.

3.1 Schwarz Method

The Schwarz method has been proposed by H.Schwarz[1] as an iterative method to show the existence of the solution of elliptical equations defined in domains whose shape does not allow the application of Fourier series. Only much later this method has been extensively used to numerically solve partial differential equations.
Let $\Omega \subset \mathbb{R}^n$ $n = 1, ..., 3$ be an open and bounded set, with a Lipschitz bondary $\partial\Omega$. Let us consider the elliptic problem, analyzed in Chapter 2:
find

$$u : \Omega \to \mathbb{R}$$

such that

$$\begin{cases} Lu = f & in\ \Omega \\ u = g & on\ \partial\Omega\ , \end{cases} \tag{3.1a}$$

where L is the linear second order elliptic operator defined in (2.1). We will consider, for sake of simplicity, homogeneous Dirichlet conditions ($g = 0$). As already seen, under

[1] The theory of this method can be found in [17] and in [20]

suitable hypotheses on coefficients of L the problem is well-posed.
By introducing the space $V = H_0^1(\Omega)$, the weak formulation of the problem (3.1a) reads:

$$find\ u \in V\ :\ a(u,v) = (f,v) \quad \forall v \in V \tag{3.1b}$$

being $a(\cdot,\cdot)$ the bilinear form associated to L.
Let us consider a decomposition of the domain into two overlapping sub-domains Ω_1 and Ω_2 such that $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$ and $\Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset$, as shown in figure (3.1). Moreover, set $\Gamma_i = \partial\Omega_i \backslash (\partial\Omega \cap \partial\Omega_i)$.

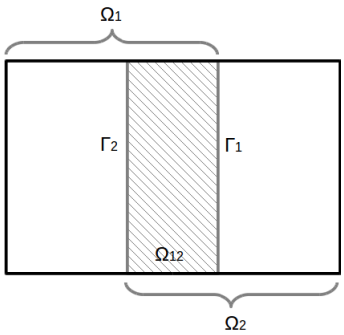


Figure 3.1: Example of domain partition

The Schwarz method is an iterative method that reads as follows: given $u_2^{(0)}$ on Γ_1, solve the following problems for $k \geq 1$:

$$\begin{cases} Lu_1^{(k)} = f & in\ \Omega_1 \\ u_1^{(k)} = u_2^{(k-1)} & on\ \Gamma_1 \\ u_1^{(k)} = 0 & on\ \partial\Omega_1 \backslash \Gamma_1 \end{cases} \tag{3.2a}$$

$$\begin{cases} Lu_2^{(k)} = f & in\ \Omega_2 \\ u_2^{(k)} = \alpha u_1^{(k)} + (1-\alpha) u_1^{(k-1)}\ ,\ \alpha \in \{0,1\} & on\ \Gamma_2 \\ u_2^{(k)} = 0 & on\ \partial\Omega_2 \backslash \Gamma_2 \end{cases} \tag{3.2b}$$

If we choose $\alpha = 1$ in (3.2b), the method is called *multiplicative Schwarz*, otherwise if $\alpha = 0$ *additive Schwarz*.
We want the sequences $u_1^{(k)}$ and $u_2^{(k)}$ to tend towards the restrictions of the solution u of the original problem (3.1a) in each sub-domain, so that

$$\lim_{k\to\infty} u_1^{(k)} = u_{|_{\Omega_1}} \quad and \quad \lim_{k\to\infty} u_2^{(k)} = u_{|_{\Omega_2}}\ . \tag{3.3}$$

It can be shown that the following Theorem holds. This result can be found in [17].

Theorem 8. *The Schwarz Method applied to problem* (3.1a) *with* Ω *split as in Figure 3.1 always converges to the solution* u *of the problem* (3.1b)*. The convergence rate increases as the measure* $|\Omega_{12}|/|\Omega|$ *increases.*

3.2 Algebraic Formulation of Schwarz method

Let us consider a regular triangulation $\mathcal{T}_h$ in finite elements of the domain Ω and the finite elements discretization of degree r $\mathbb{P}_r$ in Ω [2]. Moreover, let us suppose that the domain is split into two overlapping sub-domains Ω_1 and Ω_2, as shown in figure 3.1. We assume that in the overlapping region the nodes of the two grids match. If we set N_h^t the total number of nodes inside Ω, and N_i the total number of internal nodes of the domain Ω_i, we obtain that $N_h^t \leq N_1 + N_2$. Let us reorder the nodes, in order to have three blocks: the first one containing the nodes of $\Omega_1 \setminus \Omega_2$, the second one the ones of $\Omega_1 \cap \Omega_2$ and the third one the ones of $\Omega_2 \setminus \Omega_1$.
Discretizing the weak formulation of the original problem, we obtain the classical algebraic formulation $A\mathbf{u} = \mathbf{f}$[3], while discretizing the subproblem in the i^{th} domain, we obtain $A_i\mathbf{u}_i = \mathbf{f}_i$. The matrices A, A_1 and A_2 are linked to each other, as shown in figure 3.2, by the following relations:

$$A_1 = R_1 A R_1^T \in \mathbb{R}^{N_1 \times N_1}$$
$$A_2 = R_2 A R_2^T \in \mathbb{R}^{N_2 \times N_2}$$

where R_i and R_i^T are the restriction and extension operators from Ω to Ω_i.

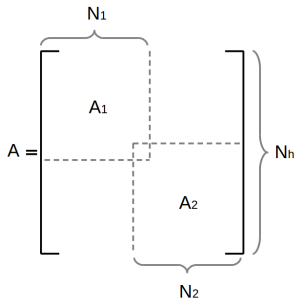


Figure 3.2: Structure of matrix A

Using these definitions, we can write an iteration of the Schwarz method applied to the system $A\mathbf{u} = \mathbf{f}$ as follows.
In the case of multiplicative Schwarz, we have: *given* $\mathbf{u}^{(0)}$, $\forall\ k \geq 0$

$$\begin{cases} \mathbf{u}^{(k+1/2)} = \mathbf{u}^{(k)} + R_1^T A_1^{-1} R_1 (\mathbf{f} - A\mathbf{u}^{(k)}) \\ \mathbf{u}^{(k+1)} = \mathbf{u}^{(k+1/2)} + R_2^T A_2^{-1} R_2 (\mathbf{f} - A\mathbf{u}^{(k+1/2)}) \end{cases} \tag{3.4}$$

On the other hand, in the case of additive Schwarz, the algebraic formulation becomes: *given* $\mathbf{u}^{(0)}$, $\forall\ k \geq 0$

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)(\mathbf{f} - A\mathbf{u}^{(k)}) \tag{3.5}$$

We can notice that, in both cases, two algebraic systems are solved at each iteration of the method. Moreover, setting

$$Q_i = R_i^T A_i^{-1} R_i \quad i = 1, 2\ , \tag{3.6}$$

[2] See Chapter 2 for details on the general theory of finite element method
[3] See Chapter 2 for details

we can rewrite the methods in the following ways:
Multiplicative Schwarz

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (Q_1 + Q_2 - Q_2AQ_1)(\mathbf{f} - A\mathbf{u}^{(k)}) \ ,$$

Additive Schwarz

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (Q_1 + Q_2)(\mathbf{f} - A\mathbf{u}^{(k)}) \ .$$

Now it is clear that an iteration of both multiplicative and additive Schwarz can be seen as an iteration of *preconditioned Richardson method* applied to solve the linear system $A\mathbf{u} = \mathbf{f}$, where the preconditioner P is:
Multiplicative Schwarz

$$P_{ms} = (Q_1 + Q_2 - Q_2AQ_1)^{-1} \ ,$$

Additive Schwarz

$$P_{as} = (Q_1 + Q_2)^{-1} \ .$$

Moreover, in the case of additive Schwarz, the following Theorem holds[4]

Theorem 9. *Let δ be the overlap thickness and $H = \max_{i=1,2}\{diam\{\Omega_i\}\}$. If we denote by $K_2(T)$ the condition number of the square matrix T with respect to the matrix norm $\|T\| = \sqrt{\rho(T^TT)}$, it holds that*

$$K_2(P_{as}^{-1}A) \leqslant \frac{C}{\delta H} \ ,$$

where C is independent of h, H and δ.

These preconditioners can also be used to precondition the algebraic system $A\mathbf{u} = \mathbf{f}$ when it is solved by *Krylov* or *Conjugate Gradient like* algorithms.
We want to underline that the number of iterations needed to solve the linear system $A\mathbf{y} = \mathbf{f}$ with *preconditioned Richardson* (it_R) and with *Preconditioned Conjugate Gradient method* (PCG) (it_{PCG}), preconditioned with a matrix P, are theoretically linked to $\kappa = K_2(P^{-1}A)$ in the following way[5]:

$$it_R \propto \kappa$$

and

$$it_{PCG} \propto \kappa^{1/2} \ .$$

Therefore, according to Theorem 9, once fixed the parameter H, the number of iterations of method (3.5) (equivalent to Richardson preconditioned with P_{as}) is

$$it \propto \delta^{-1} \ , \tag{3.7}$$

while the number of iterations of PCG preconditioned with P_{as} is

$$it \propto \delta^{-1/2} \ . \tag{3.8}$$

[4]This result can be found in [17]
[5]Details on Iterative Methods can be found in Appendix A

3.3 Numerical results

We solve the problem (3.1a) in $\Omega = (0,1) \times (0,1)$ (with $K = 2$, $\mathbf{b} = \mathbf{0}$ and $b_0 = 1$) with both additive and multiplicative Schwarz method (3.5) and (3.4), setting f and g such that the exact solution of the problem is $u = exp(x+y)+sin(x)+cos(y)$. We solve the problem also by *Preconditioned Conjugate Gradient method* (PCG) applied to the whole system $A\mathbf{u} = \mathbf{f}$, preconditioned with P_{as}. We decided to implement also this method because it is easy to be implemented and it is highly parallelizable. We firstly fix the dimension of the overlap $\delta = 0.1$ and solve the problem by decreasing the mesh size h. Then, fixed the mesh size $h = 0.01$, we decrease the overlap thickness.
We notice that, in terms of number of iterations, the behaviour of finite elements $\mathbb{P}_1$ and $\mathbb{Q}_1$ is exactly the same. Moreover, in both cases, the errors, with respect to the exact solution of the problem, in H1 and L2 norms converge to zero as the mesh size decreases in agreement with the finite element theory (Figure 3.5), namely we have a second order of convergence with respect to h in L^2 norm and a first order of convergence in H^1 norm. The number of iterations is independent of the mesh size h, but it depends on the thickness δ of the overlap. In the case of additive Schwarz the number of iterations varies according to equations (3.7) and (3.8), as we can see respectively in Figure 3.3 and 3.4, while in the case of multiplicative Schwarz it increases as δ^{-1} as δ tends to zero (Figure 3.3). Moreover, we can note (Figure 3.3) that multiplicative Schwarz is faster than the additive one.
We also see that the number of iterations of PCG method preconditioned with P_{as} is lower than the number of iterations of classical additive Schwarz. Furthermore, the number of iterations in the first case increases less than the one of the second case when δ decreases in agreement with 3.8 and 3.7, respectively.

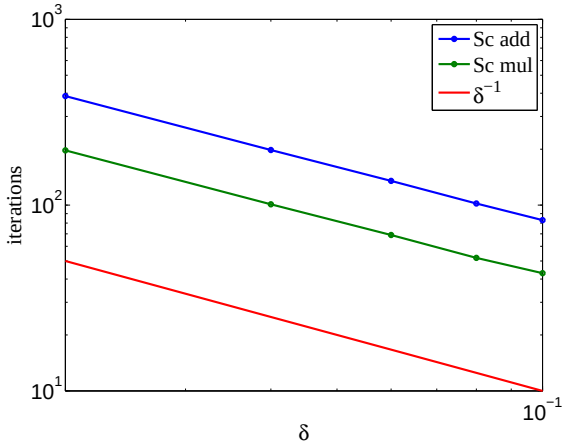


Figure 3.3: Number of iterations versus the overlap size - Schwarz methods (3.5) and (3.4)

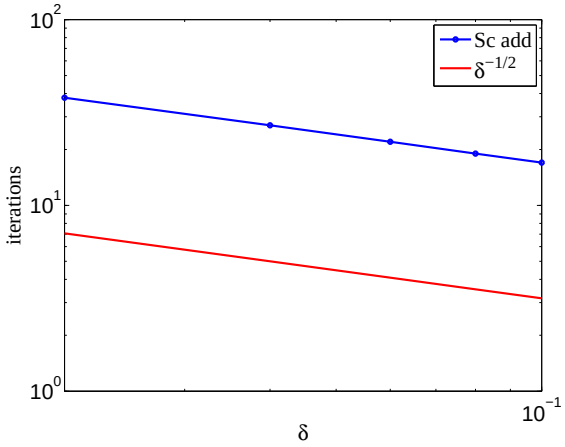


Figure 3.4: Number of iterations versus the overlap size - PCG algorithm preconditioned with P_{as}

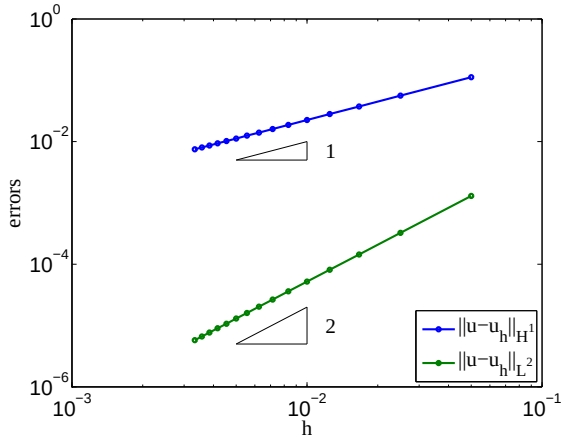


Figure 3.5: Errors versus the mesh size

Chapter 4

Domain Decomposition and Optimal Control: Interface Control Domain Decomposition Methods for Elliptic Problems

This chapter is aimed to describe *Interface Control Domain Decomposition Methods* (ICDD). ICDD methods combine domain decomposition methods with overlapping sub-domains with optimal control problem, in order to treat both homogeneous and heterogeneous problems.
Actually the basic idea of this approach consists in introducing suitable functions called *interface controls* which play the role of unknown boundary data on the interfaces of the decomposition and in minimizing, in a suitable norm, the difference between the solutions of the subproblems on the overlap.
They have been proposed and analyzed in [8]. We recall here the notations and some of the results presented in [8] for both completeness and clearness. There, the results are proven in the case of Dirichlet interface controls and in the case of homogeneous Dirichlet boundary conditions. Here we extend them to a more general setting, by considering both Dirichlet and Neumann boundary conditions. Moreover we discuss with more details the case of Robin interface controls. Proofs of Theorems and results in the case of Dirichlet interface controls can be found in [8].

4.1 The Problem Setting

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^d$ (d=1,2,3) with a Lipschitz boundary $\partial\Omega$, split into two overlapping sub-domains Ω_1 and Ω_2, such that $\Omega_1 \cap \Omega_2 = \Omega_{12} \neq \emptyset$ and $\overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}$, as shown in figure 4.1. Moreover, let $\overline{\partial\Omega} = \overline{\Gamma_N} \cup \overline{\Gamma_D}$ with $\Gamma_N \cap \Gamma_D = \emptyset$. Let us denote $\Gamma_D^i = \Gamma_D \cap \partial\Omega_i$ and $\Gamma_N^i = \Gamma_N \cap \partial\Omega_i$.

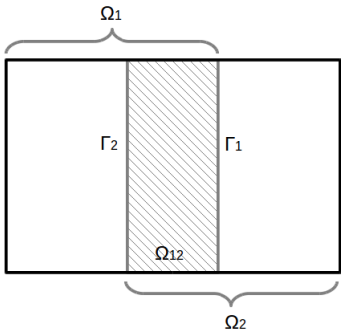


Figure 4.1: Example of domain partition

Let us consider the problem (2.6) on Ω, described and analyzed in Section 2.1:

$$\begin{cases} Lu = f & in\ \Omega \\ u = \phi_D & on\ \Gamma_D \\ \partial_{n_L} u = \phi_N & on\ \Gamma_N \end{cases}, \tag{4.1}$$

which weak formulation is given by (2.7): find $u \in H^1(\Omega)$, $u = \phi_D$ on Γ_D such that

$$a(u, v) = F(v) \quad \forall v \in V\ , \tag{4.2}$$

where $V = \{v \in H^1(\Omega)\ :\ v = 0\ on\ \Gamma_D\}$, while the bilinear form $a(\cdot, \cdot)$, and the linear functional $F(\cdot)$ are defined in (2.8).
We will refer to problem (4.1) as *global problem* $\mathcal{P}$.
We want to reformulate the problem (4.1) in an equivalent multidomain manner. In particular, we choose overlapping sub-domains. In this case, there are two different formulations.
Problem $\mathcal{P}_{\Omega_{12}}$:

$$\begin{cases} Lu_1 = f & in\ \Omega_1 \\ Lu_2 = f & in\ \Omega_2 \\ u_1 = u_2 & in\ \Omega_{12} \end{cases} \tag{4.3}$$

with boundary conditions

$$\begin{cases} u_i = \phi_{D|\Gamma_D^i} & on\ \Gamma_D^i = \partial\Omega_i \cap \Gamma_D\ ,\ i = 1, 2 \\ \partial_{n_L} u_i = \phi_{N|\Gamma_N^i} & on\ \Gamma_N^i = \partial\Omega_i \cap \Gamma_N\ ,\ i = 1, 2 \end{cases} \tag{4.4}$$

and
Problem $\mathcal{P}_{\Gamma_1 \cup \Gamma_2}$:

$$\begin{cases} Lu_1 = f & in\ \Omega_1 \\ Lu_2 = f & in\ \Omega_2 \\ \Psi(u_1) = \Psi(u_2) & on\ \Gamma_1 \cup \Gamma_2 \end{cases} \tag{4.5}$$

with the same boundary conditions given in (4.4).
We denote by $\Psi(u_i)$ either the trace of u_i on $\Gamma_1 \cup \Gamma_2$, or its conormal derivative $\partial_{n_L} u_i$ on $\Gamma_1 \cup \Gamma_2$ with respect to the same normal vector, or else a linear combination between them.
In particular, condition $(4.5)_3$ in the first case becomes

$$u_1 = u_2 \quad on\ \Gamma_1 \cup \Gamma_2\ , \tag{4.6}$$

in the second case

$$\partial_{n_L} u_1 = \partial_{n_L} u_2 \quad on\ \Gamma_1 \cup \Gamma_2\ , \tag{4.7}$$

or in the third case

$$\beta u_1 + \partial_{n_L} u_1 = \beta u_2 + \partial_{n_L} u_2 \quad on\ \Gamma_1 \cup \Gamma_2\ , \tag{4.8}$$

where $\beta > 0$ is a suitable parameter.
In the second and the third cases, we want to underline that the normal vector $\mathbf{n}$ on Γ_i is directed outward from Ω_i and that the conormal derivative of $u_j, j \neq i$ is computed upon restricting u_i to $\Omega_j \setminus \Omega_{ij}$.
We can associate the problem (4.5) with its weak formulation, as made for problem (4.1). Let us introduce the following Hilbert spaces:

$$V_i = \{v \in H^1(\Omega_i)\ :\ v = 0\ on\ \Gamma_D^i\} \quad and \quad V_i^D = \{v \in V_i\ :\ v = 0\ on\ \Gamma_i\}\ ,$$

endowed with the canonical norm of $H^1(\Omega_i)$.
In the case (4.6), the weak formulation reads: find $u_i \in H^1(\Omega_i)$, $u_i = \phi_D$ on Γ_D^i, $u_i = u_j$ on Γ^i, $i \neq j$, such that

$$a_i(u_i, v_i) = F_i(v_i) \quad \forall v_i \in V_i^D \qquad i, j = 1, 2\ ,\ i \neq j\ ,$$

where $a_i(\cdot,\cdot)$ is the restriction of $a(\cdot,\cdot)$ (defined in (2.8)) to Ω_i, and

$$F_i(v_i) = \int_{\Omega_i} f v_i d\Omega + \int_{\Gamma_N^i} \phi_N v_i d\Gamma \quad \forall v_i \in V_i^D\ .$$

In the third case (4.8) the weak formulation becomes: find $u_i \in H^1(\Omega_i)$, $u_i = \phi_D$ on Γ_D^i, such that

$$a_i(u_i, v_i) + \int_{\Gamma_i} \beta u_i v_i d\Gamma = F_i(v_i) + \int_{\Gamma_i} (\beta u_j + \partial_{n_L} u_j) v_i d\Gamma \quad \forall v_i \in V_i \qquad i, j = 1, 2\ ,\ i \neq j\ ,$$

where $a_i(\cdot,\cdot)$ and $F_i(\cdot)$ are the same as in the previous case. In particular, we obtain the second case (4.7) by setting $\beta = 0$. It can be shown that, under suitable assumptions on parameters, these problems are well posed.

We want now to prove that (4.1), (4.3)-(4.4) and (4.5)-(4.4) are equivalent problems. The Theorem 10 and Corollary 4 that follow are proven in [8]. We recall here also their proofs for completeness.
Let us firstly introduce the following Hilbert spaces:

$$V_{\phi_D} = \{v \in H^1(\Omega)\ :\ v = \phi_D\ on\ \Gamma_D\} \quad and \quad V_{i,\phi_D} = \{v \in H^1(\Omega_i)\ :\ v = \phi_D\ on\ \Gamma_D^i\}\ .$$

The following Theorem holds:

Theorem 10. *The function $u \in V_{\phi_D}$ is the weak solution of (4.1) if and only if $u_i = u_{|_{\Omega_i}} \in V_{i,\phi_D}$, $i = 1, 2$ are the weak solutions of (4.3)-(4.4). Moreover, $u_i \in V_{i,\phi_D}$, $i = 1, 2$ are the solutions of (4.3)-(4.4) if and only if they are solutions of (4.5)-(4.4).*

Proof. The proof of this Theorem is split into two steps.

- Let $u \in V_{\phi_D}$ be the solution of (4.1). If we consider the restriction of u to Ω_1 and Ω_2, $u_{|\Omega_1}$ and $u_{|\Omega_2}$, they satisfy (4.3)-(4.4) by construction. On the other hand, let u_1 and u_2 be the solutions of (4.3)-(4.4), and let us consider

$$u = \begin{cases} u_1 & in\ \Omega_1 \setminus \Omega_{12} \\ u_1 = u_2 & in\ \Omega_{12} \\ u_2 & in\ \Omega_2 \setminus \Omega_{12}\ . \end{cases}$$

  It can be easily seen that $u \in V_{\phi_D}$ and it satisfies (4.1).

- Let now u_1 and u_2 be the solutions of the problem (4.5)-(4.4), and let us set $w = u_{1|\Omega_{12}} - u_{2|\Omega_{12}}$ in Ω_{12}. It results by definition that $w \in V_{12}$, with $V_{12} = \{v \in \Omega_{12} : v = 0\ on\ \partial\Omega_{12} \cap \Gamma_D\}$. Moreover, w satisfies the following problem (with homogeneous data):

$$\begin{cases} Lw = 0 & in\ \Omega_{12} \\ \Psi(w) = 0 & on\ \Gamma_1 \cup \Gamma_2 \\ w = 0 & on\ \Gamma_D \cap \partial\Omega_{12} \\ \partial_{n_L} w = 0 & on\ \Gamma_N \cap \partial\Omega_{12}\ , \end{cases}$$

  whose unique solution is $w = 0$. It follows that $u_1 = u_2$ on Ω_{12} and they satisfy problem (4.3).
  Vice versa, if u_1 and u_2 are solutions of the problem (4.3), they also satisfy $(4.5)_1$ and $(4.5)_2$, and the boundary conditions (4.4). Moreover, since $u_1 \in V_{1,\phi_D}$ and $u_2 \in V_{2,\phi_D}$ in Ω_{12}, their image on $\Gamma_1 \cup \Gamma_2$ through the operator Ψ are also equal.

□

It is now easy to prove the following

Corollary 4. *Problems* (4.3) *and* (4.5) *with boundary conditions* (4.4) *are well-posed.*

Proof. Since (4.1) is well-posed thanks to Theorem 5, and problems (4.3)-(4.4) and (4.5)-(4.4) are equivalent to problem (4.1) thanks to Theorem 10, the well-posedness of problems (4.3) and (4.5) with boundary conditions (4.4) follows. □

4.2 The Interface Control Domain Decomposition Method

In this Section we describe the *Interface Control Domain Decomposition Methods* (ICDD) and analyze them.
The main idea of these methods consists in introducing two or more controls, one for each internal interface, which play the role of unknown Dirichlet or Neumann or Robin data at the internal interfaces of the decomposition, and in finding them by minimizing the difference between the solutions u_1 and u_2 of either problem (4.3) or (4.5) with boundary conditions (4.4) in a suitable norm defined on either Ω_{12} or $\Gamma_1 \cup \Gamma_2$.
In the following we will consider the Neumann controls as a peculiar case of Robin ones. Actually it is sufficient to set $\beta = 0$ to find Neumann case.
Let us introduce the *spaces of admissible controls*. If $\overline{\Gamma_D^i} \cap \overline{\Gamma_i} \neq \emptyset$, we denote by $\tilde{\mu}$ the trivial extension of $\mu \in H^{1/2}(\Gamma_i)$ to Γ_D^i

$$\tilde{\mu} = \begin{cases} \mu & \mathbf{x} \in \Gamma_i \\ 0 & \mathbf{x} \in \Gamma_D^i \end{cases}\ .$$

The functional spaces can be defined as follows:

- admissible Dirichlet controls:

$$\Lambda_i^D = \begin{cases} H_{00}^{1/2}(\Gamma_i) = \{\mu \in H^{1/2}(\Gamma_i) : \tilde{\mu} \in H^{1/2}(\partial\Omega)\} & if\ \Gamma_N^i = \emptyset \\ \{\mu \in H^{1/2}(\Gamma_i) : \tilde{\mu} \in H^{1/2}(\Gamma_i \cup \Gamma_D^i)\} & if\ \overline{\Gamma_D^i} \cap \overline{\Gamma_i} \neq \emptyset \\ H^{1/2}(\Gamma_i) & otherwise \end{cases} \tag{4.9}$$

- admissible Robin control ($\beta \geqslant 0$):

$$\Lambda_i^R = (H_{00}^{1/2}(\Gamma_i))' \tag{4.10}$$

It can be shown that the spaces $(\Lambda_i^D, \|\cdot\|_{H^{1/2}(\Gamma_i)})$ and $(\Lambda_i^R, \|\cdot\|_{(H_{00}^{1/2}(\Gamma_i))'})$ are Hilbert spaces.[1]
Using the notation of problem (4.5), we can formulate the *state problems* in the following way. Let us consider two unknown control functions $\lambda_i \in \Lambda_i^D$ in the case of Dirichlet controls, or $\lambda_i \in \Lambda_i^R$ in the case of Robin one. The *state problems* read: find $u_i = u_i(\lambda_i) \in H^1(\Omega_i)$, $i = 1, 2$, such that:

$$\begin{cases} Lu_i = f & in\ \Omega_i \\ \partial_{n_L} u_i = \phi_N^i & on\ \Gamma_N^i \\ u_i = \phi_D^i & on\ \Gamma_D^i \\ \Psi(u_i) = \lambda_i & on\ \Gamma_i \end{cases} \tag{4.11}$$

where $\phi_N^i = \phi_{N|\Gamma_N^i}$ and $\phi_D^i = \phi_{D|\Gamma_D^i}$.
The interface controls are determined as solutions of a minimization problem. Actually the objective is to minimize a suitable cost functional, depending on the difference, with respect to a suitable norm, between the two solutions u_1 and u_2 of problems (4.11) either on the overlapping region Ω_{12} or on the interfaces $\Gamma_1 \cup \Gamma_2$.
There are several possible cost functionals to be considered. We can minimize the difference between the two solutions in the following ways, given rise to different minimization problems:

- Case 1: Minimization in the $L^2(\Omega_{12})$ norm:

$$\begin{aligned} &\inf_{\lambda_1,\lambda_2} J_0(\lambda_1, \lambda_2), \quad with \\ &J_0(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1(\lambda_1) - u_2(\lambda_2)\|_{L^2(\Omega_{12})}^2 = \frac{1}{2}\int_{\Omega_{12}} (u_1(\lambda_1) - u_2(\lambda_2))^2 d\Omega \end{aligned} \tag{4.12}$$

- Case 2: Minimization in the $H^1(\Omega_{12})$ norm:

$$\begin{aligned} &\inf_{\lambda_1,\lambda_2} J_1(\lambda_1, \lambda_2), \quad with \\ &J_1(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1(\lambda_1) - u_2(\lambda_2)\|_{H^1(\Omega_{12})}^2 \\ &= \frac{1}{2}\int_{\Omega_{12}} (u_1(\lambda_1) - u_2(\lambda_2))^2 d\Omega + \frac{1}{2}\int_{\Omega_{12}} (\nabla u_1(\lambda_1) - \nabla u_2(\lambda_2))^2 d\Omega \end{aligned} \tag{4.13}$$

[1] See [12], section 2.7 for details

- Case 3: Minimization in the $L^2(\Gamma_1 \cup \Gamma_2)$ norm:

$$\inf_{\lambda_1,\lambda_2} J_{0,\Gamma}(\lambda_1,\lambda_2), \quad with$$
$$J_{0,\Gamma}(\lambda_1,\lambda_2) = \frac{1}{2}\|u_1(\lambda_1) - u_2(\lambda_2)\|^2_{L^2(\Gamma_1\cup\Gamma_2)} = \frac{1}{2}\int_{\Gamma_1\cup\Gamma_2}(u_1(\lambda_1) - u_2(\lambda_2))^2 d\Gamma \tag{4.14}$$

We remark that the minimization problems (4.12) - (4.14) with constraints (4.11) are optimal control problems and can be analyzed using the classical theory of optimal control described in Chapter 1. In particular, the controls are of boundary type in every case, while the observation is distributed in the first two cases and of interface type in the third one.
In order to analyze the well-posedness of the ICDD methods, it is useful to split the probem into two parts, the former depending only on the control and the latter on the data. Thanks to the linearity of the original problem, we have:

$$u_i = u_i^{\lambda_i,f} = u_i^{\lambda_i} + u_i^f \ , i = 1,2 \ ,$$

where $u_i^{\lambda_i}, i = 1,2$ are the solutions of the problems depending on the controls and $u_i^f, i = 1,2$ are the solutions of the ones depending on the data. Particularly, we have that $u_i^{\lambda_i} \in H^1(\Omega_i), i = 1,2$ satisfy:

$$\begin{cases} Lu_i^{\lambda_i} = 0 & in\ \Omega_i \\ \partial_{n_L} u_i^{\lambda_i} = 0 & on\ \Gamma_N^i \\ u_i^{\lambda_i} = 0 & on\ \Gamma_D^i \\ \Psi(u_i^{\lambda_i}) = \lambda_i & on\ \Gamma_i \end{cases} \tag{4.15}$$

while $u_i^f \in H^1(\Omega_i), i = 1,2$ are the solutions of:

$$\begin{cases} Lu_i^f = f & in\ \Omega_i \\ \partial_{n_L} u_i^f = \phi_N^i & on\ \Gamma_N^i \\ u_i^f = \phi_D^i & on\ \Gamma_D^i \\ \Psi(u_i^f) = 0 & on\ \Gamma_i \end{cases} \tag{4.16}$$

When $\Omega \subset \mathbb{R}^2, \mathbb{R}^3$, in the case of Dirichlet interface controls, let us assume that if $\overline{\Gamma_D^i} \cap \overline{\Gamma_i} \neq \emptyset$, ϕ_i^D is connected with continuity to zero at $\overline{\Gamma_D^i} \cap \overline{\Gamma_i}$.
Let us consider the notations used in Sections 2.1 and 4.1. The weak formulations of these problems read, for i=1,2:

- *Dirichlet interface controls*
  Find $u_i^{\lambda_i} \in H^1(\Omega_i)$, $u_i^{\lambda_i} = \lambda_i$ on Γ_i, $u_i^{\lambda_i} = 0$ on Γ_D^i such that

$$a_i(u_i^{\lambda_i}, v_i) = 0 \quad \forall v_i \in V_i^D \ , \tag{4.17a}$$

  and find $u_i^f \in H^1(\Omega_i)$, $u_i^f = 0$ on Γ_i, $u_i^f = \phi_D^i$ on Γ_D^i such that

$$a_i(u_i^f, v_i) = F_i(v_i) \quad \forall v_i \in V_i^D \ ; \tag{4.17b}$$

- *Robin interface controls*
  Find $u_i^{\lambda_i} \in H^1(\Omega_i)$, $u_i^{\lambda_i} = 0$ on Γ_D^i such that

$$a_i(u_i^{\lambda_i}, v_i) + \int_{\Gamma_i} \beta u_i^{\lambda_i} v_i d\Gamma = \int_{\Gamma_i} \lambda_i v_i d\Gamma \quad \forall v_i \in V_i \ , \tag{4.18a}$$

  and find $u_i^f \in H^1(\Omega_i)$, $u_i^f = \phi_D^i$ on Γ_D^i such that

$$a_i(u_i^f, v_i) + \int_{\Gamma_i} \beta u_i^f v_i d\Gamma = F_i(v_i) \quad \forall v_i \in V_i \ . \tag{4.18b}$$

Let us refer to (4.17) or to (4.18) as *primal* or *state problems*. We will specify them when required.

Correspondingly, the cost functionals can be split into the sum of two terms, a quadratic one and an affine one. Let us first introduce a unified notation. In particular, let $\mathcal{H}$ be a Hilbert space. If we denote the generic functional (to be choosen among (4.12)-(4.14)) with $J(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1 - u_2\|_{\mathcal{H}}^2$, it results:

$$J(\lambda_1, \lambda_2) = \frac{1}{2}\|u_1^{\lambda_1} - u_2^{\lambda_2}\|_{\mathcal{H}}^2 + \frac{1}{2}\|u_1^f - u_2^f\|_{\mathcal{H}}^2 + (u_1^{\lambda_1} - u_2^{\lambda_2}, u_1^f - u_2^f)_{\mathcal{H}} \ ,$$

where the first terms of the splitting is quadratic with respect to $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, while the second and the third terms compose the affine term.
Reasoning in a similar way as done in the papers [2] and [7], it can also be shown that under the definitions and hypotheses made in (2.1)-(2.6), if $\lambda_i \in \Lambda_i^D$ or $\lambda_i \in \Lambda_i^R$, in the case of Dirichlet or Robin interface controls respectively, the two problems (4.15) and (4.16) are well-posed.

4.3 The Well-Posedness of the ICDD Problem

In this Section we analyze the well-posedness of the control problems. We will use the results seen in Chapter 1.
We will split our analysis into two cases, the first one corresponding to Dirichlet interface controls and the second one to Robin interface controls.

4.3.1 Dirichlet interface controls

Let us define the Hilbert spaces $\mathbf{V} = V_{1,\phi_D} \times V_{2,\phi_D}$, $\mathbf{V}^D = V_1^D \times V_2^D$ and $\boldsymbol{\Lambda}^D = \Lambda_1^D \times \Lambda_2^D$, endowed with the corresponding graph norms, and let us set $\Lambda_{12} = L^2(\Gamma_1 \cup \Gamma_2)$.
Moreover, we set $\mathbf{u}^{\boldsymbol{\lambda},f} = (u_1^{\lambda_1,f}, u_2^{\lambda_2,f}) \in \mathbf{V}$ for any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \boldsymbol{\Lambda}^D$. Let be $\mathbf{u}^{\boldsymbol{\lambda}} = \mathbf{u}^{\boldsymbol{\lambda},0}$ and $\mathbf{u}^f = \mathbf{u}^{\mathbf{0},f}$.
We want to unify the notation in order to make the analysis simpler. We have already introduced some elements in the previous Section, but we now give a more detailed description of the notations, using the framework of optimal control.
Let $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ be a Hilbert space, and $C : \mathbf{V} \to \mathcal{H}$ a linear and continuous operator (the observation one), so that the cost functionals in (4.12)-(4.14) can be written in the following way:

$$J_{C,\mathcal{H}}(\boldsymbol{\lambda}) = \frac{1}{2}\|C\mathbf{u}^{\boldsymbol{\lambda},f}\|_{\mathcal{H}}^2 \quad \forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}^D \ , \tag{4.19}$$

where, for each case, we can identify the space, the observation operator and the norm, as shown in this table:

	$J_{C,\mathcal{H}}(\boldsymbol{\lambda})$	$\mathcal{H}$	$C\mathbf{u}^{\boldsymbol{\lambda},f}$	$\|\!\|\!\|\cdot\|\!\|\!\|_*$
Case 1:	$J_0(\lambda_1,\lambda_2)$	$L^2(\Omega_{12})$	$(u_1^{\lambda_1,f}-u_2^{\lambda_2,f})_{\|\Omega_{12}}$	$\|\!\|\!\|\boldsymbol{\lambda}\|\!\|\!\|_0=\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{L^2(\Omega_{12})}$
Case 2:	$J_1(\lambda_1,\lambda_2)$	$H^1(\Omega_{12})$	$(u_1^{\lambda_1,f}-u_2^{\lambda_2,f})_{\|\Omega_{12}}$	$\|\!\|\!\|\boldsymbol{\lambda}\|\!\|\!\|_1=\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{H^1(\Omega_{12})}$
Case 3:	$J_{0,\Gamma}(\lambda_1,\lambda_2)$	Λ_{12}	$(u_1^{\lambda_1,f}-u_2^{\lambda_2,f})_{\|\Gamma_1\cup\Gamma_2}$	$\|\!\|\!\|\boldsymbol{\lambda}\|\!\|\!\|_{0,\Gamma}=\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{L^2(\Gamma_1\cup\Gamma_2)}$

The terms $(u_1^{\lambda_1,f}-u_2^{\lambda_2,f})_{|\Gamma_1\cup\Gamma_2}$ stands for the trace in the space $H^{1/2}(\Gamma_1\cup\Gamma_2)$.
The split of the functional already introduced in the previous Section can be rewritten as:

$$J_{C,\mathcal{H}}(\boldsymbol{\lambda})=\frac{1}{2}\|C\mathbf{u}^{\boldsymbol{\lambda}}\|^2_{\mathcal{H}}+\frac{1}{2}\|C\mathbf{u}^{f}\|^2_{\mathcal{H}}+(C\mathbf{u}^{\boldsymbol{\lambda}},C\mathbf{u}^{f})_{\mathcal{H}}\ , \tag{4.20}$$

The following Lemma holds. Its proof, in the case with $\Gamma_N=\emptyset$, can be found in [8].

Lemma 2. *In all cases 1-3 $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}}$ is a norm on the control space $\boldsymbol{\Lambda}^D$.*

We want to highlight that $\boldsymbol{\Lambda}^D$ is not necessary complete[2] with respect to any of the norms $\|\!\|\!\|\cdot\|\!\|\!\|_*$, defined in the previous table. Nevertheless, it is possible to built the completion of $(\boldsymbol{\Lambda}^D,\|\!\|\!\|\cdot\|\!\|\!\|_*)$, and to look for the solution of the minimization problem in this complete space. Let us denote by $\widehat{\boldsymbol{\Lambda}}^D$ the completion of $(\boldsymbol{\Lambda}^D,\|\!\|\!\|\cdot\|\!\|\!\|_*)$. We underline that, if $(\boldsymbol{\Lambda}^D,\|\!\|\!\|\cdot\|\!\|\!\|_*)$ is complete, then it follows $\widehat{\boldsymbol{\Lambda}}^D=\boldsymbol{\Lambda}^D$.
The following Theorem holds. The proof can be found in [8].

Theorem 11. *The minimization problem*

$$\inf_{\boldsymbol{\lambda}\in\widehat{\boldsymbol{\Lambda}}^D} J_{C,\mathcal{H}}(\boldsymbol{\lambda}) \tag{4.21}$$

has a unique solution $\boldsymbol{\lambda}\in\widehat{\boldsymbol{\Lambda}}^D$. Moreover the solution satisfies

$${}_{(\widehat{\boldsymbol{\Lambda}}^D)'}\langle J'_{C,\mathcal{H}}(\boldsymbol{\lambda}),\boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^D}=(C\mathbf{u}^{\boldsymbol{\lambda},f},C\mathbf{u}^{\boldsymbol{\mu}})_{\mathcal{H}}=0\quad\forall\mu\in\widehat{\boldsymbol{\Lambda}}^D\ . \tag{4.22}$$

4.3.2 Robin interface controls

We want now to analyze the well-posedness of ICDD in the case of Robin interface controls. We will use the same notations of the previous Section and introduce new notations when necessary.
Let us set $\boldsymbol{\Lambda}^R=\Lambda_1^R\times\Lambda_2^R$ and consider $\boldsymbol{\lambda}\in\boldsymbol{\Lambda}^R$. We consider the cost functionals defined in (4.12) - (4.14), but now with $\boldsymbol{\lambda}\in\boldsymbol{\Lambda}^R$ instead of $\boldsymbol{\Lambda}^D$.
Also in this case, the following can be proven:

Lemma 3. *In all cases 1-3 $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}}$ is a norm on the control space $\boldsymbol{\Lambda}^R$.*

[2] A normed space $(V,\|\cdot\|)$ is said to be complete with respect to the norm $\|\cdot\|$ if each every Cauchy sequence in $(V,\|\cdot\|)$ is convergent in $(V,\|\cdot\|)$

Proof. Firstly, we prove that $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}}$ is a semi-norm on $\boldsymbol{\Lambda}^R$ in all cases. Then, we demonstrate that if $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}} = 0$, then $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = \mathbf{0}$.

- $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}}$ is a semi-norm on $\boldsymbol{\Lambda}^R$:

  i) It is easy to show that $\|C\mathbf{u}^{k\boldsymbol{\lambda}}\|_{\mathcal{H}} = |k|\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}}$, $\forall k \in \mathbb{R}$, $\forall \boldsymbol{\lambda} \in \boldsymbol{\Lambda}^R$.
  Thanks to the linearity of the problem (4.15) and thanks to homogeneous conditions on the external boundary, we know that: if u_i is a solution of the problem with $\partial_{n_L} u_i + \beta u_i = \lambda_i$ on Γ_i, then $v_i = ku_i$ is the solution of the problem with $\partial_{n_L} v_i + \beta v_i = k\lambda_i$ on Γ_i. Moreover, the gradient, the trace and the integral are linear operators. It follows that (for example in case 2)

$$\begin{aligned}\|\!|k\boldsymbol{\lambda}|\!\|_1 &= \left(\int_{\Omega_{12}} \left(u_1^{k\lambda_1} - u_2^{k\lambda_2}\right)^2 d\Omega + \int_{\Omega_{12}} \left(\nabla u_1^{k\lambda_1} - \nabla u_2^{k\lambda_2}\right)^2 d\Omega\right)^{1/2} \\ &= \left(\int_{\Omega_{12}} k^2 \left(u_1^{\lambda_1} - u_2^{\lambda_2}\right)^2 d\Omega + \int_{\Omega_{12}} k^2 \left(\nabla u_1^{\lambda_1} - \nabla u_2^{\lambda_2}\right)^2 d\Omega\right)^{1/2} = |k|\,\|\!|\boldsymbol{\lambda}|\!\|_1 \ .\end{aligned}$$

  In the other cases the proof is similar.

  ii) It can also be shown that the triangular inequality $\|C\mathbf{u}^{\boldsymbol{\lambda}+\eta}\|_{\mathcal{H}} \leqslant \|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}} + \|C\mathbf{u}^{\eta}\|_{\mathcal{H}}$, $\forall \boldsymbol{\lambda}, \eta \in \Lambda^R$ holds. To prove it, it is sufficient to remember the *Cauchy-Schwarz inequality*[3] and to use the linearity of the problem, and of the operators, as done before. We want to underline that the *Cauchy-Schwarz inequality* is to be used on the space $\mathcal{H}$, that is known to be a normed space with the norm $\|\cdot\|_{\mathcal{H}}$.

- $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}}$ is a norm on Λ^R:
  It is sufficient to prove that if $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{\mathcal{H}} = 0$, then $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = \mathbf{0}$. Let us split the analysis, and analyze each case separately.

  <u>*Case 1*</u>:
  $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{L^2(\Omega_{12})} = 0$ implies that $u_1^{\lambda_1} = u_2^{\lambda_2}$ a.e. in Ω_{12}. Since $u_i^{\lambda_i} \in H^1(\Omega_i)$, $i = 1, 2$, it follows that $u_1^{\lambda_1} - u_2^{\lambda_2} \in H^1(\Omega_{12})$. Therefore, we can take the trace of $u_i^{\lambda_i}$ on $\partial\Omega_{12}$, $i = 1, 2$, and it results ${u_1^{\lambda_1}}_{|\Gamma_i} = {u_2^{\lambda_2}}_{|\Gamma_i}$, $i = 1, 2$. Let us define the function

$$w = \begin{cases} u_1^{\lambda_1} & in\ \overline{\Omega_1} \setminus \overline{\Omega_{12}} \\ u_1^{\lambda_1} = u_2^{\lambda_2} & in\ \overline{\Omega_{12}} \\ u_2^{\lambda_2} & in\ \overline{\Omega_2} \setminus \overline{\Omega_{12}} \ . \end{cases}$$

  This function satisfies the homogeneous problem

$$\begin{cases} Lw = 0 & in\ \Omega \\ w = 0 & on\ \Gamma_D \\ \partial_{n_L} w = 0 & on\ \Gamma_N \ , \end{cases}$$

  whose solution is $w = 0$ in Ω. It follows that both the trace and the conormal derivative of w on Γ_i, $i = 1, 2$, are equal to zero, and so that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = \mathbf{0}$.

  <u>*Case 2*</u>:
  In this case the proof is similar to the previous one. Actually $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{H^1(\Omega_{12})} = 0$

[3] *Cauchy-Schwarz inequality*: let $(\cdot,\cdot)$ be a scalar product on a normed space V, and let $\|\cdot\|$ be the induced norm on the space V. Then $|(u, v)| \leqslant \|u\|\|v\|$, $\forall u, v \in V$

implies that $\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{L^2(\Omega_{12})} = 0$. Then, we can proceed as in *Case 1* and the thesis is proved.

Case 3:
$\|C\mathbf{u}^{\boldsymbol{\lambda}}\|_{L^2(\Gamma_1\cup\Gamma_2)} = 0$ implies that $u_1^{\lambda_1} = u_2^{\lambda_2}$ a.e. on $\Gamma_1 \cup \Gamma_2$. Let us define $w = u_1^{\lambda_1} - u_2^{\lambda_2}$ on Ω_{12}. w satisfies the following homogeneous problem:

$$\begin{cases} Lw = 0 & in\ \Omega_{12} \\ w = 0 & on\ \partial\Omega_{12} \cap \Gamma_D \\ \partial_{n_L} w = 0 & on\ \partial\Omega_{12} \cap \Gamma_N \\ w = 0 & on\ \Gamma_1 \cup \Gamma_2 \end{cases},$$

whose solution is $w = 0$ in (Ω_{12}). In particular it implies that $\|u_1^{\lambda_1} - u_2^{\lambda_2}\|_{L(\Omega_{12})} = 0$. The thesis follows from *Case 1.*

□

As in the case of Dirichlet interface controls, it is not ensured that $\boldsymbol{\Lambda}^R$ is a complete space with respect to the norms $|||\cdot|||_*$. Also in this case, it can be considered the completion of this space, $\widehat{\boldsymbol{\Lambda}}^R$, with respect to these norms, and a solution of the problem can be found in this space.
We are now ready to prove the following

Theorem 12. *The minimization problem*

$$\inf_{\boldsymbol{\lambda}\in\widehat{\boldsymbol{\Lambda}}^R} J_{C,\mathcal{H}}(\boldsymbol{\lambda}) \tag{4.23}$$

has a unique solution $\boldsymbol{\lambda} \in \widehat{\boldsymbol{\Lambda}}^R$. *Moreover, the solution satisfies*

$${}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle J'_{C,\mathcal{H}}(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^R} = (C\mathbf{u}^{\boldsymbol{\lambda},f}, C\mathbf{u}^{\boldsymbol{\mu}})_{\mathcal{H}} = 0 \quad \forall \mu \in \widehat{\boldsymbol{\Lambda}}^R\ . \tag{4.24}$$

Proof. Let us firstly suppose that $\boldsymbol{\Lambda}^R$ is complete with respect to the norm $|||\cdot|||_*$. For any $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}^R$, we set

$$\pi(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{1}{2}(C\mathbf{u}^{\boldsymbol{\lambda}}, C\mathbf{u}^{\boldsymbol{\mu}})_{\mathcal{H}} \qquad L(\boldsymbol{\mu}) = -\frac{1}{2}(C\mathbf{u}^{f}, C\mathbf{u}^{\boldsymbol{\mu}})_{\mathcal{H}}\ , \tag{4.25}$$

so that $J_{C,\mathcal{H}}(\boldsymbol{\lambda}) = \pi(\boldsymbol{\lambda}, \boldsymbol{\lambda}) - 2L(\boldsymbol{\lambda}) + \frac{1}{2}\|C\mathbf{u}^f\|^2_{\mathcal{H}}$, according to (4.20).
$\pi(\cdot,\cdot)$ is a symmetric and bilinear form on $\boldsymbol{\Lambda}^R$ and, thanks to Lemma 3, it is also continuous and coercive on $\boldsymbol{\Lambda}^R$ with respect to the norm $|||\cdot|||_*$. Moreover, L is a linear and continuous fuctional on $\boldsymbol{\Lambda}^R$. Finally, $(\boldsymbol{\Lambda}^R, |||\cdot|||_*)$ is a Hilbert space.
The existence and uniqueness of the solution are direct consequences of Theorem 1, while the Euler equation (4.24) follows from Corollary 1 and by observing that

$${}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle J'_{C,\mathcal{H}}(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^R} = 2\pi(\boldsymbol{\lambda}, \boldsymbol{\mu}) - 2L(\boldsymbol{\mu}) \quad \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \boldsymbol{\Lambda}^R\ .$$

On the other hand, let us consider the case when $\boldsymbol{\Lambda}^R$ is not complete with respect to the norm $|||\cdot|||_*$. The bilinear form $\pi(\cdot,\cdot)$ and the functional $L(\cdot)$ are continuous on $\boldsymbol{\Lambda}^R$ and they can be uniquely extended to a continuous form and functional on $\widehat{\boldsymbol{\Lambda}}^R$ thanks to *Hahn-Banach Theorem*[4]. The thesis follows in the same way as before. □

[4]It can be found in [3]

4.4 The Optimality System

Every of the minimization problem (4.12)-(4.14) with constraints (4.11) is equivalent to an optimality system, as shown in Chapter 1. In this Section the derivation of these systems is presented.
We use the classical theory of optimal control and in particular Theorem 4. We underline that if $U_{ad} = U$, as in these cases, the last inequality of the optimality system (1.20) is an equality.
As made in the previous Section, we will analyze each functional separately.

4.4.1 Case 1: functional J_0

Let us firstly consider the Gâteaux derivative of the functional. We consider the split form of the functional J_0 as given in (4.20), and we use $\mathbf{\Lambda}^*$ to refer to either $\mathbf{\Lambda}^D$ and $\mathbf{\Lambda}^R$. It is easy to show that, $\forall \boldsymbol{\mu} = (\mu_1, \mu_2) \in \widehat{\mathbf{\Lambda}}^*$, it results:

$$ {}_{(\widehat{\mathbf{\Lambda}}^*)'}\langle J_0'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\mathbf{\Lambda}}^*} = \int_{\Omega_{12}} (u_1^{\lambda_1} - u_2^{\lambda_2})(u_1^{\mu_1} - u_2^{\mu_2}) d\Omega + \int_{\Omega_{12}} (u_1^f - u_2^f)(u_1^{\mu_1} - u_2^{\mu_2}) d\Omega \ . \quad (4.26) $$

Let us define

$$ \chi_{12}(\mathbf{x}) = \begin{cases} 1 & if\ \mathbf{x} \in \Omega_{12} \\ 0 & otherwise \end{cases} $$

We can introduce the following *dual* or *adjoint problems*, for i=1,2:

$$ \begin{cases} L^* p_i = (-1)^{i+1}(u_1^{\lambda_1} - u_2^{\lambda_2})\chi_{12} & a.e.\ in\ \Omega_i \\ p_i = 0 & on\ \Gamma_D^i \\ \partial_{n_{L^*}} p_i = 0 & on\ \Gamma_N^i \\ \Psi^*(p_i) = 0 & on\ \Gamma_i \end{cases} \quad (4.27) $$

and

$$ \begin{cases} L^* \widehat{p}_i = (-1)^{i+1}(u_1^f - u_2^f)\chi_{12} & a.e.\ in\ \Omega_i \\ \widehat{p}_i = 0 & on\ \Gamma_D^i \\ \partial_{n_{L^*}} \widehat{p}_i = 0 & on\ \Gamma_N^i \\ \Psi^*(\widehat{p}_i) = 0 & on\ \Gamma_i \end{cases} \quad (4.28) $$

where $\Psi^* = \Psi$ in the case of Dirichlet interface controls, and $\Psi^*(p_i) = \partial_{n_{L^*}} p_i + \beta p_i$ in the case of Robin interface controls.
The weak formulations corresponding to these problems are, for the various cases:

- Dirichlet interface controls
  Find p_i in V_i^D such that

$$ a_i(v_i, p_i) = (-1)^{i+1} \int_{\Omega_i} \chi_{12}(u_1^{\lambda_1} - u_2^{\lambda_2}) v_i d\Omega = (-1)^{i+1} \int_{\Omega_{12}} (u_1^{\lambda_1} - u_2^{\lambda_2}) v_i d\Omega \quad \forall v_i \in V_i^D \ , \quad (4.29a) $$

  and find $\widehat{p}_i$ in V_i^D such that

$$ a_i(v_i, \widehat{p}_i) = (-1)^{i+1} \int_{\Omega_i} \chi_{12}(u_1^f - u_2^f) v_i d\Omega = (-1)^{i+1} \int_{\Omega_{12}} (u_1^f - u_2^f) v_i d\Omega \quad \forall v_i \in V_i^D \ ; \quad (4.29b) $$

- Robin interface controls
  Find p_i in V_i such that

$$\begin{aligned} &a_i(v_i,p_i)+\int_{\Gamma_i}\beta p_i v_i d\Gamma = \\ &=(-1)^{i+1}\int_{\Omega_i}\chi_{12}(u_1^{\lambda_1}-u_2^{\lambda_2})v_i d\Omega=(-1)^{i+1}\int_{\Omega_{12}}(u_1^{\lambda_1}-u_2^{\lambda_2})v_i d\Omega \quad \forall v_i\in V_i\ , \end{aligned} \tag{4.30a}$$

  and find $\widehat{p}_i$ in V_i such that

$$\begin{aligned} &a_i(v_i,\widehat{p}_i)+\int_{\Gamma_i}\beta \widehat{p}_i v_i d\Gamma = \\ &=(-1)^{i+1}\int_{\Omega_i}\chi_{12}(u_1^{f}-u_2^{f})v_i d\Omega=(-1)^{i+1}\int_{\Omega_{12}}(u_1^{f}-u_2^{f})v_i d\Omega \quad \forall v_i\in V_i\ . \end{aligned} \tag{4.30b}$$

In view of (4.27) and (4.28), the Gâteaux derivative of J_0 becomes:

$${}_{(\widehat{\boldsymbol{\Lambda}}^*)'}\langle J_0'(\boldsymbol{\lambda}),\boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^*}=\sum_{i=1}^{2}\left(\int_{\Omega_i}L^*p_iu_i^{\mu_i}d\Omega+\int_{\Omega_i}L^*\widehat{p}_iu_i^{\mu_i}d\Omega\right)\ . \tag{4.31}$$

Let us consider the terms $\int_{\Omega_i}L^*p_iu_i^{\mu_i}d\Omega$. The other one is analogous. It results:

$$\begin{aligned} &\int_{\Omega_i}L^*p_iu_i^{\mu_i}d\Omega= \\ &=\int_{\Omega_i}Lu_i^{\mu_i}p_id\Omega-\int_{\partial\Omega_i}\partial_{n_{L^*}}p_iu_i^{\mu_i}d\Gamma+\int_{\partial\Omega_i}\partial_{n_L}u_i^{\mu_i}p_id\Gamma= \\ &=-\int_{\Gamma_i}\partial_{n_{L^*}}p_iu_i^{\mu_i}d\Gamma+\int_{\Gamma_i}\partial_{n_L}u_i^{\mu_i}p_id\Gamma\ , \end{aligned}$$

thanks to the fact that $u_i^{\mu_i}$ is the weak solution of the problem (4.15) and that p_i is the solution of (4.27). Therefore, it results $Lu_i^{\mu_i}=0$ in Ω_i, $\partial_{n_L}u_i^{\mu_i}=0$ on Γ_N^i, $u_i^{\mu_i}=0$ on Γ_D^i, $p_i=0$ on Γ_D^i and $\partial_{n_L^*}p_i=0$ on Γ_N^i.
Moreover, in the case of Dirichlet interface controls, we have $u_i^{\mu_i}=\mu_i$ and $p_i=0$ on Γ_i. In this case we have

$$\int_{\Omega_i}L^*p_iu_i^{\mu_i}d\Omega=-\int_{\Gamma_i}\partial_{n_{L^*}}p_i\mu_id\Gamma\ .$$

On the other hand, in the case of Robin interface controls, we have $\partial_{n_L}u_i^{\mu_i}+\beta u_i^{\mu_i}=\mu_i$ and that $\partial_{n_L^*}p_i+\beta p_i=0$ on Γ_i. Thus, in this second case, it results:

$$\int_{\Omega_i}L^*p_iu_i^{\mu_i}d\Omega=\int_{\Gamma_i}p_i\mu_id\Gamma\ .$$

It follows that the Frechét derivatives of J_0 in the two cases are:

- Dirichlet interface controls

$${}_{(\widehat{\boldsymbol{\Lambda}}^D)'}\langle J_0'(\boldsymbol{\lambda}),\boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^D}=-\sum_{i=1}^{2}\left(\int_{\Gamma_i}\partial_{n_{L^*}}p_i\mu_id\Gamma+\int_{\Gamma_i}\partial_{n_{L^*}}\widehat{p}_i\mu_id\Gamma\right)\ ; \tag{4.32}$$

- Robin interface controls

$$ {}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle J_0'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^R} = \sum_{i=1}^{2} \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) . \tag{4.33} $$

We are now ready to formulate the following

Theorem 13. *The minimization problem* (4.12) *with constraints* (4.1)*, is equivalent to the optimality system*

$$ \begin{cases} (4.17) \\ (4.29) \\ {}_{(\widehat{\boldsymbol{\Lambda}}^D)'}\langle J_0'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^D} = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases} , \tag{4.34} $$

namely

$$ \begin{cases} solve \begin{cases} a_i(u_i^{\lambda_i}, v_i) = 0,\ u_i^{\lambda_i} = \lambda_i \text{ on } \Gamma_i & \forall v_i \in V_i^D \\ a_i(u_i^f, v_i) = F_i(v_i),\ u_i^f = \phi_D^i \text{ on } \Gamma_D^i & \forall v_i \in V_i^D \end{cases} \\ solve \begin{cases} a_i(v_i, p_i) = (-1)^{i+1} \int_{\Omega_i} \chi_{12}(u_1^{\lambda_1} - u_2^{\lambda_2}) v_i d\Omega & \forall v_i \in V_i^D \\ a_i(v_i, \widehat{p}_i) = (-1)^{i+1} \int_{\Omega_i} \chi_{12}(u_1^f - u_2^f) v_i d\Omega & \forall v_i \in V_i^D \end{cases} \\ compute\ -\sum_{i=1}^{2} \left(\int_{\Gamma_i} \partial_{n_{L^*}} p_i \mu_i d\Gamma + \int_{\Gamma_i} \partial_{n_{L^*}} \widehat{p}_i \mu_i d\Gamma \right) = 0 \qquad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases} , \tag{4.35} $$

in the case of Dirichlet interface controls, and it is equivalent to the optimality system

$$ \begin{cases} (4.18) \\ (4.30) \\ {}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle J_0'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^R} = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R \end{cases} , \tag{4.36} $$

namely

$$ \begin{cases} solve \begin{cases} a_i(u_i^{\lambda_i}, v_i) + \int_{\Gamma_i} \beta u_i^{\lambda_i} v_i d\Gamma = \int_{\Gamma_i} \lambda_i v_i d\Gamma & \forall v_i \in V_i \\ a_i(u_i^f, v_i) + \int_{\Gamma_i} \beta u_i^f v_i d\Gamma = F_i(v_i),\ u_i^f = \phi_D^i \text{ on } \Gamma_D^i & \forall v_i \in V_i \end{cases} \\ solve \begin{cases} a_i(v_i, p_i) + \int_{\Gamma_i} \beta p_i v_i d\Gamma = (-1)^{i+1} \int_{\Omega_i} \chi_{12}(u_1^{\lambda_1} - u_2^{\lambda_2}) v_i d\Omega & \forall v_i \in V_i \\ a_i(v_i, \widehat{p}_i) + \int_{\Gamma_i} \beta \widehat{p}_i v_i d\Gamma = (-1)^{i+1} \int_{\Omega_i} \chi_{12}(u_1^f - u_2^f) v_i d\Omega & \forall v_i \in V_i \end{cases} \\ compute\ \sum_{i=1}^{2} \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) = 0 \qquad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases} , \tag{4.37} $$

in the case of Robin interface controls.

We will refer to equations (4.29) and (4.30) as *adjoint* or *dual problems*.

4.4.2 Case 2: functional J_1

In this case the derivation of the optimality system is analogous to the previous case. The notation used in case 1, are used also in this Section.

The Gâteaux derivative of the functional in this case assumes the form

$$
\begin{aligned}
&{}_{(\widehat{\boldsymbol{\Lambda}}^*)'}\langle J_1'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^*} = \\
&\int_{\Omega_{12}} (u_1^{\lambda_1} - u_2^{\lambda_2})(u_1^{\mu_1} - u_2^{\mu_2}) d\Omega + \int_{\Omega_{12}} (\nabla u_1^{\lambda_1} - \nabla u_2^{\lambda_2}) \cdot (\nabla u_1^{\mu_1} - \nabla u_2^{\mu_2}) d\Omega + \\
&+ \int_{\Omega_{12}} (u_1^{f} - u_2^{f})(u_1^{\mu_1} - u_2^{\mu_2}) d\Omega + \int_{\Omega_{12}} (\nabla u_1^{f} - \nabla u_2^{f}) \cdot (\nabla u_1^{\mu_1} - \nabla u_2^{\mu_2}) d\Omega \, .
\end{aligned}
\tag{4.38}
$$

Let us introduce the following functionals:

$$
\begin{aligned}
G_i(v_i) &= (-1)^{i+1} \left(\int_{\Omega_{12}} (u_1^{\lambda_1} - u_2^{\lambda_2}) v_i d\Omega + \int_{\Omega_{12}} (\nabla u_1^{\lambda_1} - \nabla u_2^{\lambda_2}) \cdot \nabla v_i d\Omega \right) = \\
&= (-1)^{i+1} \left(\int_{\Omega_i} \chi_{12} (u_1^{\lambda_1} - u_2^{\lambda_2}) v_i d\Omega + \int_{\Omega_i} \chi_{12} (\nabla u_1^{\lambda_1} - \nabla u_2^{\lambda_2}) \cdot \nabla v_i d\Omega \right) \quad \forall v_i \in V_i \, ,
\end{aligned}
\tag{4.39}
$$

and the functional $\widehat{G}_i$ defined in an analogous way, changing in the expression for G_i the terms λ_i with f.
We can introduce the following *dual* or *adjoint problems*, for i=1,2:

- Dirichlet interface controls
  Find p_i in V_i^D such that

$$
a_i(v_i, p_i) = G_i(v_i) \quad \forall v_i \in V_i^D \, , \tag{4.40a}
$$

  and find $\widehat{p}_i$ in V_i^D such that

$$
a_i(v_i, \widehat{p}_i) = \widehat{G}_i(v_i) \quad \forall v_i \in V_i^D \, . \tag{4.40b}
$$

  Moreover the functional G_i is linear and continuous on V_i^D and so, thanks to *Riesz representation Theorem*[5], there exists a unique element $\tilde{G}_i \in V_i^D$ such that

$$
G_i(v_i) = (\tilde{G}_i, v_i)_{V_i^D} \, .
$$

  Therefore equation (4.40a) is equivalent to

$$
\begin{cases}
L^* p_i = \tilde{G}_i & in \ \Omega_i \\
p_i = 0 & on \ \Gamma_D^i \\
\partial_{n_{L^*}} p_i = 0 & on \ \Gamma_N^i \\
\partial_{n_{L^*}} p_i + \beta p_i = 0 & on \ \Gamma_i
\end{cases} \, .
\tag{4.41}
$$

  An analogous problem can be written for $\widehat{p}_i$.

- Robin interface controls
  Find p_i in V_i such that

$$
a_i(v_i, p_i) + \int_{\Gamma_i} \beta p_i v_i d\Gamma = G_i(v_i) \quad \forall v_i \in V_i \, , \tag{4.42a}
$$

  and find $\widehat{p}_i$ in V_i such that

$$
a_i(v_i, \widehat{p}_i) + \int_{\Gamma_i} \beta \widehat{p}_i v_i d\Gamma = \widehat{G}_i(v_i) \quad \forall v_i \in V_i \, . \tag{4.42b}
$$

[5]It can be found, for example, in [16]

Thanks to the functional introduced above, the Gâteaux derivative of J_1 becomes:

$$ {}_{(\widehat{\mathbf{\Lambda}}^*)'}\langle J_1'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\mathbf{\Lambda}}^*} = \sum_{i=1}^{2} \left(G_i(u_i^{\mu_i}) + \widehat{G}_i(u_i^{\mu_i}) \right) . \tag{4.43} $$

Let us focus on the first terms, the analysis of the other one is analogous.

- Dirichlet interface controls
  In this first case we can use the problem (4.41), obtaining:

$$ G_i(u_i^{\mu_i}) = {}_{(V_i^D)'}\langle \tilde{G}_i, u_i^{\mu_i}\rangle_{V_i^D} = \int_{\Omega_i} L^* p_i u_i^{\mu_i} d\Omega . \tag{4.44} $$

  Proceeding in the same way as done above for the functional J_0, we obtain that the Gâteaux derivative of the functional J_1, in the case of Dirichlet interface controls, is:

$$ {}_{(\widehat{\mathbf{\Lambda}}^D)'}\langle J_1'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\mathbf{\Lambda}}^D} = -\sum_{i=1}^{2} \left(\int_{\Gamma_i} \partial_{n_{L^*}} p_i \mu_i d\Gamma + \int_{\Gamma_i} \partial_{n_{L^*}} \widehat{p}_i \mu_i d\Gamma \right) . \tag{4.45} $$

- Robin interface controls
  In this second case, if we use the weak formulation of the dual problem (4.42), we obtain:

$$ \begin{aligned} G_i(u_i^{\mu_i}) = a_i(v_i, p_i) + \int_{\Gamma_i} \beta p_i v_i d\Gamma = \\ \int_{\Omega_i} L u_i^{\mu_i} p_i + \int_{\partial\Omega_i} \partial_{n_L} u_i^{\mu_i} p_i + \int_{\Gamma_i} \beta p_i u_i^{\mu_i} d\Gamma . \end{aligned} \tag{4.46} $$

  We recall that $Lu_i^{\mu_i} = 0$ on Ω_i, $\partial_{n_L} u_i^{\mu_i} = 0$ on Γ_N^i, $u_i^{\mu_i} = 0$ on Γ_D^i and that $\partial_{n_L} u_i^{\mu_i} + \beta u_i^{\mu_i} = \mu_i$ on Γ_i. Therefore, we obtain that the Gâteaux derivative of J_1, in the case of Robin interface controls, is:

$$ {}_{(\widehat{\mathbf{\Lambda}}^R)'}\langle J_1'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\mathbf{\Lambda}}^R} = \sum_{i=1}^{2} \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) . \tag{4.47} $$

We are now ready to formulate the following

Theorem 14. *The minimization problem* (4.13) *with constraints* (4.1)*, is equivalent to the optimality system*

$$ \begin{cases} (4.17) \\ (4.40) \\ {}_{(\widehat{\mathbf{\Lambda}}^D)'}\langle J_1'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\mathbf{\Lambda}}^D} = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\mathbf{\Lambda}}^D \end{cases} , \tag{4.48} $$

namely

$$ \begin{cases} \textit{solve} \begin{cases} a_i(u_i^{\lambda_i}, v_i) = 0,\ u_i^{\lambda_i} = \lambda_i \textit{ on } \Gamma_i & \forall v_i \in V_i^D \\ a_i(u_i^f, v_i) = F_i(v_i),\ u_i^f = \phi_D^i \textit{ on } \Gamma_D^i & \forall v_i \in V_i^D \end{cases} \\ \textit{solve} \begin{cases} a_i(v_i, p_i) = G_i(v_i) \quad \forall v_i \in V_i^D \\ a_i(v_i, \widehat{p}_i) = \widehat{G}_i(v_i) \quad \forall v_i \in V_i^D \end{cases} \\ \textit{compute } -\sum_{i=1}^{2} \left(\int_{\Gamma_i} \partial_{n_{L^*}} p_i \mu_i d\Gamma + \int_{\Gamma_i} \partial_{n_{L^*}} \widehat{p}_i \mu_i d\Gamma \right) = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\mathbf{\Lambda}}^D \end{cases} , \tag{4.49} $$

in the case of Dirichlet interface controls, and it is equivalent to the optimality system

$$\begin{cases} (4.18) \\ (4.42) \\ {}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle J_1'(\boldsymbol{\lambda}), \boldsymbol{\mu}\rangle_{\widehat{\boldsymbol{\Lambda}}^R} = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R \end{cases}, \tag{4.50}$$

namely

$$\begin{cases} solve \begin{cases} a_i(u_i^{\lambda_i}, v_i) + \int_{\Gamma_i} \beta u_i^{\lambda_i} v_i d\Gamma = \int_{\Gamma_i} \lambda_i v_i d\Gamma & \forall v_i \in V_i \\ a_i(u_i^f, v_i) + \int_{\Gamma_i} \beta u_i^f v_i d\Gamma = F_i(v_i),\ u_i^f = \phi_D^i \ on\ \Gamma_D^i & \forall v_i \in V_i \end{cases} \\ solve \begin{cases} a_i(v_i, p_i) + \int_{\Gamma_i} \beta p_i v_i d\Gamma = G_i(v_i) \quad \forall v_i \in V_i \\ a_i(v_i, \widehat{p}_i) + \int_{\Gamma_i} \beta \widehat{p}_i v_i d\Gamma = \widehat{G}_i(v_i) \quad \forall v_i \in V_i \end{cases} \\ compute\ \sum_{i=1}^2 \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) = 0 \qquad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases}, \tag{4.51}$$

in the case of Robin interface controls.

4.4.3 Case 3: functional $J_{0,\Gamma}$

Also in this third case it is possible to show the equivalence between the minimization problem (4.1), (4.14) and an optimality system. Let us consider the cases of Dirichlet and of Robin interface controls separately.

- Dirichlet interface controls
  We report the result proven in [8], without proving it.
  Let us introduce the following problems, for i=1,2:

$$\begin{cases} Lp_i = 0 & in\ \Omega_i \\ p_i = 0 & on\ \Gamma_D^i \\ \partial_{n_L} p_i = 0 & on\ \Gamma_N^i \\ p_i = u_1^{\lambda_1} - u_2^{\lambda_2} & on\ \Gamma_i \end{cases} \tag{4.52}$$

  and

$$\begin{cases} L\widehat{p}_i = 0 & in\ \Omega_i \\ \widehat{p}_i = 0 & on\ \Gamma_D^i \\ \partial_{n_L} \widehat{p}_i = 0 & on\ \Gamma_N^i \\ \widehat{p}_i = u_1^f - u_2^f & on\ \Gamma_i \end{cases} \tag{4.53}$$

  The weak formulations of these problems read:
  find p_i in V_i, $p_i = u_1^{\lambda_1} - u_2^{\lambda_2}$ on Γ_i such that

$$a_i(p_i, v_i) = 0 \quad \forall v_i \in V_i^D\ , \tag{4.54a}$$

  and find $\widehat{p}_i$ in V_i, $\widehat{p}_i = u_1^f - u_2^f$ on Γ_i such that

$$a_i(\widehat{p}_i, v_i) = 0 \quad \forall v_i \in V_i^D\ . \tag{4.54b}$$

  The following Theorem holds:

Theorem 15. *In the case of Dirichlet interface controls, the minimization problem* (4.1), (4.14) *is equivalent to the optimality system*

$$\begin{cases} (4.17) \\ (4.54) \\ \sum_{i=1}^{2} \left(\int_{\Gamma_i} (p_1 + p_2)\mu_i d\Gamma + \int_{\Gamma_i} (\widehat{p}_1 + \widehat{p}_2)\mu_i d\Gamma \right) = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases}, \tag{4.55}$$

namely

$$\begin{cases} solve \begin{cases} a_i(u_i^{\lambda_i}, v_i) = 0,\ u_i^{\lambda_i} = \lambda_i \ on\ \Gamma_i & \forall v_i \in V_i^D \\ a_i(u_i^f, v_i) = F_i(v_i),\ u_i^f = \phi_D^i \ on\ \Gamma_D^i & \forall v_i \in V_i^D \end{cases} \\ solve \begin{cases} a_i(p_i, v_i) = 0,\ p_i = u_1^{\lambda_1} - u_2^{\lambda_2} \ on\ \Gamma_i & \forall v_i \in V_i^D \\ a_i(\widehat{p}_i, v_i) = 0,\ p_i = u_1^f - u_2^f \ on\ \Gamma_i & \forall v_i \in V_i^D \end{cases} \\ compute\ -\sum_{i=1}^{2} \left(\int_{\Gamma_i} \partial_{n_{L^*}} p_i \mu_i d\Gamma + \int_{\Gamma_i} \partial_{n_{L^*}} \widehat{p}_i \mu_i d\Gamma \right) = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases} . \tag{4.56}$$

Let us refer to (4.54) and as *dual* or *adjoint* problems, even if we have to solve primal problems in this case, in order to unify the terminology.

- Robin interface controls
  In this second case it is possible to show that the minimization problem (4.1), (4.14) is equivalent to two different optimality systems. We noted a difference between them in the numerical resolution. For this reason we decided to present both of them.
  Firstly, let us define, for $i = 1, 2$,

$$p_i = u_1^{\lambda_1} - u_2^{\lambda_2} \quad and \quad \widehat{p}_i = u_1^f - u_2^f \quad on\ \Gamma_i\ . \tag{4.57}$$

We need to use the following Lemma:

Lemma 4. *Let L be a linear and continuous operator. Then, L is an injective operator if and only if $ker(L) = \{0\}$. Where $ker(L)$ is the kernel of the operator considered.*

The following Theorem holds[6].

Theorem 16. *In the case of Robin interface controls, the minimization problem* (4.1), (4.14) *is equivalent to the optimality system*

$$\begin{cases} (4.18) \\ (4.57) \\ \sum_{i=1}^{2} \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R\ , \end{cases} \tag{4.58}$$

[6]This result is proved in [9]. We prove this result here for completeness

namely

$$\begin{cases} \textit{solve} \begin{cases} a_i(u_i^{\lambda_i}, v_i) + \int_{\Gamma_i} \beta u_i^{\lambda_i} v_i d\Gamma = \int_{\Gamma_i} \lambda_i v_i d\Gamma & \forall v_i \in V_i \\ a_i(u_i^f, v_i) + \int_{\Gamma_i} \beta u_i^f v_i d\Gamma = F_i(v_i),\ u_i^f = \phi_D^i \text{ on } \Gamma_D^i & \forall v_i \in V_i \end{cases} \\ \textit{compute} \begin{cases} p_i = u_1^{\lambda_1} - u_2^{\lambda_2} & \text{on } \Gamma_i \\ \widehat{p}_i = u_1^f - u_2^f & \text{on } \Gamma_i \end{cases} \\ \textit{compute } \sum_{i=1}^2 \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) = 0 & \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases} , \tag{4.59}$$

Proof. Let us split the proof into two steps. In the first one we prove that if $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}^R$ is the solution of (4.1), (4.14), then it satisfies also (4.58). In the second step, we show that (4.58) has a unique solution.

i) If $\boldsymbol{\lambda}$ is the solution of (4.14), it follows that $u_1^{\lambda_1} = u_2^{\lambda_2}$ and that $u_1^f = u_2^f$ on $\Gamma_1 \cup \Gamma_2$ and therefore that also the optimality system (4.58) is satisfied.

ii) It remains to prove that the optimality system (4.58) admits a unique solution. Let us consider, for sake of simplicity, the case when $f = 0$, $\phi_D = 0$ and $\phi_N = 0$. In the other case the proof is analogous. Let us define the linear operator $\chi : \widehat{\boldsymbol{\Lambda}}^R \to (\widehat{\boldsymbol{\Lambda}}^R)'$ such that

$${}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle \chi(\boldsymbol{\lambda}), \boldsymbol{\mu} \rangle_{\widehat{\boldsymbol{\Lambda}}^R} = \sum_{i=1}^2 \int_{\Gamma_i} (u_1^{\lambda_1} - u_2^{\lambda_2}) \mu_i d\Gamma = \sum_{i=1}^2 \int_{\Gamma_i} p_i \mu_i d\Gamma \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R$$

We want to apply Lemma 4, in order to prove that the operator χ is injective. As consequence, thanks to the fact that

$${}_{(\widehat{\boldsymbol{\Lambda}}^R)'}\langle \chi(\boldsymbol{\lambda}), \boldsymbol{\mu} \rangle_{\widehat{\boldsymbol{\Lambda}}^R} = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R$$

has a solution, it follows that the solution is unique.
Let us show that the operator χ is injective. χ is a linear operator thanks to the linearity of the trace operator. Moreover $\chi(\boldsymbol{\lambda})$ is well defined, it is linear thanks to the linearity of the integral and it is continuous thanks to the definition of duality. It remains to prove that if $\boldsymbol{\lambda} \in ker(\chi)$, then $\boldsymbol{\lambda} = \mathbf{0}$. We have that $\boldsymbol{\lambda} \in ker(\chi)$ if and only if

$$\sum_{i=1}^2 \int_{\Gamma_i} (u_1^{\lambda_1} - u_2^{\lambda_2}) \mu_i d\Gamma = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R \, .$$

Therefore $u_1^{\lambda_1} = u_2^{\lambda_2}$ a.e. on $\Gamma_1 \cup \Gamma_2$. Let us now define $w = u_1^{\lambda_1} - u_2^{\lambda_2}$ in Ω_{12}. It satisfies the following homogeneous problem:

$$\begin{cases} Lw = 0 & in\ \Omega_{12} \\ w = 0 & on\ \Gamma_D \cap \partial\Omega_{12} \\ \partial_{n_L} w = 0 & on\ \Gamma_N \cap \partial\Omega_{12} \\ w = 0 & on\ \Gamma_1 \cup \Gamma_2 \end{cases} .$$

The unique solution of this problem is $w = 0$. In particular, it follows that $u_1^{\lambda_1} = u_2^{\lambda_2}$ in Ω_{12}. Setting now

$$w = \begin{cases} u_1^{\lambda_1} & in\ \overline{\Omega}_1 \setminus \overline{\Omega}_{12} \\ u_1^{\lambda_1} = u_2^{\lambda_2} & in \overline{\Omega}_{12} \\ u_1^{\lambda_2} & in\ \overline{\Omega}_2 \setminus \overline{\Omega}_{12} \end{cases},$$

it follows that w satisfies the homogeneous problem:

$$\begin{cases} Lw = 0 & in\ \Omega \\ w = 0 & on\ \Gamma_D \\ \partial_{n_L} w = 0 & on\ \Gamma_N \end{cases},$$

whose unique solution is $w = 0$. In particular, $u_1^{\lambda_1} = 0$ in Ω_1 and $u_2^{\lambda_2} = 0$ in Ω_{12} and therefore $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = \mathbf{0}$.

□

We will refer to this first optimality system as $J_{0,\Gamma}(1)$ method.
On the other hand, it can be proved that the minimization problem (4.1), (4.14) is analogous to another optimality system in the case of Robin interface controls, under further assumptions.
Let us define, for $i = 1, 2$,

$$p_i = u_1^{\lambda_1} - u_2^{\lambda_2} + \beta \partial_{n_L}(u_1^{\lambda_1} - u_2^{\lambda_2}) \quad on\ \Gamma_i \tag{4.60a}$$

and

$$\widehat{p}_i = u_1^f - u_2^f + \beta \partial_{n_L}(u_1^f - u_2^f) \quad on\ \Gamma_i\ . \tag{4.60b}$$

Then, the following Theorem holds[7].

Theorem 17. *Let us assume that all the previous hypotheses are still valid. Moreover, we assume that*

i) Ω is a bounded subset of $\mathbb{R}^2$ and of $\mathcal{C}^2$ class

ii) the coefficients K_{ij} and b_i, for $i, j = 1, ..., n$ are Lipschitz functions in Ω

iii) $f \in L^2(\Omega)$, $\phi_D \in H^{3/2}(\Gamma_D)$, $\phi_N \in H^{1/2}(\Gamma_N)$ and $\lambda_i \in H^{1/2}(\Gamma_i)$ for $i = 1, 2$.

In the case of Robin interface controls, the minimization problem (4.1), (4.14) *is equivalent to the optimality system*

$$\begin{cases} (4.18) \\ (4.60) \\ \sum_{i=1}^{2} \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) = 0 \quad \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^R\ , \end{cases} \tag{4.61}$$

[7] This result is proved in [9]

namely

$$\begin{cases} \textit{solve} \begin{cases} a_i(u_i^{\lambda_i}, v_i) + \int_{\Gamma_i} \beta u_i^{\lambda_i} v_i d\Gamma = \int_{\Gamma_i} \lambda_i v_i d\Gamma & \forall v_i \in V_i \\ a_i(u_i^f, v_i) + \int_{\Gamma_i} \beta u_i^f v_i d\Gamma = F_i(v_i),\ u_i^f = \phi_D^i \text{ on } \Gamma_D^i & \forall v_i \in V_i \end{cases} \\ \textit{compute} \begin{cases} p_i = u_1^{\lambda_1} - u_2^{\lambda_2} + \beta \partial_{n_L}(u_1^{\lambda_1} - u_2^{\lambda_2}) & \text{on } \Gamma_i \\ \widehat{p}_i = u_1^f - u_2^f + \beta \partial_{n_L}(u_1^f - u_2^f) & \text{on } \Gamma_i \end{cases} \\ \textit{compute } \sum_{i=1}^2 \left(\int_{\Gamma_i} p_i \mu_i d\Gamma + \int_{\Gamma_i} \widehat{p}_i \mu_i d\Gamma \right) = 0 & \forall \boldsymbol{\mu} \in \widehat{\boldsymbol{\Lambda}}^D \end{cases} , \tag{4.62}$$

The proof is analogous to the previous case. We want just to remark that the regularity assumptions assure that the third equation of (4.61) is well defined.
Let us refer to this second case as $J_{0,\Gamma}(2)$ method.
Finally, let us refer to (4.57) and to (4.60) as *dual* or *adjoint* problems, even if no problem has to be solved in these cases, in order to unify the terminology.

The optimality systems (4.34), (4.36), (4.48), (4.50), (4.55), (4.58) and (4.61) can be discretized according to Finite Element theory and the global system obtained in each of these cases can be reduced to a simpler system. This system, obtained with the Schur complement of the matrix of the optimality system with respect to the control variable, can be solved with an iterative method, as Bi-CGStab. Details can be found in Chapter 5.

Chapter 5

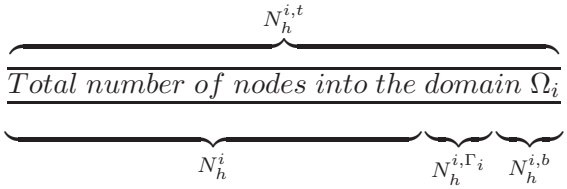
Interface Control Domain Decomposition: Finite Element Approximation of the Optimality System and Numerical Results

The aim of this Chapter is to analyze both numerical approximation and results of the ICDD methods proposed in the previous Chapter.
To reach this goal we first apply the Galerkin finite element method, described in Section 2.2 for generic elliptic equations, to numerically solve the optimality systems seen in the previous Chapter. Then we compare the numerical results obtained by applying the different ICDD methods we implemented to solve elliptic problems with both continuous and discontinuous coefficients.
We refer to [9] for the theoretical analysis of the results illustrated in this Chapter.

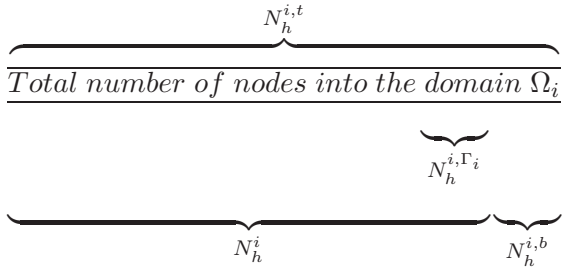
5.1 Finite Element Approximation of the Optimality System

In this Section we focus on the discretization of the optimality systems obtained in Chapter 4. We discretize the primal problems, the adjoint problems and the optimality conditions separately. In the next Section we will discuss the resolution of the algebraic optimality system obtained. Let us assume that the domain is split into two sub-domains whose number of nodes is equal to $N_h^{i,t}$, for $i = 1, 2$. Moreover, let N_h^i be the total number of nodes, except the Dirichlet ones, in the sub-domain i, for $i = 1, 2$, and let $N_h^{i,b}$ be the number of nodes on external Dirichlet boundary (numbered as last in the lists of nodes) and N_h^{i,Γ_i} be the number of nodes on the interface Γ_i, for $i = 1, 2$. In every case the nodes of the sub-domain i, for $i = 1, 2$, are listed as follows: nodes except Dirichlet ones and boundary nodes on the interface Γ_i, boundary nodes on Γ_i and finally external Dirichlet boundary nodes. In particular, in the case of Dirichlet interface control we have

the following ordering of nodes:



while in the case of Robin interface control, it results:



We are now ready to discretize the optimality system.

Primal Problems

Let us consider the weak formulation, with extensions of Dirichlet data, of primal problems (4.15) and (4.16). Moreover, let us denote with the same notations the functions from which we have removed the extensions of Dirichlet data.
Let us analyze separately the cases of Dirichlet and Robin interface controls.

Dirichlet interface controls

In the case of Dirichlet interface controls, the weak formulations of primal problems are respectively:
find $u_i^{\lambda_i} \in V_i^D$ such that

$$a_i(u_i^{\lambda_i}, v_i) = -a_i(R_{\lambda_i}, v_i) \quad \forall v_i \in V_i^D \ , \tag{5.1a}$$

where R_{λ_i} is the extension of λ_i into the domain Ω_i, and find $u_i^f \in V_i^D$ such that

$$a_i(u_i^f, v_i) = \widehat{F}_i(v_i) \quad \forall v_i \in V_i^D \quad i = 1, 2 \ , \tag{5.1b}$$

where $a_i(\cdot, \cdot)$ is the one defined in Chapter 4, and

$$\widehat{F}_i(v) = \int_{\Omega_i} f v d\Omega + \int_{\Gamma_N^i} \phi_N^i v d\Gamma - a_i(R_{\phi_D^i}, v) \ ,$$

where $R_{\phi_D^i}$ is the extension of ϕ_D^i into Ω_i.
Let us define $\boldsymbol{\lambda}_i \in \mathbb{R}^{N_h^{i,\Gamma_i}}$ as $\{\boldsymbol{\lambda}_i\}_l = \lambda_i(\mathbf{x}_{N_h^i+l})$, for $l = 1, ..., N_h^{i,\Gamma_i}$ and for $i = 1, 2$. The Galerkin Finite Element approximations of these problems are:
find $\mathbf{u}_i^{\lambda_i} \in \mathbb{R}^{N_h^i}$ such that

$$A_{Di}\mathbf{u}_i^{\lambda_i} = -B_{Di}\boldsymbol{\lambda}_i \quad i = 1, 2 \tag{5.2a}$$

and find $\mathbf{u}_i^f \in \mathbb{R}^{N_h^i}$ such that

$$A_{Di}\mathbf{u}_i^f = \mathbf{f}_{Di} \quad i = 1, 2 \,, \tag{5.2b}$$

namely, for $i = 1, 2$, we obtain the system

$$\begin{bmatrix} A_{Di} & 0 & B_{Di} \\ 0 & A_{Di} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^{\lambda_i} \\ \mathbf{u}_i^f \\ \boldsymbol{\lambda}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{Di} \end{bmatrix} \,,$$

where $\{A_{Di}\}_{lm} = a_i(\varphi_m, \varphi_l)$ for $l, m = 1, ..., N_h^i$, $\{B_{Di}\}_{lm} = a_i(\varphi_{N_h^i+m}, \varphi_l)$ for $l = 1, ..., N_h^i$ and for $m = 1, ..., N_h^{i,\Gamma_i}$, and $\{\mathbf{f}_{Di}\}_l = \widehat{F}_i(\varphi_l)$ for $l = 1, ..., N_h^i$ and for $i = 1, 2$.

Robin interface controls

In the case of Robin interface controls, the weak formulations of problems (4.15) and (4.16) are respectively:
find $u_i^{\lambda_i} \in V_i$ such that

$$a_i(u_i^{\lambda_i}, v_i) + \beta \int_{\Gamma_i} u_i^{\lambda_i} v_i d\Gamma = \int_{\Gamma_i} \lambda_i v_i d\Gamma \quad \forall v_i \in V_i \,, \tag{5.3a}$$

and find $u_i^f \in V_i$ such that

$$a_i(u_i^f, v_i) + \beta \int_{\Gamma_i} u_i^f v_i d\Gamma = \widehat{F}_i(v_i) \quad \forall v_i \in V_i \quad i = 1, 2 \,, \tag{5.3b}$$

where $a_i(\cdot, \cdot)$ and $\widehat{F}_i(\cdot)$ are defined above.
Let us define $\boldsymbol{\lambda}_i \in \mathbb{R}^{N_h^{i,\Gamma_i}}$ as $\{\boldsymbol{\lambda}_i\}_l = \int_{\Gamma_i} \lambda_i \varphi_{N_h^i - N_h^{i,\Gamma_i}+l} d\Gamma$, for $l = 1, ..., N_h^{i,\Gamma_i}$ and for $i = 1, 2$. The Galerkin Finite Element approximations of these problems are:
find $\mathbf{u}_i^{\lambda_i} \in \mathbb{R}^{N_h^i}$ such that

$$A_{Ri}\mathbf{u}_i^{\lambda_i} = -B_{Ri}\boldsymbol{\lambda}_i \quad i = 1, 2 \tag{5.4a}$$

and find $\mathbf{u}_i^f \in \mathbb{R}^{N_h^i}$ such that

$$A_{Ri}\mathbf{u}_i^f = \mathbf{f}_{Ri} \quad i = 1, 2 \,, \tag{5.4b}$$

namely, for $i = 1, 2$, we obtain the system

$$\begin{bmatrix} A_{Ri} & 0 & B_{Ri} \\ 0 & A_{Ri} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^{\lambda_i} \\ \mathbf{u}_i^f \\ \boldsymbol{\lambda}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{Ri} \end{bmatrix} \,,$$

where $\{A_{Ri}\}_{lm} = a_i(\varphi_m, \varphi_l) + \beta \int_{\Gamma_i} \varphi_{m\Gamma_i} \varphi_{l\Gamma_i} d\Gamma$ for $l, m = 1, ..., N_h^i$,
$\{B_{Ri}\}_{lm} = -\delta_{l(N_h^i - N_h^{i,\Gamma_i}+m)}$ for $l = 1, ..., N_h^i$ and for $m = 1, ..., N_h^{i,\Gamma_i}$, and $\{\mathbf{f}_{Ri}\}_l = \widehat{F}_i(\varphi_l)$ for $l = 1, ..., N_h^i$ and for $i = 1, 2$.
Let us note that the Finite element approximation of primal problems assume the same

form both in the case of Dirichlet and Robin interface controls. Actually we can write both (5.2) and (5.4) in the following way: find $\mathbf{u}_i^{\lambda_i}$ such that

$$A_i \mathbf{u}_i^{\lambda_i} = -B_i \boldsymbol{\lambda}_i \quad i = 1, 2 \tag{5.5a}$$

and find $\mathbf{u}_i^f$ such that

$$A_i \mathbf{u}_i^f = \mathbf{f}_i \quad i = 1, 2\ , \tag{5.5b}$$

namely, for $i = 1, 2$, we obtain the system

$$\begin{bmatrix} A_i & 0 & B_i \\ 0 & A_i & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^{\lambda_i} \\ \mathbf{u}_i^f \\ \boldsymbol{\lambda}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_i \end{bmatrix}\ ,$$

where A_i, B_i and $\mathbf{f}_i$ can be either A_{Di}, B_{Di} and $\mathbf{f}_{Di}$ or A_{Ri}, B_{Ri} and $\mathbf{f}_{Ri}$, respectively.

Adjoint Problems

Let us now consider the discretization of the adjoint problems of the optimality systems. As above, let us consider the weak formulation of these problems with extensions of Dirichlet data, and let us denote with the same notations the functions from whom we have removed the extensions of Dirichlet data.
In this case it can be shown that the generic forms of the discretized adjoint equations are the following: find $\mathbf{p}_i$ such that

$$\tilde{A}_i \mathbf{p}_i = G_i^1 \mathbf{u}_1^{\lambda_1} - G_i^2 \mathbf{u}_2^{\lambda_2} \quad i = 1, 2 \tag{5.6a}$$

and find $\widehat{\mathbf{p}}_i$ such that

$$\tilde{A}_i \widehat{\mathbf{p}}_i = G_i^1 \mathbf{u}_1^f - G_i^2 \mathbf{u}_2^f \quad i = 1, 2 \tag{5.6b}$$

namely, for $i = 1, 2$, we obtain the system

$$\begin{bmatrix} -G_i^1 & G_i^2 & 0 & 0 & \tilde{A}_i & 0 \\ 0 & 0 & -G_i^1 & G_i^2 & 0 & \tilde{A}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\lambda_1} \\ \mathbf{u}_2^{\lambda_2} \\ \mathbf{u}_1^f \\ \mathbf{u}_1^f \\ \mathbf{p}_i \\ \widehat{\mathbf{p}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}\ ,$$

where $\tilde{A}_i$, G_i^1 and G_i^2 are defined in the different cases, as follows.

Dirichlet interface controls

In the case of Dirichlet interface controls $\mathbf{p}_i$ and $\widehat{\mathbf{p}}_i$ belong to $\mathbb{R}^{N_h^i}$.

- Functional J_0
  We have to discretize the equations (4.29). Their Galerkin finite element approximations are systems (5.6), where

  $$\{\tilde{A}_i\}_{lm} = a_i(\varphi_l, \varphi_m) \quad l, m = 1, ..., N_h^i$$

  and

  $$\{G_i^1\}_{lm} = \{G_i^2\}_{lm} = (-1)^{i+1} \int_{\Omega_i} \chi_{12} \varphi_l \varphi_m d\Omega \quad l, m = 1, ..., N_h^i\ .$$

- Functional J_1
  In this second case we discretize the equations (4.40). The matrices of the systems (5.6) assume the following expressions:

$$\{\tilde{A}_i\}_{lm} = a_i(\varphi_l, \varphi_m) \quad l, m = 1, ..., N_h^i$$

and

$$\{G_i^1\}_{lm} = \{G_i^2\}_{lm} = (-1)^{i+1} \left(\int_{\Omega_i} \chi_{12} \varphi_m \varphi_l d\Omega + \int_{\Omega_i} \chi_{12} \nabla\varphi_m \cdot \nabla\varphi_l d\Omega \right) \quad l, m = 1, ..., N_h^i \ .$$

- Functional $J_{0,\Gamma}$
  Let us now consider the discretization of the problems (4.54). It is easy to show that the problems (4.54) are equivalent to other weak formulations with extensions of Dirichlet data on Γ_i. In this case if we consider the discretizations of these equivalent weak problems, we obtain that the Galerkin finite element approximations are given by the system (5.6), where

$$\{\tilde{A}_i\}_{lm} = a_i(\varphi_m, \varphi_l) \quad l, m = 1, ..., N_h^i \ ,$$

$$\{G_i^1\}_{lm} = -\sum_{k=1}^{N_h^{i,\Gamma_i}} a_i(\varphi_{N_h^i+k}, \varphi_l)\{\tilde{G}_i^1\}_{km} \quad l = 1, ..., N_h^i, m = 1, ..., N_h^{1,t}$$

and

$$\{G_i^2\}_{lm} = -\sum_{k=1}^{N_h^{i,\Gamma_i}} a_i(\varphi_{N_h^i+k}, \varphi_l)\{\tilde{G}_i^2\}_{km} \quad l = 1, ..., N_h^i, m = 1, ..., N_h^{2,t},$$

  where $\tilde{G}_i^j$ are the restriction operators from the domain $j - th$ to the interface Γ_i, for $i, j = 1, 2$.
  Let us observe that in this case $\tilde{A}_i = A_i$, for $i = 1, 2$.

Robin interface controls

In the case of Robin interface controls, $\mathbf{p}_i$ and $\widehat{\mathbf{p}}_i$ belong to $\mathbb{R}^{N_h^i}$ if we consider J_0 or J_1 functional, while they belong to $\mathbb{R}^{N_h^{i,\Gamma_i}}$ if we consider $J_{0,\Gamma}$ functional.

- Functional J_0
  We have to discretize the equations (4.30). Their Galerkin finite element approximations are systems (5.6), where

$$\{\tilde{A}_i\}_{lm} = a_i(\varphi_l, \varphi_m) + \int_{\Gamma_i} \beta\varphi_l\varphi_m d\Gamma \quad l, m = 1, ..., N_h^i$$

and

$$\{G_i^1\}_{lm} = \{G_i^2\}_{lm} = (-1)^{i+1} \int_{\Omega_i} \chi_{12} \varphi_l \varphi_m d\Omega \quad l, m = 1, ..., N_h^i \ .$$

- Functional J_1
  In this case we discretize the equations (4.42). The matrices of the systems (5.6) assume the following expressions:

$$\{\tilde{A}_i\}_{lm} = a_i(\varphi_l, \varphi_m) + \int_{\Gamma_i} \beta \varphi_l \varphi_m d\Gamma \quad l, m = 1, ..., N_h^i$$

  and

$$\{G_i^1\}_{lm} = \{G_i^2\}_{lm} = (-1)^{i+1} \left(\int_{\Omega_i} \chi_{12} \varphi_m \varphi_l d\Omega + \int_{\Omega_i} \chi_{12} \nabla \varphi_m \cdot \nabla \varphi_l d\Omega \right) \quad l, m = 1, ..., N_h^i \ .$$

- Functional $J_{0,\Gamma}(1)$
  Let us consider the formulas (4.57). In this case the matrix $\tilde{A}_i$ is the identity matrix $N_h^{i,\Gamma_i} \times N_h^{i,\Gamma_i}$, while the matrices G_i^j are the restriction operators defined above as $\tilde{G}_i^j$.

- Functional $J_{0,\Gamma}(2)$
  We have to consider the formulas (4.60). In this case the matrix $\tilde{A}_i$ is, as in the previous case, the identity matrix $N_h^{i,\Gamma_i} \times N_h^{i,\Gamma_i}$. The matrices G_i^j are $N_h^{i,\Gamma_i} \times N_h^j$ matrices and they are the sum of the restriction operators $\tilde{G}_i^j$ and of the restrictions of the conormal derivative with respect to primal problems on the interface Γ_i multiplied by β, for $i, j = 1, 2$.

Optimality Conditions

We want now to discretize the third equation of the optimality systems. The optimality condition, discretized with Galerkin Finite Element Method, is equivalent to the following pair of equations, for $i = 1, 2$

$$C_i \mathbf{p}_i + C_i \widehat{\mathbf{p}}_i = 0 \ , \tag{5.7}$$

namely, for $i = 1, 2$, we obtain the system

$$\begin{bmatrix} C_i & C_i \end{bmatrix} \begin{bmatrix} \mathbf{p}_i \\ \widehat{\mathbf{p}}_i \end{bmatrix} = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

in all cases except the case of functional $J_{0,\Gamma}$ and Dirichlet interface controls. Actually, in this case the equations (5.7) assume the forms

$$C_i^1 \mathbf{p}_1 + C_i^2 \mathbf{p}_2 + C_i^1 \widehat{\mathbf{p}}_1 + C_i^2 \widehat{\mathbf{p}}_2 = 0 \quad i = 1, 2 \ , \tag{5.8}$$

and therefore the system in this case is: namely, for $i = 1, 2$, we obtain the system

$$\begin{bmatrix} C_i^1 & C_i^2 & C_i^1 & C_i^2 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \widehat{\mathbf{p}}_1 \\ \widehat{\mathbf{p}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

Let us now show the expressions of C_i in the different cases.

- Dirichlet interface controls - J_0 and J_1 functionals

$$\{C_i\}_{lm} = -\int_{\Gamma_i} \partial_{n_{L^*}} \varphi_m \varphi_{N_h^i+l} d\Gamma \quad l = 1, ..., N_h^{i,\Gamma_i},\ m = 1, ..., N_h^{i,t}$$

- Dirichlet interface controls - $J_{0,\Gamma}$ functional

$$\{C_i^j\}_{lm} = \int_{\Gamma_i} \varphi_m \varphi_{N_h^i+l} d\Gamma \quad l = 1, ..., N_h^{i,\Gamma_i},\ m = 1, ..., N_h^{j,t} \quad j = 1, 2$$

- Robin interface controls - J_0 and J_1 functionals

$$\{C_i\}_{lm} = \int_{\Gamma_i} \varphi_m \varphi_{N_h^i+l} d\Gamma \quad l = 1, ..., N_h^{i,\Gamma_i},\ m = 1, ..., N_h^{i,t}$$

- Robin interface controls - $J_{0,\Gamma}$ functional

$$\{C_i\}_{lm} = \int_{\Gamma_i} \varphi_m \varphi_{N_h^i+l} d\Gamma \quad l = 1, ..., N_h^{i,\Gamma_i},\ m = 1, ..., N_h^{i,\Gamma_i}$$

Optimality system

We have discretized the equations of the optimality system. We want now to write a unique algebraic system that is the Galerkin Finite Element approximation of each of the optimality systems obtained in Chapter 4. Let us consider the equations (5.5), (5.6) and (5.7) or (5.8). Moreover, let us set $\mathbf{u}^{\boldsymbol{\lambda}} = (\mathbf{u}_1^{\lambda_1}, \mathbf{u}_2^{\lambda_2})$, $\mathbf{u}^{\boldsymbol{f}} = (\mathbf{u}_1^f, \mathbf{u}_2^f)$, $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$, $\widehat{\mathbf{p}} = (\widehat{\mathbf{p}}_1, \widehat{\mathbf{p}}_2)$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2)$ and $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$. Each of the optimality systems analyzed, once discretized, can be written in the following form:

$$\begin{bmatrix} A & 0 & 0 & 0 & B \\ 0 & A & 0 & 0 & 0 \\ G & 0 & \tilde{A} & 0 & 0 \\ 0 & G & 0 & \tilde{A} & 0 \\ 0 & 0 & C & C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u}^{\boldsymbol{\lambda}} \\ \mathbf{u}^{\mathbf{f}} \\ \mathbf{p} \\ \widehat{\mathbf{p}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{5.9}$$

where the matrices are linked to the ones defined above as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \quad G = \begin{bmatrix} -G_1^1 & G_1^2 \\ -G_2^1 & G_2^2 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix}$$

and, in the case when the optimality condition assume the form (5.7)

$$C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$$

while in the case when it is valid (5.8)

$$C = \begin{bmatrix} C_1^1 & C_1^2 \\ C_2^1 & C_2^2 \end{bmatrix}.$$

Let us highlight that the matrix G is a singular matrix.

Dirichlet interface controls: multiplicative version

In the case of Dirichlet interface controls, it is possible to implement also a different version of the methods analyzed. In particular, it is possible to solve the primal problem on Ω_1 with the control variable, and to use the trace of the solution u_1 on Γ_2 in order to solve the problem on the second sub-domain. In this way the only control is λ_1. This second version of the methods is called multiplicative version. The one presented above is the additive version. We briefly present the algebraic system to be solved in the multiplicative case. This system is formally equal to the system (5.9), but the definitions of the variables and of the matrices are different with respect to the previous case.
Let us set $\mathbf{u}_2^{\lambda_1}$ the solution of the primal problem solved in Ω_2 with $\mathbf{u}_2^{\lambda_1}{}_{|\Gamma_2} = \mathbf{u}_1^{\lambda_1}{}_{|\Gamma_2}$.
Let us firstly consider the cases of functionals J_0 and J_1. In these cases the variables and the matrices are defined as follows:

$$\mathbf{u}^{\boldsymbol{\lambda}} = \begin{bmatrix} \mathbf{u}_1^{\lambda_1} \\ \mathbf{u}_2^{\lambda_1} \end{bmatrix} \quad \mathbf{u}^{\boldsymbol{f}} = \begin{bmatrix} \mathbf{u}_1^{f} \\ \mathbf{u}_2^{f} \end{bmatrix} \quad \mathbf{p} = \mathbf{p}_1 \quad \widehat{\mathbf{p}} = \widehat{\mathbf{p}}_1 \quad \boldsymbol{\lambda} = \boldsymbol{\lambda}_1 \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 \\ B_2T_2 & A_2 \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad G = \begin{bmatrix} -G_1^1 & G_1^2 \end{bmatrix} \quad \tilde{A} = \tilde{A}_1 \quad C = C_1 \ ,$$

where T_2 is the restriction operator from Ω_1 to Γ_2. This operator is such that $T_2\mathbf{u}_1^{\lambda_1}$ is a vector of dimension N_h^{2,Γ_2} that contains the nodal values of $\mathbf{u}_1^{\lambda_1}$ on Γ_2.
In the case when we consider the functional $J_{0,\Gamma}$, on the other hand, the expression of the variables and of the matrices becomes:

$$\mathbf{u}^{\boldsymbol{\lambda}} = \begin{bmatrix} \mathbf{u}_1^{\lambda_1} \\ \mathbf{u}_2^{\lambda_1} \end{bmatrix} \quad \mathbf{u}^{\boldsymbol{f}} = \begin{bmatrix} \mathbf{u}_1^{f} \\ \mathbf{u}_2^{f} \end{bmatrix} \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{u}_2 \end{bmatrix} \quad \widehat{\mathbf{p}} = \begin{bmatrix} \widehat{\mathbf{p}}_1 \\ \widehat{\mathbf{p}}_2 \end{bmatrix} \quad \boldsymbol{\lambda} = \boldsymbol{\lambda}_1 \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_1 & 0 \\ B_2T_2 & A_2 \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \quad G = \begin{bmatrix} -G_1^1 & G_1^2 \\ 0 & 0 \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \tilde{A}_1 & 0 \\ B_2T_2 & \tilde{A}_2 \end{bmatrix} \quad C = \begin{bmatrix} C_1^1 & C_1^2 \\ 0 & 0 \end{bmatrix} \ ,$$

We want to underline that in this case, being $\tilde{A}_i = A_i$, then $\tilde{A} = A$.

5.2 Numerical Resolution of Optimality System

In this Section we show that to solve the optimality systems (5.9) is equivalent to solving one reduced linear system, the one obtained building the Schur complement of the matrix of the optimality system (5.9) with respect to $\boldsymbol{\lambda}$. In order to solve the Schur complement system, it is possible to use an iterative method, such as BiCGStab[1]. We want to underline that in an iterative method the most expensive step is the evaluation of the product between the matrix of the system and a vector. Let us show how to build the Schur complement of the matrix of the optimality system (5.9) with respect to the variable $\boldsymbol{\lambda}$. We notice that it is not possible to build the Schur complement of the matrix with respect to any other variable of the optimality system since the matrix G is a singular matrix.
In all cases, we have a system of the form (5.9).
From the first and second equations of the system, we obtain:

$$\mathbf{u}^{\boldsymbol{\lambda}} = -A^{-1}B\boldsymbol{\lambda} \quad \textit{and} \quad \mathbf{u}^{\mathbf{f}} = A^{-1}\mathbf{f} \ .$$

[1]See Appendix A for details on this method

If we use this results in the third and fourth equations, giving:

$$\mathbf{p} = -\tilde{A}^{-1}G\mathbf{u}^{\boldsymbol{\lambda}} = \tilde{A}^{-1}GA^{-1}B\boldsymbol{\lambda}$$

and

$$\widehat{\mathbf{p}} = -\tilde{A}^{-1}G\mathbf{u}^{\mathbf{f}} = -\tilde{A}^{-1}GA^{-1}\mathbf{f}\ .$$

We are now ready to rewrite the last equation of the system $C\mathbf{p} + C\widehat{\mathbf{p}} = 0$ as:

$$\Sigma\boldsymbol{\lambda} = \mathbf{F}\ , \tag{5.10}$$

where $\Sigma = C\tilde{A}^{-1}GA^{-1}B$ and $\mathbf{F} = C\tilde{A}^{-1}GA^{-1}\mathbf{f}$.
The matrix Σ is the Schur complement of the matrix of the optimality system (5.9) with respect to the variable $\boldsymbol{\lambda}$.
The idea is to solve the linear Schur complement system (5.10) with an iterative method. Let us remark that the matrix Σ is neither symmetric nor necessarily positive definite. Therefore, we need an iterative method for generic algebraic systems. For this reason we decided to use Bi-CGStab method to solve it.
We can observe that in an iterative method, at each iteration, given $\boldsymbol{\lambda}$, we have to know how to compute the product $\boldsymbol{\chi} = \Sigma\boldsymbol{\lambda}$. Moreover, we can notice that in the equation (5.10) the term $\mathbf{F}$ depends on the sole data of the problem and can be computed once, before the iterations start, while the terms $\Sigma\boldsymbol{\lambda}$ depends on the sole control.
We want now to describe how to compute $\boldsymbol{\chi}$ in the additive and in the multiplicative cases. The latter can be implemented only in the case of Dirichlet interface controls. Firstly we give a general procedure that is valid in both cases, then we will show the steps of this procedure in each case.
The evaluation of $\boldsymbol{\chi} = \Sigma\boldsymbol{\lambda}$ can be divided into three steps:
Given $\boldsymbol{\lambda}$,

1. solve the primal problems $A\mathbf{u}^{\boldsymbol{\lambda}} = -B\boldsymbol{\lambda}$ to find $\mathbf{u}^{\boldsymbol{\lambda}}$,
2. solve $\tilde{A}\mathbf{p} = -G\mathbf{u}^{\boldsymbol{\lambda}}$ to find $\mathbf{p}$,
3. compute $\boldsymbol{\chi} = C\mathbf{p}$.

The three steps of this algorithm correspond to the equations of the optimality system: the primal problems, the dual ones and the optimality condition.
Moreover, we can note that this algorithm of evaluation is composed by two linear systems to be solved and a matrix-vector product. The most expensive part of this procedure is the solution of both primal and dual problems. We highlight that in the case of Robin interface control and functional $J_{0,\Gamma}$ there are no dual problems to be solved, but only vectors to be computed. Let us explain this algorithm in the additive and multiplicative case.

- *Additive Case*
  In this case the first and the second steps of the algorithm can be split into two substeps. Actually we have to solve the equations for $\mathbf{u}_1^{\boldsymbol{\lambda}_1}$ and for $\mathbf{u}_2^{\boldsymbol{\lambda}_2}$ in the first step and for $\mathbf{p}_1$ and for $\mathbf{p}_2$ respectively in the second one. Therefore we have to solve two primal problems and two adjoint problems.
  Nevertheless in this case the primal problems , as the dual ones, are independent of

each other. So these two steps (which are the most expensive part of the evaluation procedure) could be done in parallel. This would not be true in the multiplicative case.

- *Multiplicative Case - Dirichlet interface controls*
  In this second case we have two primal problems to be solved sequentially. In the case of $J_{0,\Gamma}$ functional, moreover, we have two dual problems to be solved sequentially.
  Let us first consider the case of functionals J_0 and J_1 .
  We analyze how to compute the primal state $\mathbf{u}^{\boldsymbol{\lambda}}$.
  Given $\boldsymbol{\lambda} = \boldsymbol{\lambda}_1$,

  1. solve the primal problem $A_1\mathbf{u}_1^{\boldsymbol{\lambda_1}} = -B_1\boldsymbol{\lambda}_1$
  2. solve the primal problem $A_2\mathbf{u}_2^{\boldsymbol{\lambda_1}} = -B_2T_2\mathbf{u}_1^{\boldsymbol{\lambda_1}}$

  Let us now consider the case of $J_{0,\Gamma}$ functional. In this case the computation of the primal state is equal to the case of the other two functionals.
  Therefore, let us analyze how to compute the dual state $\mathbf{p}$.
  Given $\mathbf{u}$,

  1. solve the dual problem $\tilde{A}_1\mathbf{p}_1 = G_1^1\mathbf{u}_1^{\boldsymbol{\lambda_1}} - G_1^2\mathbf{u}_2^{\boldsymbol{\lambda_1}}$
  2. solve the dual problem $\tilde{A}_2\mathbf{p}_2 = -B_2T_2\mathbf{p}_1$

  We want to remember that in the multiplicative case the primal and the dual problems can't be parallelized because they are not independent of each other.

5.3 Numerical Results for Elliptic Problems: Efficiency and Robustness with respect to Discontinuous Coefficients

In this Section we compare the different methods we have studied. We will compare them in terms of iterations done to reach convergence. When possible, we analyze their behaviour with respect to that of the Schwarz method[2]. The comparison in terms of iteration is useful in an iterative method because each iteration could be done in parallel.

Comparison among different ICDD methods

To compare the different methods first we have considered three different test cases. The first and the second ones are with continuous coefficients and with a reference solutions, while the third one is with discontinuous data. Let us analyze them separately.
To compare the different methods, a test has been done for each of the following kind of control: Dirichlet, Neumann, Robin with $\beta = 1$. Let us consider Neumann virtual control as a special case of Robin interface controls, and distinguish between them only when required. In each of these cases, we have tested each available method, using both $\mathbb{P}_1$ and $\mathbb{Q}_1$ discretization of the problem. We have firstly fixed the overlap thickness $\delta = 0.1$, and solved the problem by decreasing the mesh size h. Then, fixed the mesh size $h = 0.01$,

[2]See Chapter 3 for details

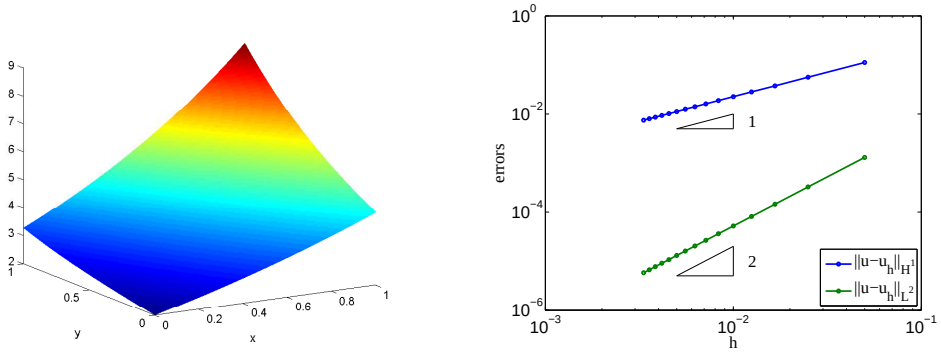
we have decreased the overlap thickness. Finally, we have fixed the number of elements inside the overlap along the direction of the thickness and we have decreased the overlap thickness, and the mesh size as consequence.
Once analyzed these test cases, we compared the different methods when, fixed both the overlap thickness and the mesh size, the coefficient K of the elliptic problem tends to zero.

Test Case 1

Consider the domain $\Omega = (0,1) \times (0,1)$ where the following global problem is defined:

$$\begin{cases} -K\triangle u + b_0 u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{5.11}$$

with g and f such that the exact solution of the problem is $u = exp(x+y)+sin(x)+cos(y)$. Moreover we have chosen $K = 2$, $b_0 = 1$. In Figure 5.1 we can see the solution of this problem and the errors behaviours with respect to the grid size. They are in accordance with the Finite Element Theory.



(a) Solution (b) Errors

Figure 5.1: Solution of the problem (5.11) with Finite Element $\mathbb{P}_1$

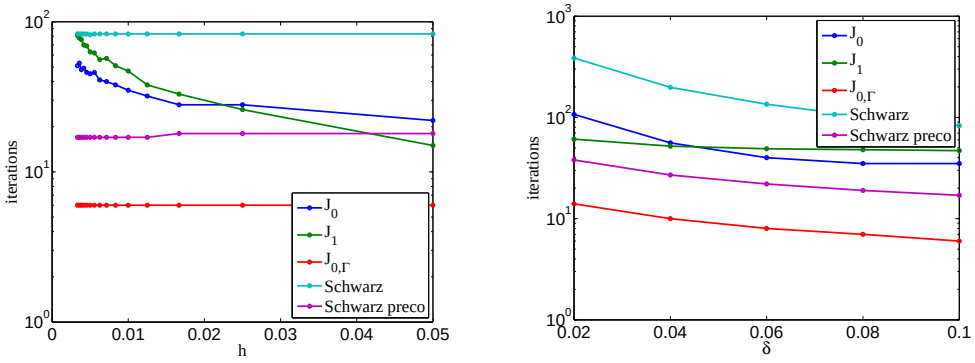
First of all, we underline the fact that Bi-CGStab number of iterations required for both $\mathbb{P}_1$ and $\mathbb{Q}_1$ Finite Element Methods (FEM) is comparable when the problem is solved with Schwarz (both in its classical version and solved with PCG[3] preconditioned with P_{as}) and with ICDD methods with all functionals, except J_0. In this last case, actually, the iterations number is higher in the case of $\mathbb{P}_1$ than in that of $\mathbb{Q}_1$ discretization.
In the following pages we will firstly analyze each case separately, and then compare them.

- *Dirichlet interface controls*
  On one hand, we can compare the methods by fixing the overlap and varying the mesh size. We can note (Figure 5.2a) that the better ICDD method seems to be $J_{0,\Gamma}$, not only because in every case its iterations are lower than others, but also because its convergence rate is independent of the overlap thickness. We can also see that this method is also better than preconditioned Schwarz ('Sc preco' in the graphs) both in terms of number of iterations and in terms of CPU time (Figure 5.4a).

[3]See Chapter 3 for details

Comparing the other ICDD methods, it can be noticed that the number of iterations of J_0 method behaves like $h^{-0.4}$ and the iterations of J_1 like $h^{-0.6}$ as the mesh size h decreases.
In Figure 5.4a we notice that $J_{0,\Gamma}$ is the best method also in terms of CPU time needed to solve the problem, even if all methods have an exponential growth of CPU time as h tends to zero.
On the other hand, we can compare the methods by fixing the mesh size and varying the overlap thickness. We can observe (figure 5.2b) that, also in this case, the method $J_{0,\Gamma}$ seems to be the better one among ICDD methods and Schwarz. Moreover, its iterations increase as the overlap thickness decreases with the same rate as preconditioned Schwarz method (as $\delta^{-0.5}$).

It can also be seen that the number of iterations of each method decreases as the overlap thickness increases, except in the case of J_1 method. In this last case the iterations are almost independent of the overlap size, while in the case of J_0, the iterations trend is δ^{-1}.
If we observe Figure 5.4b, we note that the CPU times of $J_{0,\Gamma}$ method are clearly lower than that of the other methods. It depends both on the number of PDE to be solved at each iteration of Bi-CGStab method and on the number of iterations itself. In the case of $J_{0,\Gamma}$ the number of iterations required is considerably lower with respect to the number of iterations required by the other methods. Moreover, we see that CPU time is almost independent of the overlap size δ in the case of J_1, and it increases a little as the overlap decreases in all other cases.
Finally we can compare the different method fixing the number of elements inside the overlap along the direction of the thickness $nex = 2$ and decreasing at the same time the overlap thickness and the mesh size. We notice (Figure 5.3) that the number of iterations of each method increases as the overlap thickness decreases and that its behaviour is regular as δ tends to zero. In particular, J_0 and J_1 methods are the worst method among ICDD ones. In particular their numbers of iterations increases as δ^{-1} and as $\delta^{-0.9}$ respectively. On the other hand, $J_{0,\Gamma}$ method is the best method both among ICDD methods and Schwarz. Its growth in terms of iterations is the same as preconditioned Schwarz as δ tends to zero. In particular, its number of iterations behaves as $\delta^{-0.5}$.



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.2: Test Case 1 - Dirichlet controls - additive case - iterations

In the multiplicative case there are some differences (see Figure 5.5 and 5.6). The

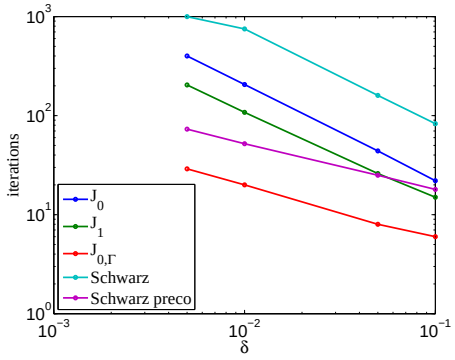
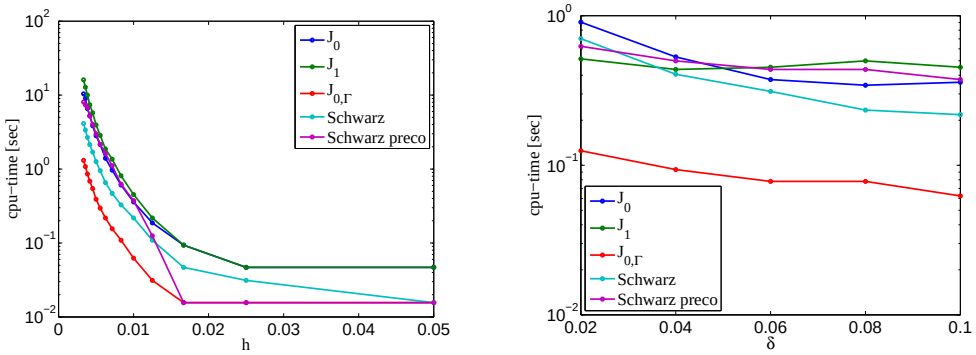


Figure 5.3: Test Case 1 - Dirichlet controls - additive case - iterations
Fixed number of elements into the overlap nex=2

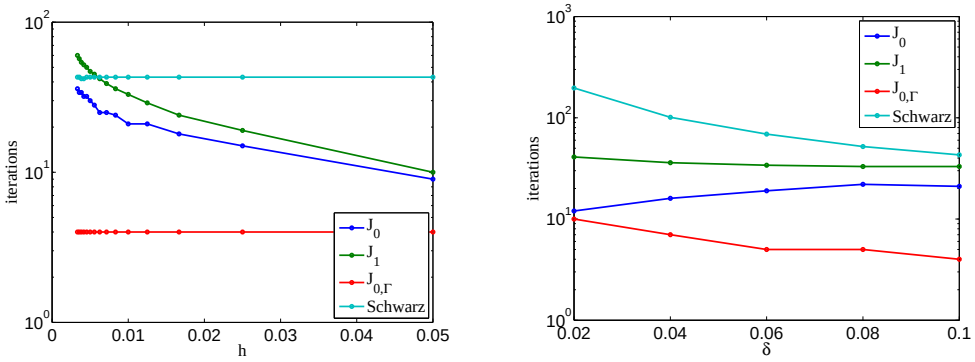


(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.4: Test Case 1 - Dirichlet controls - additive case - CPU time

number of iterations of J_1 method is always larger than the one of J_0. Moreover, when the mesh size is fixed, the number of iterations of J_0 decreases as $\delta^{0.3}$ when the overlap thickness δ tends to zero, while the number of iterations of J_1 method increases a little as δ decreases. Also the number of iterations of both $J_{0,\Gamma}$ and Schwarz increases when δ tends to zero. Finally we can note that if we fix the number of elements into the overlap, the behaviour of the numbers of iterations of all methods is the same as in the additive case, except for J_0 method. In this last case, actually, the behaviour is significantly better, even if now the dependence on δ is not uniform.

- *Robin interface controls*
  In this case, we firstly analyze the behaviour of the available methods in the case where we set $\beta = 1$. Actually the behaviour of all methods is qualitative the same for different values of β. The only differences are quantitive ones. Therefore, we will make a general comment and then compare the number of iterations for different values of β, fixed the overlap and the mesh size.
  Also in this case we can compare the different ICDD methods by either fixing the overlap or the mesh size. Behaviours of the methods with $\mathbb{P}_1$ or $\mathbb{Q}_1$ discretization are similar, except for the fact that J_0 is always better, in terms of iterations, in $\mathbb{Q}_1$ than in $\mathbb{P}_1$ case, even if the qualitative behaviour is the same in the two cases. We will focus on $\mathbb{P}_1$ discretization.



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.5: Test Case 1 - Dirichlet controls - multiplicative case - iterations

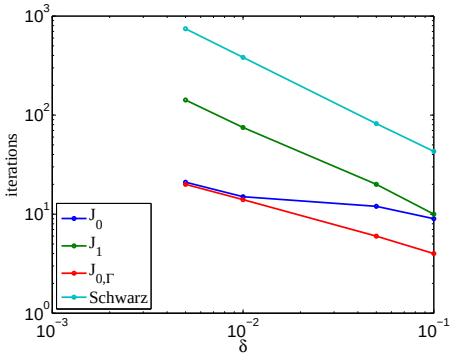
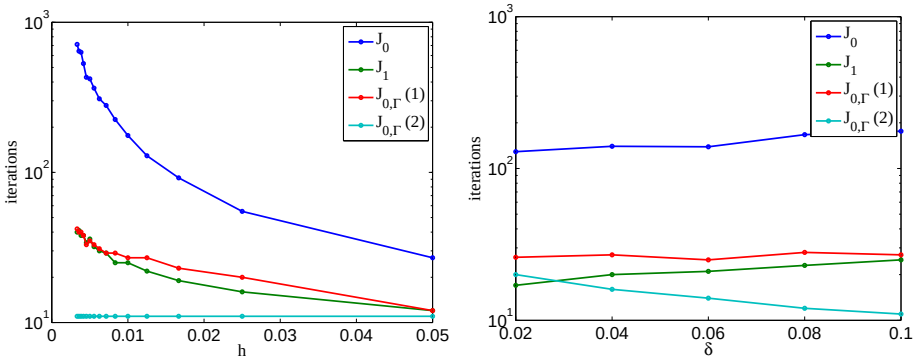


Figure 5.6: Test Case 1 - Dirichlet controls - multiplicative case - iterations
Fixed number of elements into the overlap nex=2

If we fix the overlap thickness to $\delta = 0.1$, equivalent to 10% of the whole domain size along x-direction, (figure 5.7a), we observe that the numbers of iterations of J_1 and $J_{0,\Gamma}(1)$ methods are almost the same so that these two methods seem to be comparable in terms of iterations. Their number of iterations increases as $h^{-0.5}$ as h tends to zero. However, if we set the mesh size to $h = 0.01$, we note that J_1 behaves better than $J_{0,\Gamma}(1)$, and their difference in terms of iterations increases a little as the overlap thickness decreases (figure 5.7b).
The number of iterations of method J_0, instead, is always higher than that of the other methods and grows faster (as $h^{-1.5}$). Finally, $J_{0,\Gamma}(2)$ method seems to be the best method in this case. Actually its number of iterations is constant as h decreases. It can also be seen that the number of iterations of each method increases as the mesh size decreases, once the overlap is fixed, except for $J_{0,\Gamma}(2)$ method.
We can also notice that varying the overlap thickness (Figure 5.7b), the number of iterations of each method does not change in a significative way, even if we can note that there are some trends. We observe in Figure 5.9 that if we fix the mesh size and the overlap thickness, the trend of the number of iterations of each method is not so regular, but it seems that the numbers of iterations of J_1 and $J_{0,\Gamma}(1)$ increase a little when β decreases.
In the case of Robin interface controls, if we fix the number of elements into the overlap thickness to $nex = 2$ (Figure 5.8), the behaviour of the number of iterations

of J_0 method is the same as in the case of Dirichlet interface controls. On the other hand, the number of iterations of J_1 increases as $\delta^{-0.2}$, the one of $J_{0,\Gamma}(1)$ as $\delta^{-0.5}$ and the one of $J_{0,\Gamma}(1)$ as $\delta^{-0.35}$ when the overlap thickness decreases. We notice that the behaviour of J_1 method in this case is significantly better than that of the same method in the case of Dirichlet interface controls.



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.7: Test Case 1 - Robin controls with $\beta = 1$ - iterations

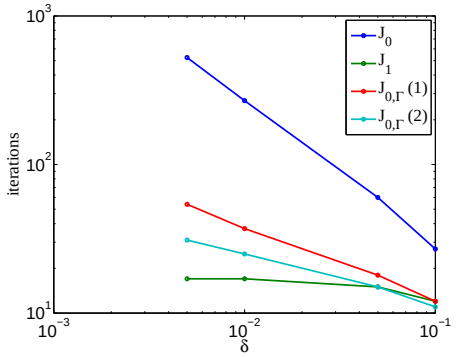


Figure 5.8: Test Case 1 - Robin controls with $\beta = 1$ - iterations
Fixed number of elements into the overlap $nex = 2$

- *Comparison among $J_{0,\Gamma}$ methods*
  Let us directly compare $J_{0,\Gamma}$ methods in the case of Dirichlet (additive case) and in the case of Robin interface controls. In the first case let us call the method $J_{0,\Gamma}$, while in the case of Robin controls we call them $J_{0,\Gamma}(1)$ and $J_{0,\Gamma}(2)$, as made above. In Figure 5.10 we can note that the behaviours of $J_{0,\Gamma}$ and of $J_{0,\Gamma}(2)$ are qualitative the same, once the overlap thickness is fixed. Actually, in both cases, the number of iterations is constant.
  On the other hand, when the mesh size is fixed, the number of iterations of $J_{0,\Gamma}$ and $J_{0,\Gamma}(2)$ increases as $\delta^{-0.55}$ and as $\delta^{-0.4}$, respectively. On the contrary, the number of iterations of $J_{0,\Gamma}(1)$ method, is almost constant.
  Finally, fixing the number of elements into the overlap along x-direction to $nex = 2$, we notice that the numbers of iterations of both $J_{0,\Gamma}$ and $J_{0,\Gamma}(1)$ methods increase as $\delta^{-0.5}$ when δ decrease, while the number of iterations of $J_{0,\Gamma}(2)$ method has a

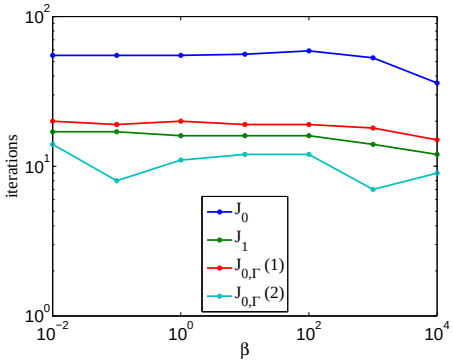
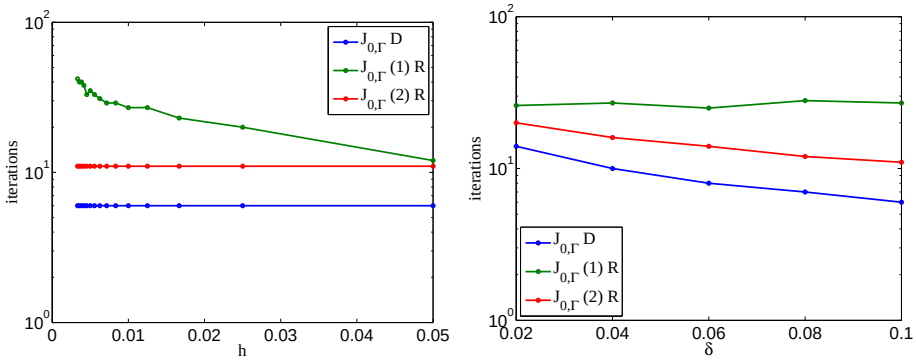


Figure 5.9: Test Case 1 - Robin controls with $\delta = 0.1$ and $h = 0.025$ - iterations

lower growth. Actually it increases as $\delta^{-0.35}$ when the overlap thickness (and the mesh size as consequence) tends to zero (Figure 5.11).
Moreover, the number of iterations of $J_{0,\Gamma}(1)$ method is always larger than that of the other methods.



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.10: Test Case 1 - Functionals $J_{0,\Gamma}$ - iterations

Conclusions:
In conclusion we can say that the best methods seem to be $J_{0,\Gamma}$ in the case of Dirichlet controls, both in its additive and multiplicative versions and $J_{0,\Gamma}(2)$ in the case of Robin interface controls. The other methods produce comparable errors on the exact solution, but they are more expensive in terms of iterations.

Test Case 2

Consider the domain $\Omega = (0,1) \times (0,1)$, and let us define $\Gamma_N = \{\mathbf{x} = (x,y) \in \Omega : y = 1\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Let us consider the following global problem:

$$\begin{cases} Lu = f & in\ \Omega \\ u = g & on\ \Gamma_D \\ \partial_{n_L} u = h & on\ \Gamma_N \end{cases} \tag{5.12}$$

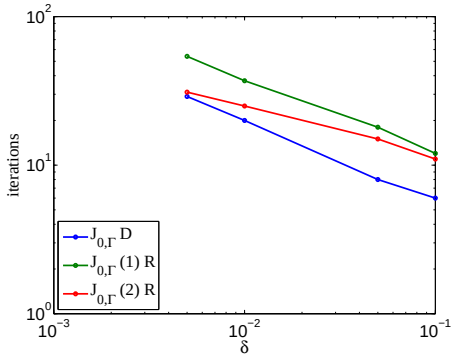


Figure 5.11: Test Case 1 - Functionals $J_{0,\Gamma}$ - iterations - Fixed number of elements into the overlap $nex = 2$

where L is the elliptic operator defined in Chapter 4. Moreover, we have chosen $K = 5$, $\mathbf{b} = (y-1, -x)'$, $b_0 = 1$, and f, g and h such that the exact solution of the problem is $u = exp(x+y) + sin(x) + cos(y)$. In Figure 5.12 we can see the errors behaviours with respect to the grid size h. They are in accordance with the Finite Element Theory.

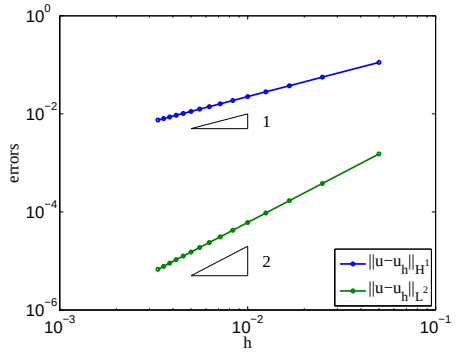
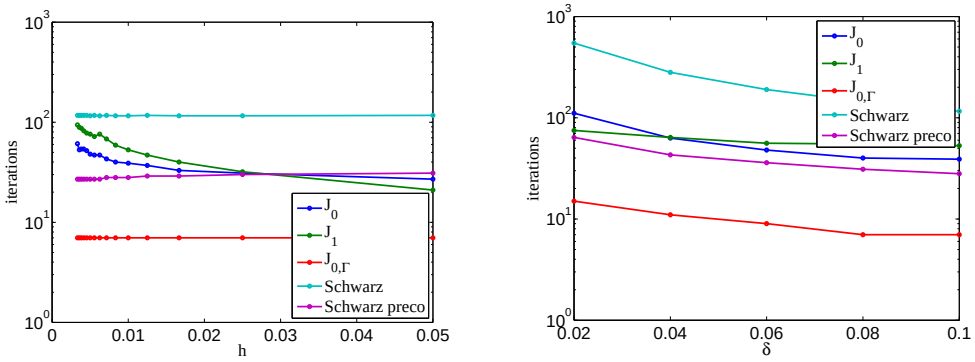


Figure 5.12: Errors of the problem (5.14) solved with Finite Element $\mathbb{P}_1$

- *Dirichlet interface controls*
  We can note in Figures 5.13 and 5.14 that the behaviour of the number of iterations of every method is similar to the behaviour already seen in Test Case 1. Nevertheless, the number of iterations of Schwarz, if compared to the previous case, grows more than those of the other methods number. All other conclusions are the same of the previous case. Let us insert some of the graphics obtained for completeness.
  On the other hand, if we consider the CPU-times (Figure 5.15), the CPU-time of preconditioned Schwarz method does not grow as the ones of the other methods.
  Finally, in the multiplicative case we can draw the same conclusions as in Test Case 1.

- *Robin interface controls*
  In these case, the behaviours of all methods is qualitative the same as in Test Case 1.

In these first two test cases, we have verified that the behaviour of all the proposed methods is independent of the problem (with continuous coefficients and data) to be solved. In the



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.13: Test Case 2 - Dirichlet controls - additive case - iterations

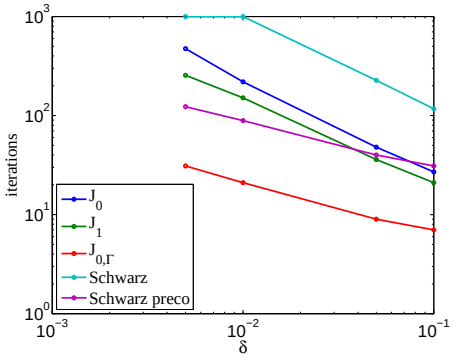


Figure 5.14: Test Case 2 - Dirichlet controls - additive case - iterations
Fixed number of elements into the overlap nex=2

following test case we want to verify the robustness (or not) of the methods with respect to discontinuous coefficients.

Test Case 3

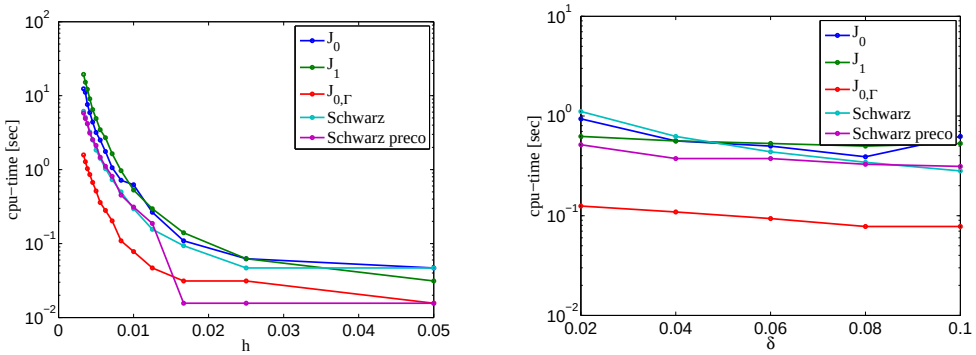
Consider the domain $\Omega = (0,1) \times (0,1)$, and let us define $\Gamma_N = \{\mathbf{x} = (x,y) \in \Omega : y = 0\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Let us consider the following global problem:

$$\begin{cases} Lu = f & in\ \Omega \\ u = g & on\ \Gamma_D \\ \partial_{n_L} u = h & on\ \Gamma_N \end{cases} \tag{5.13}$$

where L is the elliptic operator defined in Chapter 4. Moreover, we have set

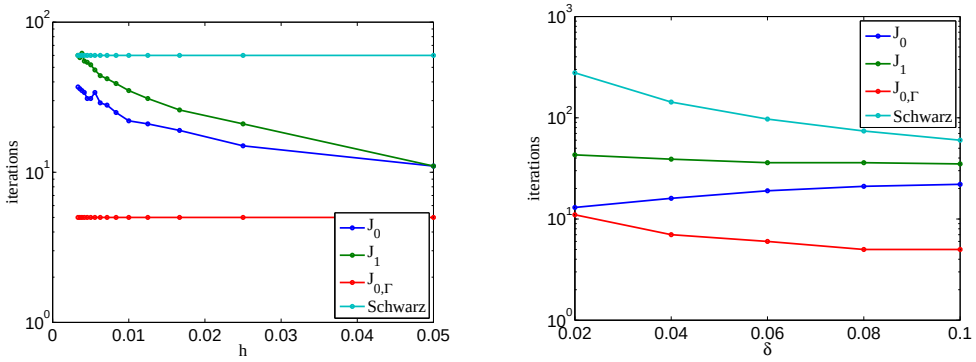
$$K = \begin{cases} 10^{-2} & if\ x < 0.5 \\ 10^{2} & if\ x \geqslant 0.5 \end{cases},$$

$$\mathbf{b} = \begin{cases} (0,0)' & if\ x < 0.5 \\ (10,5)' & if\ x \geqslant 0.5 \end{cases},$$



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.15: Test Case 2 - Dirichlet controls - additive case - CPU time



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.16: Test Case 2 - Dirichlet controls - multiplicative case - iterations

$$f(x,y) = \begin{cases} 0.3 & if\ x < 0.5 \\ -300 & if\ x \geqslant 0.5 \end{cases},$$

$b_0 = 1$, $g = 0$ and $h = 0$. In Figure 5.12 we can see the solution of this problem. It has been solved in one domain, with finite element $\mathbb{P}_1$.

We have chosen a test with discontinuous data because we want to test the robustness of ICDD methods with respect to these kinds of problems.

- *Dirichlet interface controls*
  We first want to note that (Figure 5.19 and 5.14) J_1 method, if we consider the additive form of the method, has a worse behaviour than in the cases where the problem to be solved has continuous coefficients. It could depend on the fact that the gradient of the solution is very large in the overlapping area. Therefore it is difficult to approximate it in an adequate way and to have small values of the functional. For this method, the number of iterations is very high with respect to the number of iterations of all other methods and it has not a regular behaviour with respect to the mesh size h. Moreover, when the mesh size is too small, the method J_1 does not converge to the prefixed tolerance in the maximum number of iterations, fixed at 1000. Finally the behaviour of its number of iterations when the number of elements inside the overlap along x-direction is fixed, is the same as the case of continuous coefficients, namely it behaves as $\delta^{-0.9}$ as the overlap thickness decreases.

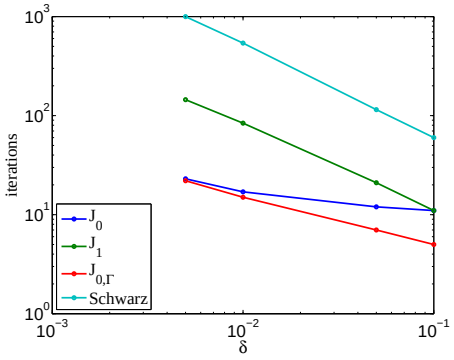


Figure 5.17: Test Case 2 - Dirichlet controls - multiplicative case - iterations
Fixed number of elements into the overlap nex=2

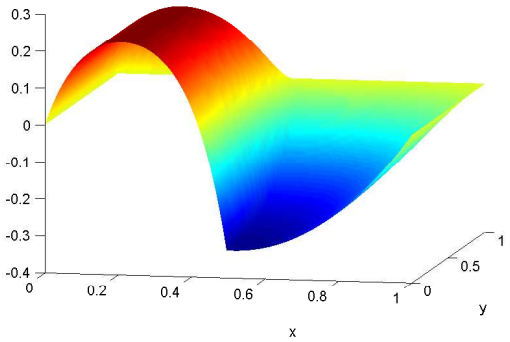
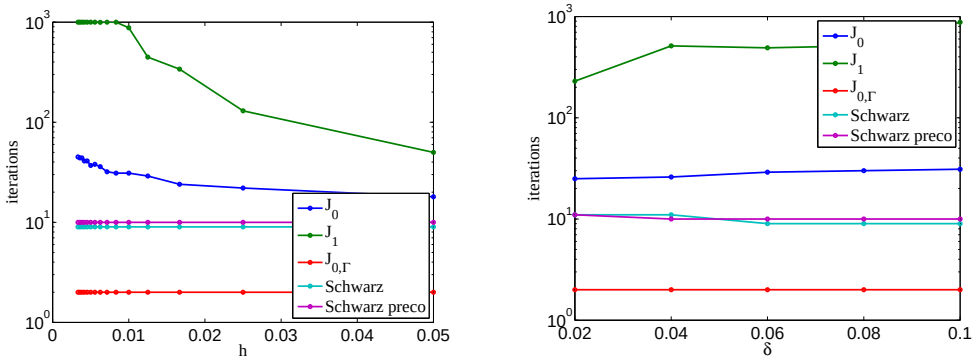


Figure 5.18: Solution of the problem (5.13) solved with Finite Element $\mathbb{P}_1$

Let us now consider the behaviour of the other methods . The number of iterations of J_0 method increases as $h^{-0.4}$ when h decreases, namely its behaviour is the same as the case when the problem to be solved has continuous coefficients. The number of iterations of J_0 is almost constant with respect to the overlap thickness both in the case when h is fixed and when nex is fixed. On the other hand, the number of iterations of $J_{0,\Gamma}$ is very small and constant with both h and δ. Also in the case when the number of elements inside the overlap is fixed at $nex = 2$ the number of iterations is almost constant with respect to the overlap thickness. Therefore, also this method is robust with respect to discontinuous coefficients and data. Moreover this method is better than Schwarz in its two forms. Moreover, we note that in this case the number of iterations of Schwarz method in its two forms is the same. It is probably due to the fact that the preconditioned version of Schwarz does not improve as the other methods in the case of discontinuous data.

If we observe the CPU-times (Figure 5.21), we can note that the CPU-times required by both $J_{0,\Gamma}$ and Schwarz are of the same order, while the CPU-times of all other methods are larger. In particular, also the CPU-time of PCG preconditioned by Schwarz is larger because one iteration of this method is more expensive than one iteration of classical Schwarz method.



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.19: Test Case 3 - Dirichlet controls - additive case - iterations

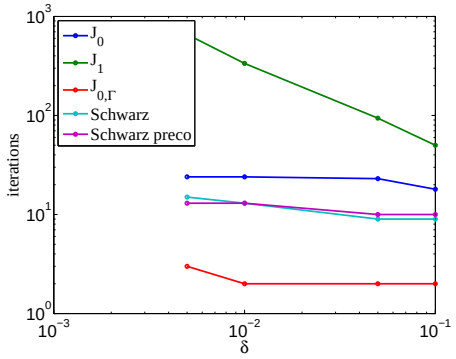
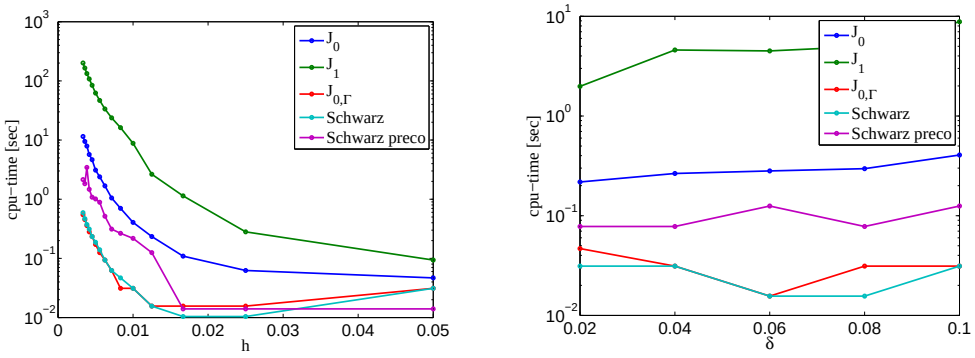


Figure 5.20: Test Case 3 - Dirichlet controls - additive case - iterations
Fixed number of elements into the overlap nex=2

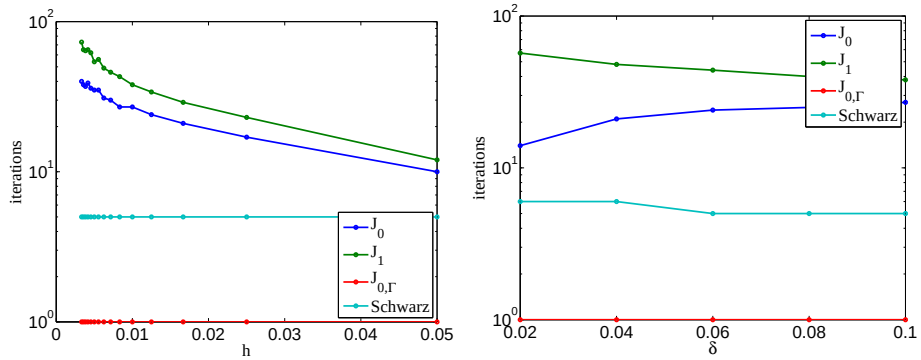
In the multiplicative case (Figure 5.22), we can see that the behaviours of the number of iterations of all methods with respect to h and δ are similar to the case when a problem with continuous coefficients is solved, except for Schwarz and $J_{0,\Gamma}$ methods, whose behaviours are significantly better. Moreover $J_{0,\Gamma}$ method is better than Schwarz and its number of iterations is independent of both the mesh size and the overlap thickness as in the additive case.
Nevertheless, we can note that by fixing the number of elements inside the overlap along x-direction, the only method that has the same behaviour as the case with continuous coefficients and data is J_1 method. The other ones are better because their numbers of iterations increase a little when the overlap thickness decreases.
Finally we can note that in this case preconditioned Schwarz and the classical one are equivalent in terms of iterations. Moreover, the CPU-times of classical Schwarz method are lower.

- *Robin interface controls*
  In the case of Robin interface controls, the only method that is robust with respect to discontinuous data of the problem to be solved is $J_{0,\Gamma}(2)$ method. Actually there are some cases when the other methods (J_0, J_1 and $J_{0,\Gamma}(1)$) do not converge to the solution with the prefixed tolerance in the maximum number of iterations. Moreover, when they converge, their numbers of iterations are too large compared with the



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.21: Test Case 3 - Dirichlet controls - additive case - CPU time



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.22: Test Case 3 - Dirichlet controls - multiplicative case - iterations

numbers of iterations of the methods with Dirichlet interface controls. For this reason J_0, J_1 and $J_{0,\Gamma}(1)$ methods can't be considered a valid choice.
In Figure 5.26 we can also note that the number of iterations of $J_{0,\Gamma}(2)$ method significantly decreases as β increases.

- *Comparison among $J_{0,\Gamma}$ methods*
  We can notice in Figures 5.27 and 5.28 that $J_{0,\Gamma}(1)$ method is not comparable with the two other methods $J_{0,\Gamma}$ and $J_{0,\Gamma}(2)$ because, as said above, it's not robust with respect to discontinuous coefficients and data of the problem. Actually its number of iterations is too large.
  On the other hand, the qualitative behaviour of both $J_{0,\Gamma}$ and $J_{0,\Gamma}(2)$ methods are similar to each other, even if the number of iterations of $J_{0,\Gamma}$ is always smaller.

Conclusions:
Finally we can say that the only robust methods with respect to discontinuous coefficients and data, among ICDD methods, are J_1 method with Dirichlet interface controls only in its multiplicative version, J_0 method with Dirichlet interface controls and $J_{0,\Gamma}$ method in its Dirichlet and in its Robin(2) version. Moreover, the better method among ICDD and Schwarz seems to be $J_{0,\Gamma}$ in the case of Dirichlet interface controls, both in its additive and multiplicative version.

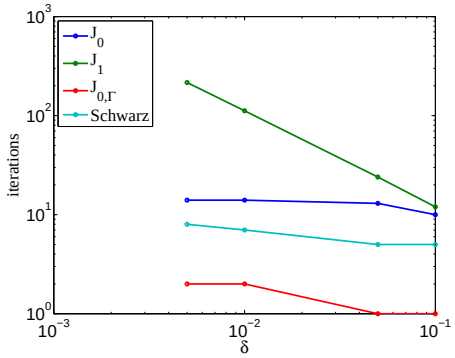
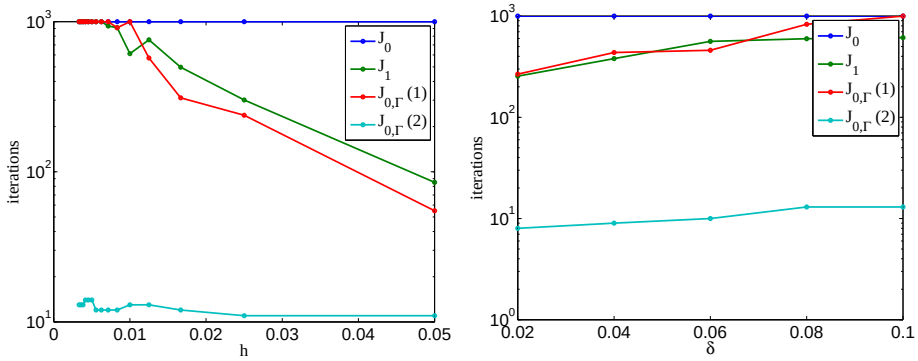


Figure 5.23: Test Case 3 - Dirichlet controls - multiplicative case - iterations
Fixed number of elements into the overlap nex=2



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.24: Test Case 3 - Robin controls with $\beta = 1$ - iterations

Test Case 4

In this Section we want to test the efficiency of the ICDD methods with respect to the coefficient K. In particular we analyze the behaviour of the methods as K tends to zero. Let us consider the domain $\Omega = (0,1) \times (0,1)$, and let us define $\Gamma_N = \{\mathbf{x} = (x,y) \in \Omega : y = 0\}$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$. Let us consider the following global problem:

$$\begin{cases} Lu = f & in\ \Omega \\ u = g & on\ \Gamma_D \\ \partial_{n_L} u = h & on\ \Gamma_N \end{cases} \tag{5.14}$$

where L is the elliptic operator defined in Chapter 4. Moreover, we have chosen $\mathbf{b} = (1,1)'$, $b_0 = 1$, and f, g and h such that the exact solution of the problem is $u = exp(x+y) + sin(x) + cos(y)$. We have chosen to solve this problem for $K = 0.01, 0.1, 1, 10$. We have chosen a triangulation $\mathcal{T}_h$ such that the Péclet number[4] is small enough that no

[4]Given an elliptic problem of the form described in Chapter 4, discretized with FEM, the Péclet number is defined as $\mathbb{P}e = \dfrac{|\mathbf{b}|h}{2K}$. If $\mathbb{P}e > 1$, then the numerical approximation is not stable and a stabilization is required in order to adequately solve the problem (for example GLS or SUPG methods can be applied). See [17] for details

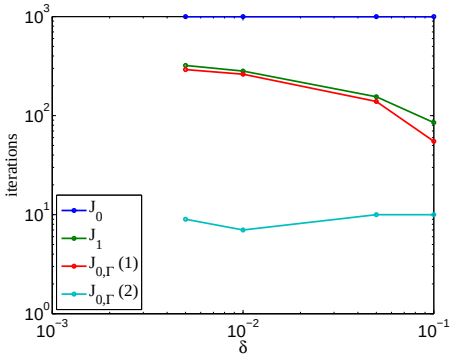


Figure 5.25: Test Case 3 - Robin controls with $\beta = 1$ - iterations - nex=2

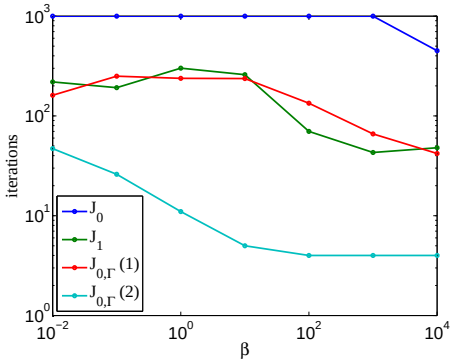
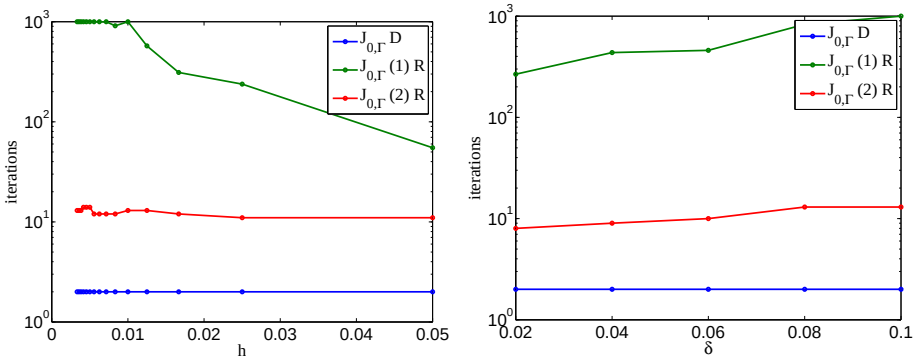


Figure 5.26: Test Case 3 - Robin controls with $\delta = 0.1$ and $h = 0.025$ - iterations

stabilization is required.
In all numerical tests the overlap thickness is fixed to $\delta = 0.04$ and the number of elements into the overlap along x-direction to $nex = 4$.

- *Dirichlet interface controls*
  In Figure 5.29 we can note that the only method whose number of iterations significantly decreases as K decreases is Schwarz method (both in its additive and muliplicative version). Preconditioned Schwarz, on the other hand, does not converge when the coefficient K is too small.
  The numbers of iterations of the other methods are almost constant if K is greater than 10^{-1}, while they increase or decrease as it becomes smaller. We can also note that if we compare ICDD methods, $J_{0,\Gamma}$ is always better than J_0 that is on its turn always better than J_1. Moreover their difference in terms of iterations increases as K decreases.

- *Robin interface controls*
  In this case (Figure 5.30), the general considerations made above on ICDD methods are still valid. We want to remark, once again, that the qualitative and quantitative behaviours of J_1 and $J_{0,\Gamma}(2)$ methods are similar.
  Moreover, in this case , $J_{0,\Gamma}(2)$ method is better than J_1 that is better than J_0. Nevertheless, in the case of Robin interface controls, the numbers of iterations of the different methods tend to be equal to each other when K tends to zero.



(a) fixed overlap δ=0.1 (b) fixed mesh size h=0.01

Figure 5.27: Test Case 3 - Functionals $J_{0,\Gamma}$ - iterations

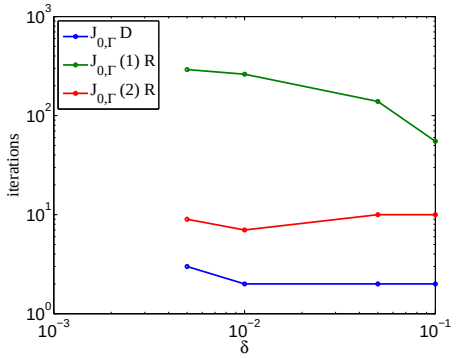
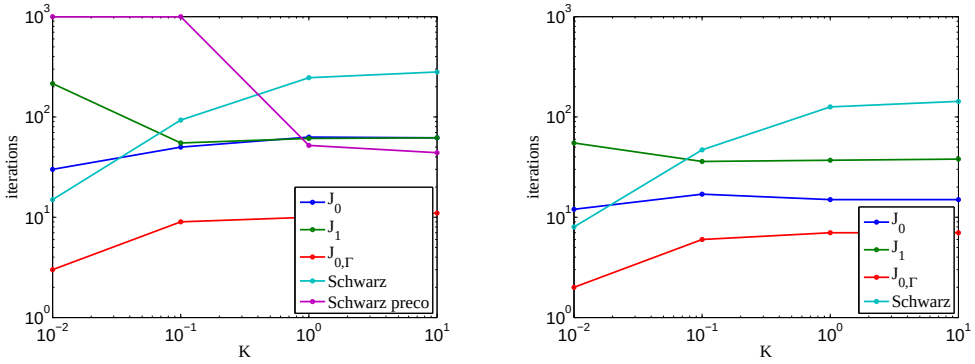


Figure 5.28: Test Case 3 - Functionals $J_{0,\Gamma}$ - iterations - Fixed number of elements into the overlap $nex = 2$

Conclusions:
After having compared the different methods in four test cases, we can conclude that the best method seems to be $J_{0,\Gamma}$ in the case of Dirichlet interface control, both in its multiplicative and additive version. Nevertheless, a valid alternative would be $J_{0,\Gamma}(2)$ in the case of Robin interface controls, with high β.



(a) additive case (b) multiplicative case

Figure 5.29: Test Case 4 - Dirichlet controls - iterations

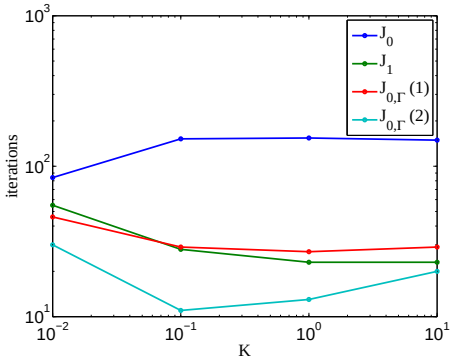


Figure 5.30: Test Case 4 - Robin controls with $\beta = 1$ - iterations

Chapter 6

Interface Control Domain Decomposition for Stokes-Darcy coupling

In this Chapter we analyze the behaviour of ICDD methods for heterogeneous problems, where two (or more) different differential models are considered in suitable subregions of the whole domain. The advantage is that we do not have to impose any interface condition between two different problems nor to fix an interface between the two sub-domains on which the problems are defined, but only to minimize an adequate functional. The choice of this functional is clearly a crucial issue of ICDD approaches.
In this Chapter we focus on Stokes-Darcy coupling. In particular, we first describe the two problems separately and then we give the formulation of the coupled problem, using either the classical formulation and the one based on ICDD methods. The description of the classical coupling of Stokes-Darcy problem with sharp interface is reported for completeness. The ICDD methods we present for Stokes-Darcy coupling can be found in [10], while the classical coupling can be found in [6] and in [11]. Finally we analyze the numerical results obtained with the proposed methods, with the goal of understanding if these methods are well-posed and consistent with the solution obtained with the classical coupling.

6.1 Stokes Problem

In this Section we introduce and analyze the Stokes problem and its discretization by finite elements. We refer to [17] and to [4] for the theoretical analysis.
Let us consider a domain $\Omega \subset \mathbb{R}^2$, with a Lipschitz boundary, as in the previous Chapters. We will use the same notations.
Let us consider the *Navier-Stokes equations* for a constant density and viscosity fluid. They read as follows: find the velocity $\mathbf{u}$ and the pressure p such that

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \triangle \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \qquad in\ \Omega\ , \tag{6.1}$$

where $\mathbf{f}$ is the force per mass unit, $\nu = \mu/\rho$ is the kinematic viscosity, ρ is the density of the fluid and μ is the dynamic viscosity of the fluid. We have denoted by $\nabla \cdot \mathbf{u}$ the divergence of the field $\mathbf{u}$. Moreover, let highlight that both the pressure and the force have

been scaled with the fluid density.
Let us consider the case when the problem is stationary ($\frac{\partial}{\partial t}\cdot = 0$) and the non linear term of the equation $(6.1)_1$ can be ignored. This is, for example, the case when the Reynolds number can be considered very small. Reynolds number is a dimensionless number defined as follows:

$$Re = \frac{|\mathbf{U}|L}{\nu} ,$$

where $\mathbf{U}$ and L are the characteristic velocity of the fluid and length of the domain, respectively, while ν is the kinematic viscosity already defined.
Under these assumptions, the Navier-Stokes equations (6.1) assume the following form:

$$\begin{cases} -\nu \triangle \mathbf{u} + \nabla p = \mathbf{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \qquad in\ \Omega\ . \tag{6.2}$$

Moreover, this problem is defined with the following boundary conditions:

$$\begin{cases} \mathbf{u} = \mathbf{g} & on\ \Gamma_D \\ \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} = \mathbf{h} & on\ \Gamma_N \end{cases} . \tag{6.3}$$

The problem (6.2) with boundary conditions (6.3) is called *Stokes problem*.

Weak formulation

We are now ready to give the weak formulation of Stokes problem. Let us define the following functional spaces:

$$Q = \begin{cases} L^2(\Omega) = \{q : \Omega \to \mathbb{R} :\ \int_\Omega q^2 d\Omega < +\infty\} & if\ \Gamma_N \neq \emptyset \\ L^2_0(\Omega) = \{q \in L^2(\Omega) :\ \int_\Omega q d\Omega = 0\} & if\ \Gamma_N = \emptyset \end{cases} ,$$

$$V = [H^1(\Omega)]^d = \{\mathbf{v} : \Omega \to \mathbb{R}^d :\ \int_\Omega (v^2 + |\nabla \mathbf{v}|^2) d\Omega < +\infty\} ,$$

$$V_0 = [H^1_{\Gamma_D}(\Omega)]^d = \{\mathbf{v} : \Omega \to \mathbb{R}^d :\ \mathbf{v} \in V,\ \mathbf{v} = \mathbf{0}\ on\ \Gamma_D\} .$$

Let us now multiply the first equation of (6.2) by $\mathbf{v} \in V_0$ and te second one by $q \in Q$, and let us integrate in Ω thus obtaining:

$$\begin{aligned} &\int_\Omega (-\nu \triangle \mathbf{u} + \nabla p) \cdot \mathbf{v} d\Omega = \\ &\int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega - \int_\Omega (\nabla \cdot \mathbf{v}) p d\Omega + \int_{\partial\Omega} (\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}) \cdot \mathbf{v} d\Gamma = \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{v} , \end{aligned}$$

and

$$\int_\Omega (\nabla \cdot \mathbf{u}) q d\Omega = 0$$

We note that $\int_{\partial\Omega} (\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n}) \cdot \mathbf{v} d\Gamma = \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v} d\Gamma$ thanks to boundary conditions (6.3).
Therefore, the weak formulation of Stokes problem (6.2) with boundary conditions (6.3),

reads as follows:
find $(\mathbf{u}, p) \in V \times Q$, $\mathbf{u} = \mathbf{g}$ on Γ_D such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v} d\Gamma & \forall \mathbf{v} \in V_0 \\ b(\mathbf{u}, q) = 0 & \forall q \in Q \, , \end{cases} \tag{6.4}$$

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined as follows:

$$a : V \times V \to \mathbb{R}, \quad a(\mathbf{u}, \mathbf{v}) = \int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega \tag{6.5a}$$

and

$$b : V \times Q \to \mathbb{R}, \quad b(\mathbf{u}, p) = -\int_\Omega (\nabla \cdot \mathbf{v}) p d\Omega \, . \tag{6.5b}$$

Let us note that the non homogeneous Dirichlet boundary conditions can be treated also by extending the Dirichlet data and building a new problem with homogeneous boundary conditions, as made in Chapter 5. If we denote with $\mathbf{R_g}$ the extension of Dirichlet data on Ω and, again with letter $\mathbf{u}$, the function from which we have removed the extension of Dirichlet data, the new weak formulation of the Stokes problem becomes:
find $(\mathbf{u}, p) \in V_0 \times Q$ such that

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = F(\mathbf{v}) & \forall \mathbf{v} \in V_0 \\ b(\mathbf{u}, q) = G(q) & \forall q \in Q \, , \end{cases} \tag{6.6}$$

where

$$F : V \to \mathbb{R}, \quad F(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} d\Omega + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v} d\Gamma - a(\mathbf{R_g}, \mathbf{v}) \tag{6.7a}$$

and

$$G : Q \to \mathbb{R}, \quad G(q) = -b(\mathbf{R_g}, q) \, . \tag{6.7b}$$

It can be shown that the following Theorem holds[1].

Theorem 18. *If $\mathbf{f} \in [L^2(\Omega)]^2$, $\mathbf{g} \in [H^{1/2}(\Gamma_D)]^2$ and $\mathbf{h} \in [H^{-1/2}(\Gamma_N)]^2$, then the Stokes problem (6.6) admits a unique solution $(\mathbf{u}, p) \in V \times Q$, $\mathbf{u} = \mathbf{g}$ on Γ_D.*
Moreover the solution satisfies the following inequality:

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leqslant C \left(\|\mathbf{f}\|_{L^2(\Omega)} + \|\mathbf{g}\|_{H^{1/2}(\Gamma_D)} + \|\mathbf{h}\|_{H^{-1/2}(\Gamma_N)} \right) \, ,$$

where C is a constant independent of the data.

Even if we don't prove this Theorem, we give the general idea of its proof. The Stokes problem can be seen as a saddle-point problem. This Theorem, therefore, is a peculiar case of the more general Theorem of existence and uniqueness of the solution for saddle-points problems (see [4] for details). In particular, it can be shown that the bilinear form $a(\cdot, \cdot)$ is continuous and coercive on V_0, namely

$$\exists \alpha > 0, \exists \gamma > 0 : \quad a(\mathbf{v}, \mathbf{v}) \geqslant \alpha \|\mathbf{v}\|^2_{V_0} \; ; \; a(\mathbf{u}, \mathbf{v}) \leqslant \gamma \|\mathbf{u}\|_{V_0} \|\mathbf{v}\|_{V_0} \quad \forall \mathbf{u}, \mathbf{v} \in V_0$$

and the bilinear form $b(\cdot, \cdot)$ is continuous on $V_0 \times Q$, namely

$$\exists \delta > 0 : \quad b(\mathbf{v}, q) \leqslant \gamma \|\mathbf{v}\|_{V_0} \|q\|_Q \quad \forall \mathbf{v} \in V_0, \; \forall q \in Q \, .$$

[1] It can be found in [4]

Moreover, the bilinear form $b(\cdot,\cdot)$ satisfies the *inf-sup* or *LBB*(Ladyzenskaya-Babuška-Brezzi) *condition*:

$$\exists \beta > 0 : \quad \inf_{q \in Q \setminus \{0\}} \sup_{\mathbf{v} \in V_0 \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{V_0} \|q\|_Q} \geqslant \beta \ . \tag{6.8}$$

Finite Element approximation of Stokes Problem

In this Section we give the finite element approximation of the Stokes problem (6.6). Assume that Ω has a polygonal shape and let us introduce a regular triangulation[2] $\mathcal{T}_h$ and two finite element spaces defined on $\mathcal{T}_h$:

$$V_h \subset V \quad and \quad Q_h \subset Q \ .$$

Moreover, let be $V_{h,0} = V_h \cap V_0$.
The finite element approximation of the Stokes problem (6.6) is:
find the approximate solution $(\mathbf{u}_h, p_h) \in V_{h,0} \times Q_h$ such that

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = F(\mathbf{v}_h) & \forall \mathbf{v}_h \in V_{h,0} \\ b(\mathbf{u}_h, q_h) = G(q_h) & \forall q_h \in Q_h \ , \end{cases} \tag{6.9}$$

where $F(\cdot)$ and $G(\cdot)$ are defined as in (6.7), using $\mathbf{R}_{\mathbf{g}_h}$ instead of $\mathbf{R}_{\mathbf{g}}$ in the two expressions. $\mathbf{g}_h$ is an approximation of the data $\mathbf{g}$ in the space $V_h(\Gamma_D)$.
Since the bilinear form $a(\cdot,\cdot)$ is still continuous and coercive on $V_{h,0}$ and $b(\cdot,\cdot)$ is still continuous on $V_{h,0} \times Q_h$, the discretized problem (6.9) is well-posed if and only if the spaces $V_{h,0}$ and Q_h satisfy the discrete LBB condition:

$$\exists \beta_h > 0 : \quad \inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in V_{h,0} \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_V \|q_h\|_Q} \geqslant \beta_h \ . \tag{6.10}$$

Actually, if LBB condition is satisfied in $V_0 \times Q$, it is not necessarily true that it is satisfied also in $V_{h,0} \times Q_h$. The fulfilment of this property depends on the choice of the discrete spaces.
Moreover, in the case when discrete LBB condition (6.10) is not satisfied, the solution of the discretized Stokes problem (6.9) could not be unique. Actually, let us suppose that

$$\inf_{q_h \in Q_h \setminus \{0\}} \sup_{\mathbf{v}_h \in V_{h,0} \setminus \{\mathbf{0}\}} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_V \|q_h\|_Q} = 0 \ .$$

Then, it could exist an element $q_h^* \in Q_h$ such that

$$b(\mathbf{v}_h, q_h^*) = 0 \quad \forall \mathbf{v}_h \in V_{h,0} \ . \tag{6.11}$$

Let now $(\mathbf{u}_h, p_h)$ be a solution of the discrete Stokes problem (6.9). If there exists $q_h^* \in Q_h$ such that (6.11) is satisfied, then also $(\mathbf{u}_h, p_h + q_h^*)$ is a solution of the discretized Stokes problem. Actually in this case the second equation of the Stokes problem (6.9) is satisfied and the first equation becomes:

$$\begin{aligned} & a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h + q_h^*) = \\ & = a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + b(\mathbf{v}_h, q_h^*) = \\ & = a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = F(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in V_{h,0} \ . \end{aligned}$$

[2] See Chapter 5 or [17]

The functions $q_h^* \in Q_h$ such that

$$b(\mathbf{v}_h, q_h^*) = 0 \quad \forall \mathbf{v}_h \in V_{h,0}$$

are called spurious modes.
It can also be shown that if we choose $V_h = [\mathbb{P}_N]^2$ and $Q_h = \mathbb{P}_{N-1}$, with $N \geqslant 2$, then the LBB discrete condition is satisfied. These couples of finite element spaces are called *Taylor-Hood* elements. The inf-sup constant β_h of these elements is independent of h which ensures an optimal convergence rate. In particular, we have chosen to use Taylor-Hood elements $([\mathbb{P}_2]^2, \mathbb{P}_1)$.
The following Theorem (Brezzi)[3] holds:

Theorem 19. *Under all the hypotheses made, if the discrete spaces V_h and Q_h satisfy the LBB discete condition* (6.10)*, then the dicrete Stokes problem* (6.9) *admits a unique solution $(\mathbf{u}_h, p_h) \in V_{h,0} \times Q_h$.*
Moreover, if $(\mathbf{u}, p) \in V_0 \times Q$ is the exact solution of the Stokes problem (6.6)*, the following convergence estimates hold true:*

$$\|\mathbf{u} - \mathbf{u}_h\|_V \leqslant \left(1 + \frac{\delta}{\beta_h}\right)\left(1 + \frac{\gamma}{\alpha}\right) \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_V + \frac{\delta}{\alpha} \inf_{q_h \in Q_h} \|p - q_h\|_Q$$

and

$$\|p - p_h\|_Q \leqslant \frac{\gamma}{\beta_h}\left(1 + \frac{\delta}{\beta_h}\right)\left(1 + \frac{\gamma}{\alpha}\right) \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_V + \left(1 + \frac{\delta}{\beta_h} + \frac{\gamma\delta}{\alpha\beta_h}\right) \inf_{q_h \in Q_h} \|p - q_h\|_Q \ .$$

Corollary 5. *In the case when $V_h = [\mathbb{P}_2]^2$ and $Q_h = \mathbb{P}_1$, if the exact solution of the Stokes problem is such that $\mathbf{u} \in [H^{p+1}(\Omega)]^2$ and $p \in H^p$ for $p > 0$, then the convergence estimates of the previous Theorem becomes:*

$$\|\mathbf{u} - \mathbf{u}_h\|_{[H^1(\Omega)]^2} \leqslant C_1 h^s |\mathbf{u}|_{[H^{s+1}(\Omega)]^2(\Omega)} + C_2 h^s |q|_{H^s(\Omega)} \tag{6.12a}$$

and

$$\|p - p_h\|_{L^2(\Omega)} \leqslant C_3 h^s |\mathbf{u}|_{[H^{s+1}(\Omega)]^2(\Omega)} + C_4 h^s |q|_{H^s(\Omega)} \ , \tag{6.12b}$$

where C_i, $i = 1, ..., 4$ are independent of both h and the exact solution.

Algebraic formulation of Stokes problem

In this Section we give the algebraic formulation of Stokes problem.
Let us introduce the basis function for the spaces $V_h = [\mathbb{P}_2]^2$ and $Q_h = \mathbb{P}_1$. First we want to highlight that the nodes required by $\mathbb{P}_1$ and $\mathbb{P}_2$ spaces are different each other, even if the triangulation $\mathcal{T}_h$ is the same. In particular, let $N_{h,p}^t$ and $N_{h,2}^t$ be the total number of nodes necessary to define a scalar function that belongs to the two spaces, respectively, and let $N_{h,p}$ and $N_{h,2}$ be the number of nodes except the Dirichlet ones. We define the basis functions of the $\mathbb{P}_1$ space as

$$\psi_i \quad i = 1, ..., N_{h,p}^t$$

and the basis functions of the $\mathbb{P}_2$ space as

$$\varphi_i \quad i = 1, ..., N_{h,2}^t \ .$$

[3] See [4] for details

The basis functions of the vectorial space $V_h = [\mathbb{P}_2]^2$ can be built in the following way: let $N^t_{h,u}$ be the total number of degrees of a function belonging to V_h and $N_{h,u}$ be the number of degrees except the ones corresponding to Dirichlet boundary conditions. Then, $N^t_{h,u} = 2N^t_{h,2}$ and $N_{h,u} = 2N_{h,2}$ because we are considering a domain $\Omega \subset \mathbb{R}^2$. The basis function of V_h can be build as:

$$\boldsymbol{\varphi}_i = \begin{bmatrix} \varphi_i \\ 0 \end{bmatrix} \quad for\ i = 1, ..., N_{h,2} \ ,$$

$$\boldsymbol{\varphi}_{i+N_{h,2}} = \begin{bmatrix} 0 \\ \varphi_i \end{bmatrix} \quad for\ i = 1, ..., N_{h,2} \ ,$$

$$\boldsymbol{\varphi}_{i+2N_{h,2}} = \begin{bmatrix} \varphi_i \\ 0 \end{bmatrix} \quad for\ i = N_{h,2}+1, ..., N^t_{h,2}$$

and

$$\boldsymbol{\varphi}_{i+N^t_{h,2}+N_{h,2}} = \begin{bmatrix} 0 \\ \varphi_i \end{bmatrix} \quad for\ i = N_{h,2}+1, ..., N^t_{h,2} \ ,$$

With these definitions, a generic function $q_h \in Q_h$ can be written as linear combination of the basis functions

$$q_h(\mathbf{x}) = \sum_{i=1}^{N^t_{h,p}} q_h(\mathbf{x}_i)\psi_i(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega$$

and a generic function $\mathbf{v}_h$ can be written in the following way

$$\mathbf{v}_h(\mathbf{x}) = \sum_{i=1}^{N^t_{h,u}} \mathbf{v}_h(\mathbf{x}_i) \cdot \boldsymbol{\varphi}_i(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \ .$$

We highlight that the choice of the basis functions in this last case allows us to reduce the vector $\mathbf{v}_h(\mathbf{x}_i)$ to a scalar value. Actually it can be only the first or the second component of the vector because each basis function has always a null component.
Let us call $\mathbf{U} = [\mathbf{U}_1^T, \mathbf{U}_2^T]^T$ the vector of nodal values, except Dirichlet ones, of the vectorial function $\mathbf{u_h} = [u_{h,1}, u_{h,2}]^T$, and $\mathbf{P}$ the vector of nodal values, except Dirichlet ones, of the function p_h, namely:

$$\{\mathbf{U}_1\}_i = \mathbf{U}_i = u_{h,1}(\mathbf{x}_i) \quad i = 1, ..., N_{h,2} \ ,$$

$$\{\mathbf{U}_2\}_i = \mathbf{U}_{i+N_{h,2}} = u_{h,2}(\mathbf{x}_i) \quad i = 1, ..., N_{h,2}$$

and

$$\mathbf{P}_i = p_h(\mathbf{x}_i) \quad i = 1, ..., N_{h,p} \ .$$

To solve the approximate Stokes problem (6.9) is equivalent to solving the following algebraic system:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \ , \tag{6.13}$$

where $A_{ij} = a(\varphi_j, \varphi_i)$, for $i, j = 1, ..., N_{h,u}$ and $B_{lj} = b(\varphi_j, \psi_l)$ for $j = 1, ..., N_{h,u}$ and $l = 1, ..., H_{h,p}$. We can note that the definition of the basis functions let us to rewrite the system (6.13) as

$$\begin{bmatrix} \tilde{A} & 0 & \tilde{B}_1^T \\ 0 & \tilde{A} & \tilde{B}_2^T \\ \tilde{B}_1 & \tilde{B}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{G} \end{bmatrix} \ , \tag{6.14}$$

where $\tilde{A}_{ij} = \int_\Omega \nu \nabla\varphi_j \cdot \nabla\varphi_i$, for $i,j = 1,...,N_{h,2}$, $\{\tilde{B}_1\}_{lj} = -\int_{\Gamma_i} \partial\varphi_j/\partial x\ \psi_l$ and $\{\tilde{B}_2\}_{lj} = -\int_{\Gamma_i} \partial\varphi_j/\partial y\ \psi_l$ for $j = 1,...,N_{h,2}$ and $l = 1,...,N_{h,p}$.
We solve the global system with a direct method[4], e.g. with LU factorization because the matrix is symmetric but not positive definite.

6.2 Darcy Problem

There are two different ways to formally derive the Darcy law. The first one is based on mixture theory and representative elementary volume (REV) technique[5], while the second one on homogenization[6]. The idea of REV approach is to consider local averages of the functions that describe the physical quantities, on a small volume of the size of a REV (hundreds or thousands of pores). The second approach, on the contrary, considers a whole family of functions depending on the spatial scale parameter $\varepsilon > 0$ that is the typical size of a pore and tries to find the limit of the involved functions as ε tends to zero. The derivation of Darcy law by these two approaches can be found in [1] and [13] respectively.
In this Section we formally derive the Darcy law with the second approach: the method of *homogenization*. This derivation can be found in [13].

Formal derivation of Darcy law

In this Section we give the general idea of the formal derivation of Darcy law from Stokes law at the porous scale ε. We assume that the porous media has a periodic substructure of cells Y. In the standard periodicity cell Y (see Figure 6.1) there is a standard obstacle $\mathfrak{S} \subset Y$ with reference size= 1 and with a piecewise smooth boundary $\partial\mathfrak{S}$. The complementary is denoted by $\mathfrak{B} = Y \setminus \mathfrak{S}$. We assume that this standard geometry is repeated periodically all over $\mathbb{R}^2$. The geometric structure within the fixed domain Ω is obtained by intersecting the ε-multiple of this periodic geometry (represented in Figure 6.2) with Ω, thus obtaining

$$\mathfrak{B}^\varepsilon = \Omega \cap (\varepsilon\mathfrak{B}) \quad and \quad \partial\mathfrak{S}^\varepsilon = \Omega \cap (\varepsilon\partial\mathfrak{S}) \ .$$

Moreover we assume that $\partial\Omega \cap \partial\mathfrak{S}^\varepsilon = \emptyset$.

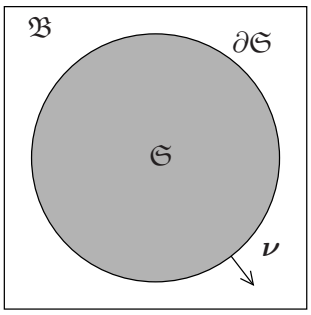


Figure 6.1: Standard microcell Y

[4]See [17] for details on other methods
[5]See [1] for details on this approach
[6]It can be found in [13]

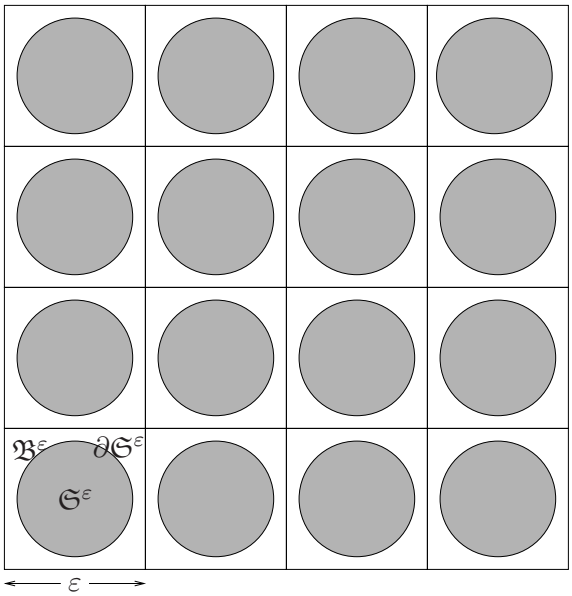


Figure 6.2: Periodic porous medium

We start from the following Stokes problem without external forces at the pore scale:

$$\begin{cases} \varepsilon^2\mu\triangle\mathbf{u}^\varepsilon(\mathbf{x}) = \nabla p^\varepsilon(\mathbf{x}) & \mathbf{x}\in\mathfrak{B}^\varepsilon \\ \nabla\cdot\mathbf{u}^\varepsilon(\mathbf{x}) = 0 & \mathbf{x}\in\mathfrak{B}^\varepsilon \\ \mathbf{u}^\varepsilon(\mathbf{x}) = 0 & \mathbf{x}\in\partial\mathfrak{S}^\varepsilon \end{cases}, \tag{6.15}$$

where μ represents the dynamic viscosity of the fluid, and $\nabla\cdot\mathbf{u}^\varepsilon$ denotes the divergence of the field $\mathbf{u}^\varepsilon$.
We have chosen ε^2 to enter into the problem because we want to scale the velocity $\mathbf{u}^\varepsilon$ so that it has a limit $\mathbf{u}$. We assume that the unknown functions $\mathbf{u}^\varepsilon$ and p^ε depend on $\mathbf{x}$ and on $\mathbf{y}=\mathbf{x}/\varepsilon$ in a separate way and that they admit asymptotic expansions of the form:

$$\mathbf{u}^\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x},\mathbf{y}) + \varepsilon\mathbf{u}_1(\mathbf{x},\mathbf{y}) + \varepsilon^2\mathbf{u}_2(\mathbf{x},\mathbf{y}) + ... \tag{6.16a}$$

and

$$p^\varepsilon(\mathbf{x}) = p_0(\mathbf{x},\mathbf{y}) + \varepsilon p_1(\mathbf{x},\mathbf{y}) + \varepsilon^2 p_2(\mathbf{x},\mathbf{y}) + ... \tag{6.16b}$$

with Y-periodicity of the coefficient functions $\mathbf{u}_i(\mathbf{x},\mathbf{y})$ and $p_i(\mathbf{x},\mathbf{y})$ with respect to the variable $\mathbf{y}$.
Moreover, the gradient taken with respect to the original variables reads

$$\nabla = \nabla_\mathbf{x} + \frac{1}{\varepsilon}\nabla_\mathbf{y} ,$$

where the subscripts indicate the gradients with respect to $\mathbf{x}$ and $\mathbf{y}$, respectively.
Using the asymptotic expansions (6.16) in the problem (6.15), we obtain:

$$\varepsilon^0\mu\triangle_\mathbf{y}\mathbf{u}_0(\mathbf{x},\mathbf{y})+\varepsilon^1(...)+... = \varepsilon^{-1}\nabla_\mathbf{y}p_0(\mathbf{x},\mathbf{y})+\varepsilon^0(\nabla_\mathbf{y}p_1(\mathbf{x},\mathbf{y})+\nabla_\mathbf{x}p_0(\mathbf{x},\mathbf{y}))+\varepsilon^1(...)+... \quad \mathbf{y}\in\mathfrak{B} ,$$

$$\varepsilon^{-1}\nabla_\mathbf{y}\cdot\mathbf{u}_0(\mathbf{x},\mathbf{y}) + \varepsilon^0(\nabla_\mathbf{y}\cdot\mathbf{u}_1(\mathbf{x},\mathbf{y}) + \nabla_\mathbf{x}\cdot\mathbf{u}_0(\mathbf{x},\mathbf{y})) + \varepsilon^1(...) + ... = 0 \quad \mathbf{y}\in\mathfrak{B} ,$$

and

$$\varepsilon^0 \mathbf{u}_0(\mathbf{x},\mathbf{y}) + \varepsilon^1 \mathbf{u}_1(\mathbf{x},\mathbf{y}) + ... = 0 \quad \mathbf{y} \in \partial\mathfrak{S} \ . \tag{6.17}$$

Comparing the terms of the same order in the equations above, we obtain:

- $O(\varepsilon^{-1})$:

$$\nabla_{\mathbf{y}} p_0(\mathbf{x},\mathbf{y}) = 0 \quad \mathbf{y} \in \mathfrak{B} \ ,$$

hence $p_0(\mathbf{x},\mathbf{y}) = p_0(\mathbf{x})$ being the function $p_0(\mathbf{x},\mathbf{y})$ Y-periodic in the variable $\mathbf{y}$, and

$$\nabla_{\mathbf{y}} \cdot \mathbf{u}_0(\mathbf{x},\mathbf{y}) = 0 \quad \mathbf{y} \in \mathfrak{B} \ ,$$

- $O(\varepsilon^0)$:

$$\mu \triangle_{\mathbf{y}} \mathbf{u}_0(\mathbf{x},\mathbf{y}) = \nabla_{\mathbf{y}} p_1(\mathbf{x},\mathbf{y}) + \nabla_{\mathbf{x}} p_0(\mathbf{x}) \quad \mathbf{y} \in \mathfrak{B} \ ,$$

and

$$\nabla_{\mathbf{y}} \cdot \mathbf{u}_1(\mathbf{x},\mathbf{y}) + \nabla_{\mathbf{x}} \cdot \mathbf{u}_0(\mathbf{x},\mathbf{y}) = 0 \quad \mathbf{y} \in \mathfrak{B} \ .$$

We can note that the pressure gradient $\nabla_{\mathbf{x}} p_0(\mathbf{x})$ can be written in the following form

$$\nabla_{\mathbf{x}} p_0(\mathbf{x}) = \sum_{j=1}^{2} \mathbf{e}_j \partial_{\mathbf{x}_j} p_0(\mathbf{x}) \ ,$$

where $\mathbf{e}_j$ is the j-th unit spatial vector.
We define the following *cell problems*: find the Y-periodic vector fields $\mathbf{w}_j(\mathbf{y})$ with components $\mathbf{w}_{ij}(\mathbf{y})$, that solve the Stokes problems, for j=1,2

$$\begin{cases} \triangle_{\mathbf{y}} \mathbf{w}_j(\mathbf{y}) = \nabla_{\mathbf{y}} \pi_j(\mathbf{y}) - \mathbf{e}_j & \mathbf{y} \in \mathfrak{B} \\ \nabla_{\mathbf{y}} \cdot \mathbf{w}_j(\mathbf{y}) = 0 & \mathbf{y} \in \mathfrak{B} \\ \mathbf{w}_j(\mathbf{y}) = 0 & \mathbf{y} \in \partial\mathfrak{S} \end{cases} ,$$

where the functions $\pi_j(\mathbf{y})$ are the corresponding Y-periodic pressure fields. Using these cell functions, we can write $\mathbf{u}_0(\mathbf{x},\mathbf{y})$ as

$$\mathbf{u}_0(\mathbf{x},\mathbf{y}) = -\frac{1}{\mu} \sum_{j=1}^{2} \mathbf{w}_j(\mathbf{y}) \partial_{\mathbf{x}_j} p_0(\mathbf{x}) \ .$$

Let us define the avaraged vector field as follows

$$\overline{\mathbf{u}}(\mathbf{x}) = \int_{\mathfrak{B}} \mathbf{u}_0(\mathbf{x},\mathbf{y}) d\mathbf{y} \ .$$

Its i-th component can be expressed as

$$\overline{\mathbf{u}}_i(\mathbf{x}) = -\frac{1}{\mu} \sum_{j=1}^{2} k_{ij} \partial_{\mathbf{x}_j} p_0(\mathbf{x}) \ ,$$

where

$$k_{ij} = \int_{\mathfrak{B}} \mathbf{w}_{ij}(\mathbf{y}) d\mathbf{y} \ .$$

Introducing the tensor $\mathrm{k} = k_{ij}$, we obtain the classical expression of Darcy law:

$$\overline{\mathbf{u}}(\mathbf{x}) = -\frac{1}{\mu}\mathrm{k}\nabla p_0(\mathbf{x}) \ .$$

Moreover, it can be shown that the *permeability tensor* k is symmetric and positive definite. The *hydraulic conductivity tensor* is defined as $\mathrm{K} = \mathrm{k}\rho g/\mu$, where ρ is the density of the fluid and g is the acceleration due to gravity. In the following Sections we will replace these tensors with scalar values, making the hypothesis that $\mathrm{k} = k\mathtt{I}$, where $\mathtt{I}$ is the identity tensor.
It remains to prove that the velocity field $\overline{\mathbf{u}}$ is divergence-free. We already know that

$$\nabla_{\mathbf{y}} \cdot \mathbf{u}_1(\mathbf{x}, \mathbf{y}) + \nabla_{\mathbf{x}} \cdot \mathbf{u}_0(\mathbf{x}, \mathbf{y}) = 0 \quad \mathbf{y} \in \mathfrak{B} \ .$$

We integrate this over $\mathfrak{B}$, thus obtaining

$$\begin{aligned}\nabla_{\mathbf{x}} \cdot \overline{\mathbf{u}}(\mathbf{x}) &= \int_{\mathfrak{B}} \nabla_{\mathbf{x}} \cdot \mathbf{u}_0(\mathbf{x}, \mathbf{y}) d\mathbf{y} = -\int_{\mathfrak{B}} \nabla_{\mathbf{y}} \cdot \mathbf{u}_1(\mathbf{x}, \mathbf{y}) d\mathbf{y} = \\ &= -\int_{\partial\mathfrak{B}} \nu \cdot \mathbf{u}_1(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{y}) = \\ &= -\int_{\partial\mathfrak{S}} \nu \cdot \mathbf{u}_1(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{y}) - \int_{\partial Y} \nu \cdot \mathbf{u}_1(\mathbf{x}, \mathbf{y}) d\Gamma(\mathbf{y}) \ .\end{aligned}$$

The boundary integral over $\partial\mathfrak{S}$ is zero due to the term that is $O(\varepsilon)$ is equation (6.17) and the boundary integral over ∂Y is zero due to the Y-periodicity of $\mathbf{u}_1(\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{y}$. Therefore, we have shown the following

Theorem 20. *Homogenization of the Stokes problem* (6.15) *is given by the Darcy problem*

$$\begin{cases} \overline{\mathbf{u}} = -\dfrac{1}{\mu}\mathrm{k}\nabla p_0 & in\ \Omega \\ \nabla \cdot \overline{\mathbf{u}} = 0 & in\ \Omega \end{cases} \tag{6.18}$$

In the following let us use $\mathbf{u}$ and p instead of $\overline{\mathbf{u}}$ and p_0, respectively.

Weak formulation and finite element approximation of Darcy law

In this Section we show how to numerically solve the Darcy problem (6.18). We can note that combining the two equations, we obtain an equation for the pressure, actually

$$0 = \nabla \cdot \mathbf{u} = -\nabla \cdot (\frac{1}{\mu}\mathrm{k}\nabla p) \quad in\ \Omega \ . \tag{6.19a}$$

Therefore the Darcy problem can be solved by computing p as the solution of (6.19) and then by setting

$$\mathbf{u} = -\frac{1}{\mu}\mathrm{k}\nabla p \quad in\ \Omega \ . \tag{6.19b}$$

We can associate this problem with the following boundary conditions. Let $\Gamma_D\ , \Gamma_N \subset \Omega$: $\overline{\partial\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with $\Gamma_D \cap \Gamma_N = \emptyset$. Then

$$\begin{cases} p = g & on\ \Gamma_D \\ \partial_n p = \dfrac{\mathrm{k}}{\mu}\nabla p \cdot \mathbf{n} = h & on\ \Gamma_N \end{cases} \tag{6.20}$$

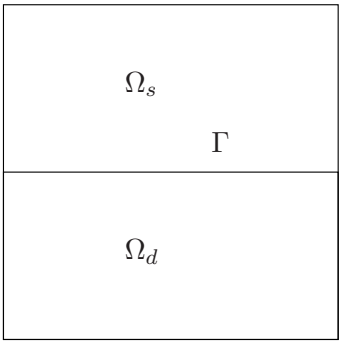


Figure 6.3: Domain partition with Sharp interface

The weak formulations of these problems read: find $p \in H^1(\Omega)$, $p = g$ on Γ_D such that

$$\int_\Omega \frac{1}{\mu} \mathrm{k} \nabla p \cdot \nabla \varphi d\Omega = \int_{\Gamma_N} h\varphi d\Gamma \quad \forall \varphi \in H^1_{\Gamma_D}(\Omega) \tag{6.21a}$$

and find $\mathbf{u} \in [L^2(\Omega)]^2$ such that

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} d\Omega = -\int_\Omega \frac{1}{\mu} \mathrm{k} \nabla p \cdot \mathbf{v} d\Omega \quad \forall \mathbf{v} \in [L^2(\Omega)]^2 \tag{6.21b}$$

The Galerkin finite element approximation of these problems is the classical one[7]. To numerically solve the Darcy problem we have chosen $[\mathbb{P}_2]^2$ elements for the velocity field[8], so that we can easily compare it with the one obtained by Stokes problem, and therefore $\mathbb{P}_3$ elements for the pressure field.

6.3 Stokes-Darcy Problem: Classical Coupling with Sharp Interface and Dimensionless Formulation

In this Section we present the classical Stokes-Darcy coupling and the dimensionless formulation of the Stokes-Darcy problem, that will be used also for the formulation of the coupled problem with ICDD methods. To achieve this goal we refer to the theory in [11] and in [6].
Let us consider a domain Ω split into two non overlapping regions, as shown in figure 6.3. We denote by Γ the interface between these two sub-domains. In the first region, Ω_s we have a fluid, while in the second one, Ω_d we have a porous medium.

Therefore we have to solve Stokes equations in Ω_s, and Darcy equations in Ω_d. Let us refer to the physical quantities of Stokes and Darcy problems with the subscripts f and d respectively. We consider a fluid with constant density ρ and constant kinematic viscosity ν. Let us remind that the dynamic viscosity μ is defined as $\mu = \nu\rho$.
The coupled Stokes-Darcy problem reads as follows:

[7] See Chapter 2 for details
[8] This space is described in the previous Section

- *Stokes problem*

$$\begin{cases} -\nu\triangle(\mathbf{u}_f) + \nabla\left(\dfrac{p_f}{\rho}\right) = \dfrac{\mathbf{f}}{\rho} & in\ \Omega_s \\ \nabla\cdot\mathbf{u}_f = 0 \end{cases} \tag{6.22a}$$

  with classical Stokes boundary conditions on $\partial\Omega_s \cap \partial\Omega$.

- *Darcy problem*

$$\begin{cases} \mathbf{u}_d = -\dfrac{k}{\nu}\nabla\left(\dfrac{p_d}{\rho}\right) & in\ \Omega_d \\ \nabla\cdot\mathbf{u}_d = 0 \end{cases} \tag{6.22b}$$

  with classical Darcy boundary conditions on $\partial\Omega_d \cap \partial\Omega$.

- *Coupling conditions*
  To couple the two problems, the following classical conditions on the interface between the two sub-domains can be considered:

  i) Continuity of fluxes:

$$\mathbf{u}_f\cdot\mathbf{n} = \mathbf{u}_d\cdot\mathbf{n} \quad on\ \Gamma\ , \tag{6.22c}$$

  where $\mathbf{n}$ denotes the unit vector normal to Γ, external to one of the two sub-domains.

  ii) Continuity of stresses:

$$-\frac{\mathbf{T}(\mathbf{u}_f, p_f)\mathbf{n}}{\rho}\cdot\mathbf{n} = \frac{p_d}{\rho} \quad on\ \Gamma\ , \tag{6.22d}$$

  where $\mathbf{T}(\mathbf{u}_f, p_f)$ is the stress tensor

$$\mathbf{T}(\mathbf{u}_f, p_f) = \mu(\nabla\mathbf{u}_f + \nabla^T\mathbf{u}_f) - p_f\mathtt{I}\ ,$$

  where $\mathtt{I}$ denotes the identity tensor.

  iii) Condition of Beavers-Joseph-Staffman (BJS):

$$-\frac{\mathbf{T}(\mathbf{u}_f, p_f)\mathbf{n}}{\rho}\cdot\boldsymbol{\tau} = \frac{\nu\alpha}{\sqrt{k}}\mathbf{u}_f\cdot\boldsymbol{\tau} \quad on\ \Gamma\ , \tag{6.22e}$$

  where α is a dimensionless experimental parameter and $\boldsymbol{\tau}$ denotes the unit vector tangential to Γ.

Dimensionless form of the Coupled problem

We now give a dimensionless form of the coupled problem (6.22), that is derived in [6]. First, let us analyze the dimension of every parameter or function that enters into the problem:

Parameter	*Dimension*
$\mathbf{u}_f$	$[LT^{-1}]$
p_f	$[ML^{-1}T^{-2}]$
$\mathbf{f}$	$[ML^{-2}T^{-2}]$
ν	$[L^2T^{-1}]$
ρ	$[ML^{-3}]$

Parameter	*Dimension*
$\mathbf{u}_d$	$[LT^{-1}]$
p_d	$[ML^{-1}T^{-2}]$
k	$[L^2]$

To write the dimensionless form of the problem, we need to introduce the following quantities:

$$\begin{array}{lll} X_f & \text{characteristic length (Stokes Domain)} & [L] \\ U_f & \text{characteristic velocity (Stokes Domain)} & [LT^{-1}] \\ \Pi_f & \text{characteristic pressure (Stokes Domain)} & [ML^{-1}T^{-2}] \, , \end{array}$$

otaining

$$\begin{aligned} x^* &= \frac{x}{X_f} && \text{dimensionless cartesian coordinate} \\ \mathbf{u}_f^* &= \frac{\mathbf{u}_f}{U_f} && \text{dimensionless Stokes velocity} \\ p_f^* &= \frac{p_f}{\Pi_f} && \text{dimensionless Stokes pressure} \\ \mathbf{u}_s^* &= \frac{\mathbf{u}_s}{U_f} && \text{dimensionless Darcy velocity} \\ p_d^* &= \frac{p_d}{\Pi_f} && \text{dimensionless Darcy pressure} \end{aligned}$$

In order to sempify the expressions of the dimensionless equations, we define also the following dimensionless numbers:

$$\begin{aligned} Re_f &= \frac{U_f X_f}{\nu} && \text{Reynolds Number} \\ E_f &= \frac{\Pi_f}{\rho U_f^2} && \text{Euler Number} \\ N_k &= \frac{k}{X_f^2} && \text{Darcy Number} \\ \tilde{N}_k &= \frac{k U_f}{\nu X_f} = N_k Re_f \\ N_{BJS} &= \frac{\alpha \nu}{\sqrt{k} U_f} = \alpha \left(N_k^{1/2} Re_f \right)^{-1} \end{aligned}$$

Using these notations we can now write the dimensionless form of (6.22).

- *Dimensionless Stokes problem*

$$\begin{cases} -a \triangle_* (\mathbf{u}_f) + \nabla_* p_f^* = \mathbf{f}^* \\ \nabla_* \cdot \mathbf{u}_f^* = 0 \end{cases} \quad in \ \Omega_s \, , \tag{6.23a}$$

  where $a = (Re_f E_f)^{-1}$ and $\mathbf{f}^* = \mathbf{f} X_f / \Pi_f$, with dimensionless Stokes boundary conditions on $\partial\Omega_d \cap \partial\Omega$.

- *Dimensionless Darcy problem*

$$\begin{cases} \mathbf{u}_d^* = -K \nabla_* p_d^* \\ \nabla_* \cdot \mathbf{u}_d^* = 0 \end{cases} \quad in \ \Omega_d \, , \tag{6.23b}$$

  where $K = Re_f E_f N_k$, with dimensionless Darcy boundary conditions on $\partial\Omega_d \cap \partial\Omega$.

- *Dimensionless Coupling conditions*

  i) Continuity of fluxes:

$$\mathbf{u}_f^* \cdot \mathbf{n} = \mathbf{u}_d^* \cdot \mathbf{n} \quad on\ \Gamma\ , \tag{6.23c}$$

  where $\mathbf{n}$ denotes the unit vector normal to Γ, external to one of the two sub-domains.

  ii) Continuity of stresses:

$$-\left(\left(a(\nabla_* \mathbf{u}_f^* + \nabla_*^T \mathbf{u}_f^*) - p_f^* \mathtt{I}\right)\mathbf{n}\right) \cdot \mathbf{n} = p_d^* \quad on\ \Gamma\ . \tag{6.23d}$$

  iii) Condition of Beavers-Joseph-Staffman (BJS):

$$-\left(\left(a(\nabla_* \mathbf{u}_f^* + \nabla_*^T \mathbf{u}_f^*) - p_f^* \mathtt{I}\right)\mathbf{n}\right) \cdot \boldsymbol{\tau} = N_{BJS} E_f^{-1} \mathbf{u}_f^* \cdot \boldsymbol{\tau} \quad on\ \Gamma\ . \tag{6.23e}$$

In the following Sections we will refer to the dimensionless form of the problem, but, for simplicity of notation, we will avoid using $*$ to indicate dimensionless quantities.

6.4 Formulation of ICDD methods for Stokes-Darcy Coupling

While in the previous Section we presented the classical formulation of Stokes-Darcy coupling, in this Section we present two different ICDD methods proposed in [10] to numerically solve Stokes-Darcy coupling. There is no theory on the well-posedness of the ICDD problems in these cases, therefore we try to apply these two methods in order to test whether they could be good methods or not.
The analysis of the numerical results obtained by solving the coupled problem with these methods are presented in the next Section.
We consider a domain Ω with a Lipschitz boundary $\partial\Omega$ such that $\overline{\partial\Omega} = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ with $\Gamma_D \cap \Gamma_N = \emptyset$, split into two overlapping sub-domains Ω_s (the Stokes domain) and Ω_d (the Darcy domain), as shown in Figure 6.4. The overlapping region Ω_{SD} is an intermediate region where neither Stokes nor Darcy law is thought to be completely valid. We define also the two internal interfaces as

$$\Gamma_s = \partial\Omega_s \setminus (\partial\Omega_s \cap \partial\Omega)$$

and

$$\Gamma_d = \partial\Omega_d \setminus (\partial\Omega_d \cap \partial\Omega)\ ,$$

and the external Neumann and Dirichlet boundaries of the two sub-domains as

$$\Gamma_D^s = \partial\Omega_s \cap \Gamma_D \quad \Gamma_N^s = \partial\Omega_s \cap \Gamma_N$$

and

$$\Gamma_D^d = \partial\Omega_d \cap \Gamma_D \quad \Gamma_N^d = \partial\Omega_d \cap \Gamma_N\ .$$

The two ICDD methods we will consider are methods with interface control and interface observation. They differs in the cost functional to be minimized and in the kind of

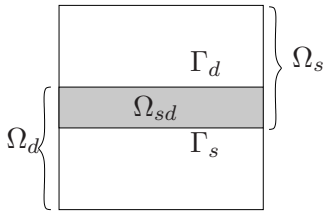


Figure 6.4: Domain partition with overlapping sub-domains

controls used. Let us consider the dimensionless formulation of Stokes and Darcy problems on Ω_s and on Ω_d respectively. We use the notation introduced in the previous Section, denoting with the subscripts f and d the quantities relatives to Stokes and Darcy problem, respectively. Let us introduce these two methods separately.

J_t method

We consider a Dirichlet interface control in the case of Stokes problem and a Neumann interface control in the case of Darcy problem. Moreover, we minimize the following cost functional:

$$J_t(\boldsymbol{\lambda}) = \frac{1}{2}\|\mathbf{u}_f - \mathbf{u}_d\|^2_{L^2(\Gamma_s)} + \frac{1}{2}\|(\mathbf{u}_f - \mathbf{u}_d)\cdot\mathbf{n}\|^2_{L^2(\Gamma_d)}\,, \tag{6.24}$$

where

$$\boldsymbol{\lambda} = \begin{bmatrix}\boldsymbol{\lambda}_s \\ \lambda_d\end{bmatrix}\,,$$

and $\boldsymbol{\lambda}_s$ and λ_d are the control functions on Γ_s and on Γ_d respectively.
The optimality system proposed in [10] for this first method is the following:

- *Primal Problems*

$$\begin{cases} -a\triangle\mathbf{u}_f + \nabla p_f = \mathbf{f} & in\ \Omega_s \\ \nabla\cdot\mathbf{u}_f = 0 & in\ \Omega_s \\ \mathbf{u}_f = \boldsymbol{\lambda}_s & on\ \Gamma_s \\ + b.c. & on\ \Gamma_D^s \cup \Gamma_N^s \end{cases} \tag{6.25a}$$

  and

$$\begin{cases} -\nabla\cdot(K\nabla p_d) = 0 & in\ \Omega_d \\ K\dfrac{\partial p_d}{\partial n} = \lambda_d & on\ \Gamma_d \\ + b.c. & on\ \Gamma_D^d \cup \Gamma_N^d \end{cases}, \quad \mathbf{u}_d = -K\nabla p_d \quad in\ \Omega_d\,. \tag{6.25b}$$

- *Adjoint Problems*

$$\begin{cases} -a\triangle\mathbf{v}_f + \nabla q_f = \mathbf{0} & in\ \Omega_s \\ \nabla\cdot\mathbf{v}_f = 0 & in\ \Omega_s \\ \mathbf{v}_f = \mathbf{u}_f - \mathbf{u}_d & on\ \Gamma_s \\ + homogeneous\ b.c. & on\ \Gamma_D^s \cup \Gamma_N^s \end{cases} \tag{6.25c}$$

and

$$\begin{cases} -\nabla\cdot(K\nabla q_d) = 0 & in\ \Omega_d \\ K\dfrac{\partial q_d}{\partial n} = (\mathbf{u}_f - \mathbf{u}_d)\cdot\mathbf{n} & on\ \Gamma_d \\ +\ homogeneous\ b.c. & on\ \Gamma_D^d \cup \Gamma_N^d \end{cases}, \quad \mathbf{v}_d = -K\nabla q_d \quad in\ \Omega_d\,. \tag{6.25d}$$

- *Optimality Conditions*

$$(\mathbf{u}_f - \mathbf{u}_d) - \mathbf{v}_d = \mathbf{0} \quad on\ \Gamma_s \tag{6.25e}$$

and

$$(\mathbf{u}_f - \mathbf{u}_d)\cdot\mathbf{n} - \mathbf{v}_f\cdot\mathbf{n} = 0 \quad on\ \Gamma_d\,. \tag{6.25f}$$

J_f method

Here we consider Neumann interface controls for both Stokes and Darcy problems. The idea of this second method is to impose the continuity of the stresses on Γ_s and the continuity of the normal component of the velocity on Γ_d. Therefore, in this case, we minimize the following cost functional:

$$J_f(\boldsymbol{\lambda}) = \frac{1}{2}\|\nu\frac{\partial\mathbf{u}_f}{\partial n} - p_f\mathbf{n} + p_d\mathbf{n}\|^2_{L^2(\Gamma_s)} + \frac{1}{2}\|(\mathbf{u}_f - \mathbf{u}_d)\cdot\mathbf{n}\|^2_{L^2(\Gamma_d)}\,. \tag{6.26}$$

The optimality system proposed in [10] for this second method is the following:

- *Primal Problems*

$$\begin{cases} -a\triangle\mathbf{u}_f + \nabla p_f = \mathbf{f} & in\ \Omega_s \\ \nabla\cdot\mathbf{u}_f = 0 & in\ \Omega_s \\ \nu\dfrac{\partial\mathbf{u}_f}{\partial n} - p_f\mathbf{n} = \boldsymbol{\lambda}_s & on\ \Gamma_s \\ +\ b.c. & on\ \Gamma_D^s \cup \Gamma_N^s \end{cases} \tag{6.27a}$$

and

$$\begin{cases} -\nabla\cdot(K\nabla p_d) = 0 & in\ \Omega_d \\ K\dfrac{\partial p_d}{\partial n} = \lambda_d & on\ \Gamma_d \\ +\ b.c. & on\ \Gamma_D^d \cup \Gamma_N^d \end{cases}, \quad \mathbf{u}_d = -K\nabla p_d \quad in\ \Omega_d\,. \tag{6.27b}$$

- *Adjoint Problems*

$$\begin{cases} -a\triangle\mathbf{v}_f + \nabla q_f = \mathbf{0} & in\ \Omega_s \\ \nabla\cdot\mathbf{v}_f = 0 & in\ \Omega_s \\ \nu\dfrac{\partial\mathbf{v_f}}{\partial n} - q_f\mathbf{n} = \nu\dfrac{\partial\mathbf{u_f}}{\partial n} - p_f\mathbf{n} + p_d\mathbf{n} & on\ \Gamma_s \\ +\ homogeneous\ b.c. & on\ \Gamma_D^s \cup \Gamma_N^s \end{cases} \tag{6.27c}$$

and

$$\begin{cases} -\nabla\cdot(K\nabla q_d) = 0 & in\ \Omega_d \\ K\dfrac{\partial q_d}{\partial n} = (\mathbf{u}_f - \mathbf{u}_d)\cdot\mathbf{n} & on\ \Gamma_d \\ +\ homogeneous\ b.c. & on\ \Gamma_D^d \cup \Gamma_N^d \end{cases}, \quad \mathbf{v}_d = -K\nabla q_d \quad in\ \Omega_d\,. \tag{6.27d}$$

- *Optimality Conditions*

$$(\nu\frac{\partial \mathbf{u}_f}{\partial n} - p_f\mathbf{n} + p_d\mathbf{n}) - q_d\mathbf{n} = \mathbf{0} \quad on\ \Gamma_s \tag{6.27e}$$

and

$$(\mathbf{u}_f - \mathbf{u}_d)\cdot\mathbf{n} - \mathbf{v}_f\cdot\mathbf{n} = 0 \quad on\ \Gamma_d\ . \tag{6.27f}$$

These two optimality systems are discretized and solved in an analogous way as done in Chapter 5.

6.5 Numerical Results for Stokes-Darcy Coupling with ICDD methods

In this Section we analize the numerical behaviour of the proposed ICDD methods.
First we analyze the two methods on two different test cases with known reference solution, then we analyze the behaviour of the numbers of iterations required by these methods varying the overlap thickness, and finally we comment their consistence with respect to two significant physical situations.

6.5.1 Errors computation and number of iterations in two test cases

We analyze the behaviour of the two ICDD methods proposed both in terms of iterations and of errors with respect to two reference solutions when the grid size h tends to zero. We split the analysis into two parts, each corresponding to one of the two methods. The reference solutions in the two cases are built in order to exactly solve these problems with the proposed methods.

- **J_t method**
  Let us consider a test case with a reference solution on the domain $\Omega = (0,1)\times(0,2)$. The domain is split into two overlapping sub-domains and the overlap thickness in δ. Stokes domain is $\Omega_s = (0,1)\times(1-\delta/2, 2)$ while Darcy domain is $\Omega_d = (0,1)\times(0,1+\delta/2)$.
  The exact solution of the problem is:

$$u_1 = -Ke^x sin(y)$$

$$u_2 = -Ke^x cos(y)$$

$$p = e^x sin(y)$$

  where K is the dimensionless Darcy coefficient. We impose Neumann conditions on vertical edges of the domain, as shown in Figure 6.5. We denote by D and N the Dirichlet and Neumann conditions, respectively.

  We have fixed the overlap thickness δ at 10% of the size of the whole domain in y-direction, namely $\delta = 0.2$, and have decreased the mesh size h. This test has been made for different values of the dimensionless Stokes and Darcy coefficients a and K, in order to test the robustness of the method with respect to the coefficients of the problem.
  In the following table there are the coefficients used in the different cases. The definitions of both a and K are given in Section 6.3.

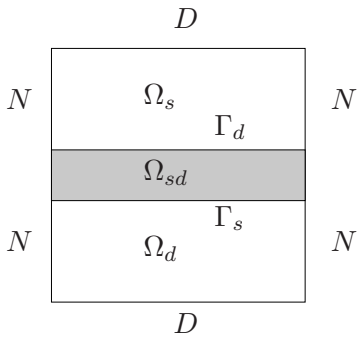


Figure 6.5: J_t method : boundary conditions

	a	K
Case 1	2	$2 \cdot 10^4$
Case 2	2	$2 \cdot 10^2$
Case 3	$2 \cdot 10^{-1}$	$2 \cdot 10^2$
Case 4	$2 \cdot 10^{-3}$	$2 \cdot 10^3$

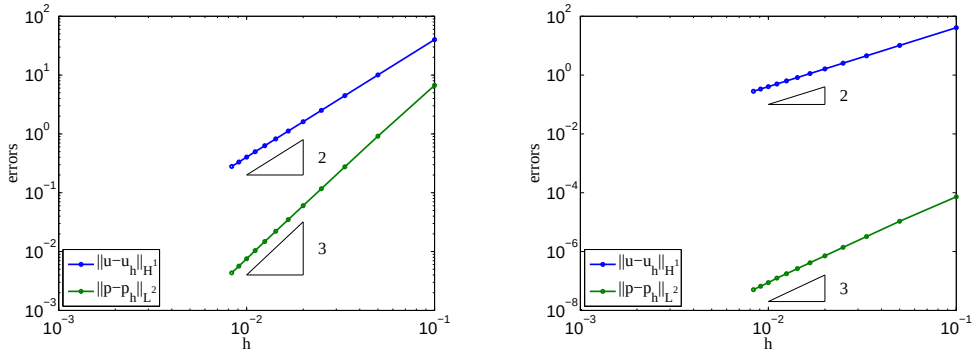
We notice that the number of iterations required by the proposed ICDD method does not depend on the coefficients. The number of iterations required, when $\delta = 0.2$, is written in the following tables:

h	1/10	1/20	1/30	1/40	1/50	1/60
number of iterations	13	10	10	9	9	9

h	1/70	1/80	1/90	1/100	1/110	1/120
number of iterations	9	9	9	8	8	8

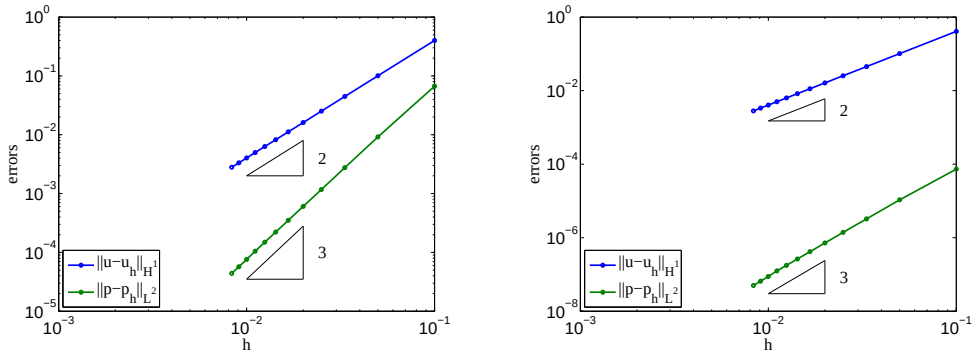
We can note that the number of iterations slightly decreases as the grid size h tends to zero. Nevertheless, even if the mesh size is large, the number of iterations required is low.

We can also observe that, once the overlap thickness is fixed, the errors of the computed solution, with respect to the reference one, decrease in accordance with the theory of finite element for the two differential problems (Stokes and Elliptic) (see Figures 6.6 - 6.9). In particular, we can note that in the case of Stokes subproblem, the errors behaviours satisfy the errors estimates (6.12), while in the case of Darcy subproblem, the finite elements estimate are satisfied. On the other hand, once fixed the mesh size, the errors increase as the overlap size decreases.



(a) Stokes domain (b) Darcy domain

Figure 6.6: Case 1 - J_t method : fixed overlap $\delta = 0.2$



(a) Stokes domain (b) Darcy domain

Figure 6.7: Case 2 - J_t method : fixed overlap $\delta = 0.2$

- **J_f method**
  Let us consider the same domain of the previous test case, and let us now consider the problem with exact solution on Stokes domain:

$$u_{f1} = -Ke^x sin(y - 0.9 + \pi/2)$$

$$u_{f2} = -Ke^x cos(y - 0.9 + \pi/2)$$

$$p_f = e^x sin(y - 0.9 + \pi/2)(1 + aK)$$

  and on Darcy domain:

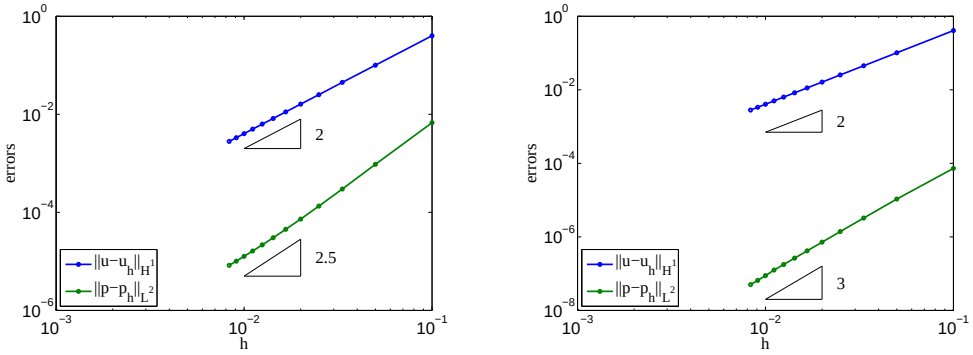
$$u_{d1} = -Ke^x sin(y - 0.9 + \pi/2)$$

$$u_{d2} = -Ke^x cos(y - 0.9 + \pi/2)$$

$$p_d = e^x sin(y - 0.9 + \pi/2)$$

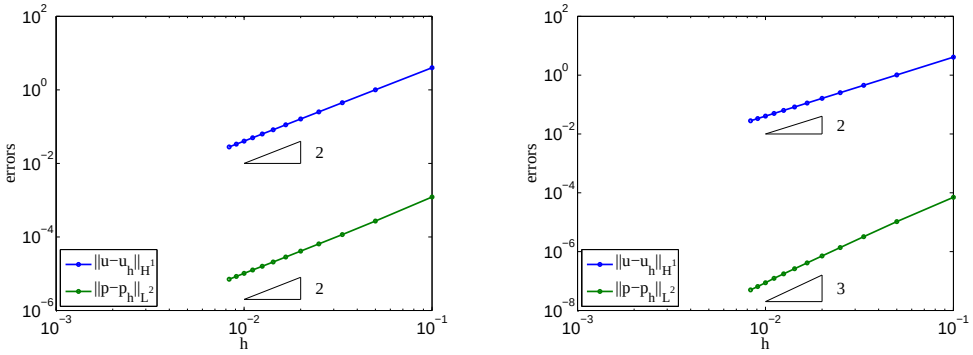
  where K and a are the dimensionless Darcy and Stokes coefficient, respectively. We impose boundary conditions as shown in Figure 6.10.

  As in the previous test case, we have fixed the overlap thickness δ at 10% of the size of the whole domain in y-direction, namely $\delta = 0.2$, and have decreased the mesh size h. In this case we report the results obtained for the following values of the dimentionless parameters:



(a) Stokes domain (b) Darcy domain

Figure 6.8: Case 3 - J_t method : fixed overlap $\delta = 0.2$



(a) Stokes domain (b) Darcy domain

Figure 6.9: Case 4 - J_t method : fixed overlap $\delta = 0.2$

a	K
$2 \cdot 10^{-3}$	$2 \cdot 10^{3}$

The number of iterations required, when $\delta = 0.2$, is written in the following tables:

h	1/10	1/20	1/30	1/40	1/50	1/60
number of iterations	9	8	8	8	8	7

h	1/70	1/80	1/90	1/100	1/110	1/120
number of iterations	7	7	7	7	6	6

We can note that, also in this case, the number of iterations decreases a little as the grid size h tends to zero. On the other hand, even if the mesh size is large, the number of iterations required is low.

We can also observe that, also with this method, once fixed the overlap thickness, the errors of the computed solution with respect to the reference one decrease in accordance with the theory of finite element for the two differential problems (Stokes and Elliptic) (see Figure 6.11).

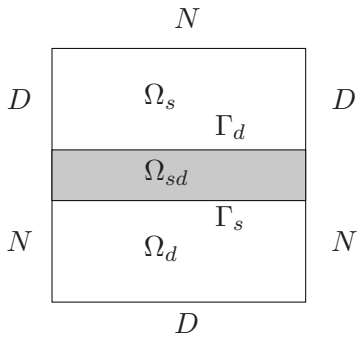
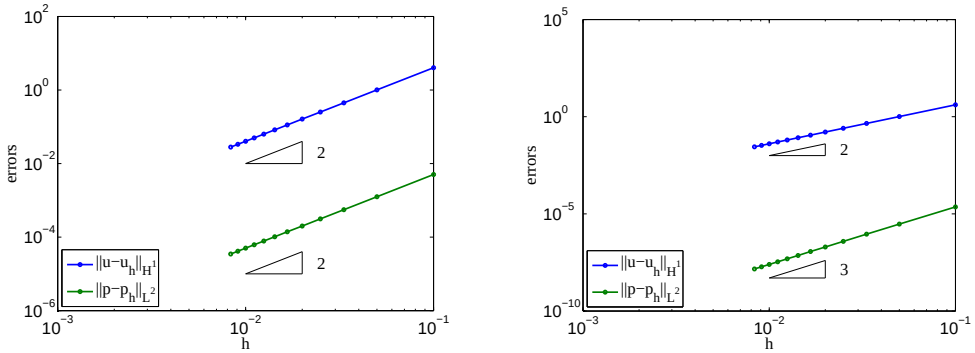


Figure 6.10: J_f method : boundary conditions



(a) Stokes domain (b) Darcy domain

Figure 6.11: J_f method : fixed overlap $\delta = 0.2$

6.5.2 Number of iterations required versus the overlap thickness

We are interested in analyzing the number of iterations required as the overlap thickness tends to zero. Let us consider two different test cases (test cases 1 and 2). In every case the domain is $\Omega = (0,1) \times (0,2)$. The real interface of the problem is set at $y = 1$ and, if δ is the overlap thickness, the Γ_s and Γ_d are set at $y = 1 - \delta/2$ and $y = 1 - \delta/2$, respectively. The coefficients of the problem are fixed to $a = 0.002$ and $K = 2000$ and the mesh size is fixed to $h = 0.02$.

- **Test case 1**
  In this first case, let us consider the boundary conditions shown in Figure 6.12, where $\mathbf{g} = (g_1, 0)$,

$$g_1(x,y) = \begin{cases} \dfrac{(0.4)^2 - (y-1.6)^2}{(0.4)^2} & for\ 1.2 < y < 2 \\ 0 & otherwise \end{cases}.$$

  In this case, the number of iterations required by the two methods to solve the problem are

δ	1/5	4/25	3/25	2/25	1/25
number of iterations J_t	16	16	18	22	32
number of iterations J_f	13	13	13	14	14

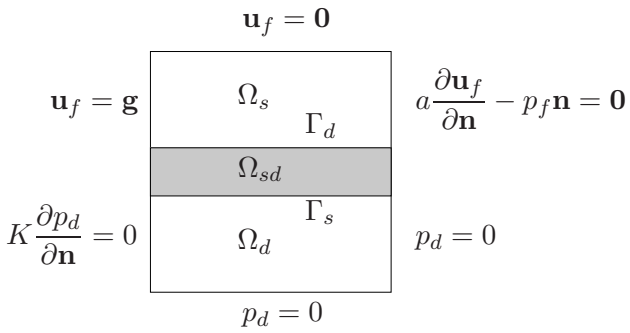


Figure 6.12: Test Case 1 - boundary conditions

We can note that the number of iterations required by J_f method is almost independent of the overlap thickness, while the number of iterations of J_t method increases as $\delta^{-1/2}$ when δ tends to zero.

- **Test case 2**
  In this case, we imposed the boundary conditions shown in Figure 6.13, where $\mathbf{g} = (0, g_2)$,

$$g_2(x, y) = \frac{(x - 0.5)^2 - (0.5)^2}{(0.5)^2} .$$

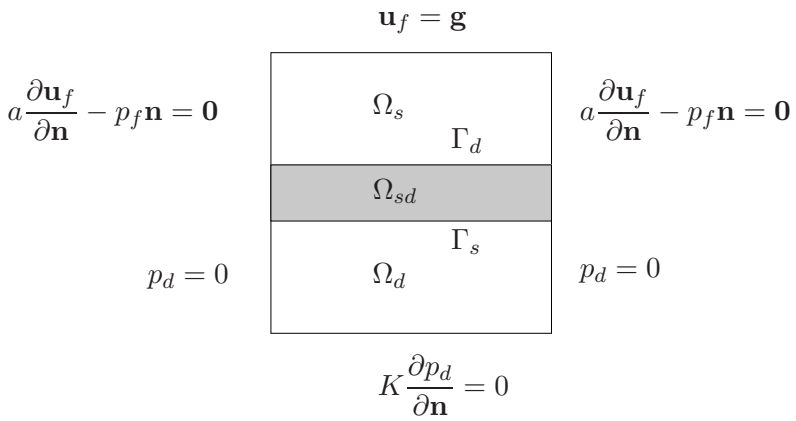


Figure 6.13: Test Case 2 - boundary conditions

In this second case the numbers of iterations required are the following:

δ	1/5	4/25	3/25	2/25	1/25
number of iterations J_t	17	17	20	21	29
number of iterations J_f	15	15	15	15	14

Also in this second case the considerations made above are still valid.

Conclusions:
In conclusion, we can say that the number of iterations of J_f method is independent of the overlap thickness and it is very low, while that required by J_t increases as $\delta^{-1/2}$ when the overlap thickness tends to zero.

6.5.3 Consistence of Solutions Computed by ICDD Methods

We want to understand if the proposed ICDD methods are consistent. More precisely, we compare the solution of the ICDD method to that of classical Stokes-Darcy problem 6.22[9]. In order to reach this goal, we solved two problems with meaningful boundary conditions (test cases 3 and 4). In each of these cases, the physical interface between the two sub-domains is set at $y = 1$, while Γ_s and Γ_d are set at $y = 0.9$ and $y = 1.1$, respectively.

- **Test case 3**
  We solved the coupled Stokes-Darcy problem in $\Omega = (0, 2) \times (0, 2)$, with $\mathbf{f} = \mathbf{0}$ and with the boundary conditions given as in Figure 6.14.

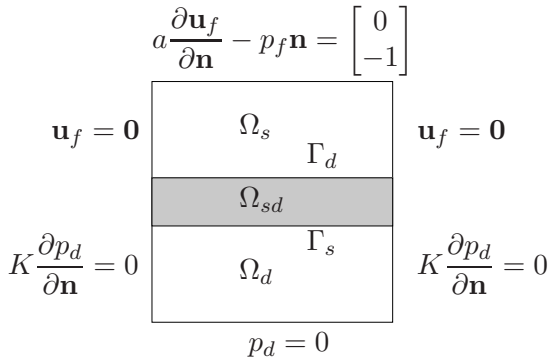
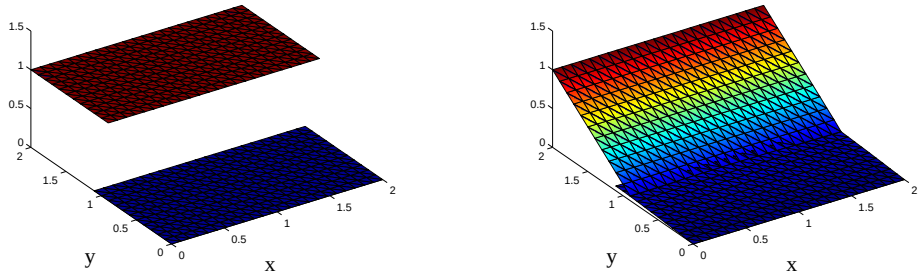


Figure 6.14: Test Case 3 - boundary conditions



(a) J_t method (b) J_f method

Figure 6.15: Test Case 3 - pressure

In Figures 6.15 and 6.16 we plot the computed solutions using both J_t and J_f methods.
We can notice that the solution computed using J_t method is meaningless. Actually the pressure is constant in each of the two sub-domains and the velocity field is identically zero. On the other hand, the solution computed using J_f method seems to be similar to what we expect.
Therefore we compare this solution with the one obtained with the classical method

[9]The numerical simulations of the classical Stokes-Darcy coupling have been provided by Professor Discacciati Marco, Universitat Politècnica de Catalunya. BarcelonaTech.

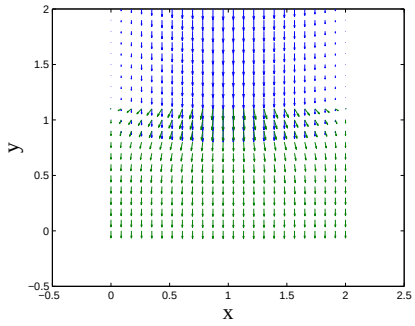
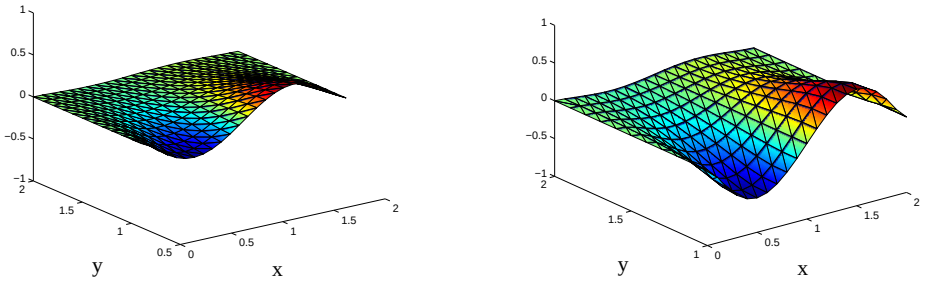
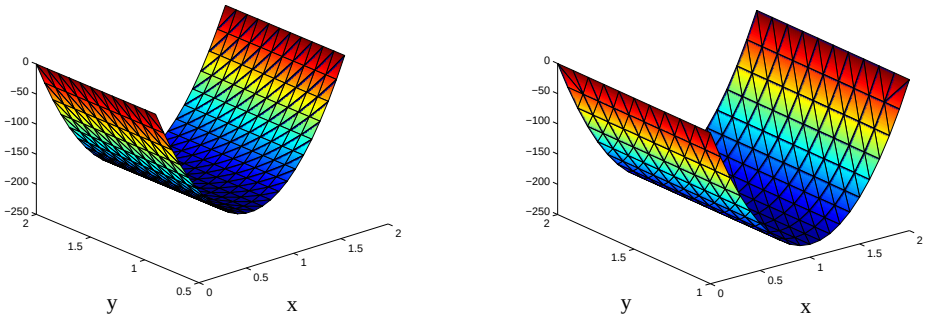


Figure 6.16: Test Case 3 - velocity - J_f method



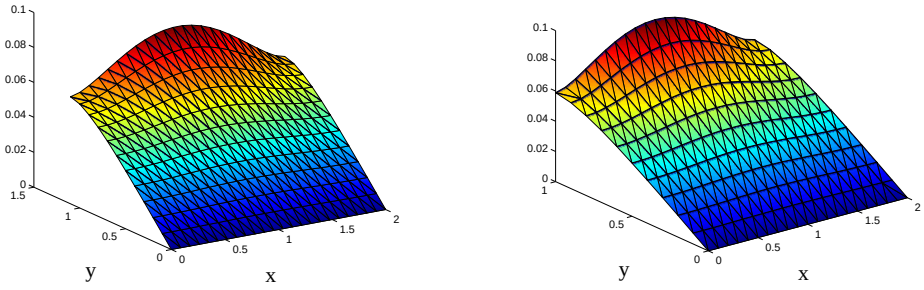
(a) J_f method (b) Classical coupling

Figure 6.17: Test Case 3 - Stokes velocity field (first component)



(a) J_f method (b) Classical coupling

Figure 6.18: Test Case 3 - Stokes velocity field (second component)



(a) J_f method (b) Classical coupling

Figure 6.19: Test Case 3 - Darcy pressure

with sharp interface (Figures 6.17-6.19). We note that the velocity fields computed in the two cases are comparable, even if it is slightly larger (in absolute value) in the case of the method with sharp interface. On the other hand, the pressures are almost equal in the two cases.
These differences can depend on the fact that the two problems are not fully equivalent. As a matter of fact we recall that the two formulations impose different boundary conditions on the internal boundary (the interface) of the Stokes sub-domains and that the position of this interface is not the same for the two methods. Moreover the ICDD solution could depend on the overlap thickness. A more in depth analysis should be carried out versus both the overlap thickness and the mesh size.

- **Test case 4**
  Consider the Stokes-Darcy coupling in the domain $\Omega = (0,5) \times (0,2)$, with $\mathbf{f} = \mathbf{0}$ and with the boundary conditions shown in figure 6.20

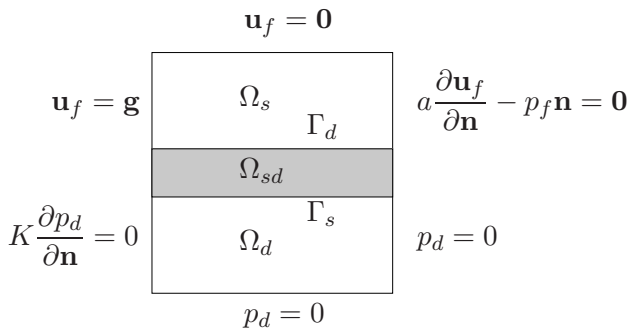
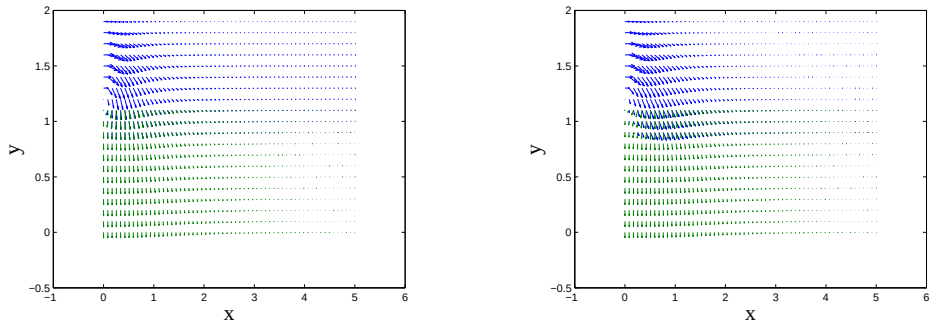


Figure 6.20: Test Case 4 - boundary conditions

where $\mathbf{g} = (g_1, 0)$,

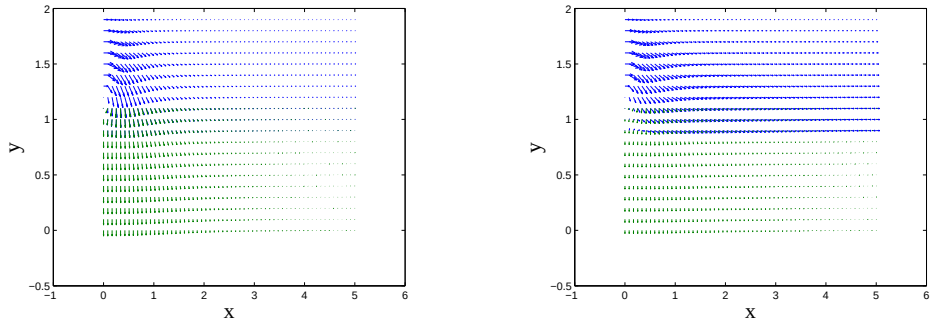
$$g_1(x,y) = \begin{cases} \dfrac{(0.4)^2 - (y-1.6)^2}{(0.4)^2} & for\ 1.2 < y < 2 \\ 0 & otherwise\ . \end{cases}$$

Moreover, we have chosen $\nu = 10^{-6} m^2/s$ and therefore the dimentionless coefficients a is $a = 0.002$.



(a) J_t method (b) J_f method

Figure 6.21: Test Case 4 - $K = 2000$

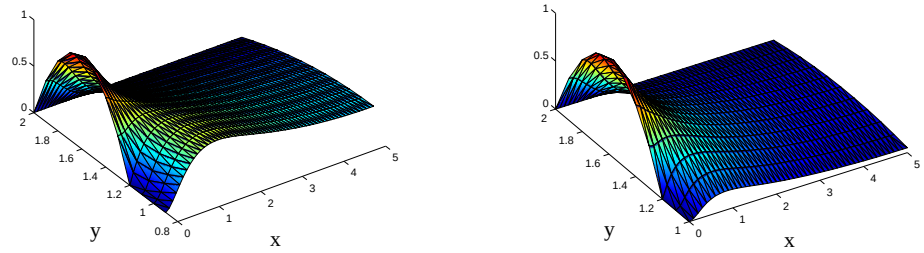


(a) J_t method (b) J_f method

Figure 6.22: Test Case 4 - $K = 20$

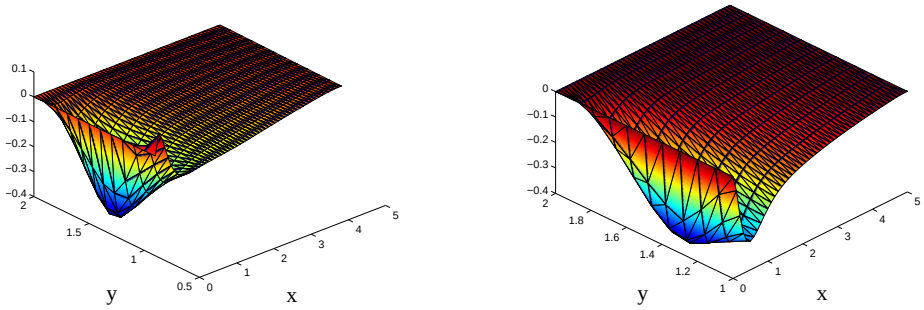
In Figures 6.21 and 6.22 we plot the velocity field computed with J_t and J_f for two different values of the coefficient K.
We can notice that the velocity field does not change for different values of K in the case of J_t method, and therefore its behaviour does not respect the physics of the problem. The analysis of the velocity field computed using J_f method is more complex. Actually in this case on one hand the velocity field changes in accordance with the physics of the problem solved because the flux that enters into the porous medium is greater when the permeability coefficient is higher. On the other hand, in the case of J_f method, it is difficult to choose the value of the velocity field on the overlapping region because the computed fields into the two sub-domains have different values on it. Moreover we can note in Figure 6.22 that, when $K = 20$, the tangential component of the Stokes velocity field with respect to the real interface, set at $y = 1$, is significantly larger than zero.

Let us compare the solution computed with the second ICDD method proposed to the one obtained with the classical Stokes-Darcy coupling.
If we choose $K = 2000$, the two solutions are almost equal in each sub-domain. On the other hand, when we set $K = 20$ (Figures 6.23-6.25) the two solutions are significantly different. Actually the Darcy pressure obtained with J_f method is smaller with respect to the other one, even if the qualitative behaviour is the same. Moreover the second component of the Stokes velocity computed using J_f method is smaller (in absolute value) then the one obtained with the classical method. The first com-



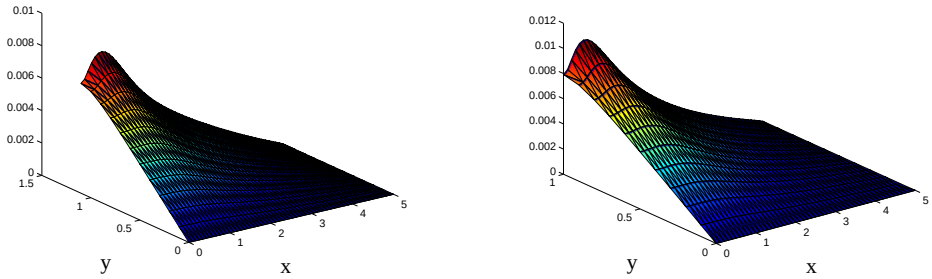
(a) J_f method (b) Classical coupling

Figure 6.23: Test Case 4 - $K = 20$ - Stokes velocity field (first component)



(a) J_f method (b) Classical coupling

Figure 6.24: Test Case 4 - $K = 20$ - Stokes velocity field (second component)



(a) J_f method (b) Classical coupling

Figure 6.25: Test Case 4 - $K = 20$ - Darcy pressure

ponent of the velocity field is larger in the first case with respect to the second one, both near the right boundary and in the whole Stokes domain.
We believe that these differences are due to the fact that J_f method in this specific case[10] tries to minimize the tangential component of the stresses on the Stokes interface, without imposing any restrictions on the value of the tangential component of the velocity field with respect to this interface, while the classical coupling makes this component to be related to the tangential stresses by means of BJS coupling condition.
Moreover, we should made these tests with different overlap thicknesses and mesh sizes in order to better understand the behaviour of J_f method.
We should also compare the solutions obtained with J_f method in meaningful physical situations with experimental data.

Conclusions:
In conclusion, according to the numerical results obtained, J_t does not seem to be suitable because it gives a velocity field independent of the permeability of the porous medium and it does not solve adequately the test case 3. This functional, in particular, considers only the difference between the velocity fields and their normal components on the interface, without considering the stresses at the interfaces and therefore the pressures are allowed to be independent of each other.
On the other hand, the numerical simulations made don't allow us to conclude whether J_f gives a solution in accordance with the real solution of the problem. Actually, on one hand, the solution of the test case 3 seems to be a good solution, and the velocity fields (as we can see in test case 4) depend on the permeability coefficient of Darcy law. On the other hand, the numerical solution obtained with $\delta = 0.2$, $h = 0.1$ and $K = 20$ is quite different from the solution computed using the classical Stokes-Darcy coupling with sharp interface.
In order to better understand the pro's and con's of J_f method, and their fields of validity, we should understand what it happens for different overlaps, mesh sizes, boundary conditions and coefficients. Moreover, in order to obtain more reliable conclusions on J_f method, we should compare the solutions obtained using this method with experimental data in meaningful physical situations. We believe that only with these comparisons we could better understand the reliability of J_f method.

[10] the normal vector to Γ_1, external to the Stokes domain, is $\mathbf{n} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Conclusions

In this thesis we analyzed interface control domain decomposition methods both analytically, when possible, and numerically.
In particular we proved the well-posedness of ICDD methods applied to the solution of elliptic problems, and we presented two ICDD methods to solve Stokes-Darcy coupling.
In the case of elliptic problems, we compared the different ICDD methods in terms of numbers of iterations required both fixing the overlap thickness or the mesh size. We made these analyses with both continuous and discontinuous coefficients, in order to understand also the robustness of these methods. We can conclude that the best method seems to be $J_{0,\Gamma}$ in the case of Dirichlet interface control, both in its multiplicative and additive version. Nevertheless, a valid alternative would be $J_{0,\Gamma}(2)$ in the case of Robin interface controls, with high β. Actually their numbers of iterations are constant when the overlap is fixed and the mesh size tends to zero, and increase a little when the mesh size is fixed and the overlap thickness decreases. Moreover, their numbers of iterations are very small both with continuous and discontinuous coefficients. In particular, we can note that the best methods are those with interface observation.
On the other hand, in the case of Stokes-Darcy coupling, we considered two different ICDD methods. After having compared the numbers of iterations required by these methods, we analyzed the solutions obtained with these methods in two significant physical problems. In particular we can conclude that the method based on the functional J_t is not valid because the solutions obtained don't respect the physics of the problem. On the other hand, the numerical simulations made don't allow us to conclude whether J_f gives a solution in accordance with the real solution of the problem. We would need more cases and we should compare the solutions obtained using this method with experimental data.
In conclusion, we found some valid methods to solve elliptic problems, and we excluded J_t method for the solution of Stokes-Darcy coupling. The analysis of J_f method is more complex and should be done with other external data. Therefore for this method we can say only that its behaviour in terms of numbers of iterations is good, but we should analyze whether it solves the problem in accordance with the physics of the problem.

Appendix A

Iterative Methods to solve linear systems

In this Appendix we want briefly to present some iterative methods used to solve algebraic linear systems and their properties in terms of convergence.
Given a linear system

$$A\mathbf{y} = \mathbf{f} \,, \tag{A.1}$$

the idea of an iterative method is to built a sequence $\{\mathbf{y}^{(k)}\}$ that converges to the exact solution $\mathbf{y}$ of the given system (A.1).
Let us introduce the error $\mathbf{e}^{(k)}$ at the k^{th} iteration with respect to the exact solution:

$$\mathbf{e}^{(k)} = \mathbf{y}^{(k)} - \mathbf{y} \,.$$

For any symmetric and positive definite matrix A, let us also define the A-norm of a vector $\mathbf{b} \in \mathbb{R}^n$ as:

$$\|\mathbf{v}\|_A = \left(\sum_{i,j=1}^{n} v_i a_{ij} v_j \right)^{1/2} .$$

A.1 Richardson and Preconditioned Gradient Methods

Let us introduce the residual $\mathbf{r}^{(k)}$ at the k^{th} iteration

$$\mathbf{r}^{(k)} = \mathbf{f} - A\mathbf{y}^{(k)} \,,$$

and a preconditioning matrix P. Richardson methods are iterative methods that compute the solution at the k^{th} iteration in the following way:

$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \alpha_k P^{-1} \mathbf{r}^{(k)} \,,$$

with α_k a parameter to be chosen.
If both A and P are symmetric positive definite matrices, we can define the Preconditioned Gradient method(PG) as a particular Richardson method, where

$$\alpha_k = \frac{{\mathbf{z}^{(k)}}^T \mathbf{r}^{(k)}}{{\mathbf{z}^{(k)}}^T A \mathbf{z}^{(k)}} \,.$$

Therefore, PG method reads as follows:
given $\mathbf{y}^{(0)}$, compute $\mathbf{r}^{(0)} = \mathbf{f} - A\mathbf{y}^{(0)}$ and $\mathbf{z}^{(0)} = P^{-1}\mathbf{r}^{(0)}$. Then, for $k = 0, 1, ...$

$$\alpha_k = \frac{{\mathbf{z}^{(k)}}^T \mathbf{r}^{(k)}}{{\mathbf{z}^{(k)}}^T A\mathbf{z}^{(k)}}$$
$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \alpha_k \mathbf{z}^{(k)}$$
$$\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} - \alpha_k A\mathbf{z}^{(k)}$$
$$P\mathbf{z}^{(k+1)} = \mathbf{r}^{(k+1)}$$

while convergence is reached.
The following Theorem holds. This Theorem can be found in [17].

Theorem 21. *Let A and P be symmetric and positive definite matrices. Then the Gradient Method converges for any value of the initial guess $\mathbf{y}^{(0)}$. Moreover,*

$$\|\mathbf{e}^{(k)}\|_A \leqslant \left(\frac{\kappa - 1}{\kappa + 1}\right)^k \|\mathbf{e}^{(0)}\|_A \ , \quad k = 0, 1, ... \ , \tag{A.2}$$

where $\kappa = K_2(P^{-1}A) \geqslant 1$.

A.2 Preconditioned Conjugate Gradient Method(PCG)

Let us consider the linear system (A.1), with A symmetric and positive definite matrix. Let P be a symmetric positive definite preconditioning matrix. PCG method is equivalent to Conjugate Gradient method applied to the following preconditioned problem:

$$P^{-1}A\mathbf{y} = P^{-1}\mathbf{f} \ .$$

PCG scheme reads as follows:
given $\mathbf{y}^{(0)}$, compute $\mathbf{r}^{(0)} = \mathbf{f} - A\mathbf{y}^{(0)}$, $\mathbf{z}^{(0)} = P^{-1}\mathbf{r}^{(0)}$ and $\mathbf{p}^{(0)} = \mathbf{z}^{(0)}$. Then, for $k = 0, 1, ...$

$$\alpha_k = \frac{{\mathbf{p}^{(k)}}^T \mathbf{r}^{(k)}}{{(A\mathbf{p}^{(k)})}^T \mathbf{p}^{(k)}}$$
$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \alpha_k \mathbf{p}^{(k)}$$
$$\mathbf{r}^{(k+1)} = \mathbf{r}^{(k)} - \alpha_k A\mathbf{p}^{(k)}$$
$$P\mathbf{z}^{(k+1)} = \mathbf{r}^{(k+1)}$$
$$\beta_k = \frac{{(A\mathbf{p}^{(k)})}^T \mathbf{z}^{(k+1)}}{{\mathbf{p}^{(k)}}^T A\mathbf{p}^{(k)}}$$
$$\mathbf{p}^{(k+1)} = \mathbf{z}^{(k+1)} - \beta_k \mathbf{p}^{(k)}$$

while convergence is reached.
The following Theorem, that can be found in [19], holds:

Theorem 22. *Let A and P be symmetric and positive definite matrices. Then the Preconditioned Conjugate Gradient Method converges for any value of the initial guess $\mathbf{y}^{(0)}$. Moreover,*

$$\|\mathbf{e}^{(k)}\|_A \leqslant 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^k \|\mathbf{e}^{(0)}\|_A \ , \quad k = 0, 1, ... \ , \tag{A.3}$$

where $\kappa = K_2(P^{-1}A) \geqslant 1$.

A.3 Number of Iterations Required by PG and PCG methods

If we want to find the solution of the system (A.1) with a prefixed tolerance $0 < \varepsilon < 1$:

$$\mathbf{y}^{(it)} \ : \ \|\mathbf{e}^{(it)}\|_A \leqslant \varepsilon \|\mathbf{e}^{(0)}\|_A \ ,$$

the number of iterations needed to reach this tolerance depends on the error estimates (A.2) and (A.3). They are both of the form

$$\|\mathbf{e}^{(k)}\|_A \leqslant 2\left(\frac{\eta-1}{\eta+1}\right)^k \|\mathbf{e}^{(0)}\|_A \ , \quad k = 0, 1, \ldots \ ,$$

where η is equal to κ in the first case and equal to $\sqrt{\kappa}$ in the second. The number of iteration is the smallest integer greater than i such that (if $\eta > 1$):

$$2\left(\frac{\eta-1}{\eta+1}\right)^i \geqslant \varepsilon$$

$$\left(\frac{\eta+1}{\eta-1}\right)^i \leqslant \frac{2}{\varepsilon}$$

$$\frac{2}{\eta} i \leqslant i \log\left(\frac{\eta+1}{\eta-1}\right) \leqslant \log\frac{2}{\varepsilon}$$

$$i \leqslant \frac{1}{2}\eta \log\frac{2}{\varepsilon} \ .$$

Therefore, we can say that the number of iterations required by PG and PCG methods, are (for any κ):

$$it_{PG} \propto \kappa$$

and

$$it_{PCG} \propto \sqrt{\kappa} \ .$$

A.4 Bi-Conjugate Gradient Stabilized Method (Bi-CGStab)

The biconjugate gradient stabilized method(Bi-CGStab) is an iterative method developed by H.A.van der Vorst (1992) for the numerical solution of nonsymmetric linear systems. Bi-CGStab is a Krilov subspace method. This theory can be found in [19]
Bi-CGStab method reads as follows:
given $\mathbf{y}^{(0)}$, compute $\mathbf{r}^{(0)} = \mathbf{f} - A\mathbf{y}^{(0)}$ and choose $\widehat{\mathbf{r}}^{(0)}$ such that $(\widehat{\mathbf{r}}^{(0)})^T \widehat{\mathbf{r}}^{(0)} \neq 0$ (for example $\widehat{\mathbf{r}}^{(0)} = \mathbf{r}^{(0)}$ if $\mathbf{r}^{(0)} \neq \mathbf{0}$). Moreover, put $\rho_{-1} = \alpha_{-1} = \eta_0 = 1$ and $\mathbf{v}^{(-1)} = \mathbf{p}^{(-1)} = \mathbf{1}$. Then, for $k = 0, 1, \ldots$

$$
\begin{aligned}
&\rho_k = (\widehat{\mathbf{r}}^{(0)})^T \mathbf{r}^{(k)} \\
&\beta_k = \frac{\rho_k}{\rho_{k-1}} \frac{\alpha_{k-1}}{\eta_k} \\
&\mathbf{p}^{(k)} = \mathbf{r}^{(k)} + \beta_k (\mathbf{p}^{(k-1)} - \eta_k \mathbf{v}^{(k-1)}) \\
&\mathbf{v}^{(k)} = A\mathbf{p}^{(k)} \\
&\alpha_k = \frac{\rho_k}{(\widehat{\mathbf{r}}^{(0)})^T \mathbf{v}^{(k)}} \\
&\mathbf{s} = \mathbf{r}^{(k)} - \alpha_k \mathbf{v}^{(k)} \\
&\mathbf{t} = A\mathbf{s} \\
&\eta_{k+1} = \frac{\mathbf{s}^T \mathbf{t}}{\mathbf{t}^T \mathbf{t}} \\
&\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \alpha_k \mathbf{p}^{(k)} + \eta_{k+1} \mathbf{s} \\
&\mathbf{r}^{(k+1)} = \mathbf{s} - \eta_{k+1} \mathbf{t}
\end{aligned}
$$

while convergence is reached.

Bibliography

[1] Bears J., Corapcioglu M.Y. (1984),*Fundamentals of Transport Phenomena in Porous Media*, Martinus Nijhoff Publishers, Dordrecht.

[2] Blanco P., Gervasio P., Quarteroni A. (2011), *Extended variational formulation for heterogeneous partial differential equations.* Comp. Meth. in Applied Math. (CMAM) no. 2, 141-172.

[3] Brezis H. (2011) *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (1st Edition) Springer.

[4] Brezzi F. (1974) *On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers*, R.A.I.R.O. Anal. Numér. 8:129-151.

[5] Canuto C., Hussaini M.Y., Quarteroni A., Zang T.A. (2006), *Spectral Methods Fundamentals in Single Domains*, Springer-Verlag Berlin Heidelberg.

[6] Discacciati M. (2012), *Preconditioners for the finite element approximation of the Darcy-Stokes problem.*, In preparation.

[7] Discacciati M., Gervasio P., Quarteroni A. (2009), *Heterogeneous mathematical models in fluid dynamics and associated solution algorithms, in Multiscale and Adaptivity: Modeling, Numerics and Applications* , C.I.M.E. Summer School, Cetraro, Italy 2009. Series: Lecture Notes in Mathematics, Vol. 2040, January, 2012.

[8] Discacciati M., Gervasio P., Quarteroni A. (2012), *Interface Control Domain Decompositions for Linear Elliptic Problems* submitted to SIAM Journal of Control and Optimization.

[9] Discacciati M., Gervasio P., Quarteroni A. (2012), *Interface Control Domain Decomposition Methods: numerical aspects and validation* MOX technical report, in preparation.

[10] Discacciati M., Gervasio P., Quarteroni A. (2012), *ICDD methods for Stokes-Darcy coupling*, MOX technical report, in preparation.

[11] Discacciati M., Quarteroni A. (2009), *Navier-Stokes/Darcy Coupling: Modeling, Analysis, and Numerical Approximation* Revista Matemática Complutense a 2009: vol. 22, num. 2, pags. 315–426

[12] Gilardi G. (1997), *Problemi variazionali per equazioni di tipo ellittico* Quaderni del Dottorato 5, Università di Milano, http://www-dimat.unipv.it/gilardi/WEBGG/PSPDF/corso97.pdf .

[13] Hornung U. (1997), *Homogeneization and Porous Media*, Springer-Verlag, New York.

[14] Lions J.L. (1971), *Optimal Control of Systems Governed by Patrial Differential Equations*, Springer-Verlag, New York.

[15] Matlab®. The MathWorks. http://www.mathworks.com .

[16] Salsa S. (2007), *Partial Differential Equations in Action*, Springer-Verlag.

[17] Quarteroni A. (2009), *Numerical Models for Differential Problems* , Series: MS&A, Vol. 2, Springer.

[18] Quarteroni A., Valli A. (1999), *Domain Decomposition Methods for Partial Differential Equations*, Oxford University Press Inc., New York.

[19] Quarteroni A., Valli A. (2008), *Numerical Approximation of Partial Differential Equations*, Springer-Verlag.

[20] Toselli A., Widlund O. (2005), *Domain Decomposition Methods,Algorithms and Theory* , Springer-Verlag, Berlin and Heidelberg GmbH & Co. .