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Optimal Control of Diffusion and Marked Point Processes

Solving the Control Problem with Backward Stochastic Differential Equations

Tesi di Laurea Magistrale in Ingegneria Matematica

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Abstract

The aim of this work is to survey the optimal control theory and the backward stochastic differential equations approach.

We extend the theory from diffusive processes to marked point processes, highlighting the significant points in common and the main differences between the two treatments. In particular we show that marked point processes are described in a natural way by the dynamic approach, which links them to the martingale theory through the compensator notion. We consider the specific case of semi-Markov processes, for which we provide an original result concerning the form of their compensator. For both diffusive and marked point processes, we show that backward stochastic differential equations can be used to represent the value function and to characterize the optimal control.

Keywords: Backward Stochastic Differential Equations, Stochastic Optimal Control, Marked Point Processes, Stochastic Differential Equations, random measure, semi-Markov processes.

Sommario

In questo lavoro si mostra come risolvere problemi di controllo ottimo stocastico attraverso un particolare approccio basato sulla teoria delle equazioni differenziali stocastiche backward (BSDEs). In particolare si presenta in parallelo il controllo ottimo di processi diffusivi ed il controllo ottimo di processi di puro salto, sottolineando i significativi punti in comune e le principali differenze tra le due trattazioni. Esiste un'ampia letteratura sui problemi di controllo ottimo stocastico, sia riguardanti processi diffusivi che processi di salto; per quanto riguarda l'approccio BSDEs, esso è stato considerato da alcuni autori nel caso diffusivo, mentre risulta meno tradizionale per i processi discreti, per i quali si opta spesso per una risoluzione attraverso programmazione dinamica. Nonostante la tesi si presenti come elaborato prevalentemente compilativo, sono presenti alcuni risultati nuovi sviluppati in previsione di un successivo avanzamento della teoria. L'obiettivo è infatti quello di fornire gli strumenti per un approfondimento futuro di alcuni specifici argomenti considerati.

Nel caso diffusivo, la soluzione di una BSDE consiste in una coppia di processi (Y, Z), soddisfacenti

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta + \int_\tau^T f(t, Y_t, Z_t)dt, \quad \tau \in [0, T]; \tag{1}$$

dove W è un processo di Wiener in $\mathbb{R}^d$, f è chiamato generatore ed η è la condizione finale dell'equazione. Cerchiamo una coppia (Y, Z) nello spazio dei processi prevedibili, a valori in $\mathbb{R}^k \times L(\mathbb{R}^d, \mathbb{R}^k)$, tale che

$$|||(Y, Z)|||^2 := \mathbb{E}\left[\int_0^T \left(|Y_t|^2 + \|Z_t\|^2\right) dt\right] < \infty.$$

Indicheremo questo spazio con $\mathbb{K}$; munito della norma $|||\cdot|||$, $\mathbb{K}$ risulta essere uno spazio di Hilbert. Per una tale soluzione mostriamo risultati di esistenza ed unicità. Detta $\mathcal{F}_t$ la σ-algebra generata dai processi W_t, indichiamo con $\mathcal{P}_{[0\,T]}$ la σ-algebra prevedibile generata a sua volta da $\mathcal{F}_t$. Si considerano le seguenti ipotesi:

Ipotesi 1. - La condizione finale $\eta : \Omega \to \mathbb{R}$ è $\mathcal{F}_T$-misurabile e $\mathbb{E}\left[|\eta|^2\right] < \infty$;

- f è misurabile rispetto a $\mathcal{P}_{[0,\,T]} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(L(\mathbb{R}^d,\, \mathbb{R}^k))$;
- Esiste $K \geqslant 0$ tale che, per ogni $\tau \in [0,\, T]$, $r, r' \in \mathbb{R}, z, z' \in L^2(\mathbb{R}^d,\, \mathbb{R}^k)$, valga
$$|f(\tau, r, z) - f(\tau, r', z')| \leqslant K\,|r - r'| + K\,\|z - z'\|\,;$$
- $\mathbb{E}\left[\int_0^T |f(t, 0, 0)|^2\, dt\right] < \infty$.

Si dimostra innanzitutto un risultato preliminare:

Lemma 1. *Assumiamo che valgano le ipotesi sopra elencate e che $f : \Omega \times [0,\, T] \to \mathbb{R}^k$ sia un processo prevedibile che soddisfa $\mathbb{E}\left[\int_0^T |f_t|^2\, dt\right] < \infty$.*
Allora l'equazione backward

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta - \int_\tau^T f_t dt, \qquad \tau \in [0,\, T].$$

ammette un'unica soluzione $(Y, Z) \in \mathbb{K}$.

Tale Lemma, assieme ad un teorema di punto fisso, permette di mostrare la buona positura dell'equazione (1). Vale infatti il seguente risultato:

Teorema 2. *Sotto le Ipotesi 1, la BSDE* (1) *ammette un'unica soluzione $(Y, Z) \in \mathbb{K}$.*

La teoria delle equazioni backward sopra schematizzata viene utilizzata per risolvere problemi di controllo ottimo applicati ai processi diffusivi. Un processo di controllo è un processo $(\mathcal{F}_t)$-prevedibile a valori in uno spazio misurabile $(U,\, \mathcal{U})$. Nel caso diffusivo, dato un processo da controllare X, il controllo u appare direttamente nella dinamica di X tramite un'opportuna funzione r:

$$dX_\tau = F(\tau, X_\tau)d\tau + G(\tau, X_\tau)\, r(\tau, X_\tau, u_\tau)d\tau + G(\tau, X_\tau)dW_\tau, \qquad \tau \in [0,\, T].$$

L'obiettivo è quello di minimizzare, al variare di tutti i possibili controlli ammissibili u, il costo funzionale della forma:

$$\mathbb{E}\left[\int_0^T l(t, X_t, u_t)dt + \phi(X_T)\right],$$

dove l e ϕ rappresentano rispettivamente il costo di esercizio ed il costo terminale del processo X. Consideriamo la formulazione debole del problema. Fissiamo un sistema di controllo ammissibile (a.c.s) $\mathbb{U} = (\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{t\geqslant 0}, \widehat{\mathbb{P}}, \widehat{u})$, dove sono considerate incognite, oltre al controllo $\widehat{u}$, anche la σ-algebra $\widehat{\mathcal{F}}$, la filtrazione $(\widehat{\mathcal{F}}_t)_{t\geqslant 0}$ e la probabilità $\widehat{\mathbb{P}}$ soggiacente al processo X. Introduciamo dunque il processo $X_\tau^{\mathbb{U}}, \tau \in [0, T]$ associato ad un tale a.c.s, soluzione dell'equazione stocastica:

$$X_\tau^{\mathbb{U}} = x + \int_0^\tau F(t, X_t^{\mathbb{U}})dt + \int_0^\tau G(t, X_t^{\mathbb{U}})\, r(t, X_t^{\mathbb{U}}, \widehat{u}_t)dt + \int_0^\tau G(t, X_t^{\mathbb{U}})d\widehat{W}_t,$$

dove $\widehat{W}$ è un processo di Wiener rispetto a $(\widehat{\mathcal{F}}_t)_{t\geqslant 0}$. Con questa formulazione, il costo che deve essere minimizzato è:

$$J(\mathbb{U}) = \widehat{\mathbb{E}}\left[\int_0^T l(t, X_t^{\mathbb{U}}, \widehat{u}_t)dt + \phi(X_T^{\mathbb{U}})\right]$$

dove indichiamo con $\hat{\mathbb{E}}$ il valore atteso rispetto alla probabilità $\hat{\mathbb{P}}$, che dipende dall'a.c.s $\mathbb{U}$; la funzione valore corrispondente è:

$$V = \inf_{\mathbb{U}} J(\mathbb{U}).$$

Osserviamo che la formulazione debole introdotta attraverso la nozione di a.c.s risulta essere molto generale, non essendo a priori fissate né la probabilità né la filtrazione considerata.

Per risolvere il problema di controllo, è possibile introdurre un'equazione backward per la coppia $(Y_\tau^{\mathbb{U}}, Z_\tau^{\mathbb{U}})$ della forma:

$$Y_\tau^{\mathbb{U}} + \int_\tau^T Z_t^{\mathbb{U}} dW^{\mathbb{U}} = \phi(X_T^{\mathbb{U}}) + \int_\tau^T \psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}})dt, \quad \tau \in [0, T]. \tag{2}$$

il cui generatore contiene la funzione hamiltoniana

$$\psi(t, x, z) = \inf_{u\in U} \Big\{ l(t, x, u) + zr(t, x, u) \Big\}.$$

Sotto opportune condizioni di limitatezza e Lipschitzianità sul generatore, si mostra che la BSDE associata al problema di controllo ammette una ed una sola soluzione $(Y_\tau^{\mathbb{U}}, Z_\tau^{\mathbb{U}})$. Si può allora provare che il problema di controllo ottimo ha una soluzione, che può essere caratterizzata attraverso la soluzione della BSDE (2). In particolare si dimostra la validità della cosiddetta relazione fondamentale:

$$Y_0^{\mathbb{U}} = J(\mathbb{U}) + \widehat{\mathbb{E}}\left[\int_0^T \left[\psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}}) - Z_t^{\mathbb{U}} r(t, X_t^{\mathbb{U}}, \widehat{u}_t) - l(t, X_t^{\mathbb{U}}, \widehat{u}_t)\right] dt\right]$$

Dalla definizione di ψ si deduce inoltre che $J(\mathbb{U}) \geqslant Y_0^{\mathbb{U}}$. Se infine assumiamo che esista una mappa misurabile $\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \to U$ tale che $\psi(t, x, z) = l(t, x, \gamma(t, x, z)) + zr(t, x, \gamma(t, x, z))$, allora esiste un a.c.s che verifica $Y_0^{\mathbb{U}} = V$.

Ci occupiamo poi del caso discreto, concentrandoci in particolare sui cosiddetti processi di punto marcato. Dato uno spazio $(\Omega,\, \mathcal{F},\, \mathbb{P})$ completo, $(T_n)_{n\geqslant 1}$ è detto processo di punto se $T_0 = 0$, $T_n < T_{n+1}$ sull'insieme $(T_n < \infty)$. Per tali processi assumiamo che non vi sia esplosione, ovvero che $T_n \to \infty$ $\mathbb{P}$-a.s. Un processo di punto marcato è allora individuato dalla coppia (T_n, ξ_n), dove T_n soddisfa la definizione precedente e ξ_n è una variabile aleatoria in uno spazio misurabile $(K, \mathcal{K})$, con ξ_0 costante. Per ogni processo di punto marcato, possiamo definire il processo a tempo continuo:

$$X_t = \sum_{n\geqslant 0} \xi_n \mathbb{1}_{[T_n,\, T_{n+1})}(t).$$

Dopo aver introdotto il processo di conteggio:

$$N_t(A) = \sum_{n\geqslant 1} \mathbb{1}_{T_n \leqslant t} \mathbb{1}_{\xi_n \in A}, \quad t > 0,\, A \in \mathcal{K},$$

costruiamo la σ-algebra da esso generata nel modo seguente:

$$\mathcal{F}_t^0 = \sigma(N_s(A),\, s \leqslant t,\, A \in \mathcal{K}), \quad \mathcal{F}_t = \sigma(\mathcal{F}_t^0,\, \mathcal{N})$$

dove $\mathcal{N}$ sono gli insiemi $\mathbb{P}$-nulli di $\mathcal{F}$. Denotiamo con $\mathcal{P}$ e *Prog*, rispettivamente, la σ-algebra prevedibile e quella progressiva per $(\mathcal{F}_t)$. Valgono le seguenti proprietà:

Proposizione 3. *(1) T_n è $\mathcal{F}_t^0$-tempo di arresto;*

(2) $\mathcal{F}_{T_n}^0 = \{A \in \mathcal{F}_\infty^0 : A \cap \{T_n \leqslant t\} \in \mathcal{F}_t, \forall t \geqslant 0\} = \sigma(T_0, \xi_0, ..., T_n, \xi_n)$;

(3) Un processo f è $(\mathcal{F}_t^0)$-prevedibile se e solo se

$$f_t(\omega) = \sum_{n\geqslant 0} f_t^{(n)}(\omega) \mathbb{1}_{(T_n,\, T_{n+1}]}(t)$$

dove f^0 è $\mathcal{B}(\mathbb{R}_+)$-misurabile , $f^{(n)}$ è $\mathcal{F}_{T_n}^0 \otimes \mathcal{B}(\mathbb{R}_+)$-misurabile.

Il seguente teorema svolge un ruolo chiave in tutta la trattazione, poiché ci permette di legare i processi di punto alla teoria delle martingale.

Teorema 4. *Esiste A crescente, $A_0 \equiv 0$, continuo a destra, prevedibile, tale che*

$$\mathbb{E}\left[\int_0^\infty H_t dN_t\right] = \mathbb{E}\left[\int_0^\infty H_t dA_t\right] \quad \forall H \geqslant 0 \quad \textit{prevedibile.}$$

In particolare questo implica che $N - A$ è una $\mathcal{F}_t$-martingala.

Per studiare i processi di punto marcato è necessario definire una nuova misura sullo spazio prodotto $(0, \infty) \times K$. Per ogni ω, introduciamo una misura $p(\omega, \cdot)$ tale che

$$p(\omega, C) = \sum_{n \geqslant 1} \mathbb{1}_{(T_n(\omega),\, \xi_n(\omega)) \in C} \quad \text{per} \quad C \in \mathcal{B}((0, \infty)) \otimes \mathcal{K}.$$

Allora si può mostrare che:

Teorema 5. *Esiste $\phi_t(A)$, $\omega \in \Omega$, $t \geqslant 0$, $A \in \mathcal{K}$, tale che*

(1) $\phi_t(\omega, \cdot)$ é una probabilità su $(K, \mathcal{K})$ $\forall t$, $\forall \omega$;

(2) $(t, \omega) \mapsto \phi_t(\omega, A)$ è prevedibile, $\forall A$;

(3) $\mathbb{E}\left[\int_0^\infty H_t(y)p(dtdy)\right] = \mathbb{E}\left[\int_0^\infty H_t(y)\phi_t(dy)dA_t\right]$ per ogni $H_t(\omega, y) \geqslant 0$ $\mathcal{P} \otimes \mathcal{K}$-misurabile.

Chiameremo $\tilde{p}(dtdy) = \phi_t(dy)dA_t$ il compensatore di p. (3) mostra che p e $\tilde{p}$ hanno la stessa restrizione su $\mathcal{P} \otimes \mathcal{K}$. Se a questo punto fissiamo un orizzonte $T > 0$, e un processo $H_t(\omega, y)$ $(K, \mathcal{K})$-misurabile che soddisfa

$$\mathbb{E}\left[\int_0^T |H_s(y)|\, \phi_s(dy)dA_s\right] < \infty,$$

possiamo definire per $t \in [0, T]$

$$\int_0^t H_s(y)q(dsdy) := \int_0^t H_s(y)p(dsdy) - \int_0^t H_s(y)\phi_s(dy)dA_s,$$

la quale risulta essere una martingala cadlag a variazione finita. Introduciamo quindi un teorema di rappresentazione delle martingale, risultato fondamentale per dedurre la teoria BSDEs associata ai processi di punto marcato.

Teorema 6. *Data M martingala cadlag rispetto a $\mathcal{F}_t$ su $[0, T]$, esiste $H_t(\omega, y)$, $\mathcal{P} \otimes \mathcal{K}$-misurabile e soddisfacente*

$$M_t = M_0 + \int_0^t \int_K H_s(y)q(dsdy).$$

A questo punto della trattazione si è rivolta una particolare attenzione a specifici processi di salto, detti processi semi-Markov. Essi costituiscono una famiglia di processi molto estesa, il cui studio è particolarmente interessante per le numerose applicazioni possibili. Per questi processi abbiamo calcolato la forma esplicita del compensatore, di cui, a nostra conoscenza, non esistono risultati analoghi in letteratura. Abbiamo iniziato la nostra analisi dalla formula di Jacod per i processi di punto marcato generali:

$$\tilde{p}(dt, dy) = \sum_{n \geqslant 1} \frac{G_n(dt - T_n, dy)}{H_n([t - T_n, \infty])} \mathbb{1}_{T_n < t \leqslant T_{n+1}}.$$

In tale formula si indica con $G_n(dt, dx)$ la legge condizionale di (S_{n+1}, ξ_{n+1}) rispetto alla σ-algebra $\mathcal{F}_{T_n}$, dove (ξ_{n+1}, T_{n+1}) è il processo di punto marcato considerato e $S_{n+1} = T_{n+1} - T_n$; $H_n(dt)$ è invece la misura marginale che corrisponde alla legge condizionale di (S_{n+1}) rispetto a $\mathcal{F}_{T_n}$. Definiamo allora i processi semi-Markov come un'opportuna generalizzazione dei processi Markov tempo-omogenei. Per i processi (S_{n+1}, ξ_{n+1}) e (S_{n+1}) introduciamo le funzioni di ripartizione rispetto alla σ-algebra $\mathcal{F}_{T_n}$, rispettivamente Q e H, e le misure ad esse associate $\tilde{H}$ e $\tilde{Q}$. Il compensatore per i processi semi-Markov assume la seguente forma:

$$\tilde{p}(dt, dy) = q(X_{t-}; dy, a(t-))\, R(X_{t-}; dt - T_{N(t-)}),$$

dove $q(x; A, t) = \mathbb{P}\{\xi_{n+1} \in A | S_{n+1} \leqslant t, \xi_n = x\}$, $x \in K$, $A \in \mathcal{K}$, $t \in \mathbb{R}_+$, $N(t)$ è il processo di conteggio introdotto, R è definita come la "hazard measure" di H:

$$R(x; dt) := \frac{\tilde{H}(x; dt)}{\tilde{H}(x; [t,\ +\infty])},$$

e

$$a(t) := t - T_{N(t)}.$$

Se infine assumiamo che la funzione di sopravvivenza di Q ammetta un rate λ, allora il compensatore assume la forma semplificata:

$$\tilde{p}(dt, dy) = q(X_{t-}; dy, a(t-))\, \lambda(X_{t-}) dt.$$

La nozione di compensatore permette di descrivere la dinamica di un processo di punto marcato, e la sua conoscenza è di fondamentale importanza sia per la teoria BSDEs che per il controllo ottimo associato a tali processi. Consideriamo innanzitutto una classe di BSDEs guidate da una misura aleatoria, senza parte diffusiva, su un intervallo finito, naturalmente associata ad un processo di punto marcato generale:

$$Y_\tau + \int_\tau^T \int_K Z_t(y) q(dt dy) = \xi + \int_\tau^T f_t(Y_t, Z_t(\cdot)) dA_t, \quad \tau \in [0,\ T], \qquad (3)$$

dove $q(dtdy) = p(dtdy) - \tilde{p}(dtdy)$, $Y : \Omega \times [0, T] \to \mathbb{R}$ è un processo progressivamente misurabile, mentre $Z : \Omega \times [0, T] \times K \to \mathbb{R}$ è $\mathcal{P} \otimes \mathcal{K}$-misurabile. In particolare, cerchiamo Y e Z nello spazio dei processi che soddisfano:

$$\|(Y, Z)\|_\beta^2 := \mathbb{E}\left[\int_0^T e^{\beta A_t} |Y_t|^2 \, dA_t\right] + \mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right] < \infty,$$

in cui β è un numero reale da scegliere in modo opportuno. Indicheremo tale spazio con $\mathbb{K}^\beta$; munito della norma $\|(Y, Z)\|_\beta^2$, $\mathbb{K}^\beta$ risulta essere uno spazio di Hilbert. Richiediamo inoltre che siano soddisfatte le seguenti proprietà:

Ipotesi 2.
- ξ è $\mathcal{F}_T$-misurabile e $\mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] < \infty$;
- $\forall t \in [0,T]$, $\forall \omega \in \Omega$, $\forall r \in K$, $f_t(\omega, r, \cdot) : \mathcal{L}^2(K, \mathcal{K}, \phi_t(\omega, dy)) \to \mathbb{R}$ e, per ogni Z, la mappa $(\omega, \tau, r) \mapsto f_\tau(\omega, r, Z_\tau(\cdot))$ è *Prog*$\otimes\mathcal{B}(\mathbb{R})$-misurabile;
- $\exists L \geqslant 0$, $L' \geqslant 0$ tale che, $\forall \omega \in \Omega$, $\tau \in [0, T]$, $r, r' \in \mathbb{R}$ e $z, z' \in L^2(K, \mathcal{K}, \phi_\tau(\omega, dy))$,

$$|f_t(r, z(\cdot)) - f_t(r', z'(\cdot))| \leqslant L' |r - r'| + L \left(|z(y) - z'(y)|^2 \phi_t(dy)\right)^{1/2};$$

- $\mathbb{E}\left[\int_0^T |f_t(0,0)|^2 e^{\beta A_t} dA_t\right] < \infty$.

Per provare esistenza ed unicità della soluzione dell'equazione backward (3), consideriamo innanzitutto un'equazione più semplice, con un generatore f_t che non dipenda ne dal processo Y ne da Z. Si ha il seguente risultato:

Lemma 7. *Assumiamo che valgano le condizioni sopra elencate e che*

$$\mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t|^2 \, dA_t\right] < \infty$$

per qualche $\beta > 0$. Allora esiste un'unica coppia (Y, Z) soluzione della BSDE

$$Y_\tau + \int_\tau^T \int_K Z_t(y) q(dtdy) = \xi + \int_\tau^T f_t dA_t, \qquad \tau \in [0, T].$$

Inoltre, per ogni $\tau \in [0, T]$ vale la seguente identità:

$$\mathbb{E}\left[e^{\beta A_\tau} |Y_\tau|^2\right] + \beta \mathbb{E}\left[\int_\tau^T e^{\beta A_t} |Y_t|^2 \, dA_t\right] + \mathbb{E}\left[\int_\tau^T \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right]$$
$$= \mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + 2\mathbb{E}\left[\int_\tau^T e^{\beta A_t} Y_t f_t dA_t\right].$$

Tramite il precedente Lemma, ed un opportuno teorema di punto fisso, si dimostra allora il risultato di buona positura cercato:

Teorema 8. *Assumiamo che valgano le Ipotesi 2 con* $\beta > L^2 + 2L'$.
Allora esiste un'unica coppia di processi (Y, Z) *in* $\mathbb{K}^\beta$ *che risolve la BSDE* (3).

Una volta introdotta la teoria delle BSDEs per processi di salto, è possibile formulare il problema di controllo per questi processi ed andarne a studiare la risolubilità tramite l'approccio BSDEs. Dato uno spazio misurabile $(U, \mathcal{U})$, definiamo processo di controllo ogni processo prevedibile $(u_\tau)_\tau$ a valori in U. Ricordiamo che, nel contesto dei processi di punto marcato, questo corrisponde a chiedere che:

$$u_t(\omega) = \sum_{n \geqslant 0} u_t^{(n)}(\omega) \mathbb{1}_{(T_n,\, T_{n+1}]}(t)$$

con $u^{(0)}$ funzione deterministica nel tempo e $\mathcal{B}(\mathbb{R}_+)$-misurabile , $u^{(n)}$ misurabile rispetto a $\mathcal{B}(\mathbb{R}_+) \otimes \sigma(T_0, \xi_0, ..., T_n, \xi_n)$. Ciò significa che il controllore decide la strategia di un controllo dopo il salto T_n avendo osservato X_t, $t \in [0,\, T_n]$ e cambia strategia solo dopo il prossimo tempo di salto T_{n+1}.

L'effetto di un controllo si esprime tramite una assegnata funzione r, della forma $r_t(\omega, y, u)$, $\omega \in \Omega$, $t \geqslant 0$, $y \in K$, $u \in U$. Richiederemo che essa sia limitata e $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$-misurabile. Anche in questo caso il problema del controllo ottimo viene formulato in modo debole. Ad ogni controllo $u(\cdot)$ si associa una probabilità $\mathbb{P}_u$ tale che il processo X, sotto $\mathbb{P}_u$, ammetta come compensatore

$$\tilde{p}^u(dtdy) := r_t(y,\, u_t)\, \tilde{p}(dtdy).$$

Il funzionale costo da minimizzare è allora:

$$J(u(\cdot)) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dA_t + g(X_T) \right]$$

dove $\mathbb{E}_u$ indica il valore atteso rispetto a $\mathbb{P}_u$. Si può mostrare che una tale probabilità esiste ed ammette la forma seguente:

$$\mathbb{P}_u(d\omega) = L_T(\omega) \mathbb{P}(d\omega)$$

dove

$$L_\tau = \exp\left(\int_0^\tau \int_K (1 - r_t(y, u_t)) \phi_t(dy) dA_t \right) \prod_{n \geqslant 1 : T_n \leqslant t} r_{T_n}(\xi_n, u_{T_n}),$$

a condizione di poter verificare che $\mathbb{E}[L_T] = 1$. Sotto le ipotesi introdotte su r si prova che L è una martingale locale positiva (rispetto a $(\mathcal{F}_t, \mathbb{P})$), soluzione unica di

$$L_\tau = 1 + \int_0^\tau \int_K L_{t-}(r_t(y, u_t) - 1)q(dtdy).$$

Inoltre vale il seguente Lemma:

Lemma 9. *Sia* $|r| \leqslant C_r$, $\gamma > 1$, $\beta = \gamma + 1 + \frac{C_r^{\gamma^2}}{\gamma - 1}$, $\mathbb{E}\left[e^{\beta A_T}\right] < \infty$. *Allora*

$$\mathbb{E}[L_T^\gamma] < \infty, \quad \mathbb{E}[L_T] = 1.$$

Sotto tali ipotesi, si può associare il costo $J(u(\cdot))$ ad ogni controllo $u(\cdot)$.

Al problema di controllo così formulato è possibile associare una opportuna equazione BSDE; definita la funzione hamiltoniana

$$f(\omega, t, x, z(\cdot)) = \inf_{u \in U} \left\{ l_t(\omega, x, u) + \int_K z(y)(r_t(\omega, y, u) - 1)\phi_t(\omega, dy) \right\}$$

per $\omega \in \Omega$, $t \geqslant 0$, $x \in K$, $z : K \to \mathbb{R}$, consideriamo la BSDE:

$$Y_\tau + \int_\tau^T \int_K Z_t(y)q(dtdy) = g(X_T) + \int_\tau^T f(t, X_t, Z_t(\cdot))dA_t, \quad \tau \in [0, T]. \quad (4)$$

Richiediamo che $Y_t(\omega)$ sia adattato e cadlag (ed in particolare Y_0 sia $\mathcal{F}_0$-misurabile, cioè deterministico), $Z_t(\omega, y)$ sia $\mathcal{P} \otimes \mathcal{K}$-misurabile $\forall \omega \in \Omega$, $t \in [0, T]$, $y \in K$, e che i processi Y, Z soddisfino opportune condizioni di integrabilità. Sotto tali ipotesi si mostra che l'equazione backward (4) ammette un'unica soluzione (Y_t, Z_t). Si può quindi risolvere il problema di controllo con l'approccio BSDEs. Per fare ciò, osserviamo innanzitutto che

$$\begin{aligned}
\mathbb{E}_u \left[\int_0^T \int_K Z_t(y)q(dtdy) \right] &= \mathbb{E}_u \left[\int_0^T \int_K Z_t(y)p(dtdy) \right] - \mathbb{E}_u \left[\int_0^T \int_K Z_t(y)\phi_t(dy)dA_t \right] \\
&= \mathbb{E}_u \left[\int_0^T \int_K Z_t(y)\tilde{p}^u(dtdy) \right] - \mathbb{E}_u \left[\int_0^T \int_K Z_t(y)\phi_t(dy)dA_t \right]
\end{aligned}$$

poichè $\tilde{p}^u$ è il compensatore di p rispetto a $\mathbb{P}_u$. Allora, ricordando che Y_0 è deterministico, si ottiene

$$Y_0 + \mathbb{E}_u \left[\int_0^T \int_K Z_t(y)(r_t(y, u_t) - 1)\phi_t(dy)dA_t \right] = \mathbb{E}_u [g(X_T)] + \mathbb{E}_u \left[\int_0^T f(t, X_t, Z_t(\cdot))dA_t \right],$$

che, dopo opportuni passaggi, fornisce la relazione fondamentale nel caso di processi di punto marcato:

$$Y_0 = J(u(\cdot)) + \mathbb{E}_u \left[\int_0^T \left[f(t, X_t, Z_t(\cdot)) - l_t(X_t, u_t) - \int_K Z_t(y)(r_t(y, u_t) - 1)\phi_t(dy) \right] dA_t \right].$$

Dalla definizione di f segue inoltre che:

$$Y_0 \leqslant J(u(\cdot)) \quad \text{per ogni controllo ammissibile} \quad u(\cdot).$$

Infine, se esiste un controllo $\underline{\mathrm{u}}^Z(\cdot)$ tale che il valore atteso a secondo termine sia esattamente uguale a 0, allora $\underline{\mathrm{u}}^Z(\cdot)$ è ottimo, e $Y_0 = \min_{u \in U} J(u(\cdot)) = J(\underline{\mathrm{u}}^Z(\cdot))$. Si arriva così al risultato conclusivo:

Teorema 10. *Sotto le ipotesi precedenti su r, g e l, supponiamo che A sia continuo e che $\mathbb{E}\left[e^{(\beta+C_r^4)A_T}\right] < \infty$. Supponiamo inoltre che esista β tale che*

$$\beta > \sup |r-1|^2, \quad \mathbb{E}\left[e^{\beta A_T}\right] < \infty, \quad \mathbb{E}\left[|g(X_T)|^2 e^{\beta A_T}\right] < \infty.$$

Allora esiste una ed una sola soluzione (Y, Z) della BSDE (4), nella classe dei processi considerata. Se inoltre supponiamo che, per tali processi Z, esista $\underline{u}^Z : \Omega \times [0, T] \to U$ prevedibile e tale che

$$\underline{u}_t^Z(\omega) \in \arg\min_{u \in U} \left\{ l_t(\omega, X_{t-}, u) + \int_K Z_t(\omega, y)(r_t(\omega, y, u) - 1)\phi_t(\omega, dy) \right\},$$

allora $\underline{u}^Z(\cdot)$ è ottimo e $Y_0 = min_{u\in U} J(u(\cdot)) = J(\underline{u}^Z(\cdot))$.

Parole Chiave: Equazioni Differenziali Stocastiche Backward, Controllo Ottimo Stocastico, Processi di Punto Marcato, Equazioni Differenziali Stocastiche, misure aleatorie, processi semi-Markov.

Contents

Introduction

In this work we deal with stochastic processes and the associated optimal control problems.

The aim of optimal control theory is to govern a given system in such a way that a certain optimality is achieved. In particular we consider systems with stochastic dynamics, namely where a random noise affects the system evolution and, in general, the observations of the controller. Although the noise dependency is unavoidable, we can act on the control process to change the state dynamics. Introducing a functional cost which depends on the state and on the control variable, we are interested in designing the optimal control that minimizes its expectation value over all possible realization of the noise process.

There exists a large literature on optimal stochastic problems; we refer the reader to [5], [4], [27], [40], [33], [37], [7]. The two main classes of controlled processes are the diffusive processes and the jump processes. Roughly speaking, in the first case the control process occurs in the controlled equation; in the jump process case, instead, the control process has the effect of modifying the initial state dynamics by a change of probability. For both processes the functional cost is defined in a similar way.

In the past many different methods have been developed to solve this kind of problems. In our work we will present a special approach based on the theory of backward stochastic differential equations, BSDEs for short; we will apply it to optimal control problems related to both diffusive and marked point processes.

BSDEs are Itô's stochastic differential equations with a final condition; this subject started with the paper [31] by Pardoux and Peng, where the authors first solved general nonlinear BSDEs driven by the Wiener process. Existence and uniqueness of the solution to BSDEs was studied under local Lipschitz hypotheses, and afterwards a systematic theory has been developed for diffusive BSDEs, we mention in particular [16], [17], [12], [29], [30].

The BSDEs approach to optimal control has been deeply studied in the diffusive case starting from [32]; the reader can consult for instance [33], [27], [40], and [17].

Later, generalizations have been considered where the Wiener process is replaced by more general processes. The general formulation of a BSDEs driven by a random measure has been introduced in [37], and has been considered in [3], [36] in the markovian case, and in [39] for general jump process.

The approach to optimal control is less traditional. Indeed, there exists a large literature on optimal control of marked point processes ([10], [18] as general references), but there are relatively few results on their connections with BSDEs. For this part our main reference consists of the recent paper [20], where the authors addressed the topic in a systematic way.

This thesis consists of a survey of the main topics, even though it has the merit of presenting some specific new results that are useful for future developments of the theory.

We give a detailed presentation of marked point processes, BSDEs and optimal control problems. BSDEs and optimal control theories are both formulated in parallel for diffusive and marked point processes. By this complementary point of view, we highlight the significant points in common and the main differences between the diffusive and the discrete treatment, and how the last one can be derived from the first by appropriate modifications. As we already pointed out, the available literature on the BSDEs approach for optimal control is inhomogeneous. For this reason, in the diffusive case we choose to present only the proofs we want to compare with the discrete discussion. In the marked point processes framework, on the contrary, we give the proof of all results, and we strive to provide every detail that was not explicit in the original articles. Moreover, we consider specific jump processes, namely semi-Markov ones, and in Section 1.4 we present the explicit form of the compensator of these processes. They define a very large family and are particularly interesting in view of the large number of applications. Although the general compensator formula for marked point processes was already pointed out in the Jacod's paper [23], we are not aware of other results on the explicit form in the semi-Markov case. This research has been done in view of a future application of the general optimal control theory based on BSDEs approach to this appealing case.

Next we shortly describe the contents of every chapter.

In **Chapter 1** we present the class of marked point processes, following the discussions in [10] and [25]. A point process is a sequence of random variables T_n satisfying $T_0 \equiv 0$, $T_n \leqslant T_{n+1}$, which can be interpreted as the times at which certain events occur. Associated to these times are some other random elements ξ_n, called marks, containing further information about events, which take values in a measurable space $(K, \mathcal{K})$. The process $(T_n, \xi_n)_n$ is said to be a marked point process, MPP for short. MPPs are widely used in a variety of fields; they may model times at which items or customers enter or leave certain manufacturing stations, queueing systems, communications network, etc. Associated with a MPP is a filtration, i.e. a flow of information representing the history of the process evolving with time. The corresponding random counting measure is $p = \sum_{n \geqslant 0} \delta_{(T_n, \xi_n)}$, where δ denotes the Dirac measure. The random measure p is fairly general, the only restriction being non explosion (i.e., $T_n \to \infty$).

There is a basic link between MPPs and the martingale theory; the corresponding results form the so called dynamic approach to MPPs. This approach is used by Last and Brandt [25] and by Brémaud [10] in their work on marked point processes and we follow it in our presentation. It is mathematically based on the concept of compensator, which describes the local dynamics of a MPP. The compensator indeed compensates the increments of a well-defined counting measure associated with the MPP in a predictable manner such that the difference becomes a martingale. We denote A_t the compensator of the counting process measure $N_t := p([0,\, t] \times K)$ and by $\phi_t(dy)dA_t$ the (random) compensator of p.

Many MPPs can be described in a natural way by the dynamic approach, as, for instance, doubly stochastic Poisson processes, marked Poisson processes and Markov chains. Martingale methods in the theory of point processes go back to Watanabe [38] (1964), who discovered the martingale characterization of the Poisson processes, but the first systematic treatment of a general MPPs using martingales was given in 1972 by Brémaud [10]. He adapted the martingale approach to MPPs and he demonstrated its usefulness in the theory of stochastic systems driven by point processes. The martingale definition of compensator gives the basis to construct a martingale calculus which has the same power as Ito calculus for diffusions.

The compensator characterizes the distribution of a marked point process, and its knowledge is very important in view of optimal control applications. Indeed, the control changes the process dynamics by an appropriate modification of the compensator for the controlled process. For these reason we devote Section 1.4 to the presentation of the particular case of semi-Markov processes, and to the calculations for their compensator.

We start our inquiry from Jacod's formula (see [23]) for the general compensator of a marked point process:

$$\tilde{p}(dt, dy) = \sum_{n \geqslant 1} \frac{G_n(dt - T_n, dy)}{H_n([t - T_n, \infty])} \mathbb{1}_{T_n < t \leqslant T_{n+1}}.$$

Here $G_n(dt, dx)$ denotes the conditional law of (S_{n+1}, ξ_{n+1}) with respect to the

σ-algebra $\mathcal{F}_{T_n}$, where (ξ_{n+1}, T_{n+1}) is the considered marked point process and $S_{n+1} = T_{n+1} - T_n$; $H_n(dt)$ is the marginal measure which corresponds to the conditional law of S_{n+1}. Then we define a semi-Markov process as an appropriate generalization of the time-homogeneous Markov one. We introduce the cumulative distribution functions (with respect to the σ-algebra $\mathcal{F}_{T_n}$) Q and H, respectively for the process (S_{n+1}, ξ_{n+1}) and (S_{n+1}), and the associated measures $\tilde{H}$ and $\tilde{Q}$. The compensator for the semi-Markov processes takes the following form:

$$\tilde{p}(dt, dy) = q(X_{t-}; dy, a(t-))\, R(X_{t-}; dt - T_{N(t-)}),$$

where $q(x; A, t) = \mathbb{P}\{\xi_{n+1} \in A | S_{n+1} \leqslant t | \xi_n = x\}$, $x \in K$, $A \in \mathcal{K}$, $t \in \mathbb{R}_+$, $N(t)$ is the counting process, R is defined as the hazard measure of H:

$$R(x; dt) := \frac{\tilde{H}(x; dt)}{\tilde{H}(x; [t,\, +\infty])},$$

and

$$a(t) := t - T_{N(t)}.$$

If we assume that the survival function of Q admits a rate λ, then the compensator assumes the simpler form:

$$\tilde{p}(dt, dy) = q(X_{t-}; dy, a(t-))\, \lambda(X_{t-})dt.$$

In **Chapter 2** we present the notion of backward stochastic equation and the associated well-posedness results. In Section 2.1 we analyse the BSDEs driven by a Wiener process, for which we mainly follow the discussion in [31] and [17]. According to these authors, the solution of a BSDE consists of a pair of adapted processes $(Y,\, Z)$ satisfying

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta + \int_\tau^T f(t, Y_t, Z_t)dt, \quad \tau \in [0,\, T],$$

where W is a Wiener motion in $\mathbb{R}^d$, f is called generator and η is the final condition. This equation is intended in the Itô sense and we look for predictable processes $(Y,\, Z)$, such that $Y : \Omega \times [0,\, T] \to \mathbb{R}^k$, $Z : \Omega \times [0,\, T] \to L(\mathbb{R}^d,\, \mathbb{R}^k)$.

The basic hypothesis on the generator f is a Lipschitz condition requiring that for a constant $K \geqslant 0$,

$$|f(\tau, y, z) - f(\tau, y', z')| \leqslant K\, |r - r'| + K\, \|z - z'\|\,,$$

for every $\tau \in [0,\, T]$, for $y,\, y' \in \mathbb{R}^k$ and $z,\, z' \in L(\mathbb{R}^d, \mathbb{R}^k)$.

In order to solve the equation, beside measurability assumptions, we require moreover the summability condition

$$\mathbb{E}\left[\int_0^T |f(t,0,0)|^2\,dt\right] < \infty.$$

All the results of the Section are without proof, except for the Lemma 2.3 and the Theorem 2.4, which together give the existence and uniqueness of the solution to the BSDE. In particular the existence and uniqueness result is achieved in Theorem 2.4, considering first a specific BSDE (as the one in Lemma 2.3), and next generalizing that result by a fixed-point theorem. The corresponding proofs play a key role, representing a valid procedure, with appropriate modifications, to prove also respective results in the marked point processes case. We finally present a priori estimates and the continuous dependence upon a given process for the considered solution.

Then in Section 2.2 we focus on a BSDE driven by a random measure, without diffusion part, on a finite time interval, of the form:

$$Y_\tau + \int_\tau^T Z_t(y) q(dt dy) = \xi + \int_\tau^T f_t(Y_t, Z_t(\cdot)) dA_t, \quad \tau \in [0,\, T],$$

where the generator f and the final condition ξ are given. In this formulation the unknown process associated with the martingale part, namely the Z-process, is a random field.

Here the basic probabilistic datum is the marked point process (T_n, ξ_n). With the same notations of Chapter 1, we call $p(dt\,dy)$ the corresponding random counting measure, and we denote A_t the compensator of the counting process measure $p([0,\,T] \times K)$ and by $\phi_t(dy)dA_t$ the (random) compensator of p. Finally, the compensated measures $q(dt\,dy) = p(dt\,dy) - \phi_t(dy)dA_t$ occurs in the above equation. The unknown process is a pair of $(Y_t, Z_t(\cdot))$, where Y is a real progressive process and $\{Z_t(y),\, t \in [0,\, T],\, y \in K\}$ is a predictable random field.

The basic hypothesis on the generator f is a Lipschitz condition requiring that for some constants $L \geqslant 0$, $L' \geqslant 0$,

$$|f_t(\omega, r, z(\cdot)) - f_t(\omega, r', z'(\cdot))| \leqslant L' \,|r - r'| + L\Big(\int_K |z(y) - z'(y)|^2\, \phi_t(\omega, dy)\Big)^{1/2}$$

for all (ω, t), for r, $r' \in \mathbb{R}$ and z, z' in appropriate functional spaces (depending on (ω, t)). We note that the generator of the BSDE can depend on the unknown Z-process in a general functional way: this is required in the applications to optimal control problems, and it is shown that our assumptions can be effectively verified in a number of cases. In order to solve the equation, beside measurability assumptions, we require (as in the diffusive case) a summability condition, namely:

$$\mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t(0,0)|^2 \, dA_t\right] + \mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] < \infty,$$

to hold for some $\beta > L^2 + 2L'$. We remark that in the Poisson case we have a deterministic compensator $\phi_t(dy)dA_t = \nu(dy)$ for some fixed measure ν on K and the summability condition reduces to a simpler form, not involving exponentials of stochastic processes.

The existence and uniqueness of the solution are provided following the diffusive case procedure: we first prove existence and uniqueness for a simplified BSDE in Lemma 2.7, then the general result is achieved in Theorem 2.8 by a fixed-point theorem. We also present a priori estimates and some results on continuous dependence upon the data.

In **Chapter 3** we finally formulate the optimal control problem and we solve it by the BSDEs approach.

We define an admissible control system, a.c.s for short, as the set

$$\mathbb{U} = (\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{t \geqslant 0}, \widehat{\mathbb{P}}, \widehat{u})$$

where $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ is a complete probability space, $(\widehat{\mathcal{F}}_t)_{t \geqslant 0}$ is a filtration that verifies the usual conditions, $\widehat{u}$ is a predictable process with respect to $(\widehat{\mathcal{F}}_t)_{t \geqslant 0}$. As we will see, the a.c.s notion plays an important role in the so called "weak" formulation of optimal control problems.

In Section 3.1 we deal with optimal control of diffusive processes. In this case, given a controlled process X, the control u occurs directly in the X dynamics by means of a function r:

$$dX_\tau = F(\tau, X_\tau)d\tau + G(\tau, X_\tau)\, r(\tau, X_\tau, u_\tau)d\tau + G(\tau, X_\tau)dW_\tau, \qquad \tau \in [0,\, T].$$

We consider the weak formulation of the control problem. We fix an a.c.s $\mathbb{U}$ and we introduce the process $X^{\mathbb{U}}_\tau, \tau \in [0,\, T]$, solution of the Ito stochastic equation:

$$X^{\mathbb{U}}_\tau = x + \int_0^\tau F(t, X^{\mathbb{U}}_t)dt + \int_0^\tau G(t, X^{\mathbb{U}}_t) r(t, X^{\mathbb{U}}_t, \widehat{u}_t)dt + \int_0^\tau G(t, X^{\mathbb{U}}_t) d\widehat{W}_t,$$

where $\widehat{W}$ is a Wiener process with respect to $(\widehat{\mathcal{F}}_t)_{t \geqslant 0}$. The state dynamics is changed by the choice of a new a.c.s, i.e., by the entire set $(\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{t \geqslant 0}, \widehat{\mathbb{P}}, \widehat{u})$.

In this formulation, the cost to be minimized is:

$$J(\mathbb{U}) = \widehat{\mathbb{E}}\left[\int_0^T l(t, X^{\mathbb{U}}_t, \widehat{u}_t)dt + \phi(X^{\mathbb{U}}_T)\right]$$

where $\hat{\mathbb{E}}$ denotes the expectation with respect to the probability $\hat{\mathbb{P}}$ depending on the admissible control system $\mathbb{U}$; the corresponding value function is

$$V = \inf_{\mathbb{U}} J(\mathbb{U}).$$

To solve this optimal control problem, we introduce the BSDE for the unknown process $(Y_\tau^{\mathbb{U}}, Z_\tau^{\mathbb{U}})$:

$$Y_\tau^{\mathbb{U}} + \int_\tau^T Z_t^{\mathbb{U}} dW^{\mathbb{U}} = \phi(X_T^{\mathbb{U}}) + \int_\tau^T \psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}})dt, \quad \tau \in [0, T].$$

where the generator contains the hamiltonian function

$$\psi(t, x, z) = \inf_{u \in U} \Big\{ l(t, x, u) + zr(t, x, u) \Big\}.$$

By the BSDEs theory, there exists only one solution this equation.

Moreover, assuming that the infimum in the ψ definition is in fact achieved, we prove that the optimal control problem has a solution, and that the optimal control can be obtained by means of the solution to the above BSDE at the initial time.

In Section 3.2 we consider the class of optimal control problems for marked point processes. We follow a classical approach, so called intensity-control: the controller can modulate the intensity but cannot directly add or erase points (see for instance [10]). As in the diffusive case, we present the weak formulation of the control problem and how to solve it with the BSDEs approach.

We consider first the jump process $X_t = \sum_{n \geqslant 0} \xi_n \mathbb{1}_{[T_n, T_{n+1})}(t)$ corresponding to a given marked point process $(\xi_n, T_n)_{n \geqslant 0}$, and an admissible control process u.

In analogy to the previous case, we would like to associate u to X by means of a certain process X^u and to minimize a cost functional with the form:

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T l_t(X_t^u, u_t) dA_t + g(X_T) \right].$$

However, in the marked point processes case, u cannot be described within a dynamics equation. Therefore, we introduce an a.c.s by an opportune change of probability. Given the uncontrolled process X and the associated filtration, we prove that there exists a Gisanov kernel L_τ associated to the marked point process such that, assuming that $\mathbb{E}[L_T] = 1$,

$$\mathbb{P}_u(d\omega) = L_T \mathbb{P}(d\omega).$$

Under this new probability, the process X admits the compensator

$$\tilde{p}^u(dt, dy) := r_t(y, u_t)\, \tilde{p}(dt, dy)$$

and the functional cost becomes:

$$J(u(\cdot)) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dA_t + g(X_T) \right].$$

where $\mathbb{E}_u$ denotes the expectation under $\mathbb{P}_u$.

To this control problem we associate the BSDE:

$$Y_\tau + \int_\tau^T \int_K Z_t(y) q(dt\, dy) = g(X_T) + \int_\tau^T f(t, X_t, Z_t(\cdot)) dA_t, \quad \tau \in [0, T].$$

where the generator contains the hamiltonian function

$$f(\omega, t, x, z(\cdot)) = \inf_{u \in U} \left\{ l_t(\omega, y, u) + \int_K z(y)(r_t(\omega, y, u) - 1)\phi_t(\omega, dy) \right\}.$$

As in the diffusive case, it can be proved that there exists only one $(Y_\tau, Z_\tau(\cdot))$ solution of the above BSDE.

Assuming that the infimum of f is in fact achieved, admitting a suitable selector, together with a summability condition of the form

$$\mathbb{E}\left[e^{\beta A_T}\right] + \mathbb{E}\left[|g(X_T)|^2 e^{\beta A_T}\right] < \infty$$

for a sufficiently large value of β, we show that the optimal control problem has a solution, and that the value function and the optimal control can be represented by means of the solution to the BSDE.

Finally, we devote a brief section to conclusions remarks, presenting in particular some suggestions for future developments of the theory; moreover, in **Appendix A**, **B**, and **C** we fix the main notations and we collect some preliminary useful results.

Chapter 1

Marked point processes

In this Chapter we present the class of marked point processes (MPPs). We follow the dynamic approach used by Last and Brandt [25] and by Brémaud [10]. In particular we give the notion of *compensator*, showing that it describes the local dynamics of a MPP.

In Section 1.1 we start by presenting marked point processes and the associated counting measures.

Section 1.2 is devoted to specific processes which admit an absolutely continuous compensator with respect to Lebesgue measure. In that case the compensator has a density, which is called *stochastic intensity*.

In Section 1.3 we present instead marked point processes with general compensators, for which the existence of a stochastic intensity is not granted. This is the general framework we address to in Section 2.2 and in Section 3.2, where we study respectively BSDEs driven by a marked point process and the associated Optimal Control problem.

Finally in Section 1.4 we introduce semi-Markov processes using the renewal Markov process theory, and we derive an explicit formula for their compensator.

1.1 Point Processes and Counting Measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(K, \mathcal{K})$ a measurable space. Assume we have a sequence $(T_n)_{n\geqslant 1}$ of jump times, T_n taking values in $[0, \infty]$. We set $T_0 = 0$ and we assume, $\mathbb{P}$-a.s.,

$$T_n < \infty \Rightarrow T_n < T_{n+1}, \qquad n \geqslant 0.$$

Such a sequence $(T_n)_{n\geqslant 1}$ is called a *point process*, and T_n is interpreted as the n-th occurrence of a given physical phenomenon. We always assume that (T_n)

is nonexplosive, i.e. $T_n \to \infty$ $\mathbb{P}$-a.s. Associated with these times we define a sequence $(\xi_n)_{n\geqslant 1}$ of random variables, called *marks*, taking values in K (mark space) and containing further informations about the events (for instance ξ_n is the number of customers in the n-th batch of arrivals). We call the double sequence $(T_n, \xi_n)_{n\geqslant 1}$ a *marked point process*. For every $A \in \mathcal{K}$ we define the counting process

$$N_t(A) = \sum_{n\geqslant 0} \mathbb{1}_{T_n \geqslant t} \mathbb{1}_{\xi_n \in A}, \qquad t \geqslant 0, \tag{1.1}$$

and we set $N_t = N_t(K)$. This process is also called a marked point process by abuse of notation (an innocuous, since N_t and (T_n, ξ_n) carry the same informations).

Generally it is difficult to study a marked point process directly in terms of the distribution of the sequence (T_n, ξ_n). We begin thus to introduce the σ-algebras

$$\mathcal{F}_t^0 = \sigma(N_s(A) : s \in [0,\, t], A \in \mathcal{K}), \qquad t \geqslant 0,$$

and we observe that each T_n is an $\mathcal{F}_t^0$-stopping time. We define the filtration generated by the counting processes setting

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^0, \mathcal{N}), \qquad t \geqslant 0,$$

where $\mathcal{N}$ denotes the family of $\mathbb{P}$-null sets in $\mathcal{F}$; it represents the history of the marked point process evolving with time. It turns out that $(\mathcal{F}_t)_{t\geqslant 0}$ is right continuous and therefore satisfies the usual conditions. In the following all the measurability concepts for stochastic processes refer to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ (see Appendix A.1).

The predictable σ-algebra (respectively, the progressive σ-algebra) on $\Omega \times [0,\, \infty)$ is denoted by $\mathcal{P}$ (respectively, by $Prog$). The same symbols also denote the restriction to $\Omega \times [0,\, T]$ for some $T > 0$.

We fix then $\xi_0 \in K$ (deterministic) and we define from $(\xi_n,\, T_n)$ the process

$$X_t = \sum_{n\geqslant 0} \xi_n \mathbb{1}_{[T_n,\, T_{n+1})}(t), \qquad t \geqslant 0. \tag{1.2}$$

We do not assume that $\mathbb{P}(\xi_n \neq \xi_{n+1}) = 1$. Therefore in general trajectories of $(T_n, \xi_n)_{n\geqslant 0}$ cannot be reconstructed from trajectories of $(X_t)_{t\geqslant 0}$ and the filtration $(\mathcal{F}_t)_{t\geqslant 0}$ is not the natural completed filtration of $(X_t)_{t\geqslant 0}$.

For $\omega \in \Omega$ we define a measure on $((0,\, \infty) \times K, \mathcal{B}(0,\, \infty) \otimes \mathcal{K})$ setting

$$p(\omega, C) = \sum_{n\geqslant 1} \mathbb{1}_{\{(T_n(\omega),\, \xi_n(\omega)) \in C\}}, \qquad C \in \mathcal{B}(0,\, \infty) \otimes \mathcal{K} \tag{1.3}$$

where $\mathcal{B}(\Lambda)$ denotes the Borel σ-algebra of any topological space Λ. p is called a random measure since $\omega \mapsto p(\omega, C)$ is $\mathcal{F}$-measurable for fixed C. We also use the notation $p(\omega, dt\, dy)$ or $p(dt\, dy)$. We remark that N is the cumulate distribution associated to p, i.e.:

$$p((0,\, t] \times A) = N_t(A) \quad \text{for} \quad t > 0,\, A \in \mathcal{K};$$

moreover, we have the following equality: for every K-indexed $\mathcal{F}_t$-predictable process H,

$$\int_0^\tau \int_K H_t(y) p(dt\, dy) = \sum_{n \geqslant 1, T_n \leqslant \tau} H_{T_n}(\xi_n),$$

which is always well defined since we are assuming that $T_n \to \infty$ $\mathbb{P}$-a.s.

We present some examples, for which we mainly refer to [25] and [10].

Example 1.1. *One-point process*
Let T be a random element of $(0, \infty])$ and put $T_1 := T$ and $T_n := \infty$, $n \geqslant 2$. Then T_n is called *one-point process.*

The next example defines a point process in terms of the differences between $T_{n+1} - T_n$, $n \in \mathbb{Z}^+ = \{0, 1, ..\}$, $(T_0 := 0)$, where we use the definition $\infty - \infty := 0$. These random variables define a point process uniquely and are sometimes referred to as the *interarrival times.*

Example 1.2. *Homogeneous Poisson process*
We consider a point process T_n and the associated counting process N_t. Suppose that T_1, $T_2 - T_1$, $T_3 - T_2$, .. are independent and distributed according to an exponential distribution with parameter λ. Moreover, for all $0 \leqslant s \leqslant t$, $N_t - N_s$ is $\mathbb{P}$-independent of $\mathcal{F}_s$ given $\mathcal{F}_0$ and, for all $k \geqslant 0$,

$$\mathbb{P}\left\{N_t - N_s = k | \mathcal{F}_s\right\} = \frac{(\lambda \cdot (t-s))^k}{k!} e^{-\lambda(t-s)}, \quad k = 0, 1, 2.. \tag{1.4}$$

Then T_n is called an *homogeneous Poisson process* with intensity λ.

If we know additional informations on the points T_n, we have the corresponding simple marked point processes:

Example 1.3. *Marked one-point process*
Let T be a random element of $(0, \infty]$, ξ a random element of K and put $(T_n, \xi_n) := (T, \xi)$ if $n = 1$ and $T < \infty$, $(T_n, \xi_n) := (\infty, x_\infty)$ otherwise. Then (T_n, ξ_n) is called *marked one-point process.*

Example 1.4. *Independently marked homogeneous Poisson process*
Let T_n an homogeneous Poisson process and ξ_n is an i.i.d.-sequence of random elements of K which is independent of (T_n). Then (T_n, ξ_n) is an *independently marked homogeneous Poisson process.*

1.2 Intensity kernels

In this Section we present the notion of stochastic intensity and its natural link with marked point processes.

Definition 1.1. *Intensity Measure*
The measure on $\mathbb{R}^+ \times K$

$$\Lambda([0,\, t] \times A) := \mathbb{E}\left[N_t(A)\right], \quad A \in \mathcal{K}. \tag{1.5}$$

is called *intensity measure* of N_t.

This new measure yields some information on the distribution of $(T_n, \xi_n)_{n \geqslant 0}$. In the univariate case, the intensity measure is a measure on $\mathbb{R}^+$. For a homogeneous Poisson process, it is multiple of Lebesgue measure, as we have seen in Example 1.2, where $\Lambda([s,\, t]) = \lambda \cdot (t - s)$.

Definition 1.2. Let N_t be a counting process associated to a point process, adapted to some filtration $\mathcal{F}_t$. Suppose that λ_t is a nonnegative $\mathcal{F}_t$-progressive process such that, for all $t \geqslant 0$,

$$\int_0^t \lambda_s ds < \infty \quad \mathbb{P} - a.s.$$

If for all nonnegative $\mathcal{F}_t$-predictable process H_t the equality

$$\mathbb{E}\left[\int_0^\infty H_s dN_s\right] = \mathbb{E}\left[\int_0^\infty H_s \lambda_s ds\right] \tag{1.6}$$

is verified, then we say that N_t admits the $(\mathbb{P}, \mathcal{F}_t)$- stochastic intensity λ_t.

Roughly speaking, the intensity describes the propensity of a process to jump at time t given the whole history up to t. This definition opens the way to a systematic analysis of point processes with the martingale approach. Indeed, as [9] pointed out, it corresponds to say that

$$N_t - \int_0^t \lambda_s ds \quad \text{is an} \quad \mathcal{F}_t - \text{martingale.}$$

and it implies that, for all $0 \leqslant s \leqslant t$,

$$\mathbb{E}\left[N_t - N_s | \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t \lambda_u du | \mathcal{F}_s\right].$$

Definition 1.3. *Intensity Kernel*
Let $(T_n, \xi_n)_n$ be a K-marked point process with filtration $\mathcal{F}_t$ and associated counting measure $p(dt\,dy)$. Suppose that for each $A \in \mathcal{K}$, $N_t(A)$ admits the $(\mathbb{P}, \mathcal{F}_t)$-predictable intensity $\lambda_t(A)$, where $\lambda_t(\omega, dy)$ is a transition measure from $(\Omega \times [0, T), \mathcal{F} \otimes \mathcal{B}_+)$ into $(K, \mathcal{K})$.
Then we say that $p(dt\,dy)$ admits the $(\mathbb{P}, \mathcal{F}_t)$-*intensity kernel* $\lambda_t(dy)$.

Remark 1.1. Let be λ_t a nonnegative $\mathcal{F}_t$-predictable process and $\phi_t(\omega, dy)$ a probability transition kernel from $(\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}_+)$ into $(K, \mathcal{K})$. If the intensity kernel $\lambda_t(dy)$ can be written in the form

$$\lambda_t(dy) = \lambda_t \cdot \phi_t(dy),$$

then the pair $(\lambda_t, \phi_t(dy))$ is called $(\mathbb{P}, \mathcal{F}_t)$-*local characteristics* of $p(dt\,dy)$.

We observe that, since $\phi_t(dy)$ is a probability, $\phi_t(K) = 1$ and then $\lambda_t = \lambda_t(K)$ is the $(\mathbb{P}, \mathcal{F}_t)$-intensity of the underlying point process $N_t = N_t(K)$.

Example 1.5. Consider the marked one-point process N_t as in Example 1.3, and let L be the distribution of (T, ξ). Assume that the function $L((0, t]) \times A)$ has, for $A \in \mathcal{K}$, a piecewise continuous derivative $f(t, A)$. The function

$$r(t, A) = \frac{f(t, A)}{L([t, \infty] \times K)}, \quad t \geqslant 0,$$

(where $0/0 := 0$) is the *hazard rate* of the random time

$$T^A := \begin{cases} T & if \quad \xi \in A \\ \infty & otherwise. \end{cases}$$

If $A = K$, then one also says that $r(t, K)$ is the hazard rate of $L(\cdot \times K)$. Then it can be shown that

$$\lambda(t, A) = \mathbb{1}_{\{t \leqslant T\}}\, r(t, A)$$

is the intensity of N_t.

We give then an important example of an intensity which is deterministic and independent of t.

Proposition 1.1. *Let N_t be a homogeneous Poisson process, with the intensity λ. Then N_t has the stochastic intensity $\lambda_t \equiv \lambda$.*

Proposition 1.1 can be generalized as follows.

Example 1.6. *Poisson process*
A point process T_n (associated to the counting process N_t) is called a *Poisson process* if T_n has independent increments (see Example 1.2) and if there is a measure Λ on $\mathbb{R}^+$, which is locally bounded (i.e. $\Lambda(t) := \Lambda([0,\, t]) < \infty$) such that for all $s < t$, $s, t \in \mathcal{B}^+$ and $k \in \mathbb{Z}^+$

$$\mathbb{P}\left\{N_t - N_s = k | \mathcal{F}_s\right\} = \frac{\Lambda([s,\, t])^k}{k!} e^{-\Lambda([s,\, t])} \quad \text{if} \quad \Lambda([s,\, t]) < \infty. \tag{1.7}$$

Obviously, Λ is then the intensity measure of N_t. If

$$\Lambda(t) = \int_0^t \lambda(s) ds$$

for some measurable function λ, then λ is called the intensity function of N_t. If $\lambda(t)$ is piecewise continuous then $\lambda(t)$ is the stochastic intensity of N_t which is in fact deterministic. As claimed in Example 1.2, a homogeneous Poisson process can indeed be shown to have independent increments and to satisfy 1.7 with $\Lambda(dt) = \lambda dt$. The intensity function is then constant.

We shall now give two examples of martingales to respect to point process filtrations. To state the first, we define the *hazard measure* of a marked point process (T, ξ) as the measure on $\mathbb{R}_+ \times K$ given by

$$R(d(t, x)) := \frac{L(d(t, x))}{L([t,\, \infty] \times K)} \tag{1.8}$$

and $R(\{0\} \times K) = 0$, where L is the distribution of (T, ξ).

Proposition 1.2. *Consider the marked one-point process associated with a marked point* (T, ξ)*, see Example 1.3. Let* R *be the hazard measure of* (T, ξ)*. Then*

$$M_t(A) := N_t(A) - R((0, t \wedge T] \times A), \quad t \in \mathbb{R}^+, A \in \mathcal{K},$$

is an $\mathcal{F}_t$*-martingale for all* $A \in \mathcal{K}$*.*

Finally we refer to Example 1.6, where we have already observed that a Poisson process is a point process with a deterministic intensity kernel $\lambda(t)$. Moreover, Watanabe [38] in 1964 pointed out that the Poisson process plays a distinguished role among point processes. Indeed the martingale property of the process $N_t - \int_0^t \lambda(s) ds$, where $\lambda(t)$ is some locally integrable deterministic function, characterizes N_t as a Poisson process with intensity $\lambda(t)$. More precisely:

Theorem 1.3. *Martingale characterization of the Poisson process*
Let N_t be a point process and let $\lambda(t)$ be a locally integrable nonnegative measurable function. Suppose that

$$N_t - \int_0^t \lambda(s)ds, \quad t \geqslant 0,$$

is an $\mathcal{F}_t$-martingale. Then N_t is a $\mathcal{F}_t$-Poisson process with intensity $\lambda(t)$, i.e., for all $0 \leqslant s \leqslant t$, $N_t - N_s$ is a Poisson random variable with parameter $\int_0^t \lambda(u)du$, independent of $\mathcal{F}_s$.

We end the section with two fundamental results, that we analyse in the following section in a more general context.

Theorem 1.4. *Projection Theorem (see [10] Theorem T3)*
Let $(T_n, \xi_n)_n$ be a K-marked point process with the counting measure $p(dt\,dy)$ and the $\mathcal{F}_t$-intensity kernel $\lambda_t(dy)$. Then, for each nonnegative $\mathcal{F}_t$.predictable K-marked process H,

$$\mathbb{E}\left[\int_0^\infty \int_K H_t(y)p(dt\,dy)\right] = \mathbb{E}\left[\int_0^\infty \int_K H_t(y)\lambda_t(dy)dt\right]. \tag{1.9}$$

Corollary 1.5. *Integration Theorem*
Let $(T_n, \xi_n)_n$ be a K-marked point process with the counting measure $p(dt\,dy)$ and the $\mathcal{F}_t$-intensity kernel $\lambda_t(dy)$. Let H be a $\mathcal{F}_t$-predictable K-indexed process such that, for all $t \geqslant 0$, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^t \int_K |H_t(y)|\,\lambda_t(dy)dt\right] &< \infty, \\ \left[\int_0^t \int_K |H_t(y)|\,\lambda_t(dy)dt\right. &< \infty \quad \mathbb{P}-a.s.,\left.\right] \end{aligned}$$

then

$$\mathbb{E}\left[\int_0^t \int_K |H_t(y)|\,q(dt\,dy)\right] \quad \text{is a} \quad (\mathbb{P}, \mathcal{F}_t) - \text{martingale} \tag{1.10}$$

where we have defined $q(dt\,dy) = p(dt\,dy) - \lambda_t(dy)dt$.

1.3 Toward a general theory of intensity

Even in the case where N_t is nonexplosive, the existence of an $\mathcal{F}_t$-stochastic intensity λ_t for N_t is not granted. For this reason, we ask ourselves what does happen in the case where there is no intensity for N_t. We start by the following theorem for the point processes (see [10] Theorem T12):

Theorem 1.6. *Let N_t be a point process with the filtration $\mathcal{F}_t$ that verifies the usual conditions. Then there exists a unique, right-continuous $\mathcal{F}_t$-predictable nondecreasing process A satisfying $A_0 \equiv 0$ such that, for all H_t $\mathcal{F}_t$-predictable processes,*

$$\mathbb{E}\left[\int_0^{T_n} H_t dN_t\right] = \mathbb{E}\left[\int_0^{T_n} H_t dA_t\right]. \tag{1.11}$$

The above stochastic integrals are defined for $\mathbb{P}$-almost every ω as ordinary (Stieltjes) integrals. A is called the *compensator*, or the dual predictable projection, of N.

Remark 1.2. To understand the meaning of the word "compensator", we present a simple calculation. Indeed, if we choose $H_t = \mathbb{1}_{[s,t]}\mathbb{1}_B$, B $\mathcal{F}_s$-measurable, the definition above becomes

$$\begin{aligned}
\mathbb{E}\left[(N_t - N_s)\mathbb{1}_B\right] &= \mathbb{E}\left[(A_t - A_s)\mathbb{1}_B\right] \\
\mathbb{E}\left[(N_t - A_t)\mathbb{1}_B\right] &= \mathbb{E}\left[(N_s - A_s)\mathbb{1}_B\right] \\
\mathbb{E}\left[(N_t - A_t)|\mathcal{F}_s\right] &= N_s - A_s.
\end{aligned}$$

i.e. $N_t - A_t$ is a $\mathcal{F}_t$-martingale. In other words, N_t itself is not a martingale, but it becomes a martingale when we subtract an increasing, right-continuous predictable process A_t. The compensator thus characterizes the distribution of a marked point process.

Remark 1.3. We do not assume N_t to be nonexplosive. In the case where N_t is nonexplosive, if A_t is absolutely continuous with respect to the Lebesgue measure in the sense that

$$A_t = \int_0^t \lambda_s ds$$

for some $\mathcal{F}_t$-progressive nonnegative process λ_t, then we are back to the situation considered in the previous section, i.e. we have a stochastic intensity kernel exists.

Remark 1.4. In the following we always make the assumptions that $\mathbb{P}$-a.s.

$$A \text{ has continuous trajectories} \tag{1.12}$$

which are in particular finite-valued.

The above Theorem can be extended to the case of marked point processes. However, we need some additional assumptions on the space of marks $(K, \mathcal{K})$.

Hypotheses 1.1. *K is a Borel subset of a compact metric space and $\mathcal{K}$ consists of the Borelians of K.*

Under Hypotheses 1.1 on K, it can be proved that there exists a function $\phi_t(\omega, A)$ such that

(1) for every $\omega \in \Omega$, $t \in [0, \infty)$, the mapping $A \mapsto \phi_t(\omega, A)$ is a probability measure on $(K, \mathcal{K})$;

(2) for every $A \in \mathcal{K}$, the process $(\omega, t) \mapsto \phi_t(\omega, A)$ is predictable;

(3) for every nonnegative $H_t(\omega, y)$, $\mathcal{P} \otimes \mathcal{K}$-measurable, we have

$$\mathbb{E}\left[\int_0^\infty \int_K H_t(y) p(dt\, dy)\right] = \mathbb{E}\left[\int_0^\infty \int_K H_t(y) \phi_t(dy) dA_t\right]. \tag{1.13}$$

The random measure $\phi_t(\omega, dy) dA_t(\omega)$ is denoted $\tilde{p}(\omega, dt\, dy)$ or simply $\tilde{p}(dt\, dy)$, and is called the compensator, or the dual predictable projection, of p.

Fix $T > 0$, and let $H_t(\omega, y)$ be a $\mathcal{P} \otimes \mathcal{K}$-measurable real function satisfying

$$\int_0^T \int_K |H_t(y)|\, \phi_t(dy) dA_t < \infty, \qquad \mathbb{P} - a.s.$$

Then the following stochastic integral can be defined, for every $\tau \in [0, T]$,

$$\int_0^\tau \int_K |H_t(y)|\, q(dt\, dy) := \int_0^\tau \int_K |H_t(y)|\, p(dt\, dy) - \int_0^\tau \int_K |H_t(y)|\, \phi_t(dy) dA_t \tag{1.14}$$

as the difference of ordinary integrals with respect to p and $\tilde{p}$. Here and in the following the symbol $\int_a^b$ is to be understood as an integral over the interval $(a, b]$. We shorten this identity writing $q(dt\, dy) = p(dt\, dy) - \tilde{p}(dt\, dy) = p(dt\, dy) - \phi_t(dy) dA_t$.

Now for all $r \geqslant 1$ we define $\mathcal{L}^{r,0}(p)$ as the space of $\mathcal{P} \otimes \mathcal{K}$-measurable real functions $H_t(\omega, y)$ such that

$$\mathbb{E}\left[\int_0^T \int_K |H_t(y)|^r\, p(dt\, dy)\right] = \mathbb{E}\left[\int_0^T \int_K |H_t(y)|^r\, \phi_t(dy) dA_t\right] < \infty$$

(the equality of the integrals follows from the definition of $\phi_t(dy)$).

Given an element H of $\mathcal{L}^{1,0}(p)$, the stochastic integral (1.14) turns out to be a finite variation martingale (see Appendix A.2).

Specifically, we have the following result:

Corollary 1.7. *We suppose that Hypotheses 1.1 on K hold and that the quantities on either side of* (1.13) *are finite. Then*

$$\int_0^{t \wedge T_n} \int_K H_t(y) q(dt\,dy)$$

is a $(\mathbb{P}, \mathcal{F}_t)$*-martingale, where* $q(dt\,dy) = p(dt\,dy) - \phi_t(dy) dA_t$.

The measure $q(dt\,dy)$ allows us to create new martingales. Moreover, there exists an integral representation theorem of marked point process martingales (see e.g. [13], [14]) which is a counterpart of the well known representation result for Brownian martingales (see e.g. [34] Ch V.3 or [18] Theorem 12.33). Recall that $(\mathcal{F}_t)$ is the filtration generated by the jump process, augmented by the usual way. Then

Theorem 1.8. *Let M be a cadlag* $(\mathcal{F}_t)$*-martingale on* $[0, T]$. *Then we have*

$$M_t = M_0 + \int_0^{\tau} \int_K H_t(y) q(dt\,dy), \qquad \tau \in [0, T],$$

for some process $H \in \mathcal{L}^{1,0}(p)$.

This theorem is the key result used in the construction of a solution to the BSDE (2.18).

1.4 Semi-Markov processes and their compensators

We want to study stochastic processes, namely the semi-Markov ones, which are specific jump processes. These form a large family and are very common in statistic applications and in control problems. In particular we make explicit the associated compensator form. As we will see in Chapter 2 and 3, its knowledge is fundamental to the application of the optimal control theory based on BSDEs; moreover, an explicit compensator formula can considerably simplify the BSDE we need to solve to find the optimal control solution.

We start to briefly present the Markov time-homogeneous processes, which can be seen as a specific case of the semi-Markov ones.

1.4.1 Markov processes

Let $X = (X(t))_{t \in \mathbb{R}_+}$ be a stochastic process with state space $(K, \mathcal{K})$.

Definition 1.4. If for all $(s,t) \in \mathbb{R}_+^2$ and $A \in \mathcal{K}$, we have

$$\mathbb{P}\{X(t+s) \in A | X(u), u \leqslant s\} = \mathbb{P}\{X(t+s) \in A | X(s)\} \quad \mathbb{P} - \text{a.s.},$$

then the stochastic process X is called a *Markov process.*

Intuitively it means that the future is independent of the past given the present. The regular conditional distributions

$$P_{s,t}(x,A) := \mathbb{P}\{X(t) \in A | X(s) = x\}, \quad A \in \mathcal{K},\ x \in K,\ (s,t) \in \mathbb{R}_+^2,\ s \leqslant t,$$

are called *transition probability functions* of the Markov process X. We assume that they always exist.

If $P_{s,t}(x,A)$ depends only on the difference $t-s$, i.e., $P_{s,t}(x,A) = P_{0,t-s}(x,A)$, for all $x \in K$, $A \in \mathcal{K}$, $(s,t) \in \mathbb{R}_+^2$, $s \leqslant t$, we say that the Markov process is *time-homogeneous.* In this case we simplify the notation, writing

$$P_t(x,A) = P_{\tau,\tau+t}(x,A).$$

Definition 1.5. *Jump process*
Let $(X(t),\ t \in \mathbb{R}_+)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(K,\ \mathcal{K})$. The process is said to be a *jump process* if, for all $\omega \in \Omega$ and all $t \in R_+$, there exists a $\delta = \delta(t,\omega)$ such that $Z(t+h) = Z(t)$ for $0 \leqslant h < \delta$ or, equivalently, if the trajectories are right-continuous in the discrete topology on the state space.

We will consider Markov jump-processes. These processes start in a state, stay there for a certain length of time, then jumps to another state, stays for a certain length of time and so on.

We introduce $0 = T_0 < T_1 < ..$ as the jump times of the Markov process X; for each $t \geqslant 0$ we define the random variable $N(t)$ by

$$N(t) = \max\{n : T_n \leqslant t\},$$

which counts the number of jumps in the time interval $(0,\ t]$.

Then we consider the discrete-time process $(\xi_n, n = 0, 1, ..)$ by

$$\xi_0 = X(0) \quad \text{and} \quad \xi_n = X(T_n).$$

We have thus $X(t) = \xi_{N(t)}$.

The random variables $S_1, S_2, ..$ defined by

$$S_{n+1} = T_{n+1} - T_n$$

for all $n \geqslant 0$, are the successive sojourn times in the states $\xi_0, \xi_1, ..$ visited by the process X. We assume that $S_\infty = \infty$ if $T_n = \infty$.
The stochastic process $(\xi_n, S_n)_{n=0,1,2,..}$ is the embedded process of $X(t)$. The process $(\xi_n)_{n \geqslant 0}$ is a Markov chain.

Markov time-homogeneous processes can be characterized in the following way:

Theorem 1.9. *Let be X_t a stochastic process specified by the marked point process $(\xi_n, T_n)_{n \geqslant 1}$. Then X_t is a Markov time-homogeneous process iff*

(1) The random variables $S_{n+1} = T_{n+1} - T_n$ and ξ_{n+1} have independent distributions conditional of $\mathcal{F}_{T_n}$;

(2) There exists a function $\lambda(x)$ such that the distribution of S_{n+1} conditional of $\mathcal{F}_{T_n}$ is given by

$$\begin{aligned} \mathbb{P}\{S_{n+1} \leqslant t | \mathcal{F}_{T_n}\} &= \mathbb{P}\{S_{n+1} \leqslant t | \xi_n\} \\ &= 1 - e^{-\lambda(\xi_n)t}, \quad t \in \mathbb{R}_+ \end{aligned}$$

i.e., S_{n+1} conditional of $\mathcal{F}_{T_n}$ has an exponential distribution with rate $\lambda(\xi_n)$.

We can introduce the transition rate $\lambda_0(x, A)$ of a Markov time-homogeneous process:

Definition 1.6. *Transition rate*
We denote $\pi(x; A)$ the probability such that $\pi(x; A) = \mathbb{P}\{\xi_{n+1} \in A | \xi_n = x\}$, and we consider the rate $\lambda(x)$ as described in Theorem 1.9. Then

$$\lambda_0(x, A) := \lambda(x)\, \pi(x; A).$$

is called the *transition rate* of X.

Remark 1.5. The time-homogeneous request in Theorem 1.9 is fundamental. Indeed, property *(1)* doesn't hold for general time non-homogeneous processes; moreover, in the general case the rate λ is dependent of the time and thus the waiting time distribution is not exponential.

In the following we mainly refer to Limnios [26] and Gikhman-Skorohod [21].

1.4.2 Markov renewal processes

The study of semi-Markov processes is closely related to the theory of Markov renewal processes, that generalizes the notion of Markov jump processes. Through all the chapter, $(K, \mathcal{K})$ is a measurable space such that $\{x\} \in \mathcal{K}$ for all $x \in K$.

Definition 1.7. A function $\pi(x; A)$, $x \in K$, $A \in \mathcal{K}$, is called a *sub-Markov transition function* (or a *sub-Markov kernel*) on $(K, \mathcal{K})$ if:

(1) $\forall x \in K$, $\pi(x; \cdot)$ is a measure on $\mathcal{K}$ such that $\pi(x; K) \leqslant 1$;

(2) $\forall A \in \mathcal{K}$, $\pi(\cdot; A)$ is a $\mathcal{K}$-measurable function;

(3) $\forall A \in \mathcal{K}$, $\forall x \in K$, $\pi(x; A)$ is a Borel measurable function.

If $\pi(x; K) = 1$ $\forall x \in K$, then $\pi(x; A)$ is a *Markov transition function* (or a *Markov kernel*) on $(K, \mathcal{K})$.

If K is finite or a countable set and if $\mathcal{K} = \mathcal{P}(K)$, a transition function on $(K, \mathcal{K})$ is determined by the matrix $(\pi_{ij}; i, j \in K)$. Hence $\pi(i; A) = \sum_{j \in A} \pi_{ij}$, $A \in \mathcal{K}$.

Definition 1.8. A function $Q(x; A, t)$, $x \in K$, $t \in \mathbb{R}_+$, $A \in \mathcal{K}$, is called a *semi-Markov kernel* on $(K, \mathcal{K})$ if:

(1) $Q(x; A, \cdot)$ $\forall x \in K$, $A \in \mathcal{K}$, is nondecreasing, right continuous real function such that $Q(x; A, 0) = 0$;

(2) $Q(\cdot; \cdot, t)$ $\forall t, \in \mathbb{R}_+$, is a sub-Markov kernel on $(K, \mathcal{K})$;

(3) $\pi(\cdot; \cdot) = Q(\cdot; \cdot, +\infty)$ is a Markov kernel on $(K, \mathcal{K})$.

The following properties of a semi-Markov kernel are straightforward consequences of the above definitions:

- For each $x \in K$, $Q(x; \cdot, \cdot)$ defines a probability measure on the σ-algebra $\mathcal{B}_+ \otimes \mathcal{K}$.
- For each $x \in K$, the function $H(x; \cdot) = Q(x; K, \cdot)$ is a distribution function such that $H(x, ; 0) = 0$.
- For each $t \in \mathbb{R}_+$, $A \in \mathcal{K}$, $Q(\cdot; A, t)$ is an $\mathcal{K}$-measurable function.

Due to the inequality $Q(x; A, t) \leqslant \pi(x; A)$, $x \in K$, $t \in \mathbb{R}_+$, $A \in \mathcal{K}$, the measure $Q(x; \cdot, t)$ is absolutely continuous with respect to the measure $\pi(x; \cdot)$ for each $t \in \mathbb{R}_+$, $x \in K$ (i.e., $\pi(x; A) = 0$ implies $Q(x; A, t) = 0$.) According to the Radon-Nikodym theorem, there exists a real $\mathcal{K}$-measurable function $F(x; y, t)$ such that

$$Q(x; A, t) = \int_A F(x; y, t)\, \pi(x; dy), \quad A \in \mathcal{K}. \tag{1.15}$$

It is easy to show that, for fixed $x, y \in K$, the function $F(x; y, \cdot)$ is nondecreasing. Hence $F(x; y, \cdot)$ can be chosen right-continuous, as is $Q(x; A, \cdot)$. Moreover, throughout we shall assume that $F(\cdot; \cdot, t)$, for fixed $t \in \mathbb{R}_+$, is $\mathcal{K} \otimes \mathcal{K}$-measurable.

The same argument can be applied to Q and H. Indeed, for fixed $x \in K$ and $A \in \mathcal{K}$, the measure $Q(x; A, \cdot)$ on $(\mathbb{R}_+, \mathcal{B}_+)$ is absolutely continuous with respect to the measure $H(x; \cdot)$. Hence there exists a real, $\mathcal{B}$-measurable function $q(x; A, \cdot)$ such that

$$Q(x; A, t) = \int_0^t q(x; A, u)\, H(x; du), \quad t, \in \mathbb{R}_+,\ x \in K,\ A \in \mathcal{K}. \tag{1.16}$$

Obviously, $q(x; \cdot, t)$ is a measure on $\mathcal{K}$ and, since $q(x; A, \cdot)$ is nondecreasing, and right continuous, the function $q(\cdot; \cdot, t)$ is $\mathcal{K} \otimes \mathcal{K}$-measurable for each fixed $A \in \mathcal{K}$.

In the following we show how new stochastic processes can be obtained by the semi-Markov kernel notion. In particular we define the Markov renewal process associated to the semi-Markov kernel Q. On the measurable space $(K \times \mathbb{R}_+, \mathcal{K} \otimes \mathcal{B}_+)$, let $P((x, s), A \times [0,\, t])$ be the Markov transition function defined by

$$P((x, s), A \times [0,\, t]) = Q(x; A, t - s) \tag{1.17}$$

for $(x, s) \in K \times \mathbb{R}_+$, $A \times [0,\, t] \in \mathcal{K} \otimes \mathcal{B}$. It is well known that, for each $(x, s) \in K \times \mathbb{R}_+$, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P}_{(x,s)})$ and a sequence of random variables $(\xi_n, T_n)_{n \in \mathbb{N}}$ such that

$$\mathbb{P}_{(x,s)} \{\xi_0 = x, T_0 = s\} = 1,$$

and

$$\begin{aligned}
\mathbb{P}_{(x,s)} \{\xi_{n+1} \in A, T_{n+1} \leqslant t | \sigma(\xi_m, T_m, m \leqslant n)\} &= \mathbb{P}_{(x,s)} \{\xi_{n+1} \in A, T_{n+1} \leqslant t | \xi_n, T_n\} \\
&= Q(\xi_n; A, t - T_n).
\end{aligned}$$

for all $n \in \mathbb{N}$, $t,\, s \in \mathbb{R}_+$, $A \in \mathcal{K}$.

Thus $(\xi_n, T_n)_{n \in \mathbb{N}}$ is a Markov process with the state space $(K \times \mathbb{R}_+, \mathcal{K} \otimes \mathcal{B}_+)$ and the transition probability given by (1.17).

Definition 1.9. The process (ξ_n, T_n) is called the *Markov renewal process* associated to the semi-Markov kernel Q.

Let us denote $S_0 = T_0$, $S_{n+1} = T_{n+1} - T_n$, $n \in N$ and let set

$$\begin{aligned}
\mathcal{F}_n &= \sigma((\xi_m, S_m),\, m \leqslant n), \\
\mathcal{M}_n &= \sigma((\xi_m,\, m \leqslant n).
\end{aligned}$$

Theorem 1.10. *For each $s \in \mathbb{R}_+$ the processes $(\xi_n, \mathcal{M}_n, \mathbb{P}_{(x,s)})$ and $((\xi_n, S_n), \mathcal{F}_n, \mathbb{P}_{(x,s)})$ are Markov chain with state spaces $(K, \mathcal{K})$ and $(K \times \mathbb{R}_+, \mathcal{K} \otimes \mathcal{B}_+)$, respectively. Their transition probabilities are given by*

$$\begin{aligned} \mathbb{P}_{(x,s)}\left\{\xi_{n+1} \in A | \mathcal{M}_n\right\} &= \pi(\xi_n; A), \qquad x \in K,\ A \in \mathcal{K}, s \in \mathbb{R}_+, \\ \mathbb{P}_{(x,s)}\left\{\xi_{n+1} \in A, S_{n+1} \leqslant t | \mathcal{F}_n\right\} &= Q(\xi_n; A, t), \quad x \in K,\ A \in \mathcal{K},\ t,\ s \in \mathbb{R}_+. \end{aligned}$$

1.4.3 Semi-Markov processes

Once we have the semi-Markov kernel Q and the associated Markov renewal process, we can achieve the definition of a semi-Markov process. Let $Q(x; A, t)$, $x \in K$, $A \in \mathcal{K}$, $t \in \mathbb{R}_+$, be a semi-Markov kernel on $(K, \mathcal{K})$ and let (ξ_n, T_n) the associate Markov renewal process. If we set

$$N(t) = \begin{cases} 0 & \text{if} \quad S_1 > t \\ \sup\{n \in \mathbb{N} : S_1 + \cdots + S_n \leqslant t\} & \text{if} \quad S_1 < t, \end{cases}$$

then we can define the jump process

$$Z(t) = Z_t = \xi_n \quad \text{for} \quad T_n \leqslant t + T_0 < T_{n+1}, \quad t \in \mathbb{R}_+,\ n \in \mathbb{N},$$

or, equivalently,

$$Z_t = \xi_{N(t)}, \quad t \in \mathbb{R}_+.$$

The jump times are $T_1 - T_0$, $T_2 - T_1$, .. and the intervals jumps are S_1, S_2, ...

Definition 1.10. The stochastic process $(Z(t))_{t \in \mathbb{R}_+}$ defined above is called a *semi-Markov process* corresponding to the semi-Markov kernel Q.

Remark 1.6. The random variable $T_\infty = \lim_{n \to \infty} T_n$ is called the *explosion time.* The semi-Markov kernel (as well as the corresponding Markov renewal and semi-Markov processes) is called *regular* if

$$T_\infty = \infty, \quad \mathbb{P}_{(x,s)} - a.s. \quad \text{for all} \quad x \in K,\ s \in \mathbb{R}_+.$$

A semi-Markov process Z_t is thus a stochastic process which makes transitions from state to state in accordance with a Markov chain, but in which the amount of time spent in each state before a transition occurs is random. We denote ξ_{n+1} as the *embedded Markov chain* of the process Z_t. In this contest, $Q(x; A, t)$ represents the probability that after making a transition in x, the process next makes a transition into a state belonging to the set A, in an amount of time less or equal to t; $\pi(x; A)$ and $H(x; t)$ are respectively the marginal distribution of ξ_{n+1} and S_{n+1}, while the functions $F(x; y, t)$ and $q(x; y, t)$ give us the conditional density function of S_{n+1} given ξ_{n+1}, and of ξ_{n+1} given S_{n+1}, respectively.

1.4.4 Derivation of the compensator

In this Section we show how to calculate the explicit form for the compensator associated to a semi-Markov process. We start from the work of Jacod [23], where a formula for the compensator is given in the general contest of marked point processes. We interpret the Jacod's formula for the particular case of semi-Markov processes, and we give a final formula consistent with the notions of Q, H and q introduced above.

In his paper Jacod defined the random measure $G_n(\omega; dt, dx)$ as the conditional law of (S_{n+1}, ξ_{n+1}) with respect to the σ-algebra $\mathcal{F}_{T_n}$:

$$\begin{aligned} G_n(\omega, dt, dx) &= \mathbb{P}\left\{S_{n+1} \in dt, \xi_{n+1} \in dx | \mathcal{F}_{T_n}\right\} \\ &= \mathbb{P}\left\{S_{n+1} \in dt, \xi_{n+1} \in dx | \sigma(\xi_0, T_0, .., \xi_n, T_n)\right\}. \end{aligned}$$

and the corresponding marginal measure $H_n(\omega; dt)$:

$$H_n(\omega; dt) = G_n(\omega; dt, K)$$

as the conditional law of S_{n+1}.

The general Jacod's formula for the compensator of a marked point processes is then:

Proposition 1.11. *([23] Prop. 3.1) The predictable projection of the measure*

$$p(dt, dy) = \sum_{n \geqslant 1} \delta_{(T_n, \xi_n)}(dt, dy)$$

is given by

$$\tilde{p}(dt, dy) = \sum_{n \geqslant 1} \frac{G_n(dt - T_n, dy)}{H_n([t - T_n, +\infty])} \mathbb{1}_{T_n < t \leqslant T_{n+1}}. \tag{1.18}$$

Now we want to understand the link between the measure G_n and the cumulative distribution functions Q and H introduced above for a semi-Markov process. We observe that in the semi-Markov case the measures G_n and H_n assume a simpler form:

$$\begin{aligned} G_n(dt, dy) &= \mathbb{P}\left\{S_{n+1} \in dt, \xi_{n+1} \in dy | \xi_n, T_n\right\} \\ &= \mathbb{P}_{(x,s)}\left\{S_{n+1} \in dt, \xi_{n+1} \in dy\right\} \end{aligned} \tag{1.19}$$

and

$$\begin{aligned} H_n(dt) &= \mathbb{P}\left\{S_{n+1} \in dt | \xi_n, T_n\right\} \\ &= \mathbb{P}_{(x,s)}\left\{S_{n+1} \in dt\right\}, \end{aligned} \tag{1.20}$$

where we are assuming that $\xi_n = x$ and $T_n = s$. We observe that the measure G_n corresponds to the cumulative distribution function Q when $C = A \times [0,\, t]$:

$$G_n(C) = Q(x; A, t).$$

We introduce then the measures $\tilde{Q}$ and $\tilde{H}$, for which Q and H are the respective cumulative distribution functions:

$$\tilde{H}(x; [0,\, t]) = H(x; t) \quad \text{or equivalently,} \quad H(x; t) = \int_0^t \tilde{H}(x; dv),$$

$$\tilde{Q}(x; A, [0,\, t]) = Q(x; A, t) \quad \text{or equivalently,} \quad Q(x; A, t) = \int_0^t \tilde{Q}(x; A, dv).$$

Thus we obtain

$$G_n(dt - T_n, dy) = \tilde{Q}(\xi_n; dt - T_n, dy).$$

On the other hand, the denominator of (1.18) is

$$H_n([t - T_n,\, +\infty]) = \mathbb{P}_{(x,s)}\left\{S_{n+1} \geqslant t - T_n\right\} = \tilde{H}(\xi_n; [t - T_n,\, +\infty]).$$

Finally we have to compute

$$\frac{G_n(dt - T_n, dy)}{H_n([t - T_n,\, +\infty])} = \frac{\tilde{Q}(\xi_n; dy, dt - T_n)}{\tilde{H}(\xi_n; [t - T_n,\, +\infty])}. \tag{1.21}$$

By the definition of $\tilde{Q}$ and $\tilde{H}$, (1.16) can be rewritten as:

$$\tilde{Q}(x; A, t) = q(x; A, t)\tilde{H}(x; dt).$$

Moreover, we introduce the hazard measure of H (see (1.8)), that is

$$R(x; dt) = \frac{\tilde{H}(x; dt)}{\tilde{H}(x; [t,\, +\infty])}. \tag{1.22}$$

With these notations (1.21) becomes:

$$\begin{aligned} \frac{G_n(dt - T_n, dy)}{H_n([t - T_n,\, +\infty])} &= \frac{q(\xi_n; dy, t - T_n)\tilde{H}(\xi_n; dt - T_n)}{\tilde{H}(\xi_n; [t - T_n,\, +\infty])} \\ &= q(\xi_n; dy, t - T_n)\, R(\xi_n; dt - T_n). \end{aligned}$$

Finally we achieve the following formula for the compensator of a semi-Markov processes:

$$\begin{aligned} \tilde{p}(dt, dy) &= \sum_{n \geqslant 0} q(\xi_n; dy, t - T_n)\, R(\xi_n; dt - T_n)\mathbb{1}_{T_n < t \leqslant T_{n+1}} \\ &= q(X_{t-}; dy, a(t-))\, R(X_{t-}; dt - T_{N(t)}), \end{aligned} \tag{1.23}$$

where

$$a(t) = t - T_{N(t)}.$$

We introduce the survival function of Q, that is:

$$h(x;t) = 1 - H(x;t) = Q(x; K, [t, +\infty])$$

If we assume that h admits a rate, i.e., there exists a $\lambda(x)$ such that

$$\lambda(x) := -\frac{h'(x;t)}{h(x;t)},$$

then the hazard measure R becomes

$$R(x; dt) = \lambda(x)dt$$

and the formula (1.23) in this case is:

$$\tilde{p}(dt, dy) = q(X_{t-}; dy, a(t-))\, \lambda(X_{t-})dt. \tag{1.24}$$

Compensator for time-homogeneous Markov processes Markov time-homogeneous processes can be seen as specific semi-Markov processes. As we already underlined, in this case ξ_{n+1} and S_{n+1} have independent distribution conditionally to $\mathcal{F}_{T_n}$ and then

$$\begin{aligned} q(x; A, a(t)) &= \mathbb{P}_{(x,s)}\{\xi_{n+1} \in A | S_{n+1} \leqslant t - T_n\} \\ &= \mathbb{P}_{(x,s)}\{\xi_{n+1} \in A\} \\ &= \pi(x; A) \\ &= \frac{\lambda_0(x; A)}{\lambda(x)}. \end{aligned}$$

where $\lambda_0(x; A)$ is the transition rate of the process introduced by (1.6).

Then for Markov time-homogeneous processes, the compensator is:

$$\begin{aligned} \tilde{p}(dt, dy) &= q(X_{t-}; dy, a(t-))\, \lambda(X_{t-})dt \\ &= \frac{\lambda_0(X_{t-}; dy)}{\lambda(X_{t-})}\, \lambda(X_{t-})dt \\ &= \lambda_0(X_{t-}; dy)dt. \end{aligned} \tag{1.25}$$

As we expected, the compensator of a Markov time-homogeneous process does not depend on t neither on $a(t) = t - T_{N(t)}$.

Remark 1.7. Let be $(Z(t), t \in \mathbb{R}_+)$ a semi-Markov process corresponding to the semi-Markov kernel Q on $(K, \mathcal{K})$. As we already pointed out in the above discussion, for such a process the Markov property doesn't hold. However, if we consider the process $(Z(t), a(t), t \in \mathbb{R}_+)$, where $a(t) = t - T_{N(t)}$, then it can be shown that this one is a Markov time-homogeneous process. In particular we have the following result:

Theorem 1.12. *([26] Theorem 3.12) The stochastic process* $(Z(t),\, a(t))$ *with values in* $K \times \mathbb{R}_+$ *is a Markov process. More precisely, for every* $(x,\, s) \in K \times \mathbb{R}_+$*:*

$$\begin{aligned}\mathbb{P}_{(x,s)}\left\{Z(t+h) \in A, a(t+h) \leqslant \alpha | \sigma(Z()\tau, a(\tau), \tau < t),\, Z(t) = y, a(t) = u\right\}\\ = \varphi((y,\, u); A, \alpha)\end{aligned}$$

where $y \in K,\ u, \alpha, t, h \in \mathbb{R}_+,\ A \in \mathcal{K},$ *and* $\varphi(\cdot, \cdot)$ *is an opportune Markov transition function.*

Chapter 2

Backward stochastic differential equations

In this Chapter we state and prove the main results on existence and uniqueness of backward stochastic differential equations.

In Section 2.1 we present the standard theory about BSDEs driven by a Wiener process; then in Section 2.2 we consider the particular class of BSDEs driven by a random measure naturally linked to a marked point process. In both cases we first give some notations and definitions about BSDEs and their solutions spaces; then we state and prove the existence and uniqueness results; finally we conclude by the analyse of the parameters dependency upon the data and upon a given process.

2.1 BSDEs driven by a Wiener process

In this Section we present the BSDE theory when the stochastic equation is driven by a Wiener process.

We start by giving the main notations and tools on BSDEs theory we will use in the next paragraphs.

2.1.1 Notation and setting

Let W a Wiener brownian motion in $\mathbb{R}^d$, $W = \{W_t^i, t \geqslant 0, i = 1, ..., d\}$, defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ be the family of elements of $\mathcal{F}$ of probability 0. We define

$$\mathcal{F}_\tau = \sigma(W_t : t \in [0,\ \tau], \mathcal{N})$$

i.e, for every $\tau \geqslant 0$, $\mathcal{F}_\tau$ is the σ-algebra generated by the random variables $\{W_t : t \in [0,\, \tau]\}$, augmented by the $\mathbb{P}$-null sets. It is well-known that the filtration $(\mathcal{F}_\tau)_{\tau\geqslant 0}$ is right-continuous (i.e ($\mathcal{F}_\tau = \cap_{t>\tau}\mathcal{F}_t$) for every τ), hence the so-called "usual conditions" hold.

We recall that a process $Z : \Omega \times \mathbb{R}_+ \to E$ is said to be predictable if it is measurable with respect to the predictable σ-field $\mathcal{P}_{[0,\,T]} = \sigma\{A \times (t, \tau] : A \in \mathcal{F}_t,\ 0 \leqslant t < \tau < T\}$.

Let Z be a predictable process with values in $L(\mathbb{R}^d, \mathbb{R}^k)$:

$$Z = \{Z_t^{ij}, t \geqslant 0, i = 1, ..., k, j = 1, ..., d\},$$

satisfying $\mathbb{P}\left\{\int_0^T \left|Z_t^{ij}\right|^2 dt < \infty\right\} = 1$ for every $T > 0$ and for every i, j. Then the Ito integral $I = \{\int_0^\tau Z_t dW_t, \tau \geqslant 0\}$ is defined as the $\mathbb{R}^k$-valued process with components

$$I_\tau^i = \sum_{j=1}^d \int_0^\tau Z_t^{ij} dW_t^j, \qquad i = 1, ..., k.$$

The process $\{I_\tau, \tau \geqslant 0\}$ is a continuous local martingale in $\mathbb{R}^k$, null at 0.

We will need the following special case of the Burkholder-Davis-Gundy inequalities (see Appendix A.2, Theorem A.8): for $p \in (0, \infty)$, there exists a constant c_p such that

$$\mathbb{E}\left[\sup_{\tau\in[0,\,T]}\left|\int_0^\tau Z_t dW_t\right|^p\right] \leqslant c_p\, \mathbb{E}\left[\left(\int_0^\tau \|Z_t\|^2 dW_t\right)^{p/2}\right], \qquad T \geqslant 0 \tag{2.1}$$

where $\|Z_t\| := \sum_{j=1}^d \sum_{i=1}^k \left|Z_t^{ij}\right|^2$.

The Ito integral process $\{I_\tau, \tau \in [0,\, T]\}$ on an interval $[0,\, T]$ is a square-integrable martingale in $\mathbb{R}^k$ if $\mathbb{E}\left[\int_0^T \|Z_t\|^2 dt\right] < \infty$ and it is a martingale if

$$\mathbb{E}\left[\left(\int_0^T \|Z_t\|^2 dt\right)^{1/2}\right] < \infty.$$

In the following we consider the BSDE on an interval $[0,\, T]$:

$$\begin{cases} dY_\tau = Z_\tau dW_\tau + f(\tau, Y_\tau, Z_\tau)d\tau, & \tau \in [0,\, T] \\ Y_T = \eta \end{cases} \tag{2.2}$$

where

$$f : \Omega \times [0,\, T] \times \mathbb{R}^k \times L(\mathbb{R}^d, \mathbb{R}^k) \to \mathbb{R}^k, \qquad \eta : \Omega \to \mathbb{R}^k$$

are given functions, and we look for unknown processes $Y : \Omega \times [0, T] \to \mathbb{R}^k$, $Z : \Omega \times [0, T] \to L(\mathbb{R}^d, \mathbb{R}^k)$. We notice that a condition at the *final* time T is given for Y and that the solution is a *pair* of processes (Y, Z). We assume that η is $\mathcal{F}_T$-measurable, and that f is measurable with respect to

$$\mathcal{P}_{[0,T]} \otimes \mathcal{B}(\mathbb{R}^k) \otimes \mathcal{B}(L(\mathbb{R}^d, \mathbb{R}^k))$$

(in particular, for any $y \in \mathbb{R}^k$ and $z \in L(\mathbb{R}^d, \mathbb{R}^k)$, the process $\{f(\tau, y, z), \tau \geqslant 0\}$ is predictable).

We stress here the fact that (2.2) is intended in the *Ito sense* and consequently we are looking for *predictable processes* (Y, Z).

The equation satisfied by Y has to be interpreted in the usual integral form: $\mathbb{P}$-a.s.,

$$Y_\tau^i = Y_0^i + \sum_{j=1}^d \int_0^\tau Z_t^{ij} dW_t^j + \int_0^\tau f^i(t, Y_t, Z_t)dt, \qquad \tau \in [0, T],\, i = 1, ..., k.$$

We adopt vector notation and write: $\mathbb{P}$-a.s.,

$$Y_\tau = Y_0 + \int_0^\tau Z_t dW_t + \int_0^\tau f(t, Y_t, Z_t)dt, \qquad \tau \in [0, T].$$

Writing the equation for $\tau = T$ and recalling that we require $Y_T = \eta$ we arrive at an equivalent formulation of (2.2): $\mathbb{P}$-a.s.,

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta - \int_\tau^T f(t, Y_t, Z_t)dt, \qquad \tau \in [0, T]. \tag{2.3}$$

Assume that a solution exists; since we require it is predictable, in particular adapted, then Y_0 is $\mathcal{F}_0$-adapted. Since the filtration $(\mathcal{F}_t)$ is generated by the brownian motion, $\mathcal{F}_0$ is trivial (i.e. its elements are sets of probability 0 or 1). It follows that Y_0 is $\mathbb{P}$-a.s. constant, hence deterministic.

2.1.2 Existence, uniqueness, regularity

The purpose of this Section is to analyse the solvability properties of BSDEs, i.e., we want to show under which conditions a BSDE is well defined; in particular we study under what hypotheses it admits a solution and this solution has a continuous dependence upon the data.

Let us consider the following assumptions.

Hypotheses 2.1. (1) *$\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^k)$, i.e. η is $\mathcal{F}_T$-measurable and* $\mathbb{E}\left[|\eta|^2\right] < \infty$.

(2) *f satisfies the measurability assumptions stated above; moreover there exists $K \geqslant 0$ such that, $\mathbb{P}$-a.s.,*

$$|f(\tau, y, z) - f(\tau, y', z')| \leqslant K\,|y - y'| + K\|z - z'\|,$$

for every $\tau \in [0, T], y, y' \in \mathbb{R}^k, z, z' \in L(\mathbb{R}^d, \mathbb{R}^k)$.

(3) $\mathbb{E}\left[\int_0^T |f(t,0,0)|^2\,dt\right] < \infty$.

We look for a solution (Y, Z) in the space of predictable processes, with values in $\mathbb{R}^k \times L(\mathbb{R}^d, \mathbb{R}^k)$, such that

$$|||(Y,Z)|||^2 := \mathbb{E}\left[\int_0^T \left(|Y_t|^2 + \|Z_t\|^2\right) dt\right] < \infty. \tag{2.4}$$

We denote this space by $\mathbb{K}$: endowed with the norm $|||\cdot|||$, it becomes a Hilbert space. Moreover by $\mathbb{K}_{cont}$ we denote the subspace given by all couples $(Y, Z) \in \mathbb{K}$ such that Y has a continuous modification and

$$|||(Y,Z)|||^2_{cont} := \mathbb{E}\left[\sup_{\tau\in[0,\,T]} |Y_\tau|^2\right] + \mathbb{E}\left[\int_0^T \|Z_t\|^2 dt\right] < \infty. \tag{2.5}$$

We first state a priori estimate, and a regularity result.

Proposition 2.1. *Let us assume that Hypotheses 2.1 holds and that $(Y, Z) \in \mathbb{K}$ is a solution. Then Y has a continuous modification and there exists a constant $c > 0$ such that*

$$|||(Y,Z)|||^2_{cont} \leqslant c\,\mathbb{E}\left[|\eta|^2\right] + c\,\mathbb{E}\left[\int_0^T |f(t,0,0)|^2\,dt\right].$$

Our aim is an existence and uniqueness theorem for the solution to the BSDE (2.3). The main tool is the following well known representation theorem (see for instance [22]):

Lemma 2.2. *Let $(\mathcal{F}_t)$ be the filtration generated by a brownian motion W in $\mathbb{R}^d$, augmented by the $\mathbb{P}$-null sets (as explained above). Given $T > 0$, $\xi : \Omega \to \mathbb{R}^k$ $\mathcal{F}_T$-measurable satisfying $\mathbb{E}\left[|\xi|^2\right] < \infty$, there exists a predictable process $\{Z_\tau, \tau \in [0, T]\}$, with values in $L(\mathbb{R}^d, \mathbb{R}^k)$, such that $\mathbb{E}\left[\int_0^T \|Z_t\|^2 dt\right] < \infty$ and*

$$\xi = \mathbb{E}\left[\xi\right] + \int_0^T Z_t dW_t.$$

We first address a simplified version of the BSDE (2.3). Indeed we consider a BSDE where we impose a restriction on the generator f, namely f does not depend on Y neither on Z; by the representation theorem we prove existence, uniqueness and an a priori estimate for this type of BSDE. Using this result and a point fixed theorem we next achieve the existence and uniqueness theorem for the general class of BSDEs we are interested in.

Lemma 2.3. Let us assume that Hypotheses 2.1 holds and let $f : \Omega \times [0, T] \to \mathbb{R}^k$ be a predictable process satisfying $\mathbb{E}\left[\int_0^T |f_t|^2\, dt\right] < \infty$.
Then the backward equation

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta - \int_\tau^T f_t dt, \qquad \tau \in [0, T]. \tag{2.6}$$

has a unique solution (Y, Z) in $\mathbb{K}$.

Proof. Assume that such a solution exists. Denoting by $\mathbb{E}^{\mathcal{F}_\tau}$ the conditional expectation with respect to $\mathcal{F}_\tau$, and noting that $I_\tau = \int_0^\tau Z_t dW_t$, $\tau \in [0, T]$, is a martingale, we have

$$\begin{aligned} Y_\tau &= \mathbb{E}^{\mathcal{F}_\tau}[Y_\tau] \\ &= \mathbb{E}^{\mathcal{F}_\tau}\left[\left(-\int_\tau^T Z_t dW_t + \eta - \int_\tau^T f_t dt\right)\right] \\ &= \mathbb{E}^{\mathcal{F}_\tau}[\eta] - \mathbb{E}^{\mathcal{F}_\tau}\left[\left(\int_0^T f_t dt - \int_0^\tau f_t dt\right)\right] \\ &= \mathbb{E}^{\mathcal{F}_\tau}[\xi] + \int_0^\tau f_t dt, \end{aligned}$$

where $\xi = \eta - \int_0^T f_t dt$.

To prove uniqueness, we first note that the equation is linear in (Y, Z), so it is sufficient to show that if $f = 0$, $\eta = 0$, then $Y = 0$, $Z = 0$. If $f = 0$, $\eta = 0$, then the last equality shows that $Y = 0$ and from the equation it follows that $\int_\tau^T Z_t dW_t = 0$, $\tau \in [0, T]$, which implies $Z = 0$ (notice that uniqueness also follows from Prop 2.1).

To show existence, let us define

$$\xi := \eta - \int_0^T f_t dt, \qquad Y_\tau := \mathbb{E}^{\mathcal{F}_\tau}[\xi] + \int_0^\tau f_t dt.$$

By Lemma 2.2, there exists Z such that $\xi = \mathbb{E}^{\mathcal{F}_T}[\xi] + \int_0^T Z_t dW_t$; consequently,

$$Y_\tau = \mathbb{E}^{\mathcal{F}_T}[\xi] + \int_0^\tau Z_t dW_t + \int_0^\tau f_t dt.$$

For $\tau = 0$ and $\tau = T$ we obtain, respectively,

$$Y_0 = \mathbb{E}^{\mathcal{F}_T}[\xi], \qquad Y_T = \xi + \int_0^T f_t dt = \eta,$$

and so the backward equation is satisfied. □

Finally we can extend the well-posedness result to the more general BSDE (2.3). We have the central result:

Theorem 2.4. *Under Hypotheses 2.1, the BSDE* (2.3) *has a unique solution* (Y, Z) *in* $\mathbb{K}$.

Proof. We define $\phi : \mathbb{K} \to \mathbb{K}$ as follows: given $(U, V) \in \mathbb{K}$, $(Y, Z) = \phi(U, V)$ is defined as the unique solution in $\mathbb{K}$ of the equation

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta - \int_\tau^T f(t, U_t, V_t)dt, \qquad \tau \in [0, T]. \tag{2.7}$$

ϕ is well defined by the Lemma 2.3. We show that ϕ is a contraction; clearly, its unique fixed point is the required solution of the BSDE.

We endow $\mathbb{K}$ with the equivalent norm

$$|||(Y, Z)|||_\beta^2 = \mathbb{E}\left[\int_0^T e^{\beta t}(|Y_t|^2 + \|Z_t\|^2)dt\right], \tag{2.8}$$

where $\beta \in \mathbb{R}$ will be fixed later. Take another pair $(U', V') \in \mathbb{K}$, and let $(Y', Z') = \phi(U', V')$, $\bar{Y} = Y - Y'$, $\bar{Z} = Z - Z'$, $\bar{U} = U - U'$, $\bar{V} = V - V'$, $\bar{f}_t = f(t, U_t, V_t) - f(t, U_t', V_t')$. Then

$$\bar{Y}_\tau + \int_\tau^T \bar{Z}_t dW_t = -\int_\tau^T \bar{f}_t dt, \qquad \tau \in [0, T].$$

By the Ito formula,

$$d[e^{\beta\tau}\left|\bar{Y}_\tau\right|^2] = \beta e^{\beta\tau}\left|\bar{Y}_\tau\right|^2 d\tau + 2e^{\beta\tau}\langle\bar{Y}_\tau, d\bar{Y}_\tau\rangle + e^{\beta\tau}\|\bar{Z}_\tau\|^2 d\tau.$$

Integrating between τ and T and noting that $\bar{Y}_T = 0$, it follows

$$e^{\beta\tau}\left|\bar{Y}_\tau\right|^2 + \int_\tau^T e^{\beta t}[\beta\left|\bar{Y}_t\right|^2 + \|\bar{Z}_t\|^2]dt = -2\int_\tau^T e^{\beta t}\langle\bar{Y}_t, \bar{Z}_t dW_t\rangle - 2\int_\tau^T e^{\beta t}\langle\bar{Y}_t, \bar{f}_t\rangle dt. \tag{2.9}$$

One verifies that $\int_0^\tau e^{\beta t}\langle \bar{Y}_t, \bar{Z}_t dW_t\rangle$, $\tau \in [0,\ T]$, is a martingale. Indeed:

$$\begin{aligned}
\mathbb{E}\left[\int_0^\tau |Z_t^* Y_t|^2\, dt\right]^{1/2} &\leqslant c\,\mathbb{E}\left[\sup_{\tau\in[0,\,T]} |Y_\tau| \int_0^\tau \|Z_t\|\, dt\right]^{1/2} \\
&\leqslant c\,\mathbb{E}\left[\sup_{\tau\in[0,\,T]} |Y_\tau|^2 + \int_0^\tau \|Z_t\|\, dt\right] \\
&< +\infty.
\end{aligned}$$

Then $\int_0^\tau e^{\beta t}\langle \bar{Y}_t, \bar{Z}_t dW_t\rangle$, $\tau \in [0,\ T]$ is a martingale with zero expectation. Next, using Hypothesis 2.1, point (2), and the inequality $ab \leqslant (a^2+b^2)/2$, we obtain

$$2\left|\langle \bar{Y}_t, \bar{f}_t\rangle\right| \leqslant 2\left|\bar{Y}_t\right| K(\left|\bar{U}_t\right| + \|\bar{V}_t\|) \leqslant (\left|\bar{U}_t\right|^2 + \|\bar{V}_t\|^2)/2 + 4K^2\left|\bar{Y}_t\right|^2.$$

Taking expectation in (2.9), setting $\tau = 0$ and using the above inequality we arrive at

$$\left|\bar{Y}_0\right|^2 + \mathbb{E}\left[\int_0^T e^{\beta t}[(\beta - 4K^2)\left|\bar{Y}_t\right|^2 + \|\bar{Z}_t\|^2]dt\right] \leqslant \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta t}(\left|\bar{U}_t\right|^2 + \|\bar{V}_t\|^2)dt\right].$$

Neglecting the first term and choosing $\beta = 4K^2+1$ we conclude that

$$|||(\bar{Y}_t, \bar{Z}_t)|||_\beta^2 \leqslant \frac{1}{2}|||(\bar{U}, \bar{V})|||_\beta^2,$$

which shows the required contraction property. □

Remark 2.1. Setting $\bar{f}_\tau = |f(\tau, 0, 0)|$, it follows from Hypotheses 2.1 that

$$|f(\tau, y, z)| \leqslant K\left(|y| + \|z\|\right) + \bar{f}_\tau, \qquad \mathbb{E}\left[\int_0^T \left|\bar{f}_t\right|^2 dt\right] < \infty \tag{2.10}$$

Remark 2.2. It can be shown that the conclusion of the previous Theorem holds true under the following, weaker assumptions. We still assume that η is $\mathcal{F}_T$-measurable, and keep the same measurability assumptions on f; moreover we assume

(i) $\mathbb{E}\left[|\eta|^2\right] < \infty$

(ii) There exists a predictable process $\{\bar{f}_\tau, \tau \in [0,\ T]\}$ such that (2.10) holds;

(iii) The inequality

$$|f(\tau, y, z) - f(\tau, y, z')| \leqslant K\|z - z'\|,$$

holds $\mathbb{P}$-a.s.,for all $\tau \in [0,\ T], y \in \mathbb{R}^k, z \in L(\mathbb{R}^d, \mathbb{R}^k)$;

(iv) The inequality

$$\langle y - y', f(\tau, y, z) - f(\tau, y, z') \rangle \geqslant \mu \left| y - y' \right|^2,$$

holds for some $\mu \in \mathbb{R}$, $\mathbb{P}$-a.s. for all $\tau \in [0, T], y, y' \in \mathbb{R}^k, z \in L(\mathbb{R}^d, \mathbb{R}^k)$,

(v) $y \to f(\tau, y, z)$ is continuous, $\mathbb{P}$-a.s. for all $\tau \in [0, T], z \in L(\mathbb{R}^d, \mathbb{R}^k)$.

Under the previous assumptions, in [11] was obtained an existence and uniqueness Theorem for BSDE (2.3). In particular, they recall the result of Pardoux [30], Theorem 2.2, where the following hypothesis was introduced:

$$|f(t, y, 0)| \leqslant |f(t, 0, 0)| + |\varphi(y)| \quad \mathbb{P} - \text{a.s.} \quad \forall (t, y) \in [0, T] \times \mathbb{R}^k, \tag{2.11}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is deterministic continuous increasing function. Under hypotheses *(i)*, *(ii)*, *(iii)*, *(iv)*, *(v)*, and the additional hypothesis (2.11), it is proved that BSDE (2.3) has a unique solution in $\mathbb{K}$.

The following two paragraphs are devoted to investigate what happens when the considered BSDE has coefficients depending on another stochastic process X. This analysis is particularly useful in view of the optimal control framework in Chapter 3. Indeed, in that Chapter we solve optimal control problems by the BSDEs theory, and doing so we deal with BSDEs depending on a particular stochastic process, namely the controlled process. This process turns out to be then the solution of another stochastic differential equation.

In Section 2.1.3 we analyse the continuous and regular dependence of the solution to the BSDE on a given general stochastic process X; then in Section 2.1.4 we consider the particular case when X is solution of a forward stochastic differential equation.

2.1.3 Equations depending on a given process: continuous and regular dependence

In this general framework we suppose that X is a process with values in $\mathbb{R}^n$ and that it belongs to the space $\mathbb{H}^p$ for every $p \in [1, \infty)$. We denote by $\mathbb{H}^p$ the space of continuous adapted (hence predictable) processes in $\mathbb{R}^n$ satisfying $\|X\|^p_{\mathbb{H}^p} := \mathbb{E}\left[\sup_{\tau \in [0, T]} |X_\tau|^p\right] < \infty$.

We consider the BSDE: $\mathbb{P}$-a.s.,

$$Y_\tau + \int_\tau^T Z_t dW_t = \phi(X_T) - \int_\tau^T \psi(t, X_t, Y_t, Z_t) dt, \qquad \tau \in [0, T]. \tag{2.12}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^k$, $\psi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times L(\mathbb{R}^d, \mathbb{R}^k) \to \mathbb{R}^k$ are given Borel functions. As before, we look for predictable processes Y and Z with values in $\mathbb{R}^k$ and in $L(\mathbb{R}^d, \mathbb{R}^k)$ respectively (we recall that W has valued in $\mathbb{R}^d$).

On the functions ϕ, ψ we fix the following assumptions.

Hypotheses 2.2. *For all $\tau \in [0, T]$, $x \in \mathbb{R}^n$, $y, y' \in \mathbb{R}^k$, $z, z' \in L(\mathbb{R}^d, \mathbb{R}^k)$, we have, for some constants $K \geqslant 0$ and $m \geqslant 0$,*

(1) $|\psi(\tau, x, y, z) - \psi(\tau, x, y', z')| \leqslant K\,|y - y'| + K\|z - z'\|$;

(2) $|\phi(x)| + |\psi(\tau, x, 0, 0)| \leqslant K(1 + |x|^m)$.

Setting $\eta = \phi(X_T)$, and $f(\tau, y, z) = \psi(\tau, X_\tau, y, z)$, we can easily check that η and f satisfy the conditions in Hypotheses 2.1. Thus there exists a unique solution $(Y, Z) \in \mathbb{K}$ to (2.12).
We concentrate now on its dependence on X.

Proposition 2.5. *Assume Hypotheses 2.2 and suppose that the mappings $x \to \phi(x)$ and $x \to \psi(\tau, x, y, z)$ are continuous. Let $X, X^n \in \mathbb{H}^p, n = 1, 2, ...$ and let (Y, Z), (Y^n, Z^n) be the corresponding solutions. If $\|X_n - X\|_{\mathbb{H}^p} \to 0$ when $n \to \infty$ and p sufficiently large then, letting $n \to \infty$,*

$$|||(Y, Z) - (Y^n, Z^n)|||^2_{cont} = \mathbb{E}\left[\sup_{\tau\in[0,\,T]} |Y_\tau - Y^n_\tau|^2\right] + \mathbb{E}\left[\int_\tau^T \|Z_t - Z^n_t\|^2 dt\right] \to 0.$$

Moreover, the dependence on X is differentiable:

Proposition 2.6. *Assume that $\phi \in C^1(\mathbb{R}^n)$ and $\psi(\tau, \cdot, \cdot, \cdot) \in C^1(\mathbb{R}^k \times \mathbb{R}^n \times L(\mathbb{R}^d, \mathbb{R}^n))$ for a.e $\tau \in [0, T]$. Moreover assume that there exist two constants $K \geqslant 0$ and $\mu \geqslant 0$ such that*

$$|\nabla\phi(x)| \leqslant K(1 + |x|^\mu), \qquad |\nabla_x\psi(\tau, x, y, z)| \leqslant K(1 + |x|^\mu), \tag{2.13}$$

for all $x \in \mathbb{R}^n, y \in \mathbb{R}^k$, $z \in L(\mathbb{R}^d, \mathbb{R}^n)$ and a.e $\tau \in [0, T]$. Then, if p is large enough, the map $X \to (Y, Z)$ is Gâteaux differentiable as a map $\mathbb{H}^p \to \mathbb{K}_{cont}$.

In fact the Gâteaux differential is strongly continuous.

2.1.4 The forward-backward system

In the previous section we have introduced the parameters dependency upon a certain stochastic process $(X_t)_t$, which is supposed to be known, and the couple of parameters of the form:

$$(\phi(X_T),\ \psi(t, X_t, Y_t, Z_t)).$$

Now we suppose that the randomness of the BSDE parameters is no longer due to a given process; conversely, we consider a process $(X_t)_t$ as itself a solution of a forward stochastic differential equation (FSDE). In this case, if we take the initial condition of the FSDE as $X_0 = x$, the solution (Y, Z) of the associated BSDE can be seen as a solution of a parametrized BSDE, with parameter x.

Let us consider again the BSDE: $\mathbb{P}$-a.s.,

$$Y_\tau + \int_\tau^T Z_t dW_t = \eta - \int_\tau^T \psi(t, X_t, Y_t, Z_t)dt, \qquad \tau \in [0,\, T]. \tag{2.14}$$

Now X is the solution of a forward stochastic differential equation. Namely, let us consider the following equation on the interval $[\tau,\, T] \subset [0,\, T]$:

$$\begin{cases} dX_\tau = F(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [0,\, T] \\ X_0 = x \end{cases} \tag{2.15}$$

The equation is understood as usual: $\mathbb{P}$-a.s.,

$$X_\tau = x + \int_0^\tau F(X_t)dt + \int_0^\tau G(X_t)dW_t \qquad \tau \in [0,\, T]. \tag{2.16}$$

Here $x \in R^n$ is given, and the functions

$$F : R^n \to R^n, \qquad G : R^n \to L(R^d, R^n)$$

are Borel measurable and satisfy the following hypotheses:

Hypotheses 2.3. *For all $x, x' \in R^n$ and for some constants K we have*

$$|F(x) - F(x')| + \|G(x) - G(x')\| \leqslant K\, |x - x'|\,.$$

It is well known that there exists a unique continuous and adapted process $X_\tau, \tau \in [0,\, T]$, solution of the forward equation. For every $p \in [1,\, \infty)$, the following inequality holds:

$$\|X\|^p_{\mathbb{H}^p} = \mathbb{E}\left[\sup_{\tau \in [0,\, T]} |X_\tau|^p\right] \leqslant c(1 + |x|)^p.$$

By the results of the previous sections, the backward equation (2.12) with X solution of (2.15) has a unique predictable solution $(Y, Z) \in \mathbb{K}$. To be more explicit, the system:

$$\begin{cases} dX_\tau = F(X_\tau)d\tau + G(X_\tau)dW_\tau, & \tau \in [0,\, T] \\ dY_\tau = \psi(\tau, X_\tau, Y_\tau, Z_\tau)d\tau + Z_\tau dW_\tau, & \tau \in [0,\, T] \\ X_0 = x \\ Y_T = \phi(X_T) \end{cases} \tag{2.17}$$

admits a unique solution with $X \in \mathbb{H}^p$ (for any $p \in [1,\, \infty)$), $(Y, Z) \in \mathbb{K}$; moreover $(Y, Z) \in \mathbb{K}_{cont}$.

Remark 2.3. We observe that, also in this case, Y_0 is deterministic. To prove this assertion, let $\mathcal{F}_{[0,\,T]}$ be the σ-algebra generated by the random variable $W_\tau, \tau \in [0,\,T]$, augmented by the $\mathbb{P}$-null sets. The process X is $\mathcal{F}_{[0,\,T]}$-measurable. Writing the backward equation on the interval $[0,\,T]$ we deduce that the solution, and in particular Y_0, is $\mathcal{F}_{[0,\,T]}$-measurable. On the other hand Y_0 is predictable, and so Y_0 is $\mathcal{F}_0$-measurable. Since $\mathcal{F}_{[0,\,T]}$ and $\mathcal{F}_0$ are independent (because W has independent increments) and since Y_0 is measurable to both the σ-algebras, the conclusion follows.

2.2 BSDEs associated to a marked point process

Now we address a class of BSDEs driven by a random measure, without diffusion part, on a finite time interval, naturally associated to a marked point process. We strongly use the notions of Chapter 1; we will see how the BSDE Wiener theory presented in Section 2.1 can be modified to achieve existence and uniqueness results in the marked point process discrete case.

As in the diffusive case (Paragraph 2.1.1), the Section starts with a brief introduction on the main notations and tools about the BSDEs theory for point processes.

2.2.1 Notation and setting

From now on, we fix a deterministic terminal time $T > 0$.
For given $\omega \in \Omega$ and $\tau \in [0,\,T]$, we denote $\mathcal{L}^r(K, \mathcal{K}, \phi_\tau(\omega, dy))$ the usual space of $\mathcal{K}$-measurable maps $z : K \to \mathbb{R}$ such that $\int_K |z(y)|^r \, \phi_\tau(\omega, dy) < \infty$ (below we will only use $r = 0$ or 1).

Next we introduce several classes of stochastic processes, depending on a parameter $\beta > 0$.

- $\mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0,\,T])$ denotes the set of real progressive processes Y such that

$$|Y|^2_\beta := \mathbb{E}\left[\int_0^T e^{\beta A_t} \, |Y_t|^2 \, dA_t\right] < \infty$$

- $\mathcal{L}^{2,\beta}(p)$ denotes the set of mappings $Z : \Omega \times [0,\,T] \times K \to \mathbb{R}$ which are $\mathcal{P} \otimes \mathcal{K}$-measurable and such that

$$\|Z\|^2_\beta := \mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} \, |Z_t(y)|^2 \, \phi_t(dy) dA_t\right] < \infty.$$

We say that $Y, Y' \in \mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0, T])$ (respectively, $Z, Z' \in \mathcal{L}^{2,\beta}(p)$) are equivalent if they coincide almost everywhere with respect to the measure $dA_t(\omega)\mathbb{P}(d\omega)$ (respectively the measure $\phi_t(\omega, dy)dA_t(\omega)\mathbb{P}(d\omega)$) and this happens if and only if $|Y - Y'|_\beta = 0$ (respectively, $\|Z - Z'\|_\beta = 0$).
We denote $\mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0, T])$ (respectively $\mathcal{L}^{2,\beta}(p)$) the corresponding set of equivalence classes, endowed with the norm $|\cdot|_\beta$ (respectively, $\|\cdot\|_\beta$).
$\mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0, T])$ and $\mathcal{L}^{2,\beta}(p)$ are Hilbert spaces,
isomorphic to $\mathcal{L}^{2,\beta}(\Omega \times [0, T], Prog, e^{\beta A_t(\omega)} dA_t(\omega)\mathbb{P}(d\omega))$ and
$\mathcal{L}^{2,\beta}(\Omega \times [0, T] \times K, \mathcal{P} \otimes \mathcal{K}, e^{\beta A_t(\omega)}\phi_t(\omega, dy)dA_t(\omega)\mathbb{P}(d\omega))$ respectively.

Finally we introduce the Hilbert space $\mathbb{K}^\beta = \mathcal{L}^{2,\beta}_{Prog}(\Omega \times [0, T]) \times \mathcal{L}^{2,\beta}(p)$, endowed with the norm $\|(Y, Z)\|^2_\beta := |Y|^2_\beta + \|Z\|^2_\beta$.

In the following we consider the backward stochastic differential equation: $\mathbb{P}$-a.s.,

$$Y_\tau + \int_\tau^T Z_t(y)q(dtdy) = \xi + \int_\tau^T f_t(Y_t, Z_t(\cdot))dA_t, \qquad \tau \in [0, T]. \tag{2.18}$$

where the generator f and the final condition ξ are given, and we look for unknown processes $(Y, Z) \in \mathbb{K}^\beta$.

2.2.2 Existence, uniqueness, regularity

In parallel to what was done in Section 2.1.2, we want to show under which conditions a BSDE driven by a marked point process is well defined. We ask ourselves when this particular class of BSDEs admits a solution, and if are there any continuous dependences upon the data.

We start to consider the following assumptions on the data f and ξ.

Hypotheses 2.4. (1) *The final condition $\xi : \Omega \to \mathbb{R}$ is $\mathcal{F}_T$-measurable and* $\mathbb{E}\left[e^{\beta A_T}|\xi|^2\right] < \infty$.

(2) *For every $\omega \in \Omega, \tau \in [0, T], r \in \mathbb{R}$, a mapping* $f_\tau(\omega, r, \cdot) : \mathcal{L}^2(K, \mathcal{K}, \phi_\tau(\omega, dy)) \to \mathbb{R}$ *is given, satisfying the following assumptions:*

(i) *for every $Z \in \mathcal{L}^{2,\beta}(p)$ the mapping*

$$(\omega, \tau, r) \mapsto f_\tau(\omega, r, Z_\tau(\omega, \cdot)) \tag{2.19}$$

is $Prog \otimes \mathcal{B}(\mathbb{R})$-measurable;

(ii) *there exists* $L \geqslant 0, L' \geqslant 0$ *such that for every* $\omega \in \Omega, \tau \in [0, T]$, $r, r' \in \mathbb{R}, z, z' \in \mathcal{L}^2(K, \mathcal{K}, \phi_\tau(\omega, dy))$ *we have*

$$\begin{aligned} &|f_\tau(\omega, r, z(\cdot)) - f_\tau(\omega, r', z'(\cdot))| \\ &\qquad \leqslant L' |r - r'| + \quad L \left(\int_K |z(y) - z'(y)|^2 \phi_\tau(\omega, dy) \right)^{1/2}; \end{aligned} \tag{2.20}$$

(iii) *we have*

$$\mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t(0,0)|^2 \, dA_t\right] < \infty. \tag{2.21}$$

Remark 2.4. (1) The slightly involved measurability condition on the generator seems unavoidable, since the mapping $f_\tau(\omega, r, \cdot)$ has a domain which depends on (ω, τ). However, in the following section, we will see how it can be effectively verified in connection with optimal control problems.

Note that if $Z \in \mathcal{L}^{2,\beta}(p)$ then $Z_\tau(\omega, \cdot)$ belongs to $\mathcal{L}^2(K, \mathcal{K}, \phi_\tau(\omega, dy))$ except possibly on a predictable set of points (ω, τ) of measure zero to $dA_\tau(\omega)\mathbb{P}(d\omega)$, so that the requirement on the measurability of the map (2.19) is meaningful.

(2) We note the inclusion

$$\mathcal{L}^{2,\beta}(p) \subset \mathcal{L}^{1,0}(p). \tag{2.22}$$

Indeed, if $Z \in \mathcal{L}^{2,\beta}(p)$, then the inequality

$$\int_0^T \int_K |Z_t(y)| \, \phi_t(dy) dA_t \leqslant \\ \left(\int_0^T \int_K |Z_t(y)|^2 \phi_t(dy) e^{\beta A_t} dA_t\right)^{1/2} \left(\int_0^T e^{-\beta A_t} dA_t\right)^{1/2}$$

and the fact that $\int_0^T e^{-\beta A_t} dA_t = \beta^{-1}(1 - e^{-\beta A_T}) \leqslant \beta^{-1}$ imply that $Z \in \mathcal{L}^{1,0}(p)$.

It follows from (2.22) that the martingale $M_\tau = \int_0^\tau \int_K Z_t(y) q(dtdy)$ is well defined for $Z \in \mathcal{L}^{2,\beta}(p)$ and has cadlag trajectories $\mathbb{P}$-a.s. It is easily checked that M only depends on the equivalence class of Z as defined above.

Just as we did in Section 2.1.2, to prove the existence and uniqueness of the solution for the BSDE (2.18) we start considering a simpler problem. Indeed we take a BSDE with a generator f_t that does not depend on Y, neither on Z, and we study the existence and uniqueness of the associated solution.

Lemma 2.7. *Suppose that* $f : \Omega \times [0, T] \to \mathbb{R}$ *is progressive,* $\xi : \Omega \to \mathbb{R}$ *is* $\mathcal{F}_T$*-measurable, and*

$$\mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + \mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t|^2 \, dA_t\right] < \infty$$

for some $\beta > 0$. Then there exists a unique pair (Y, Z) in $\mathbb{K}^\beta$ solution to the BSDE

$$Y_\tau + \int_\tau^T \int_K Z_t(y) q(dtdy) = \xi + \int_\tau^T f_t dA_t, \qquad \tau \in [0,\, T]. \tag{2.23}$$

Moreover, the following identity holds for every $\tau \in [0,\, T]$:

$$\begin{aligned} &\mathbb{E}\left[e^{\beta A_\tau} |Y_\tau|^2\right] + \beta \mathbb{E}\left[\int_\tau^T e^{\beta A_t} |Y_t|^2 dA_t\right] + \ \mathbb{E}\left[\int_\tau^T \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right] \\ &\quad = \mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + 2\mathbb{E}\left[\int_\tau^T e^{\beta A_t} Y_t f_t dA_t\right] \end{aligned} \tag{2.24}$$

and there exists two constants $c_1(\beta) = 4(1 + \frac{1}{\beta})$ and $c_2(\beta) = \frac{8}{\beta}(1 + \frac{1}{\beta})$ such that

$$\begin{aligned} &\mathbb{E}\left[\int_0^T e^{\beta A_t} |Y_t|^2 dA_t\right] + \mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right] \\ &\qquad \leqslant c_1(\beta) \mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + c_2(\beta) \mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t|^2 dA_t\right] \end{aligned} \tag{2.25}$$

Proof. Uniqueness is proved using the linearity of (2.23) in (Y, Z) and taking conditional expectation given $\mathcal{F}_\tau$. We assume that $(Y, Z) \in \mathbb{K}^\beta$ is a solution and we denote by $\mathbb{E}^{\mathcal{F}_\tau}$ the conditional expectation with respect to $\mathcal{F}_\tau$. Recalling that $M_\tau = \int_0^\tau \int_K Z_t(y) q(dtdy)$, $\tau \in [0,\, T]$, is a martingale, we have

$$\begin{aligned} Y_\tau &= \mathbb{E}^{\mathcal{F}_\tau}[Y_\tau] \\ &= \mathbb{E}^{\mathcal{F}_\tau}\left[\left(-\int_\tau^T \int_K Z_t q(dtdy) + \xi + \int_\tau^T f_t dA_t\right)\right] \\ &= \mathbb{E}^{\mathcal{F}_\tau}\left[\xi + \int_\tau^T f_t dA_t\right] \end{aligned}$$

By the linearity, it is sufficient to show that if $f = 0$, $\xi = 0$, then $Y = 0$, $Z = 0$. If $f = 0$, $\xi = 0$, then the last equality shows that $Y = 0$ and from the equation (2.23) it follows that $\int_\tau^T Z_t dq(dtdy) = 0$, $\tau \in [0,\, T]$, which implies $Z = 0$.

Still assuming a solution $(Y, Z) \in \mathbb{K}^\beta$ exists, we now prove the identity (2.24). We consider $e^{\beta A_t} |Y_t|^2$ and we apply the Itô formula for the product:

$$d(e^{\beta A_t} |Y_t|^2) = \beta e^{\beta A_t} |Y_t|^2 dA_t + e^{\beta A_t} d(|Y_t|^2).$$

We remark that the covariance is zero because we chose A_t continuous; obviously, if this hypothesis didn't hold we would also have the product of the respective jumps.

Now we recall the Itô formula for finite-variation processes (see Appendix C):

$$df(Y_t) = f'(Y_{t-})dY_t + \Delta f(Y_t) - f'(Y_{t-})\Delta(Y_t - Y_{t-}).$$

In our case we have

$$\begin{aligned} d(|Y_t|^2) &= 2Y_{t-}dY_t + Y_t^2 - Y_{t-}^2 - 2Y_{t-}(Y_t - Y_{t-}) \\ &= 2Y_{t-}dY_t + (Y_t - Y_{t-})^2 \\ &= 2Y_{t-}dY_t + \left|\Delta Y_t^2\right|, \end{aligned}$$

that motives moreover the use of a quadratic norm in the work. Finally we get

$$\begin{aligned} d(e^{\beta A_t}|Y_t|^2) &= \beta e^{\beta A_t}|Y_t|^2 dA_t + 2e^{\beta A_t}Y_{t-}dY_t + e^{\beta A_t}|\Delta Y_t|^2 \\ &= \beta e^{\beta A_t}|Y_t|^2 dA_t \\ &\quad + e^{\beta A_t}\left[2Y_{t-}\Big(Y_\tau + \int_K Z_t(y)q(dtdy) - f_t dA_t\Big) + |\Delta Y_t|^2\right]. \end{aligned}$$

So integrating on $[\tau, T]$ and recalling that A is continuous,

$$\begin{aligned} e^{\beta A_\tau}|Y_\tau|^2 = -\int_\tau^T \beta e^{\beta A_t}|Y_t|^2 dA_t - 2\int_\tau^T e^{\beta A_t}Y_{t-}\int_K Z_t(y)q(dtdy) \\ - \sum_{\tau < t \leqslant T} e^{\beta A_t}|\Delta Y_t|^2 + e^{\beta A_T}|\xi|^2 + 2\int_\tau^T e^{\beta A_t}Y_t f_t dA_t. \qquad (2.26) \end{aligned}$$

The integral process $\int_0^\tau e^{\beta A_t}Y_{t-}\int_K Z_t(y)q(dtdy)$ is a martingale, because the integrand process $e^{\beta A_t}Y_{t-}Z_t(y)$ is in $\mathcal{L}^1(p)$: in fact from the Young inequality we get

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_K e^{\beta A_t}|Y_{t-}||Z_t(y)|\phi_t(dy)dA_t\right] \\ \leqslant\ &\tfrac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta A_t}|Y_{t-}|^2(dy)dA_t\right] + \tfrac{1}{2}\mathbb{E}\left[\int_0^T \int_K e^{\beta A_t}|Z_t(y)|^2\phi_t(dy)dA_t\right] < \infty. \end{aligned}$$

Moreover we have

$$|\Delta Y_\tau|^2 = \int_{\tau-}^\tau \int_K Z_t(y)p(dtdy). \qquad (2.27)$$

In fact Y_t is the solution of (2.23) and it depends on both the measures p and $\tilde{p}$; however, being A continuous, the jump terms come only from p. Taking the absolute value and summing on t, we get

$$\begin{aligned} \sum_{0 < t \leqslant \tau} e^{\beta A_t}|\Delta Y_t|^2 &= \int_0^\tau \int_K e^{\beta A_t}|Z_t(y)|^2 p(dtdy) \\ &= \int_0^\tau \int_K e^{\beta A_t}|Z_t(y)|^2 q(dtdy) \\ &\quad + \int_0^\tau \int_K e^{\beta A_t}|Z_t(y)|^2 \phi_t(dy)dA_t \qquad (2.28) \end{aligned}$$

where the stochastic integral with respect to q is a martingale. Taking the expectation in (2.26) we obtain (2.24).

We now pass to the proof of existence of the required solution, that is based, as in the diffusive case, on the martingale representation Theorem. We start from the inequality

$$\begin{aligned}\int_\tau^T |f_t|\, dA_t &= \int_\tau^T e^{-\frac{\beta}{2}A_t} e^{\frac{\beta}{2}A_t} |f_t|\, dA_t \\ &\leqslant \left(\int_\tau^T e^{-\beta A_t} dA_t\right)^{1/2} \left(\int_\tau^T e^{\beta A_t} |f_t|^2\, dA_t\right)^{1/2}.\end{aligned}$$

Since $\beta \int_\tau^T e^{-\beta A_t} dA_t = e^{-\beta A_\tau} - e^{-\beta A_T} \leqslant e^{-\beta A_\tau}$ we arrive at

$$\left(\int_\tau^T |f_\tau|\, dA_\tau\right)^2 \leqslant \frac{e^{-\beta A_t}}{\beta} \int_\tau^T e^{\beta A_t} |f_t|^2\, dA_t. \tag{2.29}$$

That implies in particular that $\int_\tau^T |f_t|\, dA_t$ is square summable. The solution (Y, Z) is then defined by considering a cadlag version of the martingale $M_\tau = \mathbb{E}^{\mathcal{F}_\tau}\left[\xi + \int_0^T f_t dA_t\right]$. By the martingale representation Theorem 1.8, there exists a process $Z \in \mathcal{L}^{1,0}(p)$ such that

$$M_\tau = M_0 + \int_0^\tau \int_K Z_t(y) q(dydt), \qquad \tau \in [0,\, T].$$

Define the process Y by

$$Y_\tau := M_\tau - \int_0^\tau f_t(U_t, V_t) dA_t, \qquad \tau \in [0,\, T]. \tag{2.30}$$

In particular

$$Y_T = M_T - \int_0^T f_t(U_t, V_t) dA_t = \xi.$$

Then

$$\begin{aligned}Y_\tau - \xi &= M_\tau - M_T + \int_\tau^T f_t dA_t \\ &= -\int_\tau^T \int_K Z_t(dy) q(dt\, dy) + \int_\tau^T f_t dA_t\end{aligned}$$

i.e. Y_τ satisfies the equation (2.23).

It remains to show that $(Y, Z) \in \mathbb{K}^\beta$. Taking the conditional expectation, it follows from (2.23) that $Y_\tau = \mathbb{E}^{\mathcal{F}_\tau}\left[\xi + \int_\tau^T f_t dA_t\right]$ so that, using (2.29), we obtain

$$\begin{aligned} e^{\beta A_\tau} |Y_\tau|^2 &\leqslant 2e^{\beta A_\tau} \left|\mathbb{E}^{\mathcal{F}_\tau}[\xi]\right|^2 + 2e^{\beta A_\tau} \left|\mathbb{E}^{\mathcal{F}_\tau}\left[\int_\tau^T f_t dA_t\right]\right|^2 \\ &\leqslant 2\mathbb{E}^{\mathcal{F}_\tau}\left[e^{\beta A_T} |\xi|^2 + \frac{1}{\beta}\int_0^T e^{\beta A_t} |f_t|^2 dA_t\right]. \end{aligned} \tag{2.31}$$

Denoting by m_τ the right hand side of (2.31), we see that m is a martingale by the assumptions of the Lemma. In particular, for every stopping time S with values in $[0,\, T]$ (see Appendix A.3), we have

$$\mathbb{E}\left[e^{\beta A_S} |Y_S|^2\right] \leqslant \mathbb{E}[m_S] = \mathbb{E}[m_T] < \infty \tag{2.32}$$

by the Doob's optional stopping theorem (see Appendix A.3). Next we define the increasing sequence of stopping times

$$S_n = \inf\left\{\tau \in [0,\, T] : \int_0^\tau e^{\beta A_t} |Y_t|^2 dA_t + \int_0^\tau \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t > n\right\}, \tag{2.33}$$

with the convention $\inf \emptyset = T$. Computing the Ito differential $d(e^{\beta A_t} |Y_t|^2)$ on the interval $[0,\, S_n]$ and proceeding as before, we deduce

$$\begin{aligned} \beta\mathbb{E}\left[\int_0^{S_n} e^{\beta A_t} |Y_{t-}|^2 dA_t\right] + \mathbb{E}\left[\int_0^{S_n}\int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right] \\ \leqslant \mathbb{E}\left[e^{\beta A_{S_n}} |Y_{S_n}|^2\right] + 2\mathbb{E}\left[\int_0^{S_n} e^{\beta A_t} Y_t f_t dA_t\right]. \end{aligned}$$

Using the inequalities $2Y_t f_t \leqslant (\beta/2)|Y_t|^2 + (2/\beta)|f_t|^2$ and (2.32) (with $S = S_n$), we find the following estimates

$$\mathbb{E}\left[\int_0^{S_n} e^{\beta A_t} |Y_t|^2 dA_t\right] \leqslant \frac{4}{\beta}\mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + \frac{8}{\beta^2}\mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t|^2 dA_t\right],$$

$$\mathbb{E}\left[\int_0^{S_n}\int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right] \leqslant 4\mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + \frac{8}{\beta}\mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t|^2 dA_t\right],$$

from which we deduce

$$\begin{aligned} \mathbb{E}\left[\int_0^{S_n} e^{\beta A_t} |Y_t|^2 dA_t\right] + \mathbb{E}\left[\int_0^{S_n}\int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t\right] \\ \leqslant c_1(\beta)\mathbb{E}\left[e^{\beta A_T} |\xi|^2\right] + c_2(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t|^2 dA_t\right], \end{aligned} \tag{2.34}$$

where $c_1(\beta) = 4(1+\frac{1}{\beta})$ and $c_2(\beta) = \frac{8}{\beta}(1+\frac{1}{\beta})$.

Setting $S = \lim_n S_n$, we deduce

$$\int_0^S e^{\beta A_t} |Y_t|^2 dA_t + \int_0^S \int_K e^{\beta A_t} |Z_t(y)|^2 \phi_t(dy) dA_t < \infty, \qquad \mathbb{P}-a.s.$$

which implies $S = T$, $\mathbb{P}$-a.s, by the definition of S_n. Letting $n \to \infty$ in (2.34) we conclude that (2.25) holds, so that $(Y, Z) \in \mathbb{K}^\beta$. □

Theorem 2.8. *Suppose that Hypotheses* (2.4) *holds with* $\beta > L^2 + 2L'$.
Then there exists a unique pair (Y, Z) *in* $\mathbb{K}^\beta$ *which solves the BSDE* (2.18).

Proof. We use a fixed point theorem for the mapping $\Gamma : \mathbb{K}^\beta \to \mathbb{K}^\beta$ defined setting $(Y, Z) = \Gamma(U, V)$, if (Y, Z) is the pair satisfying

$$Y_\tau + \int_\tau^T Z_t(y) q(dt dy) = \xi + \int_\tau^T f_t(U_t, V_t) dA_t, \qquad \tau \in [0, T]. \tag{2.35}$$

From the assumptions on f it follows that $\mathbb{E}\left[\int_0^T e^{\beta A_t} |f_t(U_t, V_t)| dA_t\right] < \infty$, so by Lemma 2.7 there exists a unique $(Y, Z) \in \mathbb{K}^\beta$ satisfying (2.35) and Γ is a well defined map.

Let $(U^i, V^i), i = 1, 2$, be elements of $\mathbb{K}^\beta$ and let $(Y^i, Z^i) = \Gamma(U^i, V^i)$. Denote $\bar{Y} = Y^1 - Y^2$, $\bar{Z} = Z^1 - Z^2$, $\bar{U} = U^1 - U^2$, $\bar{V} = V^1 - V^2$, $\bar{f}_t = f_t(U^1_t, V^1_t) - f_t(U^2_t, V^2_t)$. Lemma 2.7 applied to $\bar{Y}, \bar{Z}, \bar{f}$ and (2.24) yields, noting that $\bar{Y}_T = 0$,

$$\begin{aligned}\mathbb{E}\left[e^{\beta A_\tau} \left|\bar{Y}_t\right|^2\right] + \beta \mathbb{E}\left[\int_\tau^T e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t\right] + \mathbb{E}\left[\int_\tau^T \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 \phi_t(dy) dA_t\right] \\ = 2\mathbb{E}\left[\int_\tau^T e^{\beta A_t} \bar{Y}_t \bar{f}_t dA_t\right] \qquad \tau \in [0, T].\end{aligned}$$

From the Lipschitz conditions of f and elementary inequalities, it follows that

$$\begin{aligned}&\beta \mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t\right] + \mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 \phi_t(dy) dA_t\right] \\ \leqslant\ & 2L\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_t\right| \left(\int_K \left|\bar{V}_t(y)\right|^2 \phi_t(dy)\right)^{1/2} dA_t\right] \\ &+2L'\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_t\right| \left|\bar{U}_t\right| dA_t\right] \\ \leqslant\ & \alpha \mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} \left|\bar{V}_t\right|^2 \phi_t(dy) dA_t\right] + \frac{L^2}{\alpha}\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t\right] \\ &+\gamma L'\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t\right] + \frac{L'}{\gamma}\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{U}_t\right|^2 dA_t\right]\end{aligned}$$

for every $\alpha > 0, \gamma > 0$. This can be written as:

$$\left(\beta - \frac{L^2}{\alpha} - \gamma L'\right) \left|\bar{Y}\right|_\beta^2 + \|\bar{Z}_\beta\|_\beta^2 \leqslant \alpha\|\bar{V}_\beta\|_\beta^2 + \frac{L'}{\gamma} \left|\bar{U}\right|_\beta^2.$$

By the assumption on β it is possible to find $\alpha \in (0,\, 1)$ such that

$$\beta > \frac{L^2}{\alpha} + \frac{2L'}{\sqrt{\alpha}}. \tag{2.36}$$

If $L' = 0$ we see that Γ is an α-concentration on $\mathbb{K}^\beta$ endowed with the equivalent norm $(Y, Z) \mapsto (\beta - (L^2/\alpha)) \left|Y\right|_\beta^2 + \|Z\|_\beta^2$. If $L' > 0$ we choose $\gamma = \sqrt{\alpha}$ and obtain

$$\frac{L'}{\sqrt{\alpha}} \left|\bar{Y}\right|_\beta^2 + \|\bar{Z}\|_\beta^2 \leqslant \alpha\|\bar{V}\|_\beta^2 + L'\sqrt{\alpha} \left|\bar{U}\right|_\beta^2 = \alpha \left(\frac{L'}{\sqrt{\alpha}} \left|\bar{U}\right|_\beta^2 + \|\bar{V}\|_\beta^2\right). \tag{2.37}$$

so that Γ is an α-concentration on $\mathbb{K}^\beta$ endowed with the equivalent norm $(Y, Z) \mapsto (L'/\sqrt{\alpha}))\left|Y\right|_\beta^2 + \|Z\|_\beta^2$. In all cases there exists a unique fixed point which is the required solution to the BSDE (2.18). □

2.2.3 Estimates and continuous dependence upon the data

We next prove some estimates on the solution of the BSDE (2.18), which show in particular the continuous dependence upon the data.

Let us consider two solutions (Y^1, Z^1), $(Y^2, Z^2) \in \mathbb{K}^\beta$ to the BSDE (2.18) associated with the drivers f^1 and f^2 and final data ξ^1 and ξ^2, respectively, which are assumed to satisfy Hypotheses 2.4. Denote $\bar{Y} = Y^1 - Y^2$, $\bar{Z} = Z^1 - Z^2$, $\bar{\xi} = \xi^1 - \xi^2$, $\bar{f}_t = f_t^1(Y_t^2, Z_t^2(\cdot)) - f_t^2(Y_t^2, Z_t^2(\cdot))$.

Proposition 2.9. Let $(\bar{Y}, \bar{Z})$ be the processes defined above.
Then, for $\beta > 2L' + L^2$, the a priori estimate hold:

$$\left|\bar{Y}\right|_\beta^2 \leqslant \frac{2}{\beta - 2L' - L^2}\mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] + \frac{4}{\beta - 2L' - L^2}\mathbb{E}\left[\int_0^T e^{\beta A_t}\left|\bar{f}_t\right|^2 dA_t\right], \tag{2.38}$$

$$\begin{aligned}\left|\bar{Z}\right|_\beta^2 \leqslant &\left(2 + \frac{16}{\beta - 2L' - L^2}\right)\mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] \\ &+ \frac{2}{\beta - 2L' - L^2}\left(1 + \frac{16}{\beta - 2L' - L^2}\right)\ \mathbb{E}\left[\int_0^T e^{\beta A_t}\left|\bar{f}_t\right|^2 dA_t\right].\end{aligned} \tag{2.39}$$

Proof. From the Ito formula applied to $e^{\beta A_t}\left|\bar{Y}_t\right|^2$ it follows that

$$d(e^{\beta A_t}\left|\bar{Y}_t\right|^2) = \beta e^{\beta A_t}\left|\bar{Y}_t\right|^2 dA_t + 2e^{\beta A_t}\bar{Y}_{t-}dY_t + e^{\beta A_t}\left|\Delta\bar{Y}_t\right|^2.$$

So integrating on $[\tau, T]$ and recalling that A is continuous,

$$\begin{aligned} e^{\beta A_\tau} \left|\bar{Y}_\tau\right|^2 &= -\int_\tau^T \beta e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t - 2\int_\tau^T e^{\beta A_t} \bar{Y}_{t-} \int_K \bar{Z}_t(y) q(dtdy) \\ &\quad - \sum_{\tau < t \leqslant T} e^{\beta A_t} \left|\Delta \bar{Y}_t\right|^2 + e^{\beta A_T} \left|\bar{\xi}\right|^2 \\ &\quad + 2\int_\tau^T e^{\beta A_t} \bar{Y}_t \left(f_t^1(Y_t^1, Z_t^1) - f_t^2(Y_t^2, Z_t^2)\right) dA_t. \end{aligned} \tag{2.40}$$

The integral process $\int_0^\tau e^{\beta A_t} \bar{Y}_{t-} \int_K \bar{Z}_t(y) q(dtdy)$ is a martingale, because the integrand process $e^{\beta A_t} \bar{Y}_{t-} \bar{Z}_t(y)$ is in $\mathcal{L}^1(p)$: in fact from the Young inequality we get

$$\begin{aligned} \mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} \left|\bar{Y}_{t-}\right| \left|\bar{Z}_t(y)\right| \phi_t(dy) dA_t\right] &\leqslant \frac{1}{2}\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_{t-}\right|^2 (dy) dA_t\right] + \\ &\quad \frac{1}{2}\mathbb{E}\left[\int_0^T \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 \phi_t(dy) dA_t\right] \\ &< \infty. \end{aligned}$$

Moreover we have

$$\begin{aligned} \sum_{0 < t \leqslant \tau} e^{\beta A_t} \left|\Delta \bar{Y}_t\right|^2 &= \int_0^\tau \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 p(dtdy) \\ &= \int_0^\tau \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 q(dtdy) \\ &\quad + \int_0^\tau \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 \phi_t(dy) dA_t \end{aligned}$$

where the stochastic integral with respect to q is a martingale. Taking the expectation in (2.40), by the Lipschitz property of the driver f^1 and using the notation $\|z(\cdot)\|_t^2 = \int_K |z(y)|^2 \phi_t(dy)$, we get

$$\begin{aligned} \mathbb{E}\left[e^{\beta A_\tau} \left|\bar{Y}_\tau\right|^2\right] &= -\mathbb{E}\left[\int_\tau^T \beta e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t\right] \\ &\quad -\mathbb{E}\left[\int_\tau^T \int_K e^{\beta A_t} \left|\bar{Z}_t(y)\right|^2 \phi_t(dy) dA_t\right] + \mathbb{E}\left[e^{\beta A_T} \left|\bar{\xi}\right|^2\right] \\ &\quad +2\mathbb{E}\left[\int_\tau^T e^{\beta A_t} \bar{Y}_t \left(f_t^1(Y_t^1, Z_t^1) - f_t^2(Y_t^2, Z_t^2)\right) dA_t\right]. \end{aligned}$$

Then we can estimate the first term in the following way:

$$\begin{aligned}
\mathbb{E}\left[e^{\beta A_\tau}\left|\bar{Y}_\tau\right|^2\right] &\leqslant -\mathbb{E}\left[\int_\tau^T \beta e^{\beta A_t}\left|\bar{Y}_t\right|^2 dA_t\right] \\
&\quad -\mathbb{E}\left[\int_\tau^T\int_K e^{\beta A_t}\left|\bar{Z}_t(y)\right|^2\phi_t(dy)dA_t\right] + \mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] \\
&\quad +2\mathbb{E}\left[\int_\tau^T e^{\beta A_t}\left|\bar{Y}_t\right|\left(\left|f_t^1(Y_t^1,Z_t^1)-f_t^2(Y_t^2,Z_t^2)\right|+\left|\bar{f}_t\right|\right)dA_t\right] \\
&\leqslant -\mathbb{E}\left[\int_\tau^T \beta e^{\beta A_t}\left|\bar{Y}_t\right|^2 dA_t\right] - \mathbb{E}\left[\int_\tau^T e^{\beta A_t}\|\bar{Z}_t\|_t^2 dA_t\right] \\
&\quad +\mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] + 2L'\mathbb{E}\left[\int_\tau^T e^{\beta A_t}\left|\bar{Y}_t\right|^2 dA_t\right] \\
&\quad +2L\mathbb{E}\left[\int_\tau^T e^{\beta A_t}\left|\bar{Y}_t\right|\|\bar{Z}_t\|_t dA_t\right] + 2\mathbb{E}\left[\int_\tau^T e^{\beta A_t}\left|\bar{Y}_t\right|\left|\bar{f}_t\right| dA_t\right]
\end{aligned}$$

We note that the quantity $Q(y,z) = -\beta|y|^2 - \|z\|_t^2 + 2L'|y|^2 + 2L|y|\|z\|_t + 2\left|\bar{f}_t\right||y|$, which occurs in the integrand terms in the right hand of the above inequality, can be written as

$$\begin{aligned}
Q(y,z) &= -\beta|y|^2 + 2L'|y|^2 + L^2|y|^2 + 2\left|\bar{f}_t\right||y| - (\|z\|_t - L|y|)^2 \\
&= -\beta_L(|y| - \beta_L^{-1}\left|\bar{f}_t\right|)^2 - (\|z\|_t - L|y|)^2 + \beta_L^{-1}\left|\bar{f}_t\right|^2
\end{aligned}$$

where $\beta_L := \beta - 2L' - L^2$ is assumed to be strictly positive. Hence

$$\begin{aligned}
&\mathbb{E}\left[\beta e^{\beta A_t}\left|\bar{Y}_t\right|^2\right] + \beta_L\mathbb{E}\left[\int_\tau^T e^{\beta A_t}(\left|\bar{Y}_t\right| - \beta_L^{-1}\left|\bar{f}_t\right|)^2 dA_t\right] \\
&\quad +\mathbb{E}\left[\int_\tau^T e^{\beta A_t}(\|\bar{Z}_t\|_t^2 - L\left|\bar{Y}_t\right|)^2 dA_t\right] \\
&\leqslant \mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] + \mathbb{E}\left[\int_\tau^T e^{\beta A_t}\frac{\left|\bar{f}_t\right|^2}{\beta_L}dA_t\right]
\end{aligned}$$

from which we deduce

$$\mathbb{E}\left[\int_0^T e^{\beta A_t}\left|\bar{Y}_t\right|^2 dA_t\right] \leqslant \frac{2}{\beta_L}\mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] + \frac{4}{\beta_L^2}\mathbb{E}\left[\int_0^T e^{\beta A_t}\left|\bar{f}_t\right|^2 dA_t\right]$$

and

$$\begin{aligned}
\mathbb{E}\left[\int_0^T e^{\beta A_t}\|\bar{Z}_t\|_t^2 dA_t\right] &\leqslant \left(2 + \tfrac{16}{\beta_L}\right)\mathbb{E}\left[e^{\beta A_T}\left|\bar{\xi}\right|^2\right] \\
&\quad + \tfrac{2}{\beta_L}\left(1 + \tfrac{16}{\beta_L}\right)\mathbb{E}\left[\int_0^T e^{\beta A_t}\frac{\left|\bar{f}_t\right|^2}{\beta_L}dA_t\right]
\end{aligned}$$

□

From the a priori estimates one can deduce the continuous dependence of the solution upon the data.

Proposition 2.10. *Suppose that Hypotheses 2.4 holds with* $\beta > L^2 + 2L'$ *and let* (Y, Z) *be the unique solution in* $\mathbb{K}^\beta$ *to the BSDE* (2.18).
Then

$$\mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\bar{Y}_t\right|^2 dA_t\right] + \mathbb{E}\left[\int_0^T e^{\beta A_t} \int_K \left|\bar{Z}_t(y)\right|^2 \phi_t(dy) dA_t\right] \\ \leqslant C_1(\beta)\mathbb{E}\left[e^{\beta A_T} \left|\bar{\xi}\right|^2\right] + C_2(\beta)\mathbb{E}\left[\int_0^T e^{\beta A_t} \frac{\left|\bar{f}_t\right|^2}{\beta_L} dA_t\right] \tag{2.41}$$

where $C_1(\beta) = (2 + \frac{18}{\beta - 2L' - L^2})$, $C_2(\beta) = \frac{2}{\beta - 2L' - L^2}(1 + \frac{18}{\beta - 2L' - L^2})$.

Proof. The thesis follows from Proposition 2.9 setting $f' = f$, $\xi' = \xi$, $f^2 = 0$ and $\xi^2 = 0$. □

Chapter 3

Optimal Control and BSDEs approach

The optimal stochastic control is a topic tightly connected to BSDEs. The control problem can be exposed in two different ways: the so called strong formulation, in which the noise process and the probability space on which it is defined are fixed, and the weak formulation, in which only the law of the noise is fixed. We present the second formulation, which can be applied both to diffusive and marked point processes. In particular, we show that BSDEs can be used in control theory to represent the value function and to characterize the optimal control. The BSDEs theory is well known in the diffusive case (see [16], [17], [27], [29]); conversely, optimal control problems for point processes are usually solved via dynamic programming, and thus, in this case, the BSDEs theory represents an innovating alternative approach.

In Section 3.1 we first present optimal control problem for Wiener processes, and their connections with the BSDEs diffusive theory. Then in Section 3.2 we extend the discussion to marked point processes, highlighting the significant points in common and the main differences between the diffusive and the discrete treatment.

3.1 Optimal control for diffusive processes

3.1.1 Weak formulation of the problem

We are given a set $U \subset \mathbb{R}^n$ and the functions

$$F : [0,\, T] \times \mathbb{R}^n \to \mathbb{R}^n$$
$$G : [0,\, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$$

$$r : [0,\, T] \times \mathbb{R}^n \times U \to \mathbb{R}^d$$
$$\phi : \mathbb{R}^n \to \mathbb{R}$$
$$l : [0,\, T] \times \mathbb{R}^n \times U \to \mathbb{R}$$

and we consider the controlled equation:

$$\begin{cases} dX_\tau = F(\tau, X_\tau)d\tau + G(\tau, X_\tau)\, r(\tau, X_\tau, u_\tau)d\tau + G(\tau, X_\tau)dW_\tau, & \tau \in [0,\, T] \\ X_0^u = x \in \mathbb{R}^n \end{cases} \tag{3.1}$$

where u is an adapted stochastic process with values in some specified set $U \subset \mathbb{R}^n$ and W is a $\mathbb{R}^d$-valued standard Wiener process.

The purpose is to minimize over all admissible controls the *cost functional*

$$\mathbb{E}\left[\int_0^T l(t, X_t, u_t)dt + \phi(X_T)\right]. \tag{3.2}$$

Remark 3.1. The data specifying the optimal control problem are the action space U, the running cost function l, the terminal cost function ϕ and the function r which describes the effect of the control process.

We work under the following general assumptions:

Hypotheses 3.1. *U is a Borel subset of $\mathbb{R}^m$, the functions F, G, r, Φ, l are Borel measurable, the function $x \mapsto F(t,x)$ is continuous on $\mathbb{R}^n$ for every $t \in [0,\, T]$, and there exists a constant C such that :*

(i) $|\phi(x) - \phi(x')| + |F(t,x) - F(t,x')| + |G(t,x) - G(t,x')| \leqslant C\,|x - x'|$,

(ii) $|r(t,x,u) - r(t,x',u')| + |l(t,x,u) - l(t,x',u')| \leqslant C(|x - x'| + |u - u'|)$,

(iii) $|G(t,x)| + |F(t,0)| + |r(t,x,u)| + |l(t,0,u)| \leqslant C$.

for every $t \in [0,\, T]$, $x, x' \in \mathbb{R}^n$, $u, u' \in U$.

We note that if $r(t,x,u) = u$ then U is required to be bounded.

An *admissible control system* (a.c.s) is given by the set

$$\mathbb{U} = (\widehat{\Omega}, \widehat{\mathcal{F}}, (\widehat{\mathcal{F}}_t)_{t \geqslant 0}, \widehat{\mathbb{P}}, \widehat{u})$$

where: $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ is a complete probability space, the filtration $(\widehat{\mathcal{F}}_t)_{t \geqslant 0}$ verifies the usual conditions and the process $\widehat{u} : [0,\, T] \times \widehat{\Omega} \to U \subset \mathbb{R}^m$ is predictable with respect to the filtration $(\widehat{\mathcal{F}}_t)_{t \geqslant 0}$.

For any a.c s $\mathbb{U}$ and fixed $x \in \mathbb{R}^n$, we consider the process $X_\tau^{\mathbb{U}}, \tau \in [0,\, T]$, solution of the Ito stochastic equation: $\mathbb{P}$-a.s, for every $\tau \in [0,\, T]$,

$$X_\tau^{\mathbb{U}} = x + \int_0^\tau F(t, X_t^{\mathbb{U}})dt + \int_0^\tau G(t, X_t^{\mathbb{U}})\, r(t, X_t^{\mathbb{U}}, \widehat{u}_t)dt + \int_0^\tau G(t, X_t^{\mathbb{U}})d\widehat{W}_t, \quad (3.3)$$

where the process $\widehat{W} : [0,\, T] \times \widehat{\Omega} \to \mathbb{R}^d$ is a Wiener process with respect to the filtration $(\widehat{\mathcal{F}}_t)_{t \geqslant 0}$. It is well known that, under Hypothesis 3.1, for all a.c.s there exists a continuous, $\widehat{\mathcal{F}}_t$-adapted solution, unique up to indistinguishability (recall that the term $G(t,x)r(t,x,u)$ is Lipschitz in x uniformly in t and u).

In this setting the cost functional depends on $\mathbb{U}$ and is given by:

$$J(\mathbb{U}) = \widehat{\mathbb{E}}\left[\int_0^T l(t, X_t^{\mathbb{U}}, \widehat{u}_t)dt + \phi(X_T^{\mathbb{U}})\right] \quad (3.4)$$

(notice that, for all a.c.s $\mathbb{U}$, $J(\mathbb{U})$ is a well defined real number). We consider the problem of minimizing $J(\mathbb{U})$ over all a.c.s $\mathbb{U}$. Any a.c.s which minimizes $J(\cdot)$, if it exists, is called optimal for the control problem starting from the fixed x at time 0 in the weak formulation. The minimal value of the cost is then called the optimal cost in the weak formulation.

Finally we introduce the value function $V \in \mathbb{R}$ corresponding to the weak formulation:

$$V = \inf_{\mathbb{U}} J(\mathbb{U}) \quad (3.5)$$

where the infimum is taken over all a.c.s $\mathbb{U}$.

3.1.2 Solving the Optimal Control problem by the BSDEs approach

To start we introduce the hamiltonian function $\psi : [0,\, T] \times \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ setting

$$\psi(t,x,z) = \inf_{u \in U} \{l(t,x,u) + zr(t,x,u)\}, \qquad t \in [0,\, T], x \in \mathbb{R}^n, z \in \mathbb{R}^d \quad (3.6)$$

and we define the following, possibly empty set:

$$\Gamma(t,x,z) = \{u \in U : l(t,x,u) + zr(t,x,u) = \psi(t,x,z),\ t \in [0,\, T], x \in \mathbb{R}^n, z \in \mathbb{R}^d\}. \quad (3.7)$$

The function ψ has some additional properties.

Lemma 3.1. *Assume Hypotheses 3.1. Then there exists a constant c such that*

$$|\psi(t,0,0)| \leqslant c, \qquad |\psi(t,x,z) - \psi(t,x',z')| \leqslant c\,|z - z'| + c\,|x - x'|\,(1 + |z| + |z'|)$$

for every $t \in [0,\, T], x, x' \in \mathbb{R}^n, z, z' \in \mathbb{R}^d$.

From now we shall always assume that Hypothesis 3.1 hold.

Let $\widetilde{W}$ be a standard Wiener process in $\mathbb{R}^d$, defined in some complete probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. For $0 \leqslant \tau \leqslant T$, we denote by $\widetilde{\mathcal{F}_\tau}$ the σ-algebra generated by $\widetilde{W}_s, s \in [0, \tau]$, and augmented by the null sets of $\widetilde{\mathcal{F}}$. For fixed $x \in \mathbb{R}^n$ we consider the equation:

$$\widetilde{X}_\tau = x + \int_0^\tau F(t, \widetilde{X}_t)dt + \int_0^\tau G(t, \widetilde{X}_t)d\widetilde{W}_t, \qquad \tau \in [0, T]. \tag{3.8}$$

The solution $\{\widetilde{X}_\tau, \tau \in [0, T]\}$ is a continuous process in $\mathbb{R}^n$, adapted to the filtration $(\widetilde{\mathcal{F}_\tau})_{\tau\in[0,T]}$. Moreover, the law of $(\widetilde{W}, \widetilde{X})$ is uniquely determined by F and G.

Next we consider the BSDE

$$\widetilde{Y}_\tau + \int_\tau^T \widetilde{Z}_t d\widetilde{W}_t = \phi(\widetilde{X}_T) + \int_\tau^T \psi(t, \widetilde{X}_t, \widetilde{Z}_t)dt, \qquad \tau \in [0, T]. \tag{3.9}$$

By Theorem 2.4 there exist a solution $(\widetilde{Y}, \widetilde{Z})$ of (3.9) on the interval $[0, T]$, where $\widetilde{Y}$ is unique up to indistinguishability and $\widetilde{Z}$ is unique up to modification. Moreover, from the proof of the Theorem 2.4 it follows that the law of $(\widetilde{Y}, \widetilde{Z})$ is uniquely determined by the law of $(\widetilde{W}, \widetilde{X})$ and by ϕ and ψ. We note that $\widetilde{Y}_0$, being measurable with respect to the degenerate σ-algebra $\widetilde{\mathcal{F}_0}$, is deterministic; in particular $\widetilde{Y}_0 = \widetilde{\mathbb{E}}\left[\widetilde{Y}_0\right]$ only depends on the law of $\widetilde{Y}$, and thus it is a functional of F, G, ϕ and ψ.

We set then

$$J^\sharp = \widetilde{Y}_0$$

The previous discussion shows that $J^\sharp$ is a number, whose value is uniquely determined by F, G, ϕ and ψ. The relevance of $J^\sharp$ to our control problem is explained in to the following proposition.

Proposition 3.2. *Assume that Hypothesis 3.1 hold. For fixed $x \in \mathbb{R}^n$ and for every a.c.s $\mathbb{U}$, we have $J^\sharp \leqslant J(\mathbb{U})$.*

Proof. We define the process

$$W_\tau^{\mathbb{U}} := \widehat{W}_\tau + \int_0^\tau r(t, X_t^{\mathbb{U}}, \widehat{u}_t)dt, \qquad \tau \in [0, T], \tag{3.10}$$

and we remark that $X^{\mathbb{U}}$ solves the equation

$$X_\tau^{\mathbb{U}} = x + \int_0^\tau F(t, X_t^{\mathbb{U}})dt + \int_0^\tau G(t, X_t^{\mathbb{U}})dW_t^{\mathbb{U}}, \qquad \tau \in [0, T]. \tag{3.11}$$

Since r is bounded, by the Girsanov theorem there exists a probability measure $\mathbb{P}^{\mathbb{U}}$ on $(\Omega, \mathcal{F})$ such that $W^{\mathbb{U}}$ is a Wiener process under $\mathbb{P}^{\mathbb{U}}$.

Let us consider the backward equation for the unknown process $\{(Y_\tau^{\mathbb{U}}, Z_\tau^{\mathbb{U}}), \tau \in [0,\, T]\}$, that we require to be predictable with respect to the filtration generated by $W^{\mathbb{U}}$ augmented with the null sets (indeed also $X^{\mathbb{U}}$ is adapted to the natural filtration of $W^{\mathbb{U}}$, as well as is adapted to the filtration of the a.c.s.):

$$Y_\tau^{\mathbb{U}} + \int_\tau^T Z_t^{\mathbb{U}} dW_t^{\mathbb{U}} = \phi(X_T^{\mathbb{U}}) + \int_\tau^T \psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}})dt, \qquad \tau \in [0,\, T] \tag{3.12}$$

By a result recalled earlier, there exists a unique solution $(Y^{\mathbb{U}}, Z^{\mathbb{U}})$ of this equation. Comparing equations (3.11)-(3.12) with (3.8)-(3.9) we note that they depend on the same functions F, G, ϕ and ψ and thus $J^\sharp = Y_0^{\mathbb{U}}$.

Now we write (3.12) with respect to $\widehat{W}$: for every $\tau \in [0,\, T]$,

$$Y_\tau^{\mathbb{U}} + \int_\tau^T Z_t^{\mathbb{U}} d\widehat{W}_t + \int_\tau^T Z_t^{\mathbb{U}} r(t, X_t^{\mathbb{U}}, \widehat{u}_t)dt = \phi(X_T^{\mathbb{U}}) + \int_\tau^T \psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}})dt. \tag{3.13}$$

We want to show that the stochastic integral in (3.13) has zero expectation with respect to the original probability $\widehat{\mathbb{P}}$. Therefore we check that $\widehat{\mathbb{E}}\left[\int_0^T \left|Z_t^{\mathbb{U}}\right|^2 dt\right]^{1/2} < \infty$. We consider (3.13) and we obtain the following inequality:

$$\sup_{\tau\in[0,\,T]} \left| \int_0^\tau Z_t^{\mathbb{U}} d\widehat{W}_t \right| \leqslant 2 \sup_{\tau\in[0,\,T]} \left|Y_t^{\mathbb{U}}\right| + \int_0^\tau \left[\left|\psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}})\right| + \left|Z_t^{\mathbb{U}}\, r(t, X_t^{\mathbb{U}}, \widehat{u}_t)\right|\right] dt.$$

By our assumptions, and taking into account the Burkholder-Davis-Gundy inequalities (see (2.1)), we have, for some constant $c > 0$,

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\int_0^T \left|Z_t^{\mathbb{U}}\right|^2 dt\right]^{1/2} &\leqslant c\,\widehat{\mathbb{E}}\left[\sup_{\tau\in[0,\,T]} \left|\int_0^\tau Z_t^{\mathbb{U}} d\widehat{W}_t\right|\right] \\
&\leqslant c\,\widehat{\mathbb{E}}\left[\sup_{\tau\in[0,\,T]} \left|Y_\tau^{\mathbb{U}}\right|\right] + c\,\widehat{\mathbb{E}}\left[\int_0^T \left[\left|\psi(t, X_t^{\mathbb{U}}, 0)\right| + \left|Z_t^{\mathbb{U}}\right|\right] dt\right] \\
&\leqslant c\left(\mathbb{E}^{\mathbb{U}}\left[\rho^2\right]\right)^{1/2} \cdot \\
&\qquad \left(\mathbb{E}^{\mathbb{U}}\left[\sup_{\tau\in[0,\,T]} \left|Y_\tau^{\mathbb{U}}\right|^2\right] + \mathbb{E}^{\mathbb{U}}\left[\int_0^T \left[\left|\psi(t, X_t^{\mathbb{U}}, 0)\right|^2 + \left|Z_t^{\mathbb{U}}\right|^2\right] dt\right]\right)^{1/2} \\
&< \infty
\end{aligned}$$

where $\rho = d\widehat{\mathbb{P}}/d\mathbb{P}^{\mathbb{U}}$ is the Girsanov density and by $\mathbb{E}^{\mathbb{U}}[\cdot]$ we denote the mean value with respect to $\mathbb{P}^{\mathbb{U}}$.

Thus $\widehat{\mathbb{E}}\left[\int_0^T \left|Z_t^{\mathbb{U}}\right|^2 dt\right]^{1/2} < \infty$ and therefore $\widehat{\mathbb{E}}\left[\int_\tau^T Z_t^{\mathbb{U}} dW_t^{\mathbb{U}}\right] = 0$.

We can now set $\tau = 0$ in (3.13) and compute expectation with respect to $\widehat{\mathbb{P}}$, obtaining:

$$J^\sharp = Y_0^{\mathbb{U}} = \widehat{\mathbb{E}}\left[\phi(X_T^{\mathbb{U}})\right] + \widehat{\mathbb{E}}\left[\int_0^T \left[\psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}}) - Z_t^{\mathbb{U}} r(t, X_t^{\mathbb{U}}, \widehat{u}_t)\right] dt\right]$$

Adding and subtracting $\widehat{\mathbb{E}}\left[\int_0^T l(t, X_t^{\mathbb{U}}, \widehat{u}_t) dt\right]$ we arrive at:

$$J^\sharp = J(\mathbb{U}) + \widehat{\mathbb{E}}\left[\int_0^T \left[\psi(t, X_t^{\mathbb{U}}, Z_t^{\mathbb{U}}) - Z_t^{\mathbb{U}} r(t, X_t^{\mathbb{U}}, \widehat{u}_t) - l(t, X_t^{\mathbb{U}}, \widehat{u}_t)\right] dt\right] \quad (3.14)$$

By the definition of ψ (formula (3.6)) the term in the square brackets is non positive and consequently $J(\mathbb{U}) \geqslant J^\sharp$. □

We notice that relation (3.14) is a backward stochastic differential equations version of the *fundamental relation.* It immediately yields important consequences:

Corollary 3.3. *For any a.c.s* $\mathbb{U}$ *it is equivalent:*

(1) $\widehat{u}_\tau \in \Gamma(\tau, X_\tau^{\mathbb{U}}, Z_\tau^{\mathbb{U}})$ $\mathbb{P}$*-a.s, for almost every* $\tau \in [0, T]$,

(2) $J(\mathbb{U}) = J^\sharp$.

Moreover, if one of the above conditions holds, then $\mathbb{U}$ *is optimal for the control problem starting from* x *at time 0.*

In order to prove the existence of an optimal a.c.s we need to require that the infimum in the definition of ψ is achieved. Namely we assume (compare (3.6)):

Hypotheses 3.2. *There exists a measurable map* $\gamma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^d \to U$ *such that:*

$$\psi(t, x, z) = l(t, x, \gamma(t, x, z)) + zr(t, x, \gamma(t, x, z))$$

Proposition 3.4. *Under Hypotheses 3.1 and 3.2 there exists an a.c.s verifying* $J(\mathbb{U}) = J^\sharp$. *Consequently* $J^\sharp = V$.

Proof. Let W be a standard Wiener process in $\mathbb{R}^d$, defined in some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $0 \leqslant \tau \leqslant T$, we denote by $\mathcal{F}_\tau$ the σ-algebra generated by W_t, $t \in [0, \tau]$, and augmented by the null sets of $\widehat{\mathcal{F}}$.

Moreover let (X, Y, Z) be the solution of the system (with respect to the natural filtration generated by W augmented by null sets):

$$\begin{cases} dX_\tau = F(\tau, X_\tau)d\tau + G(\tau, X_\tau)dW_\tau, & \tau \in [0, T] \\ X_0 = x \in \mathbb{R}^n \\ dY_\tau = -\psi(\tau, X_\tau, Z_\tau)d\tau + Z_\tau dW_\tau, & \tau \in [0, T] \\ Y_T = \phi(X_T) \end{cases}$$

We define

$$\widehat{W}_\tau := W_\tau - \int_0^\tau r(s, X_s, \gamma(s, X_s, Z_s))ds$$

the system can be rewritten: for every $\tau \in [0, T]$

$$\begin{cases} dX_\tau = F(\tau, X_\tau)d\tau + G(\tau, X_\tau)r(\tau, X_\tau, \gamma(\tau, X_\tau, Z_\tau))d\tau + G(\tau, X_\tau)d\widehat{W}_\tau \\ X_0 = x \in \mathbb{R}^n \\ dY_\tau = -\psi(\tau, X_\tau, Z_\tau)d\tau + Z_\tau r(\tau, X_\tau, \gamma(\tau, X_\tau, Z_\tau))d\tau + Z_\tau d\widehat{W}_\tau \\ Y_T = \phi(X_T) \end{cases}$$

If $\widehat{\mathbb{P}}$ is the probability under which $\widehat{W}$ is a Wiener process, $\widehat{u} = \gamma(s, X_s, Z_s)$, and

$$\mathbb{U} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \widehat{\mathbb{P}}, \widehat{u})$$

then by construction $X^{\mathbb{U}} = X$ and by uniqueness of the solution of the backward equation (3.12), $(Y^{\mathbb{U}}, Z^{\mathbb{U}}) = (Y, Z)$. Thus again by definition $\widehat{u}_\tau = \gamma(\tau, X^{\mathbb{U}}_\tau, Z^{\mathbb{U}}_\tau)$ and the claims follows immediately by Corollary 3.3.

□

3.2 Optimal control for marked point processes

3.2.1 Weak formulation of the problem

We assume that a marked point process is given, satisfying the assumptions of Chapter 1, and we denote as X the process that describes its evolution in time:

$$X_t = \sum_{n \geqslant 0} \xi_n \mathbb{1}_{[T_n, T_{n+1})}(t), \qquad t \geqslant 0.$$

In particular we suppose that $T_n \to \infty$ $\mathbb{P}$-a.s and that Hypothesis (1.12) holds.

The data specifying the optimal control problem are an action (or decision) space U, a running cost function l, a terminal cost function g, and another function r specifying the effect of the control process, exactly as in the diffusive case in Section 3.1. They are assumed to satisfy the following conditions:

Hypotheses 3.3. (1) $(U, \mathcal{U})$ *is a measurable space.*

(2) *The functions* $r, l : \Omega \times [0, T] \times K \times U \to \mathbb{R}$ *are* $\mathcal{P} \otimes \mathcal{K} \otimes \mathcal{U}$*-measurable and there exist two constants* $C_r > 1, C_l > 0$ *such that,* $\mathbb{P}$*-a.s,*

$$0 \leqslant r_\tau(x, u) \leqslant C_r, \quad |l_\tau(x, u)| \leqslant C_l, \quad \tau \in [0, T], x \in K, u \in U. \tag{3.15}$$

(3) *The function* $g : \Omega \times K \to \mathbb{R}$ *is* $\mathcal{F}_T \otimes \mathcal{K}$*-measurable.*

We define as an admissible control process, or simply a control, any predictable process $(u_\tau)_{\tau \in [0, T]}$ with values in U. The set of admissible control processes is denoted $\mathcal{A}$. We recall that a process u is $(\mathcal{F}_t)$-predictable if and only if it admits the representation

$$u(\omega, t) = \sum_{n \geqslant 0} u^{(n)}(\omega, t) \mathbb{1}_{T_n(\omega) \geqslant t \geqslant T_{n+1}(\omega)}, \tag{3.16}$$

where for each $n \geqslant 0$ the mapping $(\omega, t) \mapsto u^{(n)}(\omega, t)$ is $\mathcal{F}_{T_n} \otimes \mathcal{B}([0, \infty))$-measurable. Since we have $\mathcal{F}_{T_n} = \sigma(T_i, \xi_i, 0 \leqslant i \leqslant n)$ (see e.g. [10], Appendix A2, Theorem T30), the fact that a control is predictable can be roughly interpreted by saying that the controller, at each time T_n, based on observation of the random variables $T_i, \xi_i, 0 \leqslant i \leqslant n$, chooses his present and future control actions and updates his decisions only at time T_{n+1}.

By the choice of the control process u we can modify the compensator, and then the distribution, of the process X. We consider indeed the so called weak approach: given a control $u(\cdot) \in \mathcal{A}$ and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, we associate to them a new probability measure $\mathbb{P}_u$ on the same probability space, such that the process X admits a compensator of the form:

$$\tilde{p}^u(dtdy) = r_t(y, u_t)\tilde{p}(dtdy).$$

If we denote by $\mathbb{E}_u$ the expectation under $\mathbb{P}_u$, it corresponds to minimize the functional cost:

$$J(u(\cdot)) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t) dA_t + g(X_T) \right]. \tag{3.17}$$

We show that such a probability $\mathbb{P}_u$ exists. Actually we can find $\mathbb{P}_u$ from the original probability $\mathbb{P}$ by a change of measure of Girsanov type, as we are going to describe. We define

$$L_\tau = \exp\left(\int_0^\tau \int_K (1 - r_t(y, u_t)) \phi_t(dy) dA_t \right) \prod_{n \geqslant 1 : T_n \leqslant \tau} r_{T_n}(\xi_n, u_{T_n}), \qquad \tau \in [0, T]$$

with the convention that the last product equals to 1 if there are no indices $n \geqslant 1$ satisfying $T_n \leqslant \tau$ (similar conventions will be adopted later without further mention). It is a well-known result that L is a nonnegative supermartingale (see [23] Proposition 4.3, or [8]), solution to the equation

$$L_\tau = 1 + \int_0^\tau \int_K L_{t-}(r_t(y, u_t) - 1) q(dtdy), \qquad \tau \in [0, T].$$

If moreover $\mathbb{E}[L_T] = 1$, then the process L is a martingale and thus we can define a probability $\mathbb{P}_u$ setting

$$\mathbb{P}_u(d\omega) = L_T \mathbb{P}(d\omega). \tag{3.18}$$

The above result is a modification of the Girsanov theorem for the marked point processes (see Appendix B).

The following result gives us some conditions under which $\mathbb{E}[L_T] = 1$ holds true.

Lemma 3.5. *Let* $\gamma > 1$ *and*

$$\beta = \gamma + 1 + \frac{C_r^{\gamma^2}}{\gamma - 1}.$$

If $\mathbb{E}[\exp(\beta A_T)] < \infty$*, then we have* $\sup_{\tau \in [0, T]} \mathbb{E}[L_\tau^\gamma] < \infty$ *and* $\mathbb{E}[L_T] = 1$*.*

Proof. We follow [10], Chapter VIII, Theorem T11, with some modifications. To shorten notation we define $\rho_t(y) := r_t(y, u_t)$ and we denote $L_\tau = \mathcal{E}(\rho)_\tau$. For $\gamma > 1$ we define

$$a_t(y) := \gamma^{-1}(1 - \rho_t(y)^{\gamma^2}), \qquad b_t(y) := \gamma - \gamma\rho_t(y) - \gamma^{-1} + \gamma^{-1}\rho_t(y)^{\gamma^2}$$

so that $\gamma(1 - \rho_t(y)) = a_t(y) + b_t(y)$. Then

$$L_\tau^\gamma = \exp\left(\int_0^\tau \int_K (a_t(y) + b_t(y))\phi_t(dy)dA_t\right) \prod_{T_n \leqslant \tau} \rho_{T_n}(\xi_n)^\gamma,$$

and by Hölder's inequality

$$\begin{aligned} \mathbb{E}[L_\tau^\gamma] \leqslant & \left\{\mathbb{E}\left[\exp\left(\int_0^\tau \int_K \gamma a_t(y)\phi_t(dy)dA_t\right) \prod_{T_n \leqslant \tau} \rho_{T_n}(\xi_n)^{\gamma^2}\right]\right\}^{1/\gamma} \cdot \\ & \left\{\mathbb{E}\left[\exp\left(\int_0^\tau \int_K \frac{\gamma}{\gamma - 1} b_t(y)\phi_t(dy)dA_t\right)\right]\right\}^{\frac{\gamma-1}{\gamma}} \end{aligned}$$

Noting that $\gamma a_t(y) = 1 - \rho_t(y)^{\gamma^2}$, the first term equals $\mathcal{E}(\rho^{\gamma^2})_\tau$ and we have $\mathbb{E}\left[\mathcal{E}(\rho^{\gamma^2})_\tau\right] \leqslant 1$ by the supermartingale property. Since $b_t(y) \leqslant \gamma - \gamma^{-1} + \gamma^{-1}C_r^{\gamma^2}$ we arrive at

$$\begin{aligned}
\mathbb{E}\left[L_\tau^\gamma\right] &\leqslant \left\{\mathbb{E}\left[\exp\left(A_T\left(\gamma + 1 + \frac{C_r^{\gamma^2}}{\gamma - 1}\right)\right)\right]\right\}^{\frac{\gamma-1}{\gamma}} \\
&= \left\{\mathbb{E}\left[\exp(\beta A_T)\right]\right\}^{\frac{\gamma-1}{\gamma}} \\
&< \infty. \qquad (3.19)
\end{aligned}$$

Let $S_n := \inf\{\tau \in [0,\, T] : L_{\tau-}^n + A_\tau \geqslant n\}$ with the convention $\inf \emptyset = T$, and let $\rho_t^{(n)}(y) := \mathbb{1}_{[0,\, S_n]}(t)(\rho_t(y)) + \mathbb{1}_{[S_n,\, T]}(t)$, $L^{(n)} := \mathcal{E}(\rho^{(n)})$. Then $L^{(n)}$ satisfies:

$$L_\tau^{(n)} = 1 + \int_0^\tau \int_K L_{t-}^{(n)}(r_t^{(n)}(y) - 1)q(dtdy), \qquad \tau \in [0,\, T].$$

By the choice of $\rho^{(n)}$ we have $L_\tau^{(n)} = L_{\tau\wedge S_n}$, and by the choice of S_n it is easily proved that $\mathbb{E}\left[\int_0^\tau \int_K L_{t-}^{(n)} \left|r_t^{(n)}(y) - 1\right| \phi_t(dy)dA_t\right] < \infty$, so that $L^{(n)}$ is a martingale and $\mathbb{E}\left[L_\tau^{(n)}\right] = \mathbb{E}\left[L_{\tau\wedge S_n}\right] = 1$. The first part of the proof applied to $L^{(n)}$ and the inequality (3.19) yield in particular:

$$\sup_n \mathbb{E}\left[(L_\tau^{(n)})^\gamma\right] = \sup_n \mathbb{E}\left[(L_{\tau\wedge S_n})^\gamma\right] < \infty.$$

So $(L_{\tau\wedge S_n})_n$ is uniformly integrable and letting $n \to \infty$ we conclude that $\mathbb{E}\left[L_\tau\right] = 1$. □

Under the assumptions of the Lemma, we can thus define the probability $\mathbb{P}_u$ setting $\mathbb{P}_u(d\omega) = L_T(\omega)\mathbb{P}(d\omega)$. It can be proved (see [23] Theorem 4.5) that the compensator $\tilde{p}^u$ of p under $\mathbb{P}_u$ satisfies our claim, i.e. $\tilde{p}^u$ is related to the compensator $\tilde{p}$ of p under $\mathbb{P}$ by the formula :

$$\tilde{p}^u(dtdy) = r_t(y, u_t)\tilde{p}(dtdy) = r_t(y, u_t)\phi_t(dy)dA_t.$$

In particular, the compensator of N under $\mathbb{P}_u$ is:

$$A_\tau^u = \int_0^\tau \int_K r_t(y, u_t)\phi_t(dy)dA_t. \qquad (3.20)$$

We can finally define the cost associate to every $u(\cdot) \in \mathcal{A}$ as in (3.17): the control problem consists in minimizing $J(u(\cdot))$ over $\mathcal{A}$.

Remark 3.2. Later we will demand that

$$\mathbb{E}\left[|g(X_T)|^2 e^{\beta A_T}\right] < \infty \tag{3.21}$$

for some $\beta > 0$ fixed in such a way that the cost is finite for every admissible control (see Hypothesis 2.4).

Remark 3.3. We notice that the laws of the random coefficients r, l, g under $\mathbb{P}$ and under $\mathbb{P}_u$ are not the same in general, so that the formulation of the optimal control problem should be carefully examined when facing a specific application or modelling situation. This difficulty clearly disappears when r, l, g are deterministic.

Example 3.1. *Optimal control for Markov chains*
We assume that X is a Markov chain on $K = 1, ...N$ with matrix of transition rates $(\lambda(i,\, j))_{i,\, j\in K}$. We use the convention that $\lambda(i,\, i) = 0$. Then when X enters a state i, it stays there for an exponential time with rate $\lambda(i) := \sum_j \lambda(i, j)$ and it jumps on a state j (independent on the sojourn time) with probability $\pi(i,\, j) = \frac{\lambda(i,j)}{\lambda(i)}$ (we assume $\pi(i,\, j) = \delta_{i,j}$ if $\lambda(i) = 0$). In this case the probability p is specified by:

$$p((0,\, t] \times \{j\}) = \sum_{n\geqslant 1} \mathbb{1}_{T_n \leqslant t} \mathbb{1}_{X_{T_n} = j},$$

and his compensator is:

$$\begin{aligned}\tilde{p}(dt, \{j\}) &= \lambda(X_{t-}, j)\, dt \\ &= \pi(X_{t-}, j)\lambda(X_{t-})dt.\end{aligned}$$

In particular we can identify the measures $\phi_t(dy)$ and dA_t of the previous general discussion as:

$$\phi_t(\{j\}) = \pi(X_{t-},\, j) \quad \text{and} \quad dA_t = \lambda(X_{t-})dt,$$

remarking that the compensator $\tilde{p}$ admits the stochastic intensity $\lambda(X_{t-})$.

If now we consider the optimal control problem and the associated probability $\mathbb{P}_u$, the compensator of p becomes:

$$\tilde{p}^u(dt, \{j\}) = \lambda(X_{t-}, j)\, r_t(j\, u_t)dt.$$

We can for instance suppose that $r_t(j, u_t) = r(j, u_t)$, and consider a feedback stationary control of the form

$$u_t = \underline{\mathrm{u}}(X(t-))$$

for a given function $\underline{u} : K \to U$. Then

$$\tilde{p}^u(dt, \{j\}) = \lambda(X_{t-}, j)\, r(j,\, \underline{u}(X(t-)))dt.$$

This is the compensator of a Markov chain with transition rates $\lambda(i,\, j)r(j,\, \underline{u}(i))$: the choice of the control modifies the original transition rates, multiplying them for $r(j,\, \underline{u}(i))$.

3.2.2 Solving the Optimal Control problem by the BSDEs approach

We next proceed to the solution to the optimal control problem formulated above. A basic role is played by the BSDE:

$$Y_\tau + \int_\tau^T \int_K Z_t(y)q(dtdy) = g(X_T) + \int_\tau^T f(t, X_t, Z_t(\cdot))dA_t, \qquad \tau \in [0,\, T] \quad (3.22)$$

with terminal condition $g(X_T)$ and generator defined by means of the hamiltonian function f.
The hamiltonian function is defined for every $\omega \in \Omega$, $t \in [0,\, T]$, $x \in K$ and $z \in \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$ by the formula

$$f(\omega, t, x, z(\cdot)) = \inf_{u \in U} \left\{ l_t(\omega, x, u) + \int_K z(y)(r_t(\omega, x, u) - 1)\phi_t(\omega, dy) \right\}. \quad (3.23)$$

We assume that the infimum is in fact achieved, possibly at many points. Moreover we need to verify that the generator of the BSDE (3.22) satisfies the conditions required in the Section 2.2. It turns out that an appropriate assumption is the following one, since we will see below that it can be verified under quite general conditions. Here and in the following we set $X_{0-} = X_0$.

Hypotheses 3.4. *For every $Z \in \mathcal{L}^{1,0}(p)$ there exists a function $\underline{u}^Z : \Omega \times [0,\, T] \to U$, measurable with respect to $\mathcal{P}$ and $\mathcal{U}$, such that*

$$\begin{aligned} f(\omega, t, X_{t-}(\omega), Z_t(\omega, \cdot)) &= l_t(X_{t-}(\omega), \underline{u}^Z(\omega, t)) \\ &+ \int_K Z_t(\omega, y)(r_t(\omega, y, \underline{u}^Z(\omega, t)) - 1)\phi_t(\omega, dy) \end{aligned} \quad (3.24)$$

for almost all (ω, t) with respect to the measure $dA_t(\omega)\mathbb{P}(d\omega)$.

Note that if $Z \in \mathcal{L}^{1,0}(p)$ then $Z_t(\omega, \cdot) \in \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$ except possibly on a predictable set of points (ω, t) of measure zero respect to $dA_t(\omega)\mathbb{P}(d\omega)$, so that the equality (3.24) is meaningful. Also note that each $\underline{u}^Z$ is an admissible control.

We can now verify that all the assumptions of Hypotheses 2.4 hold true for the generator of the BSDE (3.22), which is given by the formula

$$f_t(\omega, z(\cdot)) = f(\omega, t, X_t, z(\cdot)), \qquad \omega \in \Omega, t \in [0, T], z \in \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy)).$$

Indeed, if $Z \in \mathcal{L}^{2,\beta}(p)$ then $Z \in \mathcal{L}^{1,0}(p)$ by (2.22), and (3.24) shows that the process $(\omega, t) \mapsto f(\omega, t, X_t, Z_t(\omega, \cdot))$ is progressive; since A is assumed to have continuous trajectories and X has piecewise constant paths, the progressive set $\{(\omega, t) : X_{t-}(\omega) \neq X_t(\omega)\}$ has measure zero with respect to $dA_t(\omega)\mathbb{P}(d\omega)$; it follows that the process

$$(\omega, t) \mapsto f(\omega, t, X_t, Z_t(\omega, \cdot)) = f_t(\omega, Z_t(\omega, \cdot))$$

is progressive, after modification on a set of measure zero, as required in (2.19). Next, using the boundedness assumptions (3.15), it is easy to check that (2.20) is verified with $L' = 0$ and

$$L = \sup |r - 1| = \sup\{|r_t(y, u) - 1| : \omega \in \Omega, t \in [0, T], y \in K, u \in U\}.$$

Using (3.15) again we also have:

$$\begin{aligned} \mathbb{E}\left[\int_0^T e^{\beta A_t} |f(t, X_t, 0)|^2 \, dA_t\right] &= \mathbb{E}\left[\int_0^T e^{\beta A_t} \left|\inf_{u \in U} l_t(X_t, u)\right|^2 dA_t\right] \\ &\leqslant C_l^2 \beta^{-1} \mathbb{E}\left[e^{\beta A_T}\right], \end{aligned} \tag{3.25}$$

so that (2.21) holds as we provided the right-hand side of (3.25) is finite. Assuming finally that (3.21) holds, by Theorem (2.8) the BSDE (3.22) has a unique solution $(Y, Z) \in \mathbb{K}^\beta$ if $\beta > L^2$.

The corresponding admissible control $\underline{u}^Z(\omega, t)$, whose existence is required in condition (3.21), is denoted by u^*. We are now ready to state the main result. Recall that $C_r > 1$ was introduced in (3.15).

Theorem 3.6. *Assume that conditions* (3.15) *and* (3.21) *are satisfied and that*

$$\mathbb{E}\left[\exp((3 + C_r^4) A_T)\right] < \infty. \tag{3.26}$$

Suppose that exists β *such that*

$$\beta > \sup |r - 1|^2, \quad \mathbb{E}\left[\exp(\beta A_T)\right] < \infty, \quad \mathbb{E}\left[|g(X_T)|^2 e^{\beta A_T}\right] < \infty. \tag{3.27}$$

Let $(Y, Z) \in \mathbb{K}^\beta$ *denote the solution to the BSDE* (3.22) *and* $u^* = \underline{u}^Z$ *the corresponding admissible control. Then* $u^*(\cdot)$ *is optimal and* Y_0 *is the optimal cost, i.e* $Y_0 = J(u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{A}} J(u(\cdot))$.

Remark 3.4. Note that if g is bounded then (3.27) follows from (3.26) with $\beta = 3 + C_r^4$, since $|r_t(y,u) - 1|^2 \leqslant (C_r+1)^2 < 3 + C_r^4$.

Proof. Fix $u(\cdot) \in \mathcal{A}$. Assumption (3.26) allows to apply Lemma 3.5 with $\gamma = 2$ and yields $\mathbb{E}\left[L_T^2\right] < \infty$. It follows that $g(X_T)$ is integrable under $\mathbb{P}_u$. Indeed by (3.21):

$$\mathbb{E}_u\left[|g(X_T)|\right] = \mathbb{E}\left[|L_T g(X_T)|\right] \leqslant (\mathbb{E}\left[L_T^2\right])^{1/2}(\mathbb{E}\left[g(X_T)^2\right])^{1/2} < \infty.$$

We next show that under $\mathbb{P}_u$ we have $Z \in \mathcal{L}^{1,0}(p)$, i.e $\mathbb{E}_u\left[\int_0^T \int_K |Z_t(y)|\, \tilde{p}^u(dtdy)\right] < \infty$. First, note that, by Hölder's inequality,

$$\begin{aligned}
\int_0^T \int_K |Z_t(y)|\, \phi_t(dy)dA_t &= \int_0^T \int_K e^{-\frac{\beta}{2}A_t} e^{\frac{\beta}{2}A_t} |Z_t(y)|\, \phi_t(dy)dA_t \\
&\leqslant \left(\int_0^T e^{-\beta A_t} dA_t\right)^{1/2} \left(\int_0^T \int_K e^{\beta A_t} |Z_t(y)|^2\, \phi_t(dy)dA_t\right)^{1/2} \\
&= \left(\frac{1-e^{-\beta A_T}}{\beta}\right) \left(\int_0^T \int_K e^{\beta A_t} |Z_t(y)|^2\, \phi_t(dy)dA_t\right)^{1/2}.
\end{aligned}$$

Therefore, using (3.15),

$$\begin{aligned}
\mathbb{E}_u\left[\int_0^T \int_K |Z_t(y)|\, \tilde{p}^u(dtdy)\right] &= \mathbb{E}_u\left[\int_0^T \int_K |Z_t(y)|\, r_t(y,u_t)\phi_t(dy)dA_t\right] \\
&= \mathbb{E}\left[L_T \int_0^T \int_K |Z_t(y)|\, r_t(y,u_t)\phi_t(dy)dA_t\right] \\
&\leqslant \left(\mathbb{E}\left[L_T^2\right]\right)^{1/2} \frac{C_r}{\sqrt{\beta}} \left\{\mathbb{E}\left[\int_0^T e^{\beta A_t} |Z_t(y)|^2\, \phi_t(dy)dA_t\right]\right\}^{1/2},
\end{aligned}$$

and the right-hand side of the last inequality is finite, since $(Y,Z) \in \mathcal{K}^\beta$. We have now proved that $Z \in \mathcal{L}^{1,0}(p)$ under $\mathbb{P}_u$. In particular, it follows that:

$$\begin{aligned}
\mathbb{E}_u\left[\int_0^T \int_K Z_t(y) p(dtdy)\right] &= \mathbb{E}_u\left[\int_0^T \int_K Z_t(y) \tilde{p}^u(dtdy)\right] \\
&= \mathbb{E}_u\left[\int_0^T \int_K Z_t(y) r_t(y,u_t)\phi_t(dy)dA_t\right].
\end{aligned}$$

Setting $\tau = 0$ and taking the expectation $\mathbb{E}_u$ in the BSDE (3.22), recalling that $q(dtdy) = p(dtdy) - \tilde{p}(dtdy) = p(dtdy) - \phi_t(dy)dA_t$ and that Y_0 is deterministic, we obtain

$$\begin{aligned}
&Y_0 + \mathbb{E}_u\left[\int_0^T \int_K Z_t(y)(r_t(y,u_t) - 1)\phi_t(dy)dA_t\right] \\
&= \mathbb{E}_u\left[g(X_T)\right] + \mathbb{E}_u\left[\int_0^T f(t, X_t, Z_t(\cdot))dA_t\right].
\end{aligned}$$

We finally obtain

$$
\begin{aligned}
Y_0 &= J(u(\cdot)) \\
&\quad + \mathbb{E}_u \left[\int_0^T \left(f(t, X_t, Z_t(\cdot)) - l_t(X_t, u_t) - \int_K (r_t(y, u_t) - 1)\phi_t(dy) \right) dA_t \right] \\
&= J(u(\cdot)) \\
&\quad + \mathbb{E}_u \left[\int_0^T \left(f(t, X_{t-}, Z_t(\cdot)) - l_t(X_{t-}, u_t) - \int_K (r_t(y, u_t) - 1)\phi_t(dy) \right) dA_t \right]
\end{aligned}
$$

where the last equality follows from the continuity of A. This identity is the fundamental relation in the marked point processes case. By the definition of the hamiltonian f, the term in square brackets is smaller or equal to 0, and it equals 0 if $u(\cdot) = u^*(\cdot)$. □

Hypotheses 3.4 can be verified in specific situations when it is possible to compute explicitly the functions $\underline{\mathrm{u}}^Z$. General conditions for its validity can also be formulated using appropriate selection theorems, as in the following proposition.

Proposition 3.7. *In addition to the assumptions in Hypotheses 3.3 suppose that U is a compact metric space with its Borel σ-algebra $\mathcal{U}$ and that the functions $r_t(\omega, x, \cdot), l_t(\omega, x, \cdot) : U \to \mathbb{R}$ are continuous for every $\omega \in \Omega, t \in [0, T], x \in K$ Then Hypotheses 3.4 is verified.*

Proof. Let us consider the measure $\mu(d\omega dt) = dA_t \mathbb{P}(d\omega)$ on the predictable σ-algebra $\mathcal{P}$. Let $\bar{\mathcal{P}}$ denote its μ-completion and consider the complete measure space $(\Omega \times [0, T], \bar{\mathcal{P}}, \mu)$. Fix $Z \in \mathcal{L}^{1,0}(p)$, and note that the set
$A^Z = (\omega, t) : Z_t(\omega, \cdot) \notin \mathcal{L}^1(K, \mathcal{K}, \phi_t(\omega, dy))$ has μ-measure zero and define a map $F^Z : \Omega \times [0, T] \times U \to \mathbb{R}$ setting:

$$
F^Z(\omega, t, u) = \begin{cases} l_t(\omega, X_{t-}, u) + \int_K Z_t(\omega, y) \left(r_t(\omega, y, u) - 1 \right) \phi_t(\omega, dy) & (\omega, t) \notin A^Z \\ 0 & (\omega, t) \in A^Z. \end{cases}
$$

Then $F^Z(\cdot, \cdot, u)$ is $\bar{\mathcal{P}}$-measurable for every $u \in U$, and it is easily verified that $F^Z(\omega, t, \cdot)$ is continuous for every $(\omega, t) \in \Omega \times [0, T]$. By a classical selection theorem (see [1] Theorem 8.1.3 and Theorem 8.2.11) there exists a function $\underline{\mathrm{u}}^Z : \Omega \times [0, T] \to U$, measurable with respect to $\bar{\mathcal{P}}$ and $\mathcal{U}$, such that $F^Z(\omega, t, \underline{\mathrm{u}}^Z(\omega, t)) = \min_{u \in U} F^Z(\omega, t, u)$ for every $(\omega, t) \in \Omega \times [0, T]$, so that (3.24) holds for every (ω, t). After modifications on a set of μ-measure zero, the function $\underline{\mathrm{u}}^Z$ can be made measurable with respect to $\mathcal{P}$ and $\mathcal{U}$, and (3.24) still holds, as it is understood as an equality for μ-almost all (ω, t). □

Concluding remarks

The aim of this thesis was to provide a methodological treatment of backward stochastic differential equations applied to optimal control problems. In this work we showed how to solve, in a systematic way, optimal control problems with an approach based on BSDEs. This is an alternative tool to the classical dynamic programming theory, which represents a fundamental principle in the stochastic control framework. The main idea of dynamic programming is to embed the original problem into a much larger class of problems, and then to tie all these problems together with a partial differential equation known as the Hamilton-Jacobi-Bellman (HJB) equation. The HJB equation represents a very important tool both in the diffusive case and in the context of jump processes.

Compared to dynamic programming theory, the BSDEs approach has the great advantage of solving optimal control problems also in the case of nonmarkovian processes. Conversely, for such processes there is not a systematic treatment via dynamic programming, since the generator to the HJB equation does not exist in general and the function value ceases to be deterministic. Moreover, in the diffusive case the BSDEs theory leads to useful and interesting results from a computational point of view. In fact, if we consider for instance an optimal control problem for a process taking values in $\mathbb{R}^n$, $n \gg 1$ (e.g. in a financial context), it is much more efficient to compute the solution of the associated one-dimensional BSDE instead of the solution of the n-dimensional HJB equation.

The results of this work admit several variants and generalizations: some of them are not included here for reason of brevity and some are not yet complete. For instance, the BSDE approach to optimal control of Markov jump processes deserves a specific treatment: for this topic some results are in preparation, concerning, in particular, the form of the associated BSDE and the corresponding HJB equation. Moreover, we hope that this work may lead to a better understanding of the nonmarkovian situation. In particular, we studied the specific form of the compensator in the semi-Markov case, with the intent to apply the BSDEs approach to semi-Markov optimal control problems in the future. In our study, we also considered more general processes below the semi-Markov case.

Other appealing directions of investigation could be the study of diffusive equations with jumps, and the analysis of BSDEs driven by random measures without Lipschitz assumptions on the generator, along the lines of many results available in the diffusive case. We can also consider the extensions to the case of vector-valued process Y and to random time interval. In the diffusive case, a large literature is available on these topics.

In addition, there exist a great variety of optimal control problems not discussed here. First, we can consider optimal control problems with infinite horizon; for these problems, the cost criterion in the diffusive case takes the form:

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t} l(X_t, u_t) dt\right],$$

where $\beta > 0$ is a discount factor. One of the main contributions to this topic consist of Pardoux's paper [28].

In ergodic control problems, instead, we have to deal with stochastic systems that exhibit in the long time a stationary behaviour characterized by an invariant measure. In this case, the optimal control problem consists of optimizing over the long term some criterion taking into account the invariant measure. A standard formulation corresponds to optimize over the controls u a functional of the form:

$$\lim_{T\to\infty} \sup \frac{1}{T}\mathbb{E}\left[\int_0^T l(X_t, u_t) dt\right].$$

For a detailed discussion we refer the reader to the paper [19].

We can also analyse optimal stopping problems. In the diffusive case, they consist in finding the function

$$v(x) = \sup_{\tau\in S} \mathbb{E}\left[\int_0^\tau e^{-\beta t}\, l(X_t^x) dt + e^{-\beta\tau} g(X_\tau^x)\right]$$

where S denotes the set of stopping times in $[0, +\infty]$, $e^{-\beta\tau} = 0$ if $\tau = +\infty$, and X_t^x denotes a stochastic process with x as initial value. These specific optimal control problems are investigated in depth in [15].

Optimal stopping problems can be generalized in the so called impulsive control problems. Roughly speaking, in this case the system is stopped several times, and each time it is restarted from a new state value. In this context, we deal with a series of stopping times S_n, and the functional cost depends both on the stopping times and on the associated new state values. This kind of problems are discussed by Pham in the paper [24].

Finally, it is possible to deal with more complex optimal control problems, as for instance the case of control with partial observations, or the adaptive control.

In the first case we have to control stochastic processes through observations affected by unavoidable noises. In the second, some parameters of the state dynamics are unknown: the choice of the control must be combined with an estimate of the parameters.

For all these optimal control problems there is a great literature in the diffusive case, also concerning the BSDEs approach. Conversely, few results are available in the non diffusive context. The purpose of this thesis was precisely to provide a first attempt along this direction. In fact, we showed how to extend the BSDEs approach to optimal control from diffusive to marked point processes. This can lead the path to several other applications and extension in optimal control theory.

Appendix A

Stochastic Processes

In this section we are going to recall basic notations on Filtrations, Martingales and Stopping times that we will constantly use in the rest of the work.

A.1 Filtrations, Measurability

Let $(\Omega, \mathcal{F})$ be a measurable space. A *history* $(\mathcal{F}_t, t \geqslant 0)$ on $(\Omega, \mathcal{F})$ is a family of sub-σ-algebra of $\mathcal{F}$ such that for all $0 \leqslant s \leqslant t$

$$\mathcal{F}_s \subset \mathcal{F}_t.$$

We see that a history is an increasing family of sub-σ-algebra of $\mathcal{F}$ indexed by the non negative real number. It is also called a *filtration*. We use the following notation: $\mathcal{F}_\infty = \bigvee_{t \geqslant 0} \mathcal{F}_t$. We say that the history is *right continuous* if for all $t \geqslant 0$

$$\mathcal{F}_{t+} = \bigcap_{h>0} \mathcal{F}_{t+h} = \mathcal{F}_t.$$

A family $(X_t)_{t \geqslant 0} : (\Omega, \mathcal{F}) \to (K, \mathcal{K})$ of random variables is called a K- *valued stochastic process* defined on $(\Omega, \mathcal{F})$. Associated to the process X_t, we can define for each $t \geqslant 0$ a sub-σ-algebra of $\mathcal{F}$, denoted by

$$\mathcal{F}_t^X = \sigma(X_s, s \in [0,\, t]).$$

This σ-algebra is generated by the family of random variables $(X_s, s \in [0,\, t])$ and it is called the *internal history of the process* X_t. Any history $\mathcal{F}_t$ such that

$$\mathcal{F}_t \supset \mathcal{F}_t^X \qquad t \geqslant 0$$

is called a *history* of X_t. We can say that X_t is *adapted* to $\mathcal{F}_t$. We denote the whole families $(\mathcal{F}_t, t \geqslant 0)$ and $(X_t, t \geqslant 0)$ simply by $\mathcal{F}_t$ and X_t. If X_t is a process on $(\Omega, \mathcal{F})$ and w an element of Ω, the mapping

$$t \to X_t(w)$$

is called a *trajectories* of X_t. If $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$, and K is a topological space, the process X_t is said to be *continuous* (*right continuous, left continuous*) if and only if the trajectories $t \to X_t(w)$ are continuous (right continuous, left continuous), $\mathbb{P}$-a.s.

Definition A.1. The K-valued process X_t is said to be *measurable* if and only if the mapping $(t, w) \mapsto X_t(w) : \mathbb{R}_+ \times \Omega \to K$ is $\mathcal{K}/\mathcal{B}_+ \otimes \mathcal{F}$-measurable.

Definition A.2. The K-valued process X_t is said to be $\mathcal{F}_t$-*progressive* if and only if for all $t \geqslant 0$ the mapping $(s, w) \mapsto X_s(w) : [0,\, t] \times \Omega \to K$ is $\mathcal{K}/\mathcal{B}([0,\, t]) \otimes \mathcal{F}_t$-measurable.

It follows that if X_t is $\mathcal{F}_t$-progressive, than it is adapted to $\mathcal{F}_t$ and measurable.

Theorem A.1. *If K is a metrizable topological space and X_t is an K-valued process adapted to $\mathcal{F}_t$ and right-continuous (or left continuous), then X_t is $\mathcal{F}_t$-progressive.*

Definition A.3. We define $\mathcal{P}(\mathcal{F}_t)$ to be the σ-field over $(0,\, \infty) \times \Omega$ generated by the rectangles of the form

$$(s,\, t] \times A, \qquad 0 \leqslant s \leqslant t, \quad A \in \mathcal{F}_s.$$

$\mathcal{P}(\mathcal{F}_t)$ is called the $\mathcal{F}_t$-*predictable σ-field* over $(0,\, \infty) \times \Omega$.

Theorem A.2. *If X_t is an $\mathbb{R}^n$-valued process adapted to $\mathcal{F}_t$ and left continuous, then X_t is $\mathcal{F}_t$-predictable.*

In the hierarchy of measurability, predictability stands on top: if X_t is $\mathcal{F}_t$-predictable, than it is X_t is $\mathcal{F}_t$-progressive.

A.2 Martingales

The concept of martingale is naturally linked to the concept of increasing information pattern. A $(\mathbb{P}, \mathcal{F}_t)$-*martingale over* $[0,\, c]$, $c \geqslant 0$ real number, is a real-valued stochastic process X_t such that:

(i) X_t is adapted to $\mathcal{F}_t$,

(ii) X_t is $\mathbb{P}$-integrable, i.e. $\mathbb{E}\left[|X_t|\right] < \infty \quad t \in [0,\, c]$,

(iii) for all $0 \leqslant s \leqslant t \leqslant c, \quad \mathbb{E}\left[X_t | \mathcal{F}_s\right] = X_s \quad \mathbb{P}$-a.s.

If X_t is a $(\mathbb{P}, \mathcal{F}_t)$-martingale over $[0,\, c]$ for every real $c \geqslant 0$, X_t is called a $(\mathbb{P}, \mathcal{F}_t)$-martingale.

An important property concerning the martingales is:

Theorem A.3. *Doob's inequality*
Let $(X_t)_{t\geqslant 0}$ a càd martingale. Then

$$\mathbb{E}\left[\sup_{s\in[0,\,t]} X_s^2\right] \leqslant 4\mathbb{E}\left[X_t^2\right] \quad \forall t \geqslant 0.$$

Concerning the relations between martingales and stopping times, we have the fundamental

Theorem A.4. *First Optimal stopping time theorem*
Let $(X_t)_{t\geqslant 0}$ a càd martingale, S, T bounded stopping times such that $S \leqslant T$. Then

$$\mathbb{E}^{\mathcal{F}_S}\left[X_T\right] = X_S \quad a.s.$$

In particular, for all bounded T, $\mathbb{E}\left[X_T\right] = \mathbb{E}\left[X_0\right]$.

Theorem A.5. *Second Optimal stopping time theorem*
Let $(X_t)_{t\geqslant 0}$ a càd martingale, S, T bounded stopping times such that $S \leqslant T$ and $\mathbb{E}\left[\sup_{t\geqslant 0} X_t\right] < \infty$. Then

$$\mathbb{E}^{\mathcal{F}_S}\left[X_T\right] = X_S \quad a.s..$$

In particular, for all bounded T, $\mathbb{E}\left[X_T\right] = \mathbb{E}\left[X_0\right]$.

Moreover, we can take a real-valued stochastic process X_t adapted to a history $\mathcal{F}_t$, and an increasing family of T_n-stopping times such that:

(i) $\lim_{n\to\infty} T_n = +\infty$

(ii) for each $n \geqslant 1$, $X_{t\wedge T_n}$ is a $(\mathbb{P}, \mathcal{F}_t)$-martingale.

Then X is called $(\mathbb{P}, \mathcal{F}_t)$-*local martingale* and the family $(T_n)_{n\geqslant 1}$ is a family of localizing times for X_t.

Finally we can consider the Radon-Nikodym derivative and its relation with the martingales. We introduce $\mathbb{P}$ and $\tilde{\mathbb{P}}$, two probabilities measures defined on the same measurable space $(\Omega, \mathcal{F})$, and the history $\mathcal{F}_t$. For each $t \geqslant 0$ we denote by $\mathbb{P}_t$ and $\tilde{\mathbb{P}}_t$ the restriction to $\mathcal{F}_t$ of $\mathbb{P}$ and $\tilde{\mathbb{P}}$ respectively, and we assume that for some $c \geqslant 0$, $\tilde{\mathbb{P}}_c$ is absolutely continuous with respect to $\mathbb{P}_c$. Then, for all $t \in [0,\, c]$, $\tilde{\mathbb{P}}_t$ is absolutely continuous with respect to $\mathbb{P}_t$. We consider for each

$t \in [0,\, c]$, the random variable L_t as the Radon-Nikodym derivative of $\tilde{\mathbb{P}}_t$ with respect to $\mathbb{P}_t$:

$$L_t = \frac{d\tilde{\mathbb{P}}_t}{d\mathbb{P}_t}.$$

Then L_t is a $(\mathbb{P}, \mathcal{F}_t)$- (local) martingale over $[0,\, c]$.

Quadratic variation of a local martingale

Theorem A.6. *Let M be a local (continuous) martingale. Then there exists an increasing process, unique up to indistinguishability, that we denote $\langle M \rangle := (\langle M \rangle_t,\, t \geqslant 0)$, such that*

$$M_t^2 - \langle M \rangle_t \quad \textit{is a local martingale.}$$

Moreover, for every $t > 0$,

$$\lim_{\|p\| \to 0} \sum_{k=1}^{n} (M_{t_k} - M_{t_{k-1}})^2 = \langle M \rangle_t \quad \text{in probability,} \tag{A.1}$$

where p ranges over partitions of the interval $[0,\, t]$ and the norm of the partition p is the mesh.

Remark A.1. (i) The process $\langle M \rangle$ is called "quadratic variation" of M.

(ii) If $M = W$ is the brownian motion, then $\langle W_t \rangle = t$ (it follows from the fact that $W_t^2 - t$ is a martingale).

Theorem A.7. *Kunita-Watanabe inequality*
Let M, N be two martingales and let K, H be two measurable processes. Then

$$\int_0^\infty |H_s|\, |K_s|\, |d\langle M, N \rangle_s| \leqslant \sqrt{\int_0^\infty H_s^2 d\langle M \rangle^2} \sqrt{\int_0^\infty K_s^2 d\langle N \rangle^2}.$$

The following inequalities represent the link between a local martingale and its quadratic variation. For every local (continue) martingale M, we denote $M_t^* := \sup_{s \in [0,\, t]} |M_s|$.

Theorem A.8. *Burkholder-Davis-Gundy*
Let $p \in \mathbb{R}$, $p > 0$. There exist some constants $0 < c_p < C_p < +\infty$ such that, for every continue local martingale M starting by 0,

$$c_p \, \mathbb{E}\left[\langle M \rangle_\infty^{p/2} \right] \leqslant \mathbb{E}\left[(M_\infty^*)^p \right] \leqslant C_p \mathbb{E}\left[\langle M \rangle_\infty^{p/2} \right].$$

A.3 Stopping times

Let T be a random variable which takes values in $\bar{\mathbb{R}}_+$ and let $\mathcal{F}_t$ a history. $\{T \leqslant t\} \in \mathcal{F}_t$. T is called an $\mathcal{F}_t$-*stopping time* if and only if

$$\{T \leqslant t\} \subset \mathcal{F}_t, \qquad t \geqslant 0. \tag{A.2}$$

If we regard T as the time of the first occurrence of an certain event, then (A.2) means that we can decide on the basis of $\mathcal{F}_t$ whether or not this event has occurred before t. Trivial examples of stopping times are the constants $T = t \in \mathbb{R}^+$. Most of the stopping times that we will encounter are of the following type.

Theorem A.9. *Let X be a right-continuous (left-continuous) $\mathbb{R}$-valued process adapted to $\mathcal{F}_t$, and c be a fixed real number. Define T as*

$$T = \begin{cases} \inf\{t|X_t \geqslant c\} & if\{...\} \neq \emptyset \\ +\infty & otherwise. \end{cases}$$

Then T is a stopping time.

We now list some important properties of stopping times:

(1) Any number a in $[0, \infty]$ is a stopping time (for any history $\mathcal{F}_t$).

(2) If T is an $\mathcal{F}_t$-stopping time and $a \in [0, \infty]$, then $T + a$ is an $\mathcal{F}_t$-stopping time.

(3) If T and S are $\mathcal{F}_t$-stopping times, then $S \wedge T$ and $S \vee T$ are $\mathcal{F}_t$-stopping times.

(4) If T is an $\mathcal{F}_t$-stopping time and $a \in [0, \infty)$, then $T \wedge a$ is $\mathcal{F}_a$-measurable.

Theorem A.10. *Let $(T_n)_{n\geqslant 1}$ be a family of $\mathcal{F}_t$-stopping times. Then $\sup_{n\geqslant 1} T_n$ is an $\mathcal{F}_t$-stopping time and $\inf_{n\geqslant 1} T_n$ is an $\mathcal{F}_{t+}$-stopping time.*

Corollary A.11. *Let $(T_n)_{n\geqslant 1}$ be a family of $\mathcal{F}_t$-stopping times. Then $\inf_{n\geqslant 1} T_n$, $\limsup_{n\to\infty} T_n$, $\liminf_{n\to\infty} T_n$ are $\mathcal{F}_t$-stopping times. Moreover, if $\lim_{n\to\infty} T_n$ exists, it is also an $\mathcal{F}_t$-stopping time.*

Definition A.4. Let T be a $\mathcal{F}_t$-stopping time. The *past at time T* (relative to $\mathcal{F}_t$) is the σ-algebra $\mathcal{F}_T$ defined as

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty | A \cap \{T \leqslant t\} \in \mathcal{F}_t \quad \text{for all} \quad t \geqslant 0\}$$

Under certain conditions, the expected value of a martingale at a stopping time is equal to the expected value of its initial value. Indeed

Theorem A.12. *Doob's optional stopping theorem*
Let be $(X_n)_{n\geqslant 0}$ *a martingale and* T *a stopping time with respect to* X*. If one of the following conditions holds:*

(1) $T \leqslant c$ $\mathbb{P}$*-a.s.,* $c \in \mathbb{R}$*,*

(2) $\mathbb{E}[T] < \infty$*,* $\exists c \in \mathbb{R}$ *such that* $\mathbb{E}[X_{i+1} - X_i | X_i] \leqslant c$ $\mathbb{P}$*-a.s.,* $\forall i$*,*

(3) $T < \infty$*,* $|X_n| \leqslant c$ $\forall n \leqslant T$*,* $c \in \mathbb{R}$*.*

Then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$*.*

Similarly, if $(X_n)_{n\geqslant 0}$ is a submartingale or a supermartingale and the above conditions hold, then

$$\begin{aligned} \mathbb{E}[X_T] &\geqslant \mathbb{E}[X_0] \qquad \text{for a submartingale} \\ \mathbb{E}[X_T] &\leqslant \mathbb{E}[X_0] \qquad \text{for a supermartingale.} \end{aligned}$$

A.4 Point-Process Filtrations

For the following we will refer to [10], Appendix A2. We recall our main conventions and definitions:

$$T_\infty = lim \uparrow T_n,$$

$$N_t(A) = \sum_{n\leqslant 1} \mathbb{1}_{T_n\leqslant t}\mathbb{1}_{\xi_n\in A}, \qquad A \in \mathcal{K},$$

$$N_t(K) = N_t,$$

$$p((0,\, t] \times A) = N_t(A),$$

$$\mathcal{F}_t^0 = \sigma(N_s(A) : s \in [0,\, t], A \in \mathcal{K}), \qquad t \geqslant 0.$$

Some important facts are:

Theorem A.13. *Let* $(T_n, \xi_n)_{n\geqslant 1}$ *be a* K*-marked point process defined on* $(\Omega, \mathcal{F})$*. With the above notations*

(i) $\forall n \geqslant 0$*,* T_n *is an* $\mathcal{F}_t^0$*-stopping time,*

(ii) $\forall n \geqslant 1$*,* $\sigma(T_i, \xi_i; 1 \leqslant i \leqslant n) \subset \mathcal{F}_{T_n}^0$*,*

(iii) $\forall t \geqslant 0$*,* $\mathcal{F}_t^0 = \sigma(\mathbb{1}_{\xi_n\in A}\mathbb{1}_{T_n\leqslant s}; n \geqslant 1, 0 \leqslant s \leqslant t, A \in \mathcal{K})$*.*

We have the following important results concerning the point-processes histories.

Theorem A.14. *Let $(T_n, \xi_n)_{n\geqslant 1}$ be a K-marked point process defined on $(\Omega, \mathcal{F})$. With the above notations*

(i) The process $X_t = \mathbb{1}_{T_\infty \leqslant t}$ is $\mathcal{F}_t^0$-predictable.

(ii) $\forall n \geqslant 0$, let $f^{(n)}$ be a mapping from $\omega \times [0, \infty)]$ into R_+ that is $\mathcal{F}^0_{T_n} \otimes \mathcal{B}_+$-measurable; then the process X_t defined by

$$X_t(\omega) = \sum_{n\geqslant 1} f^{(n)}(t,\omega)\mathbb{1}_{T_n(\omega)<t\leqslant T_{n+1}(\omega)}$$

is $\mathcal{F}_t^0$-predictable.

Theorem A.15. *Let $(T_n, \xi_n)_{n\geqslant 1}$ be a K-marked point process defined on $(\Omega, \mathcal{F})$. The internal history is right-continuous, i.e. $\mathcal{F}_t^0 = \cap_{h>0}\mathcal{F}^0_{t+h}$.*

This theorem is a special case of a more general result:

Theorem A.16. *Let Y_t be an $(K, \mathcal{K})$-valued process defined on $(\Omega, \mathcal{F})$, and suppose that for all $t \geqslant 0$ and all $\omega \in \Omega$, there exists a strictly positive real number $\epsilon(t,\omega)$ such that*

$$Y_{t+s}(\omega) = Y_t(\omega) \quad \textit{on}\, [t,\, t+\epsilon(t,\omega)].$$

Then the history $\mathcal{F}_t^Y$ is right-continuous.

Appendix B

Likelihood Ratios: Changes of Intensity "à la Girsanov"

We examine the relation between a certain type of absolutely continuous change of probability measures and the change of intensity that it induces in the point processes case. Such changes of probability are quite general.

B.1 Likelihood ratios and intensity changes

We first describe a particular stochastic process, namely the so called fundamental martingale.

Theorem B.1. *Let* $(N_t(1), N_t(2), ..N_t(k))$ *be a* k*-variate point process adapted to some history* $\mathcal{F}_t$*, and let* $\lambda_t(i)$*,* $1 \leqslant i \leqslant k$*, be a predictable* $(\mathbb{P}, \mathcal{F}_t)$*-intensities of* $N_t(i)$*,* $1 \leqslant i \leqslant k$*, respectively. Let* $\mu_t(i)$*,* $1 \leqslant i \leqslant k$*, be a* $\mathcal{F}_t$*-predictable processes, nonnegative, and such that for all* $t \geqslant 0$ *and all* $1 \leqslant i \leqslant k$

$$\int_0^t \mu_s(i)\lambda_s(i)ds < \infty \quad \mathbb{P} - a.s.$$

Define the process L_t *by*

$$L_t = \prod_{i=1}^{k} L_t(i),$$

where

$$L_t(i) = \begin{cases} \exp\left\{ \int_0^t (1 - \mu_s(i))\lambda_s(i)ds \right\} & \text{if } t < T_1(i) \\ \left(\prod_{n \geqslant 1} \mu_{T_n}(i) \mathbb{1}_{T_n(i) \leqslant t} \right) \exp\left\{ \int_0^t (1 - \mu_s(i))\lambda_s(i)ds \right\} & \text{if } t \geqslant T_1(i), \end{cases}$$

or equivalently,

$$L_t(i) = \Big(\prod_{n\geqslant 1} \mu_{T_n}(i)\mathbb{1}_{T_n(i)\leqslant t}\Big) \exp\Big\{\int_0^t (1-\mu_s(i))\lambda_s(i)ds\Big\}.$$

Then $L_t(i)$ is a $(\mathbb{P}, \mathcal{F}_t)$-nonnegative local martingale and a $(\mathbb{P}, \mathcal{F}_t)$-super-martingale.

In particular for all $1 \leqslant i \leqslant k$, $L_t(i)$ is a $(\mathbb{P}, \mathcal{F}_t)$-local martingale and a $(\mathbb{P}, \mathcal{F}_t)$-super-martingale.

Theorem B.2. *Direct Radon-Nikodym derivative theorem (see [4], [5])*
We use the same notations as in Theorem B.1. Moreover we suppose that

$$\mathbb{E}\left[L_1\right] = 1. \tag{B.1}$$

Define the probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = L_1.$$

Then, for each $1 \leqslant i \leqslant k$, $N_t(i)$ has the $(\tilde{\mathbb{P}}, \mathcal{F}_t)$-intensity $\tilde{\lambda}_t(i) = \mu_t(i)\lambda_t(i)$ over [0, 1].

Condition (B.1) holds true under the following requirements:

Theorem B.3. *We use the same notation as Theorem B.1. Moreover we suppose that for $1 \leqslant i \leqslant k$, $\lambda_t(i) = 1$ and $\mu_t(i)$ is bounded. Then $\mathbb{E}\left[L_1\right] = 1$.*

The last theorem, combined with Theorem B.2, can be interpreted as an existence theorem. Actually, given the existence of a probability measure $\mathbb{P}$ for which N_t is a $(\mathcal{F}_t)$-Poisson, we can construct, for any $\mathcal{F}_t$-predictable nonnegative bounded process μ_t, a probability $\tilde{\mathbb{P}}$ for which N_t has the $(\tilde{\mathbb{P}}, \mathcal{F}_t)$-intensity μ_t (take $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = L_1$).

We present now an example, for which the hypotheses of Theorem B.3 hold.

Example B.1. *Queueing Processes*
A (simple) queueing process Q_t is a $\mathbb{N}_+$-valued process, defined on some $(\Omega, \mathcal{F}, \mathbb{P})$, and the form

$$Q_t = Q_0 + A_t - D_t$$

where A_t and D_t are $\mathbb{P}$-nonexplosive point processes without common jumps. We observe that $\mathbb{P}$-a.s., $D_t \leqslant Q_0 + A_t$, $t \geqslant 0$, since Q_t is nonnegative. We call Q_t the state process and Q_0 the initial state. For each $t \geqslant 0$, we interpret the random variable Q_t as as the number of customers waiting in line, A_t as the number of arrivals in $(0,\ t]$, and D_t as the number of departures in $(0,\ t]$. A simple queue Q_t can be viewed as a bivariate point process (A_t, D_t), where A_t and D_t are the input and output processes respectively. Suppose that under the probability $\mathbb{P}$, Q_t is M/M/1 with $\mathcal{F}_t$-parameters λ and μ. Let now λ_t and μ_t be two bounded nonegative $\mathcal{F}_t$-predictable processes. Then by Theorem B.2 and Theorem B.3, we have:

$$\begin{aligned} L_t &= \exp\Big\{ \int_0^t log\left\{\frac{\lambda_s}{\lambda}\right\} dA_s - \int_0^t (\lambda_s - \lambda) ds \Big\} \\ &\times \exp\Big\{ \int_0^t log\left\{\frac{\mu_s}{\mu}\right\} dD_s - \int_0^t (\mu_s - \mu) \mathbb{1}_{Q_s>0}\, ds \Big\} \end{aligned}$$

is a $(\mathbb{P}, \mathcal{F}_t)$-martingale, and if we define $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} = L_1$, Q_t is a queue with the $(\tilde{\mathbb{P}}, \mathcal{F}_t)$-parameters λ_t and μ_t over [0, 1].

We consider then the more general case of a marked point process with the $(\mathbb{P}, \mathcal{F}_t)$-local characteristics $(\lambda_t, \phi_t(dy))$ (see Remark 1.1).

Theorem B.4. *(see [2][6]) Let $p(dt\,dy)$ be a K-marked point process with the $(\mathbb{P}, \mathcal{F}_t)$-local characteristics $(\lambda_t, \phi_t(dy))$. Let μ_t be a nonnegative $\mathcal{F}_t$-predictable process, and let $h(t, y)$ be an $\mathcal{F}_t$-predictable K-indexed nonnegative process, such that*

$$\begin{aligned} \int_0^t \mu_s \lambda_s ds < \infty \quad \mathbb{P}-a.s. \quad t \geqslant 0 \\ \int_K \mu_y h(t, y) \phi_t(dy) = 1 \quad \mathbb{P}-a.s. \quad t \geqslant 0 \end{aligned} \tag{B.2}$$

Define for each $t \geqslant 0$

$$L_t = L_0 \Big(\prod_{n \geqslant 1} \mu_{T_n} h(T_n, Y_n) \mathbb{1}_{T_n \leqslant t} \Big) \cdot \exp\Big\{ \int_0^t \int_K (1 - \mu_s h(s, y)) \lambda_s \phi_s(dy) ds \Big\},$$

where L_0 is a nonnegative $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}[L_0] = 1$ (as usual, the product $\prod_{n=1}$ is taken to be 1 if $T_1 > t$). Then:

(a) L_t is a nonnegative $(\mathbb{P}, \mathcal{F}_t)$-local martingale and a nonnegative $(\mathbb{P}, \mathcal{F}_t)$-supermartingale.

(b) If $\mathbb{E}[L_1] = 1$, L_t is a $(\mathbb{P}, \mathcal{F}_t)$-martingale over [0, 1]. Defining the probability $\tilde{\mathbb{P}}$ by $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = L_1$, $p(dt\,dy)$ admits over [0, 1] the $(\tilde{\mathbb{P}}, \mathcal{F}_t)$-local characteristics $(\mu_t \lambda_t, h(t, y) \phi_t(dy))$.

We give a sufficient condition for $\mathbb{E}[L_1] = 1$ in this more general case.

Theorem B.5. *[2] Let the conditions* (B.2) *of Theorem B.4 prevail, and suppose in addition that there exists a deterministic increasing real-valued function $B(t)$ and finite constants K_1, K_2 and α, where $\alpha > 1$, such that for all $t \in [0, 1]$,*

$$\int_K (\mu_s h(t, y))^\alpha \phi_t(dy) \leqslant K_1 + K_2 \Big(N_t + \int_0^t \lambda_s ds \Big) \quad \mathbb{P}-a.s. \tag{B.3}$$

$$\int_0^t \lambda_s ds \leqslant B(t) \quad \mathbb{P} - a.s. \tag{B.4}$$

Suppose moreover that for all $0 < M < \infty$

$$\mathbb{E}\left[L_0 \exp\{MN_1\}\right] < \infty. \tag{B.5}$$

Then $\mathbb{E}\left[L_1\right] = 1$.

Example B.2. Take $\lambda_t = \lambda(t)$, where $\lambda(t)$ is locally integrable and deterministic, and $\phi_t(\omega, dy) = F(dy)$. [Thus, under $\mathbb{P}$, $p(dtdy)$ is a marked Poisson process]. The conditions (B.4) and (B.5) are then automatically satisfied. For the condition (B.5) for instance, N_1 being independent of $\mathcal{F}_0$, $\mathbb{E}\left[L_0 \exp\{MN_1\}\right] = \mathbb{E}\left[L_0 \mathbb{E}\left[\exp\{MN_1\}\right]\right] = \mathbb{E}\left[\exp\{MN_1\}\right] < \infty$, the last inequality being a consequence of the fact that N_1 is a Poisson random variable.

B.2 An Existence Theorem

The existence of an optimal control is not always granted; we present an existence theorem that follows a method invented by Beneš (see [4]) for Wiener-driven stochastic systems and which is based on compactness arguments of Radon-Nikodym derivatives. In particular we consider this theory in the simple case of an univariate process.

Let hence N_t be an univariate point process defined on $(\Omega, \mathcal{F})$, and let U be the set of admissible controls consisting of those processes u_t that are $\mathcal{F}_t^N$-predictable and satisfy, for some positive real constant K,

$$0 \leqslant u_t \leqslant K.$$

We assume the terminal time $T = 1$. The dynamics $(\mathbb{P}_u, u \in U)$ are such that there exists a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that N_t is a $(\mathbb{P}, \mathcal{F}_t^N)$-Poisson process with intensity 1, and for each $u \in U$

$$\frac{d\mathbb{P}_u}{d\mathbb{P}} = L_1(u) = L(u),$$

where

$$L_t(u) = \Big(\prod_{n \geqslant 1} u_{T_n} \mathbb{1}_{T_n \leqslant t}\Big) \exp\Big\{ \int_0^t (1 - u_s) ds \Big\}.$$

In other words we are saying that N_t admits the $(\mathbb{P}_u, \mathcal{F}_t^N)$-intensity u_t. Now we consider ϕ as a random variable on $(\Omega, \mathcal{F})$ which is $\mathbb{P}$-square-integrable:

$$\mathbb{E}\left[|\phi|^2\right] < \infty.$$

Then ϕ is $\mathbb{P}_u$-integrable for all $u \in U$. Finally, defining

$$J(u) = \mathbb{E}_u\left[\phi\right], \quad J^* = \inf_{u \in U} J(u).$$

we have the requested existence result:

Theorem B.6. *There exists at least one control $u^* \in U$ such that $J(u^*) = J^*$.*

Proof. See [10].

Appendix C

Ito's formula for finite-variation processes

We will present the Itô's formula for finite variation (FV) functions, mainly following the discussion in [35], Chapter IV. We consider classical (deterministic) functions on $[0, \infty)$. Let x (mapping t to $x(t)$) be a FV function on $[0, \infty)$. We define

$$\Delta x_s = x_s - x_{s-}, \quad s > 0.$$

We observe that the function x induces a signed Stieltjes measure $\mu_x(u, v]) = x_v - x_u$. The measure μ_x has an atom Δx_s at any point $s > 0$, where $\Delta x_s \neq 0$. We can decompose x into the sum of its continuous and atomic parts as follows:

$$x = x_0 + x^c + x^a, \quad x_t^a = \sum_{0<s\leqslant t} \Delta x_s.$$

Integration by parts. Let x and y be two FV functions on $[0, \infty)$. We can define the product signed measure $\mu_x \times \mu_y$ on $(0, \infty)$. Taking $\mu_x \times \mu_y$ measures in the above equation, we see that

$$(x_t - x_0)(y_t - y_0) = \int_{(0,\,t]} (x_{v-} - x_0) dy_v + \int_{(0,\,t]} (y_{u-} - y_0) dx_u + [x, y]_t$$

where

$$[x, y]_t \equiv \sum_{0<s\leqslant t} \Delta x_s \Delta y_s.$$

On rearranging, we obtain the *integration by parts formula*:

$$x_t y_t - x_0 y_0 = \int_{(0,\,t]} x_{s-} dy_s + \int_{(0,\,t]} y_{s-} dx_s + [x, y]_t$$

which we write in differential notation as

$$d(xy) = x_- dy + y_- dx + d[x, y].$$

Theorem C.1. *Itô's formula for FV function*
Let $f \in C^1(\mathbb{R})$, and let x be an FV function on $[0, \infty)$. Then

$$f(x-t)-f(x-0) = \int_{(0,\,t]} f'(x_{s-})dx_s + \sum_{0<s\leqslant t} \{f(x_s)-f(x_{s-})-f'(x_{s-})\Delta x_s\}. \quad \text{(C.1)}$$

If we write F_t for $f(x_t)$, then we may write (C.1) *in differential notations as*

$$dF_t = f'(x_{t-})dx_t + \{\Delta F_t - f'(x_{t-})\Delta x_t\}.$$

Remark C.1. On an interval $[0,\, t_0]$, x is bounded, and f' is bounded on the range of x. Hence

$$\sum_{0<s\leqslant t_0} |f(x_s) - f(x_{s-})| \leqslant K \sum_{0<s\leqslant t_0} |x_s - x_{s-}| < \infty$$

and

$$\sum_{0<s\leqslant t_0} |f'(x_{s-})\Delta x_s| \leqslant K \sum_{0<s\leqslant t_0} |\Delta x_s| < \infty.$$

Proof. Fix the FV function x. Let $\mathcal{A}$ be the family of functions in $C^1(\mathbb{R})$ such that (C.1) holds. Then $\mathcal{A}$ is clearly a vector space. But it is also true that $\mathcal{A}$ is an algebra. Let $f \in \mathcal{A}$, $g \in \mathcal{A}$ and put $h = fg$. Write $F_t = f(x_t)$, $G_t = g(x_t)$ and $H_t = h(x_t)$. Then

$$\begin{aligned} dF &= f'(x_-)dx + \{\Delta F - f'(x_-)\Delta x\}, \\ dG &= g'(x_-)dx + \{\Delta G - g'(x_-)\Delta x\}. \end{aligned}$$

By the integration by parts formula,

$$dH = F_-dG + G_-dF + d[F,\, G], \quad \text{(C.2)}$$

and considerations of the atoms in (C.2) tells us that

$$\Delta H = F_-\Delta G + G_-\Delta F + \Delta F \Delta G.$$

So, from (C.2) we obtain

$$\begin{aligned} dH &= \{f(x_-)g'(x_-) + g(x_-)f'(x_-)\}dx + \{F_-\Delta G + G_-\Delta F + \Delta F\Delta G - h'(x_-)\Delta x\} \\ &= h'(x_-) + \{\Delta H - h'(x_-)\Delta x\}, \end{aligned}$$

so that $h \in \mathcal{A}$. Since it is trivial that $\mathcal{A}$ contains the function $f(x) = x$,it follows that $\mathcal{A}$ contains all polynomials.

Now let f be any element of $C^1(\mathbb{R})$. It is enough to take $t_0 \in (0, \infty)$ and prove (C.1) for all $t \leqslant t_0$. Now for some N, $x(t) \in [-N. N]$ for all $t \leqslant t_0$. Moreover, we can find polynomials p_n such that

$$\begin{array}{rcll} p_n & \to & f & \text{uniformly on} \quad [-N,\, N], \\ p_n' & \to & f' & \text{uniformly on} \quad [-N,\, N]. \end{array}$$

Since (C.1) is true for each p_n, it is now trivial that,for $t \leqslant t_0$, (C.1) is true for f. $\square$

Theorem C.1 can be generalized in n-dimensions as follows.

Theorem C.2. *n-dimensional Itô's formula*
Suppose that

$$x = (x^1, x^2, .., x^n)$$

is an $\mathbb{R}^n$-valued function, each component of which is an FV function on $[0, \infty)$, *Let f be a function on $\mathbb{R}^n$ with continuous first-order partial derivatives $D_i f$. Then*

$$f(x_t) - f(x_0) = \int_{(0,\, t]} D_i f(x_{s-}) dx_s^i + \sum_{0<s\leqslant t} \{f(x_s) - f(x_{s-}) - D_i f(x_{s-}) \Delta x_s^i\}.$$

where we sum over the repeated index i is an expression such as $D_i f(x_{s-}) dx_s^i$.

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